# Coloring Cayley Tables of Finite Groups

by

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# Abstract

The chromatic number of a latin square $L$, denoted $\chi(L)$, is defined as the minimum number of partial transversals needed to cover all of its cells. It has been conjectured that every latin square $L$ satisfies $\chi(L) \leq |L|+2$. If true, this would resolve a longstanding conjecture, commonly attributed to Brualdi, that every latin square has a partial transversal of length $|L|-1$. Restricting our attention to Cayley tables of finite groups, we prove two results. First, we constructively show that all finite Abelian groups $G$ have Cayley tables with chromatic number $|G|+2$. Second, we give an upper bound for the chromatic number of Cayley tables of arbitrary finite groups. For $|G| \geq 3$, this improves the best-known general upper bound from $2|G|$ to $\frac{3}{2}|G|$, while yielding an even stronger result in infinitely many cases.

**Keywords:** latin square (05B15); graph coloring (05C15); strongly regular graphs (05E30); Cayley table; partial transversal

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| | |
|---|---|
| $[1, n]$ | The set of integers $\{1, 2, 3, \ldots, n\}$ |
| $2^X$ | The collection of subsets of a set $X$ |
| $\chi(Y)$ | The chromatic number of the graph, latin square, or (Cayley table of the) group $Y$ |
| $C(X)$ | For a set of cells $X \subseteq L$, the multiset of columns of $L$ covered by $X$ |
| $C'(X)$ | For a set of cells $X \subseteq L$, the set of columns of $L$ covered by $X$ |
| $\Delta(G)$ | The maximum degree of a graph $G$ |
| $Dic_n$ | The dicyclic group of order $4n$ |
| $D_n$ | The dihedral group of order $2n$ |
| $G \cong H$ | The group/graph $G$ is isomorphic to the group/graph $H$ |
| $\Gamma(L)$ | Given a latin square $L$, the latin square graph of $L$ |
| $K \ltimes H$ | The semidirect product of $K$ and $H$ |
| $K_n$ | The complete graph of order $n$ |
| $\mathrm{Kn}(n, k)$ | The Kneser graph whose vertices are $k$-subsets of $[n]$ |
| $L(G)$ | Given a group $G$, the Cayley table of $G$ |
| $L_{ij}$, $L(i, j)$ | The cell in row $i$ and column $j$ of the latin square $L$ |
| $M_n$ | The Möbius ladder on 2n vertices |
| $[n]$ | The set of integers $\{0, 1, 2, \ldots, n-1\}$ |
| $N(n)$ | The maximum size of a set of MOLS of order $n$ |

| | |
|---|---|
| $O(f)$ | Given a function $f$, the set of functions $g$ for which $\limsup_{n\to\infty} \frac{g(n)}{f(n)} < \infty$ |
| $o(f)$ | Given a function $f$, the set of functions $g$ for which $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$ |
| $OA(k,n)$ | An orthogonal array with $k$ rows and $n^2$ columns |
| $\lfloor q \rfloor$ | The "floor" of (i.e. largest integer less than or equal to) $q \in \mathbb{R}$ |
| $R(X)$ | For a set of cells $X \subseteq L$, the multiset of rows of $L$ covered by $X$ |
| $R'(X)$ | For a set of cells $X \subseteq L$, the set of rows of $L$ covered by $X$ |
| $S(X)$ | For a set of cells $X \subseteq L$, the multiset of symbols contained in $X$ |
| $S'(X)$ | For a set of cells $X \subseteq L$, the set of symbols contained in $X$ |
| $Syl_2(G)$ | Given a group $G$, the isomorphism class of its Sylow 2-subgroups |
| $T_d^L$ | The $d$th right diagonal of a latin square $L$ |
| $V(\Gamma)$ | The set of vertices in a graph $\Gamma$ |
| $\mathbb{Z}_n$ | The cyclic group of order $n$ |

# Glossary

**Cayley table**

Of a group $G = \{g_0, g_1, \dots, g_{n-1}\}$, an $n \times n$ array $L = L(G)$ in which the cell $L_{i,j}$ contains the group element $g_i g_j$. We say that row $i$ and column $j$ are **bordered by** $g_i$ and $g_j$, respectively.

**chromatic index**

Of a graph or hypergraph, the minimum number of colors needed to properly color its edges.

**coloring of a latin square**

A partition of the cells in a latin square $L$ into disjoint partial transversals. The **chromatic number of** $L$ is the minimum $k$ such that $L$ can be colored with $k$ partial transversals.

**dicyclic group**

Of order $4n$, the extension with normal subgroup $\mathbb{Z}_{2n}$ and quotient group $\mathbb{Z}_2$ given by the presentation $\langle a, x \mid a^{2n} = 1, a^n = x^2, xax^{-1} = a^{-1} \rangle$.

**dihedral group**

Of order $2n$, the group of symmetries of an $n$-gon. It is defined abstractly by the presentation $\langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle$.

**equivalent latin squares**

Two latin squares $L_1, L_2$ such that $L_2$ can be obtained from $L_1$ via the composition of an isotopy and a parastrophy.

**essentially identical latin squares**

A pair of latin squares which can be obtained from one another by changing the symbols used (but not reordering rows or columns).

**finite projective plane**

Of **order** $n$, a pair $(P, L)$ containing a set $P$ of $n^2 + n + 1$ "points" and a set $L \subseteq 2^P$ of $n^2 + n + 1$ "lines," such that every point is incident to $n + 1$ lines, every line contains

$n+1$ points, any two distinct lines intersect in exactly one point, and any two distinct points are jointly contained in exactly one line.

**hypergraph**

A pair $H = (V, E)$ of vertices $V$ and **hyperedges** $E \subseteq 2^V$. A hypergraph in which each edge has size 2 is a graph.

**isotopy**

A triple of bijections $(\sigma, \tau, \phi)$ between the rows, columns, and symbols (respectively) of two latin squares of the same order. Two squares are **isotopic** if there exists an isotopy between them. The classes induced by the equivalence relation "Is isotopic to" are called **isotopy classes**.

**k-plex**

In a latin square, a collection of cells which intersects each row, column, and color class exactly $k$ times. A partition of a latin square into $k$-plexes is known as a $k$**-partition**.

**latin square**

Of **order** $n$, an $n \times n$ array of **cells** containing entries from an alphabet of size $n$ in which no entry appears more than once in any row or column. We refer to the set of cells containing any fixed symbol as a **symbol class** of $L$.

**latin square graph**

A strongly-regular graph $\Gamma(L)$, defined with respect to a latin square $L$ of order $n$, which is formed from an $n \times n$ grid of vertices by placing a clique of size $n$ on each row, column, and class of symbols in $L$.

**Möbius ladder**

A cubic circulant graph formed from the cycle $C$ of length $2n$ by adding an edge between each pair of vertices at distance $n$ in $C$. We refer to two vertices at distance $n-1$ in $C$ as **near-antipodal**.

**main class invariant**

A property of latin squares which is not changed by moving to an equivalent latin square.

**multiset**

An unordered collection of objects in which repetition is allowed. The **multiplicity** of an element in a multiset is the number of times it occurs. A multiset is **simple**, referred to as a simple set or just "a set," if every element has multiplicity 1.

**orthogonal array**

Denoted $OA(k, n)$, a $k \times n^2$ grid with symbols from an alphabet of size $n$ such that the $n^2$ ordered pairs of symbols defined by any two rows are all distinct.

**orthogonal latin squares**

Latin squares $A$ and $B$ of common order such that, when $A$ is superimposed on $B$, each possible ordered pair of symbols occurs exactly once. We refer to $(A, B)$ as an **orthogonal pair**, and say that $B$ is $A$'s **orthogonal mate.**

**parastrophy**

A map between latin squares which permutes the roles played by rows, columns, and symbols.

**partial transversal**

A collection of cells in a Latin square that intersects each row, column, and symbol class at most once. In a latin square of order $n$, a partial transversal of size $n$ is referred to simply as a **transversal**, while a transversal of size $n-1$ is a **near transversal**.

**semidirect product**

A group $G$ in which every element can be written as the product of an element of a normal subgroup $H \triangleleft G$ and an element of a (not necessarily normal) subgroup $K$ under the condition $K \cap H = \{id_G\}$. We say that $G$ is the **internal** semidirect product in this case. It is also possible to define an external semidirect product, but doing so is beyond the scope of this thesis.

**solvable group**

A group which possesses a **subnormal series** $G = G_a \triangleright G_{a-1} \triangleright \cdots \triangleright G_1 \triangleright G_0 = \{1\}$ in which all of the quotient groups $G_i/G_{i-1}$ are Abelian.

**Sylow $p$-subgroup**

Of a group $G$, where $p$ is prime and $|G| = p^k m$ with $\gcd(p^k, m) = 1$, a subgroup of order $p^k$. The famous **Sylow Theorems** state that, for any $p$ dividing $|G|$, there exists a Sylow $p$-subgroup, and furthermore any two Sylow $p$-subgroups of $G$ are isomorphic.

**vertex-transitive**

A graph $\Gamma = (V, E)$ such that, for every pair of vertices $v_1, v_2 \in V$, there exists an **automorphism** of $\Gamma$ (i.e. a bijective map $\alpha : V \to V$ in which $xy \in E$ if and only if $\alpha(x)\alpha(y) \in E$) sending $v_1$ to $v_2$.

# Chapter 1

# Introduction

Colorings of latin squares naturally generalize the notion of possessing an orthogonal mate. Determining the chromatic number of an arbitrary latin square seems to be rather difficult. In 2016, Besharati et al. conjectured that every latin square of order $n$ can be properly $(n+2)$-colored. If true, this would imply a long-standing conjecture of Brualdi that all latin squares possess a near-transversal. When we restrict our attention from general latin squares to Cayley tables of finite groups, however, the chromatic number question becomes more tractable.

This thesis presents work in two directions towards determining the chromatic number of all Cayley tables of finite groups.[1] First, we completely resolve the chromatic number question for Cayley tables of finite Abelian groups. The heretofore unknown values are established constructively, as in all of these cases the chromatic number meets a trivial lower bound. Second, we explore the relationship between colorings of a group's Cayley table and colorings of the Cayley tables corresponding to its subgroups. This work culminates in a general upper bound which restricts the chromatic number of every Cayley table to an interval of size strictly less than half the order of the underlying group. Although the chromatic number question for general Cayley tables remains open, our work suggests that its resolution is within reach.

In this thesis we use standard terminology from graph theory, group theory, and the study of latin squares. For more information, we refer the reader to [6], [41], and [29], respectively. Unless stated otherwise, we assume throughout that $L$ refers to a latin square of order $n$ and $G$ refers to a group of order $n$.

[1]Several authors have used the term "Cayley table" to describe the operation table of any set closed under a binary operation, and in this sense *all* latin squares are Cayley tables. In this work, however, when we use the term **Cayley table** we are *strictly referring to group-based latin squares.*

| | | | |
|---|---|---|---|
| $1_0$ | $2_1$ | $0_2$ | $3_3$ |
| $0_3$ | $3_2$ | $1_1$ | $2_0$ |
| $3_1$ | $0_0$ | $2_3$ | $1_2$ |
| $2_2$ | $1_3$ | $3_0$ | $0_1$ |

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| $c$ | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| $s_1$ | 1 | 2 | 0 | 3 | 0 | 3 | 1 | 2 | 3 | 0 | 2 | 1 | 2 | 1 | 3 | 0 |
| $s_2$ | 0 | 1 | 2 | 3 | 3 | 2 | 1 | 0 | 1 | 0 | 3 | 2 | 2 | 3 | 0 | 1 |

Figure 1.1: An orthogonal pair of latin squares, $L, L'$, of order 4 and the corresponding $OA(4,4)$, which contains the column $(i, j, b, c)^T$ if and only if $L_{ij} = b$ and $L'_{ij} = c$.

## 1.1 Latin squares and orthogonality

Latin squares are classical, widely utilized mathematical objects. They arise in many diverse contexts, from the construction of Steiner triple systems [11], to the design of experiments across the sciences [19], and even the recreational activities of the general public in the form of Sudoku puzzles. The study of latin squares as combinatorial objects can roughly be divided into three areas: completion, enumeration, and orthogonality. The work in this thesis falls within the third category.

Two latin squares $L, L'$ of order $n$ are **orthogonal** if, superimposing $L'$ on $L$, each possible ordered pair of symbols occurs exactly once. See Figure 1.1 for an example of superimposed, orthogonal latin squares of order 4. Orthogonal latin squares have been studied systematically since at least the time of Euler [20], who proved that there is a pair of orthogonal latin squares of order $n$ whenever $n \not\equiv 2 \pmod 4$. He then conjectured that this sufficient condition was in fact necessary for the existence of an orthogonal pair of order $n$, popularizing the idea with his famous "thirty-six officers problem." This problem, which is equivalent to asking whether there exist orthogonal latin squares of order 6, was resolved in 1900 when Tarry [46] formally established the nonexistence result predicted by Euler. This new evidence in support of Euler's conjecture likely inspired the slew of failed attempts to resolve Euler's conjecture in the early 20th century (e.g. [32, 35]).

In the end, it turns out that Euler's conjecture is false. Counterexamples were given independently by Bose and Shrikhande [8] and Parker [38] in 1959. Later that year, Bose, Shrikhande, and Parker [9] joined to show that Euler was in fact spectacularly wrong: there exists an orthogonal pair of latin squares of order $n$ whenever $n \notin \{2, 6\}$. This landmark result shifted attention to a related question around which a vibrant body of research was already growing: what is the maximum size of a set of mutually orthogonal latin squares (MOLS) of order $n$? This number is denoted $N(n)$.

There is an extensive body of research concerning the construction and extension of sets of MOLS [16]. Despite this, we know very little about the value of $N(n)$ for general $n$. Thanks to the work of Guérin, Hanani, and Wilson [53, 25], among others, we know that $N(n) \geq 4$ for all $n \geq 23$ and $N(n) \geq 6$ for all $n \geq 76$. For $n < 23$, there are numerous lower bounds known in specific cases [47, 44, 54], but much is still unknown. For example, we do not know whether $N(10)$ is greater than 2. Concerning upper bounds for $N(n)$, even less is known. A simple combinatorial argument shows that a set of MOLS of order $n$ has size at most $n-1$. We refer to a set of MOLS whose size meets this bound as **complete**. Complete sets of MOLS provide an intimate connection between latin squares and finite geometry: using a complete set of MOLS one can construct a finite projective plane, while every finite projective plane corresponds to at least one complete set of MOLS.

It has long been known that, for every prime $p$ and every positive integer $k$, there exists a finite projective plane of order $p^k$ (see [29] p. 176), and thus there exists a complete set of MOLS of order $p^k$. Conversely, it is widely believed that every projective plane has prime power order. However, there are very few known results ruling out the existence of projective planes of non-prime power order. Tarry's resolution of the 36 officer problem mentioned above implies there is no projective plane of order six, while Lam et al. [30] showed, with extensive computer assistance, that there is no projective plane of order 10. The only known result that rules out infinitely many values as potential orders of projective planes is due to Bruck and Ryser [12]: for every $n$ congruent to 1 or 2 modulo 4 that has a square free prime factor of the form $4k+3$, there is no projective plane of order $n$. This rules out projective planes of order $6, 14, 21, 22$, etc.

At times it is convenient to restate results concerning MOLS in terms of orthogonal arrays. Following [22], we define an **orthogonal array** with positive integer parameters $k$ and $n$, denoted $OA(k,n)$, as a $k \times n^2$ grid with symbols from an alphabet of size $n$ such that the $n^2$ ordered pairs of symbols defined by any two rows are all distinct. We can think of a latin square $L$ of order $n$ as an $OA(3,n)$ in which the first two rows enumerate $[n] \times [n]$ and the column with $i$ in the first row and $j$ in the second has the symbol from $L_{ij}$ in its third row (see Figure 1.1). Furthermore, a set of $m$ MOLS of order $n$ is equivalent to an $OA(m+2,n)$. For an example of how thinking in terms of orthogonal arrays can be beneficial, see [10] or Section 1.4 below.

## 1.2 Transversals, partial transversals, and colorings of latin squares

Given a pair of orthogonal latin squares $(L, L')$, consider the set $T \subseteq L$ corresponding to a fixed symbol class in $L'$. Examining $T$, we see that it (a) intersects each row and each column of $L$ exactly once, and (b) contains exactly one occurrence of each symbol in $L$. We refer to a set $T \subseteq L$ satisfying properties (a) and (b) as a **transversal of** $L$.

Transversals were first introduced to the study of latin squares as a means of simplifying the search for orthogonal mates. Indeed, a slight extension of the remark made at the start of the previous paragraph shows that a latin square has an orthogonal mate if and only if its cells can be partitioned into disjoint transversals. However, possessing a single transversal is

| | | | | | |
|---|---|---|---|---|---|
| *1* | 2 | 3 | 4 | 5 | 6 |
| 2 | 1 | 6 | 5 | *4* | 3 |
| 3 | 4 | 1 | 2 | 6 | *5* |
| 4 | *6* | 5 | 1 | 3 | 2 |
| 5 | 3 | *2* | 6 | 1 | 4 |
| 6 | 5 | 4 | *3* | 2 | 1 |

Figure 1.2: A Latin square with a transversal (whose entries are given in italics) but no orthogonal mate.

in general not sufficient for the existence of an orthogonal mate. This fact is demonstrated by Figure 1.2 (recall that there are no orthogonal latin squares of order 6). Working from a transversal framework can be useful in studying orthogonality. For example, Wanless and Webb [51] proved that, for every $n \geq 4$, there is a latin square of order $n$ with no orthogonal mate by constructing a latin square of the given order in which some entry did not appear in any transversal. More generally, it is much easier to formulate a proof that a given latin square possesses no transversals than that it has no orthogonal mate (see [39]).

Although it is much easier to find a transversal in a latin square than to find an orthogonal mate, the search for transversals is challenging in its own right. Indeed, there are several seemingly simple conjectures on the subject that have proven extraordinarily difficult to resolve. For example, consider the following conjecture of Ryser.[2]

**Conjecture 1.1** (Ryser)**.** *Every latin square of odd order possesses a transversal.*

In fact, Ryser [43] made the stronger conjecture that the number of transversals in a latin square of order $n$ is congruent to $n$ modulo 2. This is known to be true for even $n$ thanks to Balasubramanian [3]. However, as noted in [14], there are numerous known examples of latin squares of order 7 with an even number of transversals. Ryser's enumerative conjecture has therefore been weakened to the existential form of Conjecture 1.1. We will see below that colorings of latin squares are intimately tied to the existence of transversals. For more information on the enumeration of transversals in latin squares, see Section 7 of [50].

Thanks to Euler [20], it has been known since the 18th century that, for every even number $n$, there is a latin square of order $n$ that does not possess a transversal. Concerning these latin squares, it is natural to ask how close we can get to a transversal. In fact, it has been shown that we can get very close, and it has been conjectured that we can get even closer.

[2] This should not be confused with the more famous Ryser's Conjecture on matchings and coverings in $r$-partite $r$-uniform hypergraphs.

To answer this question formally, we define a **partial transversal** in a latin square as a collection of cells that intersects each row, each column, and each symbol class at most once. Note that we could define a transversal of a latin square $L$ as a partial transversal of length $n$. A partial transversal of length $n-1$, meanwhile, is commonly referred to as a **near transversal**. Many believe that near transversals are ubiquitous, as expressed by a well-known conjecture which is commonly attributed to Brualdi.

**Conjecture 1.2** (Brualdi)**.** *Every latin square possesses a near transversal.*

With multiple claimed proofs which were subsequently withdrawn (see [27] and [14]), this conjecture has become somewhat infamous for its specious simplicity. The most profitable means of attacking this conjecture has been via approximation. Starting in the 1960s there was a steady series of results pushing up the lower bound for the length of the largest partial transversal in an arbitrary latin square. This culminated in the early 1980s when Shor [45] showed that every latin square possesses a partial transversal of length $n-O(\log^2(n))$. Since then there has been work improving the constant inside the big-O [26], but Shor's bound remains the asymptotic state of the art.

We can also attack Conjecture 1.2 by generalizing the problem. We define a ***k*-coloring** of a latin square $L$ as a partition of its cells into $k$ partial transversals, and the **chromatic number** of $L$, denoted $\chi(L)$, as the minimum $k$ for which $L$ is $k$-colorable. Colorings of latin squares naturally generalize the notion of possessing an orthogonal mate; because a partial transversal has size at most $n$, $\chi(L) = n$ if and only if $L$ can be partitioned into disjoint transversals. Furthermore, showing that every latin square $L$ satisfies $\chi(L) \leq n+2$ would establish Conjecture 1.2. Indeed, if every partial transversal in $L$ has length at most $n-2$, then any set of $n+2$ partial transversals covers at most $(n-2)(n+2) = n^2-4$ cells.

Although colorings of design-theoretic objects have been studied for several decades [42, 34], latin square colorings did not appear in the literature until very recently.[3] In 2015-2016, Besharati et al. [5] and Cavenagh and Kuhl [15] independently showed that all circulant latin squares (i.e. all Cayley tables of cyclic groups) are $(n+2)$-colorable.[4] Furthermore, both groups proposed the following.

**Conjecture 1.3.** *Let $L$ be a latin square of order $n$. Then*

$$\chi(G) \leq \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n+2 & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

[3]It is worth noting that, according to one of the reviewers of [5], the chromatic number of latin squares had been considered by several different groups of researchers. The reviewer asserted that the idea had been mentioned in multiple conference talks, but no results had been published.

[4]This is a slight mischaracterization of [15]. In fact, Cavenagh and Kuhl were only able to show $\chi(L) \leq n+3$ in the case $n \equiv 6 \pmod{12}$.

This is a very strong conjecture. We have already seen that it implies Conjecture 1.2, and in fact it also implies Conjecture 1.1 by a similar argument. Conjecture 1.3 is the focal point of this thesis. As would be expected from the preceding remarks, it is currently far from being resolved. There is, however, reason to believe it is true. It has been verified in [5] for all latin squares of order at most 8, it is true asymptotically (a fact which we describe more precisely in Section 1.4), and there is an expanding list of infinite families for which it is known to hold (a list which we will add to in Chapter 2).

## 1.3 Isotopy and equivalent latin squares

So far we have been treating a latin square as an array with a fixed arrangement of symbols. However, when studying colorings – and indeed in most combinatorial analysis of latin squares – it is not necessary to impose such strict structure. Given a latin square $L$ and a partial transversal $T \subseteq L$, suppose we permute the rows, columns, and symbols of $L$ to obtain a new latin square $L'$. It is not hard to see that applying these same permutations to $T$ yields a partial transversal $T' \subseteq L'$. By extension, permuting the rows, columns, and symbols of latin square preserves the existence of a $k$-coloring.

We formalize this idea as follows. For $i = \{1, 2\}$, let $L_i$ be a latin square of order $n$ defined over the alphabets $A_i$. An **isotopy** from $L_1$ to $L_2$ is a triple of bijections $(\sigma, \tau, \phi)$ such that, applying $\sigma : [n] \to [n]$ to the rows of $L_1$, $\tau : [n] \to [n]$ to the columns, and $\phi : A_1 \to A_2$ to the symbols, we obtain $L_2$. If there exists an isotopy from $L_1$ to $L_2$, we say that the two squares are **isotopic**. It is easy to check that "is isotopic to" is an equivalence relation; we refer to the classes induced by this relation as **isotopy classes**. As mentioned in the previous paragraph, isotopies preserve partial transversals, thereby preserving colorings. Notice that the definition of isotopy allows for the introduction of new symbols. This is done to deemphasize the alphabet being used in any given definition of a latin square and focus instead on its structure.

There is an additional notion of equivalence between latin squares which is obscured by their traditional matrix representation. Thinking of a latin square as an $OA(3, n)$, we see that the roles played by rows, columns and symbols are interchangeable. We define a **parastrophy** between two latin squares $L_1, L_2$ as a permutation of the rows in the $OA(3, n)$ corresponding to $L_1$ which produces the $OA(3, n)$ corresponding to $L_2$. Notice that the definition of partial transversal is symmetric with respect to rows, columns, and symbols. Thus, latin square colorings are invariant under parastrophy.

In general, two latin squares $L_1$ and $L_2$ are said to be **equivalent** if $L_2$ can be obtained from $L_1$ by composing a isotopy with a parastrophy. An equivalence class under this relation is known as a **main class**. Many properties of latin squares depend only on a square's main class (for some examples see p. 17 of [29] ). Such properties are called **(main class)**

**invariants**. Combining the remarks made at the end of the first and third paragraph in this section, we see that the chromatic number of a latin square is a main class invariant. As such, in what follows we freely translate between members of a main class in trying to determine the chromatic number of a given latin square.

## 1.4 Graph representations of latin squares

As is often the case with combinatorial objects, there are numerous graph-theoretic representations of latin squares appearing in the literature. Perhaps most well-known is the translation between latin squares of order $n$ and $n$-edge-colorings of the complete bipartite graph $K_{n,n}$. If we label one side of the bipartition with the rows of $L$, and the other side with its columns, then we have exactly one edge for each cell. Coloring every edge with the symbol contained in the corresponding cell yields a proper edge-coloring, as no row or column can contain any symbol more than once. This idea has been utilized in determining when a partial latin squares is completable [13] and in characterizing families of latin squares with rigid structural properties [48].

Representing a latin square as an edge-colored bipartite graph can also be helpful in the study of transversals. Given an edge-colored graph, we define a **rainbow matching** as a collection of disjoint edges no two of which have the same color. A partial transversal in a latin square is then equivalent to a rainbow matching in the corresponding coloring of $K_{n,n}$. There are several conjectures stated in terms of rainbow matchings which generalize Conjecture 1.1 and Conjecture 1.2, and there are several interesting partial results known towards these conjectures [1]. The work of Aharoni and Berger cited in the previous sentence also contains an interesting generalization of latin square induced edge-colorings of $K_{n,n}$.

Given an $n$-edge coloring of $K_{n,n}$, we can define a 3-partite 3-uniform hypergraph by adding $n$ new vertices, one for each color, and turning an edge $rc$ with color $s$ into the 3-edge $(r, c, s)$. This gives us a means of representing every latin square $L$ with a hypergraph $H(L)$ on the vertex set $R \cup C \cup S$, where $R$, $C$, and $S$ are sets of size $n$ representing the rows, columns, and symbols of $L$. Observe that a partial transversal of $L$ corresponds to a set of disjoint hyperedges in $H(L)$. Thus, $\chi(L)$ is just the chromatic index of $H(L)$.

Determining the chromatic index of a hypergraph is remarkably difficult, but it is a problem which has been studied for many years. Although not much is known in terms of exact values, there have been several compelling asymptotic results on the subject. In 1989, Pippenger and Spencer [40] proved that every family of uniform hypergraphs in which the minimum degree is asymptotic to the maximum degree and the maximum codegree is asymptotically negligible compared to the maximum degree has chromatic index asymptotic to maximum degree. It was pointed out by Cavenagh and Kuhl [15] that $H(L)$ fits these conditions and has maximum degree $n$, giving us the following theorem.

**Theorem 1.4** (Pippenger, Spencer [40])**.** *For every $\delta > 0$, there exists an $n_0 > 0$ such that every latin square $L$ of order $n \geq n_0$ satisfies $\chi(L) \leq (1 + \delta)n$. In other words,*

$$\chi(L) = n + o(n).$$

This result gives us confidence in Conjecture 1.3. However, that is the extent to which the hypergraph representation of latin squares will be useful to us. Instead, we utilize a family of strongly-regular graphs first introduced by Bose in [7]. Given a latin square $L$, the associated **latin square graph** $\Gamma = \Gamma(L)$ has a vertex corresponding to each cell of $L$ and an edge between $(r_1, c_1, s_1)$ and $(r_2, c_2, s_2)$ if and only if exactly one of the equations $r_1 = r_2$, $c_1 = c_2$, or $s_1 = s_2$ holds. See Figure 1.3 for an example. These are referred to as **row edges** (denoted $E_R(\Gamma)$), **column edges** ($E_C(\Gamma)$), and **symbol edges** ($E_S(\Gamma)$), respectively. We will often refer to sets of cells in constructing induced subgraphs of $\Gamma$. For example, the first row $R_1 \subseteq L$ induces the subgraph $\Gamma[R_1] \cong K_n$.

$L =$

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 2 | 0 |
| 2 | 0 | 1 |

$\Gamma(L) =$

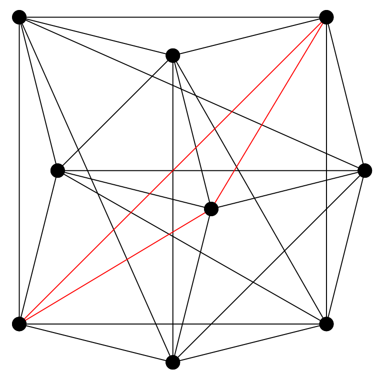

Figure 1.3: A latin square $L$ and its associated latin square graph $\Gamma(L)$ with the cells in color class 2 and the edges of the corresponding clique in red.

Informally, $\Gamma(L)$ is formed from an $n \times n$ grid of vertices by placing a clique of size $n$ on each row, column, and class of symbols in $L$. Notice that independent sets in $\Gamma = \Gamma(L)$ correspond to partial transversals of $L$. Thus, $\chi(L)$ is simply the traditional, graph-theoretic chromatic number of $\Gamma$. Thinking of colorings in terms of $\Gamma(L)$ immediately provides us with a couple of bounds on $\chi(L)$ for $n \geq 3$. Because $\Gamma(L)$ contains a clique of size $n$ (indeed it contains many cliques of size $n$), we know $\chi(L) \geq n$. On the other hand, as every vertex of $\Gamma(L)$ has degree $3n - 3$, Brooks' theorem (see Section 3.3) tells us that $\chi(L) \leq 3n - 3$.

It is worth noting why we have made the distinction $n \geq 3$ in determining these bounds. If $L$ is a latin square of order 2, then $\Gamma(L)$ is a 3-regular simple graph on 4 vertices. The only such graph is $K_4$, and we are therefore no longer able to apply Brooks' Theorem. The fact that $\chi(\Gamma(L)) = 4$ whenever $|L| = 2$ will force us to be careful about how we state several results below. However, it does not change the substance of our program; we already know that all latin squares of order less than 3 satisfy Conjecture 1.3.

Although the bounds $n \leq \chi(L) \leq 3n-3$ are easily obtained, a careful consideration of the structure of latin square graphs can tell us much more about their chromatic number. In Chapter 2 and Chapter 3 we demonstrate how this graph-theoretic framework can be used to provide significantly better upper bounds for $\chi(L)$.

## 1.5 Cayley tables and complete mappings of finite groups

For the rest of this paper we restrict our attention from general latin squares to Cayley tables of finite groups. Given a group $G = \{g_0, g_1, \ldots, g_{n-1}\}$, the **Cayley table** of $G$, denoted $L(G)$, is the $n \times n$ array in which the cell $L_{i,j}$ contains the group element $g_i g_j$. We say that the $i$th row and $i$th column of $L(G)$ are **bordered by** the group element $g_i$. It follows directly from the group axioms ([18] p. 16) that $L(G)$ is a latin square.

Observe that $L(G)$ does not depend on the ordering we have chosen for $G$, as any two orderings produce isotopic latin squares. Once we fix an ordering of $G$, we switch freely between using $(i, j)$ and using $(g_i, g_j)$ to index the cell of $L_{i,j} \in L(G)$. We refer to colorings of Cayley tables as colorings of the corresponding group, with $\chi(G)$ denoting the chromatic number of the latin square $L(G)$.

Given a set $X \subseteq L(G)$ and a group element $g \in G$, define the **shift of $X$ by $g$** as

$$Xg = \{(x, yg) \,:\, (x, y) \in X\}.$$

Shifting $X$ preserves its structure in the sense that $\Gamma[X] \cong \Gamma[Xg]$. Indeed, for any $z_1, z_2, g \in G$, we have $z_1 g = z_2 g$ if and only if $z_1 = z_2$. In particular, given any partial transversal $T \subseteq L(G)$, the shift $Tg$ is also a transversal. Actually, we can say something even stronger. Notice that any distinct pair $g_1, g_2 \in G$ satisfies $Tg_1 \cap Tg_2 = \emptyset$. Thus, whenever $L(G)$ contains a transversal $T$, it in fact possesses an $n$-coloring of the form $\{Tg \mid g \in G\}$; we need only find a single transversal in $L(G)$ to show that $\chi(G) = n$.

This observation motivated a series of papers on the combinatorial structure of finite groups. A **complete mapping** of a group $G$ is a bijection $\theta : G \to G$ such that the derived mapping $\eta : G \to G$ defined by $\eta(g) = g \cdot \theta(g)$ is also a bijection. The map $\eta$ is often referred to as an **orthomorphism**. Notice that a complete mapping of $G$, say $\theta$, is equivalent to the transversal in $T = \{(g, \theta(g)) \,:\, g \in G\} \subseteq L(G)$. Thus, a Cayley table has an orthogonal mate if and only if its underlying group possesses a complete mapping.

Complete mappings were introduced by Mann in the early 1940s [33] as a means of constructing MOLS. He noted that a group $G$ has a complete mapping if and only if its Cayley table has an orthogonal mate. Shortly thereafter, Paige proved the following [36]. As this result will be important to our arguments in Chapter 3, we present its proof here.

**Proposition 1.5.** *Every group of odd order has a complete mapping.*

*Proof.* Let $G$ be a group of order $2k+1$ for some $k \in \mathbb{N}$. We claim that the identity mapping is complete. To establish this, it is sufficient to show that the map $\eta : G \to G$ defined by $\eta(g) = g^2$ is injective. And indeed, for every $g, h \in G$ such that $g^2 = h^2$, we have

$$g = gg^{2k+1} = g^{2k+2} = (g^2)^{k+1} = (h^2)^{k+1} = h^{2k+2} = hh^{2k+1} = h.$$

□

By the comments made preceding Proposition 1.5, we have the following result on the chromatic number of Cayley tables.

**Corollary 1.6.** *Let $G$ be a group of odd order $n$. Then $\chi(G) = n$.*

The task of characterizing groups which contain complete mappings began with Paige's 1947 paper [36] in which he proved Proposition 1.5. Then, in 1950 Bateman [4] showed that every infinite group possesses a complete mapping. In the early 1950s Hall and Paige [37, 23] laid the groundwork for a complete characterization. They conjectured the following, which was finally proven by Bray, Evans, and Wilcox in 2009 [21, 52].

**Theorem 1.7.** *Let $G$ be a group of order $n$. Then the following are equivalent:*

1. $\chi(G) = n$.
2. $\chi(G) \leq n + 1$.
3. *$L(G)$ has a transversal.*
4. *$G$ has a complete mapping*
5. *There is an ordering of the elements of $G$, say $g_1, g_2, \ldots, g_n$, such that $g_1 g_2 \cdots g_n = e$ (where $e$ is the identity element of $G$).*
6. *$Syl_2(G)$ is either trivial or non-cyclic.*

*Proof.* Observe that 1 trivially implies 2, while 2 implies 3 by the contrapositive: a latin square without a transversal can only cover $(n+1)(n-1) = n^2 - 1$ of its $n^2$ cells with $n+1$ partial transversals. Having shown that 3 implies 1 and that 3 and 4 are equivalent earlier in this section, we see that 1, 2, 3, and 4 are equivalent.

Turning to the difficult part of the proof, Paige showed that 4 implies 5 in [37], before Hall and Paige [23] showed that 5 implies 6. It was also shown in [23] that 6 implies 4 for every solvable group, which was used by Dénes and Keedwell [17] to show 5 and 6 are equivalent. Finally, in 2009 Bray, Evans, and Wilcox showed that 6 implies 4 for all non-solvable groups using the classification of finite simple groups. □

In [5], Besharati et al. used this theorem to construct a $(n+2)$-coloring for every cyclic group $\mathbb{Z}_n$. We will show in Chapter 2 that this construction can be generalized to build an $(n+2)$-coloring of any finite Abelian group. Then, in Chapter 3 we provide a general upper bound for $\chi(G)$ which relies on a characterization of groups which do not satisfy property 6 of Theorem 1.7.

# Chapter 2

# The chromatic number of Abelian groups

The main result of this chapter is the following theorem which characterizes the chromatic number of every finite Abelian group.

**Theorem 2.1.** *Let $G$ be an Abelian group of order $n$. Then*

$$\chi(G) = \begin{cases} n & \text{if } Syl_2(G) \text{ is either trivial or non-cyclic,} \\ n+2 & \text{otherwise.} \end{cases}$$

Thanks to Theorem 1.7, proving this main result amounts to constructing an $(n+2)$-coloring for every Abelian group with nontrivial cyclic Sylow 2-subgroups. To this end, set $t := 2^l$ for some $l \geq 1$ and let $G$ be an Abelian group of order $n = tm$, where $m$ is odd and $Syl_2(G) = \mathbb{Z}_t$. By the fundamental theorem of finite Abelian groups, $G \cong \mathbb{Z}_t \times H$, where $H = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$ is an Abelian group of odd order $m = \prod_{i=1}^{k} m_i$.

This chapter is devoted to producing an optimal coloring of $\Gamma = \Gamma(G)$. Considered broadly, our construction can be broken into three steps. First, we find a particularly nice $m$-coloring of $H$. Then, we use this coloring to construct a partition of $V(\Gamma)$, say $\mathcal{P}$, in which the induced subgraph $\Gamma[P]$ is "nearly bipartite" for every $P \in \mathcal{P}$. Finally, we modify $\mathcal{P}$ to separate $\Gamma$ into $\frac{n}{2}+1$ bipartite induced subgraphs, and use a disjoint pair of colors for each of these subgraphs to obtain an $(n+2)$-coloring of $\Gamma$.

Throughout this chapter and the next, we make use of maps $R$, $C$, and $S$, defined as follows. Given a group $G = \{g_0, g_1, \ldots, g_{n-1}\}$ with Cayley table $L = L(G)$, we define projections $R, C : L \to [n]$ by $R(L_{ij}) = i$ and $C(L_{ij}) = j$, and $S : L \to G$ by $S(L_{ij}) = g_i g_j$. We then extend these functions to sets $A \subseteq L$ by letting $R(A) = \{R(a) : a \in A\}$ be the multiset of rows containing cells in $A$ and defining $C(A)$ and $S(A)$ similarly as multisets of columns and symbols, respectively. To simplify notation, we write $x^{(k)} \in M$ to mean $x$ occurs with multiplicity (at least) $k$ in the multiset $M$.

We also use maps $R'$, $C'$, and $S'$, which differ from $R$, $C$, and $S$ in that they ignore multiplicities, mapping sets of cells to simple sets (i.e. multisets in which each element has multiplicity 1) of rows, columns, and symbols. For example, if $X$ is the row of $L$ bordered by $g_i$, then $R(X) = \{i^{(n)}\}$, $R'(X) = \{i\}$, $C(X) = C'(X) = [n]$, and $S(X) = S'(X) = G$.

## 2.1 Coloring Abelian groups of odd order with right diagonals

Although the chromatic number of $G$ does not depend on how we order the rows and columns of its Cayley table, fixing an illustrative ordering can greatly simplify our argument. Given a $k$-fold Cartesian product of ordered sets $X = X_1 \times X_2 \times \cdots \times X_k$, a **lexicographical ordering** of $X$ is defined by the relation $(a_1, a_2, \ldots, a_k) < (b_1, b_2, \ldots, b_k)$ if and only if there is some index $j \in \{1, 2, \ldots, k\}$ such that $a_i = b_i$ for all $i \in \{1, \ldots, j-1\}$ and $a_j < b_j$. For example, the lexicographical ordering of $\{1, 2\} \times \{1, 2\}$ is $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$. In what follows, we assume that $H = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k} = \{h_0, h_1, \ldots, h_{m-1}\}$ is ordered lexicographically with respect to the canonical ordering of $\mathbb{Z}_{m_i} = \{0, 1, \ldots, m_i - 1\}$ for each $i \in \{1, 2, \ldots, k\}$.

To see why it is profitable to arrange $L$ lexicographically, consider which sets of cells are natural candidates in the search for transversals. For every group $H$ of odd order, Proposition 1.5 tells us that the main diagonal of $L(H)$ is a transversal so long as we arrange its rows and columns in the same order. With the additional assumption that $H$ is Abelian, we can say even more. Given a latin square $L$ of order $n$, define the ***d*th right diagonal of** $L$ as

$$T_d^L := \{L_{i,i+d} : 0 \leq i \leq n-1\}, \tag{2.1}$$

where indices are expressed modulo $n$. When it is clear which latin square we are discussing, we drop the superscript and simply write $T_d$. We show that the partition of $L(H)$ into its right diagonals is an $m$-coloring via the following technical lemma. In the statement and proof of this lemma, indices are expressed modulo $m$.

**Lemma 2.2.** *Let $H = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k} = \{h_0, h_1, \ldots, h_{m-1}\}$ be an Abelian group of odd order $m$ with elements ordered lexicographically, and let $s$ be a positive integer satisfying* $\gcd(s+1, m) = 1$. *Then the map $\phi : H \to H$ given by $\phi(h_i) = h_{i+c} + h_{si+d}$ is injective for every $c, d \in [m]$.*

*Proof.* Fix arbitrary integers $c, d \in [m]$. We proceed by induction on $|H|$. If $H = \mathbb{Z}_m$ then

$$\phi(h) = (s+1)h + c + d \pmod{m} \tag{2.2}$$

for all $h \in H$. Having assumed $\gcd(s+1, m) = 1$, we know $s+1$ is a generator of $\mathbb{Z}_m$. This yields the identity

$$\{(s+1)r \,:\, r \in \mathbb{Z}_m\} = \mathbb{Z}_m. \tag{2.3}$$

Now consider $a, b \in H$ such that $\phi(a) = \phi(b)$. It follows from (2.2) that

$$(s+1)a \equiv (s+1)b \pmod{m}.$$

But then (2.3) implies $a = b$, which establishes the base case of our induction and allows us to assume that $H$ is not cyclic.

We may assume $H = \mathbb{Z}_{m_1} \times H'$, where $H' = \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$ is a nontrivial Abelian group of odd order $q := \frac{m}{m_1}$. Observe that, if we order $H' = \{g_0, g_1, \ldots, g_{q-1}\}$ lexicographically, then, for every $i \in [m]$,

$$h_i = \left( \left\lfloor \frac{i}{q} \right\rfloor, g_{i \pmod{q}} \right).$$

Now, define the map $\psi : H' \to H'$ by $\psi(g_i) = g_{i+c \pmod{q}} + g_{si+d \pmod{q}}$ and consider indices $i, j \in [m]$ for which $\phi(h_i) = \phi(h_j)$. This is equivalent to the identity

$$\left( \left\lfloor \frac{i+c}{q} \right\rfloor + \left\lfloor \frac{si+d}{q} \right\rfloor, \psi(g_i) \right) = \left( \left\lfloor \frac{j+c}{q} \right\rfloor + \left\lfloor \frac{sj+d}{q} \right\rfloor, \psi(g_j) \right), \tag{2.4}$$

with the first coordinate expressed modulo $m_1$. Because $\gcd(q, s+1) = 1$, the induction hypothesis tells us that $\psi$ is injective. Thus, $i \equiv j \pmod{q}$ and we may assume $j = i + rq$ for some $r \in [m_1]$. But then the identity induced by the first coordinates in (2.4) implies

$$\begin{aligned} \left\lfloor \frac{i+c}{q} \right\rfloor + \left\lfloor \frac{si+d}{q} \right\rfloor &\equiv \left\lfloor \frac{i+rq+c}{q} \right\rfloor + \left\lfloor \frac{si+srq+d}{q} \right\rfloor \\ &= \left\lfloor \frac{i+c}{q} \right\rfloor + r + \left\lfloor \frac{si+d}{q} \right\rfloor + sr \pmod{m_1}, \end{aligned}$$

which is equivalent to $(s+1)r \equiv 0 \pmod{m_1}$. Because $0 \le r < m_1$ and $s+1$ is coprime to $m_1$, it must be the case that $r = 0$, which implies $i = j$. □

The following result, which is essentially a corollary of Lemma 2.2, shows that right diagonals form an $m$-coloring of $H$. See Figure 2.1 for an example of such a coloring in the group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

**Lemma 2.3.** *Let $H = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$ be an Abelian group of odd order $m$. If the rows and columns of the Cayley table $L = L(H)$ are ordered lexicographically then $T_d$ is a transversal for every $d \in [m]$.*

*Proof.* Index the elements of $H = \{h_0, h_1, \ldots, h_{m-1}\}$ by their lexicographical order. It follows immediately from (2.1) that $R(T_d) = C(T_d) = [n]$. To complete the proof, we must show that $S(T_d)$ is simple. This is tantamount to showing that, for each $h \in H$, there is a unique index $i \in [m]$ such that

$$h_i + h_{i+d \pmod m} = h.$$

But this follows directly from an application of Lemma 2.2 with $s = 1$ and $c = 0$. $\square$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $00_a$ | $01_b$ | $02_c$ | $10_d$ | $11_e$ | $12_f$ | $20_g$ | $21_h$ | $22_i$ |
| $01_i$ | $02_a$ | $00_b$ | $11_c$ | $12_d$ | $10_e$ | $21_f$ | $22_g$ | $20_h$ |
| $02_h$ | $00_i$ | $01_a$ | $12_b$ | $10_c$ | $11_d$ | $22_e$ | $20_f$ | $21_g$ |
| $10_g$ | $11_h$ | $12_i$ | $20_a$ | $21_b$ | $22_c$ | $00_d$ | $01_e$ | $02_f$ |
| $11_f$ | $12_g$ | $10_h$ | $21_i$ | $22_a$ | $20_b$ | $01_c$ | $02_d$ | $00_e$ |
| $12_e$ | $10_f$ | $11_g$ | $22_h$ | $20_i$ | $21_a$ | $02_b$ | $00_c$ | $01_d$ |
| $20_d$ | $21_e$ | $22_f$ | $00_g$ | $01_h$ | $02_i$ | $10_a$ | $11_b$ | $12_c$ |
| $21_c$ | $22_d$ | $20_e$ | $01_f$ | $02_g$ | $00_h$ | $11_i$ | $12_a$ | $10_b$ |
| $22_b$ | $20_c$ | $21_d$ | $02_e$ | $00_f$ | $01_g$ | $12_h$ | $10_i$ | $11_a$ |

Figure 2.1: Subscripts indicating a 9-coloring of $L(\mathbb{Z}_3 \times \mathbb{Z}_3)$ using right diagonals.

It is worth noting that, when $H$ is non-cyclic, the coloring of $L(H)$ obtained from its right diagonals cannot be realized as the set of shifts of any fixed transversal, making it fundamentally different from the colorings described in the proof of Theorem 1.7.

## 2.2 Finding Möbius ladders as induced subgraphs

We are now in a position to find a partition of $L = L(G)$ in which the subgraphs of $\Gamma = \Gamma(L)$ induced on each part are isomorphic. Specifically, each part will induce a graph isomorphic to a Möbius ladder. Expressing indices modulo $2n$, we define the **Möbius ladder** of order $2n$, denoted $M_n$, as having vertex set $\{v_0, v_1, \ldots, v_{2n-1}\}$ and edge set $E_1 \cup E_2$, where $E_1 := \{v_i v_{i+1} : i \in [2n]\}$ and $E_2 := \{v_i v_{i+n} : i \in [n]\}$. Note that $E_1$ forms a Hamilton cycle in $M_n$ while edges in $E_2$ connect opposite vertices in this cycle. We refer to $E_1$ as the "rim" of the Möbius ladder and to edges in $E_2$ as its "rungs."

For each $i \in [n]$, we refer to the pair $\{v_i, v_{i+n}\}$ as an **antipodal pair** of vertices, while $\{v_i, v_{i+n+1}\}$ and $\{v_i, v_{i+n-1}\}$ are called **near-antipodal pairs**. Alternatively, we may define a near-antipodal pair as two vertices $u, v \in V(M_n)$ such that the shortest path from $u$ to $v$ along the rim of $M_n$ has length $n - 1$. Two drawings of $M_{18}$ with a highlighted pair of near-antipodal vertices is given in Figure 2.2.

We are interested in Möbius ladders because they admit the following family of well-structured colorings.

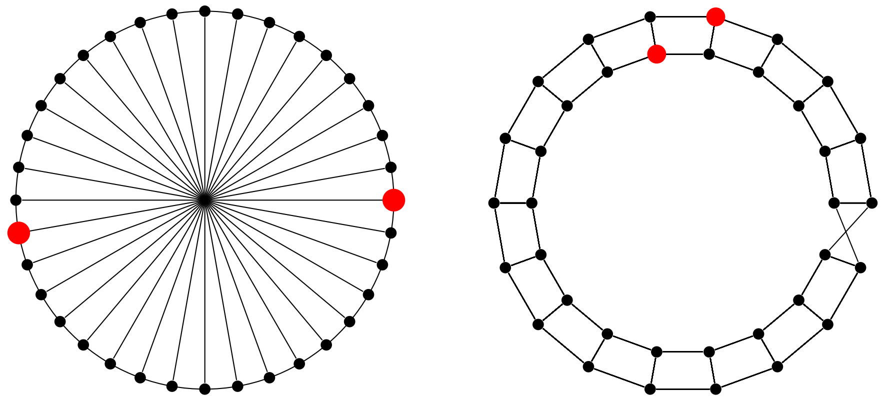

Figure 2.2: Two drawings of the Möbius ladder $M_{18}$ with a near-antipodal pair of vertices highlighted.

**Proposition 2.4.** *For $n \geq 3$, let $M_n$ be the Möbius ladder of order $2n$, and let $\{v_i, v_{i+n+1}\} \in V(M_n)$ be a near-antipodal pair. Then $M_n$ has a 3-coloring in which one of the color classes is exactly $\{v_i, v_{i+n+1}\}$.*

*Proof.* Throughout this proof we express indices modulo $2n$. By reindexing if necessary, we may assume without loss of generality that $i = 2n-1$. Consider the greedy coloring $c : M_n \to \mathbb{N}$ given by the vertex ordering $v_0, v_1, \ldots, v_{2n-1}$. This coloring will alternate

$$c(v_0) = 1,\ c(v_1) = 2,\ c(v_2) = 1, \ldots$$

until we reach $v_n$. If $n$ is odd then this alternating pattern will proceed around the rim, so that $c$ is a 2-coloring. We then construct the desired coloring $c'$ by setting $c'(v_{2n-1}) = c'(v_n) = 3$ and $c'(v_j) = c(v_j)$ for all $j \in [2n-1] \setminus \{n\}$.

If $n$ is even then $c(v_n) = 3$, after which $c$ will follow the alternating pattern

$$c(v_{n+1}) = 1,\ c(v_{n+2}) = 2,\ c(v_{n+3}) = 1,\ \ldots,\ c(v_{2n-2}) = 2.$$

Notice that $c$ maps the neighborhood of $v_{2n-1}$ to $\{1, 2\}$. Thus, $c(v_{2n-1}) = 3$ and $c$ is as desired. □

Before we can make use of Proposition 2.4, we must show how to partition $V(\Gamma)$ into sets that induce Möbius ladders. As in Lemma 2.3, this is merely a matter of finding the correct ordering for the rows and columns of $L$. Given integers $a$ and $b$ satisfying $\gcd(a, b) = 1$ and two totally ordered sets $W = \{w_0, w_1, \ldots, w_{a-1}\}$, $Z = \{z_0, z_1, \ldots, z_{b-1}\}$, we define the **mod-counting ordering** of $W \times Z$ as

$$(w_{0 \pmod a}, z_{0 \pmod b}), (w_{1 \pmod a}, z_{1 \pmod b}), \ldots, (w_{ab-1 \pmod a}, z_{ab-1 \pmod b}).$$

| | | | | | |
|---|---|---|---|---|---|
| ***00*** | ***11*** | 02 | 10 | 01 | 12 |
| 11 | ***02*** | ***10*** | 01 | 12 | 00 |
| 02 | 10 | ***01*** | ***12*** | 00 | 11 |
| 10 | 01 | 12 | ***00*** | ***11*** | 02 |
| 01 | 12 | 00 | 11 | ***02*** | ***10*** |
| ***12*** | 00 | 11 | 02 | 10 | ***01*** |

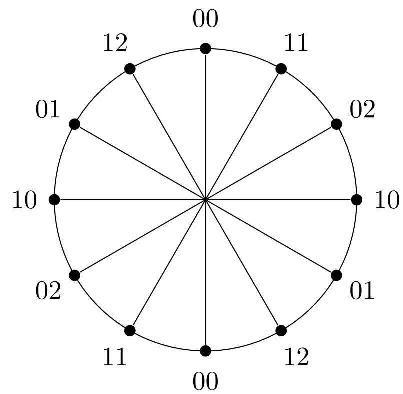


Figure 2.3: A pair of right diagonals $\mathbf{T_0} \cup \mathbf{T_1} \subseteq L(\mathbb{Z}_2 \times \mathbb{Z}_3)$ and the corresponding induced subgraph $\Gamma' \subseteq \Gamma(L(\mathbb{Z}_2 \times \mathbb{Z}_3))$, which is isomorphic to the Möbius ladder $M_6$.

Recall that $G \cong \mathbb{Z}_t \times H$, where $m = |H|$ and $t = 2^l$ are relatively prime. Thus, $G$ has a well-defined mod-counting order, and we can state the following lemma.

**Lemma 2.5.** *Let $G = \mathbb{Z}_t \times H$ be a group of order $n = tm$, where $t = 2^l$ for $l \geq 1$ and $H = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$ is an Abelian group of odd order. If we arrange the rows and columns of $L = L(G)$ in mod-counting order with respect to the lexicographical ordering of $H = \{h_0, h_1, \dots, h_{m-1}\}$ and the canonical ordering of $\mathbb{Z}_t = \{0, 1, \dots, t-1\}$, then the latin square graph $\Gamma = \Gamma(L)$ satisfies $\Gamma[T_d^L \cup T_{d+1}^L] \cong M_n$ for every $d \in [n]$.*

*Proof.* In this proof indices are expressed modulo $n$ unless otherwise stated. Fixing $d \in [n]$, let $\Gamma' := \Gamma[T_d^L \cup T_{d+1}^L]$, and apply the labels $A_i := L_{i,i+d}$, and $B_i := L_{i,i+d+1}$ to the elements of $V' = V(\Gamma')$. Recall that $E' = E(\Gamma')$ can be partitioned into the sets $E'_R$, $E'_C$, and $E'_S$, corresponding to "row edges," "column edges," and "symbol edges," respectively. It follows from (2.1) that $E'_R = \{A_i B_i : i \in [n]\}$ and $E'_C = \{A_{i+1} B_i : i \in [n]\}$. Thus, the vertex sequence

$$A_0, B_0, A_1, B_1, \dots, A_{n-1}, B_{n-1}$$

corresponds to a Hamilton cycle in $\Gamma'$ that uses all of the edges in $E'_R \cup E'_C$. An example with $G = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $d = 0$ is shown in Figure 2.3.

For $i \in [n]$, define the function $r(i) = i + \frac{n}{2} \pmod{n}$. It is left to show that $E'_S = \{A_i A_{r(i)} : i \in [n]\} \cup \{B_i B_{r(i)} : i \in [n]\}$. Let $a_i$, $b_i$ denote the symbols in the cells $A_i$ and $B_i$, respectively. By definition,

$$a_i = (2i + d, h_i + h_{i+d \pmod{m}}) \text{ and } b_i = (2i + d + 1, h_i + h_{i+d+1 \pmod{m}}), \qquad (2.5)$$

with the first coordinate expressed modulo $t$. We then see that $E'_S$ contains no edges of the form $A_i B_j$, as the first coordinates of $a_i$ and $b_j$ always have different values modulo 2.

Fixing $i \in [n]$, suppose there is some nonzero $x \in [n]$ such that $a_i = a_{i+x}$. By (2.5), this implies the identities

$$(I1)\ 2i + d \equiv 2i + 2x + d \pmod{t}, \quad (I2)\ h_i + h_{i+d} = h_{i+x} + h_{i+x+d}, \tag{2.6}$$

where in $(I2)$ indices are expressed modulo $m$. Applying Lemma 2.2 to $(I2)$ with $c = 0$ and $s = 1$, we see that $i \equiv i + x \pmod{m}$, which means $x \equiv 0 \pmod{m}$. On the other hand, $(I1)$ implies $2x \equiv 0 \pmod{2^l}$, from which we may conclude $x \equiv 0 \pmod{2^{l-1}}$. Because $m$ is odd, the only nonzero $x \in [n]$ that is divisible by both $m$ and $2^{l-1}$ is $2^{l-1}m = \frac{n}{2}$. As $x = \frac{n}{2}$ satisfies (2.6), we conclude that $A_iA_j \in E'_S$ if and only if $j = r(i)$. A nearly identical argument shows that $B_iB_j \in E'_S$ if and only if $j = r(i)$, completing the proof. □

## 2.3 Optimal colorings of Abelian groups

The results proven so far in this chapter already show Cayley tables of finite Abelian groups of order at least 3 satisfy $\chi(L) \leq \frac{3}{2}n$: simply partition $L$ into $n/2$ sets, each of which induces a Möbius ladder, and 3-color each copy of $M_n$ with a distinct set of colors. This coloring is fairly coarse, but it can be significantly improved by "recycling" colors among distinct Möbius ladders. Recall Proposition 2.4: we may pick any pair of near-antipodal vertices to be its own color class in a 3-coloring of $M_n$. If we make this selection carefully, near-antipodal pairs from many different Möbius ladders can be combined into a single color class. The following theorem shows we only need two color classes to cover a near-antipodal pair from each copy of $M_n$.

**Theorem 2.6.** *Let $G$ be an Abelian group of order $n$. Then*

$$\chi(G) \leq n + 2$$

*Proof.* Let $L$ be the Cayley table of $G$, and let $\Gamma = (V, E_R \cup E_C \cup E_S)$ be its associated latin square graph. As discussed above, Theorem 1.7 allows us to assume $G = \mathbb{Z}_t \times H$, where $t = 2^l$ for $l \geq 1$ and $H = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$ is an Abelian group of odd order $m := n/t = \prod_{i=1}^{k} m_i$. In this case $n$ is even and the integer constant

$$q := n/2$$

is well defined. Furthermore, because $\mathbb{Z}_2$ has latin square graph $K_4$, we may assume $n \geq 4$.

Ordering $H = \{h_0, h_1, \ldots, h_{m-1}\}$ lexicographically, we arrange $L$ according to the mod-counting order of $G = \mathbb{Z}_t \times H$ given by

$$g_i = \Big(i \pmod{t}, h_{i \pmod{m}}\Big) \text{ for } i \in [n]. \tag{2.7}$$

By Lemma 2.3 and Lemma 2.5, the sets

$$D_i := T_{2i} \cup T_{2i+1}$$

satisfy $\Gamma_i := \Gamma[D_i] \cong M_n$ for every $i \in [q]$.

We want to find a pair of independent sets $X, Y \subseteq V$ such that $D'_i := D_i \setminus (X \cup Y)$ induces a bipartite graph $\Gamma'_i = \Gamma[D'_i]$ for every $i \in [q]$. Given such sets $X$ and $Y$, we can properly $(n+2)$-color $\Gamma$ using a distinct pair of colors for each of the $\frac{n}{2}+1$ sets $D'_0, D'_1, \ldots, D'_{q-1}, X \cup Y$.

Towards a definition of $X$ and $Y$, let

$$k := \left\lceil \frac{n}{4} \right\rceil \text{ and } (q_0, q_1) := \begin{cases} (q, q+1) & \text{if } q \equiv 0 \pmod 3, \\ (q-1, q) & \text{otherwise,} \end{cases} \tag{2.8}$$

and let $\underline{z} := z \pmod n$ for every $z \in \mathbb{Z}$. Then, for each $i \in [k]$, we define

$$x_i := L(i, 3i),\ x'_i := L(q_0 + i, \underline{q_1 + 3i}),\ \text{and } X := \{x_i, x'_i : i \in [k]\}. \tag{2.9}$$

Similarly, for every $j \in [q-k]$, we define

$$y_j := L(j, \underline{3j + 2k}),\ y'_i := L(q_0 + j, \underline{q_1 + 3j + 2k}),\ \text{and } Y := \{y_j, y'_j : j \in [q-k]\}. \tag{2.10}$$

Figure 2.4 illustrates $L$, $X$, and $Y$ for the group $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. We observe $x_i \in T_{2i} \subseteq D_i$ and $x'_i \in T_{2i+1} \subseteq D_i$ for each $i \in [k]$. Similarly, $y_j, y'_j \in D_{k+j}$ for each $j \in [q-k]$.

Recall from the proof of Lemma 2.5 that the rim of the Möbius ladder $\Gamma_i := \Gamma[D_i]$ is formed by the row and column edges $E_R(\Gamma_i) \cup E_C(\Gamma_i)$, and observe that the shortest path from $x_i$ to $x'_i$ using these edges has length $n-1$. Thus, $x_i, x'_i$ is a near-antipodal pair for every $i \in [k]$. Similarly, $y_j, y'_j$ are near-antipodal for every $j \in [q-k]$. It follows from Proposition 2.4 that $\Gamma'_i$ is bipartite for every $i \in [q]$.

It remains to show that $X$ and $Y$ are independent sets in $\Gamma$. We begin by showing that both $R(X)$ and $C(X)$ are simple. It follows from (2.9) that

$$R(X) = [k] \cup \{q_0 + i : i \in [k]\}.$$

But $k - 1 < q_0$ and $q_0 + k - 1 < n$, which implies $R(X)$ is simple. Now, define

$$\widehat{X} := \{x_i : i \in [k]\} \text{ and } X' := \{x'_i : i \in [k]\}. \tag{2.11}$$

Looking again to (2.9), we see that

$$C(\widehat{X}) = \{3i : i \in [k]\} \text{ and } C(X') = \{\underline{q_1 + 3i} : i \in [k]\}.$$

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\widetilde{000}$ | **101** | 002 | 110 | 011 | 112 | 020 | 121 | 022 | 100 | 001 | 102 | 010 | 111 | 012 | 120 | 021 | 122 |
| 101 | **002** | **100** | $\widetilde{011}$ | 112 | 010 | 121 | 022 | 120 | 001 | 102 | 000 | 111 | 012 | 110 | 021 | 122 | 020 |
| 002 | 100 | **001** | **112** | 010 | 111 | $\widetilde{022}$ | 120 | 021 | 102 | 000 | 101 | 012 | 110 | 011 | 122 | 020 | 121 |
| 110 | 011 | 112 | **020** | **121** | 022 | 100 | 001 | 102 | $\widetilde{010}$ | 111 | 012 | 120 | 021 | 122 | 000 | 101 | 002 |
| 011 | 112 | 010 | 121 | **022** | **120** | 001 | 102 | 000 | 111 | 012 | 110 | $\widetilde{021}$ | 122 | 020 | 101 | 002 | 100 |
| 112 | 010 | 111 | 022 | 120 | **021** | **102** | 000 | 101 | 012 | 110 | 011 | 122 | 020 | 121 | 002 | 100 | 001 |
| 020 | 121 | 022 | 100 | 001 | 102 | **010** | **111** | 012 | 120 | 021 | 122 | 000 | 101 | 002 | 110 | 011 | 112 |
| 121 | 022 | 120 | 001 | 102 | 000 | 111 | **012** | **110** | 021 | 122 | 020 | 101 | 002 | 100 | 011 | 112 | 010 |
| 022 | 120 | 021 | 102 | 000 | 101 | 012 | 110 | **011** | **122** | 020 | 121 | 002 | 100 | 001 | 112 | 010 | 111 |
| 100 | 001 | 102 | 010 | 111 | 012 | 120 | 021 | 122 | **000** | $\widetilde{101}$ | 002 | 110 | 011 | 112 | 020 | 121 | 022 |
| 001 | 102 | 000 | 111 | 012 | 110 | 021 | 122 | 020 | 101 | **002** | **100** | 011 | $\widetilde{112}$ | 010 | 121 | 022 | 120 |
| 102 | 000 | 101 | 012 | 110 | 011 | 122 | 020 | 121 | 002 | 100 | **001** | **112** | 010 | 111 | 022 | $\widetilde{120}$ | 021 |
| 010 | $\widetilde{111}$ | 012 | 120 | 021 | 122 | 000 | 101 | 002 | 110 | 011 | 112 | **020** | **121** | 022 | 100 | 001 | 102 |
| 111 | 012 | 110 | 021 | $\widetilde{122}$ | 020 | 101 | 002 | 100 | 011 | 112 | 010 | 121 | **022** | **120** | 001 | 102 | 000 |
| 012 | 110 | 011 | 122 | 020 | 121 | 002 | 100 | 001 | 112 | 010 | 111 | 022 | 120 | **021** | **102** | 000 | 101 |
| 120 | 021 | 122 | 000 | 101 | 002 | 110 | 011 | 112 | 020 | 121 | 022 | 100 | 001 | 102 | **010** | **111** | 012 |
| 021 | 122 | 020 | 101 | 002 | 100 | 011 | 112 | 010 | 121 | 022 | 120 | 001 | 102 | 000 | 111 | **012** | **110** |
| **122** | 020 | 121 | 002 | 100 | 001 | 112 | 010 | 111 | 022 | 120 | 021 | 102 | 000 | 101 | 012 | 110 | **011** |

Figure 2.4: A Cayley table of $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ with elements of $\widetilde{\mathrm{X}}$, Y, and $\mathbf{D_0}$ highlighted.

Because $3(k-1) < n$, both $C(\widehat{X})$ and $C(X')$ are simple sets. Thus, $C(X) = C(\widehat{X}) \cup C(X')$ is simple unless $C(\widehat{X}) \cap C(X') \neq \emptyset$.

Suppose there were some $x \in C(\widehat{X}) \cap C(X')$. Because $x \in C(\widehat{X})$, there is an $i_0 \in [k]$ such that $x = 3i_0$, meaning $x \equiv 0 \pmod 3$. However, $x \in C(X')$ means $x = \underline{q_1 + 3i_1}$ for some $i_1 \in [k]$, which we claim implies $x \not\equiv 0 \pmod 3$.

Our proof of this claim has two cases. First, if $n$ is a multiple of 3 then so is $q$, and (2.8) tells us that $q_1 = q+1 \equiv 1 \pmod 3$. But then $x \equiv q_1 + 3i_1 \equiv 1 \pmod 3$. On the other hand, if $n$ is not divisible by 3 then (2.8) tells us that $q_1 = q \not\equiv 0 \pmod 3$. When $q + 3i_1 < n$ this yields $x = q + 3i_1 \not\equiv 0 \pmod 3$, while when $q + 3i_1 \geq n$ we have

$$x = q + 3i_1 - n \equiv q - n = -q \not\equiv 0 \pmod 3.$$

Having established that $R(X)$ and $C(X)$ are simple, we observe that $X$ and $Y$ have the same "shape" in $L$ in the sense that $R(Y) \subseteq R(X)$ and $C(Y) \subseteq \{c + 2k : c \in C(X)\}$. Thus, $R(Y)$ and $C(Y)$ must also be simple.

Next, we show $S(X)$ is simple. From here to the end of the proof indices are expressed modulo $m$. Observe that (2.8) implies $q_0 + q_1 \in \{-1, 1\}$. It therefore follows from (2.7) and (2.9) that there is some $w \in \{-1, 1\}$ such that

$$S(x_i) = (4i, h_i + h_{3i}) \text{ and } S(x'_i) = (4i + w, h_{i+q_0} + h_{3i+q_1})$$

for every $i \in [k]$. We immediately see that the first coordinate of $S(x_i) \equiv 0 \pmod 2$, while the first coordinate of $S(x'_j) \equiv 1 \pmod 2$ for every $i, j \in [k]$. Thus, $S(\widehat{X}) \cap S(X') = \emptyset$.

To see that $S(\widehat{X})$ is simple, consider $x_i, x_j \in \widehat{X}$ such that $S(x_i) = S(x_j)$. We then have $h_i + h_{3i} = h_j + h_{3j}$, and applying Lemma 2.2 with $c = d = 0$ and $s = 3$ tells us that $i \equiv j \pmod{m}$. We also have $4i \equiv 4j \pmod{t}$. If $t \leq 4$, then this identity is trivially satisfied. However, in this case (2.9) implies

$$|i - j| < k = \left\lceil \frac{mt}{4} \right\rceil \leq \left\lceil \frac{m4}{4} \right\rceil = m.$$

Thus $i = j$, as distinct numbers are congruent modulo $m$ only if their difference is at least $m$, and $x_i = x_j$. We still need to consider the case $t > 4$, which is equivalent to $t \geq 8$ because $t = 2^l$ for some integer $l \geq 1$. In this case, $4i \equiv 4j \pmod{t}$ implies $i - j \equiv 0 \pmod{2^{l-2}}$. Because $m$ is odd, it is relatively prime to $2^{l-2}$, and the Chinese Remainder Theorem tells us that $x = 0$ is the unique $x \in [2^{l-2}m]$ satisfying $x \equiv 0 \pmod{2^{l-2}}$ and $x \equiv 0 \pmod{m}$. It is easy to see in this case that the unique value is $x = 0$. Because $|i - j| < k = 2^{l-2}m$ and $i - j$ satisfies both of these these congruences, it must be the case that $i = j$.

A similar argument shows that $S(X')$ is simple. Indeed, when $S(x'_i) = S(x'_j)$, applying Lemma 2.2 with $c = q_0$, $d = q_1$, and $s = 3$ yields $i \equiv j \pmod{m}$, while $4i + w \equiv 4j + w \pmod{t}$ implies $4i \equiv 4j \pmod{t}$. From here we may proceed exactly as above.

The proof that $S(Y)$ is simple is nearly identical to the corresponding proof for $S(X)$ just given. By (2.7) and (2.10), there is some $w \in \{-1, 1\}$ such that

$$S(y_i) = (4i + 2k, h_i + h_{3i+2k}) \text{ and } S(y'_i) = (4i + 2k + w, h_{i+q_0} + h_{3i+q_1+2k})$$

for every $i \in [q - k]$. Considering the first coordinates modulo 2, we see $S(\widehat{Y}) \cap S(Y') = \emptyset$. We then check that $S(\widehat{Y})$ and $S(Y')$ are both simple by applying Lemma 2.2 and noting that $4i + z \equiv 4j + z \pmod{t}$ if and only if $4i \equiv 4j \pmod{t}$ for every $z \in \mathbb{Z}$. □

We have thus constructed an optimal coloring of every group $G$ for which $Syl_2(G)$ is nontrivial and cyclic. This chapter's main result immediately follows.

*Proof of Theorem 2.1.* If $Syl_2(G)$ is trivial or non-cyclic, Theorem 1.7 tells us that $\chi(G) = n$. Otherwise, Theorem 1.7 tells us that $\chi(G) \geq n + 2$, which can be combined with the upper bound given Theorem 2.6 to conclude $\chi(G) = n + 2$. □

# Chapter 3

# A general upper bound

In this chapter we step back into a more general setting and consider colorings of groups which are not necessarily Abelian.[1] Our ultimate goal is improve the best known upper bound for $\chi(G)$ (when $n = |G| \geq 3$) from $2n$ to $\frac{3}{2}$n. We achieve this in three steps. First, we bound the chromatic number of non-simple groups using colorings of some normal subgroup and the corresponding quotient group. Next, we combine this result with a characterization of groups with nontrivial cyclic Sylow 2-subgroups to find an upper bound for $\chi(G)$ which depends only on the order of $Syl_2(G)$. This bound is sufficient for our purposes in all cases except $n \equiv 2 \pmod 4$. Finally, we end the chapter by taking care of the outstanding case with an application of Brooks' Theorem.

## 3.1 Coloring with subgroups and block representations

Up until this point we have only considered instantiations of Cayley tables in which the rows and columns have the same ordering. This is the standard way of displaying Cayley tables, but–because the chromatic number is a main class invariant–we need not adhere to this convention. Indeed, arranging the rows and columns of a Cayley table in (possibly) distinct orders allows us to construct a highly structured block representation. More specifically, given any group $G$ and any subgroup $H \subseteq G$, we can factor $L(G)$ into $|G/H|^2$ copies of $L(H)$ (see Figure 3.1 below). To fully describe this block representation, we need to be precise about what we mean by "copies of $L(H)$."

We say that two latin squares $L_1, L_2$ are **essentially identical** if there is an isotopy between them that does not permute rows or columns. In other words, $L_1$ can be obtained from $L_2$ by changing the names of the symbols. Notice that passing between essentially identical latin squares preserves the "shape" of transversals in the sense that a set of indices which describes a transversal in $L_1$ also describes a transversal in $L_2$. This observation,

[1] As such, we will now use multiplicative notation rather than the additive notation of Chapter 2.

which will be crucial to the proof of Theorem 3.8 below, motivates our referring to the block representation in the following lemma as "highly structured."

**Lemma 3.1.** *Let $G$ be a finite group and let $H$ be a subgroup of $G$ with index $t$. Then $L = L(G)$ has the block representation*

$$L = \begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0,t-1} \\ A_{10} & A_{11} & \cdots & A_{1,t-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{t-1,0} & A_{t-1,1} & \cdots & A_{t-1,t-1} \end{pmatrix} \tag{3.1}$$

*in which $A_{00}$ is a Cayley table of $H$ and $A_{ij}$ is a latin subsquare essentially identitcal to $A_{00}$ for every $i, j \in [t]$.*

*Proof.* Let $K = \{k_0, k_1, \ldots, k_{t-1}\}$ be a set of left coset representatives and $Q = \{q_0, q_1, \ldots, q_{t-1}\}$ a set of right coset representatives for $H$ in $G$ such that $k_0 = q_0 = 1$ (the identity element of $G$). Fixing an arbitrary enumeration of $H = \{h_0, h_1, \ldots, h_{m-1}\}$, order the rows of $L$ by

$$h_0, h_1, \ldots, h_{m-1}, k_1h_0, \ldots, k_1h_{m-1}, k_2h_0, \ldots, k_2h_{m-1}, \ldots, k_{t-1}h_0, \ldots, k_{t-1}h_{m-1},$$

and order the columns of $L$ by

$$h_0, h_1, \ldots, h_{m-1}, h_0q_1, \ldots, h_{m-1}q_1, h_0q_2, \ldots, h_{m-1}q_2, \ldots, h_0q_{t-1}, \ldots, h_{m-1}q_{t-1}.$$

We may therefore define the blocks in (3.1) by

$$A_{ij} := \{(k_ih_a, h_bq_j) \,:\, a, b \in [m]\}$$

for every $i, j \in [t]$. Observe that $S(A_{00}(a, b)) = h_ah_b \in H$ for every $a, b \in [m]$. Thus, $A_{00}$ is a Cayley table of $H$. Furthermore, fixing $i, j \in [t]$, we have

$$S(A_{ij}(a, b)) = k_ih_ah_bq_j = k_iS(A_{00}(a, b))q_j. \tag{3.2}$$

for every $a, b \in [m]$. Because $S'(A_{00}) = H$, we then have $S'(A_{ij}) = k_iHq_j$. But $|k_iHq_j| = m$, meaning $A_{ij}$ is an $m \times m$ subsquare (of a latin square) containing exactly $m$ distinct symbols. This is equivalent to $A_{ij}$ being a latin subsquare. Finally, we see from (3.2) that $A_{ij}$ is essentially identical to $A_{00}$ via the isotopy $(\sigma, \tau, \phi)$ in which $\sigma$ and $\tau$ are both the identity map and $\phi : H \to k_iHq_j$ is the map $h \mapsto k_ihq_j$. □

It is well-known that every group on non-prime order contains some nontrivial proper subgroup. But all groups of prime order are cyclic, and as Cayley tables of cyclic groups are already well-understood, Lemma 3.1 allows us to at least approximately describe the structure of *every* Cayley table.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{1}$ | $h$ | $h^2$ | $p$ | $ph^2$ | $ph$ | $p^2$ | $p^2h$ | $p^2h^2$ | $p^3$ | $p^3h^2$ | $p^3h$ |
| $h$ | $\mathbf{h^2}$ | $1$ | $ph^2$ | $ph$ | $p$ | $p^2h$ | $p^2h^2$ | $p^2$ | $p^3h^2$ | $p^3h$ | $p^3$ |
| $h^2$ | $1$ | $\mathbf{h}$ | $ph$ | $p$ | $ph^2$ | $p^2h^2$ | $p^2$ | $p^2h$ | $p^3h$ | $p^3$ | $p^3h^2$ |
| $p$ | $ph$ | $ph^2$ | $\mathbf{p^2}$ | $p^2h^2$ | $p^2h$ | $p^3$ | $p^3h$ | $p^3h^2$ | $1$ | $h^2$ | $h$ |
| $ph$ | $ph^2$ | $p$ | $p^2h^2$ | $\mathbf{p^2h}$ | $p^2$ | $p^3h$ | $p^3h^2$ | $p^3$ | $h^2$ | $h$ | $1$ |
| $ph^2$ | $p$ | $ph$ | $p^2h$ | $p^2$ | $\mathbf{p^2h^2}$ | $p^3h^2$ | $p^3$ | $p^3h$ | $h$ | $1$ | $h^2$ |
| $p^2$ | $p^2h$ | $p^2h^2$ | $p^3$ | $p^3h^2$ | $p^3h$ | $1$ | $h$ | $h^2$ | $\mathbf{p}$ | $ph^2$ | $ph$ |
| $p^2h$ | $p^2h^2$ | $p^2$ | $p^3h^2$ | $p^3h$ | $p^3$ | $h$ | $h^2$ | $1$ | $ph^2$ | $\mathbf{ph}$ | $p$ |
| $p^2h^2$ | $p^2$ | $p^2h$ | $p^3h$ | $p^3$ | $p^3h^2$ | $h^2$ | $1$ | $h$ | $ph$ | $p$ | $\mathbf{ph^2}$ |
| $p^3$ | $p^3h$ | $p^3h^2$ | $1$ | $h^2$ | $h$ | $p$ | $ph$ | $ph^2$ | $p^2$ | $p^2h^2$ | $p^2h$ |
| $p^3h$ | $p^3h^2$ | $p^3$ | $h^2$ | $h$ | $1$ | $ph$ | $ph^2$ | $p$ | $p^2h^2$ | $p^2h$ | $p^2$ |
| $p^3h^2$ | $p^3$ | $p^3h$ | $h$ | $1$ | $h^2$ | $ph^2$ | $p$ | $ph$ | $p^2h$ | $p^2$ | $p^2h^2$ |

Figure 3.1: A Cayley table of $Dic_3$ divided into blocks, as per Lemma 3.1, with a color class from the proof of Theorem 3.2 in bold.

Our ability to partition $L(G)$ into blocks with similarly shaped transversals seems to offer a path towards constructing economical colorings of $G$. Unfortunately, pairs of sets in the collection $\{k_iHq_j : i, j \in [t]\}$ can in general intersect in various ways. Thus, when trying to paste together transversals from different blocks there is a constant risk of picking cells with the same symbol. When $H$ is a normal subgroup of $G$, however, we know exactly how any two blocks intersect. This allows us to color any non-simple group using colorings of smaller groups, yielding the following upper bound for $\chi(G)$.

**Theorem 3.2.** *Let $G$ be a finite group and let $H \triangleleft G$ be a normal subgroup. Then*

$$\chi(G) \leq \chi(H)\chi(G/H).$$

*Proof.* Let $F := G/H = \{f_0, f_1, \ldots, f_{t-1}\}$, and note that $F$ forms a set of left and right coset representatives of $H$ in $G$. By Lemma 3.1, $L = L(G)$ has the block representation (3.1) in which $A_{00} = L(H)$ and each $A_{ij}$ is essentially identical to $L(H)$. Recall from Section 1.3, the chromatic number of a latin square is a main class invariant. Thus, setting $m := \chi(H)$, each $A_{ij}$ has an $m$-coloring $f_{ij} : A_{ij} \to [m]$.

Just as in the proof of Lemma 3.1, we may observe that $S'(A_{ij}) = f_iHf_j$. But $H$ is normal in $G$, so we may in fact write $S'(A_{ij}) = f_if_jH$. If $K$ is the latin square formed by identifying blocks with the symbols therein contained, it then follows that $K = L(F)$. Letting $d = \chi(F)$, we may select some $d$-coloring $f_\infty : K \to [d]$.

Using $f_\infty$ and $\{f_{ij} : i, j \in [t]\}$, we construct a $(dm)$-coloring of $L$, say $f : L \to [d] \times [m]$. For each $i, j \in [t]$ and for every cell $c \in A_{ij} \subseteq L$, set $f(c) = (f_\infty(A_{ij}), f_{ij}(c))$. See Figure 3.1

for an example of a color class in $f$ when $G = Dic_3$ and $H = \mathbb{Z}_3$ . To see that $f$ is indeed a proper coloring, consider $c, c' \in L$ such that $f(c) = f(c')$. Because $f_\infty$ is a proper coloring, $c$ and $c'$ cannot lie in adjacent blocks of $V(\Gamma(K))$. They could lie in the same block, say $A_{ij}$, but because $f_{ij}$ is also a proper coloring, it is nonetheless impossible for $c$ and $c'$ to be adjacent in $\Gamma(L)$. □

It is worth noting that this theorem generalizes a result of Hall and Paige on the existence of complete mappings. Corollary 2 of [23] states: if $H$ is a normal subgroup of $G$, and both $H$ and $G/H$ possess complete mappings then $G$ possesses a complete mapping. To see that this is a special case of Theorem 3.2, recall that a group $G$ possesses a complete mapping if and only if it has chromatic number $n$.

The most obvious application of Theorem 3.2 is to recursive coloring of non-simple groups. In particular, having determined the chromatic number of every Abelian group in Chapter 2, we now have a tool for bounding the chromatic number of any solvable group. Given a solvable group $G$ with subnormal series $G = G_a \triangleright G_{a-1} \triangleright \cdots \triangleright G_1 \triangleright G_0 = \{1\}$, we now know that

$$\chi(G) \leq \chi(G/G_{a-1})\chi(G_{a-1}/G_{a-2}) \cdots \chi(G_2/G_1)\chi(G_1). \tag{3.3}$$

However, whenever a factor $G_i/G_{i-1}$ does not possess a complete mapping we pick up several redundant colors. If there are enough such factors, then the bound in (3.3) could be significantly worse than Wanless' bound of $\chi(G) \leq 2n$. For example, the dicyclic group of order 28 has a subnormal series of the form

$$Dic_7 \triangleright \mathbb{Z}_{14} \triangleright \{1\},$$

which when combined with (3.3) gives an upper bound of $\chi(Dic_7) \leq (4)(16) = 64$.

To uncover the true power of Theorem 3.2 we must make recourse to Theorem 1.7. If we want to bound the chromatic number of every Cayley table, we need only consider those which we do not already know to be $n$-colorable. Theorem 1.7 tells us that $\chi(G) = n$ if and only if $Syl_2(G)$ is either trivial or non-cyclic. Thus, for any $k > n$, showing $\chi(G) \leq k$ for *all* groups amounts to finding a $k$-coloring of every group $G$ for which $Syl_2(G)$ is nontrivial and cyclic.

In [23], Hall and Paige noticed that all groups with nontrivial cyclic Sylow 2-subgroups have a concise yet robust structural description. Given a group $G$, a normal subgroup $H \triangleleft G$ and a (not necessarily normal) subgroup $K \subseteq G$, we say that $G$ is the **(internal) semidirect product** of $H$ and $K$, written $G = K \ltimes H$, if $G = KH = \{kh : k \in K, h \in H\}$ and $K \cap H = \{1\}$. Note that $G = K \ltimes H$ implies $K$ is a collection of coset representatives for $H$ in $G$. The following result, first mentioned in this context by Hall and Paige [23], is essentially a corollary of a theorem due to Burnside (Theorem 14.3.1 in [24]).

**Lemma 3.3.** *Let $G$ be a finite group and let $P$ be a Sylow 2-subgroup of $G$. If $P$ is cyclic and nontrivial then there is a normal subgroup of odd order $H \triangleleft G$ such that $G = P \ltimes H$.*

Recall that in Chapter 2 we used the fundamental theorem of finite Abelian groups to write $G$ as the direct product of $Syl_2(G)$ and a subgroup of odd order. We now see that the existence of such a decomposition does not depend on the fundamental theorem of finite Abelian groups, but instead can be proven directly from Lemma 3.3.

We are now in a position to make full use of Theorem 3.2. By combining this theorem with Lemma 3.3, we get an upper bound on the chromatic number of every Cayley table. Because Corollary 1.6 tells us that all groups of odd order have chromatic number $n$, we state this result only for groups of even order.

**Theorem 3.4.** *Let $G$ be a group of even order $n \geq 3$ and write $n$ as $n = mt$, where $m$ is odd and $t = 2^l$ for $l \geq 1$. Then*

$$\chi(G) \leq \frac{t+2}{t}n = n + \frac{2n}{t}.$$

*Proof.* If $Syl_2(G)$ is non-cyclic then $\chi(G) = n \leq n + \frac{2n}{t}$, so we may assume $Syl_2(G) = \mathbb{Z}_t$. Then, Lemma 3.3 tells us that $G = \mathbb{Z}_t \ltimes H$ for some normal subgroup $H$ of odd order $m$. In particular, $G/H = \mathbb{Z}_t$. Combining this fact with Theorem 3.2, Corollary 1.6, and Theorem 2.1, we see that

$$\chi(G) \leq \chi(H)\chi(\mathbb{Z}_t) = m(t+2) = \frac{t+2}{t}n.$$

□

It is worth considering the degree to which this theorem improves on Wanless' upper bound of $\chi(G) \leq 2n$. As $t$ grows with respect to $m$, Theorem 3.4 approaches the conjectured best possible bound of $\chi(G) \leq n+2$. Indeed, when $n = t$ we get exactly $\chi(G) \leq n+2$. Of course, we already knew the chromatic number of all 2-groups: if the group $G$ of order $2^l$ is cyclic, then Theorem 2.1 tells us $\chi(G) = 2^l + 2$, and otherwise Theorem 1.7 tells us $\chi(G) = 2^l$. Nonetheless, Theorem 3.4 can give very strong upper bounds on $\chi(G)$. For example, if $G$ is a group whose order $n$ is divisible by 64 then $\chi(G) \leq \frac{33}{32}n$.

We are more concerned with the case when $t$ is small. When $t = 4$ Theorem 3.4 tells us that $\chi(G) \leq \frac{3}{2}n$. This is exactly the general upper bound for $\chi(G)$ which we will establish in Section 3.3. Indeed, Theorem 3.4 establishes that $\chi(G) \leq \frac{3}{2}n$ whenever $|G| = n$ is divisible by 4. We are left to consider the case $t = 2$, a case in which Theorem 3.4 reproduces Wanless' upper bound. Before dealing with this case in Section 3.3, we make a brief detour to consider possible improvements to Theorem 3.2.

## 3.2 The limits of block coloring

In the proof of Theorem 3.2 we colored the group $G$ by pasting together colorings of some normal subgroup $H$ with colorings of the quotient group $G/H$. Assuming that $L(G)$ is written as in (3.1), the latin square $K$, formed by identifying each block with the set of symbols it contains, is essentially identical to $L(G/H)$. Setting $m := \chi(H)$, we can properly color $L = L(G)$ by assigning a set of size $m$ to each block $A_{ij}$ in such a way that two blocks which are adjacent in $\Gamma(K)$ receive disjoint sets. We achieved this above by finding a coloring of $K$, say $f$, then using a disjoint set of size $m$ for each color class of $f$.

In general, this idea gives a reasonably good bound on the chromatic number of $L$. However, it is not the most efficient means of exploiting the structure in (3.1). We define an $m$**-fold coloring** of a graph $\Gamma$ as an assignment of a list of size $m$ to each of its vertices in such a way that adjacent vertices receive disjoint lists. The $m$**-fold chromatic number** of $\Gamma$, denoted $\chi_m(\Gamma)$, is the least number of colors needed to populate the lists in an $m$-fold coloring of $\Gamma$. As above, we define $m$-fold colorings and the $m$-fold chromatic number of a latin square via the corresponding latin square graph, and further extend these notions to groups by making recourse to their Cayley tables.

We can now succinctly restate the discussion in the first paragraph of this section. If $G$ is a finite group with a normal subgroup $H \triangleleft G$ satisfying $\chi(H) = m$, then

$$\chi(G) \leq \chi_m(G/H).$$

Furthermore, the argument at the end of the proof of Theorem 3.2 tells us that

$$\chi_m(G/H) \leq m\,\chi(G/H). \tag{3.4}$$

Combining these two inequalities gives us exactly the conclusion of Theorem 3.2. But equality does not always hold in (3.4), and in this case we may obtain an improved upper bound for $\chi(G)$. This compels us to ask: how small can $\chi_m(G/H)$ be?

For notational simplicity, let $\Gamma := \Gamma(L(G/H))$. It turns out that we can determine an exact lower bound for $\chi_m(\Gamma)$ when we view this quantity as a function of $m$. The first step in this process is to notice that $\chi_m$ is subadditive, i.e. that $\chi_{a+b}(\Gamma) \leq \chi_a(\Gamma) + \chi_b(\Gamma)$. This is due to the fact that the union of an $a$-fold coloring and a $b$-fold coloring is an $(a+b)$-fold coloring. We then define the **fractional chromatic number** of a graph $\Gamma$, denoted $\chi_f(\Gamma)$, as

$$\chi_f(\Gamma) := \lim_{m\to\infty} \frac{\chi_m(\Gamma)}{m}. \tag{3.5}$$

Observe that the above limit is well-defined precisely because the function $\chi_m(\Gamma)$ is subadditive. The fractional chromatic number is a well-studied graph parameter, and its value is known exactly for vertex-transitive graphs. And indeed, all Cayley tables have vertex-transitive latin square graphs.

**Proposition 3.5.** *Let $L$ be the Cayley table of a finite group $G$. Then the associated latin square graph $\Gamma = \Gamma(L)$ is vertex-transitive.*

*Proof.* We can think of $V = V(\Gamma)$ as the set of triples $\{(g, h, gh) : g, h \in G\}$. We want to show that, given any pair of vertices $v_1, v_2 \in V$, there is an automorphism $\sigma \in Aut(\Gamma)$ sending $v_1$ to $v_2$. We can write $v_1 = (g_1, h_1, g_1h_1)$ and $v_2 = (g_2, h_2, g_2h_2)$. By definition, there are unique $g_0, h_0 \in G$ such that $g_0g_1 = g_2$ and $h_1h_0 = h_2$. Furthermore, $g_0g_i = g_0g_j$ if and only if $g_i = g_j$, and a similar result holds for right multiplication by $h_0$. Thus, the map $(g, h, gh) \mapsto (g_0g, hh_0, g_0ghh_0)$, which sends $v_1$ to $v_2$, is an automorphism of $\Gamma$ with the desired property. □

Although this proposition has a straightforward proof, it marks another important distinction between Cayley tables and general latin squares. As noted by Cameron in the discussion section of [2], almost all latin square graphs are *not* vertex-transitive. In fact, this asymptotic result is even stronger: almost all latin square graphs have a trivial automorphism group. Nonetheless, that the latin square graphs corresponding to Cayley tables are among the few such graphs with this strong algebraic property should not be too surprising.

Given Proposition 3.5, the following well-known result now plays a significant role in our understanding of block colorings.

**Proposition 3.6.** *Let $\Gamma$ be a vertex-transitive graph and let $\alpha$ be the size of the largest independent set in $\Gamma$. Then*

$$\chi_f(\Gamma) = \frac{|V(\Gamma)|}{\alpha}.$$

*Proof.* We first show that $\chi_m(\Gamma) \geq \frac{m|V(\Gamma)|}{\alpha}$ for every positive integer $m$. As an $m$-fold coloring assigns a list of size $m$ to every vertex in $\Gamma$, we have to make $m|V(\Gamma)|$ color assignments in total. We can assign a single color to several different vertices, but the vertices to which any given color is assigned must form an independent set. In other words, each color can be assigned to at most $\alpha$ vertices. Thus, every $m$-fold coloring uses at least $\frac{m|V(\Gamma)|}{\alpha}$ colors.

It is left to show that, for every vertex-transitive graph $\Gamma$, there is a positive integer $m$ such that $\Gamma$ has an $m$-fold coloring using exactly $\frac{m|V(\Gamma)|}{\alpha}$ colors. To construct such a coloring, let $A$ be an independent set of size $\alpha$ in $\Gamma$. Then, because the automorphism group $Aut(\Gamma)$ acts transitively on $\Gamma$, the set

$$\mathcal{C} := \{\sigma(A) : \sigma \in Aut(\Gamma)\}$$

covers each vertex in $\Gamma$ exactly $m$ times for some $m \in \mathbb{Z}$. We can then construct an $m$-fold coloring of $\Gamma$ by assigning to each $v \in V(\Gamma)$ the list of sets in $\mathcal{C}$ which contain $v$. And in fact this coloring has the desired size, as $\sum_{\sigma \in Aug(\Gamma)} |\sigma(A)| = m|V(\Gamma)|$, so that $|\mathcal{C}| = \frac{m|V(\Gamma)|}{\alpha}$. □

It is worth reflecting on the utility of this proposition. In general, determining the size of the largest independent set in a graph is NP-Hard [28]. For latin square graphs, however, this question is equivalent to asking for the size of the largest partial transversal in the corresponding latin square. As such, we have a fairly comprehensive understanding of the possible values $\alpha$ may take in the relevant applications of Proposition 3.6.

Returning to block colorings, let $G$ be a group of order $n$ and let $H \triangleleft G$ be a normal subgroup with chromatic number $m$ such that $|G/H| = t$. Setting $\Gamma := \Gamma(L(G/H))$, we know that the largest independent set in $\Gamma$ has size $\alpha \leq k$. If $\alpha = k$ then $\chi_f(G/H) = k = \chi(G/H)$, so that passing to $k$-fold colorings of $G/H$ offers no improvement on Theorem 3.2. Otherwise $\alpha \leq k-1$, in which case $\chi_m(\Gamma) \geq \frac{mk^2}{k-1} = \frac{k}{k-1}|G|$. Thus, passing to $m$-fold colorings of a quotient group will never give us an upper bound of the form $\chi(G) \leq n + o(n)$.

Let us compare this bound to the one given in Theorem 3.4. We may assume $G = \mathbb{Z}_t \ltimes H$ is a group of order $n = tm$, where $t = 2^l \geq 2$ and $|H| = m$ is odd. The arguments in the preceding paragraph tell us that

$$\chi(G) \leq \frac{t}{t-1} n. \tag{3.6}$$

If $\chi_m(G/H)$ does in fact equal $\frac{t}{t-1}n$, we get a slight improvement on the upper bound of $\frac{t+2}{t}n$ given by Theorem 3.4. However, we may not assume that there is an $m$-fold coloring of size $m\chi_f(\Gamma)$ for every graph $\Gamma$ and every positive integer $m$. Furthermore, in the case $t = 2$ Theorem 3.4 and (3.6) give the same upper bound. This case is the barrier to proving any general upper bound for $\chi(G)$, and the best bound we can find for it aligns with Theorem 3.4. As such, the above discussion of $m$-fold colorings does little more than provide context for this chapter's central results.

## 3.3 Proof of the general upper bound

Before the work in this thesis, the best known general upper bound for $\chi(G)$ was $2n$. As noted above, Theorem 3.4 improves upon this bound for most groups. However, we don't get any improvement in the case $n \equiv 2 \pmod 4$. We take care of this case using a famous graph-theoretic result relating a graph's chromatic number to its maximum degree.

**Theorem 3.7** (Brooks' Theorem)**.** *Let $\Gamma$ be a finite connected graph that is not isomorphic to a complete graph or an odd cycle. Then*

$$\chi(\Gamma) \leq \Delta(\Gamma).$$

Looking at Brooks' Theorem, it is not immediately clear how we will use it to show that a $(3n-3)$-regular graph has a $\left(\frac{3}{2}n\right)$-coloring. There is certainly some work needed to set ourselves up to apply Brooks' Theorem. However, this work is essentially a recreation of the argument Wanless used in [49] to find a $2n$-coloring for every Cayley table. The main focus of [49] was a family of generalized transversals known as "plexes." A $k$-**plex** in a latin square $L$ is a set of cells $P \subseteq L$ which intersects each row, column, and symbol class exactly $k$ times. Associated with $k$-plexes is a generalized notion of coloring; a $k$-**partition** of a latin square is a partition of its cells into $\frac{n}{k}$ disjoint $k$-plexes.

When attributing the general upper bound $\chi(G) \leq 2n$ to Wanless, we are in fact citing his proof that every Cayley table of even order has a 2-partition. Recalling that Corollary 1.6 takes care of the chromatic number question for all groups of odd order, the connection between 2-partitions and $2n$-colorings comes from the following observation. For a latin square $L$ with latin square graph $\Gamma = \Gamma(L)$ and a 2-plex $X \subseteq L$, in the induced subgraph $\Gamma[X]$ every vertex is incident to exactly one edge of each type (row, column, symbol). Thus, a 2-partition of $L$ corresponds to a partition of $\Gamma$ into $\frac{n}{2}$ cubic induced subgraphs. Greedily 4-coloring each of these induced subgraphs with a distinct set of colors, we obtain the desired $2n$-coloring of $L$. This is where Brooks' Theorem comes in. If we are careful in defining our 2-partition, we can show that each of the cubic induced subgraphs is in fact 3-colorable.

**Theorem 3.8.** *Let $G$ be a group of order $n$. Then*

$$\chi(G) \leq \frac{3}{2}n.$$

*Proof.* Let $P$ be a Sylow 2-subgroup of $G$. We may assume $P$ is cyclic and nontrivial, as otherwise $\chi(G) = n \leq \frac{3}{2}n$ by Theorem 1.7. We know $|P| = 2^l$ for some $l \geq 1$, and there is an odd integer $m$ such that $n = 2^l m$. If $l \geq 2$, then Theorem 3.4 tells us that

$$\chi(G) \leq n + \frac{2n}{2^l} \leq n + \frac{2n}{4} = \frac{3}{2}n,$$

as desired. Thus, we may assume that $n = 2m$ and $P = \{0, p\}$. By Lemma 3.3, $G = P \ltimes H$ for some normal subgroup $H \triangleleft G$ of order $m$. Arbitrarily enumerating $H = \{h_0, h_1, \ldots, h_{m-1}\}$, we arrange $L$ as in Lemma 3.1, yielding the block representation

$$L = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

in which $A_{ij}$ is essentially identical to $A_{rs}$ for every $i, j, r, s = \{1, 2\}$ and $A_{00} = L(H)$. Furthermore, $S'(A_{01}) = S'(A_{10}) = pH$ and $S'(A_{00}) = S'(A_{11}) = H$.

By Corollary 1.6, there exists an $m$-coloring of $A_{00}$, say $(T_0, T_1, \ldots, T_{m-1})$. We can think of each $T_i$ as a complete mapping of $H$ sending $h_j$ to $h_k$ if and only if $(h_j, h_k)$ is in the set of cells comprising $T_i$. We then extend $T_i$ to a two-to-one map from $G$ to $H$ by setting $T_i(ph) = T_i(h)$ for all $h \in H$. As $A_{00}$ and $A_{10}$ are essentially identical, $(T_0|_{pH}, T_1|_{pH}, \ldots, T_{m-1}|_{pH})$ (the restrictions of the $T_i$s to $pH$) gives an $m$-coloring of $A_{10}$.

Because $A_{01}$ is also essentially identical to $A_{00}$, there is an $m$-coloring of $A_{01}$ defined by the collection of mappings $(Q_0, Q_1, \ldots, Q_{m-1})$ with domain $H$ and range $pH$. As above, we

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $1$ | $\mathbf{r}$ | $r^2$ | $r^3$ | $r^4$ | $s$ | $sr^4$ | $\mathbf{sr^3}$ | $sr^2$ | $sr$ |
| $r$ | $r^2$ | $\mathbf{r^3}$ | $r^4$ | $1$ | $sr^4$ | $sr^3$ | $sr^2$ | $\mathbf{sr}$ | $s$ |
| $r^2$ | $r^3$ | $r^4$ | $\mathbf{1}$ | $r$ | $sr^3$ | $sr^2$ | $sr$ | $s$ | $\mathbf{sr^4}$ |
| $r^3$ | $r^4$ | $1$ | $r$ | $\mathbf{r^2}$ | $\mathbf{sr^2}$ | $sr$ | $s$ | $sr^4$ | $sr^3$ |
| $\mathbf{r^4}$ | $1$ | $r$ | $r^2$ | $r^3$ | $sr$ | $\mathbf{s}$ | $sr^4$ | $sr^3$ | $sr^2$ |
| $s$ | $\mathbf{sr}$ | $sr^2$ | $sr^3$ | $sr^4$ | $1$ | $\mathbf{r^4}$ | $r^3$ | $r^2$ | $r$ |
| $sr$ | $sr^2$ | $\mathbf{sr^3}$ | $sr^4$ | $s$ | $r^4$ | $r^3$ | $\mathbf{r^2}$ | $r$ | $1$ |
| $sr^2$ | $sr^3$ | $sr^4$ | $\mathbf{s}$ | $sr$ | $r^3$ | $r^2$ | $r$ | $\mathbf{1}$ | $r^4$ |
| $sr^3$ | $sr^4$ | $s$ | $sr$ | $\mathbf{sr^2}$ | $r^2$ | $r$ | $1$ | $r^4$ | $\mathbf{r^3}$ |
| $\mathbf{sr^4}$ | $s$ | $sr$ | $sr^2$ | $sr^3$ | $\mathbf{r}$ | $1$ | $r^4$ | $r^3$ | $r^2$ |

Figure 3.2: A Cayley table of $D_5$ with the 2-plex $W_1$ in bold.

can extend the $Q_i$s to two-to-one maps from $G$ to $H$ by setting $Q_i(ph) = Q_i(h)$ for every $h \in H$, and the restrictions $(Q_0|_{pH}, Q_1|_{pH}, \ldots, Q_{m-1}|_{pH})$ will give an $m$-coloring of $A_{11}$.

Expressing indices modulo $m$, we define a set

$$W_i := \{(g, T_i(g)) \,:\, g \in G\} \cup \{(ph, Q_i(ph)), (h, Q_{i+1}(h)) \,:\, h \in H\}$$

for every $i \in [m]$. Because $W_i$ is the union of transversals from each of the blocks in (3.1), it forms a 2-plex, and it is easy to check that $\mathcal{W} = (W_0, W_1, \ldots, W_{m-1})$ is a 2-partition of $L$. We can therefore partition $\Gamma = \Gamma(L)$ into $\frac{n}{2}$ cubic, induced subgraphs $\Gamma_i = \Gamma[W_i]$ for $i \in [m]$. If we can 3-color each $\Gamma_i$, then we will have a $\left(\frac{3}{2}n\right)$-coloring of $\Gamma$.

By Brooks' Theorem, each $\Gamma_i$ is 3-colorable unless it contains a connected component isomorphic to the complete graph $K_4$. If there is such a connected component $\Lambda$, then the subgraph of $\Lambda$ formed by row and column edges must be isomorphic to a 4-cycle. We demonstrate that this cannot be the case by showing an arbitrary row/column walk of length 4 in $\Gamma_i$ is not closed. See Figure 3.2 for a visual aid in reading what follows.

Start at an arbitrary vertex $v_0 = (h_j, T_i(h_j)) \in W_i \cap A_{00}$. The only other vertex of $W_i$ in the row bordered by $h_j$ is $v_1 = (h_j, Q_{i+1}(h_j))$, so we take the edge $v_0v_1$. Following the single column edge incident to $v_1$, we get to $v_2 = (Q_i^{-1}(Q_{i+1}(h_j)), Q_{i+1}(h_j))$. The third edge in our walk is a row-edge taking us to $v_3 = (Q_i^{-1}(Q_{i+1}(h_j)), T_i(Q_i^{-1}(Q_{i+1}(h_j))))$. We are now set to follow a column edge back to $A_{00}$. If our walk is closed, then $C(v_0) = C(v_3)$. But this would mean that $T_i(Q_i^{-1}(Q_{i+1}(h_j)) = T_i(h_j)$, or, equivalently, $Q_i(h_j) = Q_{i+1}(h_j)$. But this cannot be the case, as the $Q_i$s were defined to be disjoint transversals. Our walk of length 4 therefore ends at some $v_4 \neq v_0$, and as such is not closed. □

# Chapter 4

# Future directions

The previous two chapters detailed work towards Conjecture 1.3 in the special case that $L$ is the Cayley table of a finite group. Despite our progress, the chromatic number question for Cayley tables remains open, giving us a clear beacon towards which to focus future work.

**Conjecture 4.1.** *Let $G$ be a group of order $n$. Then*

$$\chi(G) = \begin{cases} n & \text{if } Syl_2(G) \text{ is either trivial or non-cyclic,} \\ n+2 & \text{otherwise.} \end{cases}$$

This conjecture, which we believe is both true and within reach, implies the corresponding special case of Brualdi's Conjecture (see Section 1.2). Resolving even this weaker conjecture would be of significant interest.

**Conjecture 4.2.** *Let $G$ be a group of order $n$. Then $L(G)$ possesses a near transversal.*

We believe that Conjecture 4.1 needs additional algebraic insight to be solved. Contrastingly, it seems to us that Conjecture 4.2 could be solved with a novel application of known tools.

These two conjectures are interesting yet obvious extensions of the work in this thesis, and we would like to see both of them resolved in the next few years. We end by outlining a couple of more subtle means of extending our work. We hope that these discussions give the reader a better idea of the difficulties inherent in Conjecture 4.1 and the limitations of our methods.

## 4.1 The curious case of dihedral groups

With a cyclic subgroup of index 2 and a nice geometric description, dihedral groups are often the first family of non-Abelian groups introduced in an undergraduate group theory course. As such, they seem to be a good candidate for initial work towards extending

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $1_L$ | $r_E$ | $r^2_C$ | $r^3_B$ | $r^4_F$ | $s_A$ | $sr^4_I$ | $sr^3_H$ | $sr^2_D$ | $sr_K$ |
| $r_A$ | $r^2_J$ | $r^3_H$ | $r^4_K$ | $1_E$ | $sr^4_L$ | $sr^3_B$ | $sr^2_F$ | $sr_C$ | $s_I$ |
| $r^2_I$ | $r^3_G$ | $r^4_A$ | $1_C$ | $r_D$ | $sr^3_E$ | $sr^2_L$ | $sr_J$ | $s_K$ | $sr^4_B$ |
| $r^3_F$ | $r^4_H$ | $1_D$ | $r_L$ | $r^2_A$ | $sr^2_B$ | $sr_E$ | $s_C$ | $sr^4_J$ | $sr^3_G$ |
| $r^4_B$ | $1_K$ | $r_J$ | $r^2_G$ | $r^3_C$ | $sr_F$ | $s_H$ | $sr^4_D$ | $sr^3_A$ | $sr^2_E$ |
| $s_E$ | $sr_A$ | $sr^2_K$ | $sr^3_J$ | $sr^4_G$ | $1_I$ | $r^4_D$ | $r^3_L$ | $r^2_F$ | $r_H$ |
| $sr_H$ | $sr^2_I$ | $sr^3_L$ | $sr^4_A$ | $s_B$ | $r^4_C$ | $r^3_J$ | $r^2_E$ | $r_G$ | $1_F$ |
| $sr^2_G$ | $sr^3_C$ | $sr^4_E$ | $s_F$ | $sr_L$ | $r^3_D$ | $r^2_K$ | $r_I$ | $1_H$ | $r^4_J$ |
| $sr^3_K$ | $sr^4_F$ | $s_G$ | $sr_I$ | $sr^2_J$ | $r^2_H$ | $r_C$ | $1_B$ | $r^4_L$ | $r^3_A$ |
| $sr^4_C$ | $s_L$ | $sr_B$ | $sr^2_H$ | $sr^3_I$ | $r_K$ | $1_A$ | $r^4_G$ | $r^3_E$ | $r^2_D$ |

Figure 4.1: An experimentally generated 12-coloring of $D_5$ with colors given by capital letter subscripts.

Theorem 2.1. However, they were not amenable to the techniques used in Chapter 2. We were able to find several partitions of $D_n$ into 3-chromatic subgraphs, but in none of these cases could we rearrange the coloring as in the proof of Theorem 2.6.

Nonetheless, we were able to make some progress with colorings of dihedral groups. Using $D_n$ to denote the dihedral group of order $2n$, notice that Theorem 1.7 implies $\chi(D_n) = 2n$ for ever even $n$. Thus, we only need to consider dihedral groups whose order is congruent to 2 modulo 4. In [5], the authors mention computational work establishing $\chi(D_3) = 8$, a result which aligns with the prediction of Conjecture 4.1. Taking the computational verification one step further, we experimentally constructed a 12-coloring for $D_5$. This coloring is displayed in Figure 4.1. Unfortunately, experimental coloring of latin square graphs becomes infeasible very quickly. While the graph associated with $D_5$ has 100 vertices, the graph associated with $D_7$ has 144, which is already too large for brute-force methods on a computer with 128 GB of RAM.

With respect to purely theoretical approaches, it is worth mentioning that Theorem 3.8 was originally devised as a result concerning dihedral groups. And in fact, translating the proof from this context to general non-Abelian groups of order congruent to 2 modulo 4 was simply a matter of bookkeeping. We were also able to prove Brualdi's conjecture holds for dihedral groups thanks to a peculiar 2-partition. We quickly realized that this result had already been known for over a decade [31]. However, in the hopes that our ideas may inspire future breakthroughs, we present the proof here.

Let $Q_3$ denote the graph formed by the skeleton of a 3-dimensional cube. We define the graph $\Lambda_k$ for every odd $k \geq 3$ as the disjoint union of $\frac{k-1}{2}$ copies of $Q_3$ and one copy of $K_4$. See Figure 4.2 for a drawing of $\Lambda_3$. As $\alpha(Q_3) = 4$ and $\alpha(K_4) = 1$, we have $\alpha(\Lambda_k) = 2k - 1$. Thus, if $\Lambda_k$ occurs as an induced subgraph in a latin square graph $\Gamma(L)$ of order $2k$, the latin square $L$ must possess a near transversal. We show that, in dihedral groups, we can find many such induced subgraphs.

**Theorem 4.3.** *Let $n$ be an odd positive integer, let $L = L(D_n)$, and let $\Gamma = \Gamma(L)$. Then $L$ contains a 2-partition $\mathcal{P}$ in which $\Gamma[P] \cong \Lambda_n$ for every $P \in \mathcal{P}$.*

*Proof.* Defining $D_n$ by the presentation $\langle r, s \,|\, r^n = s^2 = 1, srs = r^{-1}\rangle$, order the rows and columns of $L$ as in Lemma 3.1. Then $L$ has a block representation

$$L = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix},$$

in which $A_{00}$ is a Cayley table of $\langle r \rangle$. Because $n$ is odd, Lemma 2.3 tells us that $A_{00}$ is $n$-colorable by right diagonals. As Lemma 3.1 also tells us that each of the latin subsquares $A_{ij}$ is essentially identical to $A_{00}$, we may $n$-color $A_{ij}$ with right diagonals for each $i, j \in [2]$.

Thus, for $d \in [n]$, the sets

$$P_d := \bigcup_{i,j \in [2]} T_d^{A_{ij}}$$

form a 2-partition of $L$. Fixing $d$, let $\Gamma_d = \Gamma[P_d]$. Note that the subgraph of $\Gamma_d$ which uses only row and column edges is made up of $n$ disjoint 4-cyles. We index these cycles by

$$B_j := (r^j, r^{j+d}), (r^j, sr^{-j-d}), (sr^j, sr^{-j-d}), (sr^j, r^{j+d})$$

for $j \in [n]$. To complete the proof we must show that the symbol edges of $\Gamma_d$ always behave as in Figure 4.2.

Observe that $S(r^j, r^{j+d}) = r^{2j+d}$ and $S(sr^j, r^{j+d}) = sr^{2j+d}$, while

$$S(sr^j, sr^{-j-d}) = r^{-d-2j} = sS(r^j, sr^{-d-j}).$$

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{1}$ | $r$ | $r^2$ | $\mathbf{s}$ | $sr^2$ | $sr$ |
| $r$ | $\mathbf{r^2}$ | $1$ | $sr^2$ | $\mathbf{sr}$ | $s$ |
| $r^2$ | $1$ | $\mathbf{r}$ | $sr$ | $s$ | $\mathbf{sr^2}$ |
| $\mathbf{s}$ | $sr$ | $sr^2$ | $\mathbf{1}$ | $r^2$ | $r$ |
| $sr$ | $\mathbf{sr^2}$ | $s$ | $r^2$ | $\mathbf{r}$ | $1$ |
| $sr^2$ | $s$ | $\mathbf{sr}$ | $r$ | $1$ | $\mathbf{r^2}$ |

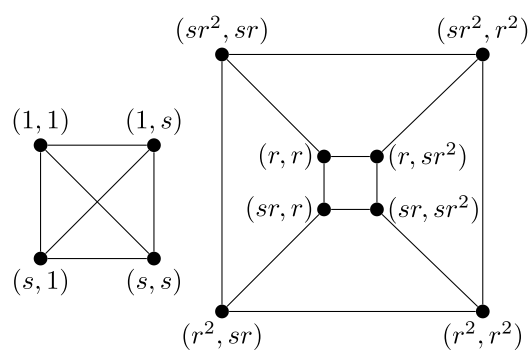


Figure 4.2: The cells of the 2-plex $P_0 \subseteq L(D_3)$ in bold, with the corresponding induced subgraph $\Gamma[P_0] \cong \Lambda_3$ on the right.

But whenever $n$ is odd, there is a unique $j_0 \in [n]$ satisfying $2j_0 + d \equiv -d - 2j_0 \pmod{n}$. Therefore there is exactly one $B_{j_0}$ corresponding to a $K_4$, and it is disconnected from the rest of $\Gamma_d$. Furthermore, $2j + d = -d - 2i$ if and only if $2i + d = -d - 2j$. Thus, whenever there is a symbol edge connecting $B_j$ to $B_i$, there must be four such symbol edges. This means that the rest of $\Gamma_d$ must break into disjoint copies of $Q_3$. □

## 4.2 Extending to equitable colorings

Given a graph $\Gamma$, an **equitable $k$-coloring** of $\Gamma$ is a $k$-coloring in which the size of any two color classes differs by at most 1. The **equitable chromatic number** of $\Gamma$, denoted $\chi_{eq}(\Gamma)$, is the minimum $k$ for which $\Gamma$ can be equitably $k$-colored. These definitions extend to latin squares via the duality between $L$ and $\Gamma(L)$.

Observe that every $n$-coloring of a latin square is an equitable coloring. And indeed, it seems natural that design-theoretic objects like latin squares should possess highly structured colorings. Such concerns likely drove Cavenagh and Kuhl to conjecture the following.

**Conjecture 4.4** (Cavenagh, Kuhl [15])**.** *Let $L$ be a latin square. Then*

$$\chi_{eq}(L) = \chi(L). \tag{4.1}$$

Cavenagh and Kuhl proved their conjecture holds for $L(\mathbb{Z}_n)$ whenever $n$ is congruent to 2 or 10 modulo 12. They did so by constructing a coloring similar to the one we constructed in Chapter 2, then extending the two smaller color classes by carefully removing cells from several of the larger ones. Can a similar process be done with our colorings? We spent some time trying to modify our colorings to equitably color $L(\mathbb{Z}_n)$ for values of $n$ with respect to which Conjecture 4.4 is still open, but were unable to find an extension. At the same time, we were also unable to show that it is not possible to extend our colorings in this way.

Perhaps our methods cannot be utilized to attack this problem. Regardless, the restriction of Conjecture 4.4 to Cayley tables deserves attention. It is not even clear to us that the conjecture should be true. However, with the chromatic number of all Cayley tables of Abelian groups now determined, there are many new cases in which a definitive counterexample could be found.

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