Magic Eulerian Hypergraphs

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Abstract

Motivated by the phenomenon of contextuality in quantum physics we study a particular family of proofs of the Kochen-Specker theorem. A proof is represented by a pair consisting of a hypergraph where each vertex has even degree, called an Eulerian hypergraph, and a labeling of its vertices. If a hypergraph admits a labeling constituting a proof, we say it is magic. We are interested in determining whether a given hypergraph is magic.

Working with the duals of Eulerian hypergraphs, we develop the parity minor relation, which allows us to establish a concept of minimality for the magic property. We introduce a splitting operation to show that certain hypergraphs are not magic. We combine the parity minor relation and the splitting operation in order to search for minimally magic hypergraphs whose duals are grafts, hypergraphs with one edge of size four and all other edges of size two.

Keywords: Kochen-Specker Theorem; Hypergraph; Graph Minor; Quantum Information Theory

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Chapter 1

Background

1.1 Introduction

The field of quantum computing relies on the exploitation of phenomena predicted by quantum physics, but not by classical physics [10, 19, 30, 34]. One such phenomenon is *contextuality*, and there have been numerous papers emphasizing its utility for quantum computation [6, 17, 30, 31].

In 1935, when the theory of quantum physics was still disputed, Einstein, Podolsky and Rosen [16] attemped to counter the probabilistic predictions of quantum mechanics by a deterministic model known as a *hidden variable model*. The essential idea of this model is that measurement outcomes exist before a measurement is made, and the process of measuring simply reveals the preexisting outcome.

The hidden variable model was shown to be inadequate first by Bell in 1966 [4] in the case that locality is assumed. Shortly thereafter, in 1967, Kochen and Specker [22] proved that the hidden variable model is also unable to predict outcomes if non-contextuality is assumed.

Non-contextuality is the assumption that the outcome of measuring some observable is independent of its context, i.e. is independent of whether the measurement is done jointly with other compatible observables. A proof of contextuality is a demonstration that, in certain circumstances, the outcome of measuring an observable is, in fact, dependent upon its context, and thus could not have existed before the measurement was made.

The first such proof (known as a Kochen-Specker proof or KS proof), in the 1967 Kochen-Specker paper [22], is a parity proof based on rays and orthogonal bases, where rays correspond to observables and orthogonal bases correspond to contexts. It necessitates 117 distinct rays and 132 bases of $\mathbb{R}^3$. As this example is quite large, it constitutes a somewhat unsatisfying proof, and is also difficult to exploit for practical applications. Due to this, much work has been done to find smaller examples, and indeed, a proof has been found with 21 rays and 7 bases [24] (this has been proven to be minimal in the number of bases -

assuming vertex-transitivity of the orthogonality graph), and another proof has been found with 18 rays and 9 bases [12]. It should also be noted that an infinite number of such proofs exist - for example, an infinite family is described in [23].

By replacing rays with corresponding projectors, bases with sets of mutually commuting projectors, and labeling vertices of hypergraphs with the given projectors (so that edges represent bases), these proofs can be displayed as labeled hypergraphs. In addition, there exist KS proofs based on labeled hypergraphs that do not have corresponding ray based proofs. The smallest such proof (in terms of vertices) found thus far was discovered by Mermin [26] and Peres [29] in 1990, and its corresponding hypergraph contains only 9 vertices (observables) and 6 edges (contexts).

It is, however, not the case that each hypergraph admits a suitable labeling, and thus we distinguish between those that do (we call these *magic* hypergraphs) and those that do not (*non-magic* hypergraphs).

The problem of deciding whether a given hypergraph is magic is an interesting, and non-trivial one. Little is known about the problem in general, and some evidence suggests that it may be undecidable in its most general form [13, 35], though we have not encountered any undecidable instances in our work. In the special case that the dual hypergraph is a graph, the problem has been completely solved by Arkhipov [3], who showed that for this special class of hypergraphs, the labeling of any magic hypergraph can be reduced to the labeling of one of two fundamental hypergraphs (one of which is the previously mentioned proof discovered by Mermin and Peres, and the other due solely to Mermin). Moreover, the duals of these two hypergraphs are exactly the graphs $K_{3,3}$ and K_5 - the two fundamental non-planar graphs.

In this thesis, we develop hypergraph minor operations in order to extend this result to hypergraphs. Additionally, we develop a tool to show that many hypergraphs are non-magic, and use a combination of these tools and other novel results to determine the magic status of some hypergraphs whose duals are hypergraphs with exactly one edge of size four (and all other edges of size two).

In the first chapter we provide an exposition of the necessary graph theory and linear algebra background for our problem. There is no novel material in this chapter. Sections 1.2 and 1.3 focus on graph theory. For these sections, we shall closely follow Bondy and Murty [8, 9], and also sometimes Diestel [15]. The purpose of Section 1.2 will be to introduce some general graph theory terminology as well as to establish the notational conventions used throughout the thesis. In Section 1.4 we introduce hypergraphs, the central objects of our study. Section 1.5 introduces the necessary linear algebra with focus on quantum physics, and will follow Nielsen and Chuang [28], as well as Kaye, Laflamme, and Mosca [21].

1.2 Basic Graph Theory

Definition 1.2.1. *A* graph G *is an ordered triple* (V, E, ψ) *consisting of a finite set* V *of* vertices*, a finite set* E*, disjoint from* V*, of* edges*, and an incidence function* ψ *that associates with each edge of* G *an unordered pair of not necessarily distinct vertices of* G*.*

Definition 1.2.2. *Let* $G = (V, E, \psi_G)$ *and* $G' = (V', E', \psi_{G'})$ *be two graphs. We say that* G *and* G' *are* isomorphic*, and write* $G \simeq G'$*, if there are bijections* $\theta : V \to V'$ *and* $\phi : E \to E'$ *such that* $\psi_G(e) = uv$ *if and only if* $\psi_{G'}(\phi(e)) = \theta(u)\theta(v)$*.*

Definition 1.2.3. *Let* $G = (V, E, \psi_G)$ *be a graph. Let* $e \in E$ *be an edge of* G*, and let* u *and* v *be vertices such that* $\psi_G(e) = uv$*. Then we say that* u *and* v *are* adjacent *(or* neighbours*), and that* u *and* v *are the* ends *of* e*. Further, we say that* e *is* incident *to both* u *and* v *and vice versa. The incidence between vertex* u *and edge* e *will be denoted either by* $u \sim e$ *or* $e \sim u$*. In addition, we shall often refer to the vertex set of* G *as* $V(G)$ *and to the edge set of* G *as* $E(G)$*.*

Definition 1.2.4. *An edge with identical ends is a* loop*. If two edges have the same ends, we say that they are* parallel edges*.*

Definition 1.2.5. *A graph is said to be a* simple graph *if it has no loops and no parallel edges.*

Oftentimes in literature, a graph as we have defined it is called a multigraph, and a simple graph is called a graph. However, as we shall make heavy use of multigraphs, this notation will be much more convenient.

Though we have defined a graph $G = (V, E, \psi)$ formally as a triple, we shall, almost without exception, refer to it informally as the tuple $G = (V, E)$. If for some edge e we have $\psi(e) = uv$, we shall simply say $e = uv$ if there is no ambiguity.

Definition 1.2.6. *Let* $G = (V, E)$ *be a graph. The* degree *of a vertex* $v \in V$ *is defined as the number of edges it is incident to, with loops counted twice. This is denoted by* $deg_G(v)$ *or simply* $deg(v)$ *if there is no ambiguity.*

Definition 1.2.7. *Let* $G = (V, E)$ *and* $G' = (V', E')$ *be two graphs. If* $V' \subseteq V$ *and* $E' \subseteq E$*, then* G' *is a* subgraph *of* G*, written as* $G' \subseteq G$*. If additionally,* $V' = V$*, then we say that* G' *is a* spanning subgraph *of* G*.*

If G' is a subgraph of G, we often say that G contains G' as a subgraph.

Definition 1.2.8. *Let* $G = (V, E)$ *be a graph. Let* $V' \subseteq V$*. By* $G[V']$ *we denote the subgraph of* G *with vertex set* V' *and whose edge set is the set of edges of* G *with both ends in* V'*. Further we say that* $G[V']$ *is the subgraph of* G induced *on* V' *and call* $G[V']$ *an* induced subgraph *of* G*.*

We now define two classes of graphs which shall appear often throughout this thesis.

Definition 1.2.9. *A* path *is a graph $P = (V, E)$ of the form $V = \{x_0, x_1, ..., x_k\}$, $E = \{e_1, e_2, ..., e_k\}$, where the x_i are distinct for $i \in \{0, \dots, k\}$, and the ends of e_i are x_{i-1} and x_i for $i \in \{1, \dots, k\}$ where $k \geq 0$. We shall refer to x_0 and x_k as the* end vertices *of P. We say that the path P is from x_0 to x_k (or from x_k to x_0), and that $x_1, \dots, x_{k-1}$ are the* internal vertices *of P. We define the* length *of P to be k.*

Definition 1.2.10. *A* cycle *is a graph $C = (V, E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$, $E = \{e_1, e_2, \dots, e_{k+1}\}$, where the x_i are distinct for $i \in \{0, \dots, k\}$, the ends of e_i are x_{i-1} and x_i for $1 \leq i \leq k$, and the ends of e_{k+1} are x_0 and x_k where $k \geq 0$. We define the* length *of a cycle to be $k + 1$. In the case that $k = 0$, the cycle is a loop.*

When there is no ambiguity we often refer to a path or a cycle simply by its underlying vertex sequence $(x_0, x_1, ..., x_k)$ or edge sequence $(e_1, e_2, ..., e_k)$.

Definition 1.2.11. *A set of paths in G is said to be* internally disjoint *if no vertex of G is an internal vertex of more than one path in the set.*

Having defined a path and a cycle allows us to define several other graph concepts which we shall need.

Definition 1.2.12. *A graph $G = (V, E)$ is said to be* connected *if there exists a path from any vertex $u \in V$ to any vertex $v \in V$. If G is not connected, we say that G is* disconnected. *A maximal connected subgraph of G is called a* component*of G.*

Definition 1.2.13. *Let $G = (V, E)$ be a graph. Let $e \in E$, and let $E' := E - \{e\}$. If $G' = (V, E')$ has more components than G, then we say that e is a* bridge.

We now present two classes of graphs which shall arise frequently when we discuss planarity.

Definition 1.2.14. *Let $G = (V, E)$ be a simple graph. If $V = V_1 \cup V_2$, where $V_1 \neq \emptyset, V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$, and $E = \{uv : u \in V_1,\ v \in V_2\}$, then G is a* complete bipartite graph. *If $|V_1| = m$ and $|V_2| = n$, then we denote G by $K_{m,n}$ (or $K_{n,m}$).*

Definition 1.2.15. *Let G be a simple graph on n vertices. If any two distinct vertices of G are adjacent, then G is the* complete graph *on n vertices, and is denoted by K_n.*

The following definition and proposition enable us to estimate the time complexity of an algorithm developed in Section 3.2.

Definition 1.2.16. *Let $G = (V, E)$ be a graph. A subset M of E is called a* matching *in G if its elements are non-loop edges and no two edges in M are incident to a common vertex in G. A matching M* saturates *a vertex v if some edge of M is incident with v. If M saturates every vertex of V, then M is a* perfect matching.

Proposition 1.2.17. *Let $G = K_{2n}$ be the complete graph on $2n$ vertices. Then there are $\frac{(2n)!}{n!2^n}$ perfect matchings of G.*

Proof. Each perfect matching is a set of n edges which are incident to no common vertices, and consequently each vertex of K_{2n} is incident to exactly one edge (since there are no loops).

We proceed by induction on n.

If $n = 1$, we are counting the number of perfect matchings in K_2, which contains a single edge - thus there is exactly $\frac{2!}{1!2^1} = 1$ perfect matching.

Now assume the result for $n = k$. Then for $n = k + 1$, we count the number of perfect matchings in K_{2k+2}. Let the vertices of K_{2k+2} be $v_1, \dots, v_{2k+2}$. Assume that the first edge appearing in the set of any matching is incident to the vertex v_1 (we can always do this by re-ordering). Then there are $2k+1$ choices for this edge, corresponding to the $2k+1$ other vertices in K_{2k+2}. Once we have chosen this edge, the set of choices for the other k edges is simply the number of perfect matchings of K_{2k} which we know to be $\frac{(2k)!}{k!2^k}$. Therefore, there are a total of $(2k+1)\frac{(2k)!}{k!2^k}$ perfect matchings of K_{2k+2}.

Notice that $(2k+1)\frac{(2k)!}{k!2^k} = \frac{2k+2}{2(k+1)}(2k+1)\frac{(2k)!}{k!2^k} = \frac{(2k+2)!}{(k+1)!2^{k+1}}$ as required. □

1.3 Graph Minors and Planarity

We now develop definitions in order to present the minor and the topological minor relation, since two of the three most important theorems discussed in this thesis pertain to the theory of graph (and hypergraph) minors.

Each relation relies on operations to transform one graph to another, and so we begin by presenting these operations.

Definition 1.3.1. *Let $G = (V, E)$ be a graph, let $v \in V$, let $V' := V - \{v\}$, and let $E' := \{e \in E : e$ is not incident to $v\}$. By $G - v$ we denote the graph (V', E'), and we say that $G - v$ was obtained from G by* deleting vertex v.

Definition 1.3.2. *Let $G = (V, E)$ be a graph, let $e \in E$, and let $E' := E - \{e\}$. By $G - e$ we denote the graph (V, E'), and we say that $G - e$ was obtained from G by* deleting edge e.

Proposition 1.3.3. *Let G and G' be graphs. Then G' is a subgraph of G if and only if G' can be obtained from G by a sequence of vertex deletions and edge deletions. Further, G' is a spanning subgraph of G if and only if G' can be obtained from G by a sequence of edge deletions.*

Definition 1.3.4. *To* identify *vertices x and y of a graph G is to replace these vertices by a single vertex incident to all the edges which were incident in G to either x or y. Any edge incident to both x and y becomes a loop incident to the new vertex.*

Definition 1.3.5. *To* contract *a non-loop edge e of a graph G is to delete the edge and then identify its ends. The resulting graph is denoted by G/e.*

Definition 1.3.6. *Let $G = (V, E)$ be a graph. To* delete *a set of vertices $V' \subseteq V$, means to delete each vertex in V'. This is denoted by $G - V'$.*

Definition 1.3.7. *Let G and G' be graphs. We say that G' is a* minor *of G if G' is isomorphic to a graph that can be obtained from G by a (possibly empty) sequence of the following operations:*

- *contracting an edge*
- *deleting an edge*
- *deleting a vertex.*

Definition 1.3.8. *Let $G' = (V', E')$ be a graph, and let $e = uv \in E'$. Let $V = V' \cup \{w\}$ for some new vertex w, and let $E = E' - \{e\} \cup \{e', e''\}$ where $e' = uw$ and $e'' = wv$. Then we say that $G = (V, E)$ was obtained from G' by* subdividing *edge e. Further, we say that any graph obtained from G' by a sequence of subdividing edges is a* subdivision *of G'.*

Definition 1.3.9. *Let G, G' be graphs so that G is a subdivision of G'. Then $V(G)$ can be partitioned into two sets of vertices, $V(G) \cap V(G')$ called the* branch vertices*, and $V(G) - V(G')$ called the* subdividing vertices*.*

Proposition 1.3.10. *Let G and G' be graphs so that G is a subdivision of G'. Then we can correspond any edge $e \in E(G')$ to a path P_e in G so that the end vertices of P_e are the ends of edge e, and all other vertices of P_e are subdividing vertices. Furthermore, the set of paths $\{P_e : e \in E(G')\}$ is internally disjoint.*

Definition 1.3.11. *Let G and G' be graphs. We say that G' is a* topological minor *of G if there exists some subgraph of G which is isomorphic to a subdivision of G'.*

Proposition 1.3.12. *[15, Proposition 1.7.1] The minor relation and topological minor relation are both partial orderings (reflexive, antisymmetric, and transitive) on the class of finite graphs.*

We have now presented two different relations, the topological minor relation and the minor relation. How do these relate?

Proposition 1.3.13. *Let G and G' be graphs. If G' is a topological minor of G, then G' is a minor of G.*

On the other hand, if G' is a minor of G, it is not necessarily true that G' is a topological minor of G, as is illustrated by the graphs in Figure 1.1. In that figure, contracting edge

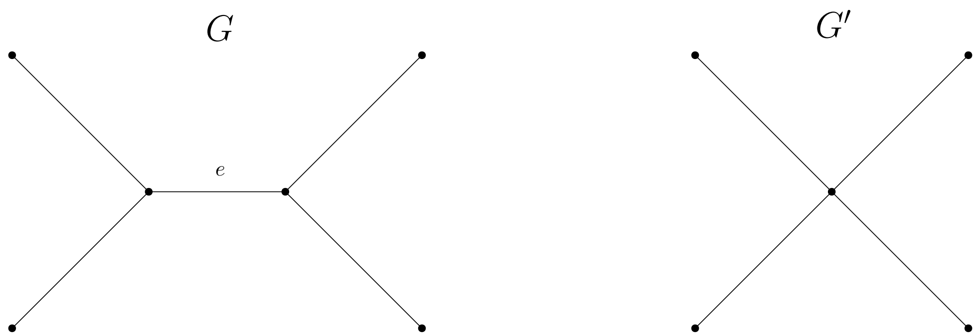

Figure 1.1: Graphs G (left) and G' (right) used to illustrate the difference between the minor and topological minor relation

e in G yields the graph G', and so G' is a minor of G. However, note that G contains no subdivision of G', and thus G' is not a topological minor of G.

We now give an introduction to planarity of graphs before discussing the connection between planarity and graph minors. Without attempting to delve into a discussion on topology which is outside the scope of this thesis, we offer the following definition of a planar graph.

Definition 1.3.14. *We say that a graph G is* planar *if G can be drawn on the plane (in $\mathbb{R}^2$) so that edges meet only at points (vertices) corresponding to their common ends. We refer to such a drawing as a* planar embedding *of G, and we call the embedded graph a* plane graph*.*

Definition 1.3.15. *Let G be a plane graph. We refer to the components of $\mathbb{R}^2 - G$ as the* faces *of G.*

Proposition 1.3.16. *A planar embedding of a graph $G = (V, E)$ can be represented by G along with a clockwise cyclic ordering of the edges in E incident to each vertex in V.*

Proposition 1.3.17. *If G is planar and G' is obtained from G by an edge contraction, deletion or subdivision, then G' is also planar.*

Proposition 1.3.18 (Euler Characteristic for Plane Graphs)**.** *For any connected plane graph $G = (V, E)$ we have that $|V| - |E| + |F| = 2$, where F denotes the set of faces of G.*

Corollary 1.3.19. *If $G = (V, E)$ is a simple planar graph with at least 3 vertices, we have that $|E| \leq 3|V| - 6$.*

Definition 1.3.20. *We say that a simple planar graph $G = (V, E)$ with at least 3 vertices is a* maximal planar graph *if for any $u, v \in V$ such that $u \neq v$ and $uv \notin E$, the graph $G' = (V, E')$ is non-planar where $E' = E \cup \{uv\}$.*

Proposition 1.3.21. *Every simple planar graph is a spanning subgraph of some maximal planar graph.*

We give the two following major theorems due to Kuratowski and Wagner. These theorems provide the link between the two graph minor relations we have presented and graph planarity. The given lemma shows that the two theorems are equivalent, but we give both formulations as they will each be useful in our work. Kuratowski's original formulation from 1930 is Theorem 1.3.22, and this result was reformulated by Wagner seven years later as it is written in Theorem 1.3.24.

Theorem 1.3.22 (Kuratowski, 1930)**.** *[15, Theorem 4.4.6] G is a planar graph if and only if G does not contain K_5 or $K_{3,3}$ as a topological minor.*

Lemma 1.3.23. *[15, Lemma 4.4.2] A graph contains K_5 or $K_{3,3}$ as a minor if and only if it contains K_5 or $K_{3,3}$ as a topological minor.*

Theorem 1.3.24 (Wagner, 1937)**.** *[15, Theorem 4.4.6] G is a planar graph if and only if G does not contain K_5 or $K_{3,3}$ as a minor.*

1.4 Hypergraphs

Hypergraphs, being the central objects of this thesis, are now defined. In general, since they are generalizations of graphs, many of these definitions and propositions shall be quite similar in nature to those appearing in Sections 1.2 and 1.3.

Definition 1.4.1. *A* hypergraph *H is an ordered triple (V, E, ψ) consisting of a finite set V of* vertices*, a finite set E, disjoint from V, of* edges*, and an incidence function ψ that associates with each edge of H a multiset of vertices of H.*

Wherever the translation is natural, the terminology used for graphs is extended to that used for hypergraphs. For example, we say $e = uvvw$ instead of $\psi(e) = uvvw$, and refer to u, v, w as the *ends* of e, for some edge e, and vertices u, v, and w.

We now give a couple more definitions which naturally extend from graphs.

Definition 1.4.2. *Let $H = (V, E)$, and $H' = (V', E')$ be two hypergraphs. If $V' \subseteq V, E' \subseteq E$, then H' is a* subhypergraph *of H, written as $H' \subseteq H$. If additionally, $V' = V$, then we say that H' is a spanning subhypergraph of H.*

Definition 1.4.3. *Let $H = (V, E)$ be a hypergraph. Let $V' \subseteq V$. By $H[V']$ we denote the subhypergraph of H with vertex set V' and whose edge set is the subset of edges of H with all ends in V'. Further we say that $H[V']$ is the subhypergraph of H* induced *on V' and call $H[V']$ an* induced subhypergraph *of H.*

Definition 1.4.4. *Let $H = (V, E)$ be a hypergraph. We define the* multiplicity *of a vertex $v \in V$ with an edge $e \in E$ to be the number of times that v appears in $\psi(e)$, and we denote this by $mult(v, e)$. When multiple hypergraphs are referenced, we use the notation $mult_H(v, e)$ to specify the hypergraph in question.*

Definition 1.4.5. *Let $H = (V, E)$ be a hypergraph, and let $v \in V$ and $e \in E$. We say that v is* incident *to e and vice versa, if $mult(v, e) > 0$. We denote this by $v \sim e$ or $e \sim v$.*

Definition 1.4.6. *Let $H = (V, E)$ be a hypergraph. We define the* degree *of a vertex $v \in V$ to be $\sum_{e \in E} mult(v, e)$, and we denote it by $deg(v)$ or $deg_H(v)$ if the vertex appears in different hypergraphs.*

Definition 1.4.7. *Let $H = (V, E)$ be a hypergraph. We define the* size *of an edge $e \in E$ to be $\sum_{v \in V} mult(v, e)$, and we denote it by $|e|$ or $|e|_H$ if the edge appears in different hypergraphs.*

The difference between graphs and hypergraphs, as we have defined them, is that hypergraphs contain edges of any size, whereas graphs contain only edges of size two. We should mention that under this definition, for some hypergraph $H = (V, E)$ we can have $\psi(e) = v$ for some edge $e \in E$ and some vertex $v \in V$. This is not the same as a loop for a graph, which would be denoted by $\psi(e) = vv$.

Definition 1.4.8. *We say that a hypergraph $H = (V, E)$ is* proper *if*

- *every vertex $v \in V$ has degree greater than zero, and every edge $e \in E$ has size greater than zero,*
- *$mult(v, e) \leq 1$ for every pair consisting of a vertex $v \in V$ and edge $e \in E$,*
- *for every pair of edges $e, f \in E$ such that $mult(v, e) = mult(v, f)$ for every vertex $v \in V$, we have $e = f$.*

Notice that a proper hypergraph is the natural generalization of a simple graph with no isolated vertices. Oftentimes, as in [5], hypergraphs are defined as proper hypergraphs, since they are used to study set systems. We call them proper, because this is for our purposes the proper definition of a hypergraph.

Our reasoning for defining these hypergraphs is that these are the hypergraphs which, when labeled, appear in quantum physics as Kochen-Specker proofs (with the addition of another condition which we shall see in Section 2.1). However, other hypergraphs naturally arise in our work, and labeling these can be useful - even though these do not directly constitute Kochen-Specker proofs. Finally, note that for a proper hypergraph, the degree of a vertex is simply the number of edges it is incident to.

One of the main results of this thesis (Theorem 3.3.18) relies upon extending graph minor operations to hypergraphs. We remark that some of the operations defined for graphs translate naturally to hypergraphs. We list these here:

1. vertex deletion

2. edge deletion

3. deletion of a set of vertices

We now introduce two structures which are intimately connected with a given hypergraph, and which we shall exploit frequently throughout this thesis: the deletion graph, and the dual hypergraph.

Definition 1.4.9. *Let $H = (V, E)$ be a hypergraph. Let $E' \subseteq E$ be the set of edges in E of size not equal to 2. Let G be the graph obtained by deleting each edge of E' from H. Then we say that G is the* deletion graph *of H.*

Definition 1.4.10. *Let $H = (V, E)$ be a hypergraph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$. The* dual *of H is the hypergraph $H^* = (X, F)$ whose vertex set is $X = \{x_1, \ldots, x_m\}$ and edge set is $F = \{f_1, f_2, \ldots, f_n\}$, where $mult(x_i, f_j) = mult(v_j, e_i)$ for any pair $1 \leq i \leq m$, $1 \leq j \leq n$.*

Example 1.4.11. Figure 1.2 displays a hypergraph H and its dual J. Here, a box around a vertex in H indicates that it is incident to edge e_1. Vertex v_1 has two boxes around it indicating that v_1 is incident to e_1 and $mult(v_1, e_1) = 2$. Notice that $mult(x_1, f_1) = 2$, so that $mult(x_1, f_1) = mult(v_1, e_1)$ as expected.

Here are the same hypergraphs in list form: $H = (V, E)$, $J = (X, F)$ where:

- $V = \{v_1, v_2, v_3\}$
- $E = \{e_1, e_2\}$
- $e_1 = v_1v_1v_2v_3, e_2 = v_2v_3$
- $X = \{x_1, x_2\}$
- $F = \{f_1, f_2, f_3\}$
- $f_1 = x_1x_1, f_2 = x_1x_2, f_3 = x_1x_2$

We often deal with a hypergraph and its dual simultaneously. In order to avoid having to refer to two different sets of vertices and edges, we do the following: we refer to the original hypergraph by $H = (P, B)$, and refer to its vertices as *points*, and its edges as *blocks* (as is common for incidence structures). We refer to the dual by $J = (V, E)$, referring to the vertices and edges of the dual simply as vertices and edges. Thus the points of the original correspond to the edges of the dual, and the blocks of the original correspond to the vertices of the dual.

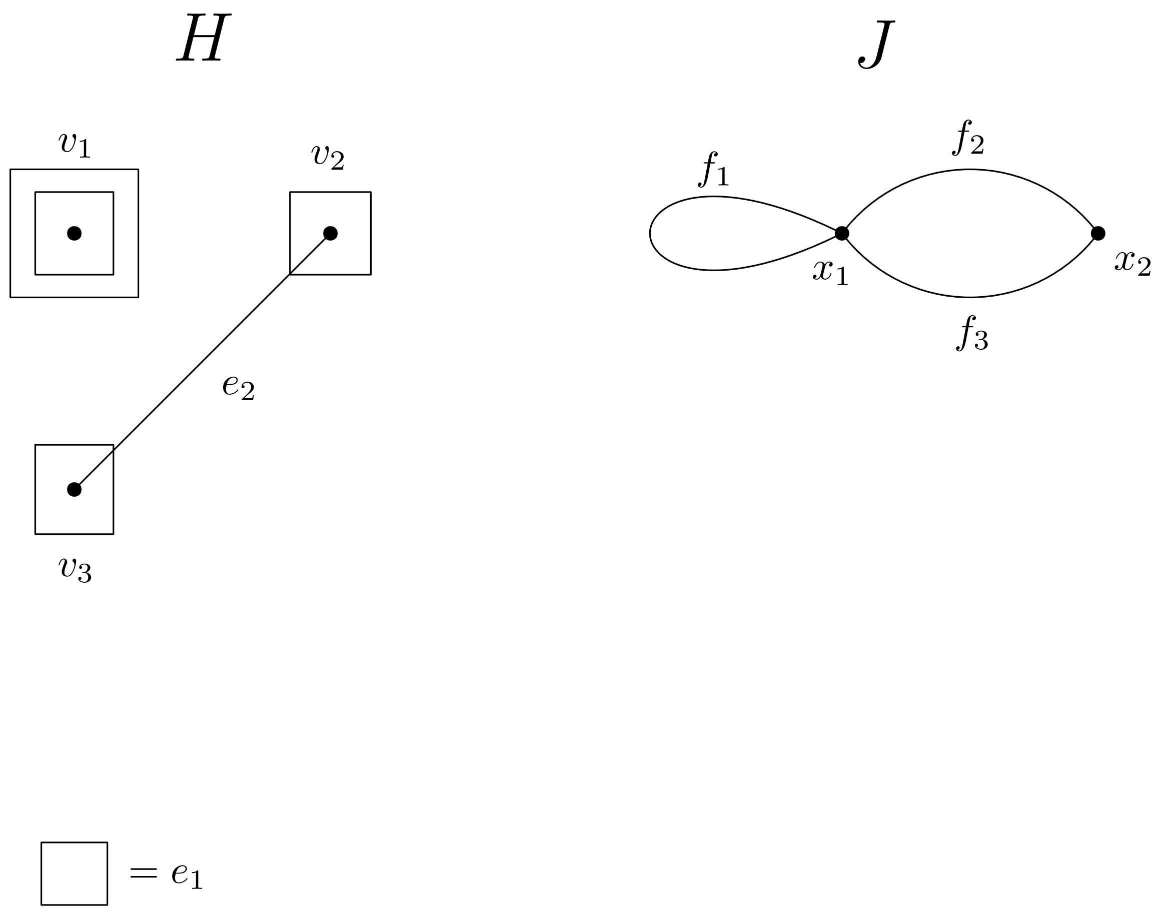

Figure 1.2: An example of the dual operation: hypergraph H on left and its dual J on right

1.5 Linear Algebra

In the introduction, we stated that we are interested in finding Kochen-Specker proofs in the form of labeled hypergraphs. The labels which appear are matrices with certain special properties. In this section, we develop the language necessary to define such matrices, and list some of their properties.

Let $\mathbb{C}$ denote the field of complex numbers.

Definition 1.5.1. *Let V be a vector space over $\mathbb{C}$. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is an* inner product *if it satisfies:*

1. $\langle v, \sum_i \lambda_i w_i \rangle = \sum_i \lambda_i \langle v, w_i \rangle$ *for $v, w_i \in V$, $\lambda_i \in \mathbb{C}$ [Linearity in the second argument]*
2. $\langle v, w \rangle = \overline{\langle w, v \rangle}$ *for $v, w \in V$ [Conjugate-commutativity]*
3. $\langle v, v \rangle \geq 0$ *with $\langle v, v \rangle = 0$ if and only if $v = 0 \in V$ [Non-negativity]*

where $\overline{z}$ denotes the complex conjugate of z.

Definition 1.5.2. *We say that a vector space over $\mathbb{C}$ equipped with an inner product is an* inner product space *over $\mathbb{C}$.*

As in [28] and [21], we forego a formal definition of Hilbert spaces. Instead, we remark that, although in the infinite dimensional case, Hilbert spaces and inner product spaces over $\mathbb{C}$ differ, in the finite dimensional case, they are equivalent. In this work, we almost exclusively speak of finite dimensional Hilbert spaces, and so no further definition will be required until necessary. However, the term Hilbert space is generally preferred in quantum physics to finite dimensional inner product space over $\mathbb{C}$, and so we follow this convention (referring to inifite dimensional Hilbert spaces explicitly when necessary).

Throughout the thesis we use $\mathcal{H}$ to denote a finite dimensional Hilbert space. Given $\mathcal{H}$, we consider the group of invertible linear operators acting on it, and we denote it by $GL(\mathcal{H})$. These operators can also be viewed as invertible matrices in $\mathbb{C}^{n\times n}$ for some positive integer n.

Definition 1.5.3. *Let $M \in \mathbb{C}^{n\times n}$ be a matrix. We say that M is* Hermitian *if $M^* = M$, where M^* denotes the conjugate transpose of M.*

Proposition 1.5.4. *Let $A, B \in \mathbb{C}^{n\times n}$ for some positive integer n. Then $(AB)^* = B^* A^*$.*

Remark 1.5.5. *Throughout this thesis, we use the symbol I to denote the identity operator on various vector spaces over $\mathbb{C}$, without explicitly defining its dimensions. The dimensions will always be apparent from the context.*

Proposition 1.5.6. *Let $A \in \mathbb{C}^{n\times n}$ for some positive integer n. If $A^2 = I$, then $A = A^{-1}$.*

Matrices which are Hermitian and square to the identity are the matrices which appear as labels for labeled hypergraph Kochen-Specker proofs. We thus provide the following proposition which shall be used often throughout our work.

Proposition 1.5.7. *Let $A, B \in \mathbb{C}^{n\times n}$ for some positive integer n. If $A^2 = I,\ B^2 = I,\ A^* = A,\ B^* = B$, and $AB = BA$, then $(AB)^2 = I$, and $(AB)^* = AB$.*

Proof. We have $(AB)^2 = ABAB = A^2B^2 = I$. In addition, we see that $(AB)^* = B^*A^* = BA = AB$. □

Contexts, referred to in the introduction, are sets of pairwise commuting matrices. Since we are interested in proving things about these objects, the following simple Proposition about sets of pairwise commuting matrices will often be of use.

Proposition 1.5.8. *Let $S \subseteq \mathbb{C}^{n\times n}$ be a set of mutually commuting matrices. Let $M \in \mathbb{C}^{n\times n}$ be a matrix that can be expressed as a product of elements from S. Then $S \cup \{M\}$ is a set of mutually commuting matrices.*

We now focus on a particular family of matrices which shall often appear as building blocks for the labels in our examples.

The following matrices, well known in quantum physics, are called *Pauli matrices.*

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We state several useful properties of Pauli matrices.

Proposition 1.5.9. *Following the notation above,*

1. *I, X, Y, Z are Hermitian,*
2. *X, Y, Z pairwise anti-commute: $XY = -YX$, $YZ = -ZY$, $XZ = -ZX$,*
3. *$X^2 = Y^2 = Z^2 = I$,*
4. *$XYZ = iI$.*

We now present an operation to build larger matrices from smaller ones.

Definition 1.5.10. *Let $A = (a_{i,j}) \in \mathbb{C}^{m\times n}, B = (b_{i,j}) \in \mathbb{C}^{p\times q}$. We define the* tensor product *of A and B, denoted by $A \otimes B$ as*

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \dots & a_{1,n}B \\ a_{2,1}B & a_{2,2}B & \dots & a_{2,n}B \\ \dots & \dots & \dots & \dots \\ a_{m,1}B & a_{m,2}B & \dots & a_{m,n}B \end{bmatrix}$$

Note that $A \otimes B \in \mathbb{C}^{mp\times nq}$.

Here are several useful properties of tensor products:

Proposition 1.5.11.

1. *Let* $A = A_1 \otimes A_2 \otimes ... \otimes A_k$, $B = B_1 \otimes B_2 \otimes ... \otimes B_k$, *where* $A_i, B_i \in \mathbb{C}^{n\times n}$ *for* $1 \le i \le k$. *Then* $AB = (A_1B_1) \otimes (A_2B_2) \otimes ... \otimes (A_kB_k)$.

2. *if* $C, D \in \mathbb{C}^{n\times n},\ r \in \mathbb{C}$, *we have* $(rC) \otimes D = C \otimes (rD) = r(C \otimes D)$

3. *if* $C, D \in \mathbb{C}^{n\times n}$ *are Hermitian, then* $C \otimes D$ *is also Hermitian. Furthermore, if* $C^2 = I$ *and* $D^2 = I$, *then* $(C \otimes D)^2 = I$.

Proof. Let us first show that $AB = (A_1B_1) \otimes (A_2B_2) \otimes \cdots \otimes (A_kB_k)$. The result clearly holds if $k = 1$. If $k = 2$, let $A_1 = (a'_{i,j}), B_1 = (b'_{i,j})$. Then

$$\begin{aligned}
AB &= (A_1 \otimes A_2)(B_1 \otimes B_2)\\
&= \begin{bmatrix} a'_{1,1}A_2 & a'_{1,2}A_2 & \dots & a'_{1,n}A_2 \\ a'_{2,1}A_2 & a'_{2,2}A_2 & \dots & a'_{2,n}A_2 \\ \vdots & \vdots & \vdots & \vdots \\ a'_{n,1}A_2 & a'_{n,2}A_2 & \dots & a'_{n,n}A_2 \end{bmatrix} \begin{bmatrix} b'_{1,1}B_2 & b'_{1,2}B_2 & \dots & b'_{1,n}B_2 \\ b'_{2,1}B_2 & b'_{2,2}B_2 & \dots & a_{2,n}B_2 \\ \vdots & \vdots & \vdots & \vdots \\ b'_{n,1}B_2 & b'_{n,2}B_2 & \dots & b'_{n,n}B_2 \end{bmatrix}\\
&= \begin{bmatrix} (a'_{1,1}b'_{1,1} + \ldots + a'_{1,n}b'_{n,1})A_2B_2 & \dots & (a'_{1,1}b'_{1,n} + \ldots + a'_{1,n}b'_{n,n})A_2B_2 \\ \vdots & \vdots & \vdots \\ (a'_{n,1}b'_{1,1} + \ldots + a'_{n,n}b'_{n,1})A_2B_2 & \dots & (a'_{n,1}b'_{1,n} + \ldots + a'_{n,n}b'_{n,n})A_2B_2 \end{bmatrix}\\
&= (A_1B_1) \otimes (A_2B_2).
\end{aligned}$$

Thus if we assume the result for $1 \le i < k$ for some $k \ge 3$ we find that $AB = (A_1 \otimes A_2 \otimes \ldots \otimes A_k)(B_1 \otimes B_2 \otimes \ldots \otimes B_k) = ((A_1 \otimes \ldots A_{k-1}) \otimes A_k)((B_1 \otimes \ldots B_{k-1}) \otimes B_k) = ((A_1 \otimes \ldots \otimes A_{k-1})(B_1 \otimes \ldots \otimes B_{k-1})) \otimes (A_kB_k)$ by the inductive hypothesis. Then by induction we get $((A_1B_1) \otimes \ldots \otimes (A_{k-1}B_{k-1})) \otimes (A_kB_k) = (A_1B_1) \otimes \ldots \otimes (A_kB_k)$ as required.

We leave the second part without proof as it is easy to verify.

For the third part, let $C = (c_{i,j})$. Then

$$\begin{aligned}
(C \otimes D)^* &= \begin{bmatrix} c_{1,1}D & \dots & c_{1,n}D \\ \vdots & \vdots & \vdots \\ c_{n,1}D & \dots & c_{n,n}D \end{bmatrix}^*\\
&= \begin{bmatrix} \overline{c_{1,1}}D^* & \dots & \overline{c_{n,1}}D^* \\ \vdots & \vdots & \vdots \\ \overline{c_{1,n}}D^* & \dots & \overline{c_{n,n}}D^* \end{bmatrix} = C^* \otimes D^* = C \otimes D.
\end{aligned}$$

as required.

Lastly, if $C^2 = I$ and $D^2 = I$, we have $(C \otimes D)^2 = C^2 \otimes D^2 = I$ where the first equality is due to part 1 of this proposition.

□

Corollary 1.5.12. *If $A = A_1 \otimes \ldots \otimes A_k$ and $B = B_1 \otimes \ldots \otimes B_k$, and $A_iB_i = \pm B_iA_i$ for $i \in \{1, ..., k\}$, then A and B commute if and only if $s := |\{i : A_iB_i = -B_iA_i\}|$ is even.*

Proof. We have

$$\begin{aligned} BA &= (B_1A_1) \otimes \ldots \otimes (B_kA_k) \\ &= (-1)^s((A_1B_1) \otimes \ldots \otimes (A_kB_k)) \qquad \text{(by property 2 of Proposition 1.5.11)} \\ &= (-1)^s(AB) \end{aligned}$$

as required. □

Definition 1.5.13. *Let $A = P_1 \otimes \ldots \otimes P_k$, where $P_i \in \{I, X, Y, Z\}$ for $i \in \{1, \ldots, k\}$. We then say that A is a* k-qubit Pauli *matrix.*

Notice that by Proposition 1.5.9 and Corollary 1.5.12, checking whether two k-qubit Pauli matrices commute can be done without ever expanding their tensor products.

Corollary 1.5.14. *Let A be a k-qubit Pauli matrix. Then $A^2 = I$, and A is Hermitian.*

Proof. This follows immediately from Proposition 1.5.11. □

This last corollary confirms that k-qubit matrices are good label candidates for Kochen-Specker proofs. They are especially useful due to the ease with which we can check commutativity conditions - something we will need to do quite often in what follows.

1.6 Thesis Outline

Recall from the introduction that we are interested in labeled hypergraphs which constitute Kochen-Specker proofs. To be more precise, for a hypergraph from a class called proper Eulerian hypergraphs, we would like to assign operators from $GL(\mathcal{H})$ to the vertices of the hypergraph such that certain conditions are satisfied. If such a vertex labeling exists, we say that the hypergraph in question is *magic* and that the labeling is a *magic labeling*. If no such labeling exists, we say that it is *non-magic*.

In Chapter 2, we present a survey of work done on the central problem of the thesis - Problem 2.1.8: Given a proper Eulerian hypergraph H, find a magic labeling of H, or prove that no such labeling exists. We place particular emphasis on a result of Arkhipov, which completely solves the central problem for the particular class of hypergraphs whose duals are graphs. It is due to this result that we can consider this problem in a graph theoretic framework instead of a purely algebraic one, and it is within this framework that we approach the problem in Chapters 3 and 4. Let us briefly remark that algebraic methods to Problem 2.1.8 do exist. We did not pursue such methods, but we mention some of them in Chapter 5.

In Chapter 3 we develop two major theorems: the Splitting Theorem and the Parity Minor Theorem. We also define an analogue of the magic property of proper Eulerian hypergraphs for edge-even hypergraphs (the duals of Eulerian hypergraphs), called the *edge magic* (or *e-magic*) property. The Splitting Theorem allows us to prove that many hypergraphs are non-magic (and thus that their duals are not e-magic). Conversely, the Parity Minor Theorem can be used to show that many edge-even hypergraphs are e-magic and to subsequently generate e-magic labelings for them. Additionally, the Parity Minor Theorem endows a concept of minimality onto the class of edge-even hypergraphs, and allows one to prove that many e-magic hypergraphs are reducible to more fundamental e-magic hypergraphs, which we call *minimally e-magic* (MEM) hypergraphs. We conclude Chapter 3 by stating some required properties of MEM hypergraphs, and we motivate Chapter 4 with a new problem - that of finding new MEM hypergraphs.

In Chapter 4, we begin a search for MEM hypergraphs within a class of edge-even hypergraphs, which we call grafts. Using the two major theorems from Chapter 3, we derive many required properties of MEM grafts. We then describe an approach used to search for such grafts based on creating grafts from maximal planar graphs, which we generate with Gunnar Brinkmann's and Brendan McKay's program *plantri* [11]. The work of Chapter 4 serves the dual purpose of solving more of Problem 2.1.8 and also of highlighting the utility of the theorems developed in Chapter 3.

As previously mentioned, there is evidence that Problem 2.1.8 is undecidable in the most general case, however the approach taken in this thesis shows that much can still be said if the class of hypergraphs is restricted. One should note that finding small minimally e-magic instances is valuable as these would constitute fundamentally new Kochen-Specker proofs. Therefore we believe that much progress can still be made, and the writer firmly believes that this is fertile territory for further research.

Chapter 2

Magic Hypergraphs

Having presented all the relevant background, we now give the central problem of the thesis, and acquaint the reader with current theory. There are no novel results in this chapter.

Section 2.1 begins with a simplified version of the central problem of this thesis, and serves to motivate it. The section concludes by presenting the central problem in full generality, and establishing notational conventions.

The purpose of Section 2.2 is twofold. On the one hand, it is a presentation of several proofs of contextuality. On the other hand, each example presented will be of use later in the thesis in establishing certain results.

Section 2.3 is a summary of the work done by Arkhipov [2, 3] on the central problem of the thesis, and presents a solution to a subproblem.

Finally, in Section 2.4 we briefly discuss some closely related problems to those of this thesis.

2.1 Problem Formulation

Recall from Definition 1.4.8 that a proper hypergraph is a hypergraph with no vertices of degree zero, no edges of size zero, where for any pair consisting of a vertex v and edge e we have $mult(v,e) \leq 1$, and two edges e, e' are the same if $mult(v,e) = mult(v,e')$ for every vertex.

We begin by presenting a fairly simple puzzle.

Problem 2.1.1. Let $H = (P, B)$ be a proper hypergraph. The problem is to find a labeling $\alpha : P \to \{1, -1\}$ of the vertices of H, so that for an odd number of $b \in B$ we have $\prod_{p:p\sim b} \alpha(p) = -1$, or determine that no such labeling exists.

For convenience of notation, we shall refer to such a labeling as a *classical* labeling, and to $\prod_{p:p\sim b} \alpha(p)$ as a *block product* henceforth. We shall also use the terms labeling and

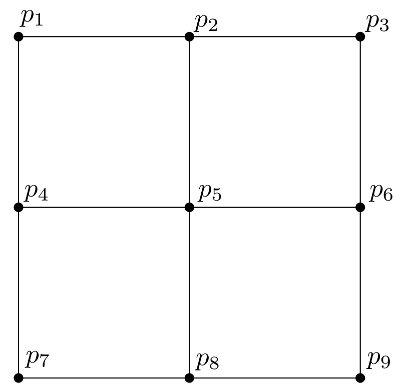

Figure 2.1: The Mermin Square

assignment interchangeably. Further recall that for hypergraphs we shall often refer to the vertices as points, and to the edges as blocks.

Ideally, we would like a polynomial-time algorithm to determine whether a classical labeling exists for a given hypergraph. The solution lies in the following observation:

Lemma 2.1.2. *If there are an odd number of blocks $b \in B$ such that $\prod_{p:p\sim b} \alpha(p) = -1$, then $\prod_{b\in B} \prod_{p:p\sim b} \alpha(p) = -1$.*

Proof. Let r be the number of blocks $b \in B$ such that $\prod_{p:p\sim b} \alpha(p) = -1$. If r is odd, then $\prod_{b\in B} \prod_{p:p\sim b} \alpha(p) = (-1)^r = -1$. □

Let us consider an example:

Example 2.1.3. $H = (P, B)$. $P = \{p_1, \ldots, p_9\}$, $B = \{b_1, \ldots, b_6\}$, $b_1 = (p_1, p_2, p_3)$, $b_2 = (p_4, p_5, p_6)$, $b_3 = (p_7, p_8, p_9)$, $b_4 = (p_1, p_4, p_7)$, $b_5 = (p_2, p_5, p_8)$, $b_6 = (p_3, p_6, p_9)$. This hypergraph (called the *Mermin Square*) is illustrated in Figure 2.1 (blocks are represented by straight lines). In the Mermin Square, each point is incident to exactly two blocks. Therefore, we find that for any $\alpha : P \to \{-1, 1\}$

$$\begin{aligned}\prod_{b\in B} \prod_{p:p\sim b} \alpha(p) &= \prod_{p\in P} \alpha(p)^2 \\ &= 1 \\ &\neq -1\end{aligned}$$

and so, by the previous lemma, we cannot come up with a classical labeling in this case.

Inspired by the previous example, we can solve this problem without too much difficulty. All we need is the following observation:

Lemma 2.1.4. *Let $H = (P, B)$ be a proper hypergraph, and let $\alpha : P \to \{1, -1\}$ be a labeling, then* $\prod_{b \in B} \prod_{p:p\sim b} \alpha(p) = \prod_{p \in P} \alpha(p)^{deg(p)}$.

Proof. Each point $p \in P$ is incident to $deg(p)$ blocks, and thus appears $deg(p)$ times in the first product. □

Proposition 2.1.5. *Let $H = (P, B)$ be a proper hypergraph. Then there exists a labeling $\alpha : P \to \{1, -1\}$ such that an odd number of blocks have block product -1 if and only if there is some point p that has odd degree.*

Proof. First, assume that each point has even degree. We have $\prod_{b \in B} \prod_{p:p\sim b} \alpha(p) = \prod_{p \in P} \alpha(p)^{deg(p)}$ by the previous lemma. Note that $\prod_{p \in P} \alpha(p)^{deg(p)} = 1$ since $deg(p)$ is even for all $p \in P$. Then by Lemma 2.1.2 no classical labeling exists.

Thus, if a labeling exists, there must be a point p of odd degree. In this scenario, simply let $\alpha(p) = -1$, and $\alpha(q) = 1$ for all $q \in P - \{p\}$. Then only the blocks incident to point p have product -1, and there are an odd number of these since $deg(p)$ is odd. □

We now state the problem in full generality. Instead of the labeling α simply mapping points to 1 or -1, it will now map points to Hermitian matrices which square to the identity matrix. In addition, we restrict the class of proper hypergraphs considered to proper Eulerian hypergraphs (defined below). The reasoning for this will appear immediately after the problem is stated in full generality (in Definition 2.1.7).

Definition 2.1.6. *Let H be a hypergraph where each point has even degree. We then say that H is an* Eulerian hypergraph.

Definition 2.1.7. *Let $H = (P, B)$ be a proper Eulerian hypergraph. If there exists some labeling $\alpha : P \to GL(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, such that*

1. *for any point $p \in P$, $\alpha(p)^2 = I$, and $\alpha(p)$ is Hermitian*
2. *for any block $b \in B$, and points $p, q \sim b$, we have $\alpha(p)\alpha(q) = \alpha(q)\alpha(p)$*
3. *for any block $b \in B$, we have*

$$\pi_H(b) := \prod_{p:p\sim b} \alpha(p) = \pm I$$

4. *for an odd number of blocks $b \in B$, we have $\pi_H(b) = -I$*

then we say that H is a magic hypergraph. *If no such assignment α exists for H, then we say that H is a* nonmagic hypergraph. *We also abbreviate these definitions by referring to H as magic or non-magic instead of as a magic or non-magic hypergraph. In the case that H is magic, and α is a labeling satisfying properties 1-4, we say that α is a* magic assignment *for H.*

Notice that Problem 2.1.1 is simply the specific case of deciding whether such an assignment exists for $\mathcal{H} = \mathbb{C}$, and H not restricted to being a proper Eulerian hypergraph. In fact, note that by Proposition 2.1.5, proper Eulerian hypergraphs are exactly the hypergraphs for which there exists no classical labeling of the vertices (since these are the proper hypergraphs where each point is incident to an even number of blocks).

However, we cannot generalize Proposition 2.1.5 to magic assignments, and conclude that no magic hypergraphs exist. This is because Lemma 2.1.4 does not generalize to this case - matrices assigned to points that are not incident to a common block need not commute.

This is the crux of this thesis, and why this problem is of interest. Recall from the introduction that we are interested in finding labelings of proper hypergraphs which display contextuality. This is manifested in the form of magic (proper) hypergraphs which do not have any corresponding classical labeling (we can think of a classical labeling roughly as a hidden variable model solution - something we do not want, since we would like to exploit scenarios where classical physics cannot predict the effects predicted by quantum physics). Therefore, proper Eulerian hypergraphs are the natural hypergraphs for us to study.

Let us now take a moment to state the main problem of this thesis.

Problem 2.1.8. Given some proper Eulerian hypergraph H, find a magic assignment for H or show that none exists.

In the following section, we begin to give some ideas on how to find assignments, and in Section 2.3, we present a solution of Problem 2.1.8 for a special class of proper Eulerian hypergraphs.

2.2 Some Magic Assignments

Let us reconsider the Mermin Square pictured in Figure 2.1. By Corollary 1.5.14, we see that tensor products of Pauli matrices are good candidates as point labels. Recall that a matrix created by taking a tensor product of k Pauli matrices is called a k-qubit matrix. If each matrix assigned in a labeling has k qubits, we say that the assignment is a *k-qubit assignment.*

In what follows, when we display labeled hypergraphs, we illustrate the blocks with block product $-I$ by thicker lines.

Example 2.2.1. Figure 2.2 depicts a 2-qubit assignment of the Mermin Square. This assignment, $\alpha : P \to \mathbb{C}^{4\times 4}$, is defined by: $\alpha(p_1) = X \otimes X$, $\alpha(p_2) = Y \otimes Y$, $\alpha(p_3) = Z \otimes Z$, $\alpha(p_4) = X \otimes I$, $\alpha(p_5) = X \otimes Z$, $\alpha(p_6) = I \otimes Z$, $\alpha(p_7) = I \otimes X$, $\alpha(p_8) = Z \otimes X$, $\alpha(p_9) = Z \otimes I$. For notational simplicity, we denote $A \otimes B$ as AB in diagrams, as all matrix multiplication will be resolved before labeling.

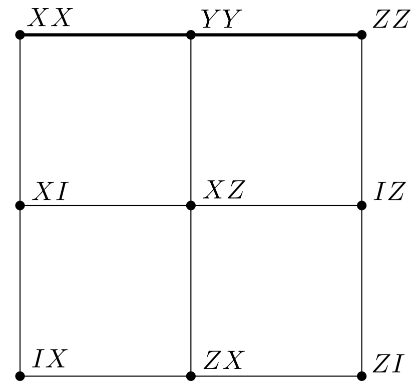

Figure 2.2: A magic assignment of the Mermin Square

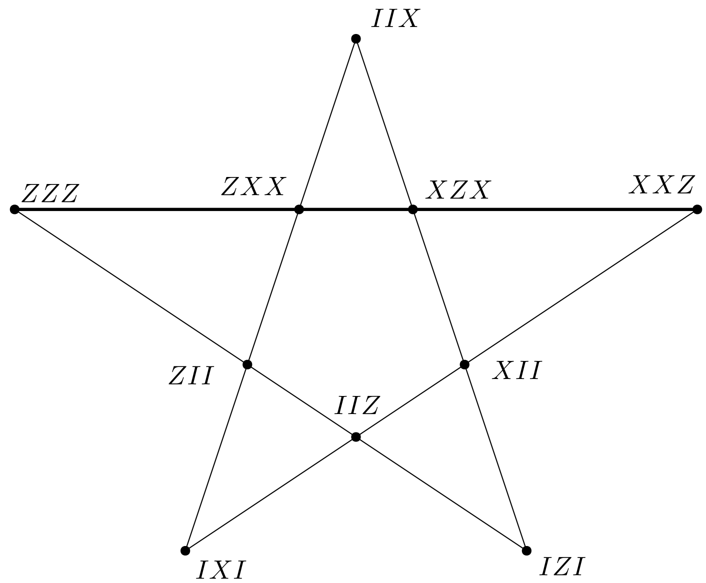

Figure 2.3: A magic assignment of the Mermin Pentagram

Figure 2.3 is an illustration of another Eulerian hypergraph (called the Mermin Pentagram) with a 3-qubit assignment.

Interestingly, it has been proven that the Mermin Pentagram has no 2-qubit assignment.

At this point, we should state a secondary problem of interest - one that expands on our main problem.

Problem 2.2.2. Let $H = (P, B)$ be a magic proper Eulerian hypergraph. Does H admit a k-qubit assignment for some positive integer k? If so, what is the smallest such k?

We will address this problem later.

Waegell and Aravind [38, 39] have also done a lot of work in finding families of proper Eulerian hypergraphs with magic assignments. We present a particular example of theirs in Figure 2.4, as it is of relevance to us in an upcoming section.

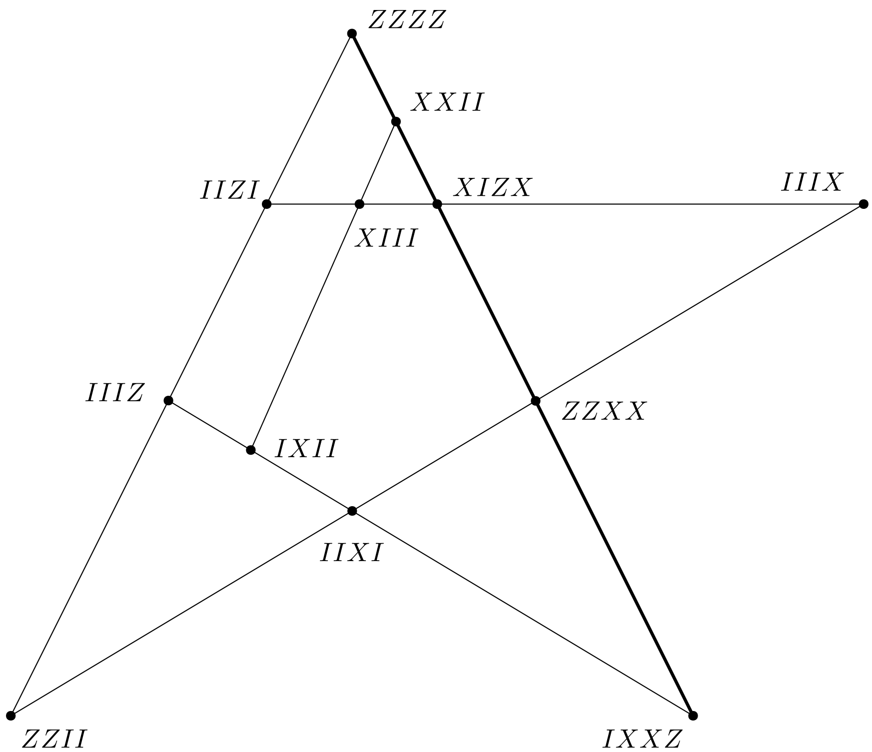

Figure 2.4: The 4-qubit Star along with a magic assignment

Example 2.2.3. The labeled Eulerian hypergraph in Figure 2.4 is called the *4-qubit star* by Waegell and Aravind in [39, Figure 1].

We should note that there exists an assignment using at most three qubits for this proper Eulerian hypergraph (as we shall see in the following section). It is a problem of interest to find irreducible (in the sense that no subset of observables or qubits can be ignored) assignments on different (non-minimal) numbers of qubits. However, in this thesis we are more concerned with solving Problems 2.1.8 and 2.2.2, than finding different assignments for hypergraphs for which these two problems have already been solved. If the reader is interested they are encouraged to see the paper referenced at the beginning of this example, and also [38]. We also mention that in [39] Waegell and Aravind give constructions for infinite families of irreducible Pauli-based assignments. We emphasize that irreducible refers to the fact that no subset of observables or qubits can be ignored. These hypergraphs are not minimally magic in our sense.

Until now we have only seen magic assignments based on Pauli matrices, however other assignments do exist. As mentioned in the introduction, KS proofs based on rays and sets of orthogonal bases can be transformed into labeled hypergraphs, and doing so for the 21 ray, 7 orthogonal bases proof appearing in [24] yields a magic assignment which maps points of a hypergraph to matrices in $\mathbb{C}^{6\times 6}$.

We remark that many more non Pauli-based proofs exist, some of which are described in [40] where the simplest known critical KS sets in Hilbert spaces of even dimensions $d \geq 10$, and odd dimensions $d \geq 7$ are derived (whereas Pauli-based proofs exist only in dimensions 2^n for $n \in \mathbb{N}$).

2.3 Magic 2-regular Hypergraphs

As of yet, we have not shown a proof that some proper Eulerian hypergraph H has no magic assignment. Furthermore, when we showed that a certain proper Eulerian hypergraph is magic, we simply displayed a labeling, and gave no insight as to how such a labeling was constructed. This is largely because the labelings presented in the previous section were generated in a rather ad hoc manner.

In this section, we present a result developed by Arkhipov in his 2012 Massachusetts Institute of Technology Master's thesis [2, 3]. This result shows that certain proper Eulerian hypergraphs have no magic assignment, and also provides a labeling method for other proper Eulerian hypergraphs which do have a magic assignment. Moreover, the result provides us with a solution to Problem 2.1.8 for a certain class of proper Eulerian hypergraphs.

Definition 2.3.1. *Let H be a hypergraph such that every point has degree k. Then we say that H is a k-regular hypergraph.*

Notice that the dual hypergraph of a 2-regular hypergraph is in fact a graph, as every point of the 2-regular hypergraph, being incident to two blocks, corresponds to an edge of size two. Thus, when referring to the dual hypergraph of a 2-regular hypergraph, we simply say *dual graph*. If a 2-regular hypergraph is also proper, then its dual graph contains neither loops nor vertices of degree zero.

Example 2.3.2. In Figure 2.5, we give the dual graphs of the Mermin Square and Mermin Pentagram. We have labeled the vertices and edges to emphasize their correspondence to blocks and points (respectively) in the original hypergraphs.

Note how properties 2-4 of Definition 2.1.7 appear in the dual graph - instead of the properties corresponding to points incident to a block, they now correspond to edges incident to a vertex. We formalize this for the sake of clarity.

Proposition 2.3.3. *Let H be a proper 2-regular hypergraph, and let $G = (V, E)$ be the dual graph of H. Then H is magic if and only if there exists some labeling $\alpha : E \to GL(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, such that*

1. *for any edge $e \in E$, $\alpha(e)^2 = I$, and $\alpha(e)$ is Hermitian*
2. *for any vertex $v \in V$, and $e, f \in E$ incident to v, we have $\alpha(e)\alpha(f) = \alpha(f)\alpha(e)$.*

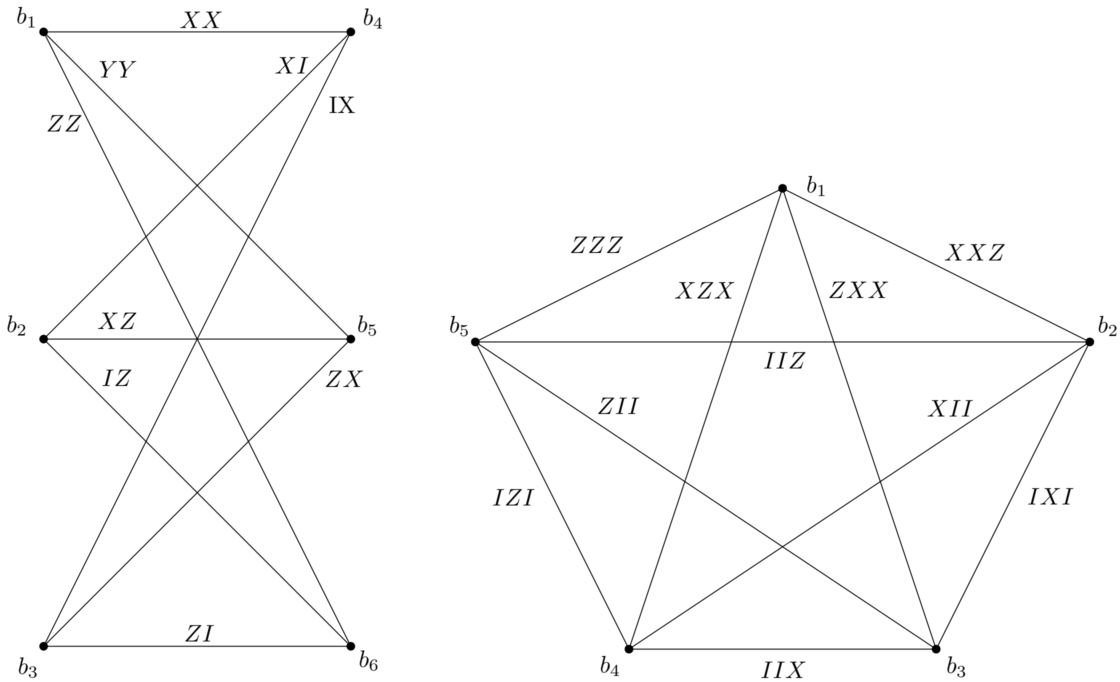

Figure 2.5: Dual graphs of the Mermin Square and Mermin Pentagram: $K_{3,3}$ (left) and K_5 (right)

3. for any vertex $v \in V$, we have

$$\pi_\alpha(v) := \prod_{e:v\sim e} \alpha(e) = \pm I$$

4. for an odd number of vertices $v \in V$, we have $\pi_\alpha(v) = -I$.

Proof. This is a direct consequence of the previous remark. $\square$

We refer to the graph G as *e-magic*, or *edge magic*. Furthermore, we shall refer to α as an *e-magic assignment* (or *e-magic labeling*) of G. Lastly, we shall sometimes refer to $\pi_\alpha(v)$ as $\pi_G(v)$ if we are dealing with multiple graphs each containing unique assignments. In addition, we may use the notation $\pi(v)$ if we are dealing with a single graph and a single assignment. We refer to this product as a *vertex product*.

We now present a technical lemma followed by the surprising and beautiful result.

Lemma 2.3.4. *[3] Let G be a planar graph, and let $\alpha : E(G) \to GL(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, such that $\alpha(e)^2 = I$ for every $e \in E(G)$. Consider a planar embedding of G which defines a clockwise cyclic ordering of the edges around each vertex. Assume that for any vertex $v \in V(G)$, for some sequence of the edges incident to v, obeying the cyclic ordering, say $e_1, \ldots, e_k$, we have $\alpha(e_1)\alpha(e_2)\ldots\alpha(e_k) \in \{I, -I\}$. Then $\alpha(e_1)\ldots\alpha(e_k) = \alpha(e_i)\alpha(e_{i+1})\ldots\alpha(e_k)\alpha(e_1)\alpha(e_2)\ldots\alpha(e_{i-1})$ for any $i \in \{1, \ldots, k\}$.*

Proof. First, let us note that a clockwise cyclic ordering as described in the statement does, in fact, exist by Proposition 1.3.16.

For $i = 1$, this result is trivial. Thus assume that the result holds for $i = j$, where $1 \leq j < k$. Let $P = \alpha(e_1)\ldots\alpha(e_k)$ (and note $P \in \{-I, I\}$, hence P commutes with e_i for $i \in \{1, \ldots, k\}$).

We have

$$\begin{aligned} \alpha(e_j)\ldots\alpha(e_k)\alpha(e_1)\ldots\alpha(e_{j-1}) &= P && \text{(by assumption)} \\ \implies \alpha(e_{j+1})\ldots\alpha(e_k)\alpha(e_1)\ldots\alpha(e_{j-1}) &= P\alpha(e_j) && \text{(since } \alpha(e_j)^2 = I) \\ \implies \alpha(e_{j+1})\ldots\alpha(e_k)\alpha(e_1)\ldots\alpha(e_j) &= P && \text{(since } \alpha(e_j)^2 = I) \\ &= \alpha(e_1)\ldots\alpha(e_k) \end{aligned}$$

as required.

Therefore, the result holds for all $i \in \{1, \ldots, k\}$.

□

Theorem 2.3.5. *(Arkhipov) [3] Let H be a proper 2-regular hypergraph. Let $G = (V, E)$ be the dual graph of H. Then H is magic if and only if G is non-planar.*

Proof. Let us first assume that G is planar. Let us also assume that G is connected (if it is not, we can simply apply this proof to each component of G).

Consider a planar embedding of G defining a clockwise cyclic ordering of the edges at each vertex v of $V(G)$ (by Proposition 1.3.16). Let α be a labeling of the edges of G obeying the assumptions of Lemma 2.3.4, noting that in the case that α is an e-magic assignment, α would obey these conditions.

Let $e_1, \ldots e_k$ be the sequence of edges incident to some $v \in V$ in clockwise cyclic order. Then, we define $\pi^*_G(v) = \alpha(e_1)\alpha(e_2)\ldots\alpha(e_k)$. By construction of α and by Lemma 2.3.4 this product is well defined and is either I or $-I$ for every vertex in V, even in the case that the $\alpha(e_i)$ do not commute.

Now let $e = uv$ be some edge in E, with $u \neq v$ (the case where no such edge exists will be treated subsequently). Let the edges incident to u be $f_1, \ldots, f_m, e$, and the edges incident to v be $e, g_1, \ldots, g_n$ in cyclic order without loss of generality (we can always put e first or last as we are dealing with a cyclic ordering). If we contract edge e to obtain a graph G_1, then the new vertex w created from the contraction preserves the previous cyclic ordering. (See Figure 2.6 for details). In other words, after the contraction of edge e, we will have $f_1, \ldots, f_m, g_1, \ldots, g_n$ as the cyclic order around w. Notice that $\pi^*_G(u) = \alpha(f_1)\ldots\alpha(f_m)\alpha(e)$, and $\pi^*_G(v) = \alpha(e)\alpha(g_1)\ldots\alpha(g_n)$.

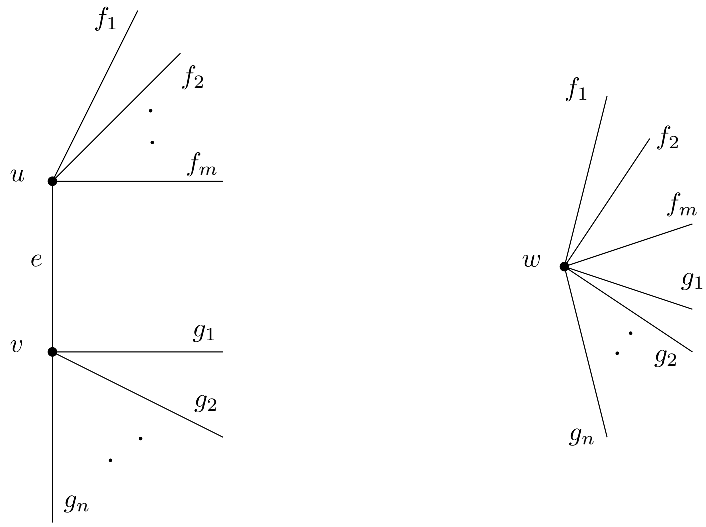

Figure 2.6: Edge contraction in Theorem 2.3.5

So

$$\begin{aligned}\pi^*_G(u)\pi^*_G(v) &= \alpha(f_1)\dots\alpha(f_m)\alpha(e)^2\alpha(g_1)\dots\alpha(g_n)\\ &= \alpha(f_1)\dots\alpha(f_m)\alpha(g_1)\dots\alpha(g_n)\\ \implies \alpha(f_1)\dots\alpha(f_m)\alpha(g_1)\dots\alpha(g_n) &\in \{-I, I\} \qquad (\text{since } \pi^*_G(u),\ \pi^*_G(v) \in \{-I, I\}).\end{aligned}$$

Therefore, since $f_1, \dots, f_m, g_1, \dots, g_n$ are the edges in clockwise cyclic order around w, and since $\alpha(f_1)\dots\alpha(f_m)\alpha(g_1)\dots\alpha(g_n) \in \{-I, I\}$ we find that by Lemma 2.3.4, $\pi^*_{G_1}(w)$ is well defined.

Note that edges between u and v that were not e become loops for w. Just as we counted loops twice when calculating the degree, the label assigned to a loop occurs twice in the product of the vertex that is the end of the loop.

Notice that the resulting assignment for the graph obtained by contracting e also obeys the assumptions of Lemma 2.3.4.

Thus, this edge contraction property ($\pi^*_{G_1}(w) = \pi^*_G(u)\pi^*_G(v)$, where w was obtained from identifying vertices u and v in the contraction) holds for subsequent contractions as well. Therefore, we can apply edge contraction inductively to finally arrive at a graph G^+ with a single vertex vertex u', and loops $e'_1, ..., e'_s$. As we also have a planar embedding of G^+, there must be an inner-most loop, and thus in the cyclic ordering, the two edge labels corresponding to that loop appear consecutively in $\pi^*_{G^+}(u')$ and multiply to identity. Therefore, $\pi^*_{G^+}(u') = I$.

Since G is connected, we have $\prod_{v \in V} \pi(v) = \pi^*_{G^+}(u') = I$ (we obtained u' by identifying all of the vertices in G). Then α cannot be an e-magic assignment of G since an e-magic assignment must have $\prod_{v \in V} \pi_G(v) = -I$ (since an odd number of vertices have $\pi_G(v) = -I$ for an e-magic assignment, by condition 4 of Proposition 2.3.3).

Now, let us assume that G is non-planar, and show that H is magic. Since G is non-planar, by Kuratowski's Theorem, it contains a $K_{3,3}$ or K_5 subdivision as a subgraph. Let G' be a subdivision of $K_{3,3}$ or K_5 appearing in G, and let K be either $K_{3,3}$ or K_5 so that G' is a subdivision of K. Then the branch vertices of G' are exactly the vertices of K, and each edge $e = uv$ in K corresponds to a path P_e from u to v in G' (by Proposition 1.3.10). We know that K is e-magic as it is the dual graph of the Mermin Square or the Mermin Pentagram, thus we have an e-magic assignment $\alpha : E(K) \to GL(\mathcal{H})$ respecting properties 1-4 of Proposition 2.3.3. Now we define $\alpha' : E(G) \to GL(\mathcal{H})$ by the following rule:

$$\alpha'(e) = \begin{cases} I & \text{if } e \notin E(G') \\ \alpha(f) & \begin{array}{l}\text{if } e \in E(G') \cap E(P_f) \text{ where } P_f \text{ is the path in} \\ K \text{ corresponding to edge } f \in E(K)\end{array} \end{cases}$$

Then we claim that α' is an e-magic assignment of G.

First, note that property 1 of Proposition 2.3.3 holds for α' since all edge labels are either I or are also edge labels of α - which is an e-magic assignment and thus obeys property 1.

We now show that properties 2 and 3 of Proposition 2.3.3 hold for each vertex $v \in V(G)$. For any vertex $v \in V(G) - V(G')$, v is only incident to edges e such that $\alpha'(e) = I$. Therefore, commutativity holds and $\pi(v) = I$ (where $\pi(v) = \prod_{e:v\sim e} \alpha'(e)$). Thus we may restrict our attention to G', since all edges outside of $E(G')$ have label I, and do not affect the product of any vertex in G'. If v is a vertex in G', then v is either a branch vertex or a subdividing vertex. If v is a subdividing vertex on some path P_f, then v is adjacent to two edges from P_f, both with label $\alpha'(f)$. Thus $\pi(v) = \alpha'(f)^2 = I$, and commutativity again holds trivially. Now assume that v is a branch vertex. Then, in G', v is incident to edges corresponding to edges in K, and with the same edge labels. Thus $\pi_G(v) = \pi_{G'}(v) = \pi_K(v)$ (where the first equality is again due to the fact that all edges in $E(G) - E(G')$ have label I).

Finally, we prove that property 4 holds. Since an odd number of vertices in K have product $-I$, we have that an odd number of vertices in G' have product $-I$. Therefore, we can finally conclude that we have an odd number of vertices with product $-I$ in G.

Therefore since properties 1-4 of Proposition 2.3.3 hold, α is an e-magic assignment of G as required.

□

Arkhipov's result answers Problem 2.1.8 for the class of proper 2-regular hypergraphs. In addition, it provides a partial answer for Problem 2.2.2 for the same class, namely, any magic proper 2-regular hypergraph does admit a k-qubit assignment. Furthermore, any 2-regular hypergraph whose dual contains a $K_{3,3}$ subdivision has $k = 2$ as the minimal value of k for a k-qubit assignment (it is easy to show that 1-qubit assignments exist only when

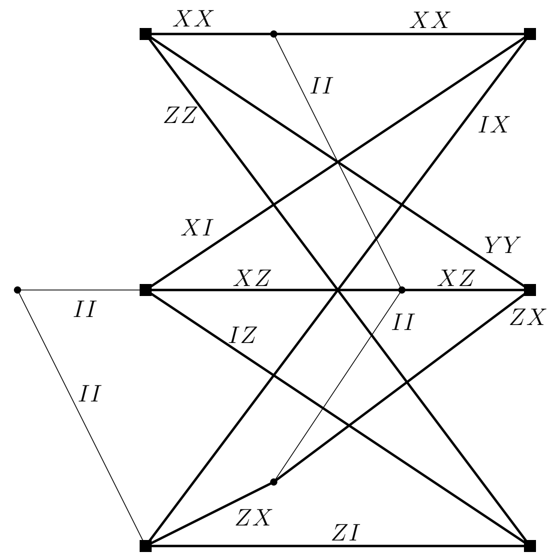

Figure 2.7: Graph labeled via the standard labeling method

classical labelings do as well). To the best of the author's knowledge, deciding whether a 2-qubit assignment can exist when only subdivisions of K_5 appear in the dual graph is an open problem.

Additionally, we can solve Problem 2.1.8 for 2-regular hypergraphs in linear time, in the number of edges of the dual graph (and thus the number of points of the original hypergraph). This can be done by applying one of the many well-known linear time planarity testing algorithms to the dual graph, which either ascertain that a given graph is planar, or output a subdivision of $K_{3,3}$ or K_5 appearing in the given graph. A number of such algorithms are listed in Section 2.7 of the text *Graphs on Surfaces* by Mohar and Thomassen [27]. One such algorithm, due to Williamson, is described in [41].

Once a $K_{3,3}$ or K_5 subdivision has been found by one of the previous planarity algorithms mentioned, we proceed by the constructive labeling suggested by the proof of Theorem 2.3.5.

Definition 2.3.6. *For a non-planar graph G, and a given subdivision of $K_{3,3}$ or K_5 appearing in G we refer to a labeling of G as in the proof Theorem 2.3.5 as a labeling obtained by the* standard labeling method*.*

Example 2.3.7. Figure 2.7 is an illustrative example of a labeling for a non-planar graph via the standard labeling method. Note that branch vertices are denoted by squares and path edges are drawn thicker to emphasize the chosen $K_{3,3}$ subdivision.

In Section 2.2 we mentioned that the duals of each of the hypergraphs presented in [39] are graphs. Therefore by Arkhipov's result, they each have magic assignments using at most three qubits. We present a particular example from that paper now.

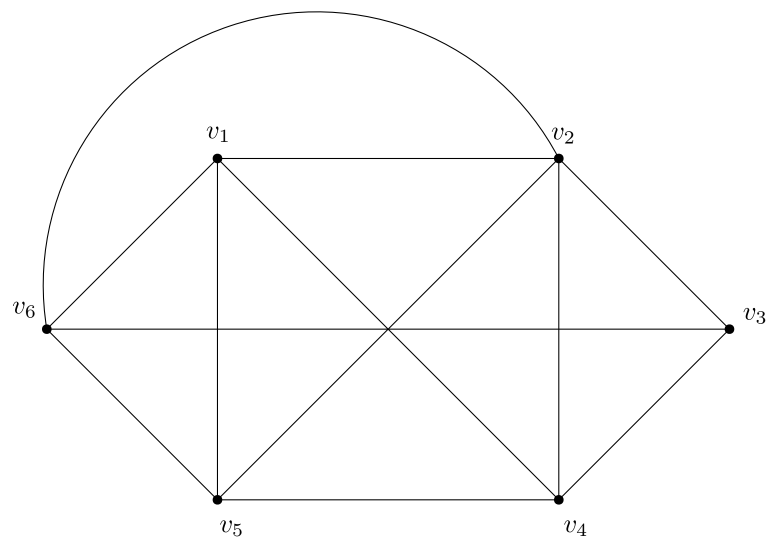

Figure 2.8: The dual of the 4-qubit Star

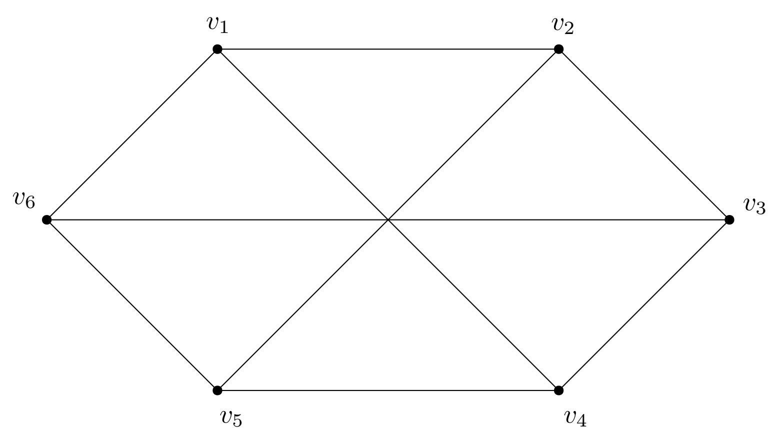

Figure 2.9: $K_{3,3}$ appearing as a subgraph in the dual

Example 2.3.8. The 4-qubit star illustrated in Figure 2.4 has a magic assignment with at most three qubits because it is a proper 2-regular hypergraph. Using Magma code given in Appendix A.1, we verified that the dual of the 4-qubit star (pictured in Figure 2.8) contains a $K_{3,3}$ subdivision, and thus the 4-qubit star has a two qubit magic assignment. In Figure 2.9 is a subgraph isomorphic to $K_{3,3}$ appearing in the dual. Here the bipartition is v_1, v_3, v_5 and v_2, v_4, v_6. A labeling obtained via the standard labeling method is displayed in Figure 2.10.

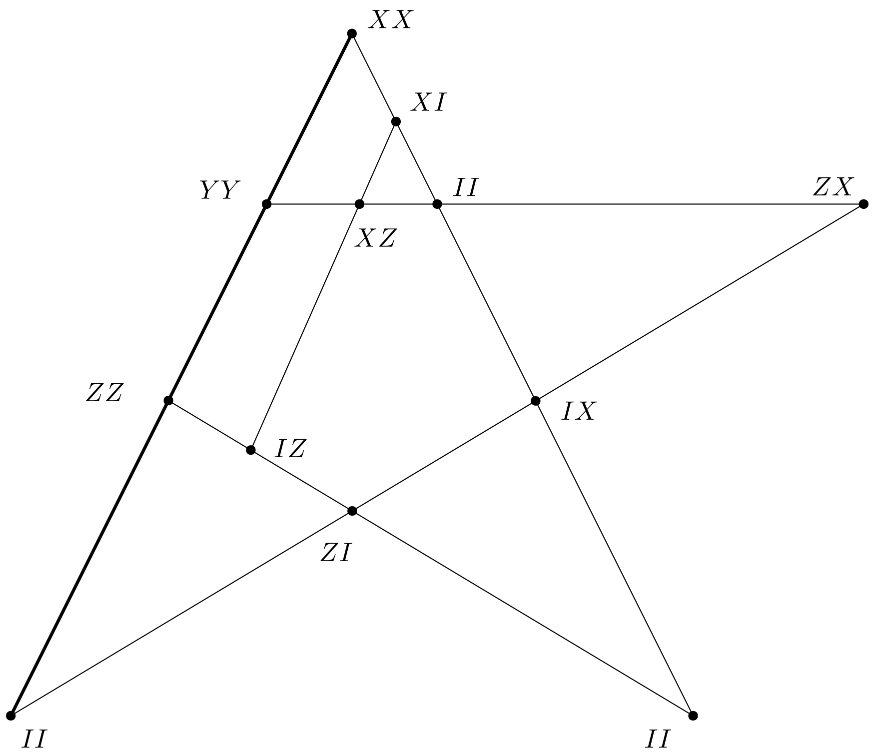

Figure 2.10: A labeling of the 4-qubit Star obtained by the standard labeling method

2.4 Parity Binary Constraint Systems

At this point we present some problems of a similar flavour to those of this thesis. The purpose of this is to give the reader some context in terms of recent developments, and also to show that the work done in this thesis has ramifications beyond the scope of Kochen-Specker sets.

Definition 2.4.1. *A* parity binary constraint system *(BCS) is a collection of constraints* $C_1, C_2, \ldots, C_m$ *over variables* $x_1, x_2, \ldots, x_n$*, taking values in* $\{-1, 1\}$*, where each constraint* C_i $(1 \le i \le m)$ *is of the form* $\prod_{x \in S_i} x = r_i$*,* S_i *is a subset of the variable set, and* $r_i \in \{-1, 1\}$*.*

Definition 2.4.2 (Cleve, Mittal)**.** *[14] A parity binary constraint system has a* quantum satisfying assignment *if there exists a finite dimensional Hilbert space* $\mathcal{H}$*, and an assignment of an operator* $A_j \in GL(\mathcal{H})$ *to each variable* x_j *such that the following conditions hold:*

1. *The operators satisfy each constraint when we substitute* x_j *with operators* A_j *and* 1 *with* I *(and so also -1 with* $-I$*) for* $1 \le j \le n$*,*

2. *For each* j*,* A_j *is Hermitian and* $A_j^2 = I$*,*

3. *Each pair of operators* A_j, A_k *that appear in the same constraint is commuting, i.e* $A_j A_k = A_k A_j$*.*

The problem of finding a quantum satisfying assignment for a parity BCS is very similar to Problem 2.1.8. Consider the proper hypergraph $H = (P, B)$ arising by representing each variable x_j by a point $p_j \in P$, and each constraint C_i by a block $b_i \in B$ where b_i is incident to a point p_j if and only if x_j appears in S_i. If we insist that the block products ($\pi_\alpha(b)$ for labeling α, block b) in Problem 2.1.8 are prespecified, we see that Problem 2.1.8 corresponds to an instance of the problem of finding a quantum satisfying assignment of a parity BCS. Moreover, any magic assignment α of a proper Eulerian hypergraph corresponds to a quantum satisfying assignment for a corresponding parity BCS if $r_i I = \pi_\alpha(b_i)$ for each $1 \leq i \leq m$.

Showing that a proper Eulerian hypergraph is not magic is therefore equivalent to proving that every parity binary constraint system for which the variables and constraints are consistent with the points and blocks of the hypergraph, and with an odd number of -1's in the right-hand sides of its constraints has no quantum satisfying assignment.

A parity BCS also has an associated two-player game. There are two cooperating players, Alice and Bob, who are allowed to agree on a strategy before the game begins but are not allowed to communicate once it has begun. Alice is given some constraint C_i and Bob is given some variable x_j appearing in S_i. Alice must assign a $\{-1, 1\}$ value for each variable in S_i so that the values satisfy constraint C_i, and Bob must assign x_j to some value in $\{-1, 1\}$. Alice and Bob win if they have assigned the same value to x_j.

In the case that Alice and Bob can win the game with probability 1, we say that the game has a *perfect strategy*. If the parity BCS has a solution (an assignment of the variables such that all constraints are simultaneously satisfied), then its associated game has a perfect strategy - Alice and Bob agree on a particular solution and choose their variable assignments based on this solution. Such a strategy is referred to as a *perfect classical strategy*. In the case that the parity BCS has no solution, Alice and Bob can still win with probability 1 with a so-called *quantum strategy* (which we call a *perfect quantum strategy* in the case that Alice and Bob do win with probability 1), which utilizes entanglement. There are different models for quantum strategies, among which are the tensor product model, and the commutative model. When we refer to a quantum strategy we are referring to the tensor product model, but we make mention of this distinction since a recent paper by Slofstra [35] showed that perfect quantum strategies exist for the commutative model where none exist for the tensor product model (and thus solved a famous conjecture of Tsirelson), and that the problem of determining whether a perfect quantum strategy exists for the commutative model is undecidable.

We will not delve into an explanation of quantum strategies or entanglement, as this venture requires background not covered in this thesis, but if the reader is interested they are referred to [14, 13, 18, 35]. Instead, we will describe the relation between this field of study and our own.

Firstly, it can be shown that if a parity BCS has a quantum satisfying assignment, then its corresponding parity BCS game has a perfect quantum strategy. Conversely, in [14], Cleve and Mittal show that for any parity BCS, if a perfect quantum strategy exists for the corresponding parity BCS game that uses finite or countably-infite dimensional entanglement, then it has a quantum satisfying assignment. Therefore, proving that a proper Eulerian hypergraph is non-magic is the same as showing that all of the corresponding parity BCS games with an odd number of -1's in the right-hand sides (of the parity BCS) have no perfect quantum strategy. Additionally, for a parity BCS (with an odd number of -1's in the right-hand sides) that has a corresponding proper Eulerian hypergraph, showing that the corresponding parity BCS game has a perfect quantum strategy using finite or countably-infinite dimensional entanglement implies that the proper Eulerian hypergraph is magic.

Secondly, for any binary constraint system with each variable in exactly two constraints, Arkhipov's result on 2-regular hypergraphs (Theorem 2.3.5) allows us decide whether its corresponding game has a perfect quantum strategy in polynomial time [13].

Lastly, in [18], Ji gives a sequence of parity binary constraint systems such that for any $d \in \mathbb{N}$, there is some BCS in the sequence which has a quantum satisfying assignment, and any quantum satisfying assignment for the BCS requires a Hilbert space of dimension at least d (and whose corresponding game requires $\log_2(d)$ dimensional entanglement). Ji also shows that a quantum satisfying assignment using tensor products of Pauli matrices (a k-qubit assignment for some $k \in \mathbb{N}$) exists for each of these binary constraint systems. Thus for any $k \in \mathbb{N}$, there exists a parity BCS that has a k-qubit quantum satisfying assignment, but no k'-qubit quantum satisfying assignment for any $k' < k$. It is natural to ask whether this property translates to our setting for the associated hypergraphs of this sequence of binary constraint systems. This would show that for Problem 2.2.2 we can find proper Eulerian hypergraphs which are magic, and for which the minimum value of k needed for a k-qubit magic assignment is arbitrarily large. As this can have a large impact on our work, we give Ji's construction now.

Definition 2.4.3. *[18] For any $N \in \mathbb{N}$, $N \geq 2$, let $G = (V, E)$ be a complete graph with vertex set $V = \{1, 2, \ldots, N\}$. To each vertex $j \in V$, associate a variable x_j, and to each edge $e = (j, k)$ for $j < k$, associate the variables $y_1^{(e)}, y_2^{(e)}, \ldots, y_7^{(e)}$. For each edge $e = (j, k)$ build the constraints: $y_1^{(e)} x_k y_2^{(e)} = 1$, $x_j y_3^{(e)} y_4^{(e)} = 1$, $y_5^{(e)} y_6^{(e)} y_7^{(e)} = 1$, $y_1^{(e)} x_j y_5^{(e)} = 1$, $x_k y_3^{(e)} y_6^{(e)} = 1$, $y_2^{(e)} y_4^{(e)} y_7^{(e)} = -1$. The parity BCS built from these variables and constraints is called the* Clifford BCS of rank N*, and the game associated to it is the* Clifford BCS game of rank N*.*

We quote Ji to indicate the inspiration of this construction and its name: 'The basic idea is to glue a bunch of magic squares and use the anti-commutativity relations implicit in the magic square to form the defining relations of the Clifford algebra.'

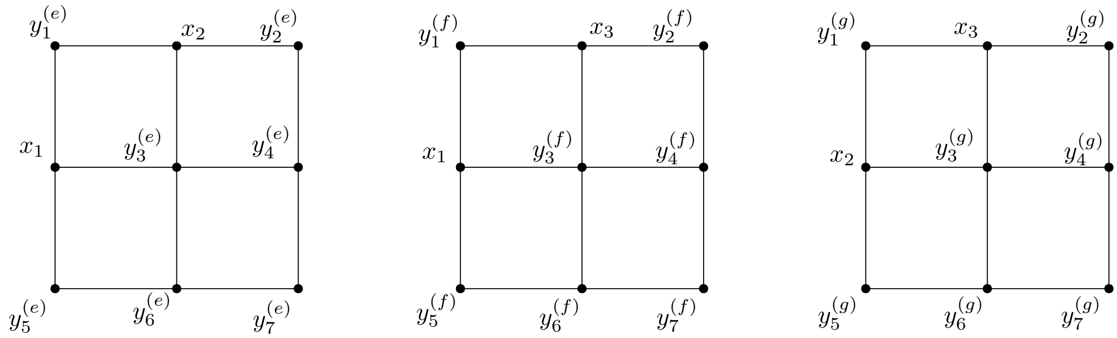

Figure 2.11: The hypergraph $\mathcal{C}_3$

Theorem 2.4.4. *[18] For each quantum satisfying assignment of the Clifford BCS of rank N, its corresponding Hilbert space has dimension at least $2^{\lfloor \frac{N}{2} \rfloor}$.*

For each N, $N \geq 2$, the Clifford BCS of rank N has the property that each variable appears in an even number of constraints. Therefore, its associated hypergraph is not only proper, but also Eulerian.

Example 2.4.5. In Figure 2.11 we display the proper Eulerian hypergraph associated to the Clifford BCS of rank 3. We refer to the proper Eulerian hypergraph associated to the Clifford BCS of rank N by $\mathcal{C}_N$. Note that all points which are labeled the same represent the same point, so that points x_1, x_2, x_3 all have degree 4. Additionally, the K_3 used in our construction has edges $e = (1,2),\ f = (1,3),\ g = (2,3)$.

Notice that for $N \in \mathbb{N}$ such that $\binom{N}{2}$ is odd (so $N \equiv 2, 3 \pmod 4$), the Clifford BCS of rank N has an odd number of constraints C_i with $r_i = -1$, and so a quantum satisfying assignment for the Clifford BCS of rank N translates directly to a magic assignment of $\mathcal{C}_N$. Thus we see that for $N \equiv 2, 3 \pmod 4$, $\mathcal{C}_N$ is magic. In Section 3.3 we study these hypergraphs further, and find the minimum k value for a magic k-qubit assignment to exist for $\mathcal{C}_N$ for all $N \in \mathbb{N}$, $N \geq 2$.

Chapter 3

New Results

As can be seen in the previous chapter, while Arkhipov's result completely characterizes the proper Eulerian hypergraphs with duals which are graphs, very little is known about the case when the duals are not graphs. Ideally one would like an extension of this result to the more general case.

The purpose of the remainder of this thesis is to use combinatorial and graph theoretic techniques to make progress on Problems 2.1.8 and 2.2.2. In Section 3.1 we introduce an algorithm called point splitting to show that certain hypergraphs are not magic using the planarity of graphs. In Section 3.2, we then recast and generalize Problem 2.1.8 in a generalized dual space of proper Eulerian hypergraphs - the space of edge-even hypergraphs. Additionally, we generalize the edge magic property defined for Arkhipov's result to all edge-even hypergraphs, and also recast the point splitting operation in the dual space. This translation immediately allows us to derive useful time complexity information about the point splitting operation. It also sets the groundwork for the approach taken in the rest of the thesis - using graph theoretic techniques in the generalized dual space to make progress on Problems 2.1.8 and 2.2.2. More precisely, in Section 3.3, we present a certain generalization of the graph minor relation to hypergraphs which we call the parity minor relation. This allows us to reduce edge magic hypergraphs to minimally edge magic hypergraphs, and thus it changes the nature of Problem 2.1.8 to one of finding minimally edge magic hypergraphs. We conclude this section and chapter by describing the properties of minimally edge magic hypergraphs.

Unless otherwise stated, any result appearing in the remainder of the thesis is novel. We make special note that a recent preprint on arXiv by William Slofstra [35] (previously referenced in Section 2.4) has results which bare a great similarity to some of our own - in particular, to the parity minor relation. A talk based on results of an earlier version of this thesis chapter was accepted for presentation [37] at the international Workshop on Algebraic Structures in Quantum Computation held at the University of British Columbia in May 2016, and the preprint [35] appeared shortly thereafter. In Section 3.3 we will elaborate on this matter, discussing the similarities and differences between Slofstra's work and our own. For the moment, we recall from Section 2.4 that Slofstra's work illustrates that undecidable instances exist in the context of finding perfect quantum strategies (using the commutative model) for parity binary constraint system games. It is possible that the method used for this proof could be used to find undecidable instances for our own problem. However, one should note that the smallest undecidable instance found by Slofstra so far has 400 variables and 300 constraints. Given that we are interested in finding new Kochen-Specker proofs in the form of minimally edge magic hypergraphs, there is still much space left for investigation. Even if no polynomial time algorithm analogous to Arkhipov's result exists for the class of all proper Eulerian hypergraphs, much can still be said, as evidenced in part by our own work.

3.1 Point Splitting

In the previous chapter, we outlined the main motivation of this thesis - Problem 2.1.8: Given a proper Eulerian hypergraph, decide whether or not it is magic, and in the case that it is magic, to find a magic assignment.

We have now seen that this problem is solved by Theorem 2.3.5 in the case that H is a proper 2-regular hypergraph i.e. each point has degree 2. In this section we present an operation called point splitting which allows us to utilize Theorem 2.3.5 to show that many proper Eulerian hypergraphs are not magic. This operation is further developed in Section 3.2, where it is also used to prove positive results.

We begin by presenting a motivating example.

Example 3.1.1. The proper Eulerian hypergraph in Figure 3.1 is called the *4-fan.* Note that point p_1 is incident to four blocks, thus we cannot simply apply Theorem 2.3.5. What if we instead considered the proper 2-regular hypergraph (called the *Split 4-fan*) illustrated in Figure 3.2? We now show that if the 4-fan is magic, then the Split 4-fan is as well. Let us assume there exists some magic labeling α of the 4-fan (pictured on the left of Figure 3.3). Let us denote $\alpha(p_i)$ by A_i. We then display a labeling α' of the Split 4-fan (on the right of Figure 3.3). Note that each block in the 4-fan has a corresponding block in the Split 4-fan whose block products contain the exact same matrices, and vice-versa. Thus α' is a magic labeling of the Split 4-fan. But the Split 4-fan is not magic, as we can see by its planar dual graph, illustrated in Figure 3.4. Thus we see that the 4-fan is not magic as otherwise the Split 4-fan would be magic as well.

We now formalize the idea illustrated by the previous example for Eulerian hypergraphs. Note that for this definition we do not insist that the given hypergraph is also proper. In the following sections (particularly Section 3.3) it is useful to consider the general class of Eulerian hypergraphs, as these will arise naturally by applying certain operations to proper

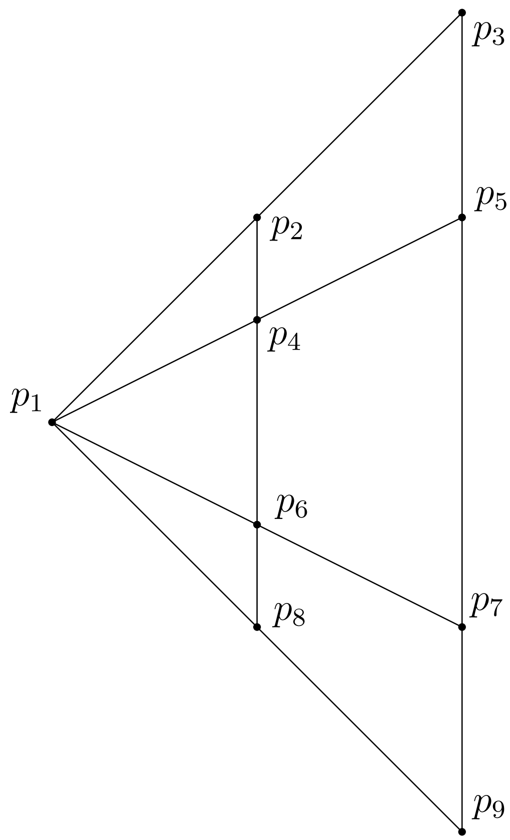

Figure 3.1: The 4-fan

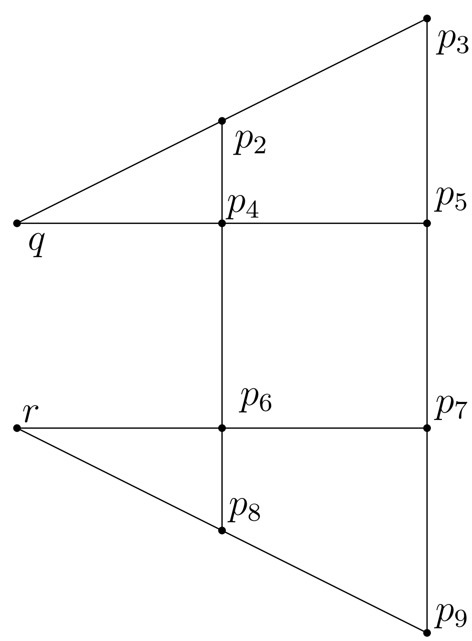

Figure 3.2: The Split 4-fan

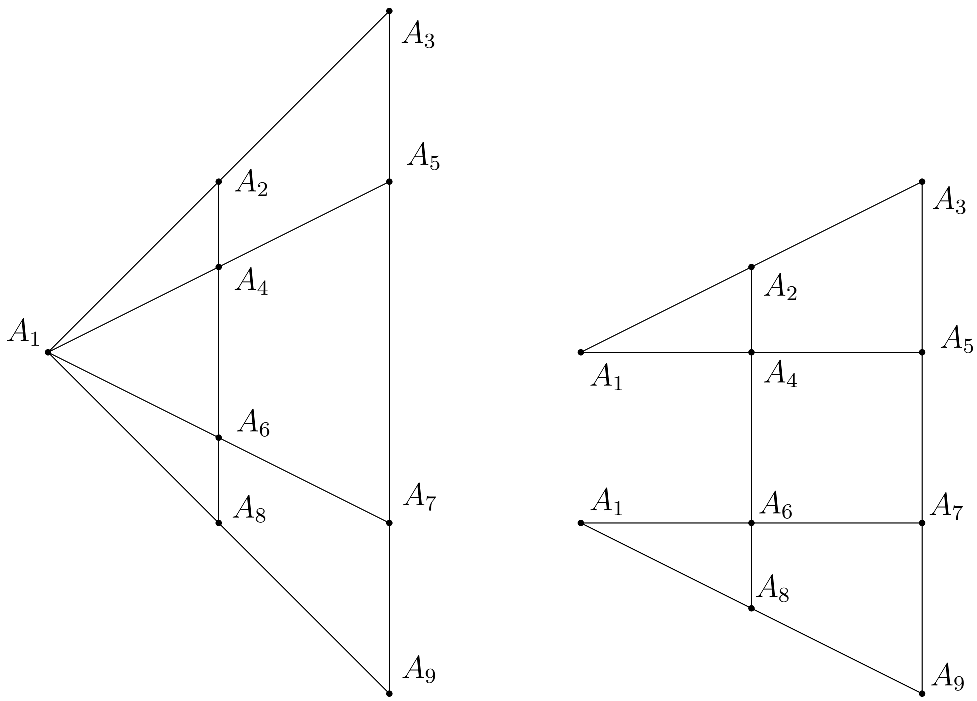

Figure 3.3: The 4-fan labeled (left) and the Split 4-fan labeled (right)

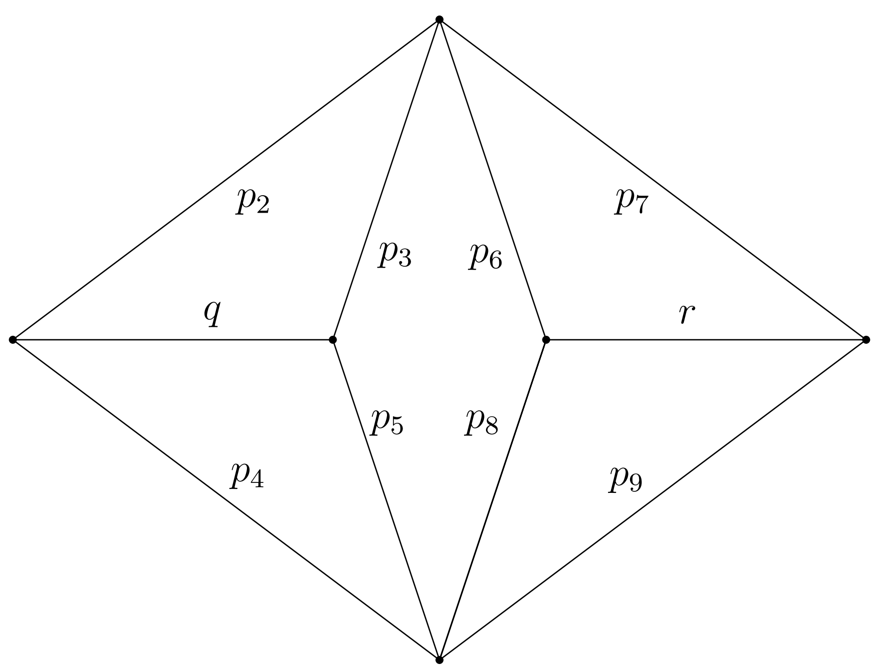

Figure 3.4: The dual graph of the Split 4-fan

Eulerian hypergraphs. In Chapter 4 we shall see the utility of defining the point splitting operation to the more general setting, when we combine the results of this chapter to search for irreducible magic proper Eulerian hypergraphs.

Definition 3.1.2. *Let $H = (P, B)$ be an Eulerian hypergraph, and let $p \in P$ such that $deg(p) \geq 4$. Let $b_1, ..., b_n$ be the set of blocks in H incident to p. For any b_i, b_j such that $1 \leq i < j \leq n$ or $i = j$ and $mult(p, b_i) \geq 2$, we define the hypergraph $H' = (P', B')$ as follows:*

- $P' = (P \cup \{q, r\}) - \{p\}$ *(where q, r are not already in P).*
- $B' = (B \cup \{b'_1, ..., b'_n\}) - \{b_1, ..., b_n\}$ *where:*

  for $t \in \{1 \ldots, n\}$, for every $s \in P \cap P'$, $mult_{H'}(s, b'_i) = mult_H(s, b_i)$

  for $t \in \{1, \ldots, n\} - \{i, j\}$, $mult_{H'}(q, b'_t) = 0$, $mult_{H'}(r, b'_t) = mult_H(p, b_t)$,

  if $i \neq j$, then for $t \in \{i, j\}$, $mult_{H'}(q, b'_t) = 1$, $mult_{H'}(r, b'_t) = mult_H(p, b_t) - 1$,

  else if $i = j$, then $mult_{H'}(q, b'_i) = 2$, $mult_{H'}(r, b'_i) = mult_H(p, b_i) - 2$.

We shall refer to this operation as splitting the point p in H, *or more simply as* splitting p. *We shall also refer to H' as a* split of H. *Additionally, we say that points q and r were* obtained by splitting p, *and that b'_i is the* block corresponding to b_i *for $i \in \{1, \ldots, n\}$. To refer to a specific split, we say that* blocks b_i and b_j were chosen in the process of splitting p.

In principle point splitting could be defined also for non-Eulerian hypergraphs, but we only use it for Eulerian hypergraphs, and thus have no need to generalize beyond this setting.

Proposition 3.1.3. *Let $H = (P, B)$ be an Eulerian hypergraph with point $p \in P$ such that $deg_H(p) \geq 4$, and let $H' = (P', B')$ be an Eulerian hypergraph obtained from H by splitting point p, such that q and r are the points obtained by splitting p. Then $deg_{H'}(q) = 2$ and $deg_{H'}(r) = deg_H(p) - 2$.*

We present an example of the splitting operation on an Eulerian hypergraph. Note that in the following example block b_6 is unaffected by the split and is thus still b_6 in the resulting hypergraph (instead of being replaced by a block b'_6).

Example 3.1.4. Consider the hypergraph H shown in Figure 3.5. Figures 3.6, 3.7, 3.8, 3.9, 3.10 display all of the non-isomorphic splits of H. In each of these figures, hypergraph $H_{i,j}$ was obtained by choosing blocks b_i and b_j in the process of splitting p.

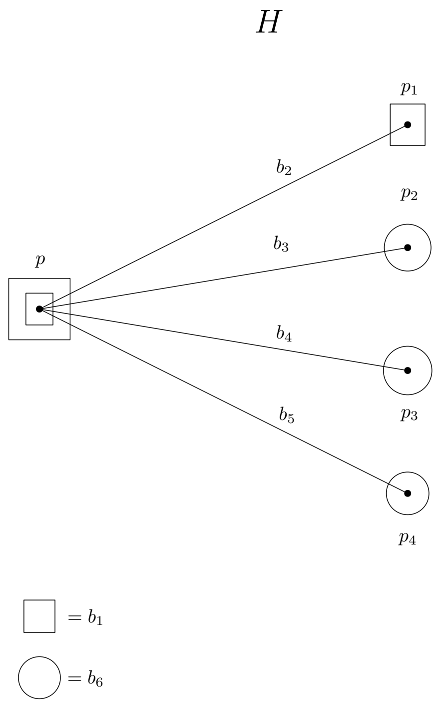

Figure 3.5: Eulerian hypergraph H used to illustrate the splitting operation

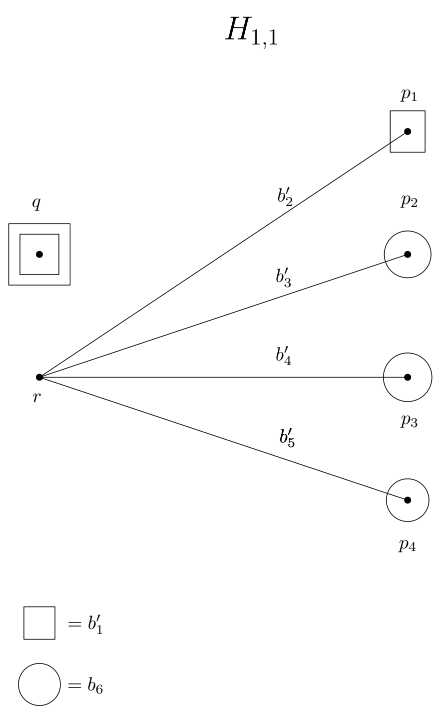

Figure 3.6: Eulerian hypergraph $H_{1,1}$ obtained by choosing blocks b_1, b_1 in the split of p in H

$H_{1,2}$

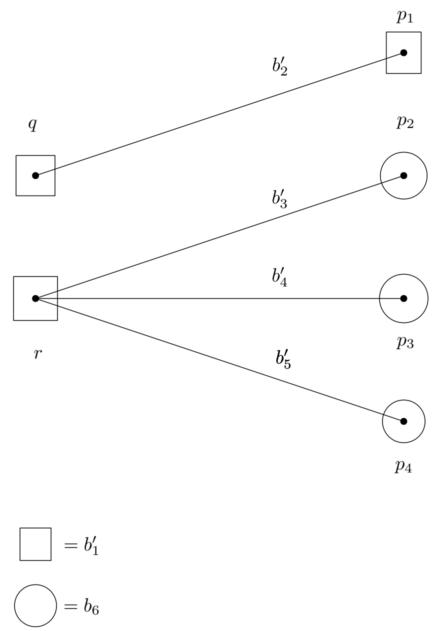

Figure 3.7: Eulerian hypergraph $H_{1,2}$ obtained by choosing blocks b_1, b_2 in the split of p in H

$H_{1,3}$

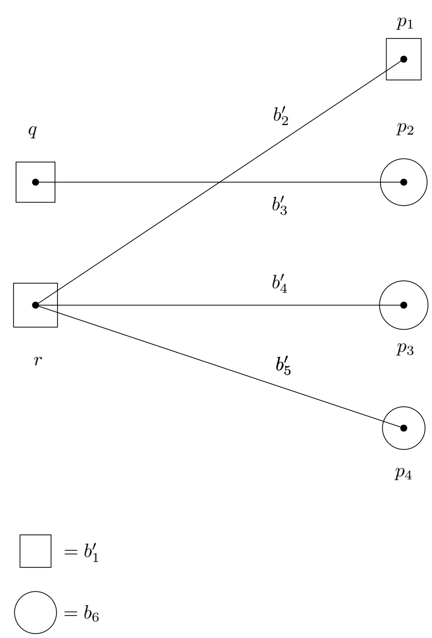

Figure 3.8: Eulerian hypergraph $H_{1,3}$ obtained by choosing blocks b_1, b_3 in the split of p in H

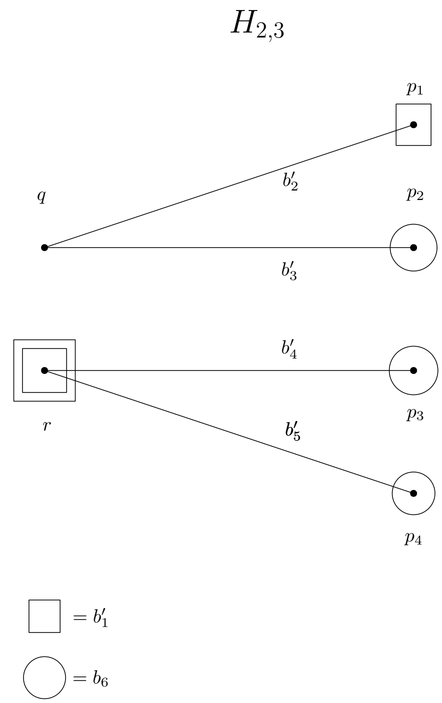

Figure 3.9: Eulerian hypergraph $H_{2,3}$ obtained by choosing blocks b_2, b_3 in the split of p in H

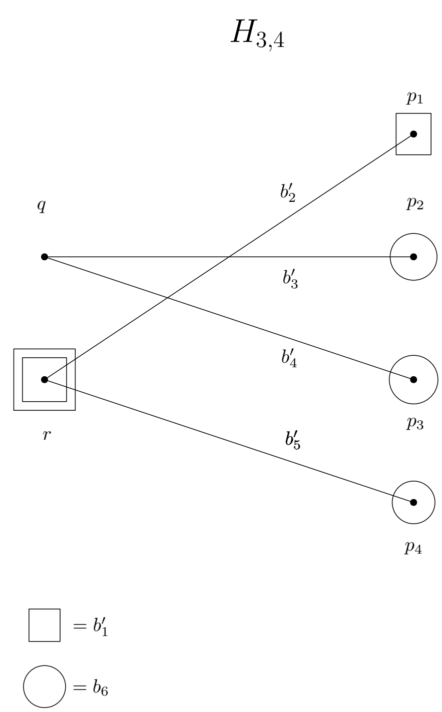

Figure 3.10: Eulerian hypergraph $H_{3,4}$ obtained by choosing blocks b_3, b_4 in the split of p in H

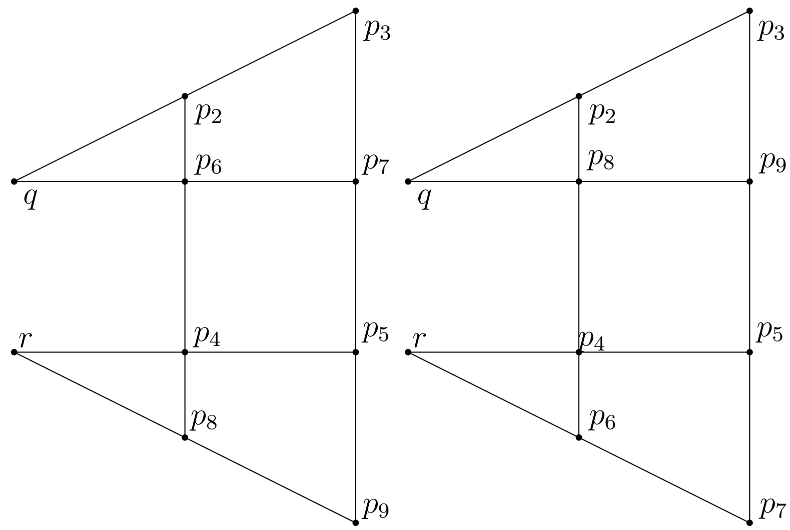

Figure 3.11: The two other hypergraphs obtained by splitting the 4-fan

As we see in the above example, the operation of splitting a point is not unique given the hypergraph and the point - it is also dependent upon the choice of b_i, b_j.

Let us consider the specific case in which $H = (P, B)$ is a proper Eulerian hypergraph. Let $p \in P$ be a point of degree at least 4. Then for any block $b \in B$, $mult_H(p, b) \leq 1$ by definition. Let H' be a hypergraph resulting from splitting p, so that q and r are the points obtained by splitting p. Since $deg_H(p) = deg_{H'}(q) + deg_{H'}(r)$ and for any block b we have $mult_H(p, b) = mult_{H'}(q, b') + mult_{H'}(r, b')$ (where b' is the block corresponding to b in H) we see that $mult(q, b') \leq 1$ and $mult(r, b') \leq 1$ as well. In addition, we see that q and r are incident to no common block in H'.

Utilizing this, if $H = (P, B)$ is a proper Eulerian hypergraph, we can calculate an upper bound on the number of non-isomorphic splits of H. If $p \in P$ is incident to $k \neq 4$ blocks, there are $\binom{k}{2}$ possible splits of p (corresponding to pairs of distinct blocks), and if p is incident to 4 blocks, then there are $\frac{\binom{4}{2}}{2} = 3$ splits (by symmetry).

Example 3.1.5. In addition to the split of the 4-fan illustrated in Figure 3.2, there are two other possible splits. They are pictured in Figure 3.11. Note that we referred to the structure before as *the Split 4-fan* (as opposed to *a Split 4-fan*) since all splits of the 4-fan are isomorphic.

The splitting operation can also be applied recursively on Eulerian hypergraphs resulting from splits. Thus to each Eulerian hypergraph H, there is associated a set of Eulerian hypergraphs that can be reached from H by recursively splitting points, and for any other Eulerian hypergraph H' we can ask the question of whether H' is in this set. We formalize this idea.

Definition 3.1.6. *Let H be an Eulerian hypergraph. If there exists a sequence of hypergraphs $H = H_1, H_2, \dots, H_n = H'$ such that H_{i+1} can be obtained by splitting some point of H_i (for $i \in \{1, \dots, n-1\}$), then we say that H' is* point split-obtainable *from H.*

We now give several properties of the splitting operation.

Proposition 3.1.7.

1. *Any hypergraph which is point split-obtainable from an Eulerian hypergraph is an Eulerian hypergraph. Any hypergraph which is point split-obtainable from a proper Eulerian hypergraph is a proper Eulerian hypergraph.*

2. *For every Eulerian hypergraph $H = (P, B)$ there is some sequence of Eulerian hypergraphs $H = H_1, H_2 \dots, H_l = H'$ such that H_{i+1} can be obtained by splitting some point of H_i (for $i \in \{1, \dots, l-1\}$) and H' is a 2-regular hypergraph. Further, any sequence of l hypergraphs beginning with H, and having the property that each hypergraph is obtained by splitting a point in the previous hypergraph, terminates with a 2-regular hypergraph. Lastly, any such sequence of less than l hypergraphs does not terminate with a* 2*-regular hypergraph, and no such sequences of length greater than l exist.*

   We shall call l the split length *of H. Explicitly, we will always have that $l = \sum_{p \in P} \frac{deg(p)-2}{2}$.*

3. *If $H = (P, B)$ is an Eulerian hypergraph, with $P = \{p_1, \dots, p_k\}$ and $H' = (P', B')$ is point split-obtainable from H, then we can partition the points of P' into sets $P'_1, \dots, P'_k$ where points in P'_i were obtained by splitting point p_i and successively splitting the points obtained (for $1 \leq i \leq k$). We shall call the points in P'_i the* set of points corresponding to p_i, *or alternatively the* set of points obtained from splitting p_i.

4. *If H is a magic proper Eulerian hypergraph and H' is point split-obtainable from H, then H' is magic (proven in Theorem 3.1.9).*

Proposition 3.1.8. *Let H be a magic proper Eulerian hypergraph, and let H' be obtained from H by splitting a point. If H is magic, then H' is magic.*

Proof. Let $H = (P, B)$ and let $H' = (P', B')$. Let $p \in P$ be the point split in H to obtain H', and let q and r be the points in P' corresponding to p.

Since H is magic, there exists some magic labeling of H, $\alpha : P \to GL(\mathcal{H})$. Then, we define a labeling $\alpha' : P' \to GL(\mathcal{H})$

$$\alpha'(s) = \begin{cases} \alpha(p) & \text{if } s \in \{q, r\} \\ \alpha(s) & \text{otherwise} \end{cases}$$

We now show that α' is a magic labeling of H'.

Thus we must show that α' obeys conditions 1-4 of Definition 2.1.7.

1. For any point $p' \in P'$, we have $\alpha'(p') = \alpha(s)$ for some $s \in P$. Since α is a magic assignment, $\alpha'(p')$ is Hermitian and squares to the identity.

2. For any block $b' \in B'$, and points $p', q' \sim b'$, either both $p', q' \in P' - \{q, r\}$, or p' is obtained from the split, so that $p' = q$ and $q' \in P' - \{q, r\}$ without loss of generality ($q' \neq r$ since the points obtained from the split are incident to no common blocks).

   In the first case we have $\alpha'(p')\alpha'(q') = \alpha(p')\alpha(q') = \alpha(q')\alpha(p') = \alpha'(q')\alpha'(p')$ as required. In the second case, we have $\alpha'(p')\alpha'(q') = \alpha(p)\alpha(q')$. Since p' is obtained from the split, p and q' must be incident to some common block in H. Thus $\alpha'(p')\alpha'(q') = \alpha(p)\alpha(q') = \alpha(q')\alpha(p) = \alpha'(q')\alpha'(p')$ as required.

3. Let $b' \in B'$. Then either $b' \in B$, or b' is incident to exactly one point from q or r.

   If $b' \in B$, then $\prod_{p':p'\sim b'} \alpha'(p') = \prod_{p:p\sim b'} \alpha(p) = \pm I$. Otherwise if $b' \notin B'$, let $b' = (q, p'_1, \ldots p'_k)$, where there is some corresponding block $b = (p, p'_1, \ldots p'_k) \in B$. Then $\prod_{p':p'\sim b'} \alpha'(p') = \alpha'(q) \prod_{i=1}^{k} \alpha'(p'_i) = \alpha(p) \prod_{i=1}^{k} \alpha(p'_i) = \prod_{p':p'\sim b} \alpha(p') = \pm I$.

4. This follows directly from the proof of 3.

Thus we see that H' is magic, as α' is a magic labeling of H'. □

We can now extend this proposition to the following theorem inductively.

Theorem 3.1.9. *If H is a magic proper Eulerian hypergraph and H' is point split-obtainable from H, then H' is magic.*

Proof. By induction on the previous proposition. □

Theorem 3.1.10 (Splitting Theorem)**.** *Let H be a proper Eulerian hypergraph. Let H' be a proper 2-regular hypergraph that is point split-obtainable from H. Let G be the dual graph of H'. If G is planar, then H is non-magic.*

Proof. Assume towards a contradiction that H is magic. Then by the previous theorem, we see that H' is magic. But then G would be non-planar by Theorem 2.3.5. This is a contradiction, therefore, H must be non-magic. □

Example 3.1.11. Consider the k-fan (defined for even k) shown in Figure 3.12, the natural generalization of the 4-fan. We can show that the k-fan is non-magic for any even k by Theorem 3.1.10. Figure 3.13 shows a split of the k-fan, and in Figure 3.14 is the planar dual graph of the split.

In general, we can test any proper Eulerian hypergraph H with Theorem 3.1.10 by checking the dual graphs of all the possible proper 2-regular hypergraphs that are point split-obtainable from H. Note that in many cases we do not have to check all of the dual

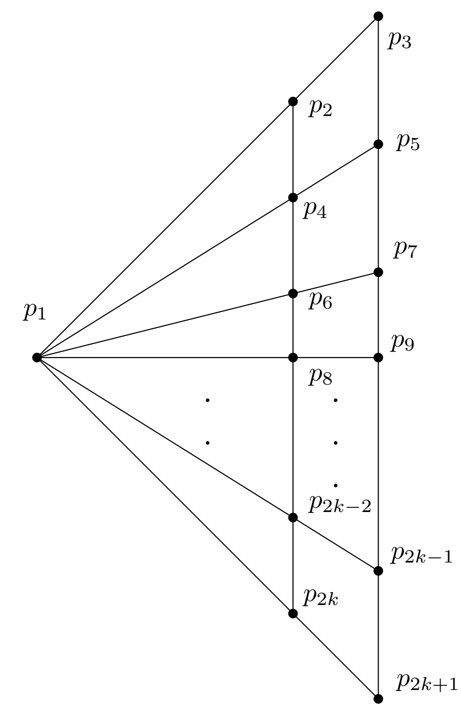

Figure 3.12: The k-fan

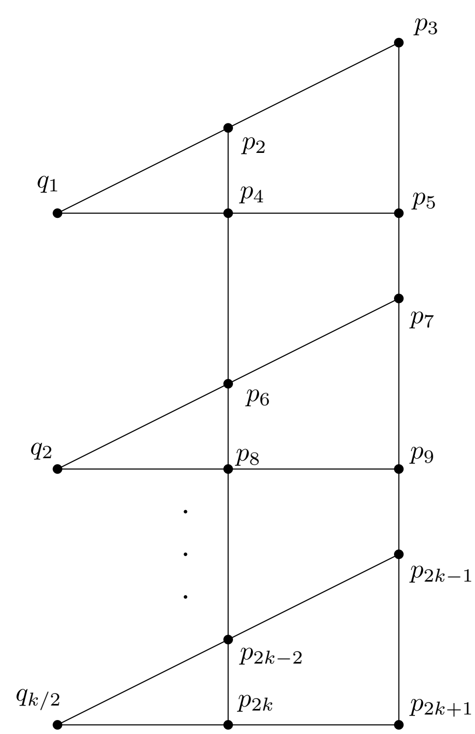

Figure 3.13: The Split k-fan

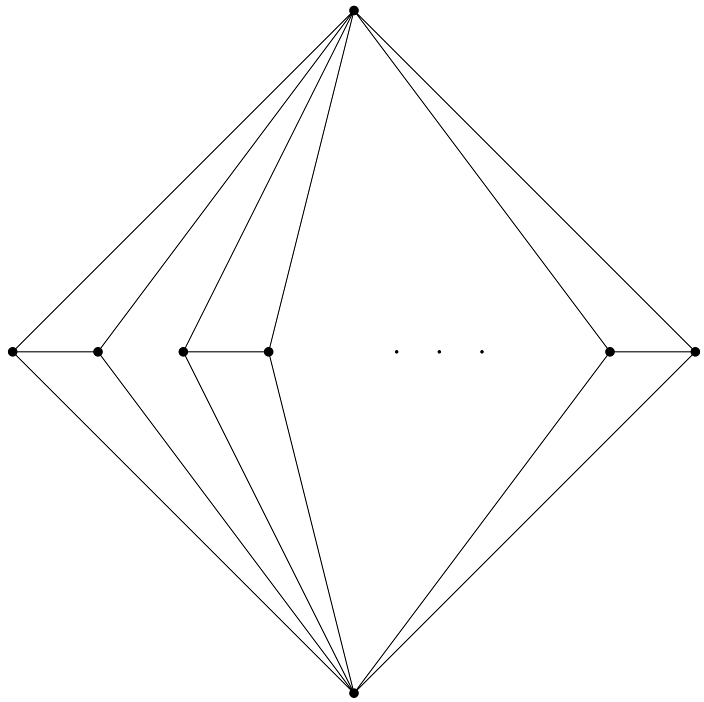

Figure 3.14: The dual graph of the Split k-fan

graphs - we can terminate when we find the first planar one. This approach suggests the two following questions:

Question 3.1.12. *If all the proper 2-regular hypergraphs which are point split-obtainable from some proper Eulerian hypergraph, H, have non-planar duals, is it true that H is magic?*

Question 3.1.13. *For a given proper Eulerian hypergraph H, what is the upper bound on the number of 2-regular hypergraphs which are point split-obtainable from H?*

In the following section, we answer both of these questions.

3.2 Dual Hypergraphs

One of the main results of this thesis is a generalization of one direction of Theorem 2.3.5 to Eulerian hypergraphs. The proof of Theorem 2.3.5 relies on dual graphs; proper 2-regular hypergraphs are never manipulated directly. Thus, it seems only natural that we should understand a little more about the duals of proper Eulerian hypergraphs if we would like to extend Theorem 2.3.5 (this is done in Section 3.3). Such a venture has the added benefit of providing a different perspective on the point splitting operation, which we shall exploit in order to answer Question 3.1.12 and Question 3.1.13.

Let us begin by describing the duals of Eulerian hypergraphs - these will be the central objects of study for the remainder of this thesis.

Definition 3.2.1. *Let J be a hypergraph such that every edge is of even size. Then we say that J is an* edge-even *hypergraph.*

Proposition 3.2.2. *The dual of an Eulerian hypergraph is an edge-even hypergraph.*

Proof. Let $H = (P, B)$ be an Eulerian hypergraph and let $J = (V, E)$ be the dual hypergraph of H. For any edge $e \in E$, and point $p \in P$ corresonding to e, we have $|e|_J = \sum_{v \in V} mult_J(v, e) = \sum_{b \in B} mult_H(p, b) = deg_H(p)$. Since H is an Eulerian hypergraph, $deg(p)$ is even and so $|e|$ is even as well. Thus we see that the dual hypergraph of an Eulerian hypergraph is an edge-even hypergraph as required.

□

Example 3.2.3. In Figures 3.15 and 3.16 we provide illustrations of an Eulerian hypergraph H and its edge-even dual J respectively. As before, edges of size greater than two are depicted by concentric shapes around the vertices to which they are incident. Each shape corresponds to a particular edge as indicated below each respective hypergraph. Note that for example $mult(v_4, e_7) = 2$, and this is depicted by two circles centered at vertex v_4. These hypergraphs are also described in list form:

- $H = (V, E)$
- $V = \{v_1, \ldots, v_5\}$
- $E = \{e_1, \ldots, e_7\}$
- $e_1 = v_1v_5, e_2 = v_2v_5, e_3 = v_1v_3, e_4 = v_2v_3, e_5 = v_1v_2, e_6 = v_1v_2v_3, e_7 = v_3v_4v_4$
- $J = (X, F)$
- $X = \{x_1, \ldots, x_7\}$
- $F = \{f_1, \ldots, f_5\}$
- $f_1 = x_1x_3x_5x_6, f_2 = x_2x_4x_5x_6, f_3 = x_3x_4x_6x_7, f_4 = x_7x_7, f_5 = x_1x_2$

We now translate the magic property to the dual space. In doing so, we will also implicitly generalize it to all Eulerian hypergraphs.

Definition 3.2.4 (Extension of Definition 2.1.7)**.** *Let $J = (V, E)$ be an edge-even hypergraph. Then we say that J is* edge magic *or* e-magic *if there exists some labeling $\alpha : E \to GL(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, such that*

1. *for any edge $e \in E$, $\alpha(e)^2 = I$, and $\alpha(e)$ is Hermitian,*
2. *for any vertex $v \in V$, and $e, f \in E$ incident to v, we have $\alpha(e)\alpha(f) = \alpha(f)\alpha(e)$,*

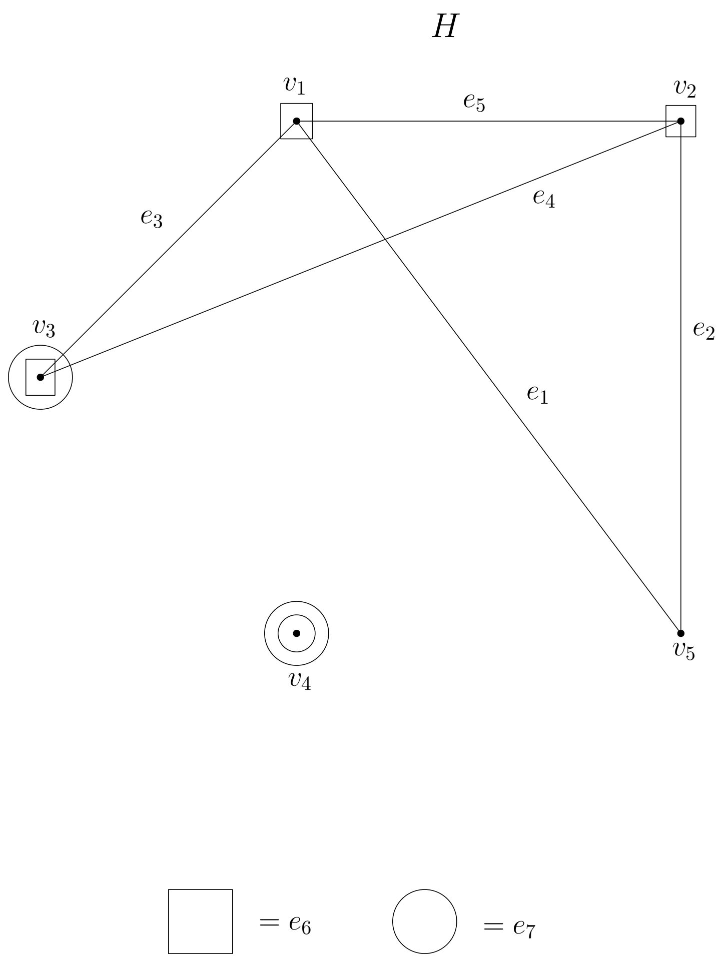

Figure 3.15: An Eulerian hypergraph H

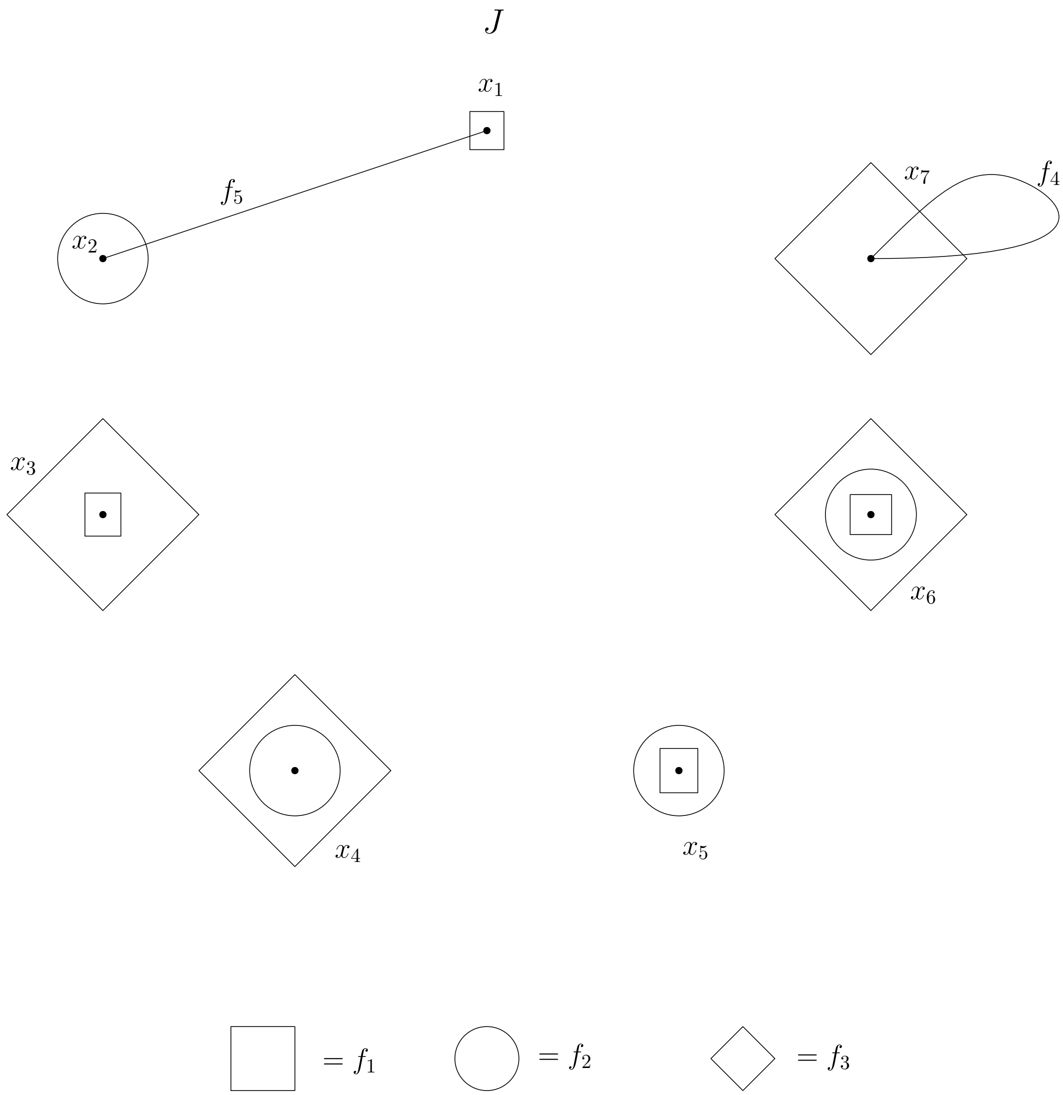

Figure 3.16: An edge-even hypergraph J - the dual of H

3. *for any vertex $v \in V$, we have*

$$\pi_\alpha(v) := \prod_{e \in E} \alpha(e)^{mult(v,e)} = \pm I,$$

4. *for an odd number of vertices $v \in V$, we have $\pi_\alpha(v) = -I$.*

If no such assignment exists, then we say that J is not e-magic. *(Note that we take $A^0 = I$ for any operator $A \in GL(\mathcal{H})$).*

Proposition 3.2.5. *Let H be a proper Eulerian hypergraph. Then H is magic if and only if the dual of H is e-magic.*

Proof. Since H is proper the multiplicities will all be zero or one. The rest follows directly by following the correspondence of blocks and points of H to vertices and edges of its dual. □

Our goal for the rest of the thesis is to deal only with edge-even hypergraphs (the duals), and do away with Eulerian hypergraphs. Thus, our current endeavour is to cast the major results of Section 3.1 in this setting. Once we have done this, the remainder of this section will be dedicated to answering Questions 3.1.12 and 3.1.13.

Definition 3.2.6 (Extension of Definition 3.1.2)**.** *Let $J = (V, E)$ be an edge-even hypergraph, and let $e \in E$ be an edge of size at least 4. Let $v_1, \dots, v_n$ be the set of vertices in J incident to e. For any v_i, v_j such that $1 \leq i < j \leq n$ or $i = j$ and $mult(v_i, e) \geq 2$, we define the hypergraph $J' = (V, E')$ as follows:*

- *$E' = E \cup \{f, g\} - \{e\}$ (where f, g are not already in E).*
- *for $t \in \{1, \dots, n\}$, for every $s \in E \cap E'$, $mult_{J'}(v_t, s) = mult_J(v_t, s)$,*

  for $t \in \{1, \dots, n\} - \{i, j\}$, $mult_{J'}(v_t, f) = 0$, $mult_{J'}(v_t, g) = mult_J(v_t, e)$,

  if $i \neq j$ for $t \in \{i, j\}$, $mult_{J'}(v_t, f) = 1$, $mult_{J'}(v_t, g) = mult_J(v_t, e) - 1$,

  else if $i = j$, $mult_{J'}(v_i, f) = 2$, $mult_{J'}(v_i, g) = mult_J(v_i, e) - 2$.

  We refer to this operation as splitting an edge *e in J, or more simply as* splitting *e, or as an* edge split *of J. Additionally, we say that edges f and g were* obtained by splitting *e. To refer to a specific edge split, we say that* vertices *v_i* and *v_j* where chosen in the process of edge splitting *e.*

Proposition 3.2.7 (Extension of Proposition 3.1.3)**.** *Let $J = (V, E)$ be an edge-even hypergraph with edge $e \in E$ such that $|e|_J \geq 4$ and let $J' = (V', E')$ be an edge-even hypergraph obtained from J by splitting edge e, such that edges f and g are the edges obtained by splitting e. Then without loss of generality $|f|_{J'} = 2$ and $|g|_{J'} = |e|_J - 2$.*

Notice that if H is an Eulerian hypergraph, J is its edge-even dual, and H' is a split of H, then the dual of H', say J', is an edge split of J.

Since for every Eulerian hypergraph H there is some 2-regular hypergraph that is point split-obtainable from H, for every edge-even hypergraph J there is some graph that is edge split-obtainable from J. Additionally, if J' is edge split-obtainable from J then the edges of J' can be partitioned so that each set of edges in the partition corresponds to a unique edge in J. Furthermore, since any hypergraph which is point split-obtainable from an Eulerian hypergraph is Eulerian, any hypergraph which is edge split-obtainable from an edge-even hypergraph is also edge-even.

We now prove a generalization of Proposition 3.1.8. The proof is similar, but contains added details since we are dealing with the duals of arbitrary Eulerian hypergraphs, not just duals of proper Eulerian hypergraphs.

Proposition 3.2.8 (Extension of Proposition 3.1.8)**.** *Let J be an edge-even hypergraph, and let J' be obtained from J by splitting an edge. If J is e-magic, then J' is e-magic.*

Proof. Let $J = (V, E)$ and $J' = (V, E')$ and let $\alpha : E \to GL(\mathcal{H})$ be an e-magic labeling of J. Let f be the edge in J which is split to obtain J', and let g and h be the edges corresponding to f in J'.

Then define $\alpha' : E' \to GL(\mathcal{H})$ as follows:

$$\alpha'(e) = \begin{cases} \alpha(f) & \text{if } e \in \{g, h\} \\ \alpha(e) & \text{if } e \in E' - \{g, h\} \end{cases}$$

We show that α' is an e-magic assignment of J' by showing that Properties $1-4$ of Definition 3.2.4 hold.

1. For any edge $e' \in E'$, $\alpha'(e') = \alpha(e)$ for some edge $e \in E$. Therefore $\alpha'(e')^2 = I$ and $\alpha'(e')$ is Hermitian since α is an e-magic assignment.

2. For any vertex $v \in V$ not incident to g or h in J', v is incident to the same edges in J' as in J, and these edges are labeled the same by α as by α'. Thus, since α is an e-magic assignment, commutativity holds.

   For any vertex $v \in V$ incident to g or h in J', notice that v is incident to f in J. Apart from that, v is incident to the same set of edges in J' as in J. Therefore, since $\alpha'(g) = \alpha'(h) = \alpha(f)$ and all other edges are labeled the same by α as by α', we see that again commutativity holds.

3. For any vertex $v \in V$ notice that $mult_J(v, f) = mult_{J'}(v, g) + mult_{J'}(v, h)$. Thus we see that:

$$
\begin{aligned}
\pi_{J'}(v) &= \prod_{e \in E'} \alpha'(e)^{mult_{J'}(v,e)} \\
&= \alpha'(g)^{mult_{J'}(v,g)} \alpha'(h)^{mult_{J'}(v,h)} \prod_{e \in E' - \{g,h\}} \alpha'(e)^{mult_{J'}(v,e)} \\
&= \alpha(f)^{mult_J(v,f)} \prod_{e \in E - \{f\}} \alpha(e)^{mult_J(v,e)} \\
&= \prod_{e \in E} \alpha(e)^{mult_J(v,e)} \\
&= \pi_J(v)
\end{aligned}
$$

4. We know from 3 that $\pi_{J'}(v) = \pi_J(v)$ for every $v \in V$. Therefore there are an odd number of vertices in J' with vertex product $-I$ since α is an e-magic assignment.

Therefore we see that α' is an e-magic assignment of J' as required. □

Continuing to transfer our knowledge to the dual framework, and implicitly generalize results to more general Eulerian hypergraphs, we give a result akin to Theorem 3.1.10 - the big result of Section 3.1.

Theorem 3.2.9 (Edge-splitting Theorem)**.** *Let J be an edge-even hypergraph. Let G be a graph which is edge split-obtainable from J. If G is planar, then J is not e-magic.*

Proof. Applying induction on Proposition 3.2.8, we see that if J is e-magic, then G must be e-magic. But G is a planar graph, and thus not e-magic. Therefore J must not be e-magic as required. □

Having cast the major results of Section 3.1 (and generalized them) in terms of the edge-even hypergraph duals, we now begin to justify the effort expended on this endeavor by presenting answers to Question 3.1.13 and Question 3.1.12. Namely, we will give an upper bound on the number of 2-regular graphs which are point split-obtainable from some proper Eulerian hypergraph, and produce a counter-example, showing that a proper Eulerian hypergraph can be non-magic, even if any point split-obtainable 2-regular hypergraph has a non-planar dual graph.

We begin by describing the set of graphs that are edge split-obtainable from a given edge-even hypergraph that is a dual of a proper Eulerian hypergraph H. This will aid us in providing an upper bound on the number of such graphs (and thus on the number of 2-regular hypergraphs which are point split-obtainable from H).

Proposition 3.2.10. *Let $J = (V, E)$ be the edge-even hypergraph defined by $V = \{v_1, \ldots, v_{2k}\}$ and $E = \{e\}$ where $e = v_1 v_2 \ldots v_{2k}$ for some positive integer k. Then the set of graphs which*

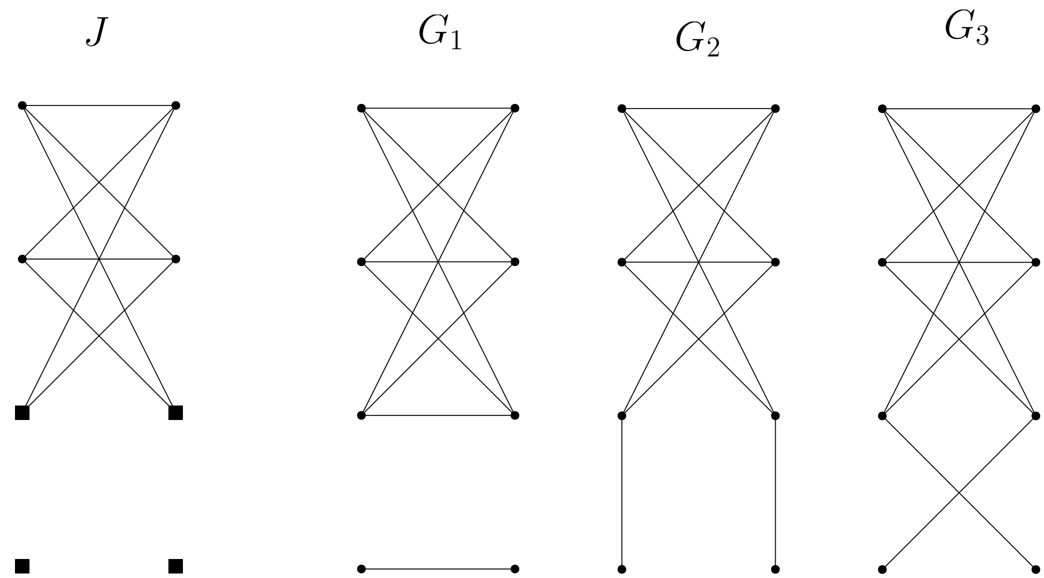

Figure 3.17: An edge-even hypergraph J (left) and the graphs arising from edge splitting in J: G_1, G_2, G_3 (right)

can be obtained by edge splits from J is the set of graphs with vertex set V and whose edges form a perfect matching of K_{2k}.

Proof. Let G be some graph which is split-obtainable from J. Since e has size $2k$, there will be k edges of size 2 in G corresponding to e. Any pair of these edges will not be incident to any common vertex, since $mult(v_i, e) = 1$ for all $1 \leq i \leq 2k$. Also by construction, none of these edges will be loops. Therefore, the set of edges will be a perfect matching.

Conversely, any perfect matching $\{e_1, \ldots, e_k\}$ can be obtained by choosing each edge sequentially when splitting - i.e split e into e_1 and f_1 where f_1 is incident to exactly those vertices that e_1 is not, then split f_1 into edges e_2 and f_2 and so on. At each step, the vertices that e_{i+1} is incident to must be a subset of the edges f_i is incident to since the e_i are disjoint. □

We give a simple example which illustrates the above proposition.

Example 3.2.11. Figure 3.17 shows the possible splits of an edge-even hypergraph J (on the left), where the edge f of size 4 is indicated by 4 large squares. J induced on the vertices incident to f is the hypergraph defined in Proposition 3.2.10 for $k = 2$. The three possible splits are displayed to the right of J, and the edge sets of the graphs induced on the vertices incident to f correspond to the three perfect matchings of a K_4. Note that, even though there is a non-planar split (the left-most split), all other splits are planar. Therefore J is not e-magic.

We now give an upper bound on the number of graphs arising from point splits for a particular edge-even hypergraph J, thus answering Question 3.1.13.

In the following proposition, take $hs(e)$ to denote half of the size of edge e.

Proposition 3.2.12. *Let $J = (V, E)$ be an edge-even hypergraph which is the dual of some proper Eulerian hypergraph. The number of graphs that are split-obtainable from J is at most* $\prod_{e \in E : |e| > 2} \frac{(2hs(e))!}{(hs(e))! 2^{hs(e)}}$.

Proof. For any edge e, we know from Proposition 3.2.10 that the set of possible resulting edges of size 2 (from successively splitting resulting edges) is exactly the set of perfect matchings on $K_{2(hs(e))}$. There are $\frac{(2(hs(e)))!}{(hs(e))! 2^{hs(e)}}$ such matchings by Proposition 1.2.17. Then in total, after splitting each edge to edges of size 2 there will be at most $\prod_{e \in E} \frac{(2(hs(e)))!}{(hs(e))! 2^{hs(e)}}$ resulting graphs. There may be less due to isomorphism. □

Thus to answer Question 3.1.13, the upper bound on the number of 2-regular hypergraphs which are point split-obtainable from some proper Eulerian hypergraph $H = (P, B)$ is:

$$\prod_{p \in P : deg(p) > 2} \left(\frac{(deg(p))!}{(\frac{deg(p)}{2})! \; 2^{\frac{deg(p)}{2}}} \right).$$

In particular, an algorithm which checks the duals of all 2-regular hypergraphs arising from some proper Eulerian hypergraph must check the planarity of at most this number of graphs.

We find easier representations in two special cases.

First we estimate an upper bound on the number of graphs that must be checked in the case that H has a single point of degree $2n$ and all other points are of degree 2. We use Stirling's approximation ($n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$) appearing in [7, Chapter VII] to show that $\frac{(2n)!}{n! 2^n} \sim \frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{\sqrt{2\pi n}(\frac{2n}{e})^n} = \sqrt{2}(\frac{2n}{e})^n$. Thus, the number of graphs whose planarity must be checked is on the order of $\Omega((\frac{2n}{e})^n)$ in this case.

Second if each point of H is of degree 4, it is easy to see that there are at most $3^{|P|}$ non-isomorphic graphs to check.

Thus we see that, in general, such an algorithm will have at least exponential time complexity in the case that all of the graphs arising as duals of 2-regular hypergraphs, which are point split obtainable from H, are non-planar.

Having answered Question 3.1.13, we now develop the tools to answer Question 3.1.12 (for a proper Eulerian hypergraph H, if every 2-regular hypergraph which is point split-obtainable from H has a non-planar dual, is it true that H is magic?). We answer this result in the negative via a counter-example.

First we give a definition and a few results which will be needed in order to show that the hypergraph in our counter-example is not magic - since our only method for doing so thus far is exactly by showing that it has some point split-obtainable 2-regular hypergraph whose dual is planar. The results we present here are also used frequently at other points in the thesis; their utility transcends our current motive.

Definition 3.2.13. *Let $J = (V, E)$ be an edge-even hypergraph. Let $\alpha : E \to GL(\mathcal{H})$. Let $J' = (V', E')$ be a subhypergraph of J. Let $\alpha' : E' \to GL(\mathcal{H})$ be defined by $\alpha'(e) = \alpha(e)$ for all $e \in E'$. Then we say that α' is the labeling obtained by* restricting *α to E' (or J').*

Proposition 3.2.14. *Let $J = (V, E)$ be an e-magic hypergraph, and let $\alpha : E \to GL(\mathcal{H})$ be an e-magic assignment of J. For some $f \in E$, let $\alpha_f : E \to GL(\mathcal{H})$ be defined by $\alpha_f(e) = \alpha(e)$ for $e \in E - \{f\}$ and $\alpha_f(f) = -\alpha(f)$. Then α_f is an e-magic assignment of J.*

Proof. Conditions 1, 2, and 3 of Definition 3.2.4 hold trivially.

For condition 4 notice that $|f|$ is even since J is edge-even. Therefore $mult(v, f)$ is odd for an even number of vertices $v \in V$. Finally, we see that an even number of vertices in V have the signs of their vertex products flipped, and so the parity of the number of vertices with vertex product $-I$ is the same for α_f as it is for α. Since α is an e-magic assignment, it follows that condition 4 holds for α_f.

Since conditions 1-4 of Definition 3.2.4 hold, α_f is an e-magic assignment of J. □

Proposition 3.2.15. *Let $J = (V, E)$ be an e-magic hypergraph. If there exists an e-magic assignment $\alpha : E \to GL(\mathcal{H})$ of J such that $\alpha(e) = I$ for some edge $e \in E$, then $J - e$ is e-magic.*

Proof. Let $J' := J - e$. Define α' to be α restricted to J'. Then we see that α' is an e-magic assignment of J' since conditions 1 and 2 of Definition 3.2.4 hold trivially and none of the vertex products are changed. □

Proposition 3.2.16. *Let $J = (V, E)$ be an e-magic hypergraph, and let $v \in V$ be a vertex of degree zero. Then $J - v$ is e-magic.*

Proof. Let $J' = J - v$. Let α be some e-magic assignment of J, and let α' be α restricted to J'. Then α' is an e-magic assignment of J' since conditions 1 and 2 of Definition 3.2.4 hold trivially, none of the vertex products in $V - \{v\}$ have changed so that condition 3 holds, and finally only a vertex of product I is removed so that the number of vertices with product $-I$ is the same for α' as it is for α - so that condition 4 also holds. □

Corollary 3.2.17. *Let $J = (V, E)$ be an edge-even hypergraph. Let $v \in V$ be a vertex such that $deg(v)$ is odd and $deg(v) = mult(v, e)$ for some edge $e \in E$. If J is e-magic, then $J - v$ is e-magic.*

Proof. Let $\alpha : E \to GL(\mathcal{H})$ be an e-magic assignment of J. Then $\pi_\alpha(v) = \prod_{f \in E} \alpha(f)^{mult(v,f)} = \alpha(e)^{mult(v,e)} = \alpha(e)$ since $mult(v, e)$ is odd and $\alpha(e)^2 = I$. But then $\alpha(e) \in \{-I, I\}$ since $\pi_\alpha(v) \in \{-I, I\}$, and so we can assume without loss of generality that $\alpha(e) = I$ since minus signs can be disregarded by Proposition 3.2.14. Then by Proposition 3.2.15 we see that $J' := J - e$ is e-magic. Lastly, in J', v is an isolated vertex, and therefore by Proposition

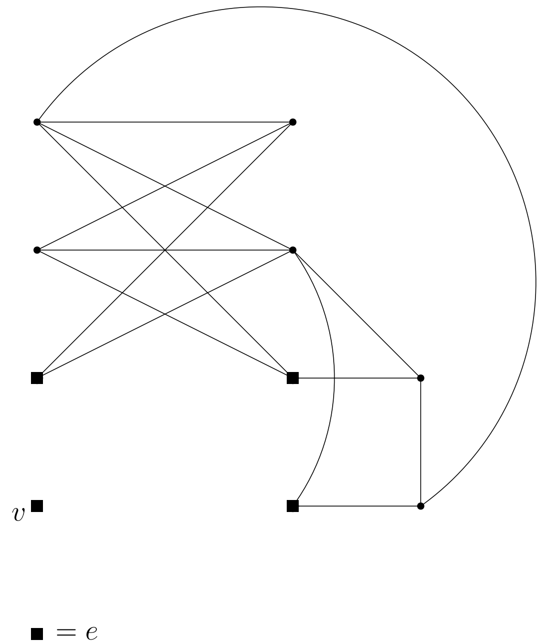

Figure 3.18: Counterexample to the converse of Theorem 3.1.10

3.2.16 $J' - v$ is e-magic. Notice that since e is incident to v in J, that $J' - v = J - v$, and so the result follows. □

Now we are finally prepared to present an answer to Question 3.1.12 in the form of an example.

Example 3.2.18. Consider the hypergraph J shown in Figure 3.18. Here the edge of size 4, e, is denoted by the 4 large square vertices. We can check that all 3 splits of edge e lead to non-planar graphs. However, vertex v is incident only to edge e, and $mult(v, e) = 1$. Thus we can apply the previous corollary (Corollary 3.2.17), and conclude that J is e-magic if and only if $J - v$ is e-magic. But $J - v$ is a planar graph, and thus is not e-magic. Therefore J is not e-magic.

The consequence of this example is that, in general, we cannot conclude whether an edge-even hypergraph J is e-magic or not based solely on splitting edges and checking the planarity of the resulting graphs. Even if every graph that is edge split-obtainable from J is non-planar, it is not guaranteed that J is magic.

To briefly recapitulate, in this section we have introduced notation in order to enable us to make direct arguments about hypergraphs arising as duals of Eulerian hyeprgraphs, instead of attempting to work with the Eulerian hypergraphs themselves - which are somewhat harder to work with. Applying this to the theory of point splitting developed in Section 3.1 led us to several results. In the upcoming section, we present our generalization of Theorem 2.3.5.

3.3 Parity Minor Theorem

Although Arkhipov's result is stated in terms of the planarity of the dual graph, Wagner's result (Theorem 1.3.24) allows us to recast one direction of Arkhipov's result in terms of graph minors. Instead of saying that if G is non-planar, then G is e-magic, we can say: Let G and G' be graphs such that G' is a minor of G. If G' is e-magic, then G is e-magic.

Therefore, one approach to generalizing Theorem 2.3.5 is to consider a generalization of the graph minor operations to hypergraphs. In other words, for some hypergraph minor relation, we would like to say: *Let J and J' be edge-even hypergraphs such that J' is a hypergraph minor of J. If J' is e-magic, then J is e-magic.*

We now provide a brief summary of several major results in graph and hypergraph minors as they pertain to this thesis. As there are many definitions of hypergraph minors developed for various applications which do not relate to the e-magic property, we do not claim this section to be a survey of this topic.

Much work has been done in the topic of graph minors, culminating in the famous result by Robertson and Seymour known as the *graph minors theorem* (see [15, Chapter 12] for a sketch of the proof), which shows that the operation of graph minors imposes a *well-quasi-ordering* on graphs. We define a well-quasi-ordering below.

Definition 3.3.1. *[15, Section 12.1] A reflexive and transitive relation is called a* quasi-ordering. *A quasi-ordering $\leq_X$ on a class X is a* well-quasi-ordering*, and the elements of X are* well-quasi-ordered *by $\leq_X$, if for every infinite sequence $x_1, x_2, \ldots$ where $x_i \in X$ for all i, there are indices $i < j$ such that $x_i \leq_X x_j$.*

In particular, in any infinite sequence of graphs $G_1, G_2, \ldots$ there must be a pair of graphs G_i, G_j so that G_i is a minor of G_j and $i < j$. Therefore, we can deduce that in any infinite set of graphs there must be a pair of distinct graphs such that one is a minor of the other. This fact can be used to show that for any class closed under the operation of graph minors (for example planar graphs), there is a finite set of so-called *forbidden minors*. This is a set $\{\mathcal{F}_i\}_{i=1}^{k}$ of graphs so that a graph G contains some $\mathcal{F}_i$ as a minor if and only if G is not in the class. Thus, in the example of planar graphs, we see that $\{K_{3,3}, K_5\}$ is exactly the set of forbidden minors. The Graph Minors Theorem is an extensive project published in over 20 articles, and in the 23rd article [33], Robertson and Seymour describe a function called a *collapse* which creates a well-quasi-order on the class of hypergraphs with $mult(v, e) \leq 1$ for any vertex and edge of the hypergraph. We now present their definition.

Definition 3.3.2. *[33, Section 1] Let J' and J be hypergraphs such that $mult(v, e) \leq 1$ for any vertex, edge pair of J' and for any vertex, edge pair of J. A* collapse *of J to J' is a function η with domain $V(J') \cup E(J')$, such that*

1. *$\eta(v)$ is a non-null connected subgraph of $K_{V(J)}$ for each $v \in V(J')$, and $\eta(u), \eta(v)$ are disjoint for all distinct $u, v \in V(J')$,*

2. $\eta(e) \in E(J)$ *for all* $e \in E(J')$, *and* $\eta(e) \neq \eta(f)$ *for all distinct* $e, f \in E(J')$,

3. *if* $v \in V(J')$ *and* $e \in E(J')$ *and* e *is incident in* J' *with* v, *then* $\eta(e)$ *is incident in* J *with a vertex of* $\eta(v)$,

4. *for each* $v \in V(J')$ *and* $f \in E(\eta(v))$ *with ends* x, y, *there is an edge* e *of* J *incident with* x *and* y,

   where $K_{V(J)}$ *denotes the complete graph on the vertex set* $V(J)$.

As previously stated, Robertson and Seymour prove that the collapse defines a well-quasi order on this class of hypergraphs.

Theorem 3.3.3. *[33, 1.2] For every countable sequence* J_i $(i = 1, 2, \dots)$ *of hypergraphs (such that* $mult(v, e) \leq 1$ *for every vertex, edge pair) there exist* $j > i \geq 1$ *such that there is a collapse of* J_j *to* J_i.

The collapse can also be viewed as a hypergraph minor relation with the following set of operations:

1. vertex deletion

2. edge deletion

3. identification of two vertices incident to a common edge

4. shrinking of an edge - the replacement of an edge by any subset of the edge

Here the identification of two vertices is done by replacing vertices $x, y \in V(H)$ with a new vertex w such $e \sim w$ if and only if $e \sim x$ or $e \sim y$.

Unfortunately, as the following example illustrates, the e-magic property is not preserved under this hypergraph minor definition.

Example 3.3.4. Consider Figure 3.19. Notice that shrinking edge h to edge e (and deleting the two vertices which were incident to h but not e) transforms a not e-magic hypergraph (this was shown in Example 3.2.11) to $K_{3,3}$, which we know to be e-magic.

Other more recent definitions of hypergraph minor also exist, for example in [1]. Of these definitions we make special mention of the hypergraph minor relation defined by Slofstra [35] as it bares a resemblance to our own. The motivation for Slofstra's work is related to perfect quantum strategies of parity binary constraint system games, and thus the hypergraph minor relation is not restricted to edge-even hypergraphs. However, if we enforce this restriction, it preserves the e-magic property. We shall return to this topic after we present our own hypergraph minor relation in order to discuss the similarities and differences in greater depth.

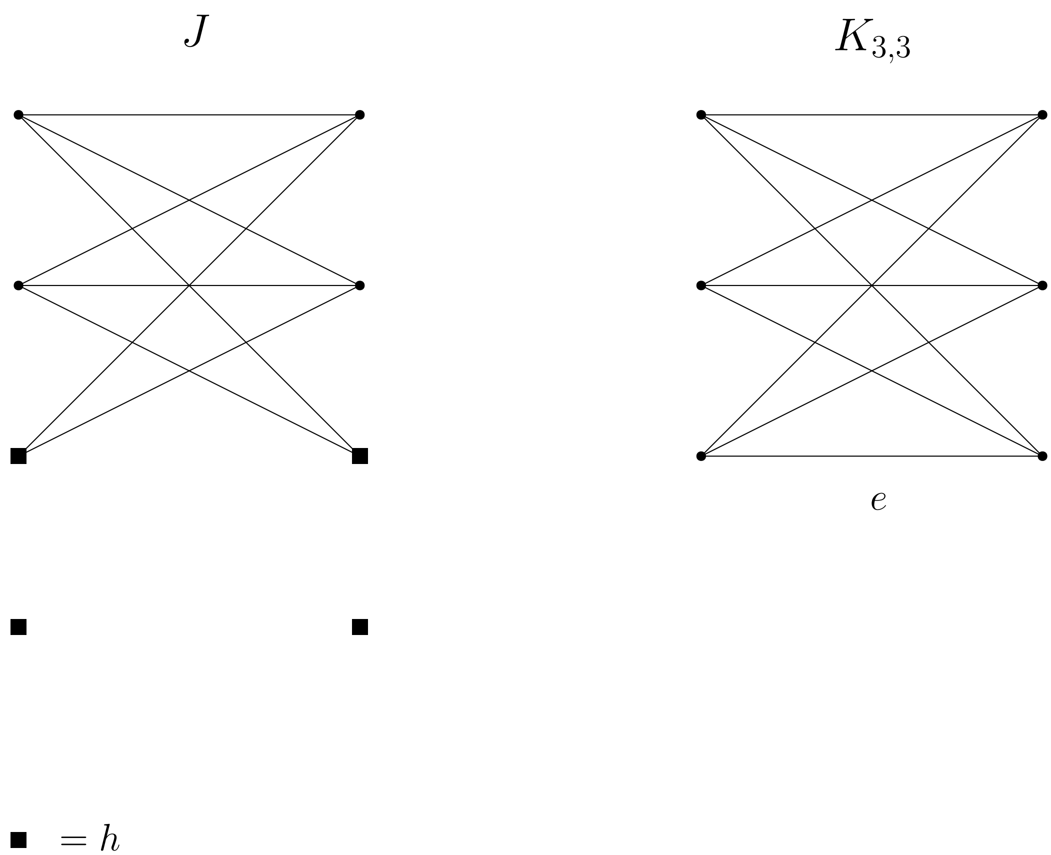

Figure 3.19: An example which shows that Robertson and Seymour hypergraph minors cannot be used for characterizing magic proper Eulerian hypergraphs.

Our current purpose is to define a set of operations to apply to an edge-even hypergraph J so that if the edge-even hypergraph resulting from these operations is e-magic, then J is also e-magic.

Towards this end, we explicitly define some operations that have already been presented for graphs - some of which we have already used for hypergraphs. We present them at this point formally because they are pivotal to this section.

Definition 3.3.5. *Let $J = (V, E)$ be a hypergraph, let $v \in V$, let $V' := V - \{v\}$, and let $E' := \{e \in E : e$ is not incident to $v\}$. By $J - v$ we denote the hypergraph (V', E'), and we say that $J - v$ is obtained from J by* deleting vertex v.

Definition 3.3.6. *Let $J = (V, E)$ be a hypergraph, let $e \in E$, and let $E' := E - \{e\}$. By $J - e$ we denote the hypergraph (V, E'), and we say that $J - e$ is obtained from J by* deleting edge e.

Definition 3.3.7. *To* identify vertices *x and y of a hypergraph $J = (V, E)$ is to replace these vertices by a single vertex w such that $mult(w, e) = mult(x, e) + mult(y, e)$ for each edge $e \in E$.*

Definition 3.3.8. *To* contract *a non-loop edge e of size 2 of a hypergraph J is to delete the edge and identify its ends. The resulting hypergraph is denoted by J/e.*

In addition to these, we define two new operations which affect the size of edges of a hypergraph.

Definition 3.3.9. *To* identify edges *e and f of a hypergraph $J = (V, E)$ is to replace these edges by a single edge g such that $mult(v, g) = mult(v, e) + mult(v, f)$ for each vertex $v \in V$.*

Definition 3.3.10. *Let $J = (V, E)$ be a hypergraph, and let $v \in V$, such that for some edge $f \in E$, we have either $mult(v, f) > 2$, or $mult(v, f) = deg(v) = 2$. Let $J' = (V, E')$, where $E' = (E - \{f\}) \cup \{f'\}$, such that $mult(w, f') = mult(w, f)$ for every $w \in V - \{v\}$, and $mult(v, f') = mult(v, f) - 2$. Then we say that J' is obtained from J by a* mod 2 reduction.

Notice that the incidence between an edge e and vertex v is only broken by mod 2 reductions if e is the only edge incident to v (since $deg(v) = mult(v, e)$). This is to avoid potential problems arising from condition 2 of Definition 3.2.4 (the commutativity condition). Additionally note that an edge can be reduced to size zero by mod 2 reductions (but is not deleted). An edge of size zero differs from a non-edge and allows us to create an isomorphism between the edge sets of J and J'. Note that edges of size zero have always been implicitly present as these arise from taking the dual of an isolated vertex.

We have now defined all the operations which we need, and thus define our new hypergraph minor relation. Note that in this thesis, the operation of contracting an edge in a hypergraph has only been defined for edges of size 2, and is thus applicable only to these edges.

Definition 3.3.11. *Let J be an edge-even hypergraph, and let J' be a hypergraph obtained from J by a (possibly empty) sequence of the following operations:*

- *deleting a vertex*
- *deleting an edge*
- *contracting an edge*
- *identification of a pair of edges*
- *mod 2 reduction applied to a vertex, edge pair.*

Then we say that J' is a parity minor *of J.*

Proposition 3.3.12. *Let J be an edge-even hypergraph and let J' be a parity minor of J. Then J' is an edge-even hypergraph.*

Proof. Out of the four operations associated to the parity minor, only two of these change the size of an edge - the identification of edges and the mod 2 reduction. Identifying edges yields an edge whose size is the sum of the two identified edges which both have even size, and since all other edges of the hypergraph remain the same, the edge-even property is retained. The mod 2 reduction operation changes the size of a single edge by two, and thus retains the parity of the size of every edge. □

Proposition 3.3.13. *The parity minor relation is a partial ordering on the class of edge-even hypergraphs.*

Proof. It is clear that the parity minor relation is reflexive.

We show that it is anti-symmetric. Consider the following three parameters of a hypergraph $J = (V, E)$:

- the number of vertices of J, i.e $|V|$.
- the number of edges of J, i.e $|E|$.
- the sum of the edge sizes of the edges of J, i.e $\sum_{e \in E} |e|$.

We note that the application of any of the five parity minor operations to J does not increase either of the three parameters. Additionally, vertex deletion strictly reduces the number of vertices, edge deletion strictly reduces the number of edges, edge contraction strictly reduces the number of vertices, edge identification reduces the number of edges, and mod 2 reduction strictly reduces the sum of the edge sizes. Therefore, we see that the parity minor relation is anti-symmetric.

Finally, it is clear that the parity minor relation is transitive.

Therefore, we see that the parity minor relation is a partial ordering on the class of edge-even hypergraphs. □

Proposition 3.3.14. *Let J be an edge-even hypergraph. Let J' be a hypergraph obtained from J by either the deletion of a vertex or the deletion of an edge. If J' is e-magic, then J is e-magic.*

Proof. Let $J = (V, E)$ and $J' = (V', E')$. In this case we can simply extend an e-magic labeling α' of J' to J by labeling all edges in $E - E'$ by I. It is trivial to show that α is then an e-magic labeling of J. □

Proposition 3.3.15. *Let J be an edge-even hypergraph. Let J' be a hypergraph obtained from J by the contraction of an edge. If J' is e-magic, then J is e-magic.*

Proof. Let $J = (V, E)$ and $J' = (V', E')$ and assume that J' was obtained from J by contracting some edge $f \in E$. Then f is a non-loop edge of size two, so assume that f is incident to vertices $u, v \in V$, and that vertex $w \in V'$ is obtained by identifying vertices u and v in the contraction process. Then by definition every edge $e \in E - \{f\}$ has a corresponding edge $e' \in E'$ such that $mult_{J'}(w, e') = mult_J(u, e) + mult_J(v, e)$ and $mult_{J'}(x, e') = mult_J(x, e)$ for each $x \in V - \{u, v\}$. Let α' be an e-magic assignment of J', and let α be defined by

$$\alpha(e) = \begin{cases} \alpha'(e') & \text{if } e \in E - \{f\} \text{ (where } e' \text{ is the edge corresponding to } e) \\ \prod_{g \in E - \{f\}} \alpha'(g')^{mult_J(u,g)} & \text{if } e = f \text{ (where } g' \text{ is the edge corresponding to } g) \end{cases}$$

We now show that α is an e-magic assignment of J (and that $\alpha(f)$ is well-defined in the sense that all factors in the product pairwise commute).

Notice that condition 1 of Definition 3.2.4 holds since all labels of α are products of labels of α', which is an e-magic assignment.

Condition 2 of Definition 3.2.4 holds for any $x \in V - \{u, v\}$ trivially since all labels are the same for α as they are for α' which is an e-magic assignment. Aside from f, each edge incident to u or v in J has a corresponding edge incident to w in J' and has the same edge label. Since all edges incident to w in J' have commuting labels, all of the edges incident to u or v in J apart from f must also have commuting labels. Further, we see that $mult_J(u, g) > 0$ implies that edge g is incident to u in J, and so its corresponding edge g' is incident to w in J'. Thus we see that $\alpha(f)$ is the well-defined product of labels of edges incident to w in J', and so must also commute with all edge labels incident to u or v in J.

Now we show that condition 3 of Definition 3.2.4 holds. By the same reasoning as in condition 2, we know that the vertex product is $\pm I$ for any vertex $x \in V - \{u, v\}$, and thus it remains to show that this holds for u and v. For u we have

$$
\begin{aligned}
\pi_J(u) &= \prod_{e\in E} \alpha(e)^{mult_J(u,e)} \\
&= \alpha(f) \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(u,e)} && (f \text{ is a non-loop edge of size two}) \\
&= \prod_{e\in E-\{f\}} \alpha'(e')^{mult_J(u,e)} \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(u,e)} \\
&= \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(u,e)} \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(u,e)} \\
&= I
\end{aligned}
$$

For v we have

$$
\begin{aligned}
\pi_J(v) &= \prod_{e\in E} \alpha(e)^{mult_J(v,e)} \\
&= \alpha(f) \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(v,e)} && (f \text{ is a non-loop edge of size two}) \\
&= \prod_{e\in E-\{f\}} \alpha'(e')^{mult_J(u,e)} \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(v,e)} \\
&= \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(u,e)} \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(v,e)} \\
&= \prod_{e\in E-\{f\}} \alpha(e)^{mult_J(u,e)+mult_J(v,e)} \\
&= \prod_{e'\in E'} \alpha'(e')^{mult_{J'}(w,e')} \\
&= \pi_{J'}(w) \in \{-I, I\}
\end{aligned}
$$

Therefore we see that condition 3 of Definition 3.2.4 holds.

Finally we show that condition 4 of Definition 3.2.4 holds. Notice that for every vertex in $x \in V - \{u, v\}$, we have $\pi_J(x) = \pi_{J'}(x)$, and we have $\pi_J(v) = \pi_{J'}(w)$, and finally $\pi_J(u) = I$. Therefore we see that the number of vertices with vertex product $-I$ is the same for α as for α', and so condition 4 holds since α' is an e-magic assignment.

Therefore we see that J is e-magic in this case. □

Proposition 3.3.16. *[35] Let J be an edge-even hypergraph. Let J' be obtained from J by identifying two edges of J. If J' is e-magic, then J is e-magic.*

Proof. Let $e_1, e_2 \in E(J)$ be the edges of J identified, and let $e' \in E(J')$ be the edge obtained from identifying e_1 and e_2. Note that if e_1 or e_2 is of size two, then J can be obtained by splitting e' in J'. In this case J is e-magic if J' is e-magic by Proposition 3.2.8 (the roles of the two hypergraphs are reversed in the statement). Note that the proof of Proposition

3.2.8 does not require that either of the edges (e_1 or e_2) is of size two, so in particular the proof that J is e-magic in this case is identical to that proof. $\square$

Proposition 3.3.17. *Let J be an edge-even hypergraph. Let J' be obtained from J by a mod 2 reduction. Then J is e-magic if and only if J' is e-magic.*

Proof. Assume that J' is e-magic, and let α' be an e-magic assignment of J'. Let $J = (V, E)$ and $J' = (V, E')$. Let $f \in E$ be the edge to which the mod 2 reduction is applied, and let $f' \in E'$ be the corresponding edge in J'. Define an e-magic assignment of J by labeling each edge of $e \in E - \{f\}$ with $\alpha'(e)$, and labeling f with $\alpha'(f')$.

The first condition of Definition 3.2.4 holds trivially since all edge labels of α are edge labels of α'.

We show that the second condition of Definition 3.2.4 holds. Let $w \in V$ be the vertex with $mult_{J'}(w, f') = mult_J(w, f) - 2$. Notice that for every vertex $v \in V - \{w\}$ commutativity holds trivially. Thus we must show that commutativity holds for w. We know that for any $e \in E \cap E'$, $mult_J(w, e) > 0$ if and only if $mult_{J'}(w, e) > 0$, and so apart from $\alpha(f)$ we know that all edge labels commute since they are labeled the same by α as by α'. For f we have two cases: either $mult_{J'}(w, f') > 0$ and so f' is incident to w so that $\alpha'(f') = \alpha(f)$ commutes with all other edge labels, or $mult_J(w, f) = deg_J(f) = 2$ in which case, f is the only edge incident to w in J, and so commutativity holds trivially.

Finally notice that for any vertex edge pair v, e we have $mult_J(v, e) \equiv mult_{J'}(v, e) \pmod 2$ so that $\pi_J(v) = \pi_{J'}(v)$ for every vertex $v \in V$. Thus we see that conditions 3 and 4 of Definition 3.2.4 hold.

The reverse direction follows the same line of reasoning except that commutativity in this case is straightforward. $\square$

Theorem 3.3.18 (Parity Minor Theorem)**.** *Let J be an edge-even hypergraph. Let J' be a parity minor of J, and assume that J' is e-magic. Then J is e-magic.*

Proof. By induction on the number of operations using Propositions 3.3.14, 3.3.15, 3.3.16 and 3.3.17. $\square$

We now give several of examples to demonstrate the utility of the Parity Minor Theorem. More examples will appear in Chapter 4.

Example 3.3.19. Consider the hypergraph, J, depicted in Figure 3.20 (the edge of size 4 is denoted by squares surrounding the vertices that it is incident to). Contracting edge e in J, results in hypergraph, J' depicted in Figure 3.21, and further contracting edge f in J', yields the hypergraph J'' shown in Figure 3.22. From J'', we can apply a mod 2 reduction on vertex v to obtain $K_{3,3}$ (shown in Figure 3.23). Thus by the Parity Minor Theorem, J is e-magic. We show the resulting magic labeling $\alpha : E \to \mathbb{C}^{4\times4}$ in Figure 3.24.

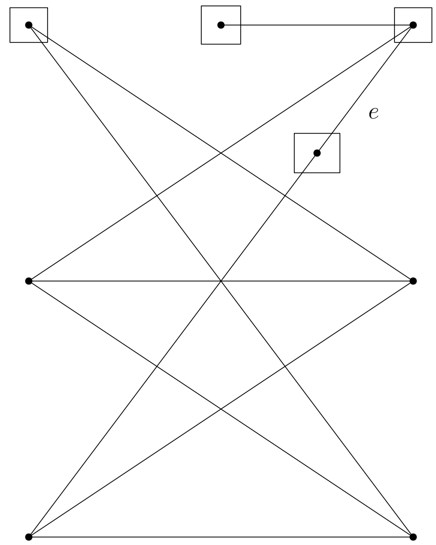

Figure 3.20: Hypergraph J with edge e

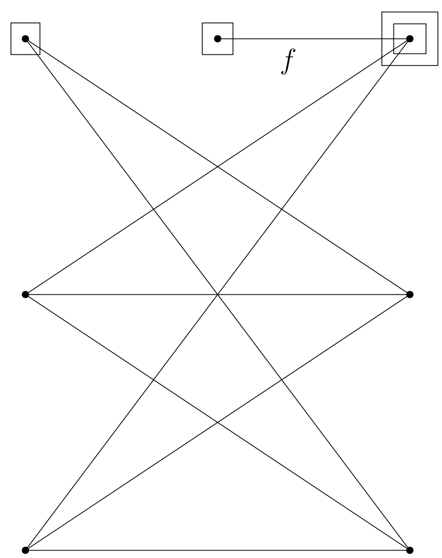

Figure 3.21: J': the result of contracting edge e in J

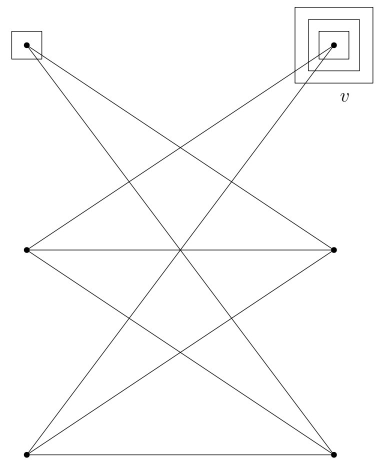

Figure 3.22: J'': the result of contracting edge f in J'

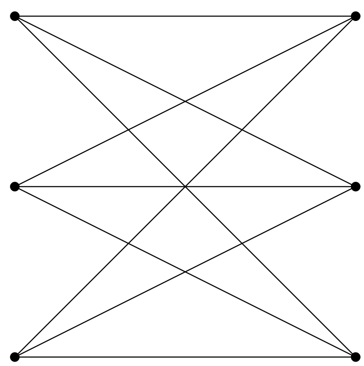

Figure 3.23: $K_{3,3}$: the result of applying a mod 2 reduction to v in J''

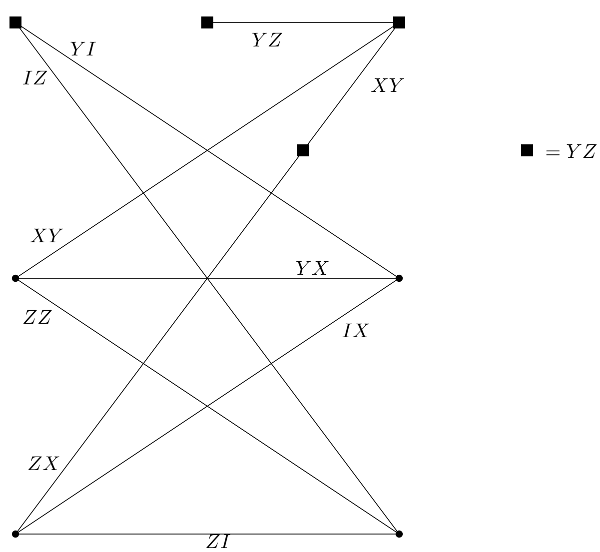

Figure 3.24: The resulting labeling of J

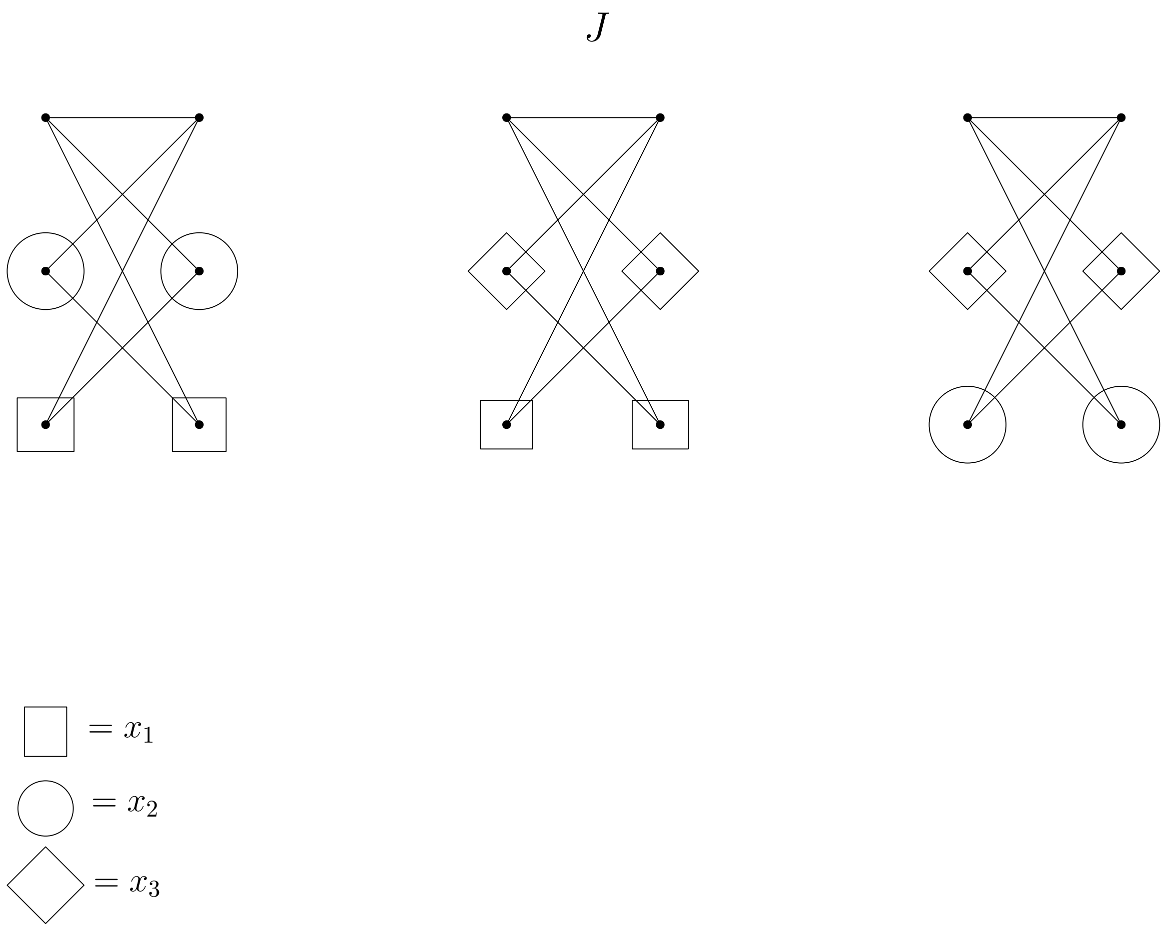

Figure 3.25: The dual of $\mathcal{C}_3$

Example 3.3.20. Recall $\mathcal{C}_3$ (shown in Figure 2.11). Its dual, J, is displayed in Figure 3.25. We know that J is e-magic since the Clifford BCS of rank 3 has a quantum satisfying assignment, but we do not know the minimum number of qubits needed for an e-magic assignment. We abstain from labeling any edges apart from x_1, x_2, x_3 in order to avoid clutter. Consider the two components of the deletion graph which have some vertex incident to x_3 in J (i.e. the middle and right pictures). If we delete edge x_3, and then contract all edges belonging to these components of the deletion graph, we obtain the hypergraph displayed in Figure 3.26. At this point we can apply mod 2 reductions to edges x_1 and x_2 (there is only one way of doing this), and then delete the arising vertices of degree zero to obtain $K_{3,3}$. Therefore we see that J has a 2 qubit e-magic assignment, and so the Clifford BCS of rank 3 has a 2 qubit magic assignment.

Proposition 3.3.21. *The proper Eulerian hypergraph $\mathcal{C}_N$ has a 2 qubit magic assignment for $N \geq 2$.*

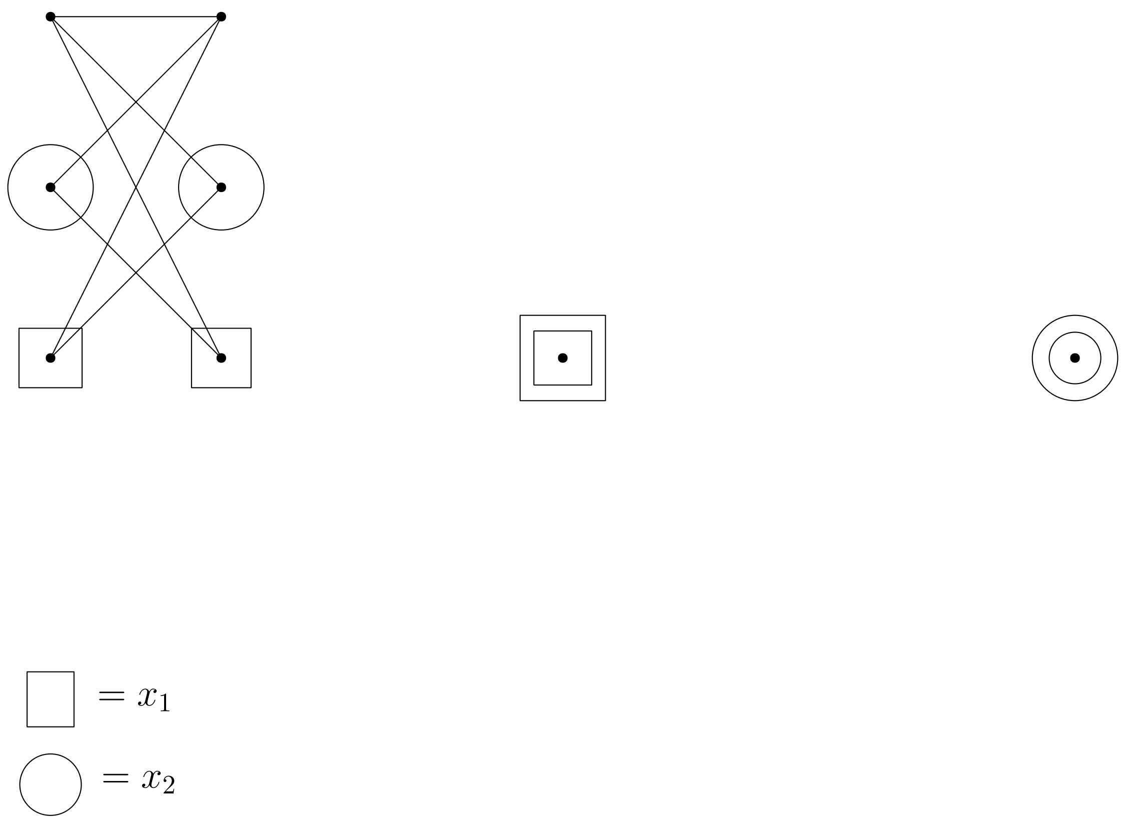

Figure 3.26: The result of deleting edge x_3 and contracting edges in the dual hypergraph of the proper Eulerian hypergraph corresponding to the Clifford BCS

Proof. We know that the result holds for $N = 2$ since the dual of the $\mathcal{C}_2$ is the Mermin square. For some other $N \geq 3$, let J be the dual hypergraph of the $\mathcal{C}_N$. Choose any two x_i edges, say x_1, x_2 without loss of generality. Delete every x_i for $i \geq 3$ to obtain a hypergraph J'. Since $N \neq 2$, there is a unique component G' of the deletion graph of J' which has four vertices incident to edges of size 4 (these edges being x_1 and x_2). Contract all edges (of size 2) in J not in $E(G')$. Then apply all possible mod 2 reductions to x_1 and x_2 to obtain a graph J''. Then J'' will be a $K_{3,3}$ along with $\binom{N}{2} - 1$ vertices of degree 0. Deleting all vertices of degree 0 yields $K_{3,3}$ which is e-magic. Therefore we see that $K_{3,3}$ is a parity minor of the dual of $\mathcal{C}_N$ for $N \geq 2$. By Theorem 3.3.18 we see that the dual hypergraph is e-magic, and thus $\mathcal{C}_N$ is also magic. In addition, we know by our constructive labeling system that a 2-qubit assignment exists. □

Therefore we see that even though the required amount of qubits grows as $\lfloor \frac{N}{2} \rfloor$ for quantum satisfying assignments in the sequence of Clifford binary constraint systems, 2 qubits is the most we need for magic assignments of their corresponding hypergraphs.

At this point we should take a brief moment to relate this result to previously known theory. The Parity Minor Theorem is a generalization of one direction of Theorem 2.3.5, and the concept of a parity minor is a generalization of the concept of graph minors to edge-even hypergraphs. We formalize this now.

Corollary 3.3.22. *Let G and G' be graphs so that G' is a minor of G. If G' is e-magic, then G is e-magic.*

Proof. Since G' is a minor of G, it can be obtained from G by edge deletions, vertex deletions, and contractions of edges of size 2. The result follows immediately. □

In Theorem 2.3.5, this result was proven using topological minors - and this is analogous to Theorem 1.3.22 (Kuratowski's Theorem). In contrast, our result is proven using the minor relation - which is analogous to Theorem 1.3.24 (Wagner's Theorem). This may seem inconsequential since the two results are equivalent, but since both of these proofs are constructive in the sense that they return a labeling, we note that the labelings given by the Parity Minor Theorem may be different than those generated by Theorem 2.3.5. Conversely, any labeling generated by that theorem can be replicated by ours. Note that, we have only proven the constructive direction of Arkhipov's result, and so our result says nothing about planar graphs - it only states that if we can reduce a graph G to $K_{3,3}$ or K_5 which we know to be e-magic by Mermin and Peres, we can construct e-magic assignments for G.

The parity minor relation also captures the edge splitting operation.

Corollary 3.3.23. *Let J be an edge-even hypergraph, and let J' be edge split-obtainable from J. Then J is a parity minor of J'.*

Proof. We show that this holds for a single edge split. The result follows by induction. If J' is obtained from J by splitting a single edge e in J to obtain edges e'_1 and e'_2 in J', then simply identify edges e'_1 and e'_2 in J' to obtain J. Therefore J is a parity minor of J'. □

We now explain some of the more technical elements of our defined parity minor operations.

The reader may be curious to know why we restrict edge contractions solely to edges of size 2 and not to all edges. Below we give a counter-example to illustrate why this would not work.

Example 3.3.24. The hypergraph on the left of Figure 3.27 is not e-magic as two of the graphs resulting from splitting are planar. However, contracting the edge of size four yields $K_{3,3}$ (on the right) which we know to be e-magic. Hence contracting edges of size greater than two in this simple manner cannot be allowed as a hypergraph minor operation for our purposes.

In a similar vein, the mod 2 reduction conditions become rather technical when reducing the multiplicity from two to zero, since we insist that $mult(v, e) = deg(v)$, instead of simply

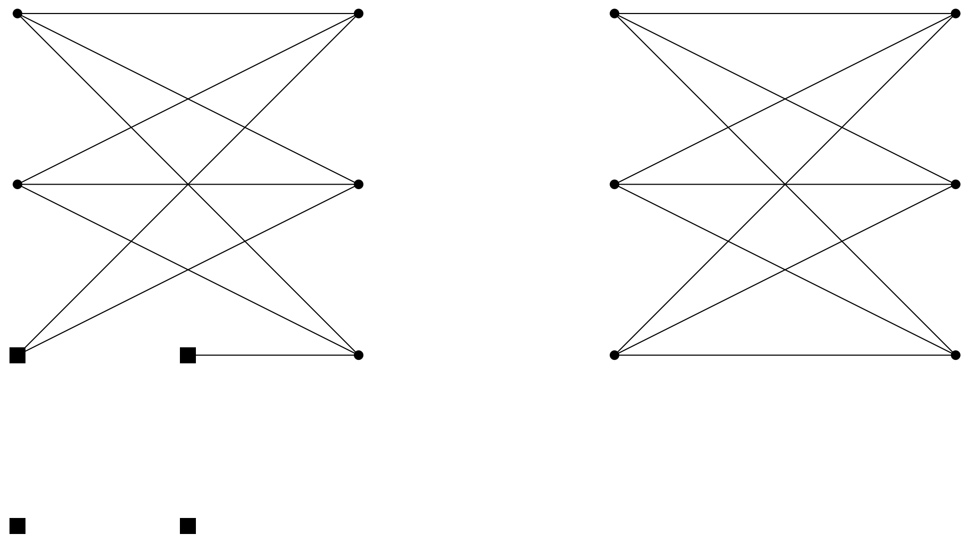

Figure 3.27: A not e-magic hypergraph H (left), an e-magic hypergraph obtained by contracting an edge of size 4 of H (right)

allowing any mod 2 reduction (after all, the vertex product will remain unchanged). We now give an example which illustrates the importance of this condition.

Example 3.3.25. The hypergraph J illustrated in Figure 3.28 is not e-magic as can be seen by the planar graph resulting from splitting the edge of size four into edges v_0v_6 and v_1v_5. However, contracting edge v_5v_6 and then applying a mod 2 reduction to the vertex resulting from the edge contraction (if this was possible) results in $K_{3,3}$ which is e-magic. Thus, without the vertex degree restriction in the mod 2 reduction, e-magic hypergraphs would be parity minors of not e-magic hypergraphs.

This example illustrates the necessity to define hypergraphs (and also the parity minor relation) as we have done, as opposed to defining only proper hypergraphs. This is analogous to the necessity of defining the complex numbers in the setting of finding roots of polynomials with real-valued coefficients. Though hypergraphs with multiplicities greater than one do not appear in the statement of Problem 2.1.8, they arise inevitably in this framework and can be used to derive valuable information about the proper hypergraphs which we are interested in. For this reason we now state a generalization of Problem 2.1.8.

Problem 3.3.26. *Given an edge-even hypergraph J, find an e-magic assignment of J or show that none exists.*

We now return to the two hypergraph minor relations defined at the beginning of this section - one of which is defined by Slofstra, and the other by Robertson and Seymour.

Slofstra's hypergraph minor relation is presented as a morphism between hypergraphs in the following way:

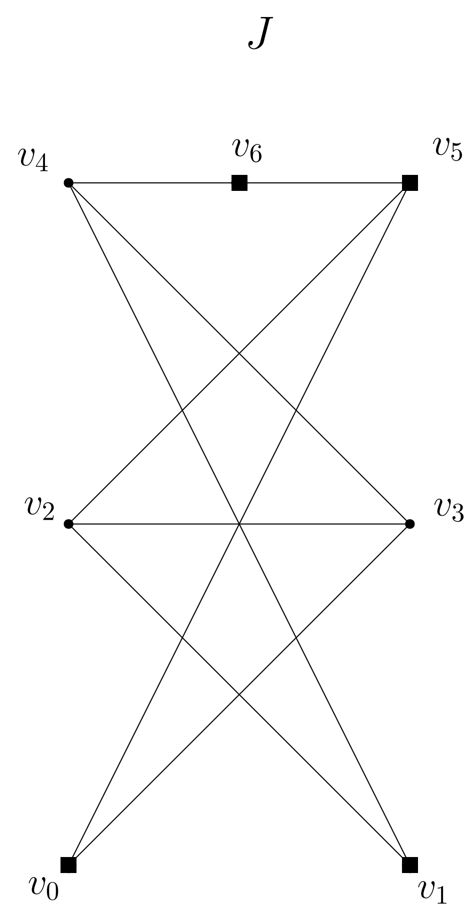

Figure 3.28: A not e-magic hypergraph J that could be reduced to $K_{3,3}$ without a degree restriction in the mod 2 reduction operation.

Definition 3.3.27. *[35, Definition 8.4] Let J and J' be hypergraphs. A* generalized morphism *$\phi : J \to J'$ consists of a pair of morphisms*

$$\phi_V : V(J) \to V(J') \cup \{\epsilon\} \text{ and } \phi_E : E(J) \to E(J') \cup \{\epsilon\},$$

such that, for all $v \in V(J)$,

1. *if $\phi_V(v) \neq \epsilon$, then*

$$\sum_{e \in \phi_E^{-1}(e')} mult_J(v, e) = mult_{J'}(\phi(v), e'),$$

   for all $e' \in E(J')$, and

2. *if $\phi_V(v) = \epsilon$, then*

$$\sum_{e \in E(J) - \phi_E^{-1}(\epsilon)} mult_J(v, e)$$

   is even, and $\phi_E(e_1) = \phi_E(e_2)$ for all edges $e_1, e_2 \in E(J) - \phi_E^{-1}(\epsilon)$ incident to v.

The effect of mapping a vertex or edge to ϵ is to delete the vertex or edge respectively. We quote from [35, Example 8.6] that the morphism allows the following operations:

1. deleting edges,
2. identifying edges,
3. deleting isolated vertices,
4. collapsing vertices incident to an even number of edges (deleting a vertex and identifying all incident edges).

Clearly, we see that there are similarities between our hypergraph minor relation and the morphism defined in Definition 3.3.27. Both utilize vertex deletion, edge deletion, and edge identification. In addition, all mod 2 reductions of even degree vertices can be done using this morphism. Our inclusion of the edge identification operation was motivated by [35], and thus Proposition 3.3.16 is attributed to Slofstra accordingly.

As suggested by the list of operations above, and also discussed in [35, Section 1], the morphism defined in Definition 3.3.27 can be viewed as a hypergraph minor relation. A notable difference exists between our Parity Minor relation and this hypergraph minor relation since they were developed for different purposes. In particular, not all edge contractions can be achieved using the morphism operations associated to Definition 3.3.27. For an edge incident to a vertex v of degree two, such contraction can be done by identifying the two edges incident to v and then collapsing v, however this cannot be done for edges incident to vertices of higher degrees.

We now recall from the introduction of this chapter, that the hypergraph minor operations defined by Robertson and Seymour form a well-quasi ordering on the class of hypergraphs where the multiplicity between any vertex, edge pair is at most one. Therefore for any class closed under their hypergraph minor operation, there is a finite list of forbidden minors. It is natural to ask if this is true of the parity minor operation and the class of not e-magic hypergraphs - in other words, is there a finite list of forbidden parity minors? Of course, we should now note that, if Problem 2.1.8 is undecidable, then there is no hope of this being the case.

The following infinite sequence of hypergraphs illustrates that the Parity Minor relation does not form a well-quasi ordering on the class of edge-even hypergraphs, thus this would not be a valid method for proving whether or not the class of not e-magic hypergraphs has a finite list of forbidden minors.

Example 3.3.28. Consider the sequence of hypergraphs $J_1, J_2, \ldots$ where J_i is a hypergraph on $2i$ vertices with a single edge of size $2i$ containing all of the vertices. Then for any $m, n \geq 1, m < n$, J_m is a hypergraph minor of J_n under the definition of Robertson and Seymour. Notice, however that J_m cannot obtained from J_n by any of the parity minor operations.

Even in the case that there are an infinite number of forbidden parity minors, finding some of them is still useful in order to characterize certain common properties of e-magic hypergraphs and to make progress on Problem 2.1.8. We refer to such forbidden parity minors as minimally e-magic hypergraphs, and define these formally now.

Definition 3.3.29. *We say that an edge-even hypergraph J is* minimally e-magic *if J is e-magic, and any parity minor of J, different from J, is not e-magic. We denote the set of minimally e-magic hypergraphs by MEM.*

Proposition 3.3.30. *$K_{3,3}$ and K_5 are in MEM.*

Proof. Both $K_{3,3}$ and K_5 are simple non-planar graphs (and therefore also proper hypergraphs) for which any vertex deletion, edge contraction or edge deletion yields a planar graph. Additionally, no mod 2 reductions are possible and identifying any pair of edges yields a hypergraph with a planar split. Hence, we conclude that both $K_{3,3}$ and K_5 are in MEM. □

Our current objective is to find other elements in MEM. We state this problem formally.

Problem 3.3.31. *Find more elements of MEM.*

To that end, we now state some properties of elements that are in MEM.

3.3.1 Some Properties of MEM Hypergraphs

Definition 3.3.32. *We say that a hypergraph $J = (V, E, \psi)$ is* simple *if for every pair of edges $e_1, e_2 \in E$ such that $\psi(e_1) = \psi(e_2)$, we have $e_1 = e_2$.*

Proposition 3.3.33. *Let $J = (V, E, \psi)$ be an edge-even hypergraph. If there exist edges $e_1, e_2 \in E$ such that $\psi(e_1) = \psi(e_2)$, then J is e-magic if and only if $J - e_2$ is e-magic.*

Proof. Assume J is e-magic. Let α be an e-magic assignment of J, and let $J' = (V, E') := J - e_2$. Then define $\alpha' : E' \to GL(\mathcal{H})$ so that

$$\alpha'(e) = \begin{cases} \alpha(e_1)\alpha(e_2) & \text{if } e = e_1 \\ \alpha(e) & \text{if } e \in E' - e_1 \end{cases}$$

Then α' is an e-magic assignment of J'.

Similarly, we can extend any e-magic assignment from J' to J simply by assigning I to edge e_2. □

Corollary 3.3.34. *Let $J \in MEM$. Then J is simple and contains no loops.*

Proof. The fact that J is simple follows directly from Proposition 3.3.33.

We show that J also contains no loops. Let $J = (V, E)$, and let α be an e-magic assignment for J. Let $e \in E$ be a loop, so that $mult(v, e) = 2$ for some vertex $v \in V$ and $mult(w, e) = 0$ for every other vertex $w \in V$.

Then $J - e$ is e-magic since e contributes $\alpha(e)^2 = I$ to the vertex product of v (the only vertex e is incident to), so that α' defined as α restricted to $J - e$ is an e-magic assignment of $J - e$. □

Proposition 3.3.35. *Let $J \in MEM$ such that J is not isomorphic to $K_{3,3}$ or K_5. Then the deletion graph of J is planar.*

Proof. Assume towards a contradiction that the deletion graph G of J is non-planar. Then G is e-magic by Theorem 2.3.5. But G can be obtained from J by deleting all edges in $E(J)$ of size not equal to two. Note that J has some edge of size not equal to two, since J is minimally e-magic and is not isomorphic to $K_{3,3}$ or K_5 and thus must not be a graph. Since G is a parity minor of J and is not isomorphic to J, we see that J cannot be minimally e-magic. This is a contradiction. Therefore the deletion graph of J is planar. □

Proposition 3.3.36. *Let $J = (V, E) \in MEM$. Then $mult(v, e) \leq 2$ for every vertex $v \in V$ and edge $e \in E$.*

Proof. Assume towards a contradiction that $mult(v, e) \geq 3$ for some vertex $v \in V$. Apply a mod 2 reduction to v and e. Then the resulting hypergraph is e-magic since J is e-magic. Therefore $J \notin MEM$. The result follows. □

Proposition 3.3.37. *Let $J = (V, E)$ be an e-magic hypergraph. If there exists some e-magic assignment $\alpha : E \to GL(\mathcal{H})$ such that $\alpha(e) = I$ for some edge $e \in E$, then $J \notin MEM$.*

Proof. We know by Proposition 3.2.15 that $J - e$ is also e-magic. But $J - e$ is a parity minor of J since it is obtained from J by an edge deletion. Therefore J is not minimally e-magic. $\square$

Proposition 3.3.38. *Let $J = (V, E) \in MEM$. Then $deg(v) \geq 3$ for any vertex $v \in V$.*

Proof. Let $\alpha : E \to GL(\mathcal{H})$ be an e-magic assignment of J. Assume towards a contradiction that there exists some vertex $v \in V$ such that $deg(v) \leq 2$.

If $deg(v) = 0$, then $\pi_\alpha(v) = I$, and so $J - v$ is e-magic, contradicting the minimality of J.

If $deg(v) = 1$, then v has odd degree, and $deg(v) = mult(v, e)$ for some edge $e \in E$. Therefore by Corollary 3.2.17 we see that $J - v$ is e-magic, contradicting the minimality of J.

If $deg(v) = 2$, then there are two cases to consider.

Case 1. v is incident to a single edge.

In this case, there is some edge $e \in E$ such that $mult(v, e) = 2$. Therefore it must be the case that $mult(v, e) = 2$ and $deg(v) = 2$. But then consider the hypergraph J' obtained by reducing e via a mod 2 reduction (note that we can do this since $mult(v, e) = deg(v)$ and $mult(v, f)$ is even). Then J' is e-magic by Proposition 3.3.17, so $J \notin MEM$ since J' is an e-magic parity minor of J.

Case 2. v is incident to two different edges.

Let the edges incident to v be e_1 and e_2. Since $\pi_J(v) = \pm I$, we know that $\alpha(e_1) = \pm\alpha(e_2)$. Furthermore, we can assume without loss of generality that $\alpha(e_1) = \alpha(e_2)$ by Proposition 3.2.14. But then the hypergraph $J' = (V', E')$ obtained by identifying edges e_1 and e_2 to obtain a new edge $e' \in E'$ is also e-magic, since we can define a labeling $\alpha' : E' \to GL(\mathcal{H})$ so that $\alpha'(e') = \alpha(e_1) = \alpha(e_2)$ and $\alpha'(e) = \alpha(e)$ for $e \in E(J') - e'$, which can easily be checked to be an e-magic labeling of J'. Thus J is not in MEM.

Therefore $deg(v) \geq 3$ for every vertex $v \in V$ as required.

$\square$

Now that we have listed some of the properties of MEM elements, we restrict ourselves to searcing for MEM's in a class of edge-even hypergraphs which we call grafts. In this setting we are able to derive additional properties of such elements, as well as to design a set of tools to search for them.

Chapter 4

Grafts

Having presented our main results (Theorem 3.2.9 and Theorem 3.3.18) we now take a systematic approach to Problem 2.1.8. Since the problem is solved for the case when the dual is a graph, we consider the case when the dual is almost a graph. In the literature (see for example [20]) a *graft* is defined as a pair (G, f) where G is a graph and f is a set of vertices of G such that $|f|$ is even. In this chapter, we use the term graft in a more restricted manner, as given in Definition 4.0.1. The main purpose of this chapter is to describe and find other elements of MEM within this class of grafts.

Definition 4.0.1. *Let $J = (V, E)$ be an edge-even hypergraph such that there is some edge $f \in E$ of size* 4*, and all other edges in E are of size 2. Then we say that J is a* graft*.*

In this chapter, unless explicitly stated otherwise, f shall always indicate the edge of size 4. Furthermore, f shall always be denoted pictorially by large square vertices. Lastly, we will denote the set of vertices incident to f by $V(f)$.

Example 4.0.2. Recall the edge-even hypergraph pictured in Figure 4.1. This edge-even hypergraph is a graft. An e-magic labeling for this graft is shown in Figure 3.24.

Definition 4.0.3. *Let the set of minimally e-magic grafts be denoted by $\mathcal{M}_4$.*

In Section 4.1 we discuss what an element of $\mathcal{M}_4$ would look like, and in Section 4.2 we develop a set of tools to find such elements.

4.1 Properties of Minimally E-magic Grafts

Given the major results of Chapter 3, we can greatly narrow down the search space for what a minimally e-magic graft could look like. The purpose of this section is to further narrow the search space by studying the properties of such an element.

We recall that the deletion graph of a hypergraph is the graph obtained by deleting all edges of size not equal to two from the hypergraph.

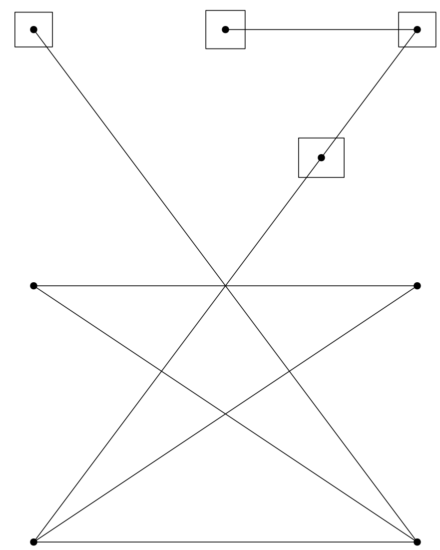

Figure 4.1: A graft

Proposition 4.1.1. *Let $J \in \mathcal{M}_4$. Let g and h be edges obtained by splitting f, and let G be the graph obtained. Let K be a subdivision of $K_{3,3}$ or K_5 appearing in G. Then $g \in E(K)$ or $h \in E(K)$.*

Proof. Assume towards a contradiction that there is some subdivision K appearing in G such that $g \notin E(K)$ and $h \notin E(K)$. By Proposition 3.3.35 we know that the deletion graph of J, G', is planar. Notice that G' is isomorphic to $(G - g) - h$. But since $g \notin E(K)$ and $h \notin E(K)$, we see that K is a subgraph of G'. This is a contradiction since G' is planar.

The result follows. □

It is not trivial to construct a graft which satisfies the conditions of Proposition 3.3.35 and Proposition 4.1.1 - in particular since for the latter, this condition must hold for all three splits of f. The problem of finding such grafts will be dealt with in Section 4.2.

Proposition 4.1.2. *Let $J = (V, E) \in \mathcal{M}_4$. Then there is at most one vertex $v \in V$ such that $mult(v, f) = 2$.*

Proof. Assume towards a contradiction that two such vertices exist in V, say u and v. Then consider splitting f into edges e_1, e_2 such that $mult(u, e_1) = 2$ and $mult(v, e_2) = 2$ to obtain some graph G. Then both e_1 and e_2 are loops. We know that G is non-planar by Theorem 3.2.9. In addition we know that $(G - e_1) - e_2$ is also non-planar since e_1 and e_2 are loops. However notice that $(G - e_1) - e_2$ is the deletion graph of J, and so by Proposition 3.3.35 must be planar. This is a contradiction.

The result follows. □

Proposition 4.1.3. *Let $G = (V, E)$ be a non-planar graph, and let $e \in E$ be a bridge. If at least one of the components of $G - e$ containing a vertex incident to e is planar, then $\alpha(e) = \pm I$ for any e-magic assignment α of G.*

Proof. Let $G' = (V', E')$ be the component of $G - e$ that is planar, and let u be the sole vertex in V' which is incident to e in G. Let α be an e-magic assignment of G, and let α' be the labeling obtained by restricting α to G'. Since G' is planar, we can consider some embedding of G' defining a cyclic ordering around each vertex of V'. Then for any vertex $v \in V'$ incident to edges $e_1, \ldots, e_k$ (where $e_1, \ldots, e_k$ are in clockwise cyclic order according to the chosen embedding), define $\pi^*_{G'}(v) = \alpha'(e_1)\alpha'(e_2)\ldots\alpha'(e_k)$. This product is well defined for G' since α is an e-magic assignment. Moreover, $\pi^*_{G'}(v) = \pm I$ for every $v \in V' - \{u\}$ (again since α is an e-magic assignment).

By the same argument as in the proof of Theorem 2.3.5 we see that if we contract any non-loop edge $e = v_1v_2$ in E' which is not incident to u to obtain a vertex v and graph $G'' = (V'', E'')$, then $\pi_{G''}(v)$ will be well defined, and $\pi_{G''}(v) = \pi_{G'}(v_1)\pi_{G'}(v_2) \in \{I, -I\}$. Thus we can successively contract all of the edges of G' which are not incident to u to obtain a graph G^+ such that every edge in $E(G^+)$ is incident to u or is a loop. As noted in Theorem 2.3.5, every planar embedding has an innermost loop, and this loop contributes I to the vertex product of the vertex it is incident to - and therefore loops have no effect on vertex products. Additionally notice that for any vertex $v \in V(G^+)$, $v \neq u$, the set of edges incident to both u and v will appear successively in the cyclic ordering of the edges around u.

Then, we see that $\pi_{G^+}(u) = \prod_{v \in V(G^+) - \{u\}} \pi_{G^+}(v)$. But notice that $\pi_{G^+}(v) \in \{I, -I\}$ for every $v \in V(G^+) - \{u\}$. Therefore $\pi_{G^+}(u) \in \{I, -I\}$. But since none of the edges incident to u have been altered, we see that $\pi_{G^+}(u) = \pi_{G'}(u)$. Finally notice that $\pi_G(u) = \alpha(e)\pi_{G'}(u)$ and since $\pi_G(u), \pi_{G'}(u) \in \{I, -I\}$, we have $\alpha(e) \in \{I, -I\}$ as required.

□

Proposition 4.1.4. *Let $J \in \mathcal{M}_4$. Let G be obtained from some split of J. Then G contains no bridges.*

Proof. Let e_1 and e_2 be the edges obtained by splitting f. Assume towards a contradiction that G contains some bridge g. Then we have two possible cases.

Case 1 At least one of the two components of $G - g$ containing a vertex incident to g is planar. Note that any e-magic assignment α of J has some corresponding e-magic assignment α' of G such that $\alpha'(e) = \alpha(e)$ for every $e \in E(G) \cap E(J)$, and $\alpha'(e_1) = \alpha'(e_2) = \alpha(f)$. Further, we know that $\alpha'(g) = \pm I$ by Proposition 4.1.3 (G is non-planar since G is obtained from some split of J, which is an e-magic graft), and so $\alpha(e) = \pm I$ for some edge $e \in E(J)$. Therefore by Proposition 3.3.37, we know that $J \notin \mathcal{M}_4$.

Case 2 Both of the two components of $G - g$ containing a vertex incident to g are non-planar.

Let G_1 and G_2 be the two components of $G - g$ containing a vertex incident to g. Then it must be the case that $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$ (or equivalently $e_1 \in E(G_2)$ and $e_2 \in E(G_1)$), since otherwise $J - f = (G - e_1) - e_2$ is non-planar which is a contradiction by Proposition 3.3.35. We can also conclude that $G_1 - e_1$ and $G_2 - e_2$ are both planar by the same token, and so e_1 must be in some subdivision of $K_{3,3}$ or K_5, $K^{(1)}$ which is a subgraph of G_1. Similarly, e_2 must be in some subdivision $K^{(2)}$ which is a subgraph of G_2. In particular, this means that e_2 must be in a cycle C in G_2. Let P be the path $C - e_2$. Then consider the graph G' obtained by contracting each edge of P in J, deleting each edge incident to the vertex obtained by the identifications from the contraction process, and then performing a mod 2 reduction to f on that vertex. The graph G' is non-planar since G_1 is a subgraph of G', and thus e-magic. Additionally, G' is a parity minor of J. Therefore $J \notin \mathcal{M}_4$.

This concludes the proof.

□

Theorem 4.1.5. *Let $J \in \mathcal{M}_4$. Let G be obtained from some split of J. Then G is connected.*

Proof. Let e_1 and e_2 be the edges obtained from the split. We prove that G is connected by contradiction.

Assume towards a contradiction that G is disconnected. Then we can conclude that one of its components is non-planar. Further, either both edges from the split are in this component, or one edge is in one component and the other is in some other component.

Case 1 If e_1 and e_2 are in the same component, G_1, then G_1 must be non-planar since every $K_{3,3}$ or K_5 subdivision of G must contain either e_1 or e_2 (by Proposition 4.1.1). Additionally, any other component of G, G' must be planar since otherwise G' would be an e-magic parity minor of J. But then, $J[V(G_1)]$ is an e-magic parity minor of J since for any e-magic assignment α of J, for any component G' of G other than G_1, there must be an even number of vertices in $V(G')$ with vertex product $-I$ (since G' is a planar component of G). Therefore α restricted to $J[V(G_1)]$ must be an e-magic assignment of $J[V(G_1)]$. Thus we see that $J[V(G_1)]$ is an e-magic parity minor of J, since $J[V(G_1)]$ can be obtained by deleting all vertices of J not in $V(G_1)$. Therefore $J \notin \mathcal{M}_4$.

Case 2 Now let us assume that e_1 is in one component, say G_1 and e_2 is in another component, say G_2. Assume that G_1 is non-planar without loss of generality. Note that e_2 must be in a cycle C in G_2, since e_2 is not a bridge by Proposition 4.1.4. Then, as in the previous proof, let $P = C - e_2$, and in J contract every edge of P, then delete every edge incident to the vertex obtained from the identification from the contraction process and finally apply a mod 2 reduction to f on that vertex. Let G^* be the graph obtained by these operations. Note that G^* is non-planar, since G_1 is a subgraph of G^*. Also, G^* is a parity minor of J. Therefore $J \notin \mathcal{M}_4$.

The result follows. □

Corollary 4.1.6. *Let $J \in \mathcal{M}_4$. Then the deletion graph of J is connected.*

Proof. Let G be the deletion graph of J, and assume towards a contradiction that G is not connected. Let G_1 and G_2 be two components of G. Let e_1 and e_2 be obtained by one of the splits of f, and let G' be the graph obtained. Then both e_1 and e_2 must have one end in $V(G_1)$ and one end in $V(G_2)$, otherwise G' is either not connected or contains a bridge (contradicting Theorem 4.1.5 or Proposition 4.1.4 respectively). Let u_1, u_2 be the vertices of G_1 such that u_1 is incident to e_1 and u_2 is incident to e_2, and let v_1, v_2 be the vertices of G_2 such that v_1 is incident to e_1 and v_2 is incident to e_2 (so that $V(f) = \{u_1, u_2, v_1, v_2\}$). Then note that the graph obtained by splitting f in J to obtain edges $e'_1 = u_1u_2$ and $e'_2 = v_1v_2$ is not connected and thus contradicts Theorem 4.1.5.

Therefore we see that the deletion graph of J must be connected as required. □

Proposition 4.1.7. *Let $J = (V, E) \in \mathcal{M}_4$. Let G be the deletion graph of J. Then either $G[V(f)]$ has at most 2 edges or $G[V(f)]$ is comprised of a cycle of size three and a vertex of degree zero.*

Proof. Assume that $G[V(f)]$ has at least three edges.

First note that $|V(f)| \geq 3$ since $mult(v, f) = 2$ for at most one vertex of V by Proposition 4.1.2, and $mult(w, f) \leq 2$ for every vertex $w \in V(f)$ by Proposition 3.3.36. In the case that $|V(f)| = 3$, if $G[V(f)]$ has 3 edges, then $G[V(f)]$ is a complete graph on 3 vertices (since J is simple by Corollary 3.3.34). But then notice that any split of f introduces no edges which are not either loops or parallel edges. Therefore, the graph obtained by any split of f is non-planar if and only if G is non-planar. But we know that G is planar since G is the deletion graph of J (by Proposition 3.3.35).

Thus we see that $|V(f)| = 4$. Let $V(f) = \{v_1, v_2, v_3, v_4\}$. Notice that up to relabeling $G[V(f)]$ must contain either edges v_1v_2, v_2v_3, v_3v_4, or edges v_1v_2, v_1v_3, v_1v_4, or edges v_1v_2, v_2v_3, v_1v_3 (corresponding to the three non-isomorphic graphs on four vertices with three edges).

Case 1 $G[V(f)]$ contains edges v_1v_2, v_2v_3, v_3v_4.

In this case, consider the split of f into edges v_1v_2 and v_3v_4. Both of these are parallel edges since edges of G already exist between these vertex pairs. Therefore the graph obtained by this split of f is non-planar if and only if G is non-planar. Since G is planar, we see that J is not e-magic in this case.

Case 2 $G[V(f)]$ contains edges $g_1 = v_1v_2, g_2 = v_1v_3, g_3 = v_1v_4$. In this case consider the split of f yielding edges $e_1 = v_1v_2$ and $e_2 = v_3v_4$, and let G' be the graph obtained from the split. For each e-magic assignment α^* of J, there is a corresponding e-magic assignment α of G' such that $\alpha(e_1) = \alpha(e_2) = \alpha^*(f)$. Note that we can create another e-magic assignment α' as follows: $\alpha'(e) = \alpha(e_2)\alpha(e)$ if $e \in \{e_2, g_2, g_3\}$, and $\alpha'(e) = \alpha(e)$ otherwise. Thus in particular we see that e_2 is labeled by I by this assignment. Notice however that e_1 is a parallel edge in G' since g_1 is incident to the same two vertices. Therefore we know that

e_2 must be in every subdivision of $K_{3,3}$ or K_5 in G'. But since $\alpha'(e_2) = I$, this means that $G' - e_2$ is e-magic which is a contradiction since $G' - e_2$ is planar.

Case 3 $G[V(f)]$ contains edges v_1v_2, v_2v_3, v_1v_3. In this case, if $G[V(f)]$ has any other edges, $G[V(f)]$ will contain a subgraph such that Case 2 will apply (up to relabeling). Therefore $G[V(f)]$ contains exactly the three edges v_1v_2, v_2v_3, v_1v_3 - and therefore is comprised of a cycle of size three and an isolated vertex.

The result follows. □

4.2 Finding E-magic Grafts

An initial method to find grafts which are minimally e-magic would be to start with edge f, and make sure that every split is non-planar by 'gluing' a subdivision of $K_{3,3}$ or K_5 to two of the vertices incident to edge f and removing the edge of the subdivision incident to the two vertices of f. In this fashion, we are guaranteed that at least one split of f is non-planar. We can then attempt to add extra edges so that all splits are non-planar. Such an approach was taken in Example 3.2.18. However, it can be quite difficult to apply this process while also ensuring that the deletion graph is planar. Furthermore, it would be challenging to classify the e-magic grafts in this fashion.

Instead of beginning with the size four edge, f, we shall work backwards.

Note that a minimally e-magic graft appearing as a dual of a proper Eulerian hypergraph will also be proper since it is simple, the multiplicities conditions are satisfied, each edge is of size at least one, and each vertex is of degree at least one. In this section we explore searching techniques for finding e-magic grafts of this sort that could potentially be in $\mathcal{M}_4$.

Example 4.2.1. Consider the graft J illustrated in Figure 4.2, with deletion graph T. Since T is a maximal planar graph, T is planar and adding any new (non-parallel) edge to T will yield a non-planar graph. Since T induced on $V(f)$ contains an independent set of size 3 (vertices 1,3 and 5), any split of f must introduce a new non-parallel edge to T, and thus every split will be non-planar.

Notice that for any proper graft with a maximal planar deletion graph T, if instead of an independent set of size 3, T induced on $V(f)$ contained an isolated vertex (a vertex of degree one), every split would still be non-planar (for example choosing f to be incident to vertices 7,3,4, and 5 instead). We record this result in a proposition.

Proposition 4.2.2. *Let J be a proper graft, such that the deletion graph of J is a maximal planar graph T. Then every split of f is non-planar if and only if T induced on $V(f)$ contains either an independent set of size 3 or a vertex of degree 0.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices in J incident to f.

Every split of f in T is non-planar if and only if each split of f introduces a new edge to T that is not a parallel edge. The possible edge pairs obtained from the split of f are:

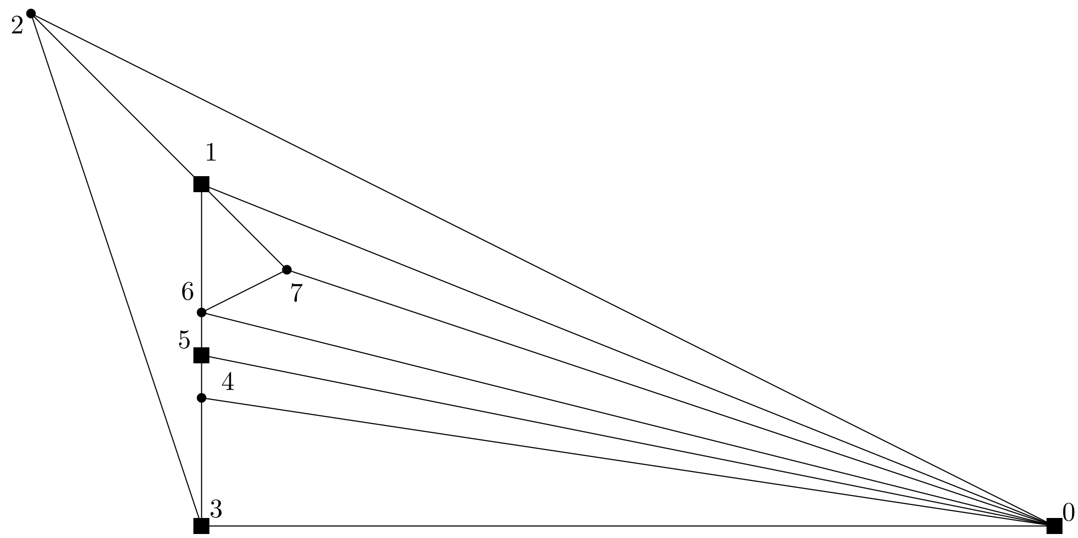

Figure 4.2: A graft J whose deletion graph is a maximal planar graph

1. v_1v_2 and v_3v_4, 2. v_1v_3 and v_2v_4, and 3. v_1v_4 and v_2v_3. Thus from each of these three splits, at least one of the two edges must not already be in T. Let us assume without loss of generality that in the first split $v_1v_2 \notin E(T)$. Notice that if $v_1v_3 \notin E(T)$ or $v_2v_4 \notin E(T)$, this is the same up to relabeling in f. Thus in case 2, assume without loss of generality that $v_1v_3 \notin T$. Lastly, either $v_1v_4 \notin T$ and so v_1 is an isolated vertex in f or $v_2v_3 \notin T$ and so $\{v_1, v_2, v_3\}$ is an independent set in T. Thus if every split of f in T is nonplanar, then T induced on $V(f)$ contains an independent set of size 3 or an isolated vertex.

Now assume T induced on $V(f)$ contains either an independent set of size three or an isolated vertex.

In the first case, let v_1, v_2, v_3 be the independent set without loss of generality. Then, notice that $v_1v_2 \notin E(T)$, $v_2v_3 \notin E(T)$, $v_1v_3 \notin E(T)$. Notice that each possible split of f must contain one of these three edges. Therefore, each split introduces a new non-parallel edge, and is thus non-planar.

In the second case, let v_1 be the vertex of degree 0. Then $v_1v_2 \notin E(T)$, $v_1v_3 \notin E(T)$, $v_1v_4 \notin E(T)$. Again, each split of f must contain one of these three edges. Therefore, each split introduces a new non-parallel edges in this case as well, and is non-planar. $\square$

Note that for the forward direction, we did not need that T is a maximal planar graph, just that every split of f is non-planar. In other words, for any proper graft J such that every split of f is non-planar, we must have that the deletion graph induced on $V(f)$ contains either an independent set of size 3 or a vertex of degree 0. From this observation, we can construct the following theorem.

Theorem 4.2.3. *Let J be a proper graft, so that the deletion graph of J is planar, and every split of J is non-planar. Then J can be obtained by a sequence of edge deletions from* *some graft R with edge r of size 4, where the deletion graph of R is a maximal planar graph, and R induced on r contains either an independent set of size 3 or a vertex of degree 0.*

Proof. Since every split of J is non-planar, G induced on $V(f)$ must contain either an independent set of size 3 or an isolated vertex by the above remark. Further G must be some spanning subgraph of a maximal planar graph T since G is planar (by Proposition 1.3.21).

Then let R be the graft obtained by adding edge f to T. Every split of f in R yields a non-planar graph, since it will contain some split of J as a subgraph. It follows that R induced on $V(f)$ must have an independent set of size 3 or an isolated vertex since every split of R is non-planar and T is a maximal planar graph. Thus R satisfies the conditions of the Theorem. □

Theorem 4.2.3 suggests a method for finding proper grafts which could potentially be in $\mathcal{M}_4$. Using the program *plantri* developed by Gunnar Brinkmann and Brendan McKay [11], we generated the maximal planar graphs up to isomorphism on a given number of vertices. Utilizing this, for a given maximal planar graph $G = (V, E)$, we found all sets of four vertices $V' \subset V$ (up to symmetry) such that the graph induced on V' contains either an independent set of size three or a vertex of degree zero (we call such a set a *fourset*). Each fourset V' along with G implicitly defines a graft J constructed by adding the edge incident to all vertices of V' to G. Following this, we calculated the maximum sets of edges E' such that each split of the graft J' obtained by deleting every edge of E' from J is non-planar. At this point we checked by hand whether $K_{3,3}$ or K_5 is a parity minor of J', since the only way to obtain $K_{3,3}$ or K_5 is through a sequence of edge contractions and mod 2 reductions - and if $|V|$ is relatively small, this is fairly simple.

We have applied this searching technique, but also taken into account an additional property which we have not proven: if J' is e-magic, then $J' - V'$ must contain a cycle. To date, we have checked all grafts obtained in this manner on up to and including ten vertices, and each of these was either not e-magic by Proposition 4.1.7 or could be reduced to $K_{3,3}$ or K_5. For the code that we wrote and additional details, the reader may consult Appendix A.2.

There is an inherent difficulty with this searching technique. The number of maximal planar graphs grows extremely quickly with number of vertices, n, the first few terms of this sequence being: 1, 2, 5, 14, 50, 233, 1249, 7595, 49566, 339722, 2406841, 17490241, 129664753, 977526957, 7475907149, 57896349553, 453382272049, 3585853662949, 28615703421545 (we started with $n = 4$). The number of grafts we have to check is also significantly larger since a single maximal planar graph could have many possible foursets. Additionally, since a planar graph does not have a unique maximal planar graph containing it as a spanning subgraph, we may check the same graft many times.

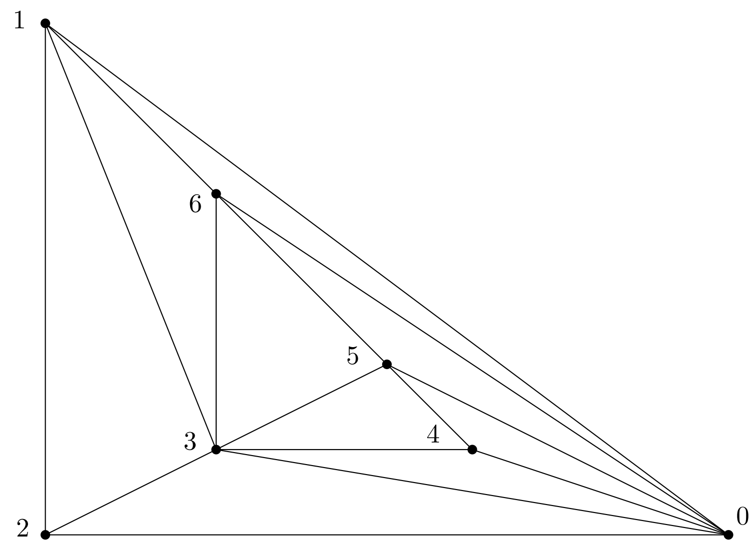

Figure 4.3: A maximal planar graph T from which we can generate grafts

However, there is much room for improvement. In Chapter 5 we discuss a conjectured proposition which could significantly reduce the number of foursets. It is also likely that some of the propositions developed at the end of Section 3.3 and in Section 4.1 can be utilized in order to simplify our work.

Example 4.2.4. The maximal planar graph shown in Figure 4.3 has four different foursets of vertices which we could use to create a proper graft: $\{2, 4, 5, 6\}$, $\{1, 2, 4, 6\}$, $\{0, 2, 4, 6\}$, $\{2, 3, 4, 6\}$. These grafts are illustrated in Figure 4.4. How do we decide which of these is e-magic and which is not? Consider the first graft, J, with f incident to $2, 4, 5, 6$ (this is the top left graft of Figure 4.4). This is e-magic by Theorem 3.3.18 since we can delete edges $34, 35, 05, 04, 56$ of J to obtain the graft illustrated in Figure 4.5, then contract edge 45 and apply a mod 2 reduction to the vertex resulting by the identification of vertices 4 and 5 to obtain K_5.

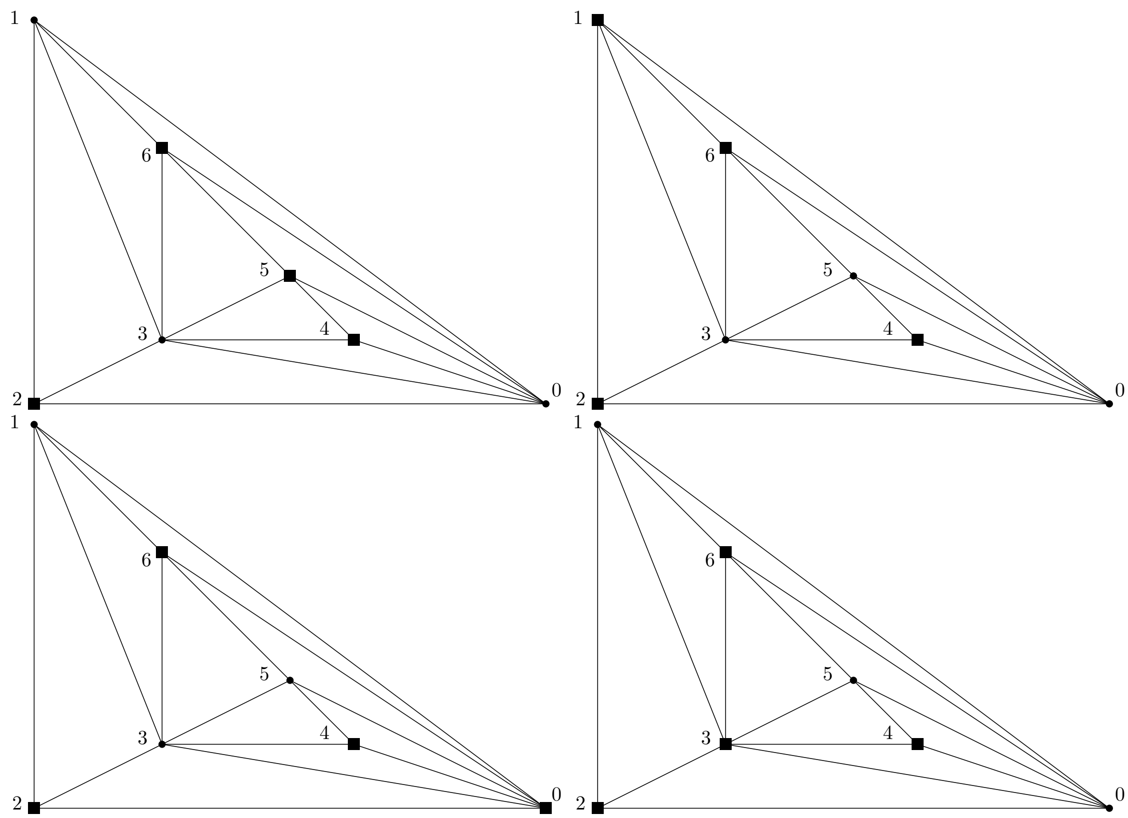

Figure 4.4: The four proper grafts that can be obtained from T

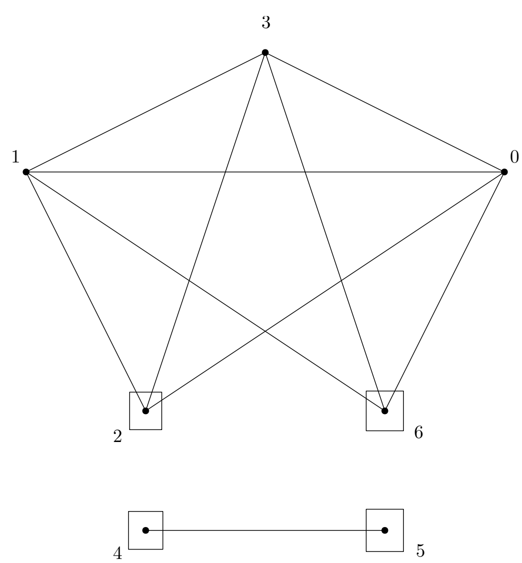

Figure 4.5: A graft obtained by deleting edges of J

Chapter 5

Outlook

In this thesis we have presented many tools to aid us in solving Problems 2.1.8 and 2.2.2. However, there is much work that can still be done on both of these problems.

I believe that there are more results which are in a similar vein to Proposition 4.1.7 that can be attained for grafts. I think that with enough work, we can show that the deletion graph of a graft J induced on the edge of size 4 of J must be an independent set of size 4 if J is proper and minimally e-magic. It could also very well be the case that $\mathcal{M}_4$ is empty, as every case which we have seen is either reducible to $K_{3,3}$ or K_5 or is not e-magic. We formalize this below as a conjecture.

Conjecture 5.0.1. *All e-magic grafts reduce to $K_{3,3}$ or K_5.*

One can consider other candidate elements for MEM apart from grafts. In general, ensuring that each split is planar can be difficult since, as we have seen, an algorithm to check this can easily be at least exponential. However, if we choose hypergraphs on n vertices, such that the graph resulting from each split will have more than $3n-6$ edges, then this condition will automatically hold. A study of such hypergraphs would be extremely interesting work for the future, and could lead to a discovery of new operations to add to the existing Parity Minor ones.

It is known [32] that for a fixed graph H, checking whether some graph G contains H as a minor can be done in cubic time (in the number of vertices of G). It would be interesting to study the complexity of this problem for the parity minor relation.

Proposition 4.1.3 suggests an approach using the theory of flows and circulations from graph theory. The fact that each bridge (with the additional planar component condition) must be labeled with identity is very reminiscent of the fact that every bridge must be labeled with the identity element in flows over groups. Additionally, if we restrict ourselves to matrices obtained by tensor products of Pauli matrices, we can attempt to make analogues with flows in some group formed by cartesian products of $\mathbb{Z}_2$.

Another approach that could be taken for Problem 2.1.8 is an algebraic one. There are several existing methods, such as expressing commutativity of Pauli matrices using the symplectic product and solving the arising quadratic equations using Groebner bases or other elimination methods. These can be used to find magic Pauli assignments for a given number of qubits, or show that no such assignments exist. For the more general case, an approach is to view the relations arising from Definition 2.1.7 as a finitely presented group, as is done in [25, 35]. In this setting, one can use the Knuth-Bendix algorithm to simplify relations for the given proper Eulerian hypergraph [25]. Oftentimes the relations can be simplified to the point where we can conclude that the hypergraph is non-magic. We have not presented such an approach in this thesis as it would have required significant additional background, however, these are certainly interesting avenues for research.

Lastly, we should note that our results have not disproved a conjecture made by Arkhipov [3] - that any magic proper Eulerian hypergraph has a 3-qubit assignment.

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Appendix A

Code

A.1 Four Qubit Star

Here we display the Magma computation that was utilized to find a $K_{3,3}$ minor occurring in the dual of the 4-qubit star, represented here by G.

```
> G := Graph< { 1 .. 6 } |
> { {1,2}, {2,3}, {1,4}, {3,4}, {2,4}, {4,5},
>   {1,6}, {2,5}, {3,6}, {5,6}, {1,5}, {2,6} } >;
>
> ispl, obstr := IsPlanar(G);
>
> ispl;
false
> obstr;
Graph
Vertex  Neighbours

1       6 4 2 ;
2       5 3 1 ;
3       6 4 2 ;
4       5 3 1 ;
5       6 4 2 ;
6       5 3 1 ;
```

A.2 Minimally E-magic Graft Search

This appendix section contains the Magma code that we wrote to search for minimally e-magic grafts. Within the main loop we access an auxiliary file populated by maximal planar graphs in g6 format generated by plantri [11]. We then loop through the graphs, calling the procedure process_triang which takes as input a single graph g. The procedure process_triang is exactly as we have described in Section 4.2. In particular, checking the foursets up to symmetry is done by choosing a set of foursets so that each fourset corresponds to a unique graft. We do this by calculating the automorphism group of the graph, grouping the set of all foursets into sets of orbits, and finally choosing a random representative from each orbit.

```
function is_indep_set(x,y,z)
return
(x notadj y) and (x notadj z) and (y notadj z);
end function;

function isFset(T,vseq)
u:=vseq[1];
v:=vseq[2];
w:=vseq[3];
x:=vseq[4];
return
(
is_indep_set(u,v,w)
or
is_indep_set(u,v,x)
or
is_indep_set(u,w,x)
or
is_indep_set(v,w,x)
or ( #(Neighbours(u) meet {v,w,x}) eq 0 )
or ( #(Neighbours(v) meet {u,w,x}) eq 0 )
or ( #(Neighbours(w) meet {u,v,x}) eq 0 )
or ( #(Neighbours(x) meet {u,v,w}) eq 0 )
)
and
not(IsForest(T - {u,v,w,x}))
;
end function;

procedure process_triang(t)
print t;
G, GV, GE := AutomorphismGroup(t);
orbit:=function(q)
return { { (Index(x))^g : x in q } : g in G };
end function;
V:=VertexSet(t);
V4:=[ [V!i,V!j,V!k,V!l] : i,j,k,l in [ 1 .. #V ]
     | (i lt j) and (j lt k) and (k lt l)
    ];
Fsets := [ q : q in V4 | isFset(t,q) ];
Forb := { orbit(SequenceToSet(f)) : f in Fsets };
Fsets := { Random(o) : o in Forb };
```

```
E := EdgeSet(t);
for f in Fsets do
fv := { V!i : i in f };
printf "\nF=%o\n",fv;
fsplits := { {a,b} : a,b in Subsets(fv,2) | #(a meet b) eq 0 };
D:=AssociativeArray();
D[1] := { {e} : e in EdgeSet(t)
  |
    (
    not(IsForest(G-fv))
    and ( { IsPlanar(G+spl) : spl in fsplits } eq {false} )
    )
    where G is t-{e}
      };
printf "#D1=%o\n",#D[1];
k:=1;
while (k lt 3*#V-6) and (#D[k] gt 0) do
k:=k+1;
D[k] := { d join {e} : d in D[k-1] , e in EdgeSet(t)
 |
    (
    ( #(d join {e}) eq k )
    and not(IsForest(G-fv))
    and ( { IsPlanar(G+spl) : spl in fsplits } eq {false} )
    )
    where G is t-(d join {e})
      };
if #D[k] eq 0 then print D[k-1]; end if;
printf "#D%o=%o\n",k,#D[k];
end while; // k
end for; // f in Fsets
end procedure;

// Main loop.

GF := OpenGraphFile("PT9.g6", 0, 0);
count := 0;
repeat
more, g := NextGraph(GF);
if more then
count +:= 1;
printf "\n\n\n___________\ncount=%o\n",count;
process_triang(g);
end if;
until not more;
```