

Game of Cops and Robbers on Eulerian Digraphs

by

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Abstract

Cops and Robbers is a well-known pursuit game played on a graph. There are two players, one controls the cops and the other controls the robber, who take turns moving along edges of the graph. The goal of the cops is to capture the robber, which is accomplished if a cop occupies the same vertex as the robber. The main question is to determine the minimum number of cops that can guarantee the robber's capture on the given graph. This problem has been widely studied for the case of undirected graphs, but very little attention has been given to finding the cop number of digraphs. In the thesis we focus on this game on Eulerian digraphs, viewed as an extension of the game on undirected graphs. Some preliminary results, which were obtained for the special case of 4-regular quadrangulations of the torus and the Klein bottle, show that there is a possibility to develop rich results in this area.

Keywords: Cops and Robbers, Directed Graphs, Eulerian Digraphs

Dedication

To my beloved wife, Maryam.

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Chapter 1

Introduction

Cops and Robbers is one of the most well-known games on graphs which was first introduced by Nowakowski and Winkler [28] and Quilliot [30]. There are several versions of this game and the most famous one (and the one that we are interested in) is as follows.

There are two players, one of them will control the cops and the other one will control the robber. The game is played on the vertices of a graph G . At the beginning of the game each cop will choose a vertex as his *initial position* and then the robber will choose his position. In each further step of the game, first each cop *moves*, where to move means either staying at the same position or changing the position to a neighbor of its current position. After that, the robber moves. Several cops can occupy the same position at any time. This is a perfect information game, meaning that each player has full knowledge about the playground (G) and all possible moves of himself and the other player.

The cops win the game if, for every strategy of the robber of selecting the initial position and his moves, they *catch* the robber, meaning that after their move some cop has the same position as the robber. Otherwise the robber wins the game.

The smallest number k of cops for which the cops win the game is called the *cop number* of G and is denoted by $c(G)$. Also when $c(G) = k$, then G is *k-cop-win*.

Let G be a finite graph with k cops moved to block the robber's access to a set B of vertices. We define the *robber territory* to be the set of vertices that the robber can reach without begin captured by any of the cops blocking B [1, 31].

If we place a cop on each vertex of G , then the robber will be caught at the very beginning of the game. Therefore $c(G) \leq n$, where n is the number of vertices of G . In a graph G , a set of vertices S is *dominating* if every vertex of G not in S is adjacent to some vertex in S . The *domination number* of a graph G is the minimum cardinality of a dominating set in G . Observe that $c(G)$ is upper bounded by the domination number of G ; however, this bound is far from tight. For example, trees are 1-cop-win but can have domination number that is linear in their order (this is the case for paths).

Aigner and Fromme [1] first considered the class of graph with cop number greater than one and got the following important and inspiring theorem.

Theorem 1.1 ([1]). *If G is a planar graph, then $c(G) \leq 3$.*

In order to prove this theorem, they proved the following lemma which is one of the main tools in studying the game of Cops and Robbers in graphs. We say that a set of cops *guards* a subgraph $S \subseteq G$ if (after some number of moves) whenever the robber enters S , he will be caught by one of the cops.

Lemma 1.2 ([1]). *If P is an isometric path in G , then one cop can guard P .*

Meyniel's conjecture is considered to be the main open problem in this area.

Conjecture 1.3 (Meyniel's conjecture). *If G is a graph of order n , then*

$$c(G) = O(\sqrt{n}).$$

Note that there exist graphs with $c(G) = \Theta(\sqrt{n})$. In order to see this, we will need the following theorem.

Theorem 1.4 ([16]). *Suppose the minimum degree of G is greater than d and its girth is at least $8t - 3$. Then*

$$c(G) > d^t.$$

When $t = 1$ (girth is 5), one cop cannot guard more than one neighbour of the robber's position. Therefore, we will need at least $\delta(G)$ cops to catch the robber.

Proposition 1.5. *The cop number of the incidence graph of a projective plane on n vertices is $\Theta(\sqrt{n})$.*

Proof. Note that the incidence graph of a projective plane of order k is a $(k + 1)$ -regular bipartite graph which has $n = 2(k^2 + k + 1)$ vertices. And since the girth of this graph G is 6 (≥ 5), the cop number of G is at least the minimum degree, which is $c(G) \geq k + 1$. It is not difficult to see that $c\sqrt{n}$ cops can actually catch the robber. Therefore $c(G) = \Theta(\sqrt{n})$. \square

The best known result on general undirected graphs is due to Lu and Peng [25] (and independently Scott and Sudakov [32]). They proved the following theorem by using random positioning of cops.

Theorem 1.6. *The cop number of any connected graph on n vertices is at most*

$$n2^{-(1+o(1))\sqrt{\log n}}.$$

Note that this bound is still far from *weak Meyniel's Conjecture*.

Conjecture 1.7 (Weak Meyniel’s conjecture). *There exist an $\varepsilon > 0$ such that every graph G of order n , has*

$$c(G) = O(n^{1-\varepsilon}).$$

Random graphs have been considered in [6, 29] and Meyniel’s conjecture has been proved for this class of graphs.

Characterization of k -cop-win graphs [13, 10] and digraphs [18], topological directions [11], computational complexity of the game [22] and capture time [8] are some other directions that recently received lots of attentions.

1.1 Directed Graphs

The game of Cops and Robbers can be played on directed graphs in an analogous way with the exception that we must move in the direction of oriented edges. As the first observation, note that if the robber occupies a source (which has no cop on it), then the cops cannot reach him and the robber wins the game. Therefore the number of sources of a digraph is a lower bound for its cop number. In order to avoid these difficulties we will only consider strongly connected digraphs.

Consider a digraph D obtained from a graph G by replacing each edge of G by a symmetric pair of inversely directed arcs. Playing the game on D is equivalent to playing the game on G . Note that D is Eulerian and hence playing the game on Eulerian digraphs is a natural generalization. We will mainly focus on Eulerian digraphs in this thesis.

One of the first results on the cop number of directed graphs is due to Hamidoune [19]. He considered the game on Abelian Cayley digraphs. It is proved that if the generating set has size d , then $d + 1$ cops are sufficient to catch the robber. This bound is often best possible.

One of the main challenges in directed graphs is the fact that Lemma 1.2 cannot be used. As mentioned before, this lemma is the base of several important results in undirected graphs and most of those results cannot (at least not easy to) be generalized.

In [24], it was proved that $c(D) = O(\sqrt{n})$ if D is strongly connected and planar. To prove this they have used the Separator Theorem of Lipton and Tarjan [23]. The same result holds for directed graphs of bounded genus.

The best known result for strongly connected directed graphs uses the probabilistic method and shows that [17]

$$c(D) = O\left(\frac{n(\log \log n)^2}{\log n}\right).$$

Another natural question that someone might ask is the relation between the cop number of a digraph and its underlying graph. It is easy to see that the cop number of a complete

graph is one. In [34], tournaments (orientated version of a complete graph) have been studied and they showed the following interesting result.

Theorem 1.8 ([34]). *For each $k \in \mathbb{N}$ there exists a tournament G with $c(G) > k$.*

The proof in [34] can be improved to $k = \log_e n$. Also it is easy to see that $\lceil \log_2 n \rceil$ is an upper bound for the dominating number of a tournament. So, we have the following corollary.

Corollary 1.9. *There exists a tournament G that $c(G) = \Theta(\log n)$.*

On the other hand, consider a graph G with $c(G) = k$. The following result shows that we can orient edges of G to obtain a digraph D in a way that $c(D) \leq 2$.

Proposition 1.10. *Orienting edges of a graph can arbitrarily decrease its cop number.*

Proof. Consider a graph G with $c(G) = k$. Let $v \in V(G)$ and orient all the edges away from it (one can consider a breadth-first search ordering to do this). Observe that the obtained digraph has no directed cycle and by starting one cop on v we can capture the robber. But this orientation is not strongly connected.

There are some sinks in this digraph. Obtain D by adding a long directed path from each of these sinks to v . The length of each of these long directed paths is sufficient to be more than the diameter of G . Note that D is strongly connected.

First note that $c(D) \leq 2$; place two cops on v and move one of them towards the robber. If the robber does not use any of these long paths, he will be caught as in the previous case. If he uses these long paths, since one of the cops is forcing him to move and there is another cop on v , he is forced to go to v or get captured. This shows that $c(D) \leq 2$.

Since we have added some new edges, we need to show that the underlying graph of D , G' , has the same cop number, k . First note that $c(G') \geq k$ because the robber can avoid using edges of long paths and escape from $k - 1$ cops. Observe that using the edges of long paths will not be beneficial to the cops (because they are too long and the robber will have enough time to adjust his position). To see that $c(G') \leq k$ note that if the robber does not use any of newly added edges (edges of long paths), then k cops (by playing the winning strategy on G) can win the game. Therefore in order to escape from k cops the robber needs to use these long paths (going into one of these paths and coming back will be a waste of time for the robber, so he might only use these paths to get to the other side). Since the length of these paths are more than the diameter of G , any of the cops will have time to get himself to the other side of the path before the robber. Therefore the robber cannot get to the other side of these long paths and using these long paths will be helpful for him. So, he k cops can eventually capture him. \square

Note that by using this technique (adding some directed long paths), we can show that this result holds when we want D to be an Eulerian digraph as well.

1.2 Goals and Outline

In Chapter 2, we will show that the game of cops and robbers on directed graphs can be inspiring and challenging. We will analyze some natural orientations of 4-regular quadrangulations. Interestingly, in one of these cases, three cops cannot capture the robber. Results of this chapter have been published in [21].

In Chapter 3, we will consider some more general orientations of 4-regular quadrangulations. We show another striking fact that the cop number of arbitrary “straight-ahead orientations” of 4-regular quadrangulations is bounded (the proved upper bound is 404). This provides evidence that studying the game in (Eulerian) digraphs is of sufficient significance. Results of this chapter is a joint work with Sebastian Gonzalez Hermosillo de la Maza, Fiachra Knox, Bruce Reed and Bojan Mohar [14].

Graphs of bounded diameter is considered in Chapter 4. We will improve the upper bound on the cop number of these graphs. This chapter is the result of a joint work with Fiachra Knox and Bojan Mohar.

In Chapter 5, we will study the affect of some graph operations on the cop number of graphs. The main result of this chapter shows that the cop number of graphs with bounded degree is not bounded. Also we show that for every ε and large enough n , there are sub-cubic graphs on n vertices with cop-number $O(n^{1/2-\varepsilon})$.

In Chapter 6, smallest k -cop-win graphs is studied. We show that the order of smallest k -cop-win graphs is monotone increasing. This confirms Bonato’s conjecture and the result is published in [20]. Some properties of the smallest planar graph with cop number equal to three is also determined.

Chapter 2

Oriented Grid I

2.1 Introduction

The main goal of this Chapter is to show that the game of Cops and Robbers in directed graphs (digraphs) can be as natural and as inspiring as the game on undirected graphs. It is known (see [12]) that the cop number of any (connected) planar graph is at most 3. It is also known that it is at most 4 for every graph embedded in the torus (and is currently not known whether four cops are ever needed). More generally, the cop number is bounded for graphs of bounded genus and, in fact, is bounded in any proper minor-closed family of graphs (see [4]). A natural question arises:

Problem 2.1. *Is the cop number of planar Eulerian digraphs bounded by a constant?*

The same question can be asked for Eulerian digraphs of bounded genus (Here we restrict our attention to Eulerian digraphs since the game on any undirected graph G is equivalent to the game on the Eulerian digraph obtained from G by replacing each edge with a pair of oppositely oriented arcs joining the same pair of vertices.). While the main tool (that of “guarding a geodesic path”, see [12]) used for undirected graphs is no longer available for digraphs, there is some hope for Problem 2.1 to have positive answer. As we show in this paper, the game can be analyzed on arbitrary 4-regular quadrangulations of the torus and the Klein bottle, at least when some “regularity” about orientations of the edges is assumed. In all treated cases, the cop number is at most 4 (see Theorems 2.3, 2.4, and 2.5) and four cops are necessary for one kind of orientation (Theorem 2.5).

2.1.1 Quotients of vertex-transitive orientations of the integer grid

Consider a 4-regular quadrangulation of a surface. It follows by Euler’s formula that the surface is either the torus or the Klein bottle and it can be shown by using the Gauss-Bonnet Theorem that the SAW (straight-ahead walks) partition the edges into cycles, all of which are noncontractible on the surface. These cycles can be split into two classes, each

class consisting of pairwise disjoint cycles (we call them *vertical cycles* and *horizontal cycles*, respectively) such that each vertical and each horizontal cycle intersect (possibly more than once). By giving each of these cycles an orientation, we obtain an Eulerian digraph in which, at each vertex, the two incoming edges and two outgoing edges are consecutive in the local rotation around the vertex. This kind of orienting the edges is said to be of *type (1)*. We will also consider *type (2)* orientations, where at each vertex, the two incoming and the two outgoing edges are opposite to each other in the local rotation. See Figure 2.1. Under this orientation, each facial quadrangle is a directed 4-cycle.

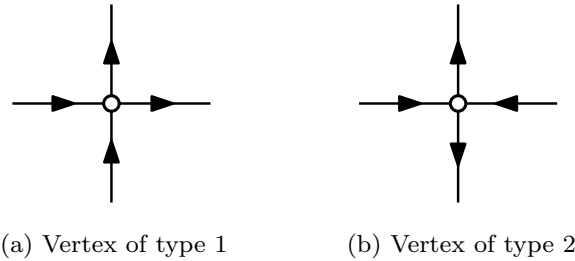


Figure 2.1: Type 1 and type 2 orientation around a vertex.

The universal cover of a 4-regular quadrangulation is the 4-regular tessellation of the plane with square faces (the integer grid), and every finite quotient of the integer grid is a 4-regular quadrangulation of the torus or the Klein bottle. An orientation of the edges of such quadrangulations is said to be *special* if its lift to the universal cover gives a vertex-transitive digraph. It is not hard to see that this means one of the cases shown in Figure 2.2. They are classified being of *type (1)* (subtypes (1a), (1b), (1c)) or (2), as indicated in the figure.

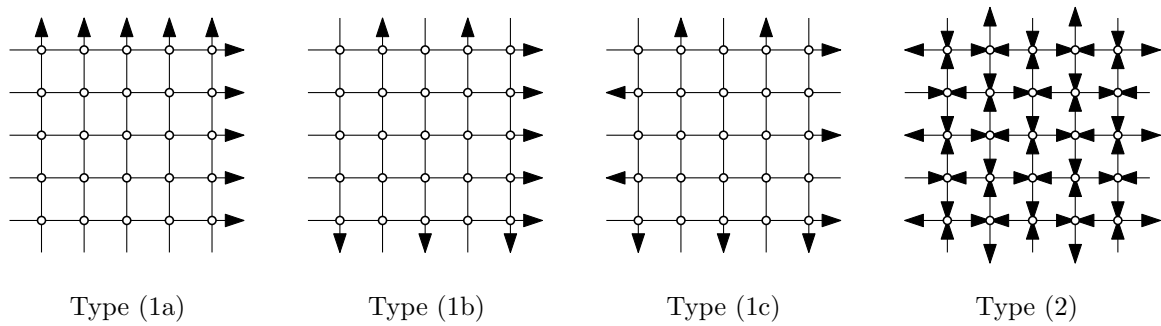


Figure 2.2: Vertex-transitive orientations of the integer grid.

Four-regular quadrangulations of the torus admit a simple description. Each such quadrangulation is of the form $Q(r, s, t)$, where r, s, t are arbitrary positive integers, $0 \leq t < r$, and $Q(r, s, t)$ is obtained from the $(r + 1) \times (s + 1)$ grid with underlying graph $P_{r+1} \square P_{s+1}$ (the cartesian product of paths on $r + 1$ and $s + 1$ vertices) by identifying the “leftmost” path of length s with the “rightmost” one (to obtain a cylinder) and identifying the bottom

r -cycle of this cylinder with the top one after rotating the top clockwise for t edges. More precisely, the left-most bottom vertex of the path P_{r+1} is identified with the t^{th} vertex in the top-side path; See Figure 2.3. In other words, the quadrangulation $Q(r, s, t)$ is the quotient of the integer grid $\mathbb{Z} \square \mathbb{Z}$ determined by the equivalence relation generated by all pairs $(x, y) \sim (x + r, y)$ and $(x, y) \sim (x + t, y + s)$. See Figure 2.3. This classification can be derived by considering appropriate fundamental polygon of the universal cover (which is isomorphic to the tessellation of the plane with squares). In graph theory, this was observed by Altschuler [2]; several later works do the same (e.g. [33]). Quadrangulations of the Klein bottle are a bit more complicated (see [26], [27], [33], or [15]). While all toroidal quadrangulations $Q(r, s, t)$ are vertex-transitive maps, this is no longer true for the Klein bottle. For our purpose it will suffice to know that the orientable double cover of such a quadrangulation Q is of the form $Q(r, s, t)$ and since it is a double cover it has $|V(Q)| = \frac{1}{2}rs$.

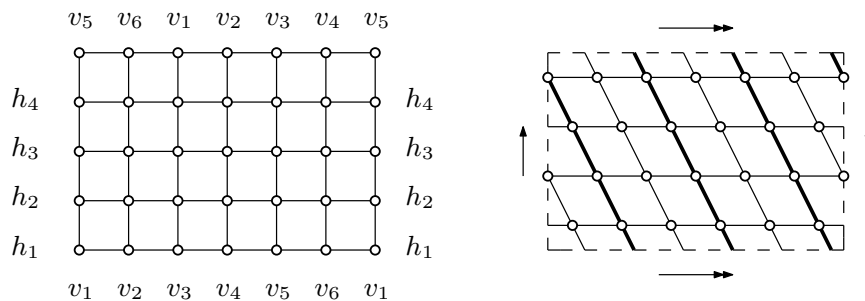


Figure 2.3: Toroidal quadrangulation $Q(6, 4, 2)$ and its drawing on the torus. One of its two vertical cycles is drawn with thick edges.

2.1.2 Some general strategies for Cops and Robbers

Here we describe two fundamental strategies that can be used on directed and undirected graphs when playing the game of Cops and Robbers.

Playing the game on a quotient. We will consider the game of k cops and a robber on Eulerian orientations of $Q(r, s, t)$. As discussed above, this covers the torus case. Any 4-regular Klein bottle quadrangulation has the orientable double cover which is a quadrangulation on the torus. Consequently, the Klein bottle case can be dealt with by the use of the following lemma. First note that a (di)graph \widehat{Q} is a *cover* over a (di)graph Q if there is a (di)graph homomorphism $\pi : \widehat{Q} \rightarrow Q$ (called the *covering projection*) which maps the edges incident with any vertex v bijectively onto the edges incident with $\pi(v)$ in Q .

Lemma 2.2. *Suppose that a graph or digraph \widehat{Q} is a cover over a (di)graph Q . If k cops have winning strategy on \widehat{Q} , then they also win on Q , that is $c(Q) \leq c(\widehat{Q})$.*

Proof. The proof is easy: Just project the winning strategy from \widehat{Q} to Q . The only thing to observe is that the starting position of R in Q lifts to \widehat{Q} in different ways, and we select

any of these to invoke the cops' strategy in \widehat{Q} , while from then on, every move of the robber in Q has a unique lift into \widehat{Q} . \square

Forcing the robber to move. In some strategies for the cops, we need to make sure that the robber does not stay at the same position. This is achieved as follows. We dedicate one of the cops to *force the robber to move*. The cop, say C , will choose a shortest path from $p(C)$ to $p(R)$. If the robber moves to a new position, then the cop moves along the path and thus keeps the same distance from R ; if R stays in the same position, then C moves along the path closer to R . Since the distance decreases every time when R does not move, R will stay at the same position at most $d - 1$ times (where d is the initial distance from $p(C)$ to $p(R)$) all together, or he will be caught. This scenario will be assumed whenever we take, as part of the strategy, to have one cop dedicated to force the robber to move. Note that the strategy works as long as one of the cops can reach the vertex $p(R)$. As our digraphs will always be connected and Eulerian (and hence strongly connected), this condition is clearly satisfied.

In the next two sections we will analyze the game on orientations of type (1) and (2) as depicted in Figure 2.2. In Section 2 we treat the orientations of type (1a), (1b) and (1c). It is shown that the cop number is always at most 3 in these cases (see Theorems 2.3 and 2.4). The orientation of type (2) is considered in Section 3. The main result here is Theorem 2.5 which shows an interesting dichotomy for this kind of orientation – the cop number is 3 or 4; the cop number 3 occurs if and only if either $s \leq 4$ or $s' = \gcd(r, t) \leq 4$ or when s and s' are both odd.

2.2 Orientations of type (1)

2.2.1 Horizontal cycles all oriented from left to right

Suppose that all horizontal cycles of $Q(r, s, t)$ are oriented from left to right, while the vertical cycles have orientations in any direction (these can be arbitrary). In particular, this covers both types (1a) and (1b).

We will call the horizontal cycles *rows* and the vertical cycles *columns*. Note that a column intersects each row precisely $r/\gcd(r, t)$ times and thus its length is $rs/\gcd(r, t)$. The minimum number of steps that is needed to catch the robber in a (di)graph G with k cops when both players play optimally is called *capture time* (for k cops).

Claim. *Three cops suffice to capture the robber and the capture time is $O(r(s + \log r))$.*

Proof. First, by using two cops we will get one of the cops in the robber's column. To achieve this, two cops will start in the same column. One of the cops (C_1) will stay in his position and the other one (C_2) will keep moving right. Since we are on a torus and the

grid is finite, after at most $O(r)$ moves either the robber will come to C_1 's column or C_2 will get to the robber's column.

Note that, after one of the cops and the robber are in the same column, the cop can always make the same move in the horizontal direction as the robber, so they will be in the same column from now on.

Using the other two cops, we can place another cop in robber's column. Note that the column is oriented cyclically, and therefore there exists an oriented path (or the whole cycle) from one of the cops to the other one which contains $p(R)$. We will call this path the *interval* of the robber. Let d ($0 \leq d \leq rs$) be the length of this interval. From now on:

1. If the robber moves horizontally, then the two cops will move horizontally as well. The direction of the interval of the robber may change, but d will stay unchanged.
2. If the robber moves vertically, then the cop who is in front of the robber will stay and the one who is behind the robber will move vertically as well, so the distance from the robber and the cop who is in front of him will decrease and the distance from the robber to the cop who is behind him stays the same, so the value of d will decrease.
3. And if the robber chooses to stay where he is, then the cop who is in front of him will stay in his position and the one who is behind him will move forward and get closer to the robber, again decreasing d .

Therefore, if the robber moves vertically d times, then he will be caught. So he can make at most rs vertical moves all together.

In order to catch him in $O(r \log(rs))$ additional "horizontal" steps, we bisect the interval of the robber by using the third cop C_3 . We position him in the row that will bisect the interval. After at most r horizontal steps, the robber will come to the position where $p(C_3)$ will bisect robber's original interval (Note that the interval may have shrunk because of some vertical moves of the robber. If this makes it shrink to a half, then we no longer need C_3 for bisection and we proceed with him to bisect the new interval). The interval of the robber will thus shrink by factor of 2. After reaching this, we can release one of the other two cops and use him to continue with bisection of the interval. The total number of steps is $O(rs + r \log(rs)) = O(r(s + \log r))$. \square

The analysis for lower bounds on the capture time would be more demanding since the results may depend on the orientations of vertical cycles. The previous analysis gives the following result.

Theorem 2.3. *Suppose that Q is an orientation of $Q(r, s, t)$ in which all horizontal cycles are oriented from left to right (while each vertical cycle has any straight-ahead orientation). Then $c(Q) \leq 3$. Moreover, if $Q(r, s, t)$ has no cycles of length at most four except for the facial quadrangles, then $c(Q) = 3$.*

Proof. The first part follows from the last claim above. It remains to argue that two cops do not suffice if $Q(r, s, t)$ has no nonfacial short cycles. Note that this assumption implies that $r \geq 5$, and if $s = 1$, then $r \geq 8$. In particular, Q has at least 8 vertices.

Suppose that we only have two cops, C_1 and C_2 . The strategy of R is as follows. Initially, R selects his position that is not occupied or attacked by any of the cops. Since $|Q| > 6$, such a position exists. During the game, R stays where he is if he is not attacked by one of the cops. Otherwise, suppose that $p(R)$ is an out-neighbor of $p(C_1)$. The assumption on short cycles implies that the two out-neighbors a, b of $p(R)$ are not attacked by C_1 . Also, a and b are diagonally opposite on a facial quadrangle of $Q(r, s, t)$. If C_2 would attack both of them, then we would have a cycle of length at most four and different from the facial quadrangle. Thus, R can move to a vertex which is not attacked by the cops. So, the two cops cannot win the game. \square

2.2.2 Vertical cycles all oriented upward

This can be dealt with in exactly the same way as in the previous subsection. Note that $Q(r, s, t)$ is isomorphic to $Q(r', s', t')$, where

$$r' = \frac{rs}{\gcd(r, t)} \quad \text{and} \quad s' = \gcd(r, t).$$

This isomorphism interchanges horizontal and vertical cycles, so three cops can catch the robber in $O(r'(s' + \log r'))$ steps.

2.2.3 Orientations of Type (1c)

In this case all edges in one row (or column) will have the same direction and the rows (columns) alternate their direction towards right and left (up and down). See Figure 2.4. In this figure, as well in the rest of them, we consider this to be a local picture in the universal cover of $Q(r, s, t)$, so that some of the vertices in the figure may actually represent the same vertex in $Q(r, s, t)$.

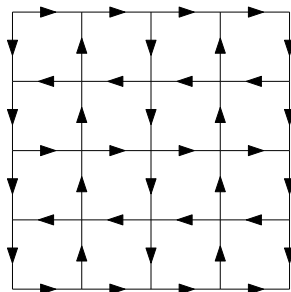


Figure 2.4: An orientation of type (1c).

Claim. *If $Q(r, s, t)$ has orientation of type (1c), then three cops can catch the robber.*

Proof. First note that if a cop can get himself to a position that is shown in Figure 2.5, then the robber cannot move or he will be caught. We will call this position a *trap* for the robber.

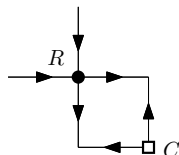


Figure 2.5: A trap, the robber cannot move anymore.

Consider Figure 2.6. If we have two cops in positions S_1 and S_3 , respectively, then as soon as R moves, he will be in a trap: if he moves up, then S_1 will move left and the robber will be in a trap; and if he moves left, then S_3 will move up and there will be another trap. Note that if our two cops are in S_2 and S_4 , then the cops will do what the robber does in the first move and after that the robber can not move or he will be in a trap (same as above). We will call S_1, S_2, S_3 and S_4 the *shadows* of R , because they can do whatever R does and can follow him and thus stay in the shadow of the current position of the robber.

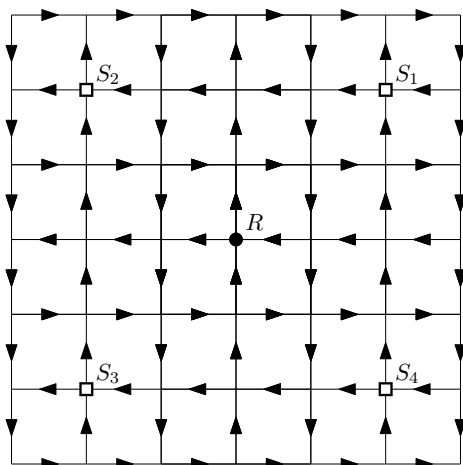


Figure 2.6: Shadows of R .

Now we define further shadows of R as follows. We start with $S^0 = \{p(R)\}$. For $i = 0, 1, 2, \dots$, let S^{i+1} be the vertices in the shadows of vertices in S^i (not containing those that are in some S^j for $j \leq i$). Using the following strategy involving three cops, we can place one of them in some S^k ($k \geq 0$) at the beginning of the game. We say that we *capture the shadow* of the robber.

By placing two cops, C_1 and C_2 , as shown in Figure 2.7, they can cover the vertices shown by empty circles by either moving onto them or having them in a trap. It is clear that there is always at least one shadow vertex among these vertices. If the robber moves, then the shadow vertices move accordingly. If one comes to the position of C_1 or C_2 or to

one of its out-neighbors, then we can stay or make a move to come to a shadow vertex. If this is not the case, then the shadow is among one of the two vertices that are trapped by C_1 or C_2 . Now we can use the third cop to force the robber to move (or to catch his shadow) and thus we are able to get C_1 or C_2 into the shadow.

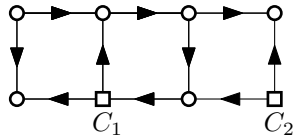


Figure 2.7: The starting position of cops to capture a shadow.

Now assume that one of the cops, say C_1 , is in a shadow $s_k \in S^k$, for some $k \geq 0$. We are done if $k = 0$, so we may assume that $k \geq 1$. One of the shadows of s_k is in S^{k-1} , call it s_{k-1} . Without loss of generality we can assume that it is the vertex two squares to the right and two squares above. As explained above, with possible exception of the first move, whenever the robber moves down or left, then so moves its shadow s_{k-1} and C_1 can get s_{k-1} into a trap and therefore capture s_{k-1} in the next move. Otherwise C_1 will move with the robber to stay in S^k until another cop gets into S^{k-1} .

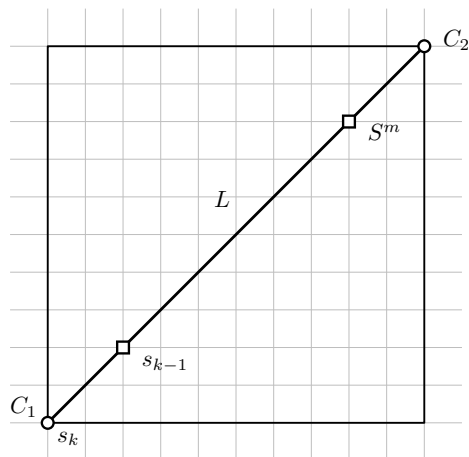


Figure 2.8: The line that connects s_k to s_{k-1} .

Consider the diagonal line L that connects s_k to s_{k-1} . (Note that this line may “wrap around” the torus handle more than once, but we consider its lift in the universal cover, where it looks like shown in the figures.) We want to get another cop on this line at an even row distance. To do this, we move C_1 as described above and at the same time we get the two remaining cops in the same column and in two consequent rows such the upper one can move left and the lower one can move right. Moving left and right (respectively), one of these two cops will get on L and if the robber stays where he is, then both cops will get it. It is clear that if both of them arrive onto the line L , then one of them will be at even row distance from s_k . If the cop, C_2 , gets on the line and he has even row distance

from s_k , then we are done and we will move into the next stage of the strategy that will be presented below. So we may assume that C_2 will get to the line at an odd row distance from s_k . Moreover, since the third cop has not arrived to L , the robber has been moving, i.e., has not stayed at his position.

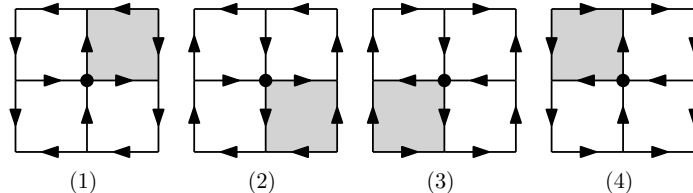


Figure 2.9: Four types of vertices.

We have four types of vertices as indicated in Figure 2.9. Note that C_2 has not arrived to a vertex of type (1), because this would mean that the robber is at a vertex of type (3) which can only move down or left. Recall that we already argued that the robber is forced to move right or up, otherwise C_1 will get to S^{k-1} . If C_2 is at a vertex of type (3), then a vertex on L at even row distance from s_k is in his trap, so he can get himself into an even row distance from s_k . The arguments for type (2) and (4) are the same and it is clear that the cop who is moving right can not get himself in a vertex of type (4). So without loss of generality we may assume that the cop who is moving right gets to the line L at a vertex of type (2). Let us consider the last moves of C_2 . There are two cases here which we shall consider. Note that C_2 is moving right, so he will alternate his position between vertices of type (1) and (2).

Case 1. C_2 is at a vertex of type (1) and moves right and will get himself on the line at a vertex of type (2). In this case, before doing the last move, C_2 was at distance 1 from L .

Case 2. C_2 is at a vertex of type (1) and moves right and will get himself at a vertex of type (2) which is not on L but the robber moves in a way that C_2 will get on the line. In this case before doing the last move, C_2 was at distance 2 from L .

In each of these cases (when C_2 is at distance 1 or 2 from L), C_2 will stay at a vertex of type (1). If the robber moves in a way that C_2 gets on the line, then he is at a vertex of type (1) and we are done. If the robber moves such that the distance between C_2 and the line becomes greater than 2, then C_2 will start moving right again. Since the third cop is moving left and becomes closer to the line, the robber can not stay where he is and can not move away from C_2 forever.

The above strategy will end up by catching L at an even row distance vertex from S^k or in one of the two cases shown in Figure 2.10.

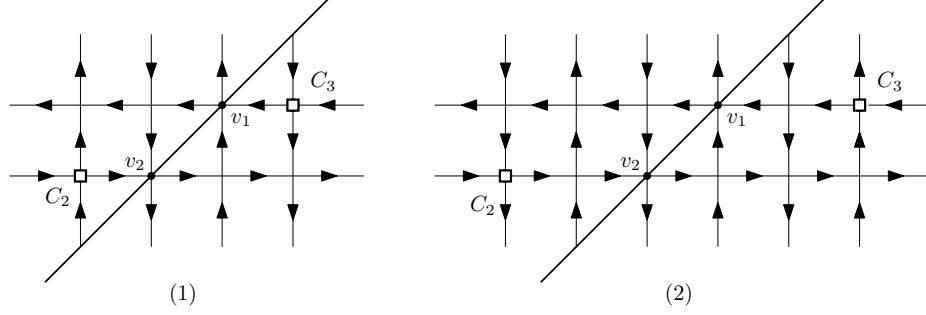


Figure 2.10: Two cases of ending the strategy.

Case 1. One of v_1 or v_2 is at even row distance from s_k , so one of the cops can catch the one which is at even distance from s_k .

Case 2. If v_1 is at even row distance from s_k , then since the robber can not move down or left and he is at a vertex of the same type as v_1 , he will move up or will stay where he is. Now C_3 can move left. If the robber does not move, then C_3 will catch v_1 as desired and if the robber moves up, then by moving left one more time we will be in case 1. If v_2 is at even row distance from s_k , then the robber is forced to move right. In this case, C_2 can move right. If the robber stays where he is, then C_2 will catch v_2 and we are done. And if the robber moves right, then C_2 will move right one more time and we will be in case 1.

Now we can assume that we are in the situation indicated in Figure 2.8. If the robber moves down or left, then C_1 will catch s_{k-1} and if he moves up or right, then C_2 will get to a shadow vertex in some S^m . Now we use the third cop to force the robber to move and therefore after some step we will have a cop in $S^{k'}$ where $k' < k$. Continuing this strategy, we will get to S^0 and catch the robber. Therefore three cops suffice to catch the robber. \square

Claim. *If $Q(r, s, t)$ has orientation of type (1c) and $r, s > 4$, then two cops cannot catch the robber.*

Proof. Note that r and s must be even, thus $r \geq 6$ and $s \geq 6$. In particular, $|Q(r, s, t)| \geq 36$.

Suppose that two cops have a strategy to capture the robber. Note that each cop guards precisely four vertices of one of the facial quadrangles: his position, two of its out-neighbors and the fourth vertex on the quadrangle is in a trap. Let $Q(C)$ be the corresponding quadrangle guarded by the cop C . Also note that each of the two out neighbors of $p(C)$ puts another vertex in a trap. In effect, this means that C “guards” up to two additional vertices; see Figure 2.11. We will denote these six vertices by $\hat{Q}(C)$.

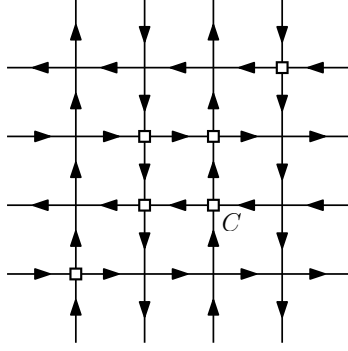


Figure 2.11: Vertices in $\hat{Q}(C)$ shown as white squares.

The strategy of the robber is the following. He selects its initial position at a vertex that is not in $\hat{Q}(C_1) \cup \hat{Q}(C_2)$ and he keeps this condition throughout the game. The size condition implies that $|Q(r, s, t)| \geq 36$ and therefore R has such a position to start with. Also the robber will not move unless he is at a vertex of $\hat{Q}(C_1) \cup \hat{Q}(C_2)$ after the move of the cops. Now, let us look at the last moves of the game before the robber gets caught or trapped. Note that by the definition of $\hat{Q}(C)$, the cop C cannot put the robber in a trap if R was not in $\hat{Q}(C)$ in the previous move. Now assume that the cop C_1 has moved and placed the robber into $\hat{Q}(C_1)$. Figure 2.12 indicates this situation. More precisely the robber is in one of the empty square vertices. By symmetry we may assume that the robber is in a position marked as R or R' . We want to show that the robber has a move to a vertex outside of the set $\hat{Q}(C_1) \cup \hat{Q}(C_2)$. The out-neighbors of R and R' are the same (a and b) and therefore the argument for these two cases will be exactly the same.

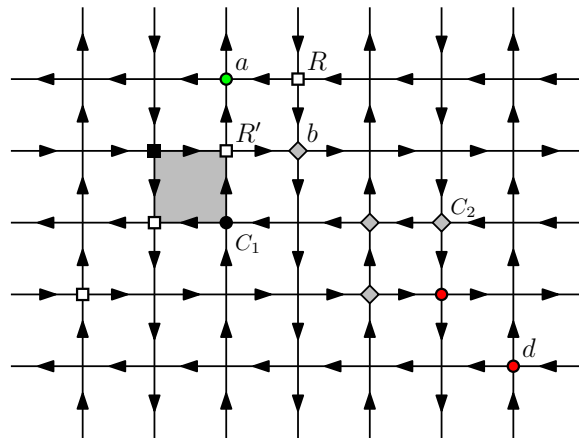


Figure 2.12: When the robber is in R or R' , he will have a move to escape.

Note that a and b cannot be in $\hat{Q}(C_1)$ since $r, s > 4$. Since the robber cannot move to b , the vertex b must be in $\hat{Q}(C_2)$. Since R cannot be in a trap, C_2 cannot be in positions R, R' and thus he is positioned at one of the four gray diamond vertices shown in Figure 2.12. In any of the four cases, $\hat{Q}(C_2)$ cannot include a because of the size condition (the

vertex denoted by d may be in $\hat{Q}(C_2)$ and d can be the same as a only if $s \leq 4$). Thus, the second cop cannot threaten both out neighbors of the robber and therefore the robber can move out of $\hat{Q}(C_1) \cup \hat{Q}(C_2)$. \square

The claims proved above imply the following.

Theorem 2.4. *If Q is an orientation of $Q(r, s, t)$ of type (1c), then $c(Q) \leq 3$; if $r, s > 4$, then $c(Q) = 3$.*

2.3 Orientation of type (2)

In this section we will discuss the orientation of type (2) of 4-regular quadrangulations. When arguing for the upper bound on the number of cops, we may assume (by Lemma 2.2) that we have the torus grid $C_n \square C_n = Q(n, n, 0)$ where each face is an oriented cycle. See Figure 2.13. Note that n must be even; and in $Q(r, s, t)$, r and $s - t$ must be even, since a graph admitting type (2) orientation is bipartite. The bipartition is given by classifying the vertices as vertices of *type LR*, whose out-neighbors are left and right form the vertex, and those of *type UD*, whose out-neighbors are up and down on the same vertical cycle. See Figure 2.14.

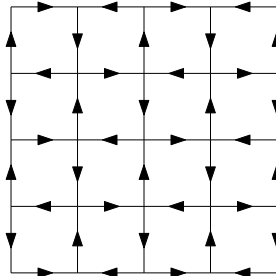


Figure 2.13: Each face is an oriented cycle.

We will prove that four cops can always capture the robber. In the first stage of the game we will put two cops in the same row as the robber, both at even distance from the robber. This can be done as described in the next claim by first bringing one cop to the same row (and following the robber after that) and then using the remaining three cops to bring the second cop into position.

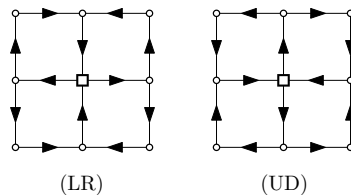


Figure 2.14: Two types of vertices.

Claim. *Three cops can achieve that one of them gets into the same row as R and is at even distance from R in that row.*

Proof. To do this, we start with two cops at the same vertex. One of them will move up and the other one will move down. They can not move up (or down) at each step, they need to repeat a pattern like up-left-up-right (or down-left-down-right). Note that there are only two types of vertices, LR and UD. First we wait, if necessary, that $p(R)$ is of the same type as the position of C_1 and C_2 . Of course we use a third cop to force the robber to move. Once $p(R)$ and $p(C_1) = p(C_2)$ are of the same type, C_1 and C_2 move with the robber to stay at the same type ever since.

The cops move as described. At each step of the robber in the vertical direction, one of them keeps the vertical distance from the row of R , while the other one decreases his vertical distance by 2. If the vertical distance becomes zero after the move of the cops, then we have reached our goal. However, if it becomes zero after the move of R (in which case R is at an LR-vertex), then the cop stays at his position. In the next move, R will have to move in horizontal direction. The cop stays at his position again, and we have reached our goal in this case as well. \square

Claim. *Four cops can catch the robber.*

Proof. By the previous claim, one of the cops can move into the same row as R at even distance from him and can stay at such a position ever since. Doing the same strategy with the remaining three cops we can get another cop in the same row at even distance from the robber.

Now one of the cops (say C_1) is (cyclically) at the right side of the robber, while C_2 is on the left side. We also have a third cop who will force the robber to move. If the robber moves up or down, then since the cops are at an even distance from him, they are at the same type of vertex and they can repeat his move and stay in the same row and keep their distances. If the robber moves right or left, then C_1 will move left and C_2 will move right. This will decrease the distance from the robber for one the cops by 2 and the other cop will remain at the same distance. Since the robber can not avoid moving both left and right, these two cops will capture him after at most $2n$ steps. \square

For a strategy of the robber we may assume that he always moves if he is being attacked by one of the cops (even when both of its out-neighbors are also attacked by cops). We say that this is a *normal strategy*.

Claim. *Suppose that $Q(r, s, t)$ has no cycles of length 2 and its only cycles of length four are the facial cycles. If Q is a type (2) orientation of $Q(r, s, t)$, then two cops cannot win the game. If the robber uses a normal strategy, then three cops win the game if and*

only if they can achieve that after one of their moves, two cops are in the same row (or the same column) as the robber and both are at the same type of vertices as the robber.

Proof. Achieving the described situation, the two cops can follow the robber and get closer and closer to him until they catch the robber as described at the end of the proof of the previous claim. Of course, we use the third cop to force the robber to move.

Suppose now that the cops have winning situation. Of course, the robber can prevent being caught in the first move of the cops. Consider the situation after the move of the cops one step before he was caught. One cop must have attacked him, otherwise, he could have stayed at the same position and not being caught. We may assume the attacking cop C is above him on the vertical cycle. Thus R is at a vertex of type LR and he can go either left or right. The condition on cycles of length 4 shows that each of the out-neighbors is either occupied with another cop or attacked from the other side by another cop. This, in particular, implies that three cops are needed. Since R uses normal strategy, he moves left or right, and now the two cops in his row can move so that both are at even distance (possibly 0) from R . \square

Claim. *Suppose that $Q(r, s, t)$ has no cycles of length 2 and its only cycles of length four are the facial cycles. Suppose that Q is a type (2) orientation of $Q(r, s, t)$ and that either $s \leq 4$ or $s' = \gcd(r, t) \leq 4$. Then $c(Q) = 3$.*

Proof. By the previous claim, it suffices to show that three cops can achieve the situation described in the claim. We may assume that $s \leq 4$; if $s' \leq 4$, then we just switch the roles of vertical and horizontal cycles since $Q(r, s, t)$ is isomorphic to $Q(r', s', t')$ with $s' = \gcd(r, t)$. Denote the horizontal cycles by D_1, \dots, D_s .

Now the cops take the following strategy. One cop is used to force the robber to move. The remaining two position themselves onto D_1 in vertices of type UD. If R ever enters D_1 , then they wait for another move of the robber to achieve the situation from the previous claim. So, the robber will try to escape this. However, if the robber enters D_2 (the row above D_1), then the two cops can move up and achieve the winning situation. Similarly, if the robber moves to the row D_s below. Thus the robber is confined to the remaining row (only when $s = 4$). But eventually, when the third cop will force him to move, he will have to leave that row. This proves that the cops win the game. \square

Let D be a row or a column. If a cop is in D or can enter D , then we say that the cop is *threatening* D . We say that D is *guarded* if there are two cops, each of which threatens D . If R ever enters a guarded row or column, then he will lose the game. Thus, we may assume that R never enters a guarded column.

Claim. *Suppose that Q is a type (2) orientation of $Q(r, s, t)$ and that $s \geq 5$ and $s' = \gcd(r, t) \geq 5$ are both odd. Then $c(Q) = 3$.*

Proof. By the previous claims it suffices to see that $c(Q) \leq 3$. Below we will describe the strategy of three cops that win the game.

Let D_1, \dots, D_s be the consecutive rows. Note that for each i , D_i is adjacent to D_{i-1} and D_{i+1} (where indices are treated modulo s) and that rows $D_{i-2}, D_{i-1}, D_i, D_{i+1}$ and D_{i+2} are all different. Similarly, we can enumerate the columns as $Q_1, \dots, Q_{s'}$ and $Q_{j-2}, Q_{j-1}, Q_j, Q_{j+1}$ and Q_{j+2} (indices modulo s') are all different.

It will be convenient to assume that in the current step of the game, $p(R) \in V(D_i) \cap V(Q_j)$. As the robber moves, the values of i and j will change (so i and j should be treated as being dependent on the step of the game we are looking at).

The strategy of the cops is divided into four phases. In phase 1, three cops achieve that one of them, say C_1 , is in the row D_i (the row of the robber) at even distance from him, say he is $2k$ steps to the left from R . After that, C_1 keeps this property by making the same moves as the robber. However, if the robber ever moves left, then C_1 moves to the right and therefore decreases the distance $2k$ by 2. When this happens, we start the strategy of the second phase all over again (even if we were already in a further phase). Since this can be done at most k times before the robber is caught, we may assume from now on that the robber never moves to the left. We may also assume that the robber never moves into a guarded row since then the cops can catch him.

The second phase of the strategy achieves in placing another cop, say C_2 , to a vertex in $D_{i-1} \cup D_{i+1}$ of the same type as the position of the robber. To achieve this, cops C_2 and C_3 approach D_i , one from above, one from below. Clearly, at some point, either one of them enters the neighboring row, say C_2 enters D_{i-1} , or R changes his row by going up or down and its new row D_i is next to the row of C_2 or C_3 (and we may assume that C_2 is in D_{i-1}). Note that if R is at a vertex of type UD, so is C_1 when the robber is to move again, and thus D_{i-1} is guarded in any of the above cases how C_2 came into D_{i-1} . Now, if the robber stays in the same row, then C_2 can adjust his position to a vertex of the same type in D_{i-1} and achieve the goal of phase 2. If the robber goes up, then C_2 can enter D_{i-1} in next 2 steps again and thus the row distance from R and C_3 will decrease. When the robber would be able to enter the row of C_3 , that row will be guarded, so it is clear that the goal of phase 2 can be achieved. So we will assume that C_2 is in row D_{i-1} and at a vertex of the same type as $p(R)$ ever since (unless we have a clear situation to catch the robber).

The third phase of the strategy achieves in placing C_2 to a vertex in $D_{i-1} \cap Q_j$ of the same type as the position of the robber. Since s' is odd, such a vertex exists: go s' steps to the right from the robber and one step down. To achieve this, the cop C_3 forces R to move. Since R cannot move left, C_2 moves to the left any time when R moves right. Thus, he will eventually reach the desired position and will stay at that relative position henceforth.

In the fourth phase of the strategy, C_3 will approach D_i from above. Once he is in D_{i+1} or in a UD vertex in D_{i+2} the row D_{i+1} is guarded and the robber needs to stay in D_i . Then C_3 repeats the following steps. If he is in D_{i+2} , then he moves left to a UD position,

and in the next move to D_{i+1} . When he is in D_{i+1} in an LR position, he moves left to the UD position. And when he is in D_{i+1} in a UD position, he goes up. At this moment, D_{i+1} is no longer guarded and R can enter it. Nevertheless, it is evident that in this way, C_3 can visit vertices in D_{i+1} that are t steps to the left of the vertex one row above the robber, for every $t = 0, 1, 2, \dots, r$. In particular, he can get to the vertex that is two steps above and one step to the right from C_2 . In that step, R must stay in D_i but can possibly move to the right. If R is at a vertex of type UD after his turn, then C_2 is also at a UD vertex and C_3 is at an LR vertex. Now C_3 can enter Q_j by stepping to the left and we have two cops, C_2 and C_3 in Q_j at even distance from R in that column. On the other hand, if R is at a vertex of type LR, then C_3 is in type UD, and he can enter D_i and the cops C_1 and C_3 will be in the row D_i at even distance from R . \square

Claim. *Suppose that Q is a type (2) orientation of $Q(r, s, t)$ and that s and $s' = \gcd(r, t)$ are both even and $s > 5$ and $s' > 5$. Then $c(Q) > 3$.*

Proof. Since $s \geq 5$ and $s' \geq 5$, $Q(r, s, t)$ has no cycles of length 2 and its only cycles of length four are the facial cycles. By the previous claims, it suffices to prove that the robber R can avoid that two of the three cops are in the same row or column as R and at even distance from him. As in the proof above, we will denote by D_1, \dots, D_s the consecutive rows and by $Q_1, \dots, Q_{s'}$ the columns.

Note that if the robber moves to a row (column) containing a cop C , then R ends up at a vertex of type LR (UD), and his next move will have to be horizontal (vertical). Thus, C can adjust his position (either by moving or staying where he is) to be in the same row (column) and have even distance from the robber after the next move of R (and ever since). But if a cop moves into the robber's row (column), then we have two cases. These cases are depicted in Figure 2.15. In Case A, R is at a vertex of type UD and can move up or down and escape. But in Case B, the cop will be in the same row as the robber and at even distance from him, and he can maintain this condition ever since.

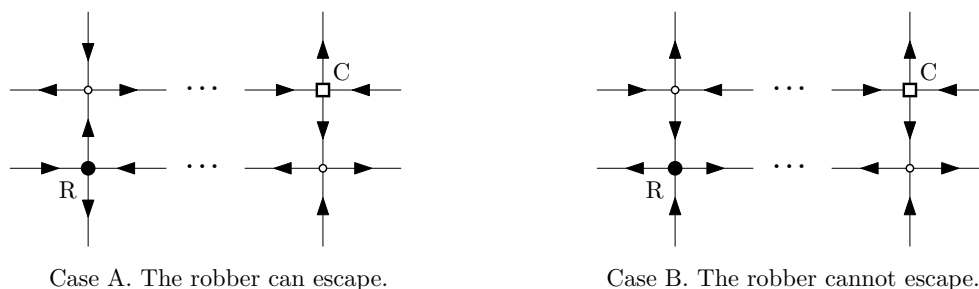


Figure 2.15: When a cop is about to move to the row of the robber.

Note that each cop is in precisely one row and in one column and that, additionally, he threatens either two adjacent rows or two adjacent columns, but not both. This implies that the number r of pairs (i, D) such that the cop C_i threatens the row D is either 3, 5,

7 or 9. The same holds for the number c of pairs (i, D) such that the cop C_i threatens the column D . Moreover, $r + c = 12$. This implies that the number of guarded rows is at most four (It is easy to see that four rows can be guarded only when $s = 5$. For example, this occurs when the cops are at vertices of type UD in rows D_1, D_2 and D_4 , in which case all rows except D_4 are guarded), and if there are four, then there is at most one guarded column. If there are three guarded rows, then there are at most 2 guarded columns, etc. If R ever enters a guarded row or column, then he will lose the game. Thus, the strategy of the robber will use the following rules:

- (1) Do not move to a vertex that is threatened by one of the cops and do not move into a guarded row or column.
- (2) Suppose that R is in a guarded row (column) after the move of cops. (It will be argued that this situation occurs either for a row or a column but not for both at the same time.) First, if R is at a vertex of type LR (UD), then move to a vertex of type UD (LR). If a cop C is in the same row (column), and he threatens the column (row) to the left/right (up/down) of R , then make the move away from the column (row) of C . Second, if R is at a vertex of type UD (LR), then move out of the row (column) unless rule (1) would need to be violated.
- (3) The robber will not move unless he is attacked or rule (2) has to be followed.

Using the discussion above, it is easy to see that R can make its first move so that his row and column will not be guarded and no cop will threaten him.

Rule (1) of the robber's strategy is that he never moves to a guarded row or column. Let us show that this is possible to obey as long as neither his row nor his column are guarded. Suppose this is the case and that a cop C_1 threatens him from the column Q_{j-1} . Then C_1 is in the same row and R is at a vertex of type UD. Since C_2 and C_3 are not both threatening the column Q_j of the robber, he can move either up or down to a safe position. Since C_1 does not threaten rows D_{i-1} and D_{i+1} , the two rows cannot be both guarded, and the robber can safely move to one of the two vertices following the rules of the strategy.

As for the continuation of the game, suppose that in the current step of the game, $p(R) \in V(D_i) \cap V(Q_j)$. (The value of i and j will therefore change depending on which step of the game we are looking at.) Let us assume for a contradiction that two of the three cops (say C_1 and C_2) placed themselves in the same row or column as the robber (without loss of generality, say it is the row D_i), that they are at even distance from him and that such a situation occurred for the first time. Rule (1) of the robber's strategy is that he never moves to a guarded row or column. This means that the row became guarded after the move of the cops, either at this stage or a few moves earlier and let us consider that situation. At that time, R was in D_i and at most one cop threatened the row. After the cops moved, D_i was guarded but at most one of the threatening cops was in the row, since

otherwise the row would be guarded already the step before. Suppose that C_2 was not yet in the row D_i , say he was in D_{i-1} . In this case, the strategy of the robber is to move to a vertex of type UD. We may also assume that C_1 is not threatening the row Q_{j+1} to the right of R . Then, we claim, R can move to the right (and also to the left if Q_{j-1} is not threatened by C_1). Since s and s' are even, the cop C_2 which threatens D_i from his position in D_{i-1} cannot be in the column Q_{j+1} and cannot threaten that column. Thus, Q_{j+1} is not guarded and R can move to the right according to his strategy. So, R can move to an unguarded column (Q_{j+1}) to a vertex of type UD. As we assumed (for contradiction) at the beginning of this discussion, R stayed in the guarded row D_i until two cops are in this row and at even distance from him. So, it must happen that the second cop stepped into the row at some point (and possibly at the same time other cops stepped into D_i). But at that point, since R was in a guarded row, by rule (2) of the strategy, he has moved and is already at a UD vertex and therefore he can escape from the row (see Figure 2.15, Case A) by entering either D_{i-1} or D_{i+1} . Note that at least one of them is not guarded. This is a contradiction which completes the proof when s and s' are even. \square

The above claims give us the exact value of the cop number on toroidal grids with type (2) orientation. Let us just observe that $Q(r, s, t)$ admits a type (2) orientation if and only if s and s' have the same parity. Now we have:

Theorem 2.5. *Suppose that Q is a type (2) orientation of $Q(r, s, t)$. Suppose that $Q(r, s, t)$ has no cycles of length 2 and its only cycles of length four are the facial cycles. If either $s \leq 4$ or $s' = \gcd(r, t) \leq 4$ or s and s' are both odd, then $c(Q) = 3$. Otherwise, $c(Q) = 4$.*

Chapter 3

Oriented Grid II

In this chapter we will consider some other orientations of 4-regular quadrangulations. First we will need some definitions.

3.1 Streams, confluxes, and traps

The grid we are working on is $C_n \square C_n$. For a vertex $v = (x, y) \in C_n \square C_n$, the digraph induced by $\{(x, z): z \in C_n\}$ will be called the *row* of v , and the subdigraph induced by $\{(z, y): z \in C_n\}$ will be called the *column* of v . A *line* containing a vertex v is either a row or a column of v . We say that two lines in $C_n \square C_n$ are consecutive if one of their coordinates correspond to consecutive vertices in one of the factors of $C_n \square C_n$. A set S of consecutive lines oriented in the same direction will be called a *stream* and its width $w(S)$ is the number of lines in the stream. If S and S' are streams such that $S' \subseteq S$, we say that S' is a substream of S .

If S_1 and S_2 are disjoint streams and the set $\mathcal{K} = V(S_1) \cap V(S_2)$ is not empty, we will call \mathcal{K} a *conflux* (see Figure 3.1). The vertices in a conflux \mathcal{K} with the minimum number of neighbours in \mathcal{K} are called *corners*. Notice that the set of corners of a conflux $\mathcal{K} = S_1 \cap S_2$ is never empty, and if $V(\mathcal{K}) \neq V(C_n \square C_n)$, then it can have four vertices (if $w(S_1), w(S_2) \geq 2$), two vertices ($w(S_i) = 1$ and $w(S_j) \geq 2$ with $\{i, j\} = \{1, 2\}$) or one vertex ($w(S_1) = w(S_2) = 1$).

If \mathcal{K} has four corners, then a corner is *main* if it has an odd number of outneighbours in \mathcal{K} and *secondary* otherwise (see Figure 3.2). However, if \mathcal{K} has one or two corners, they will all be referred to as main.

We will always assume that the robber is forced to move from its current position. We can make sure this happens by chasing him with a cop. For Lemma 3.1 and Lemma 3.2, we will assume that one cop is chasing the robber so the robber is forced to move. We will use $p(R)$ to denote the current position of the robber, and $p(C_i)$ for the position of the cop C_i .

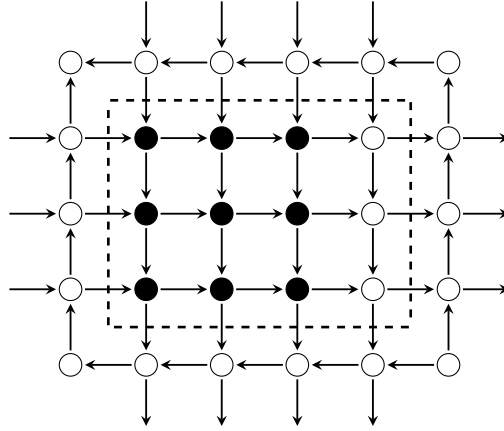


Figure 3.1: The black vertices form a conflux. The vertices inside the dashed rectangle form a maximal conflux.

Lemma 3.1. *Let \mathcal{K} be a conflux with one cop on each main corner. If $p(R) \in V(\mathcal{K})$ and $N^-(p(R)) \not\subseteq V(\mathcal{K})$, then the robber will be captured or his movements will be restricted to a stream.*

Proof. Notice that when the robber enters \mathcal{K} using a column (row), there will be a cop on the same row (column) as he is. Let us call this cop C_1 and the other one C_2 . We may assume the robber enters \mathcal{K} from above due to the symmetry of the argument. In order to leave \mathcal{K} , the robber must step on the column where C_1 is or leave through the bottom.

The strategy for C_1 and C_2 will be the following: If the robber moves towards C_1 's column, then C_1 stays where he is and C_2 copies the robber's move. If the robber moves down, then C_1 copies the robber's move, and C_2 stays in the same place if it is in the same column but different row as the robber, copies the robber's move if he is in the same row, and moves towards the robber's column otherwise.

By following this strategy, the robber and C_1 are always on the same row, so the robber cannot leave \mathcal{K} crossing C_1 's column or he will be captured. This means that the robber can move horizontally at most $w(S)$ times, where S is the stream formed by the columns containing vertices of \mathcal{K} . Therefore, the robber's movements are restricted to S . \square

Notice that in the case where the streams that form \mathcal{K} have the same width, two cops guarantee the capture of the robber. However, once the robber's movements have been restricted to a stream, one extra cop will guarantee the capture (see Chapter 2). Note that for Lemma 3.1 to work, we need to set up the trap before the robber enters it. It is possible to set a slightly different trap that works regardless of where the robber's in-neighbors are, but we need one more cop to do this.

Lemma 3.2. *Let \mathcal{K} be a conflux with one cop on each main corner and one cop in the secondary corner of \mathcal{K} without out-neighbors in \mathcal{K} if such corner exists. If the robber is in \mathcal{K} , then he will be captured or his movements will be restricted to a stream.*

Proof. Let S_1 and S_2 be the streams such that $S_1 \cap S_2 = \mathcal{K}$. If we have that $\min\{w(S_1), w(S_2)\} = 1$, then all the corners of \mathcal{K} are main and by Lemma 3.1 we are done. If $\min\{w(S_1), w(S_2)\} > 1$ and v is the secondary corner of \mathcal{K} with a cop, take S'_1 and S'_2 the minimal substreams of S_1 and S_2 respectively that contain the vertex v . After at most $\max\{w(S_1), w(S_2)\}$ moves the robber will be on a vertex of $\mathcal{K}_1 = V(S_1) \cap V(S'_2)$ or $\mathcal{K}_2 = V(S'_1) \cap V(S_2)$. Since the main corners of both \mathcal{K}_1 and \mathcal{K}_2 are covered by cops, then an application of Lemma 3.1 gives us the desired result. \square

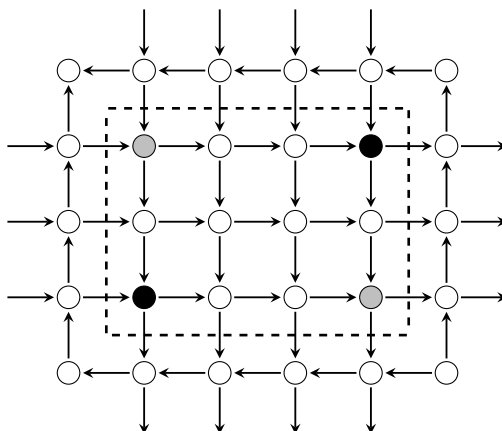


Figure 3.2: The black vertices indicate the main corners of \mathcal{K} , and the gray ones are the secondary corners.

Given a grid $G(V, A)$, we can define the conflux digraph of G , which we will denote by \mathfrak{D}_G , as the digraph whose vertex $V(\mathfrak{D}_G)$ consists of all maximal confluxes of G , and where $(\mathcal{K}_1, \mathcal{K}_2)$ is an edge of \mathfrak{D}_G whenever there exist vertices $u \in \mathcal{K}_1$ and $v \in \mathcal{K}_2$ such that $(u, v) \in A$.

There is a natural correspondence between $C_n \square C_n$ and the elements of $\mathbb{Z}_n \times \mathbb{Z}_n$. This correspondence allows us to represent each move of the robber or a cop by the addition of a vector in $\{(0, 0), (1, 0), (0, 1), (-1, 0), (0, -1)\}$ to its current position.

Given a vertex $v \in \mathbb{Z}_n \times \mathbb{Z}_n$ with $N^+(v) = \{u, w\}$, we can define the sets

$$\mathcal{SD}(v) = \{x \in \mathbb{Z}_n \times \mathbb{Z}_n : x - v = r(u + w - 2v), \text{ for some } r \in \mathbb{Z}\},$$

$$\mathcal{MD}(v) = \{x \in \mathbb{Z}_n \times \mathbb{Z}_n : x - v = r(u - w), \text{ for some } r \in \mathbb{Z}\}.$$

Observation 3.3. For any vertex v and any line L in G , $\mathcal{SD}(v) \cap L \neq \emptyset$ and $\mathcal{MD}(v) \cap L \neq \emptyset$.

The sets $\mathcal{SD}(v)$ and $\mathcal{MD}(v)$ will be called the secondary diagonal and main diagonal of v , respectively. Geometrically speaking, if we think of the arcs of the digraph as vectors, $\mathcal{SD}(v)$ is the set of all the vertices of G in the diagonal line through v defined by the sum of the arcs leaving v , and $\mathcal{MD}(v)$ is the set of vertices in the line orthogonal to that one. Notice that for every vertex $v \in V$ with $N^+(v) = \{u, w\}$, we have that $u + w - 2v$ is an element of $\{1, -1\}^2$. This value will be called the *type* of the vertex, $\tau(v)$, and two vertices will be of opposite types if their types are additive inverses in \mathbb{Z}^2 . Elements of $\{1, -1\}^2$ will be referred to as *types*. Notice that all the vertices in a conflux \mathcal{K} have the same type, so we can define $\tau(\mathcal{K}) = \tau(v)$ where $v \in \mathcal{K}$.

Let v be a vertex in G and \mathcal{K} the maximal conflux containing v . We define the horizontal escape distance of v , $\mathcal{HE}(v)$ as the length of the shortest directed path starting at v and ending at a vertex outside of \mathcal{K} using only horizontal arcs (adding $(\pm 1, 0)$). Analogously, we define the vertical escape distance v and denote it with $\mathcal{VE}(v)$. The escape distance of v is $\mathcal{E}(v) = \min\{\mathcal{HE}(v), \mathcal{VE}(v)\}$.

Lemma 3.4. Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and \mathcal{K}_4 be confluxes of G such that $N^+(\mathcal{K}_1) \cup N^+(\mathcal{K}_3) \subseteq \mathcal{K}_2 \cup \mathcal{K}_4$. If there are cops in the main corners of \mathcal{K}_2 and \mathcal{K}_4 , and the robber is in $\mathcal{K}_1 \cup \mathcal{K}_3$, then the robber will be captured or its movements will be restricted to a stream.

Proof. It is easy to see that if the robber is in $\mathcal{K}_1 \cup \mathcal{K}_3$ and is forced to move, then he will enter $\mathcal{K}_2 \cup \mathcal{K}_4$. Since the main corners of \mathcal{K}_2 and \mathcal{K}_4 are covered, the result follows from Lemma 3.1. \square

3.2 The k -regularly oriented grid

We say that a grid $G = C_n \square C_n$ is k -regularly oriented if $w(S) = k$ for every maximal stream S in G . The cases where $k \in \{1, n\}$ have been covered in Chapter 2, so in this section we will assume that G is a k -regularly oriented grid with $k < n$. Let v and w be vertices in G . We say w is a *main shadow* of v if:

- i) $w \in \mathcal{MD}(v)$.
- ii) $\tau(v) = \tau(w)$.
- iii) $\mathcal{VE}(v) = \mathcal{HE}(w)$.

We say that w is a *secondary shadow* of v if:

- i) $w \in \mathcal{SD}(v)$.

$$ii) \tau(v) = -\tau(w).$$

$$iii) \mathcal{V}\mathcal{E}(v) = \mathcal{H}\mathcal{E}(w).$$

Notice that we get equivalent definitions by changing condition *iii* for

$$iii') \mathcal{H}\mathcal{E}(v) = \mathcal{V}\mathcal{E}(w).$$

to get an equivalent definition.

In the case where $p(R) = v$, we will call w a main (or secondary) *shadow of the robber*.

We say a vertex w is a *diagonal shadow* of a vertex v if w is a secondary shadow of v or a main shadow of v . Again, if $p(R) = v$ we will use the term *diagonal shadow of the robber*. The following result states that if a cop is in a diagonal shadow of the robber and the robber moves, there is always a move that the cop can make that keeps him in a diagonal shadow of the robber. Notice that if the type of the vertex the robber is in changes when he moves, the diagonal that the cop must be in will change from secondary to main, or vice versa.

Lemma 3.5. *Let $v, u, x \in V(G)$ be vertices such that $N^+(v) = \{u, w\}$ and take $d = u - v$.*

- *If x is a secondary shadow of v , then $y = x + \tau(x) + d$ is a shadow of u and $y \in N^+(x)$.*
- *If x is a main shadow of v , then $y = x + \tau(x) - d$ is a shadow of u and $y \in N^+(x)$.*

Proof. Notice that $\tau(u) = \tau(v)$ if and only if $\tau(y) = \tau(x)$ because of condition *iii* in the definitions of diagonal shadows, and that $y - x$ is orthogonal to d .

If $x \in \mathcal{SD}(v)$, there is an integer r such that $x - v = r(u + w - 2v)$. If $\tau(u) = \tau(v)$, by substituting $x = y - \tau(x) - d$ and $v = u - d$ we get

$$\begin{aligned} x - v &= r(u + w - 2v) \\ (y - \tau(x) - d) - (u - d) &= r(u + w - 2(u - d)) \\ y - u &= r(u + d + w + d - 2u) + \tau(x) \\ y - u &= (r - 1)[(u + d) + (w + d) - 2u], \end{aligned}$$

where the last equality follows from condition *ii* in the definition of secondary shadow.

If $\tau(u) \neq \tau(v)$, notice that $\tau(v) + \tau(u) = 2d$ and $\tau(x) + \tau(y) = 2(\tau(x) + d)$, from where $\tau(u) = \tau(y)$. Also, $u - w = \pm\tau(u)$, so

$$\begin{aligned} x - v &= r(u + w - 2v) \\ (y - \tau(x) - d) - (u - d) &= r\tau(v) \\ y - u &= r\tau(v) + \tau(x) \\ y - u &= (r - 1)\tau(v), \\ y - u &= r'(u - w), \end{aligned}$$

where $r' = r - 1$ if $u - w = \tau(u)$ and $-r' = r - 1$ if $u - w = \tau(u)$.

In both cases, the escape distances equalities are given by the fact that we are in a k -regularly oriented grid and the orthogonality of $y - x$ and d . This shows y is a shadow of u .

If $x \in \mathcal{MD}(v)$, there is an integer r such that $x - v = r(u + w - 2v)$. If $\tau(u) = \tau(v)$, by substituting $x = y - \tau(x) + d$ and $v = u - d$ we get

$$\begin{aligned} x - v &= r(u - w) \\ (y - \tau(x) + d) - (u - d) &= r[(u + d) - (w + d)] \\ y - u &= r[(u + d) - (w + d)] + \tau(x) - 2d \\ y - u &= (r - 1)[(u + d) + (w + d)], \end{aligned}$$

where $\tau(x) - 2d = (u - w)$.

If $\tau(u) \neq \tau(v)$, we also get that $\tau(v) + \tau(u) = 2d$ and $\tau(x) + \tau(y) = 2(\tau(x) + d)$, from where $\tau(u) = -\tau(y)$. Again, $u - w = \pm\tau(u)$, so

$$\begin{aligned} x - v &= r(u - w) \\ (y - \tau(x) - d) - (u - d) &= r\tau(v) \\ y - u &= r\tau(v) + \tau(x) \\ y - u &= (r + 1)\tau(v), \\ y - u &= r'(u - w). \end{aligned}$$

Condition *iii* in definitions of shadows is also given by the fact that we are in a k -regularly oriented grid and the orthogonality of $y - x$ and d . This shows y is a shadow of u . \square

This next lemma will give us another way of restricting the robber's moves in the case of a k -regularly oriented grid.

Lemma 3.6. *Let u, v, s, t, x, y be vertices such that $u \neq v$, v is a diagonal shadow of u , $s \in N^+(u)$, $t \in N^+(v)$ and is a shadow of s , x is the intersection of the row of u and the column of v , and y is the intersection of the row of s and the column of t . If $d \in \{1, -1\}^2$ is orthogonal to $v - u$, then there exists an integer r such that $x = y + rd$.*

Proof. Let $A = \{(1, 0), (-1, 0)\}$ and $B = \{(0, 1), (0, -1)\}$. If $u - s \in A$, then $v - t \in B$, and therefore $x = y$ and $k = 0$. If $u - s \in B$, then $v - t \in A$. In this case, we can see that $y = x + (u - s) + (v - t)$. Since $(u - s) + (v - t) \in \{1, -1\}^2$, we get that there is an integer r such that $x = y + rd$. \square

The geometric interpretation of Lemma 3.6 is key: If a cop moves in such a way that he remains in a diagonal shadow of the robber, then there exists a set of vertices that touches

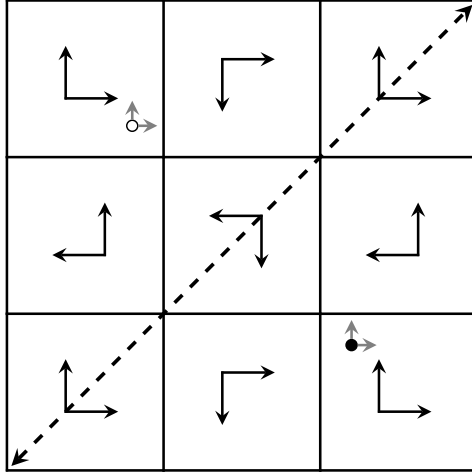


Figure 3.3: The robber (white) is not able to cross the mirror line (dashed line) or he will be caught by the cop (black).

every line in G (Observation 3.3) that the robber cannot step on (and therefore, cross) without being captured. This set corresponds to the vertices in the orthogonal bisector of the “line segment” from u to v . We will refer to this line as the *mirror* of the corresponding shadow of the robber. Two mirrors ℓ and ℓ' are parallel if the types of their vertices are parallel.

Recall that we are working on a k -regularly oriented grid. Given two parallel mirror lines ℓ and ℓ' , and d a type orthogonal to the type of a vertex in ℓ , the distance between ℓ and ℓ' , denoted by $\mathcal{A}(\ell, \ell')$, is the minimum positive integer m such that $\ell = \ell' + mkd$ or $\ell = \ell' - mkd$. Notice that $\mathcal{A}(\ell, \ell') \geq 2$ for any two different parallel mirrors, ℓ and ℓ' .

Let v be a vertex of G , $s \in \mathbb{N}$ and d a type orthogonal to $\tau(v)$. We define

$$\mathcal{MD}_s(v, d) = \{x \in V(G) : x - \frac{s(\tau(v) + d)}{2} \in \mathcal{MD}(v)\}.$$

Because of Observation 3.3, we know that every vertex $u \in G$ is in $\mathcal{MD}_r(v, d)$ for some $r \in \mathbb{N}$. Given two vertices of opposite types, u and v , the *diagonal distance*, which we will denote by $\mathfrak{d}(u, v)$, between u and v is the minimum $t \in \mathbb{N}$ such that $v \in \mathcal{MD}_t(u, d) \cup \mathcal{MD}_t(u, -d)$ (or equivalently, $u \in \mathcal{MD}_t(v, d) \cup \mathcal{MD}_t(v, -d)$). Notice that if $v \in \mathcal{MD}_t(u, d)$ if and only if $u \in \mathcal{MD}_t(v, -d)$. We define

$$\mathcal{B}(u, v) = \bigcup_{i=0}^t \mathcal{MD}_i(u, d),$$

where d has been chosen so that $v \in \mathcal{MD}_t(u, d)$. Notice that if \mathcal{K} and \mathcal{K}' are confluges of G such that $\tau(\mathcal{K}) = -\tau(\mathcal{K}')$, then $\mathcal{B}(\mathcal{K}, \mathcal{K}')$ and $\mathfrak{d}(\mathcal{K}, \mathcal{K}')$ are defined in \mathfrak{D}_G .

Observation 3.7. *If \mathcal{K} and \mathcal{K}' are maximal confluges of a k -regularly oriented grid G , and $\tau(\mathcal{K}) = -\tau(\mathcal{K}')$, then we have $\mathfrak{d}(\mathcal{K}, \mathcal{K}') < \frac{n}{k}$ and $\mathfrak{d}(\mathcal{K}, \mathcal{K}') \in 2\mathbb{Z}$.*

For the rest of the section, all the confluges will be assumed to be maximal. For the Lemmas 3.8, 3.9, 3.10 and 3.11, \mathcal{K}_i will denote a conflux whose main corners are covered by cops.

Lemma 3.8. *Suppose $\tau(\mathcal{K}_1) = -\tau(\mathcal{K}_2)$ and that the robber is in a conflux in $V(\mathfrak{D}_G) - \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ whose type is orthogonal to $\tau(\mathcal{K}_1)$. If the robber enters $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, then the cops can capture a main shadow of the robber.*

Proof. If the robber is in $V(\mathfrak{D}_G) - \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, in order to enter $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ the robber must enter a conflux \mathcal{K} such that $\mathcal{K} \in \mathcal{MD}(\mathcal{K}_1) \cup \mathcal{MD}(\mathcal{K}_2)$. In either case, a main shadow of the robber will enter $\mathcal{K}_1 \cup \mathcal{K}_2$. Since the main corners of both \mathcal{K}_1 and \mathcal{K}_2 are covered by cops, we capture a main shadow of the robber by applying Lemma 3.1 to the corresponding shadow. \square

Lemma 3.9. *Let d be a type orthogonal to $\tau(\mathcal{K}_1)$, and suppose that \mathcal{K}_2 satisfies $\tau(\mathcal{K}_1) = -\tau(\mathcal{K}_2)$ and $\mathfrak{d}(\mathcal{K}_1, \mathcal{K}_2) = 2$. If the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, then we can force him to move to a conflux in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)^c$.*

Proof. We can assume the robber is in a vertex of type d or $-d$. If the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, then by forcing him to move he must enter a conflux \mathcal{K} whose type is parallel to $\tau(\mathcal{K}_1)$. If $\tau(\mathcal{K}) = \tau(\mathcal{K}_1)$, then $\mathcal{K} \in \mathcal{MD}(\mathcal{K}_2)$, and so by forcing him to move he will exit $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. If $\tau(\mathcal{K}) = -\tau(\mathcal{K}_1)$, then we have $\mathcal{K} \in \mathcal{MD}(\mathcal{K}_1)$, so he will exit $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ if we force him to move. \square

Lemma 3.10. *Let $\mathcal{K}_1, \mathcal{K}_2$ and d be the same as in the hypothesis of Lemma 3.9 and take \mathcal{K}_3 such that $\tau(\mathcal{K}_3) = \tau(\mathcal{K}_1)$ and $\mathcal{K}_3 = \mathcal{K}_2 + \tau(\mathcal{K}_1) + r'd$ in \mathfrak{D}_G for some $r' \in \mathbb{Z}$.*

If the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_3) - \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$, then we can force him to move to a vertex in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_3)^c$ or we capture a main shadow.

Proof. Again, we can assume that the robber is in a vertex of type d or $-d$. By forcing him to move he will enter a conflux \mathcal{K} of type parallel to $\tau(\mathcal{K}_1)$. If $\mathcal{K} \in \mathcal{MD}(\mathcal{K}_3)$, the robber is forced to move to $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_3)^c$. Otherwise, an Lemma 3.8 guarantees the capture of a main shadow of the robber. \square

Lemma 3.11. *Let $\mathcal{K}_1, \mathcal{K}_2$ be such that $\mathfrak{d}(\mathcal{K}_1, \mathcal{K}_2) = \frac{|V(G)|}{k} - 2$. If the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)^c$, then we will capture a main shadow.*

Proof. This follows directly from the fact that by forcing the robber to move he must enter a conflux whose type is parallel to $\tau(\mathcal{K}_1)$, all of which are contained in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. \square

All the previous results in this section either assume that we already captured a diagonal shadow of the robber or include some assumption about the current position of the robber in their statements. The following will be the first result that makes no such assumptions.

Lemma 3.12. *Seven cops can capture a main diagonal shadow of the robber in G .*

Proof. First, one of the cops will be chasing the robber in order to force him to move, so we only need to show that six cops can capture the robber if he is forced to move. Let \mathcal{K}_1 and \mathcal{K}_2 be confluxes whose main corners are covered by cops, such that $\mathfrak{d}(\mathcal{K}_1, \mathcal{K}_2) \geq 2$ and the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)^c$ (the existence of such confluxes is given by Lemma 3.9 and the fact that $2k < |V(G)|$). The proof will be by induction on $t = \frac{|V(G)|}{k} - \frac{1}{2}\mathfrak{d}(\mathcal{K}_1, \mathcal{K}_2)$.

The case $t = 1$ is solved by applying Lemma 3.11, so we can assume $t = m \geq 2$. Let \mathcal{K}_3 be a conflux such that $\mathfrak{d}(\mathcal{K}_1, \mathcal{K}_3) = \mathfrak{d}(\mathcal{K}_1, \mathcal{K}_2) + 2$ and cover its main corners with two cops. Notice that an application of Lemma 3.10 guarantees that either we capture a diagonal shadow of the robber (in which case we are done) or that the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_3)^c$. In the latter case, the cops in \mathcal{K}_2 can be released and we can rename \mathcal{K}_3 as \mathcal{K}_2 , in which case we reduce m by one and the result follows by induction. \square

The basic idea of the proof of Theorem 3.14 is to successively capture diagonal shadows of the robber such that their mirrors get closer until the distance between them is two, and then use the remaining cops to capture the robber between those mirrors. However, it is clear that in order to effectively restrict the robber's movements we need two different mirrors. However, there is no way to guarantee that if we have a cop in a diagonal shadow of the robber and we use Lemma 3.12 again we won't capture the shadow where we already have a cop. A simple way around this problem is to use Lemma 3.12 twice at the same time. This is what the following result deals with.

Lemma 3.13. *Thirteen cops can capture two main diagonal shadows of the robber simultaneously. Moreover, we can actually guarantee that the distance between the mirrors of the diagonal shadows is two.*

Proof. Like in the proof of Lemma 3.12, a cop will force the robber to move, so we only need to show that twelve cops can archive the desired result if the robber is forced to move. For each cop C used in the strategy of Lemma 3.12, we will use one more cop C' in the following way: If $p(C) = v$, we will choose $p(C') = v'$, where $v' \in \mathcal{SD}(v) \cap \mathcal{MD}(v) - \{v\}$. Notice that $\tau(v) = \tau(v')$, so we can move C' in such a way that he stays in the shadow of C . In this way, by using the strategy of Lemma 3.12 with the first set of six cops and maintaining the copies of the cops in their shadows, we will capture two main shadows of the robber simultaneously. Let ℓ and ℓ' be the mirrors of these shadows, and C and C' the cops moving in these shadows.

If $\mathcal{A}(\ell, \ell') \leq 2$ we are done, so we can assume it is greater than two. Suppose the type of the confluxes that we used in the application of Lemma 3.12 is d . Notice that whenever

$\mathcal{A}(\ell, \ell') \geq 4$, there exist maximal confluges $\mathcal{K}_1, \mathcal{K}_2$ and \mathcal{K}_3 whose types are parallel to ℓ such that $\mathcal{MD}(\mathcal{K}_1), \mathcal{MD}(\mathcal{K}_2)$ and $\mathcal{MD}(\mathcal{K}_3)$ are mutually disjoint.

Since we are using one cop to guard each mirror, we have ten free cops. By using six of those ten cops to repeat the strategy moving positioning the cops in confluges in $\mathcal{MD}(\mathcal{K}_1) \cup \mathcal{MD}(\mathcal{K}_2) \cup \mathcal{MD}(\mathcal{K}_3)$ of type d we will capture a new diagonal shadow. If ℓ'' is the mirror corresponding to this new shadow and C'' is the cop guarding it, notice that $\max\{\mathcal{A}(\ell, \ell''), \mathcal{A}(\ell', \ell'')\} < \mathcal{A}(\ell, \ell')$ and the robber is either between ℓ'' and ℓ or between ℓ'' and ℓ' , so we can release either C or C' . Since the robber's movements are restricted to a strictly smaller set, induction over the distance between the mirrors gives us that the distance between mirrors is two. \square

With this we are ready to prove the main theorem of this section:

Theorem 3.14. *For every $k \geq 2$, if G is a k -regularly oriented grid, then $c(G) \leq 13$.*

Proof. Since we have 13 cops, we can use one to chase the robber and force him to move. That means we have twelve free cops. By Lemma 3.13, we can assume 10 of those cops are free and that the robber is restricted to the vertices between two mirrors at distance two. If we manage to capture a main diagonal shadow of the robber between the mirrors whose type is parallel to the mirrors, then we capture the robber.

Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and \mathcal{K}_4 be confluges such that $\mathfrak{d}(\mathcal{K}_1, \mathcal{K}_2) = 2$, $d = \mathcal{K}_1 - \mathcal{K}_2$, and $\mathcal{K}_3 = \mathcal{K}_1 + 2d$ and $\mathcal{K}_4 = \mathcal{K}_2 + 2d$ and guard the main diagonals of each of these four confluges with two cops. We can assume the robber is in a vertex of type orthogonal to $\tau(\mathcal{K}_1)$. Notice that $\mathfrak{d}(\mathcal{K}_2, \mathcal{K}_3) = \frac{|V(G)|}{k} - 2$. An application of Lemma 3.11 to \mathcal{K}_2 and \mathcal{K}_3 guarantees that the robber is in $\mathcal{B}(\mathcal{K}_2, \mathcal{K}_3)$. By Lemma 3.9 applied to \mathcal{K}_1 and \mathcal{K}_2 , and to \mathcal{K}_3 and \mathcal{K}_4 , we can guarantee that the robber is in $(\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2) \cup \mathcal{B}(\mathcal{K}_2, \mathcal{K}_3)^c \cup \mathcal{B}(\mathcal{K}_3, \mathcal{K}_4))^c = \mathcal{B}(\mathcal{K}_1, \mathcal{K}_4)^c$.

Let $t = \frac{|V(G)|}{k} - \frac{1}{2}\mathfrak{d}(\mathcal{K}_1, \mathcal{K}_4)$. The proof will be by induction on t . If $t = 1$, Lemma 3.11 guarantees the capture of the robber, so we can assume $t \geq 2$. Suppose $t = m$. Since the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_4)^c$, we can release the cops in \mathcal{K}_2 and \mathcal{K}_4 and move them to the main corners of the confluges $\mathcal{K}_5 = \mathcal{K}_1 + 4d$ and $\mathcal{K}_6 = \mathcal{K}_1 + 4d$. Again, an application of Lemma 3.9 with \mathcal{K}_5 and \mathcal{K}_6 guarantees that the robber is in $\mathcal{B}(\mathcal{K}_5, \mathcal{K}_6)^c$, and using Lemma 3.11 with \mathcal{K}_4 and \mathcal{K}_5 gives that the robber is in $\mathcal{B}(\mathcal{K}_4, \mathcal{K}_5)$. This now gives us that the robber is in $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_6)^c$, so if we rename \mathcal{K}_6 as \mathcal{K}_4 we get that $t = m - 1$, so the result follows by induction. \square

It is important to mention that the only part of the proof where we use 13 cops is during the application of Lemma 3.13. The rest of the proof only uses 11 cops, so finding a more efficient way of capturing two diagonal shadows simultaneously would improve the bound for the cop number of G .

3.3 Paddles

We begin by establishing more general conditions which guarantee that the robber is confined to a stream. As before, we assume that the robber is forced to move.

Lemma 3.15. *Let S be a stream and let ℓ_1 and ℓ_2 be the lines which form the boundary of S . Suppose that the robber is in S , at distance d_1 from ℓ_1 and at distance d_2 from ℓ_2 . Let v_1, v_2 be the closest vertices (using distances in the undirected grid) of ℓ_1, ℓ_2 to $p(R)$, respectively. Suppose further that there are distinct cops C_1 and C_2 , such that C_1 can move to v_1 in $m_1 \leq d_1$ moves and C_2 can move to v_2 in $m_2 \leq d_2$ moves. Then by committing C_1 and C_2 we can ensure that the robber will be caught or confined to S .*

Proof. If $d_1 = 0$ or $d_2 = 0$ then the robber is already caught. Otherwise, whatever the robber's move, we update v_1, v_2, d_1 and d_2 accordingly. Now for each $i \in \{1, 2\}$, if C_i is at v_i then he remains in place; otherwise, he moves towards v_i . We will show that the conditions of the lemma are maintained. If the robber moved in the direction of the stream, then v_1 and v_2 each move in the direction of the stream and d_1 and d_2 are unchanged. In this case the robber's move increases m_1 and m_2 by at most 1, and the cops' moves immediately decrease m_1 and m_2 by 1, to a minimum of 0; thus $m_1 \leq d_1$ and $m_2 \leq d_2$. If the robber moved towards ℓ_1 , then d_1 decreases by 1 and d_2 increases by 1. The cops' moves now decrease m_1 and m_2 by 1, to a minimum of zero, and again $m_1 \leq d_1$ and $m_2 \leq d_2$. The case in which the robber moved towards ℓ_2 is similar. \square

Given a conflux \mathcal{K} , we refer to the secondary corner with no outneighbours in \mathcal{K} as the *terminal corner* of \mathcal{K} . Let v be the main corner of \mathcal{K} with a vertical (respectively, horizontal) edge leaving \mathcal{K} (but no other edge leaving \mathcal{K} , unless \mathcal{K} has only one vertex); then we refer to the vertical (respectively, horizontal) outneighbour of v as the *vertical guard post* (respectively, *horizontal guard post*) of \mathcal{K} . If \mathcal{K} is maximal, then we refer to the vertex outside \mathcal{K} with the same outneighbours as the terminal corner as the *terminal guard post* of \mathcal{K} .

Lemma 3.16. *Let S_1 be a vertical stream, S_2 be a horizontal stream and let \mathcal{K} be the conflux $S_1 \cap S_2$. For each $i \in \{1, 2\}$, let d_i and d'_i be the distance from $p(R)$ to the boundary of \mathcal{K} in the direction of S_i and in the opposite direction, respectively. Suppose that the robber is in \mathcal{K} , and that there are distinct cops C_V, C_H and C_T such that C_V can reach the vertical guard post of \mathcal{K} in $m_V \leq d_1 + d'_2 + 1$ moves, C_H can reach the horizontal guard post of \mathcal{K} in at most $m_H \leq d_2 + d'_1 + 1$ moves, and C_T can reach either the terminal corner or, if \mathcal{K} is maximal, the terminal guard post of \mathcal{K} in $m_T \leq d_1 + d_2$ moves. Then by committing C_V, C_H and C_T we can ensure that the robber will be caught or confined to either S_1 or S_2 .*

Proof. If the robber does not leave \mathcal{K} on his move, then with the cops' moves we will decrease m_V, m_H and m_T by 1, to a minimum of zero; then it is clear that the conditions of the

lemma still hold. Since the robber can make only finitely many such moves, we may assume that the robber leaves \mathcal{K} on his move. Suppose without loss of generality that he leaves S_2 and remains in S_1 . In this case, before the robber's move we must have had $d_1 = 0$, and hence $m_V \leq d'_2 + 1$ and $m_T \leq d_2 + 1$. Let ℓ_1 and ℓ_2 be the boundary lines of S_1 , such that there is a directed path in \mathcal{K} from ℓ_2 to ℓ_1 . Let v_1 and v_2 be the closest vertices of ℓ_1 and ℓ_2 , respectively, to $p(R)$. Now observe that C_T can move to v_1 in at most $d_2 + 1$ moves (by first moving to the terminal corner or terminal guard post of \mathcal{K}), while C_V can move to v_2 in at most $d'_2 + 1$ moves (since v_2 is the vertical guard post of \mathcal{K}). We move each cop one step along the appropriate directed path; now Lemma 3.15 implies that we can ensure the robber will be caught or confined to S_1 . \square

Our strategy to catch the robber will be based on blocking streams of maximum width and sections of maximal streams which do not intersect streams of greater width, which we refer to as *brooks*.

Lemma 3.17. *Let S be either a stream of maximum width or a brook, and let $m = \lfloor w(S)/3 \rfloor + 1$. Let S' be formed by the lines of S at distance at least $m - 1$ from the boundary lines of S . Then by committing 64 cops, and temporarily using a further 64 cops, we can ensure that after some finite time, if the robber enters S' then he will be caught or confined to a stream.*

Proof. Without loss of generality we may assume that S is a vertical stream or brook, and that edges of its lines are all directed upwards. Let ℓ_1 and ℓ_2 be the boundary lines of S , and let ℓ'_1 and ℓ'_2 be the lines outside S adjacent to ℓ_1 and ℓ_2 respectively. We use two formations of cops, which we call paddles: an *inner paddle* (respectively, *outer paddle*) is a formation of cops evenly spaced at distance m (or distance 1, if $m = 0$) along each of ℓ_1 and ℓ_2 (respectively, ℓ'_1 and ℓ'_2), where each horizontal line has either two or zero cops. Given a paddle, let H be the convex hull of all of the cops in the paddle excluding the six cops which are furthest in the upwards direction. Then the *domain* of the paddle is the union of the rows intersecting H .

Claim 3.18. *If the robber enters the intersection of the domain of a paddle with S' along a horizontal edge, then he will be caught or confined to a stream.*

To prove the claim, we first observe that if the paddle is an inner paddle then there is a pair of cops below $p(R)$ at a vertical distance of at most $m - 1$; now Lemma 3.15 implies that the robber will be caught or confined to S . Hence we may assume that the paddle is an outer paddle. If $m \leq 1$ then the robber must have been at the same vertex as a cop on the previous move, which is a contradiction; hence, $m \geq 2$. Let \mathcal{K} be the maximal conflux containing the robber; then \mathcal{K} has height at most $w(S)$. Suppose without loss of generality that the horizontal edges in \mathcal{K} are directed from ℓ_2 to ℓ_1 . Then the robber is at distance $m - 1$ from ℓ_2 and at distance $w(S) - m$ from ℓ_1 . There is a cop C_T on ℓ'_1 above

the top row of \mathcal{K} , at a distance of at most m ; this cop can reach the terminal guard post of \mathcal{K} in at most $m - 1$ moves. Further, there is a pair of cops above or level with $p(R)$ at a vertical distance at most $m - 1$. Let C_V be the cop in this pair which is on ℓ'_2 , and let C be the cop in this pair which is on ℓ'_1 . We let $C_H = C$ if $C_T \neq C$; otherwise, we let C_H be the closest cop above C on ℓ'_1 . Let d_1 and d'_1 be the distances from $p(R)$ to the top and bottom rows of \mathcal{K} , respectively. Then C_V can reach the vertical guard post of \mathcal{K} in at most $(m - 1) + 1 + d_1 + 1 = (m - 1) + d_1 + 2$ moves, while C_H can reach the horizontal guard post of \mathcal{K} in at most $(2m - 1) + d'_1 \leq (w(S) - m) + d'_1 + 2$ moves, where the inequality follows from the definition of m . Now C_V , C_H and C_T each make their first moves along their respective paths, and the claim follows by Lemma 3.16.

Claim 3.19. *The cops forming an inner paddle can reform to form an outer paddle with the same domain in at most $2w(S) + 1$ moves, and vice versa.*

To prove the claim, we observe that for any vertex of ℓ_1 there is a directed path of length at most $2w(S) + 1$ to the horizontally adjacent vertex of ℓ'_1 : move up $d \leq w(S)$ times until there is an edge to ℓ'_1 , move to ℓ'_1 , and then move down d times. Similarly there is a directed path of length at most $2w(S) + 1$ from any vertex of ℓ'_1 , ℓ_2 or ℓ'_2 to the horizontally adjacent vertex of ℓ_1 , ℓ'_2 or ℓ_2 respectively; the claim now follows immediately.

Now suppose we have formed the cops into two paddles, each consisting of 32 cops. Observe that the domains P_1 and P_2 of these paddles each contain at least $12m + 1 \geq 4w(S) + 2$ rows. Since having a larger domain only helps us, we may assume that P_1 and P_2 contain $4w(S) + 2$ rows each.

We first show that once we have set up the appropriate circumstances, we can make sure that the robber remains within either P_1 or P_2 indefinitely. To achieve this we define four states, and show that regardless of the robber's move we can either stay in the same state or go to one of the other three. We say that a domain is *moving up* (respectively *moving down*) if the corresponding paddle is an inner (respectively outer) paddle. We say that a domain has t *steps to start* moving up or down if it is in the process of reforming and will complete this process in t moves, and that it is *active* otherwise. We say that we *switch* the domain when we reform the corresponding paddle to move in the opposite direction. Note that everything we say will be equally true if we reverse the vertical directions or relabel the paddles, so that we may do that at any time.

State 1: P_1 is moving up, P_2 is moving down, P_1 and P_2 occupy the same rows and the robber is on one of these rows.

In this state the robber can only force us out of State 1 by leaving the occupied rows. If he does so, without loss of generality he moves above the top row of P_1 ; then we move P_1 up, switch P_2 and enter State 2. Otherwise we remain in State 1.

State 2: P_1 is moving up, P_2 has t steps to start moving up, P_1 is d_1 rows above P_2 and the robber is d_2 rows below the top row of P_1 , where $d_1 + d_2 + t \leq 2w(S)$.

In this state if the robber moves above the top row of P_1 then we move P_1 up; then d_1 increases by at most 1 and $d_2 = 0$. Otherwise we keep P_1 stationary; then d_2 increases by at most 1 and d_1 remains the same. Hence $d_1 + d_2$ increases by at most 1; since t decreases by 1 the inequality still holds and we stay in State 2 until $t = 0$, when we enter State 3.

State 3: P_1 and P_2 are both moving up, P_1 is $d_1 \geq 1$ rows above P_2 and the robber is d_2 rows below the top row of P_1 , where $d_1 + d_2 \leq 2w(S) + 1$.

In this state if the robber moves above the top row of P_1 then we move both P_1 and P_2 up; then d_1 remains the same and $d_2 = 0$. Otherwise we move only P_2 up; then d_1 decreases by 1 while d_2 increases by at most 1. So the inequality still holds and we stay in State 3 unless $d_1 = 0$, when we enter State 4.

State 4: P_1 and P_2 are both moving up and occupy the same rows, and the robber is $d \leq 2w(S) + 1$ rows below the top row of P_1 .

If $d = 2w(S) + 1$ and the robber moves down then we switch P_2 and enter State 5. Otherwise we move P_1 and P_2 only if the robber goes above their top row, and remain in State 4.

State 5: P_1 is moving up, P_2 has t steps to start moving down, P_1 and P_2 occupy the same rows and the robber is d rows below the top row of P_1 , where $d + t \leq 4w(S) + 2$ and $d - t \geq 2$.

In this case we keep both P_1 and P_2 stationary and stay in State 5 until $t = 0$, when we enter State 1.

We next show that we can reach one of these states. We form our cops into four paddles of 32 cops each, with domains P_1, P_2, P'_1 and P'_2 . If S is a stream of maximum width then initially all of these domains occupy the same rows; otherwise, P_1 and P_2 will start just below the bottom row of S and P'_1 and P'_2 will start just above the top row of S . In either case we begin with P_1 and P_2 moving up and P'_1 and P'_2 moving down. At some finite time the robber will occupy either the top row of P_1 and P_2 or the bottom row of P'_1 and P'_2 . Without loss of generality he occupies the top row of P_1 and P_2 . At this point we enter State 3, and release the cops from P'_1 and P'_2 .

Now if at any point the robber is outside S' , Claim 3.18 implies that he cannot re-enter S' without being caught or confined to a stream. To force the robber to leave S' , we choose an arbitrary row of S and place two cops at either end of the intersection of this row with S . One cop on each end remains stationary, while the remaining two cops move in the direction of S . If the robber does not leave S' , then after some finite time he will be on the same row as one of the pairs of cops, at which point he is confined to a stream. \square

Theorem 3.20. *If G is any straight ahead orientation of a grid, then $c(G) \leq 404$.*

Proof. For each stream or brook S_i mentioned below, S'_i is formed by the lines of S_i at distance at least m from the boundary lines of S (or all the lines of S , if $m \leq 1$), where $m = \lfloor (w(S) + 1)/3 \rfloor$. For any rectangular subgraph H of G , we define $S(H)$ to be either

a stream of maximum width which is wider than any brook, or a brook, in H (making arbitrary choices where necessary).

We first aim to obtain a situation in which S_1 and S_2 are streams, S_3 is a brook, S_1 , S_2 and S_3 are all wider than any other stream which the robber can access and the robber is caught between S'_1 , S'_2 and S'_3 (all of which are blocked) in $G_2 = G \setminus (V(S_1) \cup V(S_3))$. To achieve this we will commit 202 cops. Let $S_1 = S(G)$ and let $G_1 = G \setminus V(S_1)$.

Case 1: $S(G_1)$ is a stream. Then we set $S_2 = S(G_1)$ and $G'_2 = G_1 \setminus V(S_2)$. We commit 132 cops to block S'_1 and S'_2 , as per Lemma 3.17. G'_2 falls into two components; we let G_2 be the component in which the robber lies. If $S(G_2)$ is a stream then we label it S_2^* and commit an additional 64 cops to block it; then $G_2 \setminus V(S_2^*)$ falls into two components. We relabel the component containing the robber as G_2 , release the cops blocking either S'_1 or S'_2 and relabel S_2^* as either S_2 or S_1 , as appropriate.

By repeating this process we arrive at a situation in which $S(G_2)$ is a brook. We label it S_3 , commit 64 cops to block S'_3 and commit a further 6 cops to block the confluxes formed by the intersections of the stream containing S_3 with S_1 and S_2 .

Case 2: $S(G_1)$ is a brook. Then we set $S_1^* = S(G_1)$ and $G_1^* = G_1 \setminus S_1^*$. Let S_2 be the stream of which $S(G_1^*)$ is a section. If S_1^* and S_2 are either both horizontal or both vertical streams then we relabel S_1 as S_1^* and set S_1 to be the stream of which S_1^* is a section. So we may assume that S_1 and S_2 are either both horizontal or both vertical streams, and that S_1^* is the other of horizontal or vertical. Let $G'_2 = G \setminus (V(S_1) \cup V(S_2))$. We block S'_1 and S'_2 and let G_2 be the component of G'_2 to which the robber is confined. Let S_3 be the intersection of S_1^* with G_2 . We block S'_3 and the intersections of the stream containing S_3 with S_1 and S_2 .

Without loss of generality S_1 and S_2 are vertical streams. let $G_3 = G_2 \setminus V(S_3)$. As long as $S(G_3)$ is a vertical brook we perform the following procedure to further confine the robber: Set S_2^* to be the stream containing $S(G_3)$. Block $S'(G_3)$ and the intersection of S_2^* with S_3 . Let $G'_3 = G_3 \setminus V(S(G_3))$ and relabel the component of G'_3 containing the robber as G_3 . Relabel S_2^* as S_1 or S_2 and release the cops blocking S'_1 or S'_2 , whichever no longer borders on G_3 .

Thus we may assume that $S(G_3)$ is a horizontal brook, and label it S_4 . We commit 64 cops to block S'_4 and a further 6 to block the intersections of the stream containing S_4 with S_1 and S_2 . Let $G'_4 = G_3 \setminus V(S_4)$ and let G_4 be the component of G'_4 containing the robber. We relabel the brooks which border on G_4 as S_1 , S_2 , S_3 and S_4 (this means each S_i is now a brook, whereas previously S_1 and S_2 were streams). So far we have committed 272 cops. Using the remaining 132 cops, we will shrink the robber's territory according to the following procedure.

Let $S_5 = S(G_4)$. Without loss of generality S_5 is a vertical brook. We commit 64 cops to block S'_5 and then a further 6 to block the intersections of the stream containing S_5 with S_3 and S_4 . Let $G'_5 = G_4 \setminus V(S_4)$ and let G_5 be the component of G'_5 containing the robber.

Now either S_1 or S_2 does not border on G_5 ; without loss of generality it is S_1 . We release the 70 cops blocking S'_1 and the intersections of the stream containing S_1 with S_3 and S_4 , relabel S_5 as S_1 and relabel G_5 as G_4 .

Eventually, the robber's territory is empty and he will be caught or confined to a stream. \square

Chapter 4

Graphs of Bounded Diameter

In this chapter we will try to improve the upper bound for the cop number of graphs with bounded diameter.

Lu and Peng [25] (and independently Scott and Sudakov [32]) proved the following theorem which gives the best known result on the upper bound of the cop number of general graphs.

Theorem 4.1. *The cop number of any connected n -vertex graph is at most $n2^{-(1+o(1))\sqrt{\log n}}$.*

The following is a direct corollary of their approach.

Corollary 4.2. *The cop number of any connected n -vertex graph of diameter d is at most $n^{1-\frac{1}{\lceil \log d \rceil + 1} + o(1)}$.*

They have used random positioning of cops to analyze the game and here is the main tool that they have used. We will restate their starting tool in a more general language.

Let $\mathcal{C} = \mathcal{C}(V, p)$ be a random subset of a set V with $|V| = n$, where $v \in V$ is in \mathcal{C} with probability p . Since $|\mathcal{C}|$ is binomially distributed with expectation $\mu = n \cdot p$, by the standard Chernoff-type estimate, we have that the probability that \mathcal{C} has more than $2\mu = 2n \cdot p$ vertices is at most $e^{-\mu/3}$.

For every subset A of vertices of G and integer i let $B(A, i)$ be the ball of radius i around A , that is all the vertices of G which can be reached from some vertex in A by a path of length at most i . For simplicity, when A is a single vertex v , we write $B(v, i)$. We need the following lemma.

Lemma 4.3. *Let $\mathcal{C} = \mathcal{C}(V(G), p)$, where G is a connected graph of order n . For every $n \geq 561$, the following statement holds with probability at least 0.9: for every $A \subset V(G)$ and every i such that $|B(A, i)| \geq |A| \cdot \frac{\log^2 n}{p}$, we have*

$$|B(A, i) \cap \mathcal{C}| \geq |A|.$$

Proof. Let $a = |A|$. Note that for any fixed A and i the number of vertices from \mathcal{C} in $B(A, i)$ is binomially distributed with expectation at least $a \frac{\log^2 n}{p} \cdot p = a \log^2 n$ and by the standard Chernoff-type estimate we have that the probability that $|B(A, i) \cap \mathcal{C}|$ is smaller than a (for fixed A and i) is at most $e^{-a \log^2 n/3}$. The number of sets of size a is $\binom{n}{a}$ and the number of different choices of i is at most the diameter of the graph, $d(G)$. Therefore the statement is true with probability at least

$$1 - d(G) \sum_a \binom{n}{a} e^{-a \log^2 n/3}.$$

Using Stirling's approximation we have

$$\binom{n}{a} \leq \left(\frac{e \cdot n}{a}\right)^a = \frac{e^a n^a}{a^a} = \frac{e^a}{a^a} n^a \leq e \cdot 2^{a \cdot \log n} \leq e \cdot e^{a \cdot \log n}.$$

For the second inequality above, note that for $a = 1$ or 2 , $1 < \frac{e^a}{a^a} \leq e$, but for $a \geq 3$ we have that $\frac{e^a}{a^a} < 1$. Therefore we have

$$1 - d(G) \sum_a \binom{n}{a} e^{-a \log^2 n/3} \geq 1 - e \cdot d(G) \sum_a e^{a \log n - a \log^2 n/3} \geq 1 - e \cdot d(G) \sum_a e^{-a \log^2 n/6}.$$

The last inequality holds when $\log n - \log^2 n/3 \leq -\log^2 n/6$ which is true for $n \geq 2^6$. So the statement is true with probability at least

$$1 - e \cdot d(G) \sum_a (e^{-\log^2 n/6})^a \geq 1 - e \cdot d(G) \frac{e^{-\log^2 n/6}}{1 - e^{-\log^2 n/6}} = 1 - e \cdot d(G) \frac{1}{e^{\log^2 n/6} - 1}.$$

Note that $d(G) < n$ and $e^{\log^2 n/6} > 2^{\log^2 n/6} = n^{\log n/6}$. Thus, in order to guarantee the truth of the statement with probability at least 0.9, we need to have

$$1 - e \cdot d(G) \frac{1}{e^{\log^2 n/6} - 1} > 1 - e \cdot \frac{n}{n^{\log n/6} - 1} > 0.9.$$

which is true for $n \geq 561$. □

4.1 When the diameter is at most 4

Let G be a graph of diameter at most 4 on n vertices. As discussed in the previous chapter, Lu and Peng have proved that the cop number of graphs with diameter up to 4 is at most $n^{\frac{2}{3}+o(1)}$. In this section we will first improve this result to $n^{\frac{5}{8}+o(1)}$ and then to $n^{\frac{3}{5}+o(1)}$.

Let \mathcal{C} be a random subset of vertices of G , where a vertex v is in \mathcal{C} with probability $p = n^{-\frac{3}{8}}$. As discussed above, \mathcal{C} will have with high probability $(1 - e^{-n \cdot p/3} = 1 - e^{-n^{\frac{5}{8}}/3})$ less than $2\mu = 2n \cdot p = 2n^{\frac{5}{8}}$ vertices. We will put one cop on each of the vertices in \mathcal{C} .

Let r be the position of the robber. If the size of the neighborhood of r ($|B(r, 1)|$) is greater than $n^{\frac{3}{8}} \cdot \log^2 n$, then (by Lemma 4.3) with probability at least 0.9 (we can actually prove the lemma for probability $1 - o(1)$) there is a cop in the robber's neighborhood who will capture the robber at the very beginning. So we may assume that $|B(r, 1)| < n^{\frac{3}{8}} \cdot \log^2 n$.

Consider the bipartite graph H with partition classes $B(r, 1)$ and $B(r, 3) \cap \mathcal{C}$. The edge uv exists in H if and only if there is a path of length at most 2 between (the corresponding vertices) u and v in G . If we can move some cops from $B(r, 3)$ in at most 2 moves to occupy

all vertices of $B(r, 1)$, then there is a matching in H that covers all vertices of $B(r, 1)$. This means that in one move, the cops can guard $B(r, 1)$ and the robber cannot move and therefore will be captured.

So we may assume that this matching does not exist. Therefore, by Hall's Theorem, there is a set $S_1 \subseteq B(r, 1)$ such that $|S_1| > |N_H(S_1)| = |B(S_1, 2) \cap \mathcal{C}|$. The vertices in $B(r, 1) \setminus S_1$ will be guarded by cops in at most one move and the robber cannot use them or he will get caught. As a consequence $|B(S_1, 2) \cap \mathcal{C}| < |B(r, 1)|$. If $|B(S_1, 2)| \geq n^{\frac{3}{8}} |B(r, 1)| \cdot \log^2 n$, then by Lemma 4.3, $|B(S_1, 2) \cap \mathcal{C}| \geq |B(r, 1)|$ (with probability at least 0.9). This would be a contradiction. So we may assume that $|B(S_1, 2)| < n^{\frac{3}{8}} |B(r, 1)| \cdot \log^2 n \leq n^{\frac{6}{8}} \cdot \log^4 n$.

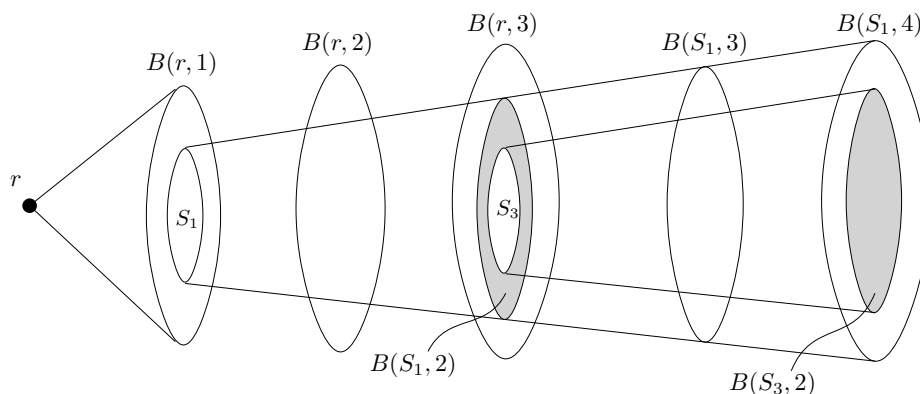


Figure 4.1: Neighbors of r .

Let us mention that this basically proves that $c(G) \leq n^{\frac{3}{4}+o(1)}$. Note that since the diameter is 4, the cops can occupy a vertex of $B(S_1, 2)$ in 4 moves and also can guard it (get to one of its neighbors) in 3 moves. Therefore, since $|B(S_1, 2)| < n^{\frac{3}{4}+o(1)}$, we can move our (another set of $n^{\frac{3}{4}+o(1)}$) cops and in 3 moves guard all vertices of $B(S_1, 2)$ and with probability at least 0.9 catch the robber. Also if we consider $p = n^{-\frac{1}{3}}$, then we can prove that $c(G) < n^{\frac{2}{3}+o(1)}$. This strategy has been used in the work of Lu and Peng [25] and Scott and Sudakov [32], as well. However, there is some improvement possible that was not discovered in [25, 32].

We may assume that the robber does not stay in r and moves to a vertex in S_1 (we can send one cop to force the robber to move). Again, we would like to send some cops from $B(S_1, 4)$ to occupy vertices in $B(S_1, 2)$.

Let \mathcal{I} be a random subset of vertices of G , where a vertex v is in \mathcal{I} with probability $n^{-\frac{1}{8}} \cdot \log^8 n$. So the probability that $|\mathcal{I}|$ is smaller than $n^{\frac{7}{8}} \cdot \log^6 n$ is at most $e^{-n^{\frac{7}{8}} \cdot \log^8 n / 3}$. Also let \mathcal{C}' be a random subset of vertices of \mathcal{I} , where a vertex v is in \mathcal{C}' (independently uniformly at random) with probability $p' = n^{-\frac{1}{4}}$. We would like to put one cop on each vertex of \mathcal{C}' and we would call vertices of \mathcal{I} *imaginary cops*. Note that the (real) cops that we are using in this step are different from the ones in the first step.

If there is a matching between vertices in $B(S_1, 2)$ and the imaginary cops in $B(S_1, 4)$ then we can occupy $B(S_1, 2)$ with imaginary cops in two moves. Otherwise, using the previous argument and Hall's Theorem, there is a set $S_3 \subseteq B(S_1, 2)$ such that $|S_3| > |B(S_3, 2) \cap \mathcal{I}|$. Thus, $|B(S_3, 2) \cap \mathcal{I}| < |B(S_1, 2)|$ and therefore (with probability at least 0.9) we have $|B(S_3, 2)| < n^{\frac{1}{8}} |B(S_1, 2)| \cdot \log^2 n < n^{\frac{7}{8}} \cdot \log^6 n$.

Note that since we have at least $n^{\frac{7}{8}} \cdot \log^6 n$ imaginary cops and the set $B(S_3, 2)$ is at distance at least 3 from r , we will have enough time to get our (imaginary) cops to occupy all vertices in $B(S_3, 2)$. Therefore after two or four moves, the robber's entire neighborhood will be occupied by imaginary cops.

Let us discuss these two cases separately. At the very beginning of the game, the robber is forced to move to S_1 and then to a vertex r' in $B(S_1, 1)$. Now he has two options. If he wants to move to $B(S_1, 2) \setminus S_3$, then, since it is occupied by imaginary cops and with probability $n^{-\frac{1}{4}}$ these cops are real cops, by Lemma 4.3,

$$|B(r', 1) \cap (B(S_1, 2) \setminus S_3)| < n^{\frac{1}{4}} \cdot \log^2 n.$$

On the other hand, if he decides to move to S_3 and then to a vertex r'' in $B(S_3, 1)$, then, since $B(S_3, 2)$ is occupied by imaginary cops, with the same argument,

$$|B(r'', 1)| < n^{\frac{1}{4}} \cdot \log^2 n.$$

So in both cases, we get to a point where the first neighborhood of the robber (with probability at least 0.9) is of size at most $n^{\frac{1}{4}} \cdot \log^2 n$. Repeating the argument of the first step for r' or r'' (instead of r), shows that $|B(S'_1, 2)|$ (or $|B(S''_1, 2)|$) $< n^{\frac{5}{8}} \cdot \log^4 n$ and since we have enough time, we can move another set of $n^{\frac{5}{8}+o(1)}$ (real) cops to occupy $B(S'_1, 2)$ (or $B(S''_1, 2)$) and therefore with high probability we can capture the robber. Therefore the cop number of graph with diameter at most 4 is bounded above by $n^{\frac{5}{8}+o(1)}$.

In the next part we will repeat this argument to improve this bound to $n^{\frac{3}{5}+o(1)}$. For simplicity, we will drop the poly-log terms and we will assume that n is sufficiently large (the exponent of $\log n$ will not depend on n and these terms will get covered by the $o(1)$ in the exponent of the final answer).

4.1.1 Repeating the argument to improve the result

In the previous section we introduced an approach to catch the robber with $n^{\frac{5}{8}+o(1)}$ cops in a graph with diameter 4. In this section we want to improve this result to $n^{\frac{3}{5}+o(1)}$ and in order to do that let us repeat the same process in a more general manner. Assume that we have $n^{1-\alpha}$ cops and they are randomly positioned throughout the graph. Therefore the probability that a vertex contains a cop is $p = n^{-\alpha}$. Using the same argument as above, it

is easy to see that $|B(r, 1)| < n^\alpha$ or with high probability we will catch the robber in the first round.

Define S_1 and $B(S_1, 2)$ as above and similarly conclude that $|B(S_1, 2) \cap C| < |B(r, 1)|$ and therefore $|B(S_1, 2)| < n^{2\alpha}$.

Now consider $n^{1-\gamma}$ imaginary cops (with probability $n^{-\gamma}$) where each imaginary cop is a real cop with probability $n^{\alpha-\gamma}$. Again by using the same strategy, conclude that $|B(S_3, 2)| < n^{2\alpha+\gamma}$ and if $2\alpha+\gamma < 1-\gamma$, then we can occupy the whole $B(S_3, 2)$ by imaginary cops. Therefore, the robber will face a neighborhood which is occupied by imaginary cops and hence the density of real cops in his neighborhood is $n^{\alpha-\gamma}$.

Now we can repeat the strategy again to get a better density of cops. Assuming that we have $n^{1-\beta}$ (new) imaginary cops, we will have:

$$|B(r, 1)| < n^{\alpha-\gamma}, \quad |B(S_1, 2)| < n^{2\alpha-\gamma}, \quad |B(S_3, 2)| < n^{2\alpha-\gamma+\beta}$$

and if $2\alpha - \gamma + \beta < 1 - \beta$, then the new set of imaginary cops can occupy $B(S_3, 2)$ to get a better density. We can continue doing this until $\gamma = \beta$ which means that $\gamma < 1 - 2\alpha$. After improving the density we need to capture the robber with real cops.

In this phase of our strategy we have $|B(r, 1)| < n^{\alpha-\gamma}$, $|B(S_1, 2)| < n^{2\alpha-\gamma}$ and we want $|B(S_1, 2)| < n^{1-\alpha}$ to be able to occupy $B(S_1, 2)$ with real cops. So we have the following two conditions to hold:

$$\gamma < 1 - 2\alpha \text{ and } 2\alpha - \gamma < 1 - \alpha.$$

Combining these conditions we get $\alpha < \frac{2}{5}$ which means that we need $n^{\frac{3}{5}+o(1)}$ cops and therefore $c(G) \leq n^{\frac{3}{5}+o(1)}$ when G is a graph with diameter at most 4.

4.2 When the diameter is 3

In this section we will consider graphs with diameter 3. Note that the argument in the previous section works and we already know that $c(G) < n^{\frac{3}{5}+o(1)}$ when G is a graph of diameter 3. In this section we will try to improve this result.

We will repeat the first phase of the strategy for graphs of diameter 4 to increase the density of cops in the first neighborhood of the robber. As mentioned in the previous section, we can increase the density of cops up to $n^{-(2\alpha-\gamma+\beta)}$ and we calculated that $\gamma < 1 - 2\alpha$.

Now consider $n^{1-\eta}$ imaginary cops and repeat the strategy but instead of occupying $B(S_3, 2)$ with imaginary cops we will occupy $B(S_3, 1)$ and $B(S_3, 2)$ (note that $B(S_3, 1) \subseteq B(S_3, 2)$). We need $|B(S_3, 2)| < n^{1-\eta}$ and we know that $|B(S_3, 2)| < n^{4\alpha+\beta-1}$. So we have $4\alpha + \beta - 1 < 1 - \eta$.

Then we will have the robber in a position, r , that both his first and second neighborhood is occupied by $n^{1-\eta}$ imaginary cops and therefore the density of real cops in the first and

second neighborhood are $n^{\eta-\alpha}$. We have:

$$|B(r, 1)| < n^{\alpha-\eta}, \quad |B(r, 2)| < n^{2\alpha-2\eta}$$

Now we can assign one (real) cop to each vertex of $B(r, 2)$ and since the diameter of the graph is 3 the cops will guard the vertices of $B(r, 2)$ before the robber can enter it. So the only condition that we need is $|B(r, 2)| < n^{1-\alpha}$, which means $2\alpha - 2\eta < 1 - \alpha$. To summarize, we need the following conditions:

$$4\alpha + \beta - 1 < 1 - \eta, \quad 2\alpha - 2\eta < 1 - \alpha, \quad 2\alpha - \beta < 1 - \alpha$$

Combining these inequalities we can get $\alpha < \frac{7}{17}$ and therefore $c(G) = O(n^{\frac{10}{17}+o(1)})$ when diameter of G is 3.

4.3 General case

Now let us consider the general case where the diameter of the graph is d . We would like to find an upper bound for the cop number of these graphs.

Theorem 4.4. *Let G be a graph of diameter d , then*

$$c(G) \leq n^{1 - \frac{2}{2^{\lceil \log d \rceil + 1}} + o(1)}.$$

Proof. Assume that we have we have $n^{1-\alpha}$ cops randomly positioned throughout vertices of G .

Using a similar argument as in section 4.1 we will get (note that we are ignoring the $\log n$ factors):

$$|B(r, 1)| < n^\alpha, \quad |B(S_1, 2)| < n^{2\alpha}, \quad \dots, \quad |B(S_{2^k-1}, 2^k)| < n^{(k+1)\alpha} \quad \text{for } k = 1, \dots, \lceil \log d \rceil.$$

Now assume that we have a set of $n^{1-\gamma}$ imaginary cops where each of them is a real cops with probability $n^{\gamma-\alpha}$.

Let $k = \lceil \log d \rceil - 1$. If there is a matching between vertices in $B(S_{2^k-1}, 2^k)$ and the imaginary cops in $B(S_{2^k-1}, 2 \cdot 2^k)$ then we can occupy $B(S_{2^k-1}, 2^k)$ with imaginary cops in 2^k moves. Otherwise, using the previous argument and by Hall's Theorem, there is a set $S_{2^{k+1}-1} \subseteq B(S_{2^k-1}, 2^{k+1})$ such that $|S_{2^{k+1}-1}| > |B(S_{2^k-1}, 2^{k+1}) \cap I|$. Thus

$$|B(S_{2^{k+1}-1}, 2^{k+1}) \cap I| < |B(S_{2^k-1}, 2^k)|,$$

and therefore $|B(S_{2^{k+1}-1}, 2^{k+1})| < n^\gamma |B(S_{2^k-1}, 2^k)| < n^{(k+1)\alpha+\gamma}$. Note that since we have $n^{1-\gamma}$ (real or imaginary) cops and the set $B(S_{2^{k+1}-1}, 2^{k+1})$ is at distance more than d from r , if $1 - \gamma > (k + 1)\alpha + \gamma$, then we can get our cops to occupy all vertices in $B(S_{2^{k+1}-1}, 2^{k+1})$.

Therefore after 2^k or 2^{k+1} moves, the robber's entire neighborhood will be occupied by imaginary cops and the density of real cops will improve from $n^{-\alpha}$ to $n^{\gamma-\alpha}$. Thus, for the new position of the robber, with probability at least 0.9, we have:

$$|B(r, 1)| < n^{\alpha-\gamma}, \quad |B(S_1, 2)| < n^{2\alpha-\gamma}, \quad \dots, \quad |B(S_{2^k-1}, 2^k)| < n^{(k+1)\alpha-\gamma}.$$

Repeating the argument with a new set of $n^{1-\beta}$ imaginary cops can improve the density of real cops in the first neighborhood of the robber and we can get

$$|B(S_{2^{k+1}-1}, 2^{k+1})| < n^{(k+1)\alpha-\gamma+\beta}.$$

And if $(k+1)\alpha - \gamma + \beta < 1 - \beta$, then the new imaginary cops can improve the density. We can continue this until $\gamma = \beta$ which means that $\gamma = \beta < 1 - (k+1)\alpha$.

Combining $(k+1)\alpha - \gamma < 1 - \alpha$ and $\gamma < 1 - (k+1)\alpha$, we have $\alpha < \frac{2}{2k+3}$. Therefore, when d is the diameter of G , then, $c(G) = O(n^{1-\frac{2}{2\lceil \log d \rceil + 1} + o(1)})$. \square

4.4 Graphs of high girth

In this section we will use girth of graphs to improve the strategy of cops and decrease the cop number. Recall that for every subset A of vertices of G and integer i we define $B(A, i)$ to be the ball of radius i around A , that is all the vertices of G which can be reached from some vertex in A by a path of length at most i . Also define $N(A, i) = B(A, i) \setminus B(A, i-1)$.

Let G be a graph with girth g , we define $\rho = \lfloor \frac{g+1}{4} \rfloor$. The following lemma is our main tool.

Lemma 4.5. *For every vertex u , two cops can guard $B(u, \rho)$.*

Proof. For $\rho = 1$ the proof is clear. So we may assume that $\rho \geq 2$ and therefore $g \geq 7$. Let us first assume that two cops are in u but the robber is already in $B(u, 2\rho - 1)$. To start our strategy we need to push the robber out of $B(u, 2\rho - 1)$. Since the robber is in $B(u, 2\rho - 1)$, there is a unique shortest path from u to the robber's position and there is no cycle in $B(u, 2\rho - 2)$. Sending a cop to follow the robber will force him to move either towards u and eventually get captured or to get out of $B(u, 2\rho - 1)$ (or enters $N(u, 2\rho - 1)$ from another vertex). Note that the cop that we need to send to follow the robber can be one of the cops in u but to avoid unnecessary complications (in Theorem 4.6 having one more cop is not important) we have used an extra cop.

Now let us assume that C_1 and C_2 are in u and the robber has entered $N(u, 2\rho - 1)$ and it is the cops' turn. Because of the girth condition, there is a unique vertex in $N(u, \rho)$ that the robber can enter in $\rho - 1$ moves. Therefore C_1 will move one step towards that vertex to be able guard it in $\rho - 1$ moves. From now on, C_1 will copy the movements of the robber, if he gets closer to $N(u, \rho)$, then C_1 will get closer to it as well and if the robber

changes his mind and gets away from that vertex in $N(u, \rho)$, then C_1 will get back (step by step) to u . Note that when the robber is in $N(u, 2\rho - 1)$, he can change the vertex that he is attacking in $N(u, \rho)$ in one move. In this case, the other cop, C_2 , will move one step towards the robber and C_1 will get back to u . So if the robber is in $N(u, 2\rho - k)$, then there is a cop in $N(u, k)$. Therefore by using this strategy the two cops can prevent the robber from entering $N(u, \rho)$ and therefore can guard $B(u, \rho)$. \square

Now we have the tool to improve the result from the previous section.

Theorem 4.6. *Let G be a graph of diameter d and girth g and let $\rho = \lfloor \frac{g+1}{4} \rfloor$. Then*

$$c(G) \leq n^{1 - \frac{2}{2\lceil \log(d/\rho) \rceil + 1} + o(1)}.$$

Proof. Let G be a graph of diameter d and girth g . We will play the same strategy as in the previous section with the exception that instead of having a stationary cop in a vertex u , we will put two cops in it and will make them guard $B(u, \rho)$, as shown in Lemma 4.5.

In the first step of our previous strategy, we used the probability of existing a cop in the first neighborhood of the position of the robber to bound the size of the first neighborhood of the position of the robber. To get a better result, assume that we have $2n^{1-\alpha}$ cops and place two cops (instead of one) randomly on each vertex with probability p .

Let r be the position of the robber. We will send a cop to follow the robber and force him to move to fulfill the requirement of Lemma 4.5. After some steps, if there is a vertex in $B(r, \rho)$ that was selected to contain (two) cops, then it means that $r \in B(u, \rho)$ where u contains two cops. By Lemma 4.5, the robber should have been captured by now. So by Lemma 4.3 (and ignoring the $\log n$ term) we may assume that $|B(r, \rho)| < n^\alpha$.

In the next step (in the previous approach) we defined the set S_1 to be (roughly) the set of vertices that cannot be guarded by the cops in $B(r, 3)$. We can redefine S_1 to be the set of vertices in $B(r, \rho)$ that cannot be guarded by the cops in $B(r, 2\rho)$. Note that although the cops are moving first, we cannot bring cops from $B(r, 3\rho)$ to cover S_1 and the reason is that ρ can be more than 1. As usual, we can see that not only $|S_1| > |B(S_1, \rho) \cap \mathcal{C}|$, but also with high probability we have $|B(S_1, \rho)| < n^{2\alpha}$. Note that in the next step we can define S_3 in $B(S_1, \rho)$ and calculate the upper bound for $|B(S_3, 2\rho)|$. The radius of the ball around S_i will grow exponentially and we have

$$|B(S_{2^k+1}, 2^k \rho)| < n^{(k+1)\alpha} \quad \text{for } k = 0, \dots, \lceil \log \frac{d}{\rho} \rceil.$$

Now we can follow the previous strategy to get $\alpha < \frac{2}{2k+3}$ and therefore (by replacing k with $\lceil \log \frac{d}{\rho} \rceil - 1$) the cop number is at most $n^{1 - \frac{2}{2\lceil \log(d/\rho) \rceil + 1} + o(1)}$. \square

4.5 Digraphs of bounded diameter

In this section we will consider digraphs of diameter two and bipartite digraphs of diameter three, which are the digraphs that between any two vertices u and v there is a directed path of length at most two (for the first case) or three (for the second case). Note that the digraph will automatically be strongly connected.

We will basically generalize the method that was introduced in [35] to digraphs.

Lemma 4.7. *Let $k > 0$ be an integer, D be a digraph of diameter 2 or a bipartite digraph of diameter 3, and let H be a sub-digraph of D , such that the maximum out-degree of H is at most k . Suppose the robber is restricted to move on the edges of H , while the cops can move on D as usual. Then $k + 1$ cops can catch the robber.*

Proof. Let r be the position of the robber, a vertex of out-degree $l \leq k$ with out-neighbors v_1, \dots, v_l and let c_1, \dots, c_{k+1} be our set of cops.

Let us first assume that D is a digraph of diameter 2. Assign c_i to cover v_i (for $i = 1, \dots, l$). Since the diameter of D is 2 (the diameter of H can be different), each c_i can get to v_i in at most two moves and therefore in one move can get to an in-neighbor of it. Thus, in one move, the cops can position themselves in a way that the robber cannot use any of its out-neighbors. So the robber cannot move. Now send another cop (we have at least one cop more than the number of out-neighbors) to capture the robber.

Now let D be a bipartite digraph of diameter 3 and let $V(D) = L \cup R$ be the bipartition of vertices of D . Move k cops to R and let the last cop to follow the robber and force him to move. Consider the position of the robber, r , when $r \in L$. Now the out-neighbors of r , v_1, \dots, v_l are in R . Assign c_i to control v_i . Since the diameter of the digraph is 3, there is a directed path of length at most 2 between the position of c_i and v_i . So each c_i by moving once towards v_i can guard it. Therefore the robber cannot use any of v_i 's (without being caught) and the cop who is following the robber will catch him. \square

Theorem 4.8. *Let D be a digraph of diameter 2, or a bipartite digraph of diameter 3, of order n . Then*

$$c(D) \leq \sqrt{2n}.$$

Proof. The proof will go by induction on m , the size of $H \subseteq D$. It is clear that $c(H) = 1$ when $|H| = 1$ or 2. Now let $m \geq 3$ and assume that there is no vertex of out-degree greater than or equal to $\lfloor \sqrt{2m} \rfloor$. Then by Lemma 4.7 we are done and $c(H) \leq \lfloor \sqrt{2m} \rfloor$. Now assume that there is a vertex v of out degree at least $\lfloor \sqrt{2m} \rfloor$. Put a stationary cop on v to protect v and its out-neighborhood. From now on the robber cannot use these vertices or he will be captured by the stationary cop. Therefore, we can remove v and its out-neighbor from H to make H' . Note that $|H'| \leq m - \lfloor \sqrt{2m} \rfloor - 1$. By the inductive hypothesis, we have $c(H) \leq 1 + c(H') \leq 1 + \sqrt{2(m - \lfloor \sqrt{2m} \rfloor - 1)} \leq \lfloor \sqrt{2m} \rfloor$.

Since this inequality holds for all subgraphs H of D , then, $c(D) \leq \sqrt{2n}$. □

Chapter 5

Graph Operations

In this chapter, we will first consider different graph operations and their effect on the cop number of graphs and then we will use them to construct sub-cubic graphs with high cop number.

5.1 Graph Operations

In this section, we will first consider the effect of subdividing an edge on the cop number of graphs and then we will briefly consider deleting an edge, contracting an edge and deleting a vertex.

5.1.1 Subdividing an edge

Let G_s be a graph obtained from subdividing an edge uv of a graph G and let the new added vertex be x . See Figure 5.1.

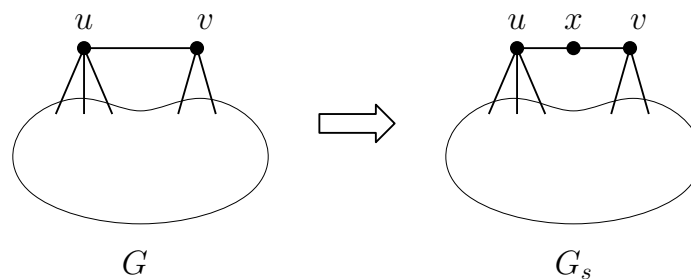


Figure 5.1: Subdividing the edge uv .

Lemma 5.1. *For every graph G , $c(G_s) \geq c(G) - 1$, in other words, by splitting an edge the cop number might decrease at most by one.*

Proof. Let us assume for contradiction that $c(G_s) < c(G) - 1$. Put a cop on u , this cop will prevent the robber to use the edge uv . Consider the strategy of cops in G_s , they can win

by $c(G_s)$ cops. Apply the same strategy in G (by $c(G_s)$ cops). Note that the robber cannot use the edge uv in G (which is equivalent to the path uxv in G_s) but the cops can use them. This restriction for the robber might decrease the cop number but cannot increase it. So we can win by $c(G_s)$ cops in $G \setminus \{uv\}$ and therefore $c(G) \leq c(G_s) + 1$, contradiction. \square

Lemma 5.2. *For every graph G , $c(G_s) \leq 2c(G)$, in other words, by splitting an edge the cop number might increase but it cannot become more than twice of the cop number of the original graph.*

Proof. Put $c(G)$ cops on the new vertex in G_s (vertex x in Figure 5.1) and consider the strategy of $c(G)$ cops to capture the robber in G . Repeat this strategy on G_s and whenever k cops are using the edge from u to v , in G_s , k cops will move from u to x and k cops that are already in x will move to v and therefore the cops can keep up with their strategy in G to catch the robber. \square

Lemma 5.3. *If G is a smallest k -cop-win graph, then $c(G_s) \leq c(G)$.*

Proof. Since G is a minimum k -cop-win graph, then removing an edge will decrease the cop number exactly by one. So, in G_s put a cop on x , then the robber cannot use uxv which is equivalent to uv in G . Since $c(G) - 1$ cops can capture the robber on $G \setminus uv$, then $c(G) - 1$ cops can capture him in $G_s \setminus uxv$ and therefore $c(G_s) \leq c(G)$. \square

Conjecture 5.4. *If G is a graph of the smallest order that is k -cop-win with $k \geq 3$, then $c(G_s) = c(G) - 1$.*

Conjecture 5.5. *For every graph G , $c(G) - 1 \leq c(G_s) \leq c(G) + 1$.*

We have some examples to support the above conjectures. C_3 is a graph with cop number 1, but when we subdivide an edge it becomes a C_4 and therefore the cop number increases to 2. On the other hand, Petersen is the smallest graph that needs 3 cops (see Theorem 6.1) but when we subdivide an edge the cop number decreases to 2. See Lemma 5.6. Let P_s be the graph obtained from Petersen by subdividing an edge, then we have the following lemma.

Lemma 5.6. $c(P_s) = 2$.

Proof. One of the properties of minimum k -cop-win graph is that $k - 1$ cops can force the robber to use any of the vertices and any of the edges while cops can avoid using that edge or vertex. Otherwise, removing the edge or vertex will not affect the cop number which is a contradiction to minimality.

Now consider the strategy of two cops to force the robber to use edge uv and repeat the strategy on P_s . So the robber will be forced to move from u (v) to x and then from x to v (u). Since the diameter of Petersen graph is 2, cops can guard any vertex by just one move. So when the robber is in x , one of the cops will move to guard u and the other one

will move to guard v and the robber will be in a trap and will get captured in the next two moves. \square

5.1.2 Subdividing all edges at the same time

Let G_a be the graph obtained from G by subdividing all edges of G the same number of times at the same time.

Lemma 5.7. *For every graph G , $c(G_a) \geq c(G)$.*

Proof. Let us assume for contradiction that $c(G_a) < c(G)$ and let k be the number that we have subdivided each edge of G to get G_a . So when we have $c(G_a)$ cops the robber can escape in G . Consider the escaping strategy of the robber in G . We will use this strategy to escape in G_a . If v is a non-subdivided vertex in G_a we will define the shadow of $v \in G_a$ to be the corresponding vertex $v \in G$. Also if v is a subdivided vertex and is closer than $\lfloor \frac{k}{2} \rfloor$ to a non-subdivided vertex, u , then v will have the same shadow as u . Note that when k is even this definition is well defined but when k is odd and v is the middle vertex of a subdivided edge, then the definition of shadow will depend on the direction of the movement of the robber (cop) and the shadow of this vertex will be the same as the shadow of the previous position of the robber (cop) in G_a . Also at the beginning of the game if a cop or robber is in such a vertex, we can define the shadow arbitrarily between one of the end points of the subdivided edge.

At the beginning of the game if a cop chooses vertex v (in G_a), the robber will assume that the cop has picked the shadow of v in G and he will play the escaping strategy in G . Note that in each $k + 1$ moves in G_a , the shadow of each cop or the robber in G will move at most once.

Every $k + 1$ moves in G_a will move each of the cops in G at most once and according to this move, the robber will move once in G . This robber's move in G can be mapped to G_a by at most $k + 1$ moves. So there was a way for the robber in G_a to move in such a way that his shadow follows the escaping strategy of the robber in G . Note that since the robber in G was not captured in his last move, the robber will not get captured in G_a in his last $k + 1$ moves.

Therefore since the robber can escape in G from $c(G_a)$ cops, by copying this strategy, the robber can escape from $c(G_a)$ cops in G_a , which is a contradiction. \square

Lemma 5.8. *For every graph G , $c(G_a) \leq c(G) + 1$.*

Proof. Using $c(G)$ cops we, can capture the robber in G . Consider this winning strategy in G . We will use the extra cop to force the robber to move forward. One of the cops will move directly towards the robber and since his distance from the robber is finite, the robber can stay where he is or move backward (use the edge that he has used in his last move) finite number of times. So we may assume that the robber will always move forward.

At the beginning of the game, put all cops in non-subdivided vertices in G_a . If the robber chooses a subdivided vertex, we wait for him to get to a non-subdivided vertex (the extra cop will force him to get there). When the robber gets to a non-subdivided vertex, the position of the shadow of the robber will be fixed in G . From now on, the cops will follow the winning strategy in G and will catch the robber. \square

Example. Subdividing all edges of C_3 once will give us C_6 and we know that $c(C_6) = 2 = c(C_3) + 1$. On the other hand, we know that cop number of any tree is one and subdividing each edge will give us another tree. So the cop number will stay the same. For a more challenging example one can consider Petersen graph. It is not difficult to show that the cop number of subdivided Petersen graph is 3.

5.1.3 Contracting an edge

Let uv be the edge that we want to contract to make G_c . Let the vertex that got created after contracting the edge uv be w . We have the following easy proposition.

Proposition 5.9. *For every graph G , $c(G_c) \leq c(G) + 1$.*

Proof. Place a cop on w on G_c and play the game on G . Note that a cop can easily use the vertex w instead of the edge uv on G (and actually save one step). Also note that the robber cannot use the edge uv on G which is equivalent to the vertex w on G_c and therefore this contraction will not help him. So in the remaining graph $c(G)$ cops can win the game and therefore in G_c we will need at most $c(G) + 1$ cops. \square

As an example, consider the Petersen graph and subdivide one of its edges once. We proved that this graph is 2-cop-win. But contracting the subdivided edge will give us the Petersen graph which is 3-cop-win.

5.1.4 Deleting an edge

Let uv be an edge that we want to remove from the edge-set of G to make G_d .

Proposition 5.10. *For every graph G , $c(G_d) \geq c(G) - 1$.*

Proof. Assume that $c(G_d) < c(G) - 1$. Then place a cop on u and play on G' . The robber cannot use the edge uv and the cops can play their strategy on G' . Therefore by using at most $c(G) - 2 + 1$ cops we can win the game which is a contradiction. \square

Since deleting an edge can make the graph disconnected, the best that we can hope for the upper bound is $2c(G)$. As an example, consider two Petersen graphs attached by a single edge. This graph needs 3 cops but deleting the connecting edge will increase the cop number to six.

Proposition 5.11. *For every graph G , $c(G_d) \leq 3c(G)$.*

Proof. If the deleted edge is a cut edge, then it is clear that $2c(G)$ cops can capture the robber. Place $c(G)$ cops on u and the other $c(G)$ cops on v . The robber will be in one of the components, use the corresponding $c(G)$ cops to capture the robber.

Now assume that uv is not a cut edge. Place $c(G)$ number of cops on u and $c(G)$ number of cops on v . Then if we need k cops to go from u to v (or similarly from v to u), when these k cops reach u then k of the cops on v will continue their move and will play their roles afterwards. Now we will start moving k cops (on a shortest path) from u to v . Note that these k cops will get to v before (or at the same time as) one or some of the cops that “used” the edge uv want to go from u to v again and we will have $c(G)$ number of cops on u and v ready to be used. Therefore $c(G_d) \leq 3c(G)$. \square

5.1.5 Deleting a vertex

Deleting a vertex can increase the cop number arbitrarily. To see this, consider an arbitrary graph G and add a vertex v and connect it to all vertices of G and make G' . It is easy to see that $c(G') = 1$ but removing one vertex can increase it to $c(G)$.

We will discuss the lower bound of this operation in the next chapter.

5.2 Graphs of bounded degree

In this section, we will show that the cop number of graphs of bounded degree is not bounded. This result was previously known [3], but our new approach is less restrictive.

Let G be a graph and v a vertex of degree k in G . Also let w_1, w_2, \dots, w_k be neighbors of v in G . We want to build a gadget, A_v , that can be replaced with v without affecting the cop number. Consider new vertices x_1, x_2, \dots, x_k and connect x_i to w_i ($i = 1, 2, \dots, k$) with a single edge.

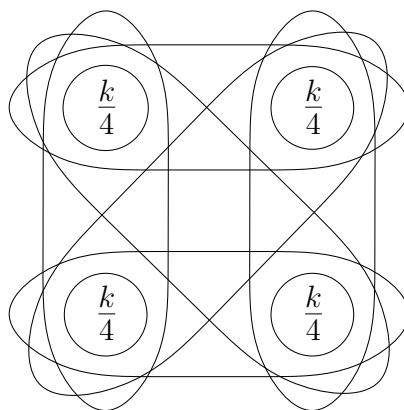


Figure 5.2: Partitioning vertices into 4 sets of size $\frac{k}{4}$ and considering each pair of them to get six sets of size $\frac{k}{2}$.

Partition x_i 's into four almost equal sets and consider each pair of them to get six sets of size (almost) $\frac{k}{2}$. See Figure 5.2. Add six (which is $\binom{4}{2}$) new vertices, y_1, y_2, \dots, y_6 and attach all x_i 's in each of these 6 sets to the corresponding y_j ($j = 1, \dots, 6$).

Replace each vertex v of G with A_v to get G' . The main property of this gadget is that one can get from x_i to x_j ($i \neq j$) in exactly two moves. Also degree of each x_i is exactly 4 and degree of each of y_i is (almost) $\frac{k}{2}$.

Also for vertices of degree $k \leq 4$ we can use the gadgets shown in Figure 5.3.

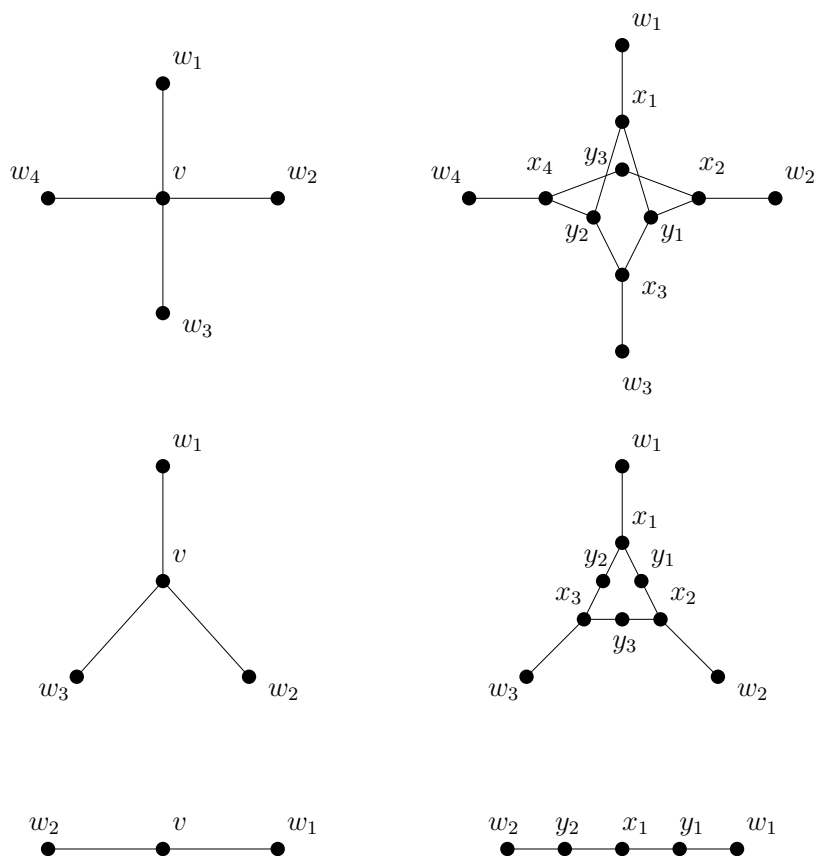


Figure 5.3: The gadget used for vertices of degree 2, 3 or 4.

Now to optimize this idea, instead of partitioning vertices into 4 parts, let's use m partitions and apply the same technique. We would like to consider every pair of these parts and attach them to a y_i . So we will have $\frac{m(m-1)}{2}$ pairs (and therefore $\frac{m(m-1)}{2}$ y -vertices). Degree of each y_i will become $\frac{2k}{m}$ and degree of each x_i becomes m . So the optimum value of m is when the degree of x_i 's and y_i 's are almost equal to each other. So $\frac{2k}{m} = m$ and therefore $m = \sqrt{2k}$. So instead of one vertex of degree k we will have less than $2k$ vertices (x_1, \dots, x_k and $y_1, \dots, y_{\frac{\lceil \sqrt{2k} \rceil (\lceil \sqrt{2k} \rceil - 1)}{2}}$) of degree $\lceil \sqrt{2k} \rceil$. Using this approach and repeating it, we can decrease the degree of each vertex down to 3.

Lemma 5.12. *For every graph G , $c(G') \geq c(G)$.*

Proof. Let us assume for a contradiction that $c(G') < c(G)$. Therefore $c(G')$ cops cannot capture the robber in G . Consider this escaping strategy, we will use it to show that the robber can escape from $c(G')$ cops in G' .

At the beginning of the game each cop will choose a vertex (in G') as their initial position. Each vertex is in a unit, A_v , which corresponds to the vertex, v , in G . The robber will assume that each cop is in the corresponding vertex in G and plays the winning strategy in G . Since the robber has a vertex v to pick in G , the robber can pick a x_i in A_v in G' . From now on, the robber will move based on the movements of (the shadow of) cops in G . Whenever shadows of cops have moved in G , the robber will have an escaping move in G which can be translated into a series of at most 3 moves in G' . Note that since the robber won't get captured in the next move of cops in G , the robber won't get captured in G' in the next 3 moves. Also note that the difference between the position of shadow of each cop in G before and after 3 moves (in G') is at most 1 and after that it will be the robber's turn to move, so he can continue the escaping strategy. Therefore the robber in G' can copy the strategy of the robber in G and escape from $c(G')$ cops in G' which is contradiction.

Another way of looking at it is that whenever the robber is captured in G' , his shadow in G has been captured as well. So if we can capture the robber in G' by using $c(G')$ cops, then we can capture the robber in G with $c(G')$ cops, which is a contradiction. \square

Now we give a new proof of a result of Andreae [3]. In this approach we will decrease the degree of vertices of the graph and by repeating it we will get Corollary 5.14 and Theorem 5.15. These results are also applicable to digraphs, see Section 5.3.

Theorem 5.13. *For any constant c and k there exist graphs of degree at most k whose cop number is at least c .*

Proof. We know that for any c there is a graph G that $c(G) \geq c$. Build G' from G (as described above) and decrease the degree of each vertex. Lemma 5.12 shows that the cop number cannot decrease. Therefore by repeating this strategy (at most $\log \log \Delta(G)$ times) we will get a graph of maximum degree k whose cop number is at least c . \square

Let us assume that G is a graph on n vertices with $c(G) = c$ and $\Delta(G) = d$, then we have:

Graph	# vertices	Δ	cop number
G	n	d	c
G'	$\leq 2dn$	$\lceil \sqrt{2d} \rceil$	$\geq c$

By repeating the strategy to get G'' from G' we will have:

G''	$\leq 2^2 d \lceil \sqrt{2d} \rceil n$	$\lceil \sqrt{2 \lceil \sqrt{2d} \rceil} \rceil$	$\geq c$
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If $2^{2^{k-2}+1} < d \leq 2^{2^{k-1}+1}$, then in k steps the maximum degree of the graph will become 3. In other words, by repeating the argument $k \leq \log \log \frac{d}{2} + 2$ times, degrees of all vertices will be at most 3 and the number of vertices of the graph will be at most

$$2^k 2^{1/2+1/4+\dots+1/2^{k-1}} d^{1+1/2+1/4+\dots+1/2^k} n \leq 2^k \cdot 2^2 \cdot d^2 \cdot n \leq O(d^2 n \log d).$$

We have essentially proved the following corollary.

Corollary 5.14. *Let G be a graph on n vertices with maximum degree d , then there exist a sub-cubic graph H on $O(d^2 n \log d)$ vertices such that $c(H) \geq c(G)$.*

5.2.1 Sub-cubic graphs

The goal of this part is to build sub-cubic graphs with high cop number. Let G be the incidence graph of a projective plane. As shown in Proposition 1.5, $c(G) = \Delta(G) = \Theta(\sqrt{n})$, where n is the number of vertices of G . Applying the above technique we will gain a sub-cubic graph on $O(n^2 \log n)$ vertices with cop number at least $\Theta(\sqrt{n})$. Now by a simple change of variable we can see that there are sub-cubic graphs on n vertices with cop number $\geq \Theta(n^{1/4-\epsilon})$.

As another example, let $\omega : \mathbb{N} \rightarrow \mathbb{N}$ be any function such that $\lim_{n \rightarrow \infty} \omega(n) = \infty$ and let $G \in \mathcal{G}_{n,p}$ for $p = \frac{\omega(n)}{n}$. It has been proved in [6, 29] that as long as $pn \rightarrow \infty$, we have a.a.s.,

$$c(G) \geq \frac{1}{(pn)^2} n^{\frac{1}{2} - \frac{9}{2 \log \log pn}} = \frac{1}{(\omega(n))^2} n^{\frac{1}{2} - \frac{9}{2 \log \log \omega(n)}} = \Theta(n^{1/2-\epsilon}).$$

In this graph the maximum degree is also a.a.s. $\leq 2pn = 2\omega(n)$. Applying the above approach we will get G' , a graph on $O(n\omega^2(n) \log \omega(n))$ vertices with $c(G') \geq \Theta(n^{1/2-\epsilon})$. Therefore, by a change of variable it is easy to check that if G' is a (sub-cubic) graph on n vertices then $c(G') \geq \Theta(n^{1/2-\epsilon})$.

Theorem 5.15. *For every value of ϵ and large enough n , there are sub-cubic graphs on n vertices with cop number at least $\Theta(n^{1/2-\epsilon})$.*

5.3 Digraphs of bounded degree

In this section we will use the same technique as in the previous section to find Eulerian digraphs of bounded degree of high cop number. The gadget that we are going to get is different but will have the same properties.

Let the maximum out-degree of the digraph be $\Delta^+ = o$ and the maximum in-degree $\Delta^- = i$. Consider x_1^-, \dots, x_i^- and x_1^+, \dots, x_o^+ . Now find the smallest k such that $2^k \geq i$ and make a complete binary tree where x_j^- 's are the leaves of the tree and direct all edges towards the root. Also find the smallest l such that $2^l \geq o$ and make a complete binary tree where x_j^+ 's are the leaves of the tree and direct all the edges away from the root. Now merge

the roots of these directed trees. Note that this gadget has less than $2^{k+1} + 2^{l+1} \leq 4(o+i)$ vertices and the distance from any x_s^- to any x_e^+ is fixed and equal to $k+l$ (for $1 \leq s \leq i$ and $1 \leq e \leq o$).

Now for any vertex v where $w_1^-, \dots, w_{d^-(v)}^-$ and $w_1^+, \dots, w_{d^+(v)}^+$ are in and out-neighbors of v , connect (with a directed edge) w_j^- to x_j^- ($j = 1, \dots, d^-(v)$) and similarly connect x_j^+ to w_j^+ ($j = 1, \dots, d^+(v)$). See Figure 5.4. Note that we can delete the unnecessary vertices.

If we replace all vertices of a digraph D with this gadget to get D' , it is easy to see that in and out-degree of vertices of D' is bounded by 2, number of vertices of D' is at most $4(\Delta^- + \Delta^+)$ times the number of vertices of D and by a similar lemma as Lemma 5.14 we have $c(D') \geq c(D)$.

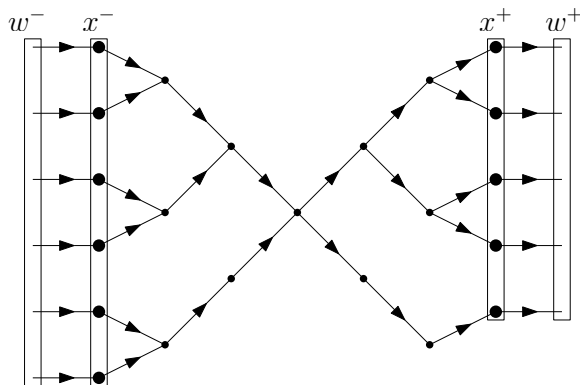


Figure 5.4: The gadget when $\Delta^- = 6$ and $\Delta^+ = 5$ (after deleting extra vertices).

Chapter 6

Minimum k -cop-win Graphs

In this chapter we will consider the smallest graphs with a given cop number. First we will see some properties of smallest general graphs and then we will discuss some properties of the smallest 3-cop-win planar graph and digraphs.

Let m_k be the minimum order of a connected graph with cop number at least k and let M_k be the minimum order of a connected graph with cop number equal to k . Clearly, $m_k \leq M_k$ and m_k is monotone increasing in k . It is known that $M_1 = m_1 = 1$, $M_2 = m_2 = 4$ and recently, it was proved that $M_3 = m_3 = 10$, and more importantly the following theorem is also known (see [5], [9]).

Theorem 6.1. *Petersen is the unique graph on at most 10 vertices with cop number 3 .*

Bonato [7] proposed the following.

Conjecture 6.2. *$M_k = m_k$ for all k . In other words, M_k is monotone increasing in k .*

6.1 Bonato's conjecture

The purpose of this section is to confirm Bonato's conjecture.

Lemma 6.3. *For every graph G and every $v \in V(G)$, $c(G \setminus v) \geq c(G) - 1$.*

Proof. Let us assume that v is a vertex of a graph G such that $c(G \setminus v) < c(G) - 1$. Place a cop on v in the entire game and play with $c(G \setminus v)$ cops on $G \setminus v$. Hence, the robber cannot occupy v throughout the game. Therefore, by $c(G \setminus v) + 1 < c(G)$ cops we can capture the robber on G which is a contradiction. \square

Observation 6.4. *For every connected graph G , there is an ordering of vertices of G such that if we remove the vertices one by one in this order, in each step the remaining graph is connected.*

This is an easy exercise which follows by considering a spanning tree of G and in each step removing a leaf vertex.

Theorem 6.5. *The values M_k are monotone increasing.*

Proof. We will show that $M_{k-1} < M_k$ for every $k \geq 2$.

Consider a graph G on M_k vertices with $c(G) = k$. By Lemma 6.4 there exists a vertex v such that $G \setminus v$ is connected. If $c(G \setminus v) = c(G)$, then this is a contradiction to the choice of G (we have a smaller graph with the same cop number).

If $c(G \setminus v) < c(G)$, then we are done because we have found a smaller graph with cop number equal to $k - 1$ (see Lemma 6.3) and therefore $M_{k-1} < M_k$.

Now assume that $c(G \setminus v) > c(G)$. Using the ordering in Lemma 6.4, we can keep removing vertices and the remaining graph will stay connected. By Lemma 6.3, removing a vertex can decrease the cop number by at most one. Since we have started from $G \setminus v$ (which has a higher cop number than G) and after removing all vertices but one we will get a graph that has cop number 1, then we will get to a graph whose cop number is equal to $c(G)$, which is again a contradiction to the choice of G . \square

This confirms Conjecture 6.2 and will give a corollary for which we need the following lemma.

Lemma 6.6. *If G is a graph of order M_k with $c(G) = k$, then G is 2-connected.*

Proof. Assume that G is a k -cop-win graph of order M_k that has a cut vertex v . Let G_1, G_2, \dots, G_l be the components of $G \setminus v$. First note that at least one of the components has cop number equal to $k - 1$, otherwise, by starting the game with $k - 1$ cops on v and moving $k - 2$ of them to the robber's component we can win the game which is a contradiction. Without loss of generality assume that G_1 is $(k - 1)$ -cop-win and all other components have cop number smaller than $k - 1$. Start the game by placing $k - 1$ cops on v , if the robber chooses a vertex in a component with cop number smaller than $k - 1$, then by moving $k - 2$ of the cops to that component we can capture the robber. Therefore, we may assume that the robber chooses G_1 . Note that by the minimality of G , the graph induced on $V(G_1) \cup v$ is also $(k - 1)$ -cop-win. So with $k - 1$ cops we can either capture the robber or he will move to another component through v . When the robber is in v or in other components we will aim to capture (his shadow on) vertex v . Therefore, we can occupy v when the robber is in other components and by moving the other $k - 2$ cops to his component, we can capture him with $k - 1$ cops which is a contradiction. Therefore, there is another component which is also $(k - 1)$ -cop-win.

Consider an escaping strategy of the robber on G when we have $k - 1$ cops and all of them start from v . The robber needs to start in a component of cop number equal to $k - 1$, say G_1 . He can escape from $k - 2$ cops in $G_1 \setminus N(v)$ and even if the last cop joins $G_1 \setminus N(v)$, the robber has a strategy to go to one of the vertices in $N(v)$, then use v to go to another

component of cop number equal to $k - 1$, say G_2 . Note that he also has a strategy to escape from G_2 as well.

Consider the graph G' obtained by disjoint union of G_1 and G_2 and adding a complete bipartite graph between $G_1 \cap N(v)$ and $G_2 \cap N(v)$. We claim that $k - 1$ cops cannot capture the robber in G' .

In G' he will select a vertex in G_1 and uses an escaping strategy as above until all the $k - 1$ cops are in $G_1 \setminus N(v)$ and then he will use the same strategy to go to one of the vertices in $N(v)$ and (without using v) goes to G_2 . Therefore, the robber can escape by using the same strategy as in G which is a contradiction to minimality of G . Thus G is 2-connected. \square

We have essentially proved the following corollary.

Corollary 6.7. *If G is a graph on M_k vertices with $c(G) = k$, then $c(G \setminus v) = c(G) - 1$ for every vertex v of G .*

Proof. Lemma 6.6 shows that $G \setminus v$ is connected and as mentioned in the proof of Theorem 6.5, $c(G \setminus v) < c(G)$. Applying Lemma 6.3 completes the proof and shows that $c(G \setminus v) = c(G) - 1$ for every vertex v of G . \square

6.2 Minimum 4-cop-win graphs

In this section we will consider the smallest graphs with cop number equal to 4, G_4 , and we will prove the following.

Theorem 6.8. $16 \leq M_4 \leq 19$.

Proof. First note that there exist a 4-regular graph on 19 vertices with girth 5 (namely the Robertson graph) and therefore the cop number of it is 4 and $M_4 \leq 19$. See Figure 6.1

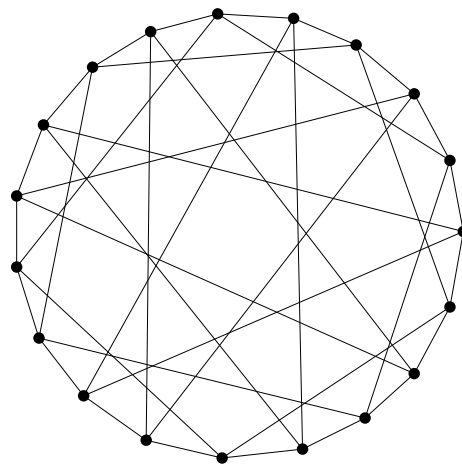


Figure 6.1: Robertson Graph, a 4-regular graph on 19 vertices with girth 5.

To get the lower bound, by Theorem 6.1, we have that $M_4 \geq 11$. It is easy to see that there are vertices of degree ≥ 3 in G_4 . Place a cop on a vertex of degree ≥ 3 , he will guard ≥ 4 vertices. Therefore if $|G_4| \leq 13$, then the territory of the robber has at most 9 vertices. By Theorem 6.1 we can see that the robber will be captured by at most 2 cops and therefore $c(G_4) \leq 3$ which is a contradiction. So we may assume that $M_4 := |G_4| \geq 14$.

If $M_4 = 14$ and there exists a vertex of degree ≥ 4 , then using a similar strategy we can show that ≤ 3 cops are enough. So assume that $M_4 = 14$ and $\Delta = 3$. Removing a vertex of degree 3, v and its neighborhood leaves 10 vertices. Since $\Delta = 3$ in the original graph, then vertices in $N^2(v)$ will be of degree ≤ 2 . Therefore these vertices cannot form a Petersen graph. Thus this graph needs at most 3 cops which is a contradiction. Therefore $M_4 \geq 15$.

First note that if $M_4 = 15$ and $\Delta \geq 5$ we are done. Now consider the case that $M_4 = 15$ and $\Delta = 3$. Note that since number of vertices is odd, then there is a vertex of even degree. So in this case there is a vertex, v of degree 2 in G_4 . If there is a vertex of degree 3, u in $N^2(v)$, then removing u will leave 11 vertices. The key point is that in the remaining graph, the degree of v is at most 1 and therefore it has no effect in the cop number of the graph. So the robber's territory will practically have 10 vertices and some of these vertices, namely $N^2(u)$, have degree less than 3 and therefore they cannot form a Petersen graph and $c(G_4) \leq 3$, a contradiction.

So if $M_4 = 15$, then $\Delta = 4$. In order to resolve it we will break the case into some claims.

Claim 1. If u and v are vertices of degree 4 that are adjacent, then G_4 is 4-regular.

Proof of Claim 1. First note that if all neighbors of u and v are vertices of degree 4, then we will move towards one of the edges to reach a pair of adjacent vertices of degree 4 who have a vertex of degree at most 3 in their neighborhood. Observe that u (and similarly v) has a neighbor, $w_u \notin N(v)$ ($w_v \notin N(u)$). If $\deg(w_u)$ ($\deg(w_v)$) is less than 4, then placing a cop on v (u) will shrink the territory of the robber to 10 vertices and one of these vertices, w_u (w_v) is of degree less than 3 and therefore cannot make a Petersen graph, a contradiction. Thus, we may assume that $\deg(w_u) = \deg(w_v) = 4$. So u and v will have a neighbor of degree at most 3, x , in common. If x is not adjacent to w_u (w_v), then by placing a cop on w_u (w_v), the territory of the robber will become 10 vertices while one of them, x , is of degree at most 2, which makes it impossible to form a Petersen graph. Therefore, we are done. So we may assume that x is adjacent to both w_u and w_v . In this case, placing a cop on u will remove at least two of the neighbors of w_v and therefore w_v will become a vertex of degree at most 2 and the remaining 10 vertices cannot form a Petersen graph and we are done. \square

Claim 2. If u and v are vertices of degree at most 3, then they are not adjacent.

Proof of Claim 2. First note that $\Delta = 4$ which guarantees the existence of a vertex of degree 4. Let us assume that u and v are vertices of degree at most 3 that are adjacent. If all neighbors of them are vertices of degree at most 3, then we will move to another pair of adjacent vertices of degree at most 3. Again, u (v) has a neighbor $w_u \notin N(v)$ ($w_v \notin N(u)$). If w_u (w_v) is a vertex of degree 4, then placing a cop on it will make v a vertex of degree at most 2 in the territory of the robber. Therefore the remaining 10 vertices cannot form a Petersen and we are done. So we may assume that $\deg(w_u) = \deg(w_v) = 3$. Now we can assume that u and v have a neighbor, x , of degree 4 in common. If w_u (w_v) is not adjacent to x , then placing a cop on x , will make w_u (w_v) a vertex of degree at most 2 in the territory of the robber and similar to above, we are done. Also if x is adjacent to both w_u and w_v , then placing a cop on w_u will remove 4 vertices from the territory of the robber and will make v a vertex of degree 1 (which does not affect the cop number). In the remaining 10 vertices, w_v will have degree 1 and therefore they cannot form a Petersen and we are done. \square

Let us assume that G_4 is not a 4-regular graph. Then, by Claim 1 and Claim 2 we have that G_4 is a bipartite graph, one part consists of all vertices of degree 4 and the other part contains only vertices of degree at most 3. Now placing a cop on a vertex of degree 4 will leave 10 vertices on the territory of the robber. But since they induce a bipartite graph they cannot form a Petersen graph and we are done. Therefore G_4 is a 4-regular graph.

Claim 3. Girth of G_4 is at least 5.

Proof of Claim 3. Assume that there is a C_3 or C_4 in G_4 , then there are at most 12 vertices in the cycle or in the first neighborhood of it. Placing a cop on a vertex other than these, leaves 10 vertices but since the remaining 10 vertices induce a C_3 or C_4 , then they cannot form a Petersen and we are done. \square

Since G_4 is 4-regular and by Claim 3 girth of G_4 is at least 5, then there should exist exactly 17 disjoint vertices that are at distance at most 2 from a fixed vertex. Since we assumed that $M_4 = 15$, this is a contradiction. This completes the proof and we have $M_4 \geq 16$. \square

Theorem 6.9. *If there is no vertex of degree 2 in G_4 , then G_4 is 3-connected.*

Proof. Let us assume that there is no vertex of degree 2 in G_4 . Since G_4 is the smallest 4-cop-win graph, it is clear that it does not have a vertex of degree 1 as well. Therefore, $\delta(G_4) \geq 3$. Also, if $M_4 = 19$ we are done, so we may assume that $M_4 \leq 18$.

Now let us assume that there is a 2-vertex-cut in G_4 and let these two vertices be u and v . Also let S_1 and S_2 (and maybe S_3, \dots) be the components of $G_4 \setminus \{u, v\}$ where $|V(S_1)| \leq |V(S_2)|$.

Claim 1. u and v are not adjacent and do not have a neighbor in common.

Proof of Claim 1. First note that since we are assuming that $M_4 \leq 18$, then at most one of the components (after removing u and v) contains up to 10 vertices and therefore only one of them can be 3-cop-win.

We will show that we can capture the robber by using only three cops. Let us assume that u and v are adjacent. We will define the shadow of vertices of the smaller component, S_1 , to be u . It means that if the robber is in any vertex of S_1 , the cops will assume that he is in u and will play accordingly. Note that if the cops can capture the shadow of the robber it means that the robber is trapped in S_1 and the other two cops can capture him. Also notice that since we have three cops we can push the robber to use vertices of S_1 and capture his shadow.

If u and v are not adjacent but have a neighbor, w , in common, a similar proof works. Let w be the shadow of vertices of S_1 and apply the same strategy. \square

Now start the game by placing one cop on u and one cop on v . The robber will choose a vertex in one of the components (S_1, S_2, \dots), if u (or v) has only one neighbor, x , in the robber's component, the cop can move to x . Note that since x has a degree of at least 3, it has at least two neighbors in the robber's component. Also observe that in this case x and v are another 2-vertex-cut and the robber cannot change his component.

Therefore without loss of generality we can assume that the cop on u (and similarly v) can guard at least two vertices in each component of $G_4 \setminus \{u, v\}$. Since by Claim 1, u and v are not adjacent and do not have a neighbor in common, the two cops on u and v guard at least 10 vertices together. See Figure 6.2.

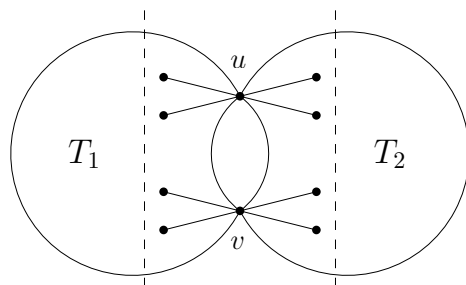


Figure 6.2: If G_4 has no vertex of degree 2 and has a 2-vertex-cut.

Now let T_1, T_2, \dots be the components of $G_4 \setminus (N[u] \cup N[v])$. Note that the smallest (sub)graph that is 2-cop-win is a 4-cycle. So any subgraph on up to three vertices are 1-cop-win.

We will consider two different cases. The first case is when each of T_1 and T_2 has four vertices. Then after the cops positioned themselves on u and v the robber will pick a vertex in T_1 (or T_2). The robber can escape from one cop in T_1 . Let the neighbors of u (similarly v) on the side of T_1 to be x_u and y_u (x_v and y_v) where $|N_{T_1}(x_u)| \geq |N_{T_1}(y_u)|$ ($|N_{T_1}(x_v)| \geq |N_{T_1}(y_v)|$). The cop on u will move to x_u and the cop on v will move to x_v .

By doing this, the cops are still covering u and v but might not cover y_u and y_v . Also each of them are covering some new vertices. If $|N_{T_1}(x_u)| = 1$ and $|N_{T_1}(x_v)| = 1$, then the territory of the robber will be at most four vertices but since $|N_{T_1}(y_u)|$ and $|N_{T_1}(y_v)| \leq 1$, they cannot form a 4-cycle. Therefore the third cop can capture the robber.

In the second case, one of the components, say T_1 , contains less than 4 vertices and therefore it is 1-cop-win. We will define the shadow of all vertices of T_1 along with x_u, y_u, x_v and y_v to be u .

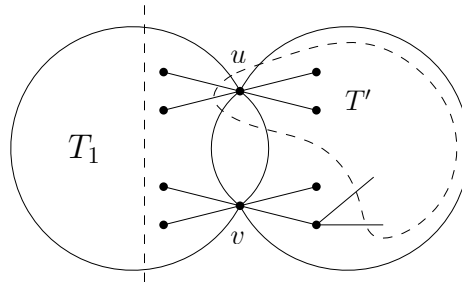


Figure 6.3: T' can contain up to 11 vertices.

The situation is depicted in Figure 6.3. In this case we will start the game by placing one cop on v (assuming that u had at most the same number of neighbors in the right side). The other two cops will try to catch the robber (or his shadow) in T' .

If v has only two neighbors on the right side, then T' can have up to 11 vertices and therefore can be 3-cop-win. Let w_1 and w_2 be neighbors of v in the right side and assume that $|N_{T'}(w_1)| \geq |N_{T'}(w_2)|$. Moving the cop on v to w_1 will change the situation. If $|N_{T'}(w_1)| = 1$, then although by moving to w_1 we cannot guard w_2 but since its degree in T' is at most one, w_2 cannot affect the cop number of the new territory of the robber and therefore we can remove them. The remaining territory will have at most 10 vertices but degree of u is 2 and therefore the new territory cannot form a Petersen graph and thus the other two cops can capture the robber. If $|N_{T'}(w_1)| \geq 2$, then moving the cop on v to w_1 will shrink the territory of the robber to 10 vertices. But again u has degree 2 in T' and by a similar argument, we are done.

If v has at least three neighbors on the right side, then T' can have up to 10 vertices. Again, moving the cop on v to his neighbor on the right side (with the most number of neighbors in T') will solve the problem. \square

6.3 Minimum planar 3-cop-win graphs

In this section we will discuss the properties of smallest planar graph on which we need 3 cops to capture the robber. We know that for planar graphs, having 3 cops is always enough [1] but might not be necessary. There exist a conjecture that the smallest graph that needs 3 cops is Dodecahedron which has 20 vertices.

It has been proved that in every planar graph of order at most 19 there exist a winning vertex for two cops. In other words, in every planar graph of order at most 19, there are 3 vertices r, c_1 and c_2 such that the closed neighborhood of c_1 and c_2 contains the closed neighborhood of r . Therefore *if* we can place our two cops in c_1 and c_2 when the robber is in r , we can capture the robber and win the game.

Theorem 6.10. *The minimum planar 3-cop-win graph is 2-connected.*

Proof. Let G be a minimum planar graph with $c(G) = 3$. Let us assume for a contradiction that v is a cut vertex of G . Removing v from G will make the graph disconnected and we will have at least two components G_1 and G_2 . A similar argument as in Lemma 6.6 works here and we can show that there are at least two components in $G \setminus v$ that are 2-cop-win, say G_1 and G_2 .

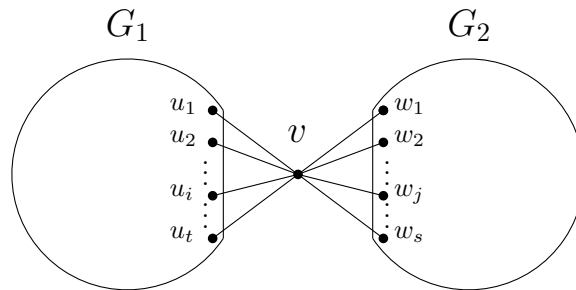


Figure 6.4: A planar graph with a cut-vertex.

Consider the escaping strategy of the robber in G when we have 2 cops and both of them start from v . The robber needs to choose a vertex v_0 in a component with cop number 2, say G_1 . Since v is a cut vertex, as long as there is a cop in v , the robber cannot change his component and needs to stay in G_1 . Note that he can escape from 1 cop in G_1 and when the second cop joins G_1 , the robber will have the strategy to go to a vertex in $N(v) \cap V(G_1)$ and then use v to go to another component, say G_2 . In the escaping strategy, he will use a vertex in $N(v) \cap G_2$, say w_j , to enter G_2 . Also similarly let $u_i \in G_1$ be the vertex that the robber uses when he needs to enter G_1 .

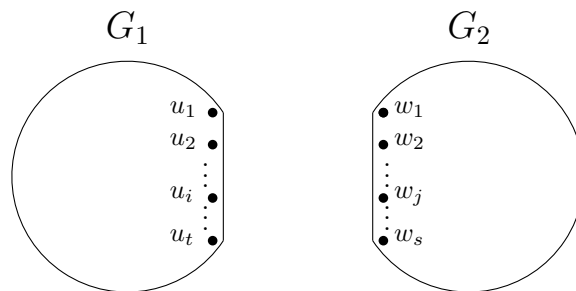


Figure 6.5: G_1 and G_2 are planar.

Since G_1 and G_2 are planar graphs, we can assume that u_1, \dots, u_t are in a line. We can draw a circle (centering on the line) that only touches u_i and u_{i-1} and apply an inversion to get the redrawing of G_1 which is still planar. We can apply the same technique to redraw G_2 and get w_j the lowest vertex in the line-up. See Figure 6.6.

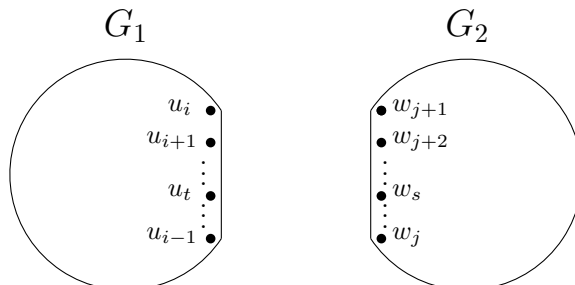


Figure 6.6: Another planar drawing of G_1 and G_2 .

Obtain G' by joining u_i to all w -vertices and also joining w_j to all u -vertices. See Figure 6.7.

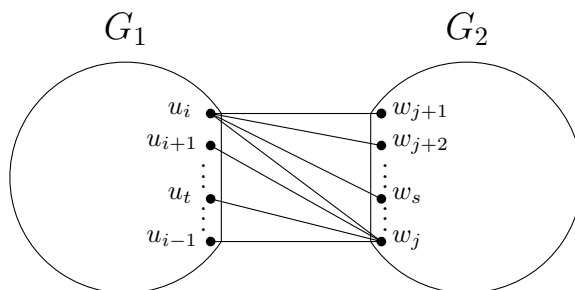


Figure 6.7: Constructing G' from G_1 and G_2 .

Start the game on G' by putting both cops on w_j and selecting the initial position of the robber to be v_0 in G_1 . From now on the robber will follow the escaping strategy in G until both cops enter G_1 . As mentioned before, in G , when the second cop enters G_1 , the robber will have an escaping strategy to go to one of u -vertices and then use v to go to $w_j \in G_2$. In G' , the robber can follow the same strategy with a little change. He will get to one u -vertices and then directly will go to w_j . The robber can again continue the escaping strategy in G and escape from 2 cops for ever. Therefore the cop number of G' is also at least 3, which is a contradiction because order of G' is smaller than G . \square

6.4 Minimum k -cop-win digraphs

In this section we will try to describe smallest k -cop-win digraphs. The smallest digraph that is 2-cop-win is a directed triangle. It is easy to see that this is the only digraph on three vertices with this property.

Observation 6.11. *If a digraph D does not have the following sub-digraphs then*

$$c(D) \geq \delta^+(D) + 1.$$

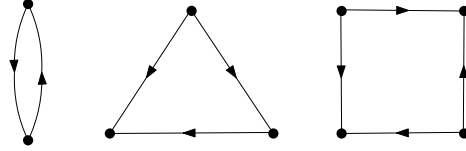


Figure 6.8: In the absence of these sub-digraphs, no vertex can guard more than one (closed) out-neighbor of another vertex.

Basically if we do not have these sub-digraphs, then there is no vertex that can guard more than one (closed) out-neighbor of another vertex. Therefore we will need one cop to force the robber to move (or capture him) and one cop to cover each out-neighbor of the robber's position.

Theorem 6.12. *The minimum strongly connected 3-cop-win digraph has 7 vertices.*

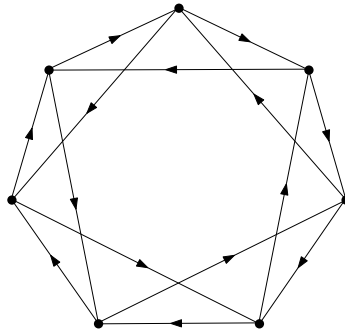


Figure 6.9: Smallest strongly connected 3-cop-win digraph.

Proof. The digraph shown in Figure 6.9 does not have any of the sub-digraphs shown in Figure 6.8 and therefore its cop number is at least three. It is easy to see that by placing three cops we can actually guard all vertices of it and therefore the cop number is equal to three.

Now we need to show that any digraph with at most 6 vertices is 1 or 2-cop-win. First observe that there should exist a vertex of out-degree at least 2, otherwise the digraph is a cycle (or is not strongly connected). If our digraph has at most 5 vertices, then placing a cop on a vertex of out-degree equal to two will leave only two vertices for the robber. The second cop can capture the robber then.

Now let us assume that our graph has 6 vertices, v_1, v_2, \dots, v_6 . Note that if there is a vertex of out-degree at least 3, using a similar argument as above, we are done. Assume that the out-neighbors of v_1 is v_2 and v_3 . By placing a cop on v_1 , v_1, v_2 and v_3 will be

guarded. The remaining vertices should induce a directed triangle, otherwise the second cop can capture the robber. Without loss of generality we can assume that the directed triangle is $v_4 \rightarrow v_5 \rightarrow v_6 \rightarrow v_4$. Since the digraph is strongly connected, then there is a way to get from $\{v_4, v_5, v_6\}$ to $\{v_1, v_2, v_3\}$. Without loss of generality we can assume that v_4 has an out-neighbor among them. We have to general cases.

Case 1, if v_1 is an out-neighbor of v_4 . Then v_2, v_3 and v_6 should form a directed triangle. By symmetry we can assume that v_3 is the out-neighbor of v_6 in the triangle. Therefore by placing a cop on v_6 we can see that v_1, v_2 and v_5 should form another directed triangle and since v_1 has two out-neighbors the directed triangle should follow the direction from v_1 to v_2 . Now placing a cop on v_5 , we can see that v_2, v_3 and v_4 should form a directed triangle but since v_2 has it two out-neighbors outside v_3 and v_4 , it is not possible.

Case 2, if v_2 (or similarly v_3) is the out-neighbor of v_4 . Then a similar argument as in case 1 will work and we are done. □

Corollary 6.13. *The minimum planar strongly connected 3-cop-win digraph has 7 or 8 vertices.*

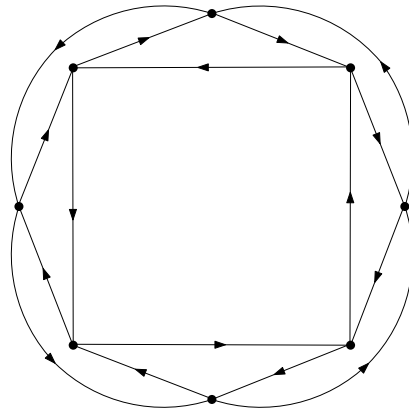


Figure 6.10: Smallest 3-cop-win planar digraph.

Proof. Using Observation 6.11, it is easy to see that the following digraph is 3-cop-win. We showed in Theorem 6.12 that there are no 3-cop-win digraphs on 6 vertices which gives the result. □

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