The Realisability of γ-Graphs

by

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Abstract

The γ-graph $\gamma \cdot G$ of a graph G is the graph whose vertices are labelled by the minimum dominating sets of G, in which two vertices are adjacent when their corresponding minimum dominating sets differ in exactly one element. We give an explicit construction of a graph having an arbitrary prescribed set of minimum dominating sets. We show as a corollary that "labellable implies realisable for γ-graphs": if the vertices of a graph H can be labelled by distinct sets of the same size, in a manner consistent with the adjacency condition for γ-graphs, then $H = \gamma \cdot G$ for some graph G. We use this corollary to extend the classification of γ-graphs, due to Lakshmanan and Vijayakumar, to all graphs on at most six vertices. We also use this corollary to relate γ-graphs both to induced subgraphs of Johnson graphs and to optimal dominating codes in graphs.

Keywords: gamma graphs; graph domination; induced subgraphs of Johnson graphs

Dedication

To all my friends, now and in the future:

> “Times of joy and times of sorrow
> We will always see it through
> Oh, I don’t care what comes tomorrow
> We can face it together
> The way old friends do”
>
> — ABBA, *The Way Old Friends Do*

Acknowledgements

I often look back at my life and think "Those past few years were definitely the most formative." While avoiding the subject of quantifying truth, I dare say that thought has never been more true than now: I am surrounded by tremendous people, as exemplified by the following.

Theorem 0.1. *My family is tremendous.*

Proof. Proof by wishful thinking: you put up with me both before and during this process, and when the inflated ego borne by a Master's degree subsides I have faith that you will still be there. □

Corollary 0.2. *My cat Emerald is tremendous.*

Proof. Proof by occupation: whenever we are together he occupies most of my phone storage and all of my lap. He is pure. □

Theorem 0.3. *My Saskatchewan friends are tremendous.*

Proof. Proof by accomplishment: you made $-40°$C tolerable while I lived there, and you make it tolerable whenever I return. That is no small feat. □

Theorem 0.4. *My Vancouver friends are tremendous.*

Proof. Proof by existence: you exist, and I love the hell out of you. I may never know how you tolerate my shenanigans, and I am not sure I need to know. □

Corollary 0.5. *My roommate George is tremendous.*

Proof. Proof by hand-wavy calculus: let D_i be game i of Dominion, and let A_j be hour j spent listening to ABBA. Transform these quantities into limits somehow; both tend to $+\infty$. Thank you for introducing me to music like I have never listened to it before—you are a gem. □

I must also single out Joan DeMars, my high school calculus teacher, who convinced me to do a mathematics degree; Karen Meagher at the University of Regina, who convinced me that doing a Master's in Vancouver might be a good idea; Sam Simon and Matt DeVos at Simon Fraser University, who comprise a partial list of fierce collaborators; Jonathan Jedwab, my supervisor through it all, whose unending wit and wisdom and patience is inspirational; and NSERC and the SFU Department of Mathematics for their financial support over the years.

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Chapter 1

Introduction

1.1 γ-graphs

We consider only finite, loop-free, unweighted, undirected graphs without multiple edges. We denote the vertex set of a graph G by $V(G)$.

Definition 1.1. *A* dominating set *of a graph G is a set $S \subseteq V(G)$ such that every vertex of G is either in S or adjacent to a vertex in S.*

Definition 1.2. *A* γ-set *of a graph G, also called a* minimum dominating set *of G, is a dominating set of smallest size. The* domination number *of G, denoted $\gamma(G)$, is the size of a γ-set of G.*

Definition 1.3 ([39])**.** *The* γ-graph *$\gamma \cdot G$ of a graph G is formed as follows:*

- *the vertices of $\gamma \cdot G$ correspond to the γ-sets of G;*
- *two vertices are adjacent if and only if their corresponding γ-sets intersect in a set of size $\gamma(G) - 1$.*

Definition 1.4. *A graph H is* realisable (as a γ-graph) *if there exists a graph G for which $H = \gamma \cdot G$. In this case, G is a* parent graph *of H.*

Definition 1.5. *A graph H is* labellable (as a γ-graph) *if, for some integer $k \geq 1$, the vertices of H can be labelled by distinct k-subsets of $\{1, 2, 3, \dots\}$ such that two vertices are adjacent if and only if their corresponding labels intersect in a set of size $k - 1$. We say that a graph admits a* (consistent) labelling *if it is labellable.*

If a graph is realisable then it is immediate from the definitions that it is labellable. Given a labellable graph H, both the labelling of Definition 1.5 and the associated integer k are not unique: adding an isolated vertex to the parent graph increases k by one, as illustrated in Figure 1.1.

Definition 1.6. *A graph is* forbidden (as a γ-graph) *if it is not realisable.*

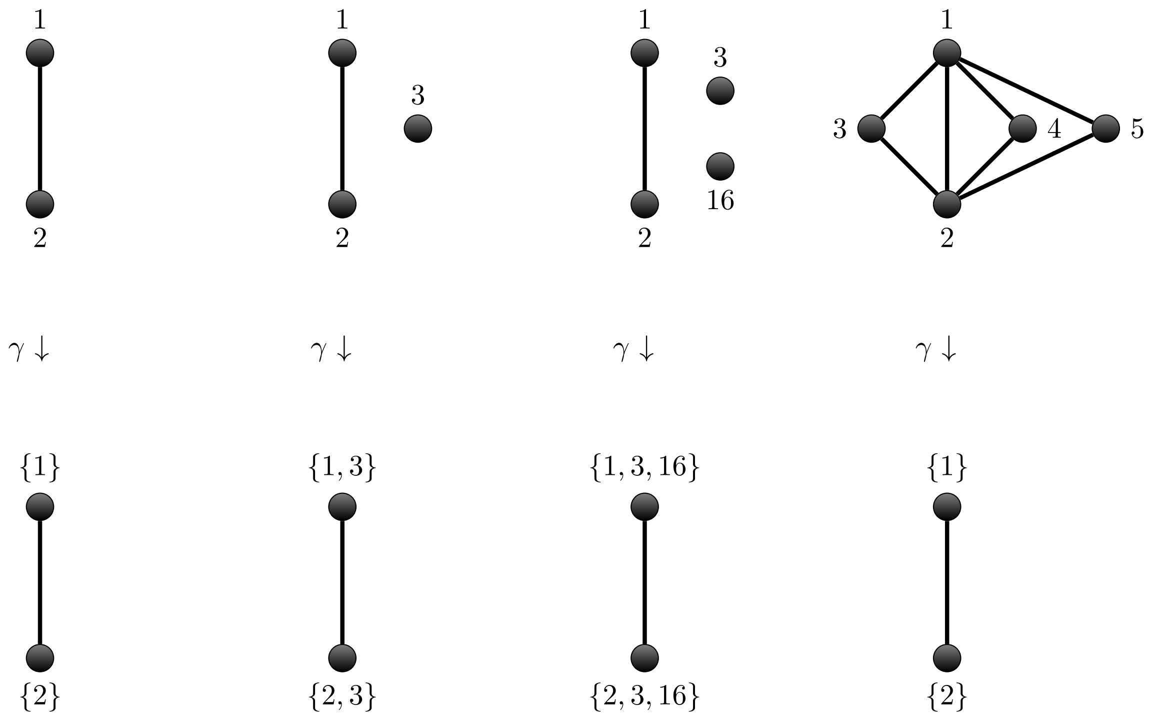


Figure 1.1: Distinct graphs with isomorphic γ-graphs.

Definition 1.7. *A graph is* minimally forbidden (as a γ-graph) *if it is forbidden but every proper induced subgraph is realisable.*

See Figure 1.2 for examples of forbidden and minimally forbidden graphs.

We consider in this thesis the following central questions:

- Which graphs are realisable as γ-graphs?
- Which graphs are minimally forbidden as γ-graphs?
- If a graph is labellable as a γ-graph, is it also realisable as a γ-graph?
- Is there an efficient algorithm to reliably determine when a graph is labellable as a γ-graph?

We also establish a relationship between γ-graphs and induced subgraphs of Johnson graphs; we define these graphs in Section 1.2.

Definition 1.8 below introduces a different object known as a γ-graph $G(\gamma)$ of a graph G, which differs from the γ-graph $\gamma \cdot G$ of Definition 1.3. This alternative definition imposes an additional restriction on the adjacency condition in $G(\gamma)$. This type of γ-graph is not the subject of this thesis, although a brief overview of results is contained in Chapter 2.

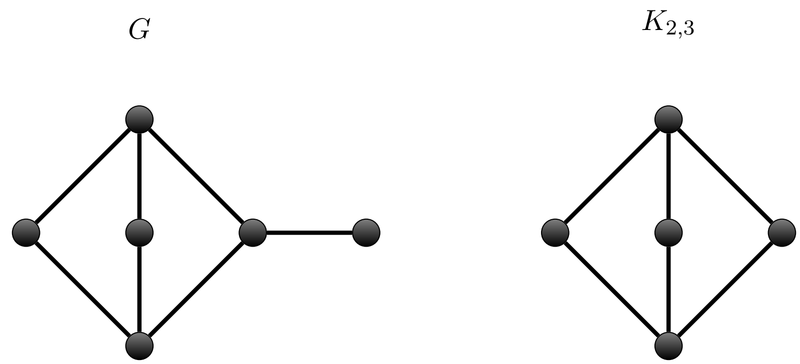


Figure 1.2: The graph G is forbidden, and it contains $K_{2,3}$ as an induced subgraph, which is minimally forbidden (see Section 6.3).

Definition 1.8 ([18])**.** *The* γ-graph $G(\gamma)$ *of a graph* G *is formed as follows:*

- *the vertices of* $G(\gamma)$ *correspond to the* γ*-sets of* G*;*
- *two vertices are adjacent if and only if their corresponding* γ*-sets* S_1 *and* S_2 *intersect in a set of size* $\gamma(G)-1$ *and the single vertex of* $S_1 \setminus S_2$ *is adjacent in* G *to the single vertex of* $S_2 \setminus S_1$*.*

Figure 1.3 gives an example, using $G = C_4$, to show that $G(\gamma)$ can differ from $\gamma \cdot G$.

1.2 Induced subgraphs of Johnson graphs

Let $[n]$ denote the set $\{1, \ldots, n\}$. We introduce two more families of graphs.

Definition 1.9 ([19, Chapter 1.6])**.** *For* $n \geq k \geq i$*, the family of graphs* $J(n, k, i)$ *is defined as follows:*

- *the vertices of* $J(n, k, i)$ *correspond to all* k*-subsets of* $[n]$*;*
- *two vertices are adjacent if and only if their corresponding* k*-subsets intersect in a set of size* i*.*

The graphs $J(n, k, k-1)$ *are known as the* Johnson graphs *and are written as* $J(n, k)$*.*

A graph G is isomorphic to an induced subgraph of a Johnson graph (for short, G is a JIS graph) if and only if for some integer $k \geq 1$ it is possible to assign distinct k-subsets of $\{1, 2, 3, \ldots\}$ to the vertices of G such that two vertices are adjacent if and only if their corresponding k-subsets intersect in a set of size $k-1$. When this happens, we say that G is *realisable (as a JIS graph)*, which follows [36].

From Figures 1.3 and 1.4 we see that $\gamma \cdot C_4$ is isomorphic to $J(4, 2)$ (which is a subgraph of $J(n, 2)$ for $n \geq 4$). More generally, a γ-graph can occur as a proper induced subgraph of a Johnson

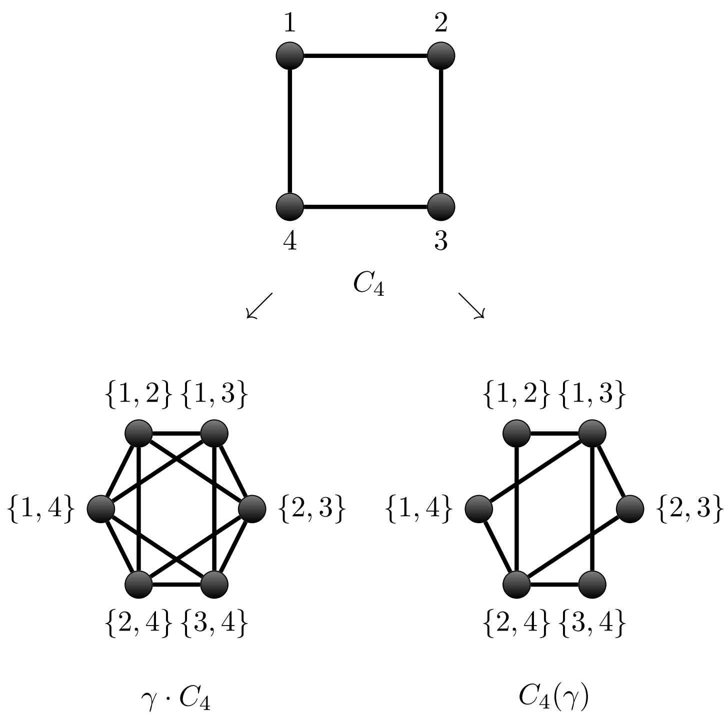


Figure 1.3: $G(\gamma)$ can differ from $\gamma \cdot G$.

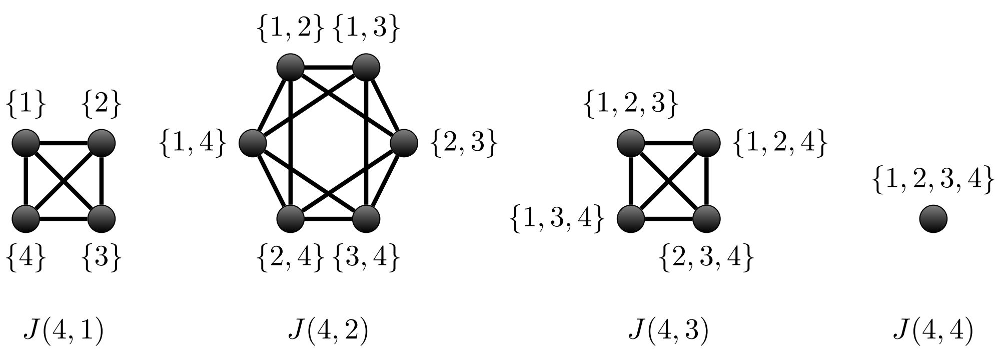


Figure 1.4: Johnson graphs $J(4, k)$ for $1 \leq k \leq 4$.

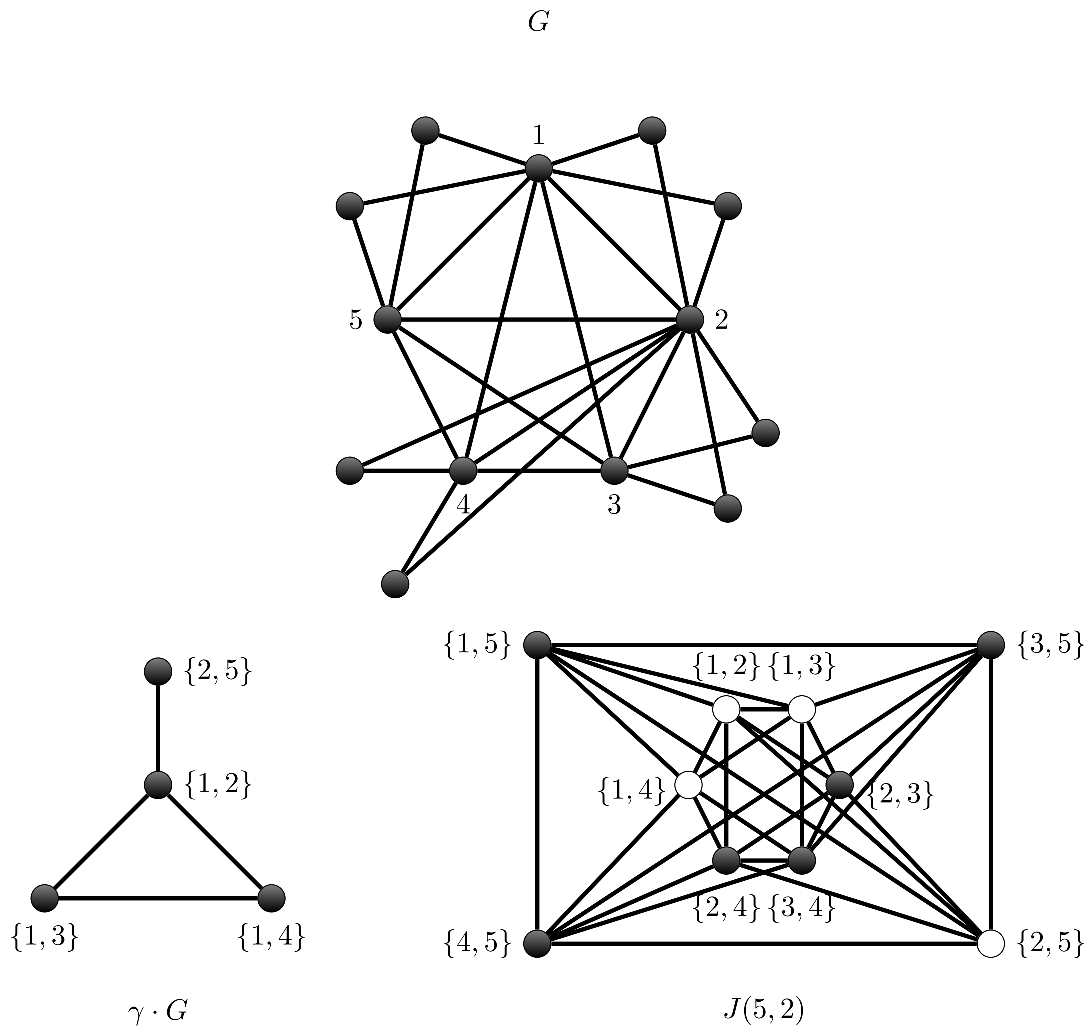


Figure 1.5: $\gamma \cdot G$ for the given graph G occurs as a proper induced subgraph of $J(5, 2)$ (indicated by white vertices).

graph (see Figure 1.5). Indeed, the adjacency condition for JIS graphs (phrased in terms of k-subsets) is equivalent to the adjacency condition for γ-graphs (phrased in terms of γ-sets of size k). This means that a graph is labellable as a γ-graph if and only if it is realisable as a JIS graph. We give a brief overview of Johnson graphs and JIS graphs in Chapter 3. We also ask the following questions:

- Are there further relationships between γ-graphs and Johnson graphs, and what are their consequences?
- Are there connections between γ-graphs and other mathematical objects, and what are their consequences?

1.3 Contributions of this thesis

The main result of this thesis is Theorem 1.10 below, which we prove in Chapter 4. Note that if a graph H is realisable as a γ-graph, having a parent graph G, then H admits a labelling by the γ-sets of G and so H is labellable as a γ-graph (equivalently, realisable as a JIS graph). Corollary 1.11 below shows that the converse is also true: if a graph H is labellable, then there exists a graph G whose γ-sets are the vertex-labels of H and so H is realisable (with $H = \gamma \cdot G$). This result yields Corollary 1.12 below, which establishes a connection between γ-graphs and JIS graphs.

Theorem 1.10. *Let $k \geq 1$ be an integer, and let $\mathcal{D}$ be a nonempty set of k-subsets of $\{1, 2, 3, \ldots\}$. Then there is a graph G whose γ-sets are the elements of $\mathcal{D}$.*

Corollary 1.11. *A graph is realisable if and only if it is labellable.*

Proof. By the definition of a γ-graph, a graph H is labellable as a γ-graph whenever it is realisable as one. Conversely, if H admits a consistent labelling using a set $\mathcal{D}$ of vertex-sets, then by Theorem 1.10 we can realise H as the γ-graph of some parent graph. $\square$

Corollary 1.12. *A graph is realisable as a γ-graph if and only if it is realisable as a JIS graph.*

Proof. By Corollary 1.11, a graph is realisable as a γ-graph if and only if it is labellable as a γ-graph. By the equivalent adjacency conditions between γ-graphs and JIS graphs, a graph is labellable as a γ-graph if and only if it is realisable as a JIS graph. $\square$

Using Definition 1.6 along with Corollaries 1.11 and 1.12, we establish that a graph is forbidden as a γ-graph (equivalently, forbidden as a JIS graph) if and only if it is not labellable. This allows us to unify many of the previously known results on γ-graphs and JIS graphs in Chapter 5. We then develop forced-labelling lemmas in Section 6.1, allowing us to explicitly show in Section 6.2 that the wheel graphs W_{2n} (see Definition 3.13) for $n \geq 3$ comprise a new family of minimally forbidden graphs. We devote the rest of Chapter 6 to classifying all graphs on up to six vertices as realisable or forbidden, and then generalise our definition of a γ-graph in Chapter 7 using distance-d domination. We conclude with some directions for future research in Chapter 8.

1.4 Relationship to previous work

We shall give two constructive proofs of Theorem 1.10. Our first construction turns out to be similar to, but more economical than, a construction in [25] presented in the context of optimal dominating codes in graphs; we discovered our first construction before becoming aware of [25]. A connection between optimal dominating codes in graphs and JIS graphs is made in both [25] and [26]; the consequence for γ-graphs is stated implicitly. We explicitly relate γ-graphs and optimal dominating codes in graphs through the intermediary JIS graphs. We also provide a second, more efficient construction for the proof of Theorem 1.10; see Figure 1.6 for a comparison of the three constructions. Until now, the consequences for γ-graphs of the constructions in [25] and [26] seem to have been overlooked; we anticipate that connecting these seemingly disparate fields will enhance future research.

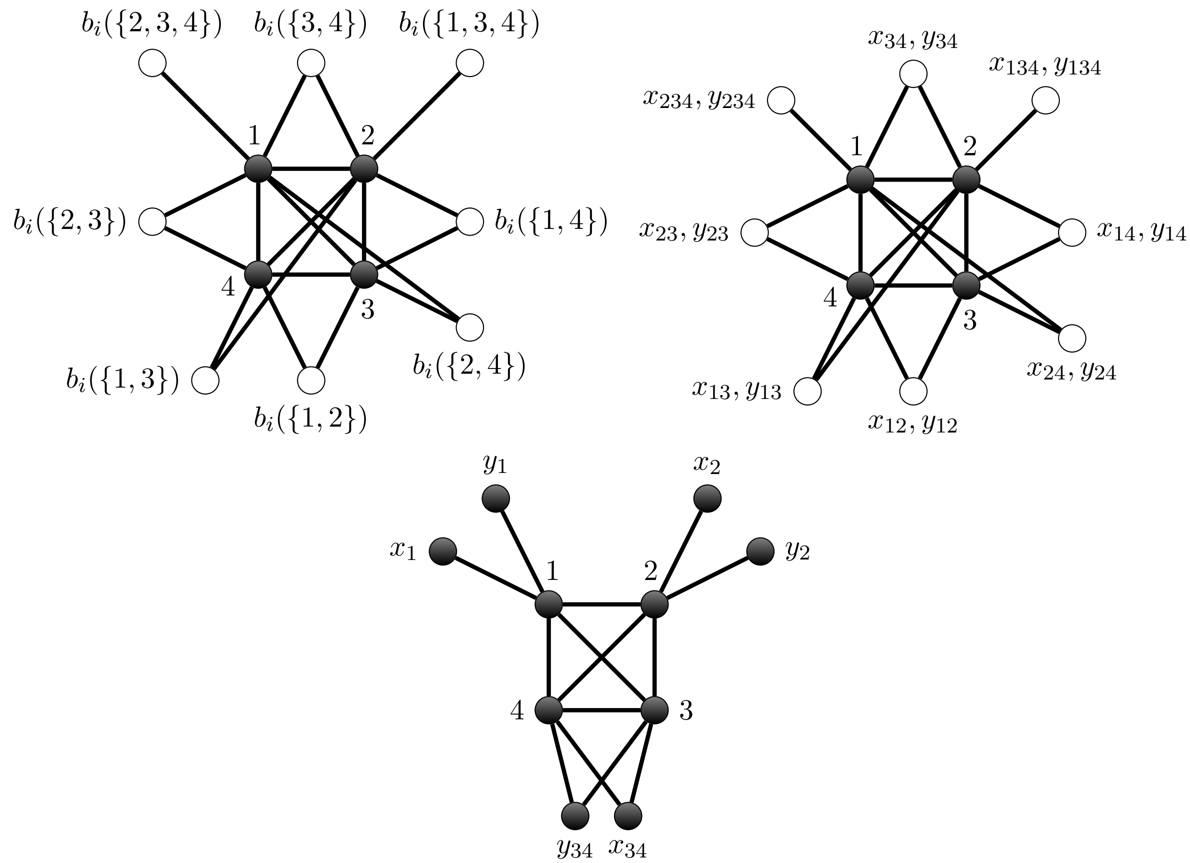


Figure 1.6: Graph G guaranteed by Theorem 1.10 for $k = 3$ and $\mathcal{D} = \{\{1,2,3\},\{1,2,4\}\}$, constructed according to the method of [25] (top left, where each white vertex represents a collection of four disjoint vertices), Section 4.1 (top right, where each white vertex represents a collection of two disjoint vertices), and Section 4.2 (bottom).

Chapter 2

Literature Survey on γ-Graphs

2.1 Overview of domination in graphs

In 1977, Cockayne and Hedetniemi [10] offered the first major survey on domination. By that time, three primary notions of domination had been established: Ore [37] related domination to the question of the minimum number of queens needed on a chessboard so that each square is accessible by a queen; Berge [4] considered domination as the strength of surveillance in a network; and Liu [31] phrased domination in terms of network communications.

Domination in graphs is detailed extensively in [23] and [22], with applications to fields such as game theory, coding theory, and matching theory. One of the first examples of a covering problem in domination is the "Five Queens Problem": how can we dominate the squares on a standard 8×8 chessboard using just five queens? This equivalently asks if a dominating set of smallest size (which we call a minimum dominating set) can be formed under certain conditions. Other qualities may be assigned to dominating sets, such as minimality (the property of containing no smaller dominating sets). Problems with these alternative formulations include finding a γ-set comprising independent vertices of a graph (known as independent coverings), and finding the maximum cardinality of an independent set in a graph. Yaglom and Yaglom [41] first considered these problems in the context of chess.

Definition 2.1. *Let $n \geq 1$ be an integer. A* clutter *of $[n]$ is a nonempty family $\mathcal{C}$ containing subsets of $[n]$ such that no element of $\mathcal{C}$ properly contains another. The corresponding* blocker *of $\mathcal{C}$ is the nonempty collection $\mathcal{B}$ of all minimal subsets of $[n]$ that contain at least one element of each subset in $\mathcal{C}$.*

Edmonds and Fulkerson [16] and Billera [6] introduce clutters and blockers and establish a relationship to domination in graphs. We may consider the set of all maximal complete subgraphs in a graph as a clutter, and the corresponding blocker contains only dominating sets of the graph. We use a clutter-blocker relationship in Section 4.2 for the second proof of Theorem 1.10.

2.2 The γ-graph $\gamma \cdot G$

The γ-graph $\gamma \cdot G$ was first defined by Sridharan and Subramanian [39], and has been studied in [40], [30], [38], and [5]. We outline the main results on the realisability of γ-graphs here. Though these results were originally obtained using a variety of clever constructions, many of them are now subsumed in this thesis by Theorem 1.10.

Theorem 2.2 ([38, Theorem 2.1])**.** *If a graph H is realisable, then every induced subgraph of H is realisable.*

Given a graph G and an induced subgraph H' of the γ-graph $H = \gamma \cdot G$, [38] establishes Theorem 2.2 by constructing a graph G' from G in such a way that $\gamma \cdot G' = H'$. We obtain this result as an immediate corollary to Theorem 1.10.

Definition 2.3. *The* Cartesian product *of two graphs G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ where, for $u, u' \in G$ and $v, v' \in H$, we have $(u, v) \sim (u', v')$ if and only if either $u = u'$ and $v \sim v'$ in H, or $v = v'$ and $u \sim u'$ in G.*

Theorem 2.4 ([30, Theorem 3.4])**.** *Let H_1 and H_2 be realisable. Then the Cartesian product $H_1 \square H_2$ is realisable.*

[30] proves Theorem 2.4 constructively. We provide our own proof in Section 5.2.

Theorem 2.5 ([30])**.** *A disconnected graph H is realisable if and only if each component of H is realisable.*

[30] proves Theorem 2.5 using the fact that if H_1 and H_2 are disjoint realisable graphs, then their disjoint union $H_1 \cup H_2$ occurs as an induced subgraph of a γ-graph obtained as a Cartesian product. Therefore, the proof of Theorem 2.5 in [30] relies on Theorems 2.2 and 2.4 above. We obtain this result as an immediate corollary to Theorem 1.10.

Theorem 2.6 ([40, Theorem 2.1])**.** *Every tree is realisable.*

Theorem 2.7 ([40, Theorem 2.4])**.** *Every cycle C_n with $n \geq 3$ is realisable.*

Definition 2.8. *A* unicyclic graph *is a connected graph containing exactly one cycle.*

Theorem 2.9 ([40, Theorem 2.6])**.** *Every unicyclic graph is realisable.*

[40] proves Theorems 2.6, 2.7, and 2.9 using a construction that applies only to parent graphs G satisfying the Private Neighbourhood Condition (PNC). Let D be a γ-set in G, and consider $x \in D$. The *private neighbourhood of x (with respect to D)* is the set $\mathrm{pn}(D, x) = \{y \in V(G) : N[y] \cap D = \{x\}\}$. If for every γ-set D in G and for each $x \in D$ the condition that $\mathrm{pn}(D, x) \neq \{x\}$ holds, we say that *G satisfies the Private Neighbourhood Condition (PNC)*; see Figure 2.1 for an example.

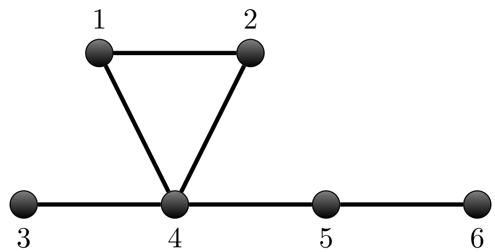
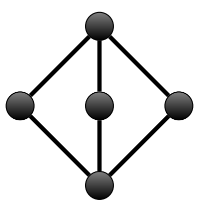


Figure 2.1: The PNC fails for this graph because $\mathrm{pn}(\{4,6\},6) = \{6\}$.

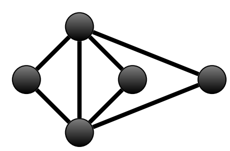
[38] amends the construction of [40] to avoid the PNC by first modifying the parent graph in a way that leaves the γ-graph unchanged.

Using Corollary 1.11, we may completely ignore the PNC. In Chapter 5 we include new proofs for the realisability of cycles, as well as for trees and unicyclic graphs phrased in terms of 2-cores (see Definition 3.4).

Theorem 2.10 ([40, Theorem 2.7][30, Theorem 2.3])**.** *A graph on at most five vertices is forbidden if and only if it is one of the following:*



(i)



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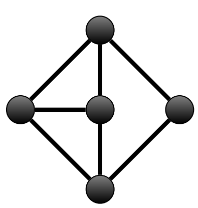
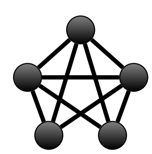


(iii)



(iv)

Each of the graphs in Theorem 2.10 fails to admit a consistent labelling. We introduce forced-labelling lemmas in Section 6.1 to help prove Theorem 2.10. We also use these lemmas in Chapter 6 to obtain a new family of forbidden graphs and to explicitly classify all graphs on six vertices as realisable or forbidden. Theorem 2.10 shows that the converse to Theorem 2.2 does not hold.

2.3 The γ-graph $G(\gamma)$

The γ-graph $G(\gamma)$ was first defined by Fricke et al. [18] and has been studied in [11], [17], and [35]. We briefly mention the main results on $G(\gamma)$ here, though this type of γ-graph is not the focus of this thesis.

Fricke et al. [18] establish many elementary results for γ-graphs $G(\gamma)$. They take a particular interest in γ-graphs $G(\gamma)$ of trees, and conclude with questions about the complexity of different parameters of such graphs, as well as about the number of minimum dominating sets in a tree. Connelly et al. [11] show constructively that all graphs are realisable as γ-graphs $G(\gamma)$, and Edwards [17] improves some of the parameters in γ-graphs $G(\gamma)$ of trees while determining that some of the results in [18] also hold for γ-graphs $\gamma \cdot G$. More recently, Mynhardt and Teshima [35] defined γ-graphs for variations on the domination parameter γ, and developed constructions to show that for each variation of the γ-graph, all graphs can be realised as that variation.

[18] also introduces the idea of forming the γ-graph sequence

$$G \to G(\gamma) \to (G(\gamma))(\gamma) \to \cdots .$$

Many γ-graph sequences stop at K_1, but some do not terminate. We can ask for which graphs does the corresponding γ-graph sequence stop or loop, and for which graphs does the sequence never terminate. Hedetniemi [24, Chapter 8] includes the γ-graph sequence problem in a top ten list of graph theory conjectures.

Chapter 3

Literature Survey on Induced Subgraphs of Johnson Graphs

3.1 Overview of Johnson graphs

Johnson graphs are well-studied as distance-regular graphs [9], in quantum probability [27], and in spectral analysis [28]. We list here some basic properties of Johnson graphs, and refer the interested reader to [20] for additional information.

It is well known that $J(n,k) \simeq J(n,n-k)$ for $1 \leq k \leq n$, so we may restrict the bounds for k and n to $1 \leq k \leq \frac{n}{2}$; although this is not strictly necessary for our γ-graph results on Johnson graphs, this relation is used often throughout the literature.

The family $J(n,k,i)$ of Definition 1.9 gives rise to the Johnson scheme [15] in association scheme theory, which we do not define here. [7] and [8] develop association schemes from an algebraic combinatorics perspective, and [3] describes association scheme theory as "group theory without groups." Association schemes have applications to orthogonal polynomials and linear programming [15], coding theory [14], and combinatorial design theory [2].

3.2 Induced subgraphs of Johnson graphs (JIS graphs)

For $1 \leq k \leq n$, a Johnson graph is formed by specifying an adjacency condition between all k-subsets of $[n]$; it is natural to consider the graphs obtained using only some of the k-subsets of $[n]$, but with the same adjacency specification. A graph obtained this way is an *induced subgraph of a Johnson graph* (for short, a *JIS graph*). The structure of JIS graphs has been studied in [36], [33], and [32].

Consider the following class of graphs.

Definition 3.1 ([36]). *Let F be a family of finite sets and let p be a fixed positive integer. The* p-intersection graph*, denoted $\Omega_p(F)$, is defined as follows:*

- *the vertices of $\Omega_p(F)$ correspond to all elements of F;*
- *two vertices are adjacent if and only if their corresponding sets share at least p elements.*

McKee and McMorris [34] provide a survey of intersection graphs and p-intersection graphs. If we restrict to families F containing sets of size $p+1$, then the theory of γ-graphs extends naturally to the theory of intersection graphs: vertices correspond to sets of the same size (namely $p+1$), and two vertices are adjacent when those sets differ by exactly one element (equivalently, they intersect in p elements). Naimi and Shaw [36] develop an algorithmic relationship between intersection graphs and JIS graphs which, when combined with Theorem 1.10 and Corollary 1.12, strengthens the relationship between γ-graphs and JIS graphs. A brief summary of results for JIS graphs follows, and Corollary 1.12 allows us to reinterpret each result in terms of γ-graphs.

Theorem 3.2 ([36])**.** *Every complete graph is realisable as a JIS graph.*

Theorem 3.3 ([36])**.** *Every cycle is realisable as a JIS graph.*

Definition 3.4. *The k-*core *of a graph G is the graph obtained by repeatedly deleting vertices of degree at most $k-1$ from G.*

To find the 2-core of G, we repeatedly delete isolated vertices and pendant vertices from G.

Theorem 3.5 ([36])**.** *A graph is realisable as a JIS graph if and only if its 2-core is empty or realisable as a JIS graph.*

The 2-core of every tree is the empty graph, which gives the following corollary to Theorem 3.5.

Corollary 3.6 ([36])**.** *Every tree is realisable as a JIS graph.*

Theorem 3.7 ([36])**.** *A disconnected graph is realisable as a JIS graph if and only if each component is realisable as a JIS graph.*

[36] proves Theorem 3.7 by induction on the number of components in the disconnected graph.

Theorem 3.8 ([36])**.** *The Cartesian product of two JIS graphs is realisable as a JIS graph.*

Theorem 3.9 ([36])**.** *For each $n \geq 5$ and for some edge e, the graph $K_n - e$ is forbidden as a JIS graph.*

Definition 3.10. *For $n \geq 1$, the* hypercube graph *Q_n is defined as follows:*

- *the vertices of Q_n correspond to all binary n-tuples;*
- *two vertices are adjacent if and only if their corresponding binary n-tuples differ in exactly one position.*

Theorem 3.11 ([33])**.** *For each $n \geq 1$, the hypercube graph Q_n is realisable as a JIS graph.*

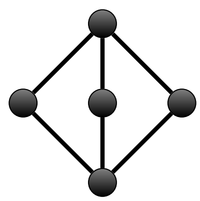
[33] proves Theorem 3.11 by induction on n.

Theorem 3.12 ([33])**.** *For $m \leq n$ and when the conditions $m \geq 2$ and $n \geq 3$ both hold, the complete bipartite graph $K_{m,n}$ is forbidden as a JIS graph.*

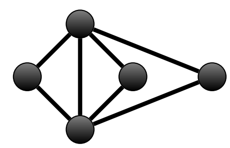
Definition 3.13. *For $n \geq 4$, the* wheel graph *W_n is the graph obtained by joining a single vertex to every vertex of the cycle C_{n-1}.*

Theorem 3.14 ([33])**.** *For each $n \geq 3$, the wheel graph W_{2n} is forbidden as a JIS graph.*

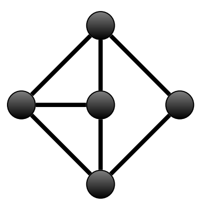
Theorem 3.15 ([32])**.** *The following graphs on five vertices are minimally forbidden as JIS graphs.*



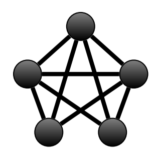
(i)



(ii)



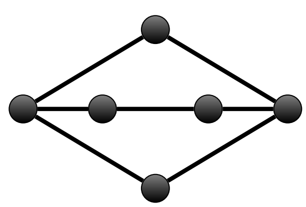
(iii)



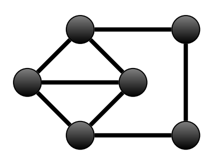
(iv)

[32] proves Theorem 3.15 by completely classifying all JIS graphs on at most five vertices.

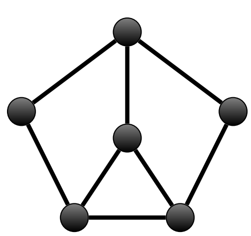
Theorem 3.16 ([33])**.** *The following graphs on six vertices are minimally forbidden as JIS graphs.*



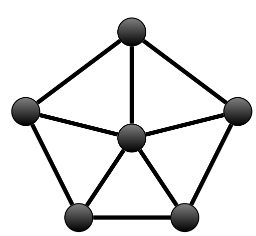
(i)



(ii)



(iii)



(iv)

[33] states Theorem 3.16 without proof, and claims that all remaining 6-vertex graphs are either JIS graphs or contain one of the 5-vertex forbidden graphs in Theorem 3.15 as an induced subgraph.

Chapter 4

Two Constructive Proofs of the Main Result

We devote this chapter to our main result, which we restate here for convenience.

Theorem 1.10. *Let $k \geq 1$ be an integer, and let $\mathcal{D}$ be a nonempty set of k-subsets of $\{1, 2, 3, \ldots\}$. Then there is a graph G whose γ-sets are the elements of $\mathcal{D}$.*

We shall give two constructive proofs of Theorem 1.10. Our first proof, given in Section 4.1, presents a construction for modifying a complete graph to individually eliminate all sets of size $k-1$, and then all sets of size k that do not appear in $\mathcal{D}$, as possible γ-sets. Our second proof, given in Section 4.2, uses a more efficient construction that introduces a blocker set and uses its properties to simultaneously eliminate many possible dominating sets from G.

Examples corresponding to the graphs in Figure 1.6 are included for each proof in order to highlight the efficiency of our constructions compared to that in [25].

4.1 First proof

Let $n = \left| \bigcup_{D \in \mathcal{D}} D \right|$, and relabel if necessary so that each element of $\mathcal{D}$ is a subset of $[n]$. Let $\mathcal{U}$ be the set of all $(k-1)$-subsets of $[n]$ along with all k-subsets of $[n]$ not contained in $\mathcal{D}$. The case $k = 1$ is trivial: take $G = K_n$. The case $k = n$ is also trivial: take $G = \overline{K_n}$. Note that in either case we may still use the following construction for G as long as, for the case $k = 1$, we interpret the 0-subset of $[n]$ to be the empty set when defining $\mathcal{U}$ and then introduce vertices $x_\emptyset$ and $y_\emptyset$ in Step (2).

(1) Initialise G to be K_n and label its vertices $1, 2, \ldots, n$.

(2) For each $U \in \mathcal{U}$, add vertices x_U and y_U to G and join each of them to the vertices of $[n] \setminus U$.

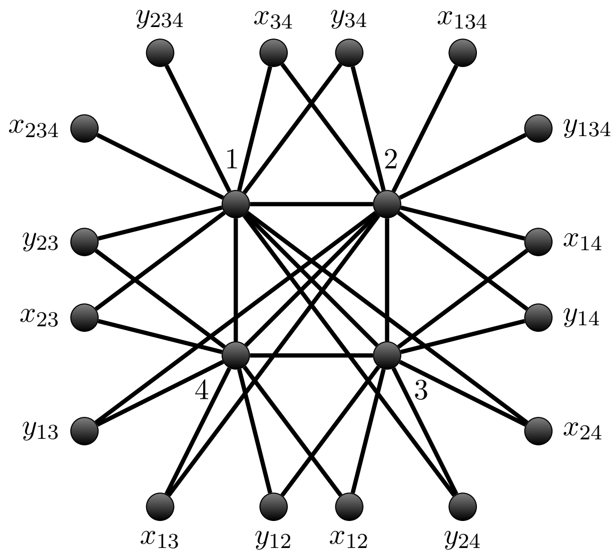


Figure 4.1: Construction of G in Section 4.1 for Theorem 1.10 with $k = 3$ and $\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$.

The resulting graph G has

$$n + 2\binom{n}{k-1} + 2\left(\binom{n}{k} - |\mathcal{D}|\right) \tag{4.1}$$

vertices and

$$\binom{n}{2} + 2(n-k+1)\binom{n}{k-1} + 2(n-k)\left(\binom{n}{k} - |\mathcal{D}|\right) \tag{4.2}$$

edges.

For example, take $k = 3$ and $\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$, so $n = 4$. Initialise G to be K_4 and label its vertices $1, 2, 3, 4$. Let $\mathcal{U} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Examine each 2-set of $\mathcal{U}$: for the set $\{1, 2\}$, add vertices x_{12}, y_{12} (where the index 12 is shorthand for $\{1, 2\}$) and join each of them to the vertices of $[4] \setminus \{1, 2\} = \{3, 4\}$. This ensures that $\{1, 2\}$, along with each of its proper subsets, is not a dominating set of G. Repeat for each other 2-set of $\mathcal{U}$. Now examine each 3-set $T \in \mathcal{U}$. For the set $\{1, 3, 4\}$, add vertices x_{134}, y_{134} and join each of them to the vertices of $[4] \setminus \{1, 3, 4\} = \{2\}$. This ensures that $\{1, 3, 4\}$, along with each of its proper subsets, is not a dominating set. Repeat for each other 3-set of $\mathcal{U}$, namely the set $\{2, 3, 4\}$. See Figure 4.1 for the resulting graph G, which has 20 vertices and 34 edges.

Let G be constructed according to Steps (1) and (2) above. We now prove Theorem 1.10 by showing that the γ-sets of G are exactly the elements of $\mathcal{D}$.

(a) *No added vertex is contained in a γ-set of G.*

Let $U \in \mathcal{U}$ and suppose, for a contradiction, that x_U is contained in a γ-set D of G. Then y_U is not contained in D, otherwise we may obtain a smaller dominating set than D by replacing the vertices x_U and y_U in D with a single vertex from the nonempty set $[n] \setminus U$: this vertex dominates x_U and y_U because $N(x_U) = N(y_U) = [n] \setminus U$ by construction, and it dominates all vertices of $[n]$ because G was initialised to K_n.

Since y_U is not contained in D, it must be dominated by one of its neighbours, namely a vertex of $[n] \setminus U$. This vertex dominates x_U because $N(x_U) = N(y_U) = [n] \setminus U$ by construction, and it dominates all vertices of $[n]$ because G was initialised to K_n. Therefore we may obtain a smaller dominating set than D by removing x_U from D, giving the required contradiction.

(b) *No element of $\mathcal{U}$ is a dominating set of G.*

By Step (2), each element $U \in \mathcal{U}$ contains no vertices adjacent to the vertices x_U and y_U, so U is not a dominating set of G.

(c) *Each element of $\mathcal{D}$ is a dominating set of G.*

Let $D \in \mathcal{D}$. For each $U \in \mathcal{U}$ the set $D \setminus U$ is nonempty, so the added vertices x_U and y_U are dominated by every element of $D \setminus U$. Furthermore, every vertex of D dominates all vertices of $[n]$ because G was initialised to K_n. Therefore D dominates G.

By part (a), the γ-sets of G contain vertices only from $[n]$. By part (b), no $(k-1)$-subset of $[n]$ dominates G, so $\gamma(G) \geq k$. By parts (b) and (c), the k-subsets of $[n]$ which dominate G are exactly the elements of $\mathcal{D}$. It follows that the γ-sets of G are the elements of $\mathcal{D}$.

This completes the first proof of Theorem 1.10.

4.2 Second proof

Take $n = \left| \bigcup_{D \in \mathcal{D}} D \right|$, and relabel if necessary so that each element of $\mathcal{D}$ is a subset of $[n]$. Regarding $\mathcal{D}$ as a clutter with respect to $[n]$ (see Definition 2.1), the blocker $\mathcal{B}$ of $\mathcal{D}$ is the collection of all minimal subsets of $[n]$ containing at least one element of each $D \in \mathcal{D}$. We use the following construction for G.

(1) Initialise G to be K_n and label its vertices $1, 2, \ldots, n$.

(2) For each $B \in \mathcal{B}$, add vertices x_B and y_B to G and join each of them to the vertices of B.

The resulting graph G has

$$n + 2|\mathcal{B}| \tag{4.3}$$

vertices and

$$\binom{n}{2} + 2 \sum_{B \in \mathcal{B}} |B| \tag{4.4}$$

edges.

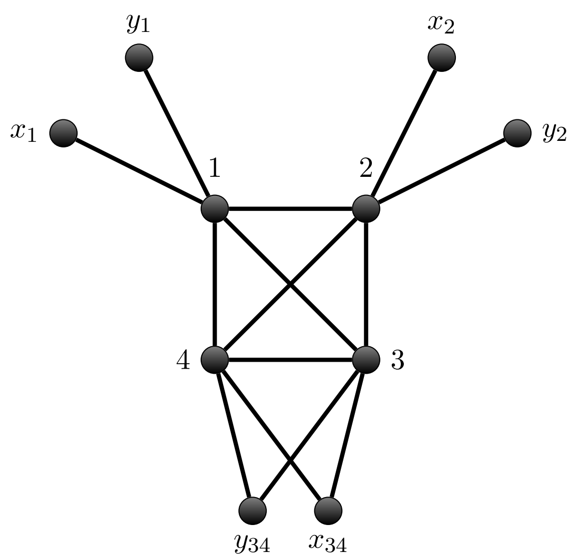


Figure 4.2: Construction of G in Section 4.2 for Theorem 1.10 with $k = 3$ and $\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$.

Consider again the example with $k = 3$ and $\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$, so $n = 4$. Initialise G to be K_4 and label its vertices $1, 2, 3, 4$. Form the blocker $\mathcal{B} = \{\{1\}, \{2\}, \{3, 4\}\}$ of $\mathcal{D}$. For the element $\{1\}$, add vertices x_1, y_1 and join each of them to the vertex 1. This ensures that $[n] \setminus \{1\} = \{2, 3, 4\}$, along with each of its proper subsets, is not a dominating set of G. Repeat for each other element of $\mathcal{B}$. See Figure 4.2 for the resulting graph G, which has 10 vertices and 14 edges.

Let G be constructed according to Steps (1) and (2) above. We now prove Theorem 1.10 by showing that the γ-sets of G are exactly the elements of $\mathcal{D}$.

(a) *No added vertex is contained in a γ-set of G.*

Consider $B \in \mathcal{B}$ and suppose, for a contradiction, that x_B is contained in a γ-set D of G. Then y_B is not contained in D, otherwise we may obtain a smaller dominating set than D by replacing the vertices x_B and y_B in D with a single vertex from the nonempty set B: this vertex dominates x_B and y_B because $N(x_B) = N(y_B) = B$ by construction, and it dominates all vertices of $[n]$ because G was initialised to K_n.

Since y_B is not contained in D, it must be dominated by one of its neighbours, namely a vertex of B. This vertex dominates x_B because $N(x_B) = N(y_B) = B$ by construction, and it dominates all vertices of $[n]$ because G was initialised to K_n. Therefore we may obtain a smaller dominating set than D by removing x_B from D, giving the required contradiction.

(b) *Each element of $\mathcal{D}$ is a dominating set of G.*

Let $D \in \mathcal{D}$. Each vertex of D dominates all vertices of $[n]$ because G was initialised to K_n. Let $B \in \mathcal{B}$; it remains to show that D dominates x_B and y_B. By the definition of $\mathcal{B}$, we may choose a vertex i in the nonempty set $B \cap D$. By construction, x_B and y_B are adjacent to $i \in D$.

(c) *No $(k-1)$-subset of $[n]$, and no k-subset of $[n]$ not contained in $\mathcal{D}$, dominates G.*

In the case $k = 1$, the statement holds vacuously. Otherwise, take $k \geq 2$ and let S be a subset of $[n]$ that either has size $k-1$, or has size k and is not contained in $\mathcal{D}$. We shall show that S does not dominate G.

Let $\widetilde{\mathcal{D}}$ be the set $\{D \setminus S : D \in \mathcal{D}\}$. By the definition of S, each element of $\widetilde{\mathcal{D}}$ is nonempty. Let $\widetilde{B}$ be an element of the blocker of $\widetilde{\mathcal{D}}$. Note that this implies $\widetilde{B} \cap S = \emptyset$. Since $\widetilde{B}$ contains at least one element of $D \setminus S$ for each $D \in \mathcal{D}$, then $\widetilde{B}$ necessarily contains at least one element of each $D \in \mathcal{D}$. By minimality of the elements of the blocker $\mathcal{B}$, we have $\widetilde{B} \supseteq B$ for some $B \in \mathcal{B}$. Using $\widetilde{B} \cap S = \emptyset$ and $B \in \mathcal{B}$, we must then have $\widetilde{B} = B$ (otherwise B would be a proper subset of $\widetilde{B}$ containing at least one element of each $D \setminus S$ in $\widetilde{\mathcal{D}}$, contradicting the minimality of $\widetilde{B}$ in the blocker of $\widetilde{\mathcal{D}}$).

We have shown that $\widetilde{B} \in \mathcal{B}$, so G contains a vertex $x_{\widetilde{B}}$. Since $\widetilde{B} \cap S = \emptyset$, it follows that S does not dominate $x_{\widetilde{B}}$ and therefore does not dominate G.

By part (a), the γ-sets of G contain vertices only from $[n]$. By part (b), the k-subsets in $\mathcal{D}$ dominate G, so $\gamma(G) \leq k$. By parts (b) and (c), we know $\gamma(G) \geq k$ and the k-subsets of $[n]$ which dominate G are exactly the elements of $\mathcal{D}$. It follows that the γ-sets of G are exactly the elements of $\mathcal{D}$.

This completes the second proof of Theorem 1.10.

4.3 Remarks

[25] connects optimal dominating codes in graphs to the set of all JIS graphs; by Corollary 1.12, it follows that we can relate graphs formed using all elements of an optimal dominating code in a graph to those formed using all γ-sets of a graph (as long as the adjacency conditions are equivalent). Note that we made this connection only by first relating γ-graphs and JIS graphs. It follows that [25, Theorem 2] is similar to Theorem 1.10, and the construction used in [25] is similar to our first construction from Section 4.1. We shall compare the efficiency of these two constructions with our second construction from Section 4.2.

The graph constructed in the proof of [25, Theorem 2] contains

$$n + (k+1)\binom{n}{k-1} + (k+1)\left(\binom{n}{k} - |\mathcal{D}|\right) \tag{4.5}$$

vertices and

$$\binom{n}{2} + (k+1)(n-k+1)\binom{n}{k-1} + (k+1)(n-k)\left(\binom{n}{k} - |\mathcal{D}|\right) \tag{4.6}$$

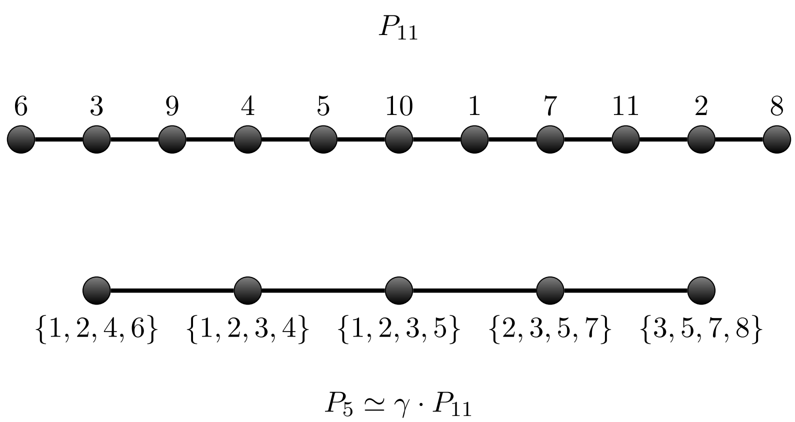


Figure 4.3: P_5 labelled by sets of (4.7) occurs as $\gamma \cdot P_{11}$ for the given labelling of P_{11}, showing that P_{11} is a smaller parent graph for P_5 than those from the constructions in [25, Theorem 2] and in Sections 4.1 and 4.2.

edges. See Figure 1.6 for a comparison of the three constructions for the example with $k = 3$ and $\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$: the construction from [25] uses 36 vertices and 62 edges.

Compare the enumeration of vertices and edges in (4.5) and (4.6) to that of (4.1) and (4.2) in our first construction: we add two vertices, rather than $k+1$, for each set in $\mathcal{U}$, so the enumerations are smaller. Our second construction is simpler, and appears to be much more economical in general according to the enumerations (4.3) and (4.4) and as illustrated in Figure 1.6.

As a larger example, take $k = 4$ and

$$\mathcal{D} = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 6\}, \{2, 3, 5, 7\}, \{3, 5, 7, 8\}\}\,, \tag{4.7}$$

so $n = 8$. Then $\mathcal{U}$ contains the 56 3-subsets of the set [8] along with the 65 4-subsets of the set [8] that are not contained in $\mathcal{D}$, whereas

$$\mathcal{B} = \{\{1, 3\}, \{1, 5\}, \{1, 7\}, \{2, 3\}, \{2, 5\}, \{2, 7\}, \{2, 8\}, \{3, 4\}, \{3, 6\}, \{4, 5\}\}$$

is the corresponding blocker of $\mathcal{D}$. Each construction uses the following numbers of vertices and edges:

(i) **Construction in Section 4.2**: 28 vertices and 68 edges.

(ii) **Construction in Section 4.1**: 250 vertices and 1108 edges.

(iii) **Construction in [25, Theorem 2]**: 613 vertices and 2728 edges.

In this example, the construction in Section 4.2 is clearly the most efficient, but further improvement is still possible. Let P_n denote the path graph on n vertices (having $n - 1$ edges), and consider P_{11},

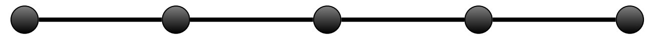


Figure 4.4: P_5 is labellable with sets of size two.

which has γ-sets of size four. Label the vertices of P_{11} as in Figure 4.3. Then $\gamma \cdot P_{11} \simeq P_5$, and the labels on $\gamma \cdot P_{11}$ are exactly the sets of (4.7). It follows that P_{11} is a parent graph of P_5 with only 11 vertices and 10 edges, which is simpler than the parent graphs obtained using the constructions of both Sections 4.1 and 4.2. Note also that P_5 has a consistent labelling using sets of size only two (see Figure 4.4), so $k = 2$ and $n = 6$. This reduces the number of vertices and edges in the constructed parent graphs, but P_{11} does not realise P_5 with γ-sets of size two.

However, the construction in Section 4.2 is not always more efficient than the construction in Section 4.1: consider the graph comprising n isolated vertices, whose vertices are labellable by the sets in $\mathcal{D} = \{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$. Using the construction in Section 4.1, the constructed graph G has $2n(2n+1)$ vertices and $n(8n^2-6n+3)$ edges. Now regard $\mathcal{D}$ as a clutter, and form the associated blocker $\mathcal{B}$. Each element of $\mathcal{B}$ is a set of size n, and $|\mathcal{B}| = 2^n$. Using the construction in Section 4.2, the constructed graph G has $2n + 2^{n+1}$ vertices and $n(2n - 1 + 2^{n+1})$ edges. It follows that for each $n \geq 8$, the graph obtained using the second construction has more vertices and more edges than the one obtained using the first construction.

These observations prompt the following questions for further study:

- Can we improve the efficiency of the constructions in Sections 4.1 and 4.2 in general?
- What is the best measure of efficiency for a construction proving Theorem 1.10?
- For a labellable graph, can we find a labelling using sets of the smallest possible size? This corresponds to minimising the number k in Definition 1.5.
- Does a labelling with sets of smaller size allow for a simpler parent graph?

Chapter 5

Simplified Approach to Previous Results on γ-Graphs

5.1 Corollaries of main result

Recall the following corollaries to Theorem 1.10.

Corollary 1.11. *A graph is realisable if and only if it is labellable.*

Corollary 1.12. *A graph is realisable as a γ-graph if and only if it is realisable as a JIS graph.*

Corollaries 1.11 and 1.12 allow us to immediately recover many previously known results on γ-graphs (equivalently, JIS graphs).

Corollary 5.1. *A disconnected graph H is realisable if and only if each component of H is realisable.*

Proof. If H is realisable then it has a labelling, so the components of H are each labellable and therefore realisable by Corollary 1.11. Similarly, if each component of H is realisable, then each component admits a labelling; label the γ-sets of each component using non-intersecting sets of symbols. It follows that H is labellable, so it is realisable by Corollary 1.11. $\square$

Corollary 5.1 was proved as a JIS graph result in [36].

Corollary 5.2. *If a graph H is realisable, then every induced subgraph of H is realisable.*

Proof. If H is realisable as a γ-graph then it is realisable as a JIS graph by Corollary 1.12. Therefore, since every induced subgraph of a JIS graph is also a JIS graph, every induced subgraph of H is also realisable by Corollary 1.12. $\square$

Corollary 5.2 becomes trivial in the context of JIS graphs, while [38, Theorem 2.1] establishes the result for γ-graphs by taking a graph G and an induced subgraph H' of the γ-graph $H =$

$\gamma \cdot G$, and then constructing G' from G in such a way that $\gamma \cdot G' = H'$. By comparison, our constructions directly realise the desired induced subgraph. Consider the labelled graphs G, H, and H' in Figures 5.1 and 5.2. Each construction uses the following number of vertices and edges:

(i) **Construction in Section 4.2:** 10 vertices and 31 edges.

(ii) **Construction in Section 4.1:** 64 vertices and 189 edges.

(iii) **Construction in [38]:** 393 vertices and at least 427 edges (the edge count must be approximated because it depends on the size of certain nonempty sets in the construction).

5.2 Realisable families

We easily recover some of the previously known families of realisable graphs. By Corollary 1.11, it is sufficient to exhibit a consistent labelling of the vertices in each graph.

Theorem 5.3. *For each $n \geq 1$, the complete graph K_n is realisable.*

Proof. Let $v_1, \ldots, v_n$ be the vertices of K_n. For $1 \leq i \leq n$, assign to v_i the label $\{i\}$. $\square$

Theorem 5.3 is trivial for γ-graphs; [36] proved the corresponding result for JIS graphs.

Theorem 5.4. *For each $n \geq 3$, the cycle graph C_n is realisable.*

Proof. Let $v_1, \ldots, v_n$ be the vertices of C_n. For $1 \leq i \leq n-1$, assign to v_i the label $\{i, i+1\}$, and assign to v_n the label $\{1, n\}$. $\square$

[40] develops complicated families of graphs to establish Theorem 5.4; [36] proves the corresponding result for JIS graphs.

Theorem 5.5. *Let H_1 and H_2 be realisable. Then the Cartesian product $H_1 \square H_2$ is realisable.*

Proof. Let $V(H_1) = \{u_1, \ldots, u_m\}$ and $V(H_2) = \{v_1, \ldots, v_n\}$. For $1 \leq i \leq m$, let S_i be the label on u_i in H_1, and for $1 \leq j \leq n$, let T_j be the label on v_j in H_2; we may assume that $S_i \cap T_j = \emptyset$. Assign to vertex (u_i, v_j) in $H_1 \square H_2$ the label $S_i \cup T_j$. This gives a consistent labelling for $H_1 \square H_2$ because, by Definition 2.3,

$$\begin{aligned}
&(u_i, v_j) \sim (u_k, v_\ell) \text{ in } H_1 \square H_2 \\
&\iff \text{either } v_j \sim v_\ell \text{ in } H_2 \text{ and } u_i = u_k, \text{ or } u_i \sim u_k \text{ in } H_1 \text{ and } v_j = v_\ell \\
&\iff \text{either } |T_j \cap T_\ell| = \gamma(H_2) - 1 \text{ and } S_i = S_k, \text{or } |S_i \cap S_k| = \gamma(H_1) - 1 \text{ and } T_j = T_\ell \\
&\iff S_i \cup T_j \text{ and } S_k \cup T_\ell \text{ differ in exactly one element,}
\end{aligned}$$

and therefore $H_1 \square H_2$ is realisable by Corollary 1.11. $\square$

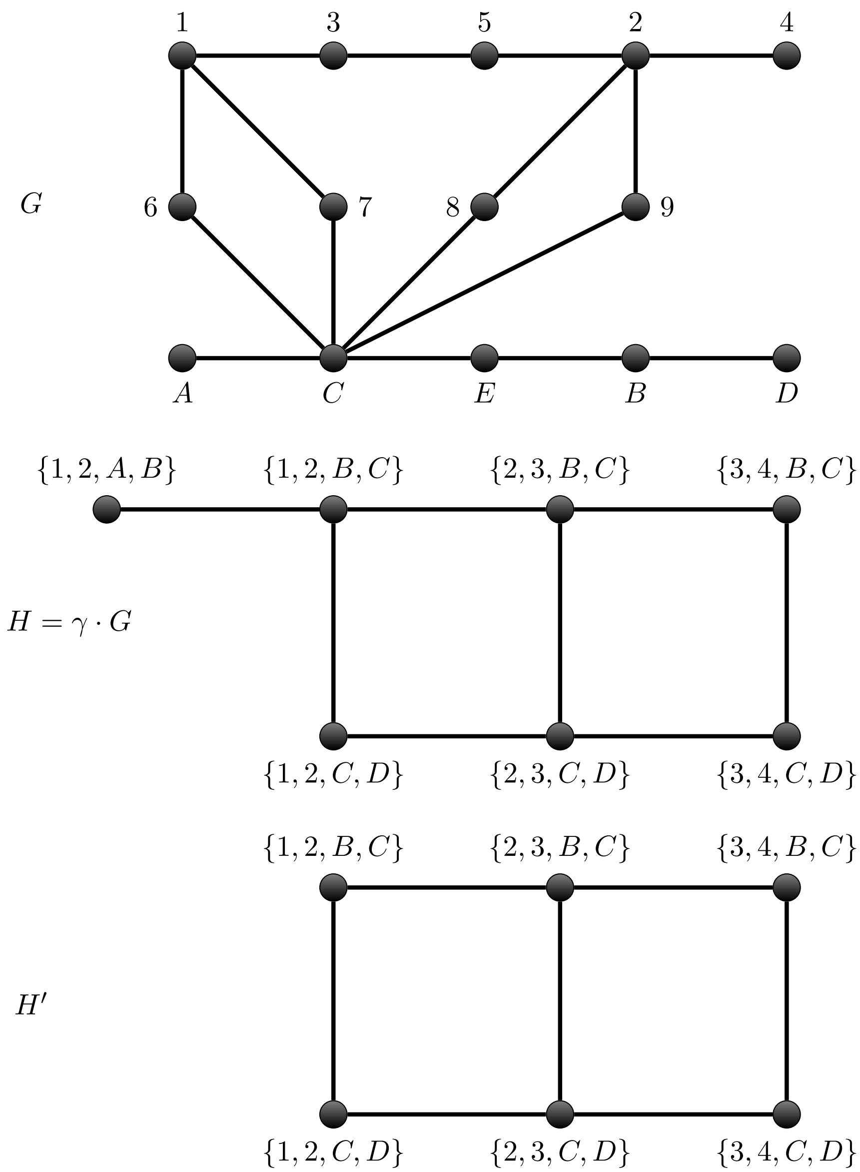


Figure 5.1: An example using the induced subgraph construction in [38], which requires 393 vertices and at least 427 edges to obtain a parent graph for the given induced subgraph of $\gamma \cdot G$.

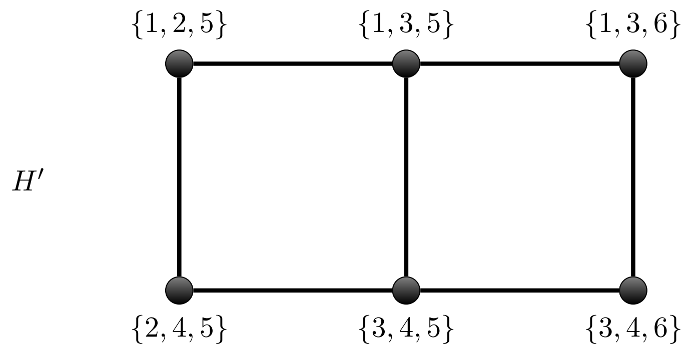


Figure 5.2: A relabelling of the induced subgraph of $\gamma \cdot G$ in Figure 5.1, which improves the efficiency of our constructions.

[30] proves Theorem 5.5 by showing that if $H_1 = \gamma \cdot G_1$ and $H_2 = \gamma \cdot G_2$, then $\gamma \cdot (G_1 \cup G_2)$ is isomorphic to $H_1 \Box H_2$; [36] proves the corresponding result for JIS graphs.

Theorem 5.6. *Every tree T is realisable.*

Proof. The proof is by induction on the number $n \geq 1$ of vertices of T. The base case $n = 1$ is a single vertex, which is labellable with a single symbol. Assume that the cases up to $n-1 \geq 1$ are true. Let w be a pendant vertex of T, and let the vertices of $T - w$ be $v_1, \ldots, v_{n-1}$. We may assume that w is adjacent to vertex v_{n-1} in T. By the inductive hypothesis, the graph $T - w$ admits a consistent labelling as a γ-graph; for $1 \leq i \leq n-1$, let S_i be the label assigned to vertex v_i in $T - w$. Let a and b be distinct symbols not contained in $\bigcup\limits_{i=1}^{n-1} S_i$. For $1 \leq i \leq n-1$, assign to v_i in T the label $S_i \cup \{a\}$, and assign to w the label $S_{n-1} \cup \{b\}$ (see Figure 5.3). □

Recall that we may find the 2-core of a graph G (see Definition 3.4) by repeatedly deleting isolated vertices and pendant vertices from G. Thus, the 2-core of a graph is empty if and only if the graph is a forest, and the 2-core of a unicyclic graph is its underlying single cycle. Theorem 5.6 can be generalised to the following.

Theorem 5.7. *A graph H is realisable if and only if the 2-core of H is either empty or realisable.*

Proof. Suppose H is realisable. If its 2-core is not empty, then the 2-core is an induced subgraph of H and so is realisable by Corollary 5.2.

Now suppose the 2-core of H is empty. Then H is a forest, and so H is realisable by Theorem 5.6 and Corollary 5.1.

Finally, suppose the 2-core K of H is not empty. We may assume that H is connected by Corollary 5.1. The proof is by induction on the number $k \geq 0$ of vertices in $H - K$. The base case $k = 0$ is realisable by assumption.

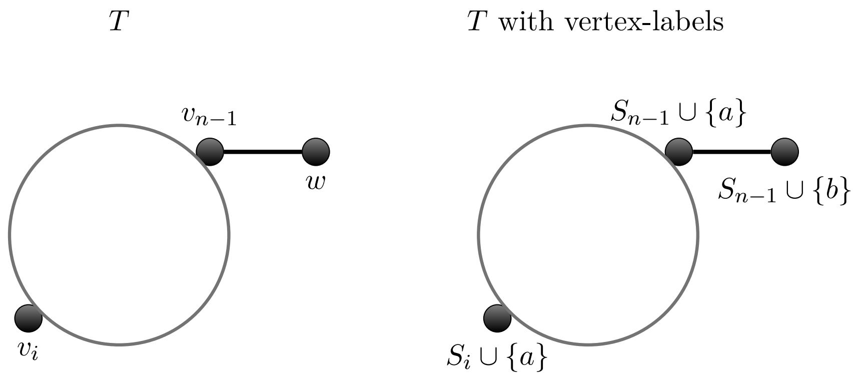


Figure 5.3: Every tree T is realisable.

Assume that the cases up to $k-1 \geq 0$ are true. Since $k \geq 1$, the graph H contains a pendant vertex w. Let $v_1, v_2, \ldots, v_{n+k-1}$ be the vertices of $H - w$, where we may assume that w is adjacent to v_{n+k-1}. By the inductive hypothesis, the graph $H-w$ admits a consistent labelling as a γ-graph; for $1 \leq i \leq n+k-1$, let S_i be the label assigned to vertex v_i in $H-w$. Let a and b be distinct symbols not contained in $\bigcup\limits_{i=1}^{n+k-1} S_i$. For $1 \leq i \leq n+k-1$, assign to v_i in H the label $S_i \cup \{a\}$, and assign to w the label $S_{n+k-1} \cup \{b\}$. □

Corollary 5.8. *Every unicyclic graph is realisable.*

[40] proves Theorem 5.6 and Corollary 5.8 using a single construction which, while relatively straightforward, can be used to modify the parent graph G of some $\gamma \cdot G$ only when G satisfies the Private Neighbourhood Condition (see Section 2.2). [36] proves Theorem 5.7 for JIS graphs, so trees and unicyclic graphs are realisable as JIS graphs.

Recall the family of wheel graphs W_n (see Definition 3.13): W_4 is isomorphic to K_4, which is realisable by Theorem 5.3. Here we establish the realisability of wheel graphs of the form W_{2n+1}, and we demonstrate in Section 6.2 that for $n \geq 3$, wheel graphs of the form W_{2n} are minimally forbidden.

Theorem 5.9. *For each $n \geq 2$, the wheel graph W_{2n+1} is realisable.*

Proof. Let $v_1, \ldots, v_{2n}$ be the vertices of W_{2n+1} that induce the cycle C_{2n}, and let v_{2n+1} be the vertex of W_{2n+1} adjacent to all other vertices. Label v_{2n+1} with the set $[n]$, and let $\{S_1, \ldots, S_n\}$ be the set of all $(n-1)$-subsets of $[n]$. Let $a_1, \ldots, a_n$ be a list of distinct symbols disjoint from $[n]$. For $1 \leq i \leq n-1$, assign to v_{2i-1} and v_{2i} the respective labels $S_i \cup \{a_i\}$ and $S_i \cup \{a_{i+1}\}$, and assign to v_{2n-1} and v_{2n} the respective labels $S_n \cup \{a_n\}$ and $S_n \cup \{a_1\}$ (see Figure 5.4 for an example with $n = 4$). □

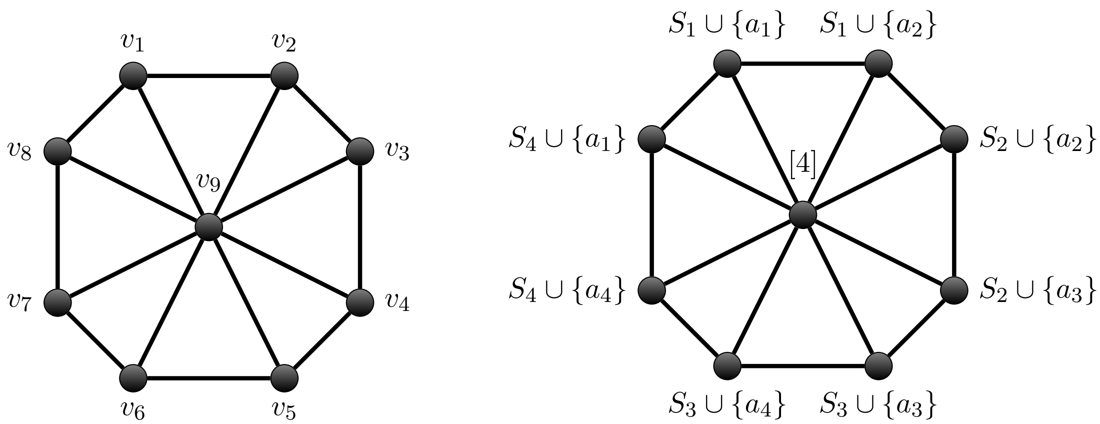


Figure 5.4: W_9 is realisable.

Theorem 5.9 is new in the context of γ-graphs. [36] hints at the result for JIS graphs by claiming (without proof) that only wheels of the form W_{2n} for $n \geq 3$ are forbidden.

Theorem 5.10. *For $1 \leq k \leq n$, the Johnson graph $J(n, k)$ is realisable.*

Proof. By definition, the vertices of $J(n, k)$ are all k-sets from $[n]$, and two k-sets are adjacent whenever they intersect in a set of size $k-1$. This corresponds to the adjacency condition for a γ-graph, so $J(n, k)$ admits a consistent labelling. $\square$

Theorem 5.10 is new for γ-graphs, and [36] proves the corresponding result for JIS graphs.

Theorem 5.11. *For each $n \geq 1$, the hypercube graph Q_n is realisable.*

Proof. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be disjoint sets of symbols. Label the vertex of Q_n corresponding to the binary n-tuple $(v_1, \ldots, v_n)$ by $(s_1, \ldots, s_n)$, where

$$s_i = \begin{cases} a_i & \text{if } v_i = 0 \\ b_i & \text{if } v_i = 1 \end{cases}$$

This is a consistent labelling of Q_n because two binary n-tuples are adjacent in Q_n only when they differ in exactly one position i, and the corresponding vertex-labels $\{s_1, \ldots, s_n\}$ and $\{t_1, \ldots, t_n\}$ differ only in the values of s_i and t_i (see Figure 5.5 for an example with $n = 3$). $\square$

We also present a short alternative proof of Theorem 5.11.

Proof. Since $Q_1 \simeq K_2$ is realisable by Theorem 5.3 and $Q_n \simeq Q_{n-1} \square K_2$ for $n \geq 2$, it follows that Q_n is realisable by Theorem 5.5. $\square$

[5] proves Theorem 5.11 by examining γ-graphs of certain families of trees; [33] proves the corresponding result for JIS graphs by induction on the number of vertices.

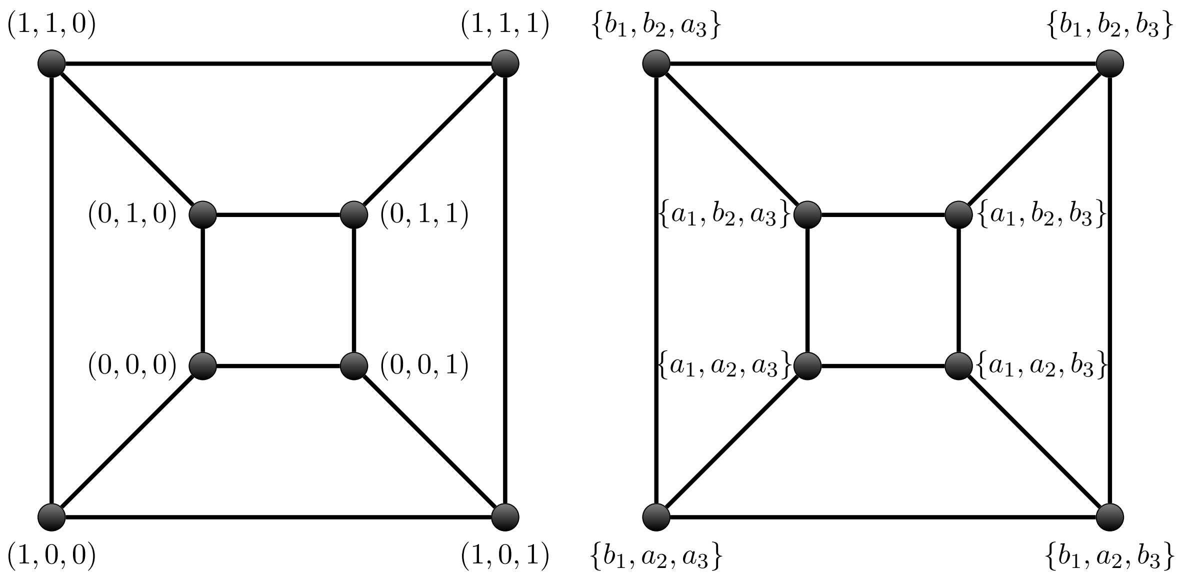


Figure 5.5: Q_3 is realisable.

5.3 Forbidden families

[38, Corollary 2.8] and [30, Theorem 2.3] give two families of forbidden graphs, which we recover here. By Corollary 5.2, it is enough to show that each graph in the family contains a forbidden graph as an induced subgraph.

Theorem 5.12. *For $m \leq n$ and when the conditions $m \geq 2$ and $n \geq 3$ hold, the complete bipartite graph $K_{m,n}$ is forbidden.*

Proof. The 2-core of $K_{1,n}$ is empty, so $K_{1,n}$ is realisable by Theorem 5.7, and the graph $K_{2,2}$ is isomorphic to C_4, which is realisable by Theorem 5.4. In all other cases, $K_{m,n}$ contains $K_{2,3}$ as an induced subgraph, which we shall see in Theorem 6.10 is forbidden, so $K_{m,n}$ is forbidden. □

Theorem 5.13. *For each $n \geq 5$ and for some edge e, the graph $K_n - e$ is forbidden.*

Proof. We shall see in Theorem 6.10 that $K_5 - e$ is forbidden. Since $K_5 - e$ occurs as an induced subgraph of $K_n - e$ for each $n \geq 5$, it follows that $K_n - e$ is forbidden. □

Chapter 6

Further Results on γ-Graphs

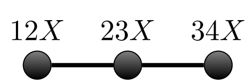
6.1 Forced-labelling lemmas

The following lemmas demonstrate that there must be a specified labelling on certain induced subgraphs within a γ-graph in order for the labelling to be consistent. These lemmas are useful as they can help prove the labellability of a graph, implying the realisability of that graph by Corollary 1.11. Alternatively, we can use these lemmas to determine which graphs are forbidden by demonstrating that a consistent labelling does not exist, and we can further demonstrate minimality if we can show that every proper induced subgraph of a forbidden graph has a labelling. In Section 6.3 we recover the minimally forbidden graphs on five vertices, and in Section 6.5 we explicitly prove that there are exactly four minimally forbidden graphs on six vertices.

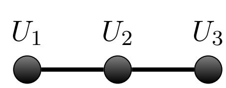
We shall label a vertex of a γ-graph $\gamma \cdot G$ as $123X$, for example, to correspond to a γ-set $\{1,2,3\} \cup X$ in the graph G, where $X \subseteq V(G)$ is a (possibly empty) set disjoint from $\{1,2,3\}$. Vertices labelled as $123X$ and $257X$, for example, involve the same set X.

We shall derive necessary conditions in the proofs of the following lemmas. It is straightforward to verify that the given conditions are sufficient.

Lemma 6.1. *If P_3 occurs as an induced subgraph of a γ-graph, then without loss of generality and for some set X, its labelling in the γ-graph must be*



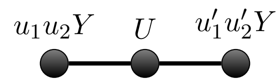
Proof. Let the vertices of the induced P_3 be labelled as



Let the label of U_1 be $\{u_1, u_2, u_3, \ldots, u_k\}$ for some $k \geq 1$. Then without loss of generality the label of U_2 is $\{u_1', u_2, u_3, \ldots, u_k\}$, where $u_1' \notin \{u_1, u_2, u_3, \ldots, u_k\}$. Since U_3 is adjacent to U_2

but not U_1, then (again without loss of generality) its label is $\{u_1', u_2', u_3, \ldots, u_k\}$, where $u_2' \notin \{u_1', u_1, u_2, u_3, \ldots, u_k\}$ and $k \geq 2$. This corresponds to the claimed labelling with $(u_1, u_2, u_1', u_2') = (1, 2, 3, 4)$ and $\{u_3, \ldots, u_k\} = X$. $\square$

Lemma 6.2. *If*



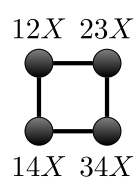
is a labelled induced subgraph of a γ-graph, where u_1, u_2, u_1', and u_2' are all distinct, then the label of U must be one of the following:

(i) $u_1u_1'Y$; *(ii)* $u_1u_2'Y$; *(iii)* $u_1'u_2Y$; *(iv)* $u_2'u_2Y$.

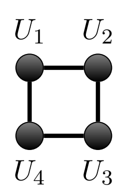
Proof. Since $\{u_1, u_2\} \cup Y$ and $\{u_1', u_2'\} \cup Y$ differ in exactly two elements and the label of U must differ from each of these sets in just one element, then the label of U must contain the set Y. If the label of U contains u_1, then for U to be adjacent to $u_1'u_2'Y$, its label must be either $u_1u_1'Y$ or $u_1u_2'Y$. Otherwise, the label of U does not contain u_1. For U to be adjacent to u_1u_2Y, its label must contain u_2. Then, for U to be adjacent to $u_1'u_2'Y$, its label must be either $u_1'u_2Y$ or $u_2'u_2Y$. $\square$

Lemma 6.1 shows that when the endpoints of an induced P_3 are labelled within a γ-graph, the outermost vertices must have labels which differ in exactly two elements. Lemma 6.2 shows that when the labels of these outermost vertices are prescribed, the central vertex then has four possible labels. This does not contradict the generality of Lemma 6.1, as we may switch the labels 1 and 2 (and independently, 3 and 4).

Lemma 6.3. *If C_4 occurs as an induced subgraph of a γ-graph, then without loss of generality and for some set X, its labelling in the γ-graph must be*



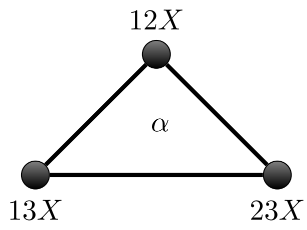
Proof. Let the vertices of the induced C_4 be labelled as



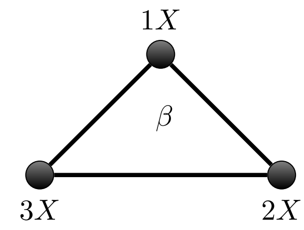
By Lemma 6.1, the labels of U_1, U_2, and U_3 must be $\{u_1, u_2, u_3, \ldots, u_k\}, \{u_1', u_2, u_3, \ldots, u_k\}$, and $\{u_1', u_2', u_3, \ldots, u_k\}$, respectively, where $k \geq 2$ and $u_1', u_2', u_1, u_2, u_3, \ldots, u_k$ are all distinct. Since U_4 is adjacent to both U_1 and U_3, one of the four cases (i) to (iv) of Lemma 6.2 with $Y = \{u_3, \ldots, u_k\}$ must apply. Since U_4 is distinct from and not adjacent to U_2, only case (ii) is possible and so the label of U_4 is $\{u_1, u_2', u_3, \ldots, u_k\}$. This corresponds to the claimed labelling with $(u_1, u_2, u_1', u_2') = (1, 2, 3, 4)$ and $\{u_3, \ldots, u_k\} = X$. $\square$

Lemma 6.4. *If K_3 occurs as an induced subgraph of some γ-graph, then without loss of generality and for some set X, its labelling in the γ-graph must have exactly one of the following forms:*

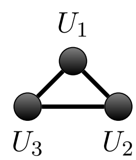
(i)



(ii)



Proof. Let the vertices of the induced K_3 be labelled as



Let the labels of U_1 and U_2 be $\{u_1, u_2, u_3, \ldots, u_k\}$ and $\{u_1', u_2, u_3, \ldots, u_k\}$ for some $k \geq 1$, where $u_1' \notin \{u_1, u_2, u_3, \ldots, u_k\}$. Since U_3 is adjacent to both U_1 and U_2, then if its label contains both u_1 and u_1', its label may be taken to be $\{u_1, u_1', u_3, \ldots, u_k\}$, where $k \geq 2$. This corresponds to the first claimed labelling with $(u_1, u_2, u_1') = (1, 2, 3)$ and $\{u_3, \ldots, u_k\} = X$.

Otherwise, without loss of generality the label of U_3 does not contain u_1. Since U_3 is adjacent to U_1 and distinct from U_2, its label must be $\{u_1'', u_2, u_3, \ldots, u_k\}$, where $u_1'' \notin \{u_1', u_1, u_2, u_3, \ldots, u_k\}$. This corresponds to the second claimed labelling with $(u_1, u_1', u_1'') = (1, 2, 3)$ and $\{u_2, \ldots, u_k\} = X$.

The forms (i) and (ii) given above are not consistent with each other (even for different X), so the induced K_3 must take exactly one of these two forms. $\square$

In Section 6.2, we shall use the fact that a labelled γ-graph containing an induced subgraph K_3 must take exactly one of the two forms α and β depicted in Lemma 6.4 (i) and (ii).

Lemma 6.5. *If $K_4 - e$ occurs as an induced subgraph of a γ-graph, then without loss of generality and for some set X, its labelling in the γ-graph must be*

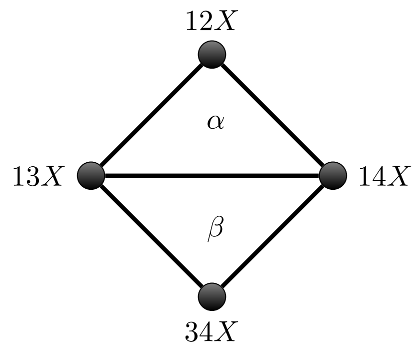
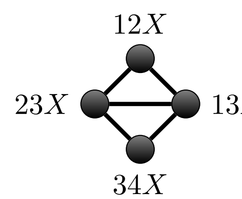
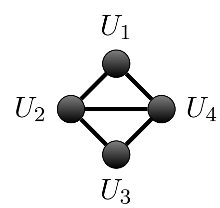


Figure 6.1: Every K_4-e labelling must have opposing label types on the two induced subgraphs K_3.



Proof. Let the vertices of the induced $K_4 - e$ be labelled as



By Lemma 6.1, the labels of U_1, U_2, and U_3 must be $\{u_1, u_2, u_3, \ldots, u_k\}, \{u_1', u_2, u_3, \ldots, u_k\}$, and $\{u_1', u_2', u_3, \ldots, u_k\}$, respectively, where $k \geq 2$, and $u_1', u_2', u_1, u_2, u_3 \ldots, u_k$ are all distinct. Since U_4 is adjacent to both U_1 and U_3, we can label it by applying Lemma 6.2 with $Y = \{u_3, \ldots, u_k\}$; because U_4 is distinct from and adjacent to U_2, only cases (i) and (iv) are applicable.

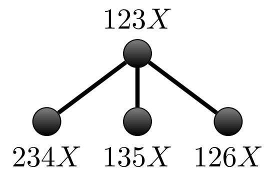
- Case (i): The label of U_4 is $\{u_1, u_1', u_3, \ldots, u_k\}$. This corresponds to the claimed labelling with $(u_1, u_2, u_1', u_2') = (1, 2, 3, 4)$ and $\{u_3, \ldots, u_k\} = X$.

- Case (iv): The label of U_4 is $\{u_2', u_2, u_3, \ldots, u_k\}$. This corresponds to the claimed labelling with $(u_1, u_2, u_1', u_2') = (4, 3, 2, 1)$ and $\{u_3, \ldots, u_k\} = X$ (after reflecting the labelled γ-graph through a horizontal axis).

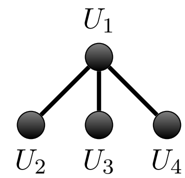
□

Remark 6.6. *The graph $K_4 - e$ contains two induced subgraphs K_3 sharing an edge. Lemma 6.5 shows that when $K_4 - e$ is an induced subgraph of some γ-graph, then the labelling of one of its induced subgraphs K_3 must have the form α and the labelling of the other must have the form β (as* *specified in Lemma 6.4). We shall use this fact in Section 6.2 to obtain a new family of minimally forbidden graphs.*

Lemma 6.7. *If $K_{1,3}$ occurs as an induced subgraph of some γ-graph, then without loss of generality and for some set X, its labelling in the γ-graph must be*



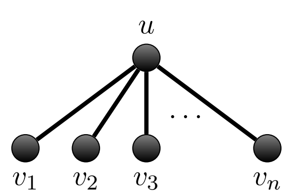
Proof. Let the vertices of the induced $K_{1,3}$ be labelled as



By Lemma 6.1, the labels of U_2, U_1, and U_3 must be $\{u_1, u_2, u_3, \ldots, u_k\}, \{u_1', u_2, u_3, \ldots, u_k\}$, and $\{u_1', u_2', u_3, \ldots, u_k\}$, respectively, where $k \geq 2$, and $u_1', u_2', u_1, u_2, u_3, \ldots, u_k$ are all distinct. Since U_4 is adjacent to U_1 but not U_2 and not U_3, then without loss of generality its label is $\{u_1', u_2, u_3', u_4, \ldots, u_k\}$, where $k \geq 3$ and $u_3' \notin \{u_1', u_2', u_1, u_2, u_3, u_4 \ldots, u_k\}$. This corresponds to the claimed labelling with $(u_1, u_2, u_3, u_1', u_2', u_3') = (4, 2, 3, 1, 5, 6)$ and $\{u_4, \ldots, u_k\} = X$. $\square$

Lemma 6.8. *For $n \geq 3$, if $K_{1,n}$ occurs as an induced subgraph of some γ-graph, then each label of the γ-graph must contain at least n elements.*

Proof. Let the vertices of $K_{1,n}$ be



and consider a consistent labelling on the vertices. The label of each v_i must differ from the label of u in exactly one element. For distinct i and j, the labels of v_i and v_j cannot differ from the label of u in the same element. Therefore the label of u contains at least n elements. $\square$

See Figure 6.2 for a labelling of $K_{1,n}$ where each label contains exactly n elements.

The consequences of these lemmas are seen throughout the rest of Chapter 6: we show that wheel graphs of the form W_{2n} for $n \geq 3$ are forbidden in Section 6.2, and we classify all graphs on up to six vertices as realisable or forbidden in Sections 6.3 to 6.6.

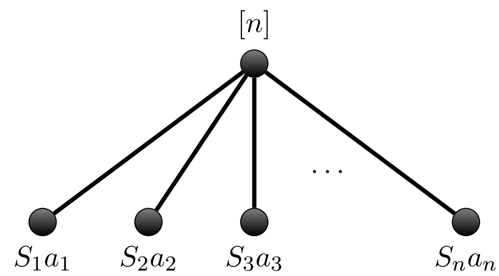


Figure 6.2: A labelling of $K_{1,n}$ with $S_i := [n] \setminus i$ for $1 \leq i \leq n$ and $a_1, a_2, \dots, a_n$ unique symbols not contained in $[n]$, so each label has exactly n elements.

6.2 Forbidden wheels

The following result is claimed without proof in [36] in the context of JIS graphs. We now prove the result in the context of γ-graphs.

Theorem 6.9. *For each $n \geq 3$, the wheel graph W_{2n} is minimally forbidden.*

Proof. Suppose, for a contradiction, that W_{2n} is not forbidden and so can be consistently labelled. Let $v_1, \dots, v_{2n-1}$ be the vertices of W_{2n} that induce the cycle C_{2n-1}, and let v_{2n} be the vertex of W_{2n} adjacent to all other vertices. Consider the sets of vertices

$$\{v_{2n}, v_1, v_2\}, \{v_{2n}, v_2, v_3\}, \dots, \{v_{2n}, v_{2n-2}, v_{2n-1}\}, \{v_{2n}, v_{2n-1}, v_1\}.$$

Each of these sets induces the subgraph K_3 in W_{2n}. Since $n \geq 3$, the vertices of adjacent induced K_3s in this (cyclic) sequence each induce a copy of $K_4 - e$, and so by Remark 6.6 the labelling of each induced K_3 must alternate between the forms α and β as the sequence is traversed. Therefore the number of induced subgraphs K_3 that occur must be even, but the number of triangles in W_{2n} is $2n - 1$, giving a contradiction. It follows that W_{2n} is forbidden.

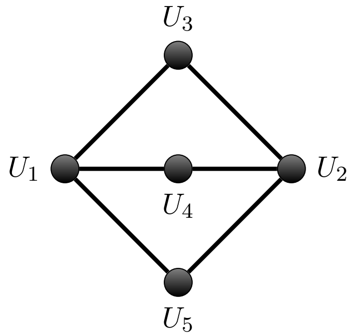
To see that W_{2n} is minimally forbidden, we must show that the graph $W_{2n} - v_i$ is realisable for each $1 \leq i \leq 2n$. If $i = 2n$, the graph $W_{2n} - v_i$ is isomorphic to C_{2n-1}, which is realisable by Theorem 5.4. Otherwise, if $i \neq 2n$, the graph $W_{2n} - v_i$ is an induced subgraph of W_{2n+1}, which is realisable by Theorem 5.9 and Corollary 5.2. $\square$

6.3 5-vertex minimally forbidden graphs

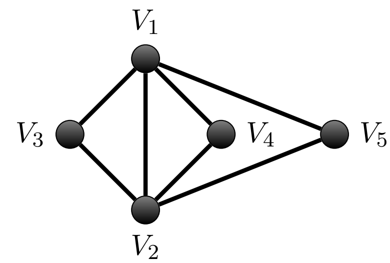
[30] determined the four minimally forbidden graphs on five vertices by showing directly that each graph does not admit a labelling, and so each graph is forbidden. We recover the four minimally forbidden graphs on five vertices here. These graphs are minimally forbidden because every graph on at most four vertices is realisable, as we verify in Section 6.4. The realisability of the remaining

5-vertex graphs was asserted in [30]; in Section 6.4, we use Corollary 1.11 to explicitly prove this result by means of consistent labellings.

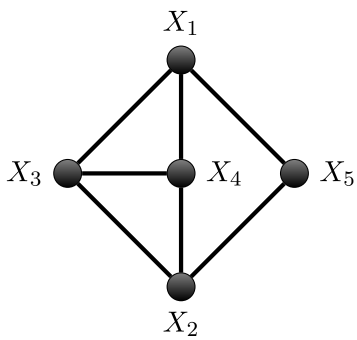
Theorem 6.10. *The following graphs on five vertices are minimally forbidden.*



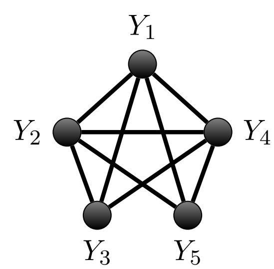
(i)



(ii)



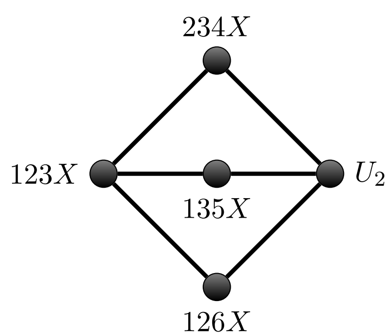
(iii)



(iv)

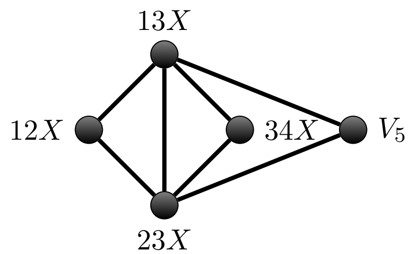
Proof. Suppose, for a contradiction, that the vertex-labels shown above are consistent.

(i) By Lemma 6.7 applied to U_1, U_3, U_4, and U_5, we must have the labels



for some set X. Apply Lemma 6.2 with $U = U_2$ and $Y = X \cup \{3\}$ and $(u_1, u_2) = (2, 4)$ and $(u_1', u_2') = (1, 5)$. Since U_2 must be distinct from $123X$ and adjacent to $126X$, each of the cases (i) to (iv) gives a contradiction.

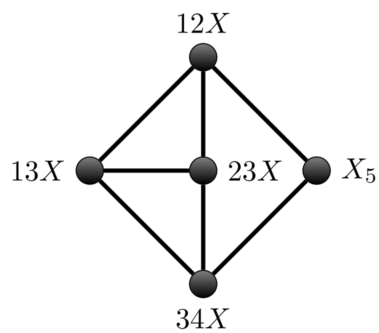
(ii) By Lemma 6.5 applied to V_1, V_2, V_3, and V_4, we must have the labels



for some set X. Since V_5 is adjacent to both $13X$ and $23X$, then by Lemma 6.4 there are two possible labels for V_5:

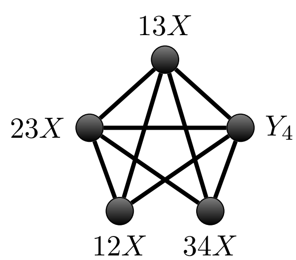
- Case 1: The label of V_5 is $12X$. This is a repeated label, giving a contradiction.
- Case 2: The label of V_5 is $35X$. Then V_5 is adjacent to $34X$, giving a contradiction.

(iii) By Lemma 6.5 applied to X_1, X_2, X_3, and X_4, we must have the labels



for some set X. Apply Lemma 6.2 with $U = X_5$ and $Y = X$ and $(u_1, u_2) = (1, 2)$ and $(u_1', u_2') = (3, 4)$. Since X_5 must be distinct from and not adjacent to both $13X$ and $23X$, each of the cases (i) to (iv) gives a contradiction.

(iv) By Lemma 6.5 applied to Y_1, Y_2, Y_3, and Y_5, we must have the labels



for some set X. Apply Lemma 6.2 with $U = Y_4$ and $Y = X$ and $(u_1, u_2) = (1, 2)$ and $(u_1', u_2') = (3, 4)$. Since Y_4 must be distinct from and adjacent to both $13X$ and $23X$, each of the cases (i) to (iv) gives a contradiction.

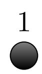
□

6.4 5-vertex classification of γ-graphs

In this section, we shall completely classify all γ-graphs on up to five vertices. This classification is asserted in [30], which states that the only forbidden graphs on up to five vertices are exactly those presented in Theorem 6.10, but does not explicitly realise all other graphs on five vertices. As we build our library of realisable graphs, we may use Corollary 5.1 to restrict our classification to connected graphs.

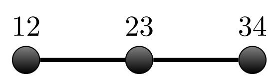
We reference [13] for all connected graphs on at most five vertices, of which there are 31. Of these, we showed in Section 6.3 that four of them are minimally forbidden. The remaining 27 are realisable, as demonstrated by the labellings in Subsections 6.4.1 to 6.4.5. All 31 graphs are presented here, ordered first by number of vertices and then by number of edges. A labelling involving smaller γ-sets is preferable because the parent graphs in the constructions of Chapter 4 are then simpler, but it is not necessary to find an optimal labelling in order to determine the realisability of a graph.

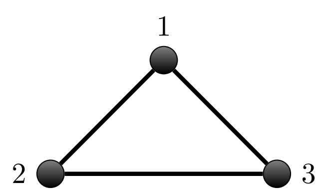
6.4.1 Realisable on 1 vertex



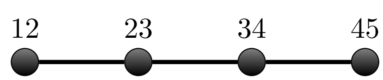
6.4.2 Realisable on 2 vertices

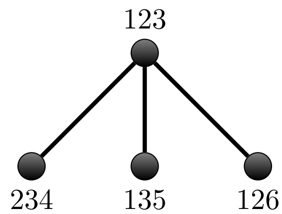
6.4.3 Realisable on 3 vertices





6.4.4 Realisable on 4 vertices




123
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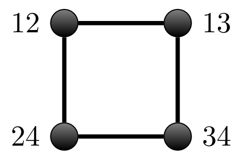
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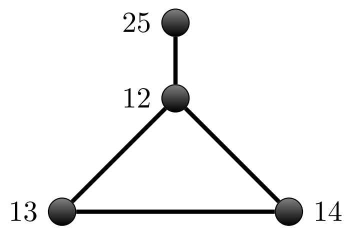
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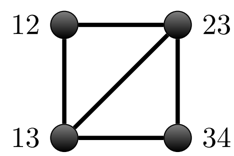
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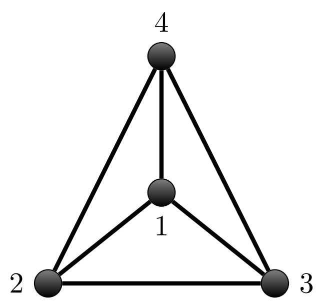
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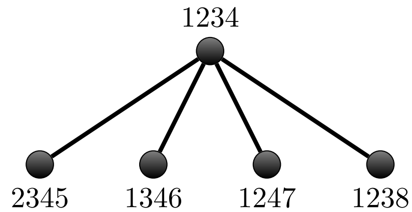
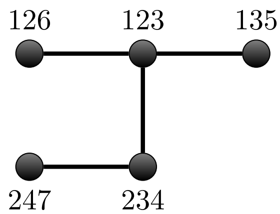
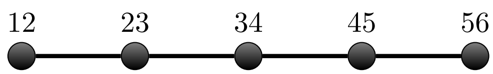
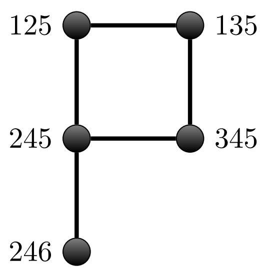
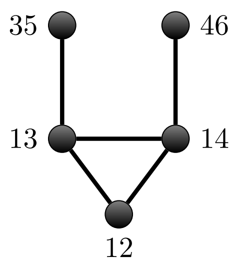
6.4.5 Realisable on 5 vertices

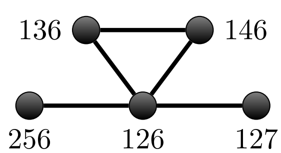


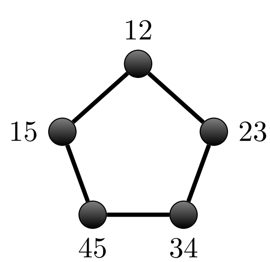


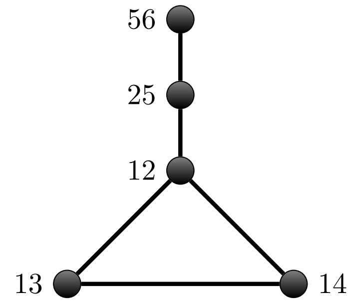


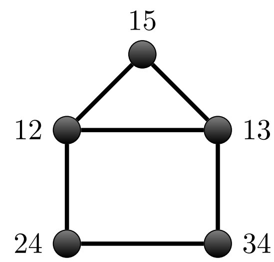


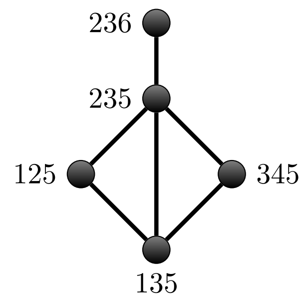


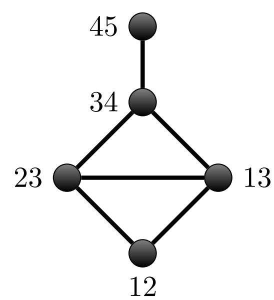


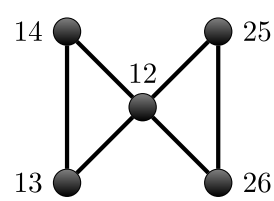


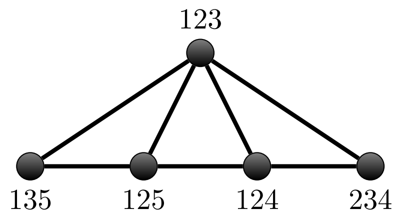








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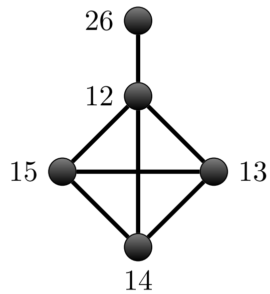
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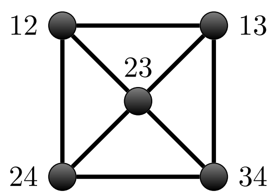
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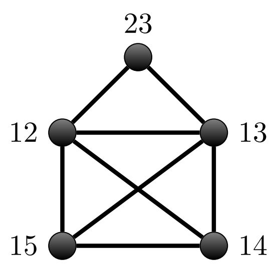
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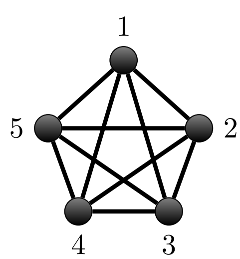
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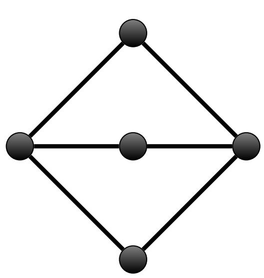
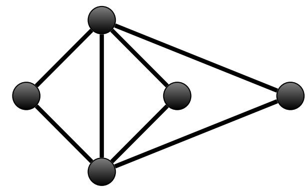
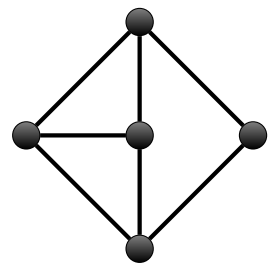
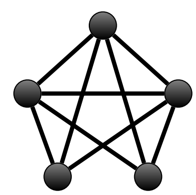




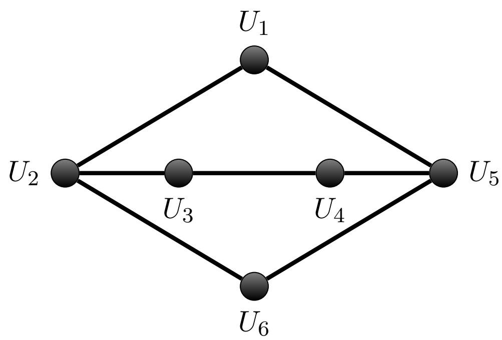




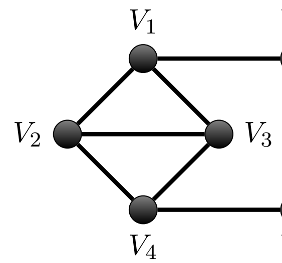
6.5 6-vertex minimally forbidden graphs

We prove here that there are exactly four minimally forbidden graphs on six vertices, which [33] states without proof in the context of JIS graphs.

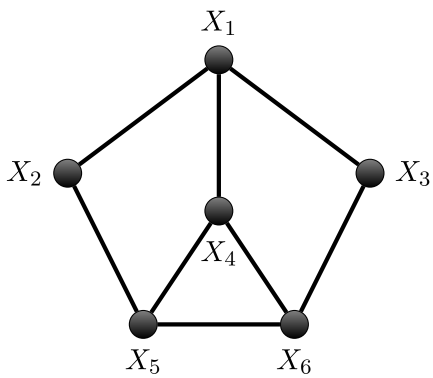
Theorem 6.11. *The following graphs on six vertices are minimally forbidden.*



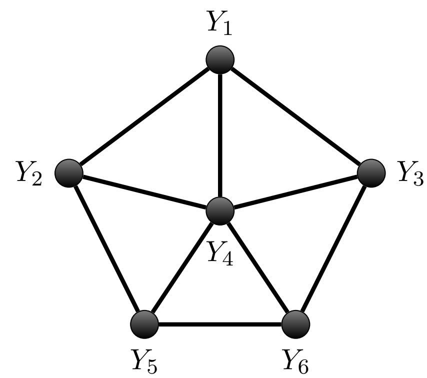
(i)



(ii)



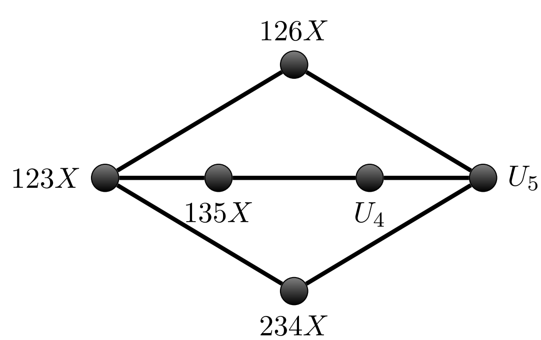
(iii)



(iv)

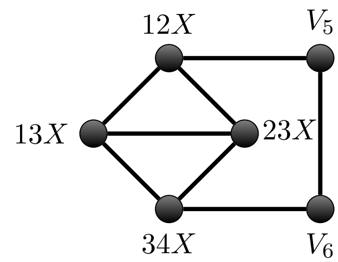
Proof. We shall show that each of the graphs (i) to (iv) is forbidden. It follows that all of these graphs are minimally forbidden because every proper induced subgraph is realisable by the 5-vertex classification of Section 6.4.

(i) By Lemma 6.7 applied to U_1, U_2, U_3, and U_6, we must have the labels



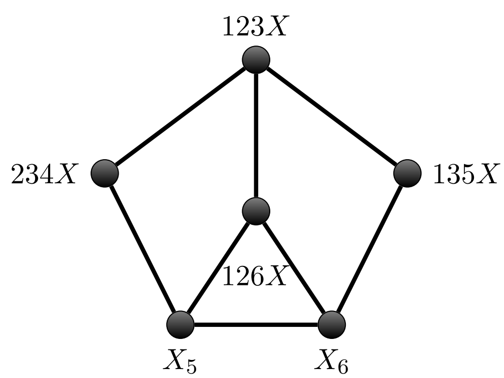
for some set X. Apply Lemma 6.2 with $U = U_5$ and $Y = X \cup \{2\}$ and $(u_1, u_2) = (1, 6)$ and $(u_1', u_2') = (3, 4)$. Since U_5 must be distinct from and not adjacent to $123X$, the only possible label for U_5 is $246X$. Then U_5 differs from $135X$ in exactly three elements, but is joined by a path of length two, giving a contradiction.

(ii) By Lemma 6.5 applied to V_1, V_2, V_3, and V_4, we must have the labels



for some set X. Since $23X, 34X$, and V_6 together induce P_3, then by Lemma 6.1 we must label V_6 by $45X$. Apply Lemma 6.2 with $U = V_5$ and $Y = X$ and $(u_1, u_2) = (1, 2)$ and $(u_1', u_2') = (4, 5)$. Since V_5 is not adjacent to $13X$ and $23X$, each of the cases (i) to (iv) gives a contradiction.

(iii) By Lemma 6.7 applied to X_1, X_2, X_3, and X_4, we must have the labels



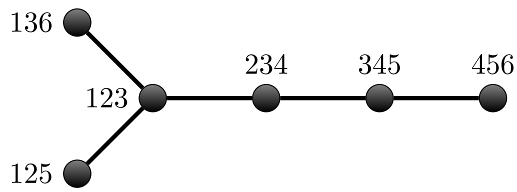
for some set X. Apply Lemma 6.2 with $U = X_5$ and $Y = X \cup \{2\}$ and $(u_1, u_2) = (3, 4)$ and $(u_1', u_2') = (1, 6)$. Since X_5 must be distinct from and not adjacent to $123X$, the only possible label for X_5 is $246X$. Then X_5 differs from $135X$ in exactly three elements but is joined by a path of length two, giving a contradiction.

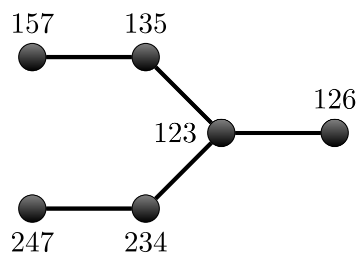
(iv) This is W_6, which is forbidden by Theorem 6.9.

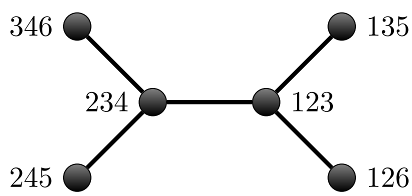
□

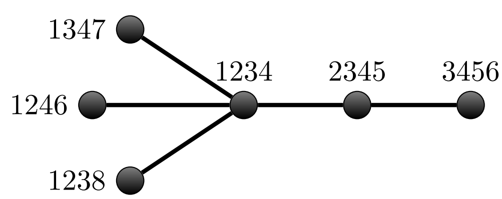
6.6 6-vertex classification of γ-graphs

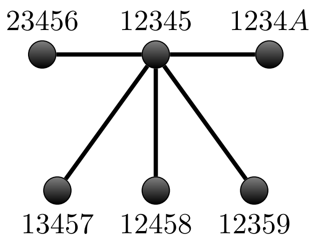
We reference [12] for the 112 connected graphs on six vertices. The 69 graphs labelled with specific sets are realisable; the 39 graphs labelled with U_i, V_i, X_i, or Y_i are forbidden because they contain as an induced subgraph one of the corresponding minimally forbidden graphs on five vertices given in Theorem 6.10; and the four unlabelled graphs are the minimally forbidden ones on six vertices given in Theorem 6.11. The graphs are ordered from least to most number of edges. When necessary, other symbols (such as capitalised letters) are used for the vertex-labels. Lemmas 6.3 and 6.8 have not been used directly until now, but they were helpful in determining some of the following labellings.







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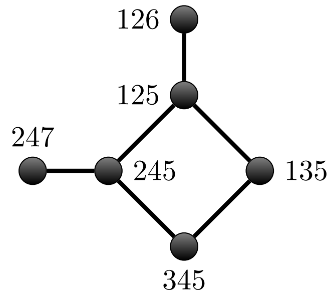
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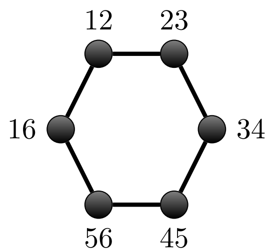
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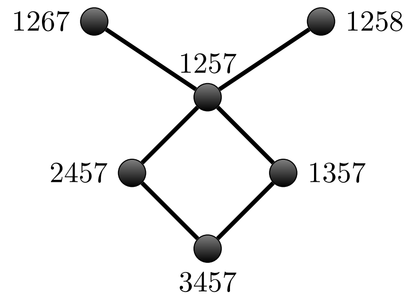
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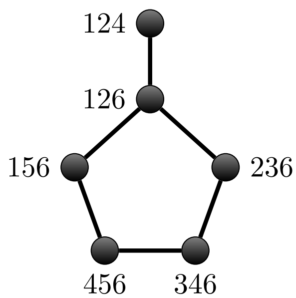
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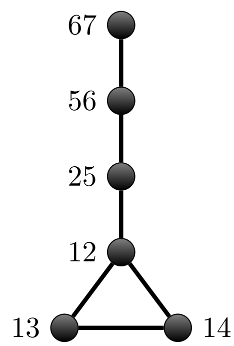
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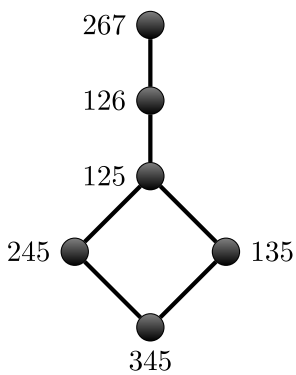
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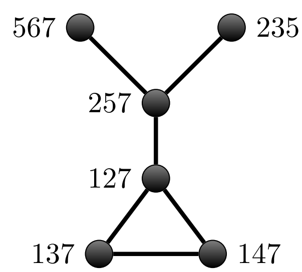
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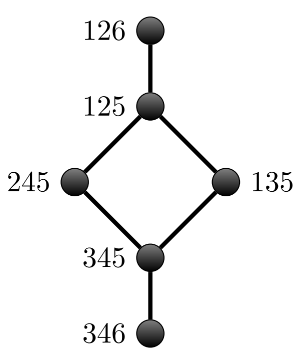
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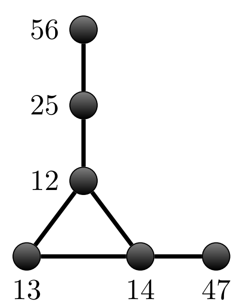
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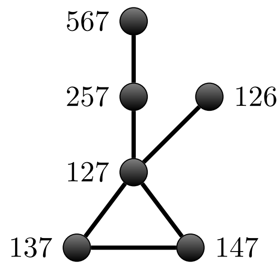
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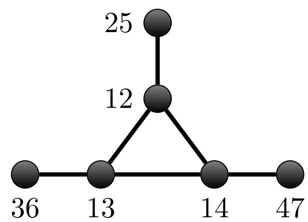
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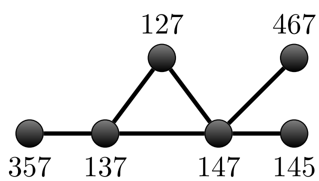
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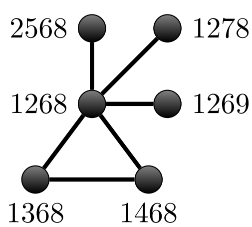
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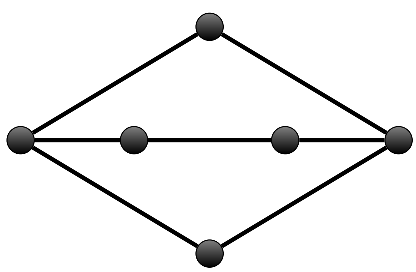

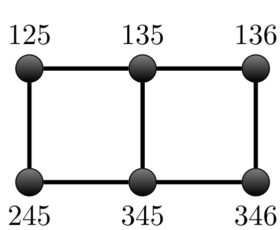
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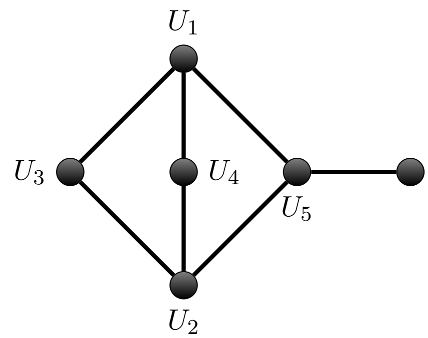
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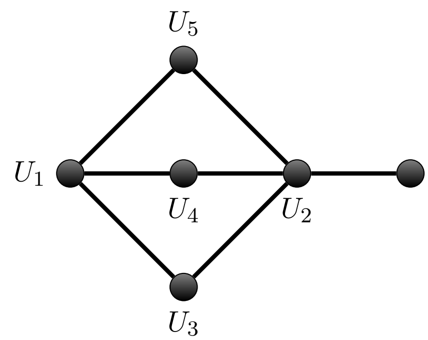
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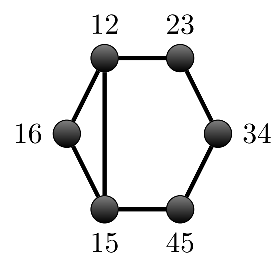
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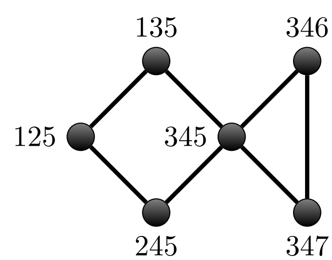
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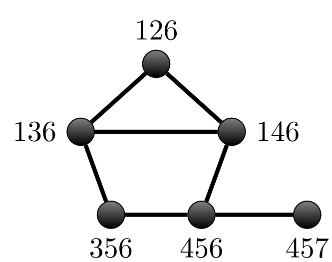
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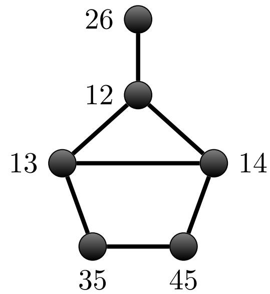
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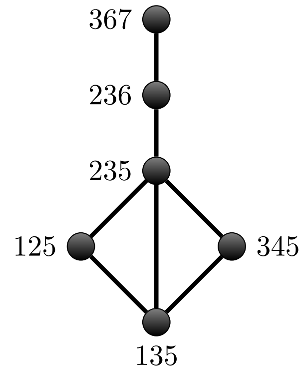
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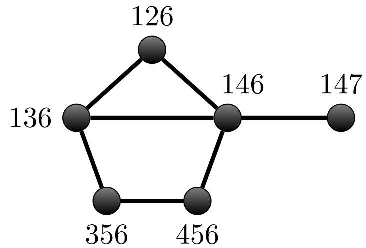
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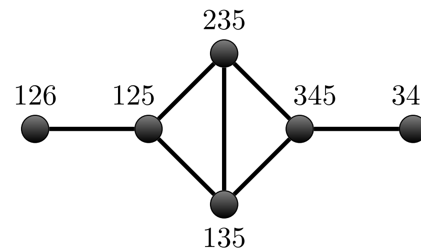
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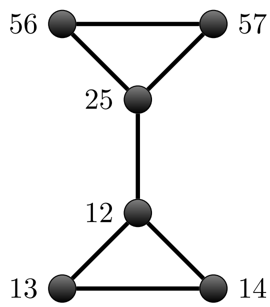
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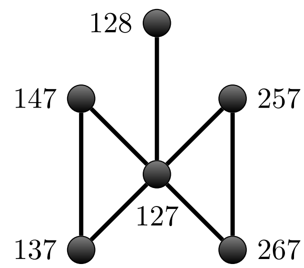
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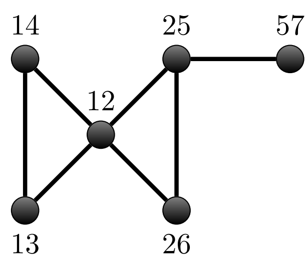
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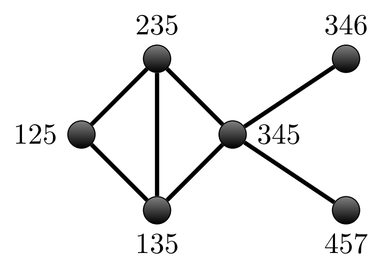
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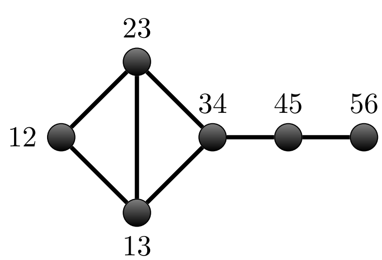
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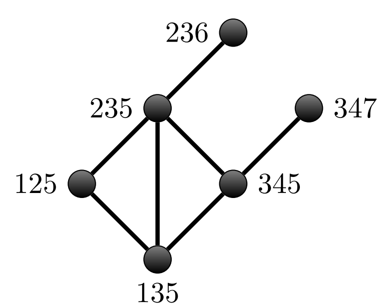
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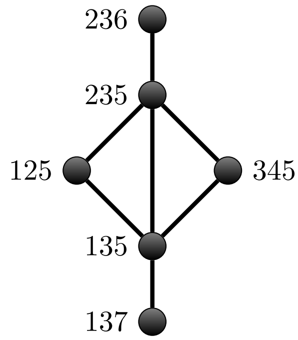
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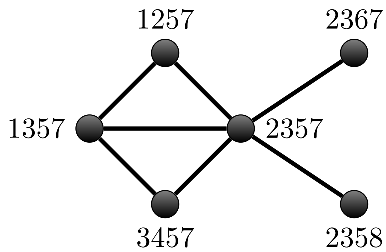
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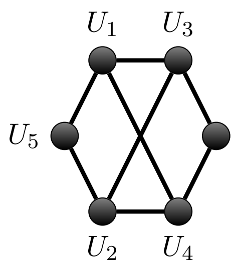
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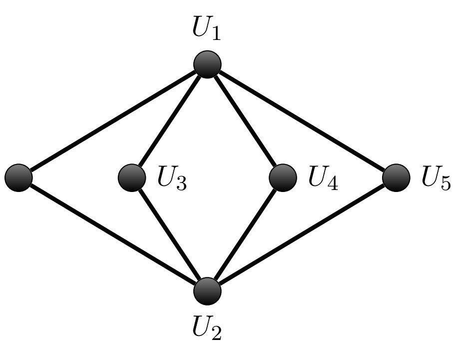
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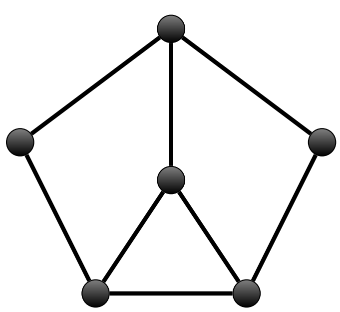

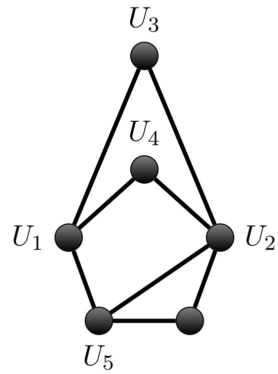
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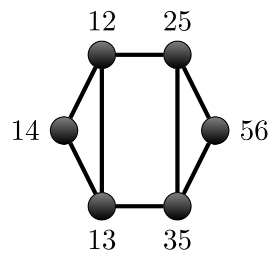
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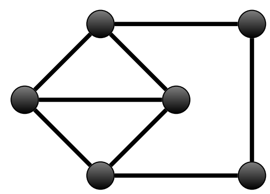

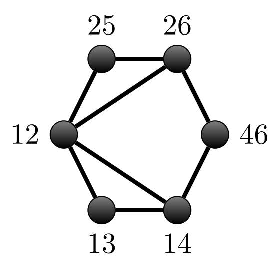
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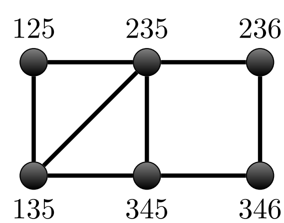
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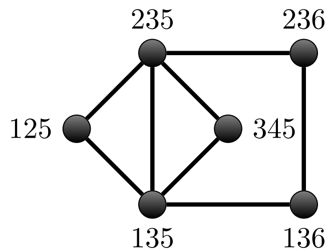
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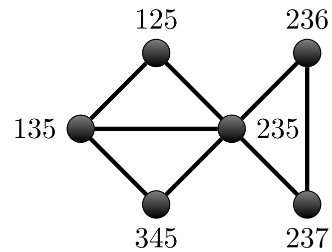
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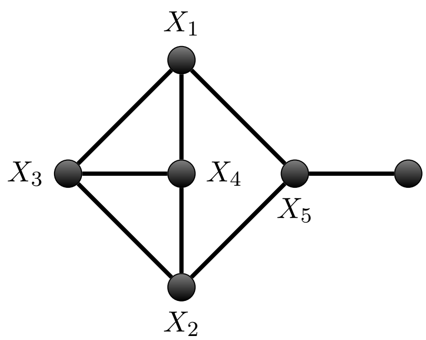
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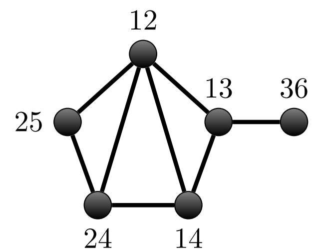
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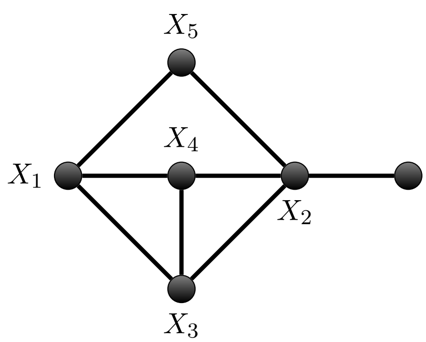
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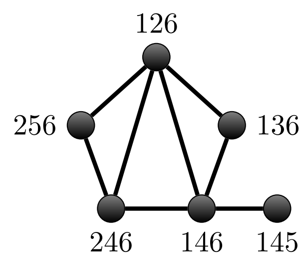
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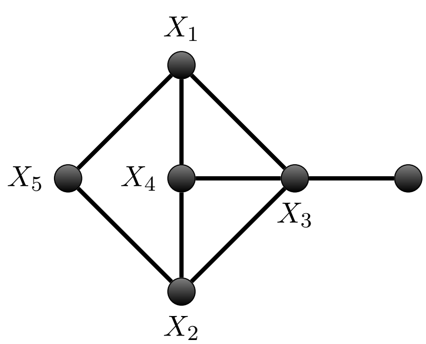
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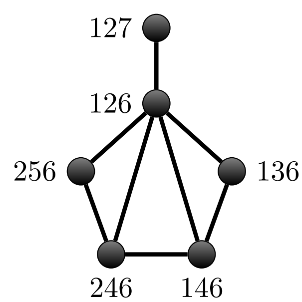
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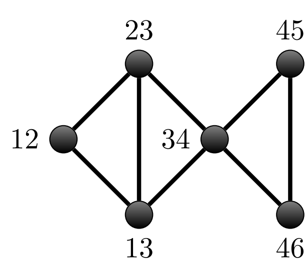
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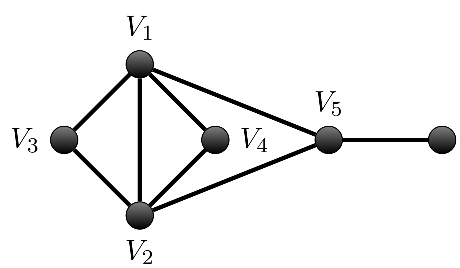
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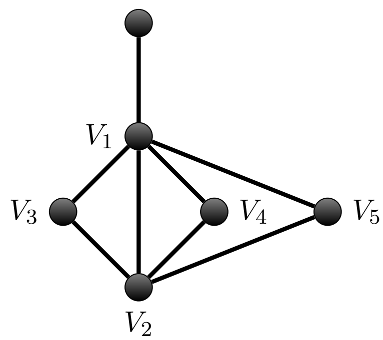
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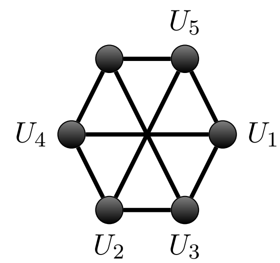
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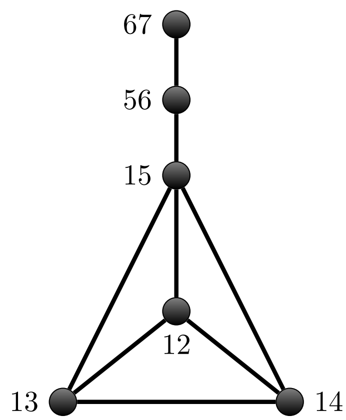
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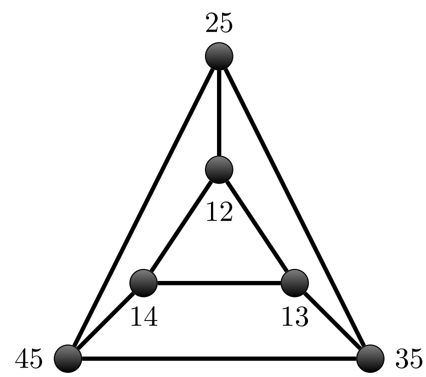
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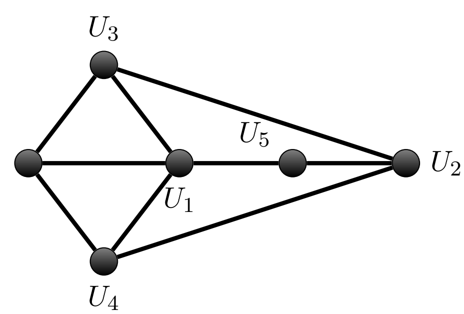
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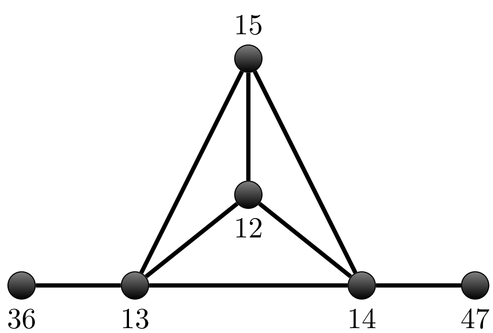
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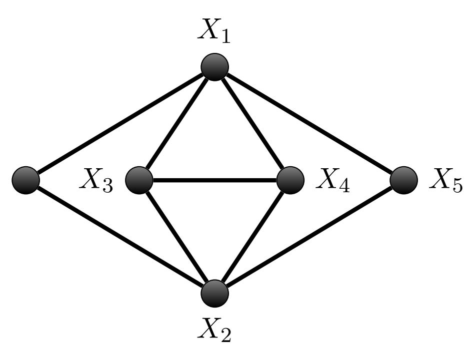
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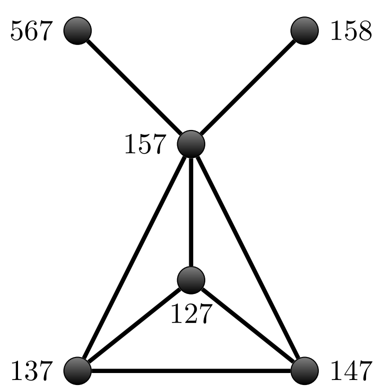
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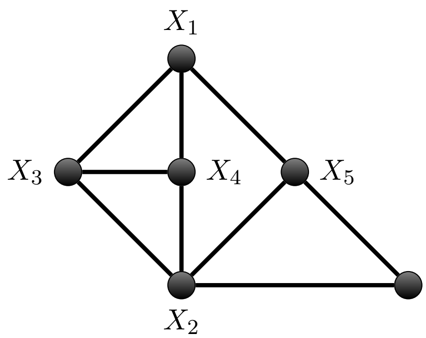
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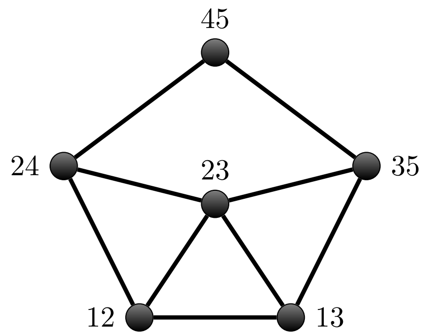
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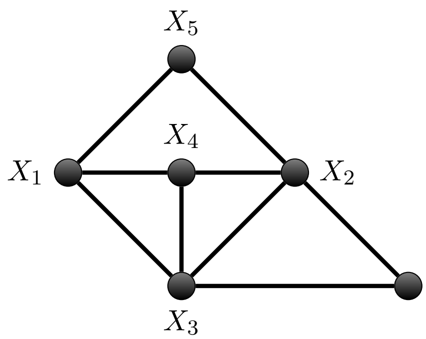
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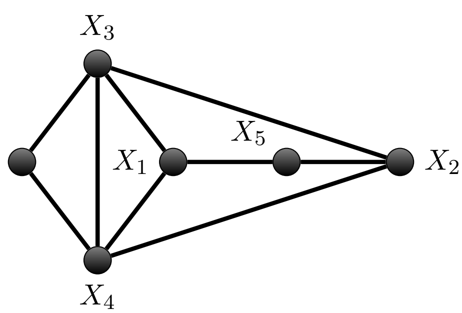
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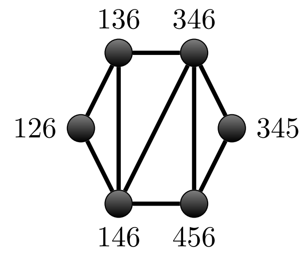
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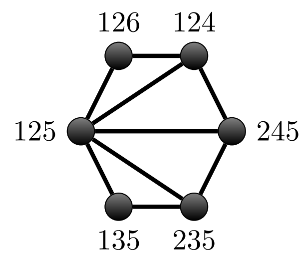
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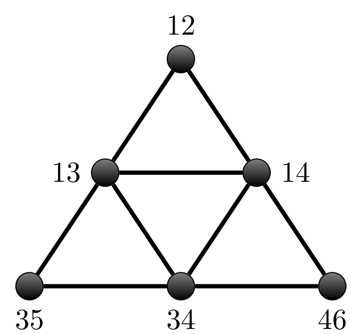
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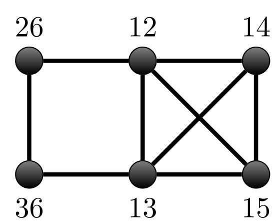
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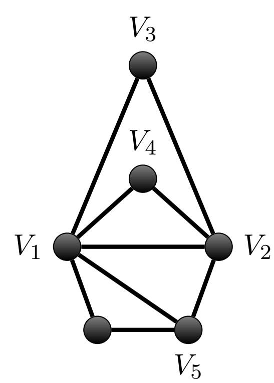
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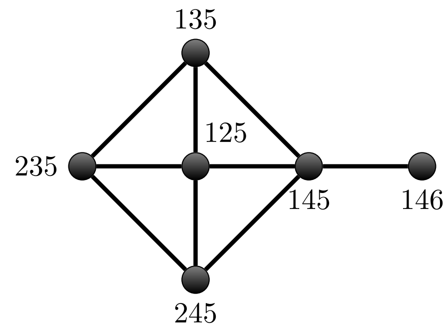
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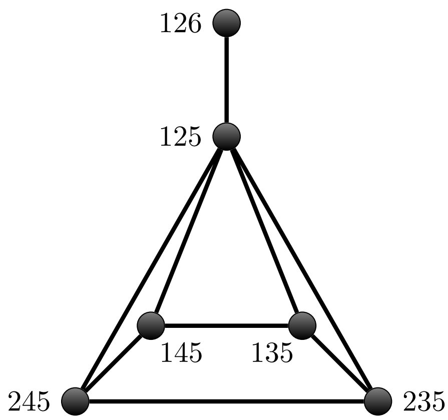
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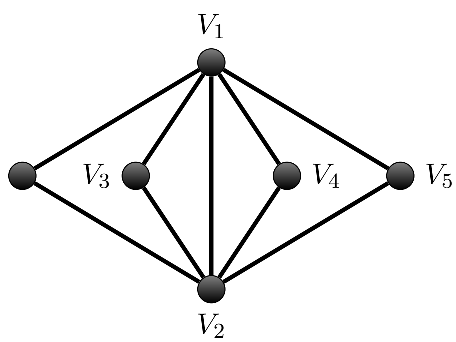
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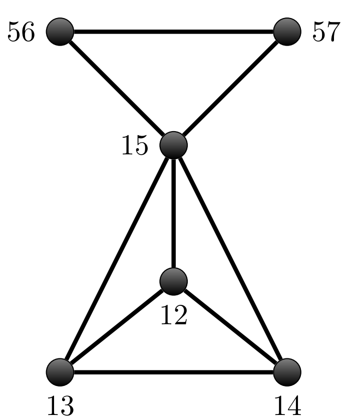
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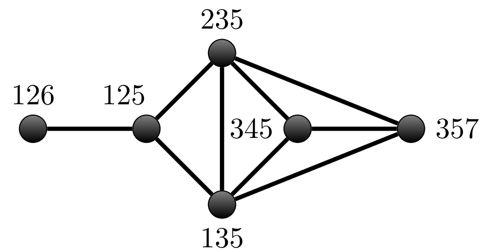
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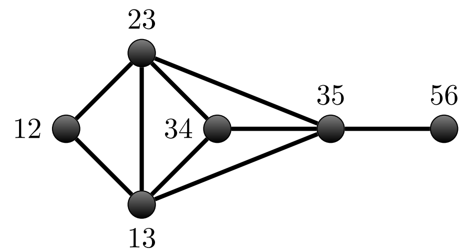
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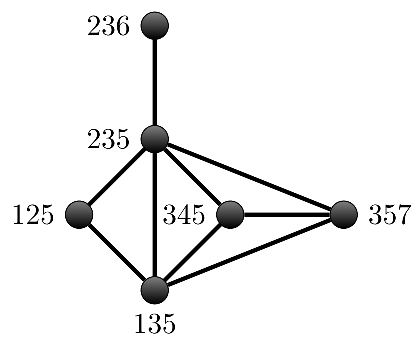
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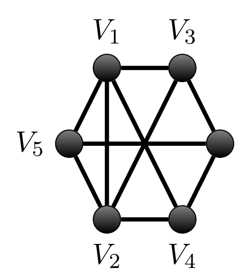
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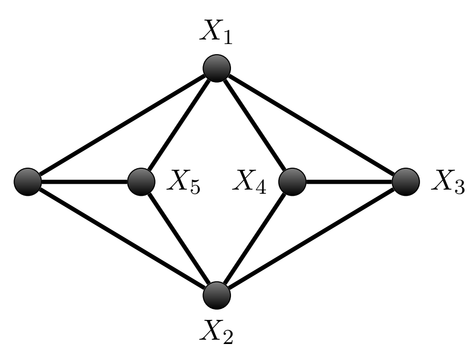
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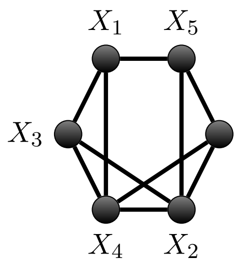
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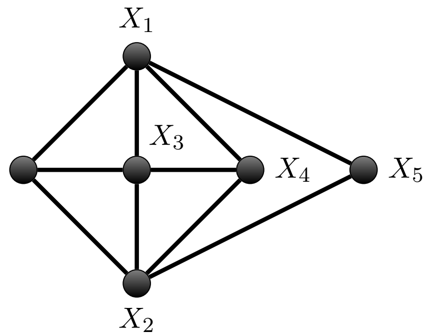
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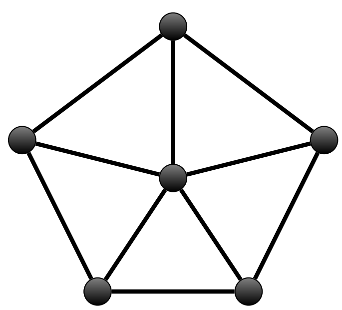

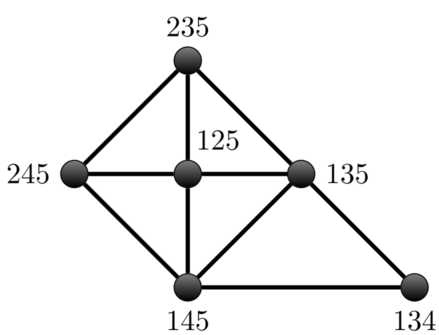
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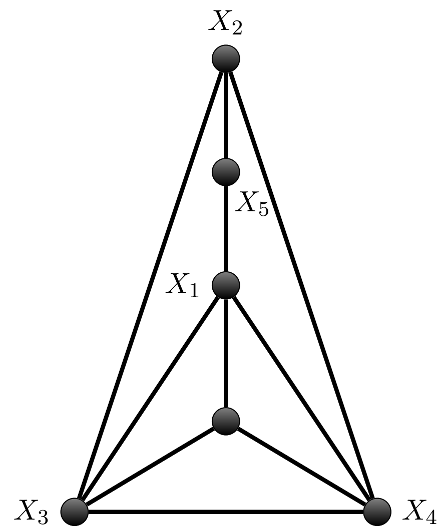
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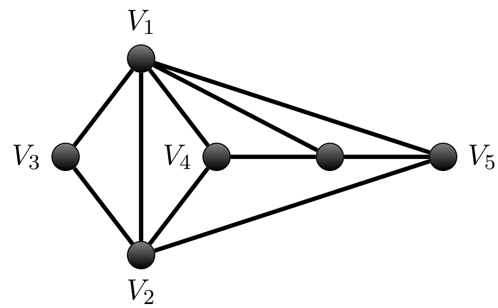
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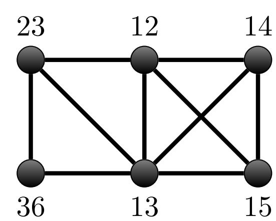
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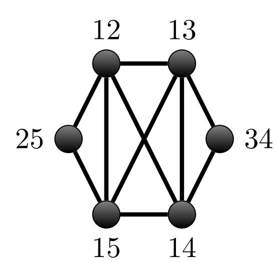
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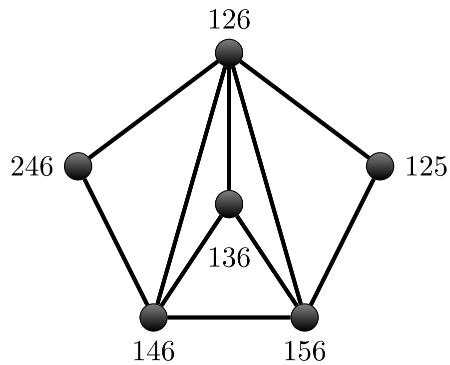
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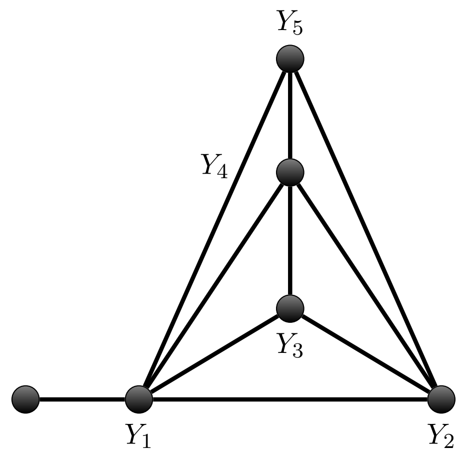
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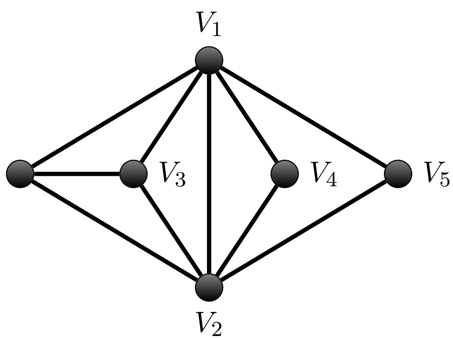
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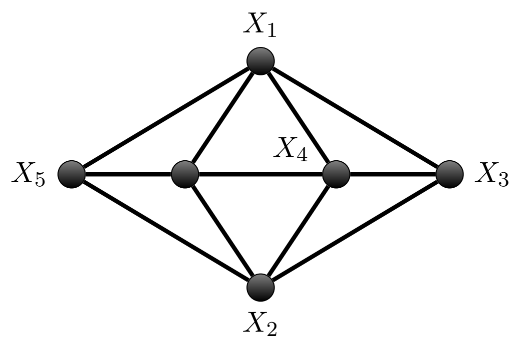
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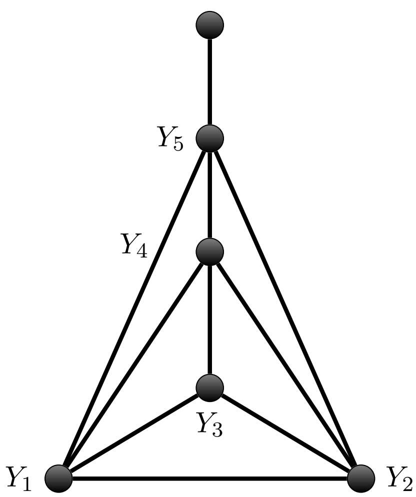
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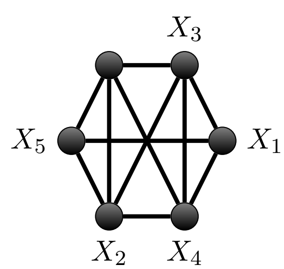
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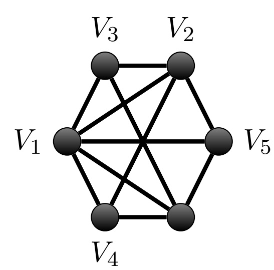
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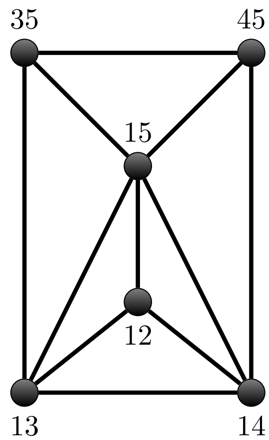
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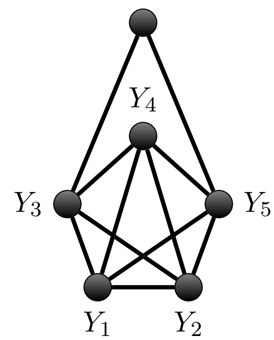
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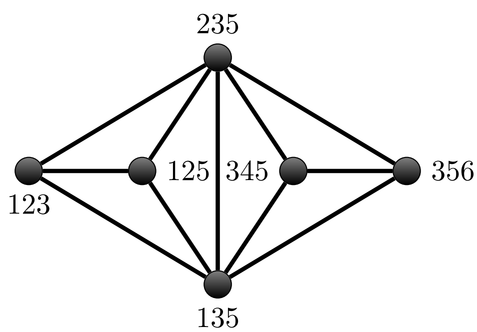
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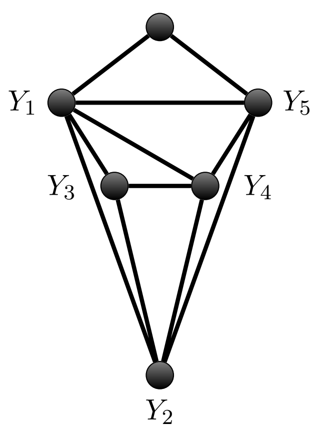
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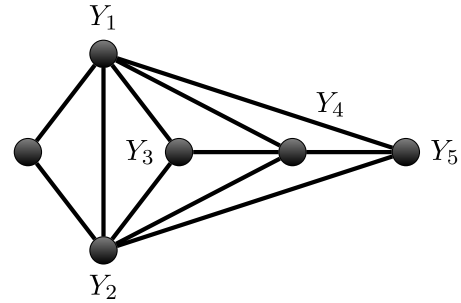
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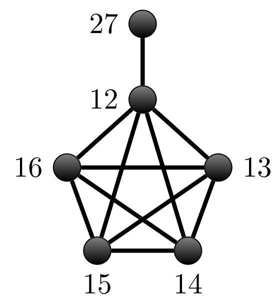
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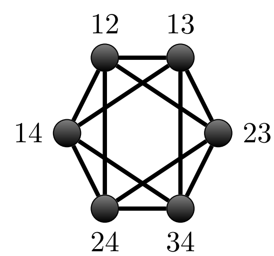
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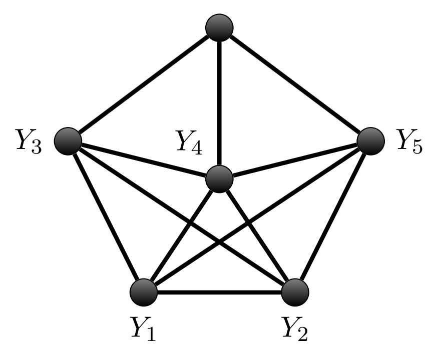
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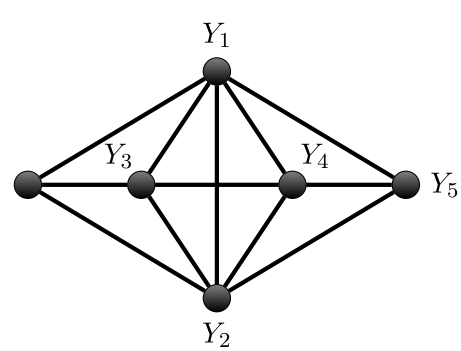
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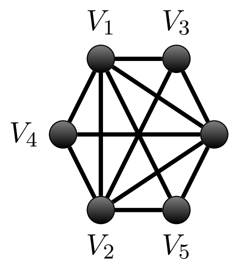
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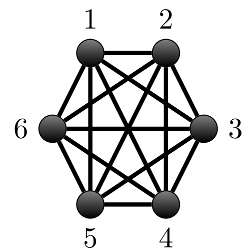
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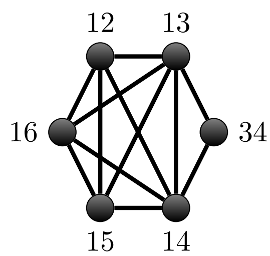
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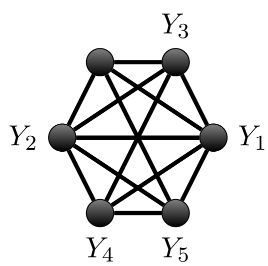
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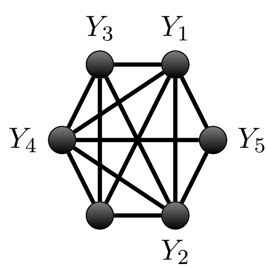
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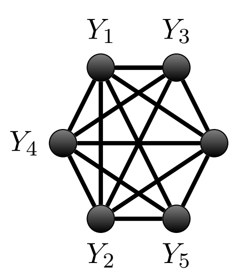
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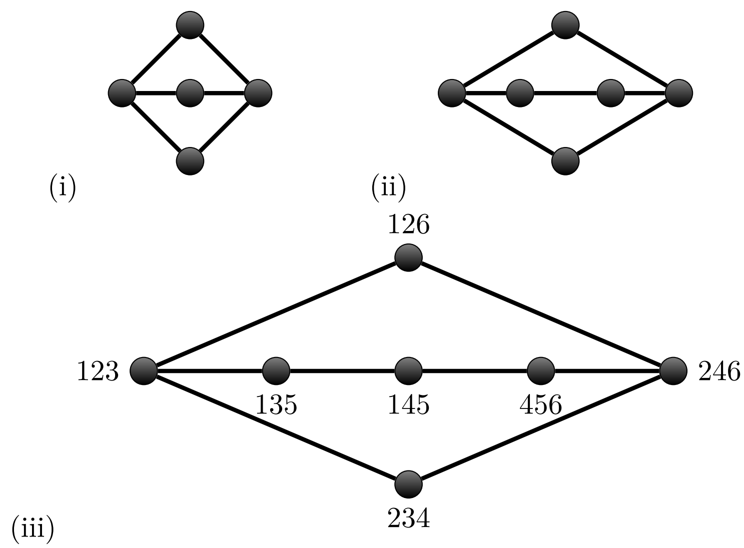


Figure 6.3: The family suggested by the graphs in (i), (ii), and (iii) is not forbidden because (iii) is realisable.

6.7 Discussion of forbidden families

It is worth investigating the minimally forbidden graphs on five and six vertices for patterns that suggest forbidden families. We recovered two forbidden families in Section 5.3 this way: the complete bipartite graphs $K_{m,n}$ that contain $K_{2,3}$ as an induced subgraph, and the graphs $K_n - e$ for $n \geq 5$. We proved in Theorem 6.10 parts (i) and (iv) that $K_{2,3}$ and $K_5 - e$ are minimally forbidden graphs on five vertices, from which the corresponding forbidden families can be deduced; no forbidden families are immediately suggested by the graphs in parts (ii) and (iii). We also used Theorem 6.11 part (iv) to make an initial conjecture that W_n is forbidden for each $n \geq 6$. We immediately disproved this by successfully labelling W_7, and then established in Section 6.2 that W_{2n} for $n \geq 3$ is in fact minimally forbidden. One must be cautious though: taken together, Theorem 6.10 part (i) and Theorem 6.11 part (i) suggest another forbidden family of graphs, but Figure 6.3 demonstrates that this family has realisable members.

[33] claims without proof that the three graphs given in Figure 6.4 are forbidden. We exhibit labellings there for two of the graphs and explicitly prove that the third is forbidden.

Lemma 6.12. *The graph H in Figure 6.4 is minimally forbidden.*

Proof. Every proper induced subgraph of H is labellable (hence realisable), so it is sufficient to show that H is forbidden. Suppose, for a contradiction, that H can be labelled as in Figure 6.4. By Lemma 6.7 applied to U_1, U_2, U_6, and U_7, we must have the labels

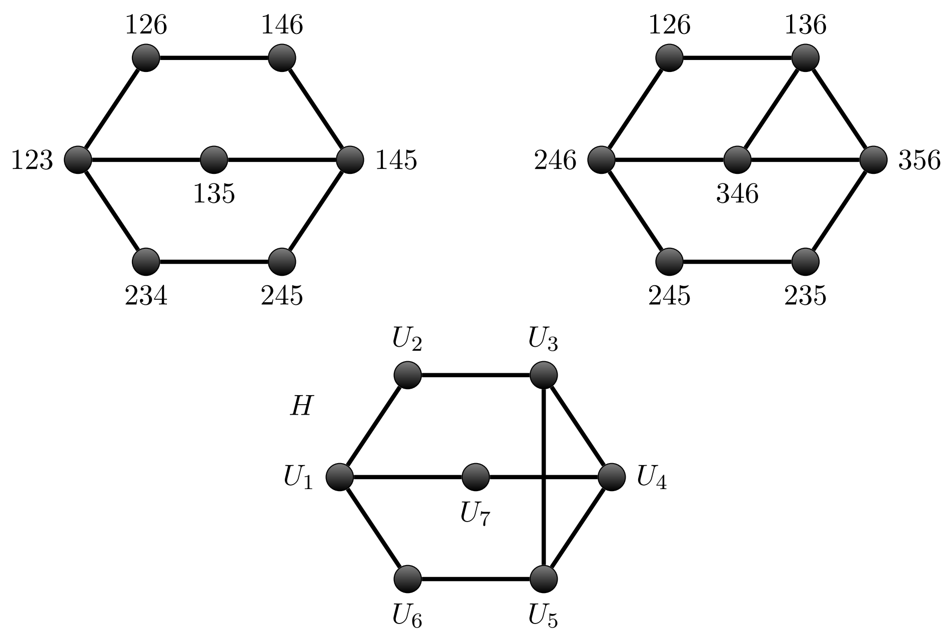
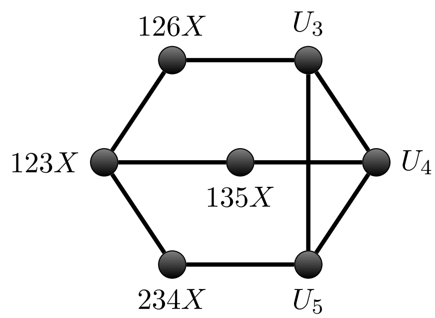


Figure 6.4: [33] claims without proof that the above graphs are forbidden as JIS graphs, but only H is forbidden.



for some set X. Apply Lemma 6.1 to the path joining $123X$ to $135X$ to U_4. This gives six possibilities for U_4: $145X, 156X, 157X, 345X, 356X$, and $357X$. We need consider only the first three of these, because the mapping that interchanges 1 with 3, and 4 with 6, maps the labelled graph to (a reflection through a horizontal axis of) itself. Therefore U_4 must be labelled $145X$ or $156X$ or $157X$. We can discard the label $156X$ because U_4 is not adjacent to $126X$, and we can discard the label $157X$ because U_4 is not adjacent to $135X$. Therefore U_4 must be labelled $145X$.

Now apply Lemma 6.2 with $U = U_3$ and $Y = X \cup \{1\}$ and $(u_1, u_2) = (2, 6)$ and $(u'_1, u'_2) = (4, 5)$. Since U_3 is adjacent to neither $123X$ nor $135X$, it must be labelled $146X$. Similarly, apply Lemma 6.2 with $U = U_5$ and $Y = X \cup \{4\}$ and $(u_1, u_2) = (2, 3)$ and $(u'_1, u'_2) = (1, 5)$. Since U_5 is adjacent to neither $123X$ nor $135X$, it must be labelled $245X$. We therefore have the labelling $U_3 = 146X$ and $U_5 = 245X$, which contradicts that U_3 and U_5 are adjacent. $\square$

Chapter 7

Distance-d γ-Graphs

In this chapter, we extend the definitions of domination and γ-sets in graphs (as, for example, in [29]) and introduce the corresponding extension of a γ-graph.

Definition 7.1. *For $d \geq 1$, a* distance-d dominating set *of a graph G is a set $S \subseteq V(G)$ such that every vertex of G is either in S or at distance at most d from a vertex in S. A* γ_d-set *of G, also called a* minimum distance-d dominating set *of G, is a distance-d dominating set of smallest size. The* distance-d domination number *of G, denoted $\gamma_d(G)$, is the size of a γ_d-set of G.*

The case $d = 1$ of Definition 7.1 corresponds to the usual definition of domination. When the context is clear, we shall say, for example, "v_1 dominates v_2" in place of "v_1 distance-d dominates v_2."

We may now extend the definition of γ-graph in accordance with distance-d domination.

Definition 7.2. *The* γ_d-graph *$\gamma_d \cdot G$ of a graph G is formed as follows:*

- *the vertices of $\gamma_d \cdot G$ correspond to the γ_d-sets of G;*
- *two vertices are adjacent if and only if their corresponding γ_d-sets intersect in a set of size $\gamma_d(G) - 1$.*

The definitions of realisable, labellable, and forbidden γ_d-graphs follow similarly. Note that the case $d = 1$ yields the γ-graph $\gamma \cdot G$.

We now generalise Theorem 1.10, the proof of which is inspired by the construction in Section 4.2.

Theorem 7.3. *Let $k \geq 1$ be an integer, and let $\mathcal{D}$ be a nonempty set of k-subsets of $\{1, 2, 3, \dots\}$. Then there is a graph G whose γ_d-sets are the elements of $\mathcal{D}$.*

Proof. Take $n = \left| \bigcup_{D \in \mathcal{D}} D \right|$, and relabel if necessary so that each element of $\mathcal{D}$ is a subset of $[n]$. Regarding $\mathcal{D}$ as a clutter with respect to $[n]$ (see Definition 2.1), the blocker $\mathcal{B}$ of $\mathcal{D}$ is the collection of all minimal subsets of $[n]$ containing at least one element of each $D \in \mathcal{D}$. We use the following construction for G.

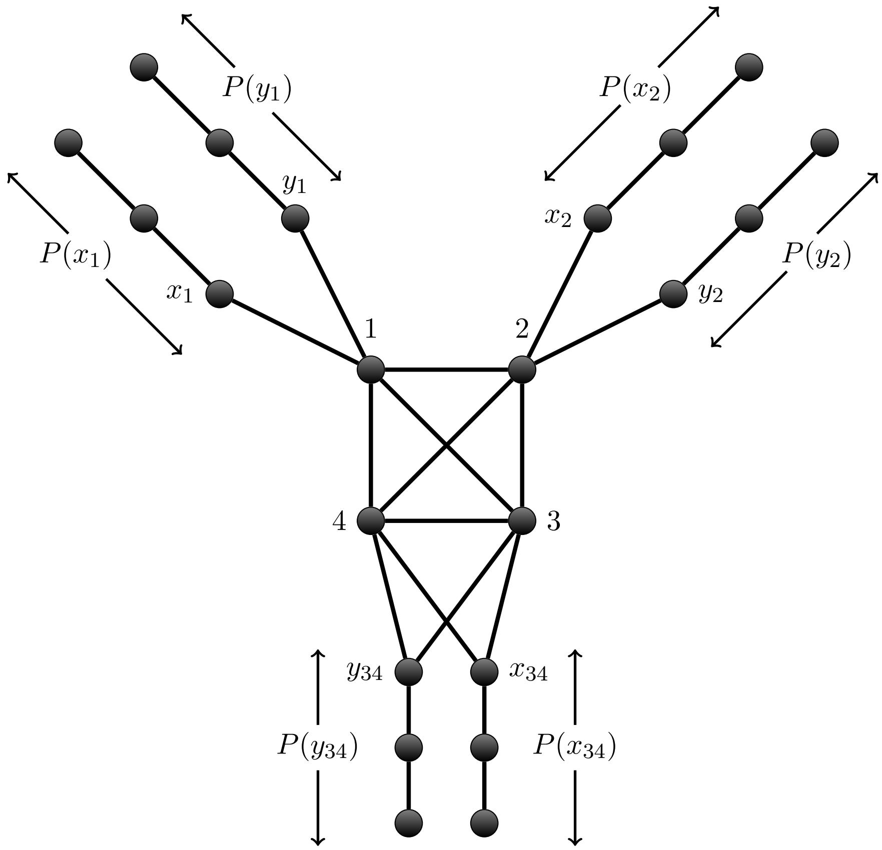


Figure 7.1: Construction of G for Theorem 7.3 with $d = 3$, $k = 3$, and $\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$.

(1) Initialise G to be K_n and label its vertices $1, 2, \ldots, n$.

(2) For each $B \in \mathcal{B}$, form paths $P(x_B)$, $P(y_B)$ of length $d - 1$ terminating in vertices x_B, y_B, respectively, and join x_B and y_B to the vertices of B.

The resulting graph G has

$$n + 2d|\mathcal{B}| \tag{7.1}$$

vertices and

$$\binom{n}{2} + 2\sum_{B\in\mathcal{B}} |B| + 2(d-1)|\mathcal{B}| \tag{7.2}$$

edges.

Let $d = 3$ and consider again the example with $k = 3$ and $\mathcal{D} = \{\{1, 2, 3\}, \{1, 2, 4\}\}$, so $n = 4$. Initialise G to be K_4 and label its vertices $1, 2, 3, 4$. Form the blocker $\mathcal{B} = \{\{1\}, \{2\}, \{3, 4\}\}$ of $\mathcal{D}$. For the element $\{1\}$, form paths $P(x_1)$ and $P(y_1)$, which terminate in vertices x_1 and y_1 respectively, and join each of x_1 and y_1 to the vertex 1. This ensures that $[n] \setminus \{1\} = \{2, 3, 4\}$, along with each of its proper subsets, is not a γ_3-dominating set of G. Repeat for each other element of $\mathcal{B}$. See Figure 7.1 for the resulting graph G, which has 22 vertices and 26 edges.

Let G be constructed according to Steps (1) and (2) above. We now show that the γ_d-sets of G are exactly the elements of $\mathcal{D}$.

(a) *No added vertex is contained in a γ_d-set of G.*

Consider $B \in \mathcal{B}$ and suppose, for a contradiction, that a vertex w in $P(x_B)$ is contained in a γ_d-set D of G. Then no vertex z in $P(y_B)$ is contained in D, otherwise we may obtain a smaller dominating set than D by replacing the vertices w and z in D with a single vertex from the nonempty set B: this vertex dominates all vertices of $P(x_B)$ and $P(y_B)$ by construction, and it dominates all vertices of $[n]$ because G was initialised to K_n.

Since no vertex z in $P(y_B)$ is contained in D, and the pendant vertex p of $P(y_B)$ is dominated by some vertex of D whose distance from p is at most d, the set D must contain some vertex of B. This vertex dominates all vertices of $P(x_B)$ and $P(y_B)$ by construction, and it dominates all vertices of $[n]$ because G was initialised to K_n, so we may obtain a smaller dominating set than D by removing w from D, giving the required contradiction.

(b) *Each element of $\mathcal{D}$ is a distance-d dominating set of G.*

Let $D \in \mathcal{D}$. Each vertex of D dominates all vertices of $[n]$ because G was initialised to K_n. Let $B \in \mathcal{B}$; it remains to show that D dominates all vertices of $P(x_B)$ and $P(y_B)$. By the definition of $\mathcal{B}$, we may choose a vertex i in the nonempty set $B \cap D$. By construction, the vertices of $P(x_B)$ and $P(y_B)$ are all dominated by $i \in D$.

(c) *No $(k-1)$-subset of $[n]$, and no k-subset of $[n]$ not contained in $\mathcal{D}$, distance-d dominates G.*

In the case $k = 1$, the statement holds vacuously. Otherwise, take $k \geq 2$ and let S be a subset of $[n]$ that either has size $k-1$, or has size k and is not contained in $\mathcal{D}$. We shall show that S does not dominate G.

Let $\widetilde{\mathcal{D}}$ be the set $\{D \setminus S : D \in \mathcal{D}\}$. By the definition of S, each element of $\widetilde{\mathcal{D}}$ is nonempty. Let $\widetilde{B}$ be an element of the blocker of $\widetilde{\mathcal{D}}$. Note that this implies $\widetilde{B} \cap S = \emptyset$. Since $\widetilde{B}$ contains at least one element of $D \setminus S$ for each $D \in \mathcal{D}$, then $\widetilde{B}$ necessarily contains at least one element of each $D \in \mathcal{D}$. By minimality of the elements of the blocker $\mathcal{B}$, we have $\widetilde{B} \supseteq B$ for some $B \in \mathcal{B}$. Using $\widetilde{B} \cap S = \emptyset$ and $B \in \mathcal{B}$, we must then have $\widetilde{B} = B$ (otherwise B would be a proper subset of $\widetilde{B}$ containing at least one element of each $D \setminus S$ in $\widetilde{\mathcal{D}}$, contradicting the minimality of $\widetilde{B}$ in the blocker of $\widetilde{\mathcal{D}}$).

We have shown that $\widetilde{B} \in \mathcal{B}$, so G contains a path $P(x_{\widetilde{B}})$. Since $\widetilde{B} \cap S = \emptyset$, it follows that S does not dominate the pendant vertex of $P(x_{\widetilde{B}})$ and therefore does not dominate G.

By part (a), the γ_d-sets of G contain vertices only from $[n]$. By part (b), the k-subsets in $\mathcal{D}$ dominate G, so $\gamma_d(G) \leq k$. By parts (b) and (c), we know $\gamma_d(G) \geq k$ and the k-subsets of $[n]$ which dominate G are exactly the elements of $\mathcal{D}$. It follows that the γ_d-sets of G are exactly the elements of $\mathcal{D}$. $\square$

This leads to the following corollary.

Corollary 7.4. *A graph is realisable as a γ_d-graph if and only if it is labellable as a γ_d-graph.*

Although Honkala et al. [26] give certain generalisations of Theorem 1.10, we believe that Theorem 7.3 has not been previously stated (even implicitly in another formalisation).

Chapter 8

Conclusions and Future Work

Here we discuss some questions that arise from this thesis, and propose some future directions for work on γ-graphs (equivalently, JIS graphs).

For each labellable graph H, Theorem 1.10 guarantees the existence of a parent graph G such that $H = \gamma \cdot G$. This greatly simplifies our ability to determine if a graph is realisable or forbidden, but it leaves many questions about efficiency. As we noted in Section 4.3 (see Figures 4.3 and 4.4), our attempts to realise P_5 constructively turned out to be less efficient than a realisation in which the parent graph is P_{11}. In general, can we improve the efficiency of the constructions in Sections 4.1 and 4.2? What is the best measure of efficiency for a construction that proves Theorem 1.10?

Figure 4.4 also depicts a labelling of P_5 with sets of size two, which are smaller than those in the labelling in Figure 4.3. Given a labellable graph, how can we label it efficiently? Can we ensure that the labels will be sets of smallest possible size? This corresponds to minimising the number k in Definition 1.5. We believe the forced-labelling lemmas of Section 6.1 would be helpful in answering this. Note also that P_{11} does not realise P_5 with γ-sets of size two. For a given labellable graph, does a labelling with sets of smaller size allow a simpler parent graph? It is not clear whether P_5 could be realised by a graph as simple as P_{11} but with γ-sets of size two.

We completely classified all graphs on up to six vertices as either realisable or forbidden. There are 853 connected graphs on seven vertices and 11117 connected graphs on eight vertices [1], so further classification would benefit from automation. The following algorithm could be used for determining which n-vertex graphs are realisable or forbidden.

- By Corollary 5.1, consider only connected graphs.
- By Theorem 5.7, consider only graphs with no vertex of degree one.
- For $5 \leq k \leq n-1$, remove each graph that contains a minimally forbidden k-vertex graph as an induced subgraph. Also remove all graphs contained in forbidden families.

- Try to label each of the remaining graphs. If a labelling can be found, then the graph is realisable by Corollary 1.11; otherwise, seek a non-existence proof using the forced-labelling lemmas for assistance.

One major advantage of further classification would be the likely emergence of new families of either realisable or forbidden graphs.

Sridharan et al. [38, Problem 2] introduce the following interesting problem: if H is a γ-graph, does there exist a graph G such that $\gamma \cdot (\gamma \cdot G) = H$? We cannot answer this using the constructions of Chapter 4 because the constructed parent graph of H often contains $K_{2,3}$ as an induced subgraph, in which case the parent graph is forbidden.

We have demonstrated a connection between γ-graphs and JIS graphs. Are there connections between γ-graphs and other mathematical objects? What are their consequences? Since the γ-graph $G(\gamma)$ (see Definition 1.8) is always a subgraph of the γ-graph $\gamma \cdot G$, are there any implications of these connections for the γ-graph $G(\gamma)$? What about for the variations of γ-graphs defined in [35]?

Finally, Haas and Seyffarth [21] define the following class of graphs.

Definition 8.1. *Let G be a graph and $k \geq \gamma(G)$ an integer. The* k-dominating graph of G*, denoted $D_k(G)$, is defined as follows:*

- *the vertices of $D_k(G)$ correspond to all dominating sets of G of size at most k;*
- *two vertices are adjacent if and only if their corresponding dominating sets differ by the addition or deletion of a single vertex.*

See Figure 8.1 for a comparison of $\gamma \cdot G$ and $D_k(G)$ for $k = 3$ and $G = P_3$. What is the relationship between the γ-graph $\gamma \cdot G$ and the k-dominating graph $D_k(G)$? Given the added complexity with dominating sets of multiple sizes and a more expansive adjacency condition, we do not anticipate a simple connection between the two classes of graphs.

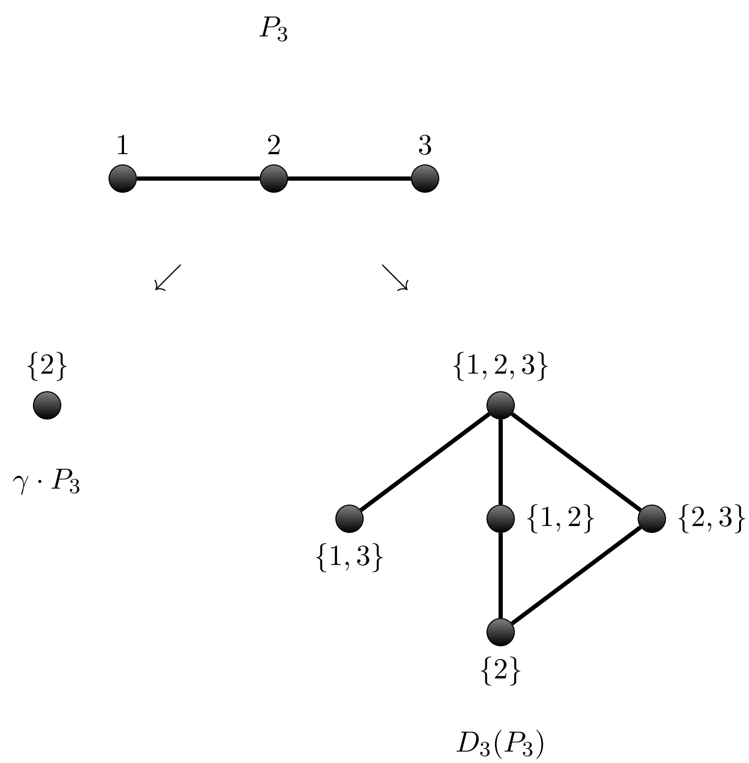


Figure 8.1: A comparison of $\gamma \cdot P_3$ and $D_3(P_3)$.

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