Students' understanding of transformations of sinusoidal functions

by

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Dissertation Submitted in Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy

in the

Mathematics Education Program

Faculty of Education

SIMON FRASER UNIVERSITY

Spring 2017

Approval

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Abstract

Trigonometry is one of the fundamental topics taught in high school and university curricula. However, it is considered as one of the most challenging subjects for teaching and learning. Contributing to research on learning trigonometry, this dissertation sheds light on aspects of undergraduate students' understanding of transformations of sinusoidal functions.
Six undergraduate students participated in the study. Two types of tasks – (A) Identifying sinusoidal functions and (B) Assigning coordinates – were presented to participants in a clinical interview.

To analyze the collected data, three theoretical frameworks, *Mason's theory of shifts of attention, Presmeg's visual imagery* and Carlson, Jacobs, Coe, Larsen, and Hsu *covariational reasoning* were used in this dissertation. Mason's theory provided opportunity to study the critical role of attention and awareness in learning and understanding mathematics, and in particular the concept of transformation of sinusoidal functions. Presmeg's classification of visual imagery was applied for investigating students' visual mental constructs since the participants applied their imagery on different occasions when they completed the interview tasks. Lastly, participants' solution approaches were evaluated using covariational reasoning, focusing on Carlson's et al. description of mental actions associated with developmental levels.

The results of this research show that undergraduate students participating in this study experienced difficulty in identifying a phase shift/ horizontal transformation of the sinusoidal functions. They, in fact, determined "BC" as the amount of the phase shift instead of "C" when they relied on the representation of sinusoids as $f(x) = A\ sin/cos((B(x + C)) + D$. Some participants were also unable to complete tasks in which coefficient of x was a fraction. I conclude this dissertation with some pedagogical suggestions in terms of learning and teaching transformations of sinusoidal functions.

Keywords: Sinusoidal function, transformation, horizontal shift, periodicity, phase shift.

To my family.

Acknowledgements

First, my sincere and high appreciations go to my senior supervisor Prof. Rina Zazkis for her academic guidance. Her careful and thoughtful revisions greatly contributed to the preparation of this thesis. I am thankful to Dr.Peter Liljedahl and my other PhD committee members, for helpful inputs and valuable discussions. Special gratitude is given to my father, my mother, my brothers, and my sister. They provided me with constant support and encouragement during my studies. Last but not least, I want to express my deepest appreciation to my husband, Mohsen, and my children, Taha and Sana. This work was not possible without their unlimited patience, co-operation, motivation and love.

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Chapter 1.

Introduction

Trigonometry has been used in various applications such as architecture, astronomy and geography. Trigonometry is one of the fundamental topics taught in high school and university curricula, but it is considered as one of the most challenging subjects for teaching and learning. Despite its importance and its complexity, research on trigonometry is sparse and quite limited. In the literature, only a small number of studies concentrate on students' learning of trigonometric concepts (e.g., Gray and Tall, 1991; Brown, 2005; Weber, 2005; Moor, 2010), and on teaching trigonometry (e.g., Akkoç and Gül, 2010 and Moor, 2012). In the succeeding chapter, the findings on aspects of trigonometric topics in the mathematics education literature are presented in greater detail.

In what follows, I begin with a brief description of my academic background and I then outline some personal experience that led my research towards trigonometric functions. I wrap up the chapter with an overview of sequencing chapters in this dissertation.

1.1. The Researcher's Background and Motivation

I came from a country where mathematics is the one of the most important topics amongst other subjects in the school curriculum. In my country, Iran, students begin to learn mathematics when they are very young, in Grade 1, and they all need to pass a final mathematics exam in order to become eligible to study the following grade. Furthermore, the majority of Iranian families are interested in sending their children to high quality schools, where students are required to successfully complete the schools' entrance exams, which mostly focus on their knowledge of various mathematics topics..

As such, there are very tough competitions amongst students to learn, practice and earn high marks on the mathematics exams. The importance of mathematics exams becomes more highlighted for high schoolers, who need to successfully pass the university entrance exam, which is comprised of the mathematics topics students learned in Grades 9-12.

The vital role that Mathematics plays in a student's future life makes mathematics classrooms boring and stressful for most of students. However, in my school life, I had a great mathematics teacher for the last three years of high school. She was the most well-educated, patient and friendly teacher I ever had in my school years. She was my inspiration to choose mathematics as my major in post-secondary education. Fortunately, I passed the university entrance exam successfully and I started my bachelors in the subject of "Pure mathematics."

Graduating from university, I became a high school mathematics teacher. As a young and energetic teacher, I often tried my best to be an effective and helpful teacher for my students. As I knew that mathematics classrooms are often boring, I attempted to create a fun and interesting learning environment for my students. I showed my students examples of the application of mathematics in real life and I provided them with in-class activities. In spite of all the efforts I made, there still were some students who hated mathematics and therefore they did not make enough effort to learn mathematics. At that stage, I noted that I needed to learn more about teaching mathematics as well.

Observing students' struggles to understand and deal with a subject in which they had no interest, motivated me to learn more about various effective teaching methods. I began my master's studies at the University of British Columbia (UBC), where I learned about teaching mathematics from different angles. Meanwhile, I tutored some high school and university students in Vancouver.

When I worked as a tutor in Vancouver, I realized that students' difficulties in some specific topics in mathematics were similar to those with which my students struggled in my home country. Not only Canadian high school students, but also university students encountered similar difficulties. Knowing university students' difficulties with some mathematics topics inspired me to explore and to do research related to teaching mathematics at the undergraduate level. Therefore, after graduating from UBC in the summer 2012, I began my PhD program in the fall 2012 at Simon Fraser University.

As a first year PhD student I worked as a Teaching Assistant (TA) for the Applied Calculus Workshop (ACW), where I helped students who were registered in Calculus I or Calculus II in doing the assignments and answering their questions about the lecture notes. Helping students in ACW made me realize that one of the difficult topics for undergraduate students is "trigonometry." Discussing with some Mathematics instructors at SFU Mathematics department about students' struggles in the calculus courses assured me that "trigonometry" is indeed one of the most challenging topics for undergraduate students. Parallel with my own experience, the SFU instructors agreed that graphing trigonometric functions is one of the hardest parts in trigonometry in a Calculus course. As such, for this PhD dissertation, I decided to investigate the way undergraduate students deal with this challenging topic.

1.2. Organization of this Study

This dissertation is comprised of eight chapters including the introduction presented in this chapter. Chapter 2 presents a review of the historical development of trigonometry. The chapter begins with the ways the ancient Egyptians and Babylonians used trigonometry in 3000 BCE, and it continues with the birth of trigonometric functions by a Persian mathematician, Al-Khawarizmi (c. 780- c. 850).

In Chapter 3, I review the research studies focused on trigonometry from different points of view. Students' understanding and misunderstanding of aspects of trigonometric functions, as well as the influence of technology on pupils' grasping of trigonometric topics are covered in the first part of Chapter 3. The rest of the chapter is devoted to the review of research studies focused on teachers' conceptions of trigonometry, as well as various methods mathematics' teachers often employ when teaching trigonometric concepts. I wrap up Chapter 3 with a brief description of gaps in the area of research on learning trigonometric concepts and the need for further research on this important topic.

The following chapter is devoted to the methodology of this study. In Chapter 4 of this dissertation, I state the main goal of this study followed by the general research questions. I then provide a review of the pilot study and describe how the interview tasks were designed for the purpose of this study. The participants for the main study and their academic backgrounds are presented next. At the end of this chapter, I describe the way I collected data and how I analyzed the data according to the three frameworks. The three theoretical frameworks (Mason's (2008) theory of shifts of attention, Presmeg's (1989) visual imagery, and Carlson's et al. (2002) covariational reasoning) which I employ for analyzing the data collected for this study. These are explained in Chapter 5.

In Chapter 6, I analyze the response of one of the participants, Andy, in detail with respect to the three theoretical frameworks. There, I describe how he completed each interview task, the difficulties he encountered and the errors he made during the interview. The analysis of the responses of the rest of the participants to each of the interview tasks is presented in the Chapter 7. To complete a comprehensive analysis; I compared the answers of five students with each other and with Andy's responses.

Finally, the conclusion of this dissertation comes in Chapter 8, where the research questions are restated and answered. In this chapter, I present a summary of the findings and then I state the contributions of this study. The limitations of this study, the applications of the findings in teaching transformations of sinusoidal functions, and the need for further research are noted at the end of this dissertation in Chapter 8.

Chapter 2.

History of Development of Trigonometry

Looking at the trigonometry history shows that it took long time to develop, the first stages of development date back to about 3000 BCE. Reading several history books about the development of mathematics, and trigonometry in particular, shows how people in ancient times used it for multiple purposes such as exploring astronomy (e.g., recording the rising and setting of stars and calculating the time of day). However, reviewing the developmental history illustrates that since the beginning, trigonometry was a difficult topic for human beings and it has taken thousands of years for mathematicians and astronomers to grasp the content. In spite of the historical difficulty of trigonometry, which caused its slow progress, teachers often expect students to understand the topic in a short time. Looking at the historical development of trigonometric concepts, however, helps in gaining appreciation of the discipline and of potential struggle of learners. In what follows the key people who have had fundamental contributions in the development of trigonometry, the difficulties they encountered and the ways they solved them are explained in this paper.

The historical development of trigonometry is described in this chapter in three different sections. In the first section, I explain the use of trigonometry in the "**Ancient Egypt and Mediterranean World"** The second one is related to the arrival of trigonometry in the "**Indian and Islamic world"** and the next section deals with the history of progression of trigonometry to "**European World.**" In the last section, I also talk about **Trigonometry in the Contemporary Curriculum**. This chapter concludes with a summary of the development of trigonometry from the earliest times to the modern period.

2.1. Ancient Egypt and Mediterranean World

The term “trigonometry” came from the Greek words trigonon, meaning “triangle,” and the Greek word -metria, meaning “measurement.” As the name implies, trigonometry emerged from the study of right triangles and the relationships among the lengths of the sides and the angles of the triangle (Steckroth, 2007). The origin of trigonometry came from the ancient **Egyptians** and **Babylonians,** who developed and used theorems on ratios of the sides of similar triangles without being actually aware of trigonometry in the modern form (Adamek, Penkalski, and Valentine, 2005). The ancient Egyptians utilized trigonometry in land surveying, constructing their pyramids, and correlating shadow lengths of a vertical stick (gnomon) with the time of day (see Figure 2.1). The shadow tables are the ancestors of cotangent and tangent (Maor, 1998).The Babylonian astronomers also related trigonometric functions to arcs of circles and to the lengths of chords subtending the arcs to develop astronomers’ records of the events of the lunar month, the rising and setting of stars and the motion of planets and of the solar system (Van Brummelen, 2009).

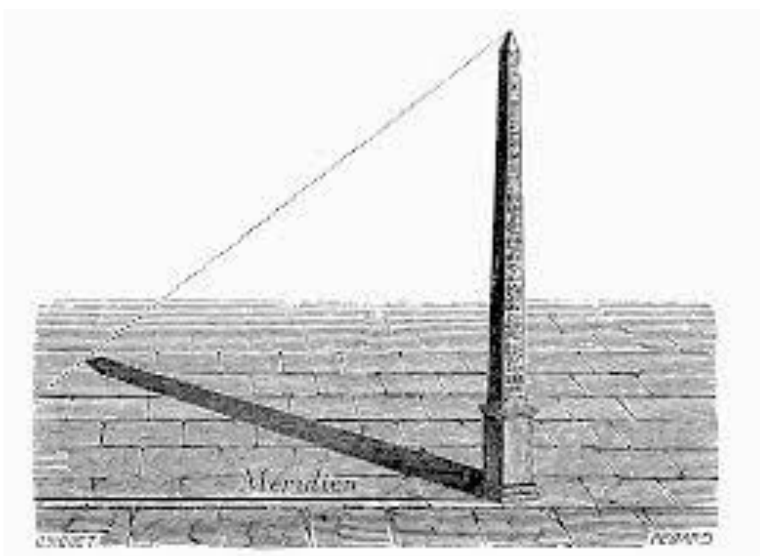


Figure 2.1. Ancient Egyptians gnomon

Although the primitive forms of trigonometry (e.g., gnomon) were in existence previously, the development of modern trigonometry into an ordered science began slowly with the introduction of the common unit of angular measure, the **degree,** originated by the Babylonians (Maor, 1998). Some historians believe that the Babylonian astronomers first divided a circle into 360 parts. The reasons for this division might be because of the intimacy of this number (360) to the number of days in the year, 365 days, or the possibility of dividing a circle naturally into six equal parts, each subtending a chord equal to the radius (Maor, 1998). What is historically clear is the fact that the system fit very well into the Babylonian sexagesimal (which is based on 60 rather than our current decimal number system based on ten) numeration system. The early **Greeks**, later, adopted the Babylonian sexagesimal number system and they introduced the **degree** for the first time (Van Brummelen, 2009). Greeks used the word μοιρα (moira) to refer to degree. The Arabs translated μοιρα into *daraja,* which, then, became the Latin word *de gradus* from which came the word degree (Maor, 1998).

Other than what is known about the Babylonian astronomers and the Greeks' adoption of the degree, there is a gap in the history of the development of trigonometry until the improvement of trigonometry by the Greek astronomer **Hipparchus of Nicaea** (ca.190-120 B.C.).

Hipparchus came to be known as "the father of trigonometry" and is the first person whose use of trigonometry is documented (Adamek et al., 2005). His work stressed the need for a system that provided a unit of measure for arcs and angles (Sozio, 2005).In astronomy, Hipparchus is credited with discovering the procession of the equinoxes (a slow circular motion of celestial poles once every 26700 years), determining the celestial longitude and latitude of 1000 stars and recording their positions on a map. He also classified stars according to their brightness by introducing a scale in which the magnitude of the brightest stars is 1 and the faintest stars have a magnitude of 6 (Maor, 1998). Hipparchus was the first person to determine exactly the times of the rising and setting of the zodiacal signs and also to estimate the size and distances of the sun and moon (Van Brummelen, 2009).

To be able to do his calculations for his astronomical work, Hipparchus needed a table of trigonometric ratios (Maor, 1998). However, since there was not such a table, he had to compute his own table (this is considered as one of the beginning difficulties of developing trigonometry). Hipparchus considered every triangle (planar or spherical) as being inscribed in a circle of large fixed radius (see Figure 2.2), so that each side of the triangle became a chord (a straight line drawn between two points on a circle) (Maor, 1998). In order to calculate the length of various parts of the triangle, Hipparchus had to figure out the length of the chord as a function of the central angle (Half of this chord later became the sine function (Van Brummelen, 2009)). The length of the chord is denoted by Crd.

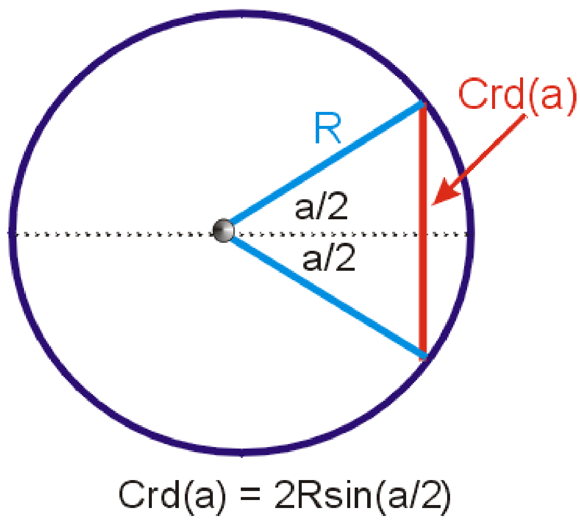


Figure 2.2. The relation between the chord function aTable 2.1nd the modern sine

Using basic circle properties, $Crd(a) = 2R\sin(a/2)$ (Or $\sin(a/2) = Crd(a)/2R$). Hipparchus chose a large fixed radius, R, to avoid fractions [when the radius is chosen large enough (R= 3438 in sexagesimal system), when divisions are made, these parts become whole numbers (Adamek et al., 2005)]. Maor (1998) states that from his calculations, Hipparchus borrowed the idea from the Babylonian astronomers who divided every circle into 360 degree and he, thus, began with the chord of 60° equal with the radius of the circle (R= 60°) when he constructed his table.

To complete his table and find other angles (except R=60), Hipparchus needed to know how to calculate the chord of the supplement of a given arc (which is written as:

$Crd(180-\alpha) = \sqrt{\left((2R)^2 - \left(Crd(\alpha)\right)^2\right)}$) and also the formula for the chord of half angle: $\left(Crd\frac{\alpha}{2}\right)^2 = R\ (2R - Crd(180-\alpha)$ (Van Brummelen, 2009). As such, although Hipparchus, as an astronomer, was mainly concerned with spherical triangle (Maor, 1998 and Hunt, 2000), he still would have needed to know many formulas of plane trigonometry (as mentioned these lack of knowledge, made the development of trigonometry a difficult task for ancient people) which are in the modern form as:

$(\sin\alpha)^2 + (\cos\alpha)^2 = 1$

(which is a trigonometric version of the Pythagorean Theorem),

$(\sin\alpha/2)^2 = (1-\cos\alpha)/2$, or

$$\sin(\alpha+\beta) = \sin\alpha \times \cos\beta + \cos\alpha \times \sin\beta$$

Hunt(2000) and Van Brummelen (2009) indicate that Hipparchus used, for example, Pythagorean Theorem [according to Thales's theorem if two angles (α and 180- α) are supplementary angles, the triangle having two sides Crd (α) and Crd (180- α) is a right triangle (see Figure 2.3)] for the supplementary formula as:

$Crd(180-\alpha) = \sqrt{(2R)^2 - \left(Crd\ (\alpha)\right)^2}$ (Formula 1)

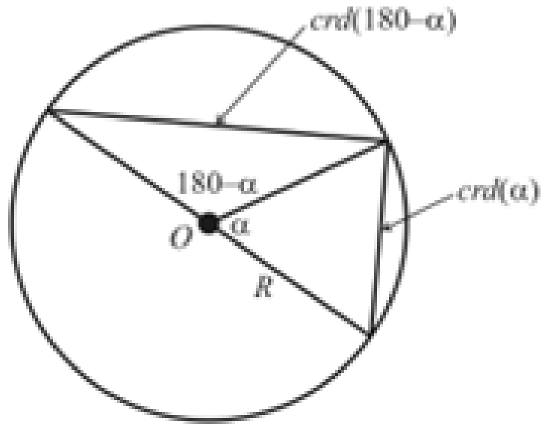


Figure 2.3. Supplementary angles

Since he knew:

$Crd(\alpha) = 2R \times \sin \alpha/2$ (Formula 2)

$Crd(180 - \alpha) = 2R \sin(180 - \alpha)/2 = 2R \times \cos \alpha/2$. (Formula 3)

Thus, from formula (1), (2) and (3):

$2R\cos(\alpha/2) = \sqrt{(2R)^2 - (2R\sin(\alpha/2))^2}$

$(2R)^2 (\cos \alpha/2)^2 = (2R)^2 - (2R \sin \alpha/2)^2$ (Formula 4)

Therefore, both sides of the equation (Formula 4) will be equal because Hipparchus knew $\sin(\alpha/2)^2 + \cos(\alpha/2)^2 = 1$ (Van Brummelen, 2009).

Hipparchus wrote twelve books on the computation of chords in a circle. Unfortunately, however, all his works were lost and most of what is known about his works is through later references in Ptolemy's Almagest (explained later in this paper), written three centuries after Hipparchus died.

The next Greek mathematician known to have made a great contribution to trigonometry and in particular to spherical trigonometry was **Menelaus of Alexandria** (c.a. 100 A.D.) (Maor, 1998). Menelaus wrote a six-book treatise on chords, but those books (e.g., *On the Triangle* and *Elements of Geometry*) have all been lost. Menelaus, however, authored a three-book work called *Spherics* which is his only surviving work. Although the Greek version of this text is lost, and all that remains is an Arabic version translated a thousand years after Menelaus wrote the original version, this work still provides a good source for the development of spherical trigonometry (Van Brummelen, 2009).

The first book of the *Sphaerica* is geometric in content and deals systematically with the conception and definition of a **spherical triangle** for the first time. Menelaus described a spherical triangle as the area included by the arcs of great circles on the surface of a sphere subject to the restriction that each of the sides or legs of the triangle is an arc less than a semicircle (Van Brummelen, 2009). He then imitated the theorems of Euclid's propositions about plane triangles and extending them to give the main propositions about spherical triangles (e.g., the two triangles on the base of any spherical triangle with two equal sides are themselves equal). In this book, however, Menelaus avoided proving any theorem (Van Brummelen, 2009).

The second book has astronomical interest only, whereas the third book contains some important information about the development of trigonometry and it deals with spherical trigonometry and includes the **Menelaus's theorem** (Van Brummelen, 2009). For plane triangles the theorem was known before Menelaus (see Figure 2.4):

... if a straight line crosses the three sides of a triangle (one of the sides is extended beyond the vertices of the triangle), then the product of three of the nonadjacent line segments thus formed is equal to the product of the three remaining line segments of the triangle.

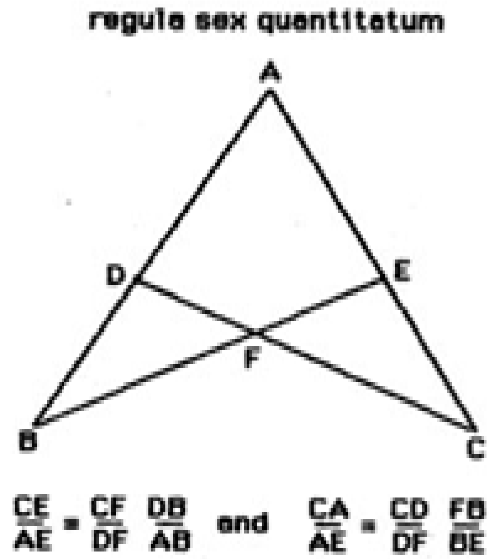


Figure 2.4. The theorem of plane triangles

Menelaus produced a spherical triangle version of this theorem, which is called Menelaus's theorem. Instead of using a spherical triangle, Menelaus indicated his proposition in terms of two intersecting great circles (see Figure 2.5):

"Between two arcs, ADB and AEC, of great circles are two other arcs of great circles DFC and BFE which intersect them and also intersect each other in F. All the arcs are less than a semicircle." (http://aleph0.clarku.edu/~djoyce/ma105/trighist.html).

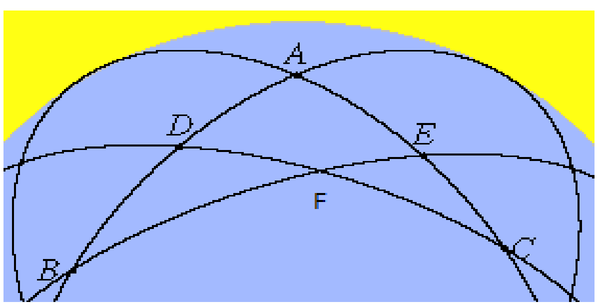


Figure 2.5. Menelaus's theorem

According to Menelaus's theorem:

$$Crd(2AE)/Crd\ (2CE) = Crd\ (2AB)/Crd(2BD) \times Crd\ (2DF)/Crd\ (2\ FC).$$

and

$$Crd(2AC)/Crd(2AE) = Crd\ (2CD)/Crd\ (2DF) \times Crd\ (2BF)/Crd\ (2BE).$$

If Menelaus's theorem for spherical trigonometry was written in terms of modern sines, they would be as follows:

$$\sin(AE)/\sin(CE) = \sin(AB)/\sin(BD) \times \sin(DF)/\sin(FC)$$

$$\sin(AC)/\sin(AE) = \sin(CD)/\sin(DF) \times \sin(BF)/\sin(BE)$$

Though Menelaus's developments were very important, the first major and the most influential work on trigonometry is *The Mathematical Syntaxis*, usually known as the *Almagest* which is a work of thirteen books by Ptolemy of Alexandria (ca. 85-ca. 165 A.D) (Maor, 1998).

Ptolemy focused on a combination of both astronomy and trigonometry in *Almagest*. The book is based upon the assumption that a motionless earth sits at the center of the universe and the heavenly bodies move around it in their prescribed orbits (Maor, 1998). In order to accomplish his goal (recording the motions of all celestial objects), Ptolemy noticed that he needed a substantial trigonometric tool. Therefore, the subject of chapters 10 and 11 of the *Almagest* is about Ptolemy's table and a detailed set of instructions (containing some of the earliest extent derivations of common trigonometric results) on how to construct the table (Van Brummelen, 2009). *Almagest* is reliant on much of the work of Hipparchus and Menelaus (Adamek et al., 2005). Some historians believe that Ptolemy completed Hipparchus's work through adding some necessary details and constructing new tables. However, since much of Hipparchus' work was lost, it is **difficult** to distinguish between what additions and modifications Ptolemy made, and what already existed (Adamek et al., 2005).

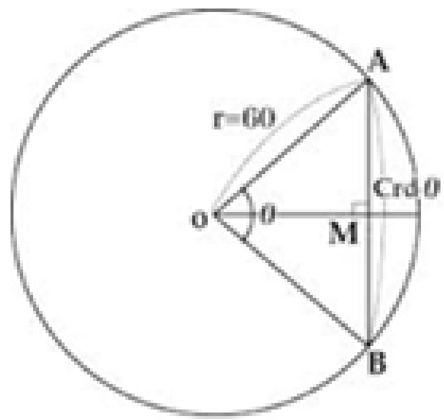


Figure 2.6. $d = Crd\theta = 2R\sin\theta/2$

Ptolemy's table gives the length of a chord in a circle as a function of the central angle (see Figure 2.6) that subtend it for angles increasing from 0 degrees to 180 degrees at intervals of half a degree (Maor, 1998). He took the diameter of the circle to be 120 units and thus R= 60. Ptolemy's table of chord is essentially a table of sines because $d = Crd\theta = 2R\sin\theta/2$ or $d = Crd\theta = 120\sin\theta/2$. As is clear from the equation, apart from proportionality factor 120, Ptolemy's table is equivalent to a table of value of sin(θ/2) and thus, by doubling the angle it is a table of sin(θ) (Maor, 1998). In other words, the table is the **earliest trigonometric table of sines.**

In his table, Ptolemy carried out his calculations to three sexagesimal places to achieve the accuracy of the chord length (Maor, 1998). Apart from two separate columns for arcs and chords, Ptolemy's table was comprised of a column of “sixtieths” (see Figure 2.7). The column of “sixtieths” allows one to incorporate between successive entries: it gives the mean increment in the chord length from one entry to the next (Maor, 1998)

arc	chord			sixtieths		
$\frac{1}{2}$	0	31	25	1	2	50
1	1	2	50	1	2	50
$1\frac{1}{2}$	1	34	15	1	2	50
⋮	⋮	⋮	⋮	⋮	⋮	⋮
109	97	41	38	0	36	23
$109\frac{1}{2}$	97	59	49	0	36	9
110	98	17	54	0	35	56
$110\frac{1}{2}$	98	35	52	0	35	42
111	98	53	43	0	35	29
$111\frac{1}{2}$	99	11	27	0	35	15
112	99	29	5	0	35	1
$112\frac{1}{2}$	99	46	35	0	34	48
113	100	3	59	0	34	34
⋮	⋮	⋮	⋮	⋮	⋮	⋮
179	119	59	44	0	0	25
$179\frac{1}{2}$	119	59	56	0	0	9
180	120	0	0	0	0	0

Figure 2.7. Ptolemy's table known as sine table

Ptolemy also developed much more powerful tools, the sum and difference formulas for chords (Van Brummelen, 2009). To determine the sum and difference formulas for chords (see Figure 2.8), Ptolemy developed the theorem that later became known as **Ptolemy's theorem** (Van Brummelen, 2009). Ptolemy's theorem is defined as:*In a quadrilateral inscribed in a circle (*see Figure 2.8*), the product of the diagonal (AC. BD) is equal to the sum of the products of the opposite sides (AB.CD+ AD. BC)"* (Hunt, 2000).

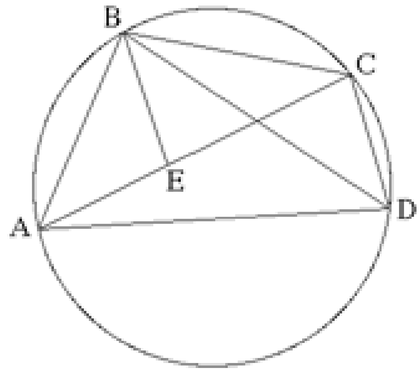


Figure 2.8. Ptolemy's theorem

When AD is a diameter of the circle, where O is the center of the circle and d the diameter, then the theorem says:

$$Crd(AOC)Crd(BOD) = Crd\ (AOB)\ Crd(COD) + d\ Cord\ (BOC).$$

Therefore, if “a” is for angle AOB and b for angle AOC, and then there is:

$$Crd(b)Crd(180 - a) = Crd\ (a)Crd(180 - b) + dCrd(b - a)$$

which gives the different formula:

$$Crd(b - a) = Crd(b)\ Crd(180 - a) - Cdr(a)Crd(180 - b)/d \quad \text{(formula (1))}.$$

and since:

$$\sin a = (1/d)Crd(2a) \quad \text{and} \quad \cos a = (1/d)Crd(180 - 2a) \quad \text{(formula (2))},$$

therefore, formula (1) corresponds to the difference formulas for modern trigonometry as:

$$\sin(a - b) = \sin a \times \cos b - \sin b \times \cos a.$$

Using Ptolemy’s theorem to find the sum of two chords is not quite as straightforward (Van Brummelen, 2009).

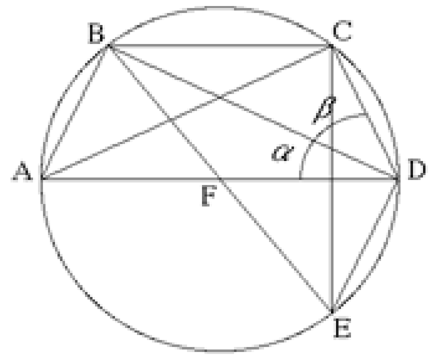


Figure 2.9. Ptolemy's theorem and sum of two angles

If BF extended to E, thus α = AB=DE and β= BC. Therefore, application of Ptolemy's theorem to BCDE gives (see Figure 2.9):

$Crd\big(180-(\alpha+\beta)\big) = Crd(180-\alpha)Crd(180-\beta) - Crd(\alpha)\,Crd(\beta)/2R$ (formula (3))

From formula (2) and (3) one could conclude the modern **sum formula for cos**:

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ (Bressoud, 2010).

Ptolemy's theorem not only leads to the equivalent of the **sum-and-difference formulas for sine and cosine** that are today known as Ptolemy's formulas, it also helps to derive the equivalent of the **half-angle formula for trigonometric functions**, $\sin(a)^2 = \sqrt{(1-\cos a)}$ (I do not explain it here due to lack of space) (Van Brummelen, 2009).

After the early trigonometry, the Indian and Islamic trigonometry will be described in detail in the next section.

2.2. Indian and Islamic World: the Age of Six Functions

After Ptolemy, Greek trigonometry had little development (Van Sickle, 2011). No documented evidence is available for the arrival of Greek astronomical models in India (Van Brummelen, 2009). The earliest Indian work believed to come from the influence of Greek trigonometry is *Siddhantas* (late fourth or early fifth century A.D.). It contains a table which is a modified version of Ptolemy's table of chord (Maor, 1998). The trigonometry of Ptolemy was based on the functional relationship between the chords of a circle and the central angles they subtend. But, the authors of the *Siddhantas* (its original version is by an unknown author) focused on a study of the relationship between half of a chord of a circle and half of the angle subtended at the center by the whole chord (Adamek et al., 2005). From this stemmed the ancestor of the modern trigonometric function known as the **sine of the angle (**although it still was not called "sine"). Therefore, the main contribution of India and particularly the *Siddhantas* is the more formal introduction of the sine function to the history of mathematics (Adamek et al., 2005).

Aryabhata (AD 476-550), an Indian mathematician and astronomer, collected and expanded upon the developments of the *Siddhantas* with his book entitled *Aryabhatiya.* It is believed that *Aryabhatiya* is the primary original Hindu work in which for the first time in the history of trigonometry the **sine** as a function of an angle was named. In his work which focused on calculating a table of "sine differences", Aryabhata used the word *ardha-jya* to refer to the half-chord and sometimes the word turned around to *jya-ardha*, chord-half. In order to shorten the word, Aryabhata used *jya* or *jiva* (Maor, 1998). Like the chord, the *jya* was defined as the length of a certain line segment in a circle (see Figure 2.10). The relationship between *jaya* and the modern sine is: *jaya (α)* = $R\sin(a)$ where R is a radius of the base circle (R= 3438). Also the *utkrama-jya* (reversed sine) or versed sine is: Vers$(a) = 1 - \cos(a)$.

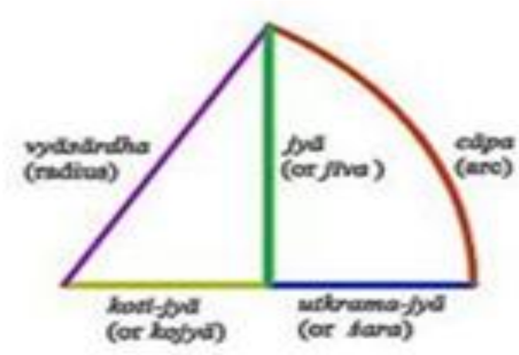


Figure 2.10. Aryabhata's definition of *jya* and *utkrama-jya*

Aryabhata's table is given in increments of 3° 45 minutes (225 minutes) for angles between 0° and 90° to four decimal places of accuracy. Since Aryabhaṭa's table is not a set of values of the trigonometric sine functions, the measurement was not of the sines themselves. Instead, it was the measurement of the differences between the sines (see Figure 2.11). Beginning with the assumption that the first entry in the table is sine (225 minute) = 225, Aryabhata used the following pattern for calculating the sines:

$(225 - the\ previous\ sine)\ +\ (225 + the\ previous\ sine)/225$. This total was then subtracted from 225 to obtain the sine table (Adamek et al., 2005).

Angle θ	jya or Hindu sine	(sin θ) × 3438	Hindu versine r − jya (90° − θ)	Hindu sine differences
3°45′	225	224·85	7	224
7°20′	449	448·95	29	222
11°15′	671	670·72	66	219
15°0	890	889·82	117	215
18°45′	1105	1105·01	182	210
22°30′	1315	1315·05	261	205
26°15′	1520	1520·58	354	199
30°0′	1719	1719·00	460	191
33°45′	1910	1910·05	579	183
37°30′	2093	2092·09	710	174
41°15′	2267	2266·08	853	164
45°0	2431	2431·01	1007	154
48°45′	2585	2584·08	1171	143
52°30′	2728	2727·55	1345	131
56°15′	2859	2858·55	1528	119
60°0′	2978	2977·04	1719	106
63°45′	3084	3083·45	1918	93
67°30′	3177	3176·06	2123	79
71°15′	3256	3255·75	2333	65
75°0′	3321	3320·85	2548	51
78°45′	3372	3371·95	2767	37
82°30′	3409	3408·75	2989	22
86°15′	3431	3430·85	3213	7
90°0′	3438	3438	3438	

Figure 2.11. Aryabhaṭa's table of sine

After Aryabhaṭa some other Indians (e.g., Bhaskara, Brahmagupta and Madhava) contributed in development of trigonometry, which are not included in this paper due to lack of space.

The Arrival of Astronomy from India into the Arab World: the Birth of Trigonometric Functions

Translations of Indian trigonometry texts as well as of Ptolmey‘s Almagest (dealing with the geometry of chords) appeared in the Muslim world by Muslim mathematicians of mostly Persian and Arab descent in the late seventh century, soon after the birth of Islam (Maor, 1998). Between the two types of trigonometry, the Greek and the Hindu, involving their tables of sines, Muslim trigonometers chose to focus on the Indian sine rather than the Greek chord for the ease of calculations (Maor, 1998).

Abu Ja'far Muhammad ibn Musa al-Khwarizmi Al-Khwarizmi, a Persian mathematician, astronomer, astrologer, geographer and a scholar**,** introduced the Indian computational and mathematical methods to Arabs for the first time.

Al-Khawarizmi (c. 780- c. 850) translated the *Aryabhatiya,* into Arabic language, in his famous book entitled *Zij al-Arjabhar*. In his translation, al-Khwarizmi kept the word *jiva* without translating its meaning. Since in Arabic words consist mostly of consonants, and vowels are interpreted by context, *jiva* could also be pronounced as *jaib*, which means bosom, or bay. Therefore, when the Arabic version of *Aryabhatiya* was translated into Latin, they translated *jaib* into *sinus* or *sine*, which means bosom or bay. The **cosine function** which first arose for the need to compute the sine of the complementary angle is a Latin translation of the word *Kotijya* of Aryabhata (Maor, 1998).

Al-Khwarizmi also wrote a book called *Zij al-Sindhind,* which contains an accurate sine and cosine table in which R=60 for the interval of 1 degree. Al-Khwarizmi's *Zij* also includes a table of versed sine to solve astronomical problems (Van Sickle, 2011). To solve attitude problems using gnomons and shadows, Al-Khwarizmi constructed the first **table of tangent** in the history of mathematics (although, he still did not call it "tangent") in his famous book *Al-Jabr wa-al-Muqabilah*. He was also the inventor of spherical trigonometry.

To calculate various astronomical coefficients with great accuracy, following al-Khwarizmi, another Arab astronomer, **Abdallah Muhammad Ibn Jabir Ibn Sinan al-Battani** (around 858-929) introduced the **trigonometric ratio** in mathematical calculation (e.g., $\tan a = \sin a/\cos a$ and $\cot a = \cos a/\sin a$) which formed the basis of modern trigonometry (Maor, 1998). Al-Battani used the following rule for finding the rise of the sun above the horizon in terms of the length "m" of the shadow by a vertical gnomon of height "h":

$m = h\ \sin(90 - \alpha)/\sin(\alpha)$ or $m = h\cot(\alpha)$.

In his book *Zij al-Sabi*, al-Battani applied the above formula to produce **the table of cotangents** for the first time, which he referred to as a "table of shadows" (in reference to the shadow of a gnomon) and he called it *zill mabsut* (or umbra recta in Latin) by degree from 1° to 90° to solve astronomical problems. He was also the first person to introduce the **reciprocal functions of secant and cosecant** (Maor, 1998). Al-Battani also provided important trigonometric formulas for right-angled triangles (with side length of a, b and c), such as the following formula (instead of using geometrical methods, as Ptolemy had done):

$$b\sin(A) = a\ \sin(90 - A)$$

Abu'l Wafa al'Buzjani (940-998), a Persian mathematician, who was an algebraist as well as a trigonometer, was known for his contributions in plane trigonometry and spherical trigonometry. He was the first mathematician who finally brought all six of the fundamental modern **trigonometric functions together** and defined them in one diagram (see Figure 2.12) in his *Almagest* (Van Brummelen, 2009; and Van Sickle, 2011). He used R=1 for the radius of the basic circle.

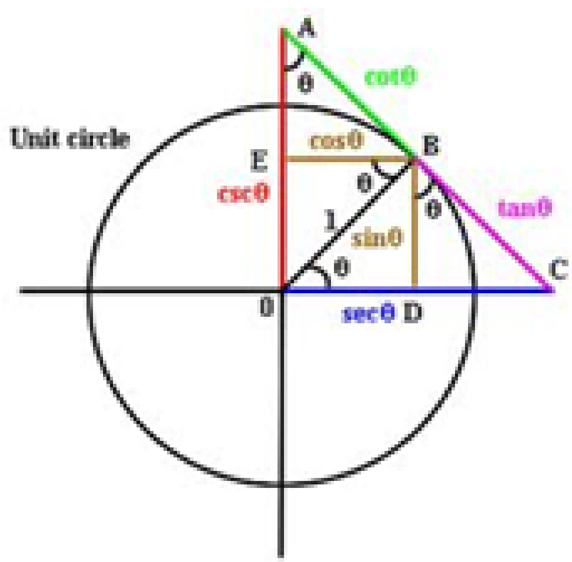


Figure 2.12. Six trigonometric functions in one diagram

$BD = \sin(\theta)$ $EB = \cos(\theta)$ $BC = \tan(\theta)$

$AB = \cot(\theta) \qquad OD = \sec(\theta) \qquad OE = \csc(\theta).$

Applying all of the modern trigonometric functions, in fact, assisted Abu‘lWafa to make trigonometric calculations much more easily and quickly than before, although initially only some of his colleagues appreciated this change (Van Sickle, 2011). He contributed a table of tangents using all six of the common trigonometric functions (Maor, 1998). Abu‘lWafa also constructed a new sine table, using eight decimal places.

Abu‘lWafa proved several identities known today as the Pythagorean identities, such as:

$\tan(a)^2 + 1 = sec(a)^2$ and $1 + \cot(\alpha)^2 = \csc(a)^2$ (Van Brummelen, 2009).

Furthermore, he developed the following trigonometric formula:

$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b).$

$\sin(2a)$= $2\sin(a)\cos(a)$ (Ptolemey had expressed the equivalent identities in terms of chords)

$\cos(2a) = 1 - 2\sin(2a)$ (Adamek et al., 2005).

Abu’l-Wefa discovered the law of sines (even though it was first introduced by Ptolemy) through applying a straightforward formulation of the **law of sines for spherical triangles** (where A, B, and C are surface angles of the spherical triangle and a, b and c are the central angles of the spherical triangle (see Figure 2.13)) (Adamek et al., 2005):

$\sin a/\sin A = \sin b/\sin B = \sin c/\sin C.$

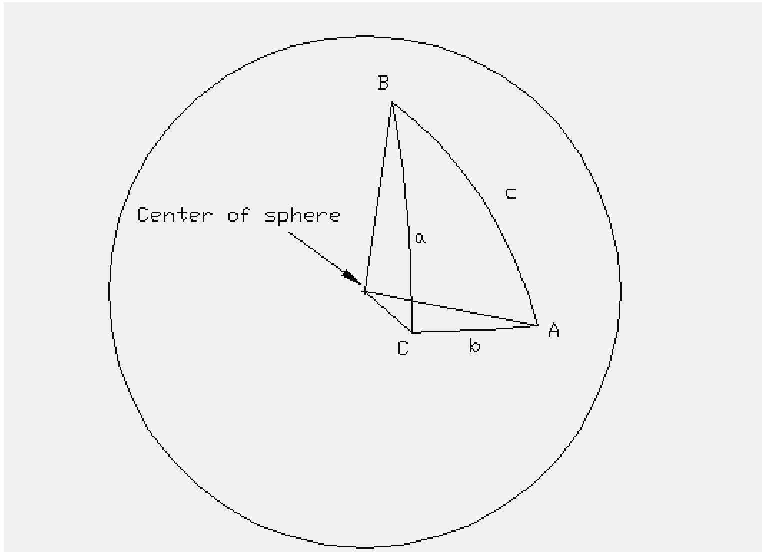


Figure 2.13. Law of sines in a spherical triangle

Not long after Abul-Wafa wrote *Almagest*, **al-Biruni** wrote a treatise entitled *Maqalid 'ilm al-hay'a,* devoted to plane trigonometry. In his book, **Abul-Rayhan al-Biruni** (973-1048), who was one of the greatest Persian scientists, discussed the following formula in **tangent** from which the half angle and multiple angle formulae had been discovered (Van Brummelen, 2011) [see the diagram below (see Figure 2.14) in which O is the center of the semicircle, and AED a right-angled triangle with a perpendicular from E to C]:

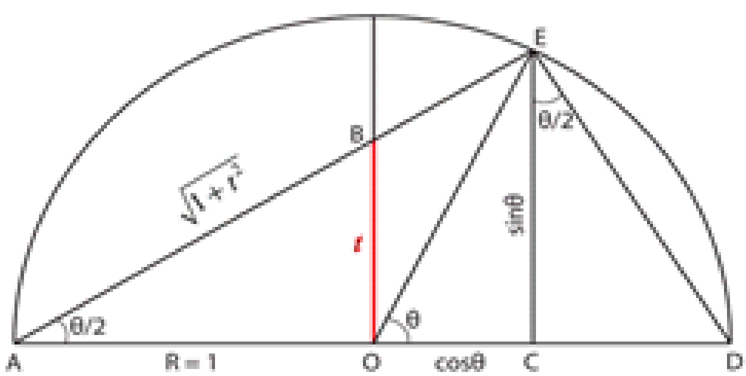


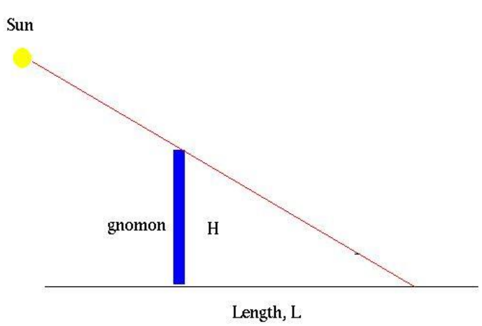
Figure 2.14. The half and multiple angle for tangent

$$\tan(\theta/2) = EC/AC = \sin\theta/(1+\cos\theta)$$

and

$$\tan(\theta)/2 = DC/EC = (1 - \cos\theta)/\sin\theta.$$

Al-Biruni also wrote three major books in trigonometry: *Qanun-i Masoodi*, *Ketāb maqālīd ʿelm al-hayʾa*, and *Ketāb fī efrād al-maqāl fī amr al-ẓelāl*. In the *Qānūn*, al-Biruni proposed trigonometric theorems equivalent to those related to the sums and differences of angles. In his *Ketāb maqālīd ʿelm al-hayʾa,* Al-Biruni focused mainly on the applications of spherical trigonometry in astronomy and evaluated Khawrazmi's results to provide more accurate detailed classification of spherical triangles and their solutions. Furthermore, in *Ketāb fī efrād al-maqāl fī amr al-ẓelāl* he discussed the familiar trigonometric definitions further and applied them to religious matters (e.g., determining times of prayer and finding the direction of Mecca). Bīrūnī, for example, used all six trigonometric functions (but in gnomonic context) to measure the time of day (for praying) using the shadows of a gnomon (in spherical triangles). As can be noticed in Figure 2.15, the gnomon is vertical; with α length of the direct shadow corresponds to cotangent, whereas the hypotenuse of the direct shadow is secant. On the other hand, when the gnomon is horizontal, the length of the reversed shadow is the tangent and the hypotenuse of the reversed shadow is the cosecant (Van Brummelen, 2009).



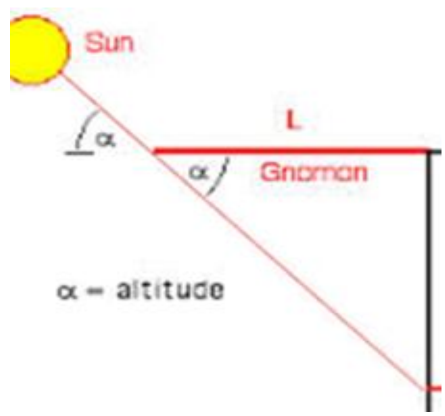


Figure 2.15. Six trigonometric functions in gnomonic context

In the 13th century, **Nasīr al-Dīn al-Tūsī** (1201-1274), a Persian mathematician, was the first person to consider trigonometry as a mathematical discipline **independent from astronomy**. He helped to **differentiate plane trigonometry and spherical** trigonometry (Adamek et al., 2005). He also stated six fundamental formulas for the solution of spherical right-angled triangles. In his book entitled, *On the Sector Figure,* Al-Tusi determined **the law of sines for plane triangle** [$a/\sin A = b/\sin B = c/\sin C$ where a, b, and c are the lengths of the sides of a triangle, and A, B, and C are the opposite angles]. He also provided proof for **the law of sines for spherical triangles,** which was identified by Abu'l-Wefa. Furthermore, he discussed and proved **the law of tangents for spherical triangles** [$\tan(A-B)/2/\tan(A+B)/2 = \tan(a-b)/2/\tan(a+B)/2$] where A, B, C are the angles at the three vertices of the triangle and lower-case a, b, c are respective lengths of the opposite sides] in his book (*On the Sector Figure*) (Lennart, 2007).

This concludes the contribution of Arab and Persian mathematicians to the development of trigonometry (Although some other Persians attempted to develop trigonometry, because of limited space they cannot be explained here). In the next section, the contribution of Europeans and their influences on the progression of trigonometry will be described in detail.

2.3. Passage to Europe

In the medieval West, knowledge of trigonometry gradually reached Europe through translation of the texts written by Muslim mathematicians and astronomers such as Mohammed ibn al-Khowarizmi and al-Battani (Maor, 1998). **Regiomontanus** (also known by his given name, Johann Muller), a German astronomer (1436-1476), wrote the **first comprehensive book on trigonometry**, *De triangulis Omnimodis libri quinque* ("of triangles of every kind in five books"). He, in fact, is the first person who removed trigonometry as a science separate from astronomy and made it into its own field (although Tusi treated trigonometry as a separate discipline in mathematics, Regiomontanus was the first who wrote a book on trigonometry).

In his *De triangulis Omnimodis* (which contains five books), Regiomontanus included all knowledge of trigonometry from Ptolemy, Hindu and Arab scholars, and in doing so created a rebirth of trigonometry in Europe (Zeller, 1941). The first book of *De triangulis Omnimodis* begins with fifty propositions on the solutions of triangles using the properties of right triangles. The second part of Regiomontanus's *De triangulis Omnimodis* includes the formula for determining the area of a plane triangle in terms of two sides and the included angle. The third book determines theorems found on Greek before the use of trigonometry, and the last two books are based on spherical trigonometry (Maor, 1998).

It is strange for historians that Regiomontanus' first trigonometry did not contain tangent although he must have been familiar with it from the Arabs' use of it in connection with shadow reckoning (Maor, 1998). Therefore, Regiomontanus' first trigonometry was not as advanced as some Arabic authors of the same time period. Later, however, when he wrote *Tabula Directionum* in 1467, Regiomontanus created a table of sines as well as a table of tangents that shows his awareness of tangents as well (Adamek et al., 2005; Zeller, 1941). Throughout this time, there was still disagreement as to the names of the trigonometric functions and whether tangent, cotangent, secant, and cosecant were proper trigonometric functions (Van Brummelen, 2009).

After Regiomontanus, **Nicholas Copernicus** (1473-1543), who was an astronomer and a trigonometer, completed a treatise known as *De revolutionibus orbium coelestium*. This work includes information on trigonometry and it is very similar to that of Regiomontanus (Van Brummelen, 2009). Copernicus' student, **George Joachim Rheticus** (1514-1574), an Indian mathematician, combined the trigonometric works of both Copernicus and Regiomontanus and eventually published his significant advances in trigonometry, a two-volume work called *Opus palatinum de triangulus* (Canon of the Science of Triangles) (Maor, 1998). In fact, this book creates a revolution in trigonometry.

The functions with respect to the arc of a circle were omitted from Rheticus's work, although most previous work had been done using spherical triangles. He, instead, constructed a right triangle where the trigonometric functions: sine, cosine, tangent, cotangent, secant and cosecant depended on the angles of the right triangle (Zeller, 1941, Adamek et.al, 2005). In his work, Rheticus, in fact, used all six trigonometric functions and he had calculated tables of them given to seven decimal places, although he never had time to finish the tables of tangents and secants (Van Brummelen, 2009). However, like Copernicus, Rheticus took a student, Valentinus Otho who supervised the calculation (by hand) of some one hundred thousand ratios to at least ten decimal places, filling some 1,500 pages and finally completing the tables in 1596. These tables are accurate enough to be used as the basis for astronomical calculations up to the early 20th century (Van Brummelen, 2009).

The next trigonometer was **Franscicus Viete** (also known as François Viete), a French mathematician in medieval trigonometry. Like his predecessors, Regiomontanus and Rheticus, Viete thought of trigonometry as an independent branch of mathematics. In his two significant books, entitled the *Canon Mathematicus* and *Universalium Inspectionum Liber Singularis,* Viete (in 1579) made tables for all six trigonometric functions for angles to the nearest minute (Adamek et al., 2005). Viete was one of the first to apply a statement similar to the formula for the law of tangents in which a, b, and c are the lengths of the three sides of the triangle, and α, β, and γ are the angles opposite those three respective sides (see Figure 2.16):

$$(a-b)/(a+b) = \tan((\alpha-\beta)/2)/\tan(\alpha+\beta)/2.$$

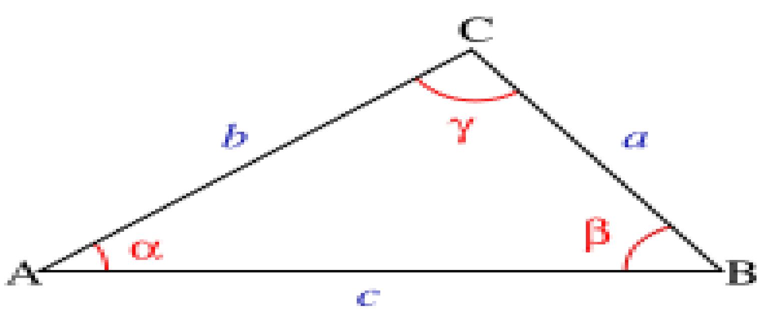


Figure 2.16. Law tangent

Viète, also, was the first who applied algebraic methods to trigonometry and he eventually founded modern **analytic trigonometry** (Adamek et al., 2005). Viete, in fact, tried to reduce the emphasis on the calculation of solutions of triangles and instead he increased the focus on analytic functional relationships (Merzbach and Boyer, 2011; Adamek et al., 2005). For instance, to derive the multiple-angle formula for $\sin(n\alpha)$and $\cos(n\alpha)$ in terms of the powers of $\sin(\alpha)$ and $\cos(\alpha)$, by letting $X = 2\cos\alpha$ and $Y_n = \cos(n\alpha)$, he obtained the recurrence formula of:

$$Y_n = XY_{n-1} - Y_{n-2}$$

which, when changed back into trigonometry, becomes the formula:

$$\cos(n\alpha) = 2\cos\alpha\ \cos(n-1)\alpha - \cos(n-2)\alpha$$ (Maor, 1998).

The term **'trigonometry'** first appears as the title of a book *Trigonometriae Sive, de dimensione triangulis, Liber* (Book of trigonometry, or the measurement of triangles) by **Bartholomaeus Pitiscus** (1561-1613). The book includes descriptions of how to construct sine and other tables, and a number of theorems on plane and spherical trigonometry with their proofs. Pitiscus also corrected Rheticus' *Opus Palatinum* which contains serious errors in the tangent and secant tables at the ends near 1° and 90° (Maor, 1998). Thereafter, Pitiscus published his own new work in 1600 incorporating that of Rheticus with a table of sines calculated to fifteen decimal places entitled the *Thesaurus Mathematicus* (Rogers, 2006). Pitiscus also was the first who discovered the formulas for $\sin 2x$, $\sin 3x$, $\cos 2x$, $\cos 3x$ (Robertson, 2006). Some historians believe that Pitiscus's discovery of $\sin 2x$, $\sin 3x$, $\cos 2x$, $\cos 3x$ later directed Viete to introduce $\sin(n\alpha)$and $\cos(n\alpha)$ in 1593 (Maor, 1998).

Next, in the early 17th century, **John Napier** (1550-1617), a Scottish mathematician, invented **logarithms** primarily for the purpose of simplifying numerical calculations in trigonometry and in 1614 he published the *Mirifici logarithmorum canonis description* (The Description of the Marvelous Rule of Logarithm).The book includes a description of a set of tables of the logarithms of trigonometric functions. Napier's book also contains some rules/propositions which unify and simplify the process of solving right-angled spherical triangles. For instance, one of the propositions is:

"*In any triangle: the sum of the Logarithms of any angle and side enclosing the same is equal to the sum of the Logarithms of the side, and the angle opposite to them". It is indicated that when Napier writes "the logarithm of an angle," he implicitly meant the "logarithm of the sine of the angle.*" (p.35, Maor, 1998, see Figure 2.17). Therefore:

$ln(\sin A) = ln(\sin C) + lna - lnc$ (Roegel, 2012).

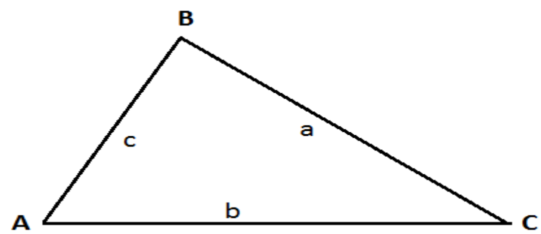


Figure 2.17. logarithm of the sine of the angle

The book also involves the proposition for the differential, which corresponds to the logarithm of the tangent. The proposition is thus:

"*In a right angled triangle the Logarithm of any leg is equal to the sum of the Deferential of the opposite angle, and the Logarithm of the leg remaining* (Roegel, 2012). As an example is:

$lnb = lnc + ln(\sin B) - ln(\cos B)$ (see Figure 2.18)

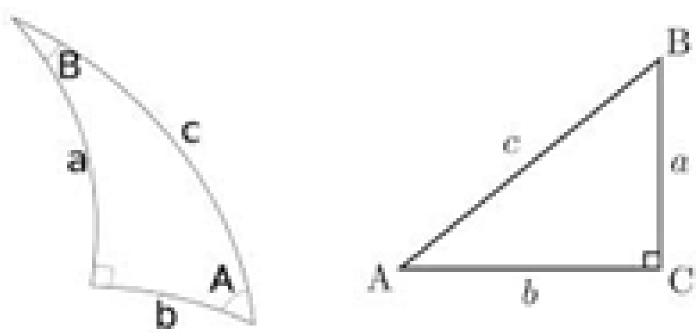


Figure 2.18. logarithm of the sine of the angle in right triangle

Around 1635, analytical trigonometry again became more prevalent with the work of **Roberval** and **Torricelli**, though Viete was one of the first mathematicians to focus on this new branch of trigonometry (Adamek et al., 2005). Torricelli and his student Roberval were the first to invent the **graph** of half an arch of a sine curve. Adamek et al., (2005) express that this invention was important in the progression of trigonometry and its moving from a computational emphasis to a functional approach.

About fifty years after Napier's publication of his logarithms, **Isaac Newton** (1643-1727), a Scottish mathematician, developed the differential and integral calculus. One of the foundations of Newton's work was his demonstration of many functions as infinite series in the powers of x:

$(X_{n+1} = X_n - [f(x)/f'(x)])$. Newton in 1676 invented the **infinite series** for **sin(x)** and similar series for **cos(x)** and **tan(x)** in his paper "*Treatise on the methods of series and fluxions*" (Ball, 2010). The discovery of infinite series representations for the trigonometric functions illustrates the influence of trigonometry on calculus which was mainly for measuring geometric figures. Later, in 1719, Isaac Newton and James Stirling, a Scottish mathematician, developed the general Newton–Stirling interpolation formula for trigonometric functions (Maor, 1998; Adamek et al., 2005; Ball, 2010).

The next trigonometer, a Swiss mathematician, **Leonhard Euler** (1707- 1783) also had great impact on the development of trigonometry in the 18th century. He developed the language of **functions** that is used today in applied mathematics (Van Brummelen, 2009). In fact, the idea of function that became an integral part of trigonometry and analysis is credited to Euler and his work, *Introduction Analysis Infinitorum* (Maor, 1998).

Around 1730, Euler argued that trigonometric functions are important in solving differential equations representing harmonic oscillations. Therefore, as a result of this invention, he noticed a significant interrelation between trigonometric functions of sines/cosines and exponential functions such as: $e^{ix} = \cos x + i \sin x$ which is now well known as **Euler's formula** (Van Brummelen, 2009). At this time, indeed, the strict analysis of trigonometric functions was established and they also introduced to the world the **hyperbolic trigonometric functions:** $(\cos x = e^{ix} + e^{-ix}/2$ and $\sin x = \left(e^{ix} - e^{-ix}\right)/2i)$.

Euler also invented the power series expansions for e and the **inverse tangent function:** $\mathrm{acrtanz} = \sum_{\mathrm{n=0}}^{\infty}(-1)\mathrm{z}^{2\mathrm{n}+1}/2\mathrm{n}+1$**.** Van Sickle (2011) states that when trigonometry became analytic and involved complex numbers, the trigonometric functions were thought of completely apart from their line representations and the circles on which they originated; rather, they transformed into a number or ratio, the ordinate point on a unit circle (Figure 2.19).

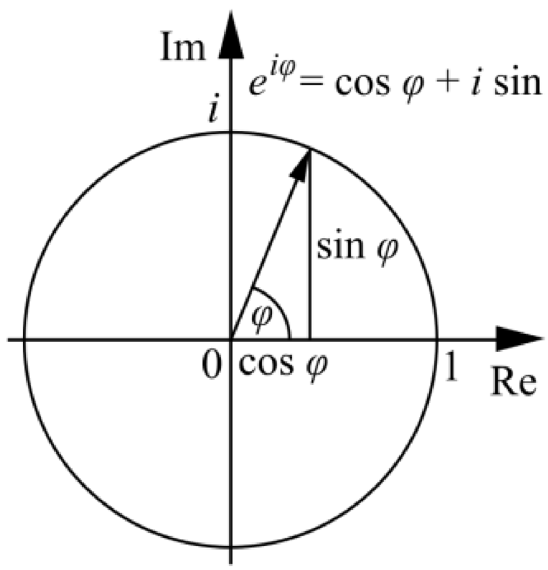


Figure 2.19. A geometric interpretation of Euler's formula

It is possible that Euler might have got the idea of trigonometric functions from **Georg Simon Klugel** (1739-1812), the author of a mathematical dictionary. He first introduced the term “trigonometric functions” in *Analytische Trigonometrie* in 1770 (Van Sickle, 2011). Klugel also defined the **trigonometric functions as ratios** for the first time (Figure 2.20):

$\sin\theta = a/c \quad \cos\theta = b/c \quad \tan\theta = a/b \quad \sec\theta = c/b \quad \csc\theta = c/a \quad \cot\theta = b/a$

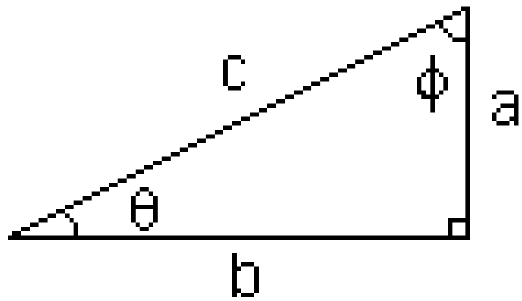


Figure 2.20. Trigonometric function as ratio

In 1715, the English mathematician **Brook Taylor** (1685-1731) defined the general **Taylor series** (a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point) and gave the series expansions and **approximations for all six trigonometric** function in his work called *Methodus incrementorum directa et inversa*. For example:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad for\ all\ x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots. \ for\ all\ x$$

2.4. Trigonometry in the Contemporary Curriculum

Later in the eighteenth century, trigonometry was made a requirement of university and high school mathematics in some countries such as the United States and Canada. It was first taught as entirely geometrical, in terms of trigonometric line, and later ratio definitions became the common practice (Adamek et al., 2005). Nowadays, trigonometric concepts are taught in grade 10. At this grade, students in British Columbia (here I will only discuss the B.C. curriculum, because in Canada each province has its own curriculum and students learn trigonometry in different levels according to their province's curriculum) understand primary trigonometric ratios (sine, cosine, tangent) through applying similarity to right triangles, generalizing patterns from similar right triangles, and solving trigonometric problems.

The following year (grade 11), students learn how to solve a contextual problem that involves two or three right triangles, using the primary trigonometric ratios. While in grade 10 and 11, B.C. students learn trigonometry in the geometry section, there is a separate section called trigonometry in Pre-calculus 11 and 12. In Pre-calculus 11, pupils establish an understanding of angles in standard position [0° to 360°], solve problems using primary trigonometric ratios for angles from 0° to 360° and then solve problems using the cosine law and sine law. In Pre-calculus 12, students gain an understanding of angles in degrees and radians, develop and apply the equation of the unit circle and solve problems through using the six trigonometric ratios for angles. At this level, students also have opportunities to graph and analyze the trigonometric functions (sine, cosine and tangent), to solve the first and second degree trigonometric equations with the domain expressed in degrees and radians and to prove trigonometric identities (e.g., reciprocal identities, sum or difference identities and double-angle identities).

In most of the current textbooks, trigonometry is initially introduced with triangle trigonometry and then circle trigonometry. However, as Bressoud (2010) indicated, this type of teaching practice (teaching first through triangle trigonometry followed by circle trigonometry) is often problematic because this practice leads to student misconceptions. For instance, students often have a difficult time conceiving of sine (which is half of that chord) as a periodically varying function when students are first taught to think of sine as opposite over hypotenuse (Van Sickle, 2011; and Bressoud, 2010). However, students are still forced to learn first through triangle trigonometry because it is thought to be simpler, even though history suggests just the opposite (Van Sickle, 2011). Bressoud (2010) expresses:

“Trigonometry arose in the study of the heavens among the classical Greeks, and this was always circle trigonometry. It took over a thousand years before the first intimations of triangle trigonometry appeared, and it was not until the 16th century that it became generally used as a tool for surveying. The switch in instructional emphasis from circle trigonometry to triangle trigonometry did not occur until the mid- to late-19th century” (p. 1).

Van Sickle, (2011) indicates that teachers assist students’ understanding of trigonometry through reviewing the history of the development of trigonometry from the past to the present. Studying the history of trigonometry and informing students about the origins of trigonometry could help students discovering where their trigonometric formulas, for instance, come from, rather than simply memorizing them. Teachers, also, by knowing about the history of trigonometry, might apply some of the earlier theorems in order to help students comprehended the present concepts.

2.5. Summary and Conclusion

Trigonometry in general and trigonometric functions in particular have a long history as thousands years ago ancient Greek and Babylonian astronomers used trigonometry for their calculations. The slow gradual development of trigonometry, in fact, illustrates the chronological difficulty of the topic. The history of trigonometry shows that trigonometry was further expanded and developed by Arab and Persian mathematicians and eventually it was noticed by Europeans as a scientific subject independent from astronomy.

According to historians’ reports, it was in the eighteenth century that trigonometry became a subject in high schools and universities. Mathematics textbooks often begin with some basic ideas such as angle and radian and then, trigonometric functions are introduced to students through right triangle strategy. Afterward, the trigonometric functions are presented to students in the unit circle. As can be seen from the history of the development of trigonometry, this way of teaching trigonometric concepts from right triangle to the unit circle is opposite to the history of trigonometry.

Having looked at the historical development, in the next chapter I turn to discuss research studies focused on teaching and learning trigonometry.

Chapter 3.

Learning and Teaching Trigonometric concepts

This literature review aims to discuss existing research on the teaching and learning of trigonometry; it is divided into **three parts**, based on the specific focus of the identified studies. The **first part** presents an overview of the studies focused on students and trigonometric concepts. This part itself is divided into three sections. The first section provides insights into students' thinking and understanding of trigonometric functions. The second section investigates a group of studies addressing students' misconceptions and difficulties of trigonometric functions and some related topics (e.g., angle measure). The final section of the first part addresses the effect of technology and students' leaning of trigonometric concepts. The **second part** of this literature review addresses studies that focus teachers and trigonometric topics. This part covers research on various teaching methods that are able to effectively support the learning of trigonometry by students. The final section of the second part discusses teachers' difficulties in grasping trigonometric concepts. This review will conclude with the **third part**, which will highlight the need for further research on teaching and learning trigonometric concepts and in particular transformations of sinusoidal functions.

Table 3.1. The literature review outline

The first part:(Students)	The second part:(Teachers)	The third part
Students' understanding of trigonometric functions	Various instructional designs for teaching trigonometric concepts	The need for further research on trigonometric concepts and in particular transformations of sinusoidal functions
Students' difficulties/ misconceptions in grasping trigonometric concepts	Teachers' difficulties in understanding trigonometric concepts	×
The influence of technology on students' learning of trigonometric concepts	×	×

3.1. Research Studies on Students and Trigonometric Concepts

As mentioned, the research studies on students and trigonometric functions are categorized into three sections: 1) students' understanding of trigonometric functions, 2) students' misconceptions in learning trigonometric concepts, and 3) technology and its influence on students' understanding of trigonometric topics.

3.1.1. Students' Understanding of Trigonometric Functions

Undergraduate students of sciences, are required to study trigonometry, especially trigonometric functions, because a strong foundation in trigonometric functions will likely strengthen their learning of various mathematical topics, such as Fourier series and integration techniques (Moor, 2010). It is shown that understanding calculus and analysis is dependent on learning of trigonometric functions (Hirsh, Weinhold and Nicolas, 1991; Demir, 2011). However, while other functions (e.g., logarithmic functions) can be computed by performing certain arithmetic calculations expressed by an algebraic formula, trigonometric functions involve geometric, algebraic and graphical concepts and procedures, simultaneously (Weber, 2005, Demir, 2011). In other words learning and understanding trigonometric functions is a difficult and challenging task for students, compared to other mathematics functions, such as polynomial functions, exponential and logarithmic functions.

To gain insights into students' understanding of trigonometric functions, Weber (2005) studied a group of college students who were taught trigonometric functions with two different instructional approaches: experimental instruction and lecture- based instruction. The experimental instruction used was based on Gray and Tall's (1994) theoretical notion of procept (the amalgam of three components: a process that produces a mathematical object and a symbol which is used to represent either process or object). The fundamental goal of this instructional approach was to provide opportunities to the students to participate in the class activities included construction of the unit circle, and drawing angles and related line segments corresponding to their trigonometric values (Weber, 2005). In the traditional, lecture-based, textbook-driven approach, the main emphasis was on developing students' procedural skills. Most of the class time was devoted to providing students with explanation on how to do particular exercises, with illustrative examples. After completing the procedure, all the students were asked a series of questions such as:

"Which is bigger sin 37° or sin 23°? Explain why."

“Without measuring, estimate the sin of 170°.”

“Is sin 145° positive and why?”

“Explain why sin θ can never equal 2.”

The findings of the study demonstrated that the traditional approach provided a limited understanding of trigonometric functions to the students; no student could give justification for the reason why sin(θ) could never be 2. Students were often unable to rationalize various properties of trigonometric functions or reasonably estimate the output values of trigonometric functions for various input values. Although the students were reminded of the definition of functions, “for each input there can be only one output,” none of students could propose reasonable answer for the question of “why sine is a function” (Weber, 2005). Consistent with these results, Challenger (2009), Gür (2009) and Marchi’s (2012) studies reported a fragmented understanding of trigonometric functions for groups of high school students taught in a traditional lecture-based course.

In Weber’s study (2005), however, the experimental student groups were able to approximate values of basic trigonometric expressions, and they could determine properties of trigonometric functions. Weber realized that at least 30 out of a class of 40 students were able to describe sine function in terms of a process between an input and output quantity. The students could also discuss why these functions have the properties that they do. Weber (2005) found that the students in the experimental group were able to describe the process of drawing the angle and defining its sine, and then clarify that for each angle there was only one possible point of intersection with the circle. He concluded that these students (the experimental group) were able to understand trigonometric functions because they could develop images of the geometric processes used to obtain those values in context of the unit circle. Similar to Moor (2010), Weber (2005) indicated that there is a close relationship between the unit circle representations of trigonometric expressions and students’ understanding of these functions. However, these results contradict the findings of Kendal and Staceys’ study (1997).

Kendal and Stacey (1997) assessed 178 high school students' understanding of trigonometric functions through two different methods: ratio and unit circle. The research shows that students had a better understanding of trigonometric functions where functions were defined in terms of ratios of sides of right angled triangles (students often taught SOHCAHTOA as a memory aid), rather than with the unit circle method. These results are consistent with those of Palmer's study (1980) in which the students were randomly assigned either a ratio or unit circle instruction to inspect which group grasped more trigonometric functions. The students in the ratio classes outperformed their unit circle counterparts. Burch (1981) also concluded that the participant students had difficulty interpreting trigonometric functions in the unite circle, recognizing that x and y coordinates of a point on the unit circle are cosine and sine values of corresponding angles compared with other determined trigonometric functions in terms of right triangle.

Brown (2005), Demir (2011) and Marchi (2012) suggest that students would better understand trigonometric functions if they have more opportunities to take part in a learning trajectory where both the unit circle and right triangles are utilized, rather than learning trigonometric functions through instructional designs which focused on only one of these two methods. To examine high school students' understanding of trigonometric functions, Brown (2005) developed a model of students' understanding of trigonometric functions within the geometry world of triangles and angles (in degrees), and within the context of the unit circle. In this model, the sine and cosine of an angle can be defined in three different ways: as ratios, as distances, and as coordinates. The results of Brown's (2005) investigations of 120 high school students revealed that those students who were able to use and connect all interpretations (ratios in right triangle, directed distances and coordinates) and moved flexibly between them, could define trigonometric functions and were also better problem solvers. However, Brown (2005) noticed that the majority of the students could still define trigonometric functions and work effectively with only one view, instead of making connections between all three representations. Brown concluded that most students had an incomplete understanding of trigonometric functions.

Following the students' lack of understanding of trigonometric functions in the learning model developed by Brown (2005), Demir (2011) modified Brown's model and he established a new learning trajectory comprised of three different representations i.e. right triangle, unit circle, and graph. Demir studied 24 students aged 16-17, and found that the new learning trajectory could support students' understanding of trigonometric functions in all aspects: the unit circle context (e.g. students knew coordinate definitions of sine and cosine, and utilized them appropriately to point to the correct position with the correct direction); the connections between unit circle and triangle context (e.g., students could determine a required reference triangle, and a proper trigonometric ratio to calculate the required trigonometric value, e.g., sin 45˚ for sin 225˚); and the graph context (e.g., students were successful in interpreting the trigonometric graphs to state the domain and range of the trigonometric functions). As reported by Weber (2005) and Challenge's (2009) studies in which the students demonstrate a certain lack of reasoning about trigonometric functions, the results of Demir's study (2011) show that even with the new learning trajectory, a number of students were still unable to explain why sine and cosine are functions.

This section of the literature review focused on the research studies (e.g., Weber 2005 and Demir, 2011) that examined students' understanding of trigonometric functions. The results of the limited number of research studies show that students often encountered various difficulties (e.g., reasoning why $\sin\theta$ could never be 2) when learning trigonometric functions. There have been some studies (e.g., Orhun, 2001 and Tuna, 2013) which connected students' misconceptions of trigonometric functions with difficulties in understanding some other trigonometric concepts such as angle measures, and graphs of trigonometric functions. In next section, studies focusing on students' conceptions of angle measure and graph of trigonometric function are reviewed.

3.1.2. Student's Difficulties/ Misconceptions in Grasping Trigonometric Concepts

In this section, the research studies focused on the difficulties students often encountered when completing trigonometric tasks are described in detail.

3.1.2.1. Students' Difficulties and the Concept of Angle Measure

A number of studies (e.g., Demir, 2011 and Tuna, 2013) have stressed that a deeper comprehension of the concept of angle measure facilitates students' understanding of trigonometric functions. Degree and radian are used as units of measurement for an angle. Some researchers (e.g., Weber, 2005; Brown, 2006; Challanger, 2009 and Demir, 2011) highlight students' difficulties in understanding angle measures in degrees, especially if they are required to work with acute angle such as $\sin 23^{\circ}$ (Weber, 2005 and Demir, 2011), negative angles such as $\sin(-45^{\circ})$ (Challanger, 2009) or if the angle is greater than 360 degrees such as $\sin 405^{\circ}$(Brown, 2006). Therefore, Martinez-Sierra (2008) and Tuna (2013) emphasize the use of radian to calculate the angle of trigonometric functions.

Tuna (2013) expressed that mathematicians often introduce radian as "the ratio of the length of the arc faced by a central angle to the length of the radius of the circle." This definition refers to the proportion of two lengths and, then, produces the radiant unit of a real number, which can be attributed to the correspondent radiant measure by the wrapping function (defined by wrapping a real number line around unit circle). Essentially, this is the way mathematicians have expressed trigonometric functions over real numbers (Akkoç and Gül, 2010). However, degree is defined by breaking a unit circle into 360 equal parts, and in turn, cannot be used as the domain of trigonometric functions. The formation of such a relationship between the concepts of radian and trigonometric functions is of crucial significance for students' understanding of trigonometric functions (Tuna, 2013).

Although the use of radian has a positive impact on understanding trigonometric functions, some studies (e.g., Fi, 2003 and Akkoc, 2008) pointed to students' difficulty in the conceptualization of angle measurements in radian. For instance, Fi (2003), in study of 14 undergraduate students, found that the students lacked understanding of radian. 11 of the 14 participants presented an understanding of the methods for converting between degree and radian measures ("2π" radians are equal to 360°), in contrast with results of the Orhun (2001), Steckroth (2007) and Akkoç's (2008) studies in which students had difficulties changing degree to radian, and vice versa. However, none of the participants gave an accurate definition for radian measure. Akkoç (2008) and Tuna's (2013) research confirms Fi's (2003) results. Akkoç (2008) indicated that the majority of 42 undergraduate students were unable to define radian, and only one defined radian as the length of an arc: "radian is the angle measurement which is equal to the arc length on the unit circle" (p. 862). Tuna (2013) also found that while 90% of undergraduate students participating in his study were unable to define radian in a circle, a large number of students were able to define the concept of angle correctly.

It was not only undergraduate students who appear to have difficulty in grasping the definition of the concept of radian, as suggested by Fi (2003), Akkoç (2008), and Tuna (2013). Orhun (2001), and Akkoc and Gül (2010) also show the same problem with high school participants. Akkoc and Gül (2010) noticed that high school students (grade 10) could not define radian as ratio of two lengths: the length of the arc of a central angle of a circle, and the radius of the circle. Orhun (2001) asked 17 students from grade 10 to answer the following two questions: "What is the arc length subtended by a central angle 60 degrees in a unit circle?" and "What is the measure of a central angle x subtends an arc of length pi/3 radian?" He found that only 19.5% were successful in calculating angles from given arc lengths. However, 70.1% could find the arc length subtended by an angle in a unit circle correctly. In other words, the students who participated in Orhun's (2001) study had difficulties similar to the participants of Akkoc and Gül (2010) and Tuna (2013).

In addition to students' difficulty in defining radian, these studies (e.g., Orhun, 2001 and Fi, 2003) reported students' misunderstanding of the symbol π. Orhun (2001) reports that many of grade 10 students determined π as the unit for radian measure, and that the students argued that 1 radian equaled to 180° instead of a number close to 3.14. Fi (2003) and Akkoç (2008) find exactly the same difficulties for participant undergraduate students. In other words, students do not view π radian as a real number when discussed in a trigonometry context (Tuna, 2013).

The aforementioned misconceptions of angle measure, which have a direct influence on the understanding of trigonometric functions, are not the only difficulties students may encounter. There are also some other important aspects such as the conception of graphs, which is a very useful tool for students in learning trigonometric functions. In what follows, students' understanding of graph of trigonometric functions is reviewed in detail.

3.1.2.2. Students' Difficulties and Graphs of Trigonometric Functions

Another important issue relating to students' difficulties in understanding trigonometric functions deals with graphs of trigonometric functions (Demir, 2011). Breslich (1928) and Orhun (2001) stated that students, without using graphs, would understand only one aspect of trigonometric functions, the ratio aspect, and they would miss the function aspect entirely. Brown (2006) and Demir (2011) expressed that if teaching trigonometric functions occurred in the context of graphs, students would have a greater chance of conceptualizing sine and cosine as functions of real numbers, and explaining why sine and cosine graphs represent functions by utilizing the formal definition "There is only one y for every x", or the process definition based on an input-output mechanism, or a combined conception of the first two, like "There is only one output for every input".

Although students' conceptions of trigonometric functions rely heavily on their comprehension in the area of graphs of trigonometric functions (Vinner, 1983 and Demir, 2011), many researchers (e.g., Baki and Kutluca, 2009; and Rose, Bruce and Sibbald, 2011) signified graphs of trigonometric functions as the most challenging topic that students typically encounter in mathematics classrooms. In a survey, 65.9% of 123 grade 10 students and even 93% of 146 mathematics teachers described trigonometric functions and graphs of trigonometric functions as the most difficult topics (for students) in the area of trigonometry (Baki and Kutluca 2009). Studies of Adamek, Penkalski, and Valentine (2005) and Tatars, Okur, and Tuna (2008) support the complexity and difficulty of this topic.

Students' difficulties in understanding graphs of trigonometric functions can be traced to the following contexts, in which students are required to 1) make connections between algebraic and graphical representations (Lambertus, 2007), and 2) make transitions between the unit circle and the graphs, which are considered to be a fundamental aspect for understanding trigonometric functions (Rose et al., 2011; Demir, 2011; Brown, 2006). Whereas the first context is common among trigonometric functions and other mathematics functions (e.g., algebraic functions as mentioned in Gagatsis, Elia & Kyriakides, 2003), the second one is only specific to trigonometric functions.

In studies involving undergraduate student participants, Leinhardt, Zaslavsky and Stein (1993), Yerushalmy and Schwartz (1993), and Knuth (2000) recognized that the majority of students were limited by their focus on using algebraic representations to solve the mathematics problems. When they were asked to complete a task by plugging values into an equation or finding points on a graph, they chose the algebraic approach. Even when students had to approximate a y-value on a graph, they plugged the value into the equation instead of using the graph. As a result, students could not develop the skills of flexibly employing, selecting, and moving between algebraic and graphical representations. In other words, many students had difficulties in understanding the links between equations and their graphs (Knuth, 2000). Although these previously mentioned studies worked on other functions instead of trigonometric functions, Challenger (2009) noticed that some high school students had common types of difficulties. For example, students had difficulty when asked to find the coordinates of the point where a given graph of a particular trigonometric function crossed the y-axis (both equation and the graph given to the students). Challenger observed that the students ignored the given graphs and they only tried to find the points where the equations equal zero. In other words, the students were unable to provide a link between graphical and algebraic representations in the context of trigonometry. Similarly, Marchi (2012), in a study of high school students, indicated that none of the high school students who participated in the study utilized a graphical representation to find the answer for tasks such as "Explain how you would solve the following equations: $\sin x = \frac{1}{3}$ and $2\sin x + 1 = \frac{2}{3}$," unless it was suggested to do so. Marchi concluded that the students, in fact, did not know that using a graph was a method for finding a solution.

As mentioned previously, the other set of research focused on graphs of trigonometric functions to explore students' abilities in making connections between the unit circle and graphs. Brown (2005), found that some honors high school students (grade 10) were unable to connect a rotation on the unit circle with a point on a graph of sine or cosine, and only a few students could conceptualize trigonometric graphs through the arc lengths on the unit circle and the corresponding horizontal position. The results of Brown's study confirmed the findings of Marchi's (2012) study in which some high school students could not correctly recall information and were not able to make the right connections when trying to connect the graph for $\sin x$ with the unit circle. Similarly, Demir (2011) noticed that in spite of some students' success, a number of high school students (grade 11) still had difficulties explaining the coordinates on the graphs as arc lengths and vertical positions on the unit circle, or marking the point on the graph corresponding to a given position on the unit circle.

To sum up, students' difficulties in the concepts of angle measure (degrees and radian) and graphs of trigonometric functions are discussed in this section. The results of a number of research studies (e.g., Tuna, 2013) illustrate that students often encountered difficulties in defining radian and grasping π as a real number. The studies (e.g., Marchi, 2012 and Brown, 2005) also reported students' inabilities to make connections between algebraic and graphical representations as well as to connect the unit circle and the trigonometric graphs when working with trigonometric curves. To overcome the reported difficulties in using graphs for trigonometric functions, Zengin, Furkan and Kutluca (2011) advise teachers to use technical devices such as GeoGebra to improve students' learning of graphs of trigonometric functions. In the next section, the effect of technology on students' understanding of trigonometry, both in general, and specifically on graphs of trigonometric functions, is described.

3.1.3. Students’ Learning of Trigonometric Concepts and the Effect of Technology

There have been limited number of research studies (e.g., Blackett and Tall, 1991; Yenitepe 2002; and Choi-Koh, 2003) on the influence of technology on the learning of trigonometric concepts. Robison (1996), for instance, investigated the impact of the use of Mathematica by 99 college students taking a course in trigonometry. The researcher divided students into two groups: a group that received instruction that included Mathematica (a computational software program used in mathematical field) and another group that was presented with static image and they were asked to use their imaginations for solving trigonometric problems. The results of the study illustrate that there is no difference in achievement between the group taught with animation (Mathematica) and the group taught with static images. However, these findings are in contrast with the results of Blackett and Tall’s (1991) study, which shows that implementing a computer software package on the topic of triangle trigonometry had a positive impact on students’ understanding of trigonometric concepts. In a study by Yenitepe (2002), 78 high school students participated in a class lecture about unit circle trigonometry without technology and with technology; Yenitepe (2002) noticed that the students involved in technology-based instruction outperformed their counterparts in the exams.

Research studies (e.g., Choi-Koh, 2003; Army, 1991; Stacey and Ball, 2001) specified that technology plays a fundamental role in understanding graphs of trigonometric functions. Choi-Koh (2003) said that graphs in traditional teaching methods have served as “a display representations mean” because teachers have had only static media. In both the table and formula environments, graphical variations in the display are not continuous; they are drawn according to the discrete data. Technology, on the other hand, provides students with an opportunity to use tools to manipulate graphic objects freely (Choi-Koh, 2003) and it release students from the tediousness of sketching time-consuming graphs by hand, conserving their mental energy for conceptual development (Kissane & Kemp, 2009 and Rose et al., 2011).

To investigate the impact of graphing calculators on students' mathematics achievements, Choi-Koh (2003) invited a 10^{th} grade student to work on seven tasks including creating a graph of trigonometric functions. The results of the case study illustrated that the student utilized the graphing calculator as a tool to successfully graph sine functions and also to detect the changes in the graphs with varying coefficients. Choi-Koh (2003) consistent with Yerushalmy (1988), Army (1991), Stacey and Ball (2001) and (Lambertus, 2007) concluded that technology-based learning environments provide a chance for students to strengthen the connection between graphs and symbolic representations, to address their main difficulties in graphing trigonometric functions regarding this connection (between analytical and graphical representation). These results contrast a few investigations such as Colgan's (1992) study on eight students in grades 11 and 12. In this study, the students were able to use a computer-based graphing tool (Zap-a-Graph) when studying the graph of trigonometric functions. Colgan (1992) noticed that this technical tool did not help students to make connections between algebraic and graphic representations of trigonometric functions and even in some cases led to misunderstandings. The students stated that Zap-a-Graph was not easy to use because it could not plot graphs of trigonometric functions in different colors and used only black and white images.

Garofalo, Drier, Harper, Timmerman, and Shockey (2000), and Demir (2011) expressed that technical tools such as GeoGebra are cognitive tools that support students in understanding connections among different representational systems in trigonometry. They mentioned that use of technology in mathematics classrooms, specifically, would allow students to establish the connections between unit circle and graphs, one of the most problematic areas for students studying trigonometry (in the area of graphing trigonometric functions). Similarly, Kissane and Kemp (2009), and Zengin, Furkan, and Kutluca (2012) suggest that Dynamic Geometry software would enable students to form relationships between graphical representations of trigonometric functions and positions of points on the unit circle. These results are consistent with Tall's (1986) study in which 'A' level students utilized computer graphics as part of a planned teaching strategy. It showed that students were able to link unit circle and trigonometric graphs successfully.

To sum up, a few research studies have focused on the influence of technology on students' learning of trigonometric concepts. While the results of some studies such as Colgan (1992), and Robison (1996) show no differences between the performances of students who have been taught with technology and those who have been taught without it, other studies found the opposite, especially in the area of graphs of trigonometric functions. All the studies (e.g., Zengin et al., 2012 and Garofalo et al., 2000) proposed that technology environments such as Geo-Gebra and Dynamic Geometry software would develop students' achievements in the area of graphing trigonometric functions by providing students with opportunities to make connections between unit circle and graph of trigonometric functions, as well as to connect analytical representations with graphical representations. In other words, technical devices are appropriate pedagogical tools that teachers could implement in classrooms to improve students' performance in depicting graphs of trigonometric functions. The next section describes in detail some other important teaching strategies researchers suggest for teaching trigonometric concepts.

3.2. Research Studies on Mathematics Teachers and Trigonometric Concepts

In this section, the research studies focused on mathematics teachers and trigonometric concepts are presented in two parts: 1) teaching trigonometric concepts through different instructional designs and 2) teachers' difficulties in teaching trigonometric concepts.

3.2.1. Teaching Trigonometric Concepts through Different Instructional Designs

There is a bulk of published research which studies how to teach trigonometry effectively; however, researchers do not agree on a specific instructional method which will guarantee students' success in learning trigonometry. In this part, I summarize important outcomes of these researches. Weber (2005) suggests that students would not benefit if instructors concentrate on teaching trigonometry through traditional, textbook-driven lessons. He says that students would conceptually understand trigonometric concepts and in particular trigonometric functions when teachers provide opportunities for reasoning with numerical values and connecting them with geometric processes, like creating the unit circle. Weber (2005) and Marchi (2012) state that this can only be achieved if various representations are introduced and utilized along with class discussion of their relationships when teaching trigonometry.

Some other researchers such as Kendal and Stacey (1997) advised teachers to teach trigonometric functions through the ratio method because students who had been taught through this method in their study earned higher marks in a trigonometry exam compared to the group who learned trigonometry by the unit circle method. By comparing the techniques and content utilized to teach trigonometry by Turkish and British teachers, Delice and Monaghan (2005) found that mathematics teachers need to develop students' performance at algebraic aspects of trigonometry like "simplifying" trigonometric expressions (because they noticed that the students typically had difficulties in simplifying trigonometric expressions). They add that those mathematics teachers who emphasize trigonometric applications in real-life situations as part of their method of instruction, better support students' success by providing students with relevant challenges. In a similar study, Gould and Schmidt (2010) invited high school teachers to ask their students to create digital story problems about real-life situations and then apply trigonometric functions to solve those problems. The authors expressed that the combination of students' motivation and this non-traditional activity approach would assist students in learning trigonometry successfully.

Barnes (1999) stated that teachers could develop students' achievements in trigonometry by giving them chances to participate in creative writing. She gave her own high school students a set of word problems (related to trigonometric concepts) to solve in groups and then asked students to explain their mathematics accurately in writing. Not only did this exercise reportedly help students to be less afraid to tackle word problems, it also improved their understanding of trigonometric concepts, whereas they had not understood it well, previously.

In a study of 21 grade 10 students over four weeks of trigonometry lessons, Gur (2009) realized that the majority of students could not justify their answers and often preferred to use a formula driven method (like the unit circle) according to mathematics textbooks instead of establishing a conceptual context for what the formula meant. After his observations, Gur (2009) encouraged teachers to allow the students to describe trigonometric definitions or concepts in their own words instead of inviting them to memorize information without understanding it conceptually. Gur (2009), similar to Orhun (2001), stated that following the memorizing formula teaching strategy would often provide students with the knowledge of trigonometry, only for a short period of time. Meanwhile, students would encounter difficulties in remembering and understanding the formula later, and in transferring the principle learned to new situations.

In addition to advising teachers to teach trigonometric functions through employing their graphs to help students better understand trigonometric concepts, Orhun (2001), Topçu et al., (2006), and Akkoc (2008) suggest that mathematics teachers begin defining trigonometric functions before they teach its relationship to angles. Orhun urge that this change in teaching trigonometry is necessary because the measurement of an angle in degree opposes the general definitions and concepts in trigonometry, and this often confuses students. Byers (2010) and Pesek and Kirshner (2000) have another suggestion for teachers. They propose that mathematics teachers avoid teaching trigonometry by beginning the instruction with definitions of trigonometric concepts first and then spending remaining class time on concept development, during which time students often experience greater difficulty when trying to construct trigonometric representations. Instead, the teachers require teaching trigonometric facts through various representations such as the unit circle, vector, ratio and functions (suggested by Weber (2005) as well).

Subsequent to their findings on students' difficulties in understanding radian (as mentioned in the section 3.1.2.1) and in order to address and eliminate this difficulty, Akkoç and Gül (2010) recommend that mathematics teachers teach radian by clearly explaining its connection to arcs. They state that if teachers provide students with opportunities to participate in a learning trajectory based on computer programs which can explore relations between arc lengths and angles in circles, and identify trigonometric functions primarily in domain of arc lengths, then, students would better understand the concept of radian. Moreover, Moor (2012), encourages mathematics teachers to use an arc approach to angle measurement by which students could develop an understanding of angle measurement as a process based on calculating an arc length subtending a fraction of circle circumference. According to this approach, radians and degrees measure the same quantity and are thus scaled versions of one another. One degree is 1/360th of circumference of any circle centered at the vertex of angle and an angle that measures, and 1 radian subtends 1/2pi of circumference of any circle centered at vertex of angle. Moor's suggestions of using an arc approach to teaching have the potential to be quite impactful.

To conclude, the different studies suggest various teaching methods, including unit circle, right triangle, story writing, real-word problems, and computer programs to better support students in understanding trigonometric concepts and eliminate related comprehension difficulties. In addition to the proposed teaching methods, the issue of mathematics teachers' difficulties in teaching students trigonometric concepts is an important part of the research literature, and will be described in the following section.

3.2.2. Mathematics Teachers' Difficulties in Teaching Trigonometric Concepts

Trigonometry is not only a challenging topic for students; it is also difficult for teachers to understand. Limited number of research studies focus on teachers and their understandings of trigonometric concepts, revealing the teachers' difficulties, which are often similar to some of the students' difficulties in understanding the concepts of trigonometry (as mentioned in the previous section). For example, radian which is one of the challenging trigonometric concepts for students is considered a difficult topic for mathematics teachers too. In a study of 14 mathematics high school teachers, Topçu, Kertil, Akkoc, Kamil, and Osman (2006) explored that most teachers had a lack of understandings of radian. For instance, in the following question:

f : R → R and f(x) = x sin x is given. Plot the following points on the Cartesian plane.

a) (30, f(30)) = ? b) $(\frac{\pi}{2}, f(\frac{\pi}{2}))$ *c)* $(\frac{\pi}{6}, f(60))$ *? d)* $(2, f(\frac{\pi}{3}))$*?*

Topçu et al., (2006) noticed that only 3.9% of the teachers considered "$\sin 30$" in radians and 90.2% of them considered "$\sin 30$" in degrees. In other words, the researchers realized that the majority of teachers did not consider radian as a real number although the trigonometric functions that were given to them were explicitly defined as a set of real numbers. These results are in line with the findings of Cizmesija and Siqus's (2013) study in which the participant teachers refer to *pi* as the unit for measuring radian, whereas real numbers not being of the form q π *(q,* π*)*, q $\in\mathbb{Q}$, are not realized as the radian measures of an angle. Furthermore, the results of both aforementioned studies illustrate that the teachers preferred to answer the given trigonometric tasks in degree rather than in radian. For example, to find a length of an arc subtending central angle given in radians, the teachers preferred to change radians into degrees. Topçu et al., (2006) and Cizmesija and Siqus's (2013) conclude that this tendency to convert radian into degrees might be because of the teachers' lack of understanding of radian, since almost none of the participant teachers could successfully define the radian as a ratio of two lengths: the length of the arc of a central angle of a circle, and the radius of the circle. These findings are consistent with Fi's (2006) study in which teachers encountered similar difficulties in understanding the concept of radian. These teachers' mistakes are in line with the misconceptions of students as reported in Moore (2010), Demir (2011), and Moor, LaForest and Kim (2012).

Teachers' misconceptions of inverse trigonometric functions are also another the difficulties noticed in Fi's (2006) study. Although the teachers recognized that the given trigonometric functions were the inverse functions, they discussed them in terms of reciprocal functions. Fi stated that the reason for these teachers' mistakes might be due to their confusion with the real number reciprocal written form. For a non-zero real number x, the multiplicative inverse is written X^{-1} and it is equal to $\frac{1}{X}$, the reciprocal of x, but this is not the case for trigonometric functions. Fi (2006) also indicated that none of the teachers who participated in the study had a conceptual understanding of the co-concept (sine - cosine, tangent - cotangent, and secant – cosecant). The researcher concluded that the teachers often assumed inverse, reciprocal, and co-function as equivalent ideas when answering the interview questions.

In this section, the results of the limited number of research studies that focused on teachers' difficulties in the area of trigonometric concepts such as conception of radian, inverse trigonometric functions and co-functions, are summarized. Some of the teachers' difficulties in understanding the definition of radian and converting radian to angle and vice versa, are similar with the students' misconceptions of trigonometric functions. Therefore, it is not surprising that one concludes that the students often make the mentioned mistakes when working with trigonometric concepts because the mathematics teachers teaching the concepts of trigonometry do not fully grasp these complex concepts themselves. What is missing from all of this literature is a closer look at transformations of trigonometric function. It is exactly this phenomenon that I am interested in.

Table 3.2. Summary of Literature Review

Researcher(s)	Focus of the research(s)	Result(s)
Kendal and Staceys,1997; Weber, 2005; Brown, 2005; Orhun, 2001; Kang, 2003.	Students' understanding of trigonometric functions	Students often have lack of ability in conceptualizing trigonometric functions
Yerushalmy and Schwartz 1993; Knuth, 2000; Fi, 2003; Martinez-Sierra, 2008; Akkoç and Akbaş Gül, 2010; Tuna, 2013; Brown, 2006	Students' conception of angle measure and graph of trigonometric functions	Students often encounter difficulties in defining radian and recognizing1 radian equaled to 180° instead of a number close to 3.14. They also are unable to make 1) connections between algebraic and graphical representations and 2) transition between unit circle and the graphs.
Blackett and Tall, 1991; Yenitepe, 2002; Choi-Koh, 2002; Kissane and Kemp, 2009; Zengin, Furkan, and Kutluca, 2012.	Effect of technology on learning trigonometric concepts	Technology often has a positive influence on students' conception of trigonometric concepts.
Pesek and Kirshner, 2000; Orhun, 2001; Weber, 2005; Akkoc, 2008; Byers, 2010; Moor, 2012.	Designing appropriate instruction for teaching trigonometric concepts	Teaching different trigonometric topics through the unit circle, right triangle, graph, real-life problems and story writing.
Topçu, et.al., 2006; Fi, 2006; Moor, LaForest and Kim, 2012, Čižmešija and Šipuš, 2013.	Teachers' misconception of trigonometric concepts	Teachers' difficulties in understanding the concept of radian; reciprocal and inverse trigonometric functions.

3.3. The Need For Further Research on Teaching/Learning Trigonometric Concepts

The review of the literature on teaching/learning trigonometric concepts (e.g., Brown, 2005 and Fi, 2006) looked at the issue through diverse lenses. A number of research studies discussed various teaching strategies (such as the unit circle and right triangle) for trigonometric concepts, while some others focused on teachers' difficulties such as the misconceptions of the concept of radian, which is interconnected with their understanding of trigonometric functions. Students' understanding of trigonometry and in particular trigonometric functions was examined by a few research studies such as Kang (2003) and Weber (2005), while other research studies (e.g., Moor, 2010) rooted the conception of trigonometric functions into students' ability to identify and reason with trigonometric functions as functions or to correctly define angle measurements, especially radian (Akkoç & Gül, 2010). There have also been some studies, such as Orhun (2001) and Demir (2011), which connect the students' identifications of trigonometric functions to their inadequate understanding of trigonometric graphs, and these authors promote teaching trigonometric functions through graphs. What is missing from all of this literature is a closer look at transformations of trigonometric functions. It is exactly this phenomenon that I am interested in.

Although the concept of transformation of mathematics functions (except trigonometric functions) in analytical and graphical contexts is subject of several studies (e.g., Eisenberg & Dreyfus,1994; Baker, Hemenway, & Trigueros, 2000; Chiu, Kessel, Moschkovich, & Munoz-Nunezby, 2001; Zazkis, Liljedahl & Gadowsky, 2003; Lage & Gaisman, 2006 and Consciência & Oliveira, 2011) that focus on students' understanding of transformations of mathematics functions and some existing obstacles, there has been limited number of research studies focus on transformation of trigonometric functions and in particular sinusoidal functions. One of the sparse studies on the transformation of trigonometric functions, Ng and Hu (2006), examined the impact of using Trigonometric Graphs, a teacher created web-based simulation, and asynchronous online discussion on students' performance in sketching transformation of trigonometric curves. Rose et al., (2011) looked at whether technology has a greater impact on student achievement and attitudes towards transformation of trigonometric functions if it is implemented before or after whole class teaching. Furthermore, Zengin et al., (2011) examined the effect of dynamic mathematics software Geogebra on student achievement in learning transformation of trigonometric functions by comparing two groups of high school students: the experimental group (subjected to lessons arranged with the GeoGebra software in computer assisted teaching), and the control group (subjected to the lessons shaped with constructivist instruction). As it appears from the above mentioned research studies, the main purpose of these researches is examining the influence of technology on students' learning of trigonometry and in particular transformation of trigonometric functions. However, these studies have not addressed questions such as, how do undergraduate students complete mathematics tasks involving the transformation of sinusoidal functions? What are the common mistakes students often encounter when they work with transformations of sinusoidal functions?

The importance of research on the transformation of sinusoidal functions is rooted in my experience as an assistant teacher at a university. When assisting students with their homework, and having discussions with some of the Calculus instructors at Simon Fraser University, I have noted that transformation of trigonometric functions and in particular sinusoidal functions is a complex topic in which numerous students face major difficulties. Students often become confused when they need to identify transformations in analytical and graphical contexts, especially if the sinusoidal functions are transformed horizontally. Even if some students could identify the transformations, the majority of them repeatedly make mistakes in graphing the transformed functions, interpreting the transformed graphs or transformed trigonometric functions. Therefore, the main purpose of my study is examining students' reasoning and abilities of transformation of sinusoidal functions since the number of studies focused on this important and challenging topic is very limited. There is not much research on how students work in situations that involve transformation of sinusoidal functions, whether they are able to recognize transformations and whether they have the skills to identify the influence of transformations on sinusoidal graphs. In other words, none of the aforementioned studies on teaching and learning of trigonometry investigated the way undergraduate students complete tasks involving transformations of sinusoidal functions, types of difficulties they might encounter, and to what degree the barriers in completing tasks are related to students' lack of understanding of trigonometric functions as cited in the literature.

Taken together, for this thesis the major goals are gaining greater insight into undergraduate students' replies to mathematics tasks involving the transformation of sinusoidal functions and the common mistakes they encounter when completing the tasks. I am interested to discover if students' difficulties are connected to algebraic or graphical representation contexts or if those difficulties are related to the way their teachers represent the key concepts. Some of these questions have answers in the context of general mathematics functions. However, there is no sufficient attention to these questions in the context of sinusoidal functions. I believe that by conducting this research study and answering these kinds of questions, I will be able to extend our knowledge to enhance teaching of trigonometry functions and improve the experience and outcomes for students.

Chapter 4.

Methodology

In what follows I review the pilot study and describe how I select and design the tasks for the main study. Then, the undergraduate students who participated in my research are introduced. Data collection and analysis are described in the last part of this chapter.

The specific research questions are in chapter 5 following the description of theoretical frameworks. The general main goal of my study is to respond to the following research questions:

1) How do undergraduate students complete mathematics tasks involving transformations of sinusoidal functions?

2) What are the common mistakes that students encounter when they work with transformations of sinusoidal functions?

4.1. Pilot study

Conducting a pilot study can provide valuable insights for researchers, and it increases the likelihood of success in the main study. Thus, I used the data collected from the interviews with eight students in the pilot study in order to design the interview tasks used in my actual research. Three male and five female students volunteered their time. Two students were registered in Calculus I, the rest of them was in Calculus II at the time of the pilot study.

In particular, eight students were required to complete five tasks that involved sinusoidal functions (transformed horizontally/vertically and both) and their sketches (I describe the sketches later in this section).

Considering the standard general form of the sinusoidal function: $f(x) = A\,sin(B(x+C)\ +D)$ or $f(x) = A\,cos(B(x+C))\ +D)$, transformations can be applied by changing the amplitude (A represents the amplitude, or steepness), the period (B helps determine the period of the graph (the length of the interval in which the graph of the sinusoidal function start repeating itself)), or by shifting the sinusoidal functions horizontally (which is determining by *C*), or vertically (which is determined by D).

For the pilot study, I gave the participants the interview tasks in which A, B, C and D was changing in the sinusoidal functions or their sketches. All eight interviews were videotaped and the interviews were carefully transcribed. To analyze the data initially I labeled the interview tasks as easy, medium and hard, based on my perceived difficulty related to the amount of change from the canonical sinusoid. For instance, I consider the task easy when the participants were required to identify the function $f(x) = cos(x) - 1$ from its given corresponding graph, or to adjust coordinates on the given sinusoidal curve to represent the function$f(x) = sin(x)$. Determining the function of $f(x) = 3sin\left(x - \frac{\pi}{6}\right)$ from its given graph, or to adjust axes on the wavy displace to show the function of $f(x) = 2cos(x) + 3$ I considered as medium-difficulty tasks. Identifying the function $f(x) = \frac{1}{2}sin\left(\frac{2}{3}x\right) + 2$ or $f(x) = 3cos\left(2x + \frac{\pi}{5}\right) - 4$ from their given corresponding graphs were considered as difficult tasks. Reviewing the transcripts, watching videos, and comparing students' responses to each task led me to realize students' varying success on some items and this led me to the detailed analysis of transformations.

The data analysis showed that the eight students could easily recognize the sinusoidal functions and their graphs in which "A" and "D" were changing, (compared with the canonical sinusoidal function). However, they encountered difficulties when the change was in coefficients "B" and "C". In other words; it was not easy for the participants to connect the parameter B in the given sinusoidal function with period and the number of repeated sine cycles in the corresponding graphs.

Furthermore, the data illustrated that the participants in the pilot study could not identify the phase shift (C) from the given sinusoidal graphs or functions properly. They were unable to link the coefficient of x in the given functions and the represented numbers for C with a phase shift. Participants unsuccessfully realized the amount of the radian by which the sinusoidal functions transformed to the right or left. Therefore, in designing tasks for my main study I focus on the changes in the two parameters B and C in the general form of a sinusoidal function.

4.2. Design of Tasks

The interview tasks selected for this dissertation are based upon reviewing the results of the pilot study, which are described in the previous section.

4.2.1. Tasks for the Main Study

For the purpose of this dissertation, based upon students' answers to the pilot study tasks and their common mistakes, I decided to modify the interview tasks for the main study. Therefore, in the next set of the interview tasks (used for this dissertation), the participants were given the sinusoidal functions transformed horizontally (in other words, those tasks in which "B" and "C" were changing).

The tasks which were designed according to the students' difficulties in the pilot study, aimed at investigating whether students are able to identify the functions transformed horizontally, how they determine the period, and how they connect the geometric representation with the symbolic representations of sinusoidal functions. For instance, in Task 4, students were given a sinusoidal curve and asked to label the coordinates such that the curve represented the function $f(x) = sin(4x)$. The successful participants would complete this task by connecting the number four (from the given function) with the point in the sinusoidal curve where the four full sine cycles would repeat.

Two different types of tasks were designed for this study. Type A) *Identifying sinusoidal functions (Tasks 1, 2 and 3)*, and Type B) *Assigning coordinates (Tasks 4 and 5)*. Both types of tasks were presented with the help of the affordances of the software: participants could drag, stretch and compress the graphs, using pens and so on. The tasks are described in detail in the next section.

4.2.2. Sketches

To design the interview tasks I constructed two sketches using the Geometer's Sketchpad software (the software provides students with opportunities to graph functions and also to manipulate the graphs by adjusting the coordinates). The first sketch was intended to study undergraduate students' recognition of sinusoidal functions. This type of sketch includes sine curves transformed vertically, horizontally or both. There were three red dots situated on the x and y-axes (see Figure 4.1 graph of $(x) = cos(3x)$). By clicking on the red dots on the x-axes, participants could stretch or squeeze the curve horizontally in order to find more points on the given curves. Similarly, students could extend/shrink the sinusoidal curves vertically by using the red dots on the y-axes.

Moreover, students had access to the tools on the left side of the screen (see Figure 4.1 as an example). The translation arrow () along with the dot at the center, for instance, give students the opportunity to move the whole graph to the left or the right side, in case if they need to access more information on the axes.

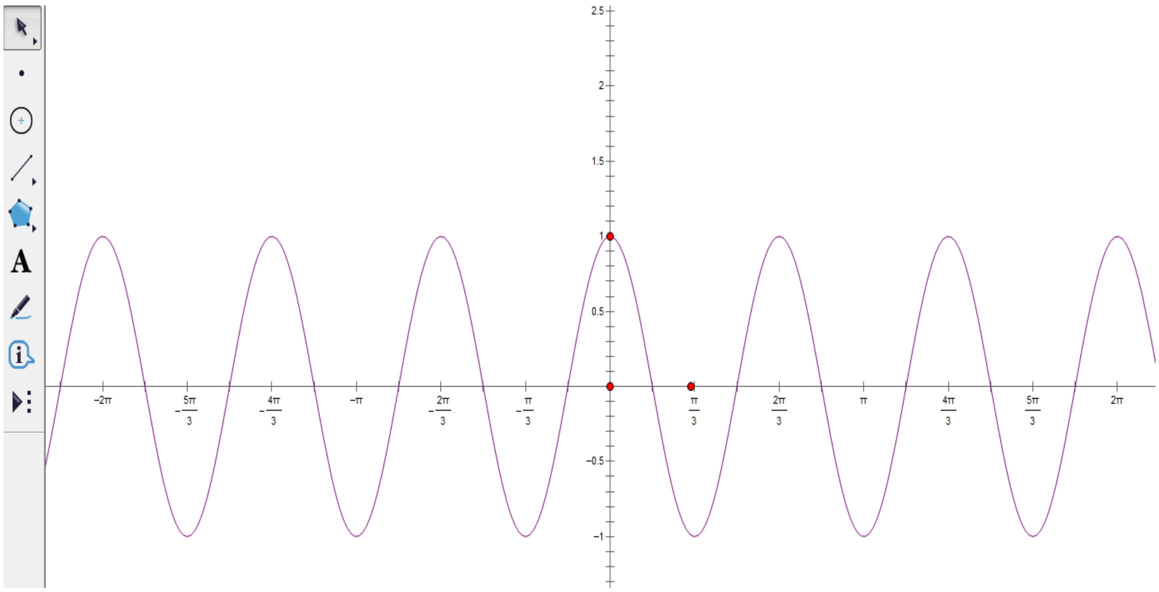
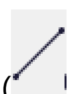


Figure.4.1. Example of the first sketch.

The second sketch was comprised of a plain sinusoidal curve and no coordinate axes (see Figure 4.2) were visible. Students could click on the given curve or the arrow on the right side to move the curve to the right/left. Using the segment straightedge tool (), participants could assign the x and y-axes on the sine cycle. To mark any point on the adjusted axes, students could also apply the tools on the right side ("A" and the pen tools). Moreover, students can use the tools to write the sinusoidal functions represented by the given graph either using the pen tool () to write the function in their own handwriting, or the  tool which helps them write in text format (see Figure 4.2, as an example). All these tools gave students opportunities to assign coordinates and the points on the curve to represent the given function.

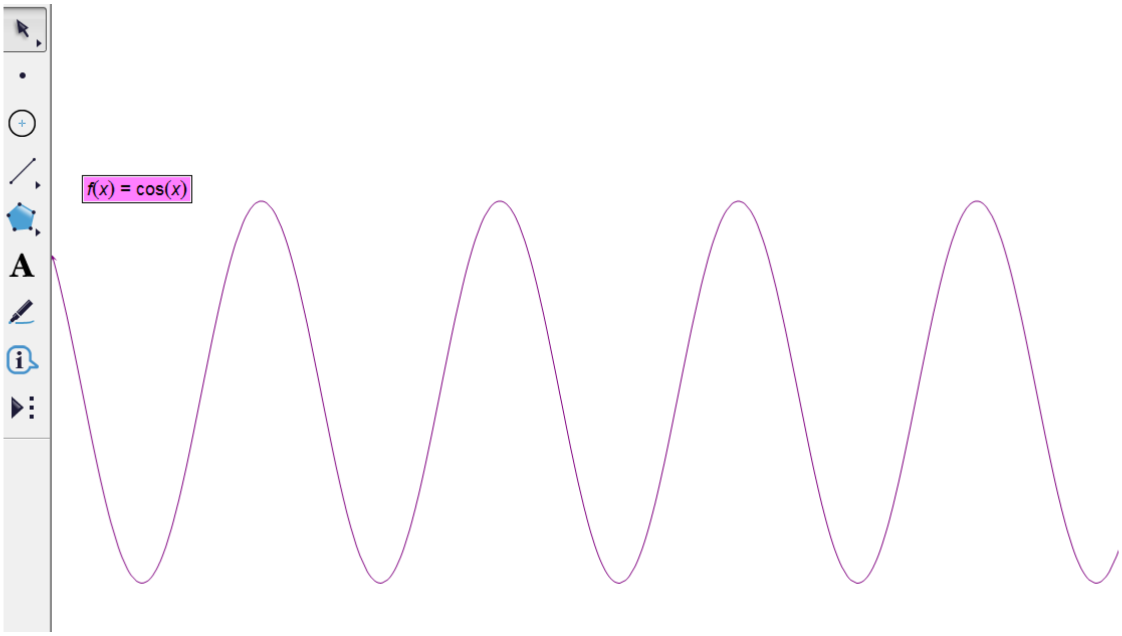


Figure.4.2. Example of the second sketch.

For the Type A tasks ('*Identifying sinusoidal function*'), the sketches indicating the sinusoidal graphs were given and the students were asked to identify the sinusoidal functions represented in the given graphs (see Figure 4.3, 4.4 and 4.5). For the Type B tasks ('*Assigning coordinates*'), the sketches comprising of the sinusoidal curves, and the particular sinusoidal functions in their algebraic form (see Figure 4.6 and 4.8), were given and the students were required to assign coordinates on the sine curve such that it described the given functions. Particular functions to be identified by their graphs in type A tasks were:

Task 1: $f(x) = sin(2x)$, Task 2: $f(x) = sin(\frac{2}{3}x)$ and Task 3: $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$. Particular functions to be represented by proper assignment of coordinates in type B tasks included:

Task 4: $f(x) = sin(4x)$ and Task 5: $f(x) = cos(3x - \frac{\pi}{4})$.

As can be noted, all interview tasks types "A" and "B" were sinusoidal functions having the coefficient of x different from 1. In other words, the periodicities of the functions were different from 2π (which is the period of the canonic sine and cosine functions). The coefficient of x in each of the tasks was either a whole number (in Tasks 1, 4 and 5) or a fraction (in tasks 2 and 3). Moreover, two interview tasks, (Tasks 3 and 5) included a phase shift.

4.2.3. Tasks 1, 2 and 3

In this part, I describe each of the Tasks 1, 2 and 3 in detail along with their graphs.

4.2.3.1. Task 1

Task 1, in which I had students identify the sinusoidal function(s), is shown in Figure 4.3. As can be seen from the figure, the function described the curve associated with the function $f(x) = sin(2x)$ (or $(x) = cos(2x - \frac{\pi}{2}$)). In fact, the sine function ($f(x) = sin(2x)$) was not presented in the task, students were asked to identify it by considering the graph (see Figure 4.3).

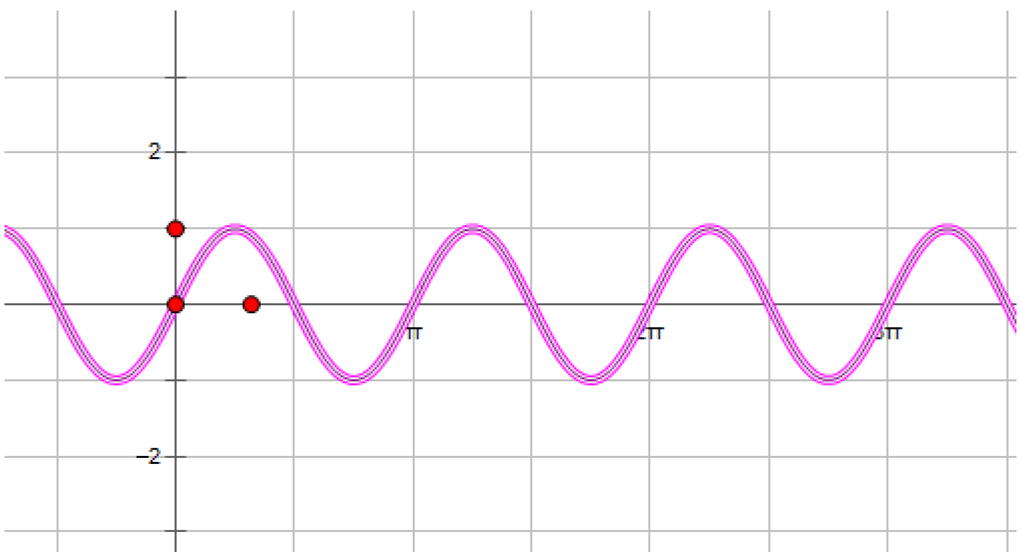


Figure.4.3. Sketch represents the function $f(x) = sin(2x)$, Task 1.

When students completed the first part of the task and they explained their thinking, I encouraged them to complete the second part of the task which was determining a cosine function for the curve if they indicated the function $f(x) = sin(2x)$, or a sine function if they determined $f(x) = cos(2x - \frac{\pi}{2})$ for the given curve.

4.2.3.2. Task 2

To complete Task 2 ($f(x) = sin(\frac{2}{3}x)$), the sketch shown in Figure 4.4 was given to the participants, and similar to Task 1, the sinusoidal functions needed to be identified from the given sinusoidal curve. Although it appears that Tasks 1 and 2 were similar, as they both represent the sinusoidal functions transformed horizontally, they had a major difference. While the coefficient of x in the sinusoidal function associated with Task 1 was a whole number, it was a fraction in Task 2. These types of tasks are not addressed in the literature focused on trigonometric functions and students' (mis)understandings. As such, in this research I aimed to investigate whether students identify the graphs representing the sinusoidal functions having whole and fractional periodicity.

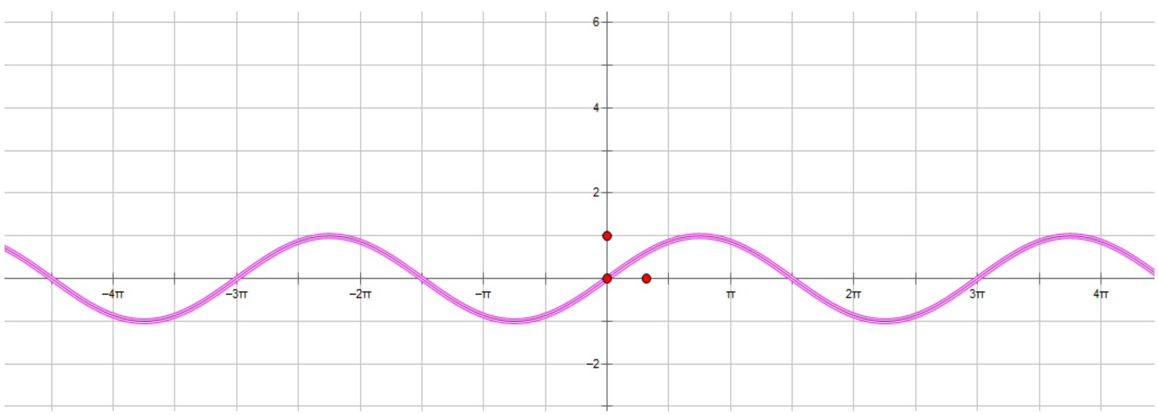


Figure.4.4. Sketch of $f(x) = sin(\frac{2}{3}x)$, Task 2.

4.2.3.3. Task 3

For Task 3 ($f(x) = \cos(\frac{2}{5}x - \frac{\pi}{5})$) students were required to deal with a fractional coefficient of x in the given function which was shifted horizontally (see Figure 4.5). The purpose of this task was to assess students' perception of horizontal/phase shifts obtained by determining the change being made to the x value. Students, in fact, needed to realize that the coefficient of x (i.e., $\frac{2}{5}$) should be factored from the whole argument (i.e., $(\frac{2}{5}x - \frac{\pi}{5})$) in order to represents the function in its canonical form, $f(x) = cos(\frac{2}{5}(x - \frac{\pi}{2}))$. In other words, the goal of the task was to investigate whether students recognize the function shifted horizontally to the right by $\frac{\pi}{2}$, or they wrongly determine $\frac{\pi}{5}$ as a phase shift.

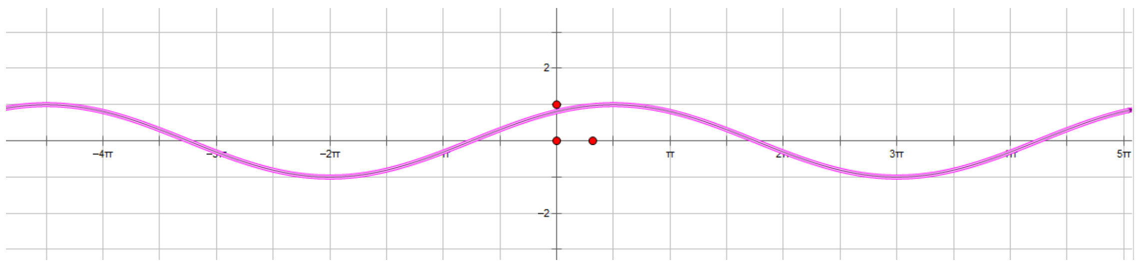


Figure 4.5. Sketch of $f(x) = cos\left(\frac{2}{5}x - \frac{\pi}{5}\right)$, Task 3.

4.2.4. Tasks 4 and 5

The next task (Task 4) was from the Type B of interview tasks, '*Assigning coordinates*'. In this task a sinusoidal curve (see Figure 4.6), along with the sinusoidal function, were given. Students were required to assign the coordinates on the sinusoidal curve to represent the graph of $f(x) = sin(4x)$ (see Figure 4.7). While Task 4 deals with the period concept (similar to Tasks 1 and 2), the last interview task (Task 5) was similar to Task 3 in which students were required to identify the period as well as the horizontal shift (see Figure 4.8 and 4.9). In Task 5, students were asked to adjust axes on the given curve to display the graph of the function $f(x) = cos(3x - \frac{\pi}{4})$.

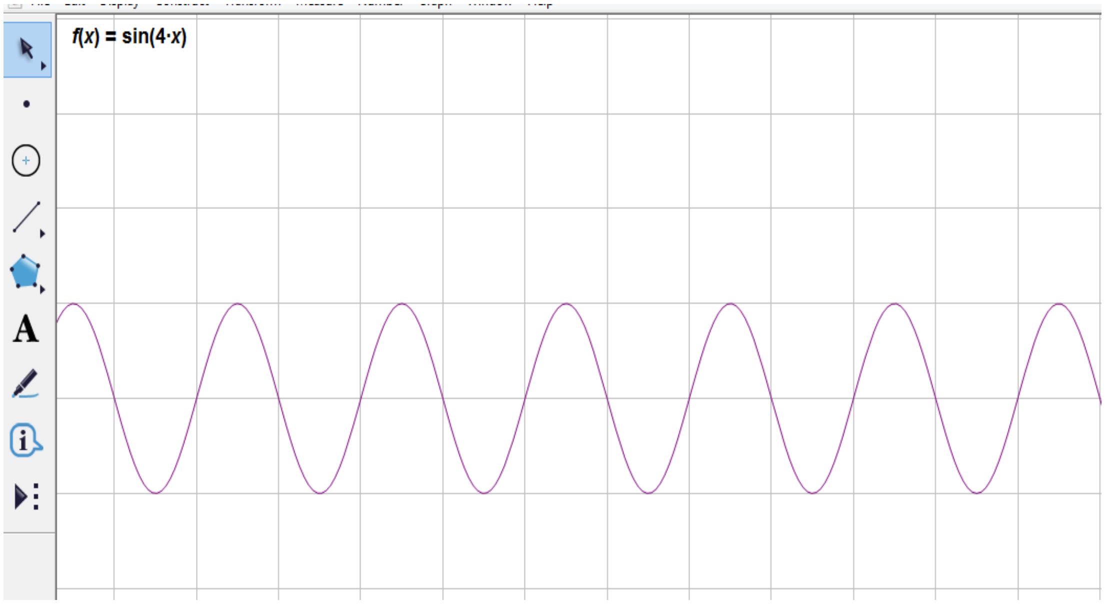


Figure 4.6. **Snapshot of Task 4,** $f(x) = sin(4x)$.

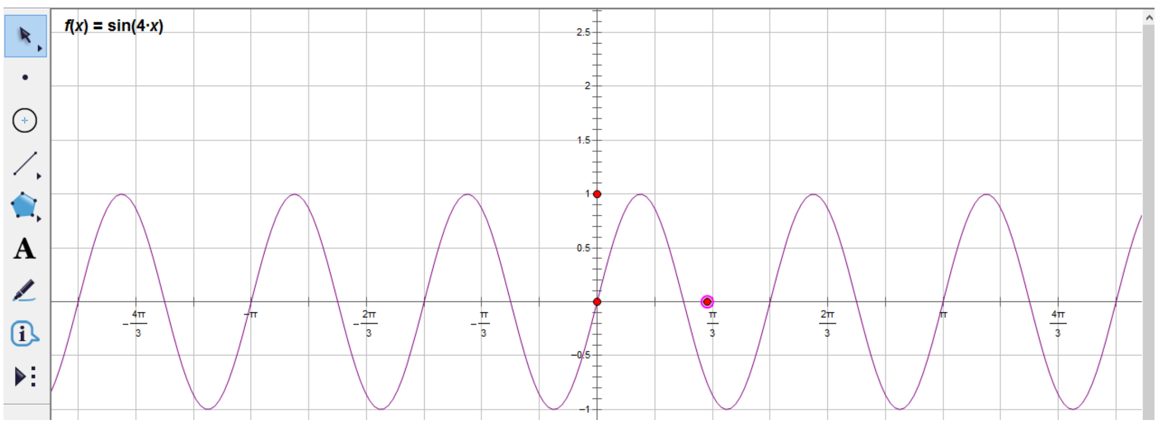


Figure 4.7. **Sketch of** $f(x) = sin(4x)$**, solution to Task 4.**

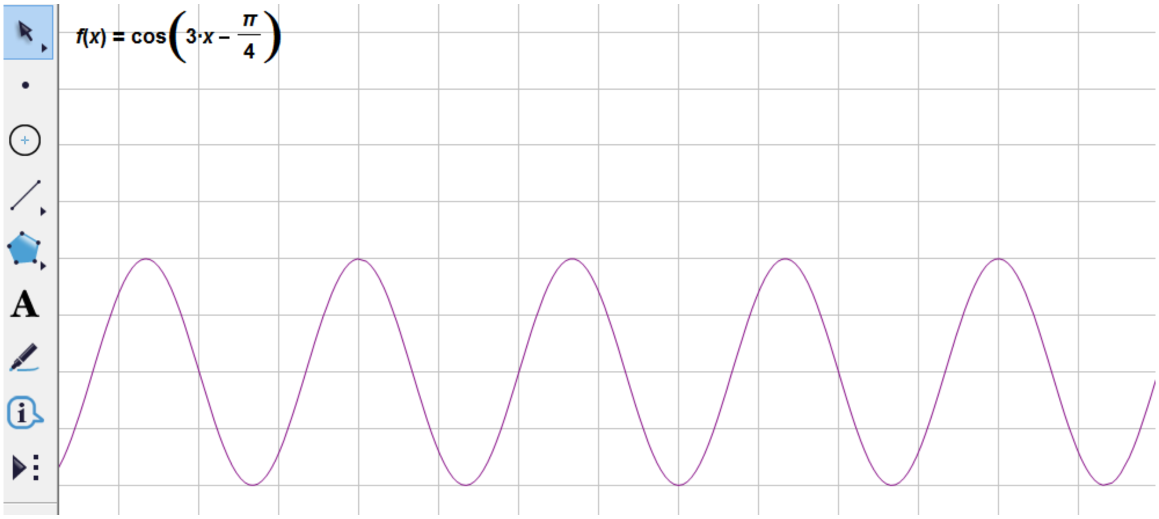


Figure 4.8. Snapshot Task 5, $f(x) = cos\left(3x - \frac{\pi}{4}\right)$.

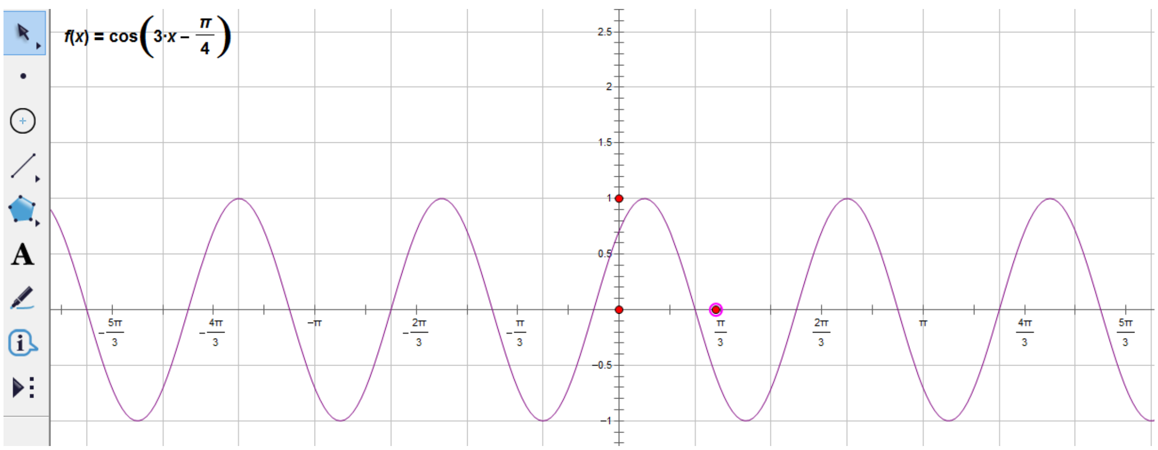


Figure 4.9. Sketch of $f(x) = cos\left(3x - \frac{\pi}{4}\right)$, solution to Task 5.

4.3. Participants

Six students participated in my main study. The participants were from a large North American university. They were selected from among students who had either completed a Calculus I course and were enrolled in a Calculus II, or were in a Calculus I course at the time of the interview. The participants were students who had volunteered to participate in the study after I had made a general request to all the Math 154/155 (Calculus I and II for the Biological Sciences) and Math 157/158 (Calculus I and II for the Social Sciences) classes in the Mathematics Department.

Six undergraduate students who participated in the main study were a male and five females. The participants all volunteered their time. None of the six participants (Andy, Emma, Sally, Rose, Mia and Kate, all pseudonyms) were familiar with Sketchpad, but they all sketched the trigonometric functions using Sketchpad without any difficulty. The participants all knew me because I was working as a Teaching Assistant in the Applied Calculus Workshops at the time of data collection.

4.3.1. Emma, Rose and Sally

Emma, Rose and Sally were all first year students registered in Calculus I at the time of the interview. It was their first semester at the University. They stated that they initially learned trigonometry when they were students in high school. While Emma found trigonometry a hard subject amongst other mathematics topics in Calculus, Sally and Rose believed that trigonometry was an interesting subject. Emma and Rose mentioned that they earned over 85% in the high school Calculus course they took in their grade 12, and they earned above 60% for the first midterm in Calculus I (the interview was conducted after the first midterm). Sally noted that she earned above 80% in both Calculus-12 and in the first midterm in Calculus I at the University.

4.3.2. Kate and Mia

Kate and Mia were second year students pursuing Bachelor's Degrees in Business and Applied Science. Kate and Mia successfully completed Calculus I and were enrolled in Calculus II at the time of the interview. It was the first time they used Sketchpad in mathematics, but they had previous experience in working with a similar Dynamic Geometry software. Before conducting the interviews, Mia stated that she might have forgotten the trigonometric facts, but she hoped that she could remember them all when working on the tasks. Kate requested to practice working with Sketchpad and reviewing some properties of transformations of sinusoidal functions before actually recording the interview.

4.3.3. Andy

Having completed AP Calculus (Advanced Placement) as a high school student, Andy was enrolled in Calculus II as well as in Applied Linear Algebra at the time of the interview. He was a first year student persuing a bachelor's degree in Applied Science. He wanted to be an engineer in the future. His grades in high school and University mathematics courses were high.

4.4. Data Collection

Data were collected using one-on-one, task-based semi-structured, clinical interviews. Each interview conducted was about 60 minutes long, and they were videotaped. In order to complete the interview tasks, a laptop was given to each participant. To provide a comfortable environment, all the interviews took place in the Science Department. To ensure students were familiar with the topic, all students participated in a brief informal conversation about sinusoidal functions as well as the transformation of sinusoidal functions before conducting the actual interviews. In particular, during the informal conservation, we discussed how sinusoidal functions might transform horizontally or vertically. The participants described how each of the elements: A, B, C and D in the general sinusoidal function $(f(x) = A\, sine/cos(B(x + C) + D)$ might affect the graphs of the functions. The students also talked about the difficulties they encountered when working with the transformation of sinusoidal functions during their Calculus courses at high school as well as at university. The participants also had time to experience working with the sketchpad.

The participants' interactions with the sketches, their dialogues, bodily movements and what the participants wrote during the interviews were captured on the videotapes.

The use of semi-structured task-based clinical interviewing was fit for data collection with respect to my research questions. It is indicated that Piaget was one of the people who used clinical interviews for the first time to determine the construction of the individuals' thoughts, the richness of their thoughts and also to assess the individuals' cognitive competence (Ginsburg, 1981). To identify the process of the individuals' thoughts, the clinical interview may involve some degree of standardization and the interviewer's prompts are dependent on the participant's involvement in the interview tasks. In this study, I used a few prompts to direct the discussion towards the purpose of the research. The prompting follow up questions were different for each participant according to the participants' interaction and their responses to the interview tasks.

4.5. Data Analysis

In order to carry out analysis, I used the three theoretical frameworks (*Mason's shift of attention, Presmeg' visual imagery and Carlson's et al. convarasional reasoning).* In the following chapter, I describe the frameworks and the way I analyse data according to the frameworks. To analyze the data, the videotapes were watched several times. The students' conversations were listened to several times and they were transcribed carefully. I reviewed the performance of each participant on each interview task. The data analysis is detailed in Chapter 5.

Chapter 5.

Theoretical Considerations

In this study, the data consist of interviewing six participants. In order to analyze the participants' responses to the interview tasks, I describe several theoretical frameworks involved in the data analysis (in this chapter). In each section of Chapter 6 and 7, the collected data are analyzed and interpreted according to three theoretical frameworks. *Mason's theory of shifts of attention* is the first framework described below. Mason's theory provides opportunity to study the critical role of attention and awareness in learning and understanding mathematics and in particular the concept of graphing sinusoidal functions. Then, *Presmeg's (1989) five different category of visual imagery* is reviewed. As a broad theory, it is used for investigating students' visual mental constructs since the participants applied their imagery skills in different occasions when they completed the interview tasks. The ability of students is evaluated by *Covariational reasoning* focusing on Carlson's et al. (2002) *collection of mental action and developmental levels.* At the end of this chapter, I describe data analysis with respect to the three frameworks.

5.1. Mason's Theory of Shifts of Attention

Among all aspects of the human psyche being discussed in psychological literature (e.g., Piaget (1954/1981)), Mason (2008) relies on two particular aspects of the human psyche, *attention* and *awareness*, in his theoretical framework. He believes that *attention* and *awareness* are human constructs in an attempt to define components of 'being human', and more specifically, 'being mathematical'. While some researchers, such as, Aguirre,Turner, Gau Bartell, Kalinec-Craig, Foote, McDuffie, and Drake, (2012) define *awareness* as understandings, insights, knowledge, and beliefs about teaching and learning mathematics, Mason (2008) refers to *awareness* as what enables people to act, calling upon our conscious and unconscious powers. He argues that *awareness* may exist in our body as we may initiate an action that our consciousness is able to build a narrative on only later in time, although we might be unable to speak about what we are doing.

To educate one's *awareness*, we need to shift our attention to actions which are being carried out and to encourage individuals to think about 'what comes to mind' in a situation (Mason, 2008). Mason argues that attention "is not an all-or-nothing experience"(p.8): its structure is comprised of a *macro* (what is attended to by an individual (i.e., what objects are in one's focus of attention)), and a *micro* level (how the objects of attention are attended to).

Mason (2008) believes that at the macro structure of attention (what is being attended to) learners can attend to more than one thing at once and that learners' macro attention is usually caught up with their current action. He states that at the macro level, *attention* can vary in multiplicity, locus, focus and sharpness. To address the "how-question (how something is being attended to)" (*micro qualities*), He distinguishes five different structures of attention. These are: *holding wholes, discerning details, recognizing relationships, perceiving properties, and reasoning on the basis of agreed properties*. He believes that when students are *holding wholes,* they may gaze at a geometric diagram or some particular part of a diagram. It can also occur when students gaze at a collection of symbols and "waiting for it to speak to you" or in the other words, "waiting for things to come to mind" (p.37). *Holding wholes* may last not too long (only a few micro-seconds) before students can *discern details*. *Discerning details* is a structure of attention, in which students' attention is caught by a specific detail that becomes recognized from the rest of the elements of the attended object. He states that *discerning details*, which often does not happen all at once, is neither algorithmic nor logically sequential. Determining details provides students opportunities to *recognize relationships* between specific elements or between symbolic and geometric representations of a concept. Mason asserts that there is an essential shift between recognizing relationships between symbolic and graphical representation of a mathematics concept to *perceiving properties.* Mason refers *perceiving properties* as "when you are aware of a possible relationship and you are looking for elements to fit it, you are *perceiving a property*" (p. 38). *Reasoning on the basis of perceived properties* is considered as the final structure of attending in which students can perceive properties as the only basis for further reasoning.

It appears that Mason's framework of shifts of attention is appropriate for analyzing the collected data in my research. Applying this framework would support me in gaining insights not only into "*what*" undergraduate students attend to when completing mathematics tasks related to transformation of sinusoidal functions, but also "*how*" they shift their attention.

Mason's terms for different structures of attention also provide a language for analyzing students' work. For example, when students gaze at the given curves or sinusoidal functions, they are *holding whole*s Students who look for particular details from the given sinusoidal functions or the given sinusoidal curve (e.g., they seek the point where the graph intersects the y-axis), they are, in fact, *discerning details*. Students are *recognizing relationship* when are able to find connection between the graphical representation of sinusoidal functions and their symbolic representations. When students consider a particular graph and recognize its shape as representing a sinusoidal function because the amplitude is 1 and a full sine cycle ended in 2π, they are *perceiving properties of sinusoidal functions.* When students determine the graph represents $f(x) = 2\sin(x)$ because its amplitude is 2 times the amplitude of a canonical sine function, they are *reasoning based on perceived properties*.

5.2. Presmeg's visual imagery

Visualization is increasingly being accepted as an important aspect of mathematical reasoning by all branches of mathematics studies (not merely of "obviously visual" branches such as geometry) (Presmeg, 2006, Wheatley and Brown, 1994). Visualization often occurs when "thinking is spontaneously accompanied and supported by images" (Mariottii & Pesci, 1994, p.45). Elliott (1998) regards visualization as the "ability to create rich, mental images which the individual can manipulate in his mind, rehearse different representations of the concept and, if necessary, use paper or a computer screen to express the idea in question" (p.45).

In studying the overall role of imagery in the conceptual understanding of mathematics concepts, Presmeg (1986a, 1986b, 1989, and 1992) identified five main types of visual imagery: (1) *concrete, pictorial imagery*; (2) *pattern imagery*; (3) *memory images of formulae*; (4) *kinesthetic imagery*; and (5) *dynamic (moving) imagery.*

Based on the Presmeg's classification, imagery is called *concrete imagery* if an individual has a picture of an object in his/her mind. For example, when students in a mathematics word problem are asked to divide a large pizza into some triangular slices, they have a *concrete image* of triangle in their mind. *Pattern imagery* is defined as an *imagery* in which pure relationships depicted of concrete details. *Pattern imagery* may appear when students required finding $sin(450°)$, which they visualize repeating pattern on the sinusoidal curve. Another type is *memory images of formulae* by which students typically "saw" a imprinted formula in their minds, written on a blackboard or in their notebooks. For instance students often used the formula of ($y = a(x - h)^2 + k$) in their minds when looking for the vertex point (h, k) of a quadratic function (e.g., $y=x^2$). *Kinaesthetic imagery* involves physical movement in evoking mathematical concepts especially when students could not remember the words. *Kinesthetic imagery* appears when a learner "walks" around imaginary quadrants with her fingers to identify where the tangent at an angle is negative. The last category of *imagery* is *Dynamic imagery* which involves moving and transforming objects in the mind like when students shift horizontally the graph of $y=x^2$.

In a study of 54 high school students, Presmeg (1997a, 1992) found that each of these five types of visualization play an important role in students' mathematical understanding. She considered the *pattern imagery* as an important essence in conceptualizing mathematics concepts because it allows learners to concentrate on the rational aspect of the mathematics problem and it lacks superfluous details. Relying on the results of her study, however, Presmeg noticed that the concrete *imagery* was the least effective type of visualization dealing with students' understanding of mathematical contents (Presmeg, 1989).

The five categories of Presmeg's visualization are applicable when considering transformations and graphing of sinusoidal functions. *Concrete imagery* could occur when students visualize a general picture of sinusoidal functions. They can visualize that a cosine graph going down, intersects the x-axes, continues going down and then goes up, intersecting the x-axes again and keeps going up, while a sine curve goes up, then down, intersecting the x-axes, goes down and up. *Pattern imagery* in the context of a sinusoidal function is identified when students can visualize a certain pattern in the graphical or symbolical representations. For instance, when students notice that in the function $f(x) = \ cos(5x)$, five sine cycles will repeat between the interval [0, 2π]. For the other category of visual *imagery*, the *memory image of formula,* undergraduate students can use the canonical formula $(f(x) = \ A\ sin(B(x + C) + D$ and $f(x) = A\cos(B(x + C)$+D) while describe their thinking, or when they are able to change a cosine function into a sine function having the formula $cos(x) = sin(x - \frac{\pi}{2})$ in their mind.

The last two categories of visual *imagery* involving movement are *kinesthetic* and *dynamic imagery. Kinesthetic imagery* occurs when students involve their body, such as their hands, to represent the expansion of the sine curve, or when they use their index finger to show the value in which the sine curve intersects the x-axes. *Dynamic imagery* involves moving and transforming a given sine curve into a cosine curve in a student's mind.

It is important to note that in this research study, I used the terms *kinesthetic imagery* and *pattern imagery* slightly different from Presmeg's definition. She applied primarily *kinesthetic imagery* in her research studies when participants cannot remember the mathematics terms and instead they expressed them by their body. However, in this study I apply *kinesthetic imagery* when students used their fingers or hands to explain their thinking, regardless of their memory of the mathematics facts related to the transformation of sinusoidal functions. I use the notion of pattern *imagery* in a similar way.

5.3. Carlson's et al. Covariational Reasoning

Covariational reasoning, defined as cognitive activities involved in coordinating two quantities, varies in tandem (Moor, 2010). In the studies of undergraduate students, for instance, Carlson, Jacobs, Coe, Larsen, and Hsu (2002) and Moor (2010) noticed that *covariational reasoning* is central for supporting undergraduate students' construction of trigonometric functions while dealing with the angle measure, the covariation of angle measure and the trigonometric ratio. The outcomes of research studies by Thompson, 1994, Carlson, 1998; Zandieh, 2000; Carlson, Jacobs, Coe, Larsen, and Hsu, 2002; Moore, and Bowling, 2008 also illustrate that *covariational reasoning* plays a critical role in developing students' understanding of graphs as representations of the relationship between two quantities' values.

Following the essential and crucial role of *covariational reasoning* in learning Calculus topics, Carlson (1998) provided a framework as a means to study multiple behaviors of a group of undergraduate students involved in interpreting and representing dynamic function situations. Carlson's (1998) framework consists of five *mental actions (MA#*) specific to the behavior displayed:

1) Coordinating the value of one variable with changes in the other (Mental action 1, M#1),

2) Coordinating the direction of the change of one variable with changes in the other variable (Mental action 2, M#2),

3) Coordinating the amount of change of one variable with changes in the other variable (Mental action 3, M#3),

4) Coordinating the average rate-of change of the function with uniform increments of change in the input variable (Mental action 4, M#4), and

5) Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function (Mental action 5, M#5).

Table 5.1 presents a summary of verbal behaviors related to general functions and trigonometric function (Carlson, et al., 2002; Carlson & Oehrtman, 2004; Oehrtman, et al., 2008, Moor, 2010).

Table .5.1. Covariational reasoning with respect to general and trigonometric functions

Mental Actions	Description of Mental Actions	Verbal Behaviors Related to General Functions	Verbal Behaviors Related to Trigonometric Functions
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	Verbal indications of coordinating the two variables (e.g., y changes with changes in x)	Verbalizing that the output of $\sin(\theta)$changes with changes in angle measure, θ)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Verbalizing an awareness of the direction of change of the output while considering changes in input	Verbalizing an awareness of the increasing output values of $\sin(\theta)$with increasing values of angle measure θ (θ between 0 and $\frac{\pi}{2}$radians
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Verbalizing an awareness of the amount of change of the output while considering changes in the input	Verbalizing that for an angle measure increasing from 0 to $\frac{\pi}{2}$ radians, the output values of $\sin(\theta)$increases from 0 to 1 length of a radius
Mental Action 4 (MA4)	Coordinating the average rate-of change of the function with uniform increments of change in the input variable.	Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input	Verbalizing that the average rate of change of the output values of $\sin(\theta)$with respect to angle measure θ decreases for successive uniform increments of angle measure θ between 0 and $\frac{\pi}{2}$ radians
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)	Verbalizing an awareness that the instantaneous rate of change of the output values of $\sin(\theta)$with respect to angle measure θ decreases over the domain of θ values from 0 to $\frac{\pi}{2}$ radians.

In spite of the fact that, previous research studies (e.g., Carlson, 1998, Zandieh, 2000) attended to *covariational reasoning* related to trigonometric functions, as it has been shown in the above table, further elaboration is essential to account for connecting sinusoidal functions to their graphs. In what follows I elaborate on mental actions 1, 2 and 3 associated with transformation of sinusoidal functions.

Mental action 1: Students are aware of the fact that the graphs of sinusoidal functions or sine curves change as periodicity (B) is changing in the sinusoidal functions. They may also recognize that as a phase shift (C) is adding or subtracting from the argument x in the sinusoidal function, the graph should shift accordingly. Or if the given curve does not intersect the origin (0, 0) or y-axes at y=1, the curve should shift horizontally and therefore the represented sinusoidal function should comprise of a phase shift.

Mental action 2: Students realize that as the coefficient of argument x in the sinusoidal functions *increases* (if it is a number more than 1), the number of repeated full cycles would increase in the interval [0, 2π]. However, if the coefficient of x in the sinusoidal function is decreasing (it is a fraction less than 1) the number of full sine curves will decrease in the interval [0, 2π]. Students verbalize that the direction of shifting the graph to the right or left depends all on the phase shift added or subtracted from the argument in the sinusoidal function (e.g., if the phase shift added to the argument the graph shifted to the left).

Mental action 3: Students connect the amount of changes in period and phase shift and their influence in the sinusoidal graphs or functions. For example, from the given function $f(x) = cos(3x - \pi)$ students acknowledge that there should be three sine cycles in the interval [0, 2π] because the coefficient of x in the given function is 3. They also identify the phase shift as $\frac{\pi}{3}$, so that the graph should be shifted to the right side of y-axes by $\frac{\pi}{3}$.

It is important to note that in my study the *mental actions #4* and *5* do not appear, because in the tasks related to a sinusoidal function there is no "average rate-of change" (M#4) or an "instantaneous rate of change" (M#5).

Four years after Carlson (1998) introduced her framework; Carlson et al. (2002) modified it. During their investigations of the complexity of high-performing 2nd-semester Calculus students' ability for the "construction of mental processes involving the rate of change as it continuously changes in a functional relationship"(p.352), Carlson et al. (2002) noticed that the mental actions determined in the Carlson's (1998) framework were insufficient to classify undergraduate students' *covariational reasoning* aptitude. They found that the *covariational reasoning* abilities can be interpreted from the collection of behaviors and mental actions shown when engaging in a mathematics problem or situation. In order to describe this collection, Carlson et al. (2002) extended the covariation framework by adding *five distinct developmental levels of covariational reasoning* (L#) that parallel the five mental actions (see Table 5.2). A student's *covariational reasoning* ability is said to reach a certain level (e.g., Level 3) when it supports not only the mental action associated with that given level, but also with all mental actions associated with lower levels (e.g., MA1 and MA2 abilities).

It is important to note that the above mentioned three frameworks affect the research questions respectively. In Chapters 3 and 4 two general research questions were stated. In what follows, the two research questions (questions #2 and 3) are added to the two previously mentioned questions (questions #1 and 4). As such, in my research I am interested to answer the following questions:

1) How do undergraduate students complete mathematics tasks involving the transformation of sinusoidal functions?

2) What is undergraduate students' covariational reasoning and visual *imagery* of the transformation of sinusoidal functions?

3) How do students shift their attention when completing interview tasks?

4) What are the common mistakes students often encounter when they work with transformations of sinusoidal functions?

In what follows, I describe in detail how the data analysis was conducted with respect to the three frameworks. .

5.4. Application of the Frameworks in Data Analysis

In order to carry out my analysis, I used the three theoretical frameworks (Mason's shift of attention, Presmeg's visual imagery and Carlson's et al. convarasional reasoning). To analyze the data, the videotapes were watched several times. The students' conversations were listened to several times and they were transcribed carefully. I reviewed the performance of each participant on each interview task. When watching the videos, I zoomed in on all the details students' focused on. For instance, if students put once the computer pointer on the origin and the point 2π, and then on the points $\frac{\pi}{2}$ and $\frac{\pi}{4}$ in the graph, I considered it important to investigate how they shifted their attention from the properties of sinusoidal functions to the details of the graph (these are important according to Mason's theory). Meanwhile, the participants' hand movements (e.g., to show the function is extended horizontally), explanations and sketches helped me to analyze their work in respect to the Presmeg's visual imagery framework. Listening to the students' discussions and watching the videotapes also provided me the opportunity to classify undergraduate students' *covariational reasoning* aptitude, which can be interpreted from the collection of behaviors and mental actions shown when engaging in the interview tasks.

When analysing students' replies to Task 1, I reviewed each student's work separately. In order to describe how students recognize the point π on the x-axes and consequently choose 2 for the coefficient of x in the sinusoidal function, I relied on the three frameworks. I found all students' words; hand written notes, sketches, and body movements essential in analysing the ways students connect the number of the full sine cycles to the periodicity of the function. Afterward, I compared the participants' approaches. In this way, I found similarities and differences among students when completing Task 1.

I followed the same procedure for Task 2 (based on the three theoretical frameworks). In Task 2, it was imperative for me to investigate how students provided a link between the length of a full sine curved ended at the point 3π and the argument of x, which was a fraction. Watching videos several times also helped me investigate whether students could identify a phase shift and periodicity from the given graph in Task 3. I was interested to inspect the points (on the x-axes) that the participants found important to zoom in, when they were identifying periodicity and phase shift. I was also curious to see which properties of sinusoidal functions they focused on and how they shifted their attention from the detail they discerned from the given graph to relationship between the graphical and symbolic representations. Moreover, I paid attention to students' hands/fingers movements and their explanations to identify their *dynamic* and some other *visual imagery* as well as their *covaritional reasoning*. Listening to the students' discussion and reviewing the transcripts directed me to examine whether students made similar mistakes in Tasks 2 and 3 when identifying the coefficient of x which was a fraction.

For Task 4, while listening to each student's discussion, I inspected whether participants found the number 4 important in the given task. I considered it significant to see where the student put the axes on the given sinusoidal curve, which points she/he put on the x and y-axes, and how and why the student might change the points or the place of the coordinates (if any). Similar to the other tasks, I was looking for each participant's explanations as well as the computer pointer's movement to determine their *mental actions*.

The approach I use to analyse Task 5 was similar to that used for Task 4, except I paid extra attention to the participants' thinking about a phase shift (as I did for Task 3). I wanted to study what aspects of the given function directed participants to realize a phase shift correctly, whether they noticed the relationship between the numbers 3, $\frac{\pi}{4}$ and the phase shift. It was also vital for me to determine what mislead them to identify inappropriate graphs for the given function.

For the purpose of this study, the data analysis is presented in the following two chapters, where Chapter 6 focuses on Andy's responses to the interview tasks, and Chapter 7 describes the rest of the students' responses to the tasks. The choice to present the data analysis in two separate chapters on the one hand introduces sufficient detail in presenting a particular case, but on the other hand does not bore the reader with unnecessary redundancy.

Table 5.2. Levels of the Covariation Framework

Levels of the Covariation Framework	Description of Levels of the Covariation Framework
Level 1. Coordination	At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1).
Level 2. Direction.	At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions MA1 and MA2 are BOTH supported by Level 2 image.
Level 3. Quantitative Coordination.	At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions MA1, MA2 and MA3 are supported by Level 3 image.
Level 4. Average Rate.	At the average rate level, the images of covariation can support the mental actions of coordinating the average rate-of-change of the function with uniform changes in the input variable. The average rate-of-change can be unpacked to coordinate the amount of change of the output variable with changes in the input variable. The mental actions MA1 through MA4 are supported by Level 4 image.
Level 5. Instantaneous Rate.	At the instantaneous rate level, the images of covariation can support the mental actions of coordinating the instantaneous rate-of-change of the function with continuous changes in the input variable. This level includes an awareness that the instantaneous rate-of-change resulted from smaller and smaller refinements of the average rate-of change. It also includes awareness that the inflection point is where the rate-of-change changes from increasing to decreasing, or decreasing to increasing. The mental actions MA1 through MA5 are supported by Level 5 image.

Chapter 6.

Data Analysis: The Case of Andy

As mentioned, the first analysis chapter focuses in detail on one student, Andy. This analysis is presented in 2 parts. In Part 1, I summarize Andy's approaches to the presented tasks. In Part 2 I analyze Andy's work on the presented tasks according to the theoretical frameworks that were described in Chapter 5: *Mason's (2008) theory of shift of attention*, *Presmeg' (1986) visual imagery* and *Carlson, Jacobs, Coe, Larsen, and Hsu's (2002) covariational reasoning*.

Using Mason's (2002) terms, Part 1 can be seen as 'account-of' Andy's solutions, while Part 2 is 'accounting-for.' The term *account-of* refers to a brief description of the key elements of the story, suspending as much as possible emotion, evaluation, judgment or explanations. This serves as data for *accounting-for*, which provides explanation, interpretation, value judgement or theory-based analysis.

Part 1: Andy's Story, Account-Of

6.1.1. Task 1: Identifying the Function of $f(x) = sin(2x)$ from the Given Graph

I showed Andy Task 1 which was the graph of $f(x) = sin(2x)$ (See Figure 6.1) and I asked him to identify a function represented by this graph.

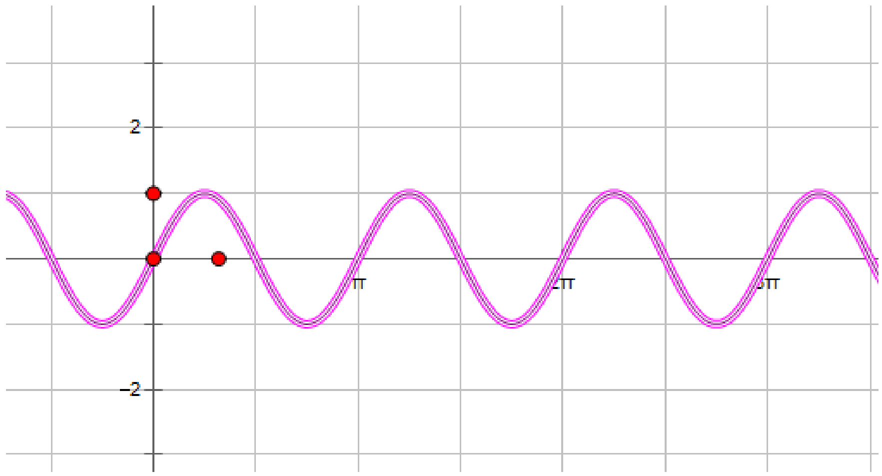


Figure 6.1. **Graph of $f(x) = sin(2x)$**

Andy focused his attention on several points on the x-axis and he recognized that there were two sinusoidal cycles in the interval [0, 2π]. Andy concluded that the given graph represented the function of $f(x) = sin(2x)$. Thereafter, I asked Andy to think about whether the same graph can be represented with a cosine function. Andy subtracted $\frac{\pi}{2}$ from the argument $2x$ (taken from the function $f(x) = sin(2x)$) and he wrote the function $f(x) = cos\left(2x - \frac{\pi}{2}\right)$. Andy stated that he actually relied on a formula $sin\ (a) = cos\left(a - \frac{\pi}{2}\right)$. I invited Andy to determine a cosine function by focusing on the given sinusoidal curve. He knew that the cosine graph is a sine curve shifted horizontally to the right; however, he struggled to find the amount of horizontal shift.

Andy, eventually, realized that the given sinusoidal curve was shifted to the right by $\frac{\pi}{4}$ (he stated "*this point* $\frac{\pi}{4}$ *is the first and smallest point in which y=1"*), and then he suggested that the cosine function was $f(x) = cos(2x + \frac{\pi}{4})$. Recognizing his mistake, I showed Andy the graph of $f(x) = cos(2x + \frac{\pi}{4})$ (see Figure 6.2). Observing the graph of $f(x) = cos\left(2x + \frac{\pi}{4}\right)$, Andy, thus, recognized his error that the suggested cosine function did not match the given curve. As a consequence, he expressed that the given graph represents the function $f(x) = cos\left(2x - \frac{\pi}{2}\right)$, the cosine function he found from the learned memorized sinusoidal formula at the beginning of the conversation.

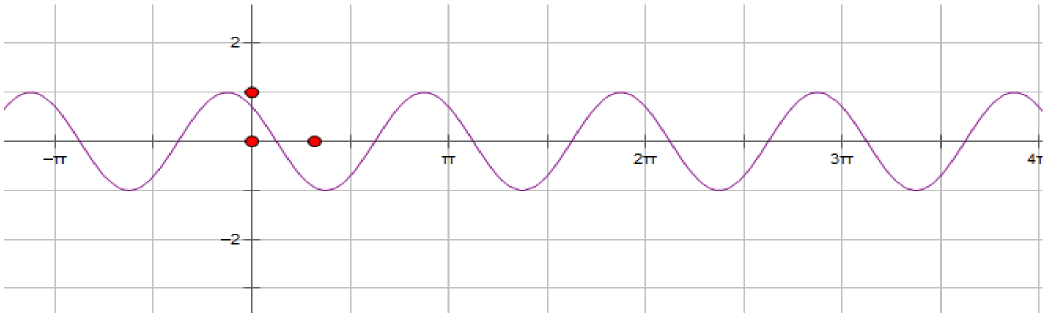


Figure 6.2. **Graph of** $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$

6.1.2. Task 2: Identifying the Function $f(x) = sin(\frac{2}{3}x)$ from the Given Graph

For Task 2, I showed Andy the graph of $f(x) = sin(\frac{2}{3}x)$ (see Figure 6.3) and I asked him to identify a sinusoidal function represented by the given curve. Following the same pattern as his response to the previous task, Andy attempted to figure out the number of full cycles within the interval [0, 2π]. Soon he realized that the given curve completed one cycle in the interval [0, 3π], about half a cycle more than the canonical sinusoidal curve completed at 2π. Andy, thus, concluded that the length of the given graph was about $\frac{3}{2} = 1.5$ times a basic canonical sinusoidal curve. In order to find a related function, he used a reciprocal of the fraction $\frac{3}{2}$ and he offered that the function was $f(x) = sin(\frac{2}{3}x)$.

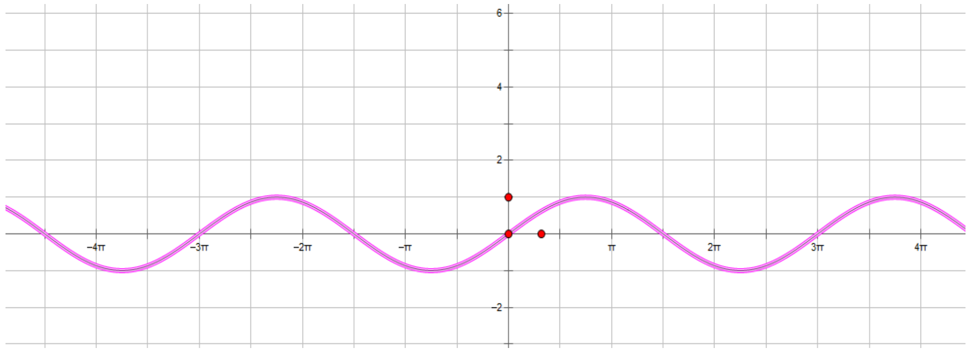


Figure 6.3. **Graph of $f(x) = sin(\frac{2}{3}x)$**

I then, asked Andy to identify a cosine function represented by the given curve. Similar to the previous task, he subtracted $\frac{\pi}{2}$ from the argument $\frac{2}{3}$x (taken from the function $f(x) = \sin(\frac{2}{3}x)$) and he wrote $f(x) = \cos\left(\frac{2}{3}\, x - \frac{\pi}{2}\right)$. However, when I asked him to determine the cosine function by focusing on the graph rather than applying his learned formula, Andy made a similar mistake as in the previous question (Task 1). Andy stretched the x-axes and then noticed that the graph was shifted to the right by $\frac{3\pi}{4}$ (since the first point (on the x-axes) by which the graph had y=1 was x=$\frac{3\pi}{4}$). He, therefore, expressed that the cosine function for the given curve should be $f(x) = cos(\frac{2}{3}x + \frac{3\pi}{4})$ which is different from his previous mentioned function. It is important to note that Andy did not consider the aforementioned suggested function in the general standard format ($f(x) = A\,cos(B(x + C) + D)$) and this interfered with his ability to identify the correct function. Andy recognized his mistake after he saw the graph of the suggested cosine function (See Figure 6.4). Then, he used his previous learned knowledge related to the transformation of sinusoidal functions and he stated that we need to multiply the coefficient of $\frac{2}{3}$ by $\frac{3\pi}{4}$. Eventually he wrote the function as $f(x) = cos(\frac{2}{3}x - (\frac{2}{3} \times \frac{3\pi}{4}))$ (see Figure 6.4) and then determined the function $f(x) = cos(\frac{2}{3}x - \frac{\pi}{2})$ which was the correct function for the graph.

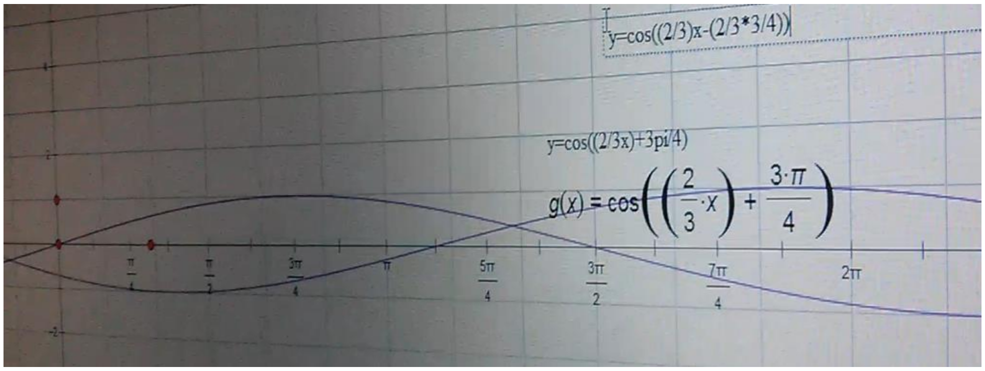


Figure 6.4. **Graph of** $\boldsymbol{f(x) = cos(\frac{2}{3}x - (\frac{2}{3} \times \frac{3\pi}{4})}$

6.1.3. Task 3: Identifying the Function of $\boldsymbol{f(x) = \ cos(\frac{2}{5}x - \frac{\pi}{5})}$ from the Given Graph.

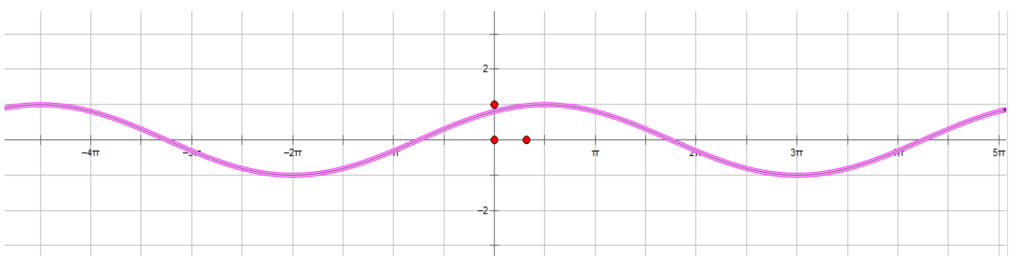


Figure 6.5. **Graph of** $\boldsymbol{f(x) = \ cos(\frac{2}{5}x - \frac{\pi}{5})}$

For the next task, Andy stated that

"…*the given graph could be written as a sine or cosine function, but I want to write a cosine function for the curve."*

In order to find the period of the cosine function, Andy looked for two consecutive points in which the curve intersects the x-axis. He expressed that the curve could complete one sinusoidal cycle (the curve going up, down and up again) from $-\frac{3\pi}{4}$ to $\frac{17\pi}{4}$ (the midpoint between 4π and $\frac{9}{2}\pi$) (see Figure 6.6). By this calculation, Andy concluded that the cosine function has a period of $5\pi = \frac{17\pi}{4} - (-\frac{3\pi}{4})$ because the cosine curve repeats every 5π radians.

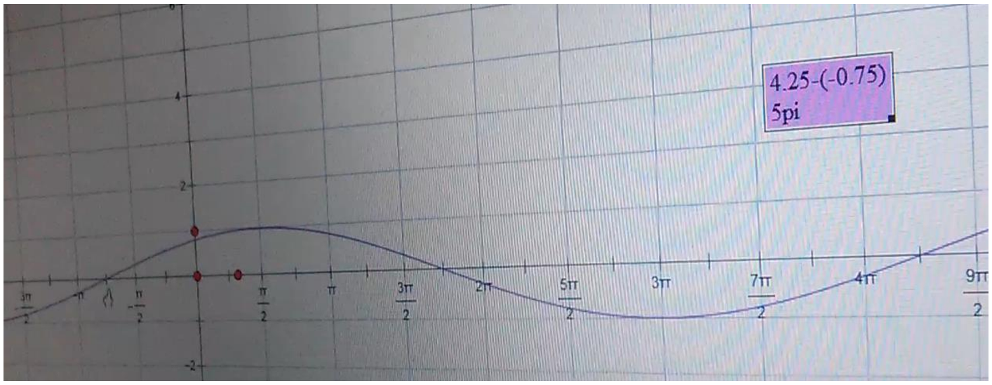


Figure 6.6. **Determining the period of the given cosine function**

After that, Andy realized that the given curve shifted horizontally because the graph did not intersect the y-axes at the point 1. To determine the amount of horizontal shift (B in canonical[1] representation), he stretched the x-axes. Andy, then stated that

> "*Ok, it is one [y=1] in between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.*"

Thus, Andy concluded that the function was $f(x) = cos\,(\frac{2}{5}x - \left(\frac{2\pi}{3} - \frac{\pi}{3}\right))$ (Function 1) (see Figure 6.7). Note that his approximation from the graph is imprecise and this influences his next decisions.

[1] I refer to the form $f(x) = A\,sin/cos\,(B(x + C)) + D$ as the canonical representation of a trigonometric function

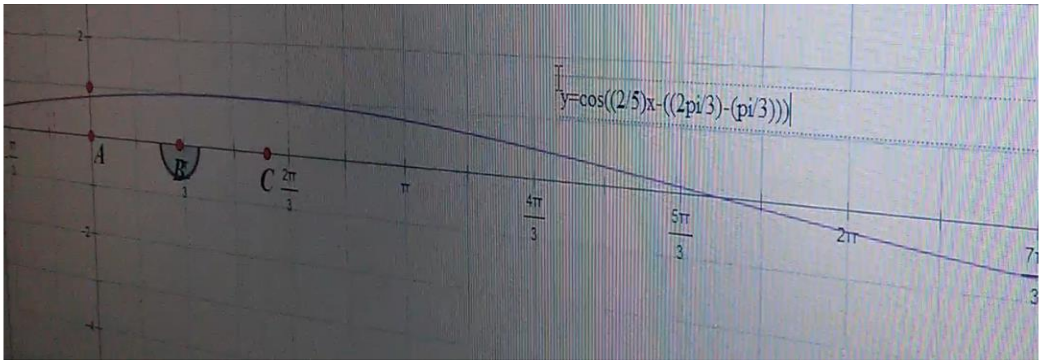


Figure 6.7. **Graph of** $f(x) = cos\,(\frac{2}{5}x - \left(\frac{2\pi}{3} - \frac{\pi}{3}\right))$

After a long pause, Andy changed the function into $f(x) = cos\,(\frac{2}{5}x - \left(\frac{\pi}{6}\right))$ and he expressed that in the previous suggested function (Function 1: $f(x) = cos\,(\frac{2}{5}x - \left(\frac{2\pi}{3} - \frac{\pi}{3}\right))$) he did not consider the midpoint between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$, the point by which the graph shifted horizontally (from his point of view). As it can be noticed from Andy's recent suggested function (or $f(x) = cos\,(\frac{2}{5}\left(x - \left(\frac{5\pi}{12}\right)\right) = cos\,(\frac{2}{5}x - \left(\frac{\pi}{6}\right))$) the horizontal shift (or B in canonical representation) was $\frac{5\pi}{12}$, whereas, the curve, actually, shifted by $\frac{\pi}{2}$ (the correct function for the given graph was $f(x) = cos\,(\frac{2}{5}x - \frac{\pi}{5}))$

Andy then stated that if we want to change the cosine function into a sine function, we should find an x-value in which the curve intersects the x-axis. He expressed that the graph shifted to the left by $\frac{3\pi}{4}$ (the first point having the value of y=0) (see Figure 6.5). Andy, then, wrote the sine function as $f(x) = sin(\frac{2}{5}x - \frac{3\pi}{4})$ which was incorrect (see Figure 6.8). Still, looking at the graph of $f(x) = sin\left(\frac{2}{5}x - \frac{3\pi}{4}\right)$ did not help Andy recognize his mistake. He stopped trying and could not continue. As such, I moved to the next task.

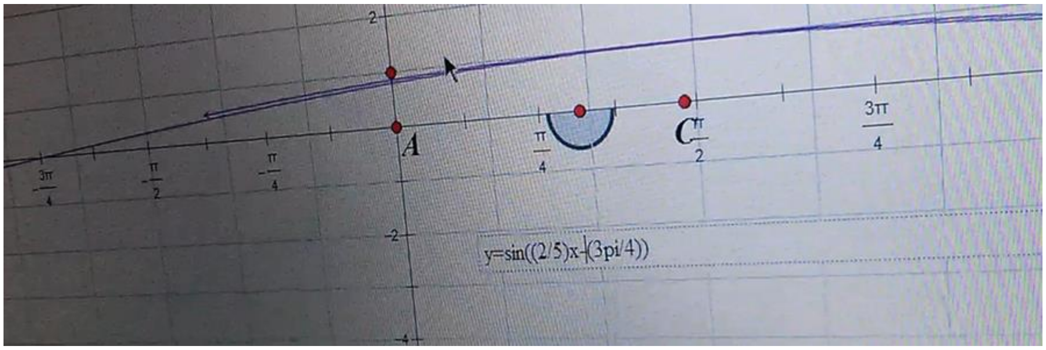


Figure 6.8. **Graph of $f(x) = sin(\frac{2}{5}x - \frac{3\pi}{4})$**

6.1.4. Task 4: Assigning Coordinates to Represent $f(x) = sin(4x)$

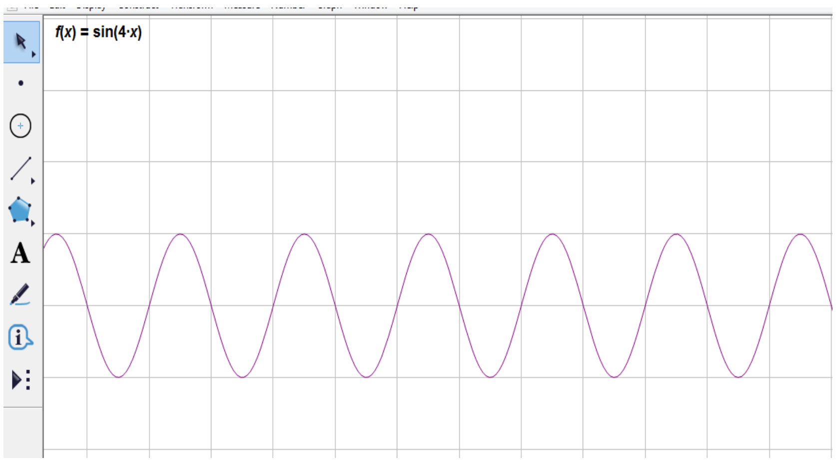


Figure 6.9. **Sinusoidal curve**

For Task 4, I showed Andy a sinusoidal curve (see Figure 6.9) and I asked him to assign the axis and coordinates such that it represents the graph of $f(x) = sin(4x)$. Since the coefficient of x was 4 in the given sine function ($f(x) = sin(4x)$), following the same method as in the previous tasks, Andy recognized that there should be 4 full cycles in the interval of [0, 2π]. To show that, he considered 2 full periods and then placed the point π on the x-axes. He then marked the points 1 and -1 on the y-axes (see Figure 6.10). Andy stated that the two consecutive cycles of the sine graph represented "*half of the graph of* $f(x) = sin(4x)$, the next half completed at (2π) should be exactly the same as the first half. When Andy was asked to identify a cosine function for the given sine function and then assign coordinated accordingly, he stated that the function should be $f(x) = cos(4x - 2\pi)$. However, he refused to assign proper coordinate on the sine cycle.

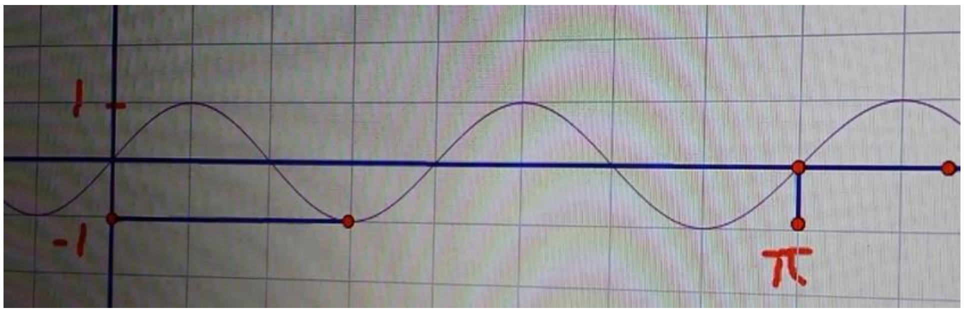


Figure 6.10. Representing graph of $f(x) = sin(4x)$

6.1.5. Task 5: Assigning Coordinates to Represent $f(x) = cos(3x - \frac{\pi}{4})$

For the last task, Andy first assigned the axes in a way that sinusoidal curve intersects the axes on the origin (0,0). He then placed the point $\frac{\pi}{4}$ on the x-axes referring (incorrectly) to the phase shift in the given cosine function ($f(x) = cos\left(3x - \frac{\pi}{4}\right)$) (see Figure 6.11). Andy then changed his mind and the position of the assigned axes as can be seen in the Figure 6.12.

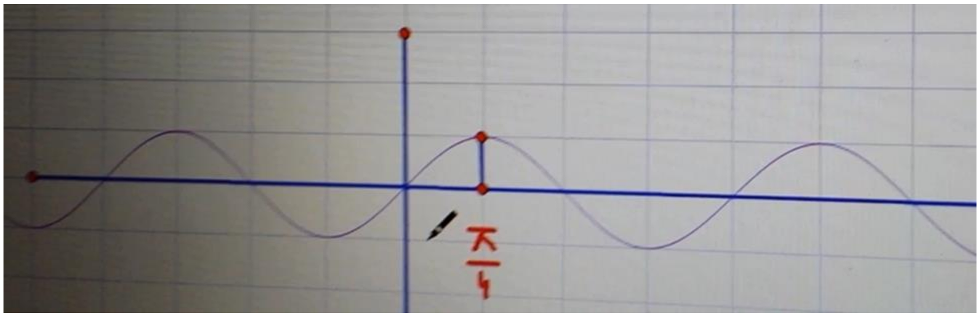


Figure 6.11. Assigning the y-axes and the point $\frac{\pi}{4}$

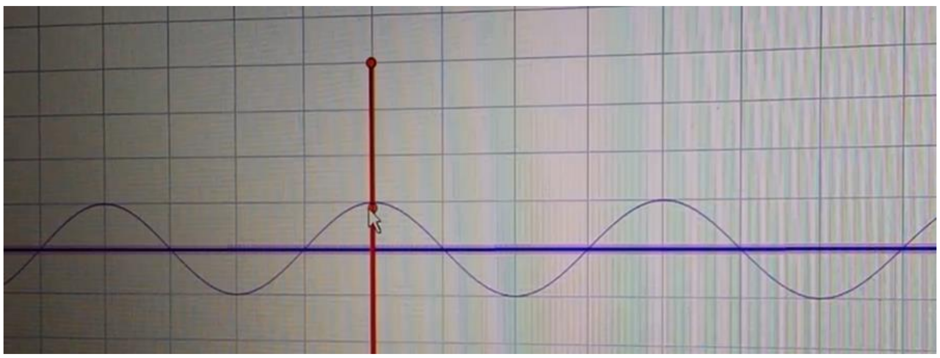


Figure 6.12. Changing the assigned axes

Thereafter, Andy counted 3 full cycles (because of having the coefficient of 3 for the argument x on the given cosine function) and then he put the point 2π on the x-axes as well as the points 1 and -1 on the y-axes. Since a phase shift is included in the cosine function($f(x) = cos\left(3x - \frac{\pi}{4}\right)$), Andy knew that he needed to shift the curve (it was movable in sketchpad) to the left (because he noticed the negative sign in the function) and he placed the y-axes on a point he labeled as $\frac{\pi}{4}$. Not only he expressed that the horizontal shift was $\frac{\pi}{4}$, but also he moved the y-axes to the left instead of the right side of the y-axes (see Figure 6.13). Andy, in fact, made similar mistakes as he did in the previous tasks. The given cosine function was shifted horizontally by $\frac{\pi}{12}$ ($\left(f(x) = cos\left(3x - \frac{\pi}{4}\right)\right) = cos\left(3(x - \frac{\pi}{12}\right)$), while Andy expressed that it was shifted by $\frac{\pi}{4}$.

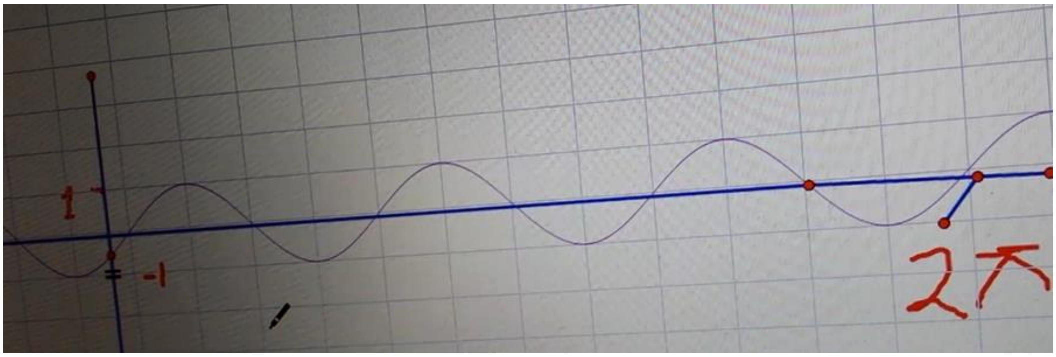


Figure 6.13. Graph of $f(x) = cos\left(3x - \frac{\pi}{4}\right)$

6.2. Part: 2 Andy's Story, Accounting-For

In Part 6.2, I focus on 'accounting for' (Mason, 2002) Andy's solutions, considering them according to three frameworks (1- Shift of attention, 2- Visualization, 3- Mental action).

6.2.1. Shifts of Attention

6.2.1.1. Task 1: Identifying the Function $f(x) = sin(2x)$ from the Given Graph.

In order to complete Task 1, Andy first focused his attention on the given graph and waited for visual feedback from the graph (his attention was *holding wholes* according to Mason's classification). He then realized that the graph went through the origin (0,0). This shows that Andy was able to *discern* a very specific *detail* from the graph. Thereafter, in order to identify the type of trigonometric function, he locked again his attention for a few seconds on the graph as *a whole*. Andy then stated that the given graph represented a sine function because it "started" from the point (0,0). One could conclude that Andy was able to *perceive properties* of a sine curve (*a cyclic graph started at (0,0))*.

Andy gazed at the graph again for a while. He then shifted his attention to *discerning more details* about the curve. He, therefore, focused on the specific points such as (0,0), ($\frac{\pi}{4}$, 1), ($\frac{\pi}{2}$, 0), ($\frac{3\pi}{4}$, -1) and (π,0) within the interval $[0, \pi]$. Andy realized that the same pattern had been repeated for the second cycle from π to 2 π (e.g., ($\frac{5\pi}{4}$, 1), ($\frac{3\pi}{2}$, 0) and so on). Matching the *details* of where the two cycles of a sine curve intersected the x-axis (one cycle from 0 to π and another one from π to 2π) and then finding the related points on the y-axis led to Andy's conclusion that the length of the period was not the same as the canonical sinusoidal curve. In other words, Andy realized that the given graph was squeezed compared with the original sine curve in the interval [0, 2 π]. Therefore he articulated that:

> "*It is a sine curve but with a factor, because a regular sine graph is a periodic function of* 2π*, but here at* 2π *there is two cycles."*

The above statement illustrates that Andy knew that there is a *relationship* between the geometric representations of the given graph and the number of repeated cycles in the given intervals. Then he indicated that the function should be $f(x) = sin(2x)$. This means that from *reasoning on the perceived properties* of the sine curve Andy was able to introduce a correct sinusoidal function for the given graph.

To write a different equation for the given graph (as a cosine function), Andy focused his attention, this time, on those *details* that would lead him to find a cosine function. To do so, Andy compared the changes in the x and y-values of the given graph with the canonical cosine graph as he indicated that:

> "*Because the cosine graph starts at the points where x=0 and y=1, so I basically need to look at the first instance on the domain that give me y=1."*

This shows that Andy was aware of the fact that the basic canonical cosine function intersects the y-axes at one. This suggests that Andy *perceived properties of cosine function*. Thereafter, he gazed at the graph as a *whole* for a while until he focused his attention on more *details* embedded in the given sinusoid graph. Andy, thus, found that the points in which the graph interconnect the x-axes were $\frac{\pi}{2}$ as well as π. Here, he again shifted his attention to y=1 and indicated that the graph did not intersect the y-axes at the point 1, as in the canonical cosine curve. Therefore, Andy interpreted the given curve as a graph shifted horizontally to the right. However, he still could not realize the amount by which the graph shifted as he stated *"… I do not know what this point is."* He gazed at the graph as a whole again. Andy finally figured out that the graph transformed horizontally by $\frac{\pi}{4}$ to the right because the value of y equals 1 for x=$\frac{\pi}{4}$. Afterwards, he determined the cosine function while *reasoning on the perceived properties* of the transformed sine function. He stated that "this one [the given curve] starts from ($\frac{\pi}{4}, 1)$, so it is $f(x) = cos(2x + \frac{\pi}{4})$ because it is shifted to the right so I need to have a positive sign."

It is clear from the above-suggested function ($f(x) = cos(2x + \frac{\pi}{4})$) that Andy knew that the coefficient of the argument of x should be 2 because there were 2 full cycles on the given graph. However, Andy made two mistakes. *First*, since the point $\left(\frac{\pi}{4}, 1\right)$ was on the right side of the y-axes, he added $\frac{\pi}{4}$ to the argument 2x in the cosine function. *Second,* he did not pay any attention to the effect of periodicity on the shifted point when he added $\frac{\pi}{4}$ to $(2x)$. These errors illustrate that Andy had some difficulties in *perceiving properties* of transformation of sinusoidal functions because if graph shifted to the right, the length of horizontal shift should be subtracted from the argument of x embedded in the sinusoidal function. Thus, I sketched $f(x) = cos\,(2x + \frac{\pi}{4})$ using a sketchpad to investigate if Andy could realize the mistakes he made in determining the cosine function.

After sketching the graph of $f(x) = cos\,(2x + \frac{\pi}{4})$ Andy immediately recognized his error, noting that

> *"Ok, I made some incorrect calculation, but I think I should do something with the coefficient of x which is 2."*

The feedback he got from the graph of his suggested function made him realize that there is a connection between the arguments of x and the point by which the graph is shifted. In other words, Andy noticed the *relationship* between the graphical representations of periodicity and the cosine function transformed horizontally. He then continued, stating

> *"Basically since the coefficient of x is 2, it needs to be multiplied by $\frac{\pi}{4}$ on 2, because the x has 2 multiplications. So the function should be $f(x) = cos(2x - \frac{\pi}{2})$."*

From Andy's suggested function ($f(x) = cos\left(2x + \frac{\pi}{4}\right)$ and $f(x) = cos(2x - \frac{\pi}{2})$), it is unclear whether Andy returned to his previous suggestion based on the known relationship between sine and cosine ($sin(a) = cos\ (a - \frac{\pi}{2})$) or whether he recognized the importance of the brackets $f(x) = cos(2x - \frac{\pi}{2})$ is indeed equivalent $f(x) = cos(2(x - \frac{\pi}{4}))$, and that is how the recognized shift by $\frac{\pi}{4}$ is featured in the formula.

6.2.1.2. Task 2: Identifying the Function $f(x) = sin(\frac{2}{3}x)$ from the Given Graph

Andy *gazed at the whole* graph for a while and then he *discerned some details* on the values of x and y on the origin. Andy noticed that the graph intersected the origin at the point (0, 0), so he knew that the curve was related to a sine function. He then looked for more *details* and he realized that the horizontal length of the graph stretched more compared with the canonical sine curve. In fact, he noticed that the graph hit the x-axes at 0, $\frac{3\pi}{2}$ and 3π. This observation of the *details* directed Andy to find a *relationship between the graphical representations and the symbolic representations of the sine function*. In other words, he knew that the argument of x was the reverse of the horizontal length or the periodicity of the graph. As a result, he introduced $\frac{2}{3}$ for the coefficient of x and thus, the sinusoidal function was $f(x) = sin(\frac{2}{3}x)$.

In order to identify a cosine function for the given graph, he initially spent several minutes finding the point on the x-axes having the value of y=1. To do so, he observed that the graph had the value of y= 1 between the points (0, 0) and ($\frac{3\pi}{2}$, 0). He then zoomed in for more *details* on the x-axes and he eventually found a point on the x-axes having the value of y=1. Andy recognized that the graph had the value of y=1 for a point between $\frac{\pi}{2}$ and π which was $\frac{3\pi}{4}$. Thus, he expressed that the graph was shifted by $\frac{3\pi}{4}$. As a result, he could provide a *relationship* between the graph shifting horizontally and its transformed function. Andy stated that the product of the horizontal transformation was $f(x) = cos\ (\frac{2}{3}x + \frac{3\pi}{4})$. Observing the graph of his suggested function, $f(x) = cos\ (\frac{2}{3}x + \frac{3\pi}{4})$, made Andy gaze at the whole graph again, while comparing both graphs (his suggested graph and the original given curve) (see Figure 6.14). Andy *discerned some details* on both graphs. For instance, he noticed that the point ($\frac{3\pi}{2}$) had the value of y=1 in the original graph (graph of $f(x) = sin\ (\frac{2}{3}x)$), but the y-value was not the same for his suggested graph (graph of $f(x) = cos\ (\frac{2}{3}x + \frac{3\pi}{4})$). He also found that the new suggested graph (graph of $f(x) = cos\ (\frac{2}{3}x + \frac{3\pi}{4})$) did not pass the origin (0,0), while the given graph (graph of $f(x) = sin\ (\frac{2}{3}x)$) intersects the origin.

Figure 6.14. Snapshot of two graph $f(x) = cos\ (\frac{2}{3}x + \frac{3\pi}{4})$ and $f(x) = sin\ (\frac{2}{3}x)$

Andy, then, noticed that there was a direct relationship between the periodicity or the horizontal length of the given sine curve and the amount of horizontal shift in its related function. Thus he multiplied the coefficient of x ($\frac{2}{3}$) by the phase shift ($\frac{3\pi}{4}$) and wrote a new function as $f(x) = cos(\frac{2}{3}x - \frac{\pi}{2})$. He also realized that the sign between the argument of x and the amount of horizontal shift should be negative because the graph had been shifted to the right. One can conclude that Andy was able to *reason on the perceived properties* of transformation of the sinusoidal function. While answering this question, Andy also was able to visualize the problem. According to Prmeseg's classification (1999) his visualization fell into three categories, which will be described in the following section.

6.2.1.3. Task 3: Identifying the Function $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$ from the Given Graph

To identify the function representing the given curve in Task 3 (see Figure 6.5), Andy *gaze*d at the curve (similar to the two previous tasks). After a long pause, he stated that

> *"…We could determine a cosine function for the curve (Figure 6.5) because it did not intersect the origin" (perceived properties of cosine functions).*

The above statement acknowledges Andy's general grasping of properties of cosine function. He then tried to *discern* the x-values in which the curve intersected thought zooming in the Figure 6.5 for a while. Andy eventually found the two points: $(-\frac{3\pi}{4}, 0)$ – (midpoint between -π and $-\frac{\pi}{2}$) and $(\frac{17\pi}{4}, 0)$ – (midpoint between 4π and $\frac{9}{2}\pi$). *Discerning* these details from the graphical representations *directed* Andy to determine the amount of B (period from the canonical representation). To compute B, he subtracted $\frac{17\pi}{4}$ from $-\frac{3\pi}{4}$ and he then determined the period for the cosine function as $B = \frac{2}{5}$. One can conclude that Andy was *reasoning on the perceived properties* of cosine function because it appears that he acknowledged that if the length of a canonical function is (2π), then it should be always divided into the length of full cycle of the given sinusoidal function (in this tasks it was 5π). It is interesting to note that finding the distance between endpoints of one cycle, in which the curve intersected the x-axes, is a strategy by which Andy often determined the periodicity in the sine function. However, he applied this method again when he wanted to find periodicity in a cosine function (Note that in a cosine function we often find the distance between the two points having the same amplitude).

Andy starred at the curve (see Figure 6.5) for several seconds again and he stated:

> *"… there should be another factor, a positive or negative after the x because it is translated by some amount to the right or left because the graph is not started basically at the point y=1…"*

This statement illustrates Andy *acknowledged properties of cosine function* because he knew that if the graph did not intersect the y-axes at 1, it means that it is shifted to the left or right. In the other words, he knew that a certain radian representing C or phase shift should be added to or subtracted from the argument $\frac{2}{5}$x. Eventually Andy provided a *relationship between the graphical representation and the cosine function* and he determined $\frac{\pi}{6}$ as the phase shift for the function $f(x) = cos\,(\frac{2}{5}x - \left(\frac{\pi}{6}\right))$ representing the given curve. However as I explained in the section 6.1, Andy this time was not able to *perceive correctly properties of cosine function* because the phase shift (C) in his suggested function (which was not representing the given curve (see Figure 6.5)) was C= $\frac{\pi}{2}$, while he found C= $\frac{\pi}{6}$. In other words, he did not consider the influence of B (period) on C.

To identify a sine function for the task (Figure 6.5), Andy stated that:

"*…the cosine function* $(f(x) = cos\left(\frac{2}{5}x - \left(\frac{\pi}{6}\right)\right)$ *should be shifted to the here by a factor of* $\frac{3\pi}{4}$ *,so the sine function is* $f(x) = sin\left(\frac{2}{5}x - \frac{3\pi}{4}\right)$*…"*

As it appears from the above statement, Andy made similar mistake as he did in the cosine function $(f(x) = cos\left(\frac{2}{5}x - \left(\frac{\pi}{6}\right)\right))$. He found the x-value $(\frac{3\pi}{4})$ at which the curve intersected the x-axis and then he subtracted it to the argument without paying attention to the value "B" and its effect on "C" in the canonical representation. This shows Andy's lack of *reasoning on the perceived properties of the sinusoidal functions* as well as inability in *connecting a graphical representation with the Canonical representation*).

6.2.1.4. Task 4: Assigning Coordinates to Represent $\boldsymbol{f(x) = sin(4x)}$

In order to assign coordinates to the given graph, Andy looked at the function for a while. It seems that he was analyzing and computing something in his mind as he whispered some unclear words. He then looked at the given sinusoidal curve and he expressed that he first needed to sketch the y and x- axes. To adjust the y and x-axes on the curve, Andy acknowledged that:

> *"…the sine curve always begins at the point (0,0)"* (as he also explained it in the previous tasks).

As such, he placed the x and y-axes as shown in Figure 6.15. One can conclude that Andy *perceived properties* of a sine curve (e.g., the sine graph begins at (0,0)).

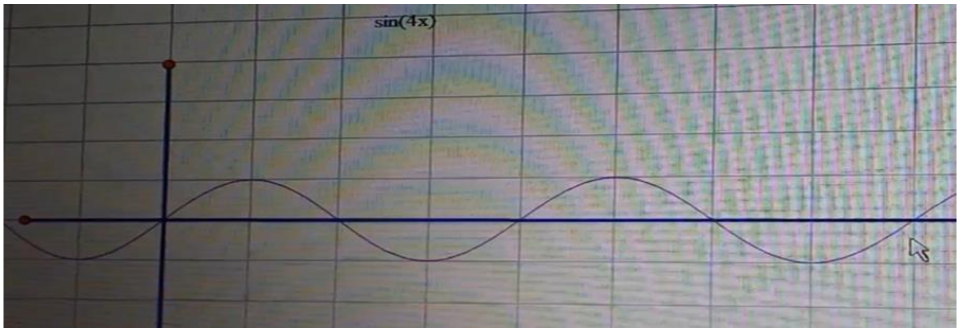


Figure 6.15. Intersecting the axes and the curve in the origin

From there, he *gazed* at the given function ($f(x) = sin(4x)$) *as the whole* and at its cycle again for a while. Andy then tried to add more details (number and radians) on the x and y-axis. Andy then zoomed on the number 4 from the function. He expressed that:

> *"…I know here [referring to x-axes] should be π because I see here [referring to the function $f(x) = sin(4x)$]) 4…"*

Andy realized that there should be four full cycles in the interval [0, 2π] or two cycles in the interval [0, π] because the coefficient of x in the sinusoidal function was 4. This illustrates that Andy was able to *reason on the perceived properties of sinusoidal functions,* when finding a *relationship* between the symbolic representation and the graphical representation (see Figure 6.16).

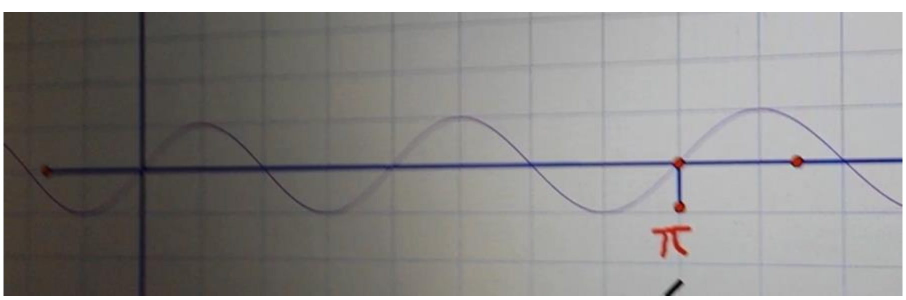


Figure 6.16. Putting π on the x-axes

To *discern more detail* from the given function, Andy again *gazed* at the graph for a while. He recognized that the maximum value of the given sine function ($f(x) = sin(4x)$) was 1. Therefore he added two points 1 and -1 on the y-axes to show the highest and the lowest points of the graph (*perceived properties* of the sine curve). After that, I asked Andy to find an alternative function for the given function and then sketch it. However, Andy refused to do that and he said that:

> *"…I do not know how to do it…It should be very hard. I just know from the sine function that the cosine function is* $(x) = cos(4x - 2\pi)$ … "

This is rather surprising, given that he was able to express given graphs as both sine and cosine functions in the previous tasks.

6.2.1.5. Task 5: Assigning Coordinates to Represent $\boldsymbol{f(x) = cos\ (3x - \frac{\pi}{4})}$

Similarly to the previous task, Andy began completing the task by focusing on the y and x-axes. He positioned the x and y-axis on one of the sinusoidal curves in a way that the slope of the graph decreased and then increased (see Figure 6.17). This illustrates that Andy *perceived properties* of the cosine graph because he acknowledged that a cosine graph initially intersect the y-axes, then its y-value becomes zero and then negative, zero again and it eventually completes one cycle at the same y-value as it begins with.

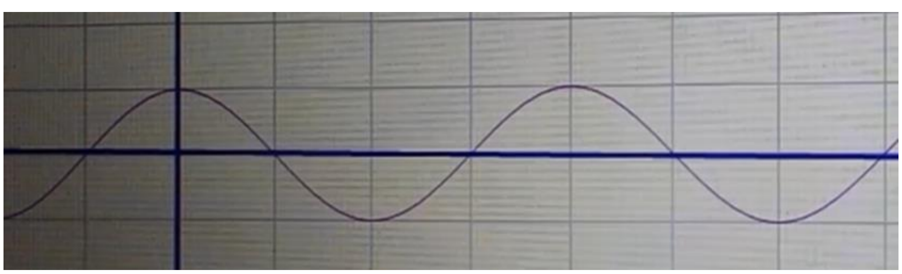


Figure 6.17. Assigning the coordinates on the sinusoidal curve

From there, he *gazed* at both the curve and the function $f(x) = cos\left(3x - \frac{\pi}{4}\right)$ in order to add more *details* to the graph. Andy then placed 2π on the x-axes where the three consecutive (cosine) cycles were completed (see Figure 6.18). Andy's action shows that he could provide a *relationship* between *the given sinusoidal function and its graphical representation,* similar to the previous tasks. One also can conclude that he was able to *reason on the perceived properties* of the transformation of sinusoidal functions. In fact, Andy was able to justify that because there was the coefficient (3) for the argument x in the given function ($f(x) = cos\ (3x - \frac{\pi}{4})$), there should be three cycles in the interval [0, 2π].

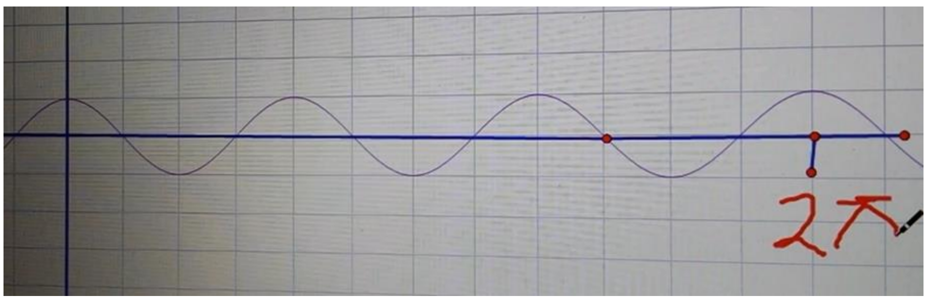


Figure 6.18. Putting 2π on the x-axis

Thereafter, Andy shifted his attention to the given function ($f(x) = cos\ (3x - \frac{\pi}{4})$) again and he noticed that its graph should shift horizontally as he realized that $\frac{\pi}{4}$ was subtracted from the argument 3x. *Discerning this detail* directed Andy to shift the y-axis from its previous place (see Figure 6.18) toward the left side (see Figure 6.19). He then shifted his attention to the coefficient of the cosine function. Andy thus added 1 and -1 on the y-axis, which shows his understanding of the *perceived properties of cosine curves*. However, Andy placed the y-axis at the wrong place and even seeing the graph of $f(x) = cos\left(3x - \frac{\pi}{4}\right)$ did not help him assign the coordinates correctly.

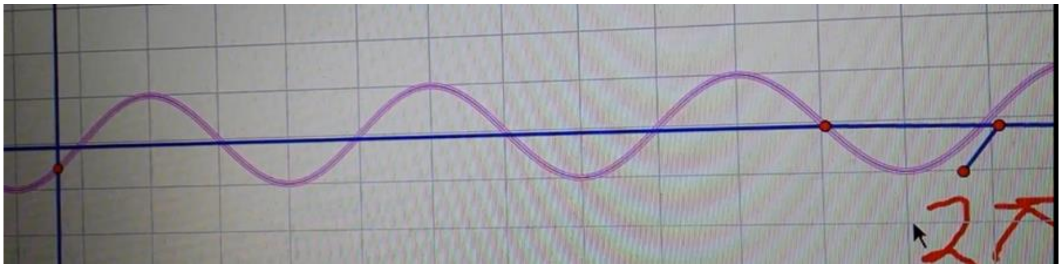


Figure 6.19. Shifting the sinusoidal curve to the left

6.2.2. Visual Imagery

6.2.2.1. Task 1: Identifying the Function of $\boldsymbol{f(x) = sin(2x)}$ from the Given Graph

In finding the sinusoidal function described by the given graph (Task 1, $f(x) = sin(2x)$), Andy repeatedly stated that because the shape of the graph is oscillating or it is "like a wave", it should be a sine function. One could conclude that Andy had a *pictorial image* of sinusoidal curves in his mind. Then, Andy described that there should be two completed sinusoidal curves within the interval [0, 2π], when he used his right index finger to show the full cycles (see Figure 6.20). He again used his index finger and pointed to the point π on the x-axis to show the period of the first cycle. He repeated the same action again for the second sinusoidal curve. He moved his hand very quickly to represent the fact that the given curve is more squeezed compared with the canonic sine curve. Andy's gesture indicates visual *kinesthetic* communication when working on the given sinusoidal functions. Eventually he stated that the graph belongs to the function of $f(x) = sin(2x)$ because "*there are two waves squeezed between 0 and 2*π." This indicated that Andy considered a certain pattern (two full cycles) repeated within the interval [0, 2π], so therefore he had the *pattern imagery* of periodicity in sine functions.

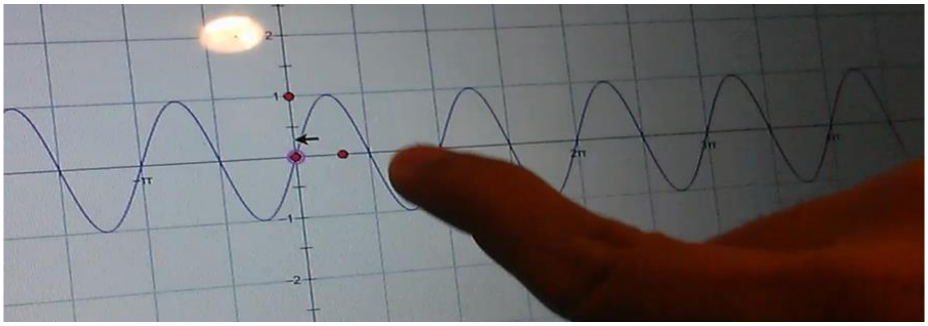


Figure 6.20. Andy's right index to represent two cycles

To determine a cosine function for the given graph, Andy pointed to a point in the graph (using his right index) to show that the graph should be translated by $\frac{\pi}{2}$ and then he wrote $f(x) = cos(2x - \frac{\pi}{2})$ (see Figure 6.21). As Figure 6.21 shows, Andy actually pointed to $\frac{\pi}{4}$, however, he referred to it as $\frac{\pi}{2}$, maybe because he had a *pictorial imagery of canonical function* in his mind (in a canonical cosine function, y=1 for x=$\frac{\pi}{2}$). He then moved his right hand across the x-axes to show the horizontal shift of the curve to the right (see Figure 6.22), while he expressed that "*a cosine graph is a sine graph translated to the right by a factor of* $\frac{\pi}{2}$."

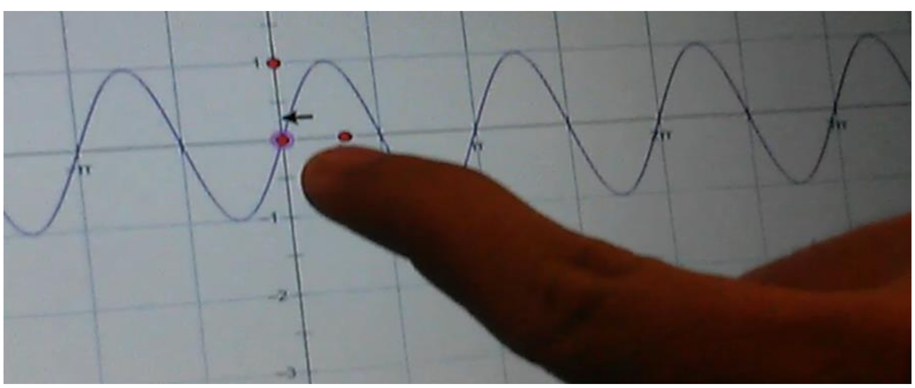


Figure 6.21. Pointing to a point on the graph

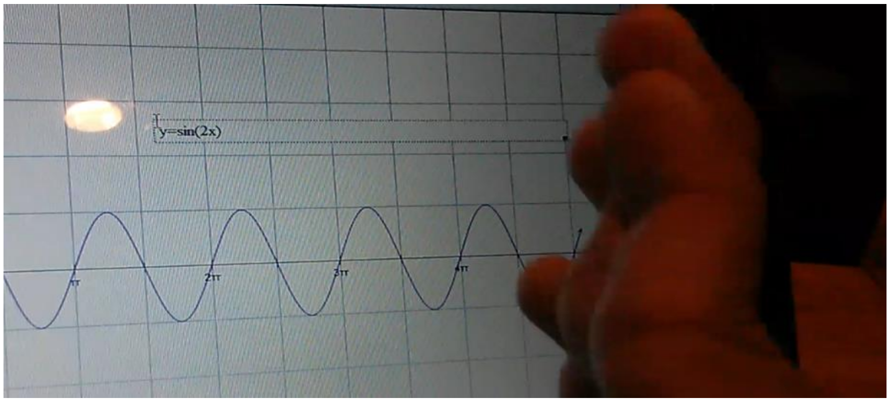


Figure 6.22. Andy moved his hand across the x-axes

One can conclude that Andy had a *kinesthetic imagery* of the horizontal transformation of a sinusoidal curve since he used his right index finger and his right hand to show his understanding of the content. Also Andy's response indicates that he had *memory image of formula* of sinusoidal function because he acknowledged that we can obtain the sine curve from the cosine curve by shifting it $\frac{\pi}{2}$ units to the right. However, when I asked him to show me this relationship in the given graph (see Figure 6.21), he changed the function from $f(x) = cos\left(2x - \frac{\pi}{2}\right)$ into $f(x) = cos\left(2x + \frac{\pi}{4}\right)$ because he noticed the point y=1 and x=$\frac{\pi}{4}$. After realizing his initial mistake in determining the cosine function (by comparing the graph of $f(x) = cos\left(2x + \frac{\pi}{4}\right)$ and $f(x) = sin(2x)$), Andy indicated that he did not consider the effect of periodicity (B in canonical representation) on phase shift (or C) . In the other words, Andy noticed that he displaced the bracket in his suggested formula. This illustrates again that Andy had a *memory image of the formula* of the transformation of a sinusoidal function since $f(x) = cos\left(2x - \frac{\pi}{2}\right)$ equals $f(x) = cos\left(2\left(x - \frac{\pi}{4}\right)\right)$.

6.2.2.2. Task 2: Identifying the Function of $f(x) = sin(\frac{2}{3}x)$ from the Given Graph

To complete Task 2 (similar to Task 1), Andy used the word oscillation repeatedly as he expressed:

“Again it is a basic oscillating function has an oscillation and its oscillation starts from 0.”

At the same time, Andy moved his two fingers like a wave across (see Figure 6.23) the given curve to demonstrate that the given curve represents an oscillating function beginning at (0,0).

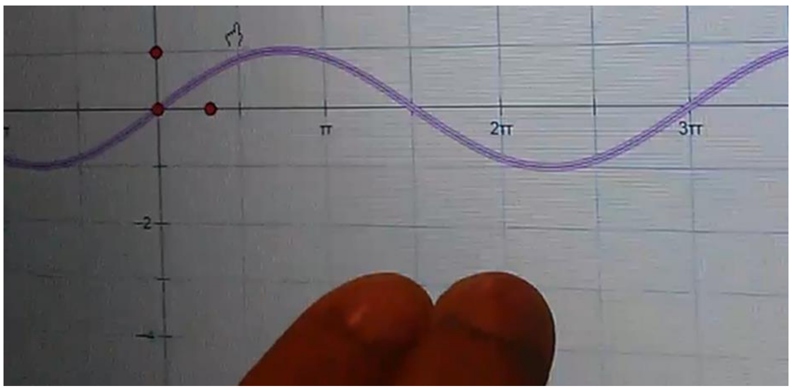


Figure 6.23. Showing the oscillating function

Andy stated that the coefficient of the argument x represents how much the graph was squeezed or extended by a factor. To exemplify a compressed graph having a whole number for the coefficient of x, he used both his hands and he put them parallel to each other (see Figure 6.24). His hands were like two closed parentheses.

Figure 6.24. Student positioned his hands parallel to each other

However he expressed that since the given graph completed between 0 and 3π, it means that the canonical sinusoidal curve was stretched and the coefficient of x is a fraction. Andy used both his hand to show the extended graph, but this time he opened his hand like an open parentheses (see Figure 6.25). This indicates that Andy identified the symbolic representation of sinusoidal functions, when focusing on the graphical representations of the related sine functions. As with the previous task, Andy's *kinesthetic imagery* led him to conclude that the sinusoidal function had a fractional argument of x because the length of a full sine cycle in the given graph is bigger than the length of full cycle in the graph of canonical function. He concluded that because the graph was stretched, the coefficient of x should be a fractional number (smaller than 1).

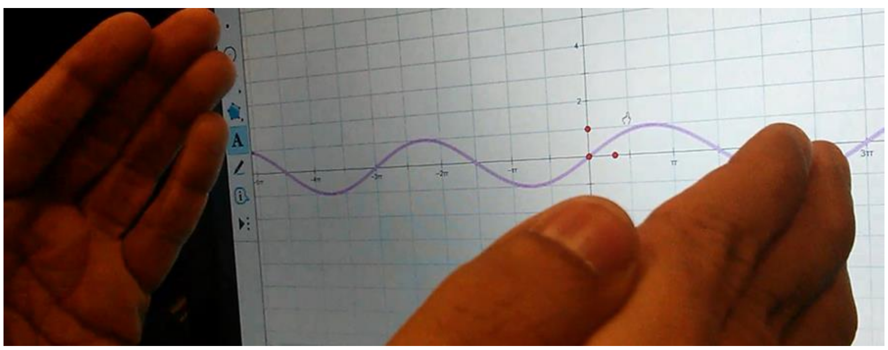


Figure 6.25. Student positioned his hands in the open parentheses position

Andy, then, expressed that in order to find periodicity (B in canonical representation); he first needed to find a point by which the sinusoidal curve had a completed cycle (3π). He stated

> *"…usually the graph complete one cycle at 2 π but here the graph finished a full cycle at 3 π so this point is 3π …".*

He used his right index finger to show me the point 3π (see Figure 6.26).

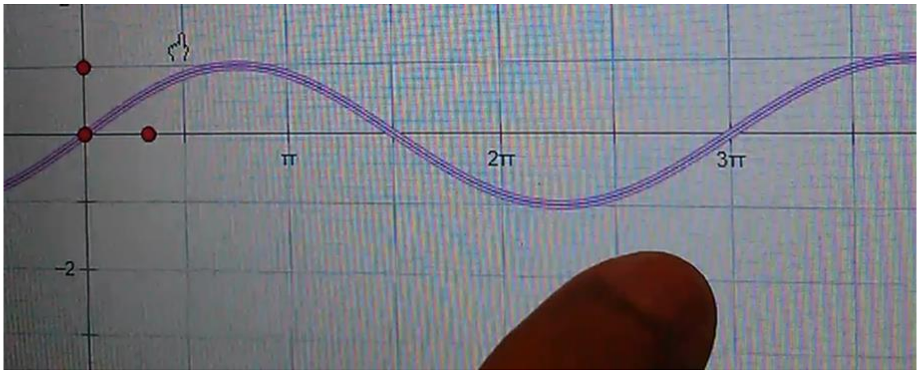


Figure 6.26. Showing the end point of one cycle of the sine curve

Then Andy moved his right hand from left to right across the x-axes and added that,

> *"…then we see that the graph is longer than the original sine graph by factor of $\frac{3\pi}{2}$." … so there is a coefficient before x in the function. This means that the coefficient of x is $\frac{2}{3}$. So the function is $f(x) = sin(\frac{2}{3}x)$"*

Andy's hand movements and his statement are evidence that not only was he able to communicate *kinaesthetically* about the sine function, he had the *memory image of the formula* of the transformation as he said "*the coefficient of x is $\frac{2}{3}$… So the function is $f(x) = sin(\frac{2}{3}x)$"*, although he did not explicitly talk about the periodicity formula: $Period = \frac{2\pi}{B}$. In other words, he knew that when the sinusoidal graph stretches horizontally, the arguments of the function change respectively.

To find an alternative function for the given graph, he expressed that

> "…*to have a cosine function we know that the graph should begin from y=1 at here…*"

In his mind, he moved the y-axes to the right side of the graph. This phase illustrates that Andy had *dynamic imagery* when solving the question. One also can state that he had *pictorial imagery* since it seems that Andy had an image of a canonical cosine graph in his mind (according to the aforementioned phase).

Along with the two aforementioned imageries, he also pointed to the place on the y-axes when he used his right index finger (See Figure 6.27). Andy expressed that the graph (which was stretched horizontally) shifted to the right by $\frac{3\pi}{4}$ (the first point he find for y=1). To show that the graph shifted horizontally, Andy moved his right hand from left to right (see Figure 6.28). He then stated the function as $f(x) = cos(\frac{2}{3}x + \frac{3\pi}{4})$. Andy's action again shows his *kinesthetic imagery* of the graphing of sinusoidal functions.

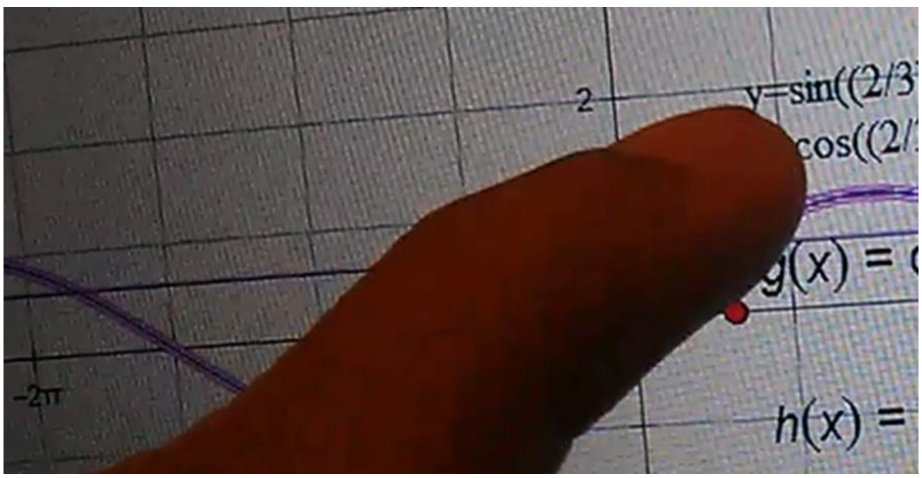


Figure 6.27. Andy's pointing to the y-axes

Observing the graph of his suggested function, Andy eventually identified the cosine function as $f(x) = cos(\frac{2}{3}x - \frac{\pi}{2})$. Andy' suggested cosine function show that he had the *memory image of the formula* of transformations of sinusoidal functions in his mind, because he subtracted the phase shift $\frac{\pi}{2}$ from the argument x.

Figure 6.28. Andy moved his hand from left to right

To sum up, it seems that Andy had a *pictorial imagery* of the original sinusoidal curves and he was able to imagine how to displace the y-axis from a given graph to have a cosine function (dynamic *imagery*). His answer to this question also indicates that he had *memory image of the formula* of a sinusoidal function because when the graph stretched or squeezed. Andy used the incorrect phase shift without considering the multiplication of the coefficient of x on the shift. He knew that periodicity would change and he therefore had to change the coefficient of the argument in the function. He also knew that if the graph shifted horizontally by a value, it should be added/subtracted to the coefficient of x (although his suggested function often had the wrong sign). In most of his answers, he often used his right index or his right hand and this shows that he was able to communicate *kinetically* when discussing the tasks.

6.2.2.3. Task 3: Identifying the Function of $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$ from the Given Graph

Similar to the other 2 previous tasks, Andy used his hands several times when he was completing Task 3. When discussing the period of the sine cycle (B), for instance, he used his right index finger to show the x-value ($-\frac{3\pi}{4}$) by which a sinusoidal function intersected the x-axes (see the yellow arrow in Figure 6.29). He considered the point ($-\frac{3\pi}{4}, 0$) as the starting point of a full sine cycle. Andy applied the same gesture to show where the particular full sine curve ended. Furthermore, Andy moved his both indexes fingers (see Figure 6.30) to show the length of half of cycle.

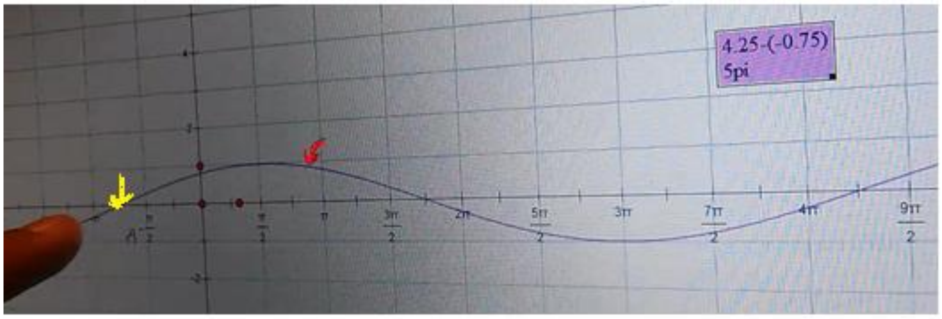


Figure 6.29. Andy was pointing to the point $-\frac{3\pi}{4}$

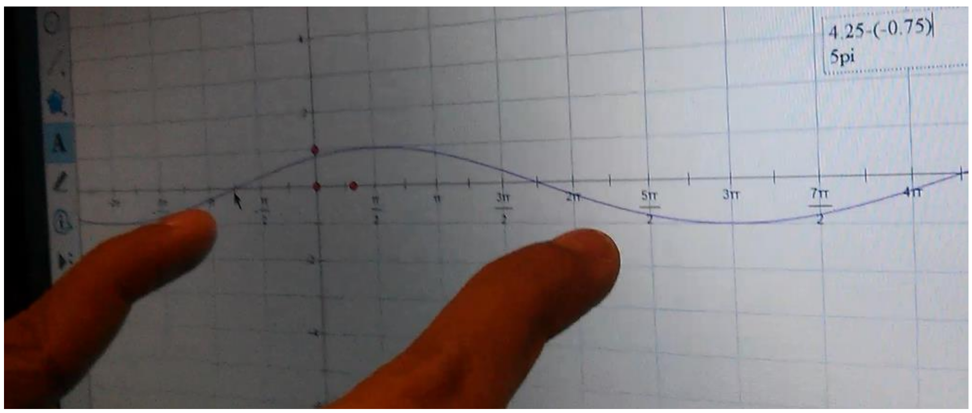


Figure 6.30. Andy was showing the half of period

To show that the graph did not intersect the y-axis at 1, and that as a result it shifted horizontally, he again used his right index finger (see Figure 6.31).

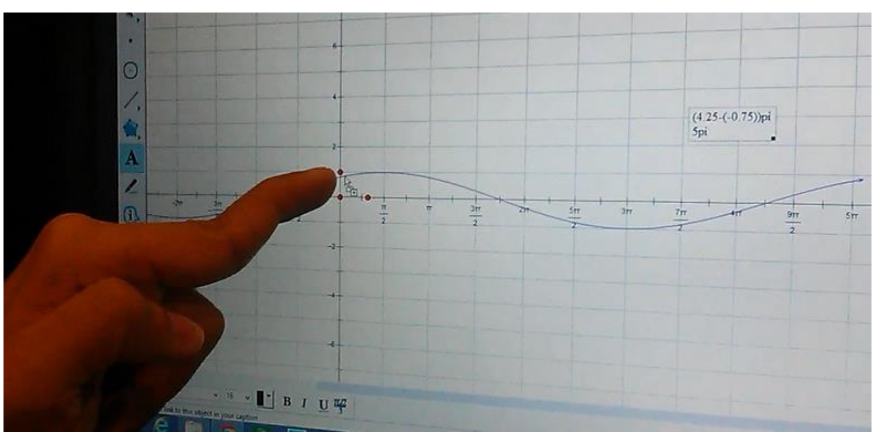


Figure 6.31. Andy showing the intersection between y-axes and the curve

All the above-mentioned instances witnessed Andy's *kinesthetic imagery* of sinusoidal function. Moreover, the last incident (see Figure 6.31) illustrates that Andy had *the formula image* of sinusoidal function. Andy identified that the curve (see Figure 6.6) did not interconnect to the y-axes at one. Therefore, a phase shift (C) needed to subtract from the argument of x in the cosine function ($f(x) = cos\,(\frac{2}{5}x - \left(\frac{\pi}{6}\right))$. However, as data show, the *memory image of formula transformations of sinusoidal function,* did not eventually support Andy to suggest a proper function for the given graph. He was also unsuccessful in suggesting a proper alternative sine function for the given task due to his limited *image of formula transformations of sinusoidal function.*

6.2.2.4. Task 4: Assigning Coordinates to Represent $\boldsymbol{f(x) = sin(4x)}$

For Task 4, Andy expressed that:

"…since there is 4 here is a factor of x, there should be 1, 2, 3 and 4 full sine curves between 0 and 2 π,"

Andy used his right index to count the number of cycles in the interval (see Figure 6.32, *kinesthetic imagery).* This statement indicates that Andy realized that there was a certain pattern to the repeated cycles. In fact he noticed that a sine curve should be repeated 4 times in a certain interval. According to Prmesg (1986), Andy had the *pattern imagery* of a sinusoidal function. He also situated the two numbers 1 and -1 on the y-axes and he expressed that:

"as the amplitude or the coefficient of $sin(4x)$ is 1, we should put 1 and -1 here…"

He again used his right index finger to pointe to the numbers 1 and -1, while he emphasized that:

"…yes, we need to show these two numbers [1,-1] here." (see Figure 6.33)

Andy's response to this task indicated that he had also a *memory image of formula* of transformation of sinusoidal function since he knew that the coefficient of x (B in the canonical representation) represents the number of sine curves appearing between 0 and 2π. Andy also knew that the coefficient of $sin(4x)$ (or "A" according to Canonical representation) represents the amplitude or the maximum and minimum values for the y-axis. Although Andy's actions and his words in the previous tasks represent his *memory image of the formula* of a sinusoidal function, he never referred to the amplitude of the sinusoidal functions in the first two tasks. It was in Task 3 when Andy discussed the coefficient of the sine function ("*as the amplitude or the coefficient of* $sin(4x)$ *is 1")*.

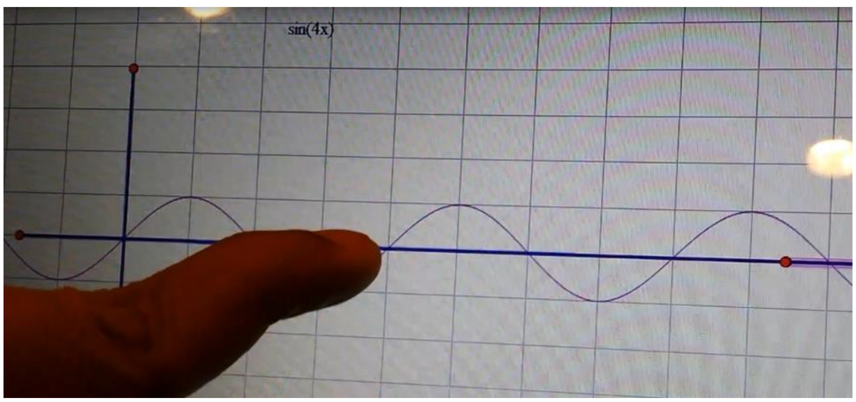


Figure 6.32. Student counted four full cycles

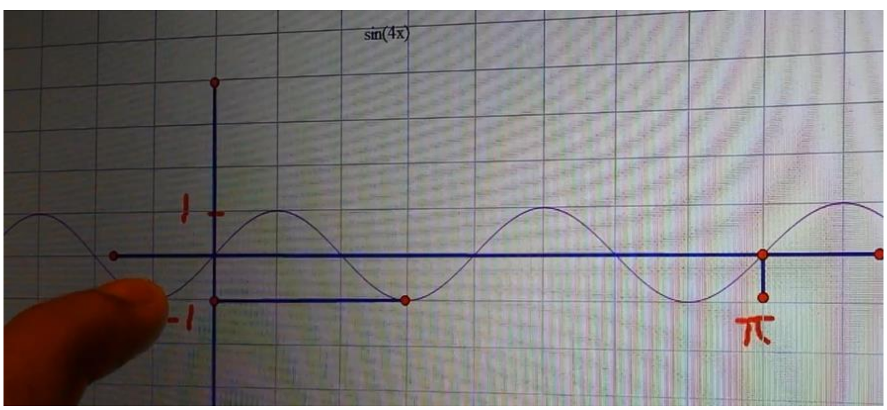


Figure 6.33. Student pointed to the number 1 and -1

He then changed $f(x) = sin(4x)$ into $f(x) = cos(4x - 2\pi)$. Since in Task 1 ($f(x) = sin(4x)$) Andy subtracted $\frac{\pi}{2}$ from the cosine function but in Task 3 he subtracted 2π, I asked him for his reason for doing so. Andy expressed that

> *"…because of having the number four behind the argument x in the cosine function, when we multiply four by $\frac{\pi}{2}$, we have 2π"*

This indicates that Andy applied his *memory image of formula* of a sinusoidal function he experienced in the previous tasks. As we saw in the previous tasks, Andy often made errors and after visiting the graph of his suggested functions, he remembered the learned formula interconnecting a sine function into a cosine function. Or one can also conclude that Andy remembered the canonical representation in which B (period) influences directly C (or phase shift).

6.2.2.5. Task 5: Assigning Coordinates to Represent $f(x) = cos\left(3x - \frac{\pi}{4}\right)$

In contrast with other tasks in which Andy used his hand and fingers in different situations, he did not use his body while he described his thoughts on the last task. However, similarly to the previous tasks, he showed his *pictorial imagery* of the cosine function coordinating when he positioned the y and x-axes on the given graph. This indicates that Andy was aware of the general shape of cosine function. Andy also focused on his *memory image of the formula* of a sinusoidal function. He recognized that there was a coefficient bigger than 1 for the x (or B> 1) in the given function. From there, he interpreted that the graph should be squeezed between the points 0 and 2π. By getting help of his *formula image*, Andy, also, found that "A" or the coefficient of $cos(3x - \frac{\pi}{4})$ was one. As such, he acknowledged that the maximum and the minimum values the function reached should be +1 and -1. In other words, since "nothing was multiplied to the $cos(3x - \frac{\pi}{4})$" – referring to the 'A' value of 1 – it means that the graph should not stretch or squeeze vertically. It was in Task 4 and 5 that Andy discussed the vertical shift. In the previous tasks, he often focused his attention to the horizontal shift and periodicity. The *formula image* also guided Andy to conclude (incorrectly) that the graph should shift horizontally because $\frac{\pi}{4}$ was subtracted from the argument 3x in the given function. However, as we saw in the sections 6.1 and 6.2 Andy was unable in recognizing correctly phase shift (C) according to canonical formula (while C was $\frac{\pi}{12}$ he talked about $\frac{\pi}{4}$ as a phase shift). Furthermore, Andy expressed that the graph should have 3 cycles in the certain interval [0, 2π] (similar to the previous tasks) as he counted "one, two, three sine cycles." This shows that Andy had *pattern imagery* of sinusoidal functions.

6.2.3. Covariational Reasoning

In this section I analyze and interpret Andy's responses to the interview tasks according to *Covariational reasoning* framework created by Carlson et al. (2002). As it was described in the section 5.3, Carlson et al's., framework comprised of *five distinct developmental levels of covariational reasoning* (L#) that parallel the five mental actions (M#).

As we saw in the previous analysis (in section 6.2), Andy recognized that there was a close relationship between the coefficient of the argument in the sine function (e.g., $f(x) = sin(2x)$) and the number of full cycles repeated in the interval [0, 2π] (*M1: Coordinating the value of one variable with changes in the other*). Andy stated that his reason for focusing on this certain interval [0, 2π] was the fact that a canonic sine function began from 0 and completed at 2π. In other words, a full cycle or periodicity happens between 0 and 2π. While completing Task 1, therefore, Andy interpreted that having more than one cycle, between 0 and π (one cycle) and from π to 2π (another cycle), in this interval means having a coefficient of x bigger than one in the sinusoidal function. Increasing the number of full sine cycles in the interval [0, 2π], in fact, coordinate increasing the coefficient of the argument x in the sinusoidal functions (*M2: Coordinating the direction of change of one variable with changes in the other variable*). As such, he expressed that the function representing the graph should be $f(x) = sin\,(2x)$. There is, in fact, a relationship between the number of full equal cycles between 0 and 2π and finding the periodicity of a sine function. Thus, if there were two cycles in the given curve (for Task 1, graph of $f(x) = sin(2x)$), the number presented the periodicity (or B) in the sine function was the number 2 (*M3: Coordinating the amount of change of one variable with changes in the other variable).* The behavior exhibited by this student when responding to this task suggested that Andy was at quantitative coordination level (*L3*: At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other variable. The mental actions MA1, MA2 and MA3 are supported by *Level 3* image). Andy's responses to this task, in fact, supported all three mental actions *M1, M2* and *M3*.

For Task 2 ($f(x) = sin\,(\frac{2}{3}x)$) Andy again verbalized that the graph of sine function and as a result the number of sine cycles in and beyond the interval [0, 2 π] changes with change in period (compared with the canonical functions) (*M1*). He stated that the smaller the coefficient of x than 1 required a larger cycle compared with the canonic sine curve (*M2*). As we saw in the previous sections (sections 6.1.2 and 6.2.2), Andy was able to recognize the amount of the periodicity or the coefficient of x and therefore the proper sine function in the task (*M3*). Similar to the task 1, his response showed that Andy had the *quantitative coordination level* (*L3*) of ability of *covariational reasoning* of sinusoidal functions.

Andy showed exactly the same level of *covariational reasoning* ability and mental action (*in identifying periodicity*) in Task 3 ($f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$), which was different from the other two tasks because a phase shift was added to the argument of this cosine function. Andy acknowledged that there is a connection between the coefficient of x ($\frac{2}{5}$) and the number cycles repeated in its relevance graph (*M1*). In addition, Andy recognized that as the coefficient of x in the cosine function decreasing ($\frac{2}{5}$ <1), the length of full sine cycle is extending (compared with the length of a canonical cosine function) (*M2*). Similar to the first two tasks, he was able to determine the amount of periodicity (*M3*). The Andy's behavior directed me to conclude that he had the *quantitative coordination level* (*L3*) of *covariational reasoning* with respect to period, because all his three mental actions *M1, M2* and *M3* supported him to coordinate the amount of change of period with the number of repeated sine cycles in interval [0, 2π].

Andy also repeated the same mental action (*M1, M2* and *M3*) (as the previous tasks) for Task 4 ($f(x) = sin\,(4x)$) and Task 5 ($f(x) = cos\,(3x - \frac{\pi}{4})$) in determining periodicity. For instance, Andy indicated that since in each of the sinusoidal functions the coefficient of x was more than one (e.g. the factor of x was 4 in the Task 4), we have more than one sine cycle in the interval [0, 2π]. In other words, he realized that the bigger the coefficient of x, the more cycles in the interval [0 and 2π] (*M2*) (this is in contrast, for Task 2 ($f(x) = sin\,(\frac{2}{3}x)$)and Task 3 ($f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$) the smaller the coefficient of x than one required a larger cycle compared with the canonic sine curve *(M2)*). As we saw in the previous sections (e.g., 6.1.4), Andy recognized the amount of the periodicity of the coefficient of x and therefore the corresponding sine functions or their related graph in Tasks 4 and 5 (*M3*). As such, Andy had the *quantitative coordination level* (*L3)* of ability of *covariational reasoning* of sinusoidal functions, when identifying periodicity in Tasks 4 and 5.

Andy was not successful in identifying the phase shift in Tasks 3 and 5. When completing Task 3, Andy acknowledged that since the graph's intersection point changed from ($\frac{\pi}{2}$,1) (the point from the canonical cosine graph) to another point, the whole argument of the function and as a result the cosine function should change (*M1*). He also admitted that a certain radian subtracted from the whole argument, because the graph shifted to the right (*M2*). However, Andy determined unsuccessfully the amount of phase shift. He identifies $\frac{\pi}{6}$ for a phase shift, instead of $\frac{\pi}{2}$. Therefore, Andy's ability for *covariational reasoning* was *direction level* (*L2*) with respect to phase shift. To deal with phase shift in Task 5, Andy realized (similar to Task 3) that since a certain radian was subtracted from the argument of x, the graph should be shifted accordingly (*M2*). However, Andy was unable to determine the amount of phase shift and to express the influence of the amount of periodicity on the phase shift and on the graph. In other words, while the phase shift was $\frac{\pi}{12}$ ($f(x) = cos(3x - \frac{\pi}{4}) = cos(3(x - \frac{\pi}{12}))$, he considered it as $\frac{\pi}{4}$ and then sketched the graph accordingly. As it appears from data, Andy's *covariational reasoning* ability was the *direction level* (*L2*) since he showed the collection of mental actions (*M1* and *M2*) when engaging in the given mathematics tasks.

When Andy was asked to write an alternative function representing the given graph for Task 1 (see Figure 6.1), he acknowledged that the given graph could not be for a canonical cosine function, because it did not intersect y-axes at the point 1 (*M1*). Andy noticed that the graph was shifted horizontally to the right and therefore the resulting cosine function should be changed accordingly. One might conclude that Andy's mental action as *M2*. However, Andy's discussion shows that he suggested adding a certain amount of radian to the argument of the function because the graph shifted to the right (so he is not at *M2 level*). He still did not talk about the amount of horizontal shift and as a result the changes in the cosine function. To find the horizontal change, he first looked for a point on the x-axes for which the y-value was 1. As we saw in the previous part of the data analysis (in the section 6.1) Andy's suggested function ($f(x) = cos\,(2x + \frac{\pi}{4})$) shows that only $\frac{\pi}{4}$ radians would influence the horizontal length of one complete cycle whereas, Andy previously expressed that there were two full cycles for the given graph. In other words, if he wanted to show a correct cosine function, he should multiply $\frac{\pi}{4}$ by 2. Thus, it appears that his *covariational reasoning* was initially at the *coordinating level* (*L1*).

After seeing the graph of $f(x) = cos\,(2x + \frac{\pi}{4})$, he realized his mistake and he said:

> *"Ok, I made some incorrect calculation, but I think I should do something with the coefficient of x which is 2. Basically since the coefficient of x is 2 it needs to be multiply $\frac{\pi}{4}$ on 2, because the x has 2 multiplications. So the function should be $f(x) = cos\,(2x - \frac{\pi}{2})$."*

The feedback Andy received from the graph of his suggested function led him to realize that the periodicity should directly affect the whole part, x and the amount of shift, if he wanted to have a proper sinusoidal function. Andy was able to realize the amount of the phase shift and its effect on the cosine function (*M3*). He also noticed that since the graph shifted to the right, the amount of phase shift should be subtracted from the argument 2x (*M2*). Observing the graph of the proposed function (the incorrect functions) supported Andy to transit the level of his *covariational reasoning* from *coordinating level* (*L1*) to *quantitative coordination (L3).* In the other words, the collection of mental actions (*M1, M2* and *M3*) helped Andy to coordinate successfully the change of the argument with the amount of change in the phase shift and then, obtaining the correct sinusoidal function.

To identify an alternative function for the given graph, Andy primarily was not able to recognize the horizontal shift in the Task 2 and he made the same error as he did (initially) in Task 1 ($f(x) = cos(\frac{2}{3}x + \frac{3\pi}{4})$). Getting feedback from the graph of his suggested function led him to find the correct horizontal shift and as a result, the corresponding cosine function representing the given graph ($f(x) = cos(\frac{2}{3}x - \frac{\pi}{2})$)(*M3*). The data illustrate that similar to Task 1, Andy had a transition from *L2* level to *L3* level in recognizing phase shift from the given graph in Task 2. However, Andy's *covariational reasoning* moved back *from L3 level* in Tasks 1 and 2 to *L1 level* for the rest of interview tasks, when he determined the alternative functions for the given graphs/functions.

6.3. Summary

In this chapter, I included a detailed analysis of one of the participants, Andy's completion of the interview tasks. At the beginning of this chapter, I described a brief overview of Andy's answers to each of the interview tasks. I then analyzed the way he completed the tasks with respect to the three theoretical frameworks applied in this study.

The data analysis shows that Andy successfully completed the Tasks 1, 4 and 5 in which the coefficient of the arguments x were whole numbers. He was also able to complete the tasks (Tasks 2 and 3) involving the fraction arguments for x. In fact, Andy determined the periodicity in the transformed functions/graphs in all interview tasks through applying visual *imagery* such as *concert, pattern, kinesthetic* and *dynamic imagery*, along with focusing on the *details discerned* from the given functions/graphs, *perceiving properties of sinusoidal functions* and *connecting the analytic representations with the graphical representations*.

The data illustrate that Andy's *covariational reasoning* ability in determining periodicity was the *quantitative coordination level* (*L3*), it was at the *direction level* (*L2*) for identifying phase shift. Andy was unable to determine the phase shift in the Tasks 3 and 5 (*L2*), even after applying all previous methods (e.g., *concrete imagery* and finding *relationship between symbolic and graphical representations*) he used to identify periodicity. Although after several attempts, he could find the alternative function for the Tasks 1 and 2 (*L3*), he was unsuccessful in Tasks 3, 4 and 5 (*L1*).

Chapter 7.

Data Analysis: The case of the “Other Five Students”

In this chapter I focus on the additional five participants and their responses to the five interview tasks. The data are analyzed in terms of the following three theoretical perspectives: 1) *shift of attention*, 2) *visual imagery* and 3) *convariation reasoning*. The students’ responses to each interview task are also compared with those of Andy (Andy’s responses to the interview tasks are discussed in Chapter 6). Table 7.1 describes the participants’ success or failure in completing the interview tasks.

Two main themes have emerged in the data analysis: “*identifying the period (B)*” and “*identifying the phase shift (C)*” $(f(x) = A\, sin/cos((B(x + C) + D))$. The students’ difficulties and their approaches to the tasks are explained in detail according to these themes.

Table 7.1 summarizes the students’ responses to the 5 interview tasks. The list of tasks is found in Chapter 4. Note that some tasks have several parts, and participants performed differently on different part of the tasks. To indicate this, the following codes are used in Table 7.1:

B Changing a sinusoidal function into a cosine function and vice versa

P Identifying the period

S Identifying the phase shift

When students completed the task correctly

◕ When students completed the task after tails/feedback

○ When students have not completed the task correctly

Table 7.1. Students' performance in the interview tasks

Task	Andy	Sally	Emma	Kate	Rose	Mia
Task 1P	⬬	◕	◕	◕	◕	◕
Task 1B	◕	◕	○	○	○	○
Task 2P	⬬	⬬	⬬	⬬	⬬	◕
Task 2B	◕	⬬	○	○	○	○
Task 3P	⬬	⬬	⬬	○	⬬	○
Task 3S	○	⬬	○	○	○	○
Task 3B	○	⬬	○	○	○	○
Task 4P	⬬	⬬	⬬	⬬	⬬	○
Task 4B	○	⬬	○	○	○	○
Task 5P	⬬	⬬	⬬	○	⬬	○
Task 5S	○	⬬	○	○	○	○
Task 5B	○	⬬	○	○	○	○

7.1. Shift of Attention and Visual Imagery: Identifying the Period (coefficient B of x)

In all the five interview tasks the participants were required to connect the period of the given sinusoidal function or the sine curves to a coefficient of x in a sine or a cosine function. For brevity, I refer to this connection as "Identifying the period" (see Figure 7.1).

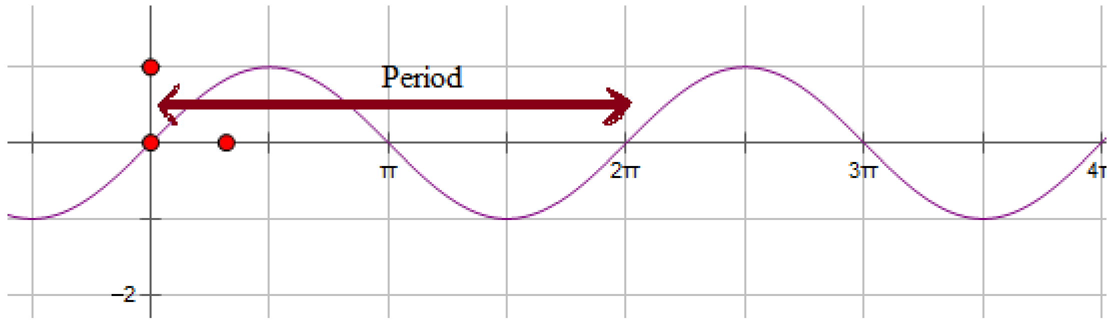


Figure 7.1. Period of a canonical function

7.1.1. Identifying Period in Task 1

7.1.1.1. Initial Confusion

Among all six students who were interviewed in this study, almost everybody (except Andy) identified the period (B) unsuccessfully at the first attempt. While Andy in Task 1 realized that the given curve represented the function $f(x) = sin(2x)$, Rose, for example, stated:

> *"It is $f(x) = sin(\frac{1}{2}x)$. It is sine graph because it starts at 0 and it should be $sin(\frac{1}{2}x)$. The sine graph start at 0 and then π and 2π but this one is o, ,$\frac{2\pi}{3}$. This is half of sine graph. Because the period here is π while it is 2π in the original sine curve."*

Kate and Emma initially made similar errors. The above statement indicates that these three participants recognized incorrectly the function for the given graph, determining it to be $f(x) = sin(\frac{1}{2}x)$. Analyzing the situation using Mason's (2008)) framework it can be concluded that Rose, Kate and Emma *reasoned on the perceived properties* of the sinusoidal functions and from there they determined *(incorrectly) relationships* between the visual representation and the symbolic representation. This is in contrast with Andy, who initially demonstrated *holding wholes* by focusing on the given graph for a while and then *discerning some details* from the sine curve (e.g., the graph intersected the x-axis at 0, $\frac{\pi}{2}$, and π). They recalled the fact that the period of a canonical sine function is 2π, whereas the period of a cycle is π in the curve given in Task 1. The students thus determined that the given curve represents the function $f(x) = sin\left(\frac{x}{2}\right)$. In other words; these three students connected the period of the sine curve, which was π radians, with the coefficient of x in their suggested sinusoidal function. The statement revealed by these participants (Kate, Rose and Emma), in fact, divided the argument x by 2 because the period of the canonical function was divided by 2.

Using Presmeg's (1989) *imagery* framework, it appears that the students, in fact, had *concrete imagery* of canonical functions in their mind since they knew what the general shape of a sinusoidal function was and where a cycle of the curve started and completed. As an example, Rose sketched a picture of sinusoidal function she had in her mind (see Figure 7.2). One might conclude the students' *memory image of the formula of sinusoidal function,* since they talked about period and the impact of it on the parameters of the suggested function ($f(x) = sin\left(\frac{x}{2}\right)$). However, while there is an opposite relationship between the length of a full-completed cycle (compared with canonical sinusoidal function) and the parameter "B" in the standard sinusoidal function, students recalled the positive/direct connection.

In contrast with Andy, who used his body (*kinesthetic imagery)* frequently when he explained his thinking, these three students never used their body (their hands or fingers). However, they had *dynamic (moving) imagery* as they were able to imagine a canonical sine function transformed into the given curve (similar to Andy). Although Kosslyn, Ganis and Thompson (2009) state that the *dynamic imagery* could lead to the *kinesthetic* sensation of making that movement, this was not the case when Rose, Kate and Emma worked on Task 1.

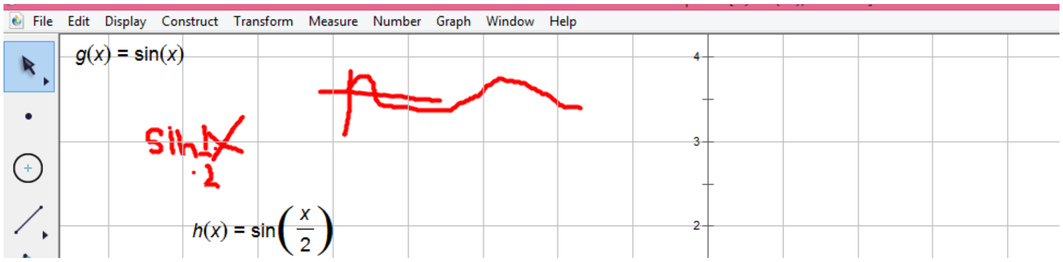


Figure 7.2. **Rose's pictorial image of sinusoidal function**

While Kate, Rose and Emma did not refer to the compression of the given curve, it was Sally and Mia's main considerations. Sally *gaze*d at the given curve for a long pause and then she expressed:

> *"….because a normal sine graph is between 0 and 2π but in this one the graph is compressed by half so it means that it is going to be* $sin(\frac{1}{2}x)$ *so it is* $f(x) = sin(\frac{1}{2}x)$*"*

Similarly, Mia stated:

> *"…this graph [canonical sine function] is compressed by half so we should have* $f(x) = sin(\frac{1}{2}x)\ldots$*"*

Holding wholes while approaching the task made students realize that the canonical curve was transformed/shrunk ("*a normal sine graph is between 0 and 2π but in this one the graph is compressed by half*"). *Holding wholes* also enabled the students to *recognize relationship* between the graphical representation of the sine graph and its relevance function (incorrectly). Compared with Rose, Kate and Emma who initially *discerned some specific details* such as the period (π) of the given curve and then focused on the *relationship* between the period and the coefficient of x in the sinusoidal function, Sally and Mia did not discuss the period of the sine curve. Instead, they focused their attention on the general shape of the curve. Meanwhile, although the five students looked at the given graph from different perspectives, they proposed the same incorrect function ($f(x) = sin(\frac{1}{2}x)$).

Similar to the other participants, Sally and Mia had *pictorial imagery* since they had concrete descriptions of canonical functions in their mind (…*this graph [canonical sine function] is compressed)*. When comparing the given curve with the graph of a canonical function, the students had, in fact, *dynamic imagery* because they were able to transform a canonical function into the given curve in their mind. This *imagery* led Sally and Mia (similar to Andy) to use their body when completing Task 1. As an example, Sally moved her two fingers (from her right hand) to illustrate the compression of the given curve (see Figure 7.3). Sally's action shows her *kinesthetic imagery* of sinusoidal functions. Mia used exactly the same gesture.

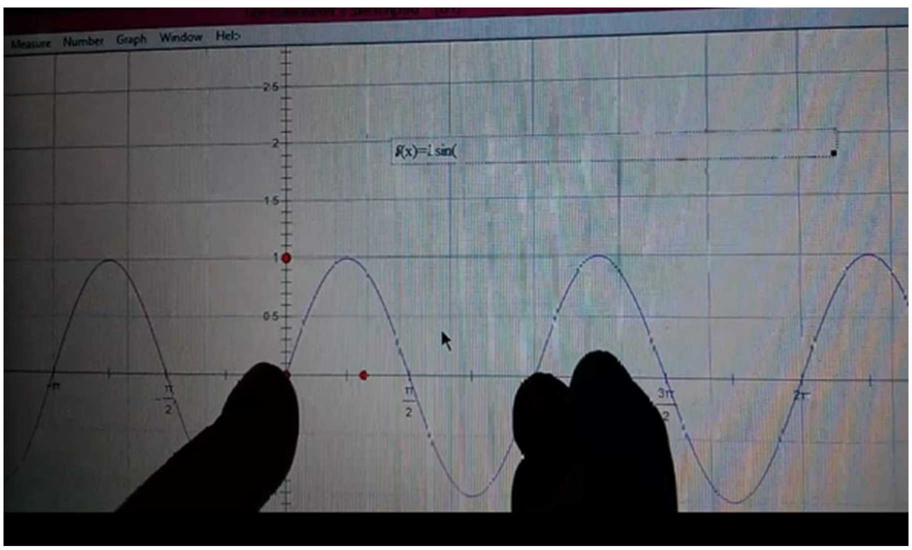


Figure 7.3. Sally's usage of her two fingers to show that the curve was shrunk

As it appears from the students' discussions that while $\frac{x}{2}$ and $\frac{1}{2}x$ are the same, they often tended to express their suggested function for Task 1 as sine of $\frac{1}{2}x$. This shows that the students attempted to establish the relationship between the period and the coefficient of x in the sinusoidal function. Since the period of the given curve (π) was half of the period of a canonical sine function (2π), the coefficient of x should be $\frac{1}{2}$. In other words, the participants' language was often in direct correspondence with the resulting algebraic expression.

The phenomenon is also reminiscent of "same A – same B" intuitive rule, as described by Stavy and Tirosh (2000). For example, "Same area – same perimeter" is a frequent confusion of learners explained by this intuitive rule. In the case of Task 1, "same A – same B" was interpreted as "half of (canonical) cycle – half of argument".

As evident from the data, the majority of students in this study had initial confusion when completing Task 1; however, they successfully found the correct function for the given curve after the graph of their suggested function was sketched. In the following section, the participants' approaches are described.

7.1.1.2. Importance of Computer Feedback

Observing the graph of the function $f(x) = sin(\frac{1}{2}x)$ made participants realize that the graph of their suggested function did not correspond to the given curve. The participants, then, used different methods to correct their initial suggestion. In the following, the students' responses are described. If a student's solution was different from others, it is described in a separate section; otherwise similar students' responses are explained in the same section.

Kate, after *gazing* at the screen for two minutes and *holding* both graphs as whole, began to compare the two sinusoidal graphs respectively (see Figure 7.4) through *discerning detail* from both curves. She stated that:

> *"….this one [#1] is half of $f(x) = sin(x)$ at 2π, but this one [#2] has one, two complete sine graphs between 0 and 2π. So, my answer was wrong. It is $f(x) = sin(2x)$."*

As it appears from the above statement, and consistent with Mason (2008) observation, identifying *details* from the curves directed Kate to *recognize relationship* between the given curve (#2 in Figure 7.4) and its appropriate sinusoidal function

$(f(x) = sin(2x))$. The above quote from Kate also illustrates her *pattern imagery* of sinusoidal function, since she was able to realize the repeated outline ("*one, two complete sine graphs between 0 and 2π*") embedded in the given curve (#2). Moreover, Kate used *concrete imagery* because she had a visual image of a canonical function when she explained her thinking ("*this one [#1] is half of* $f(x) = sin(x)$ *at* 2π").

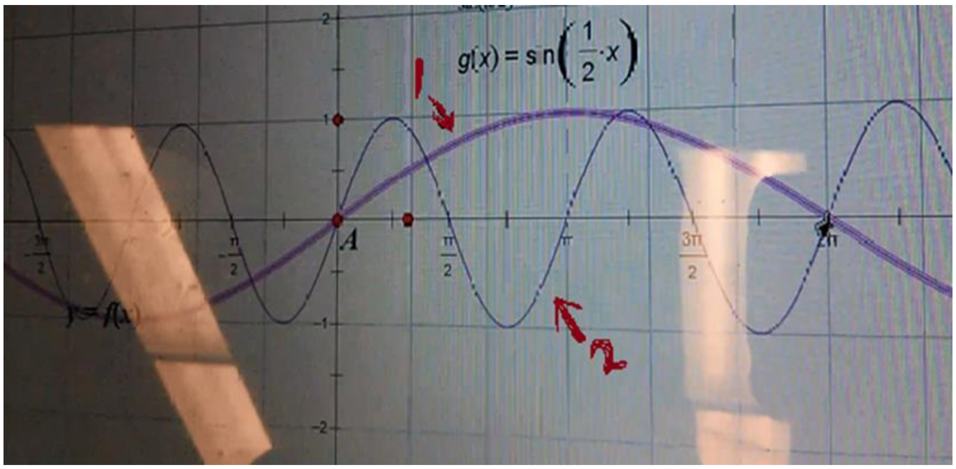


Figure 7.4. Graph of $f(x) = sin(\frac{1}{2}x)$ and $f(x) = sin(2x)$

Similarly, getting feedback from the graph of her suggested function (see Figure 7.4) shifted Sally's attention to the *connection between the graph* #2 (see Figure 7.4) *and its represented function* as she expressed:

> *"….I made a little error when I was calculating my B value which is the compression of the x-values, if it is compressed by a half…., so if it is compressed by half it is 1 over half is 2[it means* $\frac{1}{\frac{1}{2}} = 2$]. *I think the function should be* $f(x) = sin(2x)$.*"*

The above excerpt also indicates that Sally attempted *to reason on the perceived properties* of the sinusoidal functions ("… *if it is compressed by a half, so if it is compressed by half it is 1 over half is…"*). This justification led her to change her initial suggestion successfully from $f(x) = sin(\frac{1}{2}x)$ into $f(x) = sin(2x)$. In other words, she recognized the inverse relationship between the period of the given graph and the coefficient of x in the sine function.

While describing her answer, Sally used the sketchpad's pointer to refer to the coefficient of x in the sinusoidal function $f(x) = sin(\frac{1}{2}x)$ (see Figure 7.5) and she stated:

"…we actually we should calculate by going 1 over B…"

The aforementioned statements indicate that Sally had the *memory image of formula* of the canonical function because she acknowledged the coefficient of the argument x as "B." She, moreover, had *concrete imagery* of sinusoidal functions as she expressed that the graph (the canonical function) was compressed (the canonical sine curve shrunk into two full sine curves). One might also conclude that Sally had the *dynamic imagery* since she was able to move the given curve in her mind to compare it with the canonic curves.

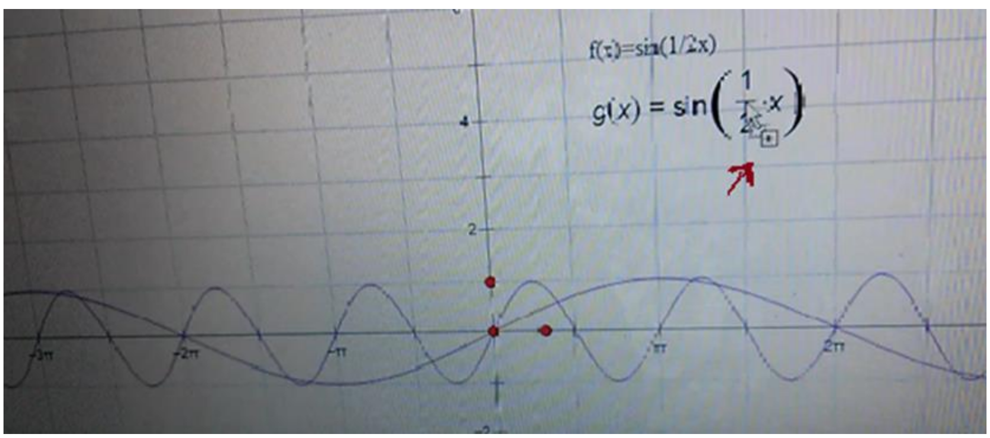


Figure 7.5. Sally's pointing to the B value in the suggested function

When she saw the graph of $f(x) = sin(\frac{1}{2}x)$, Emma, similar to Sally, attempted to find connection between the period and the coefficient of x in the sinusoidal function as she stated:

"….That mean that in $f(x) = sin(\frac{1}{2}x)$ the period is actually bigger [in curve #2], so it is $f(x) = sin(2x)$. It seems that you are taking 2π and divided it into 2 to have period of π. You suppose to get the value from here and divided it…. It has the period of π but originally it has the period of 2π so we need to divide the period by 2 so in order for you to divide it by 2 you need to have 2x here. So it is $\frac{2\pi}{2}$ [the period]."

Similar to the other two students (Sally and Kate), Emma was eager to *discern some details* from the curve #2 (in Figure 7.4) ("…*the period is bigger [see Figure 7.4, curve #2*]…"). From there she was able to *reason on the perceived properties of sinusoidal functions ("It seems that you are taking 2π and divided it into 2…")* and to *connect the visual and symbolical representations of the sinusoidal function*. She, eventually, determined a proper sinusoidal function for the graph.

To show that the period of the given function is half of the period of canonical sinusoidal curve (see Figure 7.6), Emma moved the computer pointer across the point π. This movement, in fact, led Emma to recognize the formula (it provides a connection between the coefficient of x and the factor of the argument x in the sine function) that directed her to determine a correct corresponding function. Emma, in fact, (similar to Sally) had the *memory image of formula ("…It has the period of π but originally it has the period of 2π, so in order for you to divide it by 2 you need to have 2x here… it is $\frac{2\pi}{2}$ [the period]")*. Unlike Sally, who discussed the *formula* of sinusoidal functions that connect the "B-value" with 2π (Period= $\frac{2\pi}{B}$); Emma (similar to Andy) did not explicitly talk about the component "B" and its connection with period (although later when completing the other tasks, Emma applied the same formula (Period= $\frac{2\pi}{B}$) as Sally used in Task 1). The formula Emma expressed for Task 1, showed the relationship between the length of a full sine cycle and2π. (The formula was: The length of a full sine cycle=$\frac{2\pi}{The\ coefitient\ of\ x\ in\ the\ siusoidal\ function}$, so π=$\frac{2\pi}{2}$). Similar to Andy and Sally, Emma had *concrete imagery* of canonical function("…*originally it has the period of 2π* …") as well as *dynamic imagery*.

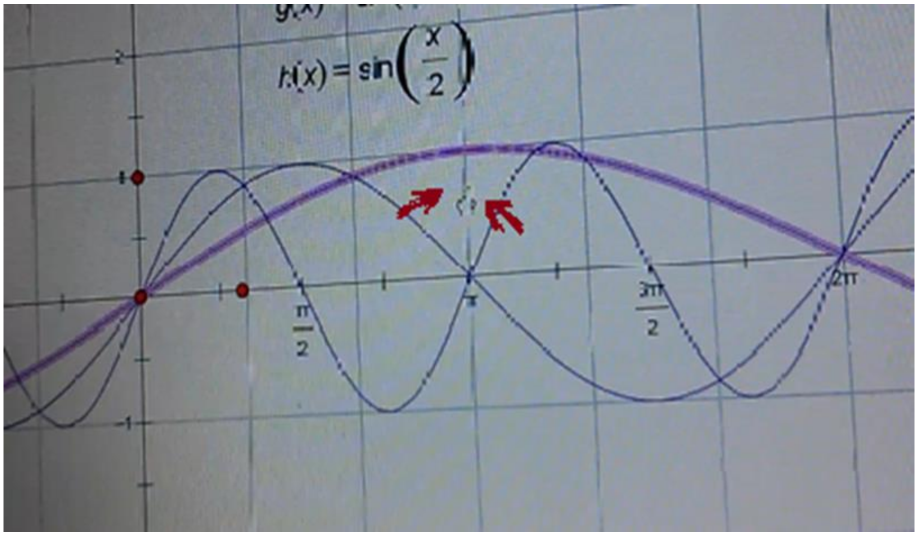


Figure 7.6. **Emma compared the period of the given function with the period of canonical function**

After seeing the graph of $f(x) = sin\left(\frac{1}{2}x\right)$, Rose and Mia (in separate interviews) stared at the graphs #1 and #2 for a while (see Figure 7.4). Then, they *held the graphs (#1 and #2) as whole*. They began to *describe in detail* the given graph (#2 in Figure 7.4) in respect with the graph of $f(x) = sin(\frac{1}{2}x)$. For example Rose stated:

> *"….so, if $f(x) = sin(\frac{1}{2}x)$ is like this, so it is going to finish at 4π. So this is going to be the whole graph. So it should not be $\frac{1}{2}x$, it should be 2x. Because when we have $\frac{1}{2}x$ we can see that it ends at 4 π. But if I put here 2x, I compressed it and I can…have this curve finishes at π...The period of sine graph is 2 π but this one is compressed, so it is $f(x) = sin(2x)$, but $\frac{1}{2}x$ is expansion in fact."*

Mia expressed:

> *"…ok this curve is compressed…, it ends here at π, but in this one is at 4π …so the function$f(x) = sin(2x)$]…."*

As mentioned previously, in order to describe that her suggested function ($f(x) = sin(\frac{1}{2}x)$) was not a proper function for the represented curve (#1, see Figure 7.4), Emma was looking for the ended points of the full sine cycles in the given graph as well as in the canonical sinusoidal graph. Meanwhile, Kate paid her attention to the number of completed cycles in the graph, while Rose and Mia compared the end point of a cycle curve #2 with that of curve #1, considering the origin as a beginning of a cycle *("….$\frac{1}{2}x$ we can see that it ends at 4 π. But…I can…have this curve finishes at π..*."). Mia used the computer pointer to show the end point of a cycle in the curve #1 (see Figure 7.7); the end point of the cycle in curve #2 ($f(x) = sin(2x)$) was displayed through the same manner (see Figure 7.8). In other words, Rose and Mia by linking the ended points of the full cycles (in the both curves and compared it with the curve of the canonical function), they *connected the visual representations with the symbolic representations*. They recalled that since a full sine cycle finished at 4π in curve #1, therefore the coefficient of x is $\frac{1}{2}$ (as $\frac{4}{2} = 2$ so the inverse of 2 is $\frac{1}{2}$ which is the factor of x in the sinusoidal function). Similarly, because a sine cycle completed at π in the given curve #2, the coefficient for x in the sinusoidal function should be the number 2 (the coefficient of π is 1. So 1 divided by 2 equals $\frac{1}{2}$ and then the inverse of $\frac{1}{2}$ is 2 which is the coefficient of x). In other words, similar to Emma and Andy, Mia and Rose applied *their memory image of formula of sinusoidal functions* when completing the task.

To illustrate their thinking, they also applied their *kinesthetic imagery.* For example, when Rose described the period of a canonical function, she used her right index finger to point at 2π (see Figure 7.9). As it was also described in the previous section (section 7.1.1), Rose and Mia had *concrete imagery* of sinusoidal function since they have a picture of a canonical function in their mind. The participants' descriptions indicate that *concrete imagery* was the most common *imagery* among the five categories of *imagery*.

As demonstrated, all the participants noticed their errors after considering the graph of the function $f(x) = sin(\frac{1}{2}x)$ and they indicated the correct function for the given curve. The students' work on Task 1 provided them with an opportunity to experience further success in the rest of the interview tasks.

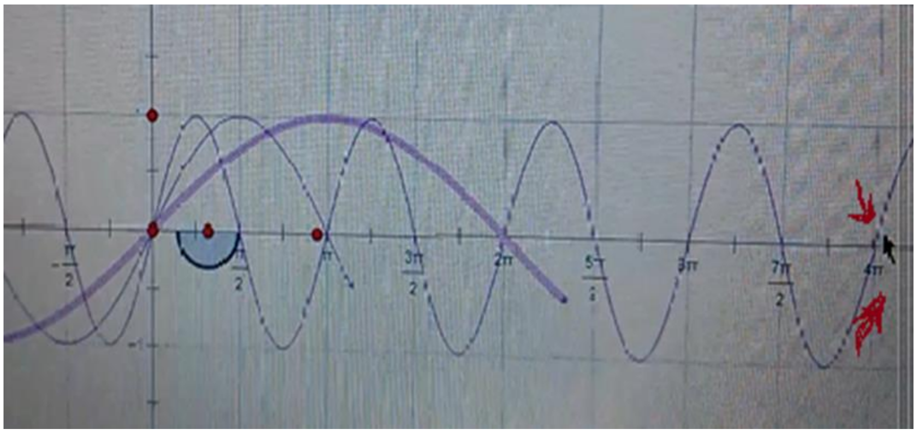


Figure 7.7. The end point of the curve #1 displaced by the computer pointer

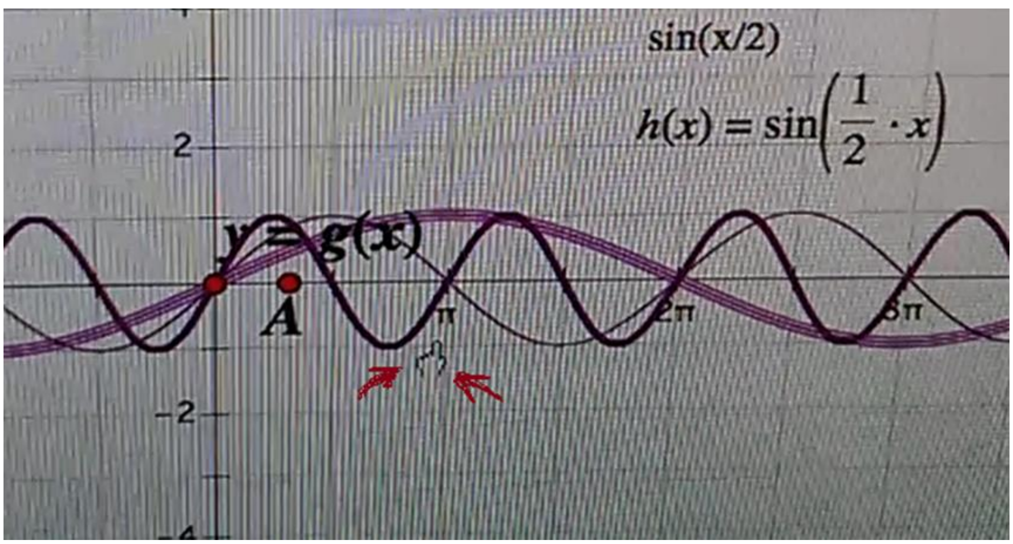


Figure 7.8. Mia used the pointer to show the end point of the given curve #2 and the compression of the canonical function

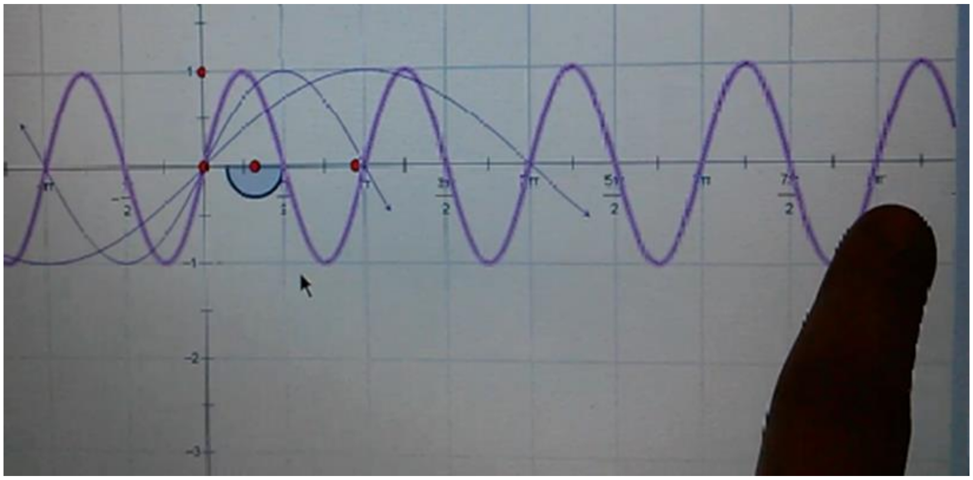


Figure 7.9. Rose was pointing to 2π

7.1.2. Identifying Period in Tasks 2 and 3

Recall that in Task 2 the participants were asked to identify the function $f(x) = sin(\frac{2}{3}x)$ from the given graph. While approaching Task 2, Sally and Emma used the same approach that led to their success for Task 1. Emma (similar to Sally in Task 1) for instance after a long pause expressed:

> *"…the graph should go over 2π… it is $f(x) = sin(\frac{2}{3}x)$ because the period is $\frac{2\pi}{B}$ and the period ends here…here is 3π... "*

Sally *gazed at the given curve* as a whole and then she stated:

> *"...in this graph we noticed…that in the normal sine graph the period is 2π, but in this graph the period is 3π so that means that the x-value are compressed… so that means we going to divide 2π by 3π, so the function should be $f(x) = sin(\frac{2}{3}x)$."*

It is evident from the above quotations that Emma and Sally's attention shifted from *holding whole* to *discerning details* from the given curve. Finding the period of the given curve and comparing it with the period of a canonical function shows that they *perceived properties* of a sinusoidal function. To interconnect the graphical representation to the numerical depiction, they used their previous experience gained from Task 1. They first found the point in which a full cycle (starting from the origin) completed (here at 3π) and then they divided the period of a canonical function (2π) by it. This illustrates that they recalled the *memory image of formula* of sinusoidal function. They also had *concrete imagery* and therefore *dynamic imagery* of sinusoidal functions since they were able to move the given graph to visualize a canonical function in their mind.

Contrary to her approach to Task 1, where Kate applied her *pattern imagery* to realize the period of the given curve *("…one and two cycles are in the interval [0, 2π]*"), she did not use the same method for Task 2. Similar to Andy, Sally and Emma; Kate also attempted to apply her *memory image of formula* of sinusoidal functions when approaching Task 2 (she expressed that the period in a sinusoidal function is $\frac{2\pi}{B}$). However, after a short pause and gazing at the given graph, Kate eventually noticed that a full cycle finished at the point 3π. She, therefore, suggested the function $f(x) = sin(3x)$ without paying attention to her mentioned formula. Apparently, observing of the graphical representation in Task 2 led Kate to identify the function inappropriately.

Recall that in Task 3 the students were asked the function of $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$ from the given graph. Emma and Sally consistently applied for Task 3 the same method they used for Tasks 1 and 2. In other words, they were able to realize a correct period for the given curve using their *memory image of formula* of sinusoidal functions. However, none of the previous approaches (*pattern imagery* and *memory image of formula*) supported Kate to determine the period of the given curve in completing Task 3 (she expressed "*I have no idea for this graph… It is a cosine function")*. Approaching Task 3, Rose applied the same method she used for Tasks 1 and 2. After gazing at the given curve, Rose noticed that a full cycle of a cosine curve, was completed at the point

5π. She then suggested $\frac{2}{5}$ as the coefficient of x in the sinusoidal function (because according to her method, ended- point of a full sinusoidal cycle (5π in this task) should be divided in to 2 and then the results should be flipped). Similar strategy also helped Rose to realize a correct period for the Task 2.

Mia's approach to Task 2 was different from her approach to Task 1. After *holding the graph as whole* for a long pause, she did *discern some details* from the x-axis. She then stated:

> *"…It is sine of x over something because if it is sine of x it would end here [at 2π]…ok, it is $f(x) = sin(\frac{x}{3})$ because there are one, two three spaces here between 0 and this point and again one, two, three here…(see Figure 7.10)"*

As it appears from the above statement, Mia counted the number of "blocks" between 0 (the point A in Figure 7.10) and the point in which the curve intersected the x-axis point B) and again from the point B to another one in which the graph intersected the x-axis (point C). Because the distance between the points A and B, and B and C was 3 blocks, Mia put the fraction $\frac{1}{3}$ for the coefficient of x in the suggested sinusoidal function. This illustrates that Mia used her *pictorial imagery* of canonical function in her mind when suggesting the incorrect function *("…if it is sine of x it would end here [at 2π]…").* Observing the graph of the function $f(x) = sin(\frac{1}{3}x)$ led her to suggest the functions $f(x) = sin\left(\frac{1}{2}x\right)$ and $f(x) = sin(\frac{1}{2}x + \frac{\pi}{2})$ as possible solutions to Task 2. Considering the visual feedback from the graphs of these suggestions, then indicated that the function should be $f(x) = sin(\frac{x}{1.5})$, which is in fact correct but unconventional notation. When she was asked to explain her thinking, she indicated:

> *"…this one was too stretched [referring to the graph of $f(x) = sin(\frac{1}{3}x)$] and this one was not stretched enough [referring to the graph of the previous suggested function $f(x) = sin(\frac{1}{2}x + \frac{\pi}{2})$], so I though it should be $f(x) = sin(\frac{x}{1.5})$ to be fitted in this curve."*

By using the “trial and error” method, she could successfully find the correct function for the given curve. Clearly, her method in approaching this task was totally different from her approach in Task 1. In this task, Mia did not use her hand while explaining her thinking and therefore, there was no sign of *kinesthetic imagery* used in completing Task 2. However, none of the approaches applied in Task 1 and 2 helped Mia to determine a correct period for the Task 3. She could only express that the curve given in Task 3 indicated a “*graph is for a cosine function expanded horizontally*”.

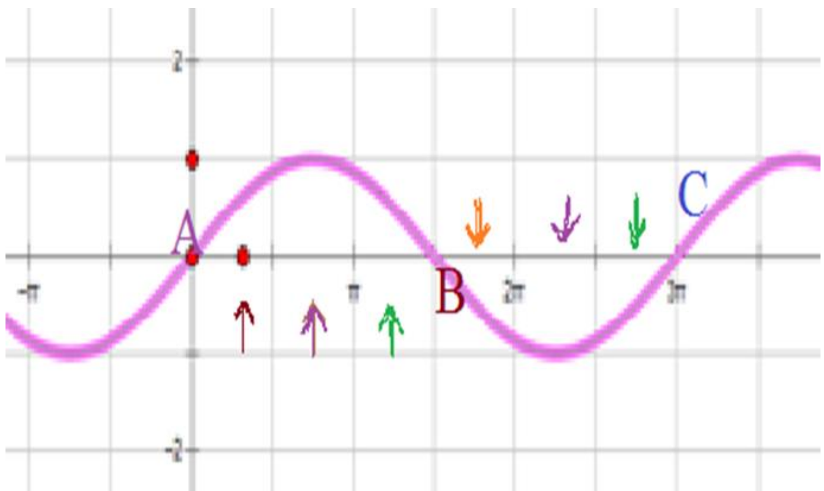


Figure 7.10. Mia counting the blocks between the points

7.1.3. Identifying the Period in Tasks 4 and 5

Recall that for Task 4, a sinusoidal curve had been shown to the participants and they were asked to assign the axis and coordinates such that it represents the graph of $f(x) = sin(4x)$. The data show that all the participants (except Mia) successfully assigned the coordinates using different strategies while approaching Task 4. Sally and Emma consistently applied their *memory image of formula* of sinusoidal functions while they determined the period in Task 4. However, the other participant, Kate changed her approach from *memory image of formula* (that she used for Tasks 2) into the *pattern* *imagery* (as she applied previously for Task 1) when completing Task 4(see Figure 7.11). She expressed:

> *"…I know that 2π is here because 1, 2, 3, 4 periods between 0,2π and here 1 and -1…"*

The above statement demonstrates that Kate also had *concrete imagery* of sinusoidal function because she knew that a canonical sine function completes one cycle in the interval [0, 2π] and the amplitude is 1. This also illustrates that Kate *perceived properties* of sinusoidal functions and she used them to *recognize relationship between the symbolic representation and the graphical representation*.

While, none of the previously used strategies helped Kate to complete successfully Task 5, in which she was asked to assign coordinates to represent $f(x) = cos(3x - \frac{\pi}{4})$, the *pattern imagery* led Rose to realize the period correctly for Task 5 (similar to Task 4). The *memory imagery* ability also supported Sally and Emma to realize correctly the period in Task 5 (similar to other interview tasks)

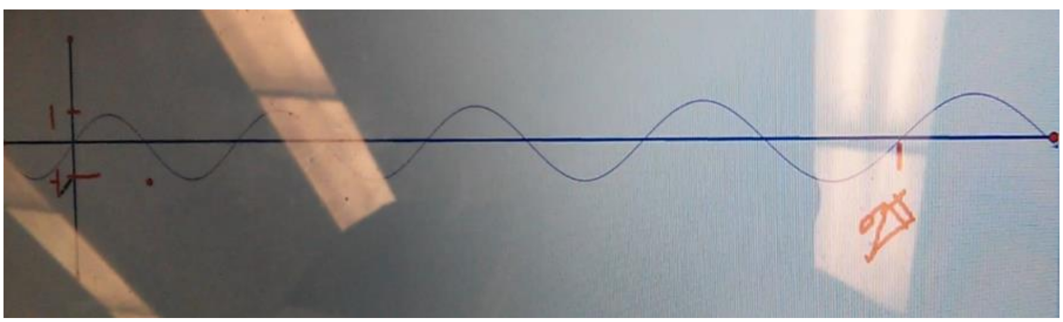


Figure 7.11. Kate placed the 4 cycles in the interval [0, 2π]

As mentioned earlier, all the participants were able to complete Task 4 successfully except Mia. When completing this task she stated:

"…I want to compress the graph, so sine starts from here and $sin(4x)$ will be half of this and this point should be a fourth so a normal sine graph ends here at 2π. In $sin(4x)$ here should be $\frac{\pi}{2}$."

As the statement reflects, Mia acknowledged that the end point of a full cycle in a canonical function is at 2π and this illustrates that she *perceived properties* of sinusoidal functions (see Figure 7.12). It also indicates that she had *concrete imagery* of sinusoidal function. However, Mia was not able *to reason on the perceived properties* of sinusoidal functions and therefore, she could not complete the task successfully. In other words, instead of assigning the axis and coordinates such that it represents the graph of $f(x) = sin(4x)$, she adjusted the coordinates for the function $f(x) = sin(x)$ (see Figure 7.12). Similarly to Kate, Mia was also unable to complete Task 5.

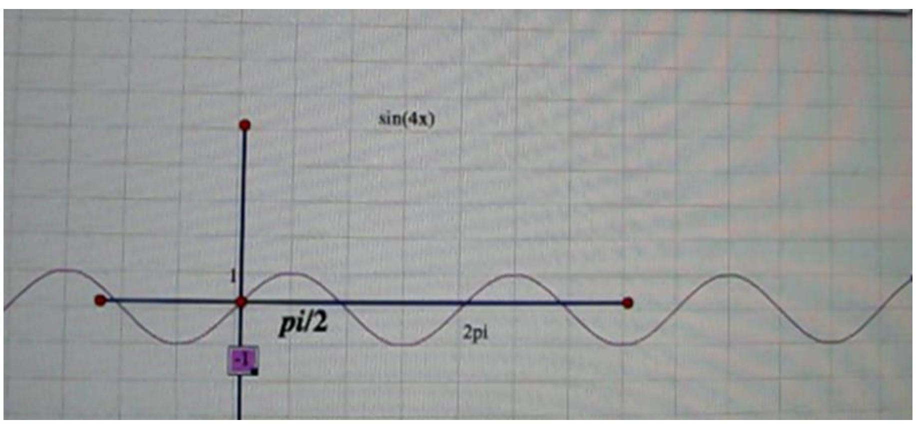


Figure 7.12. Mia misplaced the point $\frac{\pi}{2}$

7.2. Shift of Attention and Visual Imagery: Identifying the Phase Shift (C in the canonical function)

In the interview tasks (e.g., Task 3) the participants were required to recognize the phase shift/horizontal shift. From the canonical function $(f(x) = A\ sin/cos(B(x + C) + D)$ the phase shift is obtained by determining the change being made to the x-value (see Figure 7.13). The displacement will be to the left if C is positive and to the right if the phase shift is negative. For brevity, I refer to this connection as “Identifying the phase shift”.

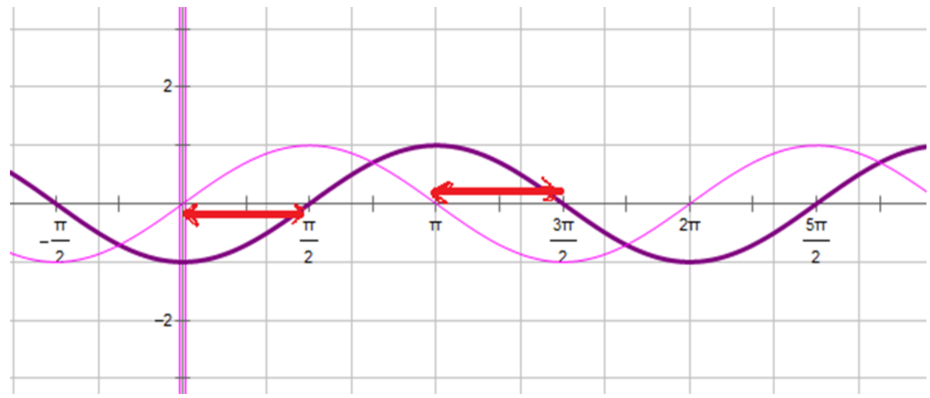


Figure 7.13. Identifying phase shift/horizontal shift

The data show that few students were able to identify the horizontal shift/phase shift when changing the sine function into the cosine functions and vice versa in the interview tasks. After successfully identifying the correct sine function ($f(x) = sin(2x)$) from the given curve in Task 1, the participants were asked to determine a cosine function for the given curve. As evident from data, the majority of participants chose the same strategy. When approaching the task, most students attempted to establish a relationship between a sine and a cosine function by adding up $\frac{\pi}{2}$ to the argument $2x$. For instance, Mia expressed:

> *“…cosine function is a flip version of sine function…it has horizontal translation to the right, so it is $f(x) = cos\left(2x + \frac{\pi}{2}\right)$.”*

From the above student’s statement one may conclude that the participants had *memory image of formula* of shifting a sine function into a cosine function because she realized that the sine and cosine functions always relate to each other by a difference of $\frac{\pi}{2}$. However, Mia (and similarly most other participants) identified incorrectly the sign of horizontal shift. In other words, while the graph shifted to the right by $\frac{\pi}{2}$, she added $\frac{\pi}{2}$ to the argument $2x$ (instead of subtracting it). The students’ discussions indicate that they have not *perceived properties* of sinusoidal functions and also have not successfully *recognized relationship between the symbolic and the graphical representation of the cosine function.*

The participants, then, were asked to describe the cosine function from the given curve (instead of focusing on the formula correctly or incorrectly). As evident from data, none of the students (except Sally) was able to determine the proper horizontal shift from the given graph. For instance after 5 minutes pausing, Rose expressed:

> *"…I know the differences between them are $\frac{\pi}{2}$….I do not know…it is shifted to the right by $\frac{\pi}{2}$…. I try to match something to match this $f(x) = cos(2x + \frac{\pi}{2})$…."*

As it is indicated from Rose's reply, she attempted to determine a horizontal shift that represents her suggested function ($f(x) = cos(2x + \frac{\pi}{2})$). However, her suggested horizontal shift again was incorrect. Even Emma, who was able to change correctly the formula of the function $f(x) = sin(2x)$ into the cosine function $f(x) = cos\left(2x - \frac{\pi}{2}\right)$, was not able to identify a cosine function from the given graph. Sally, on the other hand, completed this part of Task 1 successfully focusing on the given graph.

Unlike the other participants who relied on the formula (correctly or incorrectly) and the point $\frac{\pi}{2}$ when connecting the sine function into a cosine function, Sally focused only on the given curve in Task 1. After *holding the given curve as a whole,* Sally attempted to *discern some details* from the curve by stretching it horizontally. From there, she zoomed on the point $\frac{\pi}{4}$ on the x-axis. She stated that the curve was shifted horizontally by $\frac{\pi}{4}$, because its y-value was 1. Sally then subtracted $\frac{\pi}{4}$ from the argument (2x). She, then, considered the cosine function $f(x) = cos(2x - \frac{\pi}{4})$ for the given curve.

Observing the graph of her suggested function ($f(x) = cos(2x - \frac{\pi}{4})$) made Sally realize that the function did not fit the given curve (from Task 1). Sally expressed:

"… Ok, I think I know where my mistake was. It was that I forgot to put bracket around here…"

Recognizing her mistake led Sally to suggest successfully the function $f(x) = cos(2(x - \frac{\pi}{4}))$. Sally's success illustrates that she applied her *memory image of formula* of sinusoidal function ($f(x) = A\ cos(B(X + C) + D)$). It also shows that she was able *to* establish *a link between the visual representation of sinusoidal function and the symbolic representation.* Sally used the same approach for the rest of interview tasks when she was asked to determine alternative sinusoidal function. However, other participants consistently found difficulties in realizing the horizontal shift in the other tasks.

Similar to their initial mistakes in Task 1, the majority of students could only add up or subtract $\frac{\pi}{2}$ from the function $f(x) = sin(\frac{2}{3}x)$ in Task 2 without reflective thinking about the given graph. When asked them to focus on the given graph, participants were not able to realize the amount of radian by which the graph shifted horizontally. Kate for instance, focused her attention on the point ($\frac{3\pi}{4}$, 1) and expressed that:

"…$f(x) = cos(\frac{2}{3}x + \frac{3\pi}{4}$), because here is$\frac{3\pi}{4}$. It is shifted to the to the right…

After extending the graph, Kate zoomed on the point $\frac{3\pi}{4}$ in which the y-value was 1. The details she discerned from the graph led her to connect *the graphical representation with the incorrect symbolic representation* when focusing on the horizontal shift. The data show that Kate did not apply properly her *memory image of formula of transformation of sinusoidal functions*, because she misplaced the bracket in the suggested function (although she found a correct point from the graph). Besides, she added up the point ($\frac{3\pi}{4}$) to the argument x instead of subtracting it. The proper cosine function for the given graph was $f(x) = cos(\frac{2}{3}(x - \frac{3\pi}{4}))$*.* Similarly, other students either focused on the point $\frac{3\pi}{4}$ or $\frac{\pi}{2}$ and then added them up to the argument of the function.

The students encountered the same difficulties when attempting to determine the alternative functions in the Tasks 3, 4 and 5. To complete Task 3, for example, Rose after gazing at the graph and stretching the graph horizontally stated:

> *"….there is a transformation in x direction….the transformation… the x where starts is at ($\frac{\pi}{2}$, 1)…we should subtract this …"*

The above statement indicates that Rose focused on the point *($\frac{\pi}{2}$, 1)* which did not exist in the given graph. However because she had a *pictorial imagery* of the canonical cosine in her mind and she knew that a sine and a cosine function connecting each other with $\frac{\pi}{2}$, she subtracted $\frac{\pi}{2}$ from the arguments of the function. She, in fact, used her *memory image of formula* of sinusoidal function along with her *knowledge of properties of sinusoidal functions* without focusing on the appropriate point of given graph. The majority of students made the same error as Rose did. They just focused on the point $\frac{\pi}{2}$ and they subtracted it from the argument of the function. Meanwhile some other students extended the given graph, but they did *discern* the incorrect detail from the graph. However, they did not suggest proper phase shift and therefore a correct alternative function for Task 3. For example, Emma stated:

> *"…I think it moves this way by $\frac{\pi}{4}$…."*

Emma unsuccessfully found a point in which the graph had the y-value=1(the proper point is ($-\frac{3\pi}{4}$, *0))*. Afterward, she made the same error as Rose and most other participants did; she subtracted the incorrect point from the argument BX in the function instead of adding it to the argument x in the cosine function (because the graph shifted to the left). As evident from the data most students participated in this study mixed up both standard formula related to the transformations of sinusoidal functions together ($f(x) = A\,sin/cos(B(X+C)+D)$ and $f(x) = A\,sin/cos\,(BX+BC)+D)$), when asking to find the alternative functions for the tasks. The students' replies indicate that, in fact, the majority of participants were unable to apply their *memory image of formula* of transformation of sinusoidal functions properly. However, amongst the participants, Sally was able to determine proper alternative function for Task 3 by applying the same approach (*memory image of formula*, *discerning proper details* and *reasoning on the perceived properties of sinusoidal functions*) as the previous tasks.

Not only Sally was the only student who successfully found the alternative functions in all five-interview tasks, she determined the proper phase shifts in Tasks 3 and 5. Sally's reliance on her *memory image of formula* of sinusoidal functions supported her to *discern proper details* from the given function, for example in Task 5 ($f(x) = cos(3x - \frac{\pi}{4})$), she stated:

> *"… we first need to take out the 3 so it become $\frac{\pi}{12}$. we know it is shifted to the right by $\frac{\pi}{12}$…. So this is my normal cosine function we need to shift the graph $\frac{\pi}{12}$ to the right because that is given by formula $f(x) = cos(B(x - C))$ and C is the positive number and is $\frac{\pi}{12}$. Here is 1 and -1…."*

Identifying the amount of phase shift led Sally to assign axes and coordinates in the proper place (in order to identify phase shift in Task 3 ($f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$), Sally followed the same strategy). However, all other students were unsuccessful in identifying phase shift in Tasks 3 and 5. As an example, the *memory image of formula* (she used in previous tasks) did not help Emma to determine the horizontal shift correctly in Tasks 3 and 5. For instance when completing Task 5, she expressed:

> *"……because the amplitude is not changing so here is 1 and -1. Because the original period is 2π but here we have 3x so the period should be $\frac{2\pi}{3}$ but then we have transformation on x on the right direction. So we move the graph over by $\frac{\pi}{4}$…"*

As the above quote illustrates, Emma became confused by mixing up again conventional representations $f(x) = A\ sin/cos\ (B(x + C) + D$ (1) and $f(x) = A\ sin/cos(Bx + BC) + D$ (2). While C is the horizontal shift in the function (1) and (2), Emma made a common error as the other participants did in this task, considering BC as the shift. Whereas the phase shift was $\frac{\pi}{12}$ for the function $f(x) = cos(3x - \frac{\pi}{4})$ according to the function (2) (since $\frac{\pi}{4} = BC$ and $B = 3$ therefore $C = \frac{\pi}{12}$), Emma determined incorrectly $\frac{\pi}{4} = C$ as the amount by which the graph should be shifted horizontally. In other words, although she had *pictorial imagery* on sinusoidal functions (as she expressed "*because the amplitude is not changing so here is 1 and -1… original period is 2*π…"), she did not *perceive properties* of sinusoidal function and therefore she did not recognize the phase shift correctly. As mentioned, all the other participants made the same error as Emma did when attempting to determine phase shift in Task 3 and 5.

7.3. Covariational Reasoning ability: Identifying Period and Phase Shift

According to Carlson et. al. (2002), a students' *covariational reasoning* ability is said to reach a certain level when it supports not only the mental action associated with that given level, but also with all mental actions associated with lower levels. As explained Chapter 5, section 5.3, students' *developmental level* related to covariation could vary from *L1* (Coordination) to *L5* (Instantaneous rate). Carlson et. al. stated that at the coordination level (*L1*), the images of covariation support the mental action of coordinating the change of one variable with changes in the other variable (*MA1*). At the direction level (*L2*), the images of covariation support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions *MA1* and *MA2* are BOTH supported by *Level 2* image. *L3* includes the collection of mental actions from *M1* (coordinating the value of one variable with changes in the other) to *M3* (Coordinating the amount of change of one variable with changes in the other variable). Table 7.2 present a summary of the participants' *convariation reasoning* ability

B Changing a sinusoidal function into a cosine function and vice versa

P Identifying the period

S Identifying the phase shift

Table 7.2. Students' developmental levels

Tasks	Andy	Sally	Emma	Kate	Rose	Mia
Task1P	L3	L1/ L3	L1/ L3	L1/ L3	L1/ L3	L1/ L3
Task 1B	L1/ L3	L2/ L3	L1	L1	L1	L1
Task 2P	L3	L3	L3	L3	L3	L1/ L3
Task 2B	L1/L3	L3	L1	L1	L1	L1
Task 3P	L3	L3	L3	L1	L3	L1
Task 3S	L2	L3	L2	L2	L2	L1
Task 3B	L1	L3	L1	L1	L1	L1
Task 4P	L3	L3	L3	L3	L3	L1
Task 4B	L1	L3	L1	L1	L1	L1
Task 5P	L3	L3	L3	L3	L3	L1
Task 5S	L2	L3	L2	L2	L2	L1
Task 5B	L1	L3	L1	L1	L1	L1

As evident from data and also from Table 7.2, the students' mental actions depended on the task difficulty. Students performed at *L1* initially for Task 1 ($f(x) = sin(2x)$) (in which almost all students made similar mistakes in suggesting the function for the given curve) as they identified how the coefficient of x would change as the period changed *(M1)*. The students also stated that as the period of a sinusoidal function was decreasing (compared with the period of the canonical functions) the coefficient would decrease and it would be less than 1. One may be tempted to conclude that these students' mental action was *M2* (Coordinating the direction of change of one variable with changes in the other variable) because the students were able to coordinate the direction of change of one variable with changes in the other variable. However, students identified the direction unsuccessfully, since the period and the coefficient of the argument in the sinusoidal functions (B) had an inverse relationship (when one increases the other decreases). The same applies for *M3*. Although the students attempted to establish a connection between the amount of period and the factor of x in the sinusoidal function (*M3*), their proposed numbers for the coefficient of x were incorrect. As a result, their demonstrated developmental level was *L1* (Coordination).

Getting feedback from the graph of their suggested function $(f(x) = sin(\frac{1}{2}\,x))$ made the students realize that there is an inverse relationship between the period of the curve and the coefficient of x in the sinusoidal functions (when the period is decreasing, the coefficient of x would be greater than 1 (*M2*)). They could also identify the amount of periods and the coefficient of x correctly (*M3*). As such, upon prompting and computer feedback, they operated at *L3*.

As it has been shown in the Table 7.2, the developmental level of some participants, such as Andy and Sally, stayed consistently at *L3* (as in Task 1), while it varied for the other participants when they struggled to determine periodicity for the rest of the tasks. In other words, the students' *developmental level* varied from *L1* to *L3* when the participants attempted to identify the connection between the coefficient of x and the length of the period in the given curve (Tasks 2 and 3) or the given sinusoidal function (Tasks 4 and 5).

As described in the section 7.2 and Table 7.2, only Sally determined correctly the amount of phase shift and its influence on the function as well as its graph (*L3*) in Tasks 3 and 5. The other students (Emma, Kate and Rose) acknowledged that the curve shifted horizontally if it did not intersect the origin (0,0) and y=1 in Task 3, and if a certain amount of radian subtracted from the given function in Task 5 (*M1*). They knew that if the graph shifted to the right, a certain amount of radian should be subtracted from the argument x and vice versa in both Tasks 3 and 5 (*M2*). As such, the *covatiation reasoning* ability of Emma, Kate and Rose was at *L2* level because they could not determine the amount of phase shift correctly. However, Mia's *covariation reasoning* stayed at *L1* because not only she could not determine the direction of shifting of the curve as result of horizontal transformation, but also she could not identify the proper amount of phase shift for the Tasks 3 and 5. The majority of students did the same mistake as Mia did in Task 3 and 5, when they were required to assign the alternative sine/cosine functions for the Tasks 1-5. In fact, most participants shifted the curve incorrectly to the right when the sign of the horizontal shift was positive (If they could shift the graph correctly, their *covariation reasoning* was at *L2)*. They also did not identify the amount of phase shift correctly (If they determined the amount of phase shift successfully, their *covariation reasoning* was at *L3*). As such, the *developmental level* of the majority of students in identifying the phase shift was *L1.*

A shown in the Chapters 6 and 7, the presented tasks were challenging to the participants. They were successful to some degree, occasionally relying on a computer feedback, in identifying the periodicity of the presented functions (mostly at *L3 level*). However, the success with identifying the phase shift was rather limited (often at *L1* or *L2 level*). This main difficulty can be attributed to the inability to connect the visual representations to graphical representations of sinusoidal functions.

Chapter 8.

Discussion

In this chapter, I restate the research questions and I address the questions with respect to the findings of my study. The limitations and the difficulties I encountered during this dissertation are also described. Students' strengths and weaknesses that were revealed when completing the interview tasks support my recommendations for teaching transformations of sinusoidal functions. These recommendations are presented in the last section of this chapter.

8.1. Research Findings and Comments

In chapter 3, I reviewed the findings of prior research related to trigonometry and students' difficulties in grasping trigonometric functions. In order to deal with students' misconceptions and to help them understand the concept of trigonometric functions more efficiently, some researchers such as Tuna (2013) recommended teaching trigonometric functions in the context of graphs of functions. In accord with Tuna's suggestion, I think students would be able to sketch trigonometric functions, if they comprehend the properties of transformations of functions. Despite the significant role transformations of trigonometric functions play in grasping trigonometric functions (as Tuna (2013) proposed), there have been only sporadic research studies concentrating on this topic. Therefore, my study has shed some light on the way undergraduate students deal with transformations of sinusoidal functions.

In this research study I investigated students' *shifts of attention* as well as their forms of *visual imageries* when they attempted to match the graphical representation of functions with their analytical representations in the context of trigonometry. I was also interested in studying students' *covational reasoning* as they determined sinusoidal functions representing given graphs, and when they adjusted coordinates to match the given graphs with the presented sinusoidal functions. The difficulties students encountered when working on the tasks involving transformations of sinusoidal functions are of interest in this study. As such, the particular research questions addressed in this study are:

1) How do undergraduate students complete mathematics tasks involving the transformations of sinusoidal functions?

2) What is undergraduate students' *covational reasoning* and *visual imagery* of the transformations of sinusoidal functions?

3) How do students *shift their attention* when completing interview tasks?

4) What are the common mistakes students often encounter when they work with transformations of sinusoidal functions?

In the following section I address the research questions as I concentrate on the main themes extracted from the participants' responses to the presented tasks.

The data analysis in Chapters 6 and 7 illustrates that eventually all six undergraduate students were able to complete successfully only Task 1 ($f(x) = sin(2x)$) amongst five interview tasks. In order to complete this task, they often shifted their attention to *discerning details* of the given graph and then focused on the *reasoning on the perceived properties of sinusoidal functions*. Meanwhile, they applied their *memory image of formula*, the *pattern* as well as *concrete imagery*, when completing the task. At the first glance, one might consider students' *mental action* in grasping periodicity at advanced level, *L3* (according to Carlson's et al., (2002) *covariational reasoning* classification), since they were able to identify the period in Task 1 correctly. However, students performed differently on the other interview tasks involving periodicity (with whole number or fraction for the arguments of x).

The data show that the students' *covaritional reasoning* fluctuated when determining the periodicity of the interview tasks. Some students' *mental development* related to identifying periodicity was at the beginning *level (L1),* because the students acknowledged that there is a close relationship between the length of a full sine cycle and the periodicity of the function *(M1)*. However, they were unsuccessful in realizing that the period and the coefficient of the argument in the sinusoidal functions (B) had an inverse relationship (when one increases the other decreases, *M2*). Moreover, they were unable in determining the amount of periodicity (if they could identify periodicity, their mental action would be *M3*). However, the *covariational reasoning* of the same students shifted to the advanced level, *L3,* when completing some other tasks.

In general, the data illustrate that participants were more successful in identifying periodicity when the coefficient of x was a whole number, than when it was a fraction. As seen in Chapter 6 and 7, some students were able to determine periodicity in Task 1 and 4 (where coefficient of x was a whole number), but they encountered difficulties in completing Task 2 ($f(x) = sin(\frac{2}{3}\ x)$) and Task 3 ($f(x) = \ cos(\frac{2}{5}\ x - \frac{\pi}{5})$). For example, Kate looked at the graph in Task 2 and at the first glance she realized a full sine cycle appeared between 0 and the point 3π *(discerning details* from the given graph). Using her experience in Task 1, getting feedback from the computer and relying on the *concrete* and *memory image of formula*, however, did not support Kate to suggest the proper function for the graph in Task 2 with the fraction coefficient. Kate also hesitated to offer a function for the given graph in Task 3. She simply stated:"…*I have no idea for this graph*."

As data illustrate, according to Carlson's et al. (2002) framework, only one student's developmental level was at the advanced level (*L3*) in determining the alternative sine/cosine functions for all five interview tasks. In other words, most students were not able to identify the alternative functions for the given interview tasks and their performance was at *L1* level. As an example in Task 1 when students were asked to suggest a cosine function for the given graph, most students unsuccessfully wrote the function of $f(x) = \ cos(2x\ +\ \frac{\pi}{2})$. They knew that a cosine graph is a sine graph shifted horizontally; however, they were unable to recognize the direction of horizontal movement as well as the proper amount of the shift. The students' mistakes in the task illustrate that they were unable to determine a phase shift from the given graph. It is interesting that even some students, who applied their *memory image of the formula* of the sinusoidal functions when completing other tasks, did not use the formula to identify the phase shift in the given graph in Task 1 (when attempting to find the alternative function for the task). Even *discerning details*, *perceiving properties* of sinusoidal functions, and *reasoning on the properties* did not support them to find the *correct link* between the graph and the analytical representations and vice versa in Task 1. Similarly, the students' *covariational reasoning* stayed at *L1* level in the rest of interview tasks (Tasks 2-5), when identified the alternative functions for the tasks. Only Andy's *covariational reasoning* stayed unchanged in Task 2 (at *L3* level as in Task 1), although it moved back to *L1* from *L3* for Tasks 3, 4 and 5 (when he was asked to find the alternative functions).

Sally was the only student who identified the phase shift in Tasks 3 and 5 correctly (Recall that she also correctly found the alternative cosine/sine functions for all interview tasks). Sally, in fact, applied her *memory image of formula* as well as *dynamic* and *kinesthetic imagery* in order to match the given graph with the proper function shifted horizontally in Task 3 and 5. She performed at developmental level, *L3*. Even Andy, who determined proper alternative functions for the given graph in Task 1 and 2 (but not in the rest of tasks), did not determine phase shift in Tasks 3 and 5 successfully. As was described in Chapter 7, students often incorrectly identified $\frac{\pi}{5}$ for Task 3 (instead of $\frac{\pi}{2}$), and $\frac{\pi}{4}$ for Task 5 (instead of $\frac{\pi}{12}$). In other words, students performed poorly in both types of tasks; "*Identifying sinusoidal functions*" and "*Assigning coordinate*" tasks when completing tasks involved identifying the phase shift. In fact, the majority of participants' mental development related to realizing a phase shift is at the beginning/surface level, *L1* (according to Carlson's et al. (2002) mental action classification). One might conclude that this is the case because the majority of students, in fact, did not have a proper *memory image of formula of transformations of sinusoidal functions* in their mind. As such, they did not focus their attention on *discerning appropriate details* from the given tasks, although they *perceived properties of sinusoidal functions.* In accord with the findings of Chiu et al.,(2001) in the interview tasks involving the phase shift students, mostly misunderstood the connections between the graphical and algebraic representations.

Moreover, the students' misconceptions of recognizing phase shift affected their identification of periodicity. As mentioned previously in this chapter as well as chapters 6 and 7, determining periodicity when the coefficient of x was a fraction was a difficult task for some students. This problem is further highlighted when phase shifts are involved in the tasks. When completing Task 2 ($f(x) = sin(\frac{2}{3}x)$) in which the coefficient of x was $\frac{2}{3}$, there were some students who successfully determined periodicity, using their *memory image of formula*. However, those students became confused in identifying periodicity in Task 3 ($f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$) in which similar to Task 2, the coefficient of x was a fraction ($\frac{2}{5}$), but the amount of $\frac{\pi}{5}$ radian was subtracted from the argument x. In other words, while some students realized the given curve intersected the x-axis at the point 3π and therefore identified periodicity correctly in Task 2, determining the full length of one full cycle (the beginning and end point of a full curve) was not easy for them in Task 3 in which the sinusoid was shifted horizontally.

The results of this study in regarding identifying horizontal shift are consistent with the findings of other studies such as Oliveira (2011) and Zazkis et al., (2003). Besides students' difficulties in identifying the amount of horizontal shift, identifying the direction of the phase shift and then moving the graph to left or right accordingly was a problem for the majority of students. While the findings of previous studies indicate that participants mixed up the positive (negative) sign of the argument in a quadratic function with shifting the graph to the right (left), this problem becomes more significant in the context of sinusoidal functions. It is the case because identifying left and right shifts in a periodic sinusoidal function depends on the placement of the axes. In a periodic functions there are multiple ways of assigning coordinates on the axes to accommodate the given phase shift.

Considering students' performance on both types of interview tasks, it is evident that participants performed better in "*Identifying sinusoidal functions*" tasks compared with "*Assigning coordinate*" tasks, when students were required to determine periodicity. In other words, more students recognized period properly in the tasks a graph given and students required to identify sinusoidal function represented the graph, than in tasks in which the sinusoidal functions given and students should adjust coordinates on the given sinusoidal curve. As an example, Mia (who completed successfully Task 1, for example) assigned the axes on the given sinusoidal curve in a way that represented the graph of the function $f(x) = sin(x)$ rather than $f(x) = sin(4x)$ in Task 4. Her response to this task illustrates that she did not pay attention to the particular *details* could be *discerned* from the given function. However, due to having *concrete imagery* of a sinusoidal function in her mind, she situated the coordinates exactly at the same place as for the graph of $f(x) = sin(x)$.

In accord with the findings of Gagatsis, Elia, and Kyriakides (2003) and Lage and Gaisman (2006), it appears from the data that sometimes the task in graphical representation creates cognitive difficulties that hinder students' ability to make connections between the algebraic and the graphical representations. Meanwhile, the discrepancy in students' performance in identifying period vs. identifying phase shift illustrates that they had difficulty when considering graphs of sinusoidal functions transformed horizontally. These findings are consistent with the results of other studies by Chiu, Kessel, Moschkovich, and Munoz-Nunezby (2001) Eisenberg and Dreyfus (1994); Baker, Hemenway, and Trigueros (2000); and Consciência and Oliveira (2011) who found students' misconceptions of function transformations in general, and horizontal transformations in particular.

8.2. Contributions of the Study

As was discussed in this chapter and also previous chapters, the main focus of this dissertation is on sinusoidal functions. While some limited previous studies focused on students' difficulty in applying sine and cosine functions defined over the domain of real numbers (Challenger, 2009; Moore, 2010), on the influence of teaching trigonometric functions in context of unit circle and triangle (Brown, 2005 and Weber, 2005), on the students and teachers' difficulties in realizing angle measure (Akkoç & Akbaş Gül, 2010) and on the recognizing real numbers as radians (Orhun, 2001), this dissertation (to my knowledge) is the first research focused on students' work on the tasks involving transformations of sinusoidal functions. There are studies, such as Duval (1999) and Eisenberg and Dreyfus (1994), which examine students' difficulties in the transformation of cubic and quartic functions, however, I have not come across prior research studies that consider how students deal with tasks that require their knowledge in transforming sinusoidal functions.

In this research I focused on participants' ability to identify periodicity and phase shift in sinusoidal functions, while connecting algebraic and graphical representation of functions. There is a small number of studies, (e.g. Shama,1998 and Van Dormolen and Zaslavsky, 2003) that discuss definitions of periodicity in general. In the study by Demir (2012), students' conceptual understanding of definition of periodicity of canonical functions in the graph, and of the periodicity of a sine function ($sin(-\frac{\pi}{2})$) in the unit circle were investigated. However, my research investigated how undergraduate students completed tasks that required them to identify periodicity and phase shift from the graphs of given sinusoidal functions and to assign coordinates on the given sinusoid graph in order to match a given functions and its corresponding graph. As such, the focus and the findings of this dissertation are unique and provide a significant contribution to research in the area of trigonometry and well as in the area of function transformations.

Furthermore, my research contributed to task design. In this study I designed two types of tasks (type A and type B). While some of the prior research studies used tasks of type A, the design of tasks of type B (Assigning Coordinates) was novel and unique. None of the previous researchers required students to assign axes and coordinates so that the given sinusoidal curve would represent the given function. However, the effect of the novelty of the task on students' difficulties has to explore in further research.

8.3. Research Limitations

I recognize some limitations of my study in terms of the interview tasks, participants' familiarity with using sketchpad, as well as the number of participants.

The interview tasks I used for my research were all comprised of sinusoidal functions having fractions or whole numbers for arguments of x. However, three tasks had whole numbers for the coefficients of x and two tasks had fractions. I might have more accurate results, if I had designed more of tasks in which the sinusoidal functions comprised of fraction coefficients. This is significant since the results of this study show that students encountered more difficulties in the tasks with fraction of coefficients. Furthermore, only two tasks included a phase shift, which were challenging for the participants, when compared with the tasks involving periodicity only. These two tasks .combined phase shift and periodicity. Lack of tasks that involved a phase shift without the change in the period of the conventional sinusoid weakens the conclusions on students' difficulties in identifying a phase shift.

The use of Sketchpad gave students an opportunity to manipulate the given graphs and also to receive immediate feedback on their initial answers and therefore to correct their initial suggestions. However, it seems that the majority of students were unfamiliar with applying sketchpad in order to adjust coordinates. Task 4 and 5, in which students were required to assign the axes and place coordinates were presented at the end of interviews, when most students were tired. If I mixed the order of tasks from "Identifying sinusoidal functions" and "Assigning coordinates" students might have performed differently. I could have obtained more reliable findings if I also asked a few of the "Assigning coordinates" tasks to be completed using paper and pencil. Furthermore, although the study was done in the present of sketchpad, the specific interaction of students with sketchpad was not explored deeply in this study.

Another limitation of this study is associated with the duration of interviews. Since the interviews were one hour long, some students grew tired and they wanted to finish the interview as soon as possible no matter how appropriate their answers were.

Therefore, more accurate results could have been obtained if the interviews were conducted in two separate thirty minute sessions on two different days.

Six students from Calculus courses volunteered their time to participate in this study. Among them three were enrolled in Calculus I, where students have opportunities to refresh their knowledge about transformations of sinusoidal functions by reviewing the concepts of trigonometric functions and transformation of functions (after High School Calculus). The rest of students were from Calculus II, where they had no further chance of revising the concepts of trigonometry and transformation of trigonometric functions. The data show that the participants enrolled in Calculus I completed the interview tasks more successfully than the Calculus II students. Although it is unclear whether the stronger performance of students in Calculus I can be attributed to the more recent review of transformations of sinusoidal functions or to students' general knowledge and ability, I could have had different results if all data came from students enrolled in Calculus I.

8.4. Pedagogical Implications and Suggestions

Drawing on my findings and the fact that some of the participants in this study encountered difficulties in identifying transformations of sinusoidal functions, this concept needs to be emphasized more in the Calculus textbooks. Reviewing the Calculus textbook that has been used at this university (e.g., Barnett, Ziegler & Byleen, 2003), I noticed that there is a section in the "Functions" chapter in which horizontal and vertical transformations of functions are described, without mentioning or exemplifying transformations of trigonometric functions. In another chapter, trigonometric concepts are explained. However only the canonical sinusoidal functions are sketched. In other words, since students had limited opportunities to learn about the transformation of sinusoidal functions, their difficulties in completing the interview tasks in this study is not a surprising result (students learn transforming sinusoidal functions briefly in High School pre-Calculus). Teachers and curriculum designers need to focus on this important concept, because transformations of sinusoidal functions have application in other undergraduate courses, such as physics. Students' inability to deal with transformations of sinusoidal functions may hinder their success in other courses.

Students in my study experienced difficulties identifying phase shifts and periodicity in the given trigonometric functions. Although participants in this research were able to successfully identify some sinusoidal functions from the given graphs, they were not able to adjust coordinates on the sinusoidal curve to represent some of the given functions. This problem was highlighted even more when the coefficient of x in a given function was a fraction. However, reviewing the Calculus textbooks revealed that the authors focused mostly on whole number coefficients when included tasks on identifying functions from the given graphs. Teachers should take note of this and focus on sketching of sinusoidal functions with fractional coefficients. While the initial focus on whole number coefficients is appropriate, teachers need to enrich the repertoire of task presented to students by including fractional coefficients. This will enhance students' understanding of the connections between algebraic and graphical representation of sinusoidal functions.

After teaching periodicity, teachers can move on to the phase shift. The initial consideration of the phase shift should be for sinusoidal functions with a periodicity of 2π, if the coefficient of x is 1. That is, initial consideration of horizontal transformation should not be conflated with that of expansion or compression. Only then combined transformations should be approached.

When interview tasks in my study involved phase shift and periodicity other than 2π, the majority of students completed them unsuccessfully. This result is consistent with the prior research studies on transforming quadratic functions horizontally. This result is not surprising since, the textbook practice questions concentrate mostly on the phase shift tasks for functions having the argument of x 1 (periodicity is 2π). In these tasks students identify the phase shift more easily. Thus, teachers need to focus on the phase shift tasks in functions with periodicity other than 2π, and involving students with more practice questions in which phase shift is connected to both analytical and graphical representations.

To improve students' conceptual understanding of the transformations of sinusoidal functions, it is very important that teachers pay attention to the students' shifts of attention. Relying on the findings of this study, students are often able to *discern details* from the given function of a graph, they *perceive properties of a sinusoidal function*, but they are not able to *reason on the properties*. Teachers might shift students' attention from *details to reasoning about properties of sinusoidal functions* by asking students to describe their thinking. Using this skill would help teachers evaluate students' understanding of the transformation concept and try to improve students' learning.

8.5. Reflection of this Journey

This research journey greatly influenced the way I was thinking about teaching and learning of mathematics, and in particular transformations of sinusoidal functions. Before doing this research, I have never thought about the influence of students' imagery on their learning of mathematics. I now recognize that the students' visual imagery plays an important role in their conceptual understanding of transformations of functions. In other words, I find it important to pay attention to the students' body movements and their mental images, which might be sketched in a paper or in the computer screen when completing the mathematics tasks.

Applying Carlson's et al. (2002) *covariational reasoning* framework in this study made me realize that it significant to get insight into undergraduate students' verbal expressions related to sinusoidal functions. I note that providing students with opportunities to participate in class conversations would let me as a teacher to identify the level of their mental actions and therefore to modify my instructional approaches accordingly, where needed.

Conducting this study also changed my research approach significantly. In my previous research I relied on a single theoretical framework. I first found it a difficult task to analyze data collected for this study with respect to three different theoretical frameworks. However, I became more and more comfortable to interpret data according to each framework and this gives me richer results and a broader view about undergraduate students’ grasping of the topic. This research provided me an opportunity to enjoy doing research in one of the most difficult topics in Calculus. This motivates me to engage in future studies focusing on undergraduate mathematics topics.

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