Mathematical Tool Fluency: Learning Mathematics Via Touch-based Technology

by

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Abstract

Recent advances in the study of mathematics embodiment have given rise to renewed interest in how mathematical learning relates to our bodily actions and the sensorimotor system. In this dissertation, I explore the embodiment of mathematics learning with a particular focus on the relationship among gestures, hand and finger movements, and the use of mathematical tools. The theoretical lens of perceptuomotor integration enabled me to articulate mathematics learning through the development of tool fluency within a non-dualistic view of mathematical tools.

The dissertation is structured as three stand-alone descriptive case studies that adopt Husserl's phenomenological attitude in analysing participants' lived experience while using mathematical tools. Drawing on the work of Nemirovsky, one of the main intentions is to provide a thick description of learners' perceptual and motor activities, which may result in the emergence of perceptuomotor integration in Husserlian experiential time.

The results provide evidence for a high degree of gestural and bodily engagement while learning, communicating, and playing with mathematical tools. For example, in the first study, we discuss the process of learning cardinality for a young child in the context of mathematical explorations with a multimodal iPad application named *TouchCounts*. We identifying the development of 'finger-touching' action while the child is playing with it. In the second study, I present and discuss the notions of 'active sensation' and 'tactile perception,' in the context of a blind undergraduate student explaining the behaviour of a rational function. In the third study, which involves a prospective teacher identifying types of geometric transformation in a touchscreen geometry software (Geometer's Sketchpad (GSP) on iPad), I identify new modes of Arzarello's active interactions. Identifying, analysing, and exploring different modes of interactions with touchscreen-based mathematical tools leads me to propose a new methodological approach for analysing video data. This methodological approach enabled me to catalogue interactions in order to monitor and assess the emergence of mathematics expertise while the learner interacted with the mathematical tool.

Keywords: learning; touchscreen-based technology; cardinality; visually impaired; prospective teacher; geometric transformation; tool fluency; fingers

Dedication

To my family, and

my brother in-law Mr. Taghi Nia

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Chapter 1. Introduction

I have always loved mathematics, from kindergarten to the university level. After passing all the pure mathematics courses successfully in my undergraduate program, I decided to change my field to mathematics education and become a mathematics teacher. Since schools in Iran are single-gendered, there were few female mathematic teachers to teach in all-girls schools. Also, it was believed that the best mathematics teachers are male. So, I was hoping to become someone who loves teaching and learning mathematics and computer programming, especially for those studying in economically difficult situations and following their academic goals in STEM.

My fascination with using technology to teach and learn mathematics began when I was doing my bachelor's degree and was introduced to computer programming. It then developed through teaching mathematics and computer programming simultaneously in my home country, Iran (starting in 2000). Although there was no technology incorporated in the mathematics lesson plans and textbooks, I was always seeking new ways of introducing and visualizing mathematical concepts to my students through digital technology. At that time, there were very limited amounts of digital material and resources available, especially in Persian. So, I started making Flash animations (for example, I used Adobe Flash to show graphs of sinusoidal functions) to illustrate the dynamic nature of the mathematical concepts. However, sometimes it became tough to manage the class time, given that comprehensive and intensive mathematical concepts had to be taught in only a short period of time. Meanwhile, my innovative and creative efforts in teaching mathematics and geometry brought me several district and nationwide awards.

Followed by more than ten years of teaching mathematics and computer programming, my vast enthusiasm for learning novel ways of integrating digital technology in teaching and learning mathematics guided me to the Mathematics Education with Information Communications Technology (ICT) program at the University

Malaya in Malaysia (2009), because there was not such a program offered in Iran. During my Masters' studies, I was introduced to many mathematics educational software programs along with major theoretical frameworks. I was very captivated to learn about Radical Constructivism from Prof. Nik Aziz Nik Pa, who was a former doctoral student of Ernst von Glasersfeld. von Glasersfeld introduced "radical constructivism and spent large parts of his life on elaborating his theory upon Jean Piaget's genetic epistemology, Bishop Berkeley's theory of perception, James Joyce's Finnegan's Wake, and other important texts" (Wikipedia, 2016).

Research shows there are several reasons that educational software is not fully integrated into the classroom practices to support, expand and enhance existing teaching and learning. For example, although there was a feeling of inevitability and acceptance of the vital role of technology, teachers often employed a conservative approach, exhibited caution about changing the core subject practice, and struggled to modify their practice (Hennessy, Ruthven, & Brindley, 2005). So, I conducted my Masters' thesis on the "Effects of using Cabri3D on Geometric Thinking among Grade Ten Students" (2010). In this study, the research team and I utilized Cabri3D in actual classrooms to explore geometric 3D constructs. A set of activities based on the pedagogical goals was practiced in the classroom by students, while minimal instruction was provided to them on how to use the software. Also, I modified and adapted the instruction and textbook's exercises in Cabri3D environment. The results showed the effectiveness of using Cabri3D in teaching 3D-geometrical concepts on students' geometric level of thinking compared to a control group learning the same concepts in a traditional way. Also, from a teacher's perspective, this study gave me a sense of how utilizing Cabri3D in an actual classroom could be at the same time interesting and challenging.

Since I started my doctoral studies, I was introduced to the vital role of the body and gestures in learning mathematics and the embodiment of cognition. Especially, I was fascinated with *TouchCounts*, an iPad application that Jackiw and Sinclair had designed for young children to explore numbers (Jackiw & Sinclair, 2014). By working on this project as one of Dr. Sinclair's research assistants, I found numerous relevant articles in the field of neuroscience (Andres, Seron, & Olivier, 2007; Dehaene, 2009; Rusconi, Bueti, Walsh, & Butterworth, 2011), psychology and education (Bender &

Beller, 2012; Brannon & van de Walle, 2001, 2002; Butterworth, Varma, & Laurillard, 2011; Domahs, Moeller, Huber, Willmes, & Nuerk, 2010; Klemmer, Hartmann, & Takayama, 2006; Miller, Zanos, Fetz, den Nijs, & Ojemann, 2009; Moeller et al., 2012; Novack, Congdon, Hemani-Lopez, & Goldin-Meadow, 2014) that strongly support the role of fingers (as the primary and always available tool) for learning numbers and operations. This is also in congruence with other research indicating there is a mental link between hands and numbers (Butterworth, 2000; Gracia-Bafalluy & Noël, 2008). In Penner-Wilger's (2007) study, the three most important component abilities for counting skills (viz: subitizing, fine motor ability, and finger gnosis) were found to be a significant unique predictor of number system knowledge that are all some of the *TouchCounts* affordances. Also, consistent use of fingers positively affects the formation of number sense, and thus, the development of calculation skills (Gracia-Bafalluy & Noël, 2008). Along with this line, other researchers have suggested that finger-based counting may facilitate the establishment of number practices (Andres et al., 2007; Sato, Cattaneo, Rizzolatti, & Gallese, 2007).

Fuson's extensive research on young children learning to count resulted in documenting the path of development for children aged two to eight in producing correct sequential number words (Fuson, 1992a, 1992b, 1998). She also documented children's developmental path in connecting oral numbers to the cardinality of a set of objects. However, contrasting results of further research rejects age-related production of number strings and patterns of oral counting behaviour (Threlfall, 2008).

In addition, traditionally it is suggested that children's early encounters with the action of counting are initially performed on real-world objects and follow a one-to-one correspondence among objects, numbers, and pointing. Therefore, children's numerical ability develops when they move from associating numbers with real-world objects, to comprehending numbers as mathematical entities that can be operated on through the flexible use of symbols (Gray & Tall, 1994; Tall et al., 2001).

Furthermore, in recent years there is an apparent shift in the way that children are using touch-based devices. They learn how to play with their parents' cellphone or other touchscreen devices as young as one or two years old. *TouchCounts* benefits from the multimodal affordance of the iPad and offers one-to-one correspondence among numbers, touch, visual and audible senses. This feature could break down the suggested linkage between counting actions and real-world objects, and instead provide the one-to-one correspondence between touching, naming aloud, and symbolic representation of the number and the object. *TouchCounts* also provides children with the opportunity to do addition or subtraction by performing pinching or spreading gestures. Connecting the strong neurological, psychological, educational, and technological affordances of *TouchCounts* inspired me to learn more about the ways that touch-based interaction and digital technology contribute to learning mathematics concepts – which led to the writing of the second chapter of my thesis.

1.1. Organization and Structure of the Thesis

This dissertation consists of three stand-alone chapters written in paper format. Each paper has its own goals, framework, and participants; however, they all have some common themes, which I will discuss after introducing each of the papers.

1.1.1. First Study: Exploring Cardinality in The Era of Touchscreen-Based Technology[1]

The first article comes from Dr. Sinclair's larger research project aimed to understand how children engage with *TouchCounts*. The goal of this article is to demonstrate ways in which perceptuomotor integration emerges and partially constitutes mathematical learning. It explores how a young child named Alex builds an understanding of the cardinality principle through multimodal communicative, touchscreen-based activities involving talk, gesture, and body engagement.

[1] The study is co-authored with Dr. Stephen Campbell. Professor Campbell was instrumental in helping me tease out, illustrate, and enhance phenomenological aspects of Nemirovsky and colleagues' framework, particularly regarding the distinctions between phenomenological and natural attitudes, experiential and objective time, and the notions of retention and protention.

The study also provides an overview of the literature that argues the strong link between finger counting and bodily engagements in early childhood and numerosity. By analyzing three short episodes, it indicates how touchscreen-based interactions with *TouchCounts* provide a novel co-ordination of time, body engagement, and semiotic resources to support the development of Alex's numerical perception and motor understanding in general, and cardinality in particular. Using Nemirovsky's perceptuomotor integration theoretical lens, I show how the child develops expertise in using *TouchCounts* to operate with numbers and some ways in which the perceptual and motoric aspects of learning holistically emerge.

1.1.2. Second Study: Advanced Mathematics Communication Beyond the Modality of Sight

The second paper is part of the research project that I conducted as the principal investigator, entitled 'Advanced mathematics communication beyond the modality of sight.' While working as a tutor assistant in the department of mathematics, I was introduced to a blind undergraduate student named Anthony, who was very passionate to pursue his studies in health science. At that time, there was no special teacher and no tactile or auditory materials designed to help him with the course.

The processes of comprehending, interpreting, and visualizing mathematical and pictorial graphs all play a vital role in learning mathematics in general and pre-calculus in particular. The visual images and visual reasoning are important components that involve thinking in pictures and images "to perceive, transform and recreate different aspects" of them (Armstrong, 1993, p.10). This is not to say visually impaired students are not capable of visualizing, instead visualization goes beyond "seeing" and it can be developed in the absence of vision (Cattaneo & Vecchi, 2011). Understanding the visual aspect of a graph could be accomplished by other learners' sensorial perceptions such as touch. Also, with the verbalization and relationships with previous knowledge and experiences (Healy, 2015; Healy & Fernandes, 2014, 2011). In addition, the history of mathematics includes several visually impaired mathematicians, such as Leonhard Euler (1707–1783), Nicholas Saunderson (1682–1739), Lev Semenovich Pontryagin (1908–1988); Louis Antoine (1888–1971), and Lawrence Baggett, a blind emeritus mathematics professor who taught sighted students at Colorado University (Jackson, 2002).

Since the visuospatial understanding for people who do not see relies on auditory and tactile activities, I sought substitutes for the visual components of the course materials. Accordingly, the second paper discusses part of the journey that Anthony and I undertook to tackle difficulties (in class, in tutoring sessions, and while studying independently) encountered in learning pre-calculus concepts such as reading, writing, graphing, and comprehending graphical/pictorial materials.

The third chapter demonstrates how Anthony explains and demonstrates graphing a rational function and its behaviour. The results show Anthony's high degree of sensory and body engagement in understanding pre-calculus concepts in non-visual modes. Anthony's body coordination, words, and gestures manifested a high degree of integrated perceptual and motor activity in terms of learning mathematics, to an extent that would be culturally recognizable in the community of mathematics (Husserl, 1991; Nemirovsky, Kelton, & Rhodehamel, 2013).

1.1.3. Third Study: Touch-Based Technology in Exploring Geometric Transformation: Use of Timeline as an Analytical Tool

Prior studies of the utilization of the dragging tool in computer programs, from a cognitive perspective, suggest that dragging mediates relationships between conceptual and perceptual entities (Arzarello, Olivero, Paola & Robutti, 2002). Beyond dragging, however, touch-based manipulation requires further definition in terms of the modes of interaction, because it is different than dragging with a mouse. So, extending this perspective, in this study I suggest that each application and designed task needs its own way of identifying modes of interaction. Hence, to determine the types of touch-based interactions in a Dynamic Geometry Environment (DGE) and trace the path of identified interactions I draw on both interaction theory (Arzarello, Bairral, & Danè, 2014) and perceptuomotor integration (Nemirovsky et al., 2013).

I adapted Arzarello's theory of interaction and innovated a new methodology to analyze video data. To do so, firstly, theory-based codes for modes of touch-based interactions in a DGE are defined as 'active' or 'basic' actions (Arzarello et al., 2014). Basic actions are primary ways of interacting with a touchscreen application such as a DGE. A combination of basic actions and performed finger actions are classified as 'active action'. So, 'active actions' mostly are identified as interactions with the screen that a learner uses to reach a target or solve a given problem. An example is the use of *drag-touch-to-approach*, which involves dragging to draw a geometrical shape to justify or reason, or to deal with some particular geometric property, shape or construction.

To analyze video data, I adopted Vogel and Jung's (2013) video-coding procedure. Upon identifying the basic and active actions (codes), I utilized StudioCode[2] software to verify the codes and analyse video data. Code verification resulted in the development of an integrated coding system informed by the theoretical framework. Consequently, the integrated coding system was used to code video data and produce a coded timeline. The distribution of codes on the video timeline produced the 'paths of interaction', which enabled me to analyse and assess tool fluency and mathematical learning (Nemirovsky et al., 2013).

1.2. Tracing the Common Themes

Although the three studies presented in this thesis stand alone and follow their own goals, there are significant themes shared by them all. The common themes arise out of the perceptuomotor integration theoretical framework: embodied mathematical tool (instrument) fluency is intertwined with learning mathematics, and touch interactions play a vital role in tool fluency and mathematics learning. I elaborate on each of these common themes below.

1.2.1. Embodied Mathematics: Perceptuomotor Approach

There are recent theories that highlight the importance of embodied mathematics. Theories of embodied cognition (Lakoff & Nunez, 2000) suggest concepts are mapped onto a person's sensory-motor system and activities (Arzarello, Pezzi, & Robutti, 2007; Gallese & Lakoff, 2005; Nemirovsky & Ferrara, 2009). For example, Gallese and Lakoff (2005) write:

[2] StudioCode is a video analysis tool that lets the researcher capture, categorize, review, and analyze video data, qualitatively and quantitatively.

> Conceptual knowledge is embodied, that is mapped within our sensory-motor system… the sensory-motor system not only provides structure to the conceptual content but also categorizes the semantic content of concepts regarding the way that we function with our bodies in the world. (p. 456)

Accordingly, a given sensory arrangement engages in comprehending and classifying concepts alongside motor activities: hence "perceptuomotor." This approach questions the divisibility of knower and known, bodily activity, thinking and learning: it foregrounds sense, sensations, and motor actions in learning. This kind of learning intertwines action and perception, and is constituted by *perception* and *motor* activities such as manipulation of a mathematical instrument, bodily actions, gestures, tone of voice, facial expressions, and so on. A perceptuomotor way of learning thus comprises multimodal activities taking advantage of the conceptual learning inherent in bodily extension.

> While modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuomotor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action (Nemirovsky, 2003, p. 108).

One of the core assumptions of perceptuomotor integration is to consider meaning and matter as bound together. In this way thinking is identified as "bodily activity" (Nemirovsky et al., 2013; Gallese & Lakoff, 2005) rooted in a perceptuo-motor-sensory system. However, the assumption that the sensory-motor system is generative of conceptual knowledge perhaps implies the brain as the registrar of such bodily activities. The theory could be accused here of reductionism by attempting to study neurobiology as the fundamental architecture of learning (de Freitas & Sinclair, 2013).

In addition, within the above mathematics embodiment frameworks, the human body is recognized as an active force in learning and mathematical thinking involving various parts of the learner's body as well as multimodal ways of learning but it is not clear "what is to be a body and how mathematics itself partakes of the body" (de Freitas & Sinclair, 2013, p. 454). Without a clear view of what the "body" entails, the "integration" proposed by the theory, is difficult to parse in finer details of both biology and philosophy.

Despite acknowledging these problems, in this thesis, I look at embodiment through the lens of perceptuomotor integration, focusing on the role of the body in using ‘mathematical instruments’ such as tactile mathematical graphs or touch-based digital technology. I hope that this thesis will shed light on the practical implications for learning of embodied tool-based mathematics – opening up discussion about the potential use of touch-based interactions in learning mathematics. ‘Practical term’ here means to focus more of the application of the theory rather than the theory itself. In what follows I discuss the multimodality of mathematical instruments, followed by mathematical embodied learning in their use.

1.2.2. Multimodality of Mathematical Instruments

There is no doubt that learning is inherently multimodal because it configures the relationship among different senses and modalities such as sight, hearing, touch, and motor actions. In the touchscreen-based technology employed for teaching/learning, many of these modalities are used in combination. In two of my studies programs on iPad are employed as the mathematical tool, whereby its capabilities incite multimodal interaction. In the other study, a tactile graph serves a similar function. Here I shall summarise the theme of multimodality in the three studies below:

- First study: *TouchCounts* benefits from the iPad’s various affordances and provides a multimodal correspondence between finger touching, naming aloud, symbolising, and the object itself. Therefore, it satisfies the traditional view in mathematics education that insists on a one-to-one correspondence among counting, object-pointing, and naming.

- Second study: Working with a blind learner in this study, hearing and touch play fundamental roles. The tactile graphs help the visually impaired leaner to communicate multimodally (via talk and gesture) in explaining a mathematical function’s behaviour, as evidenced in the coordination of his body and words.

- Third Study: One of the main features of using DGEs with multitouch-screen devices is enabling a multimodal environment to communicate and support a convergence of "modes" of interactions. In the third study, GSP profits from a tracing affordance that leaves a trail behind finger movements, as well as object dragging in response to the preserved geometrical properties of the geometrical transformations.

Nemirovsky and Borba (2003), with their perceptuomotor lens, incorporate contextual ingredients with the multimodal understanding of mathematics. For example they indicate:

> The understanding of a mathematical concept rather than having a definitional essence spans diverse perceptuomotor activities, which become more or less significant depending on the circumstances. For instance, seeing a trigonometrical function as a component of a circular motion or as an infinite sum of powers may entail distinct and separate perceptuomotor activities. (p. 108)

It means that instead of learning the new concepts in terms of the theoretical definition, the learner engages via perceptuomotor activity, which differs based on the concept, and context.

The role of cultural, social, and historical ingredients along with the symbols and semiotic activities in mathematical learning processes are vital (Arzarello & Robutti, 2007). Part of these social-cultural components is defined as the *mathematical tools*. Just like de Freitas and Sinclair (2013), Nemirovsky rejects the assumption that artefacts are "inert and disposable" tools, which theories such as 'instrumental genesis' maintain. So, in the next section, I will elaborate the different definitions of the mathematical tool found in instrumental genesis (Artigue, 2002; Drijvers & Gravemeijer, 2005; Guin & Trouche, 1998; Trouche, 2005; Verillon & Rabardel, 1995), the theory of semiotic mediation (Falcade, Laborde, & Mariotti, 2007; Vygotsky, 1986), and perceptuomotor integration (Nemirovsky, 2003; 2013).

1.2.3. Approach to Mathematical instruments

In this sub-section, I discuss three well-known theoretical approaches in learning by use of a mathematical tool: instrumental genesis, semiotic mediation and perceptuomotor integration. I elaborate their main differences and similarities to justify why perceptuomotor integration was chosen to be adapted for my studies.

Instrumental genesis

Taking a didactical approach, instrumental genesis navigates correspondence between action and conceptualization. There is a distinction between *artefact* and *instrument.* A tool, which can be a material or symbolic object, is called an *artefact* unless the user becomes aware of the particular use of it. The mathematical tool is clearly considered as the "extension of the mind" rather than body (Drijvers & Trouche, 2008). For example, to understand what is the function of a pair of scissors in "the intended use" of it, the learner needs to spend some time to practice. She needs to learn how to hold it, in at least a "normal cultural way" (Butterworth, 2000): to put the thumb and fingers in the right position (thumb and index/middle finger go to the holes). After that, she likely needs to learn what the tool does, by using it to cut a shape out of cloth, paper, and other thin materials. This is all because of unfamiliarity with the artefact and its intended usages.

Once the user learns the artefact's function, it becomes an available tool for future tasks. When the mediator (teacher) asks the student to explain how the artefact (scissors) has been used for cutting thick cardboard, this can be considered as "a request of making explicit the utilization schemes" (Mariotti, 2009, p. 434). So, the mental utilization schemes define the artefact functionally via intended task accomplishment (Bartolini & Mariotti, 2008). Instrumental genesis works in two directions: toward subject and artefact. As time passes, the user will gain greater knowledge of additional features and where to use the tool. The process that the learner passes through is called *instrumentation* (Trouche, 2004). A certain amount of experience is necessary for the user to understand what the tool is not able to do, or to find other possibilities for the tool. For example, she needs to understand that an ordinary pair of scissors cannot cut a watermelon or metal. The process of *loading* *artefacts* with their potential use is called *instrumentalization (*Artigue, 2002*)*. For example, the user instrumentalizes the scissors, when she draws a circle with their sharp blade on a piece of wood, uses them as a paper weight or learns to position heavy cardboard close to the scissors' pivot to cut it more easily.

In this theory, a physically instantiated *artefact* and *instrument* are differentiated. Within the perspective of instrumental genesis, an artefact has no instrumental value originally. It is only through the process of instrumental genesis, that the *artefact* transforms into an *instrument*. That is when the learner transforms the artefact to fit the intended use and develops mental schemes identifying the artefact's use under certain conditions (Artigue, 2002; Drijvers & Gravemeijer, 2005; Guin & Trouche, 1999; Trouche, 2005; Verillon & Rabardel, 1995). Therefore in short, "instrument = artefact + scheme for a class of tasks" (Drijvers & Trouche, 2008, p. 368).

Semiotic mediation rooted in Vygotsky's mediational theory

For the genesis of human mental activity and cognitive development, Vygotsky refers to the mingling of two main streams, namely, the *natural* and the *socio-cultural*, for basic and higher mental functions, respectively. He explains that an *artefact* can be used as a tool, when the educator, who is aware of the semiotic potential of it, explores "the possibilities to guide students to connect personal meaning from the use of artefact to the mathematical meanings recognizable by an expert in such use" (Mariotti, 2009, p. 430). Vygotsky differentiates between *technical* and *psychological* tools. He defines physically and externally oriented human-made objects as technical tools. These tools usually are used in human activities, where psychological tools are the internal representations of the technical tools. Vygotsky refers to *internalization* as the "internal construction of an external operation" (1987, p. 56), which usually is directed through socially shared experience, semiotic process and dialectical engagement.

Also, Vygotsky declared "like words, tools, and non-verbal signs provide learners with ways to become more efficient in their adaptive and problem-solving efforts" (1987, p. 127). This means that for cognitive development, not only natural and socio-cultural streams are vital, but *tools* and *signs* also play a contributing role. In other words, a teacher with the use of appropriate tools and tasks offers an environment that mediates the emergence of new mathematical knowledge, which is *internally* orientated. In this process, when the learner searches for the solution, she develops new artefact signs using the tool and related tasks, which later transforms into mathematical *signs* by the teacher.

So far, I have discussed the theoretical perspectives of instrumental genesis and semiotic mediation regarding mathematical tools. The subsections highlighted how a physical artefact becomes a psychological tool or mathematical instrument through semiotic mediation and instrumental genesis. The dualisms intrinsic in the two theories are challenged by perceptuomotor theory's consideration of the mathematical instrument as both a mathematics tool and a semiotic device together (Nemirovsky et al., 2013). It rejects models of knowledge acquisition that take the form of creating mental representations or schemes in the use of tools. There are no scaffolding stages in which a teacher must define the potentiality of the artefact for the learner though a dialectical approach. Within perceptuomotor integration, the fundamental role of the teacher, as well as the lack of originally instrumental value for the mathematical tool is critiqued. In the next section, I discuss where signs, artefacts and mathematical instruments are situated in the framework of embodied cognition.

Mathematical instrument: Within an embodied perspective

It is widely agreed that signs and artefacts are culturally rooted (Radford, 2005; Arzarello & Robutti, 2008; Nemirovsky, 2003, 2013) and deeply embedded in the history of humankind by carrying signs, values, and meanings. There have been valuable efforts to promote the socio-cultural factors as well as mathematical tools involved in learning. For example, Arzarello, Paola, and Robotti (2006) introduced The Space of Action and Relations, Production, and Communication (APC-space). In ACP-space, the use of artefacts in the learning of a concept is exemplified by a mathematics laboratory. Within this embodied tool-use perspective, a set of activities for practicing, communicating, seeing and doing is needed for constructing mathematical knowledge. Arzarello et al. also add the vital mediating role of a teacher to their framework. Three main components of the APC model are the body, physical world and the cultural context, which are dynamically involved. The APC-model aims to describe and analyze didactical phenomena by way of a *semiotic bundle* (Arzarello, 2006). "A semiotic bundle is a collection of semiotic sets and the relationships between the sets of the bundle"

(Arzarello & Robutti, 2008, p. 727). For example, the semiotic representation of the unity of speech and gesture, which are the different sides of an underlying mental process, is a *semiotic bundle* that is intentionally made (McNeill, 1992). Therefore, the semiotic bundle functions to deepen the process described by Vygotsky's term "internalization". According to Vygotsky (1978), internalization is based on "the semiotic activities with tools and signs, externally oriented, which provides a new psychological tool, internally oriented, completely transformed but still maintaining some aspects of their process of their origin" (p. 727) with language as the main component of social interactions. Such assumptions, demote material processes of learning, treat the body in a simplistic way, and treat technologies used to learn mathematics as disposable and inert artefacts.

de Freitas and Sinclair's (2013) materialist approach complements previous theories, which see mathematics as the historical and culturally rooted domain of embodied cognition (a view common to perceptuomotor integration). Their proposed materialist framework aims to "embrace the body of mathematics, as well as the body of her tools/symbols/diagrams, in the 'dance of agency' that makes up mathematical activity" (p. 454). This view allows the authors to theorize mathematics in material terms. Although this theory seems fruitful in providing new rich insights into learning procedures, the role of a learner's body in and of mathematics and mathematical tools, I found Nemirovsky's perceptuomotor integration easier to operationalize at a practical level because of the microanalytical tools he provides.

Sense-making in Perceptuomotor Integration

Before discussing the perceptuomotor approach to mathematical tool use, I shall elaborate how the mathematics concept is conceptualised in this framework. First, I address what sense-making means in perceptuomotor integration.

Inspired by the process of growth, individualization and decay, he defines concepts as crystalline images, metaphorically speaking. Nemirovsky (2017) uses the crystalline metaphor of the genesis of a geological formation to describe the process of conceptualization. He explains that concepts are always under growth, decay and formation by the unpredicted flows of virtuals[3], affects[4] and senses[5] of a human being:

> Concepts offer transient shelters and places, shaping and shaped by all the varieties of life inhabiting them, such as birds and worms. Inhabiting a concept entails going roughly along — doing, making, and perceiving — meshes of trails carved out over periods spanning natural and social history. (p. 3)

Nemirovsky (2017) introduces this crystalline process in opposition to the classic class-based view of concepts[6]. In the classical model, each concept is like a tree of predication or properties. It means different predications add different branches to the tree. For example, the classical view appeals to given *a priori* definitions of shape, such as ‘a closed two-dimensional shape with four straight sides’ to define a quadrilateral versus ‘a closed two-dimensional shape with three straight sides’ to define a triangle. Each definition then can be split into new branches (classes). In the next step, to define a parallelogram in the class of quadrilaterals, the shape must be in the class of ‘quadrilaterals that have the opposite sides parallel or equal’. In contrast, the view of concepts as crystalline process rejects concepts as trees of predicated classes. This view also assumes humans do not "have" concepts but metaphorically “inhabit” them, by their ways of life (virtuals) and sense making, and that inhabiting is a matter of affective

[3] Creatures “carry out their lives by dealing with different virtuals” (Nemirovsky, 2017, p. 4). For example, choosing a place to sleep could be inspired by a sense of its virtuality such as comfortability and safeness. That is, a “virtual” is a horizon of meaning that helps guide action, but it is not a specific fixed goal.

[4] The circulation of affects is defined as multiplicities of interpretations and feelings and their mutual relationships to stimulate and displace each other.

[5] Sense is conceptualised through Nemirovsky’s use of Deleuze (1990), which is broader than Frege’s conception of sense because it is infinite: “Sense is infinite, since no finite list of determinations will suffice to fully articulate it” (Nemirovsky, 2017, p. 5). For example, having a sense of a neighbourhood being safe or dangerous can be understood in an infinite number of ways depending on different conditions, contexts and situations.

[6] To define a class here, I will appeal to set theory. So, a class could be defined as a collection of sets (or sometimes other than mathematical objects) that can be unambiguously defined by a property that all its members share.

or *perceptuomotor* activities. So, the learner needs to go through webs of trials for doing, building, creating, performing and understanding over periods of social and cultural history to “inhabit” the concept. To that end, mathematics “concepts are vast dwellings, landscapes to inhabit by virtue of their being amenable to hosting certain forms of life and not others, while, at the same time, those who inhabit a concept transform it as they go on living in it” (Nemirovsky, 2017, p. 13). Perceptuomotor integration questions a dualist view and asserts that people who “inhabit” a mathematical concept transform it. Therefore, mathematical learning implies a transformation in lived bodily experience[7] and it manifests through perceptual and motor integration and unification.

Instrumental genesis and semiotic mediation agree with the necessity of bodily and mathematical tool interaction in a cultural-historical way and follow ‘tool fluency’, however, their approach in interpreting and analysing such interactions is different. Presumably, this is the perspective that perceptuomotor integration tries to deconstruct. In other words, while semiotic mediation and instrumental genesis dig onto the process of ‘instrumentalization’ and ‘internalization’, perceptuomotor integration takes a descriptive phenomenological attitude to assess different degrees of tool fluency in temporal flows of the learners’ lived bodily experience. An in-depth analysis of transformations in the participants’ lived bodily activities is another subject that I follow through my thesis.

Mathematical instruments in a perceptuomotor integration approach: A comparison

There have been different attempts to tackle dualistic approaches toward mathematical tools; for example, through the dialectical premise in mediation theory as discussed before (Artigue, 2002; Vygotsky, 1978). In this thesis, in order to reject the ontological duality between bodily and tool-mediated expressions, and mental structures of schemas, I shall use Nemirovsky et al.’s (2013) view on the *mathematical instrument.* In perceptuomotor integration, the mathematical instrument is “a material and semiotic device *together* with a set of embodied practices that enable the user to produce,

[7] In this thesis, “experience” and “mind” terms are viewed from the perspective of embodied cognition, embodied experience, embodied mind (e.g., Varela, Thompson, & Rosch, 1991).

transform or elaborate on expressive forms (e.g., graphs, equations, diagrams, or mathematical talk) that are acknowledged within the culture of mathematics" (p. 376). In this view, there is not such a process of internalization or instrumentalization to explore mental schemes to change mathematical tools from artifacts (which would then become disposal physical objects) into signs or psychological tools. However, one of the commonalities among instrumental genesis, semiotic mediation and perceptuomotor integration is the existence of a link between using the mathematical instrument, tool fluency and sense-making. That is to say, mathematics learning involves appropriate skillful use of mathematical instruments, which are cultural tools that mediate mathematical activity. However, instrumental genesis and semiotic mediation take different approaches and look into the process of instrumentation or internalization rather than analysing learners' "lived experiences, in other words, the temporal flows of perceptuomotor activities they inhabited bodily, emotionally, and interpersonally" (Nemirovsky et al., 2013, p. 375).

The other common theme among the approaches of instrumental genesis, semiotic mediation and perceptuomotor integration is to focus on mathematical learning when the student makes use of the mathematical tool. However, instrumental genesis takes a dialectical approach and considers an instrument as a "mixed entity" (Artigue, 2002, p. 250) as part artefact and part cognitive schemes. The artefact becomes an instrument through the process of instrumental genesis. Studies taking this approach have documented mental schemes associated with instrumental genesis while using symbolic calculators and computer algebra (Drijvers & Trouche, 2008; Guin & Trouche, 1998; Trouche, 2005).

Although there are many scholars that appreciate the sensorimotor origin of mathematical thoughts such as Piaget (1962), the key components of the perceptuomotor integration aim to overcome the opposition between bodily, tool-mediated expressions and mental schemes. So, in the perceptuomotor integration approach, the phrase mathematical "instruments" is intentionally chosen to make an analogy to the culture of music. To explain, a pianist's expertise can be imagined as the quick and skillful finger, hand, body and perhaps eye movements over the piano keys. Similarly to how we cannot imagine a pianist without playing piano, "it is equally objectionable to cleave mathematical expertise from the skillful motoric and perceptual engagement with the tools of the discipline" (Nemirovsky, 2013, p. 7).

Instrumental genesis and perceptuomotor integration question understanding the relationship between technical work and conceptualisation as a "technical-conceptual cut" (Artigue, 2002, p. 247). However, perceptuomotor integration takes a non-dualist approach to overcome this theoretical need instead of dialectical one. Also, instead of privileging conceptual dimensions in meaningful learning, perceptuomotor integration considers mathematical thinking, mathematical learning and tool fluency in the process of learning. So, to track how mathematical learning emerges, perceptuomotor integration does not monitor correspondences between observed bodily and gestural activities, inferences, and processes of formal mathematical thought (Nemirovsky, 2013).

To explain, learning a new skill always goes through stages, in which perceptual and motor aspects of activities move from discordance to become synchronized and coordinated. In other words, tool fluency would be described as the interpenetration of motoric and perceptual aspects of the activity that allows the performer to "act with the holistic sense of unity and flow" (p. 373). For instance, a second language adult learner may feel discordance and incongruity with the sound that she hears and movements of her tongue, lips and vocal cords. The transition from discordance of perceptual and motor activities to fluidity is common in many other skill-learnings such as driving, playing music or soccer, prior to their integration. For example, in a given dynamic geometry environment (DGE), dragging a geometrical object could demonstrate integration of motoric aspects (dragging) and perceptual aspects (visual or haptic consequences of dragging, while the geometrical object preserves geometric properties).

There is an explicit tie between "tool fluency" and "mathematical thinking" and "mathematics learning". With perceptuomotor integration, instead of digging into the process of internalization, mental faculties, and refining/creating schemes, mathematical thinking is defined as bodily activities that involve different degrees of implicit and explicit expressions. Accordingly, mathematical learning is the transformation in lived bodily activities, coordination, and engagement when the learner participates in the mathematical activity. Thus, "First, motor activity is involuntarily enacted as part of perceiving. Second, partial motor and perceptual components have the power to elicit the enactment of the activity as a whole over time" (Nemirovsky, et al. 2013, p. 380). Therefore, mathematical learning is intertwined with tool fluency but does not necessarily follow a dialectical approach (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2011).

1.2.4. Perceptuomotor Integration in Learning Mathematics Through Touch-Based Interactions

To operationalize mathematical learning in terms of perceptuomotor integration, just like Nemirovsky I appeal to Husserl's (1991) phenomenological attitude in experiential time. This approach enables producing the phenomenological descriptions of participants' experience over a given segment of time. Husserl's perspective allows the researcher to develop a possible description of retention and protention constituting that particular experiential present and follow the tool fluency in terms of integration of perceptual and motor activities. The notions of retention (just-past) and protention (about-to) implicate recalled and anticipated aspects of lived experience within experiential time, which consists of a collection of sequential moments.

With a phenomenological attitude, I define one of the main goals of this thesis, which is common to all three studies. That is to understand the temporally extended course of actions that the participants experience. To clarify, lived experience of the learner is understood as a temporal flow of perceptuomotor activity. It is worth clarifying,

> The temporal flow of perceptuomotor activity cannot be characterized in terms of self-contained structures, schemes, or patterns, and necessitates an irreducible immersion in the particulars of the actors' gestures, tone of voice, gaze, and facial expression, steadily emerging from their creative being-in-the-world. (Nemirovsky et al., 2013, p. 375)

I also follow the path of interactions between the mathematical tool and the learner's body, seeking the emergence of mathematics expertise and learning through tool fluency. I elaborate this goal in each study below. I have explained my contributions on each study as well in chapter five.

First Study: "Exploring Cardinality in The Era of Touchscreen-Based Technology"

Working with Dr. Stephen Campbell, in this study we describe the perceptuomotor motor integration theoretical framework. The difference between natural attitude and phenomenological attitude is elaborated as well as the notion of lived experience in experiential time. This is to highlight the Husserlian phenomenological attitude that all temporal experiences have duration, consisting of an ongoing flux of retention (just-past), present time (just-now) and protention (about-to). We exemplify this ongoing continuing change in the use of *TouchCounts*.

Then we discuss and illustrate how tool fluency, and therefore mathematical learning, emerges when Alex engages with the mathematical tool (*TouchCounts*) in terms of what has just occurred and what is anticipated to occur, how the temporal flows of Alex's perceptual and motor activities integrates from an initial discordance. In this regard, we define finger-showing (known as finger-montring[8]) and finger-counting before introducing a new form of body coordination that may illustrate perceptual and motor integration as "finger-touching".

Second study: "Advanced Mathematics Communication Beyond the Modality of Sight"

Taking the same Husserlian attitude, the second study reports on and discusses a blind undergraduate student named Anthony and his journey in learning pre-calculus concepts. The mathematical tool mainly involves tactile graphs (sketch graphs and permanent graphs). The process of developing these tools is also discussed. In terms of perceptuomotor integration, I trace Anthony's learning regarding tool fluency by analysing his temporal streams when he is invited to illustrate a given rational function's behaviour for a sighted audience. This is in order to find out how he discusses the function's behaviour in a way that is culturally recognized in the mathematical community, "just like fluency with a musical instrument can be a crucial skill for certain forms of music playing and for one's membership in sociocultural communities that value and preserve such musical practices" (Nemirovsky, 2013, p. 406).

[8] Please see Di Luca & Pesenti, et al. (2008).

I also discuss Anthony's development and unifying of retentions and protentions from an early stage of discordance of motor and perceptual aspects of the concept to their integration. Anthony's early stage of perceptuomotor understanding is characterised as 'active sensation'. Later, he actively refines sensations according to the goals. I explore how Anthony's bodily performance and motor enactments combined with his verbal explanations is perceived as perceptuomotor integration. Anthony's demonstration of a fluent use of tactile graphs as well as gestural and bodily coordination in a rational function's behavioral representation will be discussed. I look for the moments when sensation is processed, organized and interpreted; so, Anthony uses the information to guide his behaviour based on his understanding of the environment. I introduce this phenomenon as 'tactile perception'. Adapting Nemirovsky et al.'s perceptuomotor integration, I hypothesize that emergence of perceptuomotor integration is evidenced by the 'tactile perception' if performed fluently and recognized by the mathematics community.

Third Study: "Touch-based technology in exploring geometric transformation: use of timeline as an analytical tool"

I pursue the same theoretical framework of perceptuomotor integration in this study, and discuss how a prospective teacher, named Anna, learns about geometric transformation using a touchscreen-based DGE. The study not only traces Anna's temporal flows of lived experience, but also analyses her touch interactions with the DGE, in terms of active and basic actions introduced by Arzarello et al.'s theory of touchscreen interaction (Arzarello et al., 2014). However, in this study I revisit basic and active codes and modify them based on this study's specific goals, tasks and specific DGE tool. I look for Anna's interpenetration of perceptual and motor activities highlighted by the performance of active actions (e.g. drag-touch-to-approach[9]).

I also use StudioCode software and suggest an innovative methodology to analyse video data. Using this methodology enables the researcher to use a video

[9] *Drag-touch* codes the touchscreen interactions when the learner drags the point to create a geometrical shape, justify and/or explain the geometrical relations. For more information please see table 4.1.

timeline as an analytical tool to analyse and assess the paths of interaction and trace emergence of tool fluency over the given stretch of time.

1.3. Summary

In this chapter, I outlined the three stand-alone papers and the common themes that go through these papers, which focus on multimodality of mathematical instruments, tool fluency and mathematical learning using the perceptuomotor integration approach and taking a Husserlian phenomenological attitude. I have also discussed two other well-known theories about the use of mathematical tools named instrumental genesis and semiotic mediation, following this discussion by describing and critiquing the perceptuomotor integration approach. Finally, I discuss the differences between instrumental genesis, semiotic mediation, and perceptuomotor integration theories.

Chapter 2. Exploring Cardinality in The Era of Touchscreen-Based Technology[10]

This paper explores how a young child (56m) builds understanding of the cardinality principle through communicative, touchscreen-based activities involving talk, gesture and body engagement working via a multimodal, touchscreen interface using contemporary mobile technology. Drawing upon Nemirovsky's perceptuomotor integration theoretical lens and other foundational aspects of Husserlian phenomenology, we present an in-depth case study of a pre-school child developing mathematical expertise and tool fluency using an iPad application called *TouchCounts* to operate with cardinal numbers. Overall, this study demonstrates that the one-on-one multimodal touch, sight, and auditory feedback via a touchscreen device can serve to assist in a child's development of cardinality.

The current generation of children have grown up in a world saturated with digital technology and media. They are, so to speak, digital natives (Prensky, 2001). Clements and Sarama (2009, 2014) suggest that the affordances of computers make them more advantageous for developing mathematical thinking than physical objects because, "computer representations may even be more manageable, flexible, extensible", and free of potentially distracting features "than their physical counterparts" (p. 324). There are many educational software programs for mathematics that have been developed for desktop computers or laptops. These programs require interaction via mouse, keyboard or/and electronic pens and not through a touchscreen. Interacting with computers via

[10] This chapter is co-authored with Dr. Stephen Campbell. Professor Campbell was instrumental in helping me tease out, illustrate, and enhance phenomenological aspects of Nemirovsky and colleagues' framework, particularly regarding the distinctions between phenomenological and natural attitudes, experiential and objective time, and the notions of retention and protention.

these devices requires hand-eye coordination, which can be a hard task for young children (Ladel & Kortenkamp, 2014).

Nowadays, mobile digital technologies are becoming much more widely used, at work, in schools, and within the home environment. The touch-sensitive interface of the tablet (such as an iPad) enables children to ‘directly’ interact and manipulate objects via their hands and all their fingers. Additionally, auditory, visual and tactile senses, as well as kinaesthetic touch through gestures, such as flicking, sliding, taping, nudging, pinching and spreading, engage children bodily in deep learning (Lane, 2015). Furthermore, preliminary research on tablets indicates a fascinating educational potential for young children. For example, Geist (2012) found that children as young as two adapt to the intuitive interface of the touch-based tablet as-easy-as with play-dough toys. Despite preschool children’s lack of familiarity with the tablets, they show a high level of independence and take to the devices quickly and persist through technical challenges (Chau, 2014; Couse & Chen, 2010; Sinclair & Sedaghatjou, 2013).

In relation to development of early number learning, researchers have shown that children first engage in rhythmic counting, and only gradually learn how the words they learn by heart in a sing-song way are related to individual ordinal or cardinal numbers (Fuson, 1988; Wynn, 1992). Also, it is asserted that most children develop knowledge of “how to count” including the one-to-one principle, the stable-order principle, and the cardinality principle before entering kindergarten (Jordan, Kaplan, Nabors Olah, & Locuniak, 2006). However, the literature also suggests that counting is considerably more complex than solely reciting counting words and requires the understanding of several underlying principles, such as the last-word rule and part-whole principles (see Fuson, 1988; Gelman & Meck, 1983; Wynn, 1992). Resnick et al. (1991) assert that the transition from ordinality to cardinality is the main key in the formation of the part-whole concept, which refers to additive compositions.

Developing a sense of how numbers decompose is important for children when they are trying to master the basic number combinations, specifically, for ‘five’ and ‘ten’. More generally, the key concepts of ordinality and cardinality further implicate the foundational perceptuomotor dimensions of time and space respectively, and thereby may have broader implications for child and adolescent development.

2.1. The Role of Fingers in Numerical Development

It has become increasingly apparent that the use of fingers fosters flexible calculation strategies, such as composing and decomposing numbers, especially with respect to five and ten, which are essential (Brissiaud & Greenbaum, 1992). Number representation with finger symbols is related to the nonverbal-symbolical form of representation (Domahs, Moeller, Huber, Willmes, & Nuerk, 2010). This is rooted in sensory and bodily experience and enhances embodied cognition, which is termed 'embodied numerosity' by Moeller et al. (2012). Butterworth (2000) notes that developmental and cross-cultural studies have shown that children use their fingers early in life, while learning basic arithmetic operations and the conventional sequence of counting words. In addition, Crollen et al. (2011) summarize the fingers contribution to number sense, specifically regarding: the iconic representation of numbers; keeping track of number words uttered; base ten and sub-base five numerical systems; and developing the one-to-one correspondence and stable-order principles by tagging fingers and countable objects with the saying of number words in sequential culturally-specific ways.

Hence, various modalities of verbal counting, numeral notation and fingers are used to represent a cardinal number. We distinguish between showing a cardinal number (1-10) on one's fingers, or *finger-showing*, from obtaining a number via *finger-counting*, which in turn implicates an ordinal process (Di Luca & Pesenti, et al., 2008, and others use less intuitive term *finger-montring* [sic] for *finger-showing*).

We note that finger-counting, and finger-showing are not necessarily the same (e.g., consider showing versus counting two), and both have been found to be culturally relative (Bender & Beller, 2012; Domahs et al., 2010). We also use the term *finger-touching* when one uses one's fingers to create a cardinal number all at once (or a 'herd') using *TouchCounts*, which may follow upon finger-showing and finger-counting. It means the child touches the screen, only once, with his stretched fingers that are showing the given number.

2.2. *Touchcounts*: Connecting Touch-Based Technology with Finger-touching

TouchCounts (Jackiw & Sinclair, 2014) supports the substantial role of body engagement and senses in education. Having indicated the important role of the fingers for numerosity, we postulate that using fingers to create numbers, when it is supported by tactile, auditory and visual modes of perception, will support and augment cardinal and ordinal understanding of numbers. *TouchCounts* utilizes the multi-touch features and gestures of the iPad. It enables young learners to summon numbers into existence and manipulate them in a digital space; it also offers visual and audible provisions in two sub-applications, namely the Enumerating and Operating Worlds. Multi-touch affordance of the iPad empowered through *TouchCounts*, enables one-to-one correspondence among tap, count and object (de Freitas & Sinclair, 2013; Sinclair & Pimm, 2014). This paper focuses on the Operating World, which offers a model of cardinality for learners to create and manipulate numbers.

2.2.1. *TouchCounts*: Operating World.

In the Operating World, learners simultaneously place one or more fingers on the screen to create a given number or 'herd', or in standard parlance, a cardinal number. If seven fingers simultaneously touch the screen, then a disc appears with the symbol 7 in the centre of the disc as well as seven smaller circles arranged around the symbol, all coloured in the same way. Additionally, the number name "seven" is announced as the herd is created (Figure 1A).

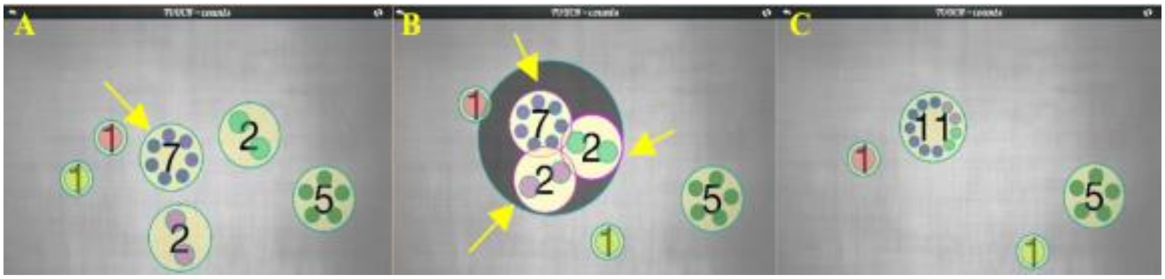


Figure 2.1 The Operating World in *TouchCounts*.

Note. After making different herds of numbers (A), with pinching herds (B) a new herd of number will be created including the inherited subset colours (C) while the resultant number is announced as the new herd is created.

If a new herd is created, its inner circles will have a different colour. Pinching herds together enacts one of the fundamental metaphors of addition, which is gathering (Lakoff & Núñez, 2000) (i.e., Figure 1B and 1C). This pinched herd inherits all the 'subset colours' and appears as a 'bigger size', which is intended to support the part-whole principle.

2.3. Theoretical Framework

We first describe our conceptualisation of cardinality. We follow Vergnaud (2009), who argues that understanding cardinality implies more than knowing that the last number-word of the counting sequence applied to a set of objects represents the numerosity of the set. Understanding cardinality also means being capable of using numbers and operations and, in particular, being able to use strategies such as 'counting on' from a given number. According to Vergnaud: a child has developed an adequate sense of cardinality if, given a set of objects, she can answer correctly the question 'How many objects are there?' and then after other objects have been added to the set, she can still answer accurately. Vergnaud also points to the movement of fingers, eyes and words as 'three different repertoires of gestures' where one-to-one correspondences amongst them help to secure the effectiveness of counting strategies, all of which can be accommodated through playing with *TouchCounts.*

For this study, we offer a phenomenological analysis of a child's interactions with *TouchCounts* using the perceptuomotor integration approach, which focuses specifically on tool fluency and how mathematical expertise develops through a systematic interpenetration of perceptual and motor aspects of working with mathematical instruments (Nemirovsky, Kelton, & Rhodehamel, 2013). Nemirovsky, et al., adopt Husserl's phenomenological framework for *experiential time* (in contrast to objective time), as a key component in their perceptuomotor approach to analysing lived experience (p. 385). Husserl's phenomenological approach was to reflect upon lived experience in manners by which one seeks to identify and describe structural features of lived experience (Campbell, 1998, 2002; Husserl, 1991).

Accordingly, we seek to identify and describe temporal flows of perceptuomotor activity in a lived experience. By lived experience, we refer to the events and transformations that occur through on-going covert and overt experiential embodied alterations infused with recall of the immediate past coupled with expectations of the immediate future; or, more specifically, what has just occurred and what is anticipated to occur as a learner engages with a mathematical tool. For instance, in using a compass to draw a circle, the learner, having anchored the pointed end of the compass on a piece of paper (immediate past), adjusts (uses their body to transform) the radius of the compass (in the present) in anticipation of creating a circle of a given size (immediate future). Also, within the perceptuomotor integration approach perceptual and motor aspects of working with a compass (as a mathematical instrument) are inextricable from the concept of circle and learning the tool is intertwined with learning mathematics and exhibits as transformations is lived bodily experience. "… concepts are vast dwellings, landscapes to inhabit by virtue of their being amenable to hosting certain forms of life and not others, while, at the same time, those who inhabit a concept transform it as they go on living in it" (Nemirovsky, 2017, p. 13).

In conducting our analysis, unlike Nemirovsky, et al., we further incorporate Husserl's foundational distinction between phenomenological and natural attitudes. Simply put, the natural attitude is the attitude one brings to bear on being in the world, as being but a part situated within a larger whole, so to speak. The phenomenological attitude is the attitude taken when one comes to realise that the natural attitude is in fact just our lived experience of the world rather than the world as it is in itself. That is to say, the phenomenological attitude is concerned with the structure of lived experience, *per se*. It is important to note that objective time implicates the natural attitude, whereas experiential time implicates the phenomenological attitude. The natural attitude of being in the world requires being in the world *now*, at the present moment in objective time, whereas within the phenomenological attitude, notions of retention and protention implicate remembered and anticipated aspects of lived experience within experiential time. Experiential time affords the simultaneous presence and collection of sequential moments in objective time, thereby rendering cardinality, and number sense more generally, possible.

Beyond lived experience, Nemirovsky et al. (2013) further suggest: “While perceptuomotor integration constitutes a transformation that is experienced by an individual, it is (a) shaped by relatively local social interaction and relatively global cultural factors and (b) socially consequential because one’s degree of instrumental fluency has bearing on one’s membership to various social groups” (p. 381). Accordingly, Nemirovsky, et al.’s framework also shares many similarities with the emerging body of theories and attention in mathematics education that moves away from a mentalist focus on structures and schemas towards a rich description of lived experiences in which learners’ activities are at once bodily, emotional and interpersonal (de Freitas & Sinclair, 2013; Radford 2011; Arzarello, 2006; Roth, 2009). This sociocultural orientation incorporates a strong position with respect to embodiment, whereby *TouchCounts* is not just an instrument, but with tool fluency becomes a quasi-extension of one’s own embodiment. Mathematics learning entails transformations within the lived body situated within a social context. In this paper, we explore ways in which cardinal aspects of numerical abilities can be developed and expedited through playing with *TouchCounts* and the impact of touch-based interactions on the development of young children’s perception and motor understanding of the cardinality principle in small group activities.

2.4. Method

The phenomenon of interest in this paper is the lived experience of a single child engaged in the use of *TouchCounts* within a classroom context. There are two interrelated aspects of the child's lived experience that provides our focus here. First, how does the child develop tool fluency with *TouchCounts*, and secondly, in what ways might tool fluency implicate and be implicated in the child's understanding of cardinality?

2.4.1. Procedure

Data for this study is part of a larger research project which aims to understand ways in which young children between three and seven years-old develop numerical abilities through embodied interactions with *TouchCounts* in small group and one-on-one activities. The two researchers set up a station in the corner of the room in a daycare facility where children were invited to play with *TouchCounts* on the iPad. See Appendix B for daycare floor plan and classroom arrangement. The children were free to come and go as they pleased, which is common practice in most daycares. This meant eight to ten children at times crowded around, but sometimes only one child. It also meant that some of the twenty-four children in the daycare did not participate in the study. Conversely, some children participated multiple times. Most of the time, there were three or four children taking turns playing with *TouchCounts*. The researchers also combined this instrument with other manipulatives such as cards (see Appendix A), so children could play with numbers in a variety of ways when not engaged with *TouchCounts*. Seventeen exploration sessions (see Appendix C for more detail) of about one hour each took place once every one or two weeks. The children tended to spend between five to fifteen minutes at a time but not necessarily the same children. One of the researchers worked with the children, by asking questions, helping them work together (so that each child had an opportunity to play) and posing various problems, while the other videoed the sessions. The interviewer-researcher is referred to herein as “N” or “researcher”, as context indicates.

2.4.2. About Alex

Because phenomenology is concerned with the lived experience of individuals, we limit our focus to a single child where the recorded data could offer a thick description (Miles, Huberman & Saldana, 2016). The child we focus on in this study is a four-year and eight-month-old boy (56m) that we shall call ‘Alex’. Alex has a twin brother ‘John’.

The twins' daycare teacher reported that Alex, along with the other children, exhibited good social interaction combined with a high desire to learn and find explanations about what he saw around himself. He enjoyed expressing his theories in drawings, clay, and other ways. Alex also liked to manipulate loose materials and make elaborated figures, and was able to count to 10 starting from one, when our study started. The teacher also reported that the children were exposed to no formal mathematics instructions in the daycare during the academic year prior or in parallel to our study.

2.4.3. Selected Episodes

Considering the descriptive nature of the research design, research aims, and the perceptuomotor framework, we selected three episodes drawn, respectively, from three separately recorded sessions at the daycare over a five-week period. These episodes demonstrate Alex's experiences in developing tool fluency and his understanding of cardinality. Although other children are involved in all three of these episodes, we focus here mainly on Alex's experiences and interactions with *TouchCounts* and the researcher. In the first episode Alex begins to gain tool fluency hampered by a dependence on ordinality. In the second episode we include two activities that contribute to Alex's evolving understanding of cardinality and perceptuomotor integration. The third episode illustrates Alex's increasingly high level of perceptuomotor expertise. Taken together, these three selected episodes provide a rich illustration of mathematical learning through the development of tool fluency.

2.5. Description and Analysis

In what follows, we begin by providing a brief description of Alex's actions, including gestures directly or not directly engaging *TouchCounts*, his interaction with the researcher, and on occasion with other children as well. We describe his "…*lived experiences* in terms of temporal flows of perceptuomotor activity, which are at once [embodied] and interpersonal" (Nemirovsky et al., 2013, p. 6). We analyse multiple streams of observed embodied activities that implicate Alex's lived experiences of mathematical thinking and learning (Campbell, 1998; Husserl, 1991; Nemirovsky et al., 2013) in terms of Alex's developing tool fluency and understanding of cardinality.

2.5.1. First Episode: Doing Six

The researcher shows a card that contains two hands, each showing three fingers. She asks Alex: “Can you do three and three?”, inviting Alex to add three plus three. Alex enumerates by touching the screen using one, then two, then three (but just two finger tips actually touch the screen), and then four fingers, making *TouchCounts* say “one”, “two”, “two”, “four” and produce the herds that each have small disks with the same colour (See figure 2.2).

The researcher notes, “We are missing a number [She points to the herds on the screen]. We have one, two, two, and four. What number is missing?”

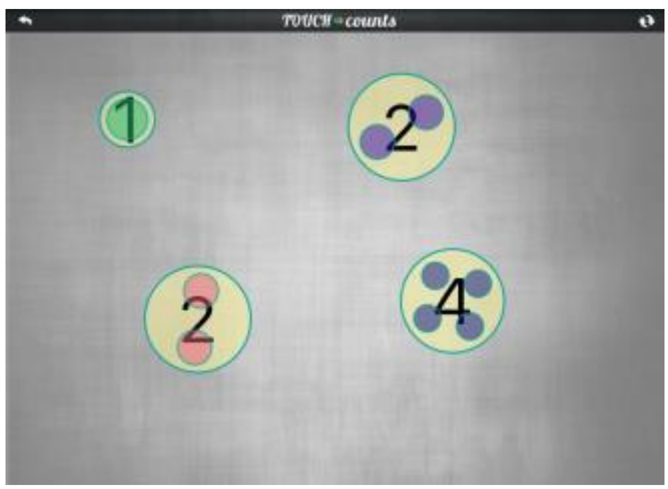


Figure 2.2 Alex makes 1, 2, 2 and 4. Each herd has small disks with the same colour.

Alex looks at the numbers on the screen and points to one of the twos:

11 Alex One, two, three [in a low pitch] … three!

Alex resets *TouchCounts*. The researcher asks Alex:

12 N Let’s see ‘*just* three’

Alex, however, follows the same enumerative steps as he did before and makes one, two (with two fingers simultaneously), three (with three fingers simultaneously), four (with four fingers simultaneously), and five (with five fingers simultaneously). *TouchCounts* says “one, two, three, four, five” (Figure 2.3) and there are five herds on the screen. Alex refused to make ‘just three’ not starting at one and count on from there to five, indicating an underdeveloped sense of cardinality.

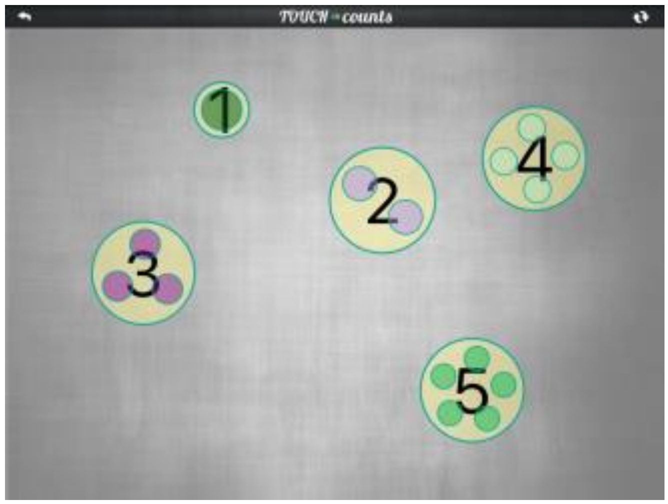


Figure 2.3 Alex makes 1, 2, 3, 4, 5 sequentially.

Moreover, up to now Alex had made all the herds using his left hand. He looks at his left hand, stretches all fingers, curls them, stretches just his left thumb, looks again at his hand, curls all his fingers and looks at the interviewer with a confused expression.

13 Alex I don’t know, I don’t know … [pause] … how to do “six”

The researcher pretends she also does not know how to make six: “Let’s think about it. How could we do six?” She asks Alex to count on his fingers up to six. Alex finger-counts by stretching his curled fingers starting from his left thumb “one”, “two”, “three”, “four”, “five” (Figure 2.4 A), again he curls all his left fingers, leaving just his left thumb. Alex looks at the researcher and says: “Six” (Figure 2.4 B).

Figure 2.4 ***(**A) Alex is not using both hands to count up to 6; (B) He shows his left thumb again as 'six'. (C) Alex counts on fingers using his lips and he always starts from one to make a six when he is asked to put 'just 6' on screen.**

Seeing that Alex refers to the last counted finger as 'six', but is no longer finger-showing "six", the researcher asks him to use both hands to make a 'six'. Alex opens his left hand and right thumb and index fingers (showing seven fingers), then counts his fingers by *touching* them one by one with his *lips* while he counts, starting from his right index finger, counting out "one", "two", "three", "four", "five", "six" (Figure 2.4 C). He looks at the researcher, notices he is showing one extra finger, so he folds his left pinkie.

14 Alex So, six [Touches his ring finger with his lips].

Alex then moves his stretched fingers towards the screen, but one of his fingers touches the screen first, making *TouchCounts* say "one". The researcher resets *TouchCounts* and invites Alex to try again and make 'just six', but Alex makes the sequence, starting from 'one' again, as he did before (Figure 2.4 C). While he produces many other numbers, he never makes a successful six on *TouchCounts* during this episode. What is evident, aside from his difficulties with finger-showing 'six', is that Alex was experiencing difficulties in moving from finger-showing numbers greater than 'two' to finger-touching the screen to generate some of the larger herds intended (i.e., three and four). Yet, in his struggle to touch the screen with all fingers shown up to five, he was clearly developing tool fluency.

Alex was able to finger-show numbers up to 'five' and at least attempt to finger-touch the screen accordingly. Making five all at once with five fingers is an important step toward cardinality, but it is not enough for understanding *fiveness* in and of itself, as a cardinal number. In trying to make ‘six’ Alex was always finger-counting and finger-touching up from one and not starting ‘just from three’ [12].

Nevertheless, in his first attempt to make ‘six’, Alex was clearly counting up from one with an aim to complete his task at ‘six’. This supports Jordan et al.’s (2006) finding that the development of ordinality precedes cardinality. It also clearly indicates the role of retention and protention, as retention is evident, proceeding from the onset of finger-counting, with protention evident in Alex’s recognising the completion of the process. So, the problems encountered thereafter were not with finger-counting, but rather, with finger-showing, further indicating a problem with cardinality.

With the broader protentions of finger-showing ‘six’ in order to finger-touch ‘six’ using *TouchCounts*, Alex’s method of finger-counting to ‘six’ with just his left hand was inadequate to the task. That is to say, his protention to finger-show ‘six’ was disrupted [13], evidently due to his underdeveloped sense of cardinality. We see further evidence of his underdeveloped sense of cardinality as well as his ordinal retention of having counted up to five on his left hand disrupted when the researcher requests that he use two hands. For, rather than re-finger-showing ‘five’ on his left hand and counting forward to ‘six’ on his right hand, he extends both the fingers on his left hand and his right thumb and index finger. Then, once again, he counts up to ‘six’, at which point he is left with a non-standard finger-showing of ‘six’, which he then unsuccessfully, due to dexterity difficulties, attempts to finger-touch on the screen. What this does show, however, is that he retained his protention of making ‘six’ on *TouchCounts*, which indicates an on-going improvement in tool fluency.

As further evidence of developing retention and protention, Alex seemed to know there were six fingers in total on the card that he was shown, but he was not successful in re-creating the manual configuration shown on the card using *TouchCounts*, which is three fingers outstretched on each hand. Nor was there much indication that he even tried to mimic this finger-showing. This may be because he still requires counting ordinality to approach cardinality for numbers greater than ‘five’, so he does have some degree of protention as an aim and outcome. This could also be evidence of a lack of one-to-one correspondence amongst fingers, words and cardinality of the set, as suggested by Vergnaud (2009), and a lack of tool fluency (Nemirovsky et al. 2013). So, we suggest, the interpenetration of perceptuomotor skills, and retentions and protentions regarding ordinality and especially cardinality, was still at an early stage.

2.5.2. Second Episode: Expanding the Sense of Cardinality

The researcher met with the children four weeks later. In this episode, we illustrate three incidents that describe multiple streams of embodied mathematical activities, which we hypothesise supports Alex's on-going co-development of tool fluency and cardinality. This episode demonstrates how guided and collaborative playing with *TouchCounts*, along with cards illustrating different finger-showings, can contribute to the enhancement of Alex's perceptuomotor skills, and his capacity for retention and protention in the development of his understanding of cardinality in three different sections entitled: Missing addend, fine motor ability and eye-hand coordination, and making six.

In this session, the researcher continued with a card playing game introduced in the previous session from which the first episode above was drawn. In this game, children were asked to make the total number of fingers that are shown on each card. Each card contained one or two hands that finger-showed numbers in non-standard ways (see Appendix 1). The broader research aim of this task was to give the children a chance to explore the world of cardinality and the extent to which the children would not mimic the hand gestures shown on the cards when using *TouchCounts*, as a higher perceptuomotor integration of the cardinal number were required to finger-touch.

Missing addend

After about twenty-minutes of the researcher card playing with a child named 'Sara', a group of six children forms around her, including the twins. The researcher explains to them "the game we were playing was that I was making something and Sara was making something [on *TouchCounts*], and when we put them together we had to get to five". The goal of this game was to explore different combinations of the number five. So, some of the children make 'fives' with different combinations: One and four, two and three, and three and two, while Alex is looking through the cards. The researcher proceeds to ask the children how to make 'five', when there is already a herd of three on screen. Alex shows two cards to the researcher and asks, "like this?" (Figure 2.5 A), illustrating that Alex was contributing to the number composition game. However, the researcher did not notice it. After a few minutes, the researcher notices Alex is playing with the cards and asks him what is he doing.

21 Alex I'm just putting the pictures… and …and putting [numbers] together to make different numbers.

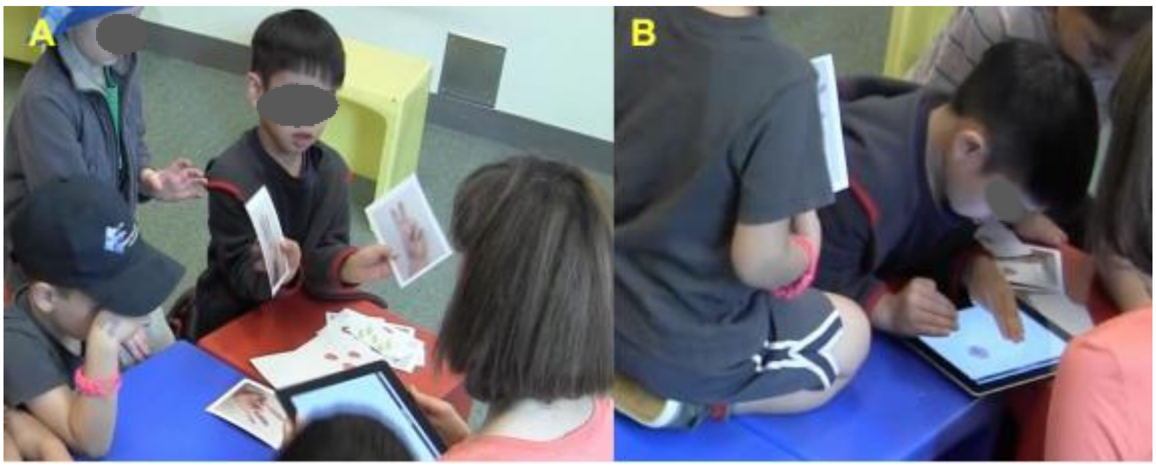


Figure 2.5 (A) Alex shows two and three asserting that it makes five. (B) Alex makes the second 'four' with the same gesture but is now careful enough to touch the screen just with his fingertips.

Perhaps Alex's understanding of cardinality, as part-part-whole, is beginning to improve, evidently by the temporal flows of tool fluency in using appropriate cards. The card game specifically targeted cardinality of the set by asking children to create the same number on *TouchCounts* regardless of the number configurations shown on each card by raised fingers. The game also aimed at subitizing, encouraging children to expand sense of finger-showing (finger showed on the cards) to finger-touching on the iPad. In this episode, although Alex did not try finger-touching yet, he was actively involved in blended finger-showing and finger-counting activities. We also observe children were actively engaged in ongoing mathematical debates which are not reported here in detail, until they developed a relatively stable social understanding of cardinality of five (Hall, Ma, & Nemirovsky, 2015). This incident, with the following one, demonstrates a smooth and concurrent trajectory of mathematical and embodied expertise.

Fine motor ability and eye-hand coordination

Because of the direct modes of number manipulation on *TouchCounts*, the "mediation" term is no longer accurate enough to capture the role of children's hands and fingers on the screen. In other words, no *interaction instrument* is needed to mediate between user and touch-interface than human senses: touch, sight and hearing.

I found the same Deleuzean relationships (2003) between hand and eye (digital, haptic, manual and tactile) as discussed earlier in Sinclair and de Freitas's article (2014). These relations[11] range from maximum to the minimum in terms of domination of eye over hand. For example, hand subordinated the observant eye in Deleuze's "digital" sense when children carefully finger-showed. Also, when Alex used his lips rather than eyes to finger-count up to six he demonstrated a "tactile" relationship between hand and eye, while his hand dominated the eye (see Figure 2.4 C). Within the perceptuomotor integration perspective, I look for coordination of eye and hand in form of "tactile", when the hand and eye mutually and equally contribute to create the given number through successful finger-touches on *TouchCounts.*

In this episode, Alex asks the researcher if he could make the numbers shown on one of the cards, which has two hands each finger-showing four. He touches the screen with four-tightly-grouped fingers and his palm and hear/sees 'seventeen' (Figure 2.5 B), because he had inadvertently touched the screen with too many parts of his hand. On the second and third tries, he successfully makes two 'fours' and pinches them well to make 'eight'.

Inspired by Alex's enthusiastic card play, the children ask the researcher to make all the cards on *TouchCounts* (Figure 2.6 A). After about five minutes, some children leave the room and the twins get another chance to play with the cards and *TouchCounts*. They line up the cards and continue the game collaboratively. They count the fingers on each card and create them with *TouchCounts* while noting the total. This

[11] I will discuss hand-eye relationships with more details in 5.1.

activity provides the twins time to become more fluent in using *TouchCounts* (Figure 2.6 B & C).

Figure 2.6 Children are actively engaged in making cards' numbers using *TouchCounts*.

Note. (A) Children ask the researcher to make cards on *TouchCounts*. (B & C) The twins make combination of card numbers on *TouchCounts*

We found playing this game, which involved finger-counting and finger-showing, in conjunction with finger-touching using *TouchCounts* could facilitate coordination of body, gestures and using mathematical tool. Comparing with the first episode, a greater interpenetration of perceptual and motor aspects was observed. Also, hand-eye relationships in terms of "digital", "tactile" and "manual" in forms of finger-show; finger-touch and finger-count offering evidence of emerging early stage of perceptuomotor tool fluency were observed (Deleuze; 2003; Sinclair & de Freitas, 2014). We agree with Andres, et al. (2007) that finger-counting contributes to structure activities, so when children make one-to-one-to-one correspondences among total fingers on cards, number names and their own finger-touching, culminating with creating the associated number herds on *TouchCounts*, produces an enriched number processing activity in a collaborative environment.

Making six

A few minutes later, with Alex looking on, Yan, a four-year-old child, makes a 'six' with two fingers and then four fingers. Alex intervenes when the researcher invites Yan to make six in another way. Alex askes:

22 Alex […] one and two and three?

The researcher does not hear the 'one' in Alex's sentence and says, "a two and a three? Do you think two and three makes six? Let's see if two and three makes six".

Alex finger-touches a herd of two and a herd of one, then without finger-touching, he says "Three!" The researcher asks him to "put them ['one' and 'two'] together". Alex *slides* the herd of one over the herd of two, but nothing happens. He then finger-showed 'three', saying "three". The researcher reminds him to pinch the herds together (instead of sliding one on top of the other). "So, you got a three and…" N says, as Alex finger-touches a 'four' on the screen.

23 N and a four? Do you think if you put them together you'll make a six?

Alex successfully pinches the herds of three and four together and makes a seven. He looks at the researcher, smiles and says, "too many".

24 N too many, you're right. So, what do you think we should do if we have three already. What do you think goes with three to make it six?

Figure 2.7 Making six using three and three. Alex thinks (A) what goes with three to make it six. (B) He finger-shows 3 and 3. (C) Alex hi-fives after a successful finger-touch for making a herd of 6.

Alex scratches his cheek, pauses and thinks (Figure 2.7 A). Nolan, who is drawing his hands in the other corner of the room, says "Another three". Alex very quickly makes two herds of three with his fingers (Figure 2.7 B); then he pinches the two herds of three together and makes a herd of six, and celebrates with a high-five (Figure 2.7 C). The researcher points to Yan, "You made a six by two and four"; and points to

Alex “and you did three and three”. The researcher continued to engage the children with all possible combinations of six.

These three consecutive incidents happened within an hour, documenting Alex’s improving tool fluency. Overall, this episode also highlights the role of researcher as a facilitator rather than a director [22 to 24]. In contrast to the first episode, Alex’s responses were more immediate and fluent. For example, in the first episode [11 to 13] Alex was following rhythmic counting, always starting from one on *TouchCounts*, to make a ‘six’, while in the above incidence Alex used his fingers to make three proficiently. He also became proficient in pinching herds together, rather than ineffectually sliding one on top of the other. In addition, an imprecise attempt to make a four, which later turned into making a successful four, points to Alex’s awareness of the need to be precise making herds with *TouchCounts* meteorically.

Alex’s increased perceptuomotor skills and improved understanding of cardinality is accompanied by more seamless continuity between retention and protention. He is better able to retain his image of the cards. In this experiential time, Alex’s action does not include retention of necessity of sequentially in making six, rather higher level of motor action and present emerge of protention regarding cardinality principle. This precision in using the tool is conjoined with the precision of gesturally articulating each of the numbers. In this sense, the process of developing a practical understanding of numbers is enhanced via perceptuomotor integration.

These activities comprising episode two happened within an hour. In contrast to the first episode, Alex’s responses were more immediate and fluent. For example, in the first episode [11 and 14] Alex was following rhythmic counting, always starting from one on *TouchCounts*, to make a ‘six’.

Later, in episode two, it was observed that Alex used his fingers to make three proficiently. He also became more proficient in finger-pinching herds together. In addition, an imprecise attempt to make a four, which later turned into making a successful four, points to Alex’s awareness of the need to be more precise in making herds with *TouchCounts.*

2.5.3. Third Episode: Playing 'I wish – I have' on *TouchCounts*

A week later, the daycare was visited once again. Alex, his twin John, and two other children were sitting beside the researcher on the carpet and waiting for their turn, while observing other children interacting with *TouchCounts*. After about 20 minutes, the researcher gave Alex a turn to play an ongoing game called: 'I wish I have'. The game was designed to further explore cardinality through the part-part-whole principle, but the children were not provided with cards or other manipulatives.

The researcher made a 'three' on the screen, then offered the iPad to Alex and asked him "Okay, this is your turn. I have three here and I wanna get up to six. What could I do?" Alex looked at his fingers and murmured to himself "six, six". Immediately, John (Alex's twin) finger-showed eight fingers (five on one hand and three on the other), curled two of them in a way that just three fingers on each hand remained (Figure 2.8 A). At that point, Alex proceeded to count John's fingers, initially starting from his bent pinky.

31 Alex One, two

Figure 2.8 (A) Alex counts on and with his twin's and his fingers and then (B) finger-touches three.

John said “No”, and looked at his left ring finger indicating to start counting from there. John accompanied Alex’s counting up to six by outstretching corresponding fingers “one, two, three”. Then, he stretched out his three right fingers and let Alex count up to six.

Alex stretched four fingers, folded his pinkie and finger-touched a herd of three to match the herd of three the researcher had placed on the screen (Figure 2.8 B). Then, he *collides* (a forceful-rapid touch-pinch of two herds using his two index fingers) the two threes (Figure 2.9 A), making *TouchCounts* form a combined herd of six and say: “Six” (Figure 2.9 B). The researcher inquires of Alex, who sits back proudly with thumbs up, “How did you get that?” to which John pipes up and responds “three, three” while pointing to the herd of six on the screen. Alex then examines the herd of six [inset to Figure 2.9 B], counting the small circles of different colours to count out the two parts of three that comprise six as a whole.

32 Alex Four, five, six.

Alex stretched four fingers, folded his pinkie and placed three fingers on the screen (Figure 2.8 B). Then, he *collide* two threes and makes *TouchCounts to* say: “Six” (Figure 2.9 A). While other children in our research pinched numbers on the screen to make a bigger set of numbers, Alex made a new herd of numbers (six here) from the collision of two threes (Figure 2.9 A).

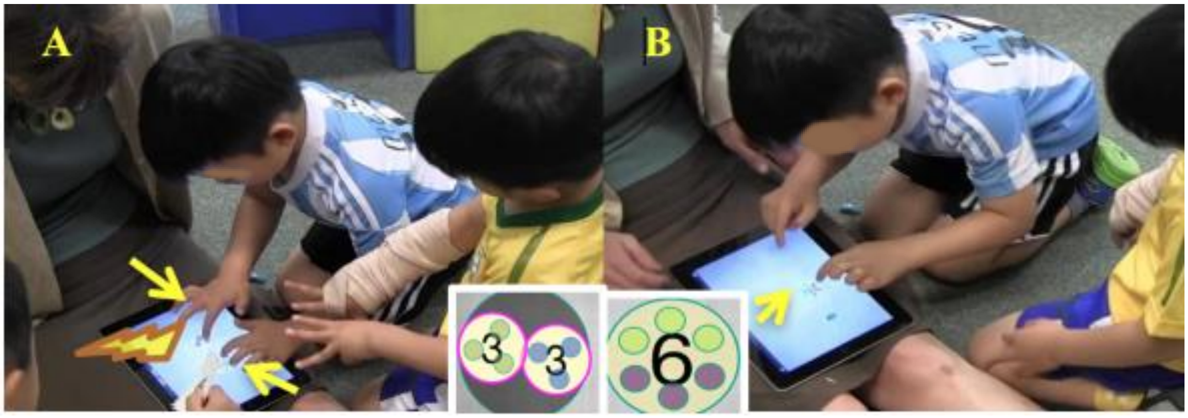


Figure 2.9 (A) Alex 'collides' two herds (B) and examines the outcome.

Evidently, his way of thinking using *TouchCounts* about how to make six involved starting with the two numbers and then operating on them, which is different than the usual written and verbal order a numbers, followed by the operation, followed by the number. More importantly, whereas he slid one number on the other before, which may indicate that his thinking of a sequential process, this colliding gesture was much more of a simultaneous process. Such a simultaneous process is different than how we speak addition and write addition, but is like how we gather things together. So, we suggest Alex's experiential present consist of emerging new pattern of motor action and perception, which is an evidence of a breakdown of the previous retention in process of emerging protention that demonstrates high level of bodily and numerical expertise.

We would argue that the colliding gesture is produced by a perceptuomotor simulation and may be infused by sociocultural norms of kids' mathematical games. The second episode and its sub-incidents, also clearly illustrates the influence of cultural and social interactions on developing tool fluency. The speed at which Alex figured out the answer and made a 'six' by *colliding* two threes may also suggest an understanding of the part-whole principle. Although he benefited from his brother's contribution, he was confident that 'three and three' makes 'six'. It seems that at this point, Alex has a well-developed *expertise* that enables him to not only *count on* his fingers, but also *count with* his twin's fingers (see Sinclair & Pimm, 2014).

The researcher congratulates Alex for 'getting to six' and asked him "How did you get to that?" Alex smiles and shows two thumbs up. Then the twins bent over the screen.

Alex counts the small green and purple circles inside the herd of six by pointing to them on the screen “by three and three!” (see Figure 7B).

Apparently, *TouchCounts* not only supported Alex's understanding of the cardinality with visual and vocal feedback, but also its features in inheriting all subsets of colours in new sets provided Alex with a record of his operations, which maintains retention; and we conjecture supports the *part-whole* principle understanding. We suggest that Alex’s emerging perceptuomotor fluency is characterized here by smaller, finer and faster overt and covert motor actions in his lived experience. His infusing of past and future in the lived present, retention and protention, respectively enabled him to articulate his brother’s participation and complete the task.

2.5.4. Triangulation

Methodological rigor for this study was attained through the application of verification and validation (Creswell, 2013). Verification is the first step in achieving validity of the research project. The standard was fulfilled through literature research and review adhering to the phenomenological descriptive method, bracketing out researchers’ past experiences, which Husserl called *epoché* was considered to suspend the authors' possible presupposition bias of effects of using *TouchCounts* on developing number sense in reflective mode (Husserl, 1991; Huberman & Saldana, 2016). Nevertheless, the researchers had to “mediate” between the phenomenological approach and the natural view that is described in the theoretical framework section (van Manen, 2016, p. 26).

We also provide an essence of experience, a combined of textural description of the experience of the child and a detailed structural description of how the child experienced it, in terms of condition, context and situation (Creswell, 2013). In addition, validation within project evaluation was accomplished by multiple methods of data collection (observation, interviews and video recording). In addition, data analysis was discussed in small group of experienced researchers (during a doctoral seminar) to check researchers’ interpretation and conclusion.

Evidently, Alex's way of thinking with *TouchCounts* about how to make six in this instance began collaboratively, starting with the task provided by the researcher, then with the help of John, observing and counting his twin's finger-showing. Thereby, he recognized that two threes constituted a total of six, and then noted that one of the two herds of three was missing from the screen. At this point, he recognized that the goal set to him was now within his reach. By finger-showing his own three and finger-touching the screen to provide the missing herd of three, he was then finally able to manifest his protention by colliding the two herds of three into a herd of six. A job well done. Thumbs up, indeed!

2.6. Discussion

We have seen in episode one that Alex is struggling with the concept of cardinality, and that his nascent understanding of number is predicated on counting on from one. Indeed, his approach seems almost ritualist, but lays bare the bare bones of retention and protention, a sequence of moments with a definite beginning flowing seamlessly toward a definite end. Until now, that is, Alex comes to finger-count beyond five and is required to finger-show six. What is particularly compelling with *TouchCounts* is that finger-showing is a prerequisite to finger-touching, which emphasises the simultaneous versus sequential nature of cardinality versus ordinality. Episode one with Alex served well to illustrate that.

In episode two, it was evident that Alex was improving his perceptuomotor skills, particularly with regard to finger-showing and finger-touching, and thereby was improving his tool fluency with *TouchCounts*. His improved tool fluency, as demonstrated by making two herds of four and combining them into a herd of eight suggested that perhaps Alex was also improving in his understanding of cardinality. At the very least, there was a seamless continuity between retention and protention. Alex knew what he wanted to do and he set about doing it quite deliberately and successfully, with only a minor disruption.

Although Alex was clearly improving his tool fluency with *TouchCounts*, was he just mimicking the finger-showing configurations on the cards when he finger-touched two herds of four, and then finger-pinched them to obtain a herd of eight? Episode three, a week later, seems to have answered that question in the affirmative. It was clear in that last episode that Alex still had work to do in transitioning from ordinality to cardinality. Remarkably, the limitations Alex had experienced in episode one with finger-showing were ameliorated vicariously with the assistance of his twin brother John, by lending his hands to finger-show a group of three fingers on each hand. This enabled Alex to use a free hand to count his brother's fingers. This collaboration enabled Alex to maintain his protentions without disruption, and eventually successfully complete the task that was set to him.

TouchCounts appears to have supported Alex's developing understanding of cardinality with tactile, visual and vocal feedback. Particularly, its feature of inheriting all subsets of colours in new herds provided Alex with a visual record of his operations, which helps support retention and, we conjecture, further supports the part-part-whole principle, which provides important support toward cardinality. We suggest that Alex's emerging perceptuomotor skills are characterized here by smaller, finer and faster overt and covert motor actions in his lived experience. His infusing of past and future in the lived present, retention and protention, respectively enabled him to articulate his brother's participation and complete his task.

2.7. Conclusion

There is a complex interplay between the natural attitude of observing objective time unfolding from moment to moment, and the phenomenological attitude, which augments the experiential present with moments retained from the immediate past, while also protentively maintaining various ends in view. Perhaps nowhere is this complex interplay more aptly illustrated, and perhaps canonically so, than with the process of developing an understanding of ordinality and cardinality.

What we see with a child's developmental understanding of ordinality and cardinality is even more complex, as it seems apparent, if not quite likely, that what is unconsciously co-developing with this understanding is one's capacity for retention and protention, per se. A significant consequence thereof is to expand experiential time beyond the bounds of objective time, laying important cognitive foundations for memory and anticipation. This is not to say that the child development of memory and anticipation is totally dependent upon developing an understanding of ordinality and cardinality, but rather only to point out that developing such an understanding is deeply implicated in those crucial cognitive functions. As such, any pedagogical innovations at hand that can encourage, enhance, and expedite such understandings should be given close consideration. We have endeavoured to do so with this case study of Alex within an embodied phenomenological framework that emphasises the role of perceptuomotor integration and tool fluency using *TouchCounts*.

Chapter 3. Advanced Mathematics Communication Beyond Modality of Sight

> *I have been totally blind since birth and have studied algebra, geometry, and calculus. I found geometry especially difficult because I lacked the understanding of many spatial concepts.... I found that I had difficulty understanding such concepts as how four walls meet the ceiling, and I actually stood on a chair to study this.*
>
> *- Bev Wieland, Programmer-Analyst, University of Delaware*

History has shown that visual impairments in general, and blindness, in particular, are not insoluble impediments for learning mathematics. Euler (1707–1783) and Saunderson (1682–1739) are two well-known mathematicians who struggled with blindness. Although representation and understanding mathematical visual cues are assumed to be at the core of understanding in mathematics, the contemporary blind mathematician Jackson (2002) suggests that the lack of access to the visual field does not diminish a person's ability to visualize but morphs it. He argues that spatial imagination amongst people who do not see with their eyes relies on tactile and auditory activities. Inspired by these possibilities, in this study I illustrate how mathematical communication and learning are inherently multimodal and embodied; hence, sight-disabled students are also able to conceptualize visuospatial information and mathematical concepts through tactile and auditory activities. Adapting Nemirovsky, Kelton and Rhodehamel's (2013) perceptuomotor integration approach, I shall show that the lack of access to the visual field in an advanced mathematics course does not obstruct a blind student's ability to visualize, but transforms it. The goal of this chapter is not to compare the visually impaired student with non-visually impaired students to address the 'differences' in understanding; instead, I will discuss the challenges that a blind student, named Anthony, has encountered and the ways that we[12] tackled those problems. I also demonstrate how the proper and precise crafted tactile materials empowered Anthony to learn about mathematical functions.

3.1. Learning Mathematics in Absence of Sight

Following the UNESCO's World Conference on Special Needs Education (1994) in Spain, the Salamanca Statement affirmed the necessity and urgency of providing education for children, youth, and adults with special educational needs within the regular education system. Within this system, most (under) graduate students with vision disabilities avoid taking mathematics courses at higher levels of education, and tend to feel anxious and negative about mathematics. As a result, these students are hindered in moving toward their desired fields of study, and careers (Janiga & Costenbader, 2015; Moon, Todd, Morton, & Ivey, 2012).

Also there are some studies on learning and teaching mathematics for learners with visual impairments from early childhood to the secondary level (Figueiras & Arcavi, 2015; Barwell et al., 2016; Dick & Kubiak, 1997; Healy, 2015; Healy & Fernandes, 2011; Marson, Harrington, & Walls, 2013.; Quek, McNeill, & Oliveira, 2006; Vygotsky, 1978). There is also limited research focused on university-level students tackling advanced mathematical concepts (Janiga & Costenbader, 2015; Marcone, 2013; Moon et al., 2012). To help redress this situation in this paper I report a blind student's journey toward understanding pre-calculus concepts.

One of the central tasks for learning mathematics in general and pre-calculus in particular, is to understand and interpret mathematical concepts, graphs, and objects (Healy, 2015; Healy & Fernandes, 2011, 2014). Healy (2015) claims that understanding the visual cues goes beyond "seeing" and can develop in the absence of vision, because such an understanding consists of other sensorial perceptions, relationships with previous experiences and knowledge, verbalization, and more. This view that blindness does not necessarily entail or result in any impairment of the visual cortex, is well

[12] In this study, "we" refers to Anthony, I, sometime his instructor and tutor.

established in the neurosciences (Burton, 2003; Hawellek, Schepers, Roeder, Engel, Siegel, & Hipp, 2013; Klinge, Eippert, Röder & Büchel, 2010; Jiang et al. 2009; Thaler, Arnott & Goodale, 2011). Moreover, there is substantial evidence indicating that the visual cortex involves and readily adapts itself to the other spatial sensory modalities of hearing and touch (for an example see: Spence, Nicholls, Gillespie, & Driver, 1998). Further explanations and more detailed discussion of neuroscience literature and instructional aspects of mathematics communication are beyond the scope of this paper.

3.2. Complexity of the Mathematical Graphs

Most mathematical and statistical graphs contain a concise, complete, and precise summary of functions, equations, and information such as: ordered pairs, axes, origin, grid lines, tick marks, intersections and labels. Obviously, these components are all visual. Thus, for students with impaired vision, understanding the mathematical concepts behind them or learning the concepts themselves becomes an extremely challenging task. It is even the case that students who can see find it hard to ‘read’ this information—being visual doesn’t necessarily mean understanding what you see. Teachers working with visually impaired students struggle to convey the concept of a graph, especially when the concept begins to get more advanced and complicated. This suggests that to understand the learning processes of visually impaired and blind mathematics learners, it is important to investigate the particular ways in which they access and process information, how this shapes their mathematical knowledge, and the learning trajectories through which that knowledge is attained.

3.2.1. Complexity of Mathematics Communication

Mathematics communication and conceptualization have strong embodied components (Arzarello et al., 2014; Arzarello, Robutti, & Thomas, 2015; Campbell, 2010; Charoenying, 2015; de Freitas, 2016; de Freitas & Sinclair, 2012, 2013; Ginsberg, 2015; Mowat, 2010; Nemirovsky & Ferrara, 2008; Nemirovsky, Kelton, & Rhodehamel, 2012; Nemirovsky et al., 2013; Radford, 2013, 2014; Tall, 2006; Wilson, 2002). Embodied communication, gesticulation, gaze, pointing, and body language in mathematics discourse have critical roles in communication between sighted individuals when discussing math concepts. In “The Emperor’s New Mind”, mathematician and physicist Roger Penrose (1989) wrote:

> Almost all my mathematical thinking is done visually and in terms of nonverbal concepts, although the thoughts are quite often accompanied by inane and almost useless verbal commentary, such as ‘that thing goes with that thing and that thing goes with that thing. (p. 424)

This is an example that shows the tremendous role played by mathematically grounded gesticulation involved with *deixis*. Deixis refers to words and phrases such as “this” and “that”, “there” and “here”, that cannot be fully understood without additional contextual information, such as pointing, gaze, and body language. Comprehending different uses of deixis is one of the main difficulties that a visually impaired student encounters during lecture time or tutoring sessions. There are many forms of deixis in verbal communication that make sense only with accompanying gestures such as pointing to a specific component of a graph. Often, sighted instructors forget that there is a student in the class who is not able to follow their gesticulations.

Despite the availability of supporting resources such as Braille/tactile coursework materials, and skilled vision teachers (vision teachers are educators who have all the skills of an ordinary teacher, but receive specialized training to work with visually impaired students) for blind students within K-12 schools, very little support is offered in higher education. For example, at the university at which study took place, there was no mathematics-vision-professor to teach the calculus course. Also, the tutor and assistant (myself) were all sighted individuals without specialized training. Therefore, no one knew how to use/read Braille or Nemeth codes with Anthony. Thus, we had to find a common way to communicate – understandable to all parties including instructor, tutor, assistant – that was also comprehensible to Anthony.

3.3. Structure of the Chapter

The chapter is organized into two main sections. In the first section, under the title methodology, I will discuss who Anthony is, what obstacles he encountered through pre-calculus course and how those challenges were tackled. In the second section, I extend Nemirovsky et al.'s perceptuomotor integration framework (2013) to explore mathematical learning and sense making using the tactile tools. I analyse Anthony's temporal flows of perceptuomotor activities that are inhabited bodily in Husserlian experiential time (1991) when he describes a rational function's behavior.

3.4. Research Questions

The research questions pertaining to this study aim at examining how a blind undergraduate student overcame obstacles in learning pre-calculus concepts. So, this study explores:

a) How do mathematical tools and resources (such as tactile graphs, screen readers, etc.) make mathematical communication and learning possible for the blind learner in a pre-calculus course?

I expected the blind student to face difficulties in graphing and explaining the function's behaviour. So, I was interested to find out if tactile graphing could be accomplished with enough precision and whether these new practices preserved the mathematical properties of the function. Accordingly, the analysis presents how a specialized mathematical tool (precise tactile graphs) supports the process of learning that coordinates the body in mathematical activity.

b) How does the emergence of the blind student's bodily activities and gestures embody and express mathematical learning?

3.5. Theoretical Framework: Perceptuomotor Integration

Generally, in learning a new skill, there are phases in which perception and motor aspects of the activity seem discordant. For example, a novice driver experiences incongruence among his bodily motor arrangements of eyes, hands, and feet when he encounters a road hazard. The same can be applied for the learning of dance, music, playing soccer, and many other skills. Nemirovsky suggests: “transition from discordance between perceptual and motor aspects to their integration is common to all learning; that is, perceptuomotor integration is a milestone for fluency in any [other] field [such as mathematics]” (p. 380). So, involuntary motor activity develops as a part of perception, while the achievement of perceptuomotor integration requires different periods of time for learning. For example, consider the backwards-brain bicycle, created for Destin Sandlin. The backwards-brain bicycle is a regular bike that has been modified, so that if the rider turns the handlebars to the right, the bike goes left. In this case, although the rider has the knowledge of operating a regular bike and the information about the unique bike, he cannot ride it because of a lack of perceptuomotor integration. Individuals cannot go even few feet without putting a foot on the ground or falling. In this experiment, the bike shifts or inverts the rider’s direction of movements. It is reported that after riding the backwards-brain bicycle over a few months, the movement starts to be accepted as normal. After adaptation, returning to a regular bicycle also demands its own period to learn, because involuntary motor participation in perception is crucial (Nemirovsky et al., 2013, p. 381).

3.5.1. Tool Fluency

Mathematical learning includes appropriating skillful use of mathematical instruments, which are cultural tools that mediate mathematical activity. A mathematical instrument is a “material and semiotic tool together with the set of embodied practices for its use within the discipline of mathematics. So, the fluent use of mathematical instrument allows for the culturally recognized creation in mathematical domain, just as members of the musical communications acknowledge” (Nemirovsky, 2013, p. 373).

The tool fluency involving perceptuomotor integration is an interpenetration of the perceptual and motor aspects of the activity that allows the performer to “act with the holistic sense of unity and flow” (p. 373). With an explicit approach to embodiment, “tool fluency” includes *mathematical thinking* and *learning*. Within perceptuomotor integration ‘*mathematical thinking’* is defined as bodily activities that involve different degrees of explicit/implicit expression. Also, ‘*mathematical learning’* is defined as the transformations and coordinations of lived bodily engagements while the subject participates in mathematics activities. Since learning mathematics relies on skillful embodied tool use – acquired in practice – the teacher's mediating intervention is not necessary (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013, 2011). This is distinct from other theories, such as instrumental genesis (Artigue, 2002; Guin & Trouche, 1998) and semiotic mediation (Bartolini Bussi & Mariotti, 2008; Falcade, Laborde, & Mariotti, 2007; Mariotti, 2013), which rely on a teacher's didactical involvement. It means, within perceptuomotor integration, there are no scaffolding stages in which a teacher must define the potentiality of the artefact for the learner though a dialectical approach.

Nemirovsky et al. (2013) exemplified tool fluency with the enactments of overt skillful motoric and perceptual engagements, in the form of finger arrangements and movements when a pianist listens to a piano piece. In contrast, lack of fluency would be described when there is not such a holistic integration. For example, a beginner soccer player may not be skillful enough to coordinate his body and pass the ball to his teammates in a proper direction with suitable speed, power, and accuracy. The lack of fluency for him could be shown through the separation between the motoric articulation of a standard pass and the perceptual qualities of that pass. So, there is a mismatch between the pass and perceptual components of it.

3.5.2. Lived Experience

The analytical approach centers on the *phenomenology of lived experience*. By *lived experience*, I mean the temporal streams of perceptuomotor activities that are at once bodily, gestural, and interpersonal. The temporal flows of perceptuomotor activities are not categorized in terms of schemas or patterns, forms of reasoning, cognitive structures, procedural strategies, or any other mentalist approach. This view informs an emerging body of literature in mathematics education that aims to find how multiple flows of embodied activities constitute the experience of mathematical learning (Arzarello, 2006; Arzarello et al., 2015; Arzarrello & Domingo, 2016; de Freitas & Sinclair, 2012; Radford, 2008, 2013, 2014; Roth & Thom, 2009).

By *temporal flows* within Husserl's phenomenological attitude, I mean any perceptuomotor activity infused with past and future: i.e., an activity at a certain moment is not isolated, rather it is comprised of partial enactments of *retentions (just-past)*, now phases, and *protentions (about-to)* (Husserl, 1991). For example, when Anthony places a Wikki Stix (wax sticks. More elaboration will be provided in 3.6.5 subsection) on the graph paper to mark a vertical asymptote of the given rational function, instead of describing his hand movements, gestures, body movements, and his words as outer manifestations of his mental schemes for the concept of rational functions, I try to understand a temporally extended course of actions that he experiences during the act of identifying vertical asymptotes. This is according to what Husserl argues about the experience of a particular note in a melody which is not solely based on the "present" sounding note, rather this temporally extended experience includes retention of the note sounding "just before", and immediate anticipation (protention) of the upcoming note, about to sound in experiential time. Taking a Husserlian perspective on Anthony's lived experience of marking vertical asymptotes with his integrated tool fluency implies the *retention* of knowing vertical asymptotes are vertical lines which correspond to the denominator zeroes of a rational function, where the function is undefined; and in anticipating (*protention*) the path that graph follows but never touches.

3.5.3. Tactile Perception Verses Active Sensation: Perceptuomotor Integration and Tool Fluency in Tactile Touch

For a blind learner, tactile touch is not a passive sensation. He actively picks, and refines sensations according to a goal (Gibson, 1962). Lepora (2016) argues:

> Sensation refers to the first stages in the functioning of the senses, related to the effect of a physical stimulus on touch receptors in the skin and their transduction and transmittal from the peripheral nervous system to the sensory areas of the brain; tactile perception refers to later stages where the sensation is processed, organized and interpreted so that [the subject] may use the information to guide its behaviour based on understanding its environment. (p.151)

Active sensing refers to deliberately controlling finger movements, while contacting a stimulus, with a goal in mind. Accordingly, lived experience of an *active sensation* for a tactile graph would refer to ongoing covert and overt experiential embodied transformations. More specifically, they reference what has just happened and adjust the bodily movements on the sensory tool according to what is anticipated to happen. For instance, in tracing a rational function on an embossed printed paper, the learner, having just placed the vertical asymptotes (immediate-past or just-past), uses his fingers to sense and place the function on the graph (present) in anticipation of graphing the function in a way that approaches the asymptote but does not intersect it (immediate future/about-to).

Consequently, active sensation transforms to *tactile perception* when the acquired learning and information guides gestures and behaviour of the subject in an environment (Lapora, 2015). Therefore, for a visually impaired individual who constantly traces different components of a tactile graph with his fingers, *active sensation* is a continuous process of transition from discordance between the procedural and motoric aspects of the stimulus in experiential time. In this case, the *active sensation* would be expressed as *mathematical thinking* infused by explicit and implicit expressions of bodily activities.

Taking a perceptuomotor integration perspective, I will follow the role of embodiment in the use of the mathematical tools. I will trace the development of Anthony's tool fluency, which emerges with body orientation and coordination, and appropriate use of the mathematical instrument. As suggested by perceptuomotor integration theory, I expect the retention and protention involved in active sensation to be clearly associated with motor or perceptual aspects. Later, when the active sensation merges with the tactile perception, it becomes an integration of perceptual and motor aspects of tool fluency.

3.6. Methodology

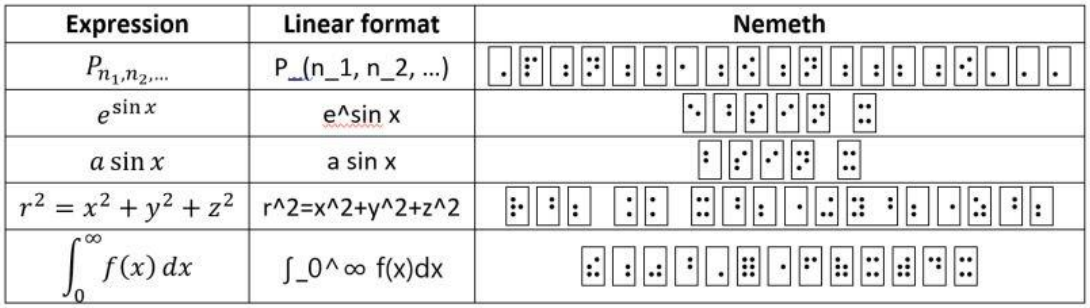
3.6.1. Who Is Anthony?

This study is part of a larger research project exploring 'Issues and Aids for Teaching and Learning Mathematics to Undergraduate Students with Visual Impairment.' Prospective participants, who are visually impaired, are identified and invited by the Centre for Students with Disabilities' (CSD) specialists to participate in the study. The participant focused on here is a blind student named Anthony, who was taking a pre-calculus course.

Anthony is a twenty-eight-year-old male, who is doing his sixth year of undergraduate studies in kinesiology and health science at a Canadian university. He is an active student both in sport and social activities, and usually participates in many school events. Anthony was born with a profound visual disability and is completely blind now. At the time of the study, he had successfully passed Mathematics Foundations the previous semester, fulfilling the prerequisite for enrolling in pre-calculus. Anthony had excellent mental calculation skills, which enabled him to do basic calculations without a calculator. He was very keen to create a pathway for other sight-disabled students to pursue their academic dreams by taking mathematics courses. What follows is a brief description of the material and software used in the study and the procedures that we employed to assist Anthony through his pre-calculus journey. In this journey my role was to assist Anthony in class and tests, to prepare class notes.

3.6.2. Written/Printed Materials

Braille is well known as a tactile writing system designed for blind and visually impaired individuals. This method of writing is commonly used in Canada up to grade twelve, but not very often afterward. One of the significant shortfalls for Braille, limiting its adaptation for mathematics courses, is that it involves strictly linear notation and is not generally useful for mathematics and its various notations (Marcone, 2013). Besides, Braille readers can only perceive what is under their fingers at the time, so it can be tough for them to obtain a general and holistic view of algebraic expressions and graphs as a whole. The Braille extension specialized for mathematical notation is known as the Nemeth Code (Nemeth, 1972). Nemeth Braille was first developed in 1952 by Abraham Nemeth and has a different coding system than Braille does. But it is still linear in nature. In this study, neither the instructor, nor the teaching assistant, nor the tutor knew how to write and read Braille codes. Considering Braille and Nemeth Code’s limitations, we chose LaTex to translate mathematical language and formulas and prepare other written and printed materials for Anthony. LaTex is used to prepare textbook content, and class and lecture notes; and then all the files were shared through DropBox among different parties.

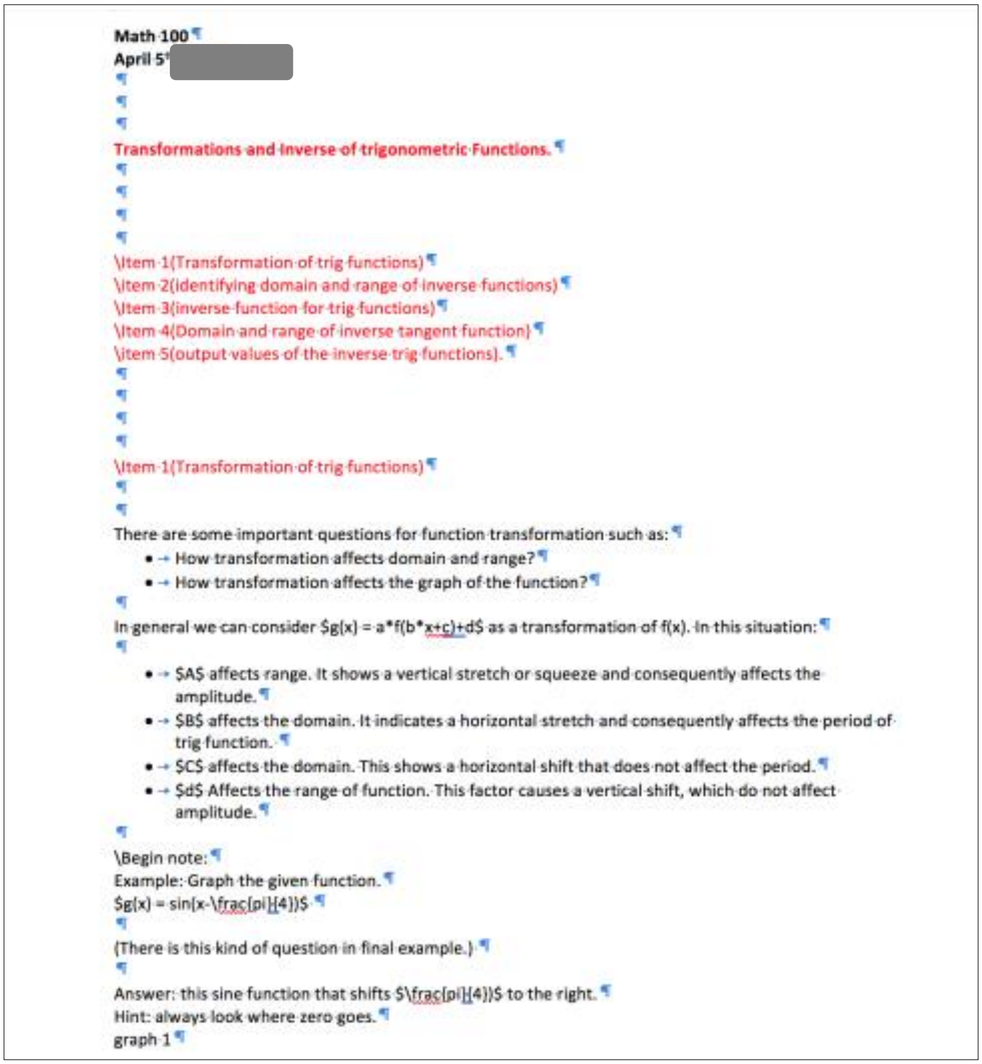


Expression	Linear format	Nemeth
$P_{n_1,n_2,\ldots}$	P_(n_1, n_2, ...)	
$e^{\sin x}$	e^sin x	
$a \sin x$	a sin x	
$r^2 = x^2 + y^2 + z^2$	r^2=x^2+y^2+z^2	
$\int_0^\infty f(x)\,dx$	∫_0^∞ f(x)dx	

Figure 3.1 An example of expressions in Nemeth. Adapted from Microsoft Developer Network

3.6.3. LaTex

Blind and visually impaired students face serious challenges when they need to translate mathematical equations and materials from print to Nemeth, or Nemeth to print, specially for sighted instructors, tutors or classmates. LaTex is a text-based and non-graphical language in nature, which can be used by anyone, blind or sighted. The most important reason that we chose to adopt LaTex was that we found LaTex to be a common language that all parties involved in teaching and learning the course (instructor, learner, tutor, and Anthony's assistant) could communicate with. So, we supplemented LaTex language with our own individual typographic codes while using Word files for mathematical communications. For example: \frac{numerator}{denominator} is used for fractions, % for comments, and all the mathematics formulas are placed between $s. So, for example, $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ means $\mathrm{d} = \sqrt{(\mathrm{x}_2 - \mathrm{x}_1)^2 + (\mathrm{y}_2 - \mathrm{y}_1)^2}$. However, employing the adapted code in the LaTex and Word files made another level of complication for Anthony. For example, in the Word files, after each section I meaningfully entered four blank lines, and marked where the explanation started or finished. Anthony became confused if three or five line spaces were accidentally entered instead of four. See Figure 3.2.



Math 100
April 5 [redacted]

Transformations and Inverse of trigonometric Functions.

\Item 1(Transformation of trig functions)
\Item 2(identifying domain and range of inverse functions)
\Item 3(inverse function for trig functions)
\Item 4(Domain and range of inverse tangent function)
\item 5(output values of the inverse trig functions).

\Item 1(Transformation of trig functions)

There are some important questions for function transformation such as:

- How transformation affects domain and range?
- How transformation affects the graph of the function?

In general we can consider $g(x) = a*f(b*x+c)+d$ as a transformation of f(x). In this situation:

- A affects range. It shows a vertical stretch or squeeze and consequently affects the amplitude.
- B affects the domain. It indicates a horizontal stretch and consequently affects the period of trig function.
- C affects the domain. This shows a horizontal shift that does not affect the period.
- d Affects the range of function. This factor causes a vertical shift, which do not affect amplitude.

\Begin note:
Example: Graph the given function.
$g(x) = sin(x-\frac{pi}{4})$

(There is this kind of question in final example.)

Answer: this sine function that shifts $\frac{pi}{4})$ to the right.
Hint: always look where zero goes.
graph 1

Figure 3.2 A screenshot from a lecture note; written in a Word document.

A table of contents (items) at the top of each LaTex and Word file was used to give insight into what was compiled in each file. So, Anthony did not have to go through the whole document to figure out what was in it. Anthony also suggested that descriptive file titles were best for identification purposes rather than the date of the class (as were previously used to record class notes in other courses) (Figure 3.3).

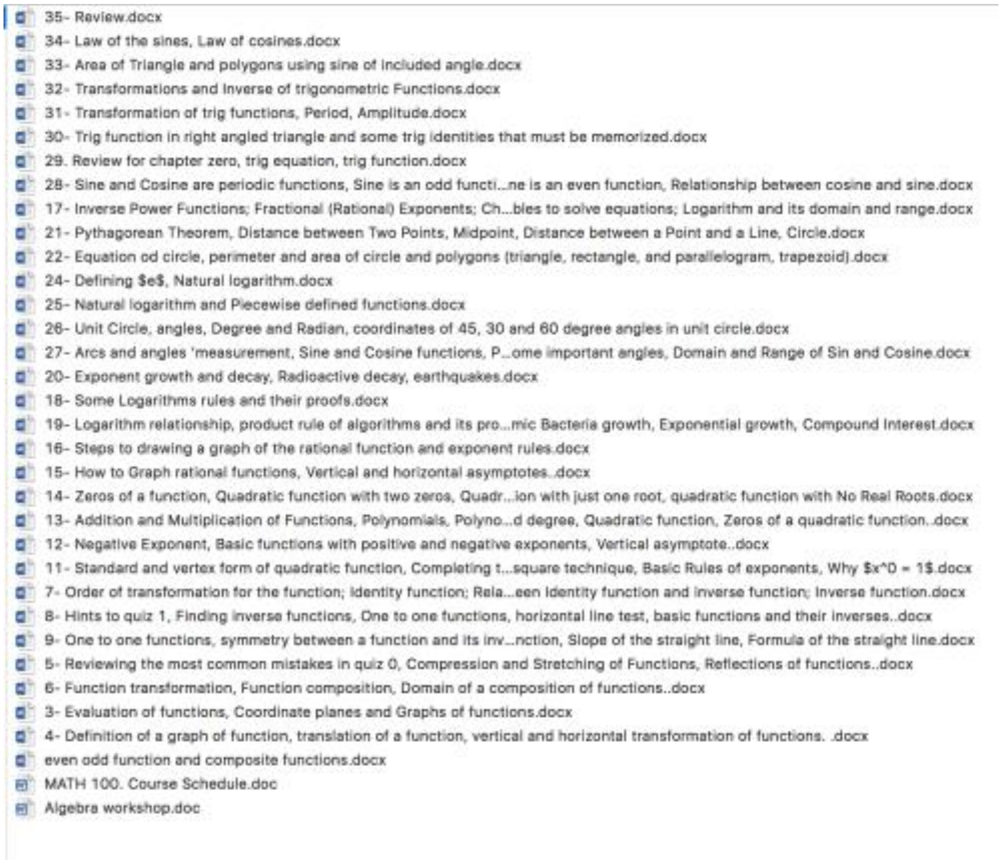


Figure 3.3 **Descriptive approach were used to name Word files**

Note. The numbers indicates order of file creation, following by the title of its contents.

3.6.4. JAWS

Digital technologies can facilitate conversions between written text spoken versions of written materials and mathematics equations. Furthermore, use of spoken rather than written materials suggests that the ears can also be used as alternates for the eyes. Having outlined the limitations and possibilities of incorporating non-visual senses for a blind learner, for conducting this research we mostly relied on auditory and tactile materials.

JAWS (Job Access with Speech), is a computer screenreader program for Microsoft Windows, which is produced by the Blind and Low Vision Group of Freedom Scientific, St. Petersburg, Florida, USA. The software allows blind and visually impaired users to read the screen either with a text-to-speech output or by a Refreshable Braille display. JAWS reads most of the programs: such as Word documents, LaTex files, browser's content and actions in Windows. Anthony's Windows laptop read using JAWS. VoiceOver is another function designed for Mac users and works in Mac iOS. In addition, the PDF reader was used when a PDF file had to be read.

3.6.5. Tactile Graphs: A Real Challenge

One of the very challenging problems in learning and teaching mathematics to visually impaired students is to help them with printed/drawn mathematical graphs and figures as the concepts become complicated. Anthony faced these difficulties in various forms, in reading a textbook's pictorial information and graphs, on sketched figures on the board during the lecture time or tutoring sessions, and when he was doing exercises at home. To tackle these challenges, we used different novel approaches such as speedy sketch graphing using a net-board and crayon, or Wikki Stix and a wheel-tracing tool (a sewing tool), as well as permanent raised graphing. In the following subsections, I will provide more details about each method.

Sketch graphing

Anthony found it difficult to follow lectures when there was an image or figure displayed on the board. The problem was worse when the lecturer used *deixis* such as "this" and "that" for describing a mathematical idea or graph pointing to different sections of the figure on the board. To tackle this trouble and assist Anthony, I rapidly made graphs using net-board and crayons (Figure 3.4 A & B) or Wikki Stix (Figure 3.4 C). I shall call this *sketch graphing.*

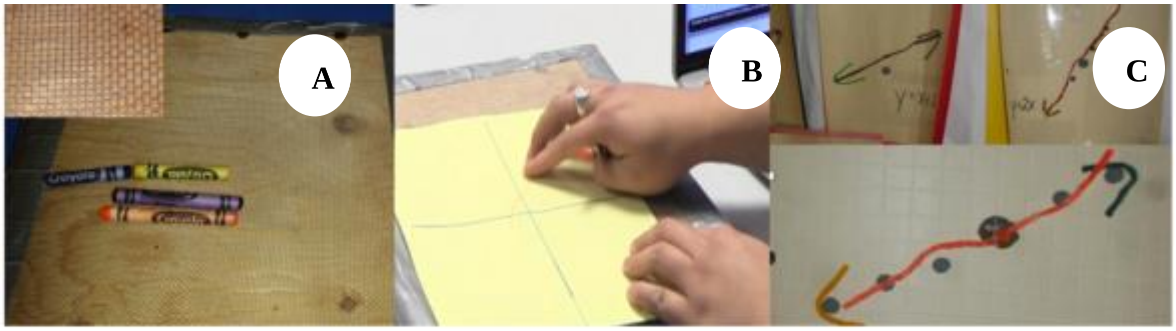


Figure 3.4 Sketch graphing

Note. (A) & (B) “sketch graphing”: An innovative way for drawing graphs during lectures and tutoring sessions using a “metal net-board”, crayons and normal paper. (C) Examples of using Wikki Stix on a dotted line graph paper at Georgia Academy for the Blind Learner. Each non-mathematical raised point on the graph (the tack used to hold down the “stix”) places a new level of confusion for the blind learner.

A net-board is a wooden board with a metal net (screen mesh) installed on it. As shown in Figure 3.4 A&B, using crayons while paper is placed on the net-board creates a texture that is raised and tangible for Anthony. I also marked axes and other relevant parts of the graph using a wheel-tracing tool that I bought from a fabric store. Figure 1B, shows a paper located on net-board and axes (in blue) that are scored by the tracing wheel.

The other approach was using Wikki Stix for graphing. Wikki Stix are wax sticks that can be pressed down with the fingertips, and no glue is needed. They are also easy to peel up and reposition. Wikki Stix were also used on embosser[13] printed graph papers, with dotted and dashed grids and axes to bring out different textures.

Wikki Stix are problematic in graphing, because there is no distinction between different components of the graph (such as asymptotes) and different parts of the function. However, using Wikki Stix was helpful and precise when Anthony was graphing on embosser-printed papers (especially during test/quiz times). Practically, sketch graphing was fast and adequate to provide a tangible ‘image’ of the graph to the learner,

[13] Braille embossers are printers for Braille. Braille embossers usually need special braille paper which is thicker and more expensive than standard paper.

and provide a sufficiently precise graph for the instructor to evaluate Anthony's understanding. See figure 3.5.

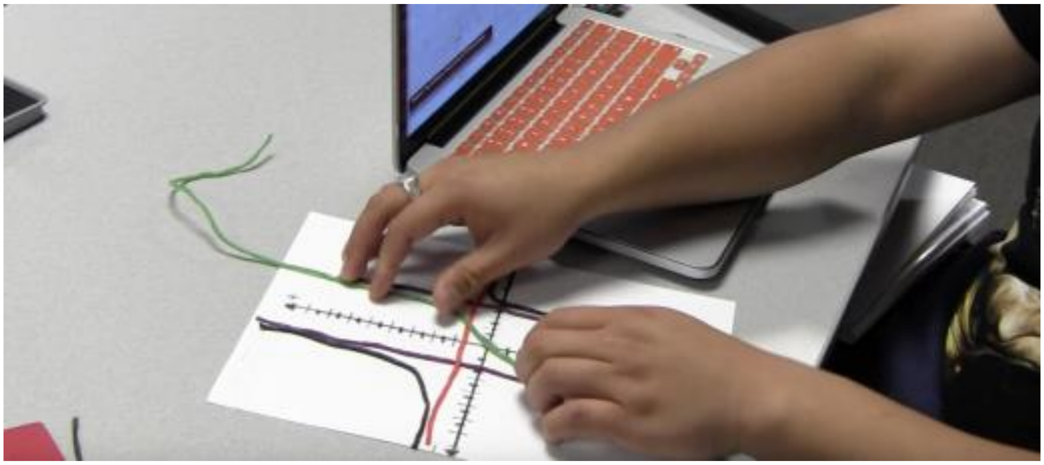


Figure 3.5 Another form of Sketch graphing using Wikki Stix on embosser-printed papers.

Permanent graphing

To present the textbook, assignments, class notes and pre-designed class diagrams and graphs, I plotted permanent graphs. To do so, firstly we examined *The Swell-Form Graphics Machine* that cooks graphs and "swells" all black inks. As the swell-touch paper goes through this machine, the heat reacts with only the black ink and causes it to "swell" or puff up, creating a tactile image. We found swelled graphs not helpful in term of providing precise tactile graphs/lines. The texture that was produced with *the Swell-Form Graphics Machine* was homogeneous, and would not allow the learner to precisely distinguish different components of the graph such as axes, function, and intersections. For instance, Figure 3.6 A shows a grid paper produced by *The Swell-Form Graphics Machine*. Here, it is not difficult to imagine all the swelled lines, curves and grid lines are homogeneous and therefore not distinguishable.

Therefore, we decided to innovate a new way to create graphs of functions. When there was enough time, I (as Anthony's assistant) created graphs using various materials and textures, trying to make them as precise as possible. For this purpose, axes and grids were first printed on a graph paper using a Braille embosser. The approach was to design and build axes and grids with different textures with the computer and send it for printing to the embosser. Then, I drew graphs using hot glue/glue gun, a wheel tracing tool, and stickers. For labeling, I used Nemeth Code on top of the graph paper, in a similar way that was represented in the original source (Figure 3.6 B).

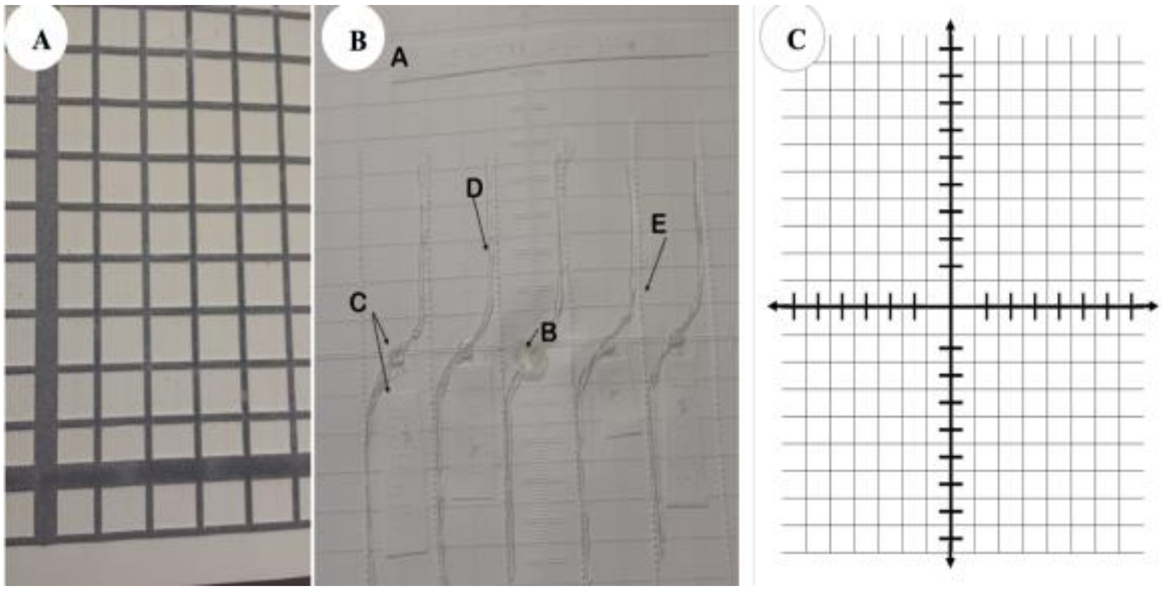


Figure 3.6 **Permanent graphing**

Note. (A) Grid paper printed by Swell-Form Graphics Machine. (B) Permanent graph – created on a grid paper printed with an embosser. X and Y-axes are marked by tick lines. In this graph: A– is the title of the graph, including section and page number (Nemeth Coded). B– Origin marked by a sticker. C– x-intercept and its coordinates (Nemeth Coded). D– Function created by a glue gun. E– Vertical asymptotes made by a tracing wheel). (C) Shows embosser grid paper with color and thick lines. Tick lines would be printed with two width dots and grid lines with a single dot.

Therefore, figures and graphs, particularly the ‘permanent graphs’, could provide all the details as precisely as the pictorial parent. Here is a summary of the characteristics of a permanent graph:

- Printed grids as well as marked *x* and *y*-axes, the origin (0,0), and x-intercepts.
- The function drawn by a glue gun (a regular line).

- Asymptotes and other guidelines drawn by glue gun or tracing tool (dashed lines).
- Function notation, figure labels, and other necessary information provided using Nemeth Codes.
- Direction of the paper was marked at the top of the paper with an arrow.

To exemplify how mathematical tools and resources (such as tactile graphs, screen readers, etc.) make mathematical communication and learning possible for the blind, I summarize how Anthony graphed $f(x) = \frac{x^2-5x}{x^2-x-6}$. In section 3.7, I elaborate how Anthony described the function behaviour.

Anthony entered the function in a Word document in LaTex as `$f(x) = \frac{(x^2 - 5*x}{x^2 - x - 6}\$` , let the VoiceOver read it several times, copied the equation and pasted in the next line, deleted the denominator, set the nominator equal to zero (`$\(x^2 - 5*x) = 0\$`) and identified *x*-intercepts at 0 and 5. Then, Anthony copied the equation again. This time he deleted the numerator, set the denominator equal to 0, and factored the equation (`$\(x^2 - x - 6) = (x – 3)*(x + 2) = 0\$`). He identified vertical asymptotes at 3 and -2. Anthony typed explanations for each step. Again, Anthony copied the function notation and identified horizontal asymptote at $y = 1$. Copying and pasting the equation to solve the given equation was a common practice when Anthony was solving equations. In each step, he copied and pasted the last form of the equation and simplified it depending on the goal (Figure 3.7 A). In this practice, Anthony utilized VoiceOver, Word and LaTex to read and work through the equation. In this case, these tools were supporting his movement back and forth between just-past and present time in anticipating the next step. For example, the tools helped Anthony to remember, read (retention) and identify zeros of the denominator (present) in anticipating where the function is undefined (protention).

After doing all the calculations, Anthony used Wikki Stix to mark horizontal and vertical asymptotes, and graph the function. In each step, Anthony went back to his computer to check if he was on the right track. After about 20 minutes Anthony could graph the function correctly (Figure 3.7 B). But this form of graph could not be used later efficiently. Wikki Stix could peel off easily. Also, unlike with permanent graphs (Figure

3.7 C), it was not easy to distinguish Wikki Stix representing graph lines from those representing asymptotes.

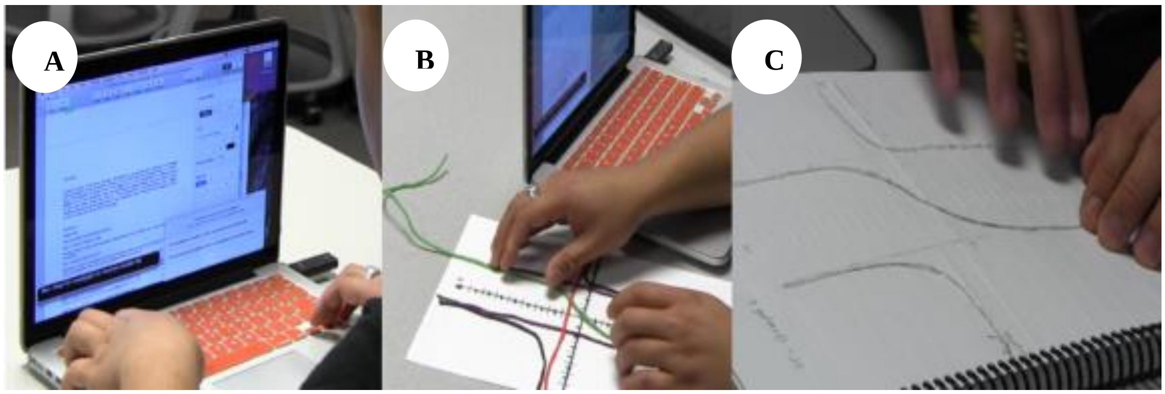


Figure 3.7 Anthony's graphing process

Note. *(A)* VoiceOver is reading the screen*, while* Anthony was continuously copying and pasting the function notation in each step to solving it. He let the VoiceOver read each step, sometimes from a few lines above. *(*B*)* Anthony graphed $f(x)= (x^2-5x)/(x^2)$ using Wikki Stix. (C) The same function illustrated as a permanent graph.

In the above section, I have discussed different obstacles encountered in teaching and learning mathematics at the university level, as well as strategies that we employed to tackle those difficulties in written, graphical and verbal materials. In particular, I have discussed how perceiving and graphing functions was challenging in the different contexts and situations. I approached the obstacles with the "sketch graphing" and "permanent graphing" strategies. During the lecture time the use of *deixis* in communication brought another level of confusion for the blind student, who had no access to visual cues.

In the next section, I will discuss how Anthony took advantage of using my innovative tactile graphs to communicate the concept. Also, I will draw attention to his high level of bodily engagement—particularly his gestures and talk.

3.7. Demonstrating the Behaviour of a Rational Functions

To examine the emergence of Anthony's coordination of motor, perception, and gestures, I asked Anthony to graph $f(x) = \frac{x^2-5x}{x^2-x-6}$ and consequently explain the graph's behaviour. Here, I first describe and analyse Anthony's lived experience in experiential time (Husserl, 1991); and then explain aspects of the episode that were especially illustrative of developing perceptuomotor integration. Accordingly, the quality of these activities could establish Anthony's tool fluency if the retentions and protentions demonstrate an excellent interpretation of both perceptual and motor aspects, especially in comparison to the early stages of the perceptuomotor integration where there is not such a holistic integration, unity and flow (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013; Nemirovsky et al. 2011).

As I reported earlier, Anthony used a Word document and VoiceOver to do the required computation and in solving relative equations. He determined vertical and horizontal asymptotes, intercepts, and marked them with Wikki Stix on raised grid paper. Then he drew the graph with the same materials. The following episode happened right after Anthony graphed the function. The researcher asked Anthony to explain the function's behaviour for her, a sighted audience. Anthony's words are underlined.

Anthony taps on the second quadrant on the permanent graph

00 <u>I'm in quadrant two, the function goes like this</u>. He moves his left hand from left to right horizontally.

01 <u>and hugs the vertical asymptote at y=1</u>. Anthony brings his right forearm parallel and just below the left forearm. So, his right forearm functions as the horizontal asymptote and left forearm as the function. It seems that the function (left forearm) moves from the -∞ and hugs the horizontal asymptote (right forearm) at y=1.

02 <u>and as it gets closer to</u> [pause]…<u>the *y*-axis</u>... [pause]...[left and right forearm move upward] <u>it goes up</u>.. [long pause, Anthony seems unsure]. This pause could be a sign of discordances between Anthony's motor and perception

activities in the experiential time. That is to say, the retentions of the y-axis as the first left-to-right vertical asymptote (perceptual) in moving his left and right forearm upward (motor activity) in anticipating the functions' behaviour – are not synchronized and integrated yet.

With an active sensation, Anthony touches the graph again, and accordingly recognizes he missed the horizontal asymptote at x = -2. This is an evidence of an "active sensation" of the tactile graph because that occurred through on-going experiential embodied adjustments infused with the recall of the experience of the past (identifying vertical asymptotes x = -2 and x = 3), and expectations of not crossing x = -2 in the immediate future. So, Anthony wanted to adjust his bodily movements (motoric activities) for what he anticipated to occur (perceptual aspects of graphing). This also explains an early perceptuomotor learning phase because of the clear association of motor and perceptual activities to retention and protention, respectively (Nemirovsky et al., 2013).

Then, Anthony "waves" his left hand to emphasize that this will demonstrate the function. He moves the left hand from left to the right horizontally. It seems that by "waving" the hand, he wipes off his last sketch.

> 03 The function [left hand-marked with yellow vector in Figure 3.8], goes horizontal; hugs up the horizontal asymptote [blue vector- Figure 3.8] at y = 1 [two forearms touch each other] (Figure 3.8 A) and then, it goes up to the vertical asymptote at x = -2 [two hands move upward, he "taps" on the left wrist with his right forearm] (Figure 3.8 B). That means that [function] doesn't cross it [vertical asymptote] and it comes up (Figure 3.8 C).

Anthony uses the "taps" (Figure 3.8 C) gesture to show the function approaching the vertical asymptote closely but not crossing it. Tapping is a motoric bodily activity and an enactment of perceptual learning where a vertical asymptote corresponds to the zeroes of the denominator for a rational function (where the function does not touch or cross it). The action of 'tapping' in present time with the retention of coordinates and characteristics of the vertical asymptote at x=-2, anticipates an approach but not a crossing. Therefore, the ‘tapping’ gesture is an emergence of integrated perceptual and motor aspects in the form of *tactile perception*.

Anthony again touches the permanent haptic graph, instead of his sketch by Wikki Stix, to refresh his short-term memory (Figure 3.8 D). He explained that it is hard to recognize different components of the graph – distinguishing asymptotes from function, for instance – on his sketch graphing because they use the same material and texture. In the permanent graph, asymptotes are drawn with a tracing tool and are dotted, while the function is in the form of raised line/curve.

> 04 …and the middle [part of the] function, is where the function is between the two vertical asymptotes [$-2<x<3$]; doesn’t cross either one [waves both his forearms to emphasize they are vertical asymptotes] (Figure 3.8 E- vertical asymptotes marked with the red dashed vectors). But, it does cross upward, over the origin, and over the horizontal asymptotes at $y= 1$ and comes toward the vertical asymptote at $x=3$ [‘Taps’ his right wrist (asymptote) to his left forearm (function)], and again, never crosses it (vertical asymptote) (Figure 3.8 F). And this is the middle part of the function [taps on the middle part of the graph].

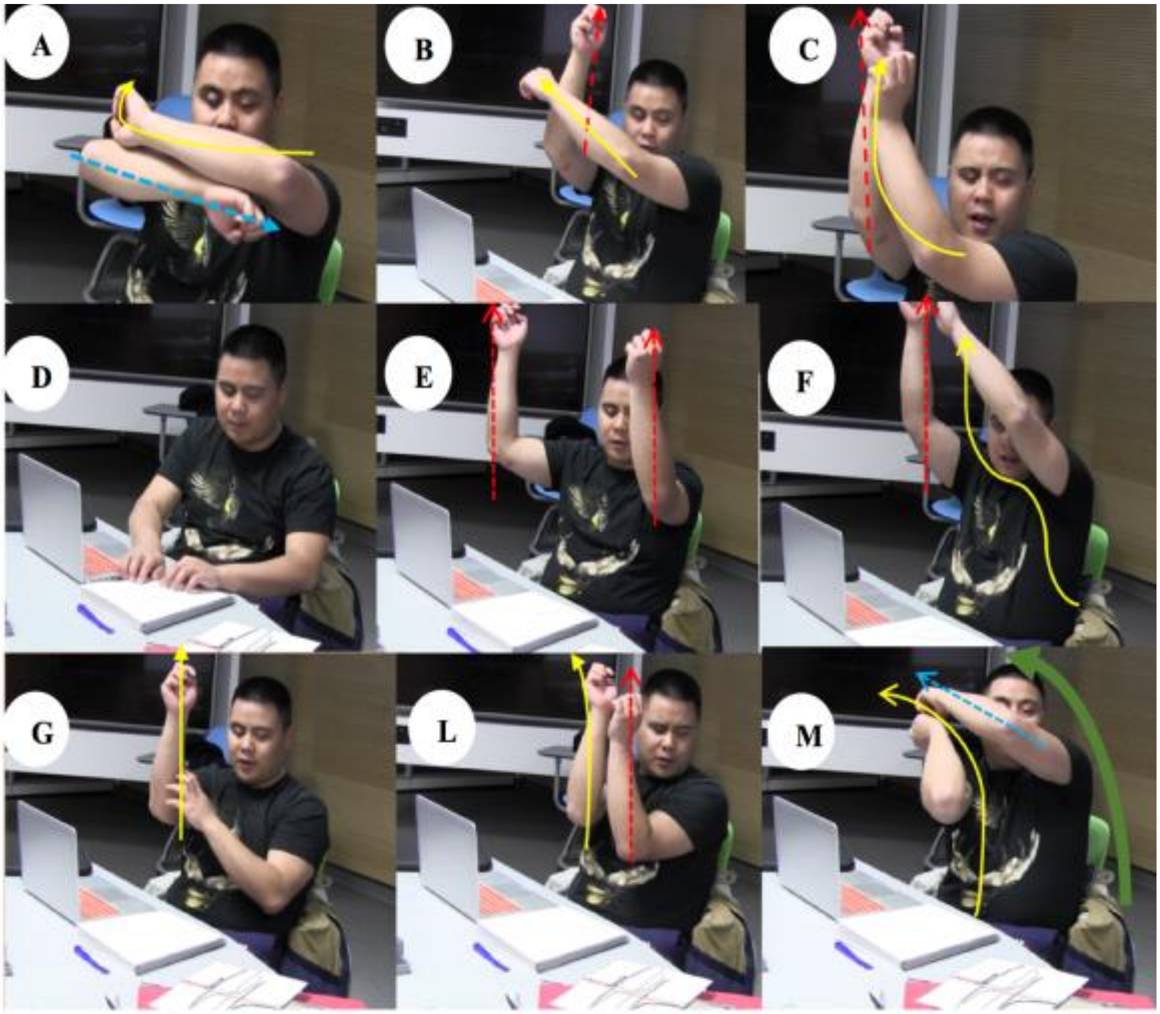


Figure 3.8 **Anthony explains a rational function's behaviour using his hands, forearms and body.**

Note. Anthony explains the rational function behaviour for $f(\mathbf{x}) = \frac{\mathbf{x}^2-5\mathbf{x}}{\mathbf{x}^2-\mathbf{x}-6}$. Each arrow shows the direction and path of the function's movement (red dashed arrows=vertical asymptote; blue dashed arrow=horizontal asymptote; yellow arrow=function; green arrow=body movement).

This is an evidence of a gradual transformation in Anthony's bodily experience. His motor and perceptual activities of showing graph behaviour, as well as its vertical and horizontal asymptotes, are now synchronized and adjusted. He also verbally explained the situation clearly with a fine-tuned perceptuomotor integration of his bodily activities. The role of tactile and 'permanent graph' and his calculations on the screen is not only to refresh his memory but also to provide an ongoing active sensation toward a tactile perception on a different scale. In other words, Anthony adapts the tactile graphs to re-scale the graph and coordinate his body.

> 05 Afterward, Anthony taps in each quadrant, while naming them. He explains the behaviour of the right side of the function, for x>3. It appears that Anthony has switched hands as needed. He 'points' and 'taps' on his right hand to stress the change of roles of this hand. So, from now on, the right-hand plays the function role and not the asymptote (Figure 3.8 G).

> 06 <u>This is my function </u>(Figure 3.8 G)<u>. It comes up along the vertical asymptote at x=3</u> [from -∞] (Figure 3.8 L). <u>And, as it comes up, instead of crossing over horizontal asymptote at y=1, it gets close to it</u> [bends both forearms, while they hug each other. Bends his hands and body to the right], <u>and does not cross it</u> (Figure 3.8 M).

During this episode, Anthony was involved in active sensation and perception because he was deliberately picking and refining active sensations to accord with his body movement. The activate sensation was undergoing a transformation into tactile perception in terms of gestural and body coordination, while Anthony was illustrating the function's behaviour (Gibson, 1962; Lepora, 2016). Evidently, Anthony's thinking and learning with the mathematical tools started from calculating vertical and horizontal asymptotes, zeros of function and x-intercepts. Then, his bodily orientations transformed from an *active sensation* to a *tactile perception* with an integrated perceptuomotor learning evidenced not only on his forearms [01-03] but his whole body [04-06]. This is also where Anthony harmonized words and *involuntary bodily activity* (motor) in bending his body along with the moving hands [Figure 3.8 M], providing evidence of his ability to anticipate the next move of the function (perceptual). Anthony's motor activities were *involuntary* and enacted as a part of perceiving.

The selected episode also presented how embodiment can work as a visual cue for a blind learner. Anthony was demonstrating the behaviour of the function with his body in a way that represented the dynamic nature of the function's graphing. In addition, the verbs that were used by Anthony (e.g.: moves, hugs, crosses, etc.) suggest that he also perceives mathematic entities as dynamic objects (Sinclair & Gol Tabaghi, 2010).

3.8. Discussion

To answer my first research question: in the first section of this paper I drew attention to the variety of obstacles that a visually impaired undergrad student, his instructor, and tutor encountered in teaching, learning and communicating pre-calculus concepts in written, graphical, and verbal materials. I also shared our experience of some solutions and tools that enabled a blind student named Anthony to overcome those struggles effectively. I explained how the lack of access to the visual cues in mathematics communication involving gestures, pointing, and *deixis* could hinder comprehension. I discussed how perceiving and graphing functions were challenging in a variety of situations (during lecture, textbook components, exams, and practice time). We tackled these obstacles with two novel strategies called ‘sketch graphing’ and ‘permanent graphing.’

In the next part of the study, I analysed temporal flows of perceptuomotor activities that are inhabited bodily and interpersonally to address the emergence of the blind student's coordination of gestures embodying and expressing mathematical learning (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013; Nemirovsky et al. 2011). I described and analysed Anthony’s responses to the given rational function’s behaviour. In this context, I focused on Anthony’s gestures and body movements, when he positioned his body in the mathematical practice. I found mathematical embodiment to be the process of learning and cultural invention that coordinated his body in the mathematical activity. This is in accordance with Hall and Greeno’s (2008) assertion that “If concepts are bundles of social and technical practice that develop over historical time, learners’ bodies are positioned or placed in these practices just as are technologies” (as cited in Hall, Ma, & Nemirovsky, 2015, p. 113).

The tactile graphs appear to have supported Anthony’s learning and enabled his *active sensation* and *tactile perception.* Anthony actively engaged in sensation, controlling the finger’s movements and course while contacting the tactile stimulus (active sensation) as a bodily enactment of mathematical thinking, which later integrated into the use of learnt information to guide his gestures and motor activities while demonstrating the behaviour of a rational function (tactile perception). So, tactile graphs maintained an on-going infusion of *retention* of perceptual and/or motor activity in anticipating the function's behaviour (protention).

In addition, the tactile graphs—along with other tools—changed the scale and modality of mathematical practices, which invited the blind learner to create and stabilize new forms of embodied mathematical activity. I found the multimodality of mathematical activity as a relatively stable, culturally understandable demonstration of a rational function's behaviour with a great deal of detail in representing its properties. So, in this case, tool fluency was attained when Anthony's demonstration of the function's behaviour was seen as "culturally recognized creation" by the members of "mathematical communities" (Nemirovsky et al., 2013, p. 373).

My study supported past research that found lack of access to the visual field does not obstruct a blind student's ability to visualize, but rather morphs it (Healy & Fernandes, 2011, 2014). In other words, Anthony's haptic exploration of the tactile graph involved exploratory procedures: active touch patterns to optimize the extraction of information the he needed to obtain (Klatzky, Lederman, & Reed, 1987). He used his fingertips to explore diverse information from tactile graphs and other written resources (active sensation). Anthony's different body parts, such as his forearms and wrists, demonstrated a high level of body engagement and mathematical embodiment, especially in terms of his gestures (wave, trace, touch, slide and tap) in the form of tactile perception, which demonstrates integrated perceptual and motor activities.

Having a non-dualistic and non-representational view of the mind and starting from the premise that cognition and environment are intertwined entities, the basic idea is that cognition is a feature of living material bodies characterized by a capacity for responsive sensation. In Anthony's case, I observed a high level of mathematical embodiment in terms of those responsive sensations, thinking and learning, which may be called cognitive constructs, but which are located in the flesh. In other words, his body gestures exhibited thinking not as a process "that takes place 'behind' or 'underneath' bodily activity, but is the bodily activity itself" (Nemirovsky & Ferrara, 2009).

In addition, mathematical tools became extensions of the learner's body. This sociocultural orientation incorporates "a strong position with respect to embodiment, whereby the mathematical tool is not just an instrument, but with tool fluency becomes a quasi-extension of one's own embodiment" (Sedaghatjou‹ Campbell, 2017, p. 7).

3.9. Conclusion

Mathematics understanding is multimodal in nature (Radford, 2013, 2014; Radford, Edwards & Arzarello, 2009). Although teaching and learning mathematics at the secondary level seems challenging and time-consuming for sight-disabled learners, it is not impossible. I found a high degree of non-visual sensory and body engagement in Anthony's comprehension of the pre-calculus concepts. This is in accordance with neurological evidence to support this position suggested by Gallese and Lakoff (2005) that "cortical pre-motor areas are endowed with sensory properties" (p. 459), and that ". . . circuitry across brain regions links modalities, infusing each with properties of others" (p. 456). I also found, as Vygotsky (1986) claims, that providing proper and timely semiotic or material tools "alters its entire structure and flow", while the learner is actively engaging in processes (Healy, 2015, p. 2).

My study took different a theoretical approach than Healy (2011, 2015, 2016, Healy & Fernandes, 2014). Instead of adopting Vygotskian mediation theory and digging into the abstraction, mental function and process, discussing artefact and sign separately I looked at the integrated motor and perceptual integration in terms of perceptuomotor tool fluency. For example, Healy & Fernandes (2011) argue "gestures are illustrative of imagined reenactions of previously experienced activities and that they emerge in instructional situations as embodied abstractions, serving a central role in the sense-making practices associated with the appropriation of mathematical meanings." (p. 157). However, the finding questions gestures as only the "imagined reenactions of previously experienced activities" as well as their emergence only in an "instructional setting". The mathematical gestures I observed could be perceived as unified perceptual and motor activities in the form of tactile perception (Lepora, 2016, Nemirovsky, et al. 2013). Specifically, the emergence of harmonized temporal flows of perceptual and motor activities in using a mathematical tool is not necessarily related to "instructional situations”. So, the mathematical tool is at once a semiotic tool, material and set of embodied practices that aided the learner to explore mathematics regardless of visually disability. Importantly, I am not arguing that the blind or sighted learner inhabits non-intersecting mathematical concepts and world. The distinguishing characteristic of this theory lies in the methods of using and exploring the mathematical instrument; or exploring mathematical concepts via instruments and therefore become fluent in their use.

This study also suggested that the digital technology that facilitated conversation between text and other written materials could support learning and practicing mathematics. This suggests a response to one of the questions raised by Healy and Fernandes (2014). However, the limitation in this study is the participation of only one student.

Future research could reveal what kind of mediators could be used to help visually disabled learners to successfully pass other courses, such as advanced biology or statistics at the university level. In addition, I found that learning advanced mathematics is very time consuming for visually disabled learners. This is not because they mentally process the information with a delay, but rather searching information in digital and tactile files, remembering each algebraic equation while working through problems, and revisiting each step takes much more time than it does for sighted learners. So, more studies are needed to figure out how university policies should be modified for those learners. Also, exploring the use of tactile graphs for students with Autism or attention disorders is recommended.

Lastly, because of budget limitations, I could not examine the role of 3D-mouse, Haptic-mouse, and 3D printers in leaning mathematics concepts or creating mathematical graphs. Other proposals could explore haptic feedback through touchscreen-based DGEs for visually impaired students.

Chapter 4. Touch-Based Technology in Exploring Geometric Transformation: Use of Timeline as an Analytical Tool

The first goal of this study is to explore how a prospective teacher learns geometric transformation by interacting with a touchscreen-based dynamic geometry environment (DGE) *The Geometer's Sketchpad- GSP.* I will analyse a four-minute excerpt from a video clip, centred on the touchscreen-based interactions of a prospective teacher as she thinks aloud to solve the problem of identifying the particular geometric transformation relating two shapes. As the second aim of the study, I introduce and employ an innovative methodology for video coding using StudioCode, which offers both qualitative and quantitative forms of analysis. To do so, I first discuss how to identify codes using Arzarello's theory of touchscreen-based interaction. I use the video timeline as an analytical tool to track the designated codes in video data, which enables tracing the path of interactions over the stretch of time. Also, I justify how this method decreases video analysis' subjectivity and enhances reliability and validity. Overall, this study demonstrates that the multimodal touch and sight feedback via a touchscreen device can serve to assist in the learning of the concept of rotation.

4.1. Geometric Transformations and a Dynamic Geometry Environment (DGE)

Geometric transformations have been advocated as an essential part of the K-12 geometry curriculum. The National Council of Teachers of Mathematics (NCTM) Standards (2015) has included transformations as one of the four central content areas in K-12 geometry that teachers should know.

> To be prepared to support the development of student mathematical proficiency, all elementary mathematics specialists should understand the following topics related to geometry and measurement with their content understanding and mathematical practices supported by appropriate technology and varied representational tools, including detailed models:
>
> C.3.2 Transformations including dilations, translations, rotations, reflections, glide reflections; compositions of transformations; and the expression of symmetry and regularity regarding transformations (NTCM, 2015, p. 2).

Also, transformations are a fundamental topic that links geometry and algebra. They provide a powerful tool for analyzing mathematical and real-life situations and make connections between different mathematical concepts and representations (Coxford, 1973). The topic has been given much attention in various provinces across Canada. For example, the Ontario Ministry of Education (2014) indicates the value of learning geometric transformation as part of spatial reasoning. The Newfoundland grade six mathematics curriculum (2015) also brings particular attention to the geometric transformation under the title of "motion geometry". BC's revised curriculum also emphasizes learning geometric transformations in primary school aiming to develop the students' core competence in communication, critical thinking, personal, and social life (BC's New Curriculum, Mathematics, 2016; Newfoundland Ministry of Education, 2015; Ontario Ministry of Education, 2014).

4.1.1. The Role of Dragging in DGEs

DGE offers construction of diverse, dynamic, and complex examples in real time that area difficult to render with paper-and-pencil representations or with concrete manipulative shapes (Presmeg, 1986). One of the most important features of DGE is dragging. Dragging figures/constructs offers *continuous* and real-time transformations that maintain the geometrical relationships integrated among the construct's components. In this case, constructs, unlike drawings, move and transform while preserving the *invariant* geometric properties. For instance, dragging a parallelogram may produce any desired orientation, shape, side-lengths parallelogram (visually apparent) but the transformed shape is always a parallelogram, even if it transforms to a rhombus, square or rectangle. The notion of "*continuous motion"* was introduced by J.V.

Poncelet (1864). The principle states that “if we suppose a given figure to change its position by having its points undergo a continuous motion without violating the conditions initially assumed to hold between them, the [...] properties which hold for the first position of the figure still hold in a generalized form for all the derived figures” (as cited in Nasim, 2008, p. 144). Battista (2008) hypothesized that geometric relationships could be perceived “as *invariants* in the *continuous* moving of the draggable figures” (p. 350), which supports human’s ability to notice of invariance. Dragging changes the shape’s representation and thus the way it is perceived, while the learner remains attendant to the visual modes of what changes or remains invariant. Furthermore, DGE prompts an environment for reasoning, and explicit descriptions of geometric relationships and shapes. Also, *direct manipulation* allows students to conceive a construction closer to the theoretical definition of a geometrical figure (Pratt & Ainley, 1997).

There are ample research findings in which the affordances of DGEs, and dragging in particular, can help to support student’s *reasoning,* and ability to *formulate conjectures* and proofs. For examples see what is cited in Leung (2008). They go so far as to claim that no counterpart in traditional learning environments could raise the same level of conjecturing, thinking and reasoning as DGEs (Battista, 2002, 2007 & 2008; Hollebrands, 2003; Mariotti, 2000; Yu & Barrett, 2002).

Research also shows learning geometric transformation can help students develop spatial ability with geometric objects (Clements & Battista, 1992). Having established the DGE’s value in learning geometry, geometric reasoning, and formulating conjectures, it has been asserted that teachers should have high levels of content as well as technological knowledge in order to create classroom environments where students develop reasoning and justification skills (Parsons, 1993).

There is no doubt that technological devices provide much flexibility in the learning of geometric transformations. DGE provides an environment enriched with multiple representations and direct, continuous, real-time and interactive manipulation (Abrahamson & Sanchez-Garcia, 2016; Arzarello, 2015; Battista, 2008; Leung, 2008; Nasim, 2008; Ng, 2016; Sedaghatjou & Norul Akmar, n.d.; Sinclair & Moss, 2012; Sinclair & Yurita, 2008; Vrahimis, 2016). In order to increase the use of multitouch-based technology in daily life and educational systems and contexts, many schools are equipped with iPads, tablets, interactive whiteboards (IWB) and so on. Therefore, each mathematics teacher is expected to take advantage of using those devices in teaching.

Nevertheless, prospective elementary teachers face difficulties in understanding various concepts related to transformations (Thaqi, Giménez & Rosich, 2011; Harper 2003; Son & Sinclair, 2010). For example, "while they could rotate the figure when a rotation axis was provided, they failed to rotate it in the absence of an axis. Also, prospective elementary mathematics teachers were unsuccessful in finding the center of the given rotated figures" (Turgut, Yenilmez, & Anapa, 2014, p. 1). Additionally, teachers often have only slight coursework experience in the topic of geometric transformations (Wang, 2011). The results of Kurtulus's (2010) study showed that performing rotations was the primary challenge to elementary prospective teachers. Therefore, in this study, I decided to place the focus only on the concept of rotation, while a prospective elementary teacher explored different forms of geometric transformations in the BlackBox task using touchscreen-based technology (GSP on iPad).

Studies show the positive impact on teaching and learning of manipulating mathematical objects virtually (Bakar, Ayub, & Tarmizi, 2010; Edwards & Zazkis, 1993; Harper, 2002; Hoyles & Healy, 1997; Laborde, 2000; Leong & Lim-Teo, 2003; McClintock, Jiang, & July, 2002; O. Ng, 2016; Rahim, 2002). However, few studies have focused on how digital technologies may contribute to prospective teachers' understanding of geometric transformations, in particular for touch-based devices. Thus, I aim to find how a prospective teacher identifies geometrical rotation while interacting with a given task in a touchscreen-based DGE.

4.2. Manipulating Geometrical Objects on Touchscreen Devices

As discussed above, prior studies on the use of DGEs have considered dragging as the main feature of DGE. Also, different types of dragging in the DGE can indicate various levels of cognitive domain of mathematical thinking (Arzarello, Olivero, Paola, & Robutti, 2002). Arzarello et al. (2002) suggest that dragging mediates the relationships between conceptual and perceptual entities: "dragging supports the production of conjectures: exploring drawings by moving them, looking at the ways after which their forms change (or do not change), [and] allows users to discover their invariant properties" (p. 66). They identified different types of dragging modalities such as wandering dragging, guided dragging, dummy-locus dragging, and line dragging. For example, guided dragging involves dragging an object to locate a particular configuration, where wandering dragging refers to moving or dragging an object randomly or without a plan to explore the relationships among the other parts of the object in the sketch. However, touch-based manipulation requires its own specific mode of interaction because dragging on a touchscreen device is different than dragging with a mouse. Also, each application and even each designed task needs its own way of identifying interactions. Nevertheless, it became a difficult task to track all the various forms of interactions while using a touch-based mathematical tool, either because of the complexity of the interactions or the method of analysis (Hulon, 2015).

Some studies have investigated the growth of knowledge of transformations by identifying changes in mental schemas (e.g., Flanagan, 2001; Yanik & Flores, 2009). The fundamental assumption for these studies is that the concepts are mental representations that develop in different stages and are located in individual minds, and that therefore learning means developing a representation (Cobb & Yackel, 1996). However, this approach supresses the learner's experiences as well as her interactions with the physical and social world by following learning trajectories and looking for mental schemas mainly in clinical interviews. For example see Yanik and Flores, (2009).

Considering the dynamic and flexible environment of digital technology, and the ability to directly interact with a geometrical shape on a touchscreen DGE, I will attend to gestures, bodily interactions with geometrical figures, and mathematical communications as institutional parts of learning (Roth, 2010; Roth & Lee, 2007; Roth & Radford, 2011, 2013; Nemirovsky, 2013). I am interested neither in studying what happens in the learner's mind when she solves a task in a DGE, nor in analysing the learner's utterances in a non-multimodal sense. Instead, as with Nemirovsky et al. (2013), I trace and analyse the temporal flows of perceptuomotor activities that constitute the experience of mathematical learning and the transition from a discordance between procedural and motor aspects to their perceptuomotor integration in experiential time (Arzarello, 2006, Radford, 2009; Roth, 2011, Nemirovsky et al., 2013). In other words, I trace the emergence of perceptuomotor integration during the interaction with a touchscreen DGE, and the role of embodiment in a learner's mathematical instrument fluency by identifying different forms of (active and basic) actions (Arzarello, 2015). Theoretically, basic actions are defined as the basic ways of interacting with the touch interface (e.g. tap & slide), while active actions are goal-oriented interactions performed by the user (e.g. drag-touch). In the theory section, I elucidate the theories of touchscreen-based interactions and perceptuomotor integration in greater depth.

4.3. Structure of the Chapter

This chapter's structure is meant to reflect the importance of my methodological approach. Though the conventional approach is to start with theory and work towards phenomena by way of research questions and methodology. In this case, the theory will be easier to elaborate by using terms from my methodology. Therefore, my theoretical approach is discussed *after* research questions and methodology.

4.4. Research Questions

In this chapter, I analyse the modes of interaction of a prospective teacher working with a BlackBox sketch in touchscreen GSP. I make a case for creating codes to extend Arzarello's touchscreen interaction framework, proposing that new codes need to be developed depending on the tool and the task. I also use an innovative methodology to analyse the prospective teacher's explorations. Therefore, this study addresses the following research questions:

- What are the modes of interactions in a touchscreen GSP for BlackBox sketch?
- How do customized expansions on Arzarello's (2014) codes of touchscreen-based GSP interactions clarify emergence of tool fluency on a mathematical instrument?

- How does a prospective teacher learn geometric transformation by interacting with a touchscreen-based GSP?

4.5. Methodology

For this descriptive case study, I invited a group of prospective teachers who had received low marks (0 to 4 out of 10) on a mathematics quiz in geometrical transformation. Another volunteer student joined simply out of interest in the topic. The case reported here is a part of a larger study that included five prospective teachers and analyzed their inactions with the touchscreen-based GSP. None of the five participants had experience working with GSP or with Sketchpad Explorer (the iPad version of GSP). The study took place in a classroom located in a Canadian university. Participants were asked to work on four tasks involving two-dimensional transformations (viz., reflection, rotation, scaling, and translation) and their interactions were videoed for future analysis. These tasks were the subject of the quiz with which they had struggled.

In this descriptive case study, I played the role of the participant-observer. I have chosen to report only on the specific part of the interaction where rotation was involved because of the reported complexity of the concept for prospective teachers (Turgut et al., 2014; Ada & Kurtuluş, 2010). In this paper, by focusing on only one prospective teacher's interactions, I aim to provide a thick description of the event. I selected to report on this one participant, whom I shall call Anna, because she spoke aloud while conjecturing and interacting with the given task. Also, the recorded video provided a full picture and rich data of her activities and interactions on and off the screen.

4.5.1. Design of the Task

The tasks were designed based on the syllabus for the university course in which the teachers were enrolled, but were different than the textbook exercises. In the textbook, two types of tasks were provided to the students. First, to decide the particular isometry that relates two shapes. Second, to find the image of a given shape under a particular isometry using a paper tracing method. For example: performing translation using a given vector (direction and distance); reflection using the line of reflection;

rotation using the centre of reflection, direction, and the angle. Later in the chapter, students were introduced to the composition of two rigid motions (for example, a glide-reflection where a translation and a reflection over a line parallel to the vector of translation are combined). The textbook chapter ended by introducing tessellation to the prospective teachers, and some problems they could use to help students visualize geometrical transformations.

Following Arzarello et al. (2014) and Leung (2011), I postulate that a mathematical task should provide conjecture, exploration, and explanations that allow multimodal communications. Therefore, for this study, the task (a BlackBox task designed by Dr. Sinclair) was used. This task consists of five different geometric transformations in the form of “mysteries.” Users are invited to drag either one or two points A and A' (where A' is the transformed image of point A). See Figure 4.1.

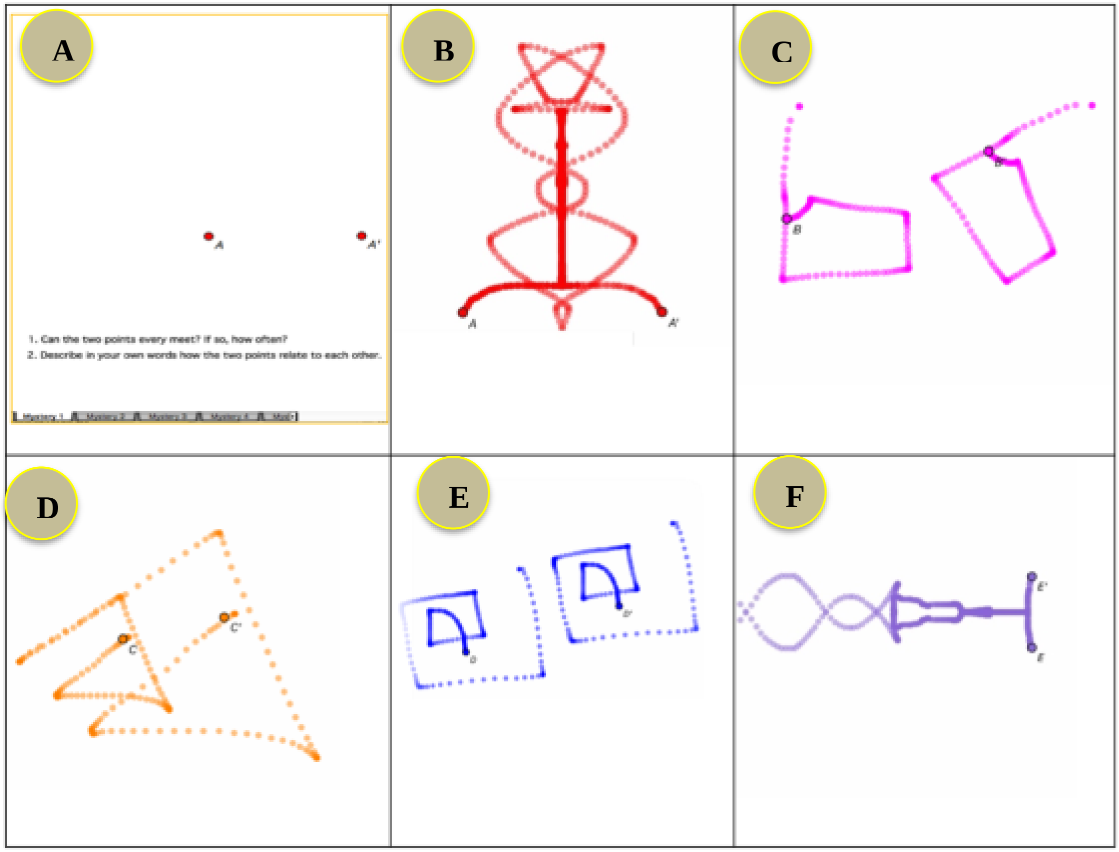


Figure 4.1 BlackBox screenshots.

Note. (A) The main page for all mysteries shows only two points visible on the screen. (B): Mystery 1- Reflection; (C): Mystery 2- Rotation; (D): Mystery 3- Scaling; (E): Mystery 4: Translation. (F): Mystery 5: Reflection.

However, in this paper, I focus only on the second mystery, which is a 45° counter-clockwise rotation (see Figure 4.2).

Figure 4.2 **Mystery two: 45° counter-clockwise rotation. Density of the trail demonstrates the speed of movement.**

The tracking feature was enabled for the task so that the participant could see the shape produced by the dragging of both B and B' (see Figure 4.1). The faster a point moves, the more spread out its trail; the slower the point moves, the denser its trail. B and B' are positioned where the movement ends. It is impossible to tell what kind of transformation relates B and B' without dragging or pushing. However, by dragging, it is possible to find the type of transformation as well as the centre and angle of rotation.

What is rotation?

Rotation is one of the rigid geometric transformations. Rotation about point M of α (-π< α < π) takes each point A in the plane to its image point A' such that A and A' lie on the same circle centre at M and the measure of angle AMA' is α, which can be written as $R_{M,\alpha}$ (A) = A'. The centre M and angle measure α are the parameters of the rotation (Figure 4.3).

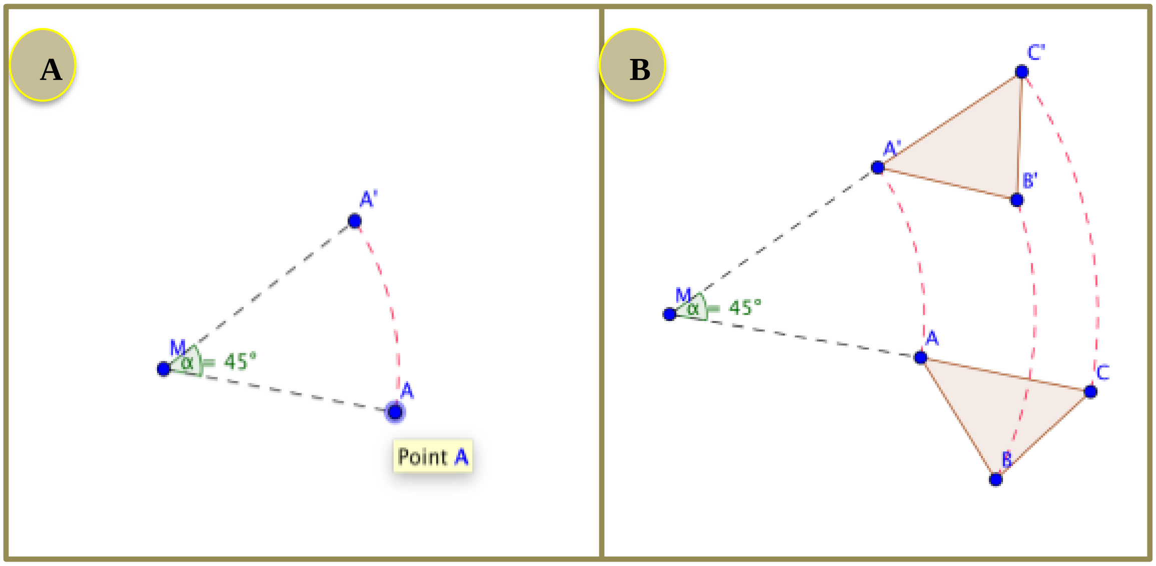


Figure 4.3 **Rotation $R_{M,\alpha}$ about centre M by α, counter clockwise (A) for point A; (B) for triangle ABC.**

Figure 4.3 shows the properties of rotation:

1) The centre of rotation is the only fixed point. In other words, in the designed task, the centre is the only spot where the two points of the screen ever meet.

2) Rotation preserves distance between points.

3) Rotation preserves the shape (lines, measure of angles and curves).

4) Rotation preserves chirality. It means the image of all non-collinear points maintain the order of points (A'B'C' has the same orientation as ABC).

So, in the second mystery, the learner was expected to identify the above statements by dragging the points on the screen. For example, the learner creates a trace or a shape and its image that may reveal: (a) the centre of the rotation is fixed on the screen, (b) the transformation preserves the shape, and (c) the rotation's attributes. I will discuss modes of interaction in the theoretical section to give a broader picture of what I am analyzing after describing what happened in the session.

4.6. Theoretical Approach: Tracing the Paths of Interaction

The theoretical frameworks that form the current study consist of two parts. The first part discusses Arzarello et al.'s (2014) theory of touchscreen-based interactions and formulation of active and basic actions. This offers an answer to my first research question: “What are the modes of interactions in a touchscreen GSP for BlackBox sketch?”

In the second part of this section, Nemirovsky et al.'s (2013) perceptuomotor integration approach will be presented alongside its application in DGE. This provides insight into my second research question: “How do customized expansions on Arzarello's (2014) codes of touchscreen-based GSP interactions clarify emergence of tool fluency on a mathematical instrument?”

4.6.1. Theory of Interaction: Active actions vs Basic Actions

In order to answer the first research question, I adapted Arzarello’s (2014) theory of interaction to identify and trace types of interactions on touchscreen-based DGE, in particular for BlackBox sketch. The theory of interaction follows a non-dualistic approach to categorize different types of touchscreen-based interactions. Arzarello et al. (2014) describe the two main categories of interactions as *basic* and *active actions*. *Basic actions* are described as the basic ways of interacting with the touch interface (e.g. to reset the program). Also, exploratory interactions made with no plan or at random are classified as *basic actions*. For example, tapping, sliding or random spinning, or pushing a point on the screen in BlackBox are categorized as *basic actions*. However, a combination of basic actions and/or non-random finger actions are classified as *active actions*. In this study, *active actions* are identified as learners’ interactions with the task, made in order to reach a target or solve a given problem. For instance, *drag-touching* and *rotating* a point to draw a shape and find the direction of rotation are active actions. That is because *drag-touch* refers to the touchscreen interactions wherein the learner drags the point to create a geometrical shape, to justify and/or to explain the geometrical relations. In BlackBox *rotating* the point also is an active action because it occurs when identifying the direction of rotation, centre of rotation, making related conjectures, etc.

The focus in this chapter represents interactions with *active actions* rather than basic ones; however, I have identified both active and basic actions in Table 4.1 and in the video timeline.

Table 4.1 Basic and active modes of action for touchscreen-based interaction (specified for BlackBox GSP task). Adapted from Arzarello et al. (2014)

	BASIC ACTIONS	
Slide	User puts finger on screen, moves in any direction without touching the points (few of these happened in a random way).	
Hold	A long tap- (more than 2 sec). Mostly happened before making a decision.	
Tap	Performed to switch between tabs/tasks/reset.	
Free	Free exploration of the application	
	ACTIVE ACTIONS	
Push	Towards	Dragging one point toward the other one (e.g.: to find the centre of rotation)
	Away	Dragging one point away from the other one (e.g.: to find the direction of the rotation)
	Along	Dragging one point along the other one (e.g.: to find the line of symmetry or to find the number of times that two points meet)
Rotate (gesturally)	Clockwise (CW)	Dragging the point in a clockwise circular rotation repeatedly (to find the type of transformation).
	Counter clockwise (CCW)	Dragging the point in a counter-clockwise circular rotation repeatedly (to find the type of transformation).
Drag-touch-to-approach	Dragging the point to draw a geometrical shape (to justify, or reason, or deal with some particular geometric property, shape, or construction).	
Drag-touch-free	Moving the point freely to create a geometrical shape (to visualize and explain).	

In this study, basic and active modes of action are classified by the specific nature of touchscreen-based interactions for the given task. This means that these modes of interaction could be identified differently in other touchscreen DGEs or different tasks. For example, Arzarello et al. (2014) observed that "push action" happened relatively few times in "a random way," while it is one of the active actions here. The process of determining active and basic actions is explained in detail in section 4.7 (video coding process).

4.6.2. Tracing Tool Fluency in Touchscreen DGE

To answer the second research question (how do customized expansions on Arzarello's (2014) codes of touchscreen-based GSP interactions clarify emergence of tool fluency on a mathematical instrument), I have adapted Nemirovsky et al.'s (2013) perceptumotor integration approach. To analyse modes of interactions, instead of digging into “geometrical thinking” as suggested by Arzarello et al. (2014), the lived experience of the learner is explored in Husserl’s experiential time. In other words, I trace the emergence of perceptuomotor integration and tool fluency in experiential time (Husserl, 1991; Nemirovsky et al., 2013). To do so, I analyse temporal flows of perceptual and motor activities regarding identified modes of action. By temporal flows, I mean lived experience infused with recall of the past and expectation of future. Perceptuomotor activity always constitutes a temporal flow, insofar as, at a given moment it is never isolated. Rather it is constituted by partial enactments of retentions (immediate past), “now phases,” and protentions (immediate future) (Husserl, 1991). For example: in the BlackBox task, when Anna pushes point B towards/away from B´ repeatedly; I avoid describing her finger movements, gestures, body movements, and words as outer manifestations of her mental schemes for the concept of rotation. Instead, I try to understand the temporally extended course of actions that she experiences in the act of identifying the place that two points meet, (i.e., centre of rotation).

Perceptuomotor integration insists that mathematical concepts are “inhabited” (Nemirovsky, 2017). Also, according to the theory, mathematical learning includes appropriate skillful use of the mathematical instrument as a cultural tool that mediates mathematical activity. Generally, in learning a new skill there are phases in which perception and motor aspects of the activity seem discordant. "… the transition from discordance between procedural and motor aspects to their integration is common to all learning; that is, perceptuomotor integration is a milestone for fluency in any field." (Nemirovsky. et al., 2013, p. 380). This means that once fluency is achieved, involuntary motor activity becomes a part of perception (Ibid). Nemirovsky et al. also consider a mathematical instrument as a "material and semiotic tool together with the set of embodied practices for its use within the discipline of mathematics. So, the fluent use of mathematical instruments allows for the culturally recognized creation in mathematical domains, just as members of the musical communications acknowledge." (p. 373).

Appealing to Nemirovsky et al.'s perceptuomotor integration theory (2013) allows me to trace the emergence of tool fluency in terms of integration of temporal streams of perceptual and motor activities using a video timeline as an analytical tool. Tool fluency is an intertwining of the perceptual and motor aspects of an activity that allows the performer to "act with the holistic sense of unity and flow" (p. 2). With an explicit approach to embodiment, tool fluency constitutes mathematical thinking and learning. Within perceptuomotor integration 'mathematical thinking' is "constituted by bodily activity at varying degrees of overt and covert expression" (p. 376). Further, transformations in lived bodily engagements while the subject performs mathematics activities are defined as 'mathematical learning'. In this sense, mathematical learning is not imparted didactically, but rather established by embodied tool fluency (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013, 2011). Therefore, the answer to the third research question on "how does a prospective teacher learn geometric transformation via interacting with the touchscreen-based GSP" is intertwined with the answer to the second research question. In other words, the answer to both lies in appealing to Nemirovsky's perceptuomotor tool fluency, which provides the foundation for learning geometric transformations. According to the theory, when tool fluency is evident, the prospective teacher has learned.

4.7. Tracking Paths of Interaction (TPI): A Methodology for Video-coding

"Without verification, you are just another researcher with a hunch" (Miles, Huberman & Saldana, 2016, p. 276). Video coding demands careful attention to language, gestures, gazes, body movements, etc., and deep reflection on the emerging patterns and meaning of human experiences. Huberman and Saldana (2016) claim when a researcher is working on a piece of video data, sometimes (s)he notes repeating patterns, themes or gestalts arising from different parts of video data, pulling together commonalities. These patterns of variables may involve similarities or differences among categories, or include connections in time and context. The patterns can also be constructed from the researchers' observations or a reoccurrence of the phenomena. It is advised that the researcher searches for additional evidence to the same pattern and remains open contradictory results. Skepticism about emerging patterns, and conducting conceptual and empirical testing for validation, will better support researchers' results. I shall add, every identified pattern should be revisited and re-examined either by the analyst or other researcher-collaborators. That is to say, the video timeline could serve as the "analytical tool" if utilized effectively. The video timeline also enables the researcher to analyse paths of interaction as temporal flows of perceptual and motor activities rather than points of interaction. Per Arzarello et al (2014, p. 43), it is "inappropriate to reduce the data of a trace to a single point."

Having discussed the importance and capability of the video timeline as an analytical tool that enables tracing of themes and codes in mathematical learning, I adapted Vogel and Jung's (2013) procedure for video coding due to its explicit cycle of verification and consolidation of data. This approach allows me to identify Arzarello's codes in the video data and trace perceptuomotor activities in the emergence of tool fluency. I postulate that using the video timeline as the analytical tool can provide a research instrument that validates emerging patterns or codes and marks them elsewhere in data. Taking this approach enables the researcher to draw procedural conclusions along the way. To identify patterns and categories – which are called active and basic actions in this study – and develop codes, I modified and adapted Arzarello's theory of touch-screen interaction. In the next section, I discuss the process of video coding and determine the categories.

4.8. Video Coding Process

In this study, categories and codes are identified based on different modes of interactions with the touchscreen application. The first step is taken by preparing video data to analyze: by watching it a few times. Preparing video data for coding provides familiarity with the content, initiates possible codes and reduces ‘pre-coding' bias. This step is called “Identifying categories and codes inductively”. Besides, preparing video data especially in a collaborative environment suggests collaborative-coding, which may reduce the subjectivity concern in solo-coding. Then, a “verification” step is taken to find if codes are informed by, or consolidated with, the chosen theory (in this case, Arzarello et al.'s (2014) theory of interaction). These steps taken together, while considering the video-data, make a cycle that helps researcher to produce, verify, and consolidate codes informed by theory. In the next phase, the researcher develops and checks the integrated system of coding and creates a standardized system of codes via “rating accordance.” “Rating accordance” rates and verifies the degree to which categories and codes conform with the theory. This could be contextually varied depending on the scope of study, theoretical framework, and/or available coding software.

For example, I considered theories of interaction (Arzarello, et al., 2014) and perceptuomotor integration (Nemirovsky, et al., 2013) in order to identify categories and codes. I identified basic and active modes of actions using the theory of interaction. I also examined active actions with respect to the emergence of tool fluency. The process of developing video coding is shown in figure 4.4.

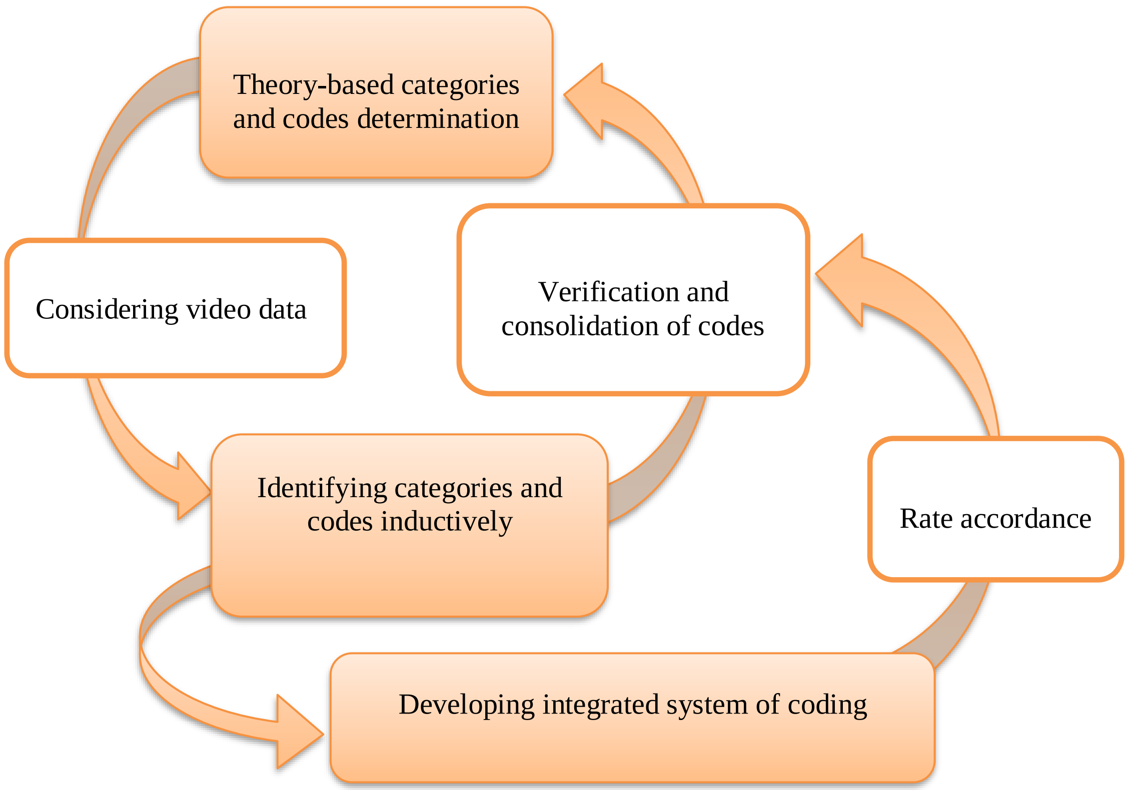


Figure 4.4. Video coding process for categories and codes determination
Note. Adapted and modified from Vogel and Jung (2013).

To establish the standard by which the interactions were assessed, the researcher and a colleague identified and examined codes using an agreed-upon list of basic and active actions. After identifying the codes, video data was coded via StudioCode[14] software for further analysis. Table 4.1 (above) shows categories of active and basic actions and codes, as well as related definitions used to code the prospective teacher's modes of interactions with the touchscreen GSP (BlackBox) in this study.

[14] StudioCode is a professional program and a video analysis tool that captures, codes and categorizes video assets to review and analyze. StudioCode 10.6 is utilized to analyze data in depth and trace actions and interaction of participants with the iPad.

After identifying the categories and codes discussed above, the video data was analysed while capturing frequency and duration of emerging actions via StudioCode. StudioCode connects codes to each segment (instance) and allows timeline analysis. Figure 4.5 shows a snapshot of the coding window in StudioCode for the BlackBox task on GSP.

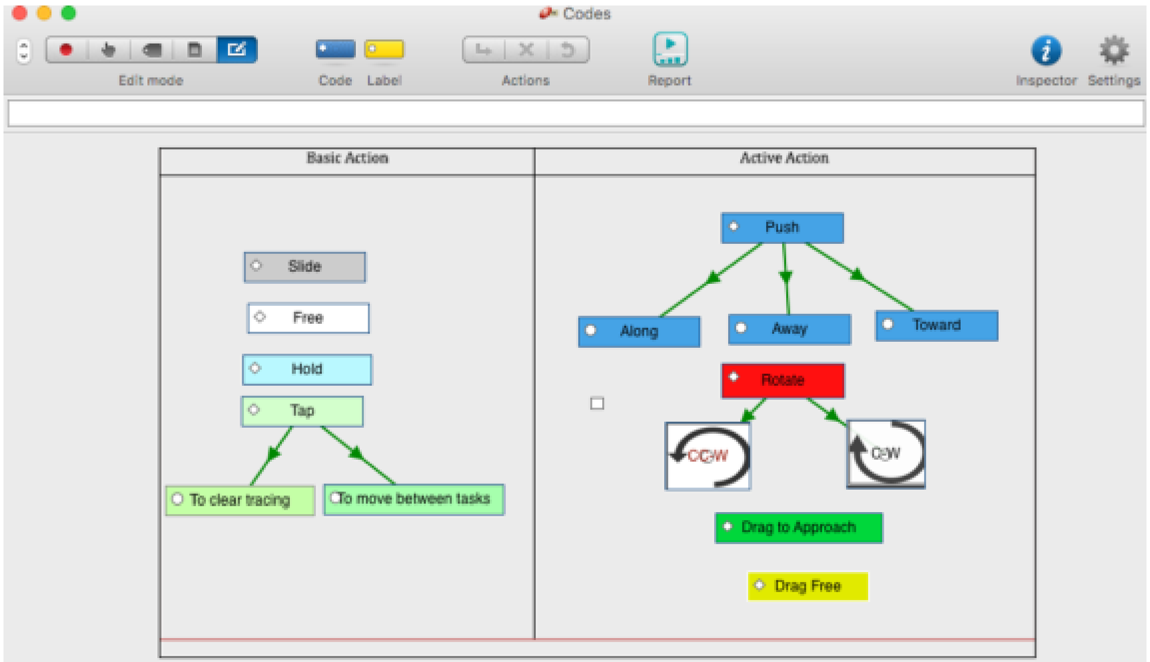


Figure 4.5. **StudioCode coding window**

Note. The screenshot shows two categories of "Basic Actions" [including: Slide; Hold; Tap; Free]; and "Active actions" [including, Push: Towards, Away & Along; Rotate: Clockwise (CW) and Counter clockwise(CCW); Drag-touch-to-approach; and Drag-touch-free]

What follows is an analysis of Anna's modes of interaction with the touchscreen GSP on rotation section of BlackBox task.

4.8.1. Video Timeline: An Analytical Tool to Trace Paths Of Interactions

In this episode, Anna was challenged with five “mysteries.” By dragging, pushing and/or rotating the given points on the screen at different speeds, she drew a variation of lines, curves and geometrical shapes to determine given geometrical transformations, in a more dynamic way than in the textbook.

In this section, with the use of StudioCode and the video coding procedure explained above, I construct a timeline illustrating Anna's modes of interactions (see Figure 4.6).

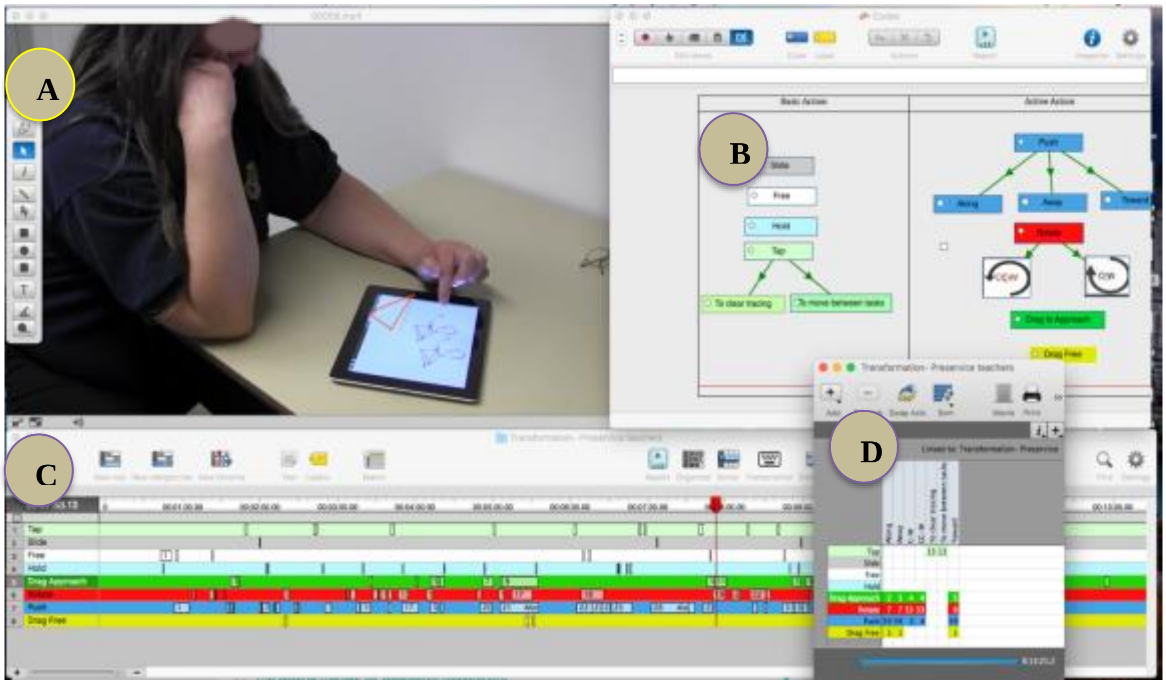


Figure 4.6 **StudioCode snapshot.**

Note. (A) shows the video that is being coded, (B) is the coding window where researcher selects code buttons and their Boolean relationships, (C) StudioCode codes video by creating corresponding coding-rows on the timeline. By clicking on the codes' name on each row all the related coded instances will be combined and played. (D) matrix window provides counts of incidence of codes, labels during the specified interval and indicates all the overlaps. Duration, time of start and end of each frequency could be exported to Excel or SPSS file too.

Before I discuss my findings in terms of timeline analysis, I shall describe Anna's lived experience while interacting with the BlackBox task. In this way, the below section also aims to answer the third research question of "how does a prospective teacher learn geometric transformation via interacting with the touchscreen-based GSP." To explain Anna's learning I appeal to Nemirovsky's perceptuomotor tool fluency (2013) and describe her lived experience in Husserl's experiential time (Husserl, 1991).

4.9. Description and Analysis

The study was conducted while the participants were enrolled in a blended geometry and pre-calculus course designed for elementary and secondary prospective teachers. Anna is an adult student who had no previous experience of working with iPads or smartphones. Anna was selected among the five participants because she showed a high level of active engagement by speaking aloud while interacting with the task. As a participant-observer, I began by inviting Anna to touch and move one of the points on the screen to examine Anna's possible conjectures. The vignette below includes only the second mystery sketch, which was a rotation. (…) indicates pauses that last more than two seconds. After exploring the first mystery, Anna read the question aloud for the second mystery.

01 "Mystery two, Okay, can the two points ever meet? If so, how often? Describe in your words how the two points relate to each other. Okay, do they ever meet?"… [Anna asks herself, pushes two points toward, away and along each other]

02 "hmm… okay… [locates two points on each other]. Okay… so... yes, they do meet… but wait [pushes points away and then towards each other. Rotates point B, so B' goes away] … they don't meet when you [drags point B along B']... Okay, so it has to be… may be… a glide translation? [be]'cause when you rotate them [rotates B counter-clockwise (CCW)] you could make them meet. But if you move them toward each other… one turns away" [pushes points towards and away repeatedly].

03 "Okay, when you move one towards another [pushes points towards]… one goes down, and one goes across...hmm…" (Figure 4.7 A).

04 "… and if you rotate counter clockwise, they both rotate counter-clockwise [Anna notices both points are approaching to the centre of rotation] ooh, ooh! How they get through?... Do they ever meet? [rotates B CW]. No, no... don't rotate them [CW]", Anna says to herself. "But if you rotate them counter-clockwise [rotates point B CCW]… oh, now they are chasing each other" [rotates and pushes points slowly and precisely trying to make them meet].

01 - 04, I observed instances of discordance between perceptual and motor aspects of Anna's course of activities when she was trying to identify the centre of

rotation. Anna conjectured and anticipated the dependency between modes of action [rotations (CCW or CW); push (along, away, toward)] and center of transformation (meeting point). In other words, Anna's temporal flows of motor arrangements (marked as active actions) in present time were incongruent with identifying the centre of rotation (protention). Her trial and error approach in identifying the point of meeting was to repeat a variety of active actions (rotating, dragging and pushing), which were continued until a possible answer emerged. Since the sketch preserved geometrical properties of rotation, it did not allow Anna to reach her protention of identifying 'the centre of rotation dependent on the type of motor action'. Also, Anna's ongoing retentions and protentions were clearly associated with the motor or perceptual aspects of learning, but not integrated with them (Nemirovsky, 2013). This indicates Anna's early stage of tool fluency. For example, Anna, expects (protention) the existence of a meeting point to depend on the direction of rotation (CCW) from her just-past remembered experience (retention). She then examines another CCW rotation (present), and finds B and B' may not meet even with a CCW rotation if not dragged to the centre of rotation. She therefore experiences a breakdown of the previous retention in the process of emerging new protention. As is evident, in the next step (05) Anna examines a CW rotation to find if the points meet.

> 05 Anna presses the reset button to erase the traces. Then she slowly turns point B clockwise and makes the points coincident. It seems she has recognized where the centre of rotation on the screen is. In addition, CW action makes the points meet. <u>"... So, these two can meet… (nods head) but how often?"</u> Anna asks herself.

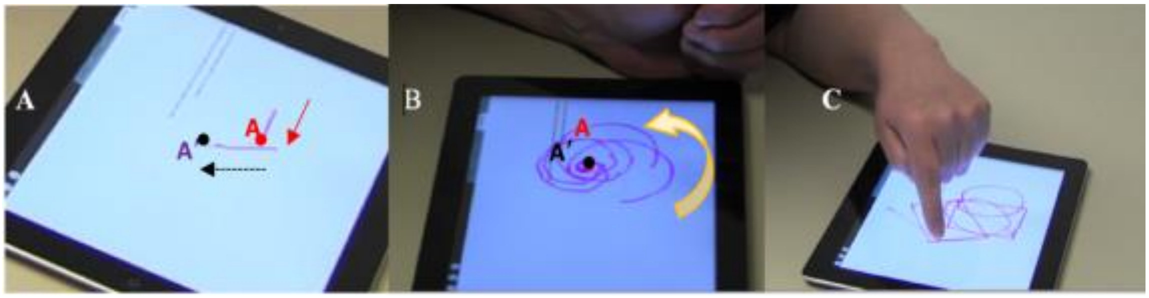


Figure 4**.7** **Some example of Anna's active actions. (A) pushing toward; (B) rotating counter-clockwise; (C) Drag-touch-to-approach**

06 Anna rotates the points counterclockwise again with a bigger radius and they meet again at the same place. "Okay, so this can meet once on a rotation [regardless of direction of rotation] … we think this is a glide-translation, but we don't know!...There is no reflection". Anna erases the traces and makes another counterclockwise rotation.

Anna anticipated the geometrical transformation as a "glide-translation", however her present activities did not support her anticipation (there is no reflection). As a result, Anna's past and future lived experience were not seamlessly infused and she experienced disruptions. She also found the centre of rotation's location on the screen, but struggled with the type of movement or active action that may cause two points to be coincident (06).

07 "Look ... this changes every time we do it" [it seems she is referring to different traces] (Figure 4.7 B) [10 second pause] … [several pushes and rotations]. I think this is a rotation. This [points meeting] only happens when it rotates …"

Apparently, touchscreen GSP's well-preserved *invariant* in the "continuous moving of the draggable" points supports Anna's noticing of the invariance (Battista, 2008, p. 350). Different actions (push, drag, rotate) changed the points' position and their rotated images; created several invariant traces and thus the way they were perceived. Meanwhile the learner remained attendant to the visual modes of what changes or remains invariant.

At the same time the dynamic nature of DGE brought a new assumption of the relation between the centre of rotation and type of active action (rotation), this may not exist in paper-and-pencil forms of tasks. "This [points meeting] only happens when it rotates", Anna said. As the researcher-observer I decide to challenge Anna's protention and conjecture of such a relation.

08 Noticing Anna's struggle to visualize geometrical transformations that she encountered from the textbook, Mina suggested: "Oh, is that the reason? The [type of] movement? What if you make a geometrical shape?"

09 Anna makes a circle [with the rotation action and the drag-touch-to-approach] and then a square [push action in different directions but same direction and the drag-touch-to-approach]. "That is, I think, a glide translation. Because one [shape] is not exactly in the same place as the other [shape]… (Figure 4.7 C).

10 Hold on... hold on, hold on here... but they also have a rotation [rotates CW; drag-touch-to-approach; drag-touch-free]… which is, may be, about 45 degrees? Anna whispers.

11 "How often do they meet then?" I said. "once, once per shape," Anna answers and makes the points coincident again [drag-touch-free].

In the related course work, Anna had learned that when rotating a shape, the centre of rotation should be given first. The hidden centre of rotation in BlackBox activity illustrated new conceptual aspects of rotation. In agreement with Arzarello et al.'s (2014), it implied that "the centre does not have to be determined (explicitly) in advance" (p. 45).

Anna remained attendant to the task and invariances in traces, which paid off in recognizing the geometric transformation as rotation. To put it more simply, Anna's infusing of past and future in the lived present, retention and protention respectively, enabled her to articulate the type of transformation at this point.

In this respect, to examine the geometric relation between two given points, Anna performed 'drag-touch-to-approach' actions followed by a 'drag-touch-free' to explain findings (see Table 4.1, above). These actions are coded to demonstrate the moments that, at the very least, there was a seamless continuity between retention and protention. Anna knew what she wanted to do and she set about doing it. In other words, with drag-touch-to-approach and drag-touch-free actions, she reached her goal quite deliberately and successfully with only a minor disruption. These active actions enabled Anna to maintain her protentions without disruption, and eventually successfully complete the task. That is where tool fluency emerged as an “interpenetration [intertwining] of the perceptual and motor aspects of an activity, allowing the performer to act with a holistic sense of unity and flow” (Nemirovsky et al, 2013, p. 373). Then Anna explored the other three mysteries one after another. She preferred to explain each mystery and the type of transformation after she explored all of them.

12 Consequently, in the discussion phase, Anna rapidly and confidently (in only 20 seconds) demonstrated and explained the geometric relationship between two points by performing active actions such as drag-touch-to-approach and rotation via drawing a few geometrical shapes. This exploring phase for the second mystery took 03:34 and Anna was unsure if the transformation was a glide-reflection or rotation. At this stage, Anna’s lived experience of performing a drag-touch-to-approach action, while supported by the integrated and holistic perceptuomotor activities, established tool fluency: the manifestation of embodied mathematical learning.

4.10. Timeline Analysis and Discussion

In this section, I use a timeline as an analytical tool to demonstrate the emergence of perceptuomotor activity integration in terms of tool fluency over a period of time. This section expands discussion to the second and third research questions. It provides evidence for the emergence of tool fluency and mathematical learning using video-timeline as an analytical tool.

Using a timeline as an analytical tool not only extends the trustworthiness of the findings but also provides visual cues for each defined code and category. I have suggested a framework for video coding, and defined categories and codes based on a suitable theoretical framework in the ‘video coding’ section: 4.7. Consequently, the timeline illustrates how the different types of touchscreen-based interactions are coded – namely as active and basic actions – and are developed through the exploration and discussion phases. This approach enabled a new way of looking at touchscreen-based interactions that may be unique for each given task or exploration. I report only on the second mystery, identified by the orange transparent box and labeled "2" in two different phases named exploration (1:58 to 5:35) and discussion (11:59 to 12:19) on the timeline (Figure 4.8). Exploration and discussion phases are named based on Anna's terms, as she decided to explore all the mysteries first and discuss them later. By timeline analysis, a series of evidence in the form of codes was built that demonstrated Anna's high level of bodily engagement with the mathematical tool.

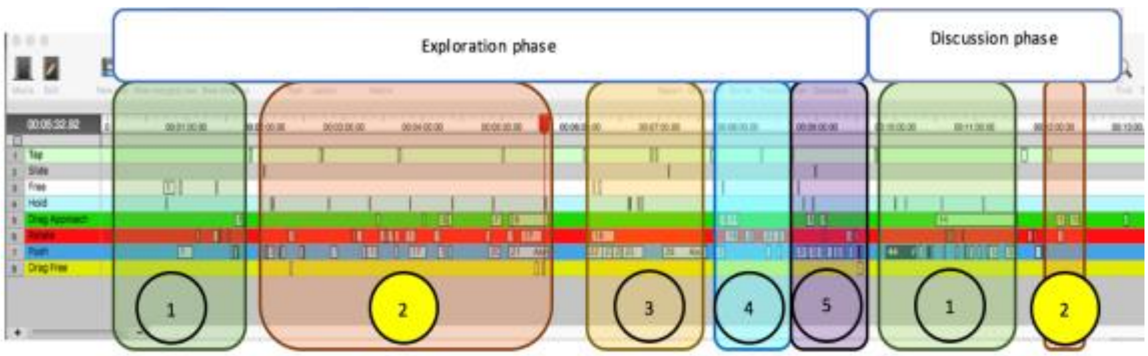


Figure 4.8 Coded Timeline in StudioCode

Note. The numbers 1-5 indicate the 5 mysteries in the BlackBox task. The orange boxes labelled (2) are the time intervals during which Anna was exploring and discussing the second mystery. Code Rows (e.g.: Tap, slide, free, etc.) are populated from code buttons. The first four codes including tap, slide, free and hold are basic actions. The next four codes including drag-touch-to-approach, rotate, push and drag-touch-free are defined as active action. The 'boxes' in each row are created in the chronological order in which codes are observed. They display actions and where they happened within the timeline.

Looking at the first orange box in Figure 4.8 from the left (4:46 to 5:33), there is a dense and high number of long active actions on the right side, that began earlier but increase in density. For example, although rotations (CCW or CW) and pushes in the red and blue rows are distributed semi-evenly in this box, the 'drag-touch-to-approach' is performed only at the end of this segment. Also, the timeline shows the last drag-touch-to-approach is a combination of rotation, push, and even drag-touch-free actions. Data in Section 4.9 shows that during this time Anna has been justifying and illustrating her answers while actively engaged with the mathematical instrument. Considering what is reported in the vignette [09 to 11], the dense and overlapping actions, the making of conjectures and the drawing of geometrical shapes by Anna, all combine to illustrate the multimodality of her communication (see Figure 4.8).

The same pattern of blended rotation action and drag-touch-to-approach is evident in the discussion phase while Anna was confidently illustrating that the mystery transformation is a rotation (11:59 to 12:19). I suggest that overlapped drag-touch-free and drag-touch-to-approach actions demonstrate the occurrence of complex combinations of active actions. These can function as a sign of mathematical embodiment regarding tool fluency in the form of holistic temporal flows of perceptuomotor integration. I have explained this in detail in Section 4.9 (09-11).

The timeline analysis enables the following interpretations: First, I observed many active actions such as push (along, away and towards), and rotate (CCW & CW), before drag-touch-to-approach and drag-touch-free actions occur. Despite Arzarello et al.'s findings (2014), I found "singularity" in the active and basic modes. "Singularity" here refers to the places where Anna used only one finger to interact with the software. Single-finger actions could be due to the nature of the designed activity, or the absence of instruction, or Anna's lack of experience in working with a touchscreen-based device. Therefore, two types of possible rotations were observed, both using a single finger: rotation clockwise (CW) and rotation counter-clockwise (CCW). Although theoretically both may seem mathematically identical, in Anna's case they provided different forms of insight about a geometric transformation. Second, rotation in various directions was also an important phenomenon because Anna conjectured a relationship between the direction of rotation and the existence, and coordinates of, the centre of rotation.

Anna conjectured about the type of transformation and then examined her conjectures by performing active actions. For example, she postulated that there was a relation between the kind of dragging/pushing and the geometric transformation [02 to 04]. Her retentions were challenged when observing unmet protentions (Husserl, 1991). Notions of retention and protention implicate remembered and anticipated aspects of her lived experience of playing in DGE. Anna restarted the activity three times to have a fresh look. Each reset meant she started either a new construction or aimed to demonstrate new traces to support her arguments or examine her conjectures.

4.10.1. The Features and Benefits of StudioCode in Video Timeline Analysis

Considering the video timeline as an analytical tool may address some concerns regarding video analysis: such as tracing certain codes and patterns over time, tracing specific codes over different videos (with same or different participants), or analyzing video data collaboratively. So, this method decreases video analysis' subjectivity, and enhances reliability and validity concurrently. Also, StudioCode can be utilized to trace categories and codes in small segments data taken from a larger sample. This helps investigators revisit the coded themes to revalidate and verify a hunch or hypothesis, and therefore keeps the investigation analytical and prevents bias. Utilizing StudioCode for analysing video data offers many more features than are covered here. These features include, but not limited to:

- visualizing difficult-to-verbalize codes and actions over a given time span,
- tracing categories and codes in small segments of the data taken from a larger sample of data,
- providing an excellent tool to count and measure the duration of occurrences of different codes over the time. This tool is named 'matrix',
- drawing shapes (arrows, lines, circles, etc.) on the video ,
- finding overlaps using Boolean operations. For example, in what segments do codes A and B but not C show up? This feature assists in identifying new emerging codes and revalidating the analysis of data in an innovative way. This also helps researchers envision the relations, such as direct association or inverse, between and among different variables or codes,
- transcribing the video and synchronizing it with the timeline,
- collaboratively coding video data and validating coding across a group of analysts (thereby minimize subjectivity).

4.11. Conclusion and Remarks

The study found that the coordinated interaction of body, tool, and talk invited a different lived experience of the concept of rotation – its properties and relations. Also, tracing the development of those lived experiences in terms of active actions on a timeline allowed us to identify the emergence of tool fluency concerning mathematical learning. In this section I elaborate how, according to Nemirovsky et al.’s (2013) perceptuomotor integration lens, tool fluency emerges in terms of embodied mathematical thinking and learning:

Mathematical thinking emerged in the form of temporal flows of bodily activities in Anna’s lived experience and proposed conjectures. In the exploration phase, although Anna noticed that the shape and its image were not positioned in the same place, she was unable to identify the form of transformation in the absence of prototypical examples that she had encountered previously in the class. She also used the term "glide-translation” rather than “glide-reflection” repeatedly, which may indicate her lack of knowledge of the concept. Anna was also unsure if the existence of the meeting point was related to the type of rotation (CCW vs. CW) or action (rotation vs. push). Those conjectures along with different types of speculative active actions can be considered early forms of tool fluency. In this early stage, she experienced breakdowns of memorized retention (an erroneous conjectured relationship between the direction of rotation and the existence, and coordinates of, the centre of rotation). She then experienced results that did not align with her expectations. Her perceptual and motor activities were not yet integrated (Nemirovsky, 2013).

I found that Anna's lived bodily engagement transformed in terms of performed active actions. That is to say, push and rotate (CW and CCW) actions that were the main active actions when she was in an early stage of tool fluency, developed into free and drag-touch-to-approach actions to illustrate how rotation influences the image of the shape to the participant-observer. The overlapped active actions were observed when Anna distinguished a rotation transformation from a “glide-translation.”

Anna constantly attempted to draw a shape to figure out the geometric transformation. Her response to the mystery within different modes of interaction (e.g.:

pushing, rotating and dragging) was productively different from simple paper-and-pencil-based drawing. This finding supports the body of literature that discusses DGE's feature of offering various complex examples in real time to the learner (Presmeg, 1986). The DGE could also offer continuous and real-time transformations that maintain the geometrical relationships among components. That is to say, constructions transform while preserving the *invariant* geometric properties. This supported Anna's recognition of invariance and patterns. Thus, it promoted an environment for conjecturing, reasoning, and describing the geometric relationships and shapes by drawing different sketches (Battista, 2008; Leung, 2008; Nasim, 2008; Ng & Sinclair, 2015; Ng, 2016; Vrahimis, 2016).

In this study, I introduced an innovative methodology to analyze video data. I explained and exemplified how to identify categories and codes. Then I used this method to analyse a particular mathematical interaction in a digital touchscreen environment. I adapted Arzarello et al.'s theory of touch-based interaction, but specified active and basic actions for the GSP BlackBox task. I utilized StudioCode software to track identified active actions on a prospective teacher's interactions with the task.

By timeline analysis, evidence was built that demonstrated the prospective teacher's high level of bodily engagement with the tool. Also, modes of active actions gradually moved from pushing and rotating (CCW or CW) to drag-touch-to-approach and drag-touch-free, when she interacted with the iPad and task to draw a conclusion, and explained her reasoning. Her conjecture of dependency between the direction of rotation and the centre of rotation failed. Evidence of tool fluency and a new form of embodied engagement with the tool consisted of explicit transformations in the learner's lived bodily engagements in mathematical practice. I illustrated how Anna's active actions transformed from discordance between perceptual and motor aspects of learning, to a holistic sense of unity and flow in terms of mathematical tool fluency.

Also, GSP appears to have supported Anna's lived experience in learning geometric transformation (rotation here) with tactile and visual feedback. For instance, a new conceptual aspect of rotation emerged from touchscreen-based interaction: that the centre of rotation does not have to be visible for learners to identify a transformation as a rotation. Notably, GSP's feature of tracing could support retention. I suggest that Anna's emerging perception and motor skills, characterized here as overlapping active actions, can become finer, overt and covert motor actions in her lived experience. This is not to argue that geometric understanding and protention are entirely dependent upon developing an understanding of rotation or use of touchscreen technology; but rather to point out that developing such an understanding could be implicated in interacting bodily with a mathematical tool.

I suggest that further research using different theoretical frameworks or DGE may extend the methodological aspects of this investigation. For example Sfard's (2008) Comognition theory of analysing mathematics discourse (word use, visual mediators, endorsed narratives, and routines) can be used to analyse mathematical thinking and learning by identifying changes in discourse over time.

Chapter 5. Conclusion

The three studies presented here focus on different aspects of the role of bodily interactions on mathematical tools, and therefore learning mathematics within a perceptuomotor integration approach. The first and last studies focused on learning mathematics via a touchscreen-based device, while the second study investigated the role of tactile graphs in teaching a pre-calculus course to a blind learner. I initially concentrated on the role of mathematical embodiment in learning mathematics. Having a non-dualistic view of mathematical instruments and beginning with an assumption of intertwined perceptual and motor aspects of tool use as perceptuomotor integration, I attended to the bodily gestures, activities and interactions with the mathematical instruments. This dissertation followed the statement that the interactions exhibited thinking not as a process that takes place 'behind' or 'underneath' bodily activity, but is the bodily activity itself (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013). Also, informed by my studies, I have broadened the notion of what constitutes a mathematical tool to include the body and extensions of embodiment in interacting with the world.

5.1. Mathematics Learning and Tool Fluency: The Role of Touch and Sight Interactions

Learning through a sense of touch and direct manipulation of a mathematical object is one of the common themes among my three papers, either by touchscreen-based device or physical manipulatives. So, I appeal to the body or research on embodiment that suggests that learning is affected by our interaction with the environment and embodied interaction involves different human senses. In addition, interacting with the digital device is rooted in embodied cognition and bodily interactions. Embodied interaction is when the users create, communicate, and share meaning through their interactions with the system. In other word, mathematical tool use intensely involves the use of human sense, and therefore, the use of human sense and body can be seen as the most natural state of user interfaces, when playing with mathematical tools (Dourish, 2001). A coordinated touch-see-bundle of sensations involved in the BlackBox and touch-see-hear in *TouchCounts* provided immediate feedback to the learner. It has been suggested that this kind of immediate feedback, received through the hands (and other senses), may allow for better learning for students who directly manipulate mathematical objects, compared with learners who could not manipulate objects on the screen directly (Chan and Black, 2006).

In this regard, new touchscreen-based technologies suggest novel opportunities including multimodal senses such as touch, auditory, and physical movement. These can benefit learning in contrast to the less direct, somewhat inert and passive mode of interactions by pencil and paper, or mouse and keyboard. For example, in my pervious chapter when the prospective teacher controls the pace, speed, direction and magnitude of the shape that she is creating, she is actively engaged and participating in the meaning-making: exploring the mysteries, which yield a better understanding of the geometrical rotation concept.

More specifically, in the first and last studies, utilizing dynamic geometry environment (DGE) on a touchscreen-based device (iPad) enabled learners to directly interact with the mathematical entities. That is because, in touchscreen-based DGEs a hand's actions are directly related to the affective modes of communications and are informed by the modes of continuous dynamic changes. Also, the objects' manipulation/creation takes place less through mediated impacts. In the touchscreen-based DGEs the "mediation" term is no longer accurate enough to capture the role of hands and fingers on the screen. That is because modes of interactions qualitatively differ from computer-based interactions' input modes. In the same way, Angel and Gibbs (2013) argue:

> ...electronic environments have a strong relationship to the affective modes of communications, highlighted through their appeal to sensory novelty through technological innovation—new media platforms proliferate the potentials for combining visibility with aural and tactile modes (as cited in Sinclair & de Freitas, 2014, p. 354).

The vital role of hand in learning mathematics is investigated greatly. As an example, Zaporozhets (2002) studied three- to five-year-old children's learning to discriminate variants of triangles and quadrilaterals. He found substantial number of errors at the beginning. So, the team of research invited children to systematically trace outline of the figures with a finger, attending to (a) directional changes of the motions at the vertices, (b) accompany such a tactile examination with a side counting. The study showed the tactile experience at this stage accomplished, while the eye performed and auxiliary role. Zaporozhets explained later: "the eye developed the ability to solve these types of perceptual tasks independently, consecutively tracing the outline of a figure, as it was earlier done by a touching hand" (2002, p. 31). Zaporozhets described the transformative change of the eye as: "initially, the eye motions have an extremely extensive nature, consecutively tracing the entire outline of the perceived figure and simulating its specifics in all details" (p. 32). Consequently, in a next stage, the eye's motions "gradually begin to decrease and to focus on the individual, most informative attributes of the object" (p. 32).

Throughout my thesis, although the first and last studies looked at different target sampling populations, both highlighted the role and the level of hand and eye engagement in response to the use of mathematical instruments. To understand the level of engagement between hand and eye, I appeal to Deleuze's (2003) idea of relationships between hand and eye. Sinclair and de Freitas (2014) also first used Deleuze to conceptualise the relationship between hand and eye in the context of touchscreen interaction.

In the book "The Logic of Sensation," Gilles Deleuze (2003) explored and analysed the work of the English painter Francis Bacon. In considering Bacon's art, Deleuze offers implicit and explicit insights into the origins and development of his own aesthetic and philosophical ideas. Enlightening Bacon's paintings and the act of painting itself, Deleuze points beyond painting toward connections with other art forms such as music and cinema. Deleuze (2003) defines four relationships between eye and hand, which he names digital, haptic, manual and tactile. These relations range from maximum to the minimum degree of subordination of the eye to hand respectively.

> To describe the relationship between the eye and the hand, and the values through which this relation passes, it is obviously not enough to say that the eye judges and the hands execute. The relationship between the hand and the eye is infinitely richer, passing through dynamic tensions, logical reversals, and organic exchanges and substitutions […]. There are several aspects in the values of the hand that must be distinguished from each other: the digital, the tactile, the manual proper, and the haptic (Deleuze, 2003, p. 124).

Adopting Deleuze (2003), and building on Sinclair and de Freitas's (2014) study, I will refer to these relationships by way of a spectrum marked with a red colour for the maximum eye domination over the hand, to the blue for the domination of hand over the eye (shown in Table 5.1). The spectrum is chosen to emphasise (a) the blurred borderlines between categories and (b) the difficulty of determining exactly when the relationship between hand and eye changes.

Table 5.1 Four relationships between hand and eye introduced by Deleuze (2003).

Digital	Haptic	Manual	Tactile
Eye dominates the hand	Equal contribution of eye and hand	Hand more dominates the eye	**Hand** dominates the eye
	First Study		
Finger-show	Finger-touch	Finger-count	
	Second Study		
N/A	N/A	N/A	Active perception & Active sensation
	Third Study		
Exploratory active actions e.g. Rotate (CCW/CW) Push (toward/along/ away)	Drag-touch-to-approach Drag-touch-free	Basic actions Or dragging a point with less focus on the traces	

The analysis of interactions on the multitouch technology regarding streams of perceptuomotor integration could provide an opportunity to examine how the relationships between eye and hand fluctuate. For example, as shown in table 1, hand subordinated the observant eye in Deleuze's "digital" sense when Alex carefully finger-showed (p. 31) in *TouchCounts,* or Anna performed exploratory interactions in BlackBox, such as rotating--CCW/CW-- and push--toward/along/away (pp. 88-89). Conversely, when the learner taps the screen accidentally or drags the point with less concentration on the consequence or effects, she enables a “manual” relationship. For example, the predominance of hand-over-eye in the studies provides a visual, tangible trace of the “manual”. However, these gestures are more than indexical. Subsequently, the parallel contribution between hand and eyes (and perhaps other senses), in Deleuze’s “haptic” (in combination with tool fluency) may indicate the integration of the motoric and perceptual aspects of learning. In addition, in *TouchCounts* and GSP, the left traces are important to highlight the ongoing process of retentions and protentions.

But of course, such a situation involves more than the eye and hand. Sinclair and de Freitas also include the ear. They exemplified how the hand of a fiver-year-old girl (Katy) was subordinated to the ear when she was tapping on *TouchCounts* and looking up. In that moment the ear dictated the next tap. So, Katy’s hands were subordinated to auditory judgment. When several fingers touched the screen at once and caused TouchCounts’ voice to suddenly jump up to a bigger number, her eyes were drawn back to evaluate the situation. Further, the authors explained that even while the eye had overlooked the initial gesture, an unexpected trace on the screen could be announced by ear.

In my second study, Anthony’s interactions with the tactile graphical tool, his hands were not subordinated at all by the eye and the relationship was a purely “tactile” one. However, other senses such as hearing and touching were involved. Therefore, in this case the haptic involvement may not refer to the sense coordination between eye and hand, but between touch, hearing, and body gestures. In this study, tactile perception and tool fluency explains this sense coordination. A visually impaired individual constantly traces different components of a tactile graph with his fingers, or listens to aural instructions. When doing this, the learner refines their finger movements accordingly. The active sensation of a discordance between perceptual and motoric activities translates to a coordinated form of ‘tactile perception.’

Sinclair & de Freitas (2014) explain learning in a multitouch environment, expanding Deleuze’s hand-eye coordination:

..This seems part of the generative nature of these multitouch environments that do not overly fix interaction (as is the case in many prescriptive educational Apps currently available) and that invite movement not only between Deleuze's different hand-eye relationships, but also with new relationships involving the ear. And we suspect that these are the kinds of encounters that allow for learning (p. 371).

In short, the expanded model of eye-hand-ear coordination in Sinclair and de Freitas (2014) can clarify another aspect of perceptuomotor tool fluency. That is to say, under perceptuomotor integration the observant eye/ear moves from being subordinated to the hand, to become integrated with other senses in the use of the mathematical tool. For example in the first study, tool fluency and perceptumotor integration can be explained when *finger-touching* emerges as coordinated finger-showing (digital) and finger-counting (manual). See table 5.1.

In the next section, I present summaries of the three studies. These summaries not only highlight the common themes across three papers but also address my research contributions in each study. Then, implications of the studies as well as limitations, and suggestions for the further researches are discussed.

5.2. Summary of the results

5.2.1. First Study

The first study "Exploring Cardinality in The Era of Touchscreen-Based Technology"[15], explored how a fifty six-month-old boy named Alex, learned cardinality through using a multimodal, touchscreen-based interface on an iPad application called *TouchCounts*.

The most prominent focus of the study was the role of hands in the development of number sense, as well as the role of *TouchCounts* in preserving this development in a digital world. In *TouchCounts* fingers and gestures are used to transform touches (and taps) to be counted, to summon numbers into existence and to operate on them.

[15] The first study was co-authored with Dr. Stephen Campbell.

TouchCounts also preserves various modalities of verbal counting, numeral notation and finger-counting to represent an ordinal or cardinal number.

The research questions pertaining to this study aimed to examine:

- How does the child develop tool fluency with *TouchCounts*?
- In what ways might tool fluency implicate and be implicated in the child's understanding of cardinality?

We utilized Nemirovsky et al.'s (2013) perceptuomotor integration, Vergnaud's (2009) definition of cardinality, and a Husserlian descriptive phenomenological attitude (1991) to conduct an in-depth case study, when Alex was developing mathematical expertise and tool fluency using *TouchCounts.* We reported on Alex's emerging perception and motor integration: from not being able to make a six out of necessity of sequentially, to the development a successful six by using his twin's fingers and *colliding* two herds of threes. The discussion progressed through analysis of three different episodes when Alex was playing with cards and *TouchCounts* in the classroom setting. We found Alex's gradual increase of perceptuomotor skills and an improved understanding of cardinality, accompanied by a unified and holistic continuity between his temporal flows in experiential time. Our study highlighted playing with *TouchCounts* preserves what Vergnaud (2009) called 'effectiveness of counting strategies' as the one-to-one-to-one correspondences amongst the movement of fingers, eyes and words represents 'three different repertoires of gestures'.

We distinguished between showing a cardinal number (1-10) on young learners' fingers, or finger-showing (known as finger-montring ["montring," is the original spelling]); obtaining a number via finger-counting, which links an ordinal process; and the development of both in creating numbers on *TouchCounts* via *Finger-touching*. The role of *TouchCounts* in transforming children's finger-showing and finger counting to finger-touching through card play also was addressed. This could be seen as the transition from Deleuze's (2003) "digital" relationship to the "tactile". *Finger-touching* in *TouchCounts* permitted Alex to coordinate his sense organs— eye, hand, ear — and gestures in different modes: e.g., number of fingers touches the screen in form of taps, number on the created herd on the screen, number name such as three and spoken “three”. So, the results highlighted the integration of a perceptual understanding of cardinality (e.g.: for 3) and motor activities that manifested in terms of successful *finger-touching*. In other words, the perceptuomotor aspects of operating on numbers on *TouchCounts* are tangled up with the concepts of ordinality and cardinality.

5.2.2. Second Study

The second study explored **“**Advanced Mathematics Communication Beyond Modality of Sight”. Being one of the few studies at university level, in this research I explored how assistive technology and an innovative method of tactile graphing could enable a blind undergraduate student, named Anthony, to learn pre-calculus concepts. In this study, I discussed some of the problems that Anthony encountered during the lecture, while being tutored or while accessing the course’s written and pictorial materials (graphs). The study aimed to answer the following questions:

A. How do mathematical tools and resources (such as tactile graphs, screen readers, etc.) make mathematical communication and learning possible for the blind learner in pre-calculus courses?

Also, to find how tactile mathematics tools support the process of learning that coordinates the body in mathematical activity, I explored:

a) How do the emergence of the blind student's bodily activities and gestures embody and express mathematical learning?

To answer the first research question, I discussed how using Braille, as the tactile writing system was not the optimal choice for the pre-calculus written materials. Then, I introduced alternatives of JAWS and VoiceOver to read the written digital materials and computer screen as well as Nemeth Coding and LaTex to communicate mathematically, when needed.

In this research, I uncovered some of the very challenging problems facing the visually impaired student in teaching and learning mathematics, as below:

- reading and comprehending printed/drawn mathematical graphs in absence of a sighted assistant;
- strong reliance of mathematical communication on gestures, body language, pointing, etc.
- enormous use of deixis by sighted mathematics community counterparts when referring to a mathematical graph or its parts.

Anthony faced these difficulties in various forms: in reading a textbook's pictorial information and graphs, on sketched figures on the board during the lecture time or tutoring sessions, when he was doing exercises at home, and tests or quizzes at school. To tackle this challenge, we invented two methods for graphing: Sketch graphing and Permanent graphing.

The Sketch graphing enabled Anthony and his assistants/teachers to quickly and efficiently draw a tactile graph during the lecture, tutoring time, or test. Permanent graphing empowered Anthony to comprehend and read a drawn graph with all given details, presented just like the original resource (textbook, class notes, etc.) in absence of a sighted assistant.

In the second part of this chapter, I explored how using the tactile graphs as a mathematical tool supported Anthony's learning. I detailed and analysed Anthony's lived experience when he was verbally and gesturally describing the given rational function graph's behavior. The emergence of Anthony's coordination of perceptual and motor activities was illustrated and his temporal flows of perceptuomotor activities – inhabited bodily and interpersonally in experiential time – were analysed (Husserl, 1991; Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013; Nemirovsky et al. 2011).

The results showed Anthony's tool fluency as the enactment of his body orientation and appropriate use of the tactile mathematical instrument. I found changes in the scale and modality of mathematical activity as an historical disruption of relatively stable, culturally-understandable demonstrations of a rational function's behaviour with great precision. That is to say, the tool fluency phenomenon was fulfilled through a fluent demonstration of a function’s behaviour that was a “culturally recognized creation” by the members of “mathematical communities” (Nemirovsky’ et al. 2013 p. 373).

My direct contribution on this study consisted in identifying some obstacles that a blind undergraduate student encountered in learning pre-calculus concepts and investigating possible aids to assist his mathematics learning at university level. I also invented tactile graphs (sketched and permanent graphs) and examined their functionality in different contexts to provide readable graphs for the blind learner.

Theoretically, the study suggested the transition from active sensations to tactile perception as a sign of tool fluency. While active sensing refers to controlling the finger movements (Gibson, 1962) while contacting a stimulus, tactile perceptions were evidenced when Anthony’s acquired information formed the tactile graphs, and his learning guided his gestures and body coordination (in the environment) when demonstrating a rational function’s behaviour (Lapora, 2015). Also, Anthony’s tactile perception revealed his understanding of a rational function as a dynamic entity that comes from $-\infty$ and moves toward $+\infty$, despite the static nature of static tactile graphs, per se.

5.2.3. Third Study

The third study, entitled ”Touchscreen-Based Technology in Exploring Geometric Transformation: Use of Timeline as an Analytical Tool” addressed following questions:

- What are the types of interactions in a touchscreen DGE geometrical context for BlackBox?
- How do customized expansions on Arzarello's (2014) codes of touchscreen-based DGE interactions clarify emergence of tool fluency on a mathematical instrument?
- How does a prospective teacher learn geometric transformation via interacting with the touchscreen-based GSP?

In this study, I discussed how a prospective teacher named Anna learned geometric transformation via direct interactions with a touchscreen-based dynamic geometry environment (The Sketchpad Explorer[16]). Anna, who had no experience playing with a touchscreen-based device, was asked to identify the type of geometric transformation in the given task, named BlackBox. I adapted Arzarello et al.'s (2015) theory of touchscreen interaction to identify different modes of actions: basic actions, and active actions. The basic actions were defined as the mode of interaction with the touch interface, while the combination of basic actions and performed finger interactions were categorized as active actions. I extended Arzarello's modes of interactions and defined new codes in terms of perceptual and motor integration, rather than cognitive domain of mathematical thinking.

The results showed touchscreen-based DGE maintains geometrical relationships between components of shapes by offering continuous and real-time transformations. It also allowed direct interactions with the geometric objects with the hand. Thus, it prompted an environment for conjecturing, reasoning, developing explicit descriptions of geometric relationships and shapes by drawing different sketches, and tracing their effects even with no explicit instruction (Battista, 2008; Leung, 2008; Nasim, 2008; Ng & Sinclair, 2015; Ng, 2016; Vrahimis, 2016).

Also, the analysed data indicated Anna's tool fluency, exhibited in the form of accelerated active actions (drag-touches and rotations) combined and co-joint with her verbal explanations. The result suggested Anna's active actions transformation from discordance between perceptual and motor aspects of learning, to a holistic sense of unity and flow in terms of mathematical tool fluency (Nemirovsky, 2013).

My contribution in this study consisted in suggesting new methodology for analysing video data. Appealing to Nemirovsky et al.'s perceptuomotor integration theory allowed me to trace the emergence of tool fluency in terms of integration of temporal streams of perceptual and motor activities, using a detailed analysis of the video's

[16] iPad application that enables users to manipulate sketches created using The Geometer's Sketchpad.

timeline as an analytical tool. I used the video timeline to trace the Arzarello's adapted codes in video data, which enabled me to trace the paths of interactions and discuss the emergence of tool fluency in terms of developing active actions over the stretch of time. With the suggested methodology, I analysed a four-minute episode concentrated on the touchscreen-based communications for Anna, as she thought aloud to solve a geometric rotation task.

Taking the three studies together, I found a high degree of embodied mathematics and temporal coordination of the capacities of sense organs (hands, ears and eyes) in response to the mathematical instrument. Bodily interactions with mathematical instruments navigate the edge of the actual and the potential: the potentiality of the body's contribution vs. the actuality of the designed instrument. Thus, in different touchscreen-based DGEs, inventive instrumental gestures tap into the potentiality of the body's engagement and reconfigure various relationships between different sensations, respectively (Sinclair & de Freitas, 2014). For example, *TouchCounts* leaves visual and aural traces on the fingers' path on the screen. It also engages the potentiality of small physical gestures: the actuality of a pinch, for instance, is a metaphor for addition. The touch gestures make concrete quantities for young children, for whom those quantities are still "abstracts" (Sinclair & de Freitas, 2014).

The BlackBox in the third study also left visual traces on the screen and involved gestures, touch and sight, however in this study, the learner engages primarily with the visual modes: to what changes or remains invariant. Also, touch gestures made rapports with unitary traces, sometimes blooming into artistic practice. More specifically, the traces manifested the geometric relationships that could be perceived "as *invariance* in the *continuous* moving of the draggable" point, which supported the learner's noticing of invariance (Battista, 2008, p. 350), for example, in this case, the centre of rotation. However, for Anthony the absence of sight and active sensation of tactile graphs was morphed into an embodied tactile perception, providing an abstract image of the graphed function. Hand gestures and tactile diagrams were mutually implicated in showing a function's behavior, which evidenced the integrated motor-sensorial and conceptual activities (Healy, 2011; 2014; 2015; Sinclair & de Freitas, 2014).

Evidently, Alex, Anna and Anthony's thinking and learning with the mathematical tools started from a discordance between motor and perceptual activities. Like Zaporozhets (2002) in this early stage, I observed the privileging of Deleuze's 'tactile' experience and 'touch' perceptions, over sight. For the participants (excepting Anthony), the eye performed an auxiliary role. The eye within this role were to coordinate perceptual and motor activities, to judge what is the next, and inform hand to adjust its movements.

Alex's bodily orientations were brought from a *finger-counting* and *showing* to a *finger-touch*; for Anthony, they developed from an *active sensation* to a *tactile perception*; and for Anna from rotation and push actions to *drag-touch-to-approach* and *drag-touch-free* active actions. This is also where the participants harmonized words and involuntary bodily activities (motoric), providing evidence of their ability to anticipate the next step of the task (perceptual). Thus, their motor activities were involuntary and enacted as a part of perceiving. This is where the perceptuomotor integration met Husserl's phenomenological attitude again.

Also, direct manipulation via touchscreen-based device allowed back and forth interactions between hands, eyes, and ears for gathering meaning, forming undergoing retentions and anticipations. These shifts in thinking could be the result of attention to the relationships between hand, eye and ears in producing gestures in DGEs. They also could be, theoretically, the result of an "unmet protention," the unanticipated result of a miscalculated "retention" in ongoing sequential time (i.e., learning from mistakes). Therefore, the continuous growth of embodied skills, and integration of perceptual and motor activities through emerging paths of lived experiences within the social and environmental contexts forms ideas and learning, without being or end (Ingold, 2016; Nemirovsky, 2017).

Such interactions highlighted the roles of mathematical instruments in the evolution of our ways of sensing and reflecting, as well as the fundamental role of instruments in the ways we come to know and arrive at fluency of use. In the first and last studies, involving DGEs, I found DGE prompts to be an environment for reasoning, conjecturing, and explicit description of geometric or arithmetic relationships. The tactile graphs played the same role for Anthony.

I also discussed touchscreen-based/tactile interactions with the mathematical objects/graphs provide flexibility in the learning of mathematics. For example, the continuous and real-time interactions with *TouchCounts* enabled Alex to develop understanding of cardinality from counting numbers sequentially. Alex's constant use of fingers in a cycle of finger-counting and finger-showings resulted in the successful finger-touching using *TouchCounts*. For the second study, the analogous continuous use of fingers in touching and creating tactile graphs were revealed in Anthony's use of arms and hands in explaining the behaviour of a rational graph through his tactile perception.

Evidently, touchscreen GSP well-treated invariants in the continuous moves of points through touch interactions. It supported Anna's noticing of the visual modes of what changed or remained invariant. Anna's "unmet protention" in examining her conjecture discontinued her touchscreen-based interactions. Such interactions guided Anna to understand that there is no relation between the direction of her active actions and centre of rotation.

5.3. Challenges and Limitations

I recognize some limitations of my studies in terms of video quality, selecting the setting, and participant recruitment. Also, I faced limitations in verifying the key-codes in the third study.

One of the common challenges that wove through all three studies regarded the quality of recorded video data. In my studies, video recording was the method of data collection. Taking a perceptuomotor integration theory lens, I needed to consider high-quality video data where the participants' interactions with the mathematical tool (and other participants, if applicable) were well-recorded and where the mathematical tool and individual interactions were displayed clearly. For example, in the first study, which took place in a daycare, sometimes there were too many children crowded around the iPad, which made recording the interactions that were happening on the screen impossible. There are also specific limitations associated with each study that I will now elaborate:

With the first study, the daycare was not randomly chosen (although I am not generalizing the findings). In the selected setting, most of the children came from educated families with a stable economic background. In addition, the interviewer (Dr. Sinclair) is a university professor and not a daycare teacher. So, conducting the study in a random classroom with a kindergarten teacher could yield different results.

For the second study, I had access to only one participant. The recruitment of visually impaired students who were willing to register for a mathematics course at the university was very challenging. So, I could not examine findings with other visually impaired students. That was because of the very limited number of undergraduate students interested in taking mathematics courses. Also, it was a very difficult task to inform prospective participants about the possible assistive materials and aids that we could offer to them if taking a mathematics course. Because of ethics restrictions, all the communications had to be made through the centre for students with disabilities (CSD). In addition, acquiring ethics permission was very time-consuming, which made us unable to start the study when the course started. Ethics restrictions also did not let me record Anthony's involvement with the course materials in session or during an actual test or quiz.

In the third study, only the students who received the lowest marks in a geometric transformation quiz were invited to participate in the study. I encountered the same limitations as in the second study, where Anthony was the only available participant. Also, the interviews did not take place in an actual classroom setting.

Another limitation of the third study was adapting and identifying the key-codes to trace their development and occurrence in the video data. I identified and analysed video data for Anna's interaction with the rotation task on the iPad (touchscreen DGE). However, these individual codes may not apply to the other participants and studies. Also, other codes may be revealed if the same task is used in the collaborative group work instead of individual exploration.

5.4. Suggestions for Further Research

The findings presented in this dissertation are far from providing a complete explanation about the contribution of the body – especially fingers, eyes, and ears – in learning mathematics. They rather contribute to a new path for the studies in the embodiment of mathematics with a focus on the coordination between finger, eyes, ears and touchscreen-based DGEs. While the case studies described here were meant to investigate mathematics learning in the form of tool fluency and body coordination, each alternative way focused on a different aspect. The chosen small sample size in my thesis helped me to focus on qualitative case study research exploring for possible phenomena of interest of emerging mathematical tool fluency in different context, not a quantitative consideration of the probability of occurrences of such phenomena. Considering larger groups of participants may result in complementary findings. So, future studies with larger sample sizes, in actual classroom settings, should further investigate the possible benefit of using touchscreen-based DGEs on teaching and learning mathematics.

Also, the first study took place in a kindergarten, while the researcher who conducted the interview was not a kindergarten teacher. So, further studies where *TouchCounts* is utilized by an elementary or kindergarten teacher are suggested. In addition, using multiple iPads, or even iPad Pro instead of iPad (with a larger screen size than iPad) may bring a greater level of collaborative engagements among the students. Also, further investigations on two-digit number combinations and creation using *TouchCounts* are suggested.

For the second study, I invited the participant by snowball sampling. I also was unable to find further participants to investigate the functionality of the suggested methods on their learning. It remains for further research to shed some light on the presented tactile inventions for blind students, and adopt and extend them in other undergraduate coursework.

In the third study, I examined mathematics learning in terms of tracing adapted codes based on Arzarello's active actions (2014) in a very small excerpt of videos, in which temporal flows of perceptual and motor activities were observed. I offer the methodology and analysis as a starting point for further studies. Adapting my suggested methodology in a large data set, would enable a team of researchers to collaboratively analyse and draw evidenced-based conclusions.

On a practical level – in light of the evidence collected and presented to date – I think that a compelling case can be made for integrating touch-based technologies into a broader curricular approach at different levels. For instance, *TouchCounts* appears to be a particularly powerful tool for teaching and learning counting and adding. Tactile graphs seem to be an effective tool in graphing or understanding complicated and advanced mathematical graphs and figures for visually impaired students. Using a video timeline as an analytical tool provides strong theoretical and practical components to be adopted for video analysis, not only in mathematics education but other fields such as medical, physical education, etc.

Theoretically speaking, I have extended the perceptuomotor integration approach to trace mathematics learning in terms of tool fluency. The chosen theoretical lens was helpful in speaking about the potential and the active role of the multitouch screen technology (*TouchCounts* and GSP) as well as describing active and dynamic sensation transformation toward tactile perception in learning mathematics for blind learners. The rationale was rooted in the idea that learning entails an interpenetration of the perceptual and motor aspects of activity with a tool, and this interpenetration is part of developing fluency with this tool. Husserl's phenomenological descriptive attitude provided a rich framework to analyse temporal flows of perceptual and motor activities. That is where I identified and adapted perceptuomotor integration in the form of unity, coordination, and harmony of bodily activities while using mathematical instruments by introducing different notations. For example, *finger-touch* and different modes of *active actions* for sighted learners are in line with Deleuze's (2003) notion of a 'tactile' relationship between eye and hand in the first and last studies, respectively. In addition, the expanded Arzerallo et al.'s (2014) modalities (modes of active actions) and the systematic video coding, could be adapted in the first study to categorize types of finger-showing, finger-counting, and finger-touching (see Sedaghatjou & Rodney (2018) for more details).

At the theoretical level, some explanations regarding what exactly learners' "body" entails and how mathematics partakes the body would be helpful. Also more clarity in terms of the implications of the "tool fluency" is needed. For example, if this perspective tool fluency could mean that a master carpenter is by definition an expert geometer by nature of the fact that they can adeptly use tools that pertain to the geometric properties of objects.

Finally, besides exploring the role of touchscreen-based DGEs on learning mathematics for different learners, from young children to prospective teachers, I expanded my knowledge about the role of human sense, especially touch and sight and their coordination in mathematics sense-making. What differentiates my exploration from many others is the minimal role of teacher and instructor as the mediator. In my studies the teacher or researcher-participant plays the role of facilitator instead of director. I hope to combine these findings in future research with my suggested video-analysis methodology to investigate relationships among mathematical tool use, human sense coordination, and identifying modes of interactions to trace mathematics learning.

5.5. Summary

In this chapter, a summary and discussion of the findings of my three studies was presented. The purpose of the studies, guiding research questions, and findings were revisited. Themes that emerged from the studies were summarized and discussed, and the relationships of the findings to the theoretical framework were explored. Also, challenges and limitations as well as and recommendations for future research, were offered.

References

Abrahamson, D., & Sanchez-Garcia, R. (2016). Learning is moving in new ways: The ecological dynamics of mathematics education. *The Journal of the Learning Sciences*, *25*(2), 36. http://doi.org/10.1017/CBO9781107415324.004

Abu Bakar, K., Ayub, M., Fauzi, A., & Ahmad Tarmizi, R. (2010). Utilization of computer technology in learning transformation. *International Journal of Education and Information Technologies*, *4*(2), 91–99.

Ada, T., & Kurtulus, A. (2010). Students' misconceptions and errors in transformation geometry. *International Journal of Mathematical Education in Science and Technology*, 41(7), 901–909.

Andres, M., Seron, X., & Olivier, E. (2007). Contribution of hand motor circuits to counting. *Journal of Cognitive Neuroscience*, *19*(4), 563–576. Retrieved from http://www.ncbi.nlm.nih.gov/pubmed/17381248

Armstrong, T. (1993). *Seven kinds of smart: Identifying and developing your many intelligences*. Plume.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274. Retrieved from http://search.ebscohost.com/login.aspx?direct=true&db=eric&AN=EJ663570&site=ehost-live

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Lati- noamericana de Investigación en Matemática Educativa. Spe- cial Issue on Semiotics, Culture, and Mathematical Thinking*, 9(1), 267-300

Arzarello, F., Bairral, M. A., & Danè, C. (2014). Moving from dragging to touchscreen: geometrical learning with geometric dynamic software. *Teaching Mathematics and Its Applications*, *33*(1), 39–51. http://doi.org/10.1093/teamat/hru002

Radford, L., Edwards, L. & Arzarello, F. (2009). Beyond words. *Educational Studies in Mathematics*, 70(3), 91 – 95. DOI: 10.1007/s10649-008-9172-y

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM*, *34*(3), 66–72. http://doi.org/10.1007/BF02655708

Arzarello, F., Paola, D., & Robutti, O. (2006). Curricular innovation: an example of a learning environment integrated with technology. *Proceedings of 17th ICMI study conference, Hanoi Institute of Technology.* Publiched on CD.

Arzarello, F., Pezzi, G., & Robutti, O. (2007). Modelling body motion: An approach to functions using measure instruments. In W. Blum, P. L. Galbraith, H. W. Henn, & M. Niss (Eds.), *Applications and modelling in mathematics education*. ICMI Study 14

Arzarello, F., & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. In L. D. English (Ed.), *Handbook of International Research In Mathematics Education* (second, pp. 216–245). New York: Routledge.

Arzarello, F., Robutti, O., & Thomas, M. (2015). Growth point and gestures: looking inside mathematical meanings. *Educational Studies in Mathematics*, *90*(1), 19–37. http://doi.org/10.1007/s10649-015-9611-5

Arzarrello, F., Domingo, P. Robutti, O. & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97–109. http://doi.org/10.1007/sl0649-008-9163-z

Bakar, K. A., Ayub, A. F. M., & Tarmizi, R. A. (2010). Exploring secondary school students' motivation using technologies in teaching and learning mathematics. In *Procedia - Social and Behavioral Sciences* (2nd ed., Vol. 2, pp. 4650-4654). DOI: 10.1016/j.sbspro.2010.03.744

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). New York: Routledge.

Battista, M. T. (2008). Development of the shape makers geometry microworld. *Research on Technology and the Teaching and Learning of Mathematics*, *2*, 131–156.*Theses*. University of Guelph (Canada), Canada. Retrieved from http://proxy.lib.sfu.ca/login?url=http://search.proquest.com/docview/304890426?accountid=13800

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 276–283). New York: Routledge.

BC's Revised Curriculum , Mathematics. (2016). Retrieved from https://curriculum.gov.bc.ca/curriculum/mathematics/5

Bell, A. H., Fecteau, J. H., & Munoz, D. P. (2004). Using auditory and visual stimuli to investigate the behavioral and neuronal consequences of reflexive covert orienting. *Journal of Neurophysiology*, 91(5), 2172–2184.

Bender, A., & Beller, S. (2012). Nature and culture of finger counting: Diversity and representational effects of an embodied cognitive tool. *Cognition*, *124*(2), 156–182. http://doi.org/10.1016/j.cognition.2012.05.005

Brannon, E. M., & Van de Walle, G. A. (2001). The development of ordinal numerical competence in young children. *Cognitive Psychology*, *43*(1), 53–81. http://doi.org/10.1006/cogp.2001.0756

Brannon, E. M., & Van de Walle, G. A. (2002). The development of ordinal numerical competence in young children. *Cognitive Psychology*, *44*(2), 191–192. http://doi.org/10.1006/cogp.2001.0780Brissiaud. R. (1992). A Toll for Number Construction: Finger Symbol Sets. In J. Bideaud, C. Meljac & J.-P. Fischer (Ed.) (1992). *Pathways to number. Children's Developing Numerical Abilities.* New Jersey. Lawrence erlbaum Associates.

Burton, H. (2003). Visual cortex activity in early and late blind people. *Journal of Neuroscience*, *23*(10), 4005–4011.

Butterworth, B. (2000). *The mathematical brain*. London: Macmillan.

Butterworth, B., Varma, S., & Laurillard, D. (2011). Dyscalculia: From brain to education. *Science*, *332*(6033), 1049–53. http://doi.org/10.1126/science.1201536

Campbell, S. R. (1998). *Preservice teachers' understanding of elementary number theory: Qualitative constructivist research situated within a Kantian framework for understanding educational inquiry*.(unpublished doctoral dissertation).Simon Fraser University, Burnaby, Canada..

Campbell, S. R. (2002). Constructivism and the limits of reason: Revisiting the Kantian problematic. *Studies in Philosophy and Education*, *21*(6), 421–445.

Campbell, S. R. (2010) Embodied minds and dancing brains: New opportunities for research in mathematics education, In: B. Sriraman & L. English (eds), *Theories of Mathematics Education: Seeking New Frontiers* (Berlin, Springer), pp. 309–331. http://doi.org/10.1007/978-3-642-00742-2

Cattaneo, Z., & Vecchi, T. (2011). *Blind vision: The neuroscience of visual impairment*. MIT Press. Cambridge, MA.

Clements, D. H., & Battista, M. T. (1990). The effects of logo on children's conceptualizations of angle and polygons. *Journal for Research in Mathematics Education*, *21*(5), 356–371. http://doi.org/10.2307/749394

Clements, D. H., &Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420– 464). New York, NY: MacMillan.

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge. http://doi.org/10.4324/9780203520574

Charoenying, Timothy. (2015). *Fostering embodied coherence: A study of the relationship between learners' physical actions and mathematical cognition*. (Doctoral dissertation). UC Berkeley: Education. Retrieved from: http://escholarship.org/uc/item/5wz5f6zh

Chau, C. L. (2014). *Positive technological development for young children in the context of children's mobile apps.* (Doctoral dissertation). Tufts University.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, 31(3-4), 175–190.

Couse, L. J., & Chen, D. W. (2010). A tablet computer for young children? exploring its viability for early childhood education. *Journal of Research on Technology in Education*, *43*(1), 75–98. http://doi.org/10.1080/15391523.2010.10782562

Creswell, J. W. (2013). *Research design: Qualitative, quantitative, and mixed methods approaches.* Sage publications.

Crollen, V., Mahe, R., Collignon, O., & Seron, X. (2011). The role of vision in the development of finger-number interactions: Finger-counting and finger-montring in blind children. *Journal of Experimental Child Psychology*, *109*(4), 525–39. http://doi.org/10.1016/j.jecp.2011.03.011

de Freitas, E. (2016). Material encounters and media events: What kind of mathematics can a body do? *Educational Studies in Mathematics*, *91*(2), 185–202. http://doi.org/10.1007/s10649-015-9657-4

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*, *80*(1), 133–152. http://doi.org/10.1007/s10649-011-9364-8

de Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: the body in/of mathematics. *Educational Studies in Mathematics*, *83*(3), 454–470. http://doi.org/http://dx.doi.org/10.1007/s10649-012-9465-z

Deleuze, G., & Bacon, F. (2003). *Francis Bacon: The logic of sensation*. Trans. by Smith, W., London, University of Minnesota Press.

Dehaene, S. (2009). Origins of mathematical intuitions: the case of arithmetic. *Annals of the New York Academy of Sciences*, *1156*, 232–59. http://doi.org/10.1111/j.1749-6632.2009.04469.x

Di Luca, S., & Pesenti, M. (2008). Masked priming effect with canonical finger numeral configurations. *Experimental Brain Research*, 185, 27–39.

Dick, T., & Kubiak, E. (1997). Issues and aids for teaching mathematics to the blind. *The Mathematics Teacher, 90*(5), 344-349. Retrieved from http://www.jstor.org.proxy.lib.sfu.ca/stable/27970181

Domahs, F., Moeller, K., Huber, S., Willmes, K., & Nuerk, H. C. (2010). Embodied numerosity: Implicit hand-based representations influence symbolic number processing across cultures. *Cognition*, *116*(2), 251–266. http://doi.org/10.1016/j.cognition.2010.05.007

Drijvers, P., & Trouche, L. (2008). From artifacts to instruments: A theoretical framework behind the orchestra metaphor. *Research on Technology and the Teaching and Learning of Mathematics, Cases and Perspectives.*, *2*, 363–391.

Edwards, L., & Zazkis, R. (1993). Transformation geometry: Naive ideas and formal embodiments. *Journal of Computers in Mathematics and Science Teaching*, *12(2)*, 121-145.

Falcade, R., Laborde, C., & Mariotti, M. A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*(3), 317–333. http://doi.org/10.1007/s10649-006-9072-y

Figueiras, L., & Arcavi, A. (2015). Learning to See: The Viewpoint of the Blind. In S. J. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. ,175–186). Cham: Springer International Publishing. http://doi.org/10.1007/978-3-319-17187-6_10

Flanagan, K. (2001). *High school students' understandings of geometric transformations in the context of a technological environment*. (Unpublished doctoral dissertation), Pennsylvania State University.

Flores, A., Samson, J., Yanik, H. B. (2006). Quotient and measurement interpretations of rational numbers. *Teaching children mathematics*, 13(1), 34-39.

Fuson, K. C. (1988). *Children's counting and concepts of number*. (C. J. Brainerd, Ed.), *Cognitive Development* (Vol. 31). Springer-Verlag. http://doi.org/10.1017/s0140525x08005724

Fuson, K. C. (1998). *Children's counting and concepts of number*. New York: Springer.

Fuson, K. C. (1992a). Relationships between counting and cardinality from age 2 to 8. In J. Bideau, C. Meljac, & J.-P. Fisher (Eds.), *Pathways to number: Children's developing numerical abilities* (pp. 127–149). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Fuson, K. C. (1992b). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York: Macmillan

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, *22*(3–4), 455–479.

Geist, E. A. (2012). A qualitative examination of two year-olds interaction with tablet based interactive technology. *Journal of Instructional Psychology*, *39*(1), 26.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Harvard University Press.

Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition*, *13*(3), 343–359. Retrieved from http://www.sciencedirect.com/science/article/pii/0010027783900148

Gibson, J. J. (1962). Observations on active touch. *Psychological Review*, *69*(6), 477.

Ginsberg, D. (2015). *Multimodal semiotics of mathematics teaching and learning*. (Doctoral dissertation). Washington, DC. Retrieved from http://hdl.handle.net/10822/1029900

Gracia-Bafalluy, M., & Noël, M.-P. (2008). Does finger training increase young children's numerical performance? *Cortex; a Journal Devoted to the Study of the Nervous System and Behavior*, *44*(4), 368–75. http://doi.org/10.1016/j.cortex.2007.08.020

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A" proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 26(2), 116–140.

Guin, D., & Trouche, L. U. C. (1998). The complex process of converting tools into mathematical instruments: the case of calculators. *International Journal of Computers for Mathematical Learning*, *3*(3), 195–227. Retrieved from http://www.springerlink.com/index/m154266063r42355.pdf

Hall, R. & Greeno, J. G. (2008). Conceptual learning. T. Good (Ed.), *21st Century Education: A Reference Handbook* (pp. 212-221). Thousand Oaks, CA: SAGE Publications Ltd. doi: 10.4135/9781412964012

Hall, R., Ma, J. Y., & Nemirovsky, R. (2015). Re-scaling bodies in/as representational Instruments in GPS drawing. In *Learning Technologies and the Body: Integration and Implementation In Formal and Informal Learning Environments* (pp. 112–132). New York: Routledge.

Harper, J (2003). Enhancing elementary pre-service teachers' knowledge of geometric transformations through the use of dynamic geometry computer software. In C. Crawford et al. (Eds.), *Proceedings of Society for Information Technology & Teacher Education International Conference* (pp. 2909-2916). Chesapeake, VA: AACE

Harper, S. R. (2002). *Enhancing elementary pre-service teachers' knowledge of geometric transformations*. (Doctoral dissertation). University of Virginia.

Hawellek, D. J., Schepers, I. M., Roeder, B., Engel, A. K., Siegel, M., & Hipp, J. F. (2013). Altered intrinsic neuronal interactions in the visual cortex of the blind. *Journal of Neuroscience*, 33(43), 17072-17080.

Healy, L. (2015). Hands that See, Hands that Speak: Investigating Relationships Between Sensory Activity, Forms of Communicating and Mathematical Cognition. In S. J. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 289–308). Cham: Springer International Publishing. http://doi.org/10.1007/978-3-319-17187-6_17

Healy, L., & Fernandes, A. (2014). Blind Students, Special Needs, and Mathematics Learning. In *Encyclopedia of Mathematics Education* (pp. 61-63). Springer Netherlands.

Healy, L., & Fernandes, A. (2011). The role of gestures in the mathematical practices of those who do not see with their eyes. *Educational Studies in Mathematics*, *77*(2–3), 157–174. http://doi.org/10.1007/s10649-010-9290-1

Healy, L., Ramos, E. B., Fernandes, A., & Peixoto, J. L. B. (2016). Mathematics in the hands of deaf learners and blind learners: Visual–gestural–somatic means of doing and expressing mathematics. In *Mathematics Education and Language Diversity* (pp. 141-162). Springer International Publishing.

Hennessy, S., Ruthven, K., & Brindley, S. (2005). Teacher perspectives on integrating ICT into subject teaching: commitment, constraints, caution, and change. *Journal of Curriculum Studies*, 37(2). 155-192. http://doi.org/10.1080/0022027032000276961

Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior*, *22*(1), 55–72. http://doi.org/10.1016/S0732-3123(03)00004-X

Hoyles, C., & Healy, L. (1997). Unfolding meanings for reflective symmetry. *International Journal of Computers for Mathematical Learning*, *2*(1), 27–59. http://doi.org/10.1023/A:1009721414546

Hulon, S. I. (2015). *Does training in instructional design practices increase a preservice teacher's ability to integrate technology? (Doctoral dissertaion).* University of South Alabama. Available from ProQuest Dissertations & Theses A&I. (1748053884).

Husserl, E. (1991). On the phenomenology of the consciousness of internal time, trans. by JB Brough. *Kluwer Academic*, *39*, 84–88.

Ingold, T. (2014). The creativity of undergoing. *Pragmatics & Cognition*, *22*(1), 124–139. http://doi.org/10.1075/pc.22.1.07ing

Jackiw, N., & Sinclair, N. (2014). *TouchCounts*. [software application for the iPad]. https://itunes.apple.com/ca/ app/touchcounts/id897302197?mt=8

Jackson, A. (2002). Communications: the world of blind mathematicians. *Notices of the American Mathematical Society*, *49*(10), 1246–1251.

Janiga, S. J., & Costenbader, V. (2015). The transition from high school to postsecondary education for students with learning disabilities: a survey of college service coordinators. *Journal of Learning Disabilities*, *35*(5), 462–468, 479.

Jiang, J., Zhu, W., Shi, F., Liu, Y., Li, J., Qin, W., ... & Jiang, T. (2009). Thick visual cortex in the early blind. *Journal of Neuroscience*, 29(7), 2205-2211.

Jordan, N. C., Kaplan, D., Nabors Olah, L., & Locuniak, M. N. (2006). Number sense growth in kindergarten: A longitudinal investigation of children at risk for mathematics difficulties. *Child Development*, *77*(1), 153–175. http://doi.org/10.1111/j.1467-8624.2006.00862.x

Kim, M., Roth, W.-M., & Thom, J. (2011). Children's gestures and the embodied knowledge of geometry. *International Journal of Science and Mathematics Education*, *9*(1), 207–238. http://doi.org/10.1007/s10763-010-9240-5

Klatzky, R. L., Lederman, S. J., & Reed, C. (1987). There's more to touch than meets the eye: The salience of object attributes for haptics with and without vision. *Journal of Experimental Psychology: General*, *116*(4), 356.

Klemmer, S. R., Hartmann, B., & Takayama, L. (2006). How Bodies Matter : Five Themes for Interaction Design, In *Proceedings of the 6th conference on Designing. Interactive systems*. ACM, New York, NY, USA, 140-149. http://dx.doi.org.proxy.lib.sfu.ca/10.1145/1142405.1142429

Klinge, C., Eippert, F., Röder, B., & Büchel, C. (2010). Corticocortical connections mediate primary visual cortex responses to auditory stimulation in the blind. *Journal of Neuroscience*, 30(38):12798-805. DOI: 10.1523/JNEUROSCI.2384-10.2010

Laborde, C. (2000). Dynamic geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics*, 44(1/2), 151–161. DOI:10.1023/A:1012793121648

Ladel, S., & Kortenkamp, U. (2014). Number concepts—Processes of internalization and externalization by the use of multi-touch technology. In *Early Mathematics Learning* (pp. 237–253). Springer. http://doi.org/10.1007/978-1-4614-4678-1_15

Leong, Y. H., & Lim-Teo, S. K. (2003). Effects of geometer's sketchpad on spatial ability and achievement in transformation geometry among secondary two students in Singapore. *The Mathematics Educator*, *7*(1), 32–48. Retrieved from http://hdl.handle.net/10497/61

Lepora, N. (2016). Active tactile perception. In *Scholarpedia of Touch,* (pp. 151–159). Springer.

Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers for Mathematical Learning*, *13*(2), 135–157. http://doi.org/10.1007/s10758-008-9130-x

Leung, A. (2011). An epistemic model of task design in dynamic geometry environment. *ZDM - International Journal on Mathematics Education*, *43*(3), 325–336. http://doi.org/10.1007/s11858-011-0329-2

Li, W. (2013). *Secondary preservice teachers' mathematical discourses on geometric transformations. (Doctoral dissertation).* (Order No. 3616940). Available from ProQuest Dissertations & Theses A&I. (1522760599).

Marcone, R. (2013). Teaching Mathematics for blind students : a challenge at the university. In *the Proceedings of the 12th International Congress on Mathematical Education* (pp. 23–35). Seoul.

Mariotti, M. 2000. Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics. 44*(1/2), 25-53. DOI:10.1023/A:1012733122556

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: The role of the teacher. *ZDM - International Journal on Mathematics Education*, *41*(4), 427–440. http://doi.org/10.1007/s11858-009-0199-z

Mariotti, M. A. (2013). Introducing students to geometric theorems: How the teacher can exploit the semiotic potential of a DGS. *ZDM - International Journal on Mathematics Education*, *45*(3), 441–452. http://doi.org/10.1007/s11858-013-0495-5

Marson, S. M., Harrington, C. F., & Walls, A. (2013). Teaching introductory statistics to blind students. *Teaching Statistics*, 35(1), 21–25.

McClintock, E., Jiang, Z., & July, R. (2002). Students' development of three-dimensional visualization in the Geometer's Sketchpad environment. P*roceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (1/4), 739-754

McNeill, D. (1992). *Hand and mind: What gestures reveal about thought.* Chicago: The University of Chicago Press.

Nemeth, A. (1972), *The Nemeth Braille Code for Mathematics and Science Notation*, (1972 rev.), Produced in braille for the Library of congress, national library service for the blind and physically handicapped by the American Printing House for the Blind.

Miles, M. B., Huberman, A. M., & Saldana, J. (2016). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Califorinia: SAGE Publications, Incorporated.

Miller, K. J., Zanos, S., Fetz, E. E., den Nijs, M., & Ojemann, J. G. (2009). Decoupling the cortical power spectrum reveals real-time representation of individual finger movements in humans. *The Journal of Neuroscience : The Official Journal of the Society for Neuroscience*, *29*(10), 3132–7. http://doi.org/10.1523/JNEUROSCI.5506-08.2009

Moeller, K., Fischer, U., Link, T., Wasner, M., Huber, S., Cress, U., & Nuerk, H. C. (2012). Learning and development of embodied numerosity. *Cognitive Processing*, *13*(1), 271–274. http://doi.org/10.1007/s10339-012-0457-9

Moon, N. W., Todd, R. L., Morton, D. L., & Ivey, E. (2012). Accommodating students with disabilities in science, technology, engineering, and mathematics (STEM). *Atlanta, GA: Center for Assistive Technology and Environmental Access, Georgia Institute of Technology*.

Mowat, E. (2010). *Making connections: network theory, embodied mathematics, and mathematical understanding*. (Doctoral dissertation). University of Alberta.

Nasim, O. (2008). *Bertrand Russell and the Edwardian philosophers: constructing the world*. Springer.

NCTM, *CAEP Mathematics Content for Secondary* (2012). Revised 3/24/2015. (2015), (2012), 3–6. Retrieved from http://www.nctm.org/uploadedFiles/Standards_and_Positions/CAEP_Standards/NCTM CAEP Standards 2012 Mathematics Content - Elementary Mathematics Specialist.pdf

Nemirovsky, R. (2011). Episodic feelings and transfer of learning. *Journal of the Learning Sciences*, *20*(2), 308–337. http://doi.org/10.1080/10508406.2011.528316

Nemirovsky, R. (2017). Inhabiting mathematical concepts. In *What are mathematical concepts? (In press)*

Nemirovsky, R., & Borba, M. (2003). Perceptuo-motor activity and imagination in mathematics learning. In *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 103–135).

Nemirovsky, R., & Ferrara, F. (2008). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*(2), 159–174. http://doi.org/10.1007/s10649-008-9150-4

Nemirovsky, R., & Ferrara, F. (2009). Mathematics imagination and embodiment cognition. *International Electronic Journal of Elementary Education*, *7*(3), 301–318. http://doi.org/10.1007/s

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2012). Gesture and imagination On the constitution and uses of phantasms, *Gesture*, *12(2*), 130–165.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing Mathematical Instruments: Emerging Perceptuomotor Integration With an Interactive Mathematics Exhibit. *Journal for Research in Mathematics Education*, *44*(2), 372–415. http://doi.org/10.5951/jresematheduc.44.2.0372

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2011). When the Classroom Floor Becomes the Complex Plane: Addition and Multiplication as Ways of Bodily Navigation. *Journal of the Learning Sciences*, *21*(2), 287–323. http://doi.org/10.1080/10508406.2011.611445

Newfoundland Minstry of Education. (2015). Motion Geometry: Newfoundland's grade 6 mathematics curriculum guide. *Motion Geometry*, 147–170. Retrived from http://www.ed.gov.nl.ca/edu/k12/curriculum/guides/mathematics/grade6/Mathematics_Grade6_Curriculum_Guide_final.pdfNg, O. (2016). *Language , Gestures And Touchscreen Dragging In School Calculus : Bilinguals ' Linguistic And Non-Linguistic Communication*. (Doctoral dissertation), Simon Fraser University, Burnaby, Canada.

Ng, O., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM*, *47*(3), 421–434. http://doi.org/10.1007/s11858-014-0660-5

Novack, M. a, Congdon, E. L., Hemani-Lopez, N., & Goldin-Meadow, S. (2014). From Action To Abstraction: Using The Hands To Learn Math. *Psychological Science*, *25*(4), 903–10. http://doi.org/10.1177/0956797613518351

Ontario Ministry of Education. (2014). *Paying Attention to Spatial Reasoning, K-12 Support Document for Paying Attention to Mathematics Education*, 1–28. Retrieved from http://www.edu.gov.on.ca/eng/literacynumeracy/LNSPayingAttention.pdf

Parsons, R. R. (1993). *Teacher beliefs and content knowledge: influences on lesson crafting of pre-service teachers during geometry instruction.* Unpublished doctoral dissertation, Washington State University.

Penner-Wilger, M., Penner-Wilger, M., Fast, L., LaFevre, J. A., Smith-Chant, B. L., Skwarchuck, S. L., Kamawar, D., & Bisanz, J. (2007). The foundations of numeracy: Subitizing, finger gnosia, and fine motor ability. In D.S. McNamara and J.G. Trafton (eds), *Proceedings of the 29th Cognitive Science Society* (pp. 1385-1390). Austin, TX: Cognitive Science Society.

Poncelet, J. V. (1864). *Applications d'analyse et de géométrie: traité de propriétés projectives des figures.* Mallet-Bachelier.

Pratt, D., and J. Ainley. 1997. The construction of meanings for geometric construction: Two contrasting cases. *International Journal of Computers for Mathematics Learning,* 1(3), 293-322.

Prensky, M. (2001). Digital natives, digital immigrants part 1. *On the Horizon*, 9(5), 1–6.

Presmeg, N.C. (1986). Visualisation and mathematical giftedness, *Educational Studies in Mathematics*, 17, 297–311.

Quek, F., McNeill, D., & Oliveira, F. (2006). Enabling Multimodal Communications for Enhancing the Ability of Learning for the Visually Impaired. *Proceedings of the 8th International Conference on Multimodal Interfaces (ICMI)*, 326–332. http://doi.org/10.1145/1180995.1181056

Radford, L. (2008). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, *70*(2), 111–126. http://doi.org/10.1007/s10649-008-9127-3

Radford, L. (2013). *Sensuous cognition*. In D. Martinovic, V. Freiman, & Z. Karadag (Eds.), *Visual mathematics and cyber* learning (pp. 141–162). New York, NY: Springer

Radford, L. (2014). Towards an embodied, cultural, and material conception of mathematics cognition. *Mathematics Education*, *46*, 349–361. http://doi.org/10.1007/s11858-014-0591-1

Radford, L., Edwards, L. & Arzarello, F. (2009). Introduction: Beyond Words Introduction. *Educational Studies in Mathematics*, *70*(2), 91–95.

Rahim, M. H. (2002). A classroom use of the Geometer’s Sketchpad in a mathematics pre-service teacher education program. *Retrieved from http://math.unipa.it/~grim/Jrahaim.PDF*.

Resnick, L. B., Bill, V., Lesgold, S., & Leer, M. (1992). Developing thinking skills in arithmetic class. In Demetriou, A,.Shayer, M.,Efklides, A., (Eds.), *Neo-Piagetian theories of cognitive development: Implications and applications for education* (pp. 210-230). London: Routledge.

Roth, W. M. (2010). Incarnation: Radicalizing the embodiment of mathematics. *For the Learning of Mathematics*, *30*(2), 8-17.

Roth, W. (2008). Where are the cultural-historical critiques of "back to the basics"? *Mind, Culture, and Activity, 15*(4), 269-278. doi:10.1080/10749030802390981

Roth, W. M. (2012). Cultural-historical activity theory: Vygotsky's forgotten and suppressed legacy and its implication for mathematics education. *Mathematics Education Research Journal*, *24*(1), 87-104.

Roth, W. M., & Lee, Y. J. (2007). "Vygotsky's neglected legacy": Cultural-historical activity theory. *Review of educational research*, *77*(2), 186-232. DOI:10.3102/0034654306298273

Roth. W. M., & Radford, L. (2011). *A cultural–historical perspective on mathematics teaching and learning*. Rotterdam; Boston;: Sense Publishers.

Roth, W. M., & Thom, J. S. (2009). Bodily experience and mathematical conceptions: from classical views to a phenomenological reconceptualization. *Educational Studies in Mathematics*, *70*(2), 175–189. http://doi.org/10.1007/s10649-008-9138-0

Rusconi, E., Bueti, D., Walsh, V., & Butterworth, B. (2011). Contribution of frontal cortex to the spatial representation of number. *Cortex, 47*(1), 2–13. http://doi.org/10.1016/j.cortex.2009.08.005

Sarama, J., & Clements, D. H. (2004). Building Blocks for early childhood mathematics. *Early Childhood Research Quarterly*, *19*(1), 181–189. http://doi.org/10.1016/j.ecresq.2004.01.014

Sarama, J., & Clements, D. H. (2009). Teaching math in the primary grades: The learning trajectories approach. *Young Children, 64*(2), 63-65.

Sato, M., Cattaneo, L., Rizzolatti, G., & Gallese, V. (2007). Numbers within our hands: modulation of corticospinal excitability of hand muscles during numerical judgment. *Journal of Cognitive Neuroscience*, *19*(4), 684–693. Retrieved from http://www.ncbi.nlm.nih.gov/pubmed/17381258

Sedaghatjou, M. (2010). *Effects of using Cabri3D on Geometric Thinking among Grade Ten Students.* (Unpublished Masters thesis). University Malaya, Kuala Lumpur, Malaysia.

Sedaghatjou, M. & Campbell, S. (2017). Exploring Cardinality in the Era of Touchscreen-based technology (in press). *International Journal of Mathematical Education in Science and Technology.*

Sedaghatjou, M., & Norul Akmar, S. (2012). The Effects of Cabri3D on Geometric Thinking. In *Practices in mathematics and science education: A reflection* (pp. 91–105). Pearson Malaysia Sdn. Bhd.

Sedaghatjou M. (in process).Touchscreen-Based Technology in Exploring Geometric Transformation: Use of Video Timeline as Analytical Too. Journal of the Learning Science.

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. *Gesture*, *14*(3), 351–374. http://doi.org/10.1017/CBO9781107415324.004

Sinclair, N. & Gol Tabaghi, S. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Education Studies in Mathematics*, 74(3), 223–240.

Sinclair, N., & Moss, J. (2012). The more it changes, the more it becomes the same: The development of the routine of shape identification in dynamic geometry environment. *International Journal of Educational Research*, *51–52*, 28–44. http://doi.org/10.1016/j.ijer.2011.12.009

Sinclair, N. & Pimm, D. (2014). Number's subtle touch: Expanding finger gnosis in the era of multi-touch technologies. *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education*, Vancouver, Canada.

Sinclair, N., & Sedaghatjou, M. (2013). Finger Counting And Adding on A Touchscreen Device. *Eighth Congress of European Research in Mathematics Education (CERME 8)*. Ankara, Turkey.

Sinclair, N., & Yurita, V. (2008). To be or to become: how dynamic geometry changes discourse. *Research in Mathematics Education*, *10*(2), 135–150. http://doi.org/10.1080/14794800802233670

Son, J. W., & Sinclair, N. (2010). How preservice teachers interpret and respond to student geometric errors. *School Science and Mathematics*, *110*(1), 31-46.

Spence, C., Nicholls, M. E., Gillespie, N., & Driver, J. (1998). Cross-modal links in exogenous covert spatial orienting between touch, audition, and vision. *Perception & Psychophysics*, 60(4), 544–557. DOI: 10.3758/BF03206045

Tall, D. (2006). A theory of mathematical growth through embodiment, symbolism and proof. In *Annales de didactique et de sciences cognitives* (11),195-215. Retrieved from http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2006e-theory-math-growth-annales.pdf

Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., … Yusof, Y. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Math, Science & Technology Education*, *1*(1), 81–104. DOI:10.1080/14926150109556452

Thaler, L., Arnott, S. R., & Goodale, M. A. (2011). Neural correlates of natural human echolocation in early and late blind echolocation experts. *PLoS One*, 6(5), e20162.

Thaqi, X., Giménez, J., & Rosich, N. (2011). Geometrical transformations as viewed by prospective teachers. In *Seventh Congress of the European Society for Research in Mathematics Education*. Retrieved January (Vol. 5, p. 2011).

Thom, J., & Roth, W. (2011). Radical embodiment and semiotics: *Towards a theory of mathematics in the flesh.* Educational Studies in Mathematics, 77(2/3), 267–284. DOI:10.1007/s10649-010-9293-y

Threlfall, J. (2008). Development in oral counting, enumeration and counting for cardinality. In I. Thompson (Ed.), *Teaching and learning early number* (pp. 61–71). Maidenhead: Open University Press.

Trouche, L. (2005). An instrumental approach to mathematics learning in symbolic calculator environments. In *The didactical challenge of symbolic calculators* (pp. 137–162). New York: Springer.

Turgut, M., Yenilmez, K., & Anapa, P. (2014). Symmetry and rotation skills of prospective elementary mathematics teachers. *Bolema: Mathematics Education Bulletin*, *28*(48), 383–402. http://doi.org/10.1590/1980-4415v28n48a19

Van Manen, M. (2016). *Phenomenology of practice: Meaning-giving methods in phenomenological research and writing*. Routledge.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Vrahimis, A. (2016). Bertrand Russell and the Edwardian philosophers: Constructing the world, by Nasim, O,. *Mind*, 125(498), 572-575. DOI:10.1093/mind/fzv14

Vérillon, P., & Raberdel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education*, *10*(1), 77–101.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, *52*(2), 83–94. http://doi.org/10.1159/000202727

Vogel, R., & Jung, J. (2013). Videocoding – a methodological research approach to mathematical activities of kindergarten children. *Eighth Congress of European Research in Mathematics Education (CERME 8)*. Ankara, Turkey.

Vygotsky, L. (1986). *Thought and language*. (A. Kozulin, Ed.), *October* (Vol. 28). MIT Press.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.

Wang, S. (2011). *The Van Hiele theory through the discursive lens: Prospective teachers ' geometric discourses*. (Doctoral Thesis). Michigan State University.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–636. http://doi.org/10.3758/BF03196322

Wynn, K. (1992). Children’s acquisition of the number words and the counting system. *Cognitive Psychology*, *24*(2), 220–251. http://doi.org/10.1016/0010-0285(92)90008-P

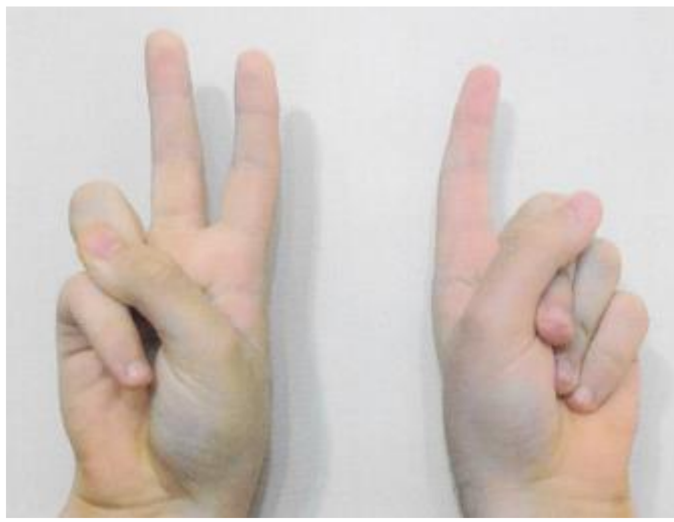
Yanik, H. B., & Flores, A. (2009). Understanding rigid geometric transformations: Jeff's learning path for translation. *The Journal of Mathematical Behavior*, 28(1), 41-57.

Yu, P., and J.E. Barrett. (2002). Shapes, actions, and relationships: A semiotic investigation of student discourse in a dynamic geometric environment. In *Proceedings of the Twenty-Fourth Conference of PME-NA*, eds. D. S. Mewborn, P. Sztajin, D.White, H.Wiegel, R. Bryant and K. Nooney, 775–84. Columbus, OH: Clearing House for Science, mathematics, and Environmental Education.

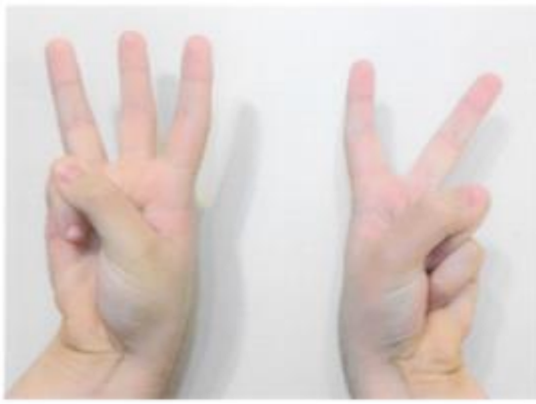
Zaporozhets, A. V. (2002). The development of sensations and perceptions in early and preschool childhood. *Journal of Russian and East European Psychology*, 40(2), 22-34

Appendix A.
Different finger's configurations (finger-showing)

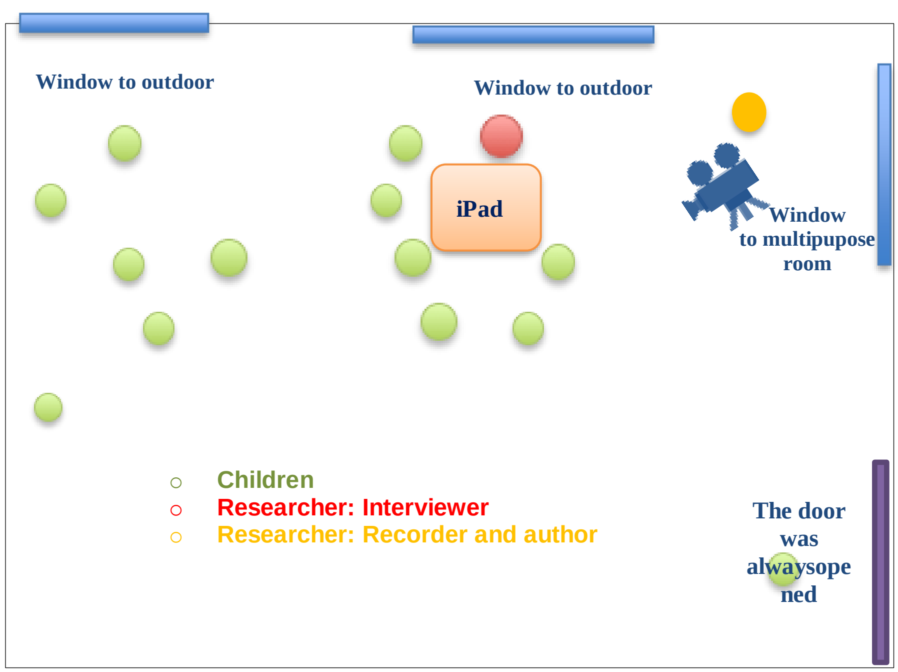
Different finger-showing gestures were used in the card play. Children were asked to create numbers shown on the card on *TouchCounts*.

Below is an example of different finger combinations of finger-showing for five.

Appendix B.
Daycare's floor plan and classroom arrangement



Appendix C.
A detailed schedule for the first study at daycare

Day Number	Date	Duration of intervention(min)
First interview	Summer 2013	
Day One	Jan 29th 2013	57
Day Two	Feb 14th 2013	56
Day Three	Feb 26th 2013	39
Day Four	Mar 14th 2013	36
Day Five	April 17th 2013	62
Day Six	May 2nd 2013	55
Day Seven	Jan 24th 2014	48
Day Eight	Feb 1st 2014	64
Day Nine	Feb 26th 2014	58
Day Ten	April 2nd 2014	65
Day Eleven	April 9th 2014	63
Day Twelve	April 23rd 2014	50
Day Thirteen	May 21st 2014	49
Day Fourteen	May 28th 2014	37
Day Fifteen	Jun 11th 2014	40
Day Sixteen	Jun 24th 2014	38
Day Seventeen	Jul 2nd 2014	74