Analysis of Target Benefit Plans with Aggregate Cost Method

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Abstract

The operational characteristics of a target benefit plan based on an aggregate pension cost method are studied through simulation under a multivariate time series model for projected interest rates and equity returns. The performance of the target benefit plan is evaluated by applying a variety of performance metrics for benefit security, benefit adequacy, benefit stability and intergenerational equity. Performance is shown to improve when the economy remains relatively stable over time and when the choice of valuation rate does not create persistent gains or losses.

Keywords: Target Benefit Plan; Benefit Security; Benefit Adequacy; Benefit Stability; Intergenerational Equity; Aggregate Cost Method; Multivariate Time Series; Simulation

Dedication

To my entire family!

Quotation

"Life is like riding a bicycle. To keep your balance, you must keep moving."
—Albert Einstein (1879 - 1955)

Acknowledgements

Honestly, this is the section that I have been avoiding since finishing it implies that I will bid farewell to my favourite campus. I love living in this city, learning on this campus and working in this department. There are so many people that supported and encouraged me during my journey at SFU.

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Chapter 1

Introduction

The two traditional pension plan designs in Canada are defined benefit (DB) and defined contribution (DC). Under the DB design, the plan promises to pay a pension benefit; this pension benefit is determined by a formula, which is often based on the employee's earnings history and the number of years of work with the company. The benefit is payable for life and is guaranteed by the plan sponsor. The sponsor's contributions fluctuate in response to the plan's economic and demographic experience. Under a DC design, the plan sponsor promises to make regular contributions to each employee's individual retirement account during the employee's working life. The account is invested and the benefit the employee receives upon retirement is the accumulated value of this account. The employee may convert this benefit to a lifetime pension, using the account value at retirement to purchase an annuity from an insurance company at prevailing market rates.

A number of alternative pension plan designs have been proposed over the years to address certain shortcomings of these traditional designs. One of these is the target benefit plan (TBP) which has received significant attention over the last 6-10 years in Canada. TBPs are a broad class of pension designs whose characteristics and performance can vary widely. The main objective of this project is to investigate the performance of a specific TBP design through simulation under a multivariate asset model, applying a variety of performance metrics to assess benefit security, benefit adequacy, benefit stability and intergenerational equity under the plan.

1.1 Background of Target Benefit Plans

The Canadian Institute of Actuaries (CIA, 2015) Task Force on TBP defines a TBP in the Canadian context, as

> *a collective, pre-funded pension plan pooling both economic and demographic risks, with a predefined retirement income goal (the "target benefit"), where the employer's financial liability is limited to predefined contributions while members' benefits may periodically be adjusted upwards or downwards relative to the original target.*

The sponsor's predefined contributions may be fixed or vary within a relatively narrow range.

A TBP is considered to be a hybrid design as it combines characteristics of both DB and DC designs. One of the key differences between DB and DC designs is their risk profile. Since the retirement income is guaranteed by the sponsor under a DB design, the sponsor bears all risks there. According to the CIA Task Force's report (CIA, 2015), more and more Canadian sponsors find the costs of DB plans unacceptable due to the combination of volatile markets and a low interest rate environment. By contrast, under a DC design, the pension income is not guaranteed and depends on the member's account value at retirement, so that all the risks are borne by each employee individually. The CIA Task Force's report (CIA, 2015) noted the following factors as contributing to the inefficiency of DC plans: ineffective management of retirement assets by employees if they choose not to annuitize; high investment management expenses if employees choose to annuitize; longevity risk (most employees choose not to purchase an annuity at retirement) and the impact of market cycles on emplyees' retirement assets during employment.

Aon Hewitt (2015a,b,c,d) claims that the risks are in an effective and acceptable way for both the sponsors and the employees under TBPs. Table 1.1, adapted from Sanders (2016b), outlines the main features of TBPs relative to both DB and DC designs.

Table 1.1: Features of target benefit plans

Similarities between TBPs and traditional designs	
Traditional DB design	Traditional DC design
The members' mortality risk is pooled	The contribution level is predefined
The financial account is collective	The sponsor bears (almost) none of the risks
The target pension benefit is predefined	The actual pension benefits are varied

TBPs are sometimes also called "collective DC plans" because the employees, as a group rather than as individuals, bear all the risks (CIA, 2015). Several developed countries have also implemented "relatives" of TBPs for their pensioners, including the US, Denmark, the Netherlands and the UK (Sanders, 2014). Currently, three Canadian provinces have detailed legislations for TBPs which are New Brunswick, Alberta and British Columbia (Sanders, 2016a). Several other Canadian jurisdictions have also passed laws to enable TBPs, but the corresponding regulations have not been finalized.

1.2 Elements of TBP Designs

Target benefit plans are extremely flexible, with an infinite number of possible designs. In the Canadian context, the four key elements of TBP design according to the CIA Task Force's report (CIA, 2015) are:

- Contribution rate: this rate is fixed (a fixed amount or a fixed percentage of payroll) and predefined. Alternatively, the contribution level may vary in a very narrow range as specified in the plan documents.
- Target benefit: it is based on a predefined formula as under a DB plan. For example, the target benefit can be set as 1% of career average earnings times years of service. The actual benefits may be greater or lower than the target, depending on plan experience.
- Affordability test: this actuarial test is used to decide whether the target benefit is affordable.
- Triggers and actions: the triggers decide whether action needs to be taken based on the results of the affordability test. The specific actions may include adjusting the pension benefit, the contribution rate and/or the investment strategy. Their priority order would normally be specified in the plan documents.

The TBP design investigated in this project can be described in terms of these four elements. We set the contribution rate as a constant percentage of salary and use a formula based on career average earnings and years of service to determine the target benefit. We employ the aggregate cost method in the affordability test and use a single trigger to adjust the pension benefits immediately at each valuation. Although we consider the case of a single trigger only, additional triggers can be easily built into our model.

1.3 Literature Review

In this project, we use a stochastic model to simulate the operation of the TBP design described above in order to understand its short-term and long-term behaviour. The first step of the project is to build an economic scenario generator (ESG) and use it to project key economic variables such as investment returns and interest rates. There are many possible choices of ESG. One class of ESGs generate equity returns and interest rates separately so that there is no relationship between the two rates. For instance, the AAA model described in Ahlgrim et al. (2008) used a stochastic volatility model to generate interest rates and equity returns separately.

Some ESGs model the inflation rate first and treat it as the main driver of all other economic variables, using a "cascade" approach to generate interest rates and equity returns. For example, the CAS-SOA model described in Ahlgrim et al. (2008) used an Ornstein-Uhlenbeck process to model inflation first. The nominal interest rate was then based on this inflation, plus the real interest rate, modelled by a two-factor Hull-White model. The equity return in excess of inflation was modelled by a regime-switching model. In this case, all nominal rates and returns were influenced by inflation. Another example is the model proposed by Wilkie (1986), which used an autoregressive model with order one to simulate inflation rates and treated this as the driving factor for all other economic series, including Consols yields (long-term interest rate) and equity returns. Wilkie (1995) updated the earlier model, again applying a "cascade" approach to model inflation-related variables. This time, a first-order vector autoregressive model was used to fit past wages and past inflation rates simultaneously. In each of these three models, the resulting economic series were correlated with each other, but only inflation could affect the other variables, not vice versa.

A third class of ESGs allows each economic variable to affect the others. They include multivariate models with stationary moments for simulating interest rates and equity returns simultaneously. Blake et al. (2001) investigated four such models (multivariate normal, mixed multivariate normal, multivariate t, and multivariate non-central t) to generate key economic variables. Hoevenaars and Ponds (2008) used a first-order vector autoregressive model to model interest rates and equity returns in order to obtain the total asset return. Beetsma et al. (2014) also used a vector autoregressive model with order one to fit interest rates and equity returns simultaneously.

In this project, in order to keep our ESG simple, we follow the idea from Hoevenaars and Ponds (2008), and Beetsma et al. (2014), and employ a first-order vector autoregressive model to project key financial time series including Canadian zero-coupon bond yields and Canadian equity returns. Once we have built an ESG, we can apply it to our selected TBP design.

Since TPBs are fairly new, not many variants have been investigated to date. Wesbroom et al. (2013) modelled the operation of a TBP using an affordability test based on the funded ratio under the traditional unit credit cost method. The triggers for action were based on this funded ratio. If the actual funded ratio fell between the upper and lower trigger points set at 90% and 110% funded ratios respectively, then no action would be taken. We call the region between the triggers the "no-action" range. If the actual funded ratio fell outside the "no-action" range, then the target benefit for current and future years would be adjusted first; the accrued benefits of current members were changed only if necessary. Wesbroom et al. (2013) used a career average earnings type benefit for the target and set the contribution from the employer as a fixed percentage of payroll.

Another way to protect accrued benefits of active members is to apply the traditional unit credit method to an open group. Open group means that some of the benefits and contributions of future members who have not yet joined the plan are included in the calculation of the funded ratio. This method is prescribed under the TBP regulation applicable in New Brunswick (CIA, 2015). Sanders (2016a,b,c) tested both the closed group and the open group approaches with a variety of trigger points and possible actions. As in Wesbroom et al. (2013), the target benefit was based on career average earnings and contributions were fixed as a percentage of payroll. Numerous performance metrics were applied to assess TBPs' benefit security, benefit adequacy , benefit stability and intergenerational equity. However, none of the designs were found suitable from a practical standpoint.

The aggregate method also allows the accrued benefits of active members to be protected but it uses a slightly different perspective to determine affordability. Sanders (2010) employed this approach, comparing the normal cost determined by the aggregate method to the fixed contribution rate. Applying a single trigger, the future service benefits of current members were adjusted at each valuation; accrued benefits were adjusted only if necessary. In contrast to the previously mentioned papers, Sanders (2010) set the target benefit as a flat benefit and the contribution was a fixed dollar amount for each employee.

Ma (2016) developed a different affordability test based on the aggregate method, borrowing the concept of the "balance ratio" from the Swedish National pension (Settergren, 2001). If the plan's balance ratio as redefined by Ma (2016) for a TBP was not equal to one, action would need to be taken to restore the balance ratio. Ma's analysis used a target benefit based on final average salary and set the contribution rate as a fixed percentage of payroll determined by the entry age normal cost method.

1.4 Outline

The rest of this project is structured as follows.

Chapter 2 introduces the asset model – a stochastic multivariate time series model. Historical financial time series are fitted and model parameters are estimated. Some properties of the model are then analyzed and a reasonable number of simulations for our asset portfolio is determined.

The liability model is discussed in Chapter 3. First, key assumptions and notation are introduced. The plan provisions are then described. We identify the sources of benefit risk and derive formulas for attributing changes in benefits to these sources. Last, the allocations of the asset portfolio is chosen based on duration matching.

In Chapter 4 we introduce our performance metrics which are used to assess our TBP's benefit security, benefit adequacy, benefit stability and intergenerational equity. The performance of our TBP based on these metrics is analyzed.

Two modifications to our TBP design are considered in Chapter 5: starting the simulation exercise from economic conditions closer to historical average rather than the current low-interest rate environment and changing from a risk-free valuation rate to a best-estimate rate. Performance metrics under these modifications are provided and analyzed.

Finally, our conclusions and areas for future exploration are provided in Chapter 6.

Chapter 2

Asset Model

In this chapter, we first introduce our asset model. Then we describe the historical data used and the modifications made to estimate the parameters of our model. Finally, we analyze key characteristics of simulations based on our fitted asset model.

2.1 Model Selection, Definition and Properties

In economics, finance and actuarial science, stochastic processes are used for modelling time series data. The main advantage of stochastic models over deterministic ones is that they allow one to study the entire range of potential outcomes. Given that our goal is to compare the variability of pension benefits across various pension designs, a stochastic process is the appropriate approach for modelling the rates of return on the asset portfolio.

Since we are also interested in studying the effect investment strategies can have on the different designs, we need to jointly model the returns for a number of asset classes. In order to keep the dimensionality of the problem under control, we consider a multivariate autoregressive process of order one (VAR(1)). We choose this process for two of its main features, namely, the rates of return on a given asset class are correlated over time; and the rates of return on different asset classes for a given time period are correlated.

We now introduce the VAR(1) model and its main properties. According to the multivariate autoregressive process of order p, VAR(p), in Reinsel (1997, pages 7 and 27), we have the following definition and theorem for VAR(1):

Definition 2.1. *An n-dimensional multivariate autoregressive model of order one, VAR(1) model, is defined as*

$$\underline{X}_t - \underline{\mu} = \underline{\Phi} \cdot (\underline{X}_{t-1} - \underline{\mu}) + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_t, \tag{2.1}$$

where $t \in \mathbb{Z}$, $\underline{X}_t$ is an n-dimensional time series, $\underline{\mu}$ is the n-dimensional mean-reverting level (long-term mean) of $\underline{X}_t$, $\underline{\Phi}$ is the $n \times n$ autoregressive coefficient matrix, $\underline{\Sigma}^{1/2}$ is the $n \times n$ Cholesky decomposition (lower triangle) of the covariance matrix $\underline{\Sigma}$, and $\underline{\varepsilon}_t$ is the n-dimensional white noise process with mean 0 and variance 1.

Theorem 2.1. *An n-dimensional VAR(1) model is stationary if all the roots (λs) of $det(\underline{\Phi} - \lambda \cdot I) = 0$ are less than 1 in absolute value, that is if all the eigenvalues of $\underline{\Phi}$ are less than 1 in absolute value, where I is the $n \times n$ identity matrix.*

To ensure that long-term results are meaningful, we want the VAR(1) model to be stationary. When valuing a pension plan, recent and current information about the asset mix and the rates of return is usually known. We are therefore interested in the conditional moments of the VAR(1) model.

Theorem 2.2. *If $\underline{X}_t$ follows an n-dimensional VAR(1) process, then $\underline{X}_t$ given $\underline{X}_0$ is given by*

$$(\underline{X}_t \mid \underline{X}_0) = \underline{\Phi}^t \cdot (\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-j} + \underline{\mu}, \tag{2.2}$$

and its conditional mean and variance are given by

$$E(\underline{X}_t \mid \underline{X}_0) = \underline{\Phi}^t \cdot (\underline{X}_0 - \underline{\mu}) + \underline{\mu}, \tag{2.3a}$$

$$Var(\underline{X}_t \mid \underline{X}_0) = \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma} \cdot (\underline{\Phi}^j)^T. \tag{2.3b}$$

Proof. Given $\underline{X}_0$, we can write $\underline{X}_1$ as follows:

$$\begin{aligned}
(\underline{X}_1 \mid \underline{X}_0) &= \Big((\underline{X}_1 - \underline{\mu}) \mid \underline{X}_0 \Big) + \underline{\mu} \\
&= \Big(\underline{\Phi} \cdot (\underline{X}_0 - \underline{\mu}) + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_1 \mid \underline{X}_0 \Big) + \underline{\mu} \\
&= \underline{\Phi} \cdot (\underline{X}_0 - \underline{\mu}) + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_1 + \underline{\mu}
\end{aligned}$$

So Equation (2.2) is true for t=1.

Assume it is true for $t-1$, that is

$$(\underline{X}_{t-1} \mid \underline{X}_0) = \underline{\Phi}^{t-1} \cdot (\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-2} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-1-j} + \underline{\mu},$$

then,

$$\begin{aligned}
(\underline{X}_t \mid \underline{X}_0) &= \Big((\underline{X}_t - \underline{\mu}) \mid \underline{X}_0\Big) + \underline{\mu} \\
&= \Big(\underline{\Phi}(\underline{X}_{t-1} - \underline{\mu}) + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_t \mid \underline{X}_0\Big) + \underline{\mu} \\
&= \underline{\Phi}\Big(\underline{\Phi}^{t-1} \cdot (\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-2} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-1-j}\Big) + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_t + \underline{\mu} \\
&= \underline{\Phi}^t(\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-2} \underline{\Phi}^{j+1} \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-1-j} + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_t + \underline{\mu} \\
&= \underline{\Phi}^t(\underline{X}_0 - \underline{\mu}) + \sum_{j=1}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-j} + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_t + \underline{\mu} \\
&= \underline{\Phi}^t(\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-j} + \underline{\mu}
\end{aligned}$$

which proves Equation (2.2).

Taking expected value, we get:

$$\begin{aligned}
E(\underline{X}_t \mid \underline{X}_0) &= E\Big(\underline{\Phi}^t \cdot (\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-j} + \underline{\mu}\Big) \\
&= E\Big(\underline{\Phi}^t \cdot (\underline{X}_0 - \underline{\mu})\Big) + E\Big(\sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-j}\Big) + \underline{\mu} \\
&= \underline{\Phi}^t \cdot (\underline{X}_0 - \underline{\mu}) + \underline{\mu}
\end{aligned}$$

which proves Equation (2.3a).

The conditional variance is obtained as follows:

$$
\begin{aligned}
Var(\underline{X}_t \mid \underline{X}_0) &= Var\Big(\underline{\Phi}^t \cdot (\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-j} + \underline{\mu}\Big) \\
&= Var\Big(\sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_{t-j}\Big) \\
&= \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot Var(\underline{\varepsilon}_{t-j}) \cdot (\underline{\Sigma}^{1/2})^T \cdot (\underline{\Phi}^j)^T \\
&= \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma}^{1/2} \cdot I \cdot (\underline{\Sigma}^{1/2})^T \cdot (\underline{\Phi}^j)^T \\
&= \sum_{j=0}^{t-1} \underline{\Phi}^j \cdot \underline{\Sigma} \cdot (\underline{\Phi}^j)^T
\end{aligned}
$$

which proves Equation (2.3b).

□

2.2 Financial Data

After choosing our asset model, we now introduce the financial data used to estimate the model parameters. We chose four Canadian financial time series corresponding to our eligible asset classes in which funds will be invested. They consist of the rates of return on three fixed income investments: short-term zero-coupon bond (3 months), medium-term zero-coupon bond (5 years), long-term zero-coupon bond (15 years), and the total return on one equity index: Toronto Stock Exchange (TSX) composite total return close index.

The Bank of Canada publishes yield curves for zero-coupon bonds. We obtained the TSX composite total return indices (including dividends) from the Canadian Financial Markets Research Centre (CFMRC) Summary Information Database. These historical data series for the period of March 1991 to February 2016 are provided in Appendix A.

The choice of March 1991 as the start date for the historical data corresponds to the effective date of a revised monetary policy aimed at keeping Canadian inflation low and stable. As a result of an agreement between the Bank of Canada and the Minister of Finance, an inflation target of 2%, varying between 1% and 3%, is now an objective of the monetary policy.

Prior to this agreement, inflation in Canada was, at times, high and volatile. Over the past 25 years (March 1991 – March 2016), the target inflation policy worked well, keeping the inflation rate fairly close to its 2% target. Figure 2.1 shows Canadian inflation rates since 1914.[1]

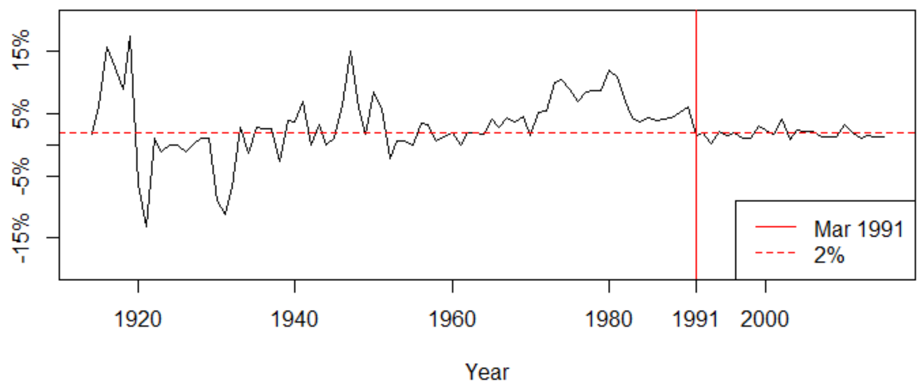

Figure 2.1: Force of annual Canadian historical inflation rates from Mar 1914 to Mar 2015

As a result, it seems reasonable to assume stable inflation rates for the future. This assumption is also not likely to affect the conclusions of this project.

For reasons that will become obvious when we discuss the investment strategy for our pension plan projections, we choose to work with monthly forces of return. This means that an investment of 1, earning a rate of r per month, will accumulate to e^r at the end of one month. Note that working with instantaneous rates guarantees that the value of an investment always remains positive ($e^r > 0$) which is desirable here.

We now describe in details the historical data collected and the manipulations made to the data before fitting the VAR(1) model.

Denoted by $\delta^y_{A,t,n}$ are the annualized instantaneous rates of return (equivalently, annual force of return) paid by asset A purchased at time t and sold at time $t+n$, where A can be: SB (short-term zero-coupon bonds), MB (medium-term zero-coupon bonds), LB (long-term zero-coupon bonds) or EQ (equities); and times t and n are measured in years.

The zero-coupon yields published by the Bank of Canada are annualized instantaneous rates of return earned on bonds purchased on the given date if held until maturity.[2]For example, an amount of 1 invested at time t into a 5-year zero-coupon bond would accumulate to $e^{5 \cdot \delta^y_{MB,t,5}}$ at time $t+5$.

[1]Retrieved from Statistics Canada (consumer price indexes: all items).
[2]Retrieved from http://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/.

For simplicity, we will assume that a zero-coupon bond purchased at time t can be sold, at any future time $t{+}n$ before maturity, for its book value. For example, a 15-year zero-coupon bond purchased at time t would sell at time $t{+}n$ ($n < 15$), for a price of $e^{n \cdot \delta^y_{LB,t,15}}$. In reality the price would depend on the prevailing rate at time $t{+}n$, $\delta^y_{LB,t+n,15-n}$, but since we do not have full yield curves at all times, we will assume that this prevailing rate is the same as the original purchase rate.

For equities, the historical data consist of values of the TSX composite total return index on a given date. Let I_t denote the value of the S&P/TSX index at time t, more specifically, it is the adjusted close index of that business day. The annualized instantaneous rate of return on an investment in the TSX made at time t and kept until time $t{+}n$ is $\delta^y_{EQ,t,n} = \dfrac{1}{n} \cdot \ln(\dfrac{I_{t+n}}{I_t})$.

Note that the rate of return earned on a bond (up until maturity) is known at the time of purchase but the rate of return is not known at the time one invests into equities (S&P/TSX index).

The forces of return per month, for investments of one month (n=1/12), are obtained by dividing the annual values by 12. Denoting the rates per month without superscript and dropping the term of the investment in the notation (no longer needed here since n=1/12 for all assets and bonds are assumed to sell for their book values), then we get the following rates of return per month at time t:

$$\underline{\delta}_t = (\delta_{SB,t}, \delta_{MB,t}, \delta_{LB,t}, \delta_{EQ,t}) = \Big(\frac{\delta^y_{SB,t,0.25}}{12}, \frac{\delta^y_{MB,t,5}}{12}, \frac{\delta^y_{LB,t,15}}{12}, \frac{\delta^y_{EQ,t,1/12}}{12} \Big).$$

Our adjusted historical dataset consists of rates of return per month, observed at a monthly frequency for a 25-year period, resulting in 300 observations for each financial time series. More precisely, we have values of the 4-dimensional vector $\underline{\delta}_t$ for t=$0, \frac{1}{12}, \frac{2}{12}, ..., 24\frac{11}{12}$ where t=0 corresponds to February 28, 1991 and t=$24\frac{11}{12}$ is January 29, 2016. Although data at a weekly or daily frequency could have been collected, we believe that 300 monthly observations are sufficient to adequately fit our asset model. Note that the TSX index on February 29, 2016 (t=25) was needed to determine the equity rate of return for the last month in our data set (i.e. the month of February 2016 corresponding to the value at t=$24\frac{11}{12}$ in the data set).

The 4-dimensional monthly forces of return series from our asset portfolio and its mean-reverting level (long-term mean) are denoted by $\underline{\delta}_t$ and $\underline{\delta}$ respectively. Hence, based on

Equation (2.1), our VAR(1) model for the four financial time series can be expressed as:

$$\underline{\delta}_t - \underline{\delta} = \underline{\Phi} \cdot (\underline{\delta}_{t-1} - \underline{\delta}) + \underline{\Sigma}^{1/2} \cdot \underline{\varepsilon}_t. \tag{2.4}$$

More specifically, we have:

$$\begin{bmatrix} \delta_{SB,t} - \overline{\delta}_{SB} \\ \delta_{MB,t} - \overline{\delta}_{MB} \\ \delta_{LB,t} - \overline{\delta}_{LB} \\ \delta_{EQ,t} - \overline{\delta}_{EQ} \end{bmatrix} = \begin{bmatrix} \phi_{SB-SB} & \phi_{SB-MB} & \phi_{SB-LB} & \phi_{SB-EQ} \\ \phi_{MB-SB} & \phi_{MB-MB} & \phi_{MB-LB} & \phi_{MB-EQ} \\ \phi_{LB-SB} & \phi_{LB-MB} & \phi_{LB-LB} & \phi_{LB-EQ} \\ \phi_{EQ-SB} & \phi_{EQ-MB} & \phi_{EQ-LB} & \phi_{EQ-EQ} \end{bmatrix} \cdot \begin{bmatrix} \delta_{SB,t-1} - \overline{\delta}_{SB} \\ \delta_{MB,t-1} - \overline{\delta}_{MB} \\ \delta_{LB,t-1} - \overline{\delta}_{LB} \\ \delta_{EQ,t-1} - \overline{\delta}_{EQ} \end{bmatrix}$$

$$+ \begin{bmatrix} \sigma_{SB-SB} & 0 & 0 & 0 \\ \sigma_{MB-SB} & \sigma_{MB-MB} & 0 & 0 \\ \sigma_{LB-SB} & \sigma_{LB-MB} & \sigma_{LB-LB} & 0 \\ \sigma_{EQ-SB} & \sigma_{EQ-MB} & \sigma_{EQ-LB} & \sigma_{EQ-EQ} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{SB,t} \\ \varepsilon_{MB,t} \\ \varepsilon_{LB,t} \\ \varepsilon_{EQ,t} \end{bmatrix} . \tag{2.5}$$

Figure 2.2 shows the monthly historical forces of return for the three selected zero-coupon bonds. We can see that the rates have followed similar trends over the past 25 years indicating a strong correlation between the rates. Also of interest, we note that the current rates stand at historical low levels.

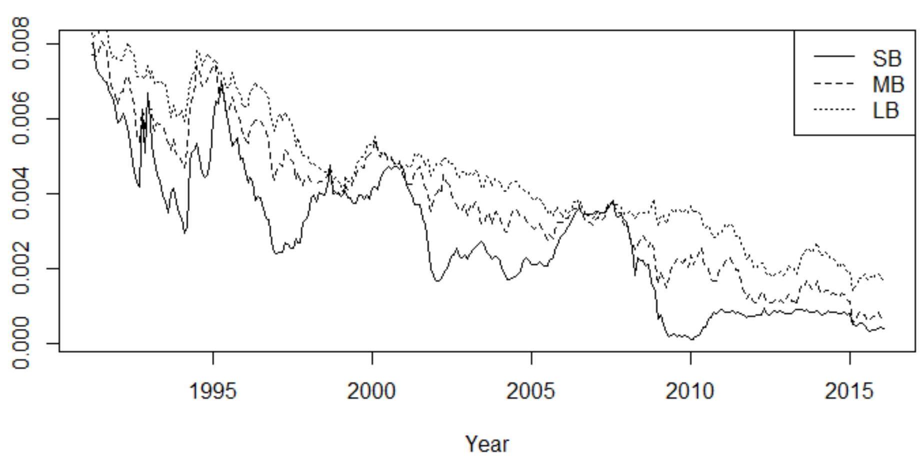

Figure 2.2: Monthly forces of return of three Canadian zero-coupon bonds from Mar 1991 to Feb 2016

The mean and standard deviation of the 300 observations (monthly forces of return) for each series of the selected bonds are shown in Table 2.1. The current rates, which are the rates on January 29, 2016, are also given in that table.

Table 2.1: Summary statistics for the monthly forces of return on three Canadian bonds from Mar 1991 to Feb 2016

	Mean	Standard Deviation	Current Value
SB	0.002824	0.001848	0.000387
MB	0.003725	0.001875	0.000552
LB	0.004478	0.001807	0.001615

Table 2.2 contains the mean and standard deviation of the monthly forces of total return (including dividends and dividends are reinvested) paid by the TSX over the past 25 years. The current rate, that is the rate of return for the month of February 2016, is also shown.

Table 2.2: Summary statistics for the monthly forces of total return on the TSX from Mar 1991 to Feb 2016

	Mean	Standard Deviation	Current Value
EQ	0.006373	0.041933	0.004669

The TSX monthly forces of total return for the past 25 years are shown in Figure 2.3.

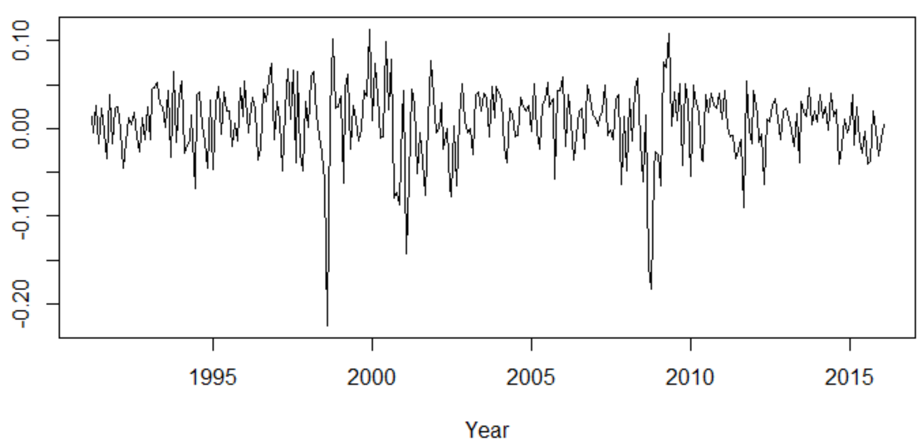

Figure 2.3: Monthly forces of total return of the TSX from Mar 1991 to Feb 2016

Based on the above description of our data collection and transformation, it can be seen that our constructed financial data consists of a matrix of four rows (asset classes) by 300 columns (dates) containing forces of return per month observed at a monthly frequency beginning with the month of March 1991 and ending with the month of February 2016.

2.3 Model Estimation

Now, we can use our constructed matrix of instantaneous rates of return to calibrate our VAR(1) model. The first step is to subtract the average rate for each asset class to obtain centered values. The average returns, as a vector for short-term, medium-term, long-term bonds and equities, in order, are:

$$\underline{\delta} = \begin{bmatrix} 0.002824 \\ 0.003725 \\ 0.004478 \\ 0.006373 \end{bmatrix}.$$

This vector of average returns will be used as the long-term mean, $\underline{\delta}$, of the VAR(1) model defined in Equation (2.4).

We used the least squares method from the MTS package in R software to estimate $\underline{\Phi}$ and $\underline{\Sigma}$ from the centered rates (see, for example, Tsay (2013, page 44)). The resulting estimates of these two matrices are:

$$\underline{\Phi} = \begin{bmatrix} 0.881060 & 0.227592 & -0.139965 & -0.000313 \\ 0.010075 & 0.942233 & 0.040185 & -0.000281 \\ 0.018911 & -0.020933 & 0.996979 & -0.000595 \\ -0.440212 & -2.362043 & 3.692235 & 0.158024 \end{bmatrix},$$

$$\underline{\Sigma} = \begin{bmatrix} 8.189273 & 3.756605 & 1.365394 & -121.1468 \\ 3.756605 & 5.579131 & 3.525399 & -69.27558 \\ 1.365394 & 3.525399 & 3.159911 & -0.358753 \\ -121.1468 & -69.27558 & -0.358753 & 170769.9 \end{bmatrix} \cdot 10^{-8}.$$

According to Theorem 2.1, our estimated VAR(1) model is stationary since the eigenvalues of $\underline{\Phi}$ are all between -1 and 1:

$$\underline{\lambda} = \begin{bmatrix} 0.9903095, & 0.9257134, & 0.9025893, & 0.1596843 \end{bmatrix}.$$

Note that we will need to add the corresponding long-term means back when using the calibrated VAR(1) model to project rates of return into the future.

Our goal here is to better understand TBP designs. In particular, we are interested in answering the following questions: (1) can the plan afford to pay the target benefits each year; (2) are the benefits large enough to provide retirees an adequate lifestyle; (3) are the benefits stable over time, and (4) what level of intergenerational equity is achieved by the plan? Before using our estimated VAR(1) model to study pension plans, we analyze some properties of the model in more details and check the consistency of these properties between the projected and the historical data sets.

First, we consider a 25-year projection of monthly rates of return generated by the model. The projections are obtained by applying Equation (2.5) recursively using a random number generator for the $\underline{\varepsilon}$'s at future times t=1,2, ..., 300 and the unique Cholesky decomposition of the positive-definite matrix $\underline{\Sigma}$ (Tsay, 2013, page 6), which is

$$\underline{\Sigma}^{1/2} = \begin{bmatrix} 2.861691 & 0 & 0 & 0 \\ 1.312722 & 1.963642 & 0 & 0 \\ 0.4771286 & 1.476370 & 0.8675204 & 0 \\ -39.18899 & -9.080735 & 36.59390 & 409.6492 \end{bmatrix} \cdot 10^{-4}.$$

As our starting vector, we use

$$\underline{\delta}_0 = \begin{bmatrix} 0.000387 \\ 0.000552 \\ 0.001615 \\ 0.004669 \end{bmatrix}$$

corresponding to the last known rates of return as of the month of February 2016.

This gives us 300 simulated rates of return for the months of March 2016 to February 2041. Then we compute some statistics for these projected rates and compare them with historical values.

Table 2.3 compares the means and standard deviations of the monthly historical rates with the projected rates generated by our fitted VAR(1) model. We observe that the average simulated returns for the three bonds are lower than the corresponding historical averages. This is due to the fact that the process starts at current values ($\underline{\delta}_0$) that are low and slowly reverts back to the long-term mean values. Given that the eigenvalues are fairly close to 1, one does expect this multivariate process to take a fairly long time to reach its mean-reverting level. The standard deviations over the projected 300 months are close to the corresponding historical ones.

Table 2.3: Moments of a 25-year projection of returns on asset compared to historical values

	Mean		**Standard Deviation**	
	Historical	Simulated	Historical	Simulated
SB	0.002824	0.002185	0.001848	0.001078
MB	0.003725	0.002628	0.001875	0.001092
LB	0.004478	0.003441	0.001807	0.000998
EQ	0.006373	0.004821	0.041933	0.039617

Second, to get some idea as to whether 300 monthly observations is a sufficiently large data set to estimate the VAR(1) model, we fit the same process, same method, to our 25-year projection of monthly rates of return. The resulting estimates, $\underline{\Phi}^{Sim}$ and $\underline{\Sigma}^{Sim}$ are:

$$\underline{\Phi}^{Sim} = \begin{bmatrix} 0.888649 & 0.168007 & -0.108618 & 0.000019 \\ -0.020020 & 0.953315 & 0.047609 & -0.000077 \\ -0.004245 & 0.000549 & 0.989990 & -0.000608 \\ -6.895796 & 6.063103 & 12.417005 & 0.065122 \end{bmatrix}$$

and

$$\underline{\Sigma}^{Sim} = \begin{bmatrix} 7.974085 & 3.508963 & 1.135781 & -161.0740 \\ 3.508963 & 5.044218 & 3.047245 & -43.89384 \\ 1.135781 & 3.047245 & 2.681872 & 26.58753 \\ -161.0740 & -43.89384 & 26.58753 & 149727.8 \end{bmatrix} \cdot 10^{-8}.$$

The dynamics of a VAR(1) model are driven by the eigenvalues of its coefficient matrix, $\underline{\Phi}$, and the Cholesky decomposition of its covariance matrix, $\underline{\Sigma}$. For example, the expected value of the process at a future time t can be expressed in terms of the eigenvalues of $\underline{\Phi}$. The eigenvalues of $\underline{\Phi}^{Sim}$, $\underline{\lambda}^{Sim}$, and the Cholesky decomposition of $\underline{\Sigma}^{Sim}$, $(\underline{\Sigma}^{Sim})^{1/2}$ are:

$$\underline{\lambda}^{Sim} = \begin{bmatrix} 0.9866788, & 0.9188677 + 0.0490099i, & 0.9188677 - 0.0490099i, & 0.0726614 \end{bmatrix}$$

and

$$(\underline{\Sigma}^{Sim})^{1/2} = \begin{bmatrix} 2.823842 & 0 & 0 & 0 \\ 1.242620 & 1.870859 & 0 & 0 \\ 0.4022111 & 1.361647 & 0.8160977 & 0 \\ -57.04073 & 14.42445 & 36.62421 & 380.6898 \end{bmatrix} \cdot 10^{-4}.$$

Based on the above values, we can see that the eigenvalues and the Cholesky decompositions under both the simulated and the historical data sets are comparable. We generated many 25-year projections and found that the fitted VAR(1) models were fairly similar, that is, they resulted in comparable eigenvalues, $\underline{\lambda}^{Sim}$, and Cholesky decompositions, $(\underline{\Sigma}^{Sim})^{1/2}$. We then conclude that 300 monthly observations constitute a large enough sample to adequately fit our financial data.

2.4 Number of Simulations

To minimize the computation time while still using enough representative scenarios for our four asset classes, we need to determine a reasonable number of simulations. This section explains how we determined a reasonable number of simulations for our asset portfolio. We come up with five steps to figure this number out and they are as follows.

Step 1: We first use the estimated $\underline{\Phi}$ and $\underline{\Sigma}^{\frac{1}{2}}$ to generate 100 scenarios of monthly forces of return over a 25-year period. Then we calculate the average X_A of the 100 generated returns at time 25 for each time series. For example, $\overline{X}_{EQ}$ is the mean of the 100 values, at time 25, for the monthly forces of total return on the TSX. We chose 25 years as the horizon because the theoretical conditional means are fairly close to the mean-reverting level for each asset class, see Table 2.4.

Table 2.4: Theoretical conditional means of return on assets at 25 years from current time

	Theoretical Mean	Mean-reversion Level
SB	0.002696	0.002824
MB	0.003564	0.003725
LB	0.004315	0.004478
EQ	0.006177	0.006373

Step 2: Repeat ***Step 1*** 50 times. Then calculate the standard deviation ($s_{\overline{X}_A}$) of these 50 sample means for each asset class. Since the values of the 100 scenarios at a given time are independent under each asset class, we can obtain an estimation of the population standard deviation of each asset class in our asset portfolio by using σ_A=($s_{\overline{X}_A}$)$\cdot\sqrt{100}$.

Step 3: Set the tolerance for each asset class. We chose 1 basis point for the monthly rates of return on Canadian zero-coupon bonds and 10 basis points for the monthly rates of total return on the TSX. Using

$$\text{tolerance}_A = \frac{\sigma_A}{\sqrt{N_A}} = \frac{(s_{\overline{X}_A}) \cdot \sqrt{100}}{\sqrt{N_A}},$$

we obtain four estimates, N_{SB}, N_{MB}, N_{LB} and N_{EQ}, for reasonable numbers of simulations for our four asset classes.

Step 4: Record N, the largest value of N_A from ***Step 3***, that is $N = \max(N_{SB}, N_{MB}, N_{LB}, N_{EQ})$. This represents one estimate of a reasonable number of simulations for our asset portfolio.

Step 5: Repeat ***Steps 1-4*** 200 times. We then have 200 independent estimates of N. Based on the distribution of these 200 values of N, we can decide on a reasonable number of simulations for our asset side. Looking at the distribution in Figure 2.4, we arbitrarily choose a conservative number of simulations equal to 5000.

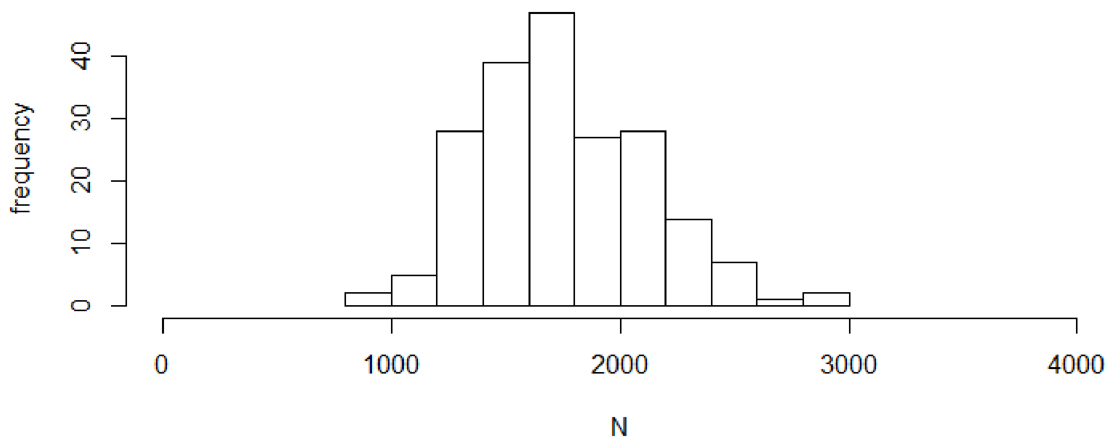

Figure 2.4: Histogram of 200 values of N based on 50 independent simulations and our tolerance criteria

Lastly, we use the fitted VAR(1) model again to generate 5000 scenarios of 25-year projection of monthly rates of return. This will give us 5000 paths of 300 simulated rates of return in our asset portfolio for the months of March 2016 to February 2041. Then we compare the first two conditional moments between the simulated results and the theoretical values at two specific times, 5 and 10 years. Current rates of return are known when valuing a pension plan, so it is more appropriate to compare conditional probabilities. The summary statistics are shown in Tables 2.5 and 2.6.

Table 2.5: Comparison between simulated and theoretical values of return on assets, 5 years from current time

	5 years from current time			
	Mean		**Standard Deviation**	
	Theoretical	Simulated	Theoretical	Simulated
SB	0.001482	0.001508	0.001364	0.001368
MB	0.002051	0.002085	0.001243	0.001229
LM	0.002799	0.002827	0.001096	0.001083
EQ	0.004389	0.004327	0.041895	0.040660

Table 2.6: Comparison between simulated and theoretical values of return on assets, 10 years from current time

	10 years from current time			
	Mean		**Standard Deviation**	
	Theoretical	Simulated	Theoretical	Simulated
SB	0.002087	0.002116	0.001462	0.001465
MB	0.002793	0.002836	0.001405	0.001407
LM	0.003539	0.003582	0.001278	0.001281
EQ	0.005244	0.006020	0.041902	0.041941

We can see that the first two moments of the 5000 paths are fairly consistent with the theoretical values.

2.5 Investment Strategy and Portfolio Returns

A quantity of interest in our simulation is the realized portfolio return during each valuation year. This return is a combination of the returns on each of the four underlying assets, and depends on the asset mix. In this project, we assume that the asset mix fixed at plan inception and the asset portfolio is rebalanced annually.

The proportion of the portfolio allocated to each asset class is denoted by w_A where $A \in \{SB, MB, LB, EQ\}$. We assume that 60% of assets are invested in Canadian equities (risk-seeking assets) and 40% in fixed income (matching assets). Therefore we have $w_{EQ} = 0.6$ and $w_{SB} + w_{MB} + w_{LB} = 0.4$. The proportion in each bond (SB, MB, LB) is determined by duration matching with the liability side, as described in Section 3.4. We now turn to the construction of the annual returns.

Since the maturity of short-term zero-coupon bonds is 3 months, new ones are assumed to be purchased four times within each year. The annualized instantaneous rate of return on short-term bonds is therefore based on 4 different yields.

Regarding the other two zero-coupon bonds (medium-term and long-term), their maturities are both greater than one year. For simplicity, we assume that both the medium-term and the long-term bonds are sold at their book values at the end of each year, and are replaced by new medium-term and long-term bonds purchased at market value, as projected by our VAR(1) model. Hence, the annualized instantaneous rate of return on medium-term and long-term bonds is just the yield rate applicable at the end of each prior year.

For equities, our VAR(1) model generates monthly returns, so the annual instantaneous rate of return on the equity portion of the portfolio is based on the 12 successive monthly rates of return within each year.

Denoting the annualized instantaneous rate of return on our portfolio during the period [t,t+1) by $\delta^{y}_{P,t}$, we have:

$$\begin{aligned} e^{\delta^{y}_{P,t}} =& w_{SB} \cdot e^{3\cdot(\delta_{SB,t}+\delta_{SB,t+3/12}+\delta_{SB,t+6/12}+\delta_{SB,t+9/12})} + w_{MB} \cdot e^{12\cdot\delta_{MB,t}} \\ &+ w_{LB} \cdot e^{12\cdot\delta_{LB,t}} + w_{EQ} \cdot e^{\sum_{s=0}^{11} \delta_{EQ,t+s/12}}. \end{aligned} \tag{2.6}$$

Chapter 3

Liability Model

In the previous chapter, we discussed our asset model. We introduce our liability model in this chapter, including our notation and key assumptions. We describe our TBP design in detail, including formula for the target benefit, the affordability test, the triggers and actions, and the contribution rate. We identify the sources of the benefit changes under our selected TBP design and derive formulas for each source of changes. Finally, we discuss how we connect our asset model and liability model.

3.1 Model Notation and Assumptions

We divide the notation and assumptions used in our liability model into three groups: plan demographics, financial factors, and others.

3.1.1 Plan Demographics

We introduce the following notation:

- e: age at entry;
- r: age at retirement;
- n_x: number of members aged x in the plan;
- ω: limiting age.

We make the following assumptions:

- The membership of the plan is mature and stationary, implying that n_x is time-invariant;
- All members have the same entry age (e=25) and retirement age (r=65);
- 100 new members enter the plan each year (n_e=100);

- There are no decrements before retirement (n_x=100 for $e \leq x \leq r$);
- All retired members are subject to deterministic mortality following the 2014 Canadian Pensioners Mortality Tables (combined) for males, without mortality improvements (reproduced in Appendix B);
- ω=116.

3.1.2 Financial Factors

- f: fixed annual inflation rate, set at 2%;
- m: annual rate of salary increase for promotion and merit, set at 0.5%;
- s: total annual salary increases, equal to 2.51%, obtained from $(1+s) = (1+m)\cdot(1+f)$;
- ep: fixed annual expenses, expressed as a percentage of plan assets, set at 0.5%;
- i_t^{gross}: gross investment return during the period [t,t+1);
- i_t: investment return during the period [t,t+1) net of expenses, obtained from $i_t = i_t^{gross} - ep$;
- j_t: annual valuation rate applicable at time t; used to discount all future obligations in the valuation performed at time t;
- v_t: one-year discount factor applicable in the valuation performed at time t, equal to $\dfrac{1}{1+j_t}$;
- $S_{x,t}$: salary over the period $[t, t+1)$ for a member aged x at time t. The starting salary, $S_{e,0}$ is assumed to be \$50,000. Therefore, different cohorts at the same time have $S_{x+1,t} = S_{x,t} \cdot (1+m)$ relationship, same aged cohorts at different times have $S_{x,t+1} = S_{x,t} \cdot (1+f)$ relationship, and same cohort at different times has $S_{x+1,t+1} = S_{x,t} \cdot (1+s)$ relationship;
- $PCE_{x,t}$: past career earnings (known) for a member aged x at time t:

$$PCE_{x,t} = \begin{cases} S_{e,t-(x-e)} + \cdots + S_{x-1,t-1}, & \text{for } e < x \leq r-1 \\ S_{e,t-(x-e)} + \cdots + S_{r-1,t-(x-r+1)}, & \text{for } x \geq r \end{cases};$$

Note: a member aged e does not have past career earnings yet.

- $FCE_{x,t}$: future career earnings (projected based on salary at time t) for a member aged x at time t:

$$FCE_{x,t} = \begin{cases} S_{x,t} + S_{x,t} \cdot (1+s) + \cdots + S_{x,t} \cdot (1+s)^{r-1-x}, & \text{for } e \leq x \leq r-1 \\ 0, & \text{for } x \geq r \end{cases};$$

  Note: retired members do not have any future career earnings.

- $CE_{x,t}$: projected career earnings at retirement for a member aged x at time t, that is

$$CE_{x,t} = PCE_{x,t} + FCE_{x,t}.$$

3.1.3 Other Notation Related to the Operation of our TBP

- α_0: the target annual accrual rate applicable to all service (accrued and future service), set at 1% in our study;

- α_t: the accrual rate applicable to all service as of time t for all members, reflecting plan experience up to and including time t;

- ρ_t: adjustment factor as of time t, that is $\rho_t = \frac{\alpha_t}{\alpha_0}$;

- β_t: year-over-year difference factor as of time t, that is $\beta_t = \alpha_t - \alpha_{t-1}$;

- γ_t: year-over-year percentage change in accrual rate as of time t, that is $\gamma_t = \frac{\alpha_t - \alpha_{t-1}}{\alpha_{t-1}}$;

- $B^T_{x,t}$: target pension benefit at retirement for a member aged x at time t, that is

$$B^T_{x,t} = \alpha_0 \cdot CE_{x,t};$$

- $B_{x,t}$: projected pension benefit at retirement for a member aged x at time t based on plan experience up to and including time t, that is

$$B_{x,t} = \alpha_t \cdot CE_{x,t} = \rho_t \cdot B^T_{x,t};$$

- $\ddot{a}_x(t)$: present value of a whole life annuity-due for a member aged x at time t, based on the valuation assumption at time t, that is

$$\ddot{a}_x(t) = 1 + \frac{1}{1+j_t} \cdot {}_1p_x + \frac{1}{(1+j_t)^2} \cdot {}_2p_x + \cdots = \sum_{k=0}^{\omega-x} v_t^k \cdot {}_kp_x;$$

- ${}_{n|}\ddot{a}_x(t)$: present value of an n-year deferred whole life annuity-due for a member aged x at time t, based on the valuation assumption at time t, that is

$$_{n|}\ddot{a}_x(t) = v_t^n \cdot {}_n p_x \cdot \Big(\sum_{k=0}^{\omega-(x+n)} v_t^k \cdot {}_k p_{x+n} \Big);$$

- $\ddot{a}^s_{x:\overline{n|}}(t)$: present value of a salary-based n-year temporary life annuity-due for a member aged x at time t, based on the valuation assumption at time t, that is

$$\ddot{a}^s_{x:\overline{n|}}(t) = \sum_{k=0}^{n-1} \Big(\frac{1+s}{1+j_t}\Big)^k \cdot {}_k p_x;$$

- $PVFSal_{x,t}$: present value at time t of all future salaries for an active member aged x at time t, that is

$$PVFSal_{x,t} = S_{x,t} \cdot \ddot{a}^s_{x:\overline{r-x|}}(t);$$

- $TPVFSal_t$: total actuarial present value at time t of all future salaries for the entire plan, that is

$$TPVFSal_t = \sum_{x=e}^{r-1} n_x \cdot PVFSal_{x,t};$$

- $PVAB_{x,t}$: actuarial present value at time t of benefits accrued by a member who is aged x at time t:

$$PVAB_{x,t} = \begin{cases} \alpha_t \cdot PCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x(t), & \text{for } x \le r-1 \\ \alpha_t \cdot PCE_{x,t} \cdot \ddot{a}_x(t), & \text{for } x \ge r \end{cases};$$

- $TPVAB_t^T$: total actuarial present value at time t of targeted benefits in respect of past service for all members at time t:

$$TPVAB_t^T = \alpha_0 \cdot \Big(\sum_{x=e+1}^{r-1} PCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x(t) \cdot n_x + \sum_{x=r}^{\omega} PCE_{x,t} \cdot \ddot{a}_x(t) \cdot n_x \Big);$$

- $TPVBPCE_t$: total actuarial present value at time t for the entire plan if pension benefit equals past career earnings for each employee:

$$TPVBPCE_t = \sum_{x=e+1}^{r-1} PCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x(t) \cdot n_x + \sum_{x=r}^{\omega} PCE_{x,t} \cdot \ddot{a}_x(t) \cdot n_x = \frac{TPVAB_t^T}{\alpha_0};$$

- $PVFB_{x,t}$: actuarial present value at time t of benefits to be accrued in respect of future service for a member who is aged x at time t, that is

$$PVFB_{x,t} = \begin{cases} \alpha_t \cdot FCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x(t), & \text{for } x \leq r-1 \\ 0, & \text{for } x \geq r \end{cases};$$

- $TPVFB_t^T$: total actuarial present value at time t of targeted benefits in respect of future service for all members at time t:

$$TPVFB_t^T = \alpha_0 \cdot \sum_{x=e}^{r-1} FCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x(t) \cdot n_x;$$

- $TPVBFCE_t$: total actuarial present value at time t for the entire plan if pension benefit equals future career earnings for each employee:

$$TPVBFCE_t = \sum_{x=e}^{r-1} FCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x(t) \cdot n_x = \frac{TPVFB_t^T}{\alpha_0};$$

- $PVB_{x,t}$: actuarial present value at time t of pension benefits (including all past and future service) for a member aged x at time t, that is

$$PVB_{x,t} = PVAB_{x,t} + PVFB_{x,t};$$

- $TPVB_t^T$: total actuarial present value at time t of the target benefits (including all past and future service) for all members at time t, that is

$$TPVB_t^T = TPVAB_t^T + TPVFB_t^T;$$

- $TPVBCE_t$: total actuarial present value at time t for the entire plan if pension benefit equals career earnings (including past and future career earnings) for each employee, that is

$$TPVBCE_t = TPVBPCE_t + TPVBFCE_t = \frac{TPVB_t^T}{\alpha_0};$$

- U: fixed contribution rate, expressed as a percentage of salary;

- C_t: total contributions received by the plan at time t (at the beginning of the year), that is

$$C_t = U \cdot \sum_{x=e}^{r-1} n_x \cdot S_{x,t};$$

- B_t: total benefit payments made from the plan at time t (at the beginning of the year), that is

$$B_t = \sum_{x=r}^{\omega} n_x \cdot B_{x,t};$$

- F_t: total fund value at time t (just before receiving C_t and paying B_t), that is

$$F_t = (F_{t-1} + C_{t-1} - B_{t-1}) \cdot (1 + i_{t-1}).$$

3.2 Aggregate Cost Method and Plan Provisions

In Chapter 1, we briefly introduced the four elements of a TBP design: target benefit, affordability test, triggers and actions, and contribution rate. Here, we discuss our specific TBP design in detail.

3.2.1 Target Benefit

The formula for the target benefit commencing at retirement under our TBP design is simple: it is expressed as the product of the target accrual rate and actual total career earnings at retirement, that is

$$B_{r,t}^T = \alpha_0 \cdot CE_{r,t}. \tag{3.1}$$

We can see that members in the same cohort have the same target benefit, but different cohorts have different target benefits as their career earnings are different.

The target benefit remains level in nominal terms after retirement. Prior to retirement, the target is updated to reflect salary experience; however, in our model salary experience always follows our assumption, so this "update" has no impact. The target is never affected by the plan's investment experience, which may differ from assumed investment returns.

3.2.2 Affordability Test

At each valuation, we test whether the accrual rate determined at the previous valuation continues to be sufficient. The affordability test in our study uses the aggregate cost method. Based on Aitken (2010), we suggest the following for the aggregate cost method.

Definition 3.1. *Under the aggregate cost method, the present value of future normal costs* *equals the difference between the present value of all pension benefits and the fund value at the valuation date. In other words, benefits not yet covered by existing funds will be financed by future normal costs of existing members. When the normal cost is expressed as a level percentage of payroll, we have:*

$$TNC_t^{old} = \frac{\alpha_{t-1} \cdot TPVBCE_t \, - \, F_t}{TPVFSal_t}; \tag{3.2}$$

where α_{t-1} is the accrual rate determined in the last valuation (time $t-1$) and TNC_t^{old} is the total normal cost for the entire plan as a percentage of payroll, right before benefit adjustment at time t.

Note that,

$$TNC_t^{old} = \frac{\alpha_{t-1} \cdot (TPVBPCE_t + TPVBFCE_t) - F_t}{TPVFSal_t}.$$

Since the contribution rate (U) is fixed at plan inception, we will compare U with TNC_t^{old} at each valuation. If TNC_t^{old} is exactly equal to U, the benefits of all cohorts are deemed to be affordable at time t. If TNC_t^{old} is higher than the contribution commitment (U), then the benefits are not affordable. If TNC_t^{old} is lower than the contribution rate (U), then the benefits are more than affordable. The affordability test is applied annually, but whether action is taken will depend on the triggers.

3.2.3 Triggers and Actions

In our study, we apply a single trigger for the actions. If TNC_t^{old} is not equal to U, the accrual rate is immediately adjusted to restore equality between future costs (TNC_t^{old}) and the contribution commitment (U). The adjusted accrual rate applies to both past and future service of every member, whether retired or active. Consequently, after each benefit adjustment we have:

$$U = TNC_t = \frac{\alpha_t \cdot (TPVBPCE_t + TPVBFCE_t) - F_t}{TPVFSal_t};$$

where TNC_t is the total normal cost as a proportion of payroll, right after benefit adjustment at time t, based on the new accrual rate α_t.

We solve for the adjusted accrual rate:

$$\begin{aligned}\alpha_t &= \frac{U \cdot TPVFSal_t + F_t}{(TPVBPCE_t + TPVBFCE_t)} \\ &= \frac{U \cdot TPVFSal_t + F_t}{TPVBCE_t}.\end{aligned} \tag{3.3}$$

The numerator represents the total assets available to finance benefits (the current asset value F_t, plus the present value of future contribution commitment for members currently in the plan), and the denominator can be thought of as a spread factor depending on economic and demographic variables.

3.2.4 Contribution Rate

Under our TBP design, the contribution rate is fixed at plan inception and is equal to U; that is, the dollar amount of contributions paid each year is $U \cdot 100\%$ of then-current payroll. At plan inception, the contribution rate (U) is chosen to be consistent with the target accrual rate (α_0) under our affordability test. Consequently, we let U equal TNC_0^{old}, the total normal cost calculated under Equation (3.2) at plan inception. Note that the value of TNC_0^{old} depends on the starting asset value, F_0. For example, a low fund value F_0 leads to a high TNC_0^{old} and a large contribution rate. Out of the infinitely many possible combinations of TNC_0^{old} and F_0, we choose the one consistent with the entry age normal (EAN) cost method. Specifically, we set F_0 to be equal the actuarial liability determined under the EAN method at plan inception. Based on Aitken (2010), we have the following definitions.

Definition 3.2. *Under the entry age normal method, the actuarial liability for each member is the difference between the present value of the member's future benefits and the present value of the member's future normal costs (calculated at entry).*

Definition 3.3. *Under the entry age normal cost method, for each member, the present value at entry of the member's future normal costs is set equal to the present value of the member's future pension benefits at entry. Specifically, for a member entering the plan at time t; the normal cost expressed as a percentage of payroll is:*

$$NC_t = \frac{B_{e,t} \cdot {}_{r-e|}\ddot{a}_e(t)}{PVFSal_{e,t}} \tag{3.4}$$

$$= \frac{B_{e,t} \cdot v_t^{r-e} \cdot {}_{r-e}p_e \cdot \ddot{a}_r(t)}{S_{e,t} \cdot \ddot{a}^s_{e:\overline{r-e|}}(t)}. \tag{3.5}$$

The normal cost as a percentage of payroll remains the same throughout a member's career under the EAN method as long as (a) the actuarial assumptions do not change, and (b) experience is exactly in line with these assumptions. Since this is true for all cohorts present at plan inception under our TBP, each member has NC_0 as their individual normal cost at t=0.

We set the starting value fund, F_0, equal to the total EAN liabilities at plan inception. Since the normal cost rate is the same for all cohorts present at plan inception, we have:

$$F_0 = \sum_{x=e}^{\omega} n_x \cdot (PVB_{x,0} - NC_0 \cdot PVFSal_{x,0}) \tag{3.6}$$

where $PVFSal_{x,0}$=0 for $x \geq r$.

Then the total normal cost calculated as a percentage of payroll under the aggregate method at plan inception is:

$$\begin{aligned}
TNC_0^{old} &= \frac{\sum_{x=e}^{\omega} n_x \cdot PVB_{x,0} - F_0}{TPVFSal_0} \\
&= \frac{\sum_{x=e}^{\omega} n_x \cdot PVB_{x,0} - (\sum_{x=e}^{\omega} n_x \cdot (PVB_{x,0} - NC_0 \cdot PVFSal_{x,0}))}{TPVFSal_0} \\
&= \frac{\sum_{x=e}^{\omega} n_x \cdot PVB_{x,0} - \sum_{x=e}^{\omega} n_x \cdot PVB_{x,0} + \sum_{x=e}^{\omega} NC_0 \cdot n_x \cdot PVFSal_{x,0}}{TPVFSal_0} \\
&= \frac{\sum_{x=e}^{\omega} NC_0 \cdot n_x \cdot PVFSal_{x,0}}{\sum_{x=e}^{r-1} n_x \cdot PVFSal_{x,0}} \\
&= NC_0
\end{aligned}$$

since members over age r do not have salaries any more.

We let

$$U = TNC_0^{old}. \tag{3.7}$$

Thus, setting the initial asset value (F_0) equal to the EAN liability produces a fixed contribution rate (U) for our target benefit plan that is consistent with the level normal cost under the EAN method. As long as the plan experience matches our assumptions, the normal cost under the EAN method does not change and the contribution commitment, U, will be sufficient to afford the target, at least for members who are already in the plan.

3.3 Sources of Benefit Risk in TBP Under Aggregate Cost Method

In our model, as in reality, experience may differ from our assumptions, giving rise to gains and losses which result in immediate benefit adjustments. In this section, we explore the resulting benefit risk.

Generally, two common sources of risks in pension plans are changes in the valuation rate and the investment return. If the valuation rates are not the same at two consecutive valuation dates, the plan will experience a gain or loss on account of the change in assumptions. If the assumed investment return is not same as the actual one, the plan would also face a gain or loss. In our TBP design, new entrants generate additional gains or losses which also need to be accounted for. Specifically, new entrants cause α_t to differ from α_{t-1} even if the experience is exactly in line with the valuation assumptions.

Our goal is to allocate the total change in the accrual rate between time $t-1$ and time t among the three sources named above: changes in the valuation rate, investment experience, and new entrants. Table 3.1 traces the evolution of α_{t-1} to α_t through three intermediate steps (α_t^*, α_t^{**}, α_t^{***}).

Table 3.1: Accrual rate from time $t-1$ to t

Accrual rate	Valuation date	Valuation rate	Investment return during $[t-1, t)$	Members included
α_{t-1}	$t-1$	j_{t-1}	j_{t-1}	aged from e to ω at time $t-1$
α_t^*	t	j_{t-1}	j_{t-1}	aged from e to ω at time $t-1$
α_t^{**}	t	j_{t-1}	j_{t-1}	aged from e to ω at time t
α_t^{***}	t	j_{t-1}	i_{t-1}	aged from e to ω at time t
α_t	t	j_t	i_{t-1}	aged from e to ω at time t

The accrual rate in effect at time $t-1$ is α_{t-1}, given by

$$\alpha_{t-1} = \frac{U \cdot TPVFSal_{t-1} + F_{t-1}}{TPVBCE_{t-1}}. \tag{3.8}$$

The accrual rate that would be determined based on the affordability test at time t if the valuation rate did not change (j_t=j_{t-1}), the investment experience matched assumptions

(i_{t-1}=j_{t-1}) and there were no new entrants, would be α_t^*.

$$\alpha_t^* = \frac{U \cdot TPVFSal_t^* + F_t^*}{TPVBCE_t^*}, \tag{3.9}$$

where

- $TPVFSal_t^*$ is the total present value of future salaries as of time t in respect of members who were present in the plan at the previous valuation (that is, excluding new entrants at time t), based on j_{t-1};
- $TPVBCE_t^*$ is the total present value of the benefit base (projected career earnings at retirement for our plan) at time t, in respect of members who were present in the plan at the previous valuation, based on j_{t-1}; and
- F_t^* is the asset value at time t assuming investment returns during the period $[t-1,t)$ match the valuation rate in the last valuation.

In Appendix C, we show that $\alpha_t^* = \alpha_{t-1}$. That is, in the absence of new entrants and without gains/losses due to investment experience or assumption changes, the same accrual rate could be maintained in each consecutive valuation under our design.

If new entrants at time t were recognized in the affordability test performed at time t, the accrual rate would change to α_t^{**}, given by

$$\alpha_t^{**} = \frac{U \cdot TPVFSal_t^{**} + F_t^*}{TPVBCE_t^{**}}. \tag{3.10}$$

Note that $TPVFSal_t^{**}$ and $TPVBCE_t^{**}$ differ from $TPVFSal_t^*$ and $TPVBCE_t^*$, respectively, only by the inclusion of new entrants at time t. Also, note that α_t^{**} does not include the impact of possible changes in the valuation rate at time t, nor does it take into account actual investment experience over the past year.

By contrast, α_t^{***} does recognize the impact of investment experience differing from assumed over the period $[t-1,\, t)$, and is given by

$$\alpha_t^{***} = \frac{U \cdot TPVFSal_t^{**} + F_t}{TPVBCE_t^{**}}. \tag{3.11}$$

Finally, α_t reflects the impact of new members entering the plan at time t, the impact of actual investment experience, and the impact of changes in the valuation rate. It is given by

$$\alpha_t = \frac{U \cdot TPVFSal_t + F_t}{TPVBCE_t}. \tag{3.12}$$

Therefore, β_t can be written as:

$$\begin{aligned}\beta_t &= \alpha_t - \alpha_{t-1} \\ &= (\alpha_t - \alpha_t^{***}) + (\alpha_t^{***} - \alpha_t^{**}) + (\alpha_t^{**} - \alpha_t^{*}) + (\alpha_t^{*} - \alpha_{t-1}) \\ &= (\alpha_t - \alpha_t^{***}) + (\alpha_t^{***} - \alpha_t^{**}) + (\alpha_t^{**} - \alpha_t^{*}).\end{aligned}$$

That is, the total change in the accrual rate from time $t-1$ to time t can be allocated as follows:

- new entrants: the difference between α_t^{**} and α_t^{*};
- investment experience: the difference between α_t^{***} and α_t^{**};
- valuation rate changes: the difference between α_t and α_t^{***}.

Regarding the impact of new entrants on the accrual rate, we have:

$$\begin{aligned}\alpha_t^{*} \cdot TPVBCE_t^{*} &= U \cdot TPVFSal_t^{*} + F_t^{*} \\ \alpha_t^{*} \cdot \Big(TPVBCE_t^{**} - n_e \cdot CE_{e,t} \cdot {}_{r-e|}\ddot{a}_e(t-1)\Big) &= U \cdot \Big(TPVFSal_t^{**} - n_e \cdot S_{e,t} \cdot \ddot{a}^s_{e:\overline{r-e|}}(t-1)\Big) + F_t^{*} \\ \alpha_t^{*} \cdot \Big(1 - \frac{n_e \cdot CE_{e,t} \cdot {}_{r-e|}\ddot{a}_e(t-1)}{TPVBCE_t^{**}}\Big) &= \frac{U \cdot TPVFSal_t^{**} + F_t^{*} - U \cdot n_e \cdot S_{e,t} \cdot \ddot{a}^s_{e:\overline{r-e|}}(t-1)}{TPVBCE_t^{**}} \\ \alpha_t^{*} - \frac{\alpha_t^{*} \cdot n_e \cdot CE_{e,t} \cdot {}_{r-e|}\ddot{a}_e(t-1)}{TPVBCE_t^{**}} &= \alpha_t^{**} - \frac{U \cdot n_e \cdot S_{e,t} \cdot \ddot{a}^s_{e:\overline{r-e|}}(t-1)}{TPVBCE_t^{**}}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\alpha_t^{**} - \alpha_t^{*} &= \frac{n_e \cdot \Big(U \cdot S_{e,t} \cdot \ddot{a}^s_{e:\overline{r-e|}}(t-1) - \alpha_t^{*} \cdot CE_{e,t} \cdot {}_{r-e|}\ddot{a}_e(t-1)\Big)}{TPVBCE_t^{**}} \\ &= \frac{n_e \cdot \Big(U \cdot S_{e,t} \cdot \ddot{a}^s_{e:\overline{r-e|}}(t-1) - \alpha_{t-1} \cdot CE_{e,t} \cdot {}_{r-e|}\ddot{a}_e(t-1)\Big)}{TPVBCE_t^{**}}. \qquad (3.13)\end{aligned}$$

That is, the increase in the accrual rate on account of new entrants is the difference between the present value of contributions to be collected from these new cohorts and the present value of benefits that would be payable to them under the old accrual rate (α_{t-1}), spread over all members (including old and new) in the plan at time t .

Note that $\alpha_t^{**} = \alpha_t^{*}$ implies

$$U \cdot S_{e,t} \cdot \ddot{a}^s_{e:\overline{r-e|}}(t-1) = \alpha_{t-1} \cdot CE_{e,t} \cdot {}_{r-e|}\ddot{a}_e(t-1),$$

which is equivalent to

$$U \cdot S_{e,0} \cdot \ddot{a}^{s}_{e:\overline{r-e|}}(t-1) = \alpha_{t-1} \cdot CE_{e,0} \cdot {}_{r-e|}\ddot{a}_e(t-1).$$

Recall that the contribution rate U was determined at plan inception based on the target benefit α_0 and the valuation rate applicable at time 0, j_0:

$$U \cdot S_{e,0} \cdot \ddot{a}^{s}_{e:\overline{r-e|}}(0) = \alpha_0 \cdot CE_{e,0} \cdot {}_{r-e|}\ddot{a}_e(0).$$

If the valuation rate at the previous valuation (j_{t-1}) was the same as the valuation rate at inception (j_0) and the accrual rate determined in the last valuation (α_{t-1}) was exactly equal to the target (α_0) then new entrants' future contributions would exactly cover their future benefits (as valued at time t) and the accrual rate would not need to change (i.e, we would have $\alpha_t^{**} = \alpha_t^{*} = \alpha_{t-1}$). In most cases, this will not be true so the addition of new entrants will in fact result in a change in the accrual rate.

The impact of actual investment returns over the period $[t-1,t)$ is:

$$\begin{aligned}\alpha_t^{***} - \alpha_t^{**} =& \frac{U \cdot TPVFSal_t^{**} + (F_{t-1} + C_{t-1} - B_{t-1}) \cdot (1 + i_{t-1})}{TPVBCE_t^{**}} \\ &- \frac{U \cdot TPVFSal_t^{**} + (F_{t-1} + C_{t-1} - B_{t-1}) \cdot (1 + j_{t-1})}{TPVBCE_t^{**}} \\ =& \frac{(F_{t-1} + C_{t-1} - B_{t-1}) \cdot (i_{t-1} - j_{t-1})}{TPVBCE_t^{**}}. \end{aligned} \tag{3.14}$$

This impact is the difference between the actual fund value at time t and the assumed fund value at time t (i.e. the investment gain or loss) spread over a quantity related to the present value of all benefits at time t (still using j_{t-1}).

Looking at the impact of valuation rate changes on the accrual rate, we have:

$$\begin{aligned} \alpha_t^{***} \cdot TPVBCE_t^{**} &= U \cdot TPVFSal_t^{**} + F_t \\ \alpha_t^{***} \cdot \Big(TPVBCE_t - \triangle\, TPVBCE_t\Big) &= U \cdot \Big(TPVFSal_t - \triangle\, TPVFSal_t\Big) + F_t \\ \alpha_t^{***} \cdot \Big(1 - \frac{\triangle\, TPVBCE_t}{TPVBCE_t}\Big) &= \frac{U \cdot TPVFSal_t + F_t}{TPVBCE_t} - \frac{U \cdot \triangle\, TPVFSal_t}{TPVBCE_t} \\ \alpha_t^{***} - \alpha_t^{***} \cdot \frac{\triangle\, TPVBCE_t}{TPVBCE_t} &= \alpha_t - \frac{U \cdot \triangle\, TPVFSal_t}{TPVBCE_t} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\alpha_t - \alpha_t^{***} &= \frac{U\cdot \bigtriangleup TPVFSal_t}{TPVBCE_t} - \alpha_t^{***}\cdot \frac{\bigtriangleup TPVBCE_t}{TPVBCE_t} \\ &= \frac{U\cdot \bigtriangleup TPVFSal_t - \alpha_t^{***}\cdot \bigtriangleup TPVBCE_t}{TPVBCE_t},\end{aligned} \tag{3.15}$$

where $\bigtriangleup TPVBCE_t = TPVBCE_t - TPVBCE_t^{**}$ is the change in the spread factor and $\bigtriangleup TPVFSal_t = TPVFSal_t - TPVFSal_t^{**}$ is the change in the present value of future salaries on account of the change in the valuation rate. The impact of a change in the valuation rate from j_{t-1} to j_t will be the increase (decrease) in the present value of future contributions for all members less the increase (decrease) in the present value of future benefits (based on α_t^{***}), divided by the new spread factor.

3.4 Selecting the Bond Portfolio: Duration Matching

In Chapter 2, we set the allocation to Canadian equities at 60% of the entire asset portfolio and the allocation to Canadian fixed income at 40%. The equity portion is considered risk-seeking: its purpose is to harness returns in excess of inflation over the long term, allowing pension benefits to grow. The purpose of the fixed income portion is to create security for the pension benefits. We achieve this by matching duration of the liability side and the fixed income portion of the asset portfolio.

The fixed income assets in our asset portfolio are all zero-coupon bonds (with maturities of 3 months, 5 years and 15 years), whose modified durations are given by

$$ModD = \frac{\text{maturity}}{1 + \text{effective annual rate}}.$$

The modified duration of the fixed income portion of the asset portfolio is then:

$$ModD_{FI} = \Big(\omega_{SB}\cdot ModD_{SB} + \omega_{MB}\cdot ModD_{MB} + \omega_{LB}\cdot ModD_{LB}\Big)\cdot 0.4.$$

Most pension plans do not hold much short-term bonds so we arbitrarily set $w_{SB} = 4\%$ representing 10% of the fixed income portfolio.

Regarding the duration of liabilities side, we calculate its effective duration of the actuarial liability (AL) under the EAN method at plan inception. We chose the effective annual valuation rates equally spaced from 0.01% to 10%, then calculated the effective duration through 10 basis point increments. We also see that the AL is a fixed number once the aggregate method is implemented, so note that it is not meaningful to test sensitivity (that is to calculate duration) of the AL under the aggregate method, because the actuarial liability is always equal to the fund value, which is not sensitive to changes in the valuation rate. From Figure 3.1, we see that the effective duration of the liability side, based on the selected valuation rates, ranges from 8 to 15.

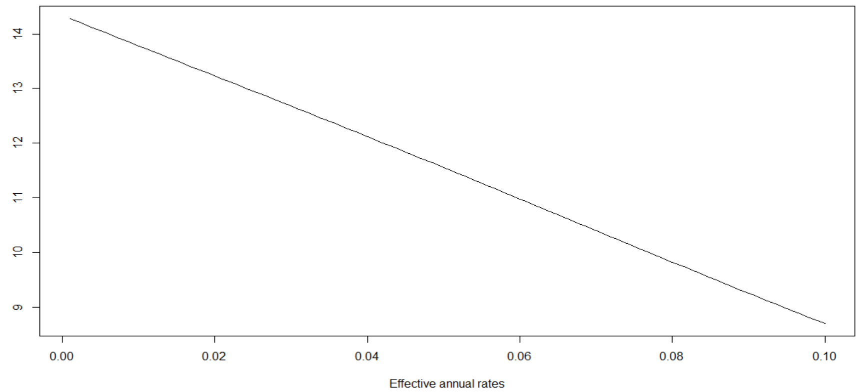

Figure 3.1: Effective duration of EAN liability

Table 3.2 displays four possible fixed income allocations based on duration matching under four different effective annual valuation rates. In order to make the allocation of each zero-coupon bond reasonable and do not let the annual valuation rate be greater than 4%, we chose the asset allocation based on the valuation rate between 3% and 4%. Rounding each allocation to the nearest percent. The resulting static asset mix is as follows:

$$\omega_{SB} = 4\%$$
$$\omega_{MB} = 3\%$$
$$\omega_{LB} = 33\%$$
$$\omega_{EQ} = 60\%.$$

Table 3.2: Fixed income allocation based on duration matching for four different valuation rate assumptions

Valuation rate	1%	2%	3%	4%
Modified duration	13.78	13.23	12.68	12.12
w_{SB} (3 months)	4.0%	4.0%	4.0%	4.0%
w_{MB} (5 years)	-1.6%	0.1%	1.9%	3.7%
w_{LB} (15 years)	37.6%	35.9%	34.1%	32.3%

Chapter 4

Performance Metrics and Simulation Results of the Base Case

After building the asset and the liability models in the previous two chapters, we can now study the operational characteristics of our TBP design by using a variety of performance metrics.

4.1 The Base Case

4.1.1 Valuation Rate

In this project, we assume that annual valuations are performed, allowing stakeholders to closely monitor the plan experience. Under the Base Case, the valuation rate applicable at a particular valuation date is equal to the annualized yield on 15-year zero-coupon bonds one month prior to the valuation date. For example, the annual valuation rate applicable at the end of February, 2016 (plan inception) equals the annual yield on 15-year zero-coupon bonds at the end of January, 2016.

Referring to the notation of Chapter 2, the valuation rate, j_t, at each annual valuation date can be defined under the Base Case as:

$$j_t = (e^{12 \cdot \delta_{LB,t-\frac{1}{12}}}) - 1. \tag{4.1}$$

Since the process $\{\delta_{LB,t}\}$ is mean-reverting, so is $\{j_t\}$. Note that our VAR(1) model from Chapter 2 can generate negative monthly yields for the selected three zero-coupon bonds. With the chosen parameters, the probability of having negative yields in any given month (under each zero-coupon bond) is less than 1%. In other words, under each zero-coupon bond, the number of scenarios with negative yield in any given month is less than 50, out of 5000. Hence, we set these negative rates to zero in order to make these yields more practical in Canada.

Figure 4.1 shows the trend in the annual effective valuation rate during valuation years 0 to 99. The top panel displays 6 basic statistics (median, 75th percentile, 25th percentile, maximum, minimum and mean) of the projected annual effective valuation rate in each year. The bottom panel is a magnified version of the top panel, cutting off the two extreme (maximum and minimum) lines. We can observe that the distribution of the valuation rate gradually drifts upward to be centered around a long-term mean, and becomes more stable after 20 years. The minimum line is approximately a straight line at 0% because most of the minimum valuation rates over the 99 valuation years are negative, adjusted to zero. The starting point of our valuation rates, 1.96%, is known as of the end of January, 2016. This rate is much lower than the long-term mean (5.52%). That is why the trend is significantly increasing during the first 20 years.

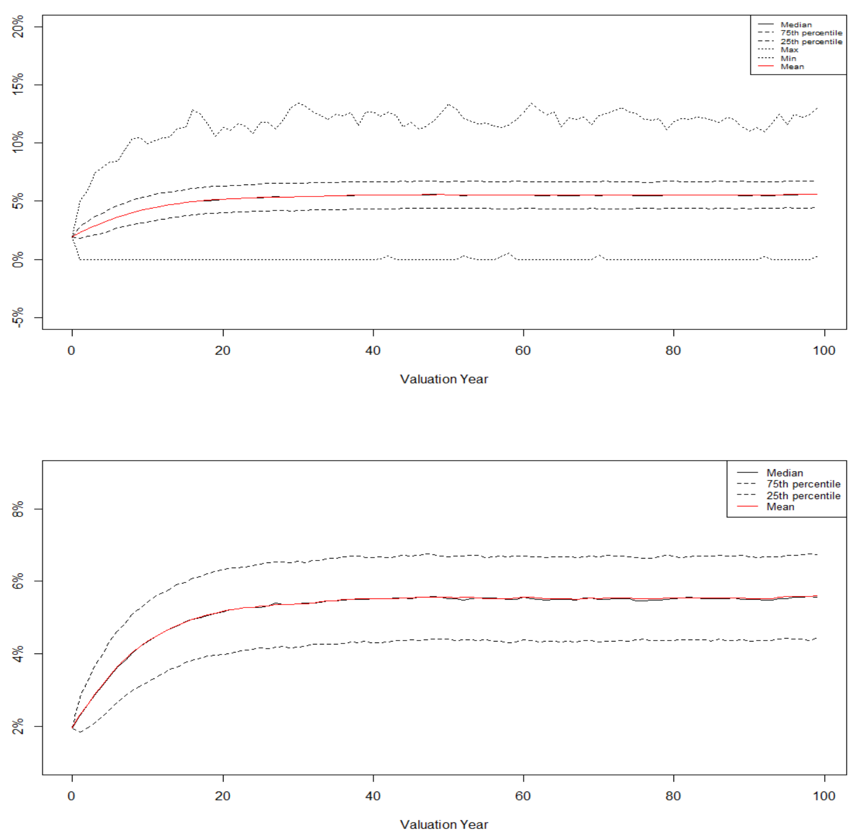

Figure 4.1: Mean, median and percentiles of annual effective valuation rates under the Base Case

There are three purposes for the valuation rate: to figure out the fixed contribution rate at plan inception (t=0) based on the target benefit; to figure out the initial fund value at plan inception based on the fixed contribution rate, and to adjust the actual pension benefit at each valuation based on the affordability test. Based on the valuation rate at plan inception (1.96%), the fixed contribution rate, U, is set at 12% using Equation (3.7), and the starting asset value, F_0, is set at \$799,114,071 using Equation (3.6).

4.1.2 Investment Return

The stylized asset portfolio consists of 4% in 3-month zero-coupon bonds, 3% in 5-year zero-coupon bonds, 33% in 15-year zero-coupon bonds and 60% in Canadian equities. The asset mix is obtained based on the method described in Chapter 3 and the effective annual investment return is obtained as follows (where the 50 basis points subtracted in the last term is for investment expenses):

$$\begin{aligned} i_t = {} & 4\% \cdot e^{3\cdot(\delta_{SB,t}+\delta_{SB,t+3/12}+\delta_{SB,t+6/12}+\delta_{SB,t+9/12})} + 3\% \cdot e^{12\cdot\delta_{MB,t}} \\ & + 33\% \cdot e^{12\cdot\delta_{LB,t}} + 60\% \cdot e^{\sum_{s=0}^{11}\delta_{EQ,t+s/12}} - 1 - 0.005. \end{aligned} \tag{4.2}$$

The rate from each component of this stylized asset portfolio is projected by the VAR(1) model in Chapter 2. As a result, the annual effective investment return, i_t, is also a mean-reverting process.

Figure 4.2 illustrates the distribution of the annual effective investment return over our simulation horizon under the Base Case. The trend of investment return is very similar to that of the valuation rate because the investment return consists of 40% bonds. The mean-reverting level of the investment return (6.37%) is higher than that of the valuation rate (5.52%), which indicates that the fund is likely to experience investment gains (see Figure 4.3).

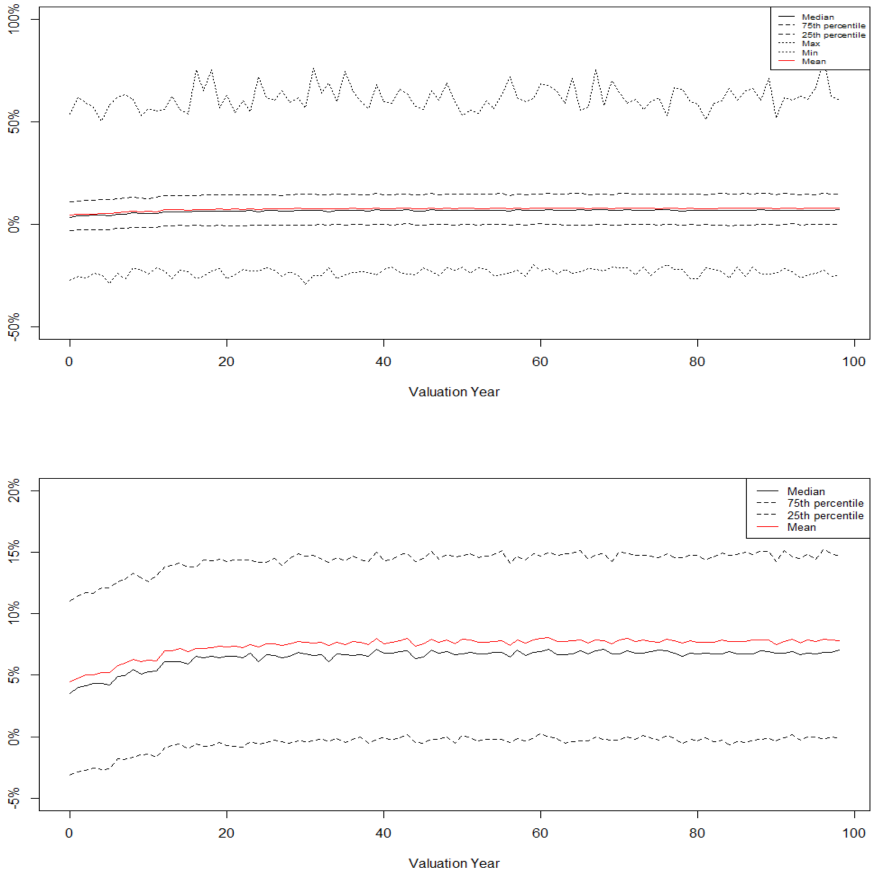

Figure 4.2: Mean, median and percentiles of annual effective investment returns under the Base Case

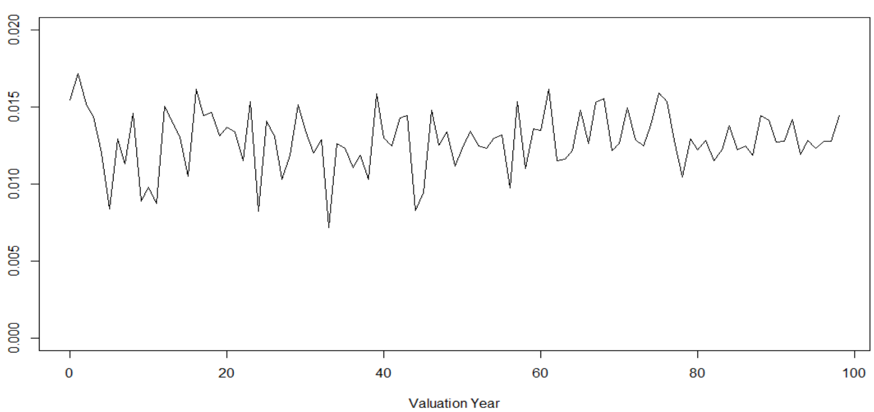

Figure 4.3: Median spread between annual investment return and annual valuation rate under the Base Case

4.1.3 Fund Value

Next, the distribution of fund value over our simulation horizon is generated. The top panel of Figure 4.4 shows the fund value over the 99-year horizon. We can see that the trend is increasing continuously without bound and it speeds up after 40 years. The bottom panel of Figure 4.4 focuses on the first 40 years and cuts off the two extreme lines.

The probability of ruin (the probability that the fund value is too small to pay all benefits due), is also calculated, and this probability is always zero at each year. Hence, this TBP design has a good feature: it does not allow paying high benefits when the fund cannot afford it. The fund value is increasing because of the 0.85% (0.85%=6.37%-5.52%) extra return and the impact of inflation.

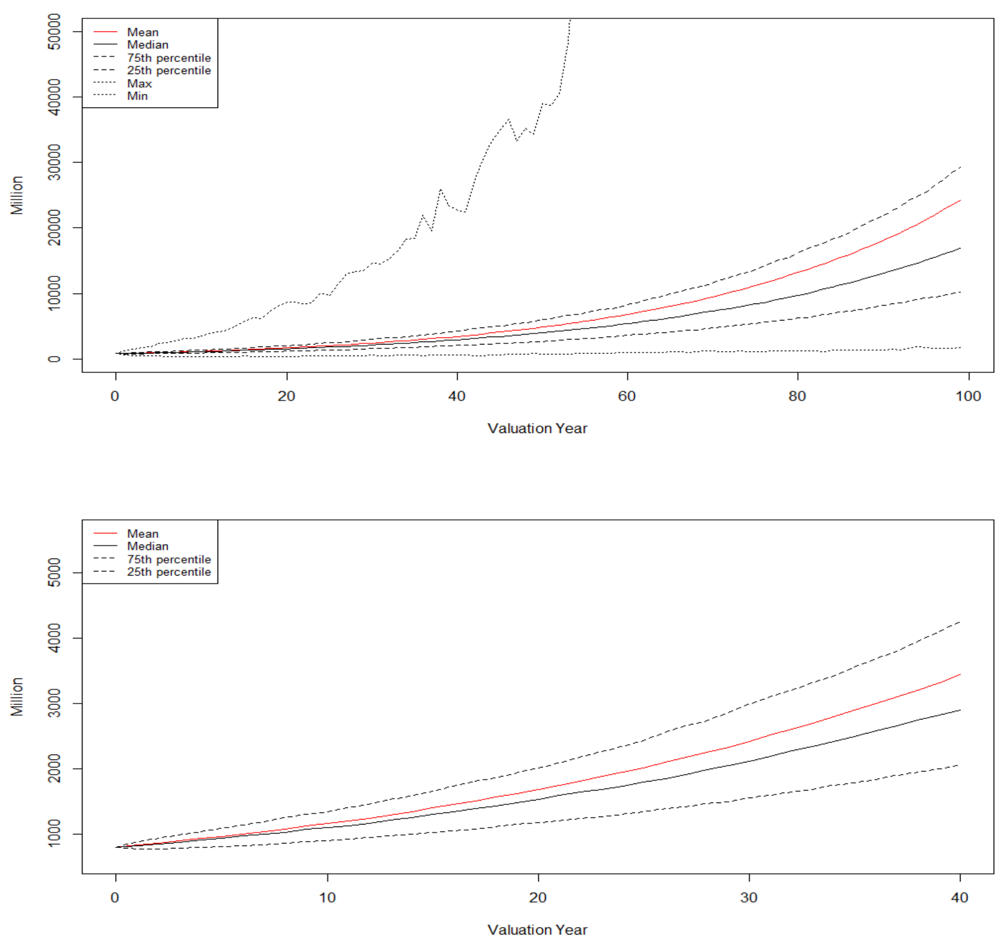

Figure 4.4: Mean, median and percentiles of the fund value under the Base Case

4.1.4 Projected Accrual Rate

Figure 4.5 illustrates the distribution of the projected accrual rate, α_t, (based on Equation (3.12)) during the 99-year horizon. Starting from a common (target) accrual rate of 1%, the accrual rate is expected to increase significantly over time with the median value reaching 2.0% after 20 years, 2.6% after 40 years, and 4.3% by the end of our horizon. These increases tend to be steepest during the first 20 years. Since later generations can get a much higher benefit, this design appears to be unfair for the members who join the plan early. Note that, in Canada, the maximum annual accrual rate is 2%, but the projected accrual rates are bigger than this upper limit most of the time.

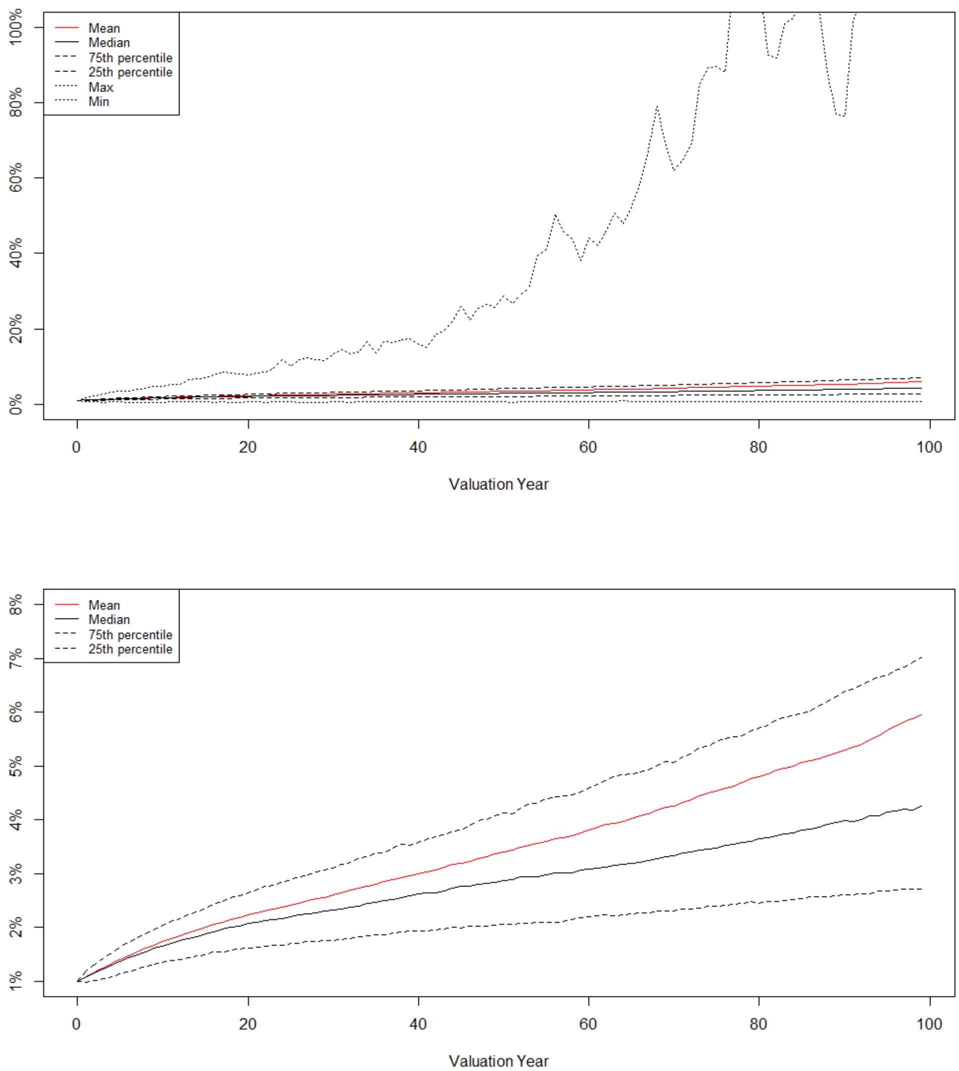

Figure 4.5: Mean, median and percentiles of projected accrual rates under the Base Case

Recall from Chapter 3 that the accrual rate is increased at a particular date if the total normal cost as a proportion of payroll, calculated based on the existing accrual rate, is lower than the fixed contribution rate, U. This total normal cost determined right before each benefit adjustment is shown in Figure 4.6. Note that the total normal cost rate after each benefit adjustment should be exactly equal to the fixed contribution rate, U, at each valuation.

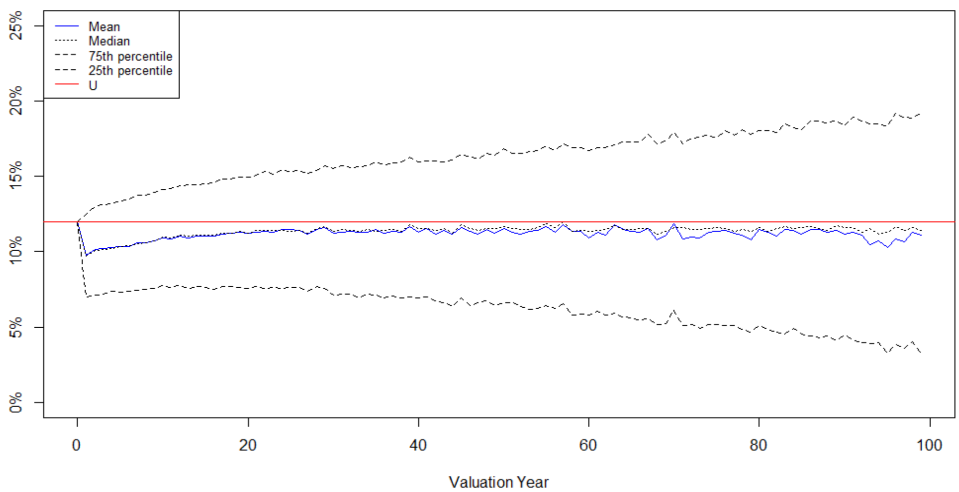

Figure 4.6: Mean, median and percentiles of the total normal cost, TNC_t^{old}, under the Base Case

The findings are consistent, both the mean and the median values of TNC_t^{old} are lower than 12% throughout the 99-year horizon, which implies that the pension benefit is more likely to be increased than decreased at each valuation. The biggest gap between the median value of TNC_t^{old} and the fixed contribution rate of 12% occur during the first 20 years, indicating strong gains experience during this time, resulting in rapid increases in the accrual rate. After 20 years, the mean and median value of TNC_t^{old} continue to be slightly below 12%, translating into continued increases in the accrual rate.

From Figures 4.5 and 4.6, we see that, on average, the pension benefits are greater than the target during the horizon, which is due to a combination of gains from investment returns and from a rising valuation rate.

4.1.5 Attribution of Changes in the Accrual Rate

In this subsection, we look in more detail at the sources of changes in the accrual rate: these being changes in the valuation rate, investment experience, and new entrants. Our focus is on γ_t, the year-over-year percentage change in the accrual rate. We divide the total change γ_t by α_{t-1}, among these three sources using the attributing framework developed in Section 3.3. Specifically we have:

$$\gamma_t = \gamma_t^{valrate} + \gamma_t^{invexp} + \gamma_t^{newent}, \tag{4.3}$$

where

$$\begin{aligned}
\gamma_t^{valrate} &= \frac{Impact\ of\ valuation\ rate\ changes}{\alpha_{t-1}} = \frac{\alpha_t - \alpha_t^{***}}{\alpha_{t-1}};\\
\gamma_t^{invexp} &= \frac{Impact\ of\ investment\ experience}{\alpha_{t-1}} = \frac{\alpha_t^{***} - \alpha_t^{**}}{\alpha_{t-1}};\\
\gamma_t^{newent} &= \frac{Impact\ of\ new\ entrants}{\alpha_{t-1}} = \frac{\alpha_t^{**} - \alpha_t^{*}}{\alpha_{t-1}} = \frac{\alpha_t^{**} - \alpha_{t-1}}{\alpha_{t-1}}.
\end{aligned}$$

The following questions will be answered: what is the relationship (positive or negative) between γ and each component; which component has the largest/smallest influence on γ, and how does the impact of each component change over time? We present three groups of scatter plots to help us answer the questions above. For each group, we chose two of the three components and used them as the x- and y-axes. This yielded three combinations which are shown in Figure 4.7 (valuation rate changes vs. investment experience), Figure 4.8 (new entrants vs. valuation rate changes), and Figure 4.9 (new entrants vs. investment experience). Within each group, we present six scatter plots corresponding to six different valuation years (T =1, 10, 20, 30, 50 and 99). Each scatter plot has 5000 datapoints, one for each of the 5000 simulation runs. The values of γ_t corresponding to each point in the scatter plot are coded as colours, with low, median and high values represented by red-orange, green and blue-purple dots, respectively. For example, the rightmost purple dot on the top-right panel (T=10) in Figure 4.7 indicates the value of γ_{10}, 0.750093, and its corresponding values of $\gamma_{10}^{valrate}$ and γ_{10}^{invexp} are 0.765855 and -0.008909, respectively. We can use these values to obtain the value of γ_{10}^{newent}, which is -0.006853. Notice that some plots (Figures 4.8 and 4.9) have a different scale for the x-axis.

Figure 4.7 can be used to trace changes in $\gamma^{valrate}$ through time. Close to inception, $\gamma^{valrate}$ has wider dispersion (values ranging from -25% to +50% at t=1), with large positive values (accounting for 25%-50% increases in the accrual rate) appearing relatively frequently. This is because the valuation rate (being linked to long-term bond yields) starts at historically low level and tends to increase rapidly in the early years towards its long-term mean. The relatively large increases in the valuation rate translate into large gains which give rise to sizeable increases in α_t in the early years. By t=20, the distribution of $\gamma^{valrate}$ narrows and becomes more closely centred around zero, meaning that increases and decreases in the accrual rate when accounting for changes in the valuation rate are equally likely. Figure 4.10 confirms that, while the difference between two consecutive valuation rates tends to be large and positive (approximately 0.3%) early on, this difference quickly diminishes (median value is near zero after 20 years). Figure 4.7 also shows positive correlation between $\gamma^{valrate}$ and γ at all times.

Figure 4.7 also provides much information about the impact of investment experience on the accrual rate, γ^{invexp}. Looking along the y-axis, we note that the distribution of γ^{invexp} is more narrow at inception (values in the range of -20% to +40%, with positive bias at t=1) but widens as time passes and stays more stable after about 20 years. This is because the numerator of Equation (3.14) does not change much while the denominator decreases significantly during the first 20 years. A slight positive bias remains even at the end of the projection horizon. This is reasonable, since investment gains are more likely than losses, given the bond-based valuation rate of the Base Case which deliberately underestimates the portfolio return each year. There is also a positive correlation between γ^{invexp} and γ visible in Figure 4.7 and, to a lesser extent, in Figure 4.9.

On the other hand, Figures 4.8 and 4.9 show that there is no relationship between γ and new entrants impact as the colorful dots are relatively symmetric around the vertical line, $x = 0$. Note that the new entrants impact is exactly zero at valuation year 1, which is consistent with Equation (3.13) of Section 3.3. After the first year, the impact of new entrants on the accrual rate falls mostly in the -1.0% to +1.0% range, which is negligible in relation to the other two components. The distribution of γ^{newent} changes subtly over the years. At t=10, the impact of new entrants has a positive bias as sharp increases in the valuation rate combined with investment gains make the new benefit level cheaper for new entrants than their corresponding contributions. The probability of a negative impact from new entrants increases slightly over time. By t=99, the once-positive bias turned negative.

Finally, in Figure 4.7, we see that there are more points with high values of γ to the right of x=0.25 than above y=0.25, which implies that the valuation rate changes have more impact on γ than the investment experience. We see that a change in the valuation rate has the largest influence on γ, then investment experience, and lastly new entrants which has barely no impact on γ.

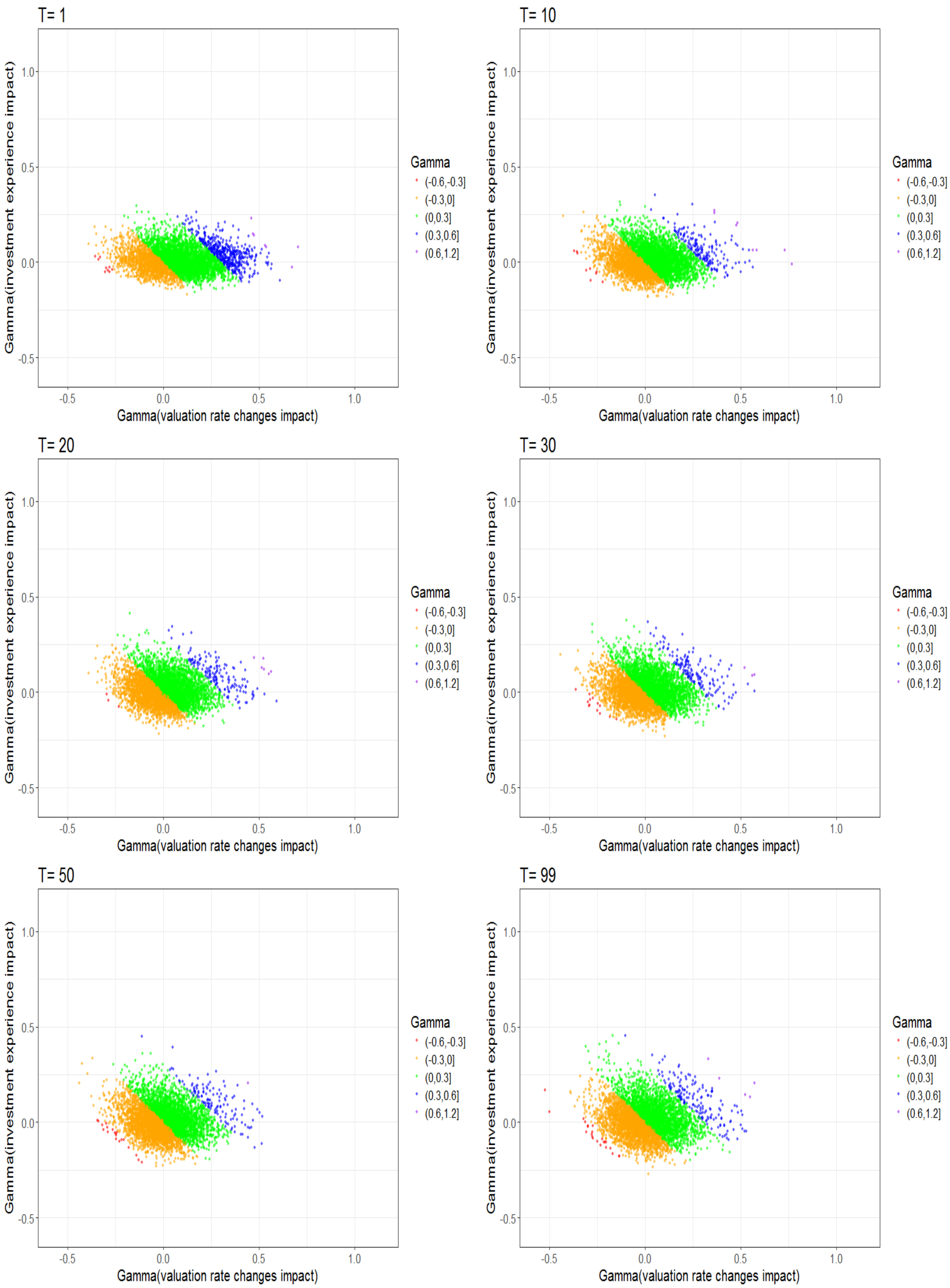

Figure 4.7: Relationship between γ and each component under the Base Case (x-axis: $\gamma^{valrate}$, y-axis: γ^{invexp})

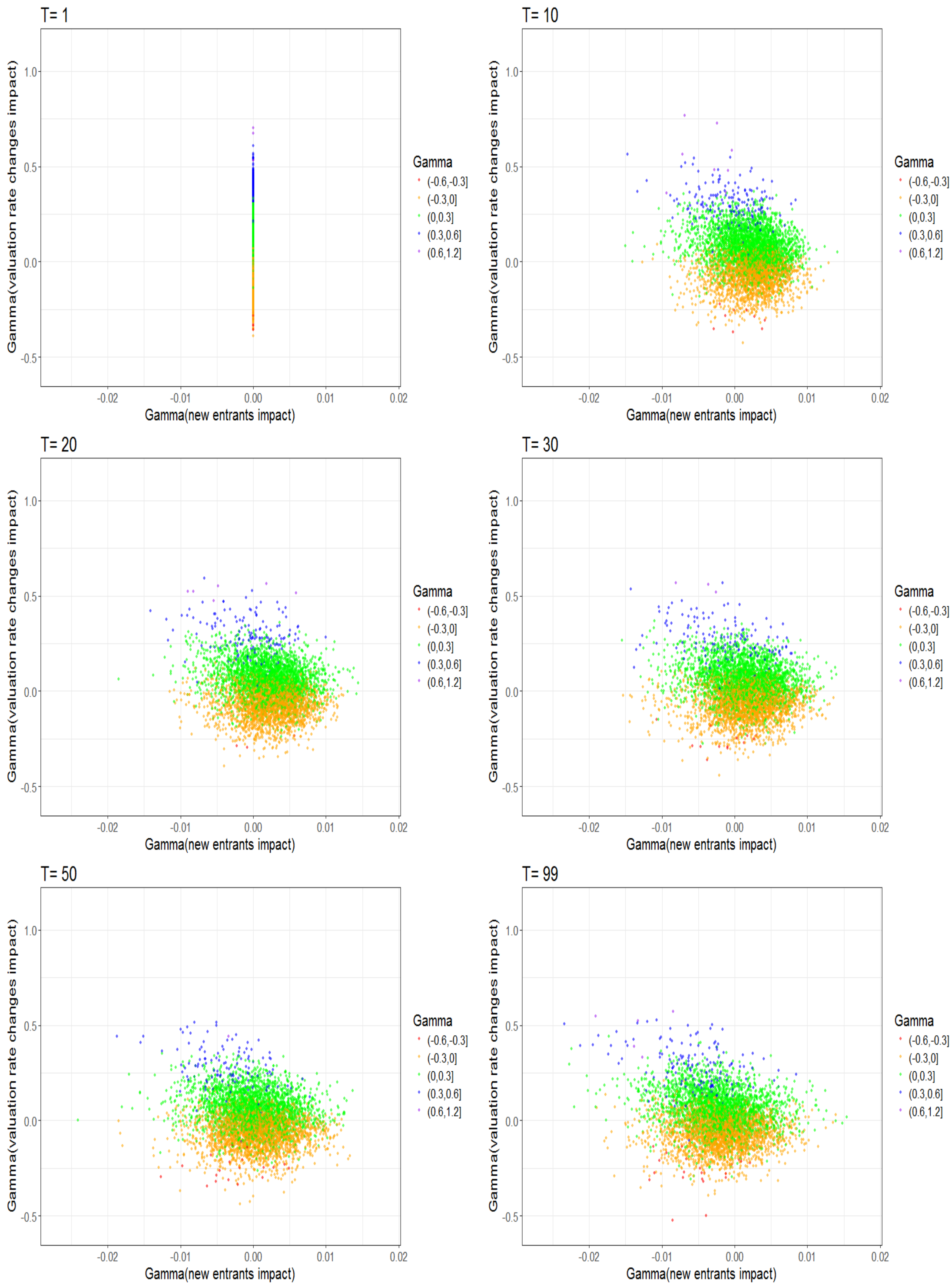

Figure 4.8: Relationship between γ and each component under the Base Case (x-axis: γ^{newent}, y-axis: $\gamma^{valrate}$)

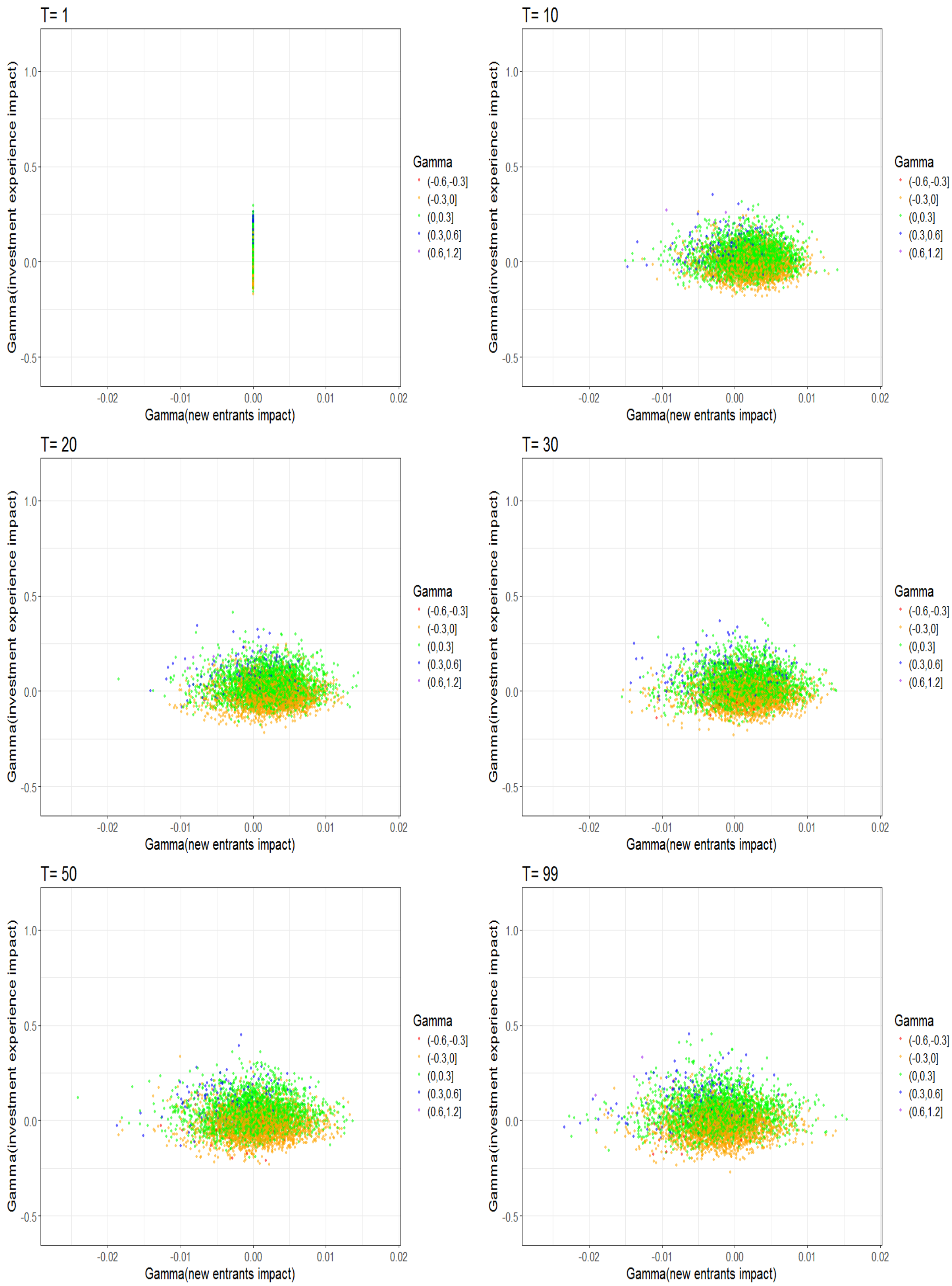

Figure 4.9: Relationship between γ and each component under the Base Case (x-axis: γ^{newent}, y-axis: γ^{invexp})

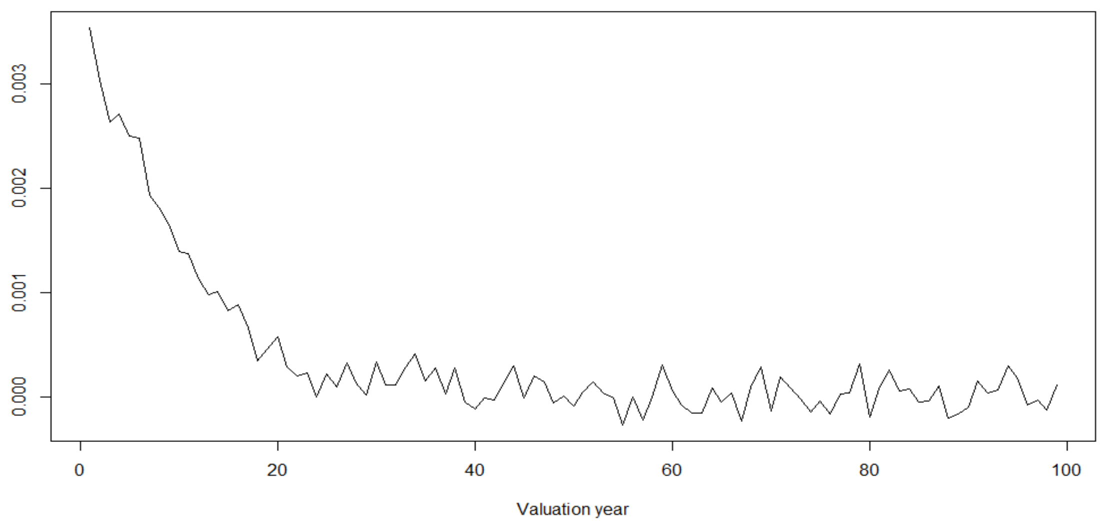

Figure 4.10: Median value of difference between two consecutive valuation rates under the Base Case

4.2 Performance Metrics

A pensioner in a TBP might have the following questions: Can the plan afford to pay the target benefit each year? Are the pension benefits large enough to provide me an adequate lifestyle? Are the pension benefits stable during my retirement years? And, are the pension benefits paid to me fair compared to other generations (past or future)? We use the following performance metrics to answer these questions.

Table 4.1: Performance metrics used to analyze a TBP

Objective	**Criteria for success**	**Performance metrics**
Benefit security	• The benefit meets the target in any given year; • The average benefit received in retirement (adjusted to take into account survival probabilities) meets the target.	1. Distribution of the factor by which benefits exceed or fall short of the target, by valuation year; 2. Probability that benefit falls below some specific proportions of the target, by valuation year; 3. Distribution of the ratio of weighted average benefit to the target, by cohort.
Benefit adequacy	• The pension benefits allow members to maintain their pre-retirement standard of living; • The pension benefits keep up with inflation after retirement.	4. Distribution of replacement ratio at retirement by cohort; 5. Distribution of weighted average replacement ratio by cohort; 6. Distribution of modified geometric average year-over-year change in pension benefit by cohort.
Benefit stability	• The pension benefits do not fluctuate much year-over-year.	7. Distribution of benefit adjustments by valuation year.
Inter-generational equity	• The pension benefits are fair among different generations.	8. Comparison of pension outcomes/ metrics among different cohorts, see metrics 3, 4, 5 and 6.

4.3 Performance Metrics of the Base Case

The seven performance metrics introduced in Table 4.1 are mainly based on those in Sanders (2016a). In 2014, Beetsma et al. proposed a methodology to assess intergenerational equity in pension plans, but we do not use this complex application in this project.

4.3.1 Benefit Security

Performance Metric 1 tracks the extent to which the benefit accrual rate in effect at each valuation date exceeds or falls short of the target, as a proportion of the target accrual rate:

$$PM_t^1 = \frac{\alpha_t}{\alpha^T} - 1, \qquad \text{for } 0 \leq t \leq 99. \tag{4.4}$$

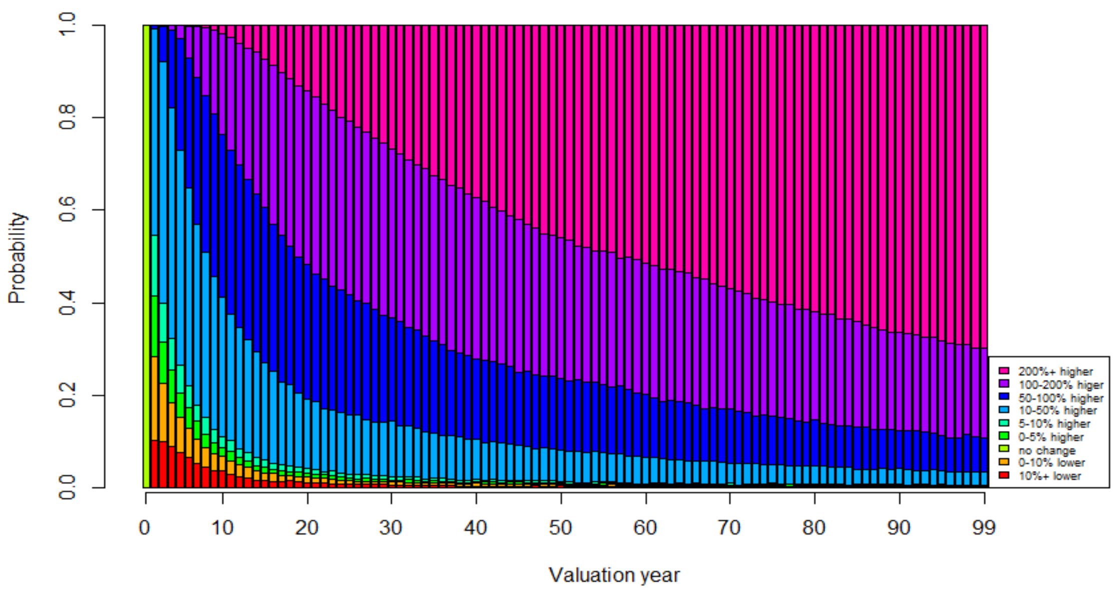

Figure 4.11: Distribution of ratio of benefit to target by valuation year under the Base Case

We break potential outcomes into 9 groups based on the size of the shortfall or excess relative to α^T. Figure 4.11 shows the probability associated with each group of outcomes at each valuation date. At plan inception (t=0), the benefit is set equal to the target. Figure 4.11 shows that the downside risk ($PM_t^1 < 0$) is reducing continuously and significantly for the first 20 years. This feature is consistent with that of the accrual rate (Figure 4.5). We also observe that the probability that the benefit exceeds the target is at least 70% in each year throughout the horizon and the probability that the benefit is at least twice the target is almost 50% after 20 years.

Performance Metric 2 calculates the probabilities of the benefit falling below 90%, 80% and 50% of the target, in each valuation year separately. Mathematically:

$$PM_t^2 = Pr(\alpha_t < C \cdot \alpha^T), \qquad \text{for } 1 \leq t \leq 99, \text{ and } C = 0.9, 0.8, 0.5. \tag{4.5}$$

This metric is a refinement of Metric 1 focusing on the downside risk. The three lines in Figure 4.12 represent the probabilities of the pension benefit falling below each specific proportion of the target. For example, the leftmost point of the solid line denotes a 10.26% probability that the benefit is below 90% of the target in valuation year 1. By comparison, the leftmost point of the dashed line corresponds to a 1.94% probability that the benefit is below 80% of the target in valuation year 1. Note that the dotted line, denoting the probability that the benefit is below half of the target during the horizon, remains at zero for most of our projection period. Figure 4.12 is consistent with Figure 4.11: downside risk decreases over time and quickly becomes negligible under the Base Case.

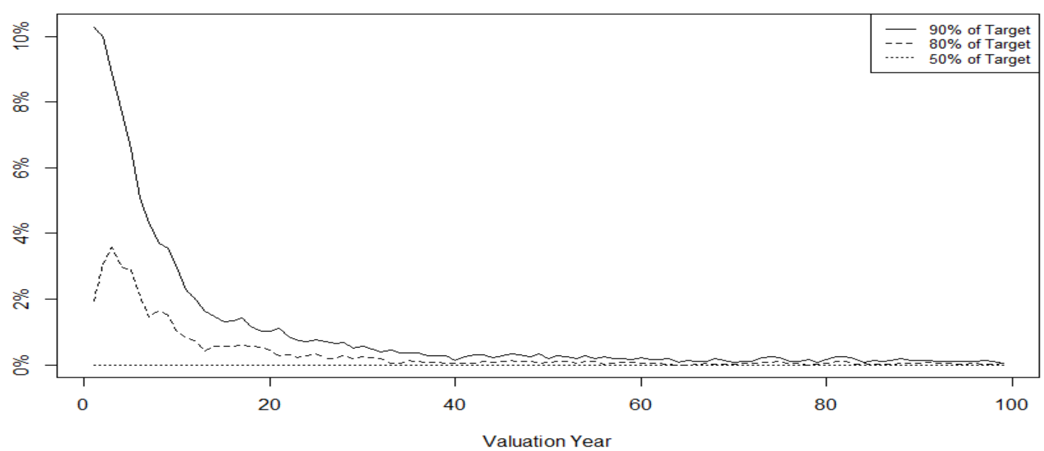

Figure 4.12: Probability of pension benefit falling below three specific proportions of the target during the valuation years under the Base Case

While Metric 2 captures the relationship between benefits and the target at each valuation date, Performance Metric 3 applies another perspective to see the cumulative impact of benefit adjustments over time. Performance Metric 3 calculates the ratio of the weighted average benefit to the target for each cohort of retirees. Here, the impact of all the pension payments after retirement is taken into account with each payment amount being weighted by its corresponding survival probability. This metric is calculated for 50 consecutive cohorts, from Cohort 0 (those who retire immediately at plan inception) to Cohort 49 (those who retire 49 years after plan inception). The weighted average benefit for each cohort is calculated as

$$WAP_t = \sum_{x=r}^{\omega} B_{x,t+x-r} \cdot \left(\frac{{}_{x-r}p_r}{\sum_{s=0}^{\omega-r} {}_s p_r} \right), \qquad t = 0, 1, \cdots, 49. \tag{4.6}$$

Hence, Performance Metric 3 is then given as:

$$PM_t^3 = \frac{WAP_t}{B_{r,t}^T}, \qquad t = 0, 1, \cdots, 49. \tag{4.7}$$

Figure 4.13 plots key percentiles of the distribution of outcomes for each cohort, and confirms that later cohorts tend to receive better pensions. This feature is consistent with that of Figure 4.5 which shows accrual rates increasing steadily after plan inception. Figure 4.13 also demonstrates that, on average, the weighted average benefits far exceed the target, even for the earlier cohorts.

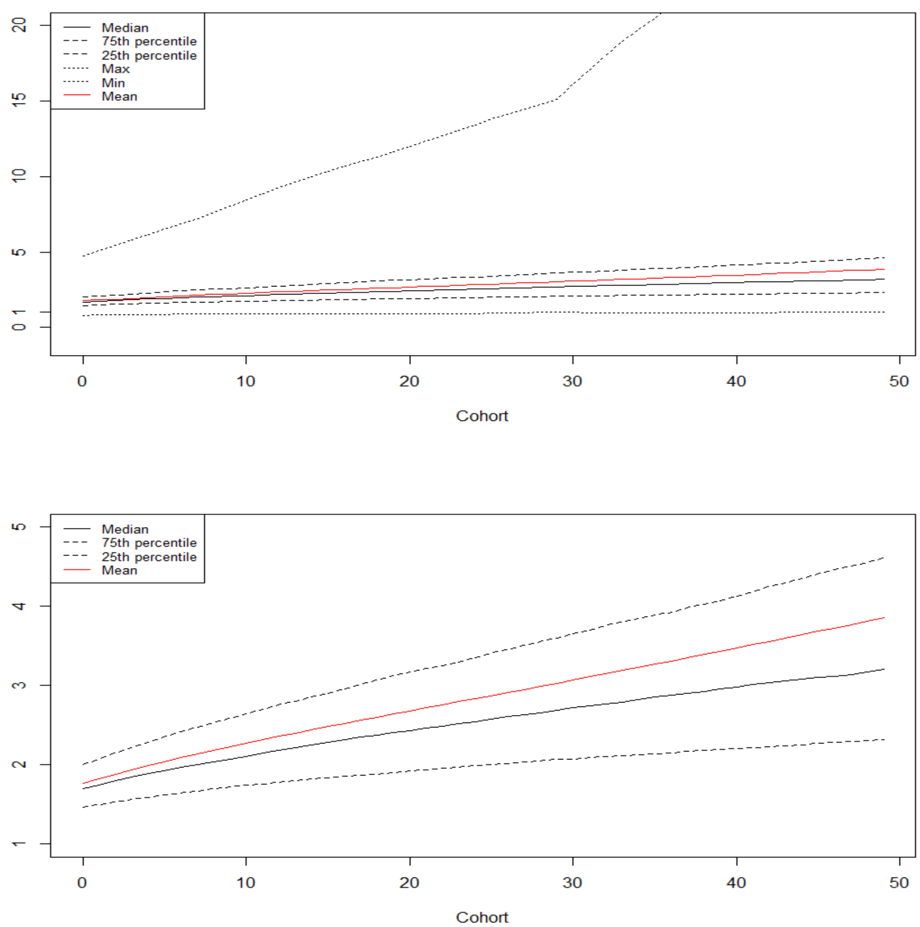

Figure 4.13: Mean, median and percentiles of the ratio of weighted average pension benefits to the target under the Base Case

4.3.2 Benefit Adequacy

Performance Metric 4 (see Figure 4.14) computes the replacement ratio at retirement for each cohort of retirees. The replacement ratio expresses the annual pension benefit payable to a new retiree as a proportion of his salary in the previous year:

$$PM_t^4 = \frac{B_{r,t}}{S_{r-1,t-1}}, \qquad t = 0, 1, \cdots, 49. \tag{4.8}$$

In general, a replacement ratio of 60%-70% is considered to be sufficient to maintain a member's lifestyle after retirement (MacDonald and Cairns, 2007).

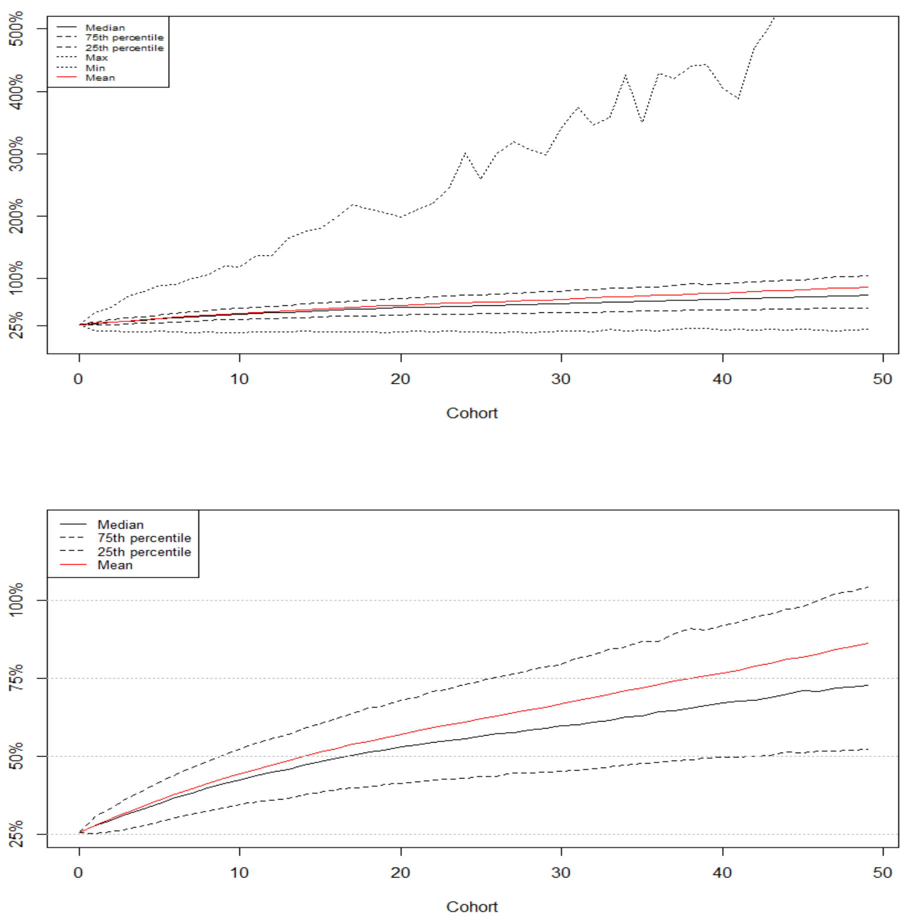

Figure 4.14: Mean, median and percentiles of the replacement ratio at retirement, by cohort, under the Base Case

Figure 4.14 plots key percentiles of the distribution of the replacement ratio for each cohort. We find that the distribution drifts upward over time. The replacement ratio at plan inception is relatively low (26%) but is expected to double in 20 years, indicating potential inequities between cohorts.

Performance Metric 5, the weighted average replacement ratio, is an enhanced version of Metric 4, applying the idea of the weighted average pension benefit used in Metric 3. Metric 4 is based on the pension benefit at retirement, while Metric 5 takes into account the impact of all the benefits after retirement for each cohort of retirees. However, inflation adjustment is not considered under this metric. Performance Metric 5 can be expressed as

$$PM_t^5 = \frac{WAP_t}{S_{r-1,t-1}} = \frac{\sum_{x=r}^{\omega} B_{x,t+x-r} \cdot \left(\frac{{}_{x-r}p_r}{\sum_{s=0}^{\omega-r} {}_s p_r} \right)}{S_{r-1,t-1}}, \qquad t = 0, 1, \cdots, 49. \tag{4.9}$$

Figure 4.15 shows the distribution of the weighted average replacement ratio, by cohort. The trend of weighed average replacement ratio is very similar to Figure 4.14, which again indicates that the later generations can receive much better pension benefits.

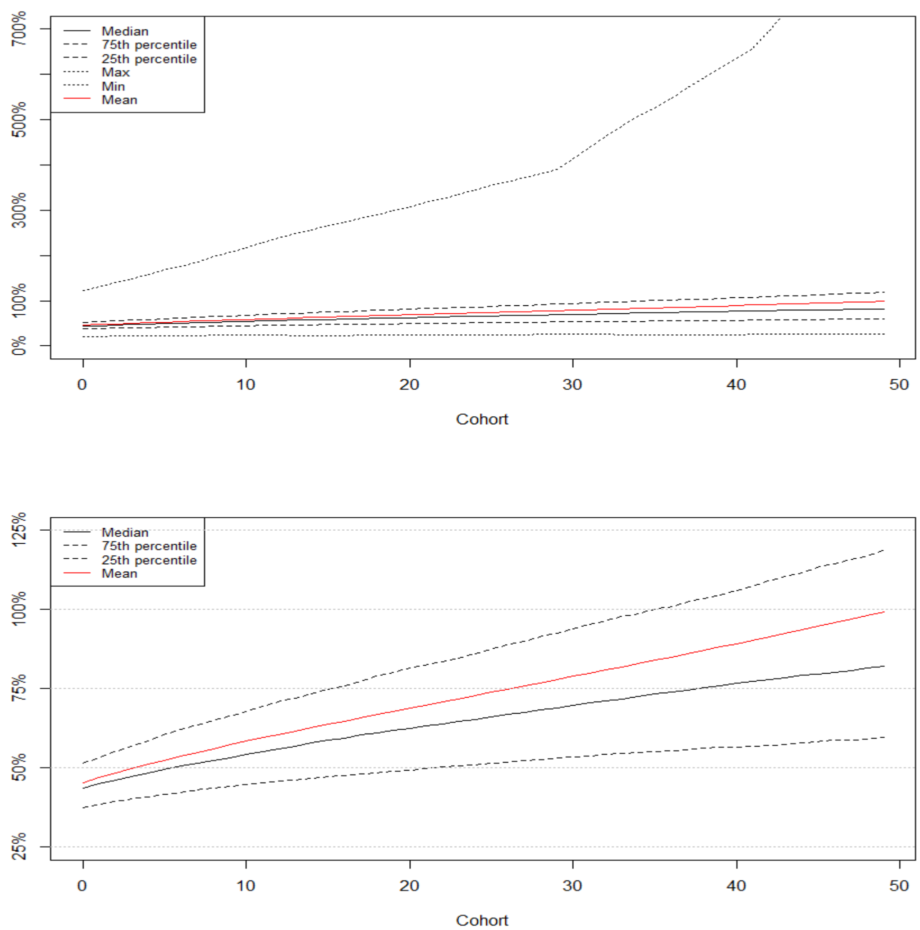

Figure 4.15: Mean, median and percentiles of the weighted average replacement ratio under the Base Case

Performance Metric 6 aims to capture the average rate of growth in the pension benefits received by a particular cohort of retirees, taking into account survival probabilities after retirement. The usual definition of geometric average growth rate (AGR) can be expressed as:

$$(1 + AGR)^{\omega-1-r} = \prod_{i=1}^{\omega-1-r} \left(\frac{B_{r+i,t+i}}{B_{r+i-1,t+i-1}} \right) = \frac{B_{\omega-1,t+\omega-1-r}}{B_{r,t}} \tag{4.10}$$

The main drawback of this geometric average growth rate is that it does not distinguish based on the time at which the growth is happening; two retirees who have exact same value of AGR can have totally different paths of pension benefits (e.g. low but sustained growth for one, no growth for the other until a large jump in the very last year). In order to capture the timing of pension benefit adjustment after retirement, we introduce the modified geometric average growth rate (MGR) which can be expressed as follows:

$$(1 + PM_t^6)^{\omega-1-r} = (1 + MGR)^{\omega-1-r} = \prod_{i=1}^{\omega-1-r} \left(\frac{B_{r+i,t+i}}{B_{r+i-1,t+i-1}} \cdot \left(\frac{{}_ip_r}{\sum_{s=1}^{\omega-r} {}_sp_r} \right) \right), \text{ for } 0 \leq t \leq 49. \tag{4.11}$$

Weighting growth by the probability of survival recognizes that increases in the early years are more useful than in the later years of retirement, while avoiding the use of an arbitrary discount factor. Figure 4.16 shows the distribution of the modified growth rate by cohort. Since the inflation rate is fixed (2% per year) in this project, this metric can also tell us whether the pension payments over each cohort's retirement years can keep up with the inflation if the survival probability after retirement is considered. From Figure 4.16, we can see that no cohort's pension payments over the retirement years can keep up with inflation. Moreover, negative values of the modified growth rate indicate that, on average, the pension benefit of the cohort decreases after retirement. We can also see that earlier cohorts have better results, which is due to the rapid increases in the accrual rate during the first 20 years.

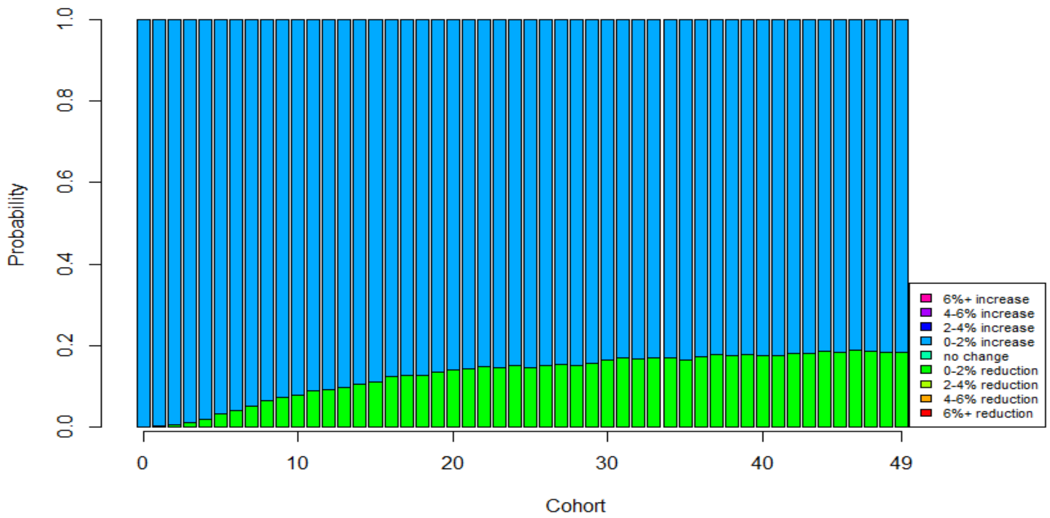

Figure 4.16: Distribution of modified geometric average growth rate by cohort under the Base Case

4.3.3 Benefit Stability

Performance Metric 7 looks at the distribution of adjustments applied to the accrual rate at each valuation. At each valuation, we calculate year-over-year percentage change in accrual rate in any given valuation date,

$$PM_t^7 = \frac{\alpha_t}{\alpha_{t-1}} - 1 = \gamma_t, \qquad t = 1, 2, \cdots, 99. \tag{4.12}$$

Then we group them into nine ranges: (20%+ higher), (10-20% higher), (2-10% higher), (0-2% higher), (exactly zero), (0-2% lower), (2-10% lower), (10-20% lower) and (20%+ lower). Pensioners do not want pension benefits to change a lot year to year. Also, from a psychological perspective, a pensioner prefers the trend of his benefits to be on the rise instead of fluctuating. Figure 4.17 summarizes the distribution of annual benefit adjustments. We see that the distribution becomes more stable after 20 years, with upside risk decreasing during the first 20 years because of the combined three impacts from Subsection 4.1.5. We also observe that the upside risk exceeds the downside risk each year, which is consistent with the fact that the accrual rate is likely to increase over time. Figure 4.17 also demonstrates that large benefit adjustments (+10%) are common in either direction, making pension benefits relatively volatile. This is not surprising, since only a single trigger is applied here. If two triggers for action are applied, such that there is a "no-action" range built by the two triggers, they could reduce the volatility of pension benefits.

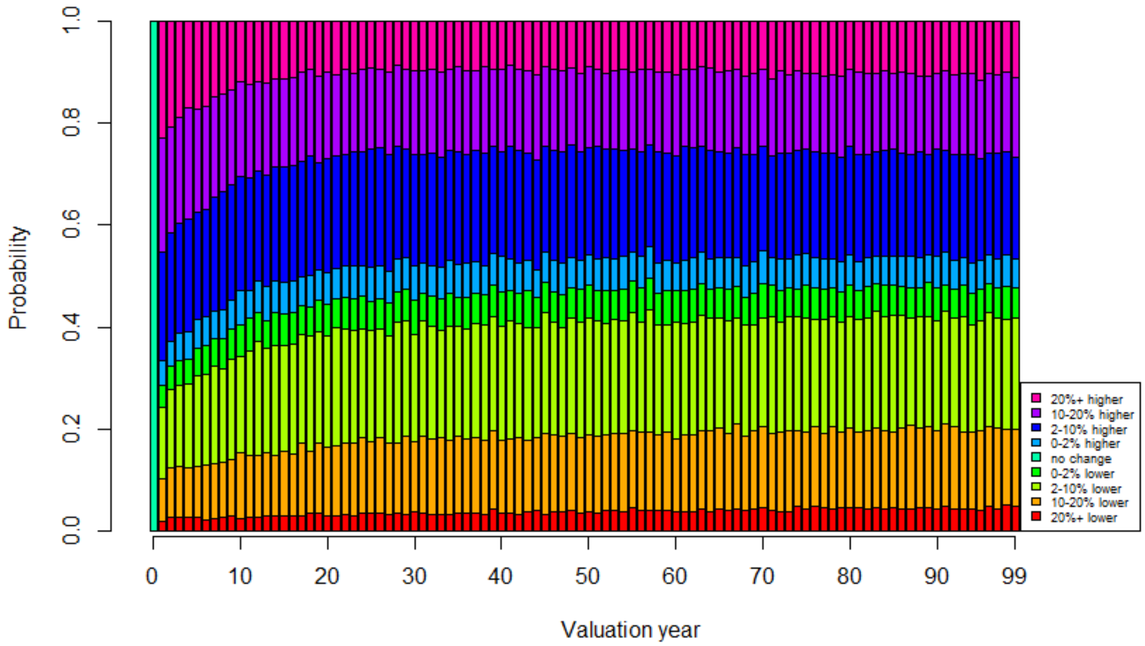

Figure 4.17: Distribution of benefit adjustments by valuation year under the Base Case

4.3.4 Summary of the Base Case

Based on the results presented in this section, we conclude that the Base Case meets the requirements of benefit security and benefit adequacy well. Since only a single trigger is applied, the benefits are unstable. The biggest issue under the Base Case is the significant difference between the outcomes for different cohorts: the later the members join the plan, the better pension benefits they can receive. There are three possible reasons for this: the fixed contribution rate at plan inception is overestimated because the valuation rate at plan inception is at historically low level, the investment return is consistently underestimated, or the design itself (using the aggregate method for the affordability test) is flawed, creating inequities over time. In the next chapter, we attempt to isolate the impact of each flaw.

Chapter 5

Modifications of the Base Case

In light of the significant generational differences we observed under the Base Case, we explore two modifications. Case 2 investigates the impact of starting the simulation exercise from economic conditions closer to historical average, instead of the current low-interest rate environment. Case 3 looks at the impact of changing from a bond-based valuation rate to one based on expected return.

5.1 Case 2

As mentioned in the previous chapter, one of the shortcomings of the Base Case is that the fixed contribution rate is set very high because current bond yields, on which the valuation rate at inception is based, are at a historically low level. Under the Base Case, the average valuation rate is projected to increase during the first 20 years because of the property of mean-reversion. In order to remove the impact of this upward trend, Case 2 follows the same approach as the Base Case, but uses a different starting point for simulation in the VAR(1) model, assuming that the starting values of the yields on bonds and the rate of return of Canadian equities are equal to their long-term means. Then our VAR(1) model is applied to project the rate of each asset class simultaneously at each month during the 99-year horizon. Hence, Case 2 can help us see the risk profile of the plan when there is no particular expected trend in the valuation rate.

Figure 5.1 illustrates the trend of annual effective valuation rate during valuation years 0 to 99. Unlike Figure 4.1, we observe that the distribution of the valuation rate is centered around a long-term mean since valuation year 0. This is due to the fact that the starting bond yields (on which the valuation rate is based) are equal to the long-term means. Furthermore, based on the valuation rate at plan inception (5.52%), the fixed contribution rate, U, is set at 4.05%, and the starting asset value, F_0, is set at $516,223,572.

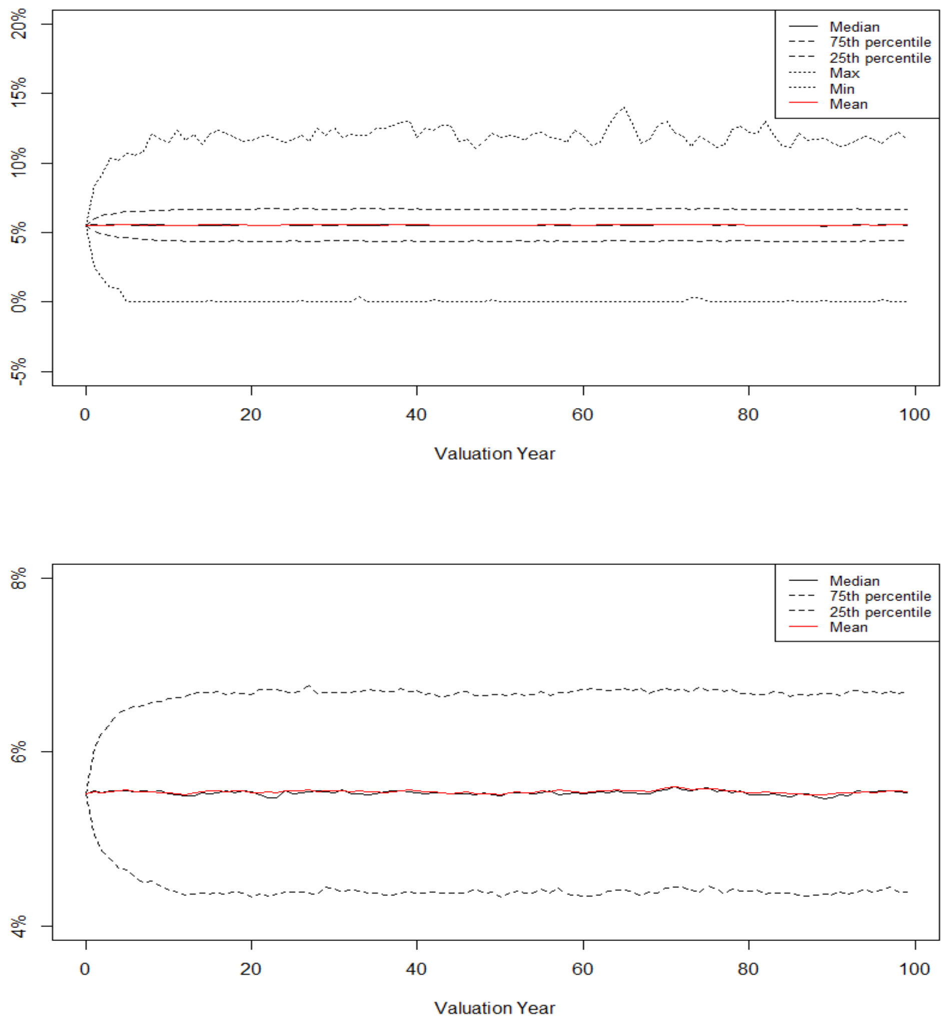

Figure 5.1: Mean, median and percentiles of annual effective valuation rates under Case 2

Figure 5.2 illustrates the distribution of the annual effective investment return over our simulation horizon under Case 2. Like the annual effective valuation rate in Figure 5.1, the annual effective investment return has a more stable distribution from plan inception. As under the Base Case, the fund is likely to experience investment gains as the long-term mean of the portfolio return (6.37%) is higher than that of the valuation rate (5.52%). The median spread is shown in Figure 5.3 and is very similar to that under the Base Case, falling between +10 and +15 basis points.

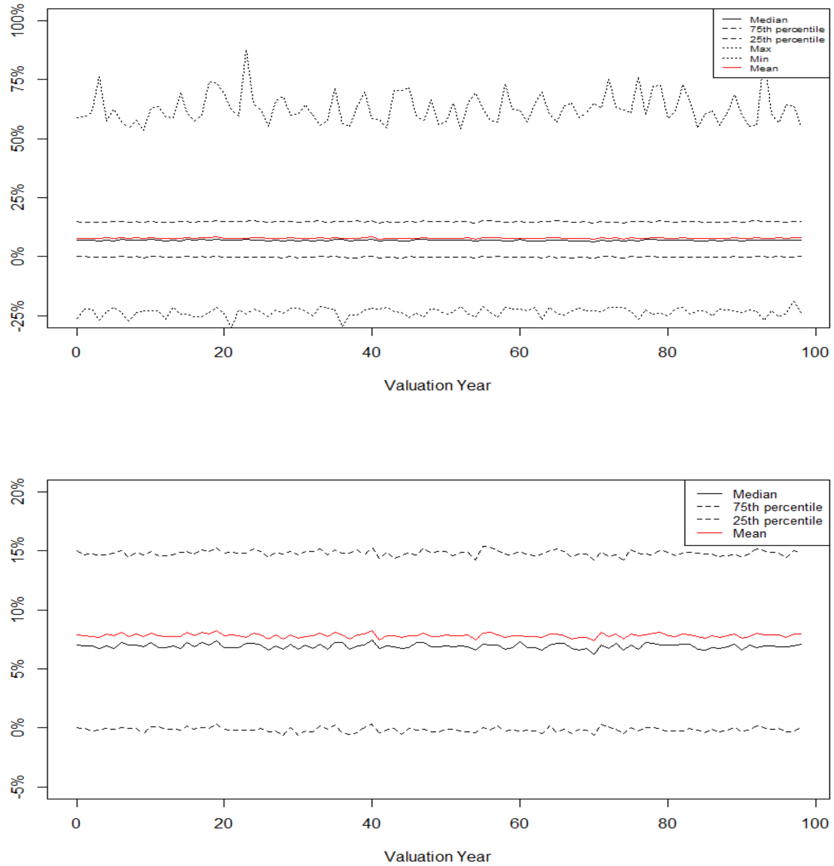

Figure 5.2: Mean, median and percentiles of annual effective investment returns under Case 2

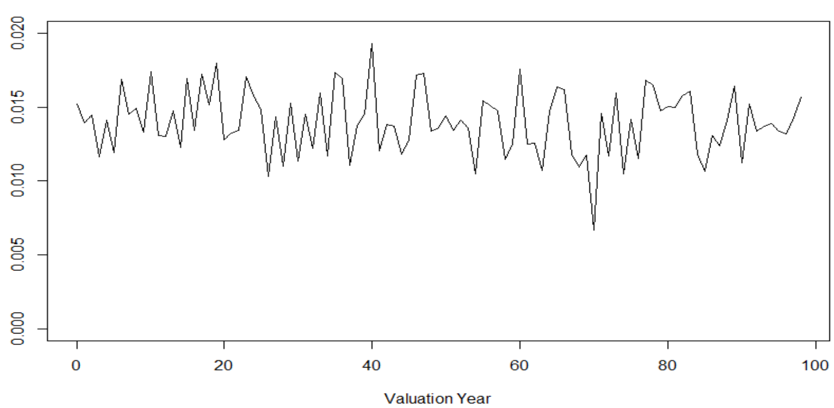

Figure 5.3: Median spread between annual effective investment return and annual effective valuation rate under Case 2

Figure 5.4 illustrates the distribution of the projected accrual rate, α_t, during the 99-year horizon, under Case 2. Compared to the corresponding graph of the accrual rate under the Base Case (Figure 4.5), the accrual rate is still projected to increase significantly over time, but much more slowly, with the median reaching only 2.0% at the end of our 99-year horizon, compared to 4.3% under the Base Case. As under the Base Case, differences still exist between different generations as those members who join the plan later can get a better pension benefit, but these differences are less pronounced.

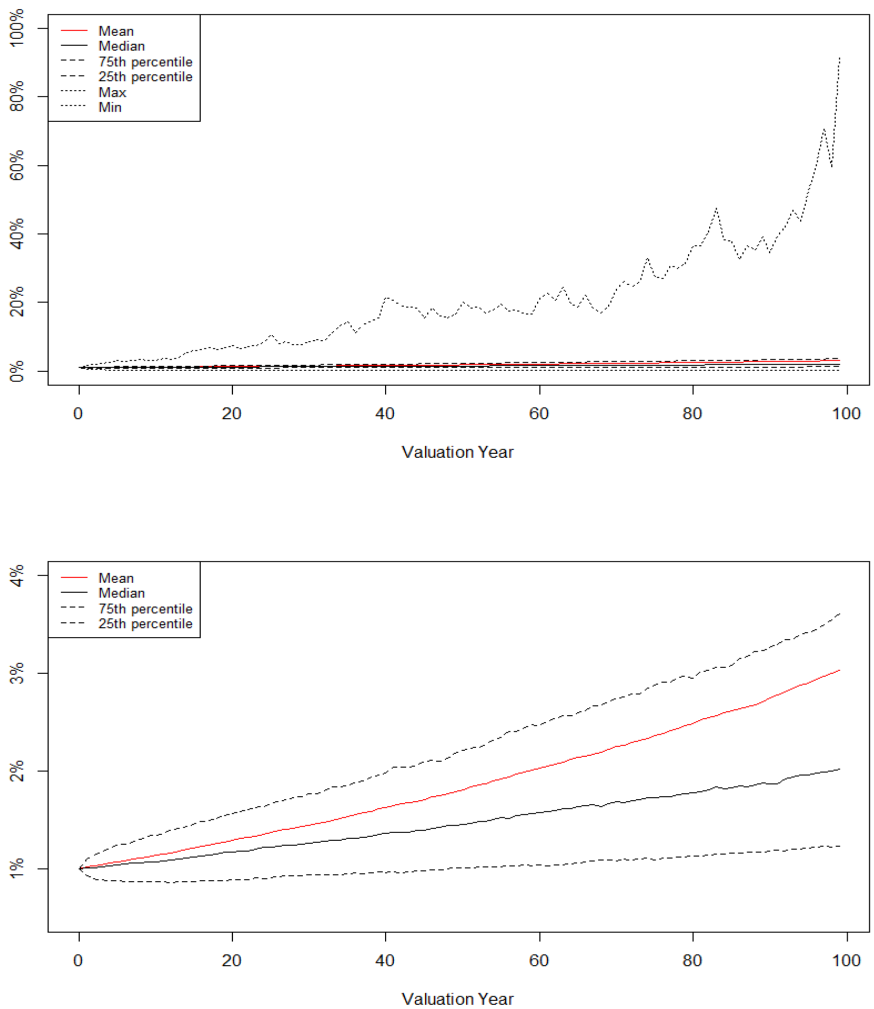

Figure 5.4: Mean, median and percentiles of projected accrual rates under Case 2

The distribution of TNC_t^{old} (the total normal cost rate as a proportion of payroll, right before each benefit adjustment) under Case 2 is shown in Figure 5.5. Both the mean and the median curves are below the fixed contribution rate (U), which is consistent with Figure 5.4 and an accrual rate that is more likely to increase than decrease. Unlike the figure of TNC_t^{old} under the Base Case (Figure 4.6), the difference between the median value of TNC_t^{old} and the fixed contribution rate of 4.05% is relatively stable since plan inception, because large gains due to increases in the valuation rate in the early years have been eliminated (see Figure 5.6).

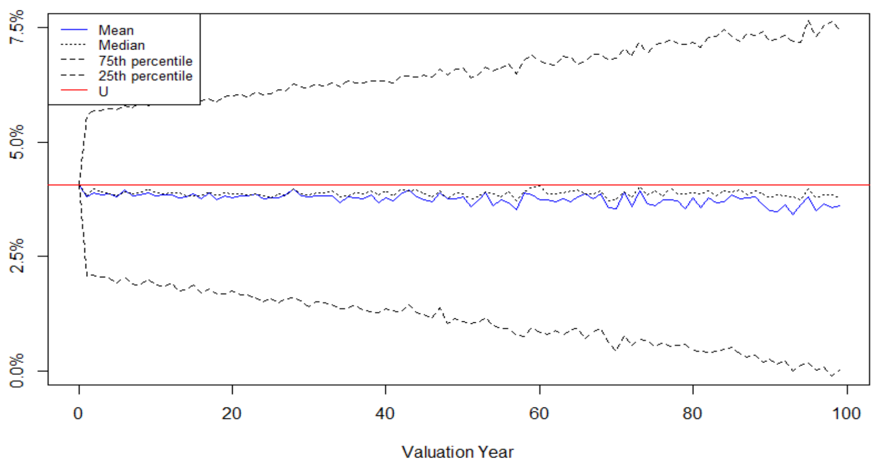

Figure 5.5: Mean, median and percentiles of the total normal cost, TNC_t^{old}, under Case 2

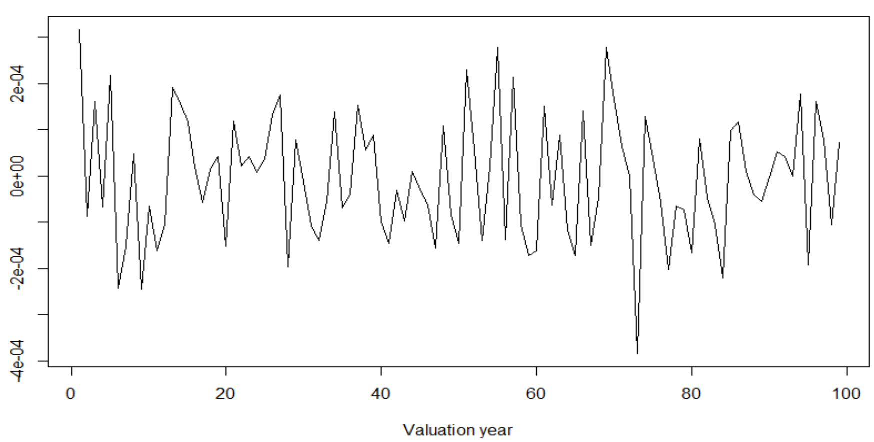

Figure 5.6: Median value of difference between two consecutive valuation rates under Case 2

From Figure 5.5, we also see that, on average, the pension benefits are greater than the target during the 99-year horizon, which is due to the fact that the investment return tends to exceed the valuation rate (see Figure 5.3).

We also analysed the sources of benefit changes under Case 2. A quick summary is provided here; for detailed charts, refer to Appendix D. As in the Base Case, we found that both valuation rate changes and investment experience had the largest impact on γ and that the new entrants impact was negligible. Not surprisingly, both the impact of valuation rate changes and the impact of investment experience were already stable from plan inception, which is due to the fact that both the valuation rate and the investment return processes started their long-term means. We also found that the entire distribution of new entrants impact shifted to the left over time. This happens because the projected accrual rate increases while the valuation rate stays very similar over time, increasing the cost of benefits relative to the initial contribution rate and leading to losses on new entrants. By contrast, under the Base Case, the dramatic increases in the accrual rate during the first 20 years are combined with (in fact driven by) rising valuation rates, so the cost difference for new entrants is less noticeable. The median value of the difference between two consecutive valuation rates under Case 2 is plotted in Figure 5.6 and is approaching stability since plan inception.

5.2 Performance Metrics of Case 2

In this section, we apply the same performance metrics as were introduced in Table 4.1 to assess benefit security, benefit adequacy, benefit stability and intergenerational equity under Case 2.

5.2.1 Benefit Security

Figure 5.7 displays the result of Performance Metric 1, the distribution of the factor by which the benefit exceeds or falls short of the target at each valuation. We can see that the upside risk is rising gradually over time, and this feature is consistent with the behaviour of the accrual rate shown in Figure 5.4. However, it is not nearly as pronounced as under the Base Case. The downside risk is not negligible at the end of our projection horizon, there is still a 13% chance that the benefits fall more than 10% short of the target.

Figure 5.8 is for Performance Metric 2, the probabilities of the benefit falling below 90%, 80% and 50% of the target in each valuation year. We can observe that each line in this figure is higher than the corresponding line under the Base Case (Figure 4.12). For example, the leftmost point of the dashed line corresponds to a 3.9% probability that the pension benefit is below 80% of the target in valuation year 1, by contrast, the corresponding probability under the Base Case is just 1.9%. Also, Figure 5.8 is consistent with Figure 5.7: downside risk decreases over time.

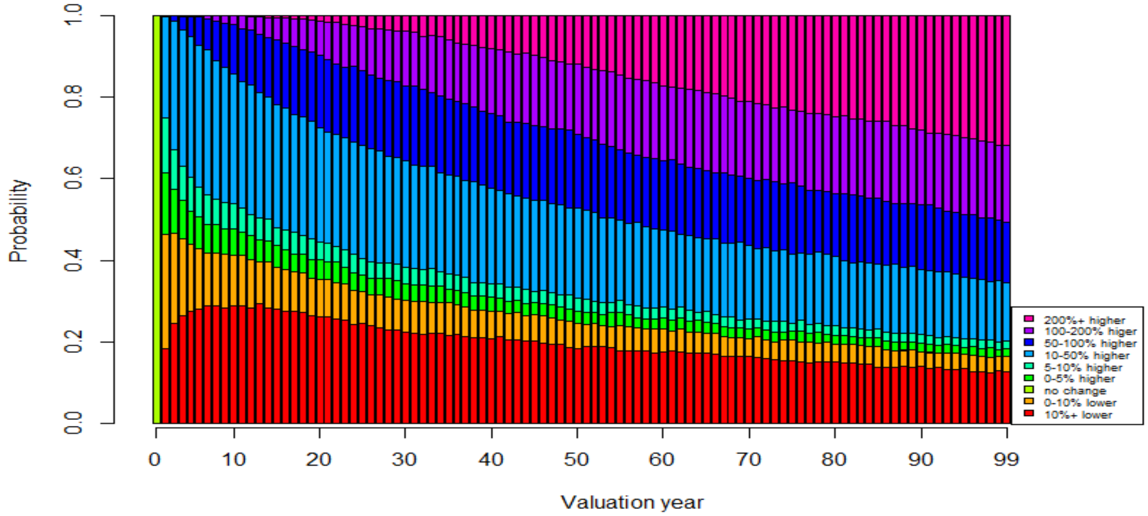

Figure 5.7: Distribution of the factor by which benefit exceeds or falls short of the target by valuation year under Case 2

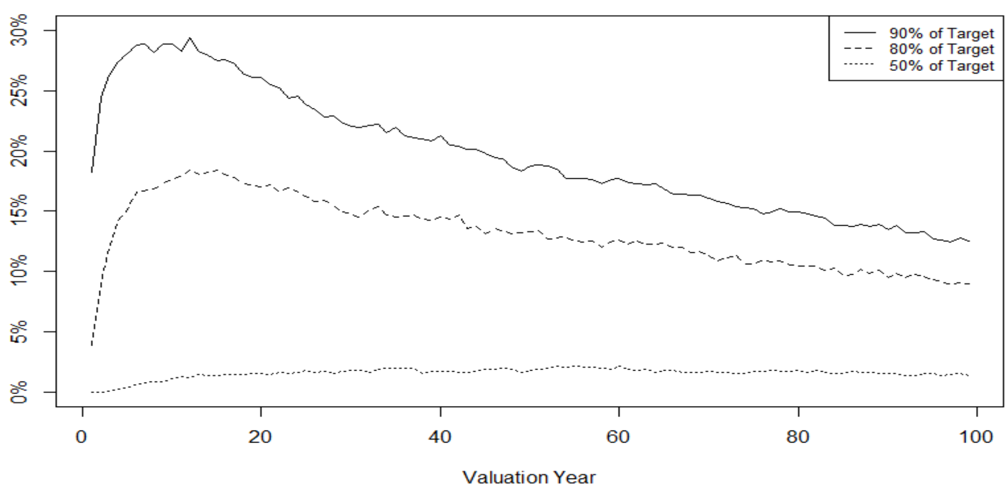

Figure 5.8: Probability of pension benefit falling below several specific proportions of the target during the valuation years under Case 2

Figure 5.9 is related to Performance Metric 3 and it confirms that later cohorts tend to receive better pension benefits. However, it also says that, on average, the weighted average benefits exceed the target, even for the earlier cohorts. Comparing to the corresponding figure under the Base Case (Figure 4.13), the distribution of the ratio of weighted average benefits to the target shifts down. This feature is consistent with the behaviour of the accrual rate (Figure 5.4): accrual rate is increasing gradually over time, but the speed is not as high as that under the Base Case.

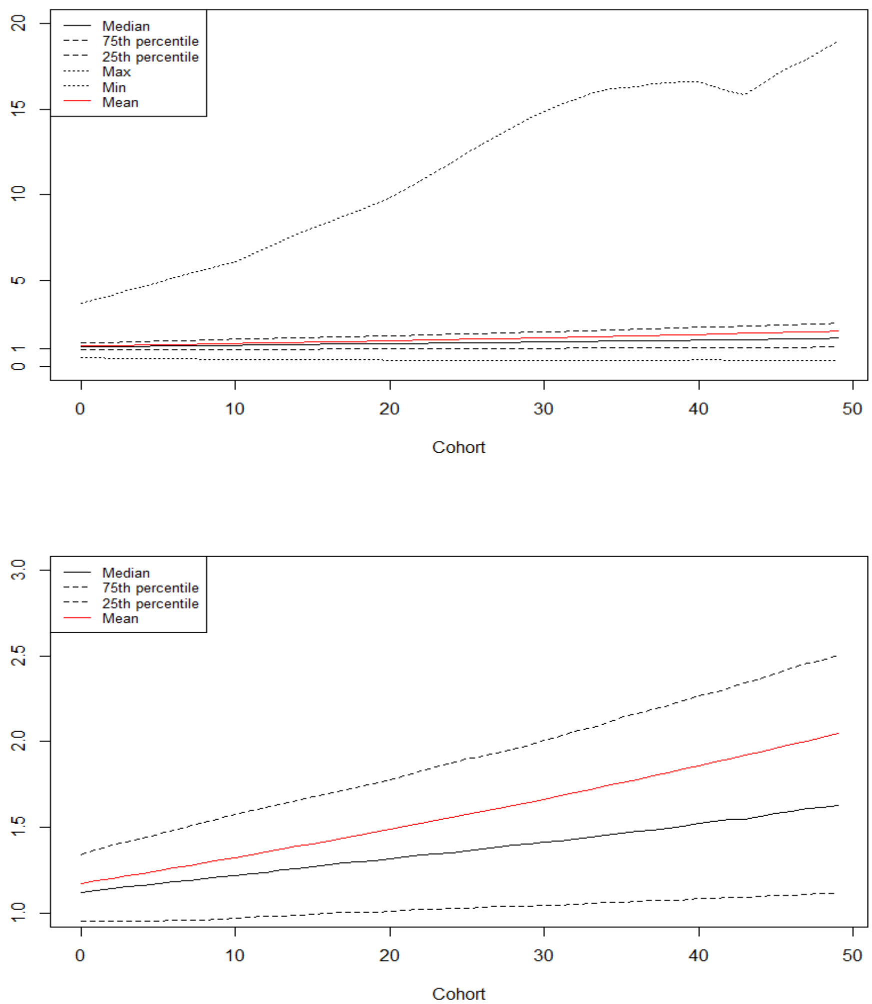

Figure 5.9: Mean, median and percentiles of the ratio of weighted average pension benefits to the target under Case 2

5.2.2 Benefit Adequacy

The result of Performance Metric 4 is shown in Figure 5.10. We see that the distribution of the replacement ratio for each cohort drifts upward with time. Comparing to Figure 4.14, the slope of the distribution of replacement ratio at retirement by each cohort is not as high as that under the Base Case but there are still sizeable differences between cohorts.

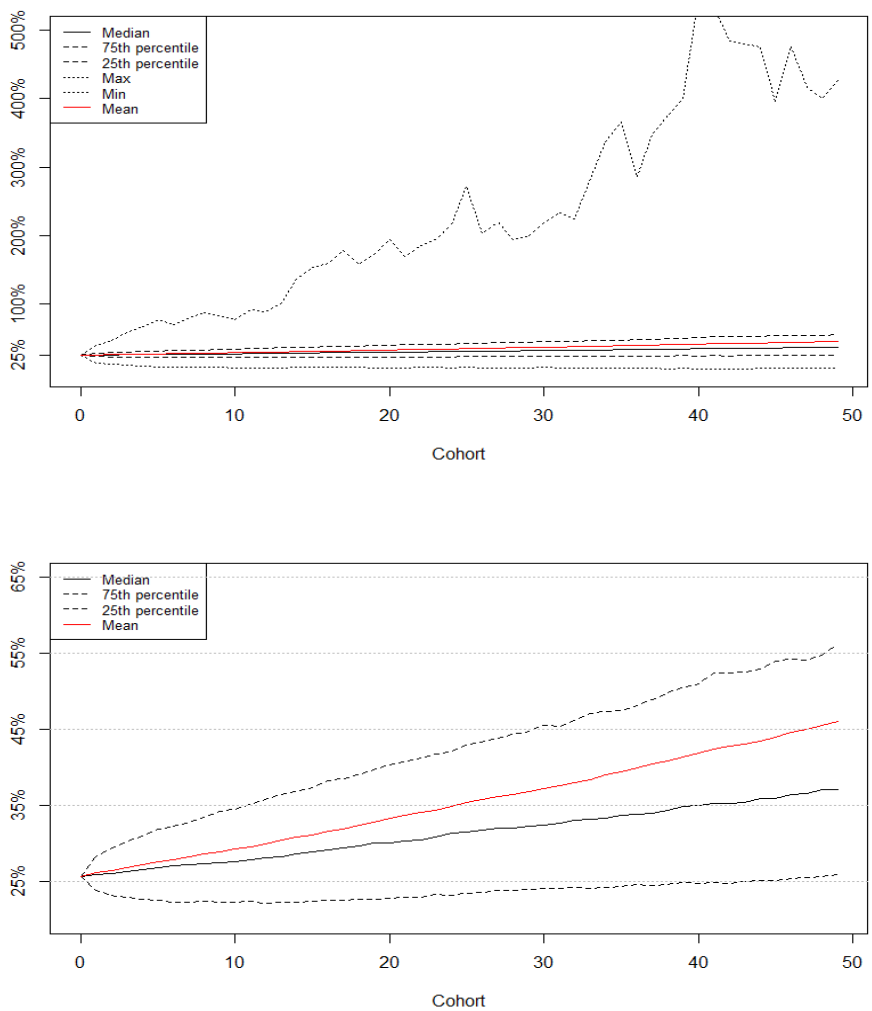

Figure 5.10: Mean, median and percentiles of the replacement ratio at retirement, by cohort, under Case 2

The enhanced version of Performance Metric 4, Performance Metric 5, is illustrated by Figure 5.11. This figure again confirms that differences exist between cohorts, and the ratios are also much lower than those under the Base Case (Figure 4.15).

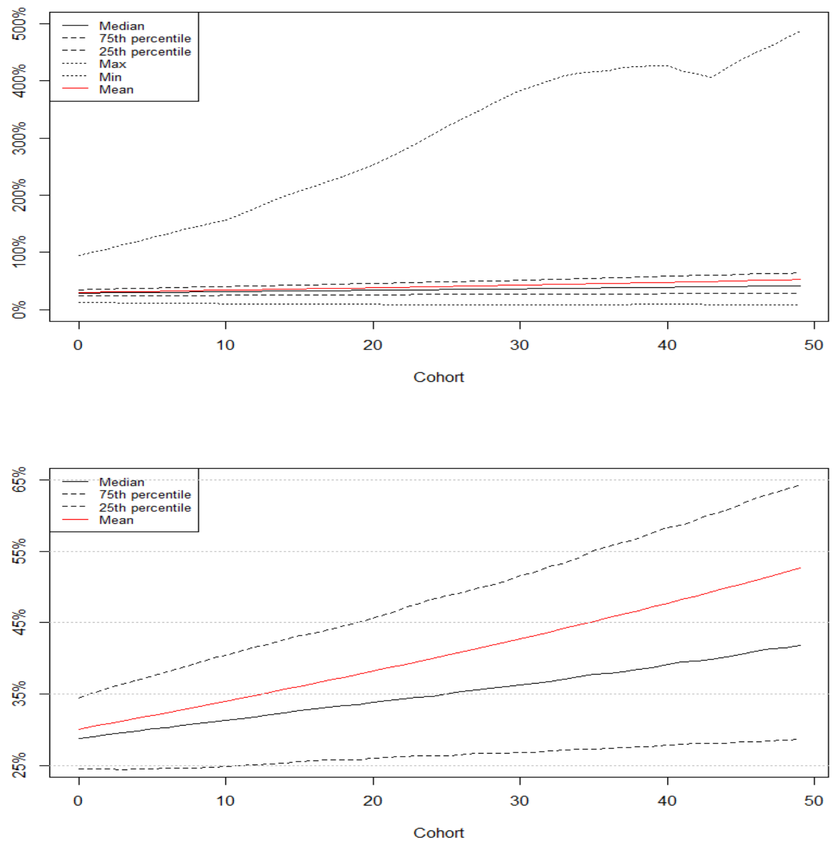

Figure 5.11: Mean, median and percentiles of the weighted average replacement ratio under Case 2

The distribution of the modified geometric average growth rate for each cohort, that is Performance Metric 6, is demonstrated in Figure 5.12. From the figure, we observe that it is impossible for each cohort's pension benefits over the retirement years to keep up with inflation. However, in contrast to the corresponding figure under the Base Case (Figure 4.16), the distribution of the modified growth rate under Case 2 is almost the same for every cohort, which is due to the fact that the accrual rate is increasing at a uniform rate. All generations have the same opportunity for reward in terms of the growth rate, even though they do not receive the same reward in terms of accrual rate under Case 2. We conclude that the generational differences present in the Base Case and visible in Figure 5.12 are due mostly to differences in economic conditions (lower bond yields at plan inception). When the situation of economy is more stable over time (as under Case 2), there is equity between generations in terms of reward for risk. Hence, we conclude that our TBP design is not inequitable by itself and the differences are due mostly to the changing economic conditions of the Base Case.

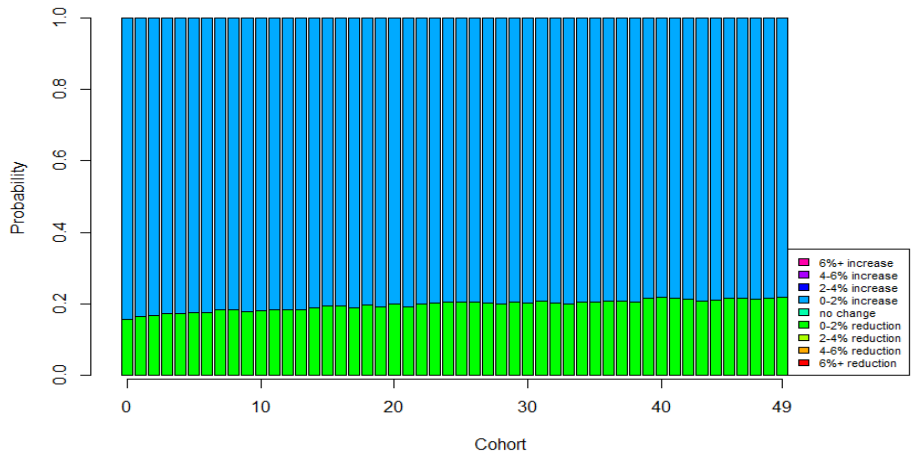

Figure 5.12: Distribution of modified geometric average growth rate by cohort under Case 2

5.2.3 Benefit Stability

The distribution of benefit adjustments by valuation year, that is Performance Metric 7, is summarized in Figure 5.13. We observe that the distribution is very stable with the upside risk exceeding the downside risk in each valuation year. This is due to the fact that the economic situation is relatively stable from plan inception onwards and investment experience is more likely to produce gains than losses.

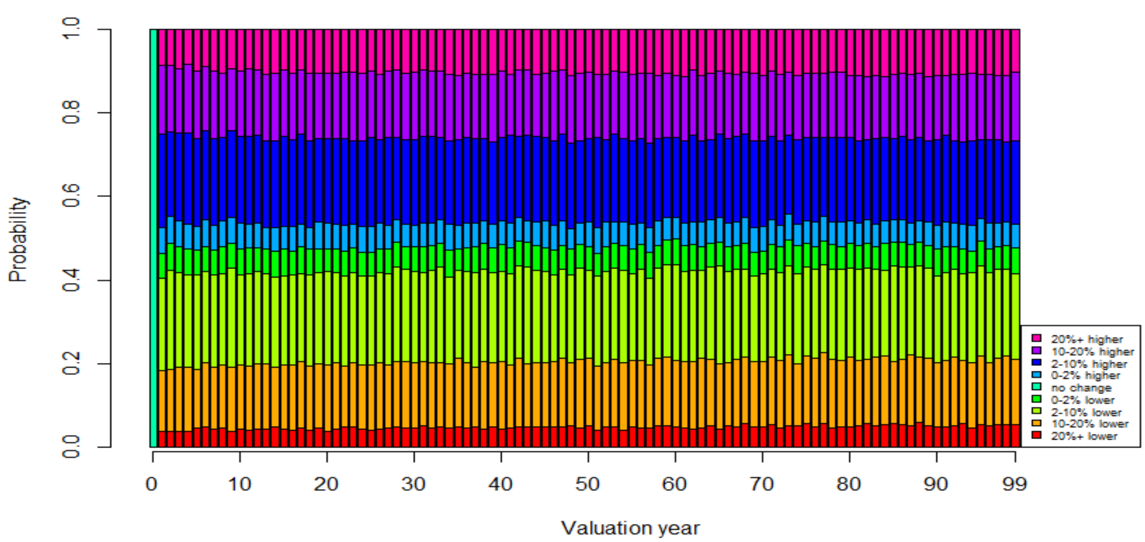

Figure 5.13: Distribution of benefit adjustments by valuation year under Case 2

5.3 Case 3

Under the Base Case the investment return is consistently underestimated. That is the investment return is higher than the assumed (valuation rate) over time. That is caused by the fact that we apply a valuation rate based on bond yields under the Base Case without an allowance for equity risk premium. Under Case 3, we take a best-estimate rate as the valuation rate. The best-estimate valuation rate projected at each valuation mimics the Benchmark Discount Rate from the British Columbia regulations of TBPs. It is a combination of the assumed returns from the four asset classes in our asset mix. The return on each bond is assumed to be the yield rate one month prior on that bond. The return on the equities is assumed to be the yield rate one month prior on the 15-year zero-coupon bond plus an equity risk-premium of 2.4% per annum, where the rate of 2.4% is the difference between the theoretical median value of annual effective returns of equities and the theoretical median value of annual effective returns on the 15-year zero-coupon bond under our VAR(1) model. These assumed returns from the four asset classes are weighted in accordance with the investment strategy in Chapter 3 and an extra 25 basis points per annum are added to take into account the benefit of diversification (see CIA (2015)). Hence, the best-estimate valuation rate, j_t^{BE}, expressed as an annual effective rate can be defined as:

$$\begin{aligned} j_t^{BE} &= 4\% \cdot e^{12\cdot\delta_{SB,t-\frac{1}{12},0.25}} + 3\% \cdot e^{12\cdot\delta_{MB,t-\frac{1}{12},5}} + 33\% \cdot e^{12\cdot\delta_{LB,t-\frac{1}{12},15}} \\ &\quad + 40\% \cdot (e^{12\cdot\delta_{LB,t-\frac{1}{12},15}} + 2.4\%) - 1 + 0.0025. \end{aligned} \tag{5.1}$$

Figure 5.14 illustrates the distribution of annual effective valuation rate during the 99-year horizon. Compared to the corresponding figure under the Base Case (Figure 4.1), the trend of annual effective valuation rate is very similar, but with the entire distribution shifted up. For example, the starting valuation rate is 1.96% under the Base Case, but 3.57% under Case 3. Based on the new valuation rate at plan inception (3.57%), the fixed contribution rate (U) becomes 7.40%, and the starting asset value (F_0) becomes \$650,361,540.

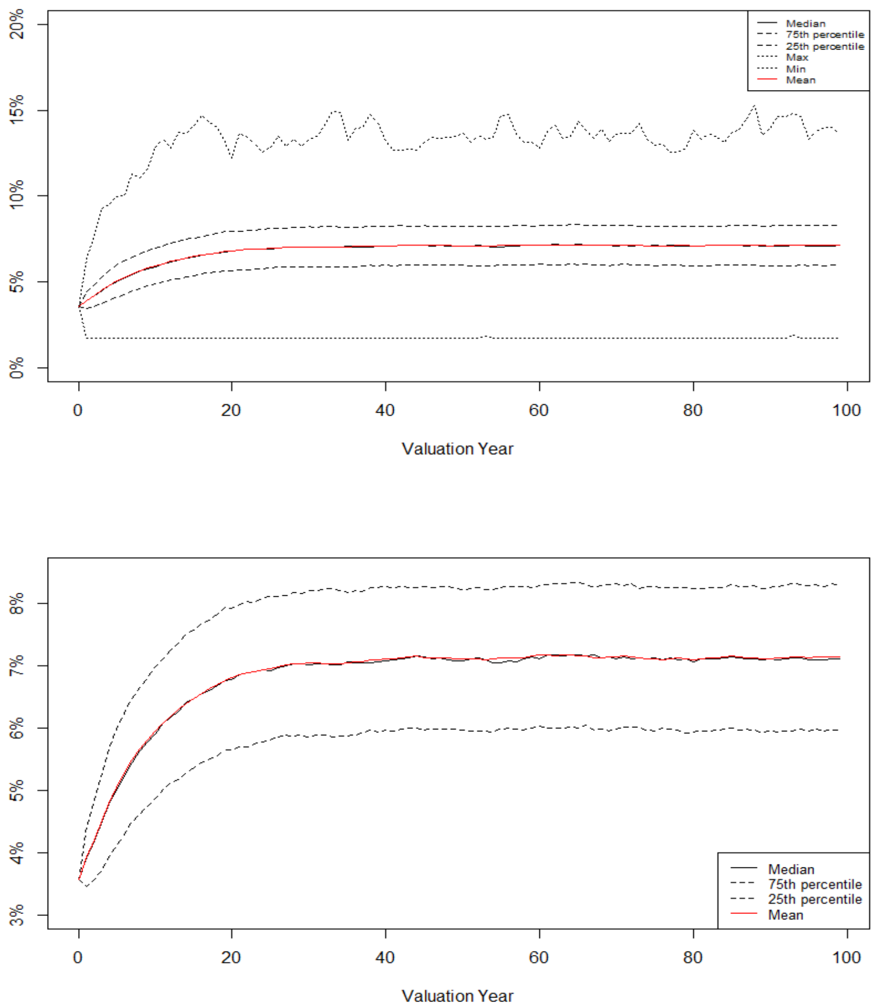

Figure 5.14: Mean, median and percentiles of annual effective valuation rates under Case 3

Figure 5.15 displays the trend of annual effective investment return over our simulation horizon under Case 3. We observe that the trend is exactly the same as under the Base Case because the investment strategy is not changed under Case 3. Unlike in the Base Case, the fund here is likely to experience investment losses as the long-term mean of the valuation rate (7.12%) is slightly higher than that of the investment return (6.37%), producing a median spread of 0 basis point to -5 basis points, shown in Figure 5.16.

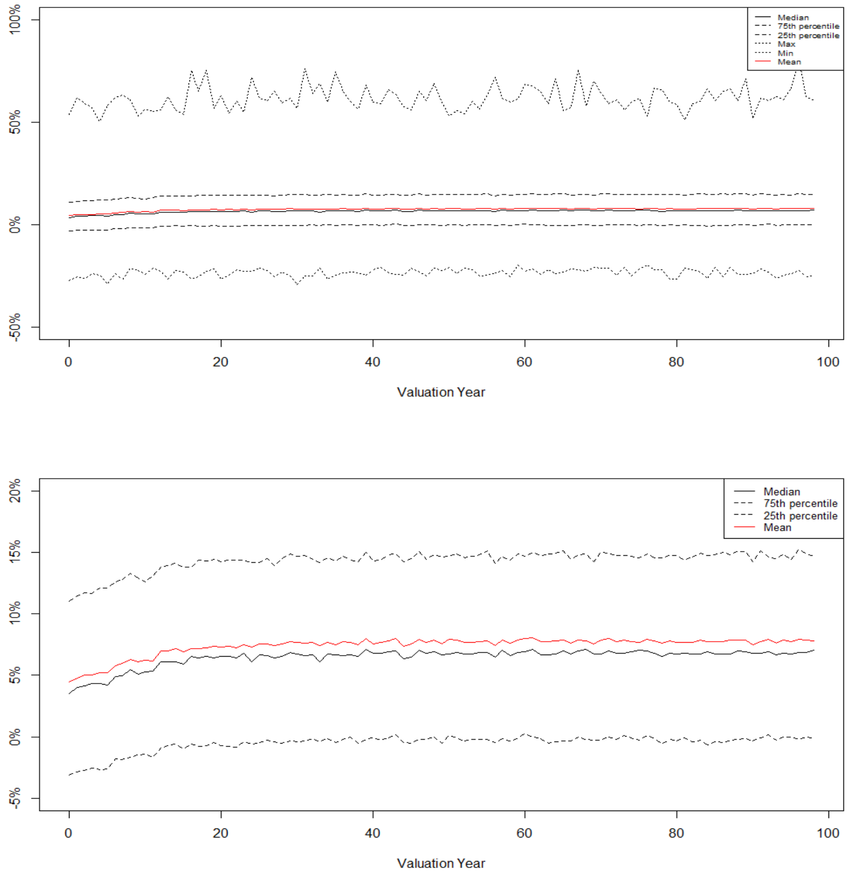

Figure 5.15: Mean, median and percentiles of annual effective investment returns under Case 3

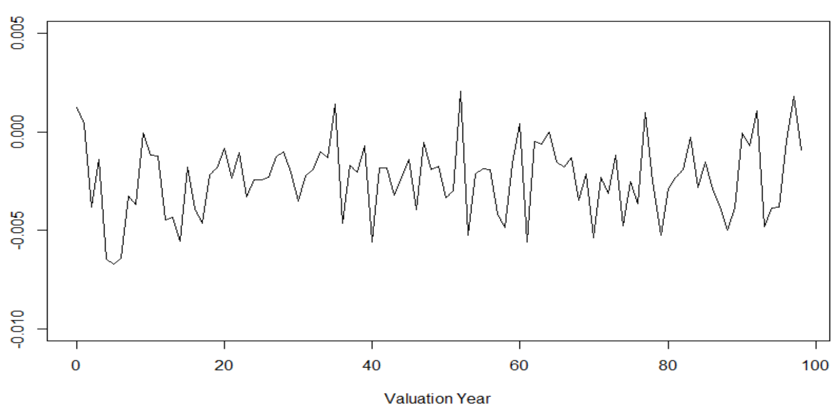

Figure 5.16: Median spread between annual investment return and annual valuation rate under Case 3

Figure 5.17 illustrates the distribution of the projected accrual rate, α_t, under Case 3. Like Figure 4.5 from the Base Case, the projected accrual rate is increasing significantly during the first 20 years. However, the median curve is almost horizontal after 20 years, which results in equity between cohorts after 20 years. The TNC_t^{old} under Case 3 is displayed in Figure 5.18. We see that both the mean and the median curves are basically at the fixed contribution rate, U, after 20 years, which implies that increases and decreases are equally likely during this period. From Figure 5.17, we also see that, on average, the pension benefits are about 150% of the target after 20 years, which is due to offsets between the three sources of benefit changes.

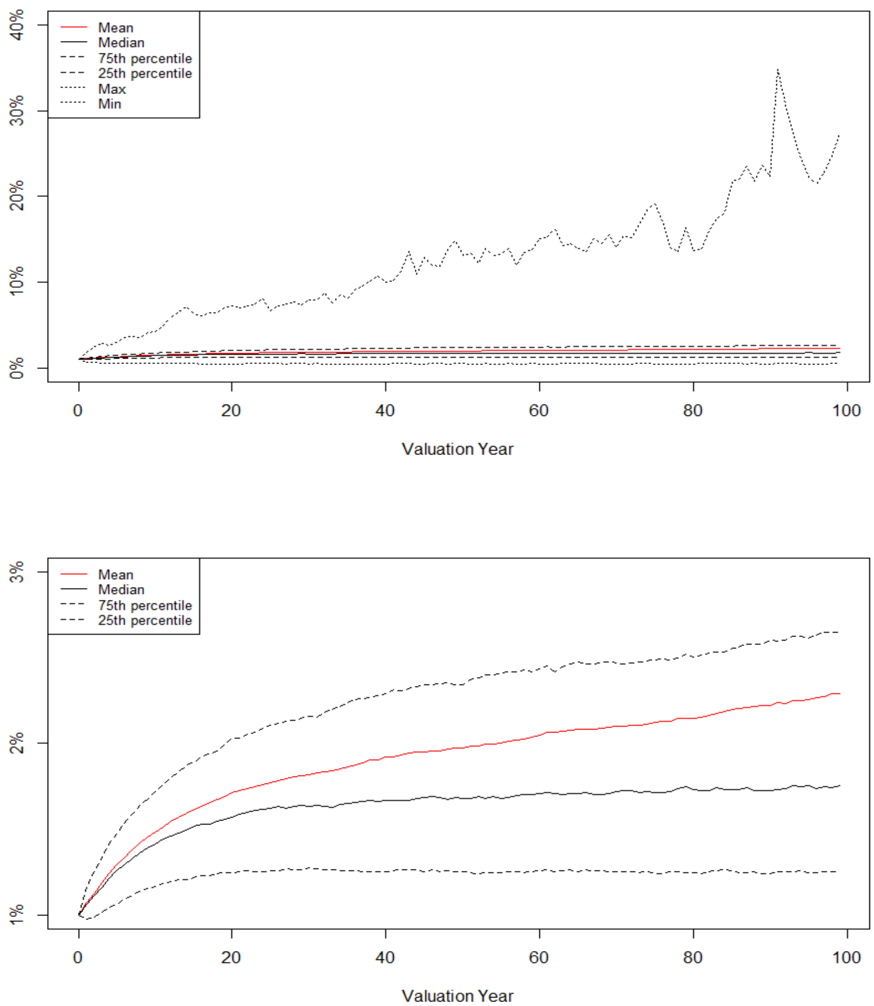

Figure 5.17: Mean, median and percentiles of projected accrual rates under Case 3

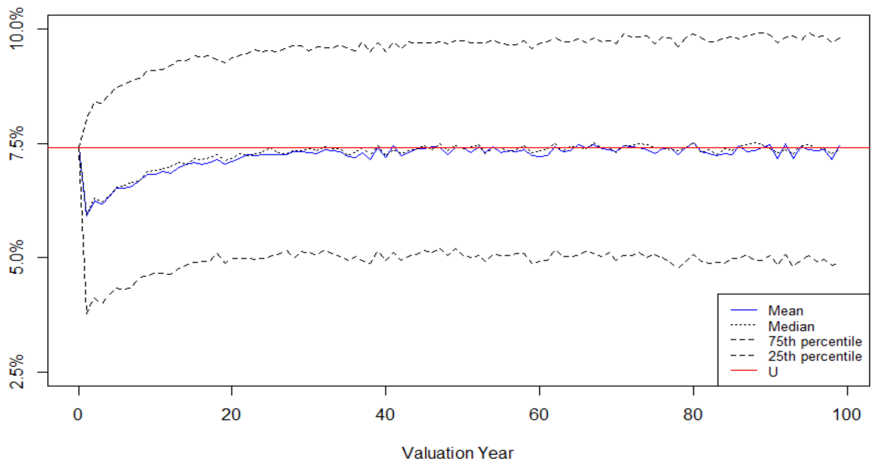

Figure 5.18: Mean, median and percentiles of the total normal cost, TNC_t^{old}, under Case 3

Attribution of the changes in the accrual rate by source resulted in largely the same conclusions as under the Base Case (for detailed charts refer to Appendix D):

- Valuation rate changes and investment experience are the most significant drivers of benefit changes.
- The impact of new entrants is negligible.
- The distribution of $\gamma^{valrate}$ shifts over time due to mean-reversion of the valuation rate: large positive impact occur in the early years but the distribution becomes more symmetric around zero as time passes.

The primary difference is that under Case 3 the distribution of γ^{newent} shifts to the right over time, indicating that new entrants tend to have a small but positive impact on the accrual rate.

5.4 Performance Metrics of Case 3

In this section, we also apply the same performance metrics as were introduced in Table 4.1 to assess benefit security, benefit adequacy, benefit stability and intergenerational equity under Case 3.

5.4.1 Benefit Security

Performance Metric 1 (the distribution of the factor by which benefits exceed or fall short of the target) is displayed in Figure 5.19. This figure is very similar to the corresponding graph under the Base Case (Figure 4.11) except the downside risk after 20 years. The probability that the benefit is less than the target stays stable (9%) after 20 years, which is consistent with the behavior of accrual rate and it is due to the fact that the valuation rate is very similar to the investment return.

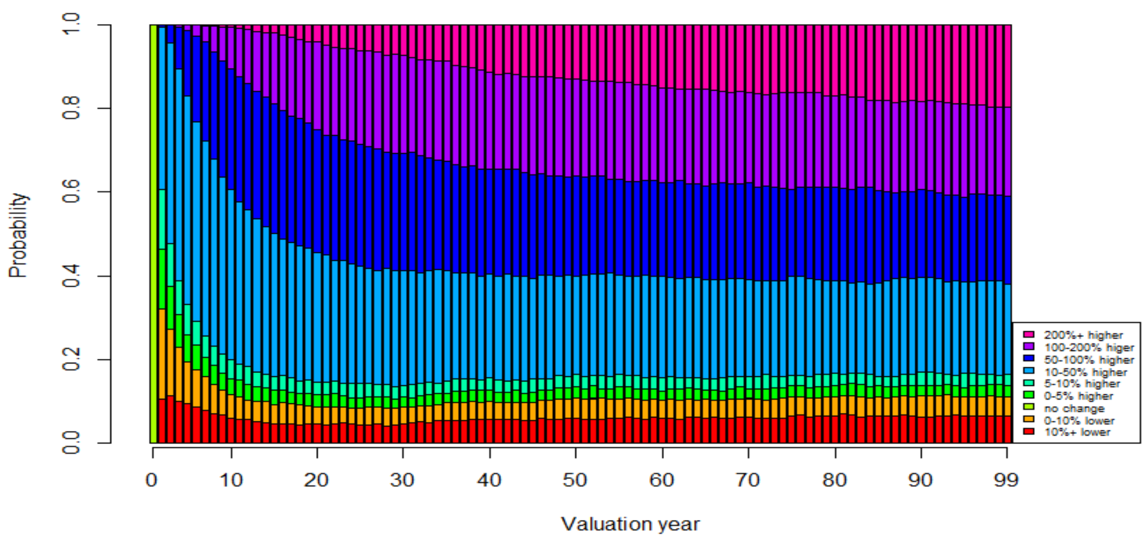

Figure 5.19: Distribution of factor by which the benefit exceeds or falls short of the target by valuation year under Case 3

Figure 5.20 shows the result of Performance Metric 2, that is the probabilities of the benefit falling below 90%, 80% and 50% of the target. The 90% and 80% of target curves, both increase slightly after 20 years, which is due to the fact that the median value of the valuation rate is larger than that of the investment return. Compared to Figure 4.12, the downside risk is not negligible, that is because the best-estimate valuation rate is very similar to the investment return. However, the probability of catastrophic outcomes (that the pension benefit being below half of the target) still remains at zero for most of our projection period.

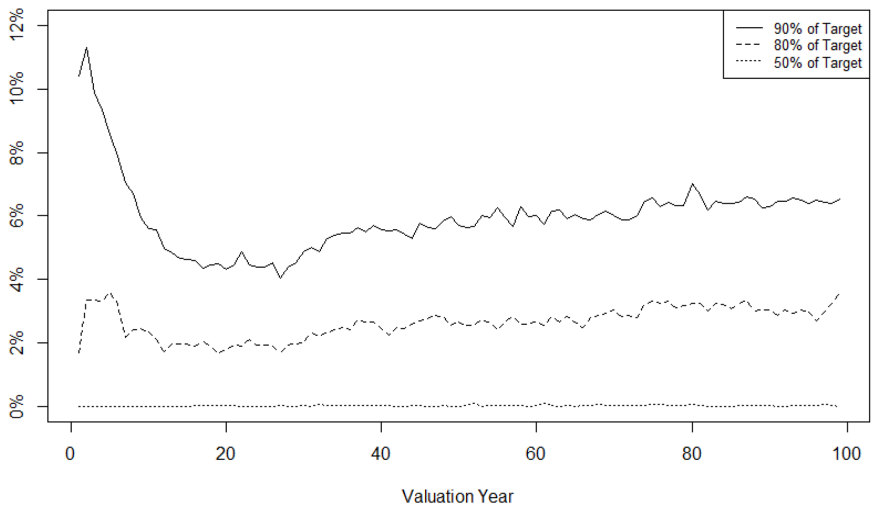

Figure 5.20: Probability of the pension benefit falling below several specific proportions of the target during the valuation years under Case 3

The distribution of the ratio of weighted average pension benefits to the target, Performance Metric 3, is illustrated in Figure 5.21. This figure is consistent with the behaviour of accrual rates shown in Figure 5.17, that is the median value of accrual rate is increasing significantly during the first 20 years and stays more stable afterwards. Compared to the median value curve in Figure 4.13, this design produces similar outcomes for different generations.

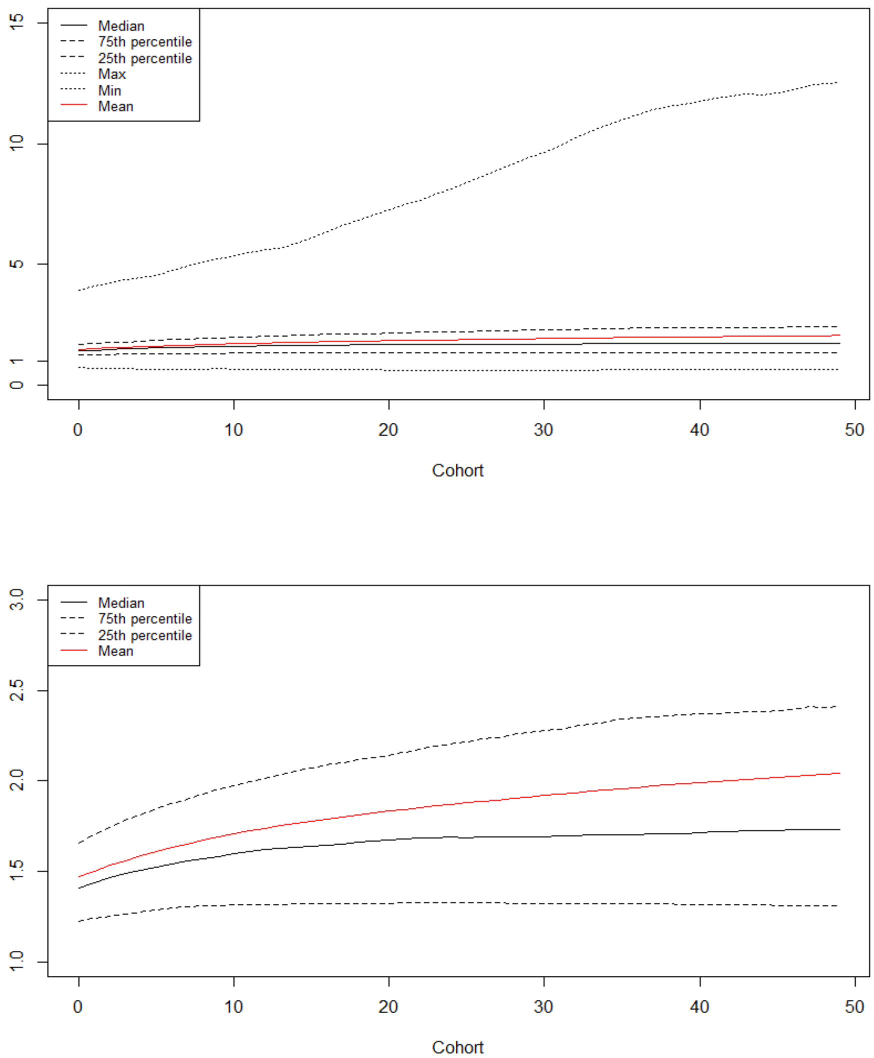

Figure 5.21: Mean, median and percentiles of the ratio of weighted average pension benefits to the target under Case 3

5.4.2 Benefit Adequacy

The distribution of the replacement ratio at retirement (Performance Metric 4) under Case 3 is plotted in Figure 5.22. Compared to the corresponding figure under the Base Case (Figure 4.14), Figure 5.22 has a similar trend, but the slopes of the curves are more gentle. Also, the potential inequities between cohorts are not obvious after the first 10 cohorts, which is consistent with the development of the accrual rate (Figure 5.17).

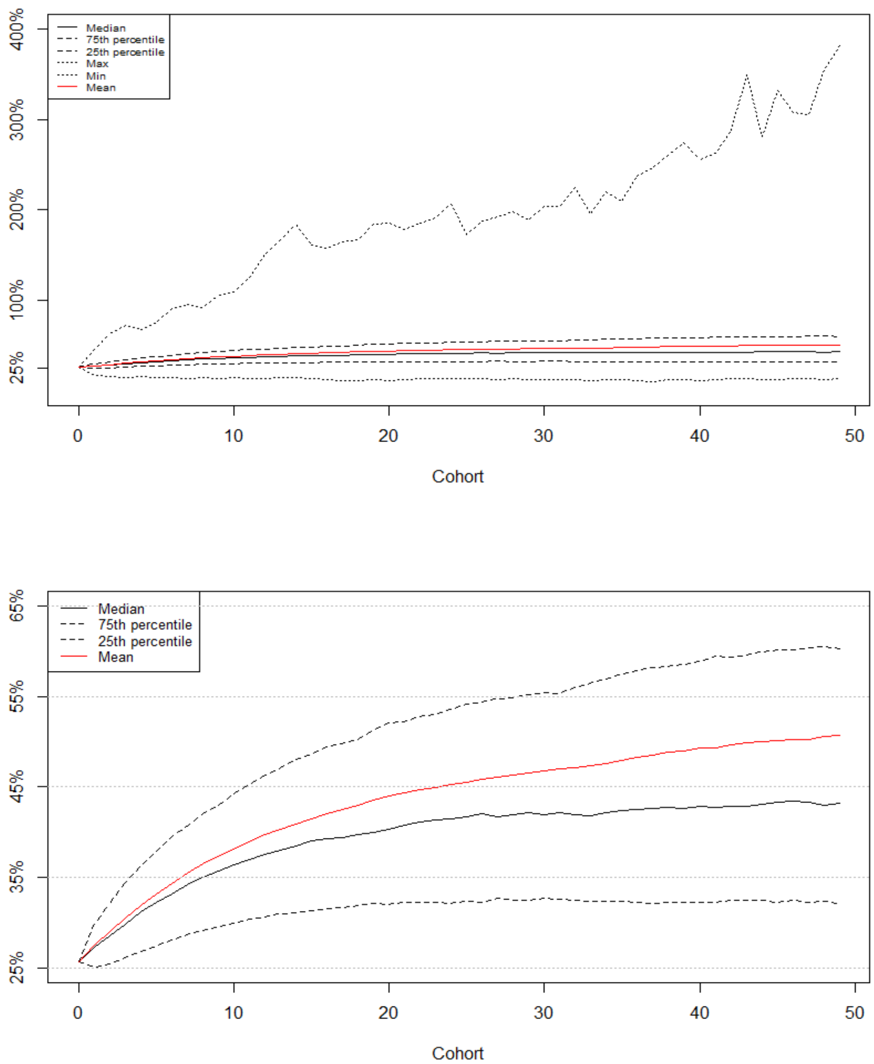

Figure 5.22: Mean, median and percentiles of the replacement ratio at retirement, by cohort, under Case 3

The enhanced version of Metric 4 (Performance Metric 5), the weighted average replacement ratio is displayed in Figure 5.23. The trend of Figure 5.23 is very similar to that of Figure 5.22, which again indicates that this design is only mildly unfair for the first 10 cohorts.

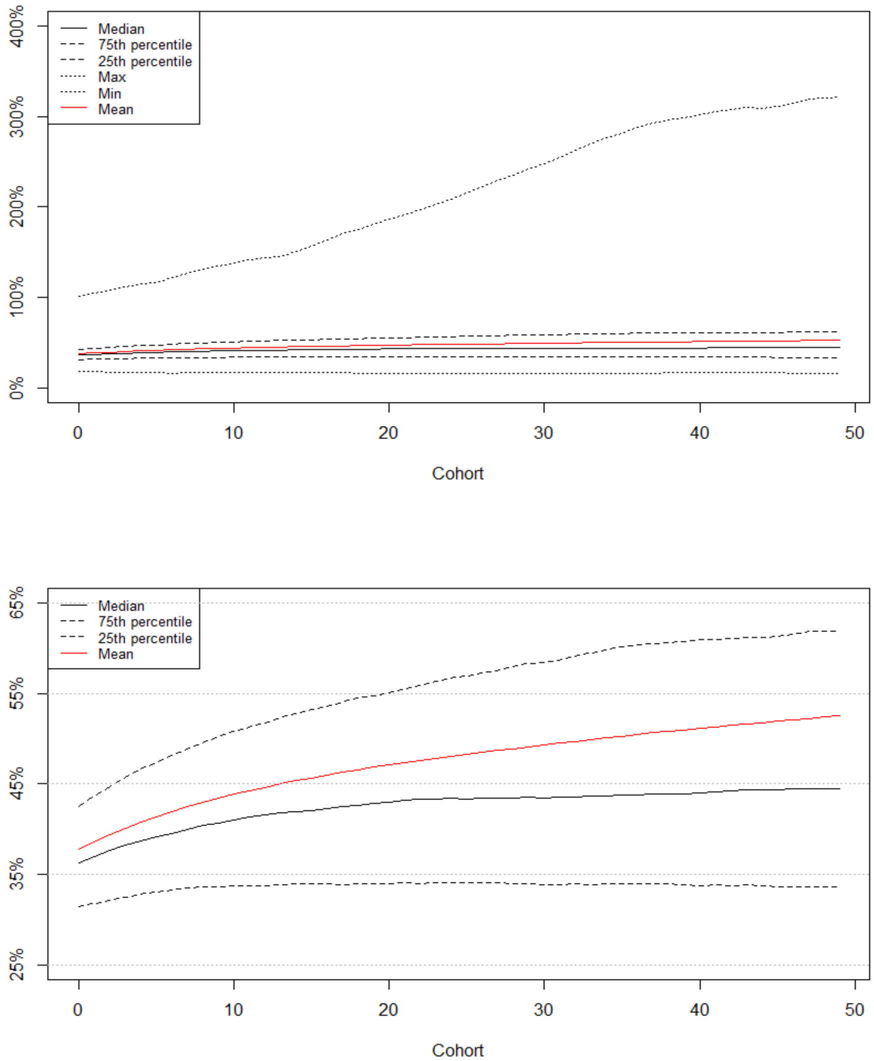

Figure 5.23: Mean, median and percentiles of the weighted average replacement ratio under Case 3

Figure 5.24 shows the distribution of the modified geometric average growth rate by cohort (Performance Metric 6). Figure 5.24 tells us that none of the cohorts can protect their pension payments over the retirement from inflation. Compared to Figure 4.16, Figure 5.24 has a similar trend, but the upside risk for the 20th cohort onwards is much lower because no gains are expected to arise from investment experience to fuel pension benefit growth. Even though Case 3 does not produce identical outcomes for all cohorts, it does produce consistent outcomes for cohorts that face the same risks, in terms of economic conditions and is therefore superior to the Base Case.

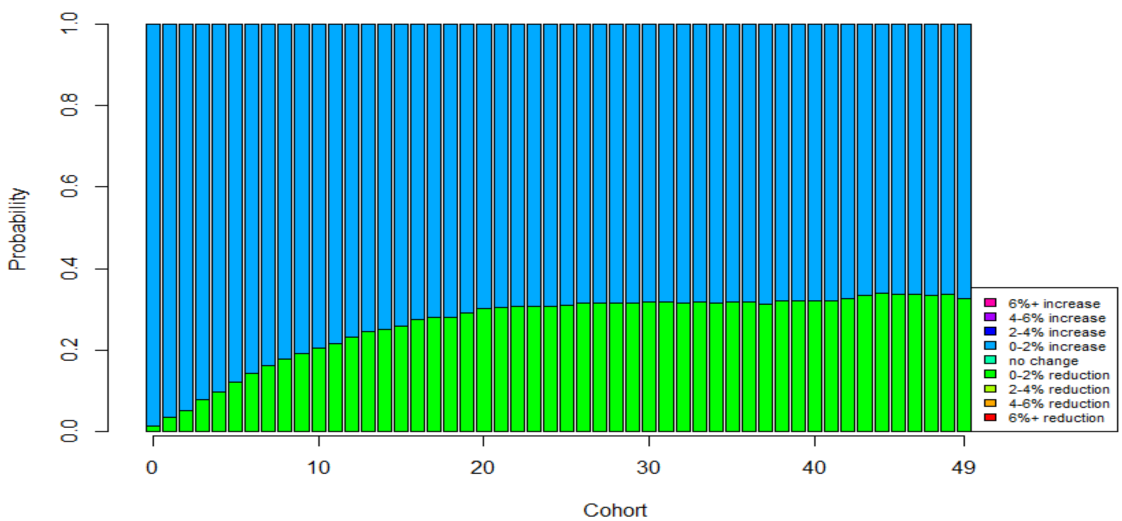

Figure 5.24: Distribution of modified geometric average growth rate by cohort under Case 3

5.4.3 Benefit Stability

The last Performance Metric, the distribution of benefit adjustments is summarized in Figure 5.25. Like Figure 4.17, the distribution becomes more stable after 20 years, with the upside risk decreasing during the first 20 years. Unlike under the Base Case, the downside risk is similar to the upside risk after 20 years, which is consistent with the plan being equally likely to experience gains or losses once bond yields return to their long-term means.

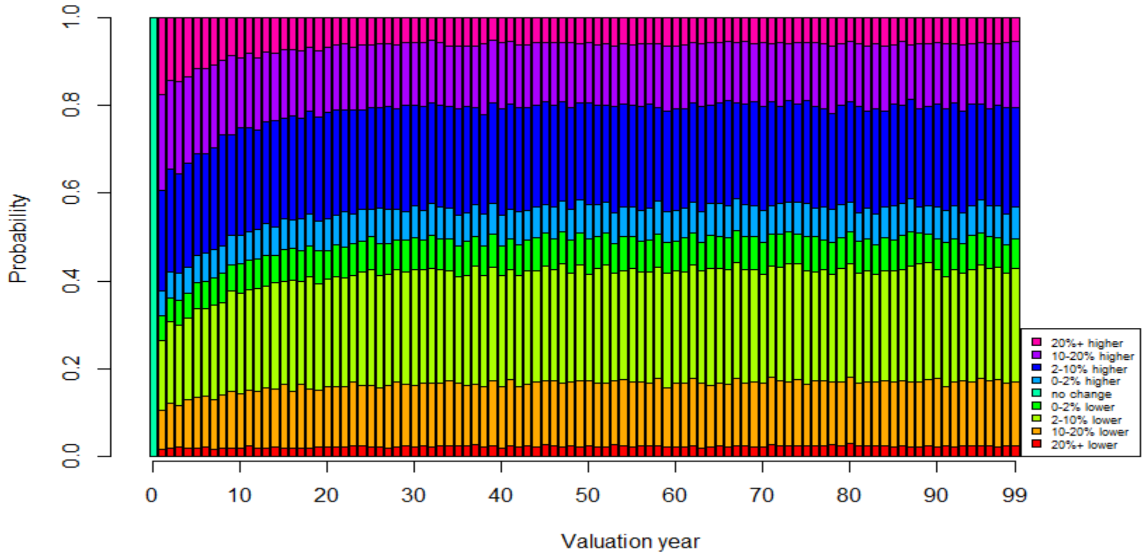

Figure 5.25: Distribution of benefit adjustments by valuation year under Case 3

Chapter 6

Conclusions

In this project, we study the operational characteristics of a target benefit plan (TBP) by simulation and apply a variety of performance metrics to assess benefit security, benefit adequacy, benefit stability and intergenerational equity. The specific design we investigate has the following four key elements: the contribution rate is set as a constant percentage of salary, the target pension benefit is based on career average earnings and years of service, the aggregate pension cost method is used in the affordability test, and a single trigger is applied to adjust both past and future benefit accruals immediately at each valuation. The asset model employs a stationary first-order vector autoregressive model to fit four Canadian historical data series (three zero-coupon bond yields and total return of TSX including dividends) in order to project Canadian interest rates and investment returns. The membership of the plan is mature and stationary with a fixed number of new entrants and retirees, and retirements are deterministic mortality.

We decompose the relative change in the accrual rate at each valuation, deriving formulas for the impact of changes in the valuation rate, investment experience, and new entrants. We show that under our specific design, the accrual rate does not stay constant at consecutive valuations even when investment experience is exactly in line with our assumptions since the last valuation date. However, our simulations show that the impact of new entrants on the accrual rate is very small when compared to the impact of changes in the valuation rate and the impact of investment experience.

The performance of our TBP is assessed and analyzed by simulation over a 99-year horizon. Empirical results indicate that when the valuation rate is based on long-term bond yields but the fund is invested partly in equities, the accrual rate tends to increase significantly over time. The sharpest increases occur during the first 20 years, coinciding with dramatic increases in the valuation rate. The distribution of the accrual rate continues to drift upward even after 20 years, with the median accrual rate increasing fourfold over our horizon.

At the same time, later cohorts of retirees have more difficulty maintaining the purchasing power of their benefits in retirement. Overall, the weighted average replacement ratio of the $50th$ cohort of retirees is double that of the first cohort, indicating that the later a member join the plan, the better pension benefits he can receive. Our simulations produce much more equitable results when the asset model assumes that bond yields and equity returns start from their long-term mean values, suggesting that most of the differences in outcomes between cohorts under our Base Case arise from changes in economic conditions.

We also investigated the impact of using a "best-estimate" valuation rate based on the expected investment return instead of long-term bond yields. We found that most metrics still shift slightly over the first 20 years but remain more stable thereafter. We consider this equitable in the sense that similar outcomes are produced for cohorts that face the same economic risks. It should be noted that this improvement in intergenerational equity comes at the cost of benefit security and benefit adequacy, both of which are slightly lower.

In future research the performance impact of the following modifications could be investigated:

- Different choices of asset mix, such as higher proportion in equities or fixed income;
- A "no-action" region between a lower and upper trigger for benefit adjustments, as described in Sanders (2016a), to stabilize benefits;
- Applying final average earnings to set the target benefit;
- Adjusting the accrual rate applicable to future service first, to protect the accrued benefits of active and retired members;
- Investing the fund separately for pensioners and active members (for example, the pensioners' fund could be invested more conservatively);
- Allowing contributions to vary in a narrow range.

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Appendix A

Original Asset Data

The following data were used to estimate the parameters of the asset model in Chapter 2. The column labeled “Date” represents the specific month. The columns labeled “SB”/“MB”/“LB” show the annualized instantaneous yields on Canadian short/medium/long-term zero-coupon bonds. The column labeled “EQ” represents the Toronto Stock Exchange composite total return close indices, including dividends. Notice that, all the data were collected at the last business day of each month.

Sources:
Yield Curves for Zero-Coupon bonds, Bank of canada; Available:
http://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/

TSX composite total return index, Canadian Financial Markets Research Centre (CFMRC) summary information database; Available:
http://www.sfu.ca/research-at-sfu/library.html

Table A.1: Original financial data

Date	SB	MB	LB	EQ	Date	SB	MB	LB	EQ
02/1991	9.6112	9.2410	9.9330	6000	02/1994	3.7377	6.2717	7.6453	8436
03/1991	9.6475	9.2433	9.8024	6084	03/1994	5.8196	7.6101	8.3507	8283
04/1991	8.8630	9.2221	9.9034	6051	04/1994	6.1205	7.7323	8.4735	8170
05/1991	8.6478	9.2738	9.9335	6213	05/1994	6.2432	8.3418	8.8042	8301
06/1991	8.5711	9.6795	10.3533	6102	06/1994	6.3910	8.8951	9.3642	7748
07/1991	8.4285	9.5759	10.2027	6242	07/1994	5.9168	8.7336	9.2830	8051
08/1991	8.3775	9.3574	9.9973	6220	08/1994	5.4314	8.2554	8.8999	8394
09/1991	8.0385	8.6367	9.4939	6014	09/1994	5.3230	8.3105	9.0614	8427
10/1991	7.8790	8.0716	9.1813	6250	10/1994	5.3998	8.5036	9.2902	8313
11/1991	7.5727	7.9677	9.2372	6146	11/1994	5.9889	8.6890	9.1662	7945
12/1991	7.0726	7.6882	9.0656	6292	12/1994	7.1724	8.8420	9.0407	8206
01/1992	7.1498	8.0139	9.0916	6453	01/1995	7.7809	8.9420	9.1005	7830
02/1992	7.3729	8.0603	9.1073	6443	02/1995	7.6735	7.9727	8.6409	8053
03/1992	7.1227	8.6047	9.4727	6163	03/1995	8.4115	8.1425	8.6770	8451
04/1992	6.7204	8.5296	9.6560	6069	04/1995	7.9331	7.8672	8.4692	8391
05/1992	6.0879	8.0368	9.2988	6143	05/1995	7.3133	7.3813	8.2076	8742
06/1992	5.4767	7.6185	9.2308	6170	06/1995	6.8762	7.4686	8.2236	8924
07/1992	5.1805	6.6089	8.5503	6279	07/1995	6.3153	7.9084	8.7024	9104
08/1992	4.9931	6.4548	8.5496	6221	08/1995	6.4562	7.5285	8.3166	8927
09/1992	7.5148	7.4876	8.5679	6055	09/1995	6.5739	7.3373	8.1183	8978
10/1992	6.1179	6.8545	8.4675	6132	10/1995	5.8826	7.1348	7.9724	8846
11/1992	8.0312	7.7627	8.9024	6053	11/1995	5.9218	6.6863	7.5519	9266
12/1992	7.1036	7.5110	8.5207	6202	12/1995	5.6760	6.5748	7.5935	9398
01/1993	6.3704	7.4826	8.7559	6125	01/1996	5.1433	6.4076	7.6262	9914
02/1993	5.8509	6.8174	8.3026	6407	02/1996	5.3174	6.8300	8.1025	9862
03/1993	5.4216	7.0673	8.3675	6715	03/1996	5.1750	7.0273	8.1776	9964
04/1993	5.1943	7.0438	8.4101	7071	04/1996	4.5666	7.1479	8.3674	10324
05/1993	4.7697	6.9776	8.3404	7272	05/1996	4.7143	7.1894	8.1824	10543
06/1993	4.5885	6.7160	8.3010	7455	06/1996	4.6472	7.1255	8.2037	10167
07/1993	4.1950	6.4681	7.8088	7464	07/1996	4.3062	7.0000	8.1103	9944
08/1993	4.8070	6.1917	7.2336	7798	08/1996	4.0099	6.7444	8.0189	10392
09/1993	4.9677	6.5203	7.6471	7545	09/1996	3.9742	6.3518	7.7952	10716
10/1993	4.5162	6.0578	7.2731	8053	10/1996	3.2846	5.6180	7.1300	11351
11/1993	4.1492	6.0281	7.3365	7926	11/1996	2.9249	5.2285	6.8254	12217
12/1993	4.0067	5.7807	7.4386	8220	12/1996	2.8604	5.6603	7.1537	12062
01/1994	3.5101	5.6379	7.1170	8670	01/1997	2.9279	5.7680	7.3660	12444

Date	SB	MB	LB	EQ	Date	SB	MB	LB	EQ
02/1997	2.9051	5.6575	7.1737	12560	02/2000	4.9150	6.1641	6.2328	19530
03/1997	3.1987	6.1727	7.4315	11961	03/2000	5.2938	5.9885	6.0908	20277
04/1997	3.1798	6.1102	7.2555	12227	04/2000	5.4786	6.1869	6.1547	20039
05/1997	3.0400	5.8922	7.1220	13079	05/2000	5.5748	6.0783	5.9954	19853
06/1997	3.0541	5.7890	6.9313	13223	06/2000	5.6727	5.8959	5.8641	21912
07/1997	3.3529	5.2779	6.3283	14135	07/2000	5.5904	5.9535	5.9063	22374
08/1997	3.2016	5.4979	6.5882	13605	08/2000	5.6852	5.7485	5.7397	24204
09/1997	3.2167	5.2423	6.3059	14514	09/2000	5.6585	5.7349	5.7855	22360
10/1997	3.8873	5.0478	6.0131	14116	10/2000	5.5788	5.8155	5.8483	20777
11/1997	3.9513	5.3198	5.9854	13455	11/2000	5.6710	5.5071	5.6338	19033
12/1997	4.0837	5.3905	5.9621	13869	12/2000	5.5652	5.2830	5.6281	19309
01/1998	4.6694	5.1961	5.7260	13882	01/2001	5.1204	5.1494	5.7190	20161
02/1998	4.7626	5.2443	5.7717	14711	02/2001	4.9152	5.0998	5.6275	17486
03/1998	4.5331	5.1342	5.6678	15706	03/2001	4.7303	5.1058	5.7048	16504
04/1998	4.8320	5.1785	5.6355	15939	04/2001	4.4374	5.4284	6.0455	17246
05/1998	4.7799	5.1694	5.5417	15800	05/2001	4.4267	5.5751	6.0228	17735
06/1998	4.7443	5.2510	5.4795	15367	06/2001	4.4227	5.6639	6.0996	16849
07/1998	5.0673	5.3751	5.4670	14469	07/2001	4.0130	5.3470	5.9429	16758
08/1998	5.5437	5.6149	5.7268	11560	08/2001	3.8385	5.0158	5.7077	16145
09/1998	4.9073	4.7762	5.1329	11762	09/2001	3.1486	4.7298	5.8849	14954
10/1998	4.8440	4.8044	5.3345	13018	10/2001	2.4011	4.2245	5.3741	15071
11/1998	4.7790	4.8389	5.1780	13319	11/2001	2.1656	4.6228	5.5568	16270
12/1998	4.7258	4.7274	5.0469	13649	12/2001	1.9790	4.7734	5.7405	16882
01/1999	4.7803	4.7606	4.7815	14169	01/2002	2.0129	4.9176	5.7430	16808
02/1999	4.8805	5.1280	5.3470	13306	02/2002	2.0891	4.7969	5.6942	16802
03/1999	4.6523	4.9392	5.1709	13938	03/2002	2.2136	5.4074	5.9753	17308
04/1999	4.6418	4.9857	5.2432	14829	04/2002	2.3800	5.1584	5.8758	16904
05/1999	4.5064	5.2932	5.3757	14481	05/2002	2.6438	5.0409	5.7527	16911
06/1999	4.4996	5.3620	5.6692	14865	06/2002	2.7636	4.9186	5.7482	15820
07/1999	4.7439	5.6239	5.9049	15029	07/2002	2.8049	4.5911	5.7252	14638
08/1999	4.7463	5.6811	5.9753	14811	08/2002	3.0284	4.4613	5.4245	14671
09/1999	4.6173	5.6211	6.0053	14812	09/2002	2.7813	4.2060	5.3176	13747
10/1999	4.7511	5.9569	6.3100	15459	10/2002	2.7253	4.2805	5.5125	13914
11/1999	4.6213	6.0662	6.3583	16045	11/2002	2.8273	4.4319	5.5380	14648
12/1999	4.9466	6.1753	6.3659	17961	12/2002	2.6924	4.0778	5.3421	14782
01/2000	5.0120	6.5174	6.6155	18130	01/2003	2.8684	4.3973	5.4623	14702

Date	SB	MB	LB	EQ	Date	SB	MB	LB	EQ
02/2003	2.9837	4.3296	5.4180	14699	02/2006	3.7527	4.0230	4.1802	27663
03/2003	3.1028	4.5201	5.5275	14262	03/2006	3.8902	4.1356	4.2903	28742
04/2003	3.1924	4.3439	5.3924	14819	04/2006	4.0777	4.3006	4.5291	28998
05/2003	3.2807	3.8213	4.9493	15459	05/2006	4.2216	4.3053	4.5075	27964
06/2003	3.1740	3.8127	5.0093	15777	06/2006	4.3067	4.4502	4.6220	27735
07/2003	2.8952	4.0775	5.3498	16409	07/2006	4.2029	4.1656	4.3838	28298
08/2003	2.8285	4.1747	5.2824	17004	08/2006	4.1462	3.9678	4.1980	28939
09/2003	2.6765	3.8248	5.0445	16834	09/2006	4.1309	3.8606	4.0951	28267
10/2003	2.7811	4.1528	5.2694	17649	10/2006	4.1473	3.8932	4.1026	29707
11/2003	2.7701	4.2132	5.2611	17869	11/2006	4.1420	3.7742	3.9969	30753
12/2003	2.6675	4.0432	5.1261	18732	12/2006	4.2293	3.9540	4.1470	31213
01/2004	2.4092	3.8121	5.0497	19436	01/2007	4.2390	4.0489	4.2368	31574
02/2004	2.2564	3.6239	4.8965	20066	02/2007	4.1999	3.9149	4.1164	31654
03/2004	2.0483	3.5736	4.8809	19643	03/2007	4.1768	3.9738	4.2098	32026
04/2004	2.0508	3.9248	5.1033	18879	04/2007	4.1627	4.0473	4.2020	32687
05/2004	2.1045	4.2085	5.1785	19304	05/2007	4.2965	4.4640	4.4000	34319
06/2004	2.1224	4.3085	5.1976	19638	06/2007	4.4312	4.5024	4.5013	34038
07/2004	2.1961	4.2447	5.1350	19457	07/2007	4.5839	4.4923	4.4781	33995
08/2004	2.2923	4.0091	4.9664	19299	08/2007	4.1053	4.3240	4.4457	33555
09/2004	2.6313	4.0740	4.9330	20007	09/2007	4.0673	4.1713	4.4278	34715
10/2004	2.7486	3.9482	4.8235	20495	10/2007	4.1307	4.1759	4.3732	36072
11/2004	2.6333	3.8371	4.8405	20893	11/2007	3.9518	3.7821	4.1529	33830
12/2004	2.5147	3.7388	4.6648	21445	12/2007	3.8868	3.8592	4.1169	34282
01/2005	2.5020	3.6517	4.5666	21360	01/2008	3.4312	3.5587	4.2133	32665
02/2005	2.5148	3.7387	4.6540	22464	02/2008	3.1067	3.2258	4.1272	33791
03/2005	2.5913	3.8428	4.6184	22379	03/2008	2.1505	2.9824	3.9987	33308
04/2005	2.5172	3.6640	4.4748	21847	04/2008	2.7420	3.1263	4.1580	34839
05/2005	2.4601	3.4663	4.3414	22434	05/2008	2.6529	3.3392	4.2078	36857
06/2005	2.4949	3.2984	4.1651	23181	06/2008	2.6537	3.4595	4.1833	36336
07/2005	2.6421	3.4723	4.2575	24412	07/2008	2.4859	3.2899	4.2398	34206
08/2005	2.7393	3.3585	4.1037	25023	08/2008	2.5177	3.0755	4.1777	34735
09/2005	2.9425	3.6013	4.2136	25878	09/2008	1.9369	3.2564	4.4027	29717
10/2005	3.1340	3.8714	4.3649	24415	10/2008	1.7830	2.9831	4.6208	24763
11/2005	3.3949	3.8550	4.1986	25494	11/2008	1.6550	2.6011	4.2510	23591
12/2005	3.4830	3.8748	4.0528	26619	12/2008	0.7806	1.8839	3.7385	22968
01/2006	3.5489	4.0091	4.2512	28232	01/2009	0.8829	2.2697	4.1063	22288

Date	SB	MB	LB	EQ	Date	SB	MB	LB	EQ
02/2009	0.5979	2.0489	4.0945	20881	02/2012	0.9179	1.4539	2.5268	35341
03/2009	0.3429	1.7834	3.8446	22508	03/2012	0.9111	1.6099	2.5763	34765
04/2009	0.2032	2.0722	4.1474	24142	04/2012	1.1334	1.6351	2.5274	34558
05/2009	0.2452	2.5155	4.2760	26909	05/2012	0.9504	1.2983	2.1710	32436
06/2009	0.3030	2.5575	4.1615	27002	06/2012	0.9069	1.3000	2.2106	32793
07/2009	0.2161	2.7165	4.3014	28141	07/2012	1.0065	1.3185	2.1345	33055
08/2009	0.2360	2.5948	4.2486	28407	08/2012	1.0263	1.3725	2.2320	33930
09/2009	0.2128	2.5709	4.1656	29868	09/2012	1.0035	1.3127	2.2147	35094
10/2009	0.2287	2.6912	4.2541	28660	10/2012	1.0228	1.3683	2.2933	35469
11/2009	0.2188	2.4166	4.1759	30138	11/2012	1.0124	1.3180	2.1888	35014
12/2009	0.1228	2.8326	4.4066	31019	12/2012	0.9629	1.4200	2.2538	35697
01/2010	0.1338	2.5288	4.1873	29361	01/2013	0.9531	1.5568	2.4718	36501
02/2010	0.1895	2.5982	4.2232	30821	02/2013	0.9607	1.3558	2.4088	36959
03/2010	0.2404	2.9156	4.1965	31994	03/2013	1.0172	1.3407	2.3723	36888
04/2010	0.3990	3.0249	4.0863	32527	04/2013	1.0599	1.2064	2.1990	36124
05/2010	0.5137	2.7885	3.7893	31396	05/2013	1.0645	1.5419	2.5189	36763
06/2010	0.5223	2.4118	3.6816	30230	06/2013	1.0590	1.8367	2.8199	35382
07/2010	0.7690	2.3738	3.7686	31427	07/2013	1.0279	1.8246	2.8797	36510
08/2010	0.8114	2.0209	3.5129	32023	08/2013	1.0622	1.9807	3.0371	37075
09/2010	0.9710	1.9924	3.4133	33332	09/2013	1.0109	1.8873	3.0078	37594
10/2010	0.9400	1.9754	3.4457	34235	10/2013	0.9852	1.7375	2.8972	39369
11/2010	1.0469	2.3673	3.5621	35047	11/2013	1.0055	1.7717	3.0620	39548
12/2010	1.0668	2.4437	3.5849	36481	12/2013	1.0307	1.9941	3.2043	40334
01/2011	0.9780	2.5541	3.8076	36840	01/2014	0.9642	1.6007	2.8247	40664
02/2011	1.0069	2.6832	3.7607	38475	02/2014	0.9163	1.5979	2.8459	42260
03/2011	0.9733	2.7411	3.8103	38523	03/2014	0.9546	1.6987	2.9000	42779
04/2011	1.0439	2.5742	3.7299	38129	04/2014	1.0286	1.6743	2.8383	43816
05/2011	0.9462	2.3607	3.4566	37799	05/2014	0.9974	1.5537	2.6905	43744
06/2011	0.9744	2.3512	3.5040	36540	06/2014	0.9884	1.5835	2.7070	45523
07/2011	0.9561	2.0400	3.2271	35627	07/2014	1.0085	1.5630	2.6008	46169
08/2011	0.9187	1.7008	3.0201	35196	08/2014	0.9651	1.4798	2.4259	47133
09/2011	0.7997	1.4210	2.6925	32148	09/2014	0.9592	1.6039	2.5581	45254
10/2011	0.8607	1.5639	2.8386	33950	10/2014	1.0016	1.5386	2.4829	44318
11/2011	0.8693	1.5046	2.6456	33879	11/2014	0.9014	1.3773	2.3103	44789
12/2011	0.8582	1.3362	2.4434	33303	12/2014	0.9274	1.3523	2.2478	44591
01/2012	0.9010	1.3095	2.4287	34759	01/2015	0.6000	0.6482	1.7068	44836

Date	SB	MB	LB	EQ	Date	SB	MB	LB	EQ
02/2015	0.5812	0.6918	1.7975	46620	02/2016				40593
03/2015	0.5444	0.7524	1.8785	45743					
04/2015	0.6606	0.9844	2.0794	46854					
05/2015	0.6308	0.9246	2.1139	46283					
06/2015	0.5719	0.8985	2.2292	44995					
07/2015	0.4062	0.7283	2.0450	44853					
08/2015	0.3740	0.7839	2.1440	43043					
09/2015	0.4312	0.8430	2.1027	41461					
10/2015	0.4345	0.9244	2.2252	42272					
11/2015	0.4861	0.9735	2.2157	42175					
12/2015	0.5050	0.8070	2.0887	40882					
01/2016	0.4648	0.6620	1.9382	40404					

Appendix B

Mortality Rates

Table B.1: Mortality Rates

Age	Rate	Age	Rate	Age	Rate	Age	Rate
1	0	30	0.0012	59	0.00587	88	0.11026
2	0	31	0.00122	60	0.00628	89	0.12454
3	0	32	0.00122	61	0.00666	90	0.14041
4	0	33	0.0012	62	0.00702	91	0.15801
5	0	34	0.0012	63	0.00743	92	0.1775
6	0	35	0.0012	64	0.0079	93	0.19909
7	0	36	0.0012	65	0.00844	94	0.22299
8	0	37	0.00122	66	0.00907	95	0.24808
9	0	38	0.00125	67	0.00981	96	0.27346
10	0	39	0.0013	68	0.01066	97	0.29848
11	0	40	0.00136	69	0.01166	98	0.32273
12	0	41	0.00144	70	0.01282	99	0.34602
13	0	42	0.00154	71	0.01417	100	0.36843
14	0	43	0.00165	72	0.01571	101	0.39026
15	0	44	0.00178	73	0.01749	102	0.41203
16	0	45	0.0019	74	0.01952	103	0.43454
17	0	46	0.00205	75	0.02183	104	0.45879
18	0.00067	47	0.00219	76	0.02449	105	0.47904
19	0.00075	48	0.00234	77	0.02754	106	0.49928
20	0.00082	49	0.0025	78	0.03105	107	0.5195
21	0.00089	50	0.00266	79	0.03511	108	0.5397
22	0.00095	51	0.00285	80	0.03981	109	0.55987
23	0.00101	52	0.00307	81	0.04522	110	0.58
24	0.00105	53	0.00333	82	0.05144	111	0.6
25	0.00108	54	0.00365	83	0.05854	112	0.62
26	0.00113	55	0.00403	84	0.0666	113	0.64
27	0.00116	56	0.00448	85	0.07571	114	0.66
28	0.00117	57	0.00495	86	0.08596	115	1
29	0.00119	58	0.00542	87	0.09744		

Appendix C

Relationship Between α_{t-1} and α_t^*

This appendix proves that the accrual rate (α_t^*) that would be determined based on our affordability test at time t if there were no new entrants at time t is equal to α_{t-1} as long as the assumptions do not change and plan experience is exactly in line with these assumptions during the period $[t-1,t)$. That is, j_{t-1}, j_t, and i_{t-1} are all equal to j. Since the valuation rate does not change, we drop the time indicator on the annuity factors so $\ddot{a}_x(t)$=$\ddot{a}_x(t-1)$=$\ddot{a}_x$.

Based on Equation (3.8), the accrual rate determined at the last valuation (time $t-1$) is

$$\alpha_{t-1} = \frac{U \cdot TPVFSal_{t-1} + F_{t-1}}{TPVBCE_{t-1}}. \tag{C.1}$$

We split the denominator into three parts:

$$TPVBCE_{t-1} = TPVBPCE_{A,t-1} + TPVBPCE_{R,t-1} + TPVBFCE_{t-1},$$

where

$$TPVBPCE_{A,t-1} = \sum_{x=e}^{r-1} n_x \cdot PCE_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x$$

relates to past earnings of active members at the last valuation (ages e to $r-1$ at time $t-1$),

$$TPVBPCE_{R,t-1} = \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} \cdot \ddot{a}_x$$

relates to past earnings of retired members at the last valuation (age r to ω at time $t-1$), and

$$TPVBFCE_{t-1} = \sum_{x=e}^{r-1} n_x \cdot FCE_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x$$

relates to future earnings of active members at the last valuation.

So the accrual rate at time $t-1$ is:

$$\alpha_{t-1} = \frac{U \cdot TPVFSal_{t-1} + F_{t-1}}{TPVBPCE_{A,t-1} + TPVBPCE_{R,t-1} + TPVBFCE_{t-1}}. \tag{C.2}$$

We now consider the accrual rate at time t for members already in the plan at the last valuation (at time $t-1$), given that our assumptions are exactly in line with the plan experience since time $t-1$. Then, we have:

$$\alpha_t^* = \frac{U \cdot TPVFSal_t^* + F_t^*}{TPVBCE_t^*} \tag{C.3}$$

$$= \frac{U \cdot TPVFSal_t^* + F_t^*}{TPVBPCE_{A,t}^* + TPVBPCE_{R,t}^* + TPVBFCE_t^*} \tag{C.4}$$

where

$$TPVFSal_t^* = \sum_{x=e+1}^{r-1} n_x \cdot S_{x,t} \cdot \ddot{a}^s_{x:\overline{r-x}|},$$

$$TPVBPCE_{A,t}^* = \sum_{x=e+1}^{r-1} n_x \cdot PCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x,$$

$$TPVBPCE_{R,t}^* = \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t} \cdot \ddot{a}_x,$$

$$TPVBFCE_t^* = \sum_{x=e+1}^{r-1} n_x \cdot FCE_{x,t} \cdot {}_{r-x|}\ddot{a}_x,$$

$$TPVBCE_t^* = TPVBPCE_{A,t}^* + TPVBPCE_{R,t}^* + TPVBFCE_t^*,$$

and F_t^* is the asset value assuming investment returns during the period $[t-1,t)$ matches the valuation rate at the last valuation,

$$F_t^* = (F_{t-1} + C_{t-1} - B_{t-1}) \cdot (1+j).$$

Note that C_{t-1} is the contribution made at time $t-1$ based on the original contribution rate U (determined at t=0 using the target α_0) and actual salaries at time $t-1$. Note also that the accrual rate determined at the last valuation (α_{t-1}) is likely to differ from the target accrual rate α_0.

In order to prove $\alpha_t^* = \alpha_{t-1}$, we need to relate $TPVFSal_{t-1}$, $TPVBPCE_{A,t-1}$, $TPVBPCE_{R,t-1}$, $TPVBFCE_{t-1}$ to $TPVFSal_t^*$, $TPVBPCE_{A,t}^*$, $TPVBPCE_{R,t}^*$, $TPVBFCE_t^*$ respectively.

Since there are no decrements before retirement, we have:

$$\ddot{a}^s_{x:\overline{r-x}|} = 1 + \frac{1+s}{1+j} + \cdots + (\frac{1+s}{1+j})^{r-x-1}$$

$$= 1 + \frac{1+s}{1+j} \cdot \ddot{a}^s_{x+1:\overline{r-(x+1)}|} \qquad \text{for } e \leq x < r.$$

Also,

$$
\begin{aligned}
TPVFSal_{t-1} &= \sum_{x=e}^{r-1} n_x \cdot S_{x,t-1} \cdot \ddot{a}^s_{x:\overline{r-x}|} \\
&= \sum_{x=e}^{r-2} n_x \cdot S_{x,t-1} \cdot \ddot{a}^s_{x:\overline{r-x}|} + n_{r-1} \cdot S_{r-1,t-1} \\
&= \sum_{x=e}^{r-2} n_x \cdot S_{x,t-1} \cdot \frac{1+s}{1+j} \cdot \ddot{a}^s_{x+1:\overline{r-(x+1)}|} + \sum_{x=e}^{r-1} n_x \cdot S_{x,t-1} \\
&= \sum_{x=e}^{r-2} n \cdot S_{x+1,t} \cdot \frac{1}{1+j} \cdot \ddot{a}^s_{x+1:\overline{r-(x+1)}|} + \sum_{x=e}^{r-1} n \cdot S_{x,t-1},
\end{aligned}
$$

where n=n_x for $e \leq x < r$ under our model.

Let y=x+1 and change the index of summation from x to y:

$$
\begin{aligned}
TPVFSal_{t-1} &= \frac{1}{1+j} \cdot \sum_{y=e+1}^{r-1} n \cdot S_{y,t} \cdot \ddot{a}^s_{y:\overline{r-y}|} + \sum_{x=e}^{r-1} n \cdot S_{x,t-1} \\
&= \frac{1}{1+j} \cdot TPVFSal^*_t + \sum_{x=e}^{r-1} n \cdot S_{x,t-1}. \qquad \text{(C.5)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&TPVBPCE_{A,t-1} \\
&= \sum_{x=e}^{r-1} n_x \cdot PCE_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x \\
&= \sum_{x=e}^{r-2} n_x \cdot PCE_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x \ + \ n_{r-1} \cdot PCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} \\
&= \sum_{x=e}^{r-2} n_x \cdot (PCE_{x,t-1} + S_{x,t-1}) \cdot {}_{r-x|}\ddot{a}_x \\
&\qquad + n_{r-1} \cdot PCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} \ - \ \sum_{x=e}^{r-2} n_x \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x \\
&= \sum_{x=e}^{r-2} n \cdot PCE_{x+1,t} \cdot {}_{r-x|}\ddot{a}_x + n \cdot PCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} \ - \ \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x
\end{aligned}
$$

We let y=x+1, and note that ${}_{r-x|}\ddot{a}_x = \frac{1}{1+j} \cdot {}_{r-y|}\ddot{a}_y$ since there are no decrements before retirement.

We change the index of summation from x to y:

$$\begin{aligned}
&TPVBPCE_{A,t-1} \\
&= \sum_{y=e+1}^{r-1} n \cdot PCE_{y,t} \cdot \frac{1}{1+j} \cdot {}_{r-y|}\ddot{a}_y \; + \; n \cdot PCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} - \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x \\
&= \frac{1}{1+j} \cdot TPVBPCE^*_{A,t} \; + \; n \cdot PCE_{r-1,t-1} \cdot \frac{1}{1+j} \cdot \ddot{a}_r - \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x. \qquad \text{(C.6)}
\end{aligned}$$

We also have

$$\begin{aligned}
TPVBPCE_{R,t-1} &= \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} \cdot \ddot{a}_x \\
&= \sum_{x=r}^{\omega-1} n_x \cdot PCE_{x,t-1} \cdot (1 + \frac{p_x}{1+j} \cdot \ddot{a}_{x+1}) \; + \; n_\omega \cdot PCE_{\omega,t-1} \\
&= \sum_{x=r}^{\omega-1} n_x \cdot PCE_{x,t-1} \cdot \frac{1}{1+j} \cdot p_x \cdot \ddot{a}_{x+1} \; + \; \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1}
\end{aligned}$$

We let y=x+1, and note that for retired members ($x \geq r$), $PCE_{x,t-1} = PCE_{x+1,t} = PCE_{y,t}$, and $n_x \cdot p_x = n_{x+1} = n_y$. Changing the index of the summation from x to y, we have

$$\begin{aligned}
&TPVBPCE_{R,t-1} \\
&= \frac{1}{1+j} \cdot \sum_{y=r+1}^{\omega} n_y \cdot PCE_{y,t} \cdot \ddot{a}_y \; + \; \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} \\
&= \frac{1}{1+j} \cdot \sum_{y=r}^{\omega} n_y \cdot PCE_{y,t} \cdot \ddot{a}_y \; - \; \frac{1}{1+j} \cdot n_r \cdot PCE_{r,t} \cdot \ddot{a}_r + \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} \\
&= \frac{1}{1+j} \cdot TPVBPCE^*_{R,t} \; - \; \frac{1}{1+j} \cdot n_r \cdot PCE_{r,t} \cdot \ddot{a}_r + \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} \\
&= \frac{1}{1+j} \cdot TPVBPCE^*_{R,t} \; - \; \frac{1}{1+j} \cdot n \cdot PCE_{r,t} \cdot \ddot{a}_r + \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} \qquad \text{(C.7)}
\end{aligned}$$

as $n = n_r$.

Finally,

$$
\begin{aligned}
&TPVBFCE_{t-1} \\
&= \sum_{x=e}^{r-1} n_x \cdot FCE_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x \\
&= \sum_{x=e}^{r-2} n_x \cdot FCE_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x \;+\; n_{r-1} \cdot FCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} \\
&= \sum_{x=e}^{r-2} n_x \cdot (FCE_{x,t-1} - S_{x,t-1}) \cdot {}_{r-x|}\ddot{a}_x + \sum_{x=e}^{r-2} n_x \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x + n_{r-1} \cdot FCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} \\
&= \sum_{x=e}^{r-2} n \cdot FCE_{x+1,t} \cdot {}_{r-x|}\ddot{a}_x + n \cdot FCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} \;+\; \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x
\end{aligned}
$$

We let y=x+1, and change the index of summation from x to y:

$$
\begin{aligned}
&TPVBFCE_{t-1} \\
&= \sum_{y=e+1}^{r-1} n \cdot FCE_{y,t} \cdot \frac{1}{1+j} \cdot {}_{r-y|}\ddot{a}_y \;+\; n \cdot FCE_{r-1,t-1} \cdot {}_{1|}\ddot{a}_{r-1} \;+\; \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x \\
&= \frac{1}{1+j} \cdot TPVBFCE_t^* \;+\; n \cdot FCE_{r-1,t-1} \cdot \frac{1}{1+j} \cdot \ddot{a}_r \;+\; \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x. \qquad \text{(C.8)}
\end{aligned}
$$

Applying Equations (C.5)-(C.8), the denominator of Equation (C.2) can be rewritten as

$$\begin{aligned}
&TPVBPCE_{A,t-1} + TPVBPCE_{R,t-1} + TPVBFCE_{t-1} \\
&= \Big(\frac{1}{1+j} \cdot TPVBPCE^*_{A,t} + n \cdot PCE_{r-1,t-1} \cdot \frac{1}{1+j} \cdot \ddot{a}_r - \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x\Big) \\
&\quad + \Big(\frac{1}{1+j} \cdot TPVBPCE^*_{R,t} - \frac{1}{1+j} \cdot n \cdot PCE_{r,t} \cdot \ddot{a}_r + \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1}\Big) \\
&\quad + \Big(\frac{1}{1+j} \cdot TPVBFCE^*_t + n \cdot FCE_{r-1,t-1} \cdot \frac{1}{1+j} \cdot \ddot{a}_r + \sum_{x=e}^{r-2} n \cdot S_{x,t-1} \cdot {}_{r-x|}\ddot{a}_x\Big) \\
&= \frac{1}{1+j} \cdot \Big(TPVBPCE^*_{A,t} + TPVBPCE^*_{R,t} + TPVBFCE^*_t\Big) \\
&\quad + n \cdot PCE_{r-1,t-1} \cdot \frac{1}{1+j} \cdot \ddot{a}_r - \frac{1}{1+j} \cdot n \cdot PCE_{r,t} \cdot \ddot{a}_r \\
&\quad + \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} + n \cdot FCE_{r-1,t-1} \cdot \frac{1}{1+j} \cdot \ddot{a}_r \\
&= \frac{1}{1+j} \cdot \Big(TPVBCE^*_t\Big) \; + \; \Big(n \cdot (PCE_{r-1,t-1} + FCE_{r-1,t-1}) \cdot \frac{1}{1+j} \cdot \ddot{a}_r \\
&\quad - \frac{1}{1+j} \cdot n \cdot PCE_{r,t} \cdot \ddot{a}_r + \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1}\Big) \\
&= \frac{1}{1+j} \cdot \Big(TPVBCE^*_t\Big) \; + \; \sum_{x=r}^{\omega} n_x \cdot PCE_{x,t-1} \\
&= \frac{1}{1+j} \cdot \Big(TPVBCE^*_t\Big) \; + \; \frac{B_{t-1}}{\alpha_{t-1}}.
\end{aligned} \tag{C.9}$$

The numerator of Equation (C.2) is:

$$\begin{aligned}
&U \cdot \Big(\frac{1}{1+j} \cdot TPVFSal^*_t + \sum_{x=e}^{r-1} n \cdot S_{x,t-1}\Big) \; + \; F_{t-1} \\
&= \frac{U}{1+j} \cdot TPVFSal^*_t \; + \; C_{t-1} \; + \; F_{t-1} \\
&= \frac{U}{1+j} \cdot TPVFSal^*_t + \frac{F^*_t}{1+j} + B_{t-1} \\
&= \frac{1}{1+j} \cdot \Big(U \cdot TPVFSal^*_t + F^*_t\Big) + B_{t-1}.
\end{aligned} \tag{C.10}$$

Equation (C.2) can then be rewritten as

$$\alpha_{t-1} \cdot \Big(\frac{TPVBCE^*_t}{1+j} + \frac{B_{t-1}}{\alpha_{t-1}}\Big) = \frac{1}{1+j} \cdot \Big(U \cdot TPVFSal^*_t + F^*_t\Big) + B_{t-1}$$

$$\Rightarrow \alpha_{t-1} \cdot TPVBCE^*_t = U \cdot TPVFSal^*_t + F^*_t.$$

Therefore, we obtain

$$\Rightarrow \alpha_{t-1} = \frac{U \cdot TPVFSal_t^* + F_t^*}{TPVBCE_t^*} = \alpha_t^*.$$

$\square$

Appendix D

Attribution of Changes in the Accrual Rate under Case 2 and Case 3

In this appendix, more details about the sources of changes in the accrual rate under Case 2 and Case 3 are provided. The three sources are changes in the valuation rate, investment experience, and new entrants. The attribution of the total change in the accrual rate to these three sources is given by Equation (4.3).

Figures D.1 to D.3 show relationships between the sources of changes under Case 2 and Figures D.4 to D.6 show these relationships under Case 3.

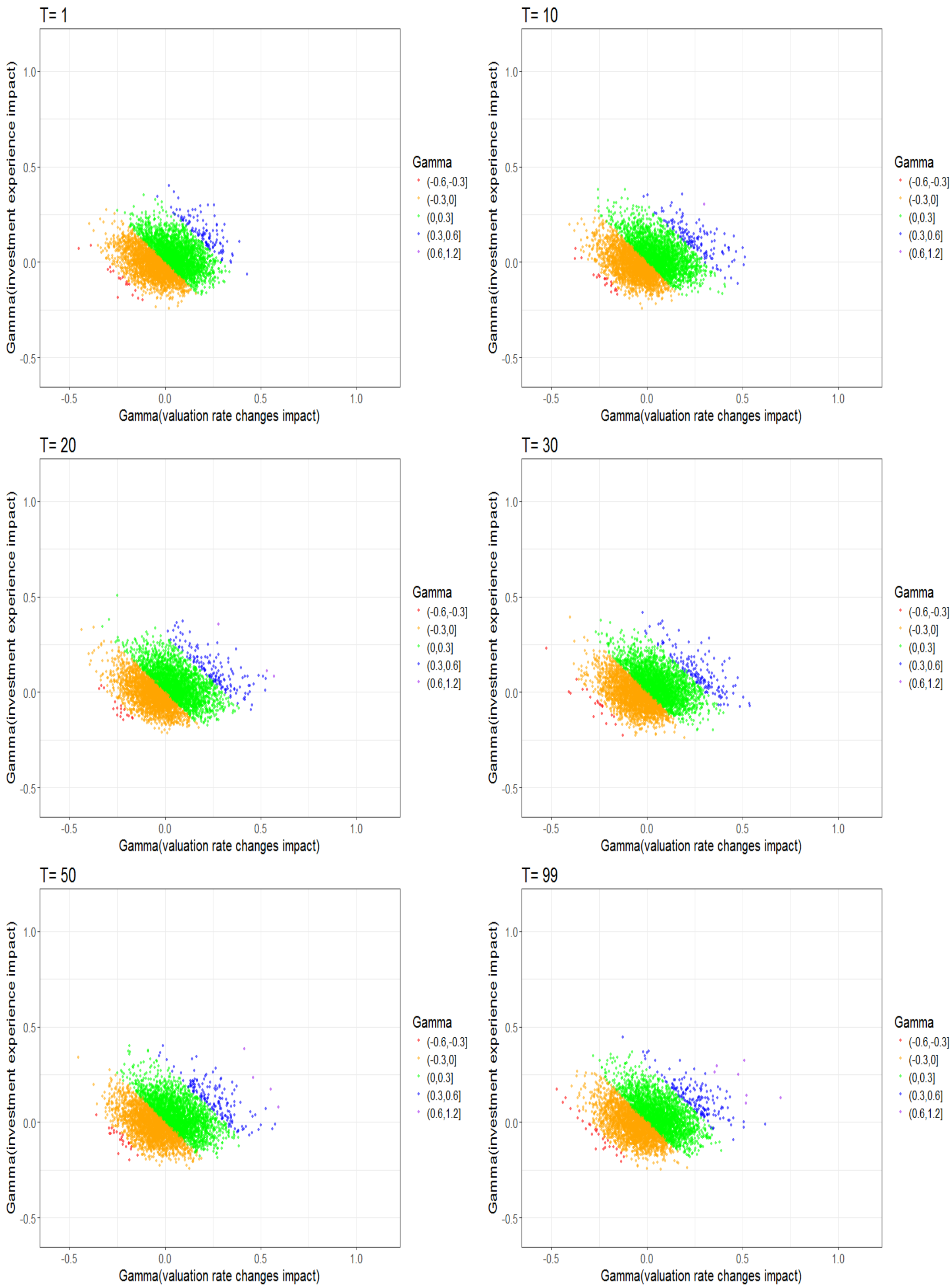

Figure D.1: Relationship between γ and each component under Case 2 (x-axis: $\gamma^{valrate}$, y-axis: γ^{invexp})

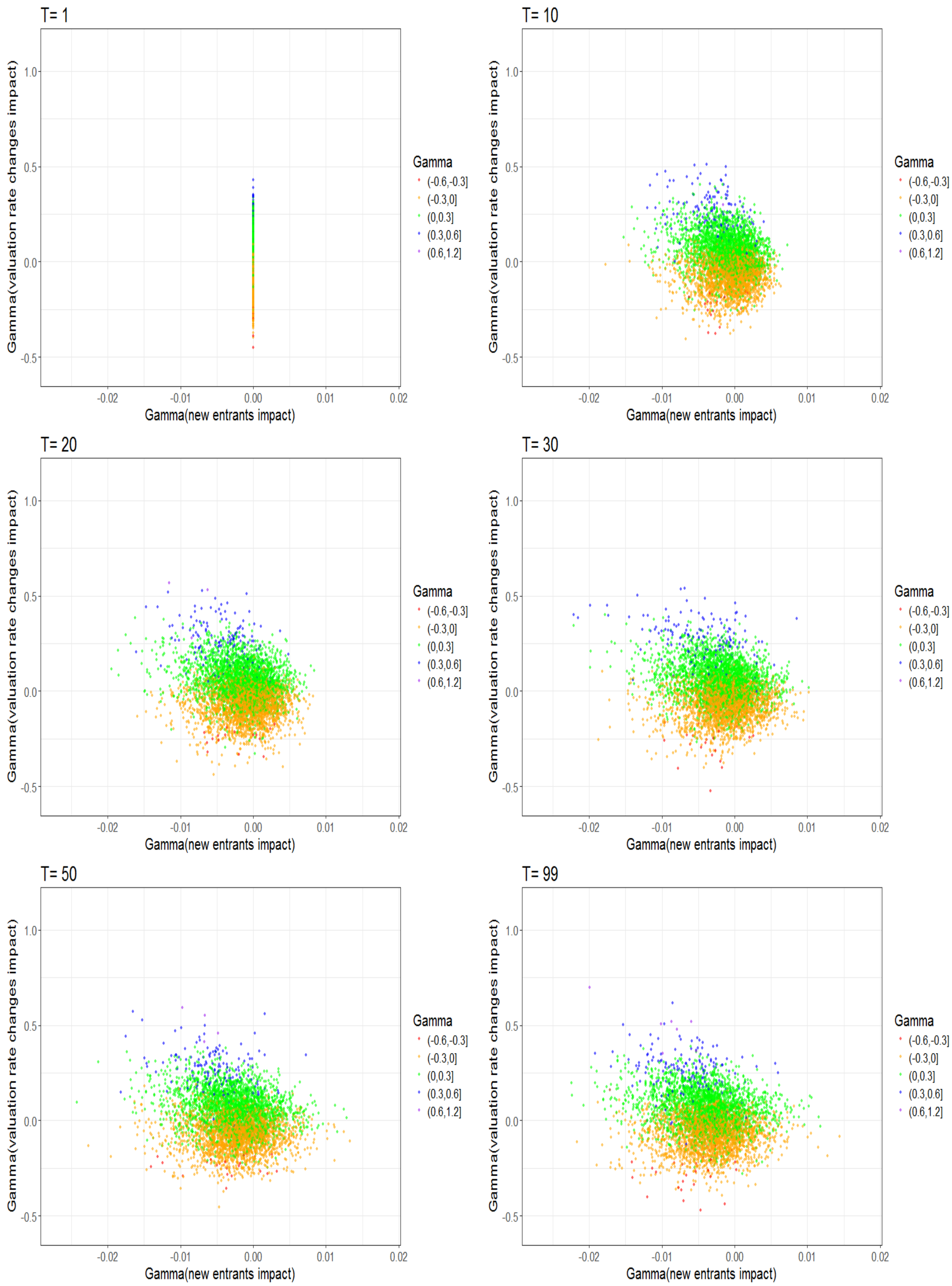

Figure D.2: Relationship between γ and each component under Case 2 (x-axis: γ^{newent}, y-axis: $\gamma^{valrate}$)

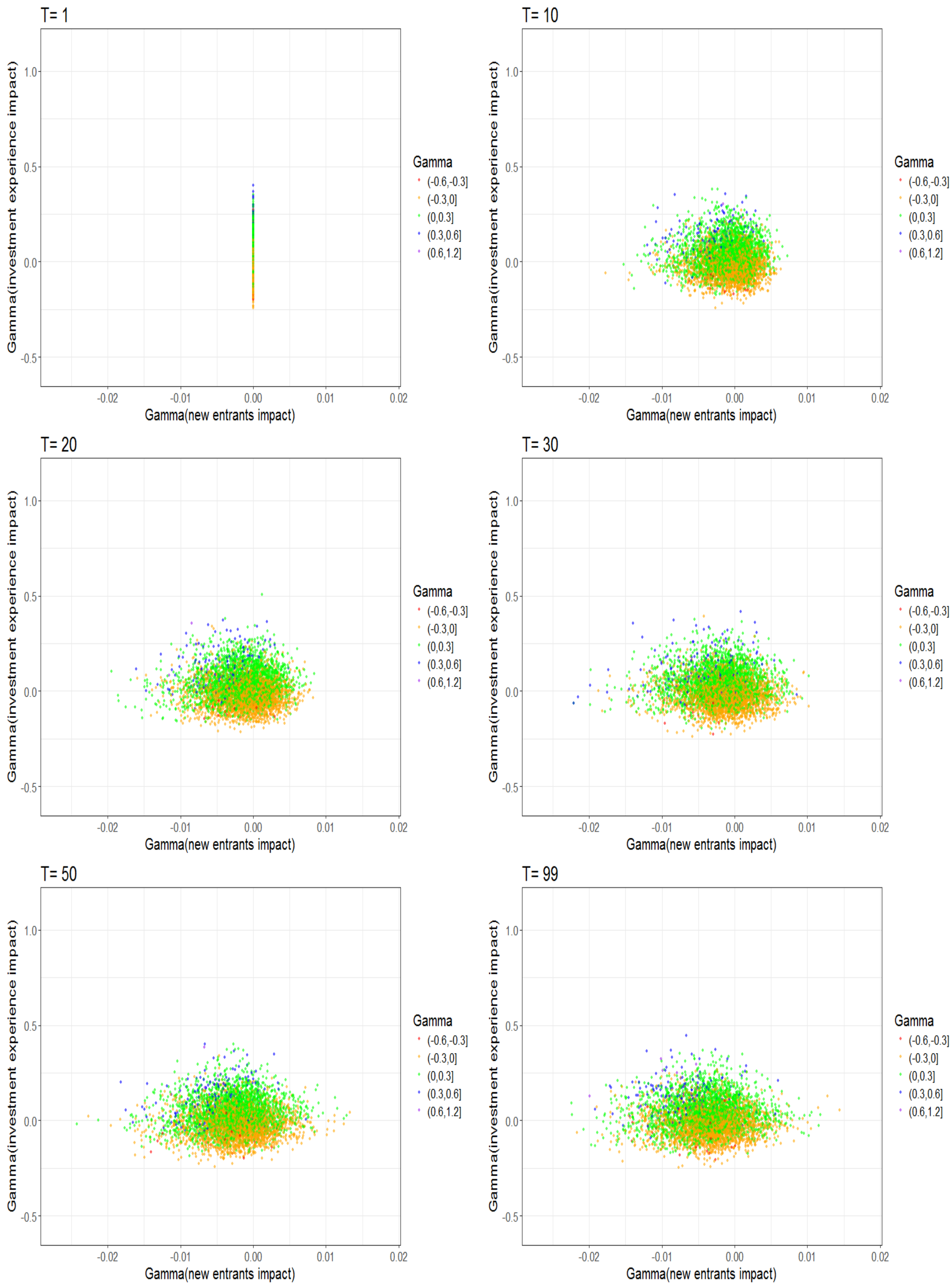

Figure D.3: Relationship between γ and each component under Case 2 (x-axis: γ^{newent}, y-axis: γ^{invexp})

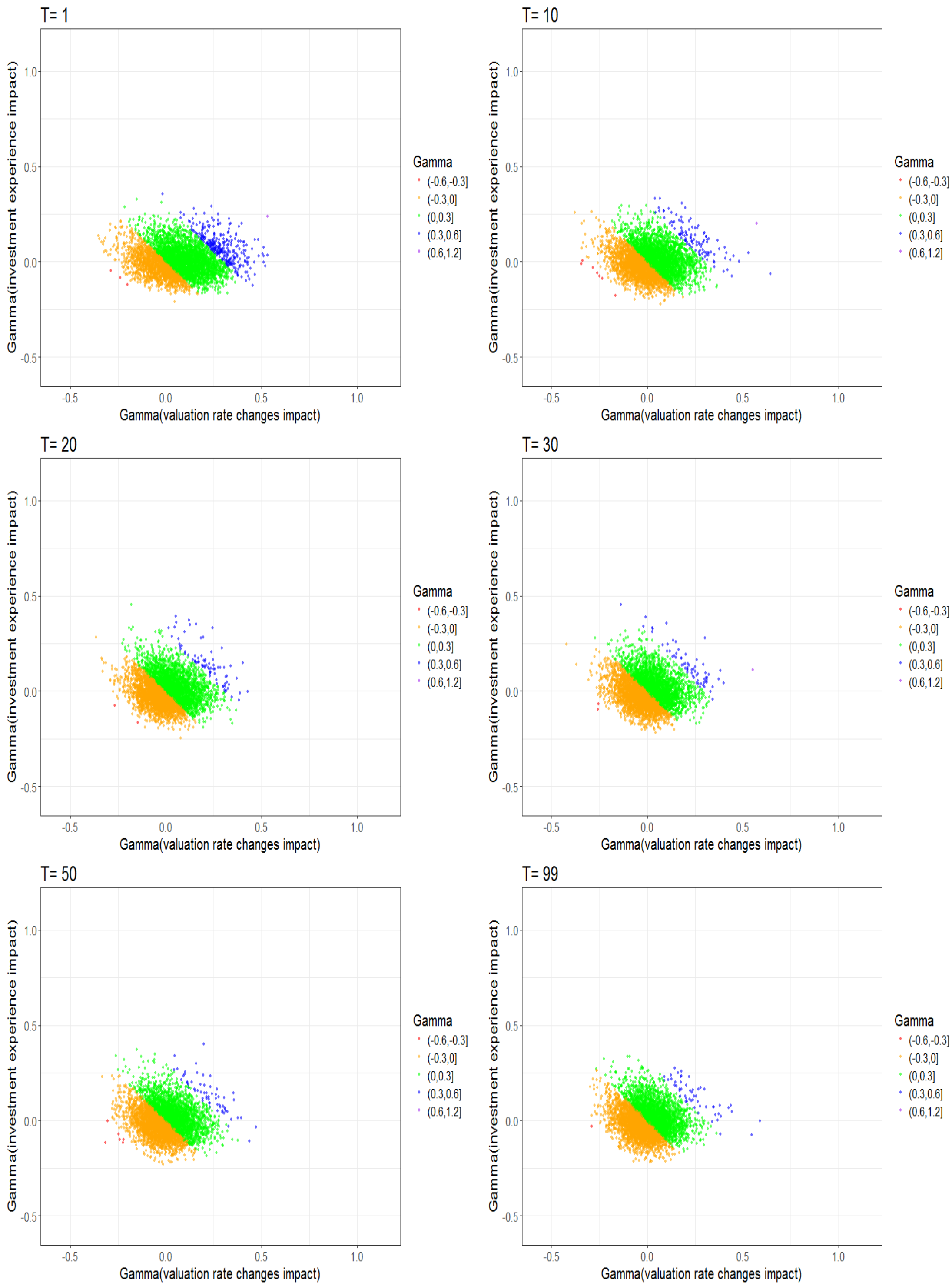

Figure D.4: Relationship between γ and each component under Case 3 (x-axis: $\gamma^{valrate}$, y-axis: γ^{invexp})

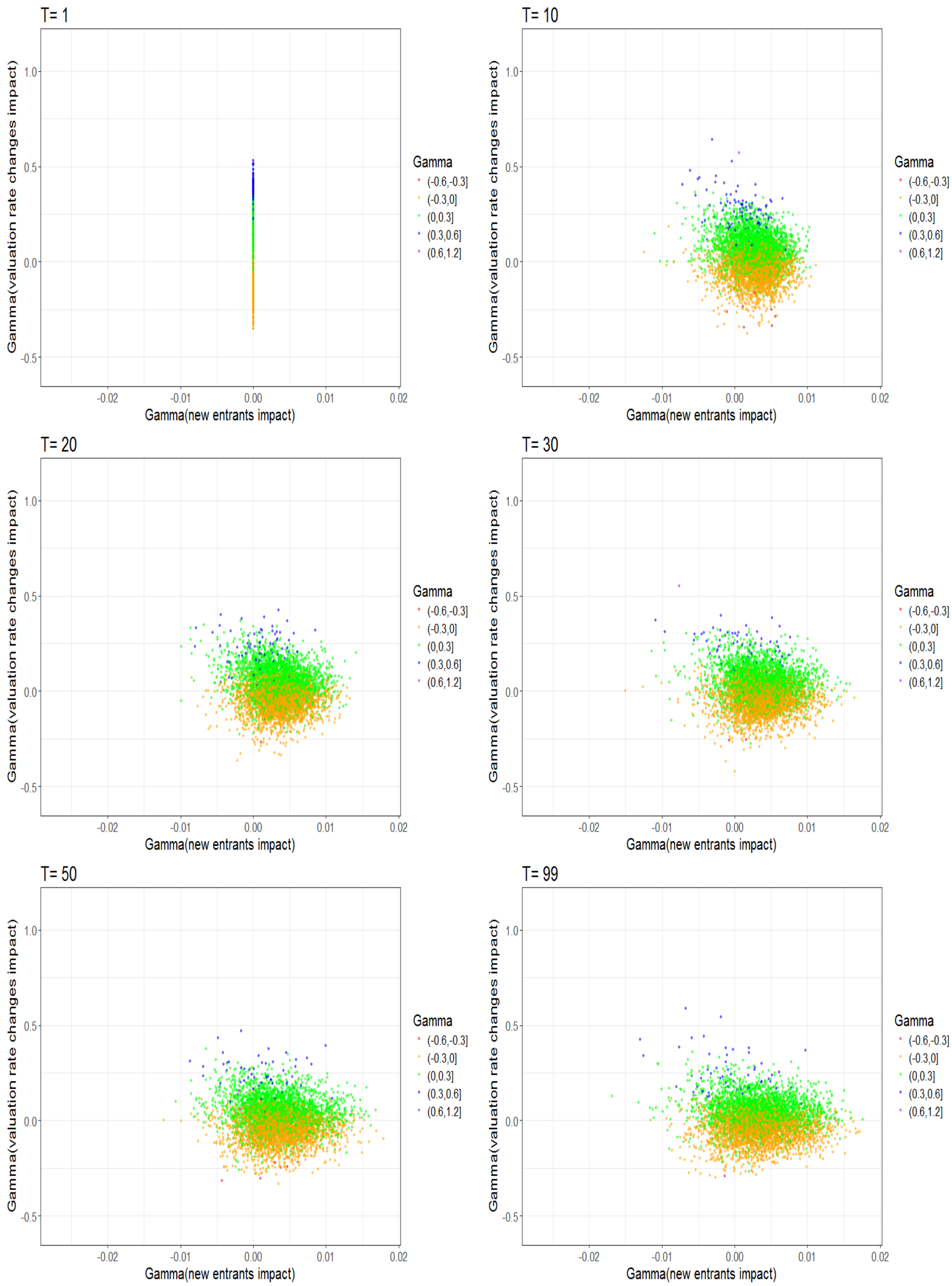

Figure D.5: Relationship between γ and each component under Case 3 (x-axis: γ^{newent}, y-axis: $\gamma^{valrate}$)

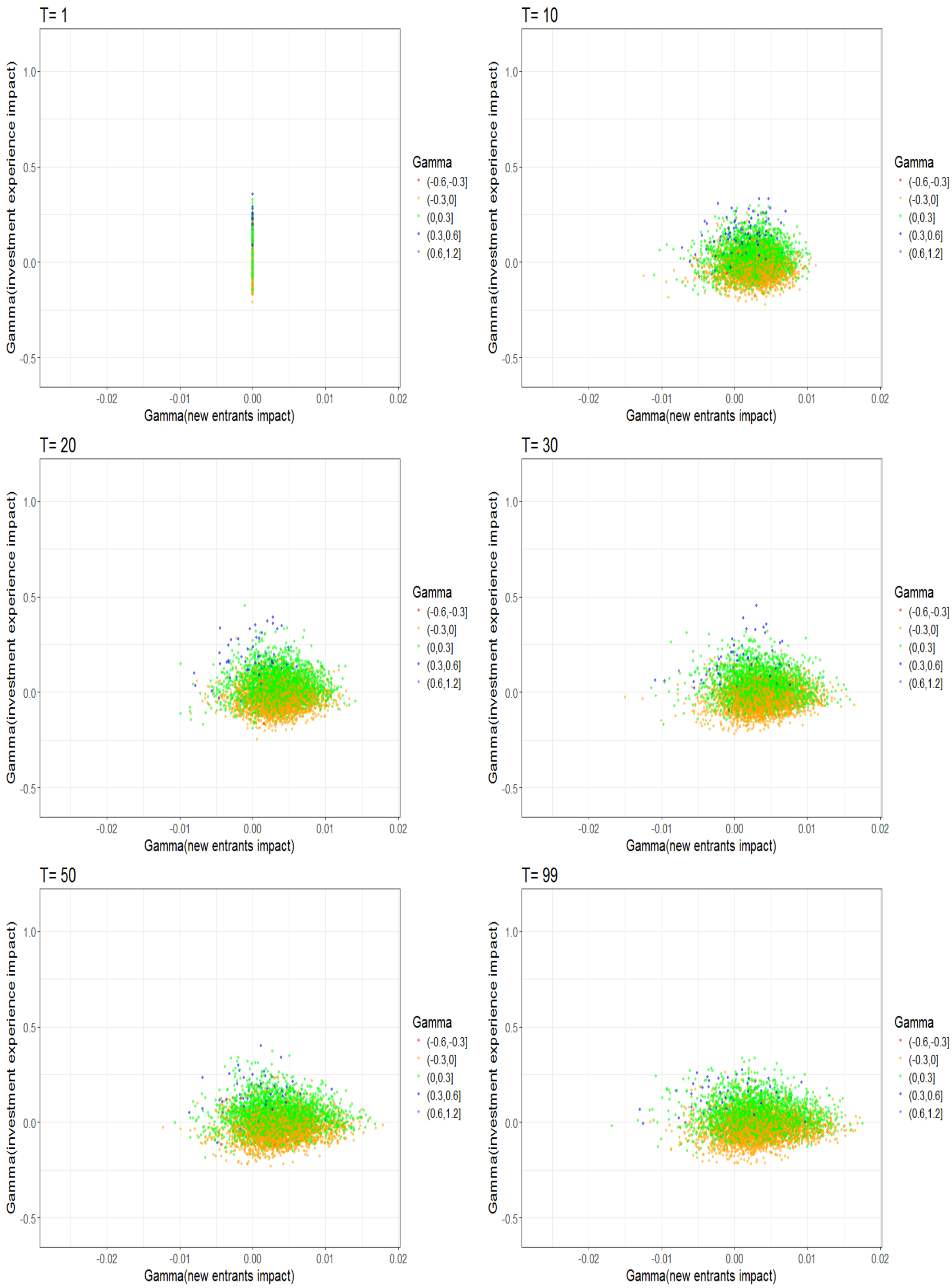

Figure D.6: Relationship between γ and each component under Case 3 (x-axis: γ^{newent}, y-axis: γ^{invexp})