Mathematical Reasoning Among Adults on the Autism Spectrum: Case Studies with Mathematically Experienced Participants

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Abstract

I investigate the unique or unusual characteristics of mathematical problem-solving among adults on the autism spectrum by conducting and analyzing three case studies. The case studies involve providing individuals with a variety of mathematical problems divided into four main groups: paradoxes of infinity, problems emphasizing algebraic or geometric solution, probability, and logic and proof. Participants are given individual interviews, intended to facilitate the communication of their thought processes when solving these problems. Results are analyzed with a variety of constructs, from a perspective that is rooted in Vygotskian ideas and supportive of neurodiversity.

Keywords: Mathematical problem-solving; mathematical reasoning; autism; disability

To my family, and to the autistic community

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List of Acronyms

ADD	Attention Deficit Disorder
ADHD	Attention Deficit Hyperactive Disorder
ASD	Autism Spectrum Disorder
CSD	Centre for Students with Disabilities
OCD	Obsessive Compulsive Disorder
SFU	Simon Fraser University

Glossary

Neurodiversity	A positive and inclusive perspective on neurological differences including but not limited to autism
Neurotypical	Typically developing, without any diagnosable neurological difference (including but not limited to autism)

Chapter 1. Introduction

I have had an unusual set of school experiences, quitting public school early on in order to be home schooled, and taking courses at the local community college in addition to home schooling for most of that time. This was a decision both my parents and I were interested in, for two primary reasons: the perceived weakness of the public school curriculum and my own negative experiences and general dislike of social activity with other elementary school students. Although I am now aware of group programs for socialization of home schooled students, I had little to no interest at the time and neither I nor my parents considered them. While I did not have much socialization in the community college environment, I did not have any negative experiences with students there, and was generally accepted. Eventually, I started attending the community college full-time, an experience which I feel was helpful to my later interest in education by exposing me to a wide variety of students with different backgrounds.

I first developed an interest in autism when I was an undergraduate, through meeting other students who were on the autism spectrum and hearing about their experiences. During my master's program I had more encounters and friendships with autistic students (most of whom would describe themselves in these terms as opposed to "with autism", emphasizing that this was a part of them and not separable), and this period contributed more to my early impressions. This was also where I first learned about organizations like the Autistic Self-Advocacy Network (ASAN). Through conversations in that community, I became aware of not only gaps in research but also negative biases that have appeared in some research directions. One of the most clear of these gaps was the focus of research on children, which is reflective of a focus in wider non-academic conversation about autism, as well as more broadly about disability generally. This gap was stated as a frustration some of my autistic friends and acquaintances had with both academic and non-academic discourse around autism, and I hope to contribute to a broader view of autism with my work.

There is also a distinction here that I have seen some argument about concerning whether autism is a disability. My answer to this is generally first to ask what definition of disability is being used, since in my experience of discourse around autism I have learned about several different perspectives on this. At first, I disliked the characterization of autism as a disability, finding that too negative. However, when looking at it with the social model of disability in mind, I found an alternative interpretation. With the social model of disability, the disability is a characteristic of the system, occurring when a person encounters a system that is designed for abilities they lack. In this model, then, saying that autism is a disability is a reflection of an autistic person's interaction with the broader structure of society, and I feel that I can agree that autism is a disability in this framework without the unfortunate implications that arise from other conceptions of disability.

I also gradually developed my interest in mathematics education through my undergraduate and graduate experiences, where I completed bachelor’s and master’s degrees in mathematics, though I did not combine the two interests immediately. Throughout my prior university career, I held various tutoring and teaching assistantship positions, and always enjoyed helping others to better understand mathematics. Eventually, I found that this interest was stronger than my interest in mathematics research itself, and decided to change my area of study. Within the area of mathematics education, not only were my teaching experiences all in a college setting, but my institutional learning experiences were mostly at the college level as well (including my enrollment in college courses while being home schooled before full-time college study). Also, I still held an interest in rigorous mathematical thinking and more advanced mathematical concepts, particularly probability and mathematical proof. Both of these factors contributed to my interest in mathematics education being focused mainly at the undergraduate level.

When I found my interest in mathematics education, it seemed natural to look at people on the autism spectrum in particular, because of my prior interest and the popular association between autism and mathematical disciplines. I knew from my prior experiences that the reality was more complex, and wanted to explore that through research. With the suspected gap in autism research involving adults, combined with the interest I already had in focusing on the undergraduate level of mathematics education, research involving the mathematical reasoning of undergraduates on the autism spectrum seemed like a natural choice. Additionally, I observed a conversation where a mathematically experienced adult on the autism spectrum was talking about the number of points in two line segments of different lengths. He framed his answer of the number of points being equal as the result of a proof immediately, and did not appear to find the result particularly objectionable or counterintuitive. By contrast, the more common intuitive inclination is to claim that the longer segment has more points and so must be larger, and most tend to hold this inclination even after being introduced to the mathematical proof that the segments are of equal size (as defined by cardinality). This difference reinforced my interest as well as pointing toward problems involving infinity and ideas about intuition as possible avenues of investigation.

Although the association between autism and mathematics is fairly common culturally, I did not find a great amount of research which addressed this for more sophisticated levels of mathematics. This is most likely at least in part due to the lack of research involving adults, but may also come in part from less sophisticated conceptions of mathematics on the part of some researchers who are not in the field of mathematics education specifically.

My research question for this work is, first, how do the experiences of mathematical problem-solving differ for adults on the autism spectrum? I am keeping in mind that this may include positive, negative, and neutral aspects. Second, what would be an explanation for those differences that might promote understanding of the autistic community? I situate this research in relation to prior work involving both autism and mathematical problem-solving. I compare results about people on the autism spectrum in a general context (not necessarily education-related) and results about unusual mathematical problems given to a more general audience (not necessarily autism-related) to the data which I have collected. I also use the Vygotskian perspective on education and development generally as an overarching framing device which informs my approach.

In the next chapter, I give an overview of some research related to autism generally and autism and education particularly. In Chapter 3, I examine the Vygotskian framework, particularly with regard to its use in research related to autism. In Chapter 4, I detail some additional theoretical constructs which I will use in my analysis. In Chapter 5, I give the specific methodology which was used in my study, as well as some details about the participants and their selection. In Chapter 6, I describe the various problems given to my participants, as well as prior research relating to those problems. Chapters 7, 8, and 9 are dedicated to results and analysis, focusing on each participant in a separate chapter. Finally, in Chapter 10, I examine connections between the participants' results in more detail, summarize the results, and draw conclusions.

Chapter 2. Background on Autism and Education

In this chapter, I present both a review of previous research related to autism and my own perspective and critique of that research. I believe that there are a significant number of studies that come in with preconceptions that they do not explicitly acknowledge, or in some cases consciously consider. I do not wish to ignore that, and acknowledge my own perspective, which tends to avoid entirely negative or deficit-focused explanations of observed results.

2.1. An Overview of Autism

The primary characteristics of autism from the fifth edition of the Diagnostic and Statistical Manual of Mental Disorders (DSM-5) are “persistent impairment in reciprocal social communication and social interaction” and “restricted, repetitive patterns of behavior, interests, or activities” (p. 50). Typical other characteristics include unusually high or low sensory sensitivities and particularly focused interests (often referred to as 'special interests' among other names, several with rather negative connotations). This diagnosis is officially termed Autism Spectrum Disorder (ASD) in the DSM-5, sometimes elsewhere called Autism Spectrum Condition (ASC). It is fused in the DSM-5 from multiple diagnoses in the fourth edition (DSM-IV) and its revised version (DSM-IV-TR), mainly 'classical' autism (“autistic disorder”), Asperger syndrome (AS), and Pervasive Developmental Disorder – Not Otherwise Specified (PDD-NOS). It is worth noting that the DSM-IV's definitions for autism and Asperger syndrome mainly differ in that the first requires an “impairment in communication” which “is marked and sustained” (DSM-IV-TR, p. 70) and can include language developmental delays, while the second requires “no clinically significant delays in early language” (p.81). The only other difference specifically mentioned in the diagnostic criteria is that in Asperger syndrome, “the lack of social reciprocity is more typically manifest by an eccentric and one-sided social approach to others (e.g. pursuing a conversational topic regardless of other reactions) rather than social and emotional indifference" (p. 80). The DSM-IV-TR also noted that the AS diagnosis, unlike the 'classical' autism diagnosis, is not usually associated with intellectual disability. However, since this is not a diagnostic criterion itself, it seems likely that this is mainly a result of the language delay criterion above. The DSM-IV-TR also states that those with an AS diagnosis often have "strengths in areas of verbal ability (e.g., vocabulary, rote auditory memory) and weaknesses in non-verbal areas (e.g., visual-motor and visual-spatial skills)" (p. 81). Additionally, it appears that the general tone of description in the DSM-5 is less negative than that of the DSM-IV-TR; in particular, passages like "In adolescence or early adult life, individuals with Autistic Disorder who have the intellectual capacity for insight may become depressed in response to the realization of their serious impairment" (DSM-IV-TR, p. 72) do not appear in the DSM-5 version. Outside of North America, psychologists and researchers may also use the International Standard Classification of Diseases and Related Health Problems (ICD-10) (World Health Organization, 2016), which has diagnoses of "childhood autism" and "Asperger syndrome" differentiated similarly to the DSM-IV-TR. The ICD-10 also associates Asperger syndrome with the phrase "autistic psychopathy", and contains several other unfortunate phrases. While the intent of creating a worldwide reference gives some reason to include a complete list of terms that may be used, even severely negative ones, this suggests that the implications and perspective from the ICD-10 are closer to that of the DSM-IV-TR than the DSM-5.

For a less clinical perspective, the Autistic Self Advocacy Network (2014) states that autism is a neurological difference with certain characteristics (which are not necessarily present in any given individual on the autism spectrum). These include differences in sensory sensitivity and experience, atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. They also list "different ways of learning" and particular focused interests (often referred to as 'special interests'), which are especially relevant for research in education. Generally, I find that the descriptions largely agree, though the ASAN's description is more positive in tone, using the phrase 'neurological variation' and avoiding words such as 'disorder' or 'impairment'. They also acknowledge but distance themselves from the term 'developmental disability', which has a variety of possible implications depending on the reader's view of disability as a whole (something I touch on in Chapter 3, in reference to

Vygotsky). The ASAN works from a perspective in support of neurodiversity, a term coined by Judy Singer in the 1990s, and generally referring to a positive and inclusive perspective on not only autism, but also Attention-Deficit Hyperactive Disorder (ADHD), dyslexia, dyscalculia, and other neurological differences (Silberman, 2015). The autistic community also often uses the term ‘neurotypical’ to describe people who are not on the autism spectrum and do not have any other commonly diagnosed neurological difference, contrasting with the term ‘neurodivergent’. It is this more positive perspective that I intend to work from in my own research and analysis.

Outside of a formal diagnosis by a psychologist, there are a variety of methods developed by researchers or people in the autistic community to evaluate to some degree whether a person is on the autism spectrum or likely to be. One of the most commonly used is the Autism Spectrum Quotient (AQ) test from Baron-Cohen, Wheelwright, Skinner, Martin, and Clubley (2001). The questions used in this test were based on the criteria from the then-current DSM-IV, and tested with populations who had received an autism diagnosis as well as those who had not; the questions used were chosen for a high correlation with the diagnoses already found, and not always directly related to the DSM-IV’s own diagnostic criteria. Specifically, they focused on five categories: “social skill, attention switching, attention to detail, communication, [and] imagination” (p. 6). Thus, any existing biases in diagnoses given by clinical psychologists related to those categories in particular are likely to be replicated in the results of the AQ test. Also, there are always some issues with using self-reports, although the authors of the test included questions regarding preferences and behaviors, rather than only questions about abilities, in order to counteract this. Figure 2.1 below gives examples of questions from each category, which are phrased in a form allowing the respondent to definitely or slightly agree or disagree (a neutral option is not listed).

Social skill	I prefer to do things with others rather than on my own.
Attention switching	I frequently get so strongly absorbed in one thing that I lose sight of other things.
Attention to detail	I usually notice car number plates or similar strings of information.
Communication	I know how to tell if someone listening to me is getting bored.
Imagination	I would rather go to the theater than a museum.

Figure 2.1: Examples of AQ questions

When reading research articles relating to the autism spectrum, one thing that becomes quickly apparent is the variance in terminology. This is partially due to the change in diagnosis in the field of psychology. Asperger syndrome, for example, was classified as separate from autism in the DSM-IV, but has been included under the DSM-V's “autism spectrum disorder”. Not much research has come out since then so far, but I have encountered people inside and outside of academic settings who continue to think of Asperger syndrome as distinct in some ways. In particular, I know of people both with and without a diagnosis who appear to have a desire to separate the set of people on the autism spectrum into one group which is simply considered a bit strange (or, in some cases, superior) and another group which is considered disabled (and given more support, regarded as less valuable, or possibly both). Thus, I expect there to be parts of the literature that reflect some of those perspectives.

There are other conceptions that have had lasting impact; the idea of an autistic person as being “wrapped up in his or her own private world” still appears in recent publications such as one from James (2010), though it may be detrimental for teaching purposes. Use of words like 'nosologic' (which relates to the study of diseases) such as by Klin, Danovitch, Mers and Volkmar (2010) in the context of ASDs can also have negative implications, as can the use of 'risk' to describe the probability of a person being on the autism spectrum, as used in Baron-Cohen, Wheelwright, Burtenshaw and Hobson (2007). In some cases, people (particularly but not necessarily children) on the autism spectrum may be also identified with their parents in ways that slant against considering them as independent persons in their own right; for example, a study by Russell, Steer, and Golding (2010) used “mothers with an ASD diagnosis” apparently to mean mothers whose children had an ASD diagnosis (possibly not considering the possibility of parents and children both having diagnoses), and I have observed similar oversights both within and outside of academic discourse. At times, studies have even claimed that some things are a defining characteristic of being human, while claiming that autistic people lack the property they are discussing. This is sometimes done in more restricted domains such as cultural learning or intentionality, and sometimes in a broader way requiring extensive interventions which start from a perceived blank slate (Dawson, Mottron, & Gernsbacher, 2008).

There is also some research, particularly associated with Simon Baron-Cohen, (e.g. Baron-Cohen, Wheelwright, Burtenshaw & Hobson, 2007) which links systemizing, defined as “the drive to analyze and/or build a system (of any kind) based on identifying input-operation-output rules” (p. 125), and autism. They also claim that this systemizing is in a sense mentally opposed to empathizing, or competes with it. However, autistic traits and systemizing appear to be measured in these studies by either simple assumptions (such as considering mathematics students against students in other disciplines) or by self-report, and these claims are notably linked to claims about sex differences and fetal testosterone levels. Additionally, it should be noted that what is tested for empathizing is recognition of what another person is feeling, and not the other common definition of caring about what another person is feeling (although empathizing is not explicitly defined by the authors). Since it is common for readers to conflate the two, and this can cause some to come away with negative conceptions of people on the autism spectrum not supported by the actual research findings, it is important to note these differing definitions of empathizing.

A study from Morsanyi, Primi, Handley, Chiesi & Galli (2012), by contrast, did not find support for a link between the systemizing and autistic trait self-report measures, nor did it find support for other aspects of this theory. Empathizing and systemizing measures were found to have either no correlation or a positive correlation, showing further problems for the empathizing-systemizing theory. However, there was some connection found between spatial thinking and performance on some mathematical problems. Some related research from Grandin, Peterson & Shaw (1998) has attempted to use music training in order to cultivate spatial reasoning in order to take advantage of this connection. One of the bases for this idea was the spatial thinking of one of the authors (Temple Grandin), who is one of the more famous people on the autism spectrum[1]. However, it can be dangerous to base too much of one's view of autism on the experiences of a few well-known people on the spectrum.

Another commonly discussed theory regarding autism is related to the idea of 'theory of mind', first related to autism by Baron-Cohen, Leslie, and Frith (1985). They define this as "knowing that other people know, want, feel, or believe things" (p. 38), and give a false-belief task in order to test this. This task starts off with two children in a room: the first places a marble into a basket and leaves the room, the second moves it elsewhere, and the first child returns after this is done. The participants are asked where the first child will look for the marble. They used three groups of children: a group with autism diagnoses, mean age around 11, and a verbal mental age (tested here with the British Picture Vocabulary Test) around 5; a group with Down's syndrome diagnoses, mean age around 10, and a verbal mental age around 3; and a group without known diagnoses and a mean age around 4 (whose verbal mental age was not tested). This study found significantly lower performance in the autism-diagnosed group than the other two on the false-belief task, without lower performance in other control tasks. Happé (1994) used a more advanced test termed 'strange stories' with groups of children and adults, which can give a more nuanced account. These tasks give a variety of stories of people making statements which may not be true with a variety of motivations (the study used categories such as "white lie", "figure of speech", and "double bluff"). In addition to simply classifying the responses as correct or incorrect, Happé also classified them as physical or mental. For example, "so he won't have to go

[1] One of Temple Grandin's most famous contributions is in the creation of more humane machines for handling livestock, which she attributes to spatial/visual reasoning. In being a famous person on the autism spectrum, she has also had some role in autism-related activism.

to the dentist" and "because it looks like a telephone" were classified as physical, and "because he doesn't like the dentist" and "she's just pretending" were classified as mental (and can correspond to the same two stories). Although there was a general finding of lower performance in the autism diagnosis group, the study found that in cases where a valid justification classified as physical was available, the difference in performance was not present. Additionally, Happé notes that she did not provide any indication of whether the answer was considered correct, incorrect, complete, or incomplete. Thus, another possibility for difference lies in how the subjects perceived the question and the goals of the interviewer, in addition to the interview subjects' beliefs.

Another study from Baron-Cohen, Jolliffe, Mortimore, and Robinson (1997) used a variety of more advanced tasks related to the idea of theory of mind, particularly one they called the "Reading the Mind in the Eyes" test. This involved showing participants a picture of a person showing only the region around their eyes as well as two emotional terms, and asking which is represented by the images. They had similar but simpler tasks as controls: one which gave pictures of the entire face (and generally simpler emotional terms) and one which gave the same eye-region pictures but asked only for the gender of the person in the image (though unstated, it seemed clear that the study was expecting and received only binary gender identification rather than anything more complex). Their study found a statistically significantly lower performance on the "Reading the Mind in the Eyes" task by their participants on the autism spectrum. However, the results of both Happé (1994) and Baron-Cohen et al. somewhat move the claim about 'theory of mind' from a conceptual issue about the minds of others to simply an issue of perceptual ability. The fact that the control tasks used by Baron-Cohen et al. show equal performance by all tested groups reinforces the difference in the type of finding. Some suggest that the participants on the autism spectrum are using some type of strategy other than the one termed "theory of mind" (Frith, Morton, & Leslie, 1991), a claim which I find difficult to test without insight into the cognitive workings of both neurotypical and autistic responses to the tasks.

Another thing to consider, which has not been remarked on in any of the theory of mind studies I have read, is the possibility of reversing the question. If a group from the general population and a group with autism diagnoses were compared in reading the faces of people on the autism spectrum (using their reports of what they were feeling as the targeted responses), I think that there may be significant differences in performance. Another possibility which I have not seen explored is the question of how developmental experiences in the success or failure of predictions of others' mental states may affect the future tendency to attempt predictions. For example, if attempts to predict what someone is like are based partly on supposing they are similar to the predictor, this will be less successful for someone outside of the norm in most cases, but more successful with others in their own subgroup. Some people on the autism spectrum may be more reluctant to speculate on mental states due to this, leading to the shorter and less mental state-based responses observed by Happé (1994), which could also be affected by interactions with peers with similar neurology. One possibility for testing the results of this would be to compare the results of children on the autism spectrum with at least one parent who also has an autism diagnosiswith children on the autism spectrum whose parents do not.

It is generally known that autism is more often diagnosed in men than in women, but whether this is a reflection of actual proportions or diagnostic procedures is disputed. A study by Russell, Steer, and Golding (2010) examined a large cohort of children to determine the prevalence of autism in the group compared to rates of an ASD diagnosis, and found that in groups of children who showed the same level of visible signs of ASD, more boys than girls were diagnosed. It is noteworthy that this finding was based on data from parental reports; since bias in those reports is more likely to fit the prevailing conception of ASD as 'male' than to go against it, this is unlikely to weaken the study's findings. However, even in their own broader diagnosis of the population, the authors found a bias toward boys in making their own determinations of which children were on the autism spectrum (though a more balanced ratio than that of the official diagnoses present in the population).

Baron-Cohen (2002, 2009) advocates an even stronger connection with the "extreme male brain" theory, which expands on the empathizing-systemizing theory by identifying a higher score in systemizing with a "male brain" and a higher score in empathizing with a "female brain". However, few of the measures considered by Baron-Cohen as sex differences can be easily separated from cultural influence, and this characterization may lead to conflation (consciously or unconsciously) of characteristics that may fall under empathizing-systemizing with other gender-associated characteristics which do not, such as personal appearance, attire, or particular interests (as relevant for the culture of the person doing the diagnosis). Framing the ‘empathizing’ and ‘systemizing’ ideas as a ‘female brain’ opposed to a ‘male brain’ also has the implicit effect of restricting a two-dimensional system to a one-dimensional line within that system, with ‘empathizing’ and ‘systemizing’ opposed in a zero-sum conception.

There are reasons to suspect that the relation between gender and ASD are more complex; multiple studies (de Vries, Noens, Cohen-Kettenis, van Berckelaer-Onnes, & Doreleijers, 2010; Strang et al., 2013) have found increased gender variance (as measured by the DSM-IV’s diagnostic criteria for Gender Identity Disorder) in individuals on the autism spectrum regardless of their assigned sex at birth. This increased rate of gender variance, among other effects, increases the likelihood of error in extrapolating general gender-related correlations with autism from a limited set of observations. Overall, while it is not surprising that many studies, such as Iuculano et al. (2014) use only male students in their sample, it is still a deficiency that should be identified.

2.2. Autism and Education

There are a variety of suggestions regarding types of learning differences among people on the autism spectrum. For instance, a study by Klinger & Dawson (2001) suggested that people on the autism spectrum did not form prototypes of objects (that is, form an idea based on one or a few typical examples) when given tasks asking about group membership, instead taking an approach based on lists of rules. Although this is presented as a problem, like many other autism-related studies, I suspect that this approach could be helpful for abstract mathematics. In particular, I have found many other students having trouble with mathematical questions that appear to result from a prototype-based approach. For instance, in integral calculus, students commonly make the mistake of believing that a right-hand Riemann sum is always greater than a left-hand Riemann sum. This holds for increasing functions, but not all functions; thus, a prototype-based approach where the prototypes are increasing functions will lead to this mistake. I found a very similar division reported in mathematics education research from Edwards & Ward (2004), phrased as lexical or extracted definitions versus stipulative definitions. The "stipulative definition" in particular is essentially a "list of rules". It also appeared to me that there was a missed opportunity in Klinger and Dawson's prototype study; their 'explicit rule' condition gave an explicit rule concerning one of many features of an object, and I think it would have been informative to compare objects that are extremely different in other parts (and thus do not look like a prototype would) but do match the one condition given.

The tendency examined by Klinger & Dawson to avoid prototype formation is also identified in some literature, such as Happé & Frith (2006), as "weak central coherence", a choice of term (originated by Frith in 1989) reflecting the tendency in autism-related research to focus on a deficit-based model and accentuate the negative. However, while the name reflects the early deficit-based conception, their more recent work has moved away from this; Happé & Frith point out that this could also be interpreted as a cognitive bias toward details rather than the 'big picture', framing it as a shift of abilities instead. The issue of deficit-based framing also applies to framing of mathematical learning disabilities (MLD), and there are interesting comparisons involving work in that direction, such as Lewis' (2014) study. Lewis suggests a framework of cognitive difference (rather than disability), but in the analysis, she still presents the problems that the MLD students' different interpretations pose for learning. Also, the data collected does not test the study's suggestions for using those different interpretations in teaching, having done only the standard tutoring approach. It appears to me that the study's use of the cognitive difference framework falls mostly in the suggestions for future research, outside the scope of the data, which causes it to feel somewhat hollow. One possibility to help move away from this would be to start with where the idea works, or where a similar idea might work, and try to delve into the differences that cause the idea to fail.

A related finding from Dawson, Mottron, & Gernsbacher (2008) is that some children on the autism spectrum may develop language by building the structure first, then learning the particular words, reversed from the typical developmental order. Whether this tendency might carry over to learning in other contexts appears to be unknown. It could be particularly interesting and relevant in a mathematics education context, because it appears to be similar to the 'definition, theorem, proof', example-avoiding structure of many higher-level university mathematics courses.

Another finding about the learning of children on the autism spectrum concerns overimitation. In the general case, overimitation is the tendency of children, when imitating the actions of an adult, to include those actions which are not actually necessary to accomplish the overall task. Marsh, Pearson, Ropar, and Hamilton (2013) found that children on the autism spectrum were significantly less likely to engage in overimitation; that is, more of them included only the necessary actions for the task's overall goal (note that they were instructed to complete the task as quickly as possible). This was tested by demonstrating to the children simple tasks, such as removing a toy from a box, and including unnecessary actions such as tapping the lid of the box in the process. In order to make some determinations as to the reasons for this, the authors also used a scale where they asked participants to rate the actions (including necessary and unnecessary ones) on a 1-5 scale from 'serious' to 'silly'. They found that the children on the autism spectrum were less likely to give answers to these questions that were consistent with whether they were necessary or unnecessary, and concluded from this that the differences in performance were "not driven by superior causal reasoning" (p. 267). In one sense, the success of the neurotypical children at answering this question is evidence that they also understood that those actions were unnecessary. It is not stated if there is a correlation between those children on the autism spectrum who did not perform well on the rating task and those who did engage in the overimitation (while they were a smaller percentage, this did still occur with some of the study participants). They conclude that the lack of overimitation is an example of a problem with social interaction in children on the autism spectrum. They do not address or measure the stated goal of their task (to reach the end result quickly) or that the behavior of the children on the autism spectrum is more suited to accomplishing that goal. Overall, I feel that their analysis is another example of being too heavily focused on accentuating the negative aspects of people on the autism spectrum, as well as the difficulties inherent in interviews with younger children who are not yet able to articulate their inner thoughts comprehensively.

Another line of inquiry concerns ‘special interests', a term with a particular meaning in the autistic community. These are intense, focused interests occurring in people on the autism spectrum (often studied in children, like much ASD research). In other work, these are variously known as circumscribed interests, splinter skills, savant skills, and a variety of other names with varying connotations within the autistic and research communities. Some of these from Klin, Danovitch, Merz & Volkmar (2007) are mathematically related, such as a focus on particular facets of arithmetic, prime numbers, or aspects of geometric shapes. The development of these interests in later life is something that does not appear to have been studied much at present, and I think it would be an interesting area to pursue. Additionally, it might also be useful to examine special interests in the context of a dialogue with an outside expert on the subject matter, since a determination of precisely what kind of interest the subject might have could be difficult for someone without knowledge of the area. Unfortunately, research in this area, particularly from an Applied Behavioral Analysis (ABA) framework, can discount these skills or even view them as detrimental to learning (Dawson et al., 2008). They may even attempt to eliminate these skills, which I find wholly contrary to the reasonable practice of education. While this might be questionable on its own, the loss of the skills related to special interests has also been connected to a loss in performance on standard measures such as IQ tests. I find this a peculiar and problematic tendency which does not have an apparent parallel in 'mainstream' education research, and a reminder of why ethical review can be necessary in this area of research.

Some of the teaching interventions and techniques suggested from autism-related research could be used at any level, in any type of course. For example, a suggestion occurring in multiple studies, such as Donaldson & Zager (2010), was to make a checklist including each step of the task. While this may present issues in cases where the intent of the problem is to figure out the steps to be used, I think that this could be helpful depending on the size of the steps that are indicated. It may also be useful to have a progression in the amount of assistance provided, moving from a given list to one the student is encouraged to create. Some aspects of this may be helpful even for the general student population, as a way to remind themselves of what the question is asking and what sort of answer they should have, something I have seen quite a few students have problems with. Other suggestions concern group work. In particular, due to the issues arising from social interaction, it is suggested by Simpson et al. (2010) that the teacher monitor the social interactions involving ASD students and to have groups assigned by the teacher instead of relying on self-selection into groups.

There are also some unusual characteristics that I noticed in the methods of some of the studies. For example, the children compared in some studies analyzed by Happé & Frith (2006) were matched not by chronological age, but by varying measures of verbal age. If this is determined by tasks similar to the one used in the study, this could be cause for concern, and the lack of a consistent standard introduces additional difficulties for a comparison. In another study by Lewis (2014) with participants characterized as having MLDs, the comparison group was normally-achieving fifth grade students, as they were at a similar level of experience with the topic. I have doubts about this approach; from what I have seen in community colleges, there are quite a few students there who would also be at any level of mathematical experience being examined, and that would remove the large age difference that could be a confounding factor. In fact, using this comparison may reflect a perception of persons with disabilities as being like children, which can lead to ethical issues in research. In addition, there was a long list of confounding factors that the author used to exclude study participants: “lack of English fluency, low socioeconomic status, anxiety, and behavior or attention issues” (Lewis, 2014, p. 357). Also, of the 11 test subjects for the study, only two qualified for the MLD group to be used in the comparison. While all of these exclusions seem reasonable individually, I find it concerning when they are combined. Another issue that I considered in this study was with the ethics of their study method. In order for their method to work, they needed to find people who would not respond to the regular tutoring methods which they were using. However, while they did not share the specifics of the study's solicitation for participants, my impression was that they were advertising it as a form of math tutoring. In this case, it appears that there could be a conflict between maintaining the tutoring protocol for the purposes of the study, and varying the protocol in order to find something more effective (which is what I would do if the tutoring were my only goal). Another study matched groups to control for age, parental occupation, and handedness, but did not control as strongly for gender, which I thought was an unusual approach (Baron-Cohen et al., 2007).

2.3. Autism and Mathematics Education

While I have touched on connections with mathematics education in discussing research in the previous section, here I will look at work that addresses mathematics education directly. Much of the research currently done on mathematics learning in people on the autism spectrum is focused on young children (e.g. Klin, Danovitch, Mers & Volkmar, 2010; Simpson, Gaus, Biggs & Williams, 2010; Iuculano et al., 2014) or looks at mostly arithmetic. There is also a notable strain of work done on the population of research mathematicians (e.g. James, 2003; Baron-Cohen, Wheelwright, Burtenshaw & Hobson, 2007), which outweighs the number of studies done on groups in the middle (mainly high school and college students, or adults other than career mathematicians). In the case of studies pertaining to young children, it should be noted that the perception of them is often reported by and thus filtered through their parents, as in Klin, Danovitch, Mers & Volkmar (2010), which could have significant effects on the results.

In the broad sense, some authors report strong mathematical interest or ability. For instance, James (2010) cites a private communication with an Irish psychiatrist who claimed that the children he diagnosed with ASD almost all had an interest in mathematics. Another recent study by Iuculano et al. (2014) identifies more effective arithmetic strategies used in the ASD sample when compared to the typically-developing sample. However, other works identify mathematical deficiencies in children on the autism spectrum. For example, one study by Donaldson, Zager & Koffler (2010) identifies difficulties with remembering operations throughout an equation, page organization, and parsing language in word problems, although it does not identify the age or grade level of the students sampled in this determination. It is also possible, however, that some of the variation could result from problems with the testing protocol being insufficiently adapted to students on the autism spectrum, as noted by Chiang (2007). Also, since most of these studies are focused on younger children, they often use only arithmetic-based questions or other questions that do not go beyond the level of elementary school mathematics, and thus they provide less insight about the possibilities for higher-level conceptual consideration of mathematics in secondary and tertiary students on the autism spectrum.

Additionally, some of the general educational differences that have been found have particular implications in a mathematical context. The suggestion of “using...literal rules” from Simpson, Gaus, Biggs & Williams (2010) to help learners on the autism spectrum (which the authors derive primarily from their case study) seems to be inevitable with the majority of mathematical coursework to some extent. Another suggestion from those authors was to clarify the ultimate purpose of the lesson. I think this is something that is not done enough in mathematics courses, and it strikes me as particularly interesting. I suspect teachers have a variety of reasons to avoid it, such as leading the students into developing it after instruction or due to concerns about students losing interest or having trouble following the explanation without context. In the latter case, the result suggests that the ASD students may take the opposite view of other students in such cases.

2.4. Autism and Gifted or Famous Individuals

There is also a strain of research focusing particularly on students who are classified as gifted and autistic. Generally, there is a pattern of particularly strong abilities in some areas, with particularly weak abilities in other areas. The severity of such variances rising to a savant level has been documented in Dawson et al. (2008) to be much more prevalent in the autistic population: 1 in 10 as opposed to 1 in 2000. While the gifted classification can help ASD students fit into a particular place, it can also create the perception that they do not need help, making it less likely for them to get the supports they need. Ruthsatz and Urbach (2012) investigated autism diagnoses in their assessment of eight “child prodigies” (one each in mathematics and art, four in music, and two in multiple areas), which they defined as having “reached professional status in a rule-based system at a remarkably young age, usually before age 10” (p. 420). They found that three of their eight participants had an autism diagnosis (including the participant identified as a prodigy in mathematics), and that the group as a whole had a higher average score on the Autism Spectrum Quotient test (Baron-Cohen et al., 2001) which was not driven solely by the three diagnosed participants.

There are also some perspectives that I find suspect, like the idea from Cash (1999) of “distinguish[ing] autism from true genius” (p. 23), which seems to suggest that genius combined with autism is somehow false. This is somewhat ironic, since Cash also pointed out the trouble that students on the autism spectrum might have with people having negative perceptions of them. Additionally, some publications regarding gifted abilities in autism make highly dubious claims, such as one from McMullen (2000) that includes “psychic dreams” as an asset of autism. It seems that on the whole, problems such as these are more common in older publications, so this appears to be something the field is improving at.

When research focuses on people who are already mathematicians, such as by James (2003), a positive correlation with autistic traits has been reported. A study from Baron-Cohen, Wheelwright, Burtenshaw & Hobson (2007) also found some evidence for this. However, it appears that these may also be based on self-report measures; further research may be warranted on not only possible reasons for the correlation, but whether it in fact exists or is particularly strong.

There are also speculations on whether certain famous mathematicians and scientists were on the autism spectrum. These naturally tend to focus on some of the most famous, such as Albert Einstein. Some others have been suggested to be on the spectrum based on some more extensive biography, such as the scientist Henry Cavendish by neurologist Oliver Sacks, and Paul Dirac by biographer Graham Farmelo (Silberman, 2015). However, these speculations are inevitably based on collections of stories and other people’s memories about a person, and thus are affected by selection bias and the natural distortion of memory. Thus, these findings cannot be conclusive, but could still be taken as a hint toward possibilities for further investigation.

2.5. Reflections

Overall, the existing research is generally sparse, particularly for high school students and adults. There are a variety of difficulties involved in finding research participants for such studies, although it is notable that part of the divide appears to be between ages where parents of children on the autism spectrum would be contacted for finding participants and getting permission to interview them and ages where people on the autism spectrum would be contacted and give permission themselves. There may be a difference in perceptions of researchers between people on the autism spectrum and their parents, and given some of the more negative things noted in some published research, it is perhaps not surprising that adults on the autism spectrum might be more reluctant to contribute to such perspectives.

It is also notable that some studies attempt to resolve the issue of finding people with an autism diagnosis directly by using tests like the AQ, which may show whether someone is more or less likely to be on the autism spectrum but does not itself give a formal diagnosis. This can be a reasonable way to compensate for the difficulty of finding participants with diagnoses, though it will generally be more effective with larger sample sizes. There are possibilities that this technique could help with identifying people who are on the autism spectrum but have not been diagnosed. However, since the AQ test was developed from people with formal diagnoses, this would only be effective for some of those who are not diagnosed, mainly those who have not had the time or opportunity to see a psychologist. Those who are undiagnosed due to some biases in perceptions could be missed by the same biases reappearing in the test.

From the review of previous research, we see that there is generally a gap in the literature involving adults, and that a significant part of publications that do address it are highly specific or speculative. I seek to help close the gap in research concerning adults on the autism spectrum in general and mathematical reasoning with my work.

Chapter 3. Vygotskian and Sociocultural Frameworks

In this chapter, I provide an overview of work by Vygotsky and researchers following his ideas which is particularly relevant to my own research. While Vygotsky's life and work occurred before the formation of the diagnosis of autism as it exists today, I demonstrate some ways that his work is applicable to autism-related research, as well as examining the work of others who have looked at autism from a Vygotskian framework.

3.1. Vygotsky's Views on Disability

In Vygotsky's writing, there is some work that directly addresses the study of "defectology". At the time, this was used to refer to studies involving children with certain disabilities (of a narrower scope than we might consider today) (Gindis, 2003). One of the main characteristics of Vygotsky's (1993) conception of "defectology" was the idea of overcompensation. Vygotsky explains this initially in a framework of physical overcompensation, such as a kidney or lung necessarily strengthening when the other one is missing or by analogy to vaccination. He argued that overcompensation also occurred in psychological development, both in its general course and in particular in the presence of various disabilities (concentrating primarily on those who were blind or deaf, as with most related efforts at the time). He thought of this as dialectical, and credited the work of Marx and that of Adler as influences on this view. Based on this, he criticized the education of children with disabilities of the time as inappropriately focusing on only the weaknesses, not the strengths, of their students. In one instance, he writes, "It is important that education aim to realize social potential fully and consider this to be a real and definite target. Education should not nurture the thought that a blind child is doomed to social inferiority" (p. 63). Additionally, much of his experimental work was done with children with disabilities, and he found this to be a way to make more general discoveries about psychology in addition to the direct benefits of the work (Gindis, 1995).

Vygotsky's emphasis on the social reasons for psychological differences among people with disabilities also has much in common with modern social constructionist views of disability. Jones (1996) contrasts the social construction model (crediting its introduction to a paper by Asch in 1984) with “functional limitations” and “minority group” conceptions. Vygotsky (1993) directly states that “a handicapped condition is only a social concept” (p. 83), and Asch's (1984) criticisms of social attitudes toward people with disabilities are remarkably similar to statements by Vygotsky (1993) such as “the task is not so much the education of blind children as it is the reeducation of the sighted” (p. 86).

While the diagnosis of autism did not exist when Vygotsky wrote, he viewed impacts on communication as a particularly important aspect of the effects of disability (with the issue coming up primarily in comparison between the blind and the deaf). In its standard definition in the DSM-5 (as discussed in Chapter 2), the first major criterion for autism spectrum disorder involves “[p]ersistent deficits in social communication and social interaction across multiple contexts”, which suggests that it would be expected to have a strong impact on the shape of individual development in the Vygotskian framework. However, it should also be noted that all of the criteria for the autism diagnosis in the DSM-5 are entirely in the deficit-focused model which Vygotsky criticized. Additionally, since Vygotsky's perspective has a social-first model of child development (as opposed to the individual-first model of Piaget and others), the developmental differences linked to autism should be expected to be more significant and far-reaching in a Vygotskian model than in the Piagetian model. Finally, it should also be noted that although phrases like “autistic thought” appear in Piaget's work and Vygotsky's criticisms of it, they do not have a direct relation to the diagnostic name (though there is an etymological similarity, as the early conceptions of autism were of being 'self-directed' or 'self-focused').

3.2. Zone of Proximal Development

One of Vygotsky's most well-known ideas in the Western academic community is the zone of proximal development (ZPD). Although many scholars do things with the idea of ZPD which are outside the scope of Vygotsky's original intention (including in studies relating to autism), the ZPD may be particularly relevant in the area of disability. Chalikin (2003) notes that for Vygotsky, ZPD was only involved in a child's development between different 'age periods' (which were not necessarily tied to a particular age, but rather to a level of development of psychological functions common in but not necessarily rigidly linked to an observed age group), and only pertains to tasks particularly involved with development toward the next period, not learning in general. Additionally, what these developmental sequences and their relations are is tied to the cultural context in which they appear, not something directly required by biological development. Due to this, impacts on development related to disabilities can come not only directly, but also through the change in social and cultural environment that often results from having a disability, something often noted as primary and secondary disability in Vygotskian discourse (Gindis, 1995). In the case of autism, the strong social element in the diagnostic criteria makes this particularly relevant. However, it should be noted that for many outside observers, the line between primary and secondary disability with regard to autism can be less clear than with some other disabilities. Because of this, some interventions may attend to more superficial behaviors rather than the underlying developmental goals.

3.3. Vygotskian Ideas of Concept Formation

I also find that the idea of everyday concepts and scientific concepts has particular relevance to my analysis. Vygotsky (1962) contrasted everyday or spontaneous concepts, which are formed through experience and observation, with scientific concepts, which are formed more directly by instruction and as part of an overall mental structure of such concepts. Due to this formation, the person in question will have a stronger conscious awareness of a scientific concept than an everyday concept (and thus be more readily able to talk or think about it directly). However, he did not view them as entirely separate, instead viewing them as influences on each other in an analogous way to a native and foreign language. Gindis (2003) points out that the formation of everyday concepts will be heavily influenced by the presentation of disability, in particular referring to sensory disabilities such as visual or auditory impairments. However, due to the common social context of everyday concept formation, it seems likely that children on the autism spectrum would also have a significant difference in the formation of everyday concepts, and that this would also lead to an increased importance of scientific concepts as Gindis suggests. While such research is typically done on children, it seems reasonable that this kind of developmental difference would continue to have an effect later in life.

There have been attempts to examine concept formation in children on the autism spectrum for some time, such as by Noach (1974). While she found weaker concept formation, her study was not able to control for intelligence measures (due to an ineffectiveness of traditional tests for subjects on the autism spectrum) and did not provide details on what sort of concept mediation they had been exposed to in the past. Klinger and Dawson (1995) examined this and several other studies which gave more specific tasks, and they did not find strong evidence of worse performance in categorization overall but speculate that there may be trouble with using categorization in less straightforward tasks. One notable specific result was that the categorization ability in one study was not related to measured receptive-language ability, although it was for both comparison groups (which they designate the "mentally retarded and normally developing groups"). However, it should be noted that all of the studies examined here were older studies, and the diagnostic profile for autism at that point in time seems to be somewhat different; specifically, most of their participants on the autism spectrum also showed significantly lower intelligence test results, which is less true with the expanding group of people with the autism spectrum diagnosis today.

Vygotskian ideas on concept formation are also relevant to the differences in prototype formation discussed in the previous chapter. Klinger and Dawson (2001) suggested that people on the autism spectrum use a rule-based approach rather than forming prototypes of objects when given tasks asking about group membership, and Edwards and Ward (2004) had similar ideas about lexical/extracted versus stipulative definitions. A related finding is that some children on the autism spectrum may develop language by building the structure first, then learning the particular words, reversed from the typical developmental order (Dawson, Mottron, and Gernsbacher, 2008). All of these, to varying extents, seem to reflect a division between everyday concept formation and scientific concept formation. In particular, the description of prototype formation as an average of observed category members sounds similar to Vygotskian descriptions of complex and everyday concept formation. A 'list of rules' approach sounds somewhat simpler than true scientific concept formation, but it may become more complex with further development, as well as possibly shifting the course of further development; the choice of phrasing and description could also be affected by researcher bias. With the idea of building the structure around it at the beginning, it seems at least more compatible with scientific concept formation, particularly in mathematics where the relevant concepts can be properly defined by an explicit listing of their properties. For example, a vector space in linear algebra is defined as a set together with two operations satisfying certain axioms, but is typically also explained by analogy with the most common examples of vector spaces, $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$. Focusing on the common examples can lead to mistakes which an approach closer to the definition would avoid.

3.4. Vygotskian Inner Speech and Autism

According to Vygotsky (1962), speech develops from a single general function to distinct functions for oneself and for others. The 'for oneself' development is first into egocentric speech (still out loud); while this refers to the same empirical phenomenon as Piaget's egocentric speech, the conception of development here is different. Instead of simply ending, Vygotsky's egocentric speech develops into inner speech, which is no longer vocalized. However, the differences are more extensive; from Vygotsky's study of egocentric speech, he found that it gradually develops into a form which has greater distinction from social speech, more idiosyncratic and less understandable to others, before it ultimately develops into inner speech. He concluded that this form of speech relies heavily on internal conceptions and senses which could not be used in social speech, and thus it is something more than just speech without vocalization. Due to the importance of inner speech in Vygotsky’s work, the difference between the autism and Asperger syndrome diagnoses in the DSM-IV (in language and communication difficulties) is one which we would expect to take particular relevance in the Vygotskian framework. However, it would not quite apply directly, since these diagnostic criteria are based on observations before inner speech is typically developed. Thus, I would not expect the combining of the diagnoses in the DSM-5 to have a significant impact for research involving adults.

Some research studies have been conducted attempting to determine the level of development of inner speech in autistic children. One study conducted by Wallace, Silvers, Martin, and Kenworthy (2009) indicated that “individuals with autism do not effectively use inner speech during the completion of cognitive tasks”, while another study from Williams, Happé, and Jarrold (2008) indicated that “individuals with ASD use inner speech to the same extent as individuals without ASD of a comparable mental age.” A third study conducted by Russell-Smith, Comerford, Maybery, & Whitehouse (2014) examined this in the particular context of executive function-related tasks (they appear to implicitly equate this to Vygotsky's 'self-regulation', and one of their citations is the Wallace study which does this explicitly), and involved a greater variety of conditions to alter the use of inner speech. They also found discrepancies in inner speech use in the same direction as Wallace et al., as well as a correlation between verbal test scores and task performance in the ASD participants. However, while the Wallace study found a statistically significant worse performance in the ASD group in the control condition, the Russell-Smith study did not.

The Wallace study had participants between 12 and 20 years old, matched participants on age and full-scale Weschler IQ scores, and was limited to participants with full-scale Weschler IQ above 80. Most of their participants fit into the DSM-IV's Asperger category. They conducted a pattern-copying task (the Tower of London), where some trials were under “articulatory suppression” (in order to suppress egocentric speech), as their way of measuring inner speech. The articulatory suppression they used was to say a particular word once per second while they were in the process of completing the pattern-copying task.

The Williams study had participants between 5 and 15 years old (with a “verbal mental age” of between 4 and 12, apparently determined by the verbal IQ score). Most of their participants were in the classical autism category for the DSM-IV. This study did similar matching to the Wallace study (although including their “verbal mental age” as a matching factor), also in an attempt to measure inner speech. However, the mean full-scale IQ of their participants was about 75 and they reported having only two participants above 100. They used a short-term memory task, asking participants to remember which of a set of pictures was in each location. There were three sets used for different trials: one where the objects had a visual similarity, one where the objects' names were similar (like 'bat' and 'cat', for example – they also talked to the participants to make sure the names they called the objects were the short and similar ones, and not something else like 'baseball bat'), and one where neither similarity occurred.

The Russell-Smith study had participants between 10 and 15 years old, with IQ (using the same Weschler tests as the others) mostly ranging from the 80s to 120s (the verbal IQ of two of the ASD participants dropped below that range, but the authors reported that their omission did not qualitatively change the results), with participants matched on both scores. Their diagnoses were also primarily in the classical autism category. The test used was the Wisconsin Card Sorting test, under four conditions: articulatory suppression similar to the Wallace study, a “concurrent mouthing” condition (the participants were asked to regularly open and close their mouths, as if they were saying something, but without actually speaking aloud), a “talk-aloud” condition (the participants were encouraged to voice their thinking aloud), and a baseline where no attempts were made to draw out or inhibit speech. Here, while more conditions are used, the overall intent of suppressing inner speech and comparing the result is the same.

Another study from Lidstone, Fernyhough, Meins, & Whitehouse (2009) found a more specific result; in the data they analyzed, they identified a subgroup of children on the autism spectrum with a profile showing greater nonverbal than verbal skill. The groups of participants had mean ages between 8 and 11, and were all male; the task they performed was simple arithmetic (alternating addition and subtraction), once with no further interference and once under auditory suppression. They found that the higher-nonverbal-performance group did not show a performance difference under auditory suppression (which may highlight less use of inner speech, or a variation that is unaffected), although the remaining group did.

It is noteworthy that the participant groups for these first two studies have very little overlap. In all four studies (Wallace et al., Williams et al., Russell-Smith et al., and Lidstone et al.), like in most autism-related studies, the vast majority of study participants were boys. In this case, there is not a sufficient sample to determine if the outcomes would be different with girls (there is a widely reported disparity in diagnosis, though not necessarily in occurrence, as discussed in Chapter 2). It should also be noted that, since verbal tasks would also particularly use inner speech, the second study may have already measured inner speech ability when matching the participants (and would thus expect no difference in inner speech after that). Also, while Wallace and colleagues cite evidence that neurotypical performance has declined under articulatory suppression, it is also possible that the inner speech of the participants on the autism spectrum takes a different form (for example, it could be more textual than auditory, or something like that) as well as or instead of using tools other than inner speech. There are other populations, such as the Deaf community, which clearly have developed inner speech not related to the typical auditory developmental trajectory, so such a variation would not be unprecedented. The Russell-Smith study's finding of correlation between their verbal tests and card sorting task performance also suggests something more complex than simply underdeveloped inner speech.

It is also notable that of the first three studies, the one where the ASD study participants had a DSM-IV diagnosis of Asperger syndrome is the Wallace study, which found a difference in both inner speech use and general performance. Given that the main difference between the autism and Asperger diagnoses was in verbal criteria, this runs counter to expectation based on those diagnostics, and provides more evidence that the developmental paths here may be more complex than these studies are able to discern.

All of these studies make the attempt to study inner speech directly, as opposed to Vygotsky's original efforts in *Thinking and Speaking* (often known by its mistranslation into English as *Thought and Language*), which examine it through the development of egocentric speech, although the Russell-Smith study's condition encouraging vocalization comes closest. None of the studies' conditions have any elements that I would expect to interact badly with other autism-related traits (one of Vygotsky's original experiments of drowning out egocentric speech with loud noise would most likely have caused severe negative reactions in some of the autistic participants related to the sound itself, and not to egocentric speech).

The model of inner speech cited by Williams and colleagues is initially a “working memory model” from Baddeley, which appears to view inner speech as simply subvocal. They then turn to the work of Vygotsky that relates inner speech to self-regulatory functions without noting the change in the definition of the term. The articulatory suppression method that the other two studies use also appears to be more consistent with the 'subvocalization' idea than with Vygotsky's model of inner speech. Wallace and colleagues do cite results that the AS technique has caused reduced performance on tests in neurotypical adults compared to other concurrent tasks not related to speech (finger-tapping, in their example), although that could have other explanations (such as the AS condition being generally more complex).

3.5. Summary

In this chapter, I have summarized several Vygotskian ideas which can be particularly relevant to examining the learning of people on the autism spectrum, such as inner speech and concept formation. Some previous researchers’ ideas on using Vygotskian principles in contexts related to autism are also addressed. However, I also use some additional framework elements in my analysis, which are discussed in the following chapter.

Chapter 4. Additional Theoretical Constructs

In my data analysis, I use a variety of theoretical constructs, including but not limited to those discussed in Chapters 2 and 3. In Chapter 2, I examined prior research and definitions regarding autism, reviewing clinical and popular definitions in describing a positive, neurodiversity-based perspective in contrast to other possibilities. I also examined several findings in the literature about autism and education from different perspectives. In particular, constructs related to prototype use (or central coherence) and special interests play a role in my analysis.

In Chapter 3, I described parts of the Vygotskian framework and related research that is useful for my analysis. I acknowledge that the diagnosis of autism did not exist when Vygotsky wrote, but relate his perspective on disability generally to perspectives on autism in particular. Vygotskian ideas about everyday and scientific concepts and inner speech are also discussed and particularly relevant in my analysis. In addition to these, I also use some other theoretical constructs which are discussed below.

4.1. Views on Intuition

In many contexts, the erroneous conclusions produced by students and the resistance to the mathematically valid solution are identified with forms of intuition. In Fischbein's (1979) use of the idea, intuition is separated into different categories, particularly "primary intuition" (developed outside of a systematic instructional setting) as opposed to "secondary intuition" (developed in a systematic instructional setting). The division of categories here has similarities to Vygotsky's distinction between everyday and scientific concepts, and I find it reasonable to consider the primary and secondary intuition used by Fischbein as identifying intuitive reasoning related to everyday or scientific concepts, respectively. Further exploration of intuition by Fischbein (1982) uses a similar division between "affirmatory intuitions" and "anticipatory intuitions", focusing primarily on the former. In this division, affirmatory intuitions are those that are "self-evident [and] intrinsically meaningful", which again stands outside the systematic instructional context.

In the context of other works, it is the primary and affirmatory definitionsthat are closest to what is typically meant when 'intuition' is named but not explicitly defined, which is useful for situating other work which mentions intuition but does not focus on it. Fischbein also argues for the importance of using intuitive ideas, which includes but is not limited to correcting those intuitive ideas which would otherwise lead to error. Here, his main focus is on developing intuitive ideas so that they are in accord with the analytic reasoning rather than in conflict. While there are still possible parallels between these theoretical constructs and Vygotskian concepts, the approach suggested by Fischbein, focused on development and adjustment of intuitive ideas, is more constructivist. Since these differences are reflected in neurological differences associated with autism, they would lead to contrasting predictions for the mathematical reasoning of people on the autism spectrum.

4.2. Grice's Maxims of Conversation

When discussing how people may tend to deviate from societal expectations for conversation, it is useful to outline explicitly what those expectations are. Grice's (1975) maxims of conversation provide one approach to doing this. They are given in four categories: Quantity, Quality, Relation, and Manner. The maxims of Quantity are to make a contribution that is as informative as required, but not more. The maxims of Quality are to not say something that one believes is false or for which one lacks sufficient evidence. Grice only puts one maxim in the Relation category, a requirement for relevance, while listing several in the category of Manner, to avoid obscure expressions and ambiguity and to be brief and orderly (in this category he allows for others). Since issues with the characteristics of conversation that Grice's maxims describe are one of the primary diagnostic criteria for autism, it is reasonable to suspect that the effects described by these maxims may be different or less in participants on the autism spectrum. Thus, outlining them explicitly can help to draw attention to areas of conversation (such as in an interview) would be affected that might otherwise be glossed over.

Chapter 5. Methodology

5.1. Formation of Research Plan

My initial idea for the project was to follow several students in mathematics courses at my university, and interview them. The interviews would involve asking about their progress, such as any particular strengths and weaknesses or other thoughts about the course material. They would also include various questions related to the course material which had been examined in other studies in the general student population, to compare results and note any interesting differences.

At first, I tried recruiting students through my university's center for students with disabilities. They agreed to tell students registered with the center about my study and invite them to participate, but after five consecutive terms I was only able to contact one student (Joshua) via this method. I also posted advertisements around the campus for the study, which did not lead to any additional responses. After this, I contacted some other nearby colleges and universities by posting advertisements on their campuses and by contacting similar centers for students with disabilities at those campuses. While they were less receptive, ultimately some were willing to accept the advertisement postings or give the information to their students in order to invite them to participate. Unfortunately, this did not produce any additional responses.

There are many possible contributing factors to the difficulties in finding participants from university campuses. First, with the original method of recruiting participants through the Centre for Students with Disabilities (CSD), I was not allowed to contact students directly, instead relying on the CSD’s staff to relay my message via email. I understand the privacy concerns involved, so I do not fault them for doing this. Also, looking for participants there inherently limits my sample to those students on the autism spectrum who are using the university's disability services, and excludes those who do not know about them, do not need them, or need services other than those the center can provide. More generally, some people may not be inclined to participate in interviews, particularly due to the social difficulties noted in the DSM-V definition of autism or due to previous negative experiences with interviews from people in the research community.

After encountering these issues, I was concerned about whether I would be able to continue in this direction of research in a reasonable amount of time. I had doubts to the point that I was considering another research direction, but considering that reinforced to me my interest in autism-related research with adults. Thus, I sought a way to continue in that general direction; even if I had to change my original idea somewhat, I hoped to shift it rather than abandon it. I realized that although I might not have found specifically students in undergraduate mathematics courses who were on the autism spectrum, I had continued to meet several people on the autism spectrum in other areas of my life, who generally seemed receptive to the idea of my research. Thus, I decided to expand the group that I was looking to interview to adults on the autism spectrum generally, in particular any with current or former college experience related to mathematics. Doing this, I was able to contact people in more ways; in particular, I was able to recruit two more participants on the autism spectrum (Cyrus and Mark) that I met outside an academic setting, who found out about my research through general conversation. My prior and continuing experiences with the autistic community were particularly helpful for this approach to finding participants.

In doing this, I also had to change my research question somewhat. My original question was, in essence, what are essential or typical characteristics of the undergraduate learning experiences of students on the autism spectrum? In expanding this to adults rather than undergraduates, I was unable to use the structure of classroom experiences as part of the framing of the question. After considering this, I decided that a focus on mathematical problem-solving would be both somewhat connected to the traditional education experience as well as something that did not have to be tied to a particular course. This led to my revised research questions (as written in Chapter 1): first, how do the experiences of mathematical problem-solving differ for adults on the autism spectrum? Second, what would be an explanation for those differences that might promote understanding of the autistic community?

This also led to a shift of focus onto what I had considered the 'additional questions' in my interviews, particularly those which are independent of a particular course's content. I found that many of the questions related to paradoxes in the literature were particularly well-suited for this, since they are mostly accessible even without an extensive mathematical background and still have the complexity to draw out interesting results in their solutions and discussion. I elaborate on the particular questions used for the interviews in Chapter 6.

5.2. Case Study Procedures

Given the overall structure of focusing in-depth on interviews with a small number of people, an approach of multiple case studies was a natural fit for my work. A case study in this sense focuses on in-depth understanding of the case in question, and only secondarily on generalizations from that understanding. Additionally, while generalization is possible, it is not of the same nature as generalization in other types of research (Stake, 1995). The focus can be on a single case or multiple cases, but the structure is still primarily that of a single case, where the multiple cases are mostly examined individually. In accordance with this, the multiple case studies can be viewed in the sense of replication rather than the sense of sampling (Yin, 2009).

Another categorization of case study design is between holistic and embedded case studies, where an embedded case study is interpreted as examining a particular feature or subset of the case in question, while a holistic case study does not use such subdivisions (Yin, 2009). In this case, my decision for a holistic case study naturally follows from my neurodiversity-informed view that the nature of being autistic is not a discrete part of the person that can be separated, and thus an embedded design does not apply.

In my own work, I examine three cases, those of each of my three interview participants (detailed in the next section). In accordance with the case study procedures above, each case is primarily considered individually, with some cross-case comparisons and inferences drawn later. In accordance with the sense of replication, the same mathematical tasks are given to multiple participants, but due to individual differences and the evolution of the interview protocol throughout the data collection process, there are still notable differences. For example, it is impossible to replicate a first-time response to a problem in the case of a participant who has seen it before.

5.3. Participants

All of the participants in the study will be named with pseudonyms throughout. The pseudonyms chosen are intended to have similar cultural connections to those of their actual names.

Joshua was recruited from the CSD as my first participant. He received an ASD diagnosis at age 18 (changed from a previous diagnosis of Obsessive-Compulsive Disorder), and was in his early twenties at the time of interview. He reported a strong interest in chemistry as well as a particularly low level of interest in subjects unrelated to the sciences, particularly chemistry (which he was majoring in), and a strong inclination to work alone. He was taking integral calculus and linear algebra courses, and I conducted interviews every week for the term of those courses. These were scheduled for one hour, but were sometimes continued for a short time past the scheduled hour. I typically started by asking the student to share any particular thoughts on the week's course materials. I also asked various questions and assigned tasks related to the covered course material, as well as additional tasks related to mathematics more generally.

Cyrus was recruited in the community outside of the university, received an ASD diagnosis at the age of 13, and was in his thirties at the time of interview. His mathematical background included a bachelor's degree in mathematics, and was working in computer programming at the time of interview. In contrast to Joshua, none of his special interests were strongly apparent in the interviews (although mathematics or computing in general may be an exception). Since he was not taking any courses at the time, the focus of my interviews was on particular tasks (including ones used in the first set), which included many mathematical paradox-related items.

Mark was also recruited in the community, received a diagnosis of Asperger syndrome at age 21 (this occurred before the release of the DSM-V), and was in his mid-twenties at the time of interview. Like Cyrus, no strong special interests appeared in the interviews. His mathematical background included a bachelor's and master's degree in mathematics. Again, the interviews focused on particular tasks. Here, some particularly geometric items were included to attempt to go against an observed algebraic inclination, as well as items aimed at showing other conflicts.

5.4. Interview Settings and Transcription

The data collection was conducted via in-person clinical interviews, which were recorded (audio-only) on a password-protected device. I had a set of pre-designed tasks (as described in Chapter 6), and allowed time for prompting, explanation, and elaboration. I attempted to elicit detailed descriptions of the participants' reasoning and methods, which were my primary focus during the interviews. While I wanted to elicit as much information as possible about the participants' thinking, I was also sensitive to their willingness to continue or abandon a line of discussion, as well as their need to end an interview at a time of their choosing. In total, I conducted eleven interview sessions with Joshua, all of which took close to one hour. I conducted ten sessions with Cyrus averaging about 45 minutes each, and two sessions with Mark taking two hours each. I have selected sessions to transcribe that illustrate the various characteristics and differences in thinking that I found in conducting the interviews, and excerpts from those interview transcripts that most strongly demonstrate the characteristics of those sessions.

In the transcripts, I use 'I' for the interviewer and 'J', 'C', or 'M' as appropriate for the interview subject. I have tried to capture how the dialogue went, although the line numbers are from a more direct transcription. '[?]' indicates something that was not clear, but by context is most likely not significant, and '(…)' indicates a significant pause. A '…' on a line indicates that some lines have been omitted, which I have also done in some gaps between sections. Other descriptions or clarifications are also bracketed, and any mathematical notation is pronounced in a standard or otherwise clear way.

5.5. Summary

In this chapter, I have described the general procedures followed in my research, from their original conceptions to the final form used in the interviews. General information about the participants as well as the form of the interviews is also included; details about the questions asked in the interviews follow in Chapter 6.

Chapter 6. Paradoxical and Counterintuitive Problems

There are a variety of tasks that were used during the interviews conducted with the three study participants. These tasks were chosen for several reasons with two main strands: one focusing on what is considered paradoxical or counterintuitive, and one focusing on what methods of solution are expected in a problem. The tasks have been separated into four categories.

Category 1: Paradoxes of Infinity

Category 2: Algebraic/Geometric Divide

Category 3: Probability

Category 4: Logic and Proof

For each category, I introduce the tasks, provide their conventional mathematical solutions, and discuss research in which these and similar tasks were used, embedding them in a broader research context. Figure 6.1 introduces each task by name, specifying which participants were given that task and which category it belongs to.

Name of Task	**Participants Interviewed**	**Category**
Ping-pong balls	Joshua, Cyrus, Mark	1
Gabriel's Horn/Painter's Paradox	Joshua, Cyrus	1
Magic Carpet problems	Joshua, Cyrus	2

Squares and triangles problem	Mark	2
Problem of Three Prisoners	Cyrus	3
Coin boxes problem	Mark	3
Four-card problem	Cyrus	4
Cardano paradox	Mark	4

Figure 6.1: Tasks and Categories

6.1. Paradoxes of Infinity

There are many results in mathematics that people may find paradoxical or counterintutitive. What these are can vary by time and context. For example, it is well-known that the Pythagoreans did not accept the existence of irrational numbers However, Fischbein, Jehiam, & Cohen (1995) found that while many contemporary students did not have much understanding of the concept of irrational numbers, they did not appear to have a strong resistance to it when exposed. However, one of the earliest and most lasting strands of such paradoxes is those related to infinity, such as Zeno's paradoxes considered by Greek philosophers. In this section, I discuss the paradoxes of infinity which were used in my interviews.

6.1.1. The Ping-Pong Ball Conundrum

Problem 1.1: The Ping-Pong Ball Conundrum
Consider an infinite set of ping-pong balls (numbered 1, 2, 3, …) being inserted into and removed from a barrel over one minute. In the first 30 seconds, the first 10 balls are inserted, and the '1' ball is removed. In the next 15 seconds, 11 through 20 are

inserted and the '2' ball is removed, and so on. How many ping-pong balls remain in the barrel at the end of the minute?

The accepted mathematical solution here is that there are no balls in the barrel, because for every possible ball, we can find a time after which it has been removed (this is because of the order the balls are removed in, and different orders can lead to different outcomes).

This problem was used by Mamolo and Zazkis (2008) with two groups of students, one undergraduate and one graduate. Both were in courses about fundamentals of mathematics at different levels which involved infinity. They were introduced to this problem after having seen the Hilbert Hotel problem, a simpler problem also involving infinity. In each case, after students' first responses to the problem, they were given the standard solution. Both groups initially gave responses that rejected things in the problem setup that seemed impossible, relating them to real-world facts such as the finite population of Earth. The undergraduate students, who were a more general population of liberal arts and social science students, continued to show resistance to the given mathematical solution in the Hilbert Hotel problem, while the graduate students (who were in a mathematics education program) did not. However, both groups continued to show disbelief in the mathematical solution for the Ping-Pong problem after instruction. One of the more common responses found in both student groups was that there were nine more balls at each step, often giving a 'nine times infinity' response. This highlights the importance of the numerical ordering in the problem, since if the balls were not ordered this way (if they were all simply generic and interchangeable balls, for instance), it would be correct to use the fact that at any step n, there were $9n$ balls in the barrel and to take the limit of that expression as n goes to infinity. In the numbered case, we can view it as essentially an arrangement of processes where a second process 'cleans up after' the first, but calculating the total at each step as $10n - n = 9n$ erases that ordering property. Without a numbering to provide order, that arrangement cannot be made; in that case, there is no information lost with the '$9n$ balls in the barrel' view.

Ely (2011) gave this problem (as the Tennis Ball Problem) to a range of participants from undergraduates who had finished college algebra to mathematics doctoral students and one mathematics professor. He used two versions, comparing the effect of asking “how many balls are left” to asking “which balls are left” (Ely, 2011, p. 8). He found that participants given the ‘which’ version were more likely to attend to the labeling (ordinal) rather than the total number (cardinal). The ‘which’ participants also showed more conflict from being presented the accepted mathematical solution of having no balls remaining, although they were still unlikely to accept it. No participants given the ‘how many’ version accepted the zero-ball solution, while the only participant given the ‘which’ version to accept it was the mathematics professor.

Disputes about the proper result of this problem have also been shown in publications where researchers discuss their own perceptions of and disagreements about the problem, rather than discussing the perceptions of students. Allis and Koetsier (1991) describe this paradox in terms of super-tasks, defined as “the execution of […] an infinite sequence of acts” (Allis & Koetsier, 1991, p. 189). They argue that this is possible not only in an abstract way, but also in a kinematic one, although their kinematic version involves motion of discs across a line rather than the formulation given above. They also give a variant where instead of the tenth ball being added, an extra zero is added to the oldest ball; this variant results in an infinite result, much like the unnumbered version above. However, I note that this could be considered to contain both results, as while there are an infinite number of balls at the end, the standard proof by contradiction can still be used to show that there are no integers marked on those balls. A later discussion by van Bendegem (1994) raises objections to both of these arguments. The objection to the abstract solution is an algebraic argument, while the objection to the kinematic one involves relativistic physical assumptions. For this argument, van Bendegem gives a more specific formulation of the discs, where the size of the discs decreases with respect to n such that the diameter of each disc is equal to a positive constant less than one multiplied by the diameter of the previous disc. Then, van Bendegem defines a point A as “the highest point of the disc that is on top of the row of discs ‘in the urn’” (van Bendegem, 1994, p. 745). He finds that while the process is running, the velocity of point A has a positive lower bound, but that it must be zero once the process finishes, and thus must have an infinite negative acceleration. The response from Allis and Koetsier

(1995) points out that the algebraic argument from van Bendegem does make an assumption of continuity (although van Bendegem asserted that it did not), and counters the kinematic argument in two ways. Their first counter is to show that this task can be mapped to a more famous paradox, Zeno's 'Achilles and the tortoise' paradox (although in this mapping the 'balls' are now signs and do not move, so this may appear unsatisfying). Their second counter is to note that in van Bendegem's kinematic objection, the point A is only a virtual point, and not a real object in the moving process. Looking at what is disputed between the authors, it is notable that the two main strands closely parallel the 'nine times infinity' solution and the objection to the real-world possibility of the problem found with the students in the study from Mamolo and Zazkis.

Ultimately, from the range of mathematical experience in responses to this task, we can see that this is a paradox that can incite argument and confusion even at high levels of academic discourse, and that few people at any level are inclined to accept the standard mathematical solution. Also, at multiple levels the nature of disputes and confusion fits into two main categories, one directly related to infinity and continuity and another related to physical properties.

In addition to its relevance as a task about a paradox of infinity, there was also a physical element given in some of the discussion from the Mamolo and Zazkis study participants, which suggested to me that this problem might be relevant from the perspective of examining algebraic or geometric solution tendencies as well (discussed further in Section 6.2).

The Ping-Pong Ball problem was used with Joshua and Cyrus, in much the same fashion as in Mamolo and Zazkis (2008), and was also discussed with Mark, although he had already seen the problem and was familiar with the conventional solution.

6.1.2. Gabriel's Horn / Painter's Paradox

Problem 1.2: Gabriel's Horn / The Painter's Paradox

Gabriel's Horn is the object created by rotating the graph of the function 1/x around the x-axis (starting from x=1), as shown in Figure 5.1 below.

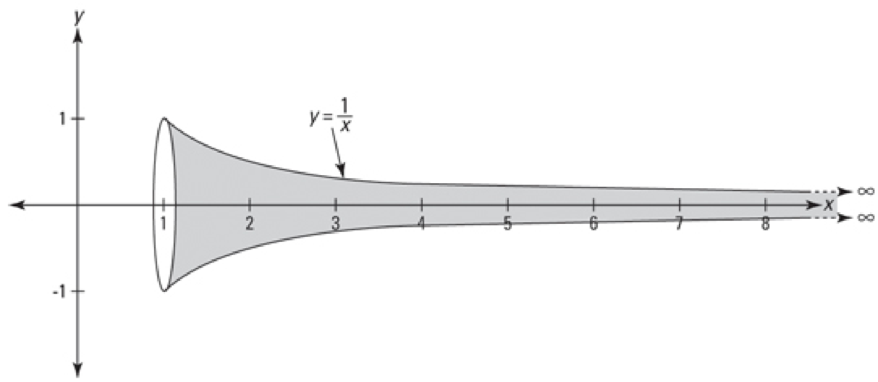

Figure 6.2: Gabriel's Horn

Its surface area and volume can be calculated as follows:

$$A = \lim_{b\to\infty} 2\pi \int_1^b \frac{1}{x}\sqrt{1+\left(\frac{-1}{x^2}\right)^2}\,dx \geq \lim_{b\to\infty} 2\pi \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} 2\pi \ln b = \infty$$

$$V = \lim_{b\to\infty} \pi \int_1^b \left(\frac{1}{x}\right)^2 dx = \lim_{b\to\infty} \pi \left(\frac{-1}{b}+1\right) = \pi$$

Since the surface area is infinite, it should require an infinite amount of paint to cover the surface. However, since the volume is finite, the horn could be painted by filling it with that amount of paint. Is this possible? How can this be resolved?

This result was discovered by Torricelli in 1641, and regarded as paradoxical by its discoverer and many later mathematicians, who used the result in discussions about the philosophical status of mathematics (Wijeratne and Zazkis, 2015). Wijeratne and Zazkis gave the problem to twelve undergraduates in a calculus course who had been presented with the relevant integration techniques in their course, and conducted interviews about the task afterward. One of the more common responses, reported from a majority of participants, was something with regard to the physical or contextual considerations of the problem (such as paint 'getting stuck' at an atomic level once the horn is small enough); because of this, this problem is another case in which I saw possibilities in both the ‘problems of infinity’ and ‘geometric and algebraic methods’ veins. Others rejected it in other ways, like speculating that it might be a type of incompleteness or flaw in mathematics, which may somehow be resolved by future developments. Only one of the twelve participants had a resolution that existed outside of the given contextualization of the problem. Rejection of the result was fairly common, as half of these students had “firmly held notions” that the volume should be infinite, contrary to the mathematical results presented.

The Gabriel’s Horn problem was used with Joshua and Cyrus, although in each case the integration was primarily done by the interviewer in order to show the paradoxical result, which I acknowledge may provide a weaker reaction than if the participants were able to complete the integrations unassisted. In the case of Joshua, the problem also came about somewhat naturally in the interview, since he was taking an integral calculus course at the time and something similar had been brought up in that class.

6.1.3. Comments on Paradoxes of Infinity

In analyzing learners’ responses to paradoxes involving infinity, researchers have commonly found the involvement of intuition, particularly in conflict with the accepted mathematical result. Since intuitive claims are an area where people on the autism spectrum often differ from others, I chose to investigate the role of intuition. The mention of intuition or of results being counterintuitive in other research findings involving paradoxes of infinity suggested to me that those tasks would be useful in examining intuition.

Fischbein (2001) considers the idea of tacit models, and suggests that some of the issues people have with paradoxes of infinity relate to a mismatch of the models they are using without examination. In particular, he suggests a conflict between the known mathematical fact of a line as a continuum and the implicit model of a line as a collection of points may cause conflict. In particular, he views mental models of time and space differently. The model he presents of space is something that is infinitely divisible into points, while his model of time is continuous and cannot be so divided, and he suggests that this could be a contributing factor to the conflict often experienced with these problems. This idea of tacit models has some similarities to the ideas about prototyping that I considered when examining autism-related research; in particular, Fischbein's idea that in this problem people are "thinking […] in terms of small spots" fits into the idea of a prototype. If this is the case, then research suggesting that people on the autism spectrum think less in terms of prototypes would imply that they may not experience the same difficulties with this paradox that other students do. This was one factor that suggested to me that problems of infinity could be a promising avenue to investigate.

As mentioned in the introduction, before any formal interviews were conducted with Mark (one of my participants), I observed a conversation where he was talking about the number of points in two line segments of different lengths. He framed his answer of the number of points being equal as the result of a proof immediately, and did not appear to find the result particularly objectionable or counterintuitive. Reflecting on this result provided additional reason for me to include problems of infinity in my interviews.

6.2. Algebraic/Geometric Divide

6.2.1. The Magic Carpet Sequence

Problem 2.1: The Magic Carpet Sequence, Part 1
You have two modes of transportation: a hoverboard and a magic carpet. The hoverboard moves along the vector (3,1) and the magic carpet moves along the vector (1,2). Can you get to a cabin at (107, 64) using these modes of transportation? If so, how? If not, why is that the case?

For this problem, a possible (and expected) solution in the context of a course in linear algebra is to find values *a* and *b* such that $a\begin{bmatrix}3\\1\end{bmatrix} + b\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}107\\64\end{bmatrix}$; in this case, those are $a = 30, b = 17$.

The work of Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012) introduced several Magic Carpet problems, and I use the first task in my interviews. The tasks as a whole were designed for and used on students in a linear algebra course who had completed at least two semesters of calculus. Since they were used at the beginning of their course, the students had not been previously exposed to the standard linear algebra solution above, although all had been introduced to the idea of a vector in some capacity. Instructionally, the intent of the problem in the context of a linear algebra context was primarily to introduce the idea of linear combinations, and to lead into other problems in the setting which introduce span and linear independence of vectors. For additional context, the second Magic Carpet task asks if there are any points that one cannot reach with a combination of the magic carpet and hoverboard from the first problem. This demonstrates part of the overall intent, to guide students toward ideas of linear independence and span as they find that there is no such point and justify their answer.

The first Magic Carpet problem was given to Joshua and Cyrus, although they did not have the same background as the students examined by Wawro et al. (2012); Joshua was in a linear algebra course which had already covered the relevant content, and Cyrus had taken such a course several years ago.

6.2.2. Squares and Triangles Problem

Problem 2.2: Squares and Triangles Problem
You are given the following figure:

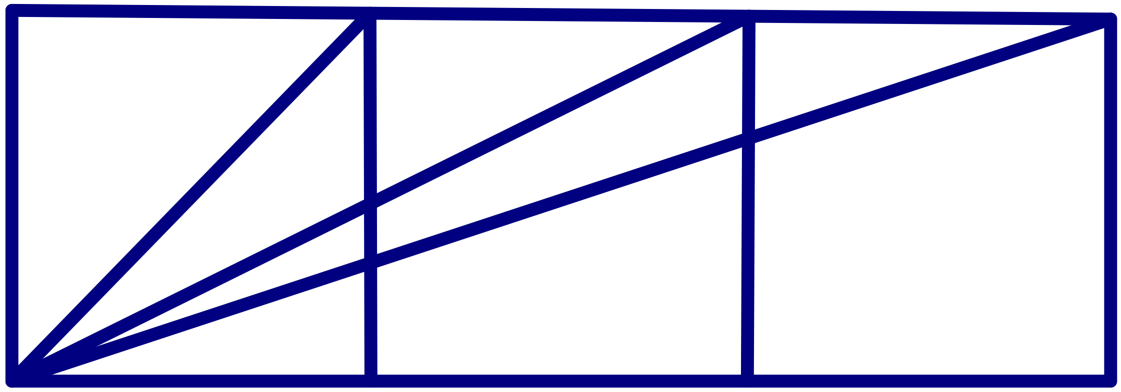

Figure 6.3: Squares and Triangles Problem Diagram

What is the sum of the three angles formed between the bottom line of the figure and the hypotenuses of the three right triangles in the figure?

The intended solution can be seen in the following figure:

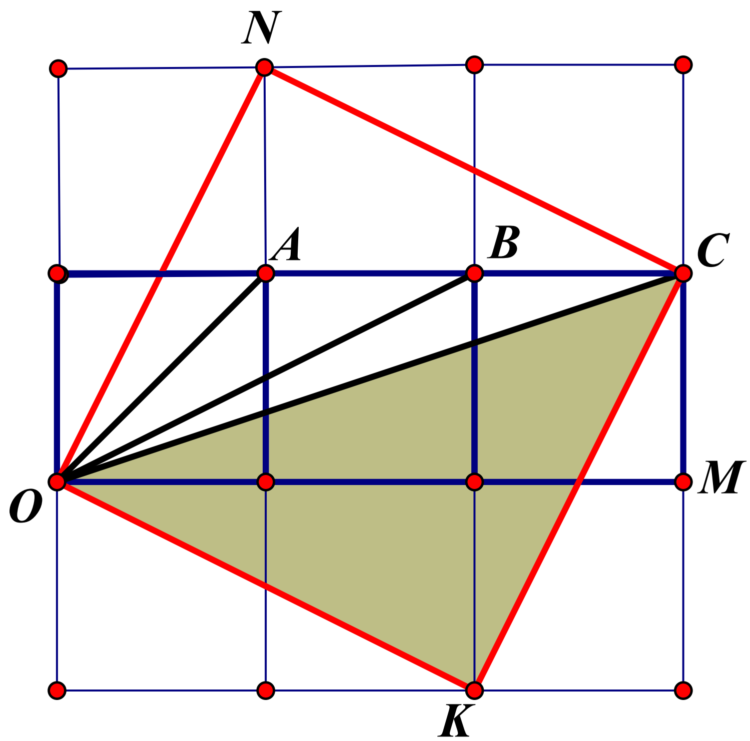

Figure 6.4: Squares and Triangles Solution Diagram

In Figure 6.3, it is easy to show that the angle ∠BOM is equal to the angle ∠MOK (for example, by considering congruent triangles), and that the sum of the measures of angles ∠COM and ∠MOK is 45 degrees (as this sum is ∠COK and OC is a diagonal in the square OKCN). Since ∠AOM is clearly also a 45 degree angle, the sum of the measures of the three angles is 90 degrees.

The squares and triangles problem was given to Mark; there, the first figure (similar to Figure 6.3) was given, as well as a statement of the problem, but nothing else aside from blank sheets of paper to work on.

6.2.3. Comments on Algebraic/Geometric Divide

One common stereotype about people on the autism spectrum is that their thinking is particularly geometric or visual. It is a main feature that Temple Grandin (1995) reports of her own thinking, and since she is one of the most well-known people on the autism spectrum and her example is often presented to people who are trying to understand autism, it is understandable that her experiences may be a source of this idea. However, to my knowledge the idea of visual thinking in people on the autism spectrum has not been studied in the context of undergraduate mathematics. Thus, my idea was to present mathematical problems which were particularly tilted either toward or away from a geometric method of solution being useful, and to observe the responses of my participants. There is also some overlap between geometrically-oriented problems and paradoxes of infinity, particularly in the Painter's Paradox, but also arising in other problems when introduced by a geometrically-minded student.

6.3. Probability

Problem 3.1a: The Problem of Three Prisoners
Tom, Dick, and Harry are awaiting execution while imprisoned in separate cells in some remote country. The monarch of that country arbitrarily decides to pardon one of the three. The decision who is the lucky one has been determined by a fair draw. He will be freed; but his name is not immediately announced, and the warden is forbidden to inform any of the prisoners of his fate. Dick argues that he already knows that at least one of Tom and Harry must be executed, thus convincing the compassionate warden that by naming one of them he will not be violating his instructions. The warden names Harry. Thereupon Dick cheers up, reasoning: "Before, my chances of a pardon were 1/3; now only Tom and myself are candidates for a pardon, and since we are both equally likely to receive it, my chance of being freed has increased to 1/2." Suppose, however, that the warden had named Tom. By the same reasoning, this piece of information would be equally encouraging for Dick. It looks like, whoever the warden names, Dick's chances are affected favorably. In fact, just imagining the potential exchange with the warden would have the same effect . . . Can all this be true? More than that, the warden need not actually exist. Just a thought experiment on Dick's part, involving a hypothetical warden, would raise Dick's probability of survival. What is true for Dick, however, is valid for Tom and Harry as well, so that each prisoner's

probability of going free is raised to 1/2, thereby violating the requirement for the sum of probabilities of all elementary events in a discrete sample space.'

Problem 3.1b: Three Prisoners, Alphabetical Warden Variant

In this variation of the problem, the warden names the first prisoner alphabetically that he is allowed to; that is, when offered a choice between naming Tom or Harry, he will always choose Harry. Does this change the conclusion? If so, how?

Problem 3.1c: Three Prisoners, Biased Pardon Variant

In this variation, the original pardon is biased: there is a known 1/2 probability that Tom will receive the pardon, and a 1/4 probability that each of the other two will receive the pardon. Does this change the conclusion? If so, how?

Problem 3.2: Problem of Gold and Silver Coins

You are given three boxes. You know that each box contains two coins: one has two gold coins, one has two silver coins, and the third has one gold and one silver coin. You choose a box, and take one of the coins, finding that it is gold. What is the probability that the other coin in your box is also gold?

The mathematical solution to the base Three Prisoners problem (3.1a), using Bayesian probability, is as follows:

$$P(\text{Dick is pardoned} \mid \text{Harry is named}) =$$

$$= \frac{P(\text{Harry is named} \mid \text{Dick is pardoned})P(\text{Dick is pardoned})}{P(\text{Harry is named})} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{3}$$

Noting that:

$$P(\text{Harry is named}) =$$

$$= P(\text{Harry is named}|\text{Tom is pardoned})P(\text{Tom is pardoned})$$

$$+ P(\text{Harry is named}|\text{Dick is pardoned})P(\text{Dick is pardoned})$$

$$+ P(\text{Harry is named}|\text{Harry is pardoned})P(\text{Harry is pardoned})$$

$$= (1)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{3}\right) = \frac{1}{2}$$

With Problem 3.1b, we get instead:

$$P(\text{Dick is pardoned } | \text{ Harry is named}) =$$

$$= \frac{P(\text{Harry is named }|\text{Dick is pardoned})P(\text{Dick is pardoned})}{P(\text{Harry is named})} = \frac{(1)\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \frac{1}{2}$$

Noting that in the denominator, we now have:

$$P(\text{Harry is named}) =$$

$$= P(\text{Harry is named}|\text{Tom is pardoned})P(\text{Tom is pardoned})$$

$$+ P(\text{Harry is named}|\text{Dick is pardoned})P(\text{Dick is pardoned})$$

$$+ P(\text{Harry is named}|\text{Harry is pardoned})P(\text{Harry is pardoned})$$

$$= (1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{3}\right) = \frac{2}{3}$$

For Problem 3.1c, the solution using this method is:

$$P(\text{Dick is pardoned} \mid \text{Harry is named}) =$$

$$= \frac{P(\text{Harry is named } |\text{Dick is pardoned})P(\text{Dick is pardoned})}{P(\text{Harry is named})} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}{\left(\frac{5}{8}\right)} = \frac{1}{5}$$

Since in the denominator we have:

$$P(\text{Harry is named}) =$$

$$= P(\text{Harry is named}|\text{Tom is pardoned})P(\text{Tom is pardoned})$$

$$+ P(\text{Harry is named}|\text{Dick is pardoned})P(\text{Dick is pardoned})$$

$$+ P(\text{Harry is named}|\text{Harry is pardoned})P(\text{Harry is pardoned})$$

$$= (1)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + (0)\left(\frac{1}{4}\right) = \frac{5}{8}$$

However, with the coin box version, using Bayesian probability with the events ‘the first coin is gold’ and ‘the second coin is gold’ as given does not tell us much. Instead, if we read it as ‘the first coin is gold’ and ‘both coins are gold’, a Bayesian solution is:

$P(\text{Both coins are gold} \mid \text{First coin is gold}) =$

$$= \frac{P(\text{First coin is gold} \mid \text{Both coins are gold})P(\text{Both coins are gold})}{P(\text{First coin is gold})} = \frac{(1)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{2}{3}$$

Another solution, not relying on conditional probability, can be outlined as follows:

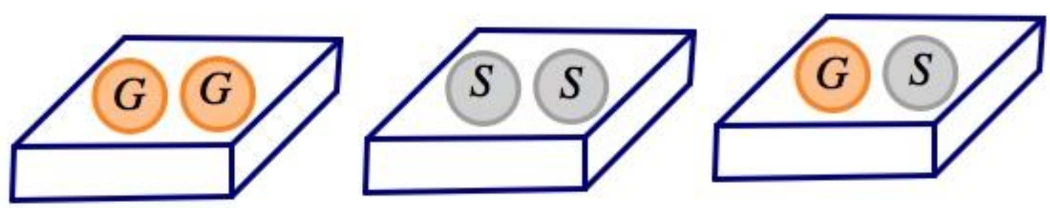

Figure 6.5: Coin Boxes Diagram

Here (in Figure 6.5), we see each coin within the three boxes. Assuming that the first coin was selected so that we have an equal chance to get each coin, adding only the knowledge that the first coin is gold establishes that we have an equal chance of it being each of the three gold coins. Since 2/3 of the gold coins are in the box with two gold coins, the probability of that being the case is 2/3, and ‘the coin is from a box with two gold coins’ is equivalent to ‘the other coin in the box is gold’.

The Problem of Three Prisoners is equivalent to the more famous, and more commonly studied, Monty Hall Problem. In that problem, a contestant is offered a choice of three doors; behind one of them is a car, and behind the other two are goats. The contestant chooses door number 1, and after this, Monty opens door number 2, revealing a goat. They are offered the choice of staying with their original door, or switching to door number 3. Looking at their chances as equivalent to the Three Prisoners problem, we would equate the pardon with the car and the three people with the three doors. The presented motivation is somewhat different, however, since in the Monty Hall problem the contestant is offered a chance to switch doors, while our version of the Three Prisoners does not offer any prisoner a chance to change fates with another.

Falk (1992) used the Problem of Three Prisoners and several variants to examine respondents' beliefs about probability concepts, including the one described as problem 3.1b, while Shimojo and Ishikawa (1989) proposed another variant which I have described as problem 3.1c. The presentation of the various forms of Three Prisoners problem included arguments which fit into two intuitive categories (and that fit common arguments given by respondents to both of these problems). The first Falk terms the "uniformity belief", that when one alternative is eliminated from equally probable alternatives, the remaining ones will stay equally probable. Shimojo and Ishikawa (1989) view this more generally as a "constant ratio" belief, that the ratios of the probabilities of the remaining outcomes will stay the same (even in cases where they were not equiprobable to begin with). These intuitions may also be a particular category of the "same A – same B" intuition discussed by Tirosh and Stavy (1999). Using this intuitive argument on each of the problems presented would result in an answer of 1/2 for Problems 3.1a and 3.1b, 2/3 for Problem 3.1c, and either 2/5 or 2/3 for Problem 3.2 (depending on whether the implicit elimination of the box with two silver coins is also accounted for or not); of these, the 1/2 answer for Problem 3.1b and the 2/3 answer for Problem 3.2 are correct, and the others are incorrect.

The second Falk terms "no-news, no-change" and Shimojo and Ishikawa term "irrelevant, therefore invariant": if no previously unknown information comes out of something, the probabilities should not change. Shimojo and Ishikawa also give the name "number of cases theorem" (using 'theorem' in the sense of a subjective theorem believed by the respondent) to the common belief that given a list of alternatives, they are equally probable to begin with. An argument presented toward the correct solution in the Monty Hall problem given in one of Marilyn vos Savant's *Parade* magazine columns about the problem, following up on the original which brought the problem some of its notoriety among the general public, relied on the “no-news” assumption (as cited in Falk, 1992). This argument said that “as I can (and will) do this regardless of what you've chosen, we've learned nothing to allow us to revise the odds”. In a sense, this also carries a kind of uniformity assumption, assuming that if there are two 'wrong' doors to choose from, those will be selected uniformly. Using this intuitive argument on each of the problems presented would result in an answer of 1/3 for Problems 3.1a and 3.1b, 1/4 for Problem 3.1c, and 1/3 for Problem 3.2. Thus, this gives the correct answer for the standard version of the Three Prisoners (3.1a), like the Monty Hall problem, but incorrect answers for all of the other variants. Thus, the possibilities for conflict with intuition with the coin-box problem are similar to those from the Three Prisoners variants. In addition, the fact that the correct answer for Problem 3.2 differs from that of the other problems makes it useful for examining the same ideas with a participant who may have seen the Monty Hall problem before and is aware of its standard solution.

Falk reports that multiple other sources have found a strong tendency toward confidence in the uniformity belief, arguing that it is a “primary intuition” following many of the characteristics in Fischbein's (1987) definition. Falk notes both contemporary and historical sources which appear to have uniformity in their definition of randomness, categorizing non-uniform distributions as 'not random'. For the Monty Hall problem, Oza (1993) examines it by expanding the sample space to include every combination of which of the three doors has the car and which door the host chooses, allowing a correct solution while maintaining uniformity, but does not directly discuss the uniformity assumption. However, there are a number of possible sources for the uniformity belief in this context. Zazkis (2008) examined student responses to several examples written to bring out basic assumptions, and some of these highlighted the differences in assumptions that occur in the 'mathematics problem' context. It is possible that this could be another source of the tendency toward the uniformity belief, both in a general sense (probability questions tend to have uniform distributions unless otherwise specified) or in a more specific sense (if the students in question have seen either this problem or a similar one in the past).

In Shimojo and Ishikawa's sample of undergraduates, their respondents gravitated strongly towards at least one of these intuitions, with very few correct responses for the more complex versions necessitating a proper Bayesian solution. Additionally, they reported that even after explaining the Bayesian probability used to solve the problem, most of their respondents still found that solution to be counterintuitive. When the Monty Hall problem and its solution were shown to a wide audience in a *Parade* magazine column, the response showed that there were several research mathematicians (even some reporting expertise in probability) who were convinced that the solution was incorrect to the extent of being motivated to publicly write in to respond (vos Savant, 1997). Later columns included more problems similar to the variations above, all of which met with similar negative responses. While this is not definitive about its prevalence, it does show that these errors are not solely the domain of the uninformed.

Another possible factor in the consideration of this problem is that the setup of each of these problems involves something which has ‘already happened’ in the problem setting: the car has already been positioned, the pardoned prisoner has already been selected, and the coin box has already been chosen. Chavoshi Jolfaee (2015) examined students’ thinking about this issue in the case of a fair coin which had been flipped but not revealed, and asking them to choose between two arguments. The first argument was that the probability that the coin is showing heads is 50%, while the second was that the probability that the coin is showing heads is either 1 or 0, but we do not know which. Undergraduate students responding to this were nearly evenly split, 10 agreeing with the first argument and 12 agreeing with the second. There were 3 more students who did not settle on either argument, one who was unable to decide and two who questioned both arguments.

Cyrus was given the Three Prisoners variants, introducing him to each of the variants in the same order they are listed in here, and Mark was given the coin box version. In this case, one of the reasons that I chose these variants of this problem was that I expected them to be significantly less likely to be recognized than the Monty Hall version.

6.3.1. Comments on Probability Problems

As reflected in research focusing on problems such as those discussed above, probability is one of the areas in mathematics where a significant conflict between intuitive reasoning and formal mathematical results can be observed. These problems are thus appealing for similar reasons to paradoxes of infinity, although the type of conflict invoked with intuition is different. In particular, I chose these problems since the closer ties to possible real-world situations have the potential to invoke different kinds of intuitive reasoning. Also, I find that these problems are accessible at a low level of formal mathematical knowledge, yet still create a conflict at a high level of formal mathematical knowledge, facilitating a reaction from varied study participants.

6.4. Logic and Proof

6.4.1. The Four-Card Problem

Problem 4.1a: Four-Card Problem, Abstract
You have four cards; each card has a letter on one side and a number on the other. The visible sides of the cards are A, B, 3, and 2. Which cards do you need to check to determine if the rule "if a card has an A on one side, then it has a 3 on the other side" is true?
Problem 4.1b: Four-Card Problem, Real-World
You have five envelopes: the first is face-up with a four-cent stamp, the second is face-up with a five-cent stamp, the third is face-down and sealed, the fourth is face-down and unsealed, and the fifth is face-up with no stamp. Which envelopes do you need to check to determine if the rule "If an envelope is sealed, then it has a five-cent stamp on it" is true?

For each version, there is a card or envelope that checks the statement directly (the A card or the sealed envelope) as well as at least one that checks the statement via the contrapositive (the 2 card or the face-up envelopes which do not have five-cent stamps). Thus, for Problem 4.1a, it is necessary to check the A and 2 cards, corresponding to the statement and its contrapositive respectively. For Problem 4.1b, it is necessary to check the first, third, and fifth envelopes. Here, the third envelope corresponds to the statement directly, and the first and fifth to its contrapositive.

Kahneman and Tversky (1982) consider the four-card problem in several studies which use different (but isomorphic) formulations of it. In the standard form they use for illustration, the respondent is shown four cards which display 'A', 'T', '4', and '7' and is asked which must be checked to determine the truth of the proposed rule “if a card has a vowel on one side, it has an even number on the other”. Johnson-Laird, Legrenzi, & Legrenzi (1972) compare a similar formulation to a logically-equivalent real-world formulation, where the subjects were given envelopes and the rule “if an envelope is sealed, then it has a 50 lire stamp on it”. Performance under this condition was much better: they tested 24 undergraduate students in each condition (giving each two trials), and 22 gave at least one correct answer (out of two equivalent formulations of the question) in the realistic condition, while only 7 (about 30%) did in the abstract condition. Other findings testing “educated adults” more broadly using the abstract condition found even lower performance, with less than 10% successful (Inglis, 2016).

More recently, Inglis (2016) found that mathematics undergraduates were more likely than undergraduates in a non-mathematical field (history) to select the standard mathematical response for the four-card problem, but still not very likely (18% in the mathematics group versus 6% in the history group). Inglis also found that while the mathematics students were significantly less likely to incorrectly select the 3 card (29% versus 52%), they were not much more likely to select the other number card (35% versus 24%). A follow-up experiment involving eye-tracking found that both groups spent much more time looking at the A and 3 cards[2] than the other two. Unexpectedly, Inglis' longitudinal study groups of both high school and undergraduate students studying

[2] Here, I have changed the card names to correspond to my formulation of Problem 4.1a.

mathematics showed a decrease in accepting the *modus tollens* inference (corresponding to the 2 card), although the question posed was slightly different, allowing responses of “conforms to the rule”, “contradicts the rule”, and “irrelevant to the rule” (p. 77) to questions of checking one card at a time where both sides are visible.

All of these findings could be influenced in various ways by the possible confounding effects of both the particular context in their trial and the broader experimental context, as Kahneman and Tversky also point out. They consider Grice's (1975) maxims of conversation, in particular the maxim that all conversational contributions be relevant to the current discussion, which are discussed in more detail in Chapter 4. From this perspective, Kahneman and Tversky note that respondents' expectations of the experimenter following these ideas (for instance, that provided information would not be irrelevant to the question asked) could influence what they consider in their responses. However, since issues with the characteristics of conversation that Grice’s maxims describe are one of the primary diagnostic criteria for autism, it is reasonable to suspect that the effects described by these maxims may be different or less present in participants on the autism spectrum.

In the interview with Cyrus, both the abstract and real-world formulations of the Four-Card Problem were used, with the abstract version coming first. However, the real-world condition was slightly altered to use the Canadian cost of stamps reported by the participant (although the cost that he gave of five cents is not accurate).

6.4.2. Cardano’s Method

Problem 4.2: Using Cardano’s Method
Solve the equation $x^3 - 15x - 4 = 0$ using both Cardano’s method and modern algebraic methods. What do you notice? How can this be resolved?

In general, Cardano’s method is as follows:

1.	$x^3 + px + q = 0.$

2. Let us look for x in the form $x = u + v$ when $3uv = -p$
3. $x^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvx$
4. Thus, our reduced equation transforms into the following one:
5. $u^3 + v^3 + (3uv + p)x + q = 0$
6. $u^3 + v^3 = -q$
7. $u^3 v^3 = -\frac{p^3}{27}$
8. This system of two equations is of the type $\begin{cases} a+b=m \\ ab=n \end{cases}$, so we can solve it as such:
9. $u^3 + v^3 = u^3 - \frac{p^3}{27u^3} = -q$
10. $u^6 + qu^3 - \frac{p^3}{27} = 0$. Let $w = u^3$.
11. $w^2 + qw - \frac{p^3}{27} = 0$
12. $u^3 = w = \frac{-q}{2} \pm \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$
13. From here, we obtain v^3 given that $u^3 + v^3 = -q$. Consider to cases. If $u^3 = \frac{-q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$ then $v^3 = \frac{-q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$. If $u^3 = \frac{-q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$ then $v^3 = \frac{-q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$.
14. That is, in both cases $x = u + v$ is the same:
 $x = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$. **This equation is known as the Cardano formula**.

In the case of the equation given above, $x^3 - 15x - 4 = 0$, the Cardano formula gives the following:

$$x = \sqrt[3]{-\frac{(-4)}{2} + \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-15}{3}\right)^2}} + \sqrt[3]{-\frac{(-4)}{2} - \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-15}{3}\right)^2}}.$$

This simplifies to $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$. However, by applying the rational roots theorem, one can find that $x = 4$ is a solution to the equation, and by polynomial long division and the quadratic formula one can find the solutions $x = -2 \pm \sqrt{3}$ as well. This appears to give four zeroes to a third-degree polynomial, which would contradict the Fundamental Theorem of Algebra. The necessary resolution here is that due to the nature of complex numbers, the result from the Cardano formula gives three real numbers, which are in fact 4, $-2 + \sqrt{3}$, and $-2 - \sqrt{3}$.

Koichu and Zazkis (in press) used this problem in a series of tasks done in a mathematics education course with both graduate and advanced undergraduate students. Among other tasks, the students were asked to check first if the Cardano formula result was equal to 4, and second if the Cardano formula result was equal to each of the other roots. Once they showed it held for all three roots, they were asked to then consider the claim that $4 = -2 \pm \sqrt{3}$, following from the implied property of transitivity after showing that $\sqrt[3]{2+11i} + \sqrt[3]{2-11i} = 4$ and $\sqrt[3]{2+11i} + \sqrt[3]{2-11i} = -2 \pm \sqrt{3}$. Their students checked the algebraic computations several times and, finding no errors, moved on to examining the reversibility of the computations used to check the first two equations. In particular, the students inspected the reversibility of cubing, leading to the recognition of the existence of multiple cube roots in the complex numbers, and that since the expression $\sqrt[3]{2+11i} + \sqrt[3]{2-11i}$ has multiple values, the law of transitivity does not apply. Some students also paid particular attention to the self-substitution used in the earlier check of the equalities involving the Cardano formula result, although this did not lead to any particularly relevant conclusions. In the way that this paradox is introduced, I see in particular a possibility to introduce something that touches on issues of mathematical logic in the context of algebra without a high level of other mathematics content required, making it possible to use with a larger range of participants (at a college algebra level or above).

This method and the paradoxes involved with it were presented to Mark, since this was a problem he was not familiar with which involved often unexamined assumptions about how mathematics works.

6.4.3. Comments on Problems of Logic and Proof

Mathematics problems inevitably involve logical reasoning at some level (probability problems perhaps more directly than some others). However, some are mostly about logical reasoning in that their content outside the logical structure in question is relatively simple, if not directly about logical reasoning in its abstract 'if P, then Q' phrasing. Others more directly involve mathematical proof or the thought process behind it. Differences in concept formation suggested by some studies, particularly those involving stipulated definitions as opposed to extracted definitions (or prototype-based definitions), led me to consider questions that were pointed at these general ideas, without other confounding factors.

6.5. Summary

In this chapter, I have given the general rationale for the choice of the interview problems, as well as detailing the problems themselves and their division into four categories: paradoxes of infinity, algebraic and geometric methods, probability, and logic and proof.

In the following three chapters, I show the application of these problems in interviews with the three study participants, starting with Joshua in Chapter 7.

Chapter 7. Interviews with Joshua

This chapter summarizes and analyzes interviews conducted with Joshua. As detailed in Chapter 5, Joshua was enrolled in integral calculus and linear algebra courses at the time of the interviews, having previously taken differential calculus. The chosen excerpts of these interviews are a representative subset of the interviews as a whole, highlighting certain elements which I want to analyze in my analysis, although it is not possible to include everything.

7.1. Magic Carpet Tasks

In this interview, Joshua is given the first of the Magic Carpet tasks from Wawro et al. (2012). As detailed in Section 6.2, this involves a hoverboard that moves along the vector (3,1) and a magic carpet that moves along the vector (1,2). The respondent is asked if one can reach a house at the point (107, 64) using those two modes of transportation.

Despite the intent of the first Magic Carpet task to work toward students' use of vectors, and Joshua's previous exposure to the vector-based solution methods, Joshua's initial approach is entirely geometric:

J.1.1	J: 107 x, 64 y, okay. [Joshua produces Figure 7.1, using a ruler.] So... isn't the 107 and 64, couldn't we just, uh, that would be equal to the determinant, wait, you can't really use, no, because what I'm thinking, we could do is we could literally draw a parallelogram, so are we allowed to do that?
J.1.2	I: Okay.
J.1.3	J: 'cause here's how I would do it. ...draw a sketch. (drawing) So, I mean, I don't know if this has to be explicitly done mathematically or whether it can be done, you know, by drawing a sketch but in physics I know that we always drew

	sketches. That way, the person knows what is going on in our heads.
J.1.4	I: Okay, well, that's what I want to know, so that's good.
J.1.5	J: Okay, so here's what's going on in my head.
J.1.6	I: Okay.
J.1.7	J: So now we've got the vector that we wanted to, which... look like so... 1, 2 we could scale up to 10, 20, so that's what I did here, and we have the vector 3, 1, which we could scale up to 30, 10. Oh, that's right, and so now, how we could approach this is we could follow this vector here, ...we could follow the vector, this is the vector 30, 10, and this is the vector 10, 20, I just scaled it by 10,
J.1.8	I: Okay.
J.1.9	J: Each vector scaled by 10, then from this point here, what we could do, is we could slide this vector up, slide it up all the way up to here and then, so what we could do, is from this point here, we could draw, so if we have 64, 107, we could- go down 20, 20 units, and 10 units to the left, like so, and then, we have the same vector, we just literally transformed the vector. I guess not [transformed], we moved the vector, the point is that it's the same thing, 10, 20, and then, this point of intersection, is where we would change what instrument we were using, so you want to look at that.
J.1.10	I: Okay. Interesting.
J.1.11	J: So that's how I would do it. I'd approach it literally geometrically. Yeah, there you go. [handing over paper]
J.1.12	I: Hm.
J.1.13	J: And so we've got two lines there, and that little point of intersection is where, and you can find that quite easily, on the x-axis, and that point of intersection is where you would switch your instruments, ...and again, we have the vector 1, 2, we can find the slope of that vector, and then we can move it over to the point 107, 64, and then, you know, it wouldn't really be that hard to find that point of intersection but that's how I'd do it, literally just play around with those vectors.

Joshua starts off by drawing the destination point, using a ruler to measure precisely where the point should be to get a drawing that is properly scaled. He mentions using the determinant as well (J.1.1), but quickly dismisses that idea and focuses on drawing a sketch. During this process, he mentions that he is unsure if this is allowed in this context, but compares it to expectations in physics (J.1.3) that reflect what is happening in his mind. For the first step in the sketch, he draws the given vectors for the magic carpet and the hoverboard, scaled up by a factor of 10 (given the rest of the scaling, they would be barely visible otherwise). After doing this, he moves the vector (20,10) so that its tip touches the destination point, and then extends the two vectors so that they intersect, concluding that the intersection point is where you should change from one mode to the other. The end result sketch is shown in Figure 7.1 below.

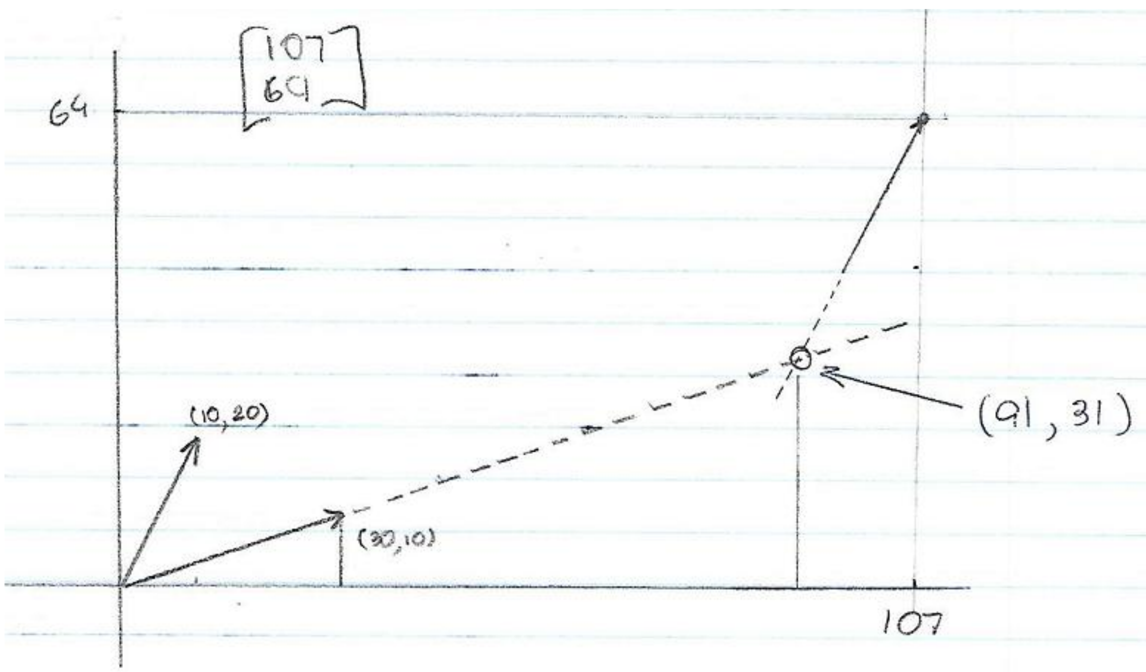

Figure 7.1: Joshua's Magic Carpet Solution

This solution is unlike those of any of the students observed by Wawro et al. in their use of this task. The drawings used by Joshua were produced using a ruler and were very precise, enough to give a correct solution (note that this was done on lined paper, not a square grid). However, it is notable that Joshua terms this a "sketch" (possibly ignoring some of that word's connotations) and seems to not regard this solution as "mathematical" (J.1.3). The drawings were measured after they were produced, and the point of intersection found by the drawings was the correct point (although the coordinates as written on the paper above are very slightly off). The interpretation as the location where the person in the problem changes from one mode to the other is also correct. Thus, this solution accomplishes the stated goal of the problem (to find a way to get to the cabin) perfectly well, although by approaching the problem this way, Joshua avoids the intent to push the student toward a standard linear algebra solution. Also, this solution is less directly related to linear combinations of vectors, though a geometric version of the idea can be brought out from the drawing used for the solution. In particular, vector scaling is used to arrive at the solution, as well as vector addition (which is geometrically accomplished by placing the start of one vector at the end of another).

Later in the interview, Joshua is asked to find an algebraic way to solve the problem:

J.1.14	I: Can you think of the algebraic way to solve the problem?
J.1.15	J: Well, like I said, I'd probably find the slope of the vector 1, 2, make that, once I have the slope, make a line with slope 1, 2, so literally it would be y is equal to m x plus b, so, our y and our x values would have been 64 and 107, we have 64, 107 with some slope, you find the b, you find the y-intercept, you'd have an equation y equals m x plus some y-intercept, you'd know the slope, you'd find the other slope, you'd find the other equation, and that point of intersection is where the x and the y values are the same. Do you know what I mean?
J.1.16	I: Okay. Wait... hm.
J.1.17	J: You find two equations for both lines, so I found two lines there, find equations for both lines and then find the common solution for both lines. [...]
J.1.22	I: So you're finding an equation that's sloped on one of the vectors and hits this point,
J.1.23	J: Yes.
J.1.24	I: and the other equation that's sloped on the other vector hits the origin.
J.1.25	J: Yeah. Exactly. And then find a common solution to those. And that's where you would switch your implements [indicating the modes of transportation]. [writing]
J.1.26	J: It's kind of nice actually, you could antiderive those lines and find curves.
J.1.39	[Joshua asks if there is an easier way, and the interviewer demonstrates the

	standard linear algebra solution.] I: And then you have your two equations, two unknowns, system.
J.1.40	J: Oh, that's a lot easier. See, I didn't think of that application. I literally, it's easier for me to just literally draw it out. Yeah no, that didn't even come to mind. Goes to show you what I'm getting out of this class, [laugh]. Which I'm not saying it's his fault, it's- it's just the way it is.

Here, when Joshua is asked to find an algebraic way of solving the problem, he starts by finding the slope of one of the vectors on his previous drawing, which he corrects to finding a line with the same slope as that vector and that hits the destination point. He continues to doing this with the other vector, then saying to set them equal to each other in order to find the intersection point. After this, he also mentions being able to antiderive the lines to find curves (although he doesn’t claim this is directly related and it is not pursued by the interviewer). Then, Joshua asks if there is an easier way to solve the problem. At this point, the interviewer outlines the standard linear algebra solution with a system of equations, which Joshua states did not occur to him at all.

As Joshua acknowledges, he had been presented with the linear algebra material that one could use to solve it in the intended linear algebra way, but it appears that he adapted his geometric solution to its algebraic counterpart rather than trying to find an algebraic solution. For comparison, Wawro et al. (2012) state in their research that the student solution attempts they observed fell into three categories of “guess and check”, “system of equations first”, and “vector equation first”. The third category fits most closely with the standard linear algebra solution, and its presence for the original study's students highlights the differences in Joshua's approach (which fits into none of the three): some students with no prior linear algebra instruction presented a vector-based solution, while Joshua did not, despite linear algebra instruction from the course he was taking at the time.

Another possible effect of the unusual tendency seen here is that it may pose a difficulty for an instructor's plan to confront students with a problem that would ordinarily necessitate a particular approach (the introduction of which is the goal of the activity), as happened with this problem. The intent seen in the problem design by Wawro et al. was to push students into the use of a vector-based approach, which notably did not occur here (note particularly J.1.40).

One possible line of explanation for Joshua's choice of method here is that his tendency toward geometric thinking about the problem results from an instance of overcompensation, as defined by Vygotsky (1993). Joshua may have particular strengths in areas related to the geometric reasoning he uses here, which he is using to compensate for weaknesses in areas related to the algebraic reasoning that would be involved in the 'standard' solution to the problem. However, since overcompensation is an idea defined in relation to individuals' development, the observations in interviews with adult students will most likely be of the end result of the compensation process that Vygotsky described (and not show the process itself).

The use of a geometric/visual approach notably fits with what we see in other sources, such as a description of the thought process in Temple Grandin's work. Grandin (1995) describes her own memory as being based on remembering static or moving images, and being able to both understand others' information and express her own better in writing than verbally (which may suggest an issue with the interview process). She also describes thinking of abstract ideas in terms of images or sequences of images. However, the range of variation in autism as well as other interview experiences lead me to believe that the underlying principle for differences in problem-solving methods is more complex and does not always push toward a geometric approach (although it may be more common).

The remark about finding curves from these lines (J.1.26) as an offhand remark is not followed up on, and does not directly apply to Joshua's problem-solving process, but is a reflection of his general inclination to investigate and feel positively about geometric or graphical findings which may or may not be directly related to the given task; in other discussions, Joshua had stated that he looks into things like this on his own, particularly when they fall into areas of his interest.

Also, the concluding remarks by the student (J.1.40) not only point to a need for instructional attention in the linear algebra class, but also suggest that there could be a more general pattern across multiple courses of using unexpected approaches that may avoid the general intent of the lesson. I suggest that while this can certainly be a problem if it goes unnoticed, with a well-tuned approach it could be turned to an advantage. This is much like Vygotsky's concept of compensation, although strictly speaking Vygotsky's original conception of compensation was for development of more general reasoning abilities as well as child development. While the strict conception would not apply to adults or to specific linear algebra skills, I think that the general idea of using strengths to reinforce weaknesses, possibly in ways that have a different form than the expected one, is useful here. It could also involve the use by the student of Vygotskian compensation that was already done during development, although viewing students only as adults we cannot be certain.

7.2. The Gabriel's Horn Paradox

In another interview with Joshua, I presented him with the Gabriel's Horn or Painter's Paradox. As discussed in Section 6.1, this paradox involves the surface of revolution of the function 1/x from one to infinity about the x-axis. The area of this surface is infinite, but the volume is finite (equal to π); the term 'painter's paradox' comes from problem phrasing asking how much paint would be required to paint the inside of the surface. The surface area suggests it would be infinite, but the finite volume suggests that it would be possible to pour in that volume of paint to cover the entirety of the object (thus painting the object after the paint is poured out).

The first interview about this topic began by giving Joshua the setup to the problem, eventually presenting (as in Wijeratne and Zazkis, 2015) the standard mathematical solution (the integrals shown above) when Joshua did not succeed in arriving at it himself. At this point, the interviewer asked for Joshua's perspective on the problem and (primarily) its result:

J.2.1	I: The volume. Okay. So, this does, it turns out, [have] a finite volume.
J.2.2	J: Oh, wow.
J.2.3	I: So we can fill it with
J.2.4	J: Pi...

J.2.5	I: pi gallons of paint.
J.2.6	J: Oh. Wow. See, that's beautiful.
J.2.7	I: So, now, let's look at the surface area. [The interviewer shows the surface area calculation.]
J.2.8	I: So we conclude that the surface area is infinite.
J.2.9	J: Well, that's very interesting, because, from the integral test I would think that, since we see that the volume is convergent, that the surface area would be convergent, and that's very interesting. Isn't it beautiful? I think it's kind of beautiful.
J.2.10	I: Okay. So, in terms of the paint thing, we've got an infinite area on this horn to paint.
J.2.11	J: Mm-hm.
J.2.12	I: And we can paint all of it by filling it up with pi gallons of paint.
J.2.13	J: That's... see, through rigorous analysis, my theorizing off the top of my head is proven wrong. Very interesting.
J.2.14	J: I wonder if he'll mention- if he'll give me any marks for that pi volume, or, yeah, Gabriel's Horn volume thingy. On the midterm. [Part of this problem had been presented to Joshua shortly before the interview on a calculus exam.] That's very interesting.
J.2.17	I: So, now my question is, ah... ah, now that I've presented this, um, does it seem strange or wrong, do you have a problem with it?
J.2.18	J: Well, it does seem kind of strange, um, because, like I said [?] the integral test, we learned that if the volume underneath the series is divergent, then the series itself is divergent, but here you've clearly shown that, you know, we've got a convergent volume, but the series itself must be divergent, because you need an infinite amount of paint to paint the trumpet, the horn, whatever you want to call it. So, that's how I would think about it. Is there something that you would add?
J.2.19	I: Well, I'm just curious as to whether you feel this is a conflict or not.
J.2.20	J: I do actually feel like it's a conflict. Because we've, again, when I work this out in my head, it would seem kind of weird that the volume is smaller than the

	surface area, and the surface area, if we take surface area to be in $\mathbb{R}^2$, should be smaller than the volume, which we take as being in $\mathbb{R}^3$, so yes, it does seem kind of conflicting. Do you know what I mean?
J.2.21	I: Okay.
J.2.22	J: How can a volume be smaller than an area? For the same- and, I, you know, now I'm thinking about that orange paint thing, and it seems kind of bizarre. Ah, so yeah, I see what you're saying.

After presenting the solution for the volume, Joshua calls it “beautiful” (J.2.6), which he says again after the solution for the surface area (J.2.9). He also acknowledges that his earlier ideas about the problem were proven wrong. After asking specifically, he acknowledges that it is strange (J.2.18) and that he could see some kind of conflict, but this is mostly following the prompting and it does not appear to be a strong feeling, or one that he considers particularly important. Instead, the perception of mathematical beauty appears to be the predominant emotional reaction.

Surprisingly, given the general history of the interviews, he did not present any physical objections or analogies in the initial discussion, but they did appear in a later interview that revisited the topic:

J.2.23	J: I think it's interesting. Like I said, I think it's kind of weird how you can have infinite surface area but a defined volume, it makes no sense. Is that what you're- is that what you mean?
J.2.24	I: Yeah. Yeah, I wanted you to maybe, expand on your thoughts.
J.2.25	J: What I think I did wrong?
J.2.26	I: No, that's- that's not what I'm saying.
J.2.27	J: Okay, well, could you repeat the question?
J.2.28	I: Well, phrasing it as 'what you think you did wrong' is too directed.
J.2.29	J: Okay, um,
J.2.30	I: And I'm not making that statement, I'm just sort of asking, um...
J.2.31	J: What went wrong?
J.2.32	I: No.

J.2.33	I: No, not necessarily... Okay, so, do you- of- of the two things that are in conflict, which one of them feels more correct?
J.2.34	J: The volume being defined. Um, it feels more correct because when... okay, I visualize a horn in real life, and I visualize literally filling the horn, tipping the horn vertically and filling it with paint, and, to me, um, if I filled the horn, which went all the way, you know, the little thing got- the tip went all the way to infinity, to me, you know, in real life that's not possible, and so the horn has to end somewhere, and so, to me, the volume having some defined volume makes more sense than, you know, the surface area being infinite, which, you know, kind of [?]. So I'm literally going with a real-life, a real-life, uh, interpretation of- of this. Literally, you can't fill the horn more than- with more paint than when the tip of the horn reaches the diameter of the paint molecule. You know, once the diameter- once the-the horn- the tip of the horn moves further out to space than the diameter of a paint molecule, you can't fit any more. You can't fit any more paint, and so it's got to have some defined volume.
J.2.35	I: Okay. What if we were to simply consider the, uh, abstract mathematical volume without putting any paint in it?
J.2.36	J: I wouldn't really say that any of them are correct or incorrect, because- because mathematically they're, you know, that's... the law, I don't know if you want to call it the law, but the equations show what they show, and if they show what they show, you know, if they show that the surface area is infinite but the volume is defined, and not infinite, then that's what they show, you know, there's nothing we can really do about it, I mean, I'm sorry. So, graphically speaking, you know, neither of them seem really, you know, it's just like one over infinity. You know, infinity isn't a number, it's a concept, so how do you take the inverse of infinity? Or, how do you take the inverse of a concept? Well, it would seem kind of not correct to- to me, but I think it's perfectly fine to do that, and so I accept one over infinity as being zero, considering that if I plug in a really big number into my calculator, and take the inverse, it spits out some really tiny number that I more or less could consider zero. So I can't say that either [?] correct or not correct.

In this later interview portion, Joshua is asked what he thinks of the Gabriel's Horn problem, and characterizes it as not making sense. When asked, he says that the volume being defined makes more sense than the surface area being infinite from a physical perspective, reasoning that infinite objects do not exist in real life. Prompted to consider it from an abstract mathematical perspective, he says that he could not say that it is correct or incorrect, but that the equations show what they show; he does not reject it.

Overall, Joshua demonstrates a particularly high level of trust in mathematics, and perhaps in rules and systems generally. He also shows a lower level of trust in intuition, although it is unclear whether this is from having less of a sense of intuition or simply less trust or value put on it. The results here are quite different from those of WIjeratne and Zazkis; most of the students in their research did not readily separate physical or 'real-life' considerations (typically expressed geometrically) from the mathematics. Here, while Joshua does produce some contextual geometric analogies (J.2.34) as expected, he does not resist separating them from the abstract mathematical result, and again voices acceptance of its validity (J.2.35). The displayed preferences for structured formal results arising from rules over intuitive ideas fit with findings of people on the autism spectrum being less disposed to interpret concepts using prototypes and instead using a more systematic approach (discussed in Section 2.2). This could also be viewed in terms of Vygotskian compensation, where stronger scientific concepts are used in place of weaker everyday concepts. However, it also may be tied to Joshua's ideas of what counts as a mathematical solution (J.1.3), and thus could also be influenced by the mathematical context. Joshua's initial reaction to the re-examination of the problem is also somewhat suggestive of his view of mathematics, as he first thinks of it being about 'how he was wrong' (J.2.25), and continues this for several turns (J.2.31).

7.3. The Ping-Pong Ball Conundrum

As discussed in Section 6.1, the Ping-Pong Ball Conundrum involves an infinite set of ping-pong balls (numbered 1, 2, 3, …) being inserted into and removed from a barrel over one minute. In the first 30 seconds, the first 10 balls are inserted, and the '1' ball is removed. In the next 15 seconds, 11 through 20 are inserted and the '2' ball is removed, and so on. Joshua was told this and asked how many ping-pong balls remain in the barrel at the end of the minute.

Once the interviewer reads the ping-pong paradox to Joshua, he considers it:

J.3.1	J: Kind of hard to visualize that without, seeing the question, but I think I sort of get a grasp. So we got the first 10, uh, [short silence]
J.3.2	I: It's at the bottom here. [Interviewer shows the question printed on a screen, and Joshua reads it; the text shown is below.] Screen: Imagine you have an infinite set of ping-pong balls numbered 1, 2, 3, . . ., and a very large barrel; you are about to embark on an experiment. The experiment will last for exactly 1 minute, no more, no less. Your task is to place the first 10 balls into the barrel and then remove number 1 in 30 seconds. In half of the remaining time, you place balls 11 to 20 into the barrel, and remove ball number 2. Next, in half the remaining time (and working more and more quickly), place balls 21 to 30 into the barrel, and remove ball number 3. Continue this task ad infinitum. After 60 seconds, at the end of the experiment, how many ping-pong balls remain in the barrel?
J.3.3	J: Wouldn't you have no ping-pong balls at the end in the barrel?
J.3.4	I: Okay, why?
J.3.5	J: Because, how to approach this, because what I'm seeing is that the time is shrinking, and, the first thing that comes to mind is, as we move to infinity, the limit as time is approaching zero, and, by that time you'll reach a point where you're removing, ah, how do I explain this? I'm kind of having difficulty putting it in words but, but, the point is, it's like, take in, remove one, or, put in, put one in, take one out, put one in, and then, as you shrink, as the time interval gets so tiny, what happens is that, eventually, you reach a point where, it's one negative zero- negative one, negative, you know
J.3.6	I: Well, we put in ten, and take out one.
J.3.7	J: Yeah, we put in ten, take out one, and then in half the time, we put in, ten again? […]

J.3.8	I: Yeah. We're doing the same operation, in smaller and smaller windows of time.
J.3.9	J: Oh. Well, sorry, I guess I misread the question, wouldn't technically you then have infinity balls in the barrel? Infinity minus, whatever, you know, you're taking out, which is still equal to infinity?
J.3.10	I: Okay, and explain why that would be.
J.3.11	J: Because at a certain point, you're going so quick that, the number of balls, the way I'm picturing this is that it's basically the reverse of what I said at the beginning, the number of balls is rising exponentially, minus whatever you're subtracting, but the point is that the net gain, is somehow always exponential, to the point where it reaches infinity. That's how I'm picturing it. Because remember that as you shrink those time intervals, you're still putting in ten balls, and whatnot, minus one, so, nine balls.
J.3.12	I: Okay.
J.3.13	J: You're putting in nine balls, but the point is that you're shrinking the time, uh, you're shrinking the time intervals so tiny that, you get to a point where, yeah, it's basically infinity.
J.3.14	I: Okay.
J.3.15	J: That's a cool one, I actually like that one. More than the [Hilbert] hotel.

Here, Joshua initially believes that no balls remain, but this turns out to be based on a misreading of the question (that the number of balls added and removed at each step were both one); once it is clarified, he concludes that the number is rising exponentially, and that it reaches infinity.

While Joshua asks for a visualization of the question (J.3.1), what he was shown was the question simply written in text. That he found this helpful, and did not ask for any further visualization, suggests that Joshua may process the written problem (or writing in general) more visually, which could tie in to his general disposition toward visualization.

The interpretation Joshua gives here that the number of balls is rising exponentially appears to be based on the size of the time intervals (J.3.11). This contrasts with a common similar response from the work of Mamolo and Zazkis (2008) where respondents got to infinity in the form of “nine infinity”, which reflects linear growth. It also fits with the responses found by Ely (2011), where students also generally came to infinite conclusions. However, despite this increased attention to the time-based setup of the problem, Joshua does not give a physical objection to the inherent plausibility of the problem.

Next, the interviewer introduces the paradox in the standard solution to Joshua:

J.3.16	I: Okay, so, if- so if we’ve, got an infinite number of balls in there, which are numbered, all the balls are numbered, 1, 2, 3, dot dot dot, right?
J.3.17	J: Mm-hm.
J.3.18	I: So, name me a ball that’s in there.
J.3.19	J: 998? […]
J.3.20	I: Well, ball number 998 isn’t in there. We took it out. At the 998th iteration of our action. [Joshua suggests ball number 1 and ball number 3 in similar exchanges.]
J.3.21	J: Oh, I see, it’s a sequences and series, ah. I see. Okay. So technically, there’s really no defined ball that we haven’t taken out. We have, I guess, but even then, maybe my first answer was correct, because if you’re saying there’s no defined ball in there, then, it’s a question as to whether something is defined in there. And if you’re saying that every defined ball was removed, as the time shrunk to zero or approaching zero, then technically there’s infinity minus, whatever’s defined, which is zero.
J.3.22	I: Well, we can’t really do infinity minus infinity type stuff.
J.3.23	J: Okay.
J.3.24	I: And, well, weird things like this are sort of part of why you can’t do that. Because there isn’t a simple answer to it.
J.3.25	J: That’s really interesting. Um, yep. So technically, it would be zero. I mean, because you’re taking out every defined ball. […]
J.3.26	I: And yet, at every step, we’re putting in more than we’re taking out. So this seems sort of strange, doesn’t it?

J.3.27	J: It seems very strange, yeah, it's very, ah, something that would introduce a lot of arguments in the math community. As to what the correct answer is. But, I mean, yeah, I'm kind of, I see exactly what you're saying.
J.3.28	I: Okay.
J.3.29	J: I totally see the point that you're trying to make.
J.3.30	I: What do you think the point I'm trying to make is?
J.3.31	J: That whatever you put in, eventually you take out.
J.3.32	I: Yeah.
J.3.33	J: So, there's no real net gain, because whatever you're putting in, eventually is coming out. It almost actually kind of makes me think of the world population. And how each person is given a name and whatnot. And, okay, they're getting put in, but eventually, they're getting taken out. That's actually, see, but then again, the world population example, so you'd think that there'd be zero people, but even though one person is getting put in and eventually getting taken out, there's always people.
J.3.34	I: Well, we haven't reached the metaphorical end of the 60-second experiment yet, you see.
J.3.35	J: And see, that's what I'm thinking about now, now it's a question of the time frame. Mm, that's a good one. I like that one.

Here, Joshua is asked to name a ball that is present in the barrel, and he tries 998, 1, and 3. After this, he sees it as a sequences and series problem, and concludes that there are no balls left in the barrel. Joshua agrees that it is strange when asked, and suggest that it would lead to arguments in the mathematical community. He gives an analogy that there is no net gain, and compares it to the world population.

Here, while Joshua only tries relatively small natural numbers, after three tries he starts to see the problem in terms of sequences and series (J.3.21). This may be somewhat expected, since Joshua was taking the part of the calculus sequence that includes sequences and series at the time of the interview. It is notable that Joshua takes little prompting and no formal explanation of the proof by contradiction to arrive at the correct conclusion, that all of the balls have been removed. This combined with his remark that the solution would cause arguments in the mathematical community (J.3.27) suggests that he sees a kind of conflict here. However, it appears that conflict is with something that he expects from other people but does not have a problem with himself, possibly something typically categorized as intuitive reasoning. The remark at J.3.27 is also correct; as discussed in Section 6.1.1, there are in fact mathematical publications that could be construed as "arguments in the math[ematical] community." This accurate prediction goes against some of the more severely negative predictions based on a theory of mind deficit conception of autism, that people on the autism spectrum might be unable to consider the thoughts and arguments of others in this way. Joshua's positive view of the problem (J.3.35) also contrasts with the students of Mamolo and Zazkis, so his perspective appears unusual in multiple ways.

7.4. Summary of Joshua's Interviews

Two separate instances of compensation could be extrapolated from the analysis of Joshua's interviews. First, the use of the geometric interpretations could be using a strong skill at geometric/visual reasoning to compensate for problems in skills related to algebra (which Joshua hints at in J.1.40, although from the interview portions examined here it is not entirely clear what the troublesome parts are). This could serve as a strong tool for learning if the connections between geometric and algebraic interpretations are reinforced instead of dismissed, and such an approach in a classroom setting has the potential to enrich learning for the class as a whole.

Second, a general problem with or mistrust of intuitive reasoning may lead to a kind of compensation where knowledge that is learned explicitly as a scientific concept, instead of formed as an everyday concept, is emphasized. Since mathematics coursework in a traditional academic setting uses scientific concepts explicitly, and the role of everyday concepts is often viewed as interference to be minimized, this trait would most likely be helpful in the context of a standard mathematics classroom. By contrast, some alternative ideas for mathematics instruction involving the use of students' pre-existing real-world concepts might produce unusual and less helpful results for some students on the autism spectrum.

Chapter 8. Interviews with Cyrus

This chapter summarizes and analyzes interviews conducted with Cyrus. As detailed in Chapter 5, Cyrus had completed a bachelor's degree in mathematics at the time of the interviews. Again, the chosen excerpts of these interviews are a representative subset of the interviews as a whole, highlighting certain elements which I want to analyze in my analysis.

8.1. Magic Carpet Tasks Revisited

Here, Cyrus was given the same Magic Carpet tasks from Wawro et al. (2012) as were given to Joshua in the previous chapter. While Cyrus had taken courses in linear algebra, this occurred several years prior to the interviews. However, he immediately went to trying an algebraic method of solution. In fact, at no point did he produce a sketch of any kind, even though the problem was presented initially with a drawing of the destination point (although the original presentation from Wawro et al. did not do this). All written work from Cyrus was algebraic, as seen in Figure 8.1; the axes and point were presented by the interviewer and not used by Cyrus.

C.1.1	C: No, you c- I think you only can with the second one, because a hundred and seven is not divisible by- that's a hundred and seven, right?
C.1.2	I: Yeah.
C.1.3	C: A hundred and seven is not divisible by three.
C.1.4	I: We can use these in combination.
C.1.5	C: Oh, we can use,
C.1.6	I: Like we can use part of this one and part of that one. […]
C.1.7	C: Okay, um, so then that means... hmm... I'm not sure it can be done, hold on, think, ...okay. [?] am I allowed to write down on this?

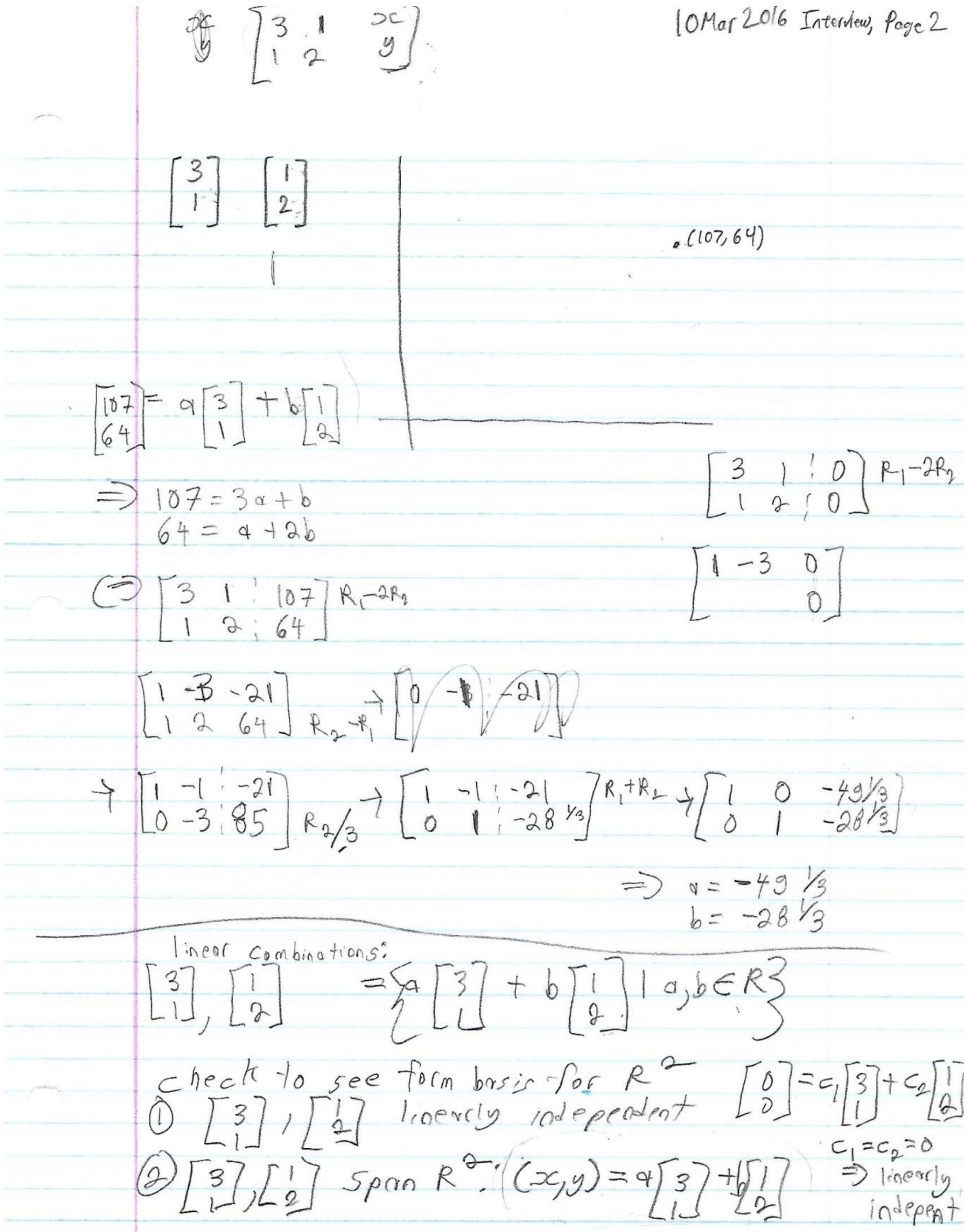

Figure 8.1: Cyrus' Magic Carpet Solution

In this segment, Cyrus suggests first that you would need to use the second mode of transportation (the magic carpet), because 107 (the x-coordinate of the destination) is not divisible by 3 (the x component of the hoverboard's vector). The interviewer clarifies that the two modes can be used in combination, and then Cyrus starts to calculate in writing.

With this first attempt at thinking about the problem, when Cyrus erroneously believes that the magic carpet and hoverboard cannot be used in combination, it appears that he is thinking about eliminating the first one as a possibility. However, for someone investigating the problem that way, it is somewhat surprising to see the concept of divisibility invoked (C.1.3) more readily than thinking about an extended form of the vectors.

Once Cyrus has done some of the calculations, he explains his thought process:

C.1.8	C: I'm treating it as, a, matrix algebra problem. Solving systems of linear equations. I'm going through the- uh, what do you call it, doing row operations to find out if this system actually has a solution or not.
C.1.9	I: Okay.
C.1.10	C: And for this problem I think both a and b have to be, they both have to be whole numbers, and definitely nothing negative, for it to work.
C.1.11	I: Oh, ah, I should mention that, ah, the context of our problem actually doesn't require that. Uh, we could ride the magic carpet for, like, half a time unit,
C.1.12	C: Oh, you can. [?]
C.1.13	I: or we could ride it backwards. […]
C.1.14	C: Okay, um, […] that's a mistake, alright, so...
C.1.15	I: This is- is this a solution method that you, recall from, a course, or something like that?
C.1.16	C: Yeah, this is a solution method I recall when I took a second year linear algebra course.

Here, Cyrus says that he is using matrix algebra, and speculates that both of his coefficients should be whole numbers, and not negative. The interviewer says that neither of those are required, and asks if he recalls this method from a particular course. Cyrus confirms that he remembers it from a linear algebra course.

It is notable that although Cyrus hasn’t been exposed to linear algebra for some time, the matrix solution is his first thought (C.1.8). In this and other problems, it generally appears that if Cyrus at least partially recalls a solution method that he has used previously, he tends to start by recalling and using it first, rather than trying to work it out again from more basic principles.

Cyrus arrived at a single solution, which he considers:

C.1.17	C: Okay, I got one unique solution [a = -49 1/3 and b = -28 1/3, written as mixed numbers], but I'm really not sure it makes sense. Either I did something wrong somewhere along the way, or, I don't know, just based on the- I'm just gonna guess it's not possible to do this.
C.1.18	I: Hmm... ah, if you got a solution that did seem reasonable, then would your conclusion be different?
C.1.19	C: Yes. If these two numbers [from the solution above] were, um, if these two numbers were different, I would say it would make sense.
C.1.20	I: Okay, but the- these numbers that you have, they don't make sense, ah, why in particular?
C.1.21	C: Well they're both- they're both negative, that's why.
C.1.22	I: Okay. So that would take you off, over there.
C.1.23	C: Yeah.
C.1.24	I: Quadrant three. And that's right, that doesn't make sense.
C.1.25	C: Mm-hm. I suppose that's a little bit of geometric thinking on my part there, 'cause you'd be all the way in quadrant, I forget it's one, two, three, four I think isn't it?
C.1.26	I: Yeah.
C.1.27	C: You'd be all the way in quadrant three.

Cyrus says that his result does not make sense, and concludes from this that reaching the house as the problem asks is not possible. Asked what about the solution does not make sense, he says that it is that both values are negative, and that this would go into quadrant III on the graph.

The discussion of what it implied, and what a reasonable solution might imply, marked the only instance of any use of geometric methods of reasoning, which Cyrus remarks on (C.1.25). However, Cyrus only discusses it in this way after the interviewer has used those terms (C.1.22, C.1.24), so he may have thought about it in another form initially. However, he does check the naming of the quadrants himself (C.1.25) rather than using the interviewer's earlier naming of the third quadrant (C.1.24), using the input as more of a suggestion of direction than an answer in itself.

The concluding discussion of what sort of solution would indicate that reaching the point was possible returns to purely algebraic terms:

C.1.28	C: Okay, that's fine. Okay, so, yeah, that's pretty much what I make of this problem. If we were able to find- there would be two cases. If these had both been positive numbers and there was a unique solution, then definitely there is- this is- the problem was asking if you can- you can reach this point. Using either the carpet or the hover-thingy.
C.1.29	I: Not either-or. We can use part of one then part of the other. […]
C.1.30	C: But can we reach this as our destination and, if there was either one unique solution and it must be positive, or if there was infinitely many solutions, in either of those cases it would be possible to do this.

Here, we see that even approaching the problem from a hypothetical standpoint, the interpretations provided are still entirely algebraic. Since Cyrus stated (C.1.16) that this was a method recalled from a linear algebra course, we can see evidence for both an inclination toward algebraic methods and an inclination for the use of recollection of previous solution techniques, which both stand in contrast to Joshua's earlier solution. This approach fits with the majority of students in the study from Wawro et al. (2012), although those students presumably did not have previous exposure to the method that Cyrus is recalling here. Cyrus' reaction to the negative solution he obtained, combined with his correct interpretation of the hypotheticals that he gives, shows that he is not approaching the algebraic problem in a way entirely detached from the original problem's context. However, since he does not bring up the geometric interpretation of that context independently, it is uncertain whether he is more inclined to translate the solution into terms of the geometric context, or to mentally translate the context into algebraic terms.

8.2. Card-Sorting Tasks

In this task, I gave Cyrus the four-card sorting problem as used by Johnson-Laird, Legrenzi & Legrenzi (1972). As described in Chapter 5, this task involves a participant being given sets of cards with two distinct sides where some cards display the ‘front’ and some display the ‘back’, and a set of logical rules to check given in an ‘if-then’ format. Both the abstract and real-world versions of the problem were used in the interview, in that order. For the abstract version, the rule given to test was “If a card has an A on one side, then it has a 3 on the other side” and drew cards with ‘A’, ‘B’, ‘3’, and ‘2’ on them (it was stated that each card had a letter on one side and a number on the other). For the real-world version, the rule given was “if an envelope is sealed, then it has a five-cent stamp on it”.

Cyrus was given the first version, where cards all had letters on one side and numbers on the other, drawings of the four cards, and the rule (both orally and written on the board):

C.2.1	I: And the question is, which of the cards do I need to flip over in order to check if this rule is true?
C.2.2	C: Oh, if a card has A on one, then it has a 3 on the other side. Off the top of my head, I think there’s only one card you can turn over to check this rule, and that’s A. Turn it over to see if there’s a 3 on the other side.
C.2.3	I: Can? Or should or have to? Because there’s some nuances there, and I’m not quite sure how you mean it.
C.2.4	C: According to that rule, there’s no guarantee that, if you have a 3 on one side, then there’s going to be an A. Because this rule is not transitive, from the way it’s being described. So, what we want to do is, I’ll say we must check the A, to make sure there’s a 3 on the other side.
C.2.5	I: Okay.

C.2.6	C: It wouldn't do us any good to check if the 3, if there's an A on the other side of the 3 because, the rule didn't stipulate about the 3s. There might not necessarily be, what's the word, in terms of functions I guess this would be, one-to-one thing.
C.2.7	I: Okay.
C.2.8	C: Right. But on the flip side, hold on, the way you phrased it in the first place was, what should I do to check this rule. If I can only, can I only do one thing, or? […]
C.2.9	I: Ah, you can check as many cards as you like.
C.2.10	C: Okay.
C.2.11	I: I'm just asking which cards could you check that would give you useful information.
C.2.12	C: Okay, so the, first one is the must, you have to check, the A on the other side. And then, after that, checking, well, that's the thing. Checking the 3 would help, but it wouldn't confirm the rule, if there was an A on the other side of the 3.
C.2.13	I: Hm.
C.2.14	C: 'Cause there might not be.
C.2.15	I: That's an interesting thing to say. Ah, if it couldn't confirm it, then how does it help?
C.2.16	C: Okay, now… okay, well, the other, okay, so the other thing to do would be to check, the other two cards, B and 2, to see if they have an A on the other side. Supposing that, supposing one of those had an A on the other side, then that would- that would break the, that would violate the rule.
C.2.17	I: Um, one thing, ah, that, it sounds, from the way you're phrasing that it sounds like there's something that may have slipped about the problem conditions. We're told that each of these cards has a letter on one side and a number on one side.
C.2.18	C: Oh, okay.
C.2.19	I: So, the B card cannot have an A on the other side.

C.2.20	C: Okay, okay, right. So then, you would check the, okay. You would check the A, to see if there's a 3 on the other side, and then you would check the 2 to make sure there's no A on the other side. 'Cause then that would- if you found an A on the other side of the 2 then that would violate the rule it- in itself.
C.2.21	I: Okay. And... the other cards?
C.2.22	C: Well, checking the B doesn't confirm the rule either because, if, let's say there was a 3 on the other side of the B, then that's still valid. That's still consistent with the rule. And then checking the A, that there's an A or not an A on the other side of the 3, again, it doesn't confirm the rule, because whether there's an A or not an A on the other side of 3, both situations are allowed.
C.2.23	I: Mm. Okay.
C.2.24	C: Yeah, so, checking if there's an A on the other side of the 2, that would break the hypothesis, right there.

In this section, Cyrus is given the problem and immediately goes to checking the A to see if a 3 is on the other side. He says it is not necessary to check the 3, and asks if the problem allows checking more than one card. Told that it does, he says checking the 3 would help, but wouldn't confirm the rule. Asked how it would help, he doesn't respond directly, but goes to suggesting that we check the other two cards, B and 2, to see if there is an A on the other side. The interviewer reminds him that the cards all have one side as a letter and one as a number, and he concludes that we only need to check the 2.

Cyrus's assertion that checking the 3 would "help" but not confirm the rule (C.2.12) seems unusual, but the request for clarification is not answered and it appears to have been dropped. This is essentially the only deviation from the 'standard' mathematical solution in Cyrus' answer. By contrast, only seven of the twenty-four students in Johnson-Laird, Legrenzi, and Legrenzi's study produced a correct answer in this formulation.

Next, the interviewer introduces the applied version of the problem. It is presented mostly in the same way as the original in Johnson-Laird, Legrenzi, and Legrezi's work, but changes the stamp cost into local currency (Cyrus was asked to give the value for this). The problem is drawn and explained, with the rule "If an envelope is sealed, then it has a five-cent stamp on it" written. There are five envelopes: the first is face-up with a four-cent stamp, the second is face-up with a five-cent stamp, the third is face-down and sealed, the fourth is face-down and unsealed, and the fifth is face-up with no stamp. Cyrus is asked which envelopes to check, as before:

C.2.25	C: Okay, and the question is how do we go about checking this rule?
C.2.26	I: Yeah. Which ones do we need to check?
C.2.27	C: Okay. So, for, for number, five, uh, sorry, for sure we need to check the third one, on the other side, to see if there's a five-cent stamp on it. That one's a must.
C.2.28	I: Okay.
C.2.29	C: And then the other one is… we also need to check the first one, to see if it's sealed, because if it is sealed, then that violates the rule.
C.2.30	I: Okay.
C.2.31	C: And then, same with number five, actually, yeah, we need to check that one as well, just to make sure, hold on, yeah, we need to check the fifth one, to make sure that it is not sealed. Otherwise, that- that would violate the rule, if it was not sealed. Sorry, if number five was sealed, then it violates the rule.
C.2.32	I: Okay.
C.2.33	C: And then, number four doesn't really matter, because it could have, either a five-cent stamp or a different stamp on the other side. So if it's open, then it doesn't matter.
C.2.34	I: Okay. Hmm, it seemed like the only envelope you haven't talked about either way is number 2. So what do you think of envelope number 2?
C.2.35	C: Oh, it doesn't matter, because, hold on, actually, yeah it wouldn't matter if it's sealed or not sealed. It could be in either state.

Here, Cyrus says that we need to check envelopes three, one, and five, but that four and two don't matter (and don't need to be checked).

This generally proceeds the same way as the first version, even using the same terms for the same kinds of logical constructs; the A card (C.2.12) and the sealed envelope (C.2.27) are termed “a must” while the 2 card and the envelopes without five-cent stamps can “violate the rule” (C.2.20, C.2.29). So, it appears that these two constructs are each thought of differently, and each is thought of consistently across the two variations of the problem. Unlike with the students from Johnson-Laird, Legrenzi, and Legrenzi’s study, it appears that Cyrus views both the abstract and applied versions of the problem in the same way. This contrasts somewhat with a part of the previous problem (C.1.19) where Cyrus rejects a solution based on its contextual problems.

Next, the interviewer adds another version of the statement, “An envelope is sealed only if it has a five-cent stamp on it”, writing this on the board:

C.2.36	C: An envelope is sealed only if it has a five-cent stamp on it. Okay, so in this case, an envelope is sealed only if it has a five-cent stamp on it, okay so for this one you still have to check number 3, to see if there’s a five-cent stamp on the other side. And then number 2, you could still be either way.
C.2.37	I: Okay, and, any others?
C.2.38	C: Um, the first one, okay, so, if the first one was sealed, then, that would violate the rule.
C.2.39	I: Okay.
C.2.40	C: And then, and then number 4 could be either way, it could have a five-cent stamp, or a different stamp, and then, number 5, an envelope is sealed, oh wait, actually, number 5, that would violate the rule if that was sealed, since it doesn’t have a five-cent stamp.
C.2.41	I: Okay.
C.2.42	C: So both number 1 and number 5 violate the rule if they were sealed.
C.2.43	I: I think you started off mentioning number 3?
C.2.44	C: Yeah, and number 3 must be sealed, or else it violates the rule.
C.2.45	I: Well, number 3 is sealed. It’s-
C.2.46	C: Sorry, if number 3, doesn’t have a five-cent stamp on it, on the other side, then it violates the rule.

C.2.47	I: Okay, so how does this rule compare to the other one, then?
C.2.48	C: Oh, shoot.
C.2.49	I: Because it seems like you checked the same things.
C.2.50	C: Yeah. They can't be exactly the same, is that, ah, I almost forget,
C.2.51	I: Should I write down the first one again?
C.2.52	C: Yeah.
C.2.53	I: Okay. [The interviewer rewrites the first condition.] […]
C.2.54	C: Okay, right, so for this one, number 4, the difference is number 4 it doesn't matter what's on the other side. Or, sorry, I mean, actually, okay, so this would be the same regarding the first one, and the third one, yeah, so, again, with this one, the first one, if it is sealed on the other side, then it violates the rule. And then number 2 can be either sealed or not sealed on the other side, for both of them.
C.2.55	I: Okay.
C.2.56	C: And then number 4, hold on, wait, number 4 doesn't matter either way, actually, okay, I think the difference is number 5. So, for the first one, an envelope is sealed only if it has a five-cent stamp on it, if number 5 was sealed, then that would violate the rule, according to the first one.
C.2.57	I: Uh-huh.
C.2.58	C: But according to the second one written there, if an envelope is sealed then it has a five-cent stamp on it, actually, hold on. Actually, number 5 can't be s- never mind, number 5 can't be s- I think they're both exactly the same, then.
C.2.59	I: Okay. Mm, so, that's sort of, part of what I was going to ask you. So it seems like, they're equivalent for all of these.
C.2.60	C: Yeah.
C.2.61	I: So if they're equivalent for all of these, ah, is there any way in which they're different?
C.2.62	C: An envelope is sealed only if it has, I think the wording's just different. Actually, no, they would be, I think one is the converse of the other one, isn't it? And then, but they just happen to be the same in this case.

C.2.63	I: Hmm. Ah, well, the lower one is, easier to, take the converse of, because it's an if-then.
C.2.64	C: Yeah.
C.2.65	I: What would the converse of that be?
C.2.66	C: If- it would be, if, if an envelope has a five-cent stamp on it, then it is sealed.
C.2.67	I: Okay. But is that equivalent to your statement, on these envelopes?
C.2.68	C: No, it isn't, actually. That would not be the same as the first one.
C.2.69	I: Okay. So it's not the converse.
C.2.70	C: Well, I think they're equivalent, then.

Cyrus checks all of the envelopes against this new rule, and comes to the same conclusion as the first time. Asked to compare this with the first rule, Cyrus first says that they can't be exactly the same. The first rule is written again, and he checks several envelopes against both of them and concludes that they are the same. Asked if there is any way in which they are different, he first supposes that it is just the wording, and then that one is the converse of the other and they only happen to coincide. The interviewer then asks him to find the converse of the first statement, and he does, and finds that it is not equivalent, returning to the conclusion that the two given statements are the same.

The method used to check this rule and if it is equivalent at first (C.2.36-58) by checking each envelope individually has some resemblance to methods using truth tables, although Cyrus does not explicitly state that the cases presented fill out all of the logical possibilities to check.

After being given both tasks, Cyrus was asked to compare the experiences:

C.2.71	I: Alright. And how would you compare looking at these rules to the first one we did with the A, B, 3, 2?
C.2.72	C: How would I compare these rules,
C.2.73	I: Yeah, like, the experience of checking them.
C.2.74	C: It's pretty similar. Only this is a bit more complicated, there's more envelopes,

C.2.75	I: Uh, well if I ditched the last one.
C.2.76	C: Okay, it's roughly the same thing, because you're only checking for, one thing on the other side.
C.2.77	I: Okay. While you were doing it, did the process feel like, any more or less complicated with the envelope?
C.2.78	C: It felt a bit more challenging with the envelopes, than with the cards.
C.2.79	I: Do you have any idea as to why that might be the case?
C.2.80	C: I'm not sure, it seems, when you look at the way the rules are phrased, it seems like it should be more or less the same, yeah I don't really know.

After the interviewer asks Cyrus to compare the experiences of doing the two problems, he states that they are generally similar, noting that the second problem (the real-world version) had more envelopes. The interviewer asks him to ignore the last one, and given this Cyrus says that they are roughly the same, except that the envelopes felt more challenging.

While the result found in the study with a typical population was that the real-world condition was generally easier, this did not appear to be the case here. Cyrus appeared equally successful at both versions, and in fact stated (C.2.78) that he thought the real-world version was more difficult. While the increase in number of envelopes and the fact that multiple questions were asked in that formulation would contribute to this, given the large effect in the other direction found by Johnson-Laird, Legrenzi, and Legrenzi, this is still a rather unusual result. One possibility for the effect in the general population is that the real-world situation shifted their interpretation toward a more conversational one (as described by Grice's maxims) rather than an approach informed by a more direct attempt at mathematical logic. If so, this could be an instance of Vygotskian compensation using a stronger understanding of mathematical logic in both cases.

8.3. The Ping-Pong Ball Conundrum (II)

As discussed in Section 6.1, and used previously with Joshua, the Ping-Pong Ball Conundrum involves an infinite set of ping-pong balls (numbered 1, 2, 3, …) being inserted into and removed from a barrel over one minute. In the first 30 seconds, the first 10 balls are inserted, and the '1' ball is removed. In the next 15 seconds, 11 through 20 are inserted and the '2' ball is removed, and so on. The participant is told this and asked how many ping-pong balls remain in the barrel at the end of the minute.

Here, the interviewer reads the problem to Cyrus, and then shows him a printed version of the problem. There are multiple rounds of explanation as the problem is presented to Cyrus before he understands the problem correctly and addresses it.

C.3.33	C: Okay, but if that keeps going on, then, but we're eventually going to have an infinite number of balls in there, but it just depends on how many, if we go through n successions of this, we're just not going to have the first n in the basket.
C.3.34	I: Alright.
C.3.35	C: So, but it's still an infinite number of balls, we're just not going to have, 1 through n in there.
C.3.36	I: If n goes to infinity, what does not having the first n mean?
C.3.41	C: I don't know. That means we've got no balls in there whatsoever. That sounds like the intuitive answer, but I'm just not certain about that.
C.3.42	I: Hm. Ah, that's the intuitive answer for what reason?
C.3.43	C: Um, that still doesn't sound right, but you'd still, even if n went to infinity, you'd still have an infinite number in there. Yeah, that's what I would say. I don't know what my reasoning is for it, I think just like you could, just like in [the Hilbert] hotel problem you could keep placing more and more in there. Well, okay, something similar to that. In this case you could just keep adding more and more balls in there, even after, going through an infinite number of times. But except, oh my god it doesn't, yet it still seems to kind of contradict itself. Because if n now encompasses all the numbers, and you're taking them all away, then how can you have an infinite number of balls still in the basket? So,

	I don't know, I'm going to say there's zero balls in there, at that point, if you take the limit as n goes to infinity, I'm going to say there's zero balls.
C.3.44	I: Okay.
C.3.45	C: I know I sort of changed around on that one.
C.3.46	I: Alright. Mm, but at this point would you say that that is what you're sticking with?
C.3.47	C: Yeah, I'm going to stick with that, if n goes to infinity, then you have zero balls left.
C.3.48	I: Okay, and does that seem like, a reasonable sort of thing, or does it seem kind of weird, or, ah, is it something that you would accept?
C.3.49	C: I think it seems weird, but I would accept it.

At first, Cyrus looks at the time intervals, finding the length of the first four and suggesting that they diverge as a series. The interviewer clarifies that the focus of the problem is on the number of balls, and describes what is in the bin during the first two time intervals. Cyrus extrapolates from this that as each time interval progresses, more and more balls will be in the barrel, but the first n will be missing. The interviewer asks what this means as n approaches infinity (which may be considered either the number of steps performed or the number of balls removed: since these are always equal, it is not clear from the statements which conception Cyrus is focusing on), and Cyrus says that means there will be no balls in the barrel. At first he calls this an intuitive answer; thinking further, he switches to there being an infinite number and then back to zero (without any intervention). He describes it as weird, but is willing to accept it.

The brief characterization of the zero answer as intuitive (C.3.41) is unusual, and may suggest multiple layers of reasoning that Cyrus considers 'intuitive'. However, while this does not last, the intuitive conclusions are still not held by Cyrus to be particularly important for a final conclusion (C.3.49).

Next, the interviewer presents Cyrus with some alternative arguments made by other students:

C.3.50	I: Okay. And, there are, I think we have them here, ah, couple of arguments

	that, ah, different students had, ah, trying to work out this particular problem, and I'd like to tell you about a couple of those, and see what you think about them.
C.3.51	C: Okay.
C.3.52	I: Okay, so, one argument about this was that, for each chunk, you're essentially adding nine balls.
C.3.53	C: Mm-hm.
C.3.54	I: So the total amount at any time should be nine times, the amount of chunks you've gone through.
C.3.55	C: Mm-hm.
C.3.56	I: And, but that goes to infinity.
C.3.57	C: Right, okay, so, you add nine but, right, you add nine balls, okay.
C.3.58	I: So does that seem correct or incorrect, and, why? How does that argument sound to you?
C.3.59	C: So, it's like, nine times n, okay. And if n goes to infinity, from that one, the answer would clearly be it just blows up, if n goes to infinity, it would be infinite. You'd have infinite number of balls left in there. So, but it still doesn't seem to make sense to me when I try to actually predict in my head what's going on there, it doesn't really seem to make sense with that. I would say that answer doesn't make as much sense. To me it makes sense as long as you have a finite number of time intervals.
C.3.60	I: Okay. And why doesn't this work in the infinite case?
C.3.61	C: My only reasoning is somehow it doesn't make sense to me, once you've already taken away, essentially, once you've taken away every single natural number, then you can't have anything left.
C.3.62	I: Mm. Okay, well,
C.3.63	C: Yeah, now I'm really confused, I'm just not sure if there's a correct answer to this or not.
C.3.64	I: That's sort of, ah, the second argument, where we ask, okay, if there are balls remaining, ah, all our balls are numbered by natural numbers,

C.3.65	C: Mm-hm.
C.3.66	I: So, if there's some balls remaining, what are they? Name one.
C.3.67	C: Okay. So, if there are some balls remaining, then what are they. Name one. Okay. Um, so then, I don't know, my- my reasoning followed the case where you'd have nothing left at the end.
C.3.68	I: Right. Mm, which- yeah, that's- that's correct. And that's why that works, is it- there aren't any.
C.3.69	C: Mm-hm.
C.3.70	I: You can't name one.
C.3.71	C: Right.
C.3.72	I: This- this is sort of, the, ah, contradiction proof version of proving this.
C.3.73	C: Okay.
C.3.74	I: Where you go, okay, suppose by way of contradiction, that there are some balls left.
C.3.75	C: Mm-hm.
C.3.76	I: Ah, then- since this is a set of natural numbers, it must have a least element. Call it n.
C.3.77	C: Right, okay.
C.3.78	I: But then, in the nth step, we've removed that.
C.3.79	C: Mm-hm.
C.3.80	I: So we don't have that.
C.3.81	C: Right, okay.
C.3.82	I: Contradiction.
C.3.83	C: That's a proof by contradiction, I see.
C.3.84	I: Yeah. Therefore, there are no balls.
C.3.85	C: Mm. Ah, okay. And, okay, but I wouldn't have really thought of that- not in that way, at least, but it makes sense, once you've- go to the next step, you've just removed the one that's remaining.
C.3.86	I: Okay.

In this segment, Cyrus is first presented with the argument that, since we add nine balls each time, there should be infinitely many at the end, and asked what he thinks of it. He finds it to make less sense, but after being presented with it, is uncertain if there is a correct answer. He is then presented with the proof by contradiction argument for there being zero balls, and agrees that one cannot name a single ball remaining, though he says he would not have considered it in that way.

While Cyrus' conclusion agrees with the proof by contradiction conclusion, he says that he would not have viewed it that way (C.3.85). This may be related to the first intuitive answer he gave earlier (C.3.41), not having the 'infinity' conclusion as something to start off with as reasonable to contradict, which suggests that Cyrus may view the problem in an unusual way which is more conducive to the ultimately correct solution. In fact, not only does Cyrus not reach that as a conclusion, he unusually characterizes it as making less sense (C.3.59), while most typical students have the opposite view. However, he does agree that such a solution is valid for any finite case. From this, it appears that Cyrus does not have the unexamined continuity assumption that many students use to extrapolate to the infinite case in this problem, or may have learned to ignore it. If it is ignored, this is also noteworthy, since while Cyrus does have formal mathematical training, others at the same or higher level of formal mathematical training still did not have this response, as found by Ely (2011).

8.4. Gabriel's Horn (II)

This paradox, as described in previous chapters, involves the surface of revolution of the function 1/x from one to infinity about the x-axis. The area of this surface is infinite, but the volume is finite. This task was presented to Cyrus in a similar way to how it was presented to Joshua: while both were given the opportunity to attempt the calculations, neither completed them independently, so the result was shown (as the techniques of integration were not considered a fundamental part of the paradox).

As the task is presented to Cyrus, he says that he remembers the relevant integrals (for area and volume), although it was not recent.

C.4.1	I: Gabriel's Horn is the shape generated by rotating one over x about the x axis from one to infinity.
C.4.2	C: Rotating it from about one to infinity, okay.
C.4.3	I: So we've got that space there.
C.4.4	C: Mm-hm.
C.4.5	I: And, we have this object and it has a surface area and volume. Ah, do you happen to recall the methods of integration that would find you the surface area and volume of this object?
C.4.6	C: Um, kind of, I just vaguely remember a triple integral.
C.4.7	I: Ah.
C.4.8	C: I don't remember for this exact one, but for other, when you rotate stuff and you find volumes.
C.4.9	I: Yeah. Well, that's the Calc III method.
C.4.10	C: Mm-hm.
C.4.11	I: Typically, this one is done using a more Calc II method [of rotation]. We have, well,
C.4.12	C: Okay, so something to do with the area, okay, it looks like you're copying down a formula,
C.4.13	I: Right. Yeah, that's true.
C.4.14	I: Okay. So, we have a volume and a surface area integral.
C.4.15	C: Okay. Volume and a surface area integral. Okay.
C.4.16	I: Yeah. Now, our sort of, question that we look at, involving this, is, ah, let's say we wanted to paint Gabriel's Horn.
C.4.17	C: Mm-hm.
C.4.18	I: How much paint could we need? Ah- well, um,
C.4.19	C: If we wanted to paint Gabriel's Horn, how much paint would we need?
C.4.20	I: Yeah.
C.4.21	C: Hold on, let me think. I can't think straight right now. My guess is that all we really need is the surface area's amount.
C.4.22	I: Okay. And what if we were to do it based on the volume? Like, if we filled up

	the entire volume with it, would that work also?
C.4.23	C: Um, yes, but wouldn't that be excessive?
C.4.24	I: Sure, sure, granted. But it seems like it would work, yeah?
C.4.25	C: I think it would work, yeah.

Here, the interviewer introduces Gabriel's Horn, and asks what Cyrus recalls that could find the surface area and volume. He mentions triple integrals, but says he only recalls them vaguely. The interviewer indicates two ways of considering the problem, to paint the surface area or fill the volume of the horn. To show these algebraically, the interviewer gives the formulas for the surface area and volume of Gabriel's Horn (as raw integrals, without their values calculated), and asks before calculation what amount of paint might be needed to paint the entire horn. Cyrus says that it should be the surface area amount, and when asked, says the amount based on the volume would be excessive but would also work.

Next, the interviewer shows the two integrals being calculated, and that the volume is equal to π and that the surface area is infinite:

C.4.26	C: Oh, okay. So, then the answer would be- would definitely be the volume. That would be sufficient to paint all of Gabriel's Horn.
C.4.27	I: And yet we've painted an infinite surface area.
C.4.28	C: Which, there lies the contradiction. This is a paradox, then.
C.4.29	I: Well, sort of, yeah.
C.4.30	C: Okay.
C.4.31	I: So, does it seem reasonable that this object would have a finite volume and an infinite surface area?
C.4.32	C: Does it seem reasonable, um,
C.4.33	I: Does it make sense?
C.4.34	C: Not at first. But it seems like it is mathematically sound, so, it does seem like the right answer.
C.4.35	I: Okay.
C.4.36	C: So I would say yes, it makes sense overall, when you find the limits. But it

	still kind of contradicts intuition.
C.4.37	I: Hm. Okay. So, that's an interesting thing you said there, ah, is contradicting intuition a particular problem? Ah, how do you feel about that?
C.4.38	C: It isn't a problem when you're dealing with mathematics.
C.4.39	I: Okay.
C.4.40	C: It happens, I don't want to say frequently, but often enough.
C.4.41	I: Okay. So, ah, how do you think of or use intuition when you look at problems like this?
C.4.42	C: How do I use intuition?
C.4.43	I: Yeah.
C.4.44	C: Um, well often, if I was figuring these limits out for myself, I would sort of use it, I would probably end up second-guessing myself if I found out that the limit, um, the limit, um, does not converge, for the surface- if the limit diverges for the surface area, I would probably use my intuition to sort of second-guess myself. And then just make sure I've done everything, correctly with, calculating the surface area.
C.4.45	I: Okay. And, if you go and you double-check it all, and it all checks out, ah, then what?
C.4.46	C: Um, then if I, was, if I had made sure that I checked my mathematics thoroughly enough, then I would probably accept that answer.
C.4.47	I: Okay.
C.4.48	C: Mm, so, for this one, I pretty much accept it based on the calculation that was done to figure out the limits.

Here, once Cyrus is given the integral calculations (which he reads over but does not work through independently), he concludes that the volume gives the amount of paint necessary to paint the horn, and recognizes this as a type of paradox. He says that it does not seem reasonable at first, but is mathematically sound. While he views it as contradicting intuition, he doesn't see that as a problem. Asked about his own use of intuition, he says that it could lead him to recheck his answer, but if confirmed, he would accept the formal mathematical result.

Given the results of the integrals, Cyrus does not show any signs of doubt, but instead immediately states that the volume is sufficient to paint the horn (C.4.26). While he recognizes it as a paradox (C.4.28), he doesn't seem to have any problem with this, and concludes that he would accept the mathematical answer (C.4.48). While the reaction is not positive as Joshua's was to this task, it is still not negative, unlike trends in the general student population.

From the discussion of using volume or surface area, we saw that Cyrus started off with an intuitive idea that the volume should be larger than the surface area (C.4.23), an idea which Joshua also held. Compared to Joshua's case, however, Cyrus seems to be less attached to the idea that a volume should be larger than a surface area, referencing it only once (C.4.34).

Cyrus' responses when asked about the role of intuition suggest that he uses it to some extent (C.4.44), but considers the formal result more significant (C.4.46). In particular, he does not consider "contradict[ing] intuition" to be a barrier to "mak[ing] sense overall" (C.4.36). He also suggests that this is particularly true in the area of mathematics (C.4.38), suggesting that he may not view other domains the same way or to the same extent. The use of 'mathematics' specifically suggests that this is an idea about mathematics specifically, rather than about academic disciplines or formal reasoning involving scientific concepts generally. Still, this fits with the other interview results, as well as other research done with people on the autism spectrum that hints at a preference for more formal, rule-based reasoning over what is generally considered intuitive.

8.5. Problem of Three Prisoners

In this problem (as described in Section 6.3), one of three prisoners (initially named Tom, Dick, and Harry) is given a pardon, but none are allowed to be told of their own fate. The warden talks to the first prisoner, who convinces the warden to give the name of one other prisoner who will be executed. The question posed is whether this first prisoner's chances have now been improved or not.

The interviewer starts by giving the explanation of the problem to Cyrus, including the prisoner's claims about the response not making a difference (to the warden) and cheering up by perceiving improved chances (to himself), and asks what Cyrus thinks about those claims:

C.5.1	C: I think it is correct in the sense that once the warden declares the first's person's going to be hanged, then he declares the next person's going to be hanged, and then the remaining individual, I think it's correct in his assumption that as soon as he hears those other people are going to be hanged, that's going to increase his chances of getting a pardon.
C.5.2	C: I think, but I'm not exactly sure.
C.5.3	I: Okay, so, in the model you're thinking of, how do you think the warden decides what name to give?
C.5.4	C: Hmm... how does he decide what name to give, he doesn't actually decide it at random, does he? Well, okay, I think with the last person, the way it was stated was that it's basically like rolling dice whether he gets the pardon or not, isn't it?
C.5.5	I: Well, it's a fair draw between these three people. So you could, like, roll a regular six-sided die and go 'okay, it's Tom on one or two, Dick on three or four, Harry on five or six' or something like that.
C.5.6	C: I'm not exactly sure how it was stated in that thing- I wasn't exactly clear how it is that the warden determines who's going to die for sure, and who's going to...
C.5.7	I: (…) The warden is tasked to go and tell these prisoners about this pardon scheme and what's happening, but is not allowed to inform a prisoner of their own fate.
C.5.8	C: Oh, he can't inform a prisoner of his own fate?
C.5.9	I: Right.
C.5.10	C: Okay, so he could only tell... okay. Well then if that's the case I think being told that this individual or that individuals' going to be executed and you get your chance at a pardon, I think it does affect it. I think it does affect the draw.
C.5.11	I: Okay. Uh, why or how?

C.5.12	C: Um, oh, geez, (sigh). I think the...hmm.
C.5.13	C: Okay, you know what, I'm not really sure about the reasoning for my answer, but that's what I'm leaning towards right now.
C.5.14	I: Okay.
C.5.15	C: Hmm. There must be some other bit of information that I'm overlooking right now.
C.5.16	I: Um, well, the warden can only say the name of somebody that is going to be executed, because if the warden says that such-and-so is going to be freed, then you know that you're going to die, and that violates his rule.
C.5.17	C: Right, okay, mm-hm.
C.5.18	I: Since there's only one pardon to go around.
C.5.19	C: We don't really- was there any info about, that fair draw, what kind of probability distribution does it have? There was no info about that, was there?
C.5.20	I: I would think that typically 'fair' would be taken to mean 'equally distributed'.
C.5.21	C: Right, okay. But then the real question is, are his chances of getting a pardon equal to that of not getting a pardon.
C.5.22	I: Well, originally, there were three of them, so that's one third each. Originally.
C.5.23	C: Right, okay.
C.5.24	I: And that's the base probability that may or may not be affected by this other stuff.
C.5.25	C: I'm just going to go out on a limb and guess, I think it would affect, um, hmm.
C.5.26	C: Okay, you know what, I'm really not sure about this one now. I just think the fact that the warden tells the guy, if he tells the person that this person or that person is going to be executed, then that would affect his chances of a pardon, I think. I don't know how to reason that.
C.5.27	I: Okay. In that case, is it something of an intutitive belief? Or,
C.5.28	C: I think so. Yeah, it's an intuitive belief.

At first, Cyrus agrees with the reasoning presented that it will be more likely for the prisoner who is having the conversation with the warden to get a pardon, but is not sure why that is. After some clarification of the terms, Cyrus still holds this belief, and agrees with the suggestion that it is intuitive. However, he does not have confidence in it.

The appearance of this intuitive conclusion (C.5.26) shows that Cyrus will use intuition by itself to come to a preliminary conclusion, but does not have much confidence in it.

Next, the interviewer introduces the second part of the problem statement, suggesting that any of the three prisoners could have this same conversation.

C.5.29	I: I don't think we've quite reached the end of the problem statement. Because in our original statement, if we're going with the version of alright, doing this is going to increase the probability for Dick, who you're talking to,
C.5.30	C: Mm-hm.
C.5.31	I: and there's no particular reason that you couldn't do this for Tom or Harry, too, because it's symmetric.
C.5.32	C: Right, okay.
C.5.33	I: Which means they all have increased probability.
C.5.34	C: Which is...
C.5.35	I: And then it adds to more than one.
C.5.36	C: Right, okay.
C.5.37	I: So how would you reconcile that?
C.5.38	C: That sounds like a paradox to me.
C.5.39	C: So in this case, if the probabilities add up to more than one, then, well that means it's not a probability distribution, but at the same time, (sigh) hmm, so it works for each of those guys, ...sorry, what was the next question?
C.5.40	C: How do you reconcile it? Okay,
C.5.41	I: Yeah, I mean, you're agreeing with the first part of the presented reasoning, of, that-
C.5.42	C: It's symmetric, right.
C.5.43	I: Doing, that doing this is going to increase Dick's chances.

C.5.44	C: Mm-hm.
C.5.45	I: But Dick isn't special in any way, so if we did the same stuff for Tom, it should come out the same, and if we do the same stuff for Harry it should come out the same.
C.5.46	C: Well then, if that's the case, then I would think it doesn't affect it. They all have an equal chance of getting the pardon.
C.5.47	I: Okay.
C.5.48	C: The only reason I say that is because then it would still be a probability distribution. That's the way to reconcile it.
C.5.49	I: Okay.
C.5.50	C: And, yeah, so then the conclusion is that it doesn't affect things.
C.5.51	I: Okay. So in that case, what's wrong with the first bit of reasoning from earlier? Where did that go wrong?
C.5.52	C: What's wrong with the first bit of reasoning from earlier... um, oh dear, well isn't it something to do with the fact that they don't ... hmm. Well, it has to do with conditional probability, so if he's telling Dick that the other two have been hanged, they're going to get hanged, then there's no way he can tell,
C.5.53	I: Not two. He can only say one.
C.5.54	C: Oh, he can only say one of them that's getting hanged?
C.5.55	I: Well, if he says that both of them are getting hanged, then that gives the thing away.
C.5.56	C: Right, um, okay, so, if he tells it to the one, what's wrong with that reasoning?
C.5.57	C: Okay. I'm- I'm not really sure, I'm drawing blanks here.

When asked to reconcile the symmetry of this argument with other facts about probability, Cyrus reverses his earlier belief and concludes that the probability of the prisoner talking to the warden being executed is still one-third. The interviewer then asks what the flaw was in the earlier reasoning which he had agreed with, and is unable to find a specific reason to reject it.

In this session, although Cyrus does not find specific flaws in either argument, he still sides with the more general probability-based reasoning over the ‘increased chances’ reasoning for the specific problem. Being unable to find a reason for the argument specifically failing does not stop him from believing that its contradiction with the rules of probability means that it must be incorrect. For Cyrus, it appears that the intuitive reasoning used earlier is the weakest method for finding a conclusion, and he does not seem to search for ways to hold on to his intuitive answer, unlike many other students.

After this point, the conversation about the three prisoners was stopped for that interview session, and continued later in another interview session. The interviewer reviews the setup of the problem with Cyrus, although the problem statement has been slightly modified to call the prisoners simply A, B, and C instead of the names from before.

C.5.58	C: I think the fact that B has now been eliminated increases A’s chances of a pardon. But, obviously you could say the same for C. Yeah, my intuition tells me that, actually, I’m going to say it this way. My intuition says that it doesn’t affect it, but then when I think about it and my experience with problems similar to this, I would say, it improves his chances of getting a pardon. For both A and C.
C.5.59	I: Okay. Um, do you have any sort of particular reason or codified principle that you would say it happens because of, this general idea, something like that?
C.5.60	C: Um, I would be willing to bet it has something to do with conditional probabilities.
C.5.61	I: Okay, anything else? Any more…?
C.5.62	C: Hm, let me think, it’s a bit tricky. Actually, you know, I’m using my memory now, based on a similar problem I saw years ago, it’s like a game show thing, where somebody’s showing a bunch of doors being opened, I’m going to say that person A shouldn’t get excited. In reality it’s person C who gets the improved chances of a pardon.
C.5.63	I: Hmm. I was wondering if you’d think of that other problem or not.

C.5.64	C: I don't really have it reasoned out. I'm just guessing the calculation for that other problem might have had something to do with conditional probabilities.

Returning to the problem, Cyrus now says that his intuition suggests that the prisoner's chances are not affected, but that experience with previous problems of this type suggests otherwise. He suggests that it is related to conditional probability, and compares it to the Monty Hall problem, saying that C's chances would improve but not A's.

Here, once Cyrus remembers his encounter with the Monty Hall problem (likely due at least in part to the break between sessions), his first attempt at giving an answer to the question is based on his recollection of their structure. From this, it appears that Cyrus gives more credence to results from systematic reasoning (presumably based on scientific concepts) to the extent that he prefers the recollection of a system where this was used to applying reasoning based on everyday concepts. Compared to the earlier session, Cyrus' attribution of a conclusion to intuition is also reversed (comparing C.5.26 to C.5.58), which appears to be based on this recollection. As his thinking progresses on the question, however, he moves away from and then returns to the new 'intuitive' conclusion (C.5.62), which is ultimately the correct answer for the standard version of the problem, with no reference to the old 'intuitive' conclusion.

The interviewer then asks how the jailer decides to name another prisoner, setting up the difference between Problem 3.1a (where another prisoner is named randomly) and Problem 3.1b (where prisoners are named with a fixed order).

C.5.65	I: Okay. So, one question, there, is, how does the jailer work? Like, A presents this argument to the jailer, and the jailer is going to say either B or C.
C.5.66	C: Right.
C.5.67	I: So, if the people to be executed are A, B or A, C, then the jailer has one choice.
C.5.68	C: Right, so.
C.5.69	I: But, if the people to be executed are B and C, then the jailer has to pick. [Another round of clarification follows.]

C.5.70	C: Yes, he does, so,
C.5.71	I: Which is made how?
C.5.72	C: Right, that's a really good question, would he be one half likely to pick, if it's B and C who are going to be executed, then, presumably the chances of picking B or C would be the same, wouldn't they?
C.5.73	I: Well, possibly. Uh, it's actually not stated in the problem, uh, how the jailer thinks.
C.5.74	C: Right. Um,
C.5.75	I: So, how about this. Consider, ah, version 1, where the jailer picks by coin flip.
C.5.76	C: Mm-hm.
C.5.77	I: And a version 2, where the jailer picks alphabetically.
C.5.78	C: Ah, okay.
C.5.79	I: Can't say A, because we're talking to A, but will say B if B is available else will say C.
C.5.80	C: Right, okay.
C.5.81	I: So, are the implications of these two versions different?
C.5.82	C: Hold on, let me work that out, in my head.
C.5.83	I: Would it help if I gave you some paper?
C.5.84	C: Sure. Might as well have some, just in case.
C.5.85	C: Okay, so [writing] okay, so, A, A is- B is chosen, A, C is chosen, B and C means, (?) likely for A, or C, and alphabetical, so that's, (?) B, C, A is chosen
C.5.86	C: Oh, it would be different, then. They would have different implications. In the case where it's alphabetical, then, this is how the jailer thinks, if it's alphabetical then, if B and C are the ones who are going to be executed, then he's going to tell person A that B is the one who's going to be executed, whereas if it's a fair coin toss, it's going to be, in the case where B and C are going to be executed, there's a one-half chance the jailer's going to tell them B, and a one-half chance the jailer's going to tell them it's C.
C.5.87	I: Okay. So what effect does that have on the overall conclusions that A

	should make? Like, in the alphabetical version, what should A conclude from hearing B or from hearing C?
C.5.88	C: Okay. So in the alphabetical version, right, even the sample space is different now. Um, so in the alphabetical version, if A hears B, then he should assume that either, okay, hold on, he should assume that either himself and B are going to be executed or B and C are going to be executed. And if he hears C, then he knows for sure that he's going to be executed himself.
C.5.89	I: Hm. Okay. Whereas, in the coin toss version,
C.5.90	C: Right. Whereas in the coin toss version, if he hears B, then, it's uncertain. There's a chance any of them could be executed. But if he hears C, then, right. So it's much less certain. Whether the jailer tells him B or C, then he's still just as likely to be executed himself.
C.5.91	I: Okay. Uh, so, you're saying that it's still one-third in the coin toss case either way, then.
C.5.92	C: Hold on, let me think about that for a second, based on this. Hmm. [writing]
C.5.93	C: I'm going to say yes. In the coin toss situation, it just stays the same at one third.
C.5.94	I: Okay. And what about-
C.5.95	C: Actually, actually, no, hold on. Wait a sec. Let me just make sure, uh, yeah, it sounds right. I think it's, um, I think it's a, um, yeah, I think it's one third.
C.5.96	I: It sounds right, why?
C.5.97	C: Mm, because there's no other indication as to whether B gets called or C, he has no information about whether or not his chances get improved.
C.5.98	I: Okay. And, ah, so what are A's chances in the alphabetical one if he hears B?
C.5.99	C: If he hears B, his chances are, I'm going to say his chances of getting the pardon is one half. So I'm going to say it this way, probability of A gets pardon given B is named is going to be equal to one half.
C.5.100	I: Alright, and if C gets named, then the probability is?
C.5.101	C: Probability of A… [writing] I already know what it is. It's going to be zero, exactly zero.

C.5.102	C: This doesn't make sense, then, does it? Because, actually, if we know that C is named, then he's dead for sure, and then, then this sample space would not add up to 1. If you counted the probabilities.
C.5.103	C: That's what I make of this so far.
C.5.104	I: What exactly wouldn't add up to 1?
C.5.105	C: If you added up the probability of each of these three situations, so like, the probability of AB, what was it, sorry, the probability of A and B getting executed, given that B is named, added up with the probability of these two other cases. So there's three things in the sample space here.
C.5.106	I: Yeah.
C.5.107	C: Hmm, actually, no, I take that back, this could still make sense, because then, isn't each of these situations a third likely?
C.5.108	I: Yeah.
C.5.109	C: Okay, so then this makes sense. So the probability that A gets pardoned given that B gets named is exactly one half.
C.5.110	I: Okay.

Here, Cyrus is asked about how the jailer operates, and suggests that he would be equally likely to name B or C if both were possible, but does not appear certain of that. The interviewer suggests multiple ways for the jailer to deal with this, one random like Cyrus suggested and one alphabetical (where the jailer will always say B over C if both are possible), and asks Cyrus what would happen in each case. Cyrus works each of these cases out on paper, concluding that prisoner A's chances are still one-third with the random jailer whether B or C is named, while with the alphabetical jailer, A's chances are one-half if B is named and zero is C is named.

The change to the phrasing of the problem, made at the beginning of this section, was something that Cyrus requested in order to better follow the problem. While somewhat unusual, this is consistent with Cyrus' reactions to the two versions of the card-sorting task, where he preferred the abstract version to the real-world version.

At this point in the problem, Cyrus feels that the result in the original case that the probability is still one third "sounds right" (C.5.95). This appears to be based on a 'no new information' interpretation (C.5.97), one of the common forms of reasoning described in earlier studies of this problem. However, while this is an interpretation that can lead to a problem with the 3.1b variant, such a problem does not occur (C.5.109). This suggests either a more appropriately nuanced interpretation of 'no information' or that this is mainly an after-the-fact explanation (or possibly both). In fact, when Cyrus takes issue with the result in working out the problem (C.5.102), this interpretation is not considered further.

Next, the interviewer introduces another version (Problem 3.1c). Here, the probability that C is pardoned is 1/2 while the probabilities for A or B being pardoned are 1/4 each; prisoner A makes the same arguments to the jailer as before.

C.5.111	C: Okay, so, in this case we're trying to find out probability of A gets pardon given B is named [writes this]. So then it could be, okay so hold on, the sample space, AB, AC, BC. Okay, so, and then, okay so the probability of this is, no, it's one half, if C survives.
C.5.112	I: Because now we're biased.
C.5.113	C: And so then, the original probability that B survives, is, a quarter. Then the probability that A survives is a quarter as well, right?
C.5.114	I: Yeah.
C.5.115	C: The probability that A gets the pardon given that B is named. So B is named, so then, we can use conditional probability, so [the second] situation is gone, so it's either out of this or that. So it's going to be, the probability that A gets pardoned given B is named. So, wouldn't it be probability that A pardoned and B is named over the probability that B is named. So the probability that A is pardoned and B is named, is, simply BC. So a quarter, divided by, and the probability that B is named is, a half plus a quarter, which is, three quarters, which makes it, okay, I totally forget how to do fractions for the moment, so that's going to be, a third.
C.5.116	I: Hm. Now you said that the probability that B is named is three quarters. But then you're saying that B is named all the time in the BC case.

C.5.117	C: Right. Sorry about that?
C.5.118	I: Well, if you say the probability that B is named is three quarters,
C.5.119	C: Right.
C.5.120	I: Then it's the AB situation and the BC situation.
C.5.121	C: Right, so wouldn't it be these two probabilities added up?
C.5.122	I: Ah, we don't have the alphabetical jailer. So in the BC situation, the jailer flips a coin.
C.5.123	C: Oh, right, so that makes it more complicated, then. Right, so then it's not three quarters, it's something else.
C.5.124	C: Wait, but, wouldn't the probability still be the same? Of B being named?
C.5.125	I: Uh, would it? How would we find the probability of B being named?
C.5.126	C: It would be both these situa- oh, right, but even in this situation that doesn't mean, okay, I see what you're saying. In this situation, okay, for this one, B would be named for sure. But then in this one, it's half likely to be B and half likely to be C.
C.5.127	I: Right.
C.5.128	C: Oh, right, okay. So then you have to consider that. So then would it be, instead, a quarter times a half, so one eighth?
C.5.129	I: That seems reasonable.
C.5.130	C: Mm-hm. So the probability that B is named, is going to be a half plus an eighth. [writing] It's going to be five eighths. Am I right? Yes, five eighths. So, then, this thing would come out to a quarter divided by five eighths. So then that's going to be, how do I do these fractions, anyway? It's been a long time since I actually did any, like, real arithmetic or fractions, uh, [?] five, [?] two, so that's two over five. So two-fifths is a little bit better than a quarter. So he should be happy about that.
C.5.131	I: Mm.
C.5.132	C: Oh, wait, I see, I just realized a mistake I made in this.
C.5.133	I: Yeah?
C.5.134	C: Okay, the probability that B is named is now five eighths, then, this

	probability here, what do you call it? The-
C.5.135	I: Intersection?
C.5.136	C: The intersection, that's going to be affected as well. So, okay, [writing], so then instead it's going to be, the probability that A is pardoned and B is named, wait, no, that can't be, so the probability that A gets pardoned, and B is named, is going to be, [writing] okay, now I'm confused, so the probability that A gets pardoned and B is named is, it's going to be a quarter times a half. [writing] Over, what's the denominator here? Over five eighths. So then it's going to be, one over eight, divided by five eighths, is, equals one fifth. So in reality, his chances just got worse.
C.5.137	I: Hmm. Interesting.

Cyrus starts by working out the problem in terms of conditional probability. The interviewer notices that his calculations are consistent with the jailer behaviour for the previous version of the problem, and clarifies that the jailer has reverted to being random. After this, he completes a second version of the calculations, finding an increased probability. He then notices a mistake (without interviewer prompting) and revises his calculations again, concluding that the probability the prisoner will be pardoned has gone down.

Despite Cyrus' inclination toward using algebraic methods and calculations, he still mentions trouble with fractions (C.5.115, C.5.130), like many students.

After this, Cyrus asks for a review of the problem, now that he has arrived at a conclusion.

C.5.138	C: Can you just summarize what exactly happened? I'm sorry but I actually forget what the original question was, I got so caught up doing this calculation.
C.5.139	[The interviewer summarizes version 3.1c of the problem (see Chapter 5), with the biased pardon.]
C.5.140	C: Right. And because it's decreased, then he should be sad. Okay. I got you.
C.5.141	I: And then we could look similarly, if, uh, if the jailer had said C.
C.5.142	C: Right.

C.5.143	I: And, ah, before calculating it, what do you think should happen in that case?
C.5.144	C: Um, if the jailer names C, I think, hold on. The jailer names C, let me just think about it for a sec.
C.5.145	I: Okay. [a bit of time passes]
C.5.146	C: So, I think he should be sad, as well, if that's the case. That's what I'm going to say. This is a little bit of intuition here.
C.5.147	I: Okay, ah, how come?
C.5.148	C: My guess, based on the previous one, is it's going to be a similar calculation, that's going to make the probability lower, when you multiply some things.
C.5.149	I: Hm, okay. But, remember, that, C is, the one who, is more likely to get pardoned, than B, so would that have an effect on what happens?
C.5.150	C: Yeah, you're right, it doesn't make sense with that. No, I'm going to, you know what, I really think it's going to be, okay, I think what you just said is what intuitively sounds right. But, I'm going to say that, B's chances, that's my prediction, B's chances get increased, of getting a pardon. Once you name C.
C.5.151	I: But, A's chances don't, or they go down, or what?
C.5.152	C: They go down.
C.5.153	I: Hmm.
C.5.154	C: So it's going to decrease from a quarter to something less than a quarter.
C.5.155	I: Alright. Well, let's take a look at the calculation.
C.5.156	C: I just want to make, okay, so, just so I know what we're doing, the probability that A gets, pardon, given C is named. [writing] named, over, probability that C is the one who is named. Okay. So now, okay, let's find out what's the probability that C is named. So, in this case it would be, probability that C is named, is equal to a quarter, plus, an eighth. Equals to three eighths. Hold on, is that- yeah, two plus- okay. Then, top part, the probability that A gets pardoned and C is named, is, [writing] okay, so, probability that A gets a pardon and, the probability that A gets a pardon and C is named is only this case. So, that's a quarter. So, [?] to think about that, a quarter times, C is named is three eighths, divided by three eighths, no, wait, that's not right, wait,

	okay that sounds- I think that's the case. It changes to, hold on, [?], that's the probability that B is named, hold on, the probability that A gets a pardon, and, C is named is, no, that's, [crossing out] that's going to be, 'cause those events are not independent, that's why you can't just multiply them by each other, okay, I didn't make that mistake the first time, a quarter times one half, again, divided by three eighths, can equal, an eighth divided by three eighths, a third. So, okay, so I was wrong. In this case, A should be happy. His chances just improved. Although, B's chances are still better than A's.
C.5.157	I: Ah, why would B's chances be better than A's?
C.5.158	C: According to the calculation, B would have a two-third chance of getting pardoned if C was the one that was named.
C.5.159	I: Okay, right.
C.5.160	C: So, I guess that's both good and bad in a way. He knows that his chances are better than, well, okay, here's the question. Did he know that, did he know about the judge's bias in the first place? That, he's only got a quarter- a one quarter chance of getting the pardon?
C.5.161	I: Yeah.
C.5.162	C: Okay, so if he knew about that, then, I don't know, he should be both- I guess he should be- he should be a little bit happier than he was. Knowing that his chances improved even though they're still worse than B's.

Now, Cyrus is asked to consider the problem if C (the prisoner more likely to get the pardon) is named. Asked to speculate on the answer before calculation, Cyrus suggests that A's probability of a pardon will go down here also. Reminded that C is more likely to get the pardon, Cyrus is unsure, but says that A's probability of a pardon will go down and B's will go up. Cyrus makes the calculation, and (self-correcting another error) finds that A's probability of a pardon has gone up, although he notes that it is still less than B's probability of getting a pardon (not independently calculated).

Here, it seems notable that Cyrus asks for a review of the problem (C.5.138) after the long calculation. Among other things, this suggests that the algebraic calculation is somewhat absorbing for Cyrus in and of itself, consistent with his other interview responses.

Also, we again see (C.5.148) Cyrus' initial speculation without calculation of the result of the problem is based on the algebraic structure of the previous calculations. This is somewhat ineffective, as he makes claims about A and B, one of which is true and one of which is false. However, more typical intuitions are also ineffective, and Cyrus' version does not appear to provide him with any conflicts or reasons to reject the final answer (C.5.156). The idea is also somewhat in line with what Fischbein calls secondary intuition; in that framework, it appears that Cyrus prioritizes secondary intuition over primary intuition.

Cyrus also asks a question about the knowledge of the person in the problem (C.5.162). However, since he only asks this in terms of the prisoner's conclusion instead of earlier with the calculation of the problem, it appears that Cyrus views this knowledge on a separate level from the rest of the problem.

At the end of the interview, Cyrus is asked to reflect on the problem.

C.5.163	I: What was it that made you decide, when you did, to resort to, okay, I'm actually going to calculate this and write it out and all that.
C.5.164	C: The reason for that, was, when I saw the problems, with the game show, it was like, a game show where a whole bunch of doors,
C.5.165	I: Yeah, the Monty Hall Problem.
C.5.166	C: Mm-hm. Yeah, when I saw that, I remember a calculation using conditional probabilities. So then I thought, if I show the sample space, and, then, do the conditional probabilities based on, I think it's called the basic Bayesian formula, that I
C.5.167	I: Yeah.
C.5.168	C: know about. Then I thought I might arrive at an answer that makes sense in terms of the probability.
C.5.169	I: Okay.
C.5.170	C: Because I was kind of stumped in terms of, arriving at an answer just

	thinking in terms of intuition. It was like, I got to the point where I really just had absolutely no idea what the outcome would be. So I felt I needed to work it out in order to come to any kind of conclusion. [At this point, Cyrus asks for confirmation that the answer he gave is correct, which is given by the interviewer.]

In his reflection, Cyrus says that he remembered that the Monty Hall Problem was calculated using conditional probability, and based on its similarities to this problem, decided to use that technique in this problem. He also says that he was not able to come to an answer intuitively, and thus felt that he needed to work out the problem. He is still not entirely sure his answer is correct, and asks the interviewer to confirm it, which the interviewer does.

One thing that Cyrus touches on in his reflection is not being able to come to the answer via intuition (C.5.170). This may appear somewhat contradictory, since he had attributed several responses to intuition throughout the interview, but it could be that Cyrus views those responses as something less than 'answers'. That view fits with his trend of rejecting them when they were contradicted by more formal reasoning with little to no resistance. This could also be evidence of compensation, using stronger skills of systematic reasoning in place of weaker intuitive reasoning skills. If so, this could be a particularly helpful form of compensation in many mathematical contexts, such as those where the typical intuitive responses are incorrect, as seen here.

8.6. Summary of Cyrus' Interviews

One overall trend apparent from Cyrus' interviews is his inclination toward algebraic and formal methods of solution, and a distrust of or disinclination toward informal or intuitive methods. This can fit as another form of compensation, where stronger algebraic skills are used in place of weaker geometric or informal ones. However, this combined with the previous chapter suggests that any effect of compensation related to autism is more complex and individual rather than people on the autism spectrum all fitting a certain type, since Cyrus' and Joshua's forms of compensation here contain both points of similarity (mistrust of intuitive reasoning) and points of opposition (algebraic or geometric methods). Another trend is the inclination to use comparisons to previously solved problems and previously used methods as the beginning of the approach to a problem, although this is at least in part due to previous mathematical experience.

Chapter 9. Interviews with Mark

In this chapter, I will examine the results of interviews conducted with Mark. These interviews involve a different set of tasks, with a higher average level of mathematical sophistication, since Mark held a master's degree in mathematics at the time of the interviews. Additionally, new techniques were employed in the interviews with Mark to encourage more of the participant's thought processes to be spoken aloud. Drawing from the work of Ericsson and Simon (1993) on think-aloud protocols, Mark was instructed to not explain how he was thinking, but to state it as if he were talking to himself. It should be noted that since one of the primary diagnostic criteria for autism is based on problems with social communication, primarily coming from observations of language, there may be some additional differences from the use of think-aloud protocols with participants on the autism spectrum, somewhat comparable to the inner speech tasks discussed in Chapter 3.

9.1. Sum of Angles Task

Here, the task presented is to find the sum of the angles from the bottom line of the figure below (presented to Mark) to the hypotenuses of each of the three right triangles.

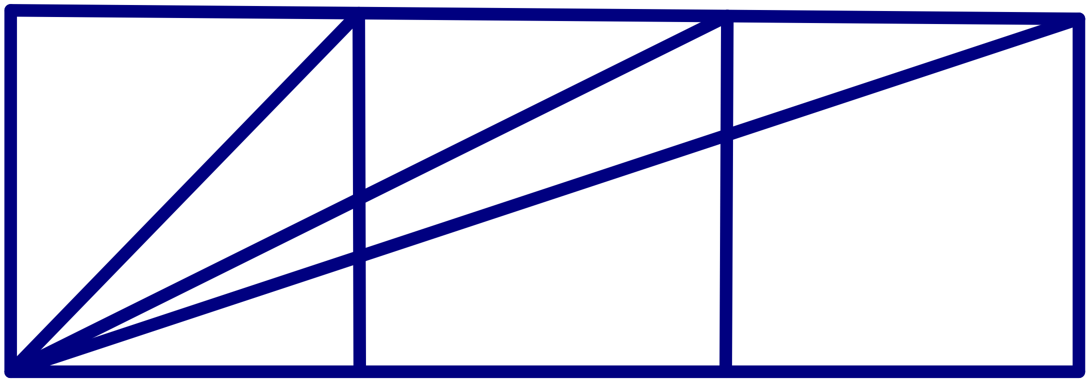

Figure 9.1: Squares and Triangles Problem Diagram

M.1.1	I: The task is to find a sum of these three angles. [showing Figure 9.1]
M.1.2	M: Hmm. Okay, so, well, I'd like to keep track of what I'm doing, so I'll call that theta one, theta two, theta three, and I want to find the sum, and okay, this is a square, so I'll just note that theta one is pi over four. Okay, so, there's probably a more elegant way to do this than oh, calculate all the angles and what they are, and these two aren't nice, anyway. Are they? Mm... no.
M.1.3	M: Hmm. So, wait, what kind of thing am I supposed to be getting for the sum? I mean, I could say, like, ah, arcsine of this plus arcsine of this plus arcsine of this but that seems to sort of go against the spirit of the problem. That seems to sort of 'not count', if you will.
M.1.4	M: So, probably, the sum should be a nice number and I should be able to find that. The sum is a nice number.
M.1.5	I: Please keep talking.
M.1.6	M: Mm. Okay... wait. Uh, so, well what is the ratio of these angles to... this. Let's see. I had this sort of intuitive idea that maybe this angle is half of this one, and I'm not sure if that's actually true. But, well, it would be pretty nice if it were true, and I could definitely use it. So, could I try and show if that were true in some way? Let’s see. This is a slope of one, and this is a slope of one half, so this is halfway up. Uh, yeah, so this thing is, we've got one half, one, theta two, and, wait, no, if that were true, then sine would look like a bunch of jumping lines or something. Would it? Because sine is the ratio, of these. And

	this is staying the same, so yeah, that's probably not true. Hmm. But I think this should come out to be something nice. Well, if it doesn't, then, I don't know what you could do other than 'oh, here's a mess of arcsines'. That's not the sort of nice elegant problem that people look at, that gives you 'some mess of arcsines', so, uh, it can't be that.
M.1.7	M: Hmm.
M.1.8	I: Please keep talking.
M.1.9	M: Wait, is this ratio something useful and I forgot? No. Hmm. Well, I could say that, these angles up here are also theta one, theta two, theta three and then, well, I'll just call these complementary angles phi one, phi two, phi three, and the sums of the theta plus phis are all ninety, or pi over two, hmm, does that help anything? I don't see how that's going to help anything.
M.1.10	M: Hmm. What else could I do with this? Huh. That's useful, oh.
M.1.11	I: Louder, please.
M.1.12	M: Okay, I feel like I don't have anything else to say, but there's got to be something. I can do, do something like this, um, hmm. What useful facts about geometry could I use here? There's some trig identities about sums of angles, no, hmm. What if I think of this in a sort of polar coordinate radial thing? Like, what would these points be in polar coordinates and would that give me anything useful? Well, if I assume that the square side length is, one, might as well, then this is root 2, pi over 4 well, okay, this one is, this is, root 5, something, this is, root 10, something else, well, no, that doesn't seem to be a useful fact. Hmm. It must be one of those things that's simple when you see it. But I'm never quite sure what to do with those. Not some kind of, orderly procedure or sequence of building blocks of smaller facts or something. [Mark continues to make attempts at solving the problem for about 15 minutes, without success.] …Okay, now I just feel, kind of embarrassed that I'm not really sure what to do.
M.1.13	I: So it's your decision, if you would like to continue, you can, if you would like to stop, you can.
M.1.14	M: Uh,

M.1.15	I: And, ah, compliments for thinking aloud.
M.1.16	M: Okay, well, I guess, at this point, I'd probably start to think, well I don't seem to be getting anywhere with the tools that I have and start like, looking up some relevant fact or formula or something that I could, well, at the very least, use to get out of the rut that I'm in right now and do something else.
M.1.17	I: Sometimes to stop is just a smart move.
M.1.18	M: Yeah.
M.1.19	I: So, this is your decision?
M.1.20	M: I think so, yeah.

In this first part of the task, after the interviewer presents the question, Mark begins by giving the angles variable names and noting that the value of the first one is $\pi/4$ (immediately starting in radians). He then states that the other two angles are not 'nice' values, and while they could be calculated by the arcsine function, this would be against the intent of the problem. He then investigates the idea of the next angle being half of the first (which doesn't work out), and then again reflects on the arcsine solution existing and going against the intent, saying that it is not elegant. Next, Mark designates the complementary angles to the ones which are to be found with different variable names, and considers if those have use. He also considers trigonometric identities and polar coordinates without initial success. He then speculates that the problem is one that would be simple if one were to see something (which is unidentified), but not one that has an orderly procedure associated with it. Mark continues to speculate on adding the angles geometrically, as well as establishing some other relations between them, none of which he considers to be going toward a result. Eventually, he decides that he would normally go to look up additional information, and stops at that point.

Here, we can see two characteristics demonstrated in Mark (particularly at M.1.3): an inclination toward algebraic solutions, and a tendency to think about the problem at least in part via considering the hypothetical problem designer's intent. In this particular case, the second leads to the quick dismissal of the first, though it is still mentioned again (M.1.6). The reoccurrence of mentioning the rejected solution suggests there may be some difficulty in switching away from the preferred method, even once it is recognized as necessary. This is somewhat demonstrated in the next few steps, as Mark puts some known facts about the problem into algebraic terms (M.1.9) and searches for geometric facts with algebraic forms of expression (M.1.12). Outside of algebra specifically, there is also a desire to see things in a systematic way, or to fit them inside of an existing system (M.1.12). There is little to no mention of intuition in this consideration of the problem.

Next, the interviewer offers Mark a hint on the problem:

M.1.21	I: Okay. Would you like me to tell something about this problem?
M.1.22	M: Yeah.
M.1.23	I: Okay. So, it was so difficult for me to say nothing, because at one point you were really close to the solution that I am familiar with.
M.1.24	M: Oh?
M.1.25	I: This was the point when you wanted to create some geometric construction, that would be, that would represent the sum of the angles in a geometric way.
M.1.26	M: Oh?
M.1.27	I: Because your guess, that, well, it should be something nice, it is not about, sum of arctans of course, and what can be something nice here, it is that, oh, the sum equals 90 degrees. All of this is completely reasonable for me. But the problem is that it is inconvenient on this drawing to make these calculations. Because you have these differences and so on and so forth, and sum of angles is better, is more visual than, the difference of angles. To our needs. So, you articulated this idea, that you need somehow to visualize a sum of angles in geometric way.
M.1.28	M: Mm-hm.
M.1.29	I: But then, you went to rotation. Which was very close to a construction that would really solve the problem. The construction that would really give us, a sum of the angle instead of the difference of the angle, is not rotation, but symmetry construction.
M.1.30	M: Okay.
M.1.31	I: I'll take another side, to show what I want to show.

M.1.32	M: So, wait, if we made a mirror of these down there?
M.1.33	I: Yep. Which one is reasonable to mirror if your goal is to have a sum of the angles instead of the difference?
M.1.34	M: Um, well let's see. If I mirror the, theta 1, then, I've got a 90 degree angle. Which is what I claim my sum is.
M.1.35	I: Yeah
M.1.36	M: But, wait, then, ah, what do I do with the other ones?
M.1.37	I: Okay. Keep going.
M.1.38	M: Uh, let's see. …If, well if I just mirror all of them, then, I, have this sort of weird radial-looking thing, of, well, it looks like the NBC logo or something.
M.1.40	M: But, that just seems like a bit of a mess.
M.1.41	I: Okay, so, if you'd like to have more time, and with this keyword mirroring, you would find the right option. I'm sure. But because you decided to stop, I can just show it to you.
M.1.42	M: Okay.
M.1.43	I: You, again, you are very close. Your candidates, this, this, and this.
M.1.44	M: Yeah.
M.1.45	I: First of all. And, you consider this one, and say that, ah, this is not a good way. Then somehow you jump to all three of them. But if you'd continued systematically, so your next candidate is this one.
M.1.46	M: Yeah.
M.1.47	I: And in this case, we would have, the following situation, so, this is, number 1, number 2, number 3. This one, this one, and this one. So, I am trying to mirror this angle, the second one. [The interviewer walks Mark through the solution described in Chapter 6. This includes several prompts for Mark to continue the solution past the point the interviewer has described, which are generally unproductive.]
M.1.119	I: So, can you please, summarize this way of solution for me?
M.1.120	M: Um,
M.1.121	I: In one [or] two sentences. Not in detail. What is the main idea of this

	solution?
M.1.122	M: I guess, well, it feels more like, uh, well if you take this- take these triangles, and, mirror them, and, find, ah, some nice geometric things that work together, um,
M.1.123	I: It's a good summary. You put to work together several nice geometric things. Okay. Ah, and if I'll ask you, where everything is clear here, in this solution? And Mark, I see that you are smiling. Why are you smiling?
M.1.124	M: Well, yes and no, I mean, it's clear in that, uh, okay, I see that, ah, all of these things are true and they produce the result, ah, but it's one of those, okay how am I supposed to think of doing this?
M.1.128	I: Okay. Eh, so you think that it's just a trick out of the pocket, or
M.1.129	M: Um, well there must be some sort of way that someone thinks of doing this, but it doesn't quite mesh with the sort of way I usually think about things.
M.1.130	I: I see. I see.
M.1.131	M: I don't know if I can quite use it.
M.1.132	I: For me, the way how to discover this construction was that they want a better representation of the sum. Because the initial drawing gives us, the difference, what can I do in order to have instead of the difference, an angle that would represent the sum? But, I understand that it can be sort of artificial.

In this section, the interviewer presents the idea of mirroring the figure, and shows this solution to Mark in several steps. When asked to summarize, Mark says that he considers this a process of finding multiple geometric facts that work together to arrive at the solution. He says that he sees that all of those facts are true, and that their combination produces the result, but that he doesn't see how one would think of them initially, although he recognizes that such a way must exist.

Despite the confidence expressed in the interview that Mark would get all of the presented parts of the solution eventually (M.1.41, M.1.73, M.1.85, M.1.113), given the general patterns presented here, I am not sure how long doing so might take, or what conditions would need to be present to motivate Mark into doing this. Particularly, unlike the later problems in this chapter, Mark had already conceded the point of doing it himself (M.1.16). That the problem is ultimately not solved independently is a piece of evidence for the supposition that problems that are strongly skewed toward solution methods the participant disfavors can pose a particular challenge, which is most likely reinforced by that preference's leading to having had less prior practice in using those methods successfully. This could be particularly prominent in problems which require particular cognitive jumps, or have unusual solutions in a way that particularly draws on the problem solver's weak points.

Although Mark does not have confidence in understanding the rationale behind the solution himself (M.1.124), for him this does not translate into a lack of confidence that such a rationale exists (M.1.129). This continues the trend shown with Joshua and Cyrus where trust and confidence in mathematics is generally high and not cast into doubt by individual problems with a task.

There is also an example here (M.1.104) where Mark checks something minor (here, whether a line is really known to be straight) that would often be assumed and not reconsidered when talking about the problem. Comparing with other interview results, Mark has a general trend of rechecking calculations that might otherwise be assumed. He may also feel more cautious in the geometric context, particularly since he mentions being embarrassed at earlier issues with finding a solution to this problem (M.1.12).

Finally, the interviewer concludes by asking Mark how much he liked the problem (one of the standard questions used here):

M.1.133	I: So, the same question as about the previous problem, the statement I like this problem, please, to which extent do you agree on the scale from one to ten?
M.1.134	M: Hm, Maybe, four.
M.1.135	I: Wow.
M.1.136	M: It feels like, uh, it feels a little like one of those things that, you say it is good for you, and, I'm like, well, yes, okay, it's probably good for me, and, it would probably be helpful for me to know how to, do some more stuff like this, but, it, doesn't really mesh with me and I don't really like doing it.

M.1.137	I: Okay. Okay. Very clear, and just to be sure, the previous problem, okay you had an algebraic solution, and then you had this hint, rotation, which helped you to solve the problem. The situation here is similar, so you tried some algebraic way, then you have this hint, mirror, and, you could complete the problem.
M.1.138	M: Sort of.
M.1.139	M: With the, ah, triangle, angle, sort of problem, I'm, sort of, ah, well, one, I'm mostly forced to resort to, geometric fiddling with things, and two, there seem to be, ah, quite a few different- more different pieces of the geometric fiddling with things.
M.1.140	I: (…) here, after mirroring trick, you should continue thinking about something else.
M.1.141	M: And I feel like, okay, I can, piece these things together, and, I don't see the solution, I see the idea that I could work out the solution if, I, ah, paid close attention to putting these different fiddly bits together.
M.1.142	I: So, to summarize, ah, you like the problem on the page 2 [another problem] more than problem on the page 3 [this problem], and you explained your reasoning.
M.1.143	M: Well, maybe, I mean, I guess I'm not entirely sure what is meant by like.
M.1.144	I: Well, how would you define like, I like something, what does it mean?
M.1.145	M: Ah, well, hm. I guess it's less of that, and more of, it feels, like, ah, I'm saying that I like the problem that was easier more and I don't want to say that.
M.1.146	I: Okay.
M.1.147	M: Because, that seems like an embarrassing thing to say.
M.1.148	I: Probably, a way to rephrase the question would be, which problem looks more beautiful for you?
M.1.149	M: Ah. Well, yeah, I think that in that sense, I would go with my, sort of, earlier reasoning, about, how, when I see the trick here [in another problem], I see the solution,
M.1.150	I: Uh-huh.

M.1.151	M: Whereas, when I see the trick here, I see, okay, you could put this together with this and this would lead to a solution, eventually, I don't see the solution just clearly from the thing.
M.1.152	I: I see. Mark, you made it very clear. I have no further question about these problems.

When asked to rate the problem, Mark says that he did not like it (4 out of 10), though he suggests that the problem is useful mathematically if not enjoyable. He views the problem as assembling different pieces of geometric facts to arrive at the solution, and says that although he sees the idea of being able to work out a solution, he does not see the solution itself. He also questions what is meant by 'liking' a problem in this context, saying that he does not want to decide on liking a problem more because it is easier. When asked which is more beautiful, he says that he appreciated another problem (where he did see the geometric solution) more than this one, saying that he only sees that one could use it for a solution, not the solution itself.

Note that the problem mentioned in M.1.142 as 'another problem' is given by Figure 9.2 below, where the question is to determine the ratio of the areas of the two squares. Mark approached both problems algebraically, but followed and replicated the geometric solution for the squares-and-circle problem to a much greater extent than the squares and triangles problem here.

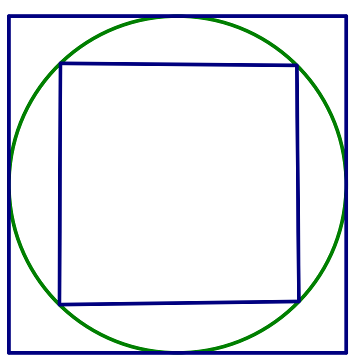

Figure 9.2: Squares-and-circle problem diagram

From the reaction Mark gives here (M.1.138), it appears that he does not feel like, by following the solution given by the interviewer, he has really 'completed' the problem. This appears to be a significant factor in the low rating (M.1.134), since we see that Mark does see mathematical value in the problem (M.1.136). With the way it is described (M.1.151), it may be that Mark does not perceive the solution as complete in his own mind, and that contributes to disliking it. Considering it in terms of mathematical beauty, Mark puts his reasoning in 'I' statements, suggesting that he is not ruling out the problem having a mathematical beauty or elegance which he does not currently perceive.

9.2. Cardano's Method

Another task given was to solve the equation x^3-15x+4=0 by a version of Cardano's method given in modern notation. The solution from Cardano's method, in conjunction with the solutions found using modern methods, gives an apparent paradox which is discussed in more detail in Chapter 6: $x = 4, -2+\sqrt{3}, -2-\sqrt{3}, \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$ appear to be four different zeroes to a third-degree polynomial. Cardano's method was presented to Mark with the following explanation:

1. $x^3 + px + q = 0.$
2. Let us look for x in the form $x = u + v$ when $3uv = -p$
3. $x^3 = u^3 + v^3 + 3uv(u+v) = u^3 + v^3 + 3uvx$
4. Thus, our reduced equation transforms into the following one:
5. $u^3 + v^3 + (3uv + p)x + q = 0$
6. $u^3 + v^3 = -q$
7. $u^3 v^3 = -\frac{p^3}{27}$
8. This system of two equations is of the type $\begin{cases} a+b=m \\ ab=n \end{cases}$, so we can solve it as such:
9. $u^3 + v^3 = u^3 - \frac{p^3}{27u^3} = -q$
10. $u^6 + qu^3 - \frac{p^3}{27} = 0$. Let $w = u^3$.
11. $w^2 + qw - \frac{p^3}{27} = 0$
12. $u^3 = w = \frac{-q}{2} \pm \sqrt{(\frac{q}{2})^2 + (\frac{p}{3})^3}$

13. From here, we obtain v^3 given that $u^3+v^3=-q$. Consider to cases. If $u^3=\frac{-q}{2}+\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}$ then $v^3=\frac{-q}{2}-\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}$. If $u^3=\frac{-q}{2}-\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}$ then $v^3=\frac{-q}{2}+\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}$. 14. That is, in both cases $x=u+v$ is the same: $x=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^2}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^2}}$. **This equation is known as Cardano formula**.

Note that there is a small error in the last step (corrected in the version presented in chapter 6, but preserved here); both instances of $\left(\frac{p}{3}\right)^2$ should be $\left(\frac{p}{3}\right)^3$. Once this was provided, Mark was asked if he wanted to use the formula or go through the algebra, and he decided to go through it himself:

M.2.1	M: Let's see. Okay. So we have, x cubed plus p being our negative fifteen, x plus q being our negative four, equals zero, and, hm. Okay. This is still sort of a, final result phrase type thing, what I mean is that it's a sort of, okay we did the scratch work for how to do a mathematical proof off to the side somewhere else and now we've written it up, because it just pulls out this let us look for x in the form x equals u plus v when 3 u v equals minus p out of nowhere, okay what's u and v? [Mark works through the presented algebra, replicating it.]
M.2.3	[Mark comes to the error mentioned above.] M: Okay. Wait. They were cubes here. Now they're squares down here. That shouldn't happen.
M.2.4	I: You're right. It's supposed to be cubed. Please check- please, uh, change it.
M.2.5	M: Okay. … Alright. So, let's see. [Mark finishes the solution up to the last step.] Hey. That’s going to produce an imaginary number. Well, I suppose it's still a solution.
M.2.6	M: [Mark finishes simplifying the result, getting $\sqrt[3]{2+11i}+\sqrt[3]{2-11i}$.] Okay, that seems reasonable that that would be a solution, if someone's trying to solve this in the real numbers it's probably not the solution they want, but, okay.

In this part, Mark follows the full explanation of Cardano's method, and replicates the majority of it on his own paper. After this, he uses the values from the specific equation given by the interviewer, and in doing this, notices the error in the last step. After it is resolved, he gets the expression $\sqrt[3]{2+11i}+\sqrt[3]{2-11i}$ as a solution, stating that this will produce an imaginary number and that this is a valid solution, although it is not the solution someone solving the given equation in the real numbers is likely to be looking for. The algebraic processes here take about 13 minutes. Mark's written work is given below in Figure 9.3.

$$x^3+(-15)x+(-4)=0$$

Want $x=u+v$ where $3uv=15$

$$x^3=u^3+v^3+3uv(u+v)=u^3+v^3+3uvx$$

$$u^3+v^3+3uvx+px+q=0$$

$$u^3+v^3+q=0,\quad u^3+v^3=-q$$

$$u^3v^3=\frac{-p^3}{27}$$

$$u^3+v^3=u^3-\frac{p^3}{27u^3}=-q$$

$$u^6+qu^3-\frac{p^3}{27}=0 \qquad \text{Let } w=u^3$$

$$w^2+qw-\frac{p^3}{27}=0$$

$$u^3=w=\frac{-q}{2}\pm\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$$

Case 1: $u^3=\frac{-q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$

Then $v^3=\frac{-q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$

Case 2: $u^3=\frac{-q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$

Then $v^3=\frac{-q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}$

So in either case $x=u+v=\sqrt[3]{2+\sqrt{\left(\frac{-4}{2}\right)^2+\left(\frac{-15}{3}\right)^3}}+\sqrt[3]{2-\sqrt{\left(\frac{-4}{2}\right)^2+\left(\frac{-15}{3}\right)^3}}=$

$$=\sqrt[3]{2+\sqrt{4+(-125)}}+\sqrt[3]{2-\sqrt{4-125}}=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}=$$

$$=\sqrt[3]{2+11i}+\sqrt[3]{2-11i}$$

Figure 9.3: Mark's Work for Cardano's Method

Here, the work that Mark does involves not just reading through the explanation provided of Cardano's method, but working through it himself and rewriting the majority of it. According to colleagues who have used this task in their work, this is particularly unusual; typically, interviewees given this task in a similar context will not check the algebra, and start by only attending to the bottom line which presents the paradox of four apparently distinct solutions to a third-degree polynomial (Koichu, 2016, personal communication). This is also unusual compared to other participants in this study (particularly considering the Gabriel's Horn task). Mark is clearly willing to do algebraic computations that many would avoid (a pattern which will be visible later), in particular when there are parts that appear unclear to him (as discussed in the next interview section). This shows some focus on doing mathematics as a process, as well as some unwillingness for Mark to accept a given result without working through and verifying it himself. Considering the reaction when such verification does not happen (M.1.138), it is clear that Mark places significant value on having done the verification himself.

Also, in the work itself, we see a focus on the general algebraic result and somewhat less concern for any particular values or real-world applications, particularly in M.2.3 and M.2.6. This is something that reflects a general tendency in Mark for abstraction that will also be seen in later sections.

After this, there is some discussion of issues that came up in following the first solution.

M.2.7	I: First of all my next request will be solve it in real numbers. But, before that, um, you mentioned at the beginning of reading this page that the substitution for x for u + v with special condition comes out of nowhere.
M.2.8	M: Yeah.
M.2.9	I: Now you carefully read and, even recreated all the chain. So can you guess how the inventor of this method thought about this condition?
M.2.10	M: Um... hmm, I think part of it might be, I ended up with these two cases, I really wish they were the same, how could I make that happen? Because that would give me a nice result. Can I do that? What else. I'm not really quite sure, let me see, what they're doing with, all of this. A lot of it felt kind of, pull something out of a hat well, okay, I suppose all these steps are true, I guess you could do that, I'm not quite sure why you'd want to but alright. Kind of,
M.2.12	I: So, is it still for you out of nowhere, or you see some logic behind the- the substitution.

M.2.13	M: I don't really see where it came from, no.
M.2.14	I: Okay. Okay, I don't want to push it, and [it’s] just a question with no wrong answer. Because I asked about something that cannot be verified. How do you feel about.
M.2.15	M: Yeah. I'm also sort of curious about this, step 8, this system of two equations is of this type, uh, I... don't really, oh, they're doing just a method of substitution, they solve the second one and put it into the first one, okay.
M.2.16	I: I think that they wanted just to re-denote this pair, to see the structure in this pair.
M.2.17	M: Okay, wait, the structure of, oh well, we can solve this for v cubed and plug it into the first one and you get this, well, okay.
M.2.18	I: Okay, so, do you feel comfortable now with the Cardano method?
M.2.19	M: Depends on what you mean by comfortable.
M.2.20	I: You know how to solve the given equation.
M.2.21	M: I see that all the steps are perfectly valid and this is a reasonable solution to an equation like this, it looks like, well, this being a formula that produces a unique number, it's only going to produce one of the roots, and based on this example, not necessarily going to be, ah, the root you're most likely to want.
M.2.22	I: Ah...
M.2.23	M: Which is, which I'm a little surprised by, considering that I generally expect the old-time solution methods to be less inclined to things like imaginary numbers, because I know that historically people really didn't like imaginary numbers. But I guess I don't know exactly when Cardano's from, so...
M.2.24	I: This equation was, uh, a historical one, from the book, from Bombelli, and this equation is associated with the history of mathematics with the invention of complex numbers.
M.2.25	M: Oh.
M.2.26	I: Out of this equation, these complex numbers were considered almost the first time in history.
M.2.27	M: Interesting.

M.2.28	I: But. This equation is special.

In this portion of the interview, some of the portions of the previous part where Mark showed some surprise or confusion were revisited. He speculated that the substitution used in the version of Cardano's method presented to him was thought of in the reverse order (that its producer started with the goal and worked backwards), but that he ultimately didn't see where the substitution came from. He said that he saw that all the steps in it were valid, and was surprised by the apparent imaginary solution produced, since he did not expect historical mathematicians to want such solutions. After this, the interviewer notes that the solution is associated with the early history of the use of complex numbers.

As happened multiple times in the previous section, Mark finds a component where he sees that the move is valid, but not where it came from (M.2.13). This is similar to his perspective on the solution to the squares-and-triangles problem, but having a smaller instance of this in a component of the solution does not appear to be an issue for him in the way that the multiple instances in the previous section were.

Note that here (at M.2.21), the statement that "a formula...produces a unique number" is taken as clearly true, and applicable to the formula in Cardano's method, which becomes relevant later as a piece of the paradox.

At this point in the interview, Mark is asked to temporarily disregard Cardano's method, and solve the same equation in a more standard way.

M.2.29	I: Let's say that a modern mathematician is now asked to solve the equation x^3-15x-4=0.
M.2.30	M: [Mark writes the equation again.]
M.2.31	I: How would you solve it? Without knowing all this stuff.
M.2.32	M: Okay. So, let's see, what can I do, well, ah, the first thing I've got is, ah, the first thing that comes to mind is the rational roots theorem, and, which I'm never entirely sure which way the rational roots theorem goes, but I'm pretty sure it's factors of the constant term over factors of the leading degree term, plus or

	minus. So, that gives me as my candidates plus or minus 1, 2, and 4. Well, it's definitely not plus or minus 1, I can see that just by looking at it. No, that won't work. And then, plus or minus, 2, well, okay, the x^3 and the 15x will have opposite signs, so that's reasonable, ah, 2 cubed is 8, that's too small, so out of these, plus or minus 4 are the most plausible ones. Oh, 15, ah, yeah 15 times 4, and the x is 16 times 4, oh, I see, that's nice. So plus 4 will give me 4 cubed minus 15 times 4 minus 4, which is 16 times 4 minus 15 times 4 minus 4 and yes, that is zero. So, okay, one root is plus 4.
M.2.33	I: That's great.
M.2.34	M: So, do we want a root, or do we want all of them?
M.2.35	I: Your decision.
M.2.36	[Mark uses polynomial long division and the quadratic formula to find the other two roots of the polynomial.] M: Well, I can get them. By, okay, plus 4 is a root, so x-4 is a factor of this, I can, do, $x^3 + 0x^2 – 15x - 4$, so this polynomial long division stuff, x goes into x^3 x^2, x^3 times x, that gives me, $x^3 – 4x^2$, subtract $4x^2$ -15x - 4, okay, so $4x^2$ divided by x is +4x, and 4x times x is $4x^2$, 4x times negative 4 is minus 16x, and negative 15 plus 16, we have x-4 and, okay, that's clearly +1. Okay. So we have x-4 times $x^2 + 4x + 1$, and if I want the other two roots, ah, well, we ex- well, okay, I was going to say we expect from the other method that they won't be real numbers, but I'm not supposed to pay attention to that, um, this does not look very factorable anyway, so well just go, okay, x = -4 plus or minus the square root of 16 minus 4 times 1 times 1, all over 2, and, oh, x equals negative 2 plus or minus, um, root 12 over 2. Hm. Ah, I see where we're going in terms of paradox now. Here we have three roots in the real numbers. And this is, well, hm. This appears to be, anyway, um, a root that is not in the real numbers. But we know from the fundamental theorem of algebra that there are only three roots for a third-degree polynomial. So, I suppose, uh, the most reasonable way to reconcile this that occurs to me is that this thing that I got over here actually is a real number and if I hacked through a big mess of simplifying complex numbers and all that stuff then I would see what real number it is.
M.2.37	I: (?)

M.2.38	M: Which I don't entirely recall all of the techniques to do off the top of my head.

Mark uses the Rational Roots Theorem to find the possible rational solutions, tests them to find one that is a solution, and uses polynomial long division from that to get a quadratic polynomial. Then, he uses the quadratic formula to find the two remaining solutions. Mark notices that he appears to have found four solutions, notes that this is against the Fundamental Theorem of Algebra, and concludes that the result from Cardano's method must actually be a real number which he could find via some methods of simplification.

The work here using the modern method is complete, and we see that given the opportunity, Mark does not tend to avoid longer algebraic computations (M.2.36). I see some parallels between this tendency in an algebraic method of solution and Joshua's measured drawing in a geometric method of solution. This also appears to be a necessary component of the displayed need to go through algebraic computations given from others, which led to Mark finding an error in the given computation (M.2.3).

Next, Mark is asked to show that the result from Cardano's method is equal to four (setting up for the paradox), and then to show that it is also equal to one of the other roots, demonstrating the paradox.

M.2.39	I: Okay. It's a very complete and understandable answer. And thank you for understanding the paradox before it was explained to you. Anyway, let me add now a piece of historical information.
M.2.40	M: Okay.
M.2.41	I: Bombelli, who solved this equation, ah, and obtained this root in complex numbers, knew also about the root of four. And the breaking through in the history of complex numbers happened when Bombelli showed that this terrible number equals four. Bombelli didn't know about the other two roots. For him it was just the root of the equation.
M.2.42	M: Hm.
M.2.43	I: So, can you please try and show that four equals this one?
M.2.44	[Mark does a series of algebraic computations to show $4 = \sqrt[3]{2+11i} +$

	$\sqrt[3]{2-11i}$, starting with the claim and working to a true statement.] M: Hmm. Alright. Claim. $4 = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$. Alright. Well, uh, it should be, uh, somewhat easier to work with, 'I think it's four', than, 'I think it's some real number but I don't know what it is'. Alright. Hm. So, this is true if and only if, ah, well, I could solve it for one of the cube roots, well, uh, this is sort of, not quite valid, assuming my answer, but if all of my steps are reversible, then I can say it goes the other way and it's valid at the end. (…) Um, let's see, what could I do to this, that would be useful. Oh, I could, complete the square on it, would that be useful? I mean, I don't want to square root stuff, because that's not reversible. (…) Therefore, the cube root of 2 + 11 i, is equal to 2 + i, and I can go to this step, and then I square both sides and get my $(u-2)^2 = -1$, and from there each of these things is a perfectly valid move.
M.2.45	I: Great.
M.2.46	M: And I can get there.
M.2.47	I: So you really proved that number four equals this complex number.
M.2.48	M: I'm pretty sure I did, yeah.
M.2.49	I: You remember our standard question, on the scale from one to ten, how sure you are.
M.2.50	M: Maybe, nine point five?
M.2.51	I: Good.
M.2.52	M: Not quite as sure as I was about that, ah, Bayesian probability stuff.
M.2.53	I: Ah, any particular place that you are unsure about? In this proof?
M.2.54	M: Ah, well I guess the thing that makes me worry is this bit where I'm taking the square root of this stuff because it has a similar structure to a lot of student mistakes I'm familiar with, and that makes me want to be extra careful about it.
M.2.55	I: I see. I see, but, right now you don't see any mistake in this computation.
M.2.56	M: Mm, no. (…)

The interviewer starts by mentioning that Bombelli historically found that the root from Cardano's method is equal to 4, and asks Mark to show this. Mark proceeds though a long series of algebraic manipulations (M.2.44 takes about 12 minutes) to get to this result, making note that the steps used are reversible (to justify starting from the conclusion). He reports a very high level of confidence in this solution (9.5 out of 10). The interviewer then asks Mark to show that the result from Cardano's method is equal to one of the other solutions, that is, $\sqrt[3]{2+11i}+\sqrt[3]{2-11i}=-2\pm\sqrt{3}$. There is some initial resistance from Mark, saying that since it is equal to four, it can't be equal to something else. Still, at the interviewer's request, Mark does some of the algebraic work again (reusing some from before, and with some simplifying facts pointed out by the interviewer), finding that the equality holds there as well. The interviewer notes that at this point, they have arrived at the paradox.

We see that at the end of the algebraic manipulations to show that the Cardano's method result is equal to 4 (M.2.44), Mark makes a point of noting the reversibility and validity of his last set of algebraic moves (M.2.43). This is consistent with a general tendency seen in Mark's interviews to place importance on visible mathematical justifications and to vocalize or otherwise indicate them.

Also, the confidence expressed here (M.2.50) is very high, though not absolute and lower than some of Mark's other answers to the confidence question (as will be seen in future sections). While he mentions a resemblance to known student mistakes (M.2.54), there don't appear to be any significant doubts at this stage.

Next, the interviewer brings in the additional roots to the equation, and asks Mark to go through a similar process with one of those roots:

M.2.57	I: Okay. So, as I mentioned, Bombelli knew only about the root 4. But we know about two additional roots.
M.2.58	M: Mm-hm.
M.2.59	I: I'll probably ask you to simplify it a bit more, not to get too complicated. Eh, expressions. So if you simplify it a bit more, you will have what?
M.2.60	M: Okay, let's see, this is minus 2, plus or minus, 12 is 4 times 3, so we have 2 root 3 over 2, minus 2, plus or minus root 3. Okay.
M.2.61	I: Good. Exactly. I just don't want to carry on this square root of 12. Eh, if

	Bombelli would knew about the existence of three roots, (?) root 4 and these two ones, of course he would like to check whether these two roots can be equal to this one.
M.2.62	M: Uh-huh.
M.2.63	I: And my question is, after you check that this expression equals four, how would you check that- if this expression equals this one? [indicating $-2 \pm \sqrt{3}$]
M.2.64	M: And, well, it seems to me, probably the simplest way, seeing as I've already done this, is to say, well look it's equal to four, it's obviously not something else.
M.2.66	M: Is four equal to minus two minus root three? No. Equality can't work like that.
M.2.67	I: So, I'm hesitating.
M.2.68	I: Do it.
M.2.69	I: To ask you to do it. Okay, do it, let's say, from one of the roots, just not to be too complicated, let's say that we choose which one. Minus 2, what do we want?
M.2.70	M: Eh, minus root 3.
M.2.71	I: Minus. Good. So you pick up, eh, you pick minus 2 minus square root of 3, and now your task is to check if this one equals to the root in complex numbers.
M.2.72	M: Okay, so, I want to show this. Well, this isn't true, so, okay. But if I'm starting with this, presumably I don't know that, and I would want to approach it the same way. And, I guess that means what I would do is do this thing that I did, except with minus 2 minus root 3 instead of 4. Yech. Ah, okay.
M.2.73	I: (…)
M.2.74	M: Okay, so is there, well, I'm guessing by the way you said that that there is a nicer thing to do than hack through this again.
M.2.75	I: Oh, yes, I can show you. Ah, it's no problem. Ah, I can do, let's say this. So if I'll need to do this one, 4 equals, eh, 2 plus 11 i plus cube root 2 minus 11 i, I can cube it but without, eh, transforming it to this form. Besides like you are lazy and you don't like to write again and again the same stuff, I am lazy. That's why I'd like to remind myself the formula of cubic before doing it just to simplify the

M.2.76	M: Okay.
M.2.77	I: solution. If I remember, $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.
M.2.78	M: Okay. Yeah, I suppose that-
M.2.79	I: I just don't want to do what you done that is to square and then to multiply once more so I just recall myself this formula.
M.2.82	M: Well, I could try it with, this,
M.2.83	I: Okay.
M.2.84	M: That'd probably be nicer to do than that other stuff, so, okay, then we have,
M.2.85	I: Would you like, a really quick exercise, to do in this way, this four stuff.
M.2.86	M: Ah, then, I don't-
M.2.87	I: You don't.
M.2.88	M: I don't really think it's necessary.
M.2.89	I: Okay.
M.2.90	M: I mean, ah, I'd get the same thing on the right-hand side.
M.2.91	I: Mm-hm.
M.2.92	M: And on the left-hand side I'd get 64. I know that. So, that should be fine. [Mark works through some calculations.] Well, on the one hand, I know that this is not correct, because it's actually equal to four, but on the other hand, I don't see how- Oh! Oh, ah, this is not equal to this. [Mark does more calculations, aiming to get to a clearly false statement.]
M.2.93	I: (?) about this formula.
M.2.94	M: But, wait. I thought something else, though, because, if I subtract this, from both sides, then what I get is, minus two minus root three, cubed, minus fifteen, times that thing, minus four. Which has to be zero, because that's another root. Which means that this should be true.
M.2.95	M: Now,
M.2.96	I: And here we come to the paradox. Would you like to complete this computation in order to see that minus two minus square root of three is as good as four in this computation?

M.2.97	M: Okay. Okay, so, let's see, this is, wait. Okay. My a is minus 2 and my b is minus root 3, so we'll have, ah, minus 2 cubed is minus 8, plus er, minus root 3, cubed, plus 3 times negative 2, times negative root 3, and that's minus 2 minus root 3, and then minus 4, does that equal 15 times minus 2 minus root 3, well, presumably it does, but let's see here, minus 8 minus 4 is minus 12, then, ah, minus it'd be 3 root 3, um, plus 6 root 3 times minus 2 minus root 3, is that equal to 15 minus 2 minus root 3? Well, let's see. Minus 12 minus 3 root 3, minus 12 root 3, minus 6 times 3 so 18 equals minus 30 minus 15 root 3, yes that's true. Okay.

Here, Mark is asked to show that show $\sqrt[3]{2+11i}+\sqrt[3]{2-11i}$ is equal to one of the other roots of the polynomial. Mark is initially confident that this is not true, but still checks it algebraically. He is willing to work through it the same way, but the interviewer demonstrates a shortcut, and Mark comes to the result that it is equal once he recognizes the form of the original polynomial.

After this point where Mark has come to the paradox, we proceed to a direct discussion of how it is possible, and the mathematical reasoning behind what has just been found.

M.2.98	I: So now the real question. How it's possible. All this was preparation for this question, sorry it took so long, but I think that you didn't suffer.
M.2.99	M: Okay. So, this, as I recall, this is to do with, some weird business about complex roots and stuff like that.
M.2.100	M: More precisely, in general, if you take the root of some complex number, you get multiple different complex numbers, and stuff like that. And, as far as whether this seems like a reasonable thing, ah, I suppose that it's- something that kind of has to come out of, uh, saying that i squared is negative one. If you square, well if you take negative i and you square it, you also get negative one, so, the decision of which i is plus i is, I suppose entirely arbitrary. And, since you've just declared i, you can't even know which one it is, kind of? Well, it's the one you declared it was, but, there's nothing that necessarily makes that one more positive than the other one. So, for me I suppose that,

	uh, it's because of things like that that it seems like, ah, this stuff about there being multiple roots of things in complex numbers and all of that, is reasonable. I think it's also somewhat more apparent to see with the, ah, r e to the i theta version of complex numbers, and then we sort of see a commonality with, periodicity of angles, trig functions, stuff like that, like, if you go around halfway, or you go around all the way and then go around halfway, you end up pointing in the same direction, so, well, I guess by that reasoning there should be infinitely many, but, they eventually start repeating their directions in the complex plane, too, and thus end up being the same real numbers.
M.2.101	I: Mark, if, I'd say this explanation, and you would be in my shoes. On the scale from one to ten, to which extent this explanation would be convincing for you. What you just said. Or simply,
M.2.102	M: Hmm.
M.2.103	I: to which extent are you convinced by your explanation?
M.2.104	M: Ah… maybe, maybe that one would be more, like, ah, seven. Well, I think that it's, less of that being the explanation in and of itself and that's more of, well, I've seen all of this stuff done, with the complex numbers and complex plane and conversions and like all of those things, that are sort of the real proof that I know is there. And, I can't just like, reproduce all of that. That would take forever.
M.2.105	I: I know that when you began, these transformations, you were convinced that it's impossible. And by the end, you are now convinced that it is possible, and you tried to produce some explanation based on some general laws related to the complex root.
M.2.106	M: Yeah.
M.2.107	I: Can you please point out exactly, when you changed your opinion? And why, on this process.
M.2.108	M: Um, I suppose, uh, that would be, uh, when I got to, well, I'm going to say this step, I'm going to mark it with a little star here [pointing to where he wrote $\left(-2-\sqrt{3}\right)^3-4=15\left(-2\sqrt{3}\right)$ as a step], uh, and, this is when I noticed that

	this has an equivalent form to that equation x cubed minus fifteen x minus four equals zero, ah, that we started with.
M.2.109	I: Mm-hm.
M.2.110	M: And so, ah, it must be true, because this is a root of that equation. That's what it's for.
M.2.111	I: Something amazes me right now. Let me ask a psychological question about what you have done now. You were quite confident that it is not true here.
M.2.112	M: Yeah.
M.2.113	I: You are quite confident, okay, seven out of ten, but it's quite confident that, ah, the opposite is true.
M.2.114	M: Yeah.
M.2.115	I: So were you inconfident in some place here? Or one confidence just turned into another confidence?
M.2.116	M: Not really.
M.2.117	I: Without any hesitation in the middle.
M.2.118	M: I think, well, there was a bit in the middle I'm like, wait a second, and, I think that in my reasoning process, there was a part of, ah, remembering this stuff about non-uniqueness of complex roots.
M.2.119	I: Mm-hm.
M.2.120	M: Which, went to my earlier thing of, well it's a number it has to be unique, went, oh, right, complex roots don't do that.
M.2.121	I: I think we should stop.
M.2.122	I: I would like to interfere because you want to stop, but there is something that I am puzzled with. Because you talk about complex roots, but I see real root of this, and I see a real root of four, and you are talking about complex roots, and I am lost, because I see two real roots. The both of them you just showed equal this strange expression. So, something doesn't sit well with my understanding of transitivity.
M.2.123	M: Uh,

M.2.124	I: I don’t see anything complex.
M.2.125	M: Well, the cube root of the two plus eleven i, cube root of two minus eleven i, like, these are complex numbers. And,
M.2.126	I: But we were talking about the equation, the starting equation, and two real roots of it.
M.2.127	M: Hmm. Okay, so, I suppose that what we would want to do there is something like, uh, hmm. We can say that they’re both equal to this sum of cube roots thing, ah, although they’re not equal to each other, because the cube roots of complex numbers and those kinds of things aren’t unique, and, unlike something like square root of four, we don’t necessarily have a way to declare, okay, this is the primary one, to make it a function all the time. So, hmm. I guess what we have to do is say that for complex numbers, there’s an implicit understanding, well, somewhat implicit, mentioned in initial discussions of complex numbers, hopefully unless you’ve got a really lousy book that leaves it out or something, that says, okay, expressions like this are not unique, be careful with what saying it is equal to something means, it is not quite, an equivalence relation the same way the normal equals sign is.
M.2.128	I: So, an equivalence relation, what is it?
M.2.129	M: Hmm. I’m not quite sure what to call it. There’s probably an agreed-on name somewhere.

Once confronted with the paradox, Mark says that he recalls some facts about complex roots which can explain what happened, saying that the distinction of which root of -1 is ‘positive’ is arbitrary, as well as the repetition in the polar form of complex numbers, using these as examples of non-uniqueness in complex numbers which he says generalizes. When asked, he gives a moderately high (7 out of 10) rating for how convincing this explanation is, saying that it is part of a more in-depth explanation that he can’t replicate during the interview. Asked about his change of belief on the paradox, he says that it occurred when he noticed a form similar to the original equation given in the problem. The interviewer then mentions that this change of belief appeared to be very fast, and Mark says that he had a short hesitation, which is what prompted his recall of the facts mentioned about complex numbers. The interviewer then points out that none of the solutions are complex, leading Mark to say a bit more about non-uniqueness and the implicit parts of the notation.

As the interviewer remarks (M.2.98), Mark appears to at least not mind doing algebraic computations, and probably enjoys doing them to some extent. Some of the computations took significant amounts of time (two in this task took at least 12 minutes each), and this does not appear to pose any deterrent to Mark. For most of the computation periods, the general appearance is of an experience of flow, or absorption in the task at hand, and the uninterrupted stretches of time spent on a task are characteristic of these experiences. This is somewhat similar to Cyrus' experience of getting lost in algebraic computation, although Mark does not appear to lose sight of the overall problem.

By pointing out that none of the roots are complex (M.2.122), the interviewer tries to create a conflict in Mark's thinking, and while Mark elaborates somewhat about non-uniqueness in complex numbers, he does not appear to see the ultimate results all being real numbers to be a problem for this discussion at all. This reinforces the appearance that Mark has high confidence in this conclusion, and that it is fairly difficult to introduce doubt at such a stage (which will recur in later sections).

Finally, there is a short discussion about Mark's feelings about the paradox, which is considered particularly in light of the unusual affective responses found in other interview participants.

M.2.130	I: On a scale from one to ten, how do you like this paradox?
M.2.131	M: …Eight or nine?
M.2.132	I: Explain, please.
M.2.133	M: Well, there's sort of a gradual process of seeing things and figuring things out through working through it. And I thought that some of the things we found here, were pretty neat, and I don't really know how to put these kinds of things into words that well. Um…
M.2.134	I: What did you like about it, then? What don't you like about it?
M.2.135	M: Hmm. I liked these kinds of, realizations, about- how complex numbers

	work, and I think it might have helped me under- get a- bit of a more complete understanding of why some of this complex number weirdness does what it does. This, this sort of thing is, a little frustrating to nail down, like, okay, uh, so we can see that in this case, cube root of two plus eleven i plus cube root of two minus eleven i, isn't a number as such.
M.2.136	I: Mm-hm.
M.2.137	M: Which- feels a little wrong, like, here's an expression, there's one expression, I wrote it down, it should be a number.
M.2.138	I: Hmhm. Thank you.
M.2.139	M: But it's not.

When asked, Mark indicates that he generally likes the paradox ("8 or 9" out of 10), saying that he liked the gradual process of figuring things out and some of the findings themselves (but saying he could not express it well). He said that while he liked some of those realizations, he had some frustration with the expression (from Cardano's formula) which was found not to be a number.

We see here that Mark does not hold to the previous idea once it has been shown incorrect (M.2.92 to M.2.97), despite reporting high confidence in it before (M.2.50); there is no time spent trying to justify or hold to the previous idea once it has failed mathematically. This is consistent with the types of responses to paradoxes seen in previous interviews, and thus again differs from responses to paradoxes in the general population. The situation here is not as linked to a conflict with intuition compared with other paradox encounters, however, since the idea that was dropped was also justified algebraically (in terms of the definition of equality, at M.2.66), rather than as something intuitive. The reason for the paradox being initially noticed was also one of algebraic structure (M.2.108). Additionally, the most doubt displayed is in expressions like "that's interesting" (M.2.92) or "wait" (M.2.94). It does not appear that there is a significant portion of the interview where Mark has doubts, which is consistent with his own self-assessment (M.2.118). Overall, there appears to be a general confidence in mathematical reasoning; keeping in mind the long periods of algebraic work, even for parts already given (M.2.1-6), this confidence appears to be in mathematics as a process, rather than in mathematics as a result or in mathematicians generally. Due to the explicit addition of the questions about confidence in this version of the interview protocols, this is more difficult to directly compare with the results from Joshua and Cyrus. However, it does fit Cyrus' expressions of confidence in mathematical reasoning relative to his intuitive speculation.

We also note (in M.2.99) that Mark has encountered the idea of complex numbers having multiple roots before (in the context of a course on complex variables), although he did not recall this before being confronted with the apparent paradox. Thus, it appears that this particular facet of Mark's mathematical knowledge probably did not significantly impact the reasoning presented before this point.

Mark here shows a generally very positive perception of encountering a mathematical paradox (high rating at M.2.131, only negative elements are minimized in M.2.135). This is, again, somewhat unusual compared to participants in such tasks in the general population, but it is not limited to paradox-related tasks, as will be shown in later sections.

At some points (such as M.2.44), I also notice that Mark preemptively avoids the appearance of saying something mathematically invalid, such as by talking about the reversibility of algebraic steps and using "quantity squared" despite also writing on paper. On the other hand, some of the dialogue shows particularly casual language ("square root thing", M.2.2), so this tendency is probably not simply an issue of formality. These are naturally both affected by the 'speak-aloud' protocol, and it is noteworthy to see what is filtered despite the protocol and what is not.

While Mark does engage in algebraic work that most respondents would avoid, I also notice that at times (M.2.44, M.2.92), Mark gives variable names to longer expressions, in places where it would not generally be considered necessary. This is one of the few things that deviates from the general structure of lengthy, written-out algebraic work, but is still firmly within the systematic algebraic context; it suggests that Mark is not entirely against shortening answers, but needs to have his own justification for doing so. This is most likely at least partly a result of Mark's exposure to various advanced mathematics courses.

A notable common element in Mark's solutions is the consideration of the intent of the problem (in M.1.6 and M.2.6). Among other things, this works against the conception of autism promoted by researchers that give the lack of theory of mind as a characteristic trait of autism (as discussed in Section 2.1), particularly since the consideration of intent is unprompted by the interviewer. The consideration of the patterns of thought of historical figures (M.2.23) also works against this.

9.3. The Ping-Pong Ball Conundrum (III)

In this part of the interview, Mark was asked about the ping-pong ball paradox (presented in more detail in Section 6.1). However, this went somewhat differently than most of the problems, since Mark had seen it recently.

At the beginning of this interview, the ping-pong ball paradox is mentioned, and Mark is asked what he recalls of it:

M.3.1	I: What do you remember about this paradox? Are you familiar with it?
M.3.2	M: I think so. Um, the, ten balls in, one ball out, that thing?
M.3.3	I: Yeah.
M.3.4	M: Okay.
M.3.5	I: So, say more please.
M.3.6	M: Uh, okay.
M.3.7	I: What is the paradox? What are the conditions, and what is the paradox?
M.3.8	M: The way I remember it working is, you take an interval of time, from time zero to some fixed time t, and, in the first half of the interval you put in balls numbered 1 through 10, and you take out ball number 1.
M.3.9	I: Mm-hm.
M.3.10	M: And in the first half of the remaining half, you put in balls 11 through 20 and you take out number 2.
M.3.11	I: Where is number 2?
M.3.12	M: Well, it's the one that we put in last time.

M.3.13	I: So, ah, your enumeration of the balls is 1, 2, 3, 4, 5, 10, 11, 12, so on?
M.3.14	M: Yeah.
M.3.15	I: And number 1 you mean here, and number 2 you mean here.
M.3.16	M: Yeah.
M.3.17	I: Okay.
M.3.18	M: And then in the next remaining half chunk I put in 21 through 30 and take out 3. [marking on paper] 11 through 20, 21 through 30, 3, and, you sort of repeat this.
M.3.19	I: Just to be clear, I understand that, the ball you take out, number 1 is from the 1 to 10,
M.3.20	M: Yeah.
M.3.21	I: And number 2 is?
M.3.22	M: It's in here, well,
M.3.23	I: Also here.
M.3.24	M: 2 is between 1 through-
M.3.25	I: Because you denoted it here, so,
M.3.26	M: Yeah.
M.3.27	I: you take the balls from the beginning, the ball number 1, the ball number 2, the ball number 3,
M.3.28	M: Well, in- I take this one out in this time interval, I take this one out in this time interval,
M.3.29	I: But which ball do you take? In- let's imagine all the-
M.3.30	M: Number 2!
M.3.31	I: Number 2 in-
M.3.32	M: That's the point of numbering all the balls.
M.3.33	I: Number 2 from the second 10?
M.3.34	M: No.
M.3.35	I: Number 2 from the first 10.
M.3.36	M: No, number 2. There's only one number 2.

M.3.37	I: Oh, okay. Okay.
M.3.38	M: These are 11 through 20.
M.3.39	I: Okay, I see. Perfectly clear, thanks. And, what is the paradox?
M.3.40	M: That's the point of numbering all the balls in the first place.
M.3.41	I: Okay.
M.3.42	M: Because if you do it with unnumbered balls you get a different result.
M.3.43	I: Mm-hm, mm-hm.
M.3.44	M: And then, the question is, ah, what are you left with at the end?
M.3.45	I: Mm-hm.
M.3.46	M: Which is, nothing.
M.3.47	I: Why?
M.3.48	M: Well, because every ball has been removed.
M.3.49	I: Okay.
M.3.50	M: Like, suppose by way of contradiction that you have some stuff left. Well, all your balls are numbered.
M.3.51	I: Mm-hm.
M.3.52	M: So, your smallest one has some number. Call it n.
M.3.54	M: But in the nth step you removed it.
M.3.55	I: Mm-hm.
M.3.56	M: Contradiction.
M.3.57	I: Okay.
M.3.58	M: So you don't have anything.
M.3.59	I: Mark, now you're really smiling. There is something that you like about this paradox?
M.3.60	M: Yeah. Well, I like things like, proof by contradiction and mathematical proofs and, nice orderly structured things like that.

Here, Mark outlines his general understanding of what the ping-pong ball paradox is, as well as the solution. He explains the solution via proof by contradiction, and states a like for ordered structures, including in this both mathematical proof generally and proof by contradiction specifically.

Here, there is a possible conflict questioned by the interviewer (M.3.29), which Mark does not see as a conflict; he appears to be taking the “number 2” in a more formal way, and in this interpretation there is only one. There appears to be some frustration from the apparent problem with communication as well; this exchange shows the most variation in tone from Mark across all of his interviews. Among other things, this frustration may suggest higher than average investment in the problem and its result.

When prompted to explain the paradox (M.3.50), we see that Mark starts off by immediately going to proof by contradiction, which has been shown by other research to be generally unintuitive for students and not typically a first choice. The appreciation Mark states for it here (M.3.60) is considerably outside the norm for students of mathematical proof.

In the next part of the interview, Mark is given a modified version of the paradox to consider:

M.3.61	I: But now, when you put 10 balls in, you put 1 ball out, and this is the first ball out of these 10.
M.3.62	M: Okay.
M.3.63	I: You put the next 10 balls, and you take out the first one out of these 10 balls.
M.3.64	M: Okay, so you take out number 11.
M.3.65	I: Yeah.
M.3.66	M: Like number 1, number 11, number 21,
M.3.67	I: Mm-hm.
M.3.68	M: And so on.
M.3.69	I: Yeah.
M.3.70	M: Okay.
M.3.71	I: What would be at the end?
M.3.72	M: Well, you’ll have infinitely many. You’ll have 2 through 10, and 12 through

	20, and all of those. Since you never take them out.
M.3.73	I: Okay. Ah, if you compare these two versions of the paradox, which one is more paradoxal? Which one is more intuitive or counterintuitive?
M.3.74	M: Hmm. Well, I think that the first one is more counterintuitive by itself, but the really counterintuitive thing is putting the two together.
M.3.75	I: And what about the fact that, ah, actually, the huge difference in the results, for the first time, for the original version of the paradox you have nothing as an answer, and for the modified version of the paradox you have what for the answer?
M.3.76	M: Ah, infinity.
M.3.77	I: Infinity for the answer. But, ah, in this and in that case, what you do, you put 10 in and 1 out, 10 in and 1 out, 10 in and 1 out, and only difference is in the order, which balls you put out. How does it make you feel, this thing? To which extent is it surprising for somebody, or not?
M.3.78	M: Well, I can certainly see that it would be generally surprising, and pretty weird, like, if you just tell people about it in general, I mean, to me it seems like, well, okay, yes, this is the result of doing this mathematical proof and, this mathematical proof, and the results seem intuitively kind of weird, but, that’s also kind of what you expect because, you’re messing around with infinity, and of course you’re going to get something that’s intuitively weird, because intuition is based on people’s life experiences, and people’s life experiences don’t have infinity in them anywhere!
M.3.79	I: I agree. Okay, let me play a character, who is shocked by the difference between the paradoxes. So, it cannot be. You take 10 in, 1 out, 10 in, 1 out, 10 in, 1 out. So whatever, infinity or not, it’s supposed to be the same result. And your explanation, that it is just something strange and weird about infinity, is just not convincing for me.
M.3.80	M: Okay.
M.3.81	I: So how would you convince such a person that the results are so different in these two cases?
M.3.82	M: Well, partially, I guess, the important difference is that you’re doing things in

	order. Like, in the first case, your removing process is slower, well, okay, in both cases the removing process is slower, but in your first case the removing process isn't skipping anything, so, every ball is eventually going to be removed.
M.3.83	I: Mm-hm.
M.3.84	M: I think the second case is, partially more of the intuitive one, like, you put in more than you take out, and you'll end up with an infinite pile. That seems reasonable enough. And, you can see what-
M.3.85	I: I am playing that person. The first task is not intuitive for me, because you put in more than you put out. And when you're saying that nothing remained, it's very not intuitive.
M.3.86	M: That's not unreasonable, it makes sense that, well, it doesn't fit with the sort of stuff that we do in general, in life, in finite cases, I guess, well, the sort of mathematical proof way of going about it is, well, okay, it seems like you should be left with something, but if you're left with something, then, well all the balls have numbers, so whatever you're left with has to have a number. And, if we assume, alright, I'm left with this one, and we can't have been left with a specific one, well, let's call it n. But, we have infinitely many steps, and in the nth step, we removed this one that we called n. So whatever we're left with it can't be that one.
M.3.87	I: Mm-hm.
M.3.88	M: But it can't be that one, no matter which one we pick.
M.3.89	I: Well,
M.3.90	M: So there's nothing we can be left with.
M.3.92	I: Ah, it is very difficult for me to play that person, because I completely agree with you. So let's stop. Okay.

Here, the interviewer outlines another version of the paradox, which retains the process of adding ten balls and removing one, but in this case, the ball removed is always the first of the ten that were added; Mark restates this as being balls numbered 1, 11, 21, and so on. Mark then says that the end result for this version is infinity; when asked to compare then, he says that the first version is more counterintuitive, but the combination of both versions is more counterintuitive than either individually. He says that while this would be surprising in a general context, in the context of a mathematical proof involving infinity it is not unusual. Mark is then asked to convince a hypothetical character who does not believe the results. His first response emphasizes the importance of doing the operations in a particular order, and the second is a more informal version of the proof by contradiction argument.

In the later response to the interviewer's hypothetical objections (M.3.86), we see more strongly the gravitation toward using proof by contradiction. The use of proof by contradiction outside a formal mathematical context is another indicator of an unusually strong predilection for using it, as well as a general acceptance of it significantly higher than most students exposed to it.

9.4. Gold and Silver Coins (A Probability Problem)

Here, Mark was presented with a short probability problem: There are three boxes, each of which contains two coins. Box 1 has two silver coins, box 2 has two gold coins, and box 3 had one silver and one gold coin.

The interviewer presents the basic problem setup to Mark, and then asks for the probability of getting a second gold coin:

M.4.1	I: So the question is, what is the probability that, if you take another coin from the same box, it will be also a gold coin?
M.4.2	M: Okay. So, well, the sort of, first instinctual thought is a half, but I don't really think that that's right. So, wait a second. How did I get the first gold coin? What was the procedure there?
M.4.3	I: You look at these three boxes, and you take this coin from one of the boxes.
M.4.4	M: Okay. So I uniformly pick one of the boxes,
M.4.5	I: Yeah.
M.4.6	M: And uniformly pick a coin from the box that I picked.
M.4.7	I: Yeah.

M.4.8	M: Okay. …Alright, so I want the probability that- second coin in the same box?
M.4.9	I: Yes, the second coin should be picked from the same box as the first one.
M.4.10	M: Okay, so that's the probability that, that's equal to the probability that I, picked a coin from this box, because this will happen if and only if I picked from here, and that probability is one third. Wait. That seems off. Okay, I want the probability that I'll get another gold coin. And, I'll get another gold coin if I picked this box. Wait. That's not quite right. Okay, I should probably draw out the whole thing. Um, [writing a chart of possible outcomes], so the possibilities are, I picked box 1 [with both coins silver] and get a silver coin, I picked box 2 [with both coins gold] and I get a gold coin, I picked box 3 [with one silver and one gold coin] and I get a gold coin, and I picked box 3 and get a silver coin. I could say the probability that I picked box 1 and get a gold coin, well, that's zero, I don't really need to consider that. Okay, so the probability I picked box 1 and got a silver coin, that's one third. The probability that I picked box 2 and got a gold coin, now, that's, that's also one third, the probability that I'm in box 3 and I got a gold coin that's, one sixth, the probability that I'm here and I got a silver coin, well that's, also one sixth. Okay. So now, I'm told, alright I've got a gold coin. So now I've narrowed it down to, here, this didn't happen, this didn't happen. And so, I want to know what's the probability that, my other coin is also gold, well, it's the probability that I'm in that situation, over the total probability that the thing that I know happens, happened. And that's two thirds. Okay. And, I could, write it in a more elegant mathematical formula way, with the Bayes' formula stuff, but I don't really feel I need to do that.
M.4.11	I: So just to be clear, (…) your first intuition was half, but you know it is wrong.
M.4.12	M: Yeah.
M.4.13	I: Your second thing was one third,
M.4.14	M: Which also seemed wrong.
M.4.15	I: Okay.
M.4.16	M: Like, I'm not, exactly quite sure how to put my finger on thinking that's wrong, but it definitely seemed wrong, like, okay, I definitely missed something here.

M.4.17	I: With one third.
M.4.18	M: Yeah.
M.4.19	I: Yeah. And, ah, your third approach led you to two thirds and this is the answer that you are confident in?
M.4.20	M: Yeah.
M.4.21	I: Okay. Let's say that, on the scale from 1 to 10, to which extent are you confident in your answer two thirds?
M.4.22	M: Mmm… maybe like, nine point nine?
M.4.23	I: Mm.
M.4.24	M: I'm not entirely comfortable with claiming a hundred percent confidence about anything.
M.4.25	I: About anything.
M.4.26	M: Yeah.
M.4.27	I: Okay, because, ah, when you ask such a question and get something else but not 10, the next natural question is, what should ha- what can happen in order to increase your confidence?
M.4.28	M: Well, I think that, if I wrote out the Bayes' formula stuff, I'd put my confidence at maybe like, nine point nine nine nine or so.

Here, Mark gives a first response of one half, which he says he thinks is wrong, and then asks for the specifics of the procedure by which the coins are selected. After the explanation, he gives a second response of one third, but rejects this as well, and proceeds to work out the possible outcomes on paper, ultimately giving the result of two thirds that Mark settles on, reporting very high confidence (9.9 out of 10). He then says that he is not comfortable with claiming full confidence, and says that if he were to write out the solution using Bayes' formula, his confidence would be higher (about 9.999).

The rejected responses given here (1/2 at M.4.2, 1/3 at M.4.10) show that Mark does use some intuitive reasoning, although he does not put much confidence in it, and this appears to be the only case of an intuitive response in Mark's interviews. The first response of 1/2 is a common response to the equivalent Monty Hall problem, and one which many people, including university mathematicians, have argued for. The second response of 1/3 is the answer to the standard Monty Hall problem, and the recognition of its similarities could be what led to its suggestion. However, this version is phrased in such a way that the probability asked for is the complement of the standard Monty Hall result.

This combined with his suggestion of writing out the formula to increase confidence more (M.4.28) suggests that his confidence in an argument is directly related to how close it is to what is recognized academically as a proper form for mathematics.

Next, Mark is presented with an alternate solution to the problem, and asked to compare:

M.4.29	I: Okay. Okay. Great! Okay, so, let me present for you another solution.
M.4.30	M: Okay.
M.4.31	I: To this problem. Because you probably realized the pattern, this is what we do with every problem. The question, with which probability the second coin would be gold, is equivalent to another question. With which probability, from the beginning, you pick the box with two gold coins.
M.4.32	M: Okay?
M.4.33	I: And in this case, the solution is one third.
M.4.34	M: …Right. Which is, I guess goes back to, what is missing between here and here. Which is, that we're not doing it from the beginning, we were asking in the middle. Like, okay, we know that this is what happened, we drew the first gold coin. And we're asking from that point. And, which means that part of the universe is now barred to us.
M.4.36	M: We're not here, and we're not here. [indicating the 'silver coin' parts on the chart] A, more extreme example, if you do it at the end, like, you pick a box and you get a gold coin, and you take out the second coin, and it's also gold, what's the probability that you're going to pick the box with two gold coins? Well, it's one, you already did it.
M.4.37	I: Which solution is right, and which solution is wrong? What about my solution that leads to the answer one third, and your solution, which leads to the answer two thirds, because you understand that they both cannot be right.

M.4.38	M: Right. The two thirds solution is the correct one from how you phrased the problem.
M.4.39	I: Can you repeat please, how did I phrase the problem?
M.4.41	I: Then you can compare your phrasing and my phrasing.
M.4.42	M: Okay, your phrasing was, something like, ah, that we drew one coin, and it was gold, and now we're going to pick the second coin from the box that we drew our first coin from, and what's the probability that that one is gold.
M.4.43	I: Mm-hm. Yeah, I think so.
M.4.44	M: And,
M.4.45	I: And, you think that the right solution to this is two thirds.
M.4.46	M: Yeah.
M.4.47	I: And your level of confidence now, is?
M.4.48	M: Mm… nine point nine.
M.4.49	I: Okay. Which is about the same as it was.
M.4.50	M: Yeah.
M.4.51	I: So, which problem do you think fits the solution that I presented?
M.4.52	M: Ah, if we- if we were to ask something like, we have these three boxes, if we, pick one of the boxes, what is the probability that we get two gold coins?
M.4.53	I: Yeah.
M.4.54	M: Or, that we pick the box with two gold coins, which is the same thing.
M.4.55	I: And, why these two problems are different?
M.4.56	M: Because, the-
M.4.57	I: I'm just repeating because you've just, but I want to be, everything very clear
M.4.58	M: Because, the original formulation is asking in the middle of the process, when some of the things have already been decided.
M.4.59	I: Mm-hm.
M.4.60	M: And, if, you just sort of ignore what happened or are somehow unaware of what happened, then, I suppose then you'd have to stick with your original probability estimate because you don't have any information to revise it from.

M.4.61	I: But could you point out, the exact place in which the difference in formulations lead to different solutions, because your explanation is quite general. You say that this is the process, that this is just one act, and that's why there is a difference, but probably there is a difference and probably there is not.
M.4.62	M: Right here,
M.4.63	I: Mathematically. In (?) of course there is a difference, what's the mathematical difference?
M.4.64	M: Ah, the, math- hm. That, by stating that the first coin is gold, some of the, events in the probability space are ruled out.
M.4.65	I: Okay. Uh-huh.
M.4.66	M: And so that narrows our consideration, to the subset of the probability space that we know that we're in.
M.4.67	I: So probably I don't understand the table in this case. What is Box 1, silver?
M.4.68	M: Well, this is, ah, well these are the, four possible things that can happen with your first coin.
M.4.69	I: Okay. This row is about the first coin.
M.4.70	M: Yeah. You either picked box 1 or 2 or 3. And, it's either, well, this isn't entirely complete, there are the two other sort of, theoretical possibilities, well, you could have had, box 1 and a gold coin, but the probability of that is zero, so, we, and you could have had box 2 and a silver coin, and the probability of that is also zero.
M.4.71	I: Mm-hm.
M.4.72	M: So, well, we don't really care about those, because they aren't, really there to begin with.
M.4.73	I: I see, but, uh, what's written here is, as a matter of fact, is, isomorphic, to my formulation of the problem, because, what are the probability of picking one, box 1 is a third, what is the probability of picking, of choosing box number 2 is one third,
M.4.74	M: Right.

M.4.75	I: And what the probability of choosing box number 3, it is one third.
M.4.76	M: Right.
M.4.77	I: So let's say, at the level of the first row, there is no contradiction between original formulation and my second formulation. So the difference, I'm trying to pinpoint the mathematical difference between two formulations. So, if there is a difference, it's supposed to be at the next step.
M.4.78	M: Yeah.
M.4.79	I: Okay, so what it is?
M.4.80	M: Ah, the difference is, we know that our first coin is gold. Which means that we're either in this bit or this bit. [indicating 'box 2, gold' and 'box 3, gold']
M.4.81	I: Mm-hm.
M.4.82	M: And, since we know we're somewhere in here,
M.4.83	I: Mm-hm.
M.4.84	M: and we originally had, well, okay we had this equal distribution among all of the, well, maybe not, equal distribution's not the right word. I mean, we had this given distribution among these possible outcomes.
M.4.85	I: Mm-hm.
M.4.86	M: And now, since, we know that these outcomes didn't happen.
M.4.87	I: Yeah. So you rule them out. It's clear.
M.4.88	M: So, those are out of our new probability space.
M.4.89	I: So what does it mean? In this case?
M.4.90	M: So we have, the one third probability that we're in the box 2 case, ah, proportional to the overall probability that we would end up in, the subset that we know we're in.
M.4.91	I: Mm-hm.
M.4.92	M: Which is, one third plus one sixth.
M.4.93	I: Okay. So this is explanation, my okay means, do you want to add something else, or you're done with, ah…
M.4.94	M: That… seems like it's more or less it.

Here, the interviewer presents another solution, which starts from the beginning (before any coins are picked) and gets to the answer of one third. Given this, Mark says that they differ in starting from the beginning or the middle, and that in one version, some possibilities are ruled out. When asked, he says that the solution giving two thirds is the correct one based on the initial phrasing of the problem, and reports the same confidence (9.9) as before. Mark is then asked to give a problem which fits the one third solution, and poses the question starting from the beginning, without any coins chosen, asking the probability of getting two gold coins. Asked to explain the difference, he states that the original version started in the middle, where some of the relevant events had been decided. Asked to explain it mathematically, Mark states this in terms of a probability space where some events are ruled out and we are left with a subset of the original probability space. Mark explains the table that he produced, and indicates on that table which parts correspond to the subset that he described. The table is similar to Figure 9.4 below.

Box 1, S	Box 1, G	Box 2, S	Box 2, G	Box 3, G	Box 3, S
1/3	0	0	1/3	1/6	1/6

Figure 9.4: Mark's chart of probabilities for coin boxes

The solution the interviewer is presenting here (M.4.31) appears to be similar to the second solution that Mark rejected (M.4.10). While Mark does not reference this directly, he does not seem to spend noticeable time considering the alternate solution before rejecting it. It also does not appear to affect his confidence much, as the interviewer notes (M.4.49); in fact, the reported confidence levels are exactly the same.

One of the parts of Mark's interpretation of the problem relies on being "in the middle" (M.4.34), which is unusual in comparison to common ways to describe probability. To explain this interpretation, Mark uses a reduction-to-absurdity argument, comparing the extreme case of both coin choices having already happened (M.4.36). Compared to the interpretations of decided and undecided events examined by Chavoshi Jolfaee (2015), this appears to not fit neatly into either common version.

When asked to elaborate on a solution, Mark's response is initially general (M.4.58), without any specifics about boxes or coins. This is consistent with his work in the Cardano's method problem, where the algebra is mostly done without the problem's numbers substituted in; this suggests that the generality displayed there is part of an overall pattern of thinking and presenting solutions generally. This fits generally with the tendency found by some researchers to avoid prototype-based thinking (discussed in Section 2.2) in favor of processes based on general rules or principles.

Next, the interviewer proposes another possible alternative:

M.4.95	I: Okay. So let me add something to, to this explanation of this case.
M.4.96	M: Okay.
M.4.97	I: Eh, the probability that you are here,
M.4.98	M: Yeah.
M.4.99	I: Is a half.
M.4.100	M: Right. Well,
M.4.101	I: And this probably should be, added to the formula. Because there are consequences of this statement, that two options are impossible.
M.4.102	M: Well, that is a half [indicating 1/3+1/6]. I didn't simplify it, but, it's there.
M.4.103	I: I mean, the chance, that you are in this situation,
M.4.104	M: Yeah.
M.4.105	I: is one half of all the possible chances.
M.4.106	M: Right.
M.4.107	I: In this case, this expression, should be multiplied by one half. And the final answer would be one third. What do you think about it?
M.4.108	M: Well, the probability that we're in this situation is one, because we were told we're there.
M.4.109	I: Okay.
M.4.110	M: We know it's true.
M.4.111	I: Uh-huh.
M.4.112	M: We had, okay, the probability is one half, when we started, but then we

	drew the coin,
M.4.113	I: Which one here? One half, what's when we started?
M.4.114	M: Well, before we drew anything.
M.4.115	I: Okay.
M.4.116	M: Then we can say, okay, the probability that we end up in this situation is one half.
M.4.117	I: Mm-hm.
M.4.118	M: And, then we did it. And it happened.
M.4.119	I: Mm-hm.
M.4.120	M: And, now it's- something known, and it- it's not uncertain anymore.

In this excerpt, the interviewer states that the probability of being in the initial situation (of having the first gold coin) is one half, and suggests that this should be added to the formula, asking what Mark thinks of this idea. Mark says that the probability of having the first gold coin is one, since it was given in the problem, and argues that because of this, the first coin is not uncertain.

The introduction of the idea here (M.4.107) is another example of an attempt to introduce a conflict that gains very little traction before being rejected. It does not appear that this argument affects Mark's confidence, although in this case it does not seem similar to something Mark considered and rejected on his own. Mark's argument for rejecting it also echoes his previous "in the middle" explanation from the previous segment, showing a consistent consideration of the problem.

Finally, Mark is asked how much he liked the problem:

M.4.121	I: The standard question, on the scale of one to ten, how much do you like it? Or, as we reformulated it, how beautiful is the problem for you?
M.4.122	M: Hmm, maybe, nine? I very much like these probability kinds of things
M.4.123	I: Okay. So, the explanation to, high score nine is, you like probability.
M.4.124	M: Partly. Yeah, it's, well,
M.4.125	I: Can you be more specific because from, from this answer

M.4.126	M: I think,
M.4.127	I: I can conclude that any probability problem would be, very highly appreciated by you.
M.4.128	M: That's partly true, but, I think that, now that I think about it, this is, perhaps, one of the simplest possible formulations of a thing that hits all of the probability issues that it does.

Here, Mark rates the problem quite positively (9 out of 10), and attributes this to both liking probability generally and that this problem evokes specific ideas in probability and is one of the simplest ways to hit on those ideas.

This conclusion shows a positive perception that is not restricted to the paradox-related tasks, and Mark's reasoning (M.4.128) hints at an appreciation both for the concepts of probability used by this task and the relative simplicity or elegance of its structure.

9.5. Summary of Mark's Interviews

One common trait in Mark's interviews is a focus on algebraic methods of solution, even with problems presented entirely geometrically (M.1.6). This is a trait shared in common with Cyrus, but in opposition to Joshua. Another trait apparent in Mark's solutions is a consideration of the intent of the problem's author (such as M.1.3) or characteristics of its construction (M.4.128). This again contradicts some of the more severely negative versions of a theory of mind deficit conception of autism, particularly since the considerations of the mind of the person constructing the problem are unprompted.

Another feature apparent in Mark's interviews is a strong desire to independently confirm calculations and theorems which are presented. Considering the difference in results when this does not occur (M.1.151), it appears that this is in fact important for Mark to progress on the problems. In a sense, this is the opposite of a common problem with undergraduate students, where they may attempt to apply procedures without working them out in order to understand them.

Chapter 10. Conclusions and Reflections

In this chapter, I focus on comparing the results of the participants in my interviews, and draw more general conclusions. I also consider limitations of my current research and propose suggestions for future research directions.

10.1. Summary

My particular research questions in this work were, first, how do the experiences of mathematical problem-solving differ for adults on the autism spectrum? I am keeping in mind that this may include positive, negative, and neutral aspects. Second, what would be an explanation for those differences that might promote understanding of the autistic community? To address these questions, I have examined the mathematical problem-solving of three adults – Joshua, Cyrus, and Mark – on the autism spectrum by conducting a series of individual interviews. The participants had varying degrees of mathematical preparation, ranging from undergraduate mathematics coursework towards a degree in science to holding a master's degree in mathematics. By providing a range of mathematical tasks in different areas, including algebra, geometry, probability, logic, and paradoxes of infinity, I was able to observe different aspects of their approaches to problem-solving. I have examined the approaches shown by these interviews with a variety of theoretical constructs and with reference to prior mathematics education and autism research, from a perspective that is informed by Vygotsky and supportive of neurodiversity.

10.2. Findings

One common factor I have seen in the three participants is an inclination toward particular methods while solving problems. Joshua explicitly reported a preference for visual interpretation and explanation that was reflected throughout the interviews, as well as a preference for real-world examples. In contrast, both Cyrus and Mark stated and showed a fairly strong preference toward algebraic solutions rather than geometric ones. These methods appear to be strongly embedded in participants' minds, even in some cases where they are consciously trying to avoid them (M.1.6), suggesting that their underlying tendency is fairly strong. There is also in some of Cyrus and Mark's interviews a tendency toward more abstract and general solutions (C.2.78, M.2.3, M.2.6, M.4.58) and away from real-world illustrations and applications, which may be linked to the algebraic tendency or may be separate (although abstraction is typically considered easier in an algebraic context than it is in a geometric one). These could be effects of different forms of compensation, but the variations in presentation here suggest that the nature of that compensation may be more complex then something directly related to spatial reasoning.

All of the participants display unusually positive reactions to paradox-related tasks. Mark displays less outward enthusiasm (compare "pretty neat" at M.2.133 with, for example, "beautiful" in J.2.9), but it is still positive, and difficult to tell whether this is more a difference in participants' perception or their expressiveness. Also, terms like "beautiful" are more readily associated with geometric ideas than algebraic ones. This may also hint at broader differences in perception that the algebraic/geometric difference is only a part of. These broader differences could be linked to the formation and structure of inner speech, but this is more difficult to determine with adult participants.

A third commonality between participants is that they tend to have a high level of trust in the truth and consistency of mathematics, not displaying many of the typical objections to paradoxical tasks outlined in prior research. When faced with apparent contradictions, they are more likely to question an intuitive response rather than a formal mathematical result. Additionally, contextual considerations of problems phrased in a 'real-world' setting appear to be given less relevance. This may be related to the observation with Cyrus in the four-card problem that having a real-world context did not help to solve the problem more easily; the other participants do not have a direct comparison for this. This also appears to be a logical result of an orientation toward structure, or in Vygotskian terms, systematic over intuitive reasoning.

Both Joshua's use of diagrams to solve the Magic Carpet task and Mark's attention to presented algebraic manipulations show an unusual level of precision for lower-level problem-solving tasks. While this does not have a direct parallel in Cyrus' interviews, he was not averse to doing these calculations either. This suggests some possibility for a tendency toward completing these calculations instead of skipping steps, which is consistent with other observations of more systematic and less intuitive reasoning.

A more broad similarity, seen in instances like Joshua's use of diagrams (J.1.1), Cyrus' use of divisibility (C.1.3), or Mark's "in the middle" probability description (M.4.34), is that problem-solvers on the autism spectrum are likely to come up with some unexpected ideas in their approaches to problem solving. In a classroom setting, these sorts of responses could be used as a starting point to show aspects of the mathematical content in a new light, but the teacher would need to be prepared to do this. In the context of a specific student, it could thus be helpful to find the pattern of their responses (any special interests could be helpful here) in order to be more prepared to take advantage of such opportunities. Taken as a whole, the tendencies shown here provide an illustration of some differences in learning seen in students on the autism spectrum; this begins to answer the question of what differences are present in the experience of mathematical problem-solving in adults on the autism spectrum, but the variety of results suggests that more research is required. I believe that these differences and the framework used to interpret them can help promote some understanding of people on the autism spectrum and of the autistic community, and that the variation of the participants is also helpful in demonstrating variation in the autistic community more broadly. The characteristics observed here suggest some parallels in other areas of life that can contribute to understanding, but the observation of variations is also important in order to avoid undue generalization and an incorrect understanding of people on the autism spectrum generally.

10.3. Limitations

Due to the overall problems with finding a wide range of participants, as well as the gender difference in diagnosis (discussed in Section 2.1), I only found male (or male-presenting; I did not ask them for their own identification) participants for my study, which I acknowledge as a shortcoming which I plan to rectify in future research. There were also difficulties in finding participants overall, which led to the small number of interviewees with varying mathematical backgrounds. I was unable to follow multiple undergraduates through their mathematics courses, which was my original intent. However, this did allow for a greater depth of study of the interviews from each individual participant, and led to a focus on certain types of problems which I find has been fruitful.

10.4. Contributions

As seen in Chapter 2, there is little research concerning adults on the autism spectrum, and much of what is present focuses on a small subset of particularly extraordinary individuals. This research contributes toward filling the gap concerning adults on the autism spectrum selected from the general population, rather than by public notability. I believe that the results in this area show a need for more research related to various aspects of the experiences of adults on the autism spectrum.

The results of my research show a variety of different ways that adults on the autism spectrum can approach and solve mathematical problems. Those results and the differences between them show that there is not necessarily a single approach that students on the autism spectrum can be expected to use, and that a variety of forms which may be considered unusual can produce successful results. This highlights the importance of being able to see validity in unusual student work and interacting with students without deficit-based preconceptions, something which holds importance across a variety of forms of disability-related education research and beyond.

In a broader mathematics education context, the responses to these problems demonstrate both a broader range of possibilities for those specific problems as well as evidence of the importance of considering unusual solutions more generally. A wider variety of successful approaches can be used both in a general sense and particularly when responding to student ideas that may be similar but less successful.

In addition to particular findings regarding mathematical problem-solving, the experiences in the case studies presented here can be viewed in comparison to more general results in autism-related research. For instance, my research findings are consistent with the theory that people on the autism spectrum learn in a manner that relies less on prototypes (Klinger & Dawson, 2001). I believe that the examples of problem-solving in the interviews show that this can produce positive results, and does not need to be viewed as a deficiency. They are also consistent with the systemizing theory (e.g. Baron-Cohen, Wheelwright, Burtenshaw, & Hobson, 2007), where systemizing is viewed as an inclination to create or analyze a system based on the formulation of rules. However, this could also come from the mathematical inclinations already known about and sought in the participants. This should not be taken as support for the suggestion by many proponents of the systemizing theory of its opposition with empathizing (viewed here as the recognition of what someone else is feeling), since the nature of the interviews shows very little about any skills in that category, either positively or negatively. However, some speculations about the intent of the question and its designer do push back against some more extreme interpretations of theory of mind deficiency claims (which suppose that people on the autism spectrum are to various degrees unaware of the minds of others), while more complex uses of theory of mind again do not occur in the context of the interviews.

10.5. Future Research Directions

In extending the research which I present here, I believe that a helpful comparison group for these problems would be adults who have taken some mathematics courses, but who are focusing their studies in a field which is not heavily mathematics-based. This could help to separate the effects of mathematical training from more fundamental inclinations. Additionally, it would be helpful to find participants of different genders and cultural or geographic backgrounds, as well as increasing the number of participants overall.

I have found some evidence that a significant portion of people on the autism spectrum, while unwilling to participate in verbal interviews, might be open to answering similar questions given on paper or electronically; this is one possibility for increasing the number of participants. I think that this may mitigate the discomfort with social interaction, and might provide some (though less) help in dealing with prior negative experiences with researchers. There may also be more complex results of neurological differences and past experiences that contribute to a preference for non-verbal interviews. I adapted some of my questions to a written format for this reason, but did not succeed in getting responses to this version during the time for this study.

Other notable traits which are not common to all participants, such as Mark's strong inclination to confirm calculations, can be viewed as possibilities for further inquiry. It could be that other situations were less conducive to that effect appearing with the other participants, so it would be something to consider when designing further studies.

Another possibility is to extend similar questions across the broader range of neurodiversity, to see if there are distinctive traits in the problem-solving of people with other neurological differences. However, this may be better examined by researchers with a more in-depth familiarity with those neurological differences.

10.6. Personal Reflections

When I started this project, coming from an academic background of pure mathematics, I was unfamiliar with qualitative research methods of the kind I have employed in my research here. At first, I was not entirely trusting of something that appeared unscientific and subjective. However, even large-sample quantitative research, which I might have been more inclined to use at the time, did not produce the confidence of a mathematical proof. I became more open to the idea by reading relevant published works by others, but I have gained much more confidence in it as I have engaged in qualitative research myself and seen how it can produce results that would not be feasible to obtain via other methods.

However, my reading into the research literature on autism was a more mixed experience. While I found some of the theoretical constructs enlightening and useful in my own work, I also found some work that showed a pattern of authors who were unaware of their own negative biases, something I aim to improve on with my own work both now and in the future. I was encouraged by the change over time in these perceptions, both in the overall trends in the literature and on the part of some individual authors (e.g. Happé & Frith, 2006). I also found, in academic work related to disabilities more generally, that the move away from deficit-focused models toward perspectives emphasizing diversity is something that has occurred to varying extents in other areas of disability as well. This experience was something I partially expected, but I still felt a stronger impact by reading the publications myself.

In the area of conducting research via interviews, I believe that I have gained some appreciation for the challenges and rewards of such an approach, and have observed changes in my approach as I gained experience in interviewing. In particular, having a body of experience to contrast with my one-on-one tutoring experience has been helpful. In a setting of talking to a single student about mathematical problems I find myself no longer as prone to falling back on responses from a tutoring context, being more able to refrain from giving explanations too soon and observing more of what the student does.

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