A multi-state model for a life insurance product with integrated health rewards program

by

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Abstract

With the prevalence of chronic diseases that account for a significant portion of deaths, a new approach to life insurance has emerged to address this issue. The new approach integrates health rewards programs with life insurance products; the insureds are classified by fitness statuses according to their level of participation and would get premium reductions at the superior statuses. We introduce a Markov chain process to model the dynamic transition of the fitness statuses, which are linked to corresponding levels of mortality risks reduction. We then embed this transition process into a stochastic multi-state model to describe the new life insurance product. Formulas are given for calculating its benefit, premium, reserve and surplus. These results are compared with those of the traditional life insurance. Numerical examples are given for illustration.

Keywords: Health rewards program; Mortality risks reduction; Life insurance; Multi-state model; Markov chain process

Dedication

To my beloved parents!

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Chapter 1

Introduction

1.1 Background knowledge of health issues

One of the biggest threat to people's life and health in the 21st century is non-communicable diseases (NCDs). Also known as chronic diseases, NCDs are of long duration and slow progression. Four main types of NCDs are cardiovascular diseases, cancers, chronic respiratory diseases and diabetes. Though not as fatal as acute infectious diseases, NCDs collectively contribute to nearly 63% of deaths worldwide and greatly impact the well-being of people's life (Bloom et al., 2011). The World Health Organization has projected that deaths caused by NCDs will increase by 15% globally from 2010 to 2020, to cause 44 million deaths by 2020 (Alwan, 2011). As more and more people spend their lives in poor health suffering from chronic conditions, the economic burden has been soaring over the decades.

Among other factors that cause the prevalence of NCDs, lifestyle choices drive 80% of the disease burden and 60% of mortality (Discovery, 2015). Compared to other forces such as ageing and heredity, unhealthy lifestyle is one of the modifiable risk factors. For example, tobacco use, unhealthy diet, physical inactivity and harmful use of alcohol can largely increase the risk of NCDs. Since these behavioural factors are preventable, opportunities exist to lessen the economic and health burden by reducing the risk factors associated with these diseases. By addressing the aforesaid modifiable behavioural risk factors, we would effectively reduce the incidence of NCDs. Other solutions include vaccines, early detection, adherence to chronic diseases medication. Evidences have shown that these interventions are a cost-effective economic investment. If applied early, the expense of these preventions and interventions largely reduce the need for more expensive treatment.

However, armed with better knowledge of how to lead a healthy lifestyle and how to reduce the risks harming one's well-being, we are still slow to adopt healthy practices. Studies rooted in behavioural economics could bring some insight into this phenomenon, highlighting how economic decisions are made and which mechanisms drive individual's choices. Present bias, an important concept in behavioural economics, defines human's tendency to prefer short-term wins over long-term benefits. Individuals often require additional incentive to make better decisions regarding their longer-term health. Wellness incentive programs are a practical tool to provide immediate rewards for behaviours to encourage healthy living that affects long-term outcomes.

1.2 Overview of existing solutions

To lower the health care cost and also to increase employees' morale, retention and productivity, workplace wellness programs emerged and became increasingly popular. Discovery Limited, a South Africa-based financial services group, launched "Vitality" in 1997, an innovative wellness program; the participants are encouraged, supported and rewarded to have health-enhancing activities such as eating healthy and doing exercises.

Emerging in the recent past, innovative technologies bring unprecedented levels of convenient information access and personal data records to the public at large. For example, wearable technologies such as activity trackers and smartwatches enable dynamic records of physical activities, online data collection through mobile apps and statistics sharing via social media. The interaction among populations and groups brings extra motivation to keep on going towards fitness goals of individuals.

Governments have begun to make efforts to encourage individuals to take initiatives in proactive behaviours towards health and well-being. In early 2016, Canada launched its first wellness program in the province of British Columbia (BC). Thanks to a new mobile app called Carrot Rewards, BC residents are rewarded with loyalty points of selected partners by completing activities focused on healthy living. The activities include completing a health profile, participating in learning activities and quizzes, and in later phases going to the gym, visiting a flu clinic, or buying produce from the grocery store. The app is also designed to link up with wearable devices in later stages to track daily workouts data.

As the development of new technologies makes it possible to collect dynamic data for a variety of health improvement activities, there is a growing interest for insurance companies to introduce new products by integrating health promotion programs into life insurance. In 2000, Discovery Life launched an innovative product that integrates their wellness program into life insurance products for Vitality members; the more Vitality points one earns, the higher one's rewards level and the larger premium discount is applied.

Insurance businesses with a built-in Vitality program started to gain a widespread popularity in areas of Africa, Japan, Australia and the United Kingdom. AIA launched Vitality Program in Australia in March 2014. In 2015, John Hancock Life Insurance Company (U.S.A.), a Manulife's subsidiary, introduced a whole new shared value approach to their life insurance products by integrating with the health promotion Vitality program. Manulife Financial Corp. also launched a similar insurance program "Manulife Vitality" in 2016, which essentially offers Canadians discounts for healthy behaviours. Sumitomo Life Insurance Co. started Japan Vitality project jointly with Discovery Ltd. and SoftBank Corp. in July 2016, introducing insurance products incorporated with Vitality into the Japanese life insurance market.

1.3 Motivations

To keep up with the growing popularity of life insurance with an integrated health rewards program, comprehensive studies from an actuarial perspective of the insurance products are needed. In this project, we intend to propose a statistical model which enables the modelling of the dynamic transitions of individuals' status in the wellness program. The statuses are closely related to the performance of healthy behaviours, and thus represent the health conditions of the members. We embed this into a multi-state model to describe the state transitions. Under this framework, we could analyze the new life insurance products from many aspects.

Traditional insurance products view the mortality risk as static. However, diseases and mortality risks are variable depending on one's nutrition intake, physical activities, smoking and drinking habits, etc. The traditional insurance policies charge constant premiums from policyholders and pay the benefit upon disease and death. This product design presents no incentives for clients to improve their health and reduce mortality. Moreover, potential problems exist such as more hospital visits than necessary and negative attitudes towards chronic disease prevention. The new insurance products, on the other hand, measure the health risk of a policyholder dynamically on an on-going basis. This is not only more realistic and reasonable, but also serves to encourage the clients to proactively improve their health and reduce risks. This innovation would benefit participants by bonuses and rewards, reduction on mortality risks, and overall better health. For insurers, it means fewer claims, lower lapses, and advantageous selection.

A comparison between this new type of life insurance with the traditional life insurance product would give an idea of the strengths and weaknesses of the new product to both insureds and insurers. If future studies would be able to point out the reasons behind those drawbacks and advantages, it would shed light on how to design the insurance products to make them more profitable while staying attractive to customers. From the insurer's perspective, they may avoid potential harm to the business, and make the advantages to the best. From insureds' point of view, if the new products are proved to be more affordable thanks to the improved mortality rates, more people would be persuaded to make a purchase and to join the wellness program. By the power of the market, effort could be made by a majority of the population to address the concerns of the chronic diseases.

1.4 Literature Review

The natural alignment of the benefit for life insurance and health improvement has long been recognized. The idea of integrating health promotion with life insurance actually has deep historical roots. As far back as the early 1900s, researchers and industry leaders studied and outlined methods to promote healthier living and extend healthy life expectancy. For example, it was pointed out by Life Extension Institute, Inc. (1919) that

"We see an opportunity not only for important financial returns, materially lowering the cost of insurance, but concomitantly what we consider more important, opportunities for making valuable contributions to the health of the policyholders and to the vitality of the nation."

However, it was found that "very little progress has been made in delivering life insurance products that reflect or actively incentivize the goal of improving healthy life expectancy." (see John Hancock Life Insurance Company (2015)). Premium and benefit calculations were based on static individual risk information at inception. Along with the development of theories and technologies in medical science, inventions are made relating to systems and methods for pricing insurance products based on evaluation of insured members' compliance in health-promoting measures. Sirmans (2004) applied patent for this invention. Policyholders' participation in wellness promotion program is continuously monitored and used as a basis for establishing incentives. For example, exercise and activity monitors are worn by insured members to record their heart rate, type of activity, exercise intensity and duration, etc. These records together with other measures such as physical examination results can be used to determine rewards and incentives to insured members. Incentives and rewards include reductions on life insurance premiums, subsidies on membership fees for health club, points earning on loyalty cards of the collaborative brands. Such insurance products have appeared in the market. Detailed policy features and participation rules are described in insurers' manuals and reports (see Discovery Vitality Ltd. (2016) and John Hancock Life Insurance Company (2015)).

Along with this rising popularity, discreet reflections and critical thinking are made on the controversial issues of incentive health promotion programs. Gorin and Schmidt (2015) argued that extrinsic rewards can "crowd out" intrinsic motivation, leading to an overall reduction in healthy behaviours in the long run. Sharing personal information and health data also raises concerns of privacy protection. The issue was discussed by Slomovic (2015).

No research papers are found in the actuarial literature devoted to modelling and analyzing such insurance products. To develop a statistical framework for the new life insurance, models and methodologies of the related works in the existing literature are reviewed.

Long before the fitness rating in wellness programs was adopted in life insurance, rating systems have been developed in automobile insurance to encourage policyholders to drive carefully and better assess individual risks. Known as Bonus-Malus systems, policyholders who are responsible for accidents are penalized by premium surcharges and claim-free policyholders are rewarded by discounts. Lemaire (2012) designed statistical tools to analyze and evaluate 30 Bonus-Malus systems in 22 countries. Markov chain process is one of the most popular models to describe the transition among different rankings.

As examination and assessment of mortality risks become more professional and delicate, further classification of mortality risks among survivors becomes possible. Multi-state models are developed to model life insurance where insured individuals at any time occupy one of a few possible states. Hougaard (1999) reviewed the application of multi-state models in survival data subject to different levels of mortality. Kwon and Jones (2008) presented a discrete-time, multi-state model for life insurance with mortality rates influenced by risk factor changes; it allowed a more accurate description of dynamic changes of mortality risks. Xia (2012) analyzed a long-term disability insurance portfolio with a multi-state model, where the states transitions are modelled by a continuous-time Markov process.

Nolde and Parker (2014) defined two types of insurance surplus, accounting surplus and stochastic surplus. They examined the stochastic behaviour of the insurance surplus over time for a portfolio of homogeneous life policies. Cash flow method is adopted by the authors to calculate the moments of the surplus. Xia (2012) also applied the cash flow method in calculating the moments of present value of future benefit payments for policies under a multi-state model setting. Formulas for moments of cash flows are derived for various cases appeared in a multi-state stochastic framework.

To understand the effect of health promotion program on life-expectancy improvement, the dependence of mortality rates on behavioural risk factors is studied by researchers in many organizations and institutes. A report, by the World Health Organization (2002), described the amount of disease, disability and death in the world today that can be attributed to a selected number of risk factors. It also showed how much this burden could be lowered in the next 20 years if the same risk factors were reduced. Mathers et al. (2008) reported their studies on the global burden of disease, providing a comprehensive assessment of the health of the world's population. It also provided detailed global and regional estimates of premature mortality, disability and loss of health for 135 causes by age and sex. Ezzati et al. (2004) carried out a more extensive analysis of the mortality and burden of disease attributable to 26 global risk factors using a consistent analytic framework known as Comparative Risk Assessment (CRA). Detailed tables for individual risk factors were provided.

1.5 Outline

The rest of the project report is organized as follows. Chapter 2 introduces the Markov chain process to model the transition of health rewards systems. Application of the model is illustrated on two types of fitness rating system appeared in the market. In Chapter 3, a multi-state model is tailored to have the health rewards system embedded, which together enable the modelling of the new life insurance product. Expressions are derived for future benefits, premiums and reserves of the new life insurance. It then introduces the cash flow method to define the surplus of the new life insurance. Computing formulas for the moments of surplus are derived. Chapter 4 compares the traditional life insurance with the new product from the aspects of total liability, premiums and reserves. In Chapter 5, numerical examples are given for illustration. Applying the formulas derived in Chapters 3 and 4, theoretical results are tested and numerical analyses of the comparisons are conducted for further insight.

Chapter 2

Health Rewards System

A typical health rewards system in a wellness program integrated in life insurance includes several components. To make a plan work, it needs pre-determined standards of performance in participation, fitness status rating system measuring the health conditions of members and financial incentives. These standards and measurements are based on scientific results and experts' knowledge. Members would accumulate points on meeting the requirements of the established protocols. More points lead to higher status in the fitness rating system and better bonuses. The rationale under this design is that the fitness rating system classifies the policyholders by their health conditions. People with different health conditions are subject to different levels of mortality risks, which determine the probability of life insurance claims.

In practice, performance of participation is assessed on many aspects. A member can do exercise, quit smoking, go on a healthy diet and do regular examinations to earn points in the fitness rating system. To simplify the model in this project, we consider only one type of health-promoting behaviour in the wellness program, physical activity. This simplification also helps to link the health improvement with the reductions on the attributable mortality risk due to physical inactivity.

Table 2.1 gives a made-up example of a fitness rating system of a health rewards program. According to the established protocols, the fitness rating system divides the members into five different statuses named as Blue, Bronze, Silver, Gold and Diamond based on their performance. The fitness levels from 0-4 correspond to health conditions from poor to good. Members earn points by executing plans of exercise with the required intensity and duration. The points required for each status are displayed in the table.

Table 2.1: Example: a five-status fitness rating system

Status	Blue	Bronze	Silver	Gold	Diamond
Level #	0	1	2	3	4
Points	≤ 1000	$1000 \sim 2000$	$2000 \sim 3000$	$3000 \sim 5000$	≥ 5000

Participants are classified into different statuses once they enter the health rewards program, based on the information such as medical history and reported habits of exercise. Starting from the second year, their fitness statuses are reassessed every year based on the performance in the last year. Different statuses offer different levels of rewards.

Members at each status naturally have their own behavioural pattern of keeping and changing the exercise habits, resulting in different probabilities of transferring to each status in the next year. Generally, members at superior statuses are more likely to meet higher standards and gain more points, and hence have higher probability of transferring to superior statuses than those at inferior levels. We believe that for a general population, there is a hidden transition process among different statuses reaching its equilibrium state, which keeps the average mortality risks at the level suggested by the life table. The health rewards program is designed to encourage more people to develop and keep better exercise habits, and thus increases the probability of transferring to higher statuses over all the statuses. The transition probabilities can be estimated in practice by observed data.

2.1 Assumptions

To model the transition process of the fitness statuses in a health rewards program, we make the following assumptions:

- Members of the program are independent from each other regarding their behaviour of the health activities.
- A Member's probabilities of meeting the requirements (points) for each status depend only on his/her current status.

2.2 Markov chain process and transition probability

Inspired by the Markovian modelling framework for bonus-malus systems of automobile insurance, we adopt the Markov chain process to describe the transition process of the health rewards program. Suppose that the fitness rating system has L levels, $0, 1, 2, \cdots, L-1$. Let L_t be the fitness level occupied by a participant in the health rewards program during the $(t+1)$-th year. The status trajectory of the participant is modelled by a sequence $\{L_0, L_1, L_2, \cdots.\}$ of random variables valued in $\{0, 1, \cdots, L-1\}$. It starts from some level l_0, that is $L_0 = l_0$.

As we have described in Section 2.1, the transition of a participant within the system depends only on the his/her current status. In this sense, the process $\{L_0, L_1, L_2, \cdots.\}$ satisfies the key property of Markov chain process, that is, conditioning on knowing the current level, the future trajectory is independent of the statuses occupied in the past. We model the fitness status transition process by a discrete-time Markov chain process. We further assume that it is a time-homogenous Markov chain process, which means that the transition probabilities are independent of time and of age of the participants.

By Markov property, the probability distribution of L_{t+1} depends only on the current level L_t and not on previous levels, namely,

$$Pr[L_{t+1} = l_{t+1} | L_0 = l_0, L_1 = l_1, \cdots, L_t = l_t] = Pr[L_{t+1} = l_{t+1} | L_t = l_t].$$

Let $p_{l_1 l_2}$ be the probability of moving to level l_2 given the current level is l_1, that is

$$Pr[L_{t+1} = l_2 | L_t = l_1] = p_{l_1 l_2},$$

where $l_1, l_2 \in \{0, 1, 2, \ldots L-1\}$, $p_{l_1 l_2} \geq 0$, satisfying

$$\sum_{l_2=0}^{L-1} p_{l_1 l_2} = 1. \tag{2.1}$$

The one-step transition matrix P is then expressed as

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,L-1} \\ p_{10} & p_{11} & \cdots & p_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L-1,0} & p_{L-1,1} & \cdots & p_{L-1,L-1} \end{bmatrix}.$$

By (2.1), we get one property of the transition matrix P,

$$P \cdot \mathbf{1} = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,L-1} \\ p_{10} & p_{11} & \cdots & p_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L-1,0} & p_{L-1,1} & \cdots & p_{L-1,L-1} \end{bmatrix} \cdot \mathbf{1} = \mathbf{1},$$

where $\mathbf{1} = [1, 1, \cdots, 1]^T$ is a L dimensional column vector with all entries equal to 1.

The probability of a trajectory $\{L_0 = l_0, L_1 = l_1, \cdots, L_n = l_n\}$ is given by

$$\begin{aligned} Pr[L_1 = l_1, \ldots, L_n = l_n | L_0 = l_0] = \; & Pr[L_1 = l_1 | L_0 = l_0] \cdot Pr[L_2 = l_2 | L_1 = l_1] \cdot \ldots \\ & \cdot Pr[L_n = l_n | L_{n-1} = l_{n-1}] \\ = \; & p_{l_0 l_1} \times p_{l_1 l_2} \cdots p_{l_{n-1} l_n}. \end{aligned} \tag{2.2}$$

The n-step transition probability can be computed as the n-th power of the transition matrix, P^n. If the Markov chain process is irreducible and aperiodic, then the limit of P^n

exists and there is a unique stationary distribution $\boldsymbol{\varpi}$. $\boldsymbol{\varpi}$ is a vector satisfying

$$\begin{aligned}
&0 \le \varpi_i \le 1, \quad i = 0, 1, \cdots, L-1; \\
&\sum_{i=0}^{L-1} \varpi_i = 1; \\
&\boldsymbol{\varpi}^T \cdot P = \boldsymbol{\varpi}^T.
\end{aligned}$$

If the status transition process modelled by the Markov chain process turns out to have a stationary distribution, we would expect this distribution to have longer tail on higher statuses than the stationary distribution of the general population. The difference is the effect of the health rewards program, which has encouraged a larger portion of the program members to maintain a good exercise habit.

2.3 Examples of fitness rating systems

There are mainly two types of fitness rating system adopted by the existing health rewards programs in the market. Type I leads to status transitions naturally satisfying the Markov property, while Type II does not. However, we could transform the transition process under Type II rating system and make it fit with the model of Markov chain process.

2.3.1 Type I fitness rating system

A type I fitness rating system is adopted by the wellness program launched in Australia, by a life insurance company AIA (see AIA Australia Limited (2015)). The status transition is decided by the accumulation of points in the past one year only. Based on the assumptions we made at the beginning of Chapter 2, the one-year status transition depends only on the current status. A made-up example of it is presented in Table 2.1. The transition among fitness levels is described in Table 2.2.

Table 2.2: Transition of type I fitness rating system

Current levels	Levels after earning k points in the year				
	$k \le 1000$	$1000 \le k \le 2000$	$2000 \le k \le 3000$	$3000 \le k \le 5000$	$k \ge 5000$
0	0	1	2	3	4
1	0	1	2	3	4
2	0	1	2	3	4
3	0	1	2	3	4
4	0	1	2	3	4

The status transition process can be directly described by a Markov chain process, represented by its transition matrix P_I:

$$P_I = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} & p_{04} \\ p_{10} & p_{11} & p_{12} & p_{13} & p_{14} \\ p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\ p_{30} & p_{31} & p_{32} & p_{33} & p_{34} \\ p_{40} & p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix}.$$

2.3.2 Type II fitness rating system

Type II fitness rating system is applied in the original Vitality program in South Africa (see Discovery Vitality Ltd. (2016)). An example of it is shown by Table 2.3. The status transition to status Diamond, is determined by 4-year history of points accumulation. As we have assumed, the probability of achieving the goal of earning more than 3000 points in one year depends on the current status. The transition from level 3 to level 4, however, depends not only on the current status but also on the previous 3-year history of the status, which does not satisfy the Markov property by itself.

Table 2.3: An example of type II fitness rating system standards

kStatus	Blue	Bronze	Silver	Gold	Diamond
Level #	0	1	2	3	4
Points	≤ 1000	$1000 \sim 2000$	$2000 \sim 3000$	≥ 3000	≥ 3000 for 4 consecutive years or longer

Applying the techniques used in Bonus-Malus system in car insurance(see Lemaire (2012)), we could transform the transition into a Markov chain process. We introduce fictitious sub-levels between level 3 and level 4 to meet the Markov property. Table 2.4 demonstrates the transformed transition among different levels based on points accumulation of one year.

Table 2.4: Transformed transition of type II fitness rating system

Current levels	Levels after earning k points in the year			
	$k \leq 1000$	$1000 \leq k \leq 2000$	$2000 \leq k \leq 3000$	$k \geq 3000$
0	0	1	2	3.1
1	0	1	2	3.1
2	0	1	2	3.1
3.1	0	1	2	3.2
3.2	0	1	2	3.3
3.3	0	1	2	4
4	0	1	2	4

The transformed status transition can now be modelled by a Markov chain process, whose transition matrix is denoted by P_{II} as

$$P_{II} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{0,3.1} & 0 & 0 & 0 \\ p_{10} & p_{11} & p_{12} & p_{1,3.1} & 0 & 0 & 0 \\ p_{20} & p_{21} & p_{22} & p_{2,3.1} & 0 & 0 & 0 \\ p_{3.1,0} & p_{3.1,1} & p_{3.1,2} & 0 & p_{3.1,3.2} & 0 & 0 \\ p_{3.2,0} & p_{3.2,1} & p_{3.2,2} & 0 & 0 & p_{3.2,3.3} & 0 \\ p_{3.3,0} & p_{3.3,1} & p_{3.3,2} & 0 & 0 & 0 & p_{3.3,4} \\ p_{40} & p_{41} & p_{42} & 0 & 0 & 0 & p_{44} \end{bmatrix}.$$

Chapter 3

Multi-state Model

Adding death as a terminal state, our health rewards system can be naturally embedded into a multi-state model. Mutual transitions among several healthy states are allowed, while the transition from death to other states is disallowed. States 0 to $L-1$ in the multi-state model correspond to fitness rating levels 0 to $L-1$, and state L is death. Figure 3.1 shows the diagram of an example when $L = 5$.

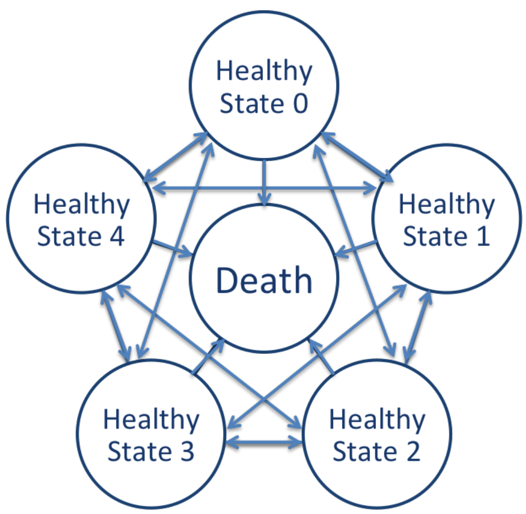


Figure 3.1: Example: Multi-state model with $L = 5$

3.1 Basic assumptions

Before studying the model in details, we make several assumptions below.

- Individuals of the wellness program group have independent mortality risk, which is determined by their status in the fitness rating system, age and gender (if applicable).

- Survivors transfer to different health statuses each year according to the transition process described in Chapter 2.
- We suppose that people at state $L-1$ have completely removed the attributable fraction of mortality risk due to the given risk factor (in our case, the physical inactivity).
- For the purpose of simplifying the illustration and focusing on the study of new product features, we assume that the annual interest rate is constant throughout the study period. That is, the discount factor v is set as a constant.
- Expenses and other possible sources of decrements (e.g., lapses) are ignored.

3.2 Mortality risks in the health rewards program

The statuses in the fitness rating system represent different levels of mortality risk. As assumed, participants at Diamond status, the highest fitness level, have the lowest mortality rates while those at the starting status have the highest mortality rates. The states 0 to $L-1$ in the multi-state model suggest the order of mortality risk. State 0 has the highest mortality risk, while State $L-1$ has the lowest mortality risk.

For the general population, an unknown distribution of fitness statuses exists. This unknown distribution of the general population gives an average mortality rate suggested by the life table. Basically, a life table used to price traditional life insurance products measures the average mortality risk of the general population. For convenience, we call the mortality risk suggested by the life table the normal mortality. Naturally, in the general population, some bear a higher than the normal mortality risk while others have a lower mortality risk than the normal level. We call states with higher than normal mortality rates the unfit states, while states with lower than normal mortality rates the fit states.

As assumed, state $L-1$ completely removes the attributable portion of mortality rate. We assume that people in the fit states, except the state $L-1$, have partially removed the attributable fraction of mortality rates. People in the unfit states, on the other hand, have "negative" elimination of the removable fraction of mortality rate. That is, a percentage of attributable fraction of the mortality rate is added on to the normal mortality rate. The precise estimate of the partially reduced (or additional) attributable mortality at each state relies on conclusions of scientific research in the related fields. In this project, we simply assume that the partial reductions and additions of attributable mortality rate precisely reflect the mortality risk in each state.

In order to describe these conditions and assumptions, we introduce the following notations.

- q_x: the normal mortality rate at age x in the life table of the general population.
- q_x^i: the mortality rate of an x-year-old insured at fitness level $i(i = 0, 1, \ldots, L-1)$.

- δ_x: the attributable fraction of mortality rate at age x due to a given health risk factor. We assume that $0 \leq \delta_x < 1$ for all x.
- θ_i: the percentage of removed attributable fraction of mortality of state i ($i = 0, 1, \ldots, L-1$). We let $\theta_0 < 0$, $\theta_{L-1} = 1$ and $\theta_i < \theta_j$, for any $i < j$. For negative θ_i, we assume that $1 - \delta_x \cdot \theta_i \leq 1/q_x$, for all age x.
- $\{\omega_i\}_{i=0}^{L-1}$: the initial distribution of statuses for the wellness program group, satisfying the condition that $\sum_{i=0}^{L-1} \omega_i = 1$.

In the multi-state model, we assume that the mortality rate of a policyholder aged x in state i is

$$q_x^i = q_x \cdot (1 - \theta_i \cdot \delta_x),$$

for $i = 0, 1, \ldots, L-1$. The assumptions on θ_i and δ_x suggest the following properties:

$$\begin{aligned} q_x^{L-1} &= q_x \cdot (1 - \delta_x), \\ q_x^0 &= q_x \cdot (1 - \theta_0 \cdot \delta_x) > q_x, \end{aligned}$$

and for any $i < j$,

$$\begin{aligned} q_x^i &= q_x \cdot (1 - \theta_i \cdot \delta_x) \\ &> q_x \cdot (1 - \theta_j \cdot \delta_x) \\ &= q_x^j. \end{aligned}$$

Basically, we have $q_x^0 > q_x^1 > \cdots > q_x^{L-1}$. For mortality rate of the fit states, we have $\theta_i \geq 0$, implying

$$q_x^i = q_x \cdot (1 - \delta_x \cdot \theta_i) \leq q_x,$$

while for mortality rates of the unfit states, we have $\theta_i < 0\%$, resulting in

$$q_x^i = q_x \cdot (1 - \delta_x \cdot \theta_i) > q_x.$$

Since q_x^i is a mortality rate, it has to be within the range of $[0, 1]$. When q_x and δ_x are large enough, it could happen that $q_x \cdot (1 - \delta_x \cdot \theta_i) > 1$, which we assume would not occur within the age range of our interest.

Let $Q_x = [q_x^0, q_x^1, \ldots, q_x^{L-1}]^T$ and $\Theta = [\theta_0, \theta_1, \ldots, \theta_{L-1}]^T$. Then

$$\begin{aligned} Q_x &= [q_x^0, q_x^1, \cdots, q_x^{L-1}]^T \\ &= [q_x \cdot (1 - \delta_x \cdot \theta_0), q_x \cdot (1 - \delta_x \cdot \theta_1), \cdots, q_x \cdot (1 - \delta_x \cdot \theta_{L-1})]^T \\ &= q_x \cdot \mathbf{1} - q_x \cdot \delta_x \cdot \Theta, \end{aligned} \tag{3.1}$$

where $\mathbf{1} = [1, \cdots, 1]^T$, as has been mentioned in Chapter 2.

For any vector $\mathbf{f} = [f_0, f_1, \dots, f_{L-1}]^T$ representing a distribution of statuses, we define the average reduction factor of the attributable mortality risk for a population with status distribution as $\mathbf{f}$, denoted by $\Delta_\Theta(\mathbf{f})$, as

$$\Delta_\Theta(\mathbf{f}) = \frac{q_x - \mathbf{f}^T \cdot Q_x}{q_x \cdot \delta_x}, \tag{3.2}$$

where Θ represents the established classification of fitness statuses, $\mathbf{f}^T \cdot Q_x$ is the average mortality rate of the population with status distribution $\mathbf{f}$. Replacing Q_x by (3.1) and noting that $\sum_{j=0}^{L-1} f_j = 1$, we get

$$\begin{aligned} \Delta_\Theta(\mathbf{f}) &= \frac{q_x - \mathbf{f}^T \cdot (q_x \cdot \mathbf{1} - q_x \cdot \delta_x \cdot \Theta)}{q_x \cdot \delta_x} \\ &= \frac{(q_x - \mathbf{f}^T \cdot q_x \cdot \mathbf{1}) + \mathbf{f}^T \cdot q_x \cdot \delta_x \cdot \Theta}{q_x \cdot \delta_x} \\ &= \mathbf{f}^T \cdot \Theta, \end{aligned} \tag{3.3}$$

which is a function of $\mathbf{f}$ and is independent of age x.

Proposition 3.2.1. *For any given distribution vectors $\mathbf{f}_1$ and $\mathbf{f}_2$, we have that*

$$\mathbf{f}_1^T \cdot Q_x \leq \mathbf{f}_2^T \cdot Q_x \quad \Leftrightarrow \quad \mathbf{f}_1^T \cdot \Theta \geq \mathbf{f}_2^T \cdot \Theta,$$

where $\mathbf{f}_1$ and $\mathbf{f}_2$ are interpreted in this project as the distribution of fitness status of their respective populations.

Proof. By the definition of $\Delta_\Theta(\mathbf{f})$ in (3.2) or (3.3), we have

$$\begin{aligned} & \mathbf{f}_1^T \cdot Q_x \leq \mathbf{f}_2^T \cdot Q_x \\ \Leftrightarrow \quad & \frac{q_x - \mathbf{f}_1^T \cdot Q_x}{q_x} \geq \frac{q_x - \mathbf{f}_2^T \cdot Q_x}{q_x} \\ \Leftrightarrow \quad & \Delta_\Theta(\mathbf{f}_1) \geq \Delta_\Theta(\mathbf{f}_2) \\ \Leftrightarrow \quad & \mathbf{f}_1^T \cdot \Theta \geq \mathbf{f}_2^T \cdot \Theta. \end{aligned}$$

□

Considering situations in practice, we make further assumptions on the initial distribution of fitness statuses of the group of participants.

Assumption 3.2.2. *For a given group of participants with initial distribution of fitness status* $\Omega = [\omega_0, \omega_1, \cdots, \omega_{L-1}]^T$*, we assume*

$$\sum_{i=0}^{L-1} \omega_i \cdot q_x^i \leq q_x, \quad \text{for all age } x. \tag{3.4}$$

The assumption (3.4) can be expressed in vector forms, which is,

$$\Omega^T \cdot Q_x \leq q_x, \quad \text{for all age } x \ . \tag{3.5}$$

The practical meanings of (3.4) is that the participants of the integrated wellness program in the new life insurance has an average mortality rate at issue which is less than or equal to the normal level. Before insurers accept an insured and sign the contract, they assess the health condition and mortality risk of the potential policyholders. The insurance company usually sets some criteria favoring healthy candidates. Moreover, accounting for the adverse selection, people with better exercise habits are more likely to purchase the new life insurance rewarding physical activities. As a result, the average mortality risk should be no higher than the normal level.

Proposition 3.2.3. *The condition (3.5) is equivalent to*

$$\Delta_\Theta(\Omega) = \Omega^T \cdot \Theta \geq 0.$$

Proof. Using the definition of $\Delta_\Theta(\Omega)$ in (3.2) and (3.3), we have

$$\begin{aligned} & \Omega^T \cdot Q_x \leq q_x \\ \Leftrightarrow \quad & q_x - \Omega^T \cdot Q_x \geq 0 \\ \Leftrightarrow \quad & \frac{q_x - \Omega^T \cdot Q_x}{q_x \cdot \delta_x} \geq 0 \\ \Leftrightarrow \quad & \Delta_\Theta(\Omega) \geq 0 \\ \Leftrightarrow \quad & \Omega^T \cdot \Theta \geq 0. \end{aligned}$$

□

The Proposition 3.2.3 provides a necessary and sufficient condition of (3.4) which depends only on the distribution Ω for a given Θ (representing the classification of mortality risk). In addition, Proposition 3.2.3 shows that Assumption 3.2.2 is independent of age x.

We further make some reasonable assumptions on the transition matrix P.

Assumption 3.2.4. *Given two populations with initial status distribution* Ω_1 *and* Ω_2*, respectively, the order of the average mortality rates remains unchanged after one year of* *participation in the wellness program with transition matrix P. That is,*

$$\text{If} \quad \Omega_1^T \cdot Q_x \leq \Omega_2^T \cdot Q_x, \ \text{then} \quad \Omega_1^T \cdot P \cdot Q_x \leq \Omega_2^T \cdot P \cdot Q_x,$$

or equivalently, by Proposition 3.2.1,

$$\Omega_1^T \cdot \Theta \geq \Omega_2^T \cdot \Theta \quad \Rightarrow \quad \Omega_1^T \cdot P \cdot \Theta \geq \Omega_2^T \cdot P \cdot \Theta. \tag{3.6}$$

Assumption 3.2.5. *Let Ω_0 be the initial status distribution of the policyholders, satisfying $\Omega_0^T \cdot Q_x = q_x$. We assume that*

$$\Omega_0^T \cdot P \cdot Q_x \leq \Omega_0^T \cdot Q_x = q_x,$$

or equivalently, by Proposition 3.2.1,

$$\Omega_0^T \cdot P \cdot \Theta \geq \Omega_0^T \cdot \Theta = 0.$$

Actually, $(\Omega_0^T \cdot P)^T$ is the status distribution for all the participants in their second year. Assumption 3.2.5 states that after one year of participation in this health rewards program, the average mortality rate of the population is reduced. If the underlying Markov chain process is irreducible and aperiodic, as we have mentioned in Chapter 2, its stationary distribution exists.

Corollary 3.2.5.1. *Let $\boldsymbol{\varpi} = [\varpi_0, \cdots, \varpi_{L-1}]^T$ be the stationary distribution of the underlying Markov chain process. Under Assumption 3.2.5 and Assumption 3.2.4, we have*

$$\boldsymbol{\varpi}^T \cdot Q_x \leq \Omega_0^T \cdot Q_x = q_x, \tag{3.7}$$

where $\boldsymbol{\varpi}^T \cdot Q_x$ is the average mortality rate of the population in the long-term stationary state of the health rewards system.

Proof. In Assumption 3.2.4, let $\Omega_1^T = \Omega_0^T \cdot P$ and $\Omega_2^T = \Omega_0^T$. By repeatedly applying (3.6), and noting that $(\Omega_0^T \cdot P^k)^T$, for $k = 2, 3, \cdots$ are all distribution vectors (in fact, they are the status distribution of the third year, fourth year, ... in the health rewards program), we get

$$q_x = \Omega_0^T \cdot Q_x \geq \Omega_0^T \cdot P \cdot Q_x \geq \cdots \geq \Omega_0^T \cdot P^n \cdot Q_x.$$

By taking the limit, when $n \to \infty$, (3.7) holds. $\square$

The stationary distribution of the homogeneous Markov chain process $\boldsymbol{\varpi}$, if exists, is decided only by the transition matrix P. Therefore, for any initial distribution Ω, the transition process reverts it back to its stationary state. For the general population with the normal mortality rate, the wellness program aims at improving the overall health conditions. In its long-term equilibrium state, the average mortality rate should be smaller than the normal mortality rate.

3.3 Transition probabilities in the multi-state model

In the multi-state model, we define ${}_tp_x^{ij}$ as t-year transition probability in the multi-state model for an x-year-old insured from state i to state j, where $i, j \in \{0, 1, 2, \ldots, L\}$ and t is a non-negative integer. The transition probabilities ${}_tp_x^{ij}$ satisfy the condition that

$$\sum_{j=0}^{L} {}_tp_x^{ij} = 1,$$

for $i = 0, 1, \cdots, L$, $t = 0, 1, 2, \cdots$. When $i = L$, since state L is an absorbing state, for all $t \geq 0$,

$${}_tp_x^{Lj} = \begin{cases} 1, & \text{if } j = L; \\ 0, & \text{otherwise.} \end{cases}$$

When $t = 0$, we have

$${}_0p_x^{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

When $t = 1$, it is the one-year transition probability as

$$p_x^{ij} = \begin{cases} q_x^i, & \text{for } j = L, \\ (1 - q_x^i) \cdot p_{ij}, & \text{for } j \leq L - 1, \end{cases} \tag{3.8}$$

for $i = 0, 1, \cdots, L - 1$. The one-year transition probability defined by (3.8) is based on the assumption that policyholder in state i bears a mortality rate of q_x^i and given that he/she survives after one year, his/her transition from state i to state $j (j < L)$ follows the Markov chain process described in Chapter 2.

When $t > 0$, the t-year transition probability, ${}_tp_x^{ij}$, can be calculated recursively through the Chapman-Kolmogorov equation given by

$${}_tp_x^{ij} = \sum_{l=0}^{L-1} {}_{t-1}p_x^{il} \cdot p_{x+t-1}^{lj},$$

where $i, j = 0, 1, \cdots, L - 1$.

The transition process in the multi-state model can be described by a non-homogeneous Markov chain process. We denote the one-step transition matrix by $P_x = \{p_x^{ij}\}$, which is a $(L+1) \times (L+1)$ matrix, given by

$$\begin{aligned} P_x &= \begin{bmatrix} (1-q_x^0)\cdot p_{00} & (1-q_x^0)\cdot p_{01} & \cdots & (1-q_x^0)\cdot p_{0,L-1} & q_x^0 \\ (1-q_x^1)\cdot p_{10} & (1-q_x^1)\cdot p_{11} & \cdots & (1-q_x^1)\cdot p_{1,L-1} & q_x^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-q_x^{L-1})\cdot p_{L-1,0} & (1-q_x^{L-1})\cdot p_{L-1,1} & \cdots & (1-q_x^{L-1})\cdot p_{L-1,L-1} & q_x^{L-1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \Phi_x \cdot P & Q_x \\ \mathbf{0} & 1 \end{bmatrix}, \end{aligned}$$

where Q_x is given by (3.1), $\mathbf{0}$ is a L-dimension vector with all elements being 0, and Φ_x is a diagonal matrix defined as

$$\Phi_x = \begin{bmatrix} 1-q_x^0 & 0 & \cdots & 0 \\ 0 & 1-q_x^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-q_x^{L-1} \end{bmatrix}. \tag{3.9}$$

3.4 Premium and benefit payments

In this section, we study the annual premium and death benefit for an n-year insurance in the discrete time multi-state model that we have introduced. The death benefit is payable at the end of the year of death. When n goes to infinity, the policy becomes whole life insurance.

Assume that the starting state is s for a policyholder, we wish to value an annuity of 1 per year payable while the insured is in state i within n years, denoted by $\ddot{a}_x^{si}$ as

$$\ddot{a}_{x:\overline{n}|}^{si} = \sum_{t=0}^{n-1} v^t \cdot {}_tp_x^{si}. \tag{3.10}$$

Suppose that the premium rate in state 0, corresponding to Blue status in fitness rating system, is π_h. The premium rate of each state has different reductions on π_h, $\{r_l\}_{l=0}^{L-1}$. We have $0 = r_0 < r_1 < \cdots < r_{L-1}$. The distribution of the initial status for a randomly selected policyholder in the group is Ω. We denote the expected present value of premiums to be collected from a randomly selected person at age x as C_x. We assume that the first year premium rate is determined by the first-year assessment of the insured's fitness status, then C_x is given by

$$C_x = \sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} \ddot{a}_{x:\overline{n}|}^{sl} \cdot \pi_h \cdot (1 - r_l). \tag{3.11}$$

3.4.1 Temporary insurance

The expected present value of unit benefit payable to an insured with starting state s, is denoted by $B_{x:\overline{n}|}^{sL}$, where x is the age at issue, and the superscript sL indicates that starting state is s at issue of the policy and the final state is death. By definition of death benefit, we have

$$B_{x:\overline{n}|}^{sL} = \sum_{k=0}^{n-1} \sum_{i=0}^{L-1} {}_k p_x^{si} \cdot q_{x+k}^{i} \cdot v^{k+1}. \tag{3.12}$$

Let Z be a random variable denoting the present value of the benefit of an n-year temporary insurance payable to a randomly selected policyholder from a group with Ω as the initial status distribution. The expectation is

$$\begin{aligned} E[Z] &= \sum_{s=0}^{L-1} \omega_s \cdot B_{x:\overline{n}|}^{sL} \\ &= \sum_{s=0}^{L-1} \omega_s \sum_{k=0}^{n-1} \sum_{i=0}^{L-1} {}_k p_x^{si} \cdot q_{x+k}^{i} \cdot v^{k+1}. \end{aligned} \tag{3.13}$$

By the equivalence principle, equaling C_x and $E[Z]$, given by (3.11) and (3.13), respectively, we obtain the premium of state 0 for a member aged x at issue as

$$\pi_h = \frac{\sum_{s=0}^{L-1} \omega_s \cdot B_{x:\overline{n}|}^{sL}}{\sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} \ddot{a}_{x:\overline{n}|}^{sl} \cdot (1 - r_l)}.$$

3.4.2 Endowment insurance

For an n-year endowment insurance policy, we use ${}_nE_x^s$ to denote the expected benefit for an insured starting at state s. That is,

$${}_nE_x^s = \sum_{l=0}^{L-1} v^n \cdot {}_n p_x^{sl}. \tag{3.14}$$

Let W be the random variable of the present value of a pure endowment with unit benefit payable to a randomly selected policyholder in the group with initial distribution being Ω. Then we have

$$E[W] = \sum_{s=0}^{L-1} \omega_s \cdot {}_nE_x^s$$

$$= \sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} v^n \cdot {}_n p_x^{sl}.$$

For an endowment insurance, the expected present value of benefit is given by

$$\begin{aligned} E[Z+W] &= \sum_{s=0}^{L-1} \omega_s \cdot \left(B_{x:\overline{n}|}^{sL} + {}_n E_x^s \right) \\ &= \sum_{s=0}^{L-1} \omega_s \cdot \left(\sum_{k=0}^{n-1} \sum_{i=0}^{L-1} {}_k p_x^{si} \cdot q_{x+k}^i \cdot v^{k+1} + \sum_{i=0}^{L-1} v^n \cdot {}_n p_x^{si} \right). \end{aligned} \tag{3.15}$$

To distinguish from temporary insurance, we use π'_h to denote the premium at state 0. By the equivalence principle, we have

$$\pi'_h = \frac{\sum_{s=0}^{L-1} \omega_s \cdot \left(B_{x:\overline{n}|}^{sL} + {}_n E_x^s \right)}{\sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} \ddot{a}_{x:\overline{n}|}^{sl} \cdot (1 - r_l)},$$

where $B_{x:\overline{n}|}^{sL}$ and ${}_n E_x^s$ are given by (3.12) and (3.14), respectively.

3.5 Benefit reserve

Benefit reserve or reserve at time t after issue is the difference between the conditional expected costs of mortality in the future and the expected future premiums to be collected. Under our multi-state life insurance model, premium rates and mortality risks are determined by members' status in the fitness rating system. Thus the reserve at time t of a life insurance depends on the state at time t. When $i = L$ at time t, the reserve simply equals to zero. For $i < L$ at time t, we clearify several notations below to describe reserves:

- ${}_t V^{(i)}$: the reserve of the new life insurance product for a policyholder in state i at time t;
- $B_{x+t:\overline{n-t}|}^{iL}$: the expected present value of future benefit at time t of a n-year temporary insurance, payable to a policyholder in state i at time t.
- ${}_{n-t} E_{x+t}^i$: the expected present value of future benefit at time t of a $(n-t)$-year pure endowment for a policyholder in state i at time t.

The reserve at time t for an insured at state i holding an n-year temporary insurance with the health rewards program is

$${}_t V^{(i)} = B_{x+t:\overline{n-t}|}^{iL} - \sum_{j=0}^{L-1} \pi_h \cdot (1 - r_j) \cdot \ddot{a}_{x+t:\overline{n-t}|}^{ij}. \tag{3.16}$$

For an n-year endowment insurance with health rewards program, the reserve at time t for a policyholder at state i is given by

$$
{}_tV^{(i)} \quad = B^{iL}_{x+t:\overline{n-t}|} + {}_{n-t}E^{i}_{x+t} - \textstyle\sum_{j=0}^{L-1} \pi'_h \cdot (1-r_j) \cdot \ddot{a}^{ij}_{x+t:\overline{n-t}|} \tag{3.17}
$$

3.6 Insurance surplus of a homogeneous portfolio

The surplus is defined as the assets exceeding liabilities. It serves as an important indicator of the financial position for insurance companies. However, surplus is influenced by a variety of factors including mortality, interest, premiums, benefits and reserves etc. In this section, we study the stochastic behaviour of the surplus for the new life insurance products.

Nolde and Parker (2014) provides surplus analysis for traditional life insurance products under the assumption of stochastic interest rate and mortality following a non-parametric life table. We extend the framework in Nolde and Parker (2014) to analyze the surplus of life insurance products under a multi-state model. We focus on the impact of the new product design and hence use constant interest rate in our analysis.

3.6.1 Cash flows

Consider a homogeneous portfolio of size m of the n-year new life insurance contract, which is described in the previous sections. Each policy pays a death benefit of b at the end of the year of death if the death occurs within n years since the issue date. A pure endowment benefit of c is paid if the policyholder survives to the end of year n since the issue date. The annual premium at the initial level is π_h. At the beginning of each year, survivors in the level $l(l = 0, 1, \ldots, L-1)$ pay a premium of $\pi_h \cdot (1 - r_l)$. As the discount rate, v, is constant, the only uncertainty is the mortality.

Consider a valuation date at the end of policy year $r(r < n)$, referred to as time r. For $i = 1, \ldots, m$ and $j = 0, \ldots, n$, we introduce the following indicator variables.

$$
\begin{aligned}
\mathscr{L}^l_{ij} \quad &= \begin{cases} 1, & \text{if the policyholder } i \text{ is alive in health level } l \text{ at time } j, \\ 0, & \text{otherwise;} \end{cases} \\
\mathscr{D}_{ij} \quad &= \begin{cases} 1, & \text{if the policyholder } i \text{ dies in policy year } j, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Let $\mathscr{L}^l_j := \sum_{i=1}^m \mathscr{L}^l_{ij}$ and $\mathscr{D}_j := \sum_{i=1}^m \mathscr{D}_{ij}$. That is, $\mathscr{L}^l_j$ is the number of in-force policies at time j held by the survivors at health level l, and $\mathscr{D}_j$ is the number of deaths in policy year j. Moments of these indicator random variables are needed in the analysis later. We introduce the following lemma.

Lemma 3.6.1. *The moments of the indicator variables are computed as*

$$\begin{aligned}
E[\mathscr{L}_j^l] &= m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_jp_x^{\alpha l}, \\
Var[\mathscr{L}_j^l] &= m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_jp_x^{\alpha l}\cdot(1-{}_jp_x^{\alpha l}), \\
E[\mathscr{D}_j] &= m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_jp_x^{\alpha L}, \\
Var[\mathscr{D}_j] &= m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_jp_x^{\alpha L}\cdot(1-{}_jp_x^{\alpha L}).
\end{aligned}$$

For $0 \le i < j \le n, 0 \le \{l,s,t\} \le L-1$, *we have*

$$\begin{aligned}
Cov[\mathscr{D}_i,\mathscr{D}_j] &= -m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_ip_x^{\alpha L}\cdot {}_jp_x^{\alpha L}, \\
Cov[\mathscr{D}_i,\mathscr{L}_i^l] &= -m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_ip_x^{\alpha L}\cdot {}_ip_x^{\alpha l}, \\
Cov[\mathscr{D}_i,\mathscr{L}_j^l] &= -m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_ip_x^{\alpha L}\cdot {}_jp_x^{\alpha l}, \\
Cov[\mathscr{L}_i^s,\mathscr{L}_i^t] &= -m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_ip_x^{\alpha s}\cdot {}_ip_x^{\alpha t}.\ (s \neq t) \\
Cov[\mathscr{L}_i^l,\mathscr{D}_j] &= m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_ip_x^{\alpha l}\cdot ({}_{j-i}p_{x+i}^{lL}-{}_jp_x^{\alpha L}), \\
Cov[\mathscr{L}_i^s,\mathscr{L}_j^t] &= m\sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_ip_x^{\alpha s}\cdot ({}_{j-i}p_{x+i}^{st}-{}_jp_x^{\alpha t}).
\end{aligned}$$

Proof. Under the assumption of the homogeneous portfolio and that policyholders have independent mortality risks, we have the following facts.

- $\{\mathscr{D}_j\}_{j=1}^r \cup \{\mathscr{L}_r^l\}_{l=0}^{L-1}$
  $\sim$ Multinomial $(m; q_x(0), q_x(1), \ldots, q_x(r-1), {}_rp_x(0), {}_rp_x(1), \ldots, {}_rp_x(L-1))$;
- $\{\mathscr{D}_j\} \sim$ Binomial $(m; q_x(j-1))$;
- $\{\mathscr{L}_j^l\} \sim$ Binomial $(m; {}_jp_x(l))$;

where $q_x(t) = \sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_tp_x^{\alpha L}$ and ${}_tp_x(l) = \sum_{\alpha=0}^{L-1}\omega_\alpha \cdot {}_tp_x^{\alpha l}$, for $t = 0, 1, \cdots$, $l = 0, 1, \cdots, L-1$. If two indicator variables are not within the same multinomial distribution, their covariances are calculated from first principles. □

For the purpose of surplus analysis, we distinguish between the cash flows that occur before time r and those that occur after. To be consistent with the assets and liabilities of the company, we study the net cash inflows prior to time r and the net cash outflows after time r. For the valuation at time r, annual net cash inflows are evaluated retrospectively, so we call it retrospective cash inflow at time $j(0 \le j < r)$, denoted by RC_j^r as

$$\begin{aligned}
RC_j^r &= \sum_{i=1}^{m} \left[\sum_{l=0}^{L-1} \pi_h \cdot (1-r_l) \cdot \mathscr{L}_{ij}^l \cdot \mathbb{1}_{\{j<r\}} - b \cdot \mathscr{D}_{ij} \cdot \mathbb{1}_{\{j>0\}} \right] \\
&= \pi_h \cdot \left[\sum_{l=0}^{L-1} (1-r_l) \cdot \sum_{i=1}^{m} \mathscr{L}_{ij}^l \right] \cdot \mathbb{1}_{\{j<r\}} - b \cdot \left(\sum_{i=1}^{m} \mathscr{D}_{ij} \right) \cdot \mathbb{1}_{\{j>0\}} \\
&= \pi_h \cdot \sum_{l=0}^{L-1} (1-r_l) \cdot \mathscr{L}_j^l \cdot \mathbb{1}_{\{j<r\}} - b \cdot \mathscr{D}_j \cdot \mathbb{1}_{\{j>0\}}, \qquad (3.18)
\end{aligned}$$

where $\mathbb{1}_{\{\mathcal{A}\}}$ is an indicator function; it takes the value of 1 if the condition $\mathcal{A}$ is true and 0 otherwise. Obviously, RC_j^r is the income brought by the premium collection from the survivors at time j net of the death benefit payout to those who die in the jth policy year. Notice that at special occasions, when $t = 0$, RC_0^r is simply the sum of premium collected at the issue date, while when $t = r$, RC_r^r is the outflow of death benefit payment.

Similarly, we derive the expression of the net cash outflows prospectively evaluated at time r. The prospective cash outflow, PC_j^r denotes the cash outflow in the j-th year after the valuation date r. It is given by

$$\begin{aligned}
PC_j^r = \sum_{i=1}^{m} \Bigg[& b \cdot \mathscr{D}_{i,r+j} \cdot \mathbb{1}_{\{j>0\}} + c \cdot \sum_{l=0}^{L-1} \mathscr{L}_{i,r+j}^l \cdot \mathbb{1}_{\{j=n-r\}} \\
& - \pi_h \cdot \sum_{l=0}^{L-1} (1-r_l) \cdot \mathscr{L}_{i,r+j}^l \cdot \mathbb{1}_{\{j<n-r\}} \Bigg] \\
= \; & b \cdot \mathscr{D}_{r+j} \cdot \mathbb{1}_{\{j>0\}} + c \cdot \sum_{l=0}^{L-1} \mathscr{L}_{r+j}^l \cdot \mathbb{1}_{\{j=n-r\}} - \pi_h \cdot \sum_{l=0}^{L-1} (1-r_l) \cdot \mathscr{L}_{r+j}^l \cdot \mathbb{1}_{\{j<n-r\}} \\
= \; & b \cdot \mathscr{D}_{r+j} \cdot \mathbb{1}_{\{j>0\}} + \sum_{l=0}^{L-1} \left[c \cdot \mathbb{1}_{\{j=n-r\}} - \pi_h \cdot (1-r_l) \cdot \mathbb{1}_{\{j<n-r\}} \right] \cdot \mathscr{L}_{r+j}^l. \qquad (3.19)
\end{aligned}$$

Obviously, PC_j^r is the insurer's liability of death benefit and pure endowment benefit payment at time $r + j$ net of the premiums collected from the survivors.

The calculation of the moments of RC_j^r and PC_j^r is needed in the analysis of surplus later in the chapter. The formulas for calculating these moments are expressed in terms of moments of the indicator variables defined earlier, which we discuss in Lemma 3.6.1. In the following theorem, we list the formulas for calculating moments of RC_j and PC_j without proof as they are straightforward.

Theorem 3.6.2. *By (3.18) and (3.19), and Lemma 3.6.1, we have the following results for the moments of RC_j^r and PC_j^r:*

$$E[RC_j^r] = \pi_h \cdot \sum_{l=0}^{L-1} (1-r_l) \cdot E[\mathscr{L}_j^l] \cdot \mathbb{1}_{\{j<r\}} - b \cdot E[\mathscr{D}_j] \cdot \mathbb{1}_{\{j>0\}};$$

$$E[PC_j^r] = b \cdot E[\mathscr{D}_{r+j}] \cdot \mathbb{1}_{\{j>0\}} + \sum_{l=0}^{L-1} \beta_l(j) \cdot E[\mathscr{L}_{r+j}^l];$$

$$\begin{aligned}
Cov[RC_i^r, RC_j^r] &= \pi_h^2 \cdot \mathbb{1}_{\{i<r\}} \cdot \mathbb{1}_{\{j<r\}} \cdot \sum_{s=0}^{L-1}\sum_{t=0}^{L-1} (1-r_s)(1-r_t) \cdot Cov[\mathscr{L}_i^s, \mathscr{L}_j^t] \\
&+ b^2 \cdot Cov[\mathscr{D}_i, \mathscr{D}_j] \cdot \mathbb{1}_{\{i>0\}} \cdot \mathbb{1}_{\{j>0\}} - \pi_h \cdot b \cdot \mathbb{1}_{\{i<r\}} \cdot \mathbb{1}_{\{j>0\}} \cdot \sum_{l=0}^{L-1} (1-r_l) \cdot Cov[\mathscr{L}_i^l, \mathscr{D}_j] \\
&- b \cdot \pi_h \cdot \mathbb{1}_{\{i>0\}} \cdot \mathbb{1}_{\{j<r\}} \cdot \sum_{l=0}^{L-1} (1-r_l) \cdot Cov[\mathscr{D}_i, \mathscr{L}_j^l];
\end{aligned}$$

$$\begin{aligned}
Cov[PC_i^r, PC_j^r] &= b^2 \cdot Cov[\mathscr{D}_{r+i}, \mathscr{D}_{r+j}] \cdot \mathbb{1}_{\{i>0\}} \cdot \mathbb{1}_{\{j>0\}} - \sum_{l=0}^{L-1} b \cdot \mathbb{1}_{\{i>0\}} \cdot \beta_l(j) \cdot Cov[\mathscr{D}_{r+i}, \mathscr{L}_{r+j}^l] \\
&+ \sum_{s=0}^{L-1}\sum_{t=0}^{L-1} Cov[\mathscr{L}_{r+i}^s, \mathscr{L}_{r+j}^t] \cdot \beta_s(i) \cdot \beta_t(j) - \sum_{l=0}^{L-1} \beta_l(i) \cdot b \cdot \mathbb{1}_{\{j>0\}} \cdot Cov[\mathscr{L}_{r+i}^l, \mathscr{D}_{r+j}];
\end{aligned}$$

$$\begin{aligned}
Cov[RC_i^r, PC_j^r] &= \pi_h \cdot b \cdot \mathbb{1}_{\{i<r\}} \cdot \mathbb{1}_{\{j>0\}} \sum_{l=0}^{L-1} (1-r_l) \cdot Cov[\mathscr{L}_i^l, \mathscr{D}_{r+j}] \\
&+ \sum_{s=0}^{L-1}\sum_{t=0}^{L-1} (1-r_s) \cdot \pi_h \cdot \mathbb{1}_{\{i<r\}} \cdot \beta_t(j) \cdot Cov[\mathscr{L}_i^s, \mathscr{L}_{r+j}^t] - b^2 \cdot \mathbb{1}_{\{i>0\}} \cdot \mathbb{1}_{\{j>0\}} \cdot Cov[\mathscr{D}_i, \mathscr{D}_{r+j}] \\
&- \sum_{l=0}^{L-1} b \cdot \mathbb{1}_{\{i>0\}} \cdot \beta_l(j) \cdot Cov[\mathscr{D}_i, \mathscr{L}_{r+j}^l];
\end{aligned}$$

where $\beta_l(j) = \left[c \cdot \mathbb{1}_{\{j=n-r\}} - \pi_h \cdot (1-r_l) \cdot \mathbb{1}_{\{j<n-r\}}\right]$, *for* $l = 0, 1, \cdots, L-1$.

3.6.2 Retrospective gain and prospective loss

Now we further introduce the random variables, RG_r and PL_r, which are the retrospective gain and the prospective loss at valuation time r, respectively. The retrospective gain at time r is the accumulated value of the collected premiums before time r net of the benefit paid. It can be calculated as the accumulated sum of the retrospective cash inflows before time r, that is

$$RG_r = \sum_{j=0}^{r} RC_j^r \cdot v^{-(r-j)}.$$

The prospective loss, PL_r is sum of the discounted value of the premiums to be collected net of the benefits to be paid after time r. It can be expressed in terms of the prospective cash outflows as follows,

$$PL_r = \sum_{j=0}^{n-r} PC_j^r \cdot v^j.$$

Since RG_r and PL_r are respectively the functions of the cash flow variables RC_j^r and PC_j^r, their moments are readily computed as the functions of the moments of RC_j^r and PC_j^r, which are presented in Theorem 3.6.2.

The expected value of the prospective loss conditional on the number of in-force policies at the valuation date is known as the actuarial reserve in the context of traditional life insurance. We extend this concept to fit in our framework of new life insurance product. We define the actuarial reserve of the portfolio of the new life insurance policies to be the expected value of the prospective loss conditional on the number of in-force policies at each fitness level. The randomness of this aggregate actuarial reserve comes from the uncertainty of the in-force policies at each level by the valuation time r. Since the actuarial reserve is a function of the random variables $\{\mathscr{L}_r^l\}_{l=0}^{L-1}$, we denote it by ${}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right)$ with the expression

$${}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right) \quad = \quad E[PL_r|\{\mathscr{L}_r^l\}_{l=0}^{L-1}] = \sum_{j=0}^{n-r} v^j \cdot E[PC_j^r|\{\mathscr{L}_r^l\}_{l=0}^{L-1}],$$

where by definition of PC_j^r given in (3.19),

$$\begin{aligned} E[PC_j^r|\{\mathscr{L}_r^l\}_{l=0}^{L-1}] &= E\left[\sum_{s=0}^{L-1}(c \cdot \mathbb{1}_{\{j=n-r\}} - \pi_h \cdot (1-r_s) \cdot \mathbb{1}_{\{j<n-r\}}) \cdot \mathscr{L}_{r+j}^s \right. \\ &\quad \left. + b \cdot \mathscr{D}_{r+j} \cdot \mathbb{1}_{\{j>0\}} \mid \{\mathscr{L}_r^l\}_{l=0}^{L-1}\right] \\ &= \sum_{s=0}^{L-1}(c \cdot \mathbb{1}_{\{j=n-r\}} - \pi_h \cdot (1-r_s) \cdot \mathbb{1}_{\{j<n-r\}}) \cdot E\left[\mathscr{L}_{r+j}^s \mid \{\mathscr{L}_r^l\}_{l=0}^{L-1}\right] \\ &\quad + b \cdot \mathbb{1}_{\{j>0\}} \cdot E\left[\mathscr{D}_{r+j} \mid \{\mathscr{L}_r^l\}_{l=0}^{L-1}\right] \end{aligned}$$

Notice that the conditional expectations of the indicator variables, $E[\mathscr{L}_{r+j}^s \mid \{\mathscr{L}_r^l\}_{l=0}^{L-1}]$ and $E[\mathscr{D}_{r+j}|\{\mathscr{L}_r^l\}_{l=0}^{L-1}]$, are functions of the random variables $\{\mathscr{L}_r^l\}_{l=0}^{L-1}$. Specifically,

$$\begin{aligned} E\left[\mathscr{L}_{r+j}^s \mid \{\mathscr{L}_r^l\}_{l=0}^{L-1}\right] &= \sum_{l=0}^{L-1} \mathscr{L}_r^l \cdot {}_jp_{x+r}^{ls}, \\ E\left[\mathscr{D}_{r+j} \mid \{\mathscr{L}_r^l\}_{l=0}^{L-1}\right] &= \sum_{l=0}^{L-1} \mathscr{L}_r^l \cdot {}_jp_{x+r}^{lL}. \end{aligned}$$

Thus (3.20) can now be expressed in terms of $\{\mathscr{L}_r^l\}_{l=0}^{L-1}$ as

$$
\begin{aligned}
E[PC_j^r|\{\mathscr{L}_r^l\}_{l=0}^{L-1}] &= \sum_{s=0}^{L-1}(c\cdot \mathbb{1}_{\{j=n-r\}} - \pi_h\cdot(1-r_s)\cdot \mathbb{1}_{\{j<n-r\}})\cdot \sum_{l=0}^{L-1}\mathscr{L}_r^l\cdot {}_jp_{x+r}^{ls} \\
&\quad + b\cdot \mathbb{1}_{\{j>0\}}\cdot \sum_{l=0}^{L-1}\mathscr{L}_r^l\cdot {}_jp_{x+r}^{lL}.
\end{aligned}
$$

Similar to RC_j^r and PC_j^r, the moments of $E[PC_j^r|\{\mathscr{L}_r^l\}_{l=0}^{L-1}]$ can be computed with the moments of the indicator variables derived previously. Thus the moments of the aggregate actuarial surplus, ${}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right)$, are available.

3.6.3 Surplus

The insurance surplus is defined to be the difference between the assets and the liabilities at a given valuation date. In our framework, the retrospective gain can be viewed as the assets at the valuation date, while the prospective loss can be taken as the future liabilities. By viewing the uncertainty of both RG_r and PL_r caused by mortality risk at the issue date and valuating the value of their difference at time r, we define the stochastic surplus, denoted by S_r^{stoch}, as

$$
S_r^{stoch} = RG_r - PL_r. \tag{3.20}
$$

An alternative definition of surplus is to replace the prospective loss by the actuarial reserve, which is the amount that an insurer needs to set aside on each valuation date based on the expected future obligation for all the in-force policies at that time. We call it the accounting surplus, denoted by S_r^{acct} with

$$
S_r^{acct} = RG_r - {}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right).
$$

The expected value of these two types of surplus are the same, since by definitions

$$
\begin{aligned}
E[S_r^{acct}] &= E[RG_r] - E\left[{}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right)\right] \\
&= E[RG_r] - E\left[E[PL_r|\{\mathscr{L}_r^l\}_{l=0}^{L-1}]\right] \\
&= E[RG_r] - E[PL_r] \\
&= E[RG_r - PL_r] \\
&= E[S_r^{stoch}].
\end{aligned}
$$

If the premium is determined by the equivalence principle, it can be easily shown that the expected value of surplus is zero. We show this result below using the expression of stochastic surplus given by (3.20). By definitions, we have

$$
\begin{aligned}
E[S_r^{stoch}] &= E[RG_r - PL_r] \\
&= E\left[\sum_{j=0}^{r} RC_j^r \cdot v^{-(r-j)} - \sum_{j=0}^{n-r} PC_j^r \cdot v^j\right] \\
&= v^{-r} \cdot E\left[\sum_{j=0}^{r} RC_j^r \cdot v^j - \sum_{j=0}^{n-r} PC_j^r \cdot v^{r+j}\right] \\
&= v^{-r} \cdot E\Bigg[\sum_{j=0}^{r}\left(\sum_{l=0}^{L-1}(1-r_l)\cdot \pi_h \cdot \mathscr{L}_j^l \cdot \mathbb{1}_{\{j<r\}} - b \cdot \mathscr{D}_j \cdot \mathbb{1}_{\{j>0\}}\right)\cdot v^j - \sum_{j=0}^{n-r}\Big(b\cdot \mathscr{D}_{r+j}\cdot \mathbb{1}_{\{j>0\}} \\
&\quad + \sum_{l=0}^{L-1}\big(c\cdot \mathbb{1}_{\{j=n-r\}} - (1-r_l)\cdot \pi_h \cdot \mathbb{1}_{\{j<n-r\}}\big)\cdot \mathscr{L}_{r+j}^l\Big)\cdot v^{r+j}\Bigg] \\
&= v^{-r} \cdot E\Bigg[\sum_{j=0}^{r}\left(\sum_{l=0}^{L-1}(1-r_l)\cdot \pi_h \cdot \mathscr{L}_j^l \cdot \mathbb{1}_{\{j<r\}} - b \cdot \mathscr{D}_j \cdot \mathbb{1}_{\{j>0\}}\right)\cdot v^j - \sum_{j=r}^{n}\Bigg(b\cdot \mathscr{D}_j\cdot \mathbb{1}_{\{j-r>0\}} \\
&\quad + \sum_{l=0}^{L-1}\big(c\cdot \mathbb{1}_{\{j=n\}} - (1-r_l)\cdot \pi_h \cdot \mathbb{1}_{\{j<n\}}\big)\cdot \mathscr{L}_j^l\Bigg)\cdot v^j\Bigg] \\
&= v^{-r} \cdot E\left[\sum_{j=0}^{n}\left(\sum_{l=0}^{L-1}(1-r_l)\pi_h \cdot \mathscr{L}_j^l \cdot \mathbb{1}_{\{j<n\}} - b\cdot \mathscr{D}_j \cdot \mathbb{1}_{\{j>0\}} - \sum_{l=0}^{L-1} c\cdot \mathscr{L}_j^l \cdot \mathbb{1}_{\{j=n\}}\right)\cdot v^j\right] \\
&= 0.
\end{aligned}
$$

Basically, the expected value of the surplus defined is the accumulated value at time r of the expected present value of the difference between the assets and the liabilities over the entire portfolio life. According to the equivalence principle, the above expectation equals to zero.

With our derivation of the variance and covariance of RG_r, PL_r and ${}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right)$ previously, the variance calculation for the surplus is straightforward as

$$
\begin{aligned}
Var[S_r^{stoch}] &= Var[RG_r - PL_r] \\
&= Var[RG_r] + Var[PL_r] - 2\cdot Cov[RG_r, PL_r]; \\
Var[S_r^{acct}] &= Var\left[RG_r - {}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right)\right] \\
&= Var[RG_r] + Var\left[{}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right)\right] - 2\cdot Cov\left[RG_r, {}_r\mathcal{V}\left(\{\mathscr{L}_r^l\}_{l=0}^{L-1}\right)\right].
\end{aligned}
$$

Chapter 4

Comparison with traditional insurance product

In this chapter, we compare our new life insurance with the traditional insurance products. For the traditional life insurance without wellness program, we could assume that some hidden transition matrix exists, and its stationary distribution has been reached by the general population. The average level of mortality rate is the mortality rate suggested by the life table. To make the two types of insurance products comparable, we assume that the group of policyholders of the new life insurance start with average mortality risks at the normal level. That is, the initial status distribution Ω_0 is under Assumption 3.2.5.

In fact, a traditional insurance product is modelled by a two-state model, with only two states, survived-0 and dead-1. We adopt the traditional notations described below of a two-state model and review the general results of the traditional life insurance.

- ${}_tp_x$: the t-year survival probability for an x-year-old.
- $\ddot{a}_{x:\overline{n}|}$: the annuity factor for a traditional n-year insurance policy.
- $A^1_{x:\overline{n}|}$: the expected present value of unit benefit for a traditional n-year temporary insurance.
- $A_{x:\overline{n}|}$: the expected present value of unit benefit for a traditional n-year endowment insurance.
- ${}_nE_x$: the expected value of unit benefit for a traditional n-year pure endowment.
- Y: the random variable denoting the present value of benefit payable to an insured.

4.1 Mean and variance of the benefit

4.1.1 Expected present value of the benefit

The benefit of a life insurance payable to policyholders is the total liability of a insurance company. If the new life insurance with an integrated wellness program improves health condition and well-being of the population and reduces the mortality risk, then intuitively, the total liability for insurers would be lower. However, this conclusion is not that obvious mathematically.

In a two-state model, the expected present value of benefit payable for n-year temporary and n-year endowment life insurance are given by

$$\begin{aligned} A^1_{x:\overline{n}|} &= \sum_{k=0}^{n-1} {}_kp_x \cdot q_{x+k} v^{k+1}, \\ A_{x:\overline{n}|} &= \sum_{k=0}^{n-1} {}_kp_x \cdot q_{x+k} v^{k+1} + {}_nE_x, \end{aligned}$$

where ${}_nE_x = v^n \cdot {}_np_x$. For endowment insurance, the annuity and endowment life insurance have the following relationship:

$$A_{x:\overline{n}|} = 1 - d \cdot \ddot{a}_{x:\overline{n}|},$$

where $d = 1 - v$, is the annual effective discount rate.

Let $\Omega_0 = [\omega_0, \omega_1, \cdots, \omega_{L-1}]$ under Assumption 3.2.5 be the initial distribution of status of the group holding the new life insurance. We compare $\sum_{s=0}^{L-1} \omega_s \cdot B^{sL}_{x:\overline{n}|}$ with $A^1_{x:\overline{n}|}$, and $\sum_{s=0}^{L-1} \omega_s \cdot \left(B^{sL}_{x:\overline{n}|} + {}_nE^s_x\right)$ with $A_{x:\overline{n}|}$. First, we point out the relationship between the expected future benefit and the annuity factor for the new life insurance in the following lemma.

Lemma 4.1.1. *Similar to the relationship of life annuity and life insurance under the traditional framework, we have*

$$B^{sL}_{x:\overline{n}|} + {}_nE^s_x = 1 - d \sum_{l=0}^{L-1} \ddot{a}^{sl}_{x:\overline{n}|},$$

where $s \in \{0, 1, \ldots, L-1\}$ and $\ddot{a}^{sl}_{x:\overline{n}|}$ is defined in (3.10).

Proof. The following analysis is within the framework of new life insurance with integrated wellness program. Let K^s_x be the curtate future lifetime for an x-year-old with initial state s. The present value of an n-year life annuity-due is $\ddot{a}_{\overline{min\{n, K^s_x+1\}}|}$, which can be expressed as

$$\ddot{a}_{\overline{min\{n, K^s_x+1\}}|} = \begin{cases} \ddot{a}_{\overline{K_x+1}|} = \frac{1-v^{K_x+1}}{d}, & K^s_x \leq n-1 \\ \ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}, & K^s_x \geq n \end{cases},$$

or, we can write

$$\ddot{a}_{\overline{min\{n,K_x^s+1\}}|} = \frac{1 - v^{min\{K_x^s+1,n\}}}{d}. \tag{4.1}$$

Since $v^{min\{K_x^s+1,n\}}$ is the present value of an n-year endowment insurance, we have the following formula after taking the expectation with respect to K_x^s on both sides of the equation (4.1):

$$\sum_{l=0}^{L-1} \ddot{a}_{x:\overline{n}|}^{sl} = \frac{1 - (B_{x:\overline{n}|}^{s,L} + {}_nE_x^s)}{d},$$

where $\ddot{a}_{x:\overline{n}|}^{sl}$, $B_{x:\overline{n}|}^{s,L}$ and ${}_nE_x^s$ are given by (3.10), (3.12) and (3.14), respectively. □

Lemma 4.1.2. *If Ω_0 and P satisfy Assumption 3.2.5 and Assumption 3.2.4, respectively, then for $t > 0$ that takes an integer value,*

$${}_tp_x \leq \sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} {}_tp_x^{si}.$$

Proof. The proof of Lemma 4.1.2 is given in Appendix A. □

Lemma 4.1.2 indicates that the average t-year survival probabilities of the participants (with initial distribution Ω_0) of the health rewards program is greater than or equal to that of the general population.

Corollary 4.1.2.1. *For n-year annuity factors, the following inequality holds:*

$$\ddot{a}_{x:\overline{n}|} \leq \sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} \ddot{a}_{x:\overline{n}|}^{sl}.$$

Proof. By the definition of $\ddot{a}_{x:\overline{n}|}$ and Lemma 4.1.2, we have

$$\begin{aligned}
\ddot{a}_{x:\overline{n}|} &= \sum_{t=0}^{n-1} v^t \cdot {}_tp_x \\
&\leq \sum_{t=0}^{n-1} v^t \sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} {}_tp_x^{si} x \\
&= \sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} \ddot{a}_{x:\overline{n}|}^{sl}. \qquad \text{(by (3.10))}
\end{aligned}$$

□

Theorem 4.1.3. *Under Assumption 3.2.5 and Assumption 3.2.4, life insurance with integrated wellness program has a smaller expected present value of benefit for both endowment* *insurance and temporary insurance, compared to the corresponding traditional life insurance. That is,*

$$\begin{aligned} A_{x:\overline{n}|} &\geq \sum_{s=0}^{L-1} \omega_s \cdot \left(B^{sL}_{x:\overline{n}|} + {}_nE^s_x \right); \\ A^1_{x:\overline{n}|} &\geq \sum_{s=0}^{L-1} \omega_s \cdot B^{sL}_{x:\overline{n}|}. \end{aligned} \tag{4.2}$$

Proof. By the definition of $A_{x:\overline{n}|}$ and Corollary 4.1.2.1, we have

$$\begin{aligned} A_{x:\overline{n}|} &= 1 - d \cdot \ddot{a}_{x:\overline{n}|} \\ &\geq 1 - d \cdot \sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} \ddot{a}^{sl}_{x:\overline{n}|} \\ &= \sum_{s=0}^{L-1} \omega_s - d \cdot \sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} \ddot{a}^{sl}_{x:\overline{n}|} \\ &= \sum_{s=0}^{L-1} \omega_s \left(1 - d \cdot \sum_{l=0}^{L-1} \ddot{a}^{sl}_{x:\overline{n}|} \right) \\ &= \sum_{s=0}^{L-1} \omega_s \cdot \left(B^{sL}_{x:\overline{n}|} + {}_nE^s_x \right). \qquad \text{(by Lemma 4.1.1)} \end{aligned}$$

Similarly, starting from the definition of $A^1_{x:\overline{n}|}$ and ${}_nE^s_x$ by (3.14), we have

$$\begin{aligned} A^1_{x:\overline{n}|} &= A_{x:\overline{n}|} - {}_nE_x \\ &\geq \sum_{s=0}^{L-1} \omega_s \cdot \left(B^{sL}_{x:\overline{n}|} + {}_nE^s_x \right) - {}_nE_x \qquad \text{(by (4.2))} \\ &= \sum_{s=0}^{L-1} \omega_s \cdot B^{sL}_{x:\overline{n}|} + \sum_{s=0}^{L-1} \omega_s \cdot {}_nE^s_x - v^n \cdot {}_np_x \\ &= \sum_{s=0}^{L-1} \omega_s \cdot B^{sL}_{x:\overline{n}|} + \sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} v^n \cdot {}_np^{sl}_x - v^n \cdot {}_np_x \\ &= \sum_{s=0}^{L-1} \omega_s \cdot B^{sL}_{x:\overline{n}|} + v^n \cdot \left(\sum_{s=0}^{L-1} \omega_s \sum_{l=0}^{L-1} {}_np^{sl}_x - {}_np_x \right) \\ &\geq \sum_{s=0}^{L-1} \omega_s \cdot B^{sL}_{x:\overline{n}|}. \qquad \text{(by Lemma 4.1.2)} \end{aligned}$$

□

The reduced expected benefit payment of the new life insurance product, compared to that of the traditional life insurance, is due to the improvement of policyholders' health which results in the reduction of mortality risks. The difference can be viewed as shared value created by the insured together with the insurers. The insurers design the product and provide a platform and environment that systematically record and encourage the healthy behaviours of the policyholders, and the insured are those who take actions and actually enhance the overall well-being of the group. Here, we define the relative absolute difference of expected benefit payment as a share value margin, denoted by η and η' for temporary insurance and endowment insurance, respectively, as

$$\begin{aligned}\eta &= \frac{\left|\sum_{s=0}^{L-1}\omega_s \cdot B^{sL}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|}\right|}{A^{1}_{x:\overline{n}|}}, \\ \eta' &= \frac{\left|\sum_{s=0}^{L-1}\omega_s \cdot \left(B^{sL}_{x:\overline{n}|} + {}_nE^s_x\right) - A_{x:\overline{n}|}\right|}{A_{x:\overline{n}|}}.\end{aligned} \tag{4.3}$$

For premiums determined by the equivalence principle, if we add a loading, ψ, which is less than the share value margin (η for temporary insurance and η' for endowment insurance), it can be explained that the insured and insurers share the advantage of this new approach to life insurance together. Instead of expense loading or loss loading, we can call it shared value loading. The fraction, ψ/η (or ψ/η'), is the insurers' portion of the shared value.

4.1.2 Variance of present value of the benefit

The variance of the benefit for an n-year temporary insurance policy can be obtained by the second moment and the first moment.

$$\begin{aligned}Var(Z) &= E[Z^2] - E[Z]^2 \\ &= \sum_{s=0}^{L-1}\omega_s \cdot {}^2B^{sL}_{x:\overline{n}|} - \left(\sum_{s=0}^{L-1}\omega_s B^{sL}_{x:\overline{n}|}\right)^2, \\ Var(Y) &= E[Y^2] - E[Y]^2 \\ &= {}^2A^{1}_{x:\overline{n}|} - (A^{1}_{x:\overline{n}|})^2,\end{aligned} \tag{4.4}$$

where ${}^2B^{sL}_{x:\overline{n}|}$ is the expected present value of unit benefit of an n-year new temporary insurance calculated at twice the force of interest. Similarly, for an n-year endowment insurance, variances are given by

$$\begin{aligned}Var(Z) &= E[Z^2] - E[Z]^2 \\ &= \sum_{s=0}^{L-1}\omega_s \left({}^2B^{sL}_{x:\overline{n}|} + {}^2_nE^s_x\right) - \left(\sum_{s=0}^{L-1}\omega_s \left(B^{sL}_{x:\overline{n}|} + {}_nE^s_x\right)\right)^2, \\ Var(Y) &= E[Y^2] - E[Y]^2 \\ &= {}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2,\end{aligned} \tag{4.5}$$

where ${}_n^2E_x^s$ is the expected present value of unit benefit of an n-year pure endowment of the new life insurance product calculated at twice the force of interest. Obviously, at twice the force of interest, we also have

$$
\begin{aligned}
{}^2A^1_{x:\overline{n}|} &> \sum_{s=0}^{L-1} \omega_s \cdot {}^2B^{sL}_{x:\overline{n}|}, \\
{}^2A_{x:\overline{n}|} &> \sum_{s=0}^{L-1} \omega_s \left({}^2B^{sL}_{x:\overline{n}|} + {}_n^2E_x^s \right).
\end{aligned}
$$

However, the same relationship holds for the deduction, the squared expectation. As a result, the comparison between the variance of benefit of two insurance products becomes unclear. We will study this later by numerical examples.

4.2 Average Premium

The annual net premiums for traditional life insurance products are

$$
\begin{aligned}
\pi &= \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}; \qquad \text{(for temporary insurance)} \\
\pi' &= \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}. \qquad \text{(for endowment insurance)} \qquad (4.6)
\end{aligned}
$$

The health rewards program improves the mortality experience for the population, which results in lower benefit payment due and higher probabilities of premium collection. However, because of the reduction applied at different health levels, the initial-level premiums of life insurance with integrated wellness program, π_h and π'_h, are usually higher than the corresponding premiums of a traditional life insurance, π and π'. It is more reasonable to compare the premium of a traditional insurance with the expected average premium at time t paid by a randomly selected policyholder given that he/she is alive at time t. The latter is also the average premium over the group of survivors at time t in the portfolio of the new insurance product. As the distribution of statuses among survivors changes with time t, the average premium of the new life insurance is a function of t. Taking the temporary insurance as an example, the group average of the premium among survivors at time t for the new insurance product, denoted by $\bar{\pi}_h(t)$, is given by

$$
\bar{\pi}_h(t) = \pi_h \cdot \sum_{l=0}^{L-1} (1 - r_l) \cdot Pr[L_t = l | L_t < L], \qquad (4.7)
$$

where

$$\begin{aligned}
Pr[L_t = l | L_t < L] &= \frac{Pr[L_t = l, L_t < L]}{Pr[L_t < L]} \\
&= \frac{\sum_{s=0}^{L-1} Pr[L_t = l, L_t < L | L_0 = s] \cdot Pr[L_0 = s]}{\sum_{s=0}^{L-1} Pr[L_t < L | L_0 = s] \cdot Pr[L_0 = s]} \\
&= \frac{\sum_{s=0}^{L-1} \omega_s \cdot {}_tp_x^{sl}}{\sum_{s=0}^{L-1} \omega_s \cdot \sum_{m=0}^{L-1} {}_tp_x^{sm}}. \qquad (4.8)
\end{aligned}$$

Substitute (4.8) into (4.7), we get

$$\bar{\pi}_h(t) = \pi_h \cdot \sum_{l=0}^{L-1} (1 - r_l) \frac{\sum_{s=0}^{L-1} \omega_s \cdot {}_tp_x^{sl}}{\sum_{m=0}^{L-1} \sum_{s=0}^{L-1} \omega_s \cdot {}_tp_x^{sm}}. \qquad (4.9)$$

No direct conclusion can be reached at this point on the comparison of two premiums, $\bar{\pi}_h(t)$ and π. The results really depend on the design of premium reduction factors, $\{r_l; l = 0, 1, \ldots, L-1\}$, and the distribution of the fitness statuses of survivors at time t. Another interesting quantity is the percentage of the group members paying a premium less than π. We will discuss this later by numerical illustration.

4.3 Reserves

In the framework of a traditional life insurance product, the reserve, ${}_tV$, of a temporary insurance policy at time t is given by

$${}_tV = A^1_{x+t:\overline{n-t|}} - \pi \cdot \ddot{a}_{x+t:\overline{n-t|}}.$$

For the n-year temporary insurance with health rewards program, the expression of the reserve at time t for an insured being in state i is given by equation (3.16), that is

$${}_tV^{(i)} = B^{iL}_{x+t:\overline{n-t|}} - \sum_{j=0}^{L-1} \pi_h \cdot (1 - r_j) \cdot \ddot{a}^{ij}_{x+t:\overline{n-t|}},$$

for $i = 0, 1, \cdots, L-1$, and $t > 0$.

For endowment insurance, the reserve of a traditional life insurance is given by

$${}_tV = A_{x+t:\overline{n-t|}} - \pi' \cdot \ddot{a}_{x+t:\overline{n-t|}}.$$

By equation (3.17), we have the following expression for reserves of the new life insurance:

$${}_tV^{(i)} = B^{iL}_{x+t:\overline{n-t|}} + {}_{n-t}E^i_{x+t} - \sum_{j=0}^{L-1} \pi'_h \cdot (1 - r_j) \cdot \ddot{a}^{ij}_{x+t:\overline{n-t|}},$$

for $i = 0, 1, \cdots, L-1$, and $t > 0$.

From the insurers' point of view, it is more relevant to compare the average reserve over the in-force policies with the reserve of the traditional life insurance, as they decide the average funding requirements for an in-force policy. We introduce the average reserve for the new life insurance product ${}_t\bar{V}_h$ as

$$
\begin{aligned}
{}_t\bar{V}_h &= \sum_{l=0}^{L-1} {}_tV^{(i)} \cdot Pr[L_t = i | L_t < L] \\
&= \sum_{l=0}^{L-1} {}_tV^{(i)} \cdot \frac{\sum_{s=0}^{L-1} \omega_s \cdot {}_tp_x^{si}}{\sum_{m=0}^{L-1} \sum_{s=0}^{L-1} \omega_s \cdot {}_tp_x^{sm}} \quad \text{(by (4.8))}.
\end{aligned}
$$

Again, comparisons of the reserves described above are hard to be done theoretically. For the new life insurance, though the expected future liability is reduced, the higher probabilities of premium payments and different levels of premium reductions make it complicated to analyze. We will discuss the comparison of reserves in numerical examples.

4.4 Surplus

We derive the expressions and the calculation formulas of surplus for the new life insurance in Chapter 3. For surplus of the traditional life insurance products, one is referred to Nolde and Parker (2014) for details. The comparisons will be analyzed through numerical examples in Chapter 5.

Chapter 5

Numerical illustration

In this chapter, we use numerical examples to illustrate and further study the properties of the new insurance product modelled by a multi-state model. The mortality table used in the illustration is 1997-04 Canadian Institute of Actuaries (CIA), male, age last birthday, insured lives mortality table. See Table B.1 in Appendix B for mortality rates.

In our examples, we suppose that the integrated health rewards program aims at reducing the risk factor of physical inactivity. The health rewards system adopts the status transition standards shown in Table 2.1. The transition matrix P is chosen as

$$P = \begin{bmatrix} 0.30 & 0.40 & 0.15 & 0.10 & 0.05 \\ 0.20 & 0.25 & 0.30 & 0.15 & 0.10 \\ 0.10 & 0.20 & 0.30 & 0.25 & 0.15 \\ 0.05 & 0.10 & 0.20 & 0.45 & 0.20 \\ 0.01 & 0.05 & 0.14 & 0.40 & 0.40 \end{bmatrix}.$$

We assume that P reflects the common physical activity habits and the effect of the incentives of the program. Other parameters are chosen as follows:

- The percentage of removed attributable fraction of mortality $\Theta = [-0.6, 0, 0.3, 0.75, 1.0]^T$;
- The initial distribution of statuses at issue $\Omega_0 = [0.3, 0.4, 0.15, 0.10, 0.05]^T$;
- Premium reduction percentage at each level $[r_0, r_1, r_2, r_3, r_4] = [0, 2.5\%, 5.0\%, 7.5\%, 10\%]$;
- The discount factor $v = \frac{1}{1.06}$.

Notice that mortality risk deviation factor at Bronze status is 0, indicating that this is a class representing the normal level of mortality risk suggested by a life table. It is easy to verify that $\Omega_0^T \cdot \Theta = 0$, suggesting that the population has the normal mortality rate on average at issue. Assumption 3.2.5 is also satisfied as $\Omega_0^T \cdot P \cdot \Theta \geq \Omega_0^T \cdot \Theta$.

The attributable fraction of mortality rate δ_x is specified in Table 5.1. The data is available online[1], provided by the study of Ezzati et al. (2004). We use the attributable fraction of mortality rate due to physical inactivity in the region of North American. The attributable fraction δ_x can be viewed as a piece-wise continuous function of age x.

Table 5.1: Attributable fraction of mortality by age group

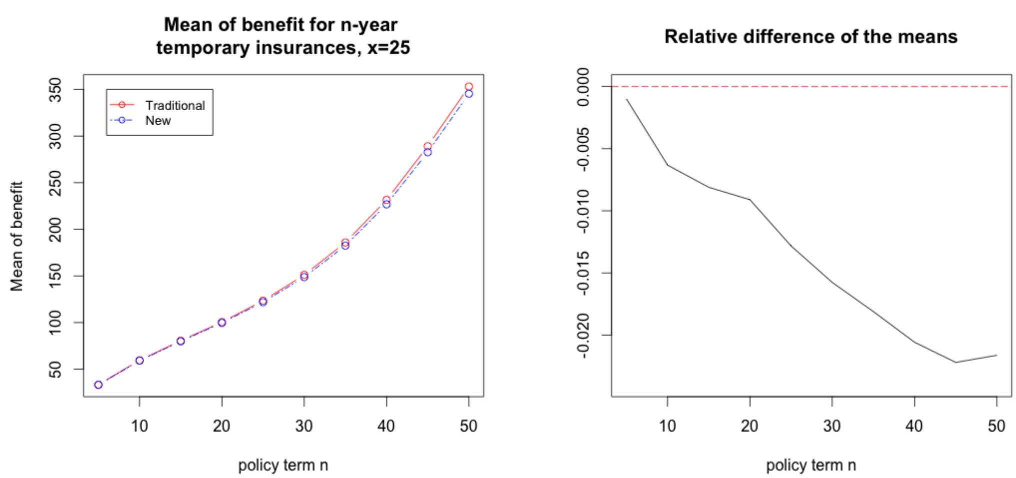
Age Group	15-29	30-44	45-59	60-69	70-79	$\geq$80
Attributable Fraction of Mortality (%)	0.67	3.21	5.54	7.31	6.55	5.67

5.1 Mean and standard deviation of benefit payments

For both temporary insurance and endowment insurance, we study the difference of the traditional life insurance product and our new life insurance product in terms of their mean and standard deviation of the present value of benefit payments. Formulas used in the calculation are (3.13), (3.15), (4.4) and (4.5).

For temporary insurance, we first fix the policyholder's age to be 25 and study the change of the moments of the present value of benefit along with the term of policy n.

Figure 5.1: Mean of benefit for a 25-year-old with n-year temporary insurance



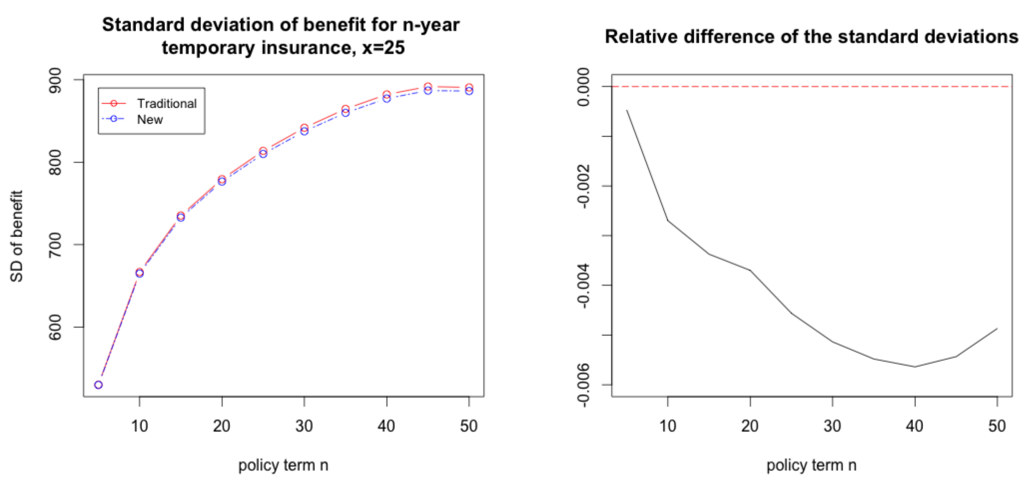
As is proved in Chapter 4, mean of the present value of benefit payments is reduced for the new life insurance product compared to the traditional one. In Figure 5.1, the significance of the difference, which is shown by the absolute value of relative difference, increases with the term of policy until $n = 45$. Actually, the absolute value of the relative difference is the shared value margin, η, defined by (4.3) in Chapter 4. That means the

[1] Comparative quantification of health risks for the year 2000: http://www.who.int/healthinfo/global_burden_disease/risk_factors_2000/en/

temporary insurance with longer term tends to have a larger shared value margin. This is caused by the combination of two factors. One is that the attributable fraction of mortality rate, δ_x, is larger for the middle age than for young people. The temporary insurance with the highest relative difference has term of 45, which covers the age range of 60-69 with the highest δ_x. The other factor is the effect of the wellness program, which improves the average health condition of the group. As can be seen in Table 5.2, the statuses distribution of the group (for young policyholders) becomes relatively stable after year 10. Therefore, the increase of relative difference after $n = 10$ is mainly due to the higher attributable fraction of mortality rate.

The unsmooth changes in the chart of relative difference are due to jump in the values of δ_x at bounds of each age range, which is shown in Table 5.1. The same explanation is applicable for the following charts with this issue.

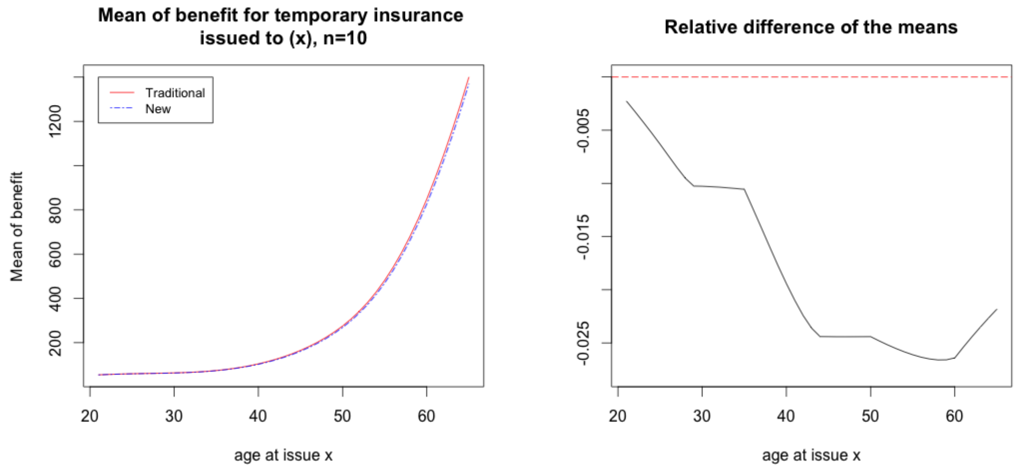
Figure 5.2: Standard deviation of benefit for a 25-year-old with n-year temporary insurance



The comparison of the standard deviation for the two insurance products is shown in Figure 5.2, although it is hard to confirm theoretically. The numerical examples here show the new insurance product has smaller standard deviation than the traditional one does. The significance of the difference increases with the term first, then reaches its highest at 40, and then decreases afterwards.

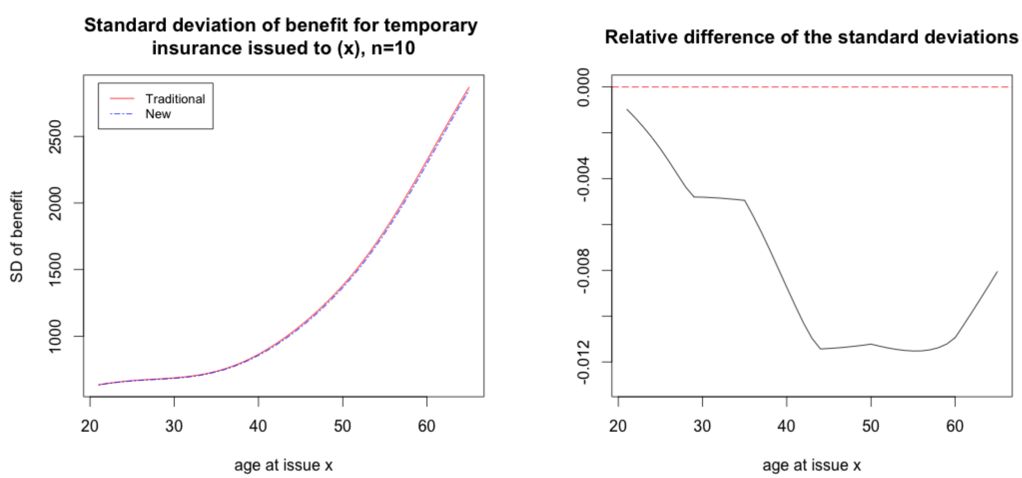
Now we fix the term of policy to study the impact of age at issue. Given that the term of policy is 10, the change of moments of the benefit over age x is shown by Figure 5.3 and Figure 5.4. As expected, the mean of benefit for the new life insurance is smaller than the traditional one. The significance of the difference increases with age first and then decreases. The fitness rating system reaches its stationary state around year 10. For a 10-year temporary insurance, the shared value margin reaches its maximum at age of 55-60, whose policy life covers the age range with the highest δ_x.

Figure 5.3: Mean of benefit of a 10-year temporary insurance for age x from 20 to 65



The negative relative differences of the standard deviation show the reduced volatility of the benefit payment for the new life insurance. The significance of difference has the same trend as that of the mean. Age range of 45-60 has the highest significance of difference of standard deviation. We could say that the late middle age have a relatively more stable death benefit than other age groups.

Figure 5.4: Standard deviation of benefit of a 10-year temporary insurance for age x from 20 to 65



For endowment insurance, the corresponding comparisons are displayed in Figures 5.5-5.8. From Figure 5.5 and Figure 5.6, we observe that with age at issue fixed at 25, endowment insurance under the new product design, also has a reduction in both mean and standard deviation of the benefit. The relative difference increases with term of policy and age at issue. However, compared with their corresponds for temporary insurance case, the significance of difference is generally smaller. This is intuitively true. A reduced mortality rate implies an increased survival probability, resulting in a higher amount of survival benefit which counteracts with the reduction in death benefit.

Figure 5.5: Mean of benefit for a 25-year-old with n-year endowment insurance

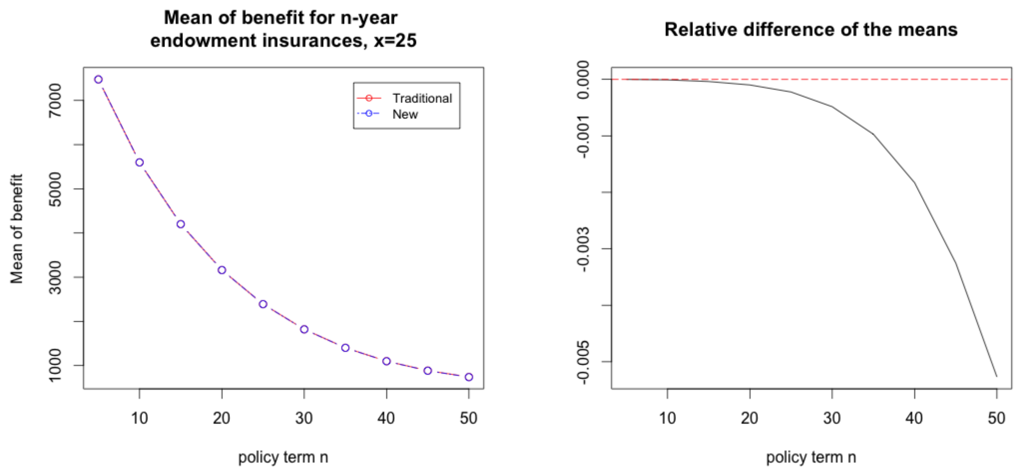
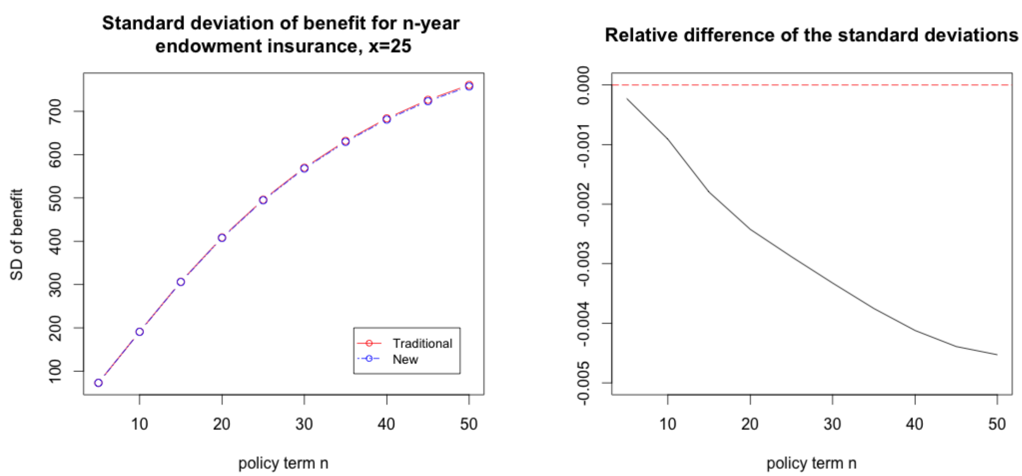


Figure 5.6: Standard deviation of benefit for a 25-year-old with n-year endowment insurance



Next, in Figure 5.7 and Figure 5.8, we fix the term of endowment insurance to be 20, and investigate the moments of benefit of policies issued to the insured of 20-65. Again, the endowment insurance with all issue ages shows a reduction in mean and standard deviation of the benefit for the new product. The significance of difference of means increases with age at issue, while that of standard deviation increases with age first and then decreases, reaching its maximum at the age range of 45-60.

From Figure 5.5 and Figure 5.7, we can also remark that for endowment insurance, the shared value margin of the new life insurance product increases with term of policy and age at issue.

Figure 5.7: Mean of benefit of a 20-year endowment insurance for age x

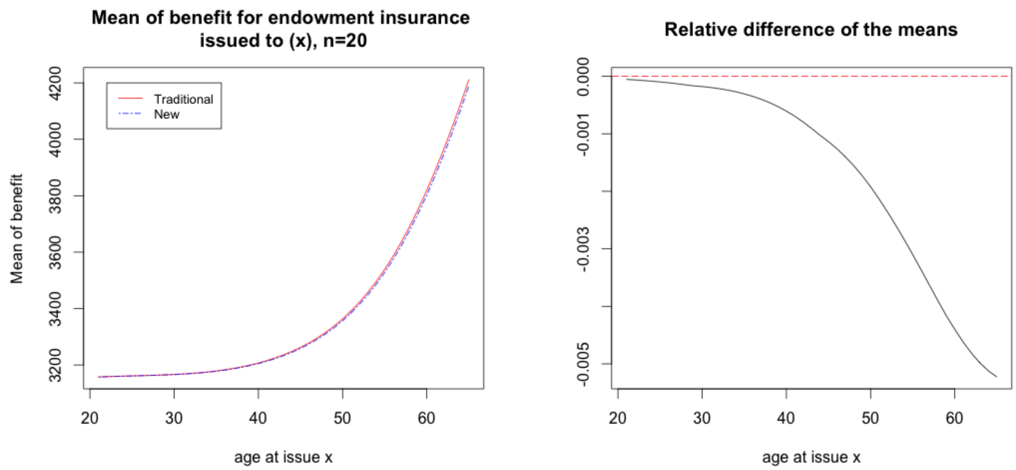
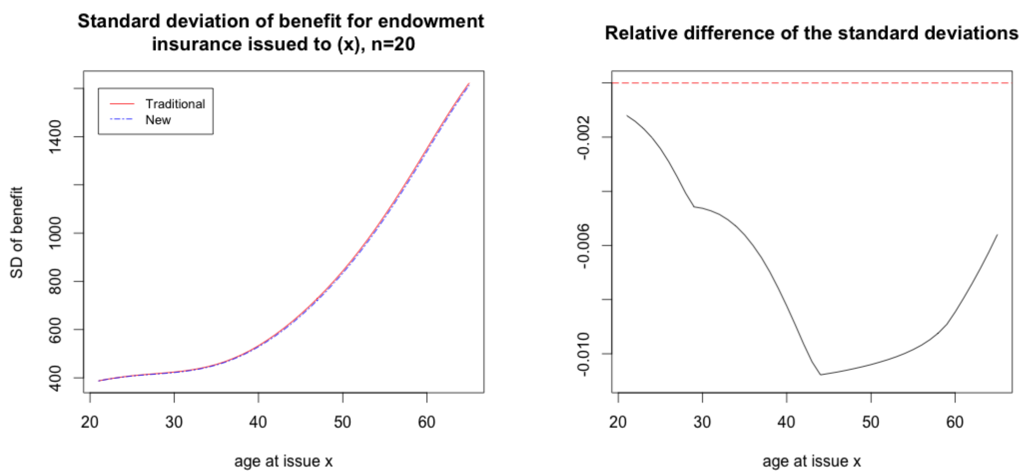


Figure 5.8: Standard deviation of benefit of a 20-year endowment insurance for age x



5.2 Premium

In this section, premium comparisons of the new life insurance products and traditional life insurance products are illustrated by numerical results. Table 5.2 and Table 5.3 display in the temporary insurance case and in the endowment insurance case, respectively, the details of the new life insurance premium paid by each status and the status distribution of the survived policyholders at each policy year. The status distributions of the survivors at each policy year are calculated by (4.8), and the average premiums are computed by (4.9). Premiums of the traditional life insurance products are calculated by (4.6).

For temporary insurance, we consider a 10-year policy issued to an insured at age of 30. From Table 5.2, we observe that for members at Silver status or above, they pay a premium less than the premium of a traditional life insurance ($\pi = 8.1558$). This percentage increases to 72% from 30% in the beginning. In the end, more than 70% of the insured with the new life insurance pay a premium lower than they do with the traditional insurance, which is an attractive feature to policyholders. For members at Bronze status, bearing an average mortality risk of the general population, they pay more in this new life insurance. Basically, people are penalized for being average. Though it sounds less attractive to them, the surcharge works to urge them to be more active in exercise. Regarding the aim of the new life insurance, this is reasonable. For people who are willing to join the wellness program, they are expected to improve their health conditions and reduce mortality risks with a goal lower than the normal level. In this sense, staying where they were is not satisfactory in the standards of the new health rewards system.

Table 5.2: Premium and status distribution of the survivors
Temporary insurance, $x = 30, n = 10$

Status	Blue	Bronze	Silver	Gold	Diamond	Average Premium ($\pi =$ 8.1558)	% of the group (premium $\leq \pi$)
Premium Reduction r_l	0.00%	2.50%	5.00%	7.50%	10.00%		
Year \ Premium	8.5094	8.2967	8.0840	7.8712	7.6585		
1	30%	40%	15%	10%	5%	8.2541	30.00%
2	19.05%	26.25%	23.70%	19.25%	11.75%	8.1299	54.70%
3	14.41%	21.43%	23.34%	25.13%	15.68%	8.0707	64.15%
4	12.36%	19.09%	22.82%	28.07%	17.66%	8.0423	68.55%
5	11.39%	17.97%	22.51%	29.50%	18.63%	8.0286	70.64%
6	10.92%	17.43%	22.36%	30.19%	19.10%	8.0220	71.65%
7	10.70%	17.17%	22.29%	30.52%	19.32%	8.0189	72.13%
8	10.59%	17.05%	22.25%	30.68%	19.43%	8.0174	72.36%
9	10.54%	16.99%	22.23%	30.76%	19.48%	8.0166	72.47%
10	10.52%	16.96%	22.23%	30.79%	19.50%	8.0163	72.52%

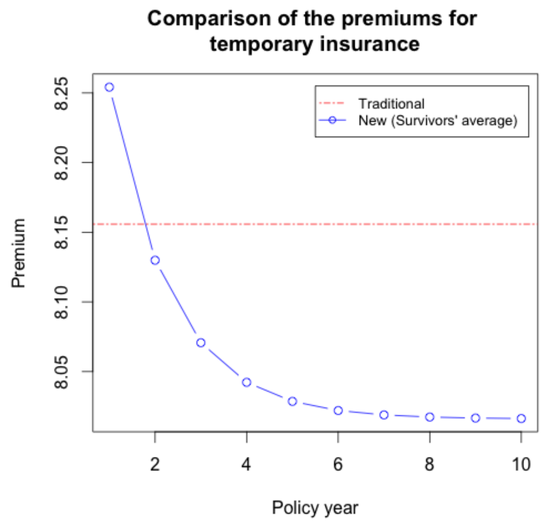
We consider a 20-year policy issued to an insured at age of 30 for endowment insurance. Table 5.3 shows that members would only pay a lower premium than that of the traditional product if they achieve Gold status or higher. The premium rate at Bronze status is slightly higher than traditional premium rate ($\pi' = 262.29$). The percentage of members getting a lower premium rate increases from 15% to 50%. Compared to the case of temporary insurance, the percentage is smaller for the endowment insurance case due to the smaller shared value margin discussed in Section 5.1.

Table 5.3: Premium and status distribution of the survivors
Endowment insurance, $x = 30, n = 20$

Status	Blue	Bronze	Silver	Gold	Diamond	Average Premium ($\pi' =$ 262.29)	% of the group (premium $\leq \pi'$)
Premium Reduction r_l	0.00%	2.50%	5.00%	7.50%	10.00%		
Year \ Premium	277.12	270.19	263.27	256.34	249.41		
1	30.00%	40.00%	15.00%	10.00%	5.00%	268.81	15.00%
2	19.05%	26.25%	23.70%	19.25%	11.75%	264.76	31.00%
3	14.42%	21.44%	23.34%	25.13%	15.68%	262.83	40.81%
4	12.36%	19.09%	22.82%	28.07%	17.66%	261.91	45.74%
5	11.39%	17.97%	22.51%	29.50%	18.63%	261.46	48.13%
6	10.92%	17.43%	22.36%	30.19%	19.10%	261.25	49.28%
7	10.70%	17.17%	22.29%	30.52%	19.32%	261.15	49.84%
8	10.59%	17.05%	22.25%	30.68%	19.43%	261.10	50.11%
9	10.54%	16.99%	22.23%	30.76%	19.48%	261.07	50.24%
10	10.52%	16.96%	22.23%	30.79%	19.50%	261.06	50.30%
11	10.50%	16.95%	22.22%	30.81%	19.52%	261.06	50.33%
12	10.50%	16.94%	22.22%	30.82%	19.52%	261.05	50.34%
13	10.50%	16.94%	22.22%	30.82%	19.52%	261.05	50.35%
14	10.49%	16.93%	22.22%	30.83%	19.53%	261.05	50.35%
15	10.49%	16.93%	22.22%	30.83%	19.53%	261.05	50.35%
16	10.49%	16.93%	22.22%	30.83%	19.53%	261.05	50.35%
17	10.49%	16.93%	22.22%	30.83%	19.53%	261.05	50.35%
18	10.49%	16.93%	22.22%	30.83%	19.53%	261.05	50.35%
19	10.49%	16.93%	22.22%	30.83%	19.53%	261.05	50.35%
20	10.49%	16.93%	22.22%	30.83%	19.53%	261.05	50.35%

The distribution of the different statuses among the survivors gradually drifts towards the higher statuses, indicating a healthier lifestyle of the group overall. The evolution of the status distribution of the survivors in the multi-state model (through P_x) is very close to that of the embedded homogeneous Markov chain process (through P). The evolution of the distribution through P is shown in Table C.1 in Appendix C, which can be used to approximate the distribution of fitness statuses for survivors of the multi-state model. This is especially true for young people, whose low mortality risks and small attributable fraction of mortality rate minimize the difference of survival probability among different statuses.

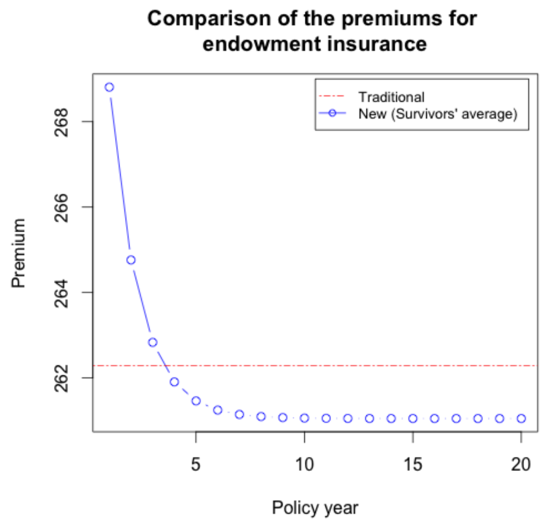
Figure 5.9: Average premium of the new life insurance compared with premium of the traditional counterparty (Temporary insurance, $x = 30, n = 10$)



For more intuitive illustration, Figure 5.9 shows that for the new temporary insurance, the average premium paid by policyholders of the in-force policies starts to drop down below the premium ($\pi = 8.1558$) of a traditional product from the second policy year. Or in other words, starting from the second policy year, the expected premium charged from a randomly selected policyholders (given that he/she is alive) is lower than the premium of a traditional temporary policy.

Figure 5.10 shows the corresponding case of endowment insurance. The average premium paid by the survivors of the new insurance policies remains above the premium of the traditional endowment insurance ($\pi' = 262.29$) in the first three years. Starting from the fourth policy year, the group of survivors of the new insurance are charged a lower premium on average than what they would be charged for the traditional life insurance.

Figure 5.10: Average premium of the new life insurance compared with premium of the traditional counterparty (Endowment insurance, $x = 30, n = 20$)

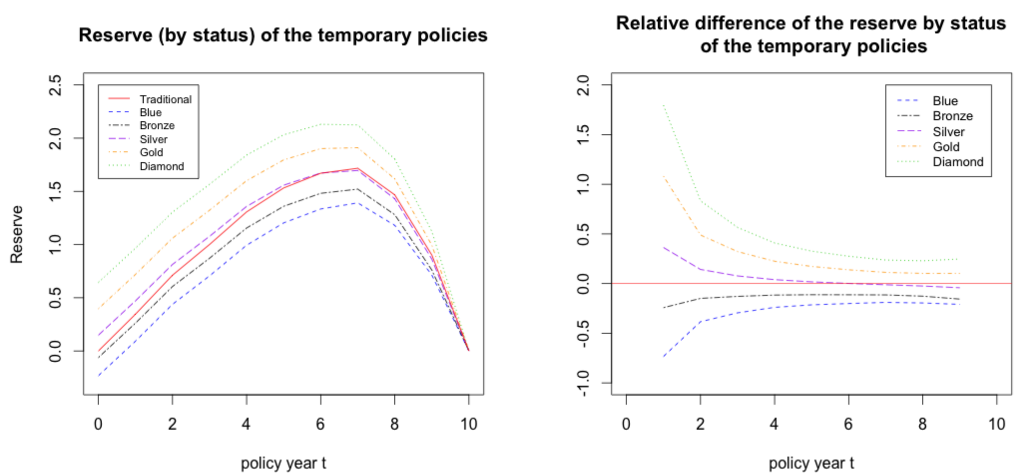


5.3 Reserve

In this section, we study the reserves of the new life insurance product for different terms and ages at issue compared to their counterparts of the traditional life insurance.

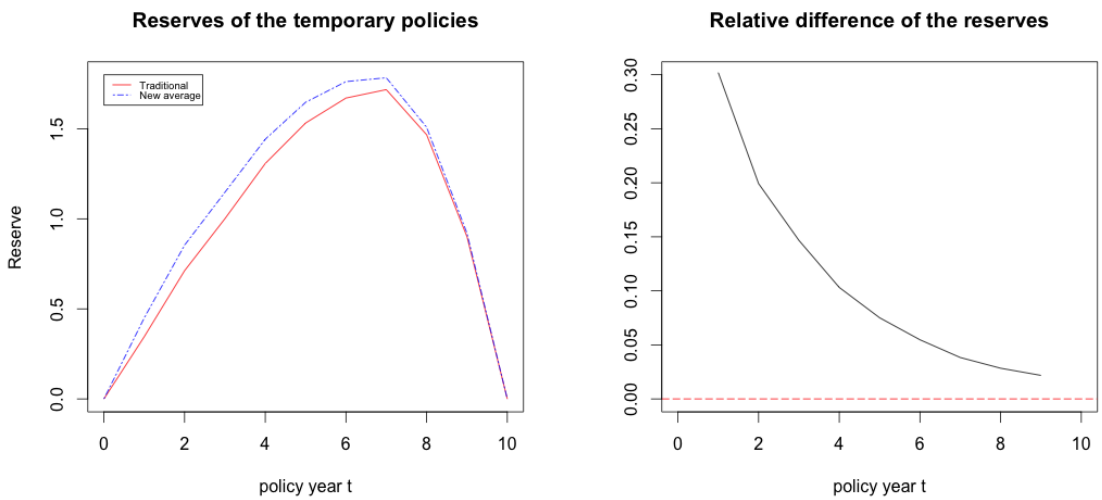
For temporary insurance, we first consider a 10-year contract issued to a 30-year-old policyholder, and then see the changes for a longer term of the policy (n=20) and for the older age at issue (x=50).

Figure 5.11: Reserve by status of the new product vs. the traditional reserve temporary insurance, x=30, n=10



From Figure 5.11, we observe that for temporary insurance, when $n = 10$ and $x = 30$, compared to the reserves of a traditional policy, the relative difference on reserves of the new life insurance vary a lot at different statuses. The superior statuses like Gold and Diamond tend to have higher reserves than those of a traditional policy, while those lower statuses like Blue and Bronze have smaller reserves. Figure 5.12 shows that the average reserve, weighted with distribution of each status among survivors, is higher than the reserve of the traditional insurance. That means, the insurer of the new life insurance needs to set aside larger amount of reserve for each in-force policy on average than for each of the traditional life insurance contract.

Figure 5.12: Average reserve of the new product vs. the traditional reserve
temporary insurance, x=30, n=10



For a 10-year temporary insurance issued to a 50-year-old, reserves by status compared to the reserve of the traditional contract are shown in Figure 5.13. Reserves at different statuses are pretty close to each other. The relative difference for each status is smaller than that of the policies issued to the insureds aged 30. Another feature is that the reserves at superior statuses are higher than traditional reserve at the beginning and then become lower in later stage, while the inferior status, Blue, is the other way around. Silver and Bronze have lower reserves over the entire policy life. This is because compared to the younger insureds, for late middle age policyholders, the positive effect of the mortality risk reduction on the liability prevails over the effect of the premium reduction on the asset.

Figure 5.14 illustrates that the average reserves of the different statuses among survivors stay below the reserve of the traditional policies all the time. Compared to the traditional insurance policies, the insurer needs to retain less reserves on average for each of the in-force policy of the new life insurance.

Figure 5.13: Reserve by status of the new product vs. the traditional reserve, temporary insurance, x=50, n=10

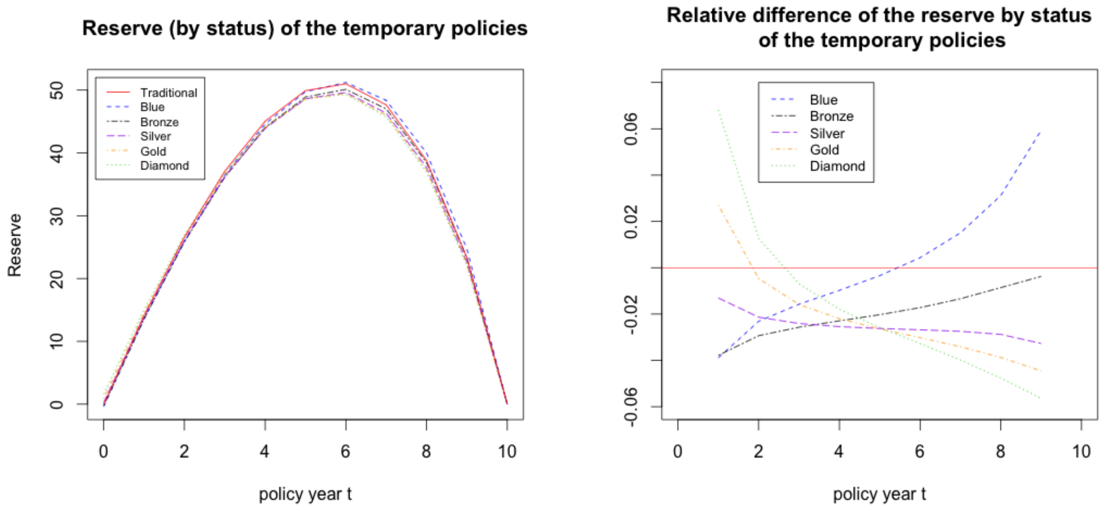
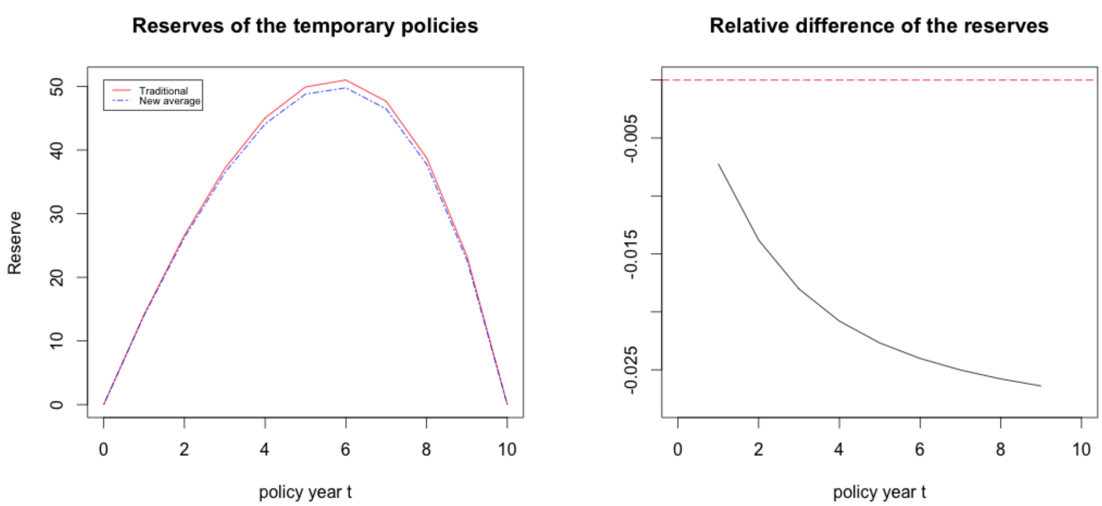


Figure 5.14: Average reserve of the new product vs. the traditional reserve, temporary insurance, x=50, n=10



For a temporary insurance contract with $n = 20$ and $x = 30$, the comparisons are demonstrated in Figure 5.15 and Figure 5.16. The effect of the longer term of policy is similar to that of the older age at issue. At early stages of the policy life, the reserves of superior statuses are higher than the traditional reserve, while at the middle and late stages, reserves at most statuses become lower. The average reserve is larger than the reserve for the traditional life insurance at the beginning and drop down below that since the 5th policy year. Comparing Figure 5.15 with Figure 5.11, the significance of difference is smaller for a long-term policy than for a short-term policy.

Figure 5.15: Reserve by status of the new product vs. the traditional reserve, temporary insurance, x=30, n=20

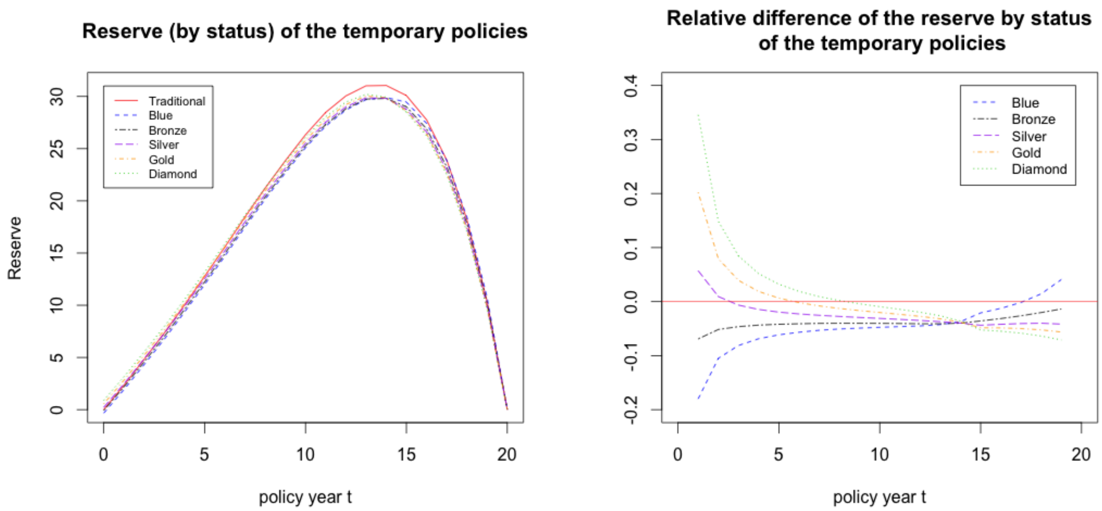
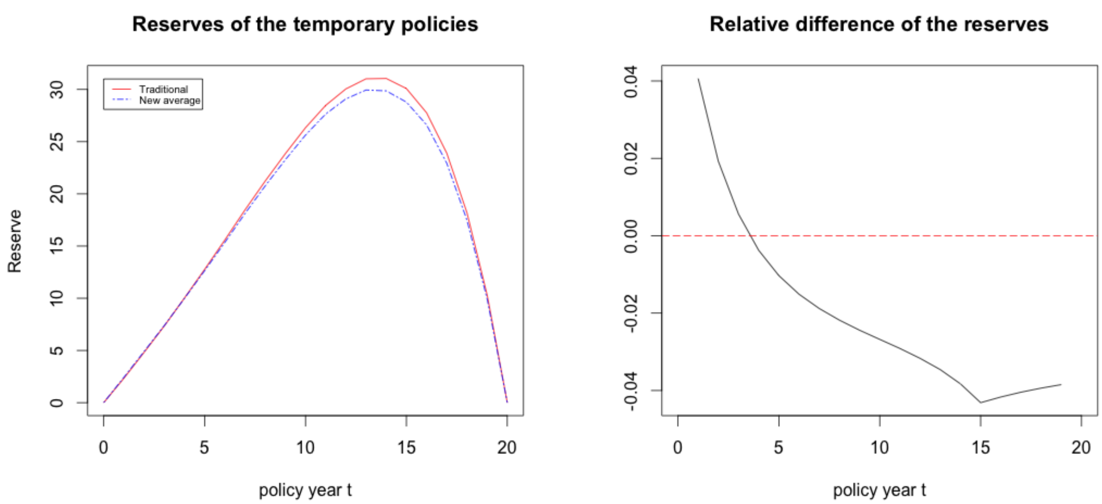


Figure 5.16: Average reserve of the new product vs. the traditional reserve, temporary insurance, x=30, n=20



For endowment insurance, as the reserves at different statuses overlap in the chart, we only display the charts of the relative difference between reserves at each status and the traditional insurance reserve. Results are shown in Figure 5.17, Figure 5.19 and Figure 5.21. Regardless of ages at issue and terms of policy, the changes of reserves by status follow the same pattern. Superior statuses like Diamond or Gold have higher reserve than the traditional one in the beginning, while lower statuses start with smaller reserves. All reserves gradually converge to the traditional level in the end. The significance of difference for higher statuses is larger than that for lower statuses. The weighted average of the reserves at each status ends up to be higher than the reserve of a traditional endowment insurance in the entire policy life. Therefore, insurers selling new endowment insurance are required to retain more reserves for each in-force policies than the reserve of the traditional endowment insurance. The relative differences of the reserves for endowment insurance, compared to temporary insurance, are generally smaller in significance.

Figure 5.17: Reserve by status of the new product vs. the traditional reserve, endowment insurance, x=30, n=20

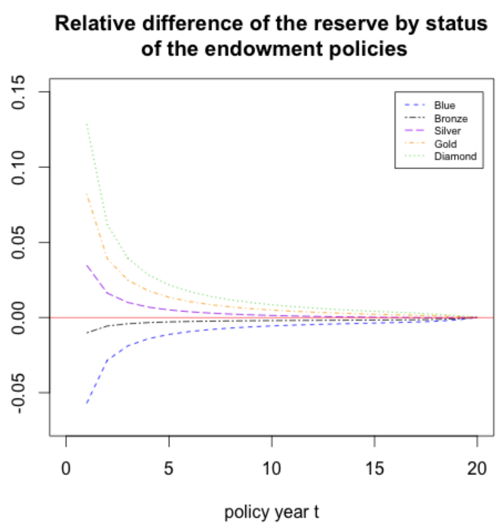


Figure 5.18: Average reserve of the new product vs. the traditional reserve, endowment insurance, x=30, n=20

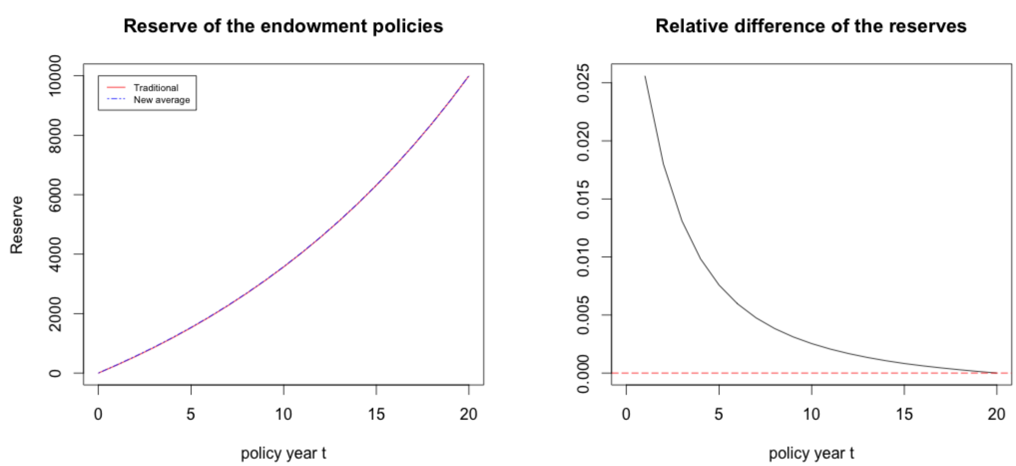


Figure 5.19: Reserve by status of the new product vs. the traditional reserve, endowment insurance, x=30, n=30

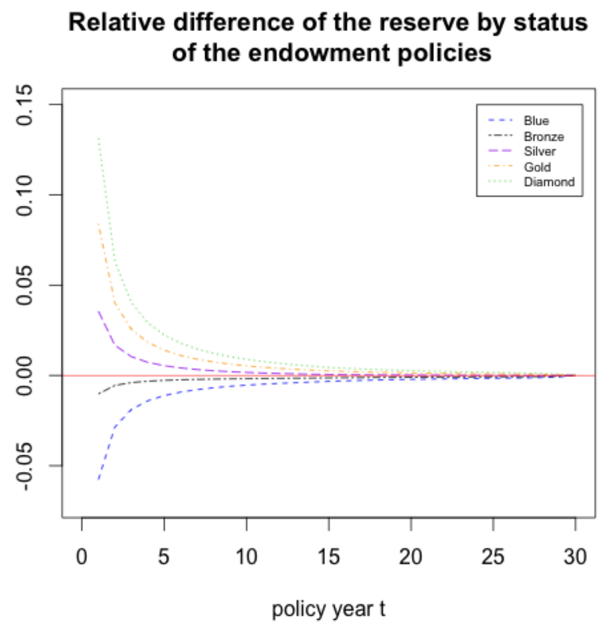


Figure 5.20: Average reserve of the new product vs. the traditional reserve, endowment insurance, x=30, n=30

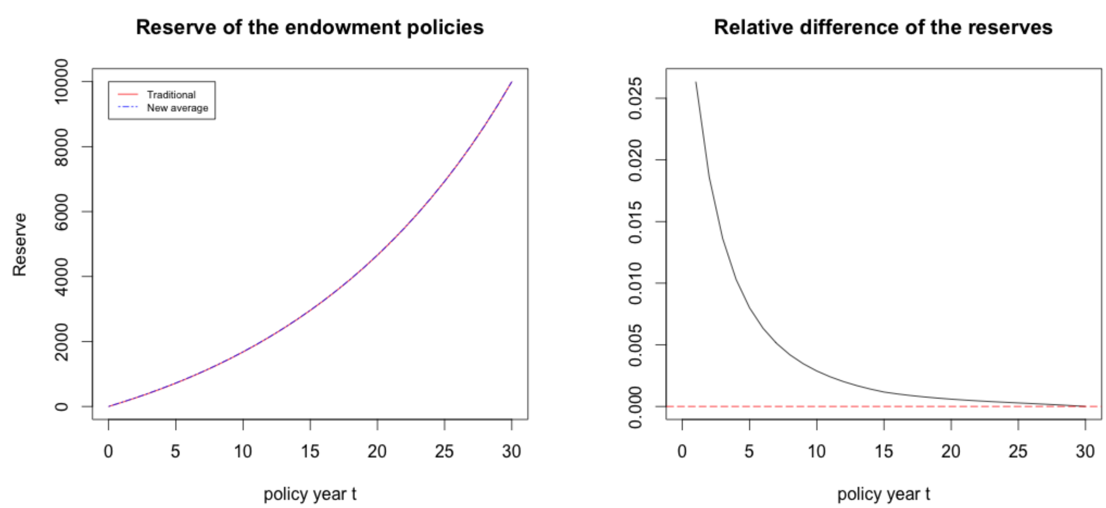


Figure 5.21: Reserve by status of the new product vs. the traditional reserve, endowment insurance, x=50, n=20

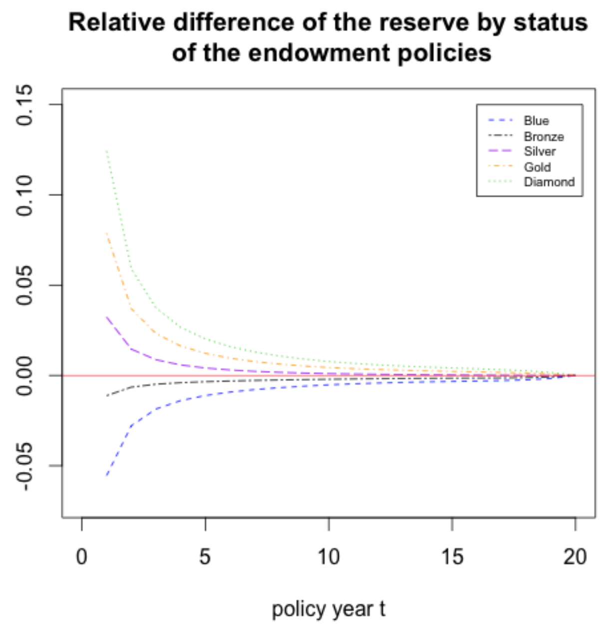
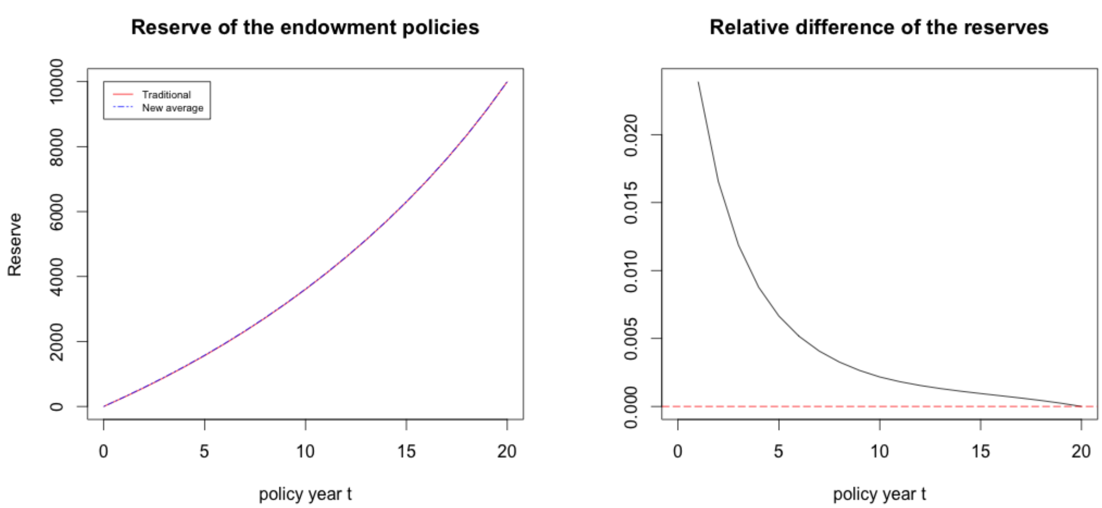


Figure 5.22: Average reserve of the new product vs. the traditional reserve, endowment insurance, x=50, n=20



5.4 Surplus

We use numerical examples to study the standard deviations of stochastic surplus and accounting surplus for temporary insurance and endowment insurance. Consider a homogeneous portfolio of 100 policies. The initial status distribution of the policyholders is Ω_0. We choose 10-year policies for temporary insurance, and 20-year policies for endowment insurance. Both are issued to insureds aged 30. The calculations for the new life insurance product use the formulas derived in Section 3.6. We refer to Nolde and Parker (2014) for the formulas to calculate the traditional life insurance surplus.

From Table 5.4 and Table 5.5, we could observe that for temporary insurance, both stochastic surplus and accounting surplus have smaller standard deviations for the new life insurance portfolio than that for the traditional portfolio. Thus for insurers maintaining a portfolio of temporary insurance with integrated wellness program, they would have a more stable financial position than with the traditional product.

Table 5.4: Standard deviation of stochastic surplus for 10-year temporary insurance issued to insureds at age 30; m=100.

t	Traditional product	New product	Relative difference
1	7,379.97	7,279.03	-1.3678%
2	7,821.95	7,715.77	-1.3575%
3	8,290.54	8,178.71	-1.3488%
4	8,787.34	8,669.44	-1.3417%
5	9,314.04	9,189.60	-1.3360%
6	9,872.42	9,740.98	-1.3314%
7	10,464.39	10,325.44	-1.3279%
8	11,091.93	10,944.96	-1.3250%
9	11,757.18	11,601.66	-1.3227%
10	0	0	-

Table 5.5: Standard deviation of accounting surplus for 10-year temporary insurance issued to insureds at age 30; m=100.

t	Traditional product	New product	Relative difference
1	2,905.45	2,879.71	-0.8860%
2	4,233.18	4,188.67	-1.0514%
3	5,353.87	5,292.00	-1.1556%
4	6,381.92	6,303.82	-1.2237%
5	7,374.23	7,280.52	-1.2707%
6	8,355.29	8,246.31	-1.3044%
7	9,340.44	9,216.28	-1.3293%
8	10,350.26	10,210.67	-1.3487%
9	11,391.50	11,236.09	-1.3643%
10	0	0	-

For the endowment insurance portfolio, Table 5.6 shows that the stochastic surplus has smaller standard deviations for the new life insurance than that for the traditional insurance in early stage. The relative difference of standard deviation increases with time and becomes positive in the end. Thus for insurers maintaining a portfolio of endowment insurance with integrated wellness program, their financial position is more stable in the early years while becoming more volatile in later periods, compared to insurers of the traditional insurance product. In terms of accounting surplus, the situations are more complicated. Table 5.7 shows that relative difference of the standard deviations of the accounting surplus is positive in the early and late stage and is negative in the middle.

Table 5.6: Standard deviation of stochastic surplus for 20-year endowment insurance issued to insureds at age 30; m=100.

t	Traditional product	New product	Relative difference
1	7,213.03	6,582.21	-8.7456%
2	7,609.22	6,977.07	-8.3078%
3	8,022.42	7,395.62	-7.8131%
4	8,453.51	7,839.30	-7.2657%
5	8,902.87	8,309.63	-6.6634%
6	9,371.23	8,808.19	-6.0081%
7	9,859.39	9,336.68	-5.3017%
8	10,367.77	9,896.88	-4.5419%
9	10,897.27	10,490.69	-3.7310%
10	11,448.96	11,120.14	-2.8721%
11	12,023.40	11,787.35	-1.9632%
12	12,621.24	12,494.60	-1.0034%
13	13,243.61	13,244.29	0.0051%
14	13,890.90	14,038.96	1.0658%
15	14,564.02	14,881.31	2.1786%
16	15,263.05	15,774.21	3.3490%
17	15,988.92	16,720.68	4.5767%
18	16,741.89	17,723.95	5.8659%
19	17,522.11	18,787.40	7.2211%
20	0	0	-

Table 5.7: Standard deviation of accounting surplus for 20-year endowment insurance issued to insureds at age 30; m=100.

t	Traditional product	New product	Relative difference
1	2,822.41	2,836.45	0.4974%
2	4,055.32	4,067.67	0.3046%
3	5,057.05	5,061.31	0.0842%
4	5,943.80	5,939.48	-0.0727%
5	6,771.18	6,759.05	-0.1791%
6	7,563.53	7,544.73	-0.2486%
7	8,335.55	8,311.59	-0.2874%
8	9,103.00	9,075.82	-0.2986%
9	9,870.84	9,843.34	-0.2786%
10	10,643.88	10,619.13	-0.2325%
11	11,427.55	11,410.74	-0.1471%
12	12,224.13	12,223.62	-0.0042%
13	13,035.45	13,060.12	0.1892%
14	13,860.72	13,925.12	0.4646%
15	14,698.49	14,821.10	0.8342%
16	15,542.90	15,751.76	1.3437%
17	16,395.95	16,722.09	1.9892%
18	17,252.26	17,737.45	2.8124%
19	18,111.68	18804.56	3.8256%
20	0	0	-

Chapter 6

Conclusions and discussions

This project, on the first attempt, models a new type of life insurance with an integrated health rewards program. We believe the contributions to the related fields in actuarial science are three-fold. First, it models the health rewards program by a homogeneous Markov chain process, which is adapted to describe the dynamics determined by two main types of health rating systems in the market. Second, it builds up a framework of multi-state life insurance model, with the embedded homogeneous Markov chain process describing the impact of the health rewards system on state transitions. Quantities of interest such as future benefit, premium, reserve and surplus are studied under the framework. Third, the comparison of the new type of life insurance with its traditional counterparts provides knowledge to insurers and policyholders about the advantages (e.g., improved health condition, reduced benefit payment, less average premium, etc.) and shortcomings (e.g., larger premium at Blue and Silver status, possibly higher reserve for each in-force policy on average, etc.) of the innovation.

Under the assumptions that the new life insurance portfolio starts with a group whose average mortality risks equal to those suggested in the life table (see Assumption 3.2.5 for details), and that the health rewards program effectively and reasonably promotes healthier lifestyle of the participants (see Assumption 3.2.4 for details), it has been proved that the expected future benefit payment is reduced for both temporary insurance and endowment insurance, compared to their corresponding traditional life insurance products. The fraction of the reduced benefit payment, which demonstrates the significance of reduction, can be viewed as the shared value margin created by the reduced mortality risks. The shared value margin is larger for temporary insurance than for endowment insurance, as is confirmed by the numerical results.

We further analyze the comparison on premium, reserve and surplus through numerical examples and reach the following conclusions with the specific parameters we choose. First, with pricing under the equivalent principle, the average premium of the new life insurance is higher in the beginning than the premium of the corresponding traditional life insurance, and then drops below that at a later stage. Policyholders at the superior health statuses benefit from lower premiums, while policyholders at the average and lower statuses have to pay premium surcharges. During the policy life of the new life insurance, an increasing portion of policyholders achieve the superior statuses. Second, for endowment policies, the insurer of the new life insurance is responsible for higher reserves on average for each of the in-force policies than the reserves of the traditional life insurance. For temporary insurance, the reserves are likely to be lower for policies with longer term and older age at issue. Third, the insurer of the new temporary insurance has a more stable financial position indicated by the less volatile surplus, while the situation for the insurer of the new endowment insurance varies along the policy life. These conclusions are limited to the assumptions we made and the premium structure we chose.

The work in this project is limited by the assumption of independent behaviour of the participants in the health rewards program, and by the assumption on the transition matrix P that it keeps the order of the health conditions of two groups. Further research is needed to understand its mathematical nature and its reasonableness in practice. In addition, only a single risk factor is considered, while in the market, several risk factors are handled together in one wellness program. Moreover, rewards and bonuses are actually more flexible. Our work can be extended by relieving the aforementioned limitations. Also, the model can be made more realistic by including expenses, lapses and stochastic interest rates.

With more effective tracking of the dynamic evolution of mortality risks, life insurance begins to develop features of non-life insurance. The further research of the insurance products with an integrated health rewards program depends not only on the developments of more advanced statistical models, but also on the progress of studies in various related fields. To better estimate the key model parameters, we would require a more solid method to quantify the effect of healthy behaviours on risk factors reduction, a more reliable approach to classify health conditions, a better estimate of the mortality risks of each class, and a more precise understanding of human's behaviour patterns and the impact of incentives. These were beyond the scope of this project but point to a valuable area for further research.

Bibliography

[1] AIA Australia Limited. (2015). *AIA Vitality Premium Adjustment Rules*. [online] Available at: https://login.aiavitality.com.au/AIA-Vitality-Premium-Adjustment-Rules.pdf [Accessed 20 Nov. 2016].

[2] Alwan, A. (2011). Burden: mortality, morbidity and risk factors. *Global status report on noncommunicable diseases 2010*, Geneva: World Health Organization, pp.9-31.

[3] Bloom, D.E., Cafiero, E.T., Jané-Llopis, E., Abrahams-Gessel, S., Bloom, L.R., Fathima, S., Feigl, A.B., Gaziano, T., Mowafi, M., Pandya, A., Prettner, K., Rosenberg, L., Seligman, B., Stein, A.Z., and Weinstein, C. (2011). *The Global Economic Burden of Noncommunicable Diseases*. Geneva: World Economic Forum.

[4] Discovery Vitality Ltd. (2016). *Main rules for Discovery Vitality.* [online] Available at: https://www.discovery.co.za/discovery_coza/web/linked_content/pdfs/vitality/benefit_rules/new_vitality_main_rules.pdf [Accessed 20 Nov. 2016].

[5] Discovery Vitality Ltd. (2015). *Summary of points.* [online] Available at: https://www.discovery.co.za/discovery_coza/web/linked_content/pdfs/vitality/points_and_partners.pdf [Accessed 20 Nov. 2016].

[6] Discovery. (2015). *People and Sustainable Development Report 2015.* [online] Available at: https://www.discovery.co.za/discovery_coza/web/investor_relations/results_and_reports/annual_reports/2015/downloads/People_and_Sustainable_Development_Report_2015.pdf [Accessed 20 Nov. 2016].

[7] Ezzati, M., Lopez, A.D., Rodgers, A. and Murray, C.J. (2004). Comparative quantification of health risks. *Global and regional burden of disease attributable to selected major risk factors.* Geneva: World Health Organization, pp.1987-1997.

[8] Gorin, M. and Schmidt, H. (2015). 'I Did it For the Money': Incentives, Rationalizations and Health. *Public Health Ethics*, 8(1), pp.34-41.

[9] Hougaard, P. (1999). Multi-state models: a review. *Lifetime data analysis*, 5(3), pp.239-264.

[10] John Hancock Life Insurance Company (U.S.A.). (2015). *CREATING SHARED VALUE FROM BETTER HEALTH: A Whole New Approach to Life Insurance.* [online] Available at: http://www.thevitalitygroup.com/wp-content/uploads/2015/04/John-Hancock.pdf [Accessed 28 Sept. 2016].

[11] Kwon, H.S. and Jones, B.L. (2008). Applications of a multi-state risk factor/mortality model in life insurance. *Insurance: Mathematics and Economics*, 43(3), pp.394-402.

[12] Lemaire, J. (2012). *Bonus-malus systems in automobile insurance.* Boston/Dordrecht/London: Kluwer Academic Publishers.

[13] Life Extension Institute, Inc. (1919). *Prolonging life as a function of life insurance, five years' experience of the Life Extension Institute.* New York: The Marchbanks Press, pp.7.

[14] Mathers, C., Fat, D.M. and Boerma, J.T. (2008). *The global burden of disease: 2004 update.* World Health Organization. Available at: `http://www.who.int/healthinfo/global_burden_disease/GBD_report_2004update_full.pdf` [Accessed 20 Nov. 2016].

[15] Nolde, N. and Parker, G. (2014). Stochastic analysis of life insurance surplus. *Insurance: Mathematics and Economics*, 56, pp.1-13.

[16] Sirmans, J.R. Jr. (2004). *System and method for evaluating insurance member activity and pricing insurance products.* U.S. Patent Application 10/974,186.

[17] Slomovic, A. (2015). eHealth and Privacy in US Employer Wellness Programs. *"Under Observation-The Interplay between eHealth and Surveillance", Ronald Leenes, Nadezhda Purtova, Samantha Adams (eds.), Springer, Forthcoming.* [online] Available at: `https://ssrn.com/abstract=2613452` [Accessed 20 Nov. 2016].

[18] Slomovic, A. (2015). Life Insurance, Wellness and Privacy. [online] Available at: `http://www.annaslomovic.com/single-post/2015/04/14/Life-Insurance-Wellness-and-Privacy` [Accessed 20 Nov. 2016].

[19] World Health Organization. (2002). *The world health report 2002: reducing risks, promoting healthy life.* [online] Available at: `http://www.who.int/whr/2002/en/` [Accessed 20 Nov. 2016].

[20] Xia, Y. (2012). *Analysis of long-term disability insurance portfolios with stochastic interest rates and multi-state transition models.* MSc. project report. Simon Fraser University.

Appendix A

Proof

Before proving Lemma 4.1.2, we introduce the following lemma.

Lemma A.0.1. *For a transition matrix P satisfying Assumption 3.2.4, Ω_0 is a distribution vector assumed in Assumption 3.2.5, we have*

$$\Omega_0^T \cdot {}_kP_x \cdot P \cdot \Theta \geq 0, \quad k = 1, 2, \cdots, \tag{A.1}$$

where ${}_kP_x$ can be computed recursively by the following formulas:

$$\begin{cases} {}_kP_x = {}_{k-1}P_x \cdot \Phi_{x+k-1} \cdot P, & \text{for } k = 2, 3, \cdots, \\ {}_1P_x = \Phi_x \cdot P, & \text{for } k = 1, \end{cases}$$

where Φ_x is defined by (3.9).

Proof. We first define $g(\cdot)$ as a function of a matrix $H = \{h_{ij}\}_{n\times m}$ calculating the sum of all entries, that is

$$g(H) = \sum_{i=1}^{n}\sum_{j=1}^{m} h_{ij}.$$

For a given vector $\mathbf{f} = (f_1, ..., f_L)^T$ such that $0 \leq f_i \leq 1, i = 1, 2, \ldots, L$, if $g(\mathbf{f}) = 1$, we can conclude that $\mathbf{f}$ is a distribution vector.

We now use the method of mathematical induction to prove (A.1).
STEP 1: For k=1,

$$\Omega_0^T \cdot {}_1P_x \cdot \Theta = \Omega_0^T \cdot \Phi_x \cdot P \cdot \Theta.$$

Since $(1 - q_x^i) > 0, i = 0, 1, \cdots, L-1$, and at least one of the entries of Ω_0^T is positive, $g(\Omega_0^T \cdot \Phi_x) = \sum_{i=0}^{L-1} \omega_i \cdot (1 - q_x^i) \triangleq \bar{q}_x^\omega > 0$. Further,

$$g\left(\frac{\Omega_0^T \cdot \Phi_x}{\bar{q}_x^\omega}\right) = \frac{\bar{q}_x^\omega}{\bar{q}_x^\omega} = 1$$

implies that $\Omega_0^T \cdot \Phi_x / \bar{q}_x^\omega$ is a distribution vector.

First, we prove that $(\Omega_0^T \cdot \Phi_x / \bar{q}_x^\omega) \cdot \Theta \geq \Omega_0^T \cdot \Theta$. As $\bar{q}_x^\omega$ being positive, we would have $\Omega_0^T \cdot \Phi_x \cdot \Theta \geq \Omega_0^T \cdot \Theta$. Actually, we have

$$
\begin{aligned}
& \frac{\Omega_0^T \cdot \Phi_x}{\bar{q}_x^\omega} \cdot \Theta - \Omega_0^T \cdot \Theta \\
= \quad & \left(\frac{\Omega_0^T \cdot \Phi_x}{\bar{q}_x^\omega} - \Omega_0^T \right) \cdot \Theta \\
= \quad & \frac{1}{\bar{q}_x^\omega} \sum_{i=0}^{L-1} \left[\omega_i \cdot (1 - q_x^i) - \omega_i \cdot \sum_{j=0}^{L-1} \omega_j \cdot (1 - q_x^j) \right] \cdot \theta_i \\
= \quad & \frac{1}{\bar{q}_x^\omega} \left[\sum_{i=0}^{L-1} \theta_i \cdot \omega_i \cdot (1 - q_x^i) - \sum_{i=0}^{L-1} \theta_i \cdot \omega_i \cdot (\sum_{j=0}^{L-1} \omega_j - \sum_{j=0}^{L-1} \omega_j \cdot q_x^j) \right] \\
= \quad & \frac{1}{\bar{q}_x^\omega} \left[\sum_{i=0}^{L-1} \theta_i \cdot \omega_i \cdot (-q_x^i) + \sum_{i=0}^{L-1} \theta_i \cdot \omega_i \cdot \sum_{j=0}^{L-1} \omega_j \cdot q_x^j \right] \qquad \left(\text{by } \sum_{j=0}^{L-1} \omega_j = 1 \right) \\
= \quad & \frac{1}{\bar{q}_x^\omega} \left[- \sum_{i=0}^{L-1} \theta_i \cdot \omega_i \cdot q_x \cdot (1 - \delta_x \cdot \theta_i) + \sum_{i=0}^{L-1} \theta_i \cdot \omega_i \cdot \sum_{j=0}^{L-1} \omega_j \cdot q_x \cdot (1 - \delta_x \cdot \theta_j) \right] \\
= \quad & \frac{1}{\bar{q}_x^\omega} \left[-q_x \sum_{i=0}^{L-1} \theta_i \cdot \omega_i + q_x \cdot \delta_x \sum_{i=0}^{L-1} \theta_i^2 \cdot \omega_i + (q_x - q_x \cdot \delta_x \sum_{j=0}^{L-1} \omega_j \cdot \theta_j) \sum_{i=0}^{L-1} \theta_i \cdot \omega_i \right] \\
= \quad & \frac{1}{\bar{q}_x^\omega} \left[q_x \cdot \delta_x \cdot \left[\sum_{i=0}^{L-1} \theta_i^2 \cdot \omega_i - \left(\sum_{i=0}^{L-1} \theta_i \cdot \omega_i \right)^2 \right] \right] \\
\geq \quad & 0,
\end{aligned}
\tag{A.2}
$$

where $\left[\sum_{i=0}^{L-1} \theta_i^2 \cdot \omega_i - \left(\sum_{i=0}^{L-1} \theta_i \cdot \omega_i \right)^2 \right]$ can be viewed as a variance, since $\Omega_0 = [\omega_0, \cdots, \omega_{L-1}]^T$ is a distribution vector. Then by (A.2), we have

$$
\begin{aligned}
& \frac{\Omega_0^T \cdot \Phi_x}{\bar{q}_x^\omega} \cdot \Theta \geq \Omega_0^T \cdot \Theta = 0 \\
\Rightarrow \quad & \frac{\Omega_0^T \cdot \Phi_x}{\bar{q}_x^\omega} \cdot P \cdot \Theta \geq \Omega_0^T \cdot P \cdot \Theta \geq \Omega_0^T \cdot \Theta = 0 \qquad \text{(by Assumption 3.2.4 and 3.2.5)} \\
\Rightarrow \quad & \frac{\Omega_0^T \cdot \Phi_x}{\bar{q}_x^\omega} \cdot P^2 \cdot \Theta \geq \Omega_0^T \cdot P^2 \cdot \Theta \geq \Omega_0^T \cdot P \cdot \Theta \geq \Omega_0^T \cdot \Theta = 0 \quad \text{(by Assumption 3.2.5)} \\
\Rightarrow \quad & \Omega_0^T \cdot \Phi_x \cdot P^2 \cdot \Theta \geq 0 \\
\Leftrightarrow \quad & \Omega_0^T \cdot {}_1P_x \cdot P \cdot \Theta \geq 0.
\end{aligned}
$$

STEP 2: Suppose that the inequality equation holds for some $t = k \geq 1$, that is,

$$\Omega_0^T \cdot {}_kP_x \cdot P \cdot \Theta \geq 0. \tag{A.3}$$

STEP 3: Prove that the inequality equation holds for $t = k + 1$. First, we have

$$\Omega_0^T \cdot {}_{k+1}P_x \cdot P \cdot \Theta = \Omega_0^T \cdot {}_kP_x \cdot \Phi_{x+k} \cdot P \cdot \Theta.$$

Further by (A.3), we get

$$\frac{\Omega_0^T \cdot {}_kP_x}{g(\Omega_0^T \cdot {}_kP_x)} \cdot P \cdot \Theta \geq 0. \tag{A.4}$$

Let $\Omega_k^T = (\Omega_0^T \cdot {}_kP_x)/g(\Omega_0^T \cdot {}_kP_x)$. Since

$$g(\Omega_k^T) = \frac{g(\Omega_0^T \cdot {}_kP_x)}{g(\Omega_0^T \cdot {}_kP_x)} = 1,$$

and entries of Ω_k^T are all non-negative, Ω_k^T is a distribution vector. Similar to the process of (A.2), it can be proved that

$$\frac{\Omega_k^T \cdot \Phi_{x+k}}{g(\Omega_k^T \cdot \Phi_{x+k})} \cdot \Theta \geq \Omega_k^T \cdot \Theta. \tag{A.5}$$

Then by (A.5), we have

$$\begin{aligned}
& \frac{\Omega_k^T \cdot \Phi_{x+k}}{g(\Omega_k^T \cdot \Phi_{x+k})} \cdot P \cdot \Theta \geq \Omega_k^T \cdot P \cdot \Theta \qquad \text{(by Assumption 3.2.4)} \\
\Rightarrow \quad & \frac{\Omega_0^T \cdot {}_kP_x \cdot \Phi_{x+k} \cdot P \cdot \Theta}{g(\Omega_0^T \cdot {}_kP_x) \cdot g(\Omega_k^T \cdot \Phi_{x+k})} \geq \frac{\Omega_0^T \cdot {}_kP_x}{g(\Omega_0^T \cdot {}_kP_x)} \cdot P \cdot \Theta \geq 0 \quad \left(\text{by } \Omega_k^T = \frac{\Omega_0^T \cdot {}_kP_x}{g(\Omega_0^T \cdot {}_kP_x)} \text{ and (A.4)}\right) \\
\Rightarrow \quad & \Omega_0^T \cdot {}_kP_x \cdot \Phi_{x+k} \cdot P \cdot \Theta \geq 0 = \Omega_0^T \cdot \Theta \\
\Rightarrow \quad & \Omega_0^T \cdot {}_kP_x \cdot \Phi_{x+k} \cdot P^2 \cdot \Theta \geq \Omega_0^T \cdot P \cdot \Theta \geq \Omega_0^T \cdot \Theta = 0 \\
\Leftrightarrow \quad & \Omega_0^T \cdot {}_{k+1}P_x \cdot P \cdot \Theta \geq 0.
\end{aligned}$$

Therefore, by the mathematical induction, it has been proved that

$$\Omega_0^T \cdot {}_kP_x \cdot P \cdot \Theta \geq 0, \quad k = 1, 2, \cdots.$$

$\square$

A.1 Proof of Lemma 4.1.2

Proof. We use the method of mathematical induction to prove it.
STEP 1: For t=1, by the definition of p_x^{si} in equation (3.8), we have

$$\sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} p_x^{si} = \sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} (1 - q_x^i) p_{si}$$

$$
\begin{aligned}
&= [\omega_0, \omega_1, \ldots, \omega_{L-1}] \cdot \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,L-1} \\ p_{10} & p_{11} & \cdots & p_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L-1,0} & p_{L-1,1} & \cdots & p_{L-1,L-1} \end{bmatrix} \cdot \begin{bmatrix} 1 - q_x^0 \\ 1 - q_x^1 \\ \vdots \\ 1 - q_x^{L-1} \end{bmatrix} \\
&= \Omega^T \cdot P \cdot (\mathbf{1} - Q_x) \\
&= \Omega^T \cdot P \cdot \mathbf{1} - \Omega^T \cdot P \cdot Q_x \\
&= 1 - \Omega^T \cdot P \cdot Q_x \\
&\geq 1 - \Omega^T \cdot Q_x \qquad \text{(by Assumption 3.2.5)} \\
&= 1 - q_x \\
&= p_x,
\end{aligned}
$$

where $\mathbf{1} = [1, 1, \ldots, 1]^T$ is of dimension $(L-1)$.

STEP 2: Suppose that the inequality equation holds for some $t = k \geq 1$, that is,

$$
\begin{aligned}
{}_kp_x &\leq \sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} {}_kp_x^{si} \\
&= [\omega_0, \omega_1, \ldots, \omega_{L-1}] \cdot \begin{bmatrix} {}_kp_x^{00} & {}_kp_x^{01} & \cdots & {}_kp_x^{0,L-1} \\ {}_kp_x^{10} & {}_kp_x^{11} & \cdots & {}_kp_x^{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ {}_kp_x^{L-1,0} & {}_kp_x^{L-1,1} & \cdots & {}_kp_x^{L-1,L-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\
&= \Omega^T \cdot {}_kP_x \cdot \mathbf{1},
\end{aligned}
$$

where ${}_kP_x$ is a matrix composed of $\{{}_kp_x^{ij}\}, i, j \in \{0, 1, \ldots, L-1\}$.

STEP 3: Prove that the inequality equation holds for $t = k+1$. We have

$$
\begin{aligned}
\sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} {}_{k+1}p_x^{si} &= \sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} {}_kp_x^{sj} \cdot p_{x+k}^{ji} \\
&= \sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} {}_kp_x^{sj} \cdot p_{ji} \cdot (1 - q_{x+k}^i) \\
&= \Omega^T \cdot \begin{bmatrix} {}_kp_x^{00} & {}_kp_x^{01} & \cdots & {}_kp_x^{0,L-1} \\ {}_kp_x^{10} & {}_kp_x^{11} & \cdots & {}_kp_x^{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ {}_kp_x^{L-1,0} & {}_kp_x^{L-1,1} & \cdots & {}_kp_x^{L-1,L-1} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,L-1} \\ p_{10} & p_{11} & \cdots & p_{1,L-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L-1,0} & p_{L-1,1} & \cdots & p_{L-1,L-1} \end{bmatrix} \cdot \begin{bmatrix} 1 - q_{x+k}^0 \\ 1 - q_{x+k}^1 \\ \vdots \\ 1 - q_{x+k}^{L-1} \end{bmatrix} \\
&= \Omega^T \cdot {}_kP_x \cdot P \cdot (\mathbf{1} - Q_{x+k}) \\
&= \Omega^T \cdot {}_kP_x \cdot \mathbf{1} - \Omega^T \cdot {}_kP_x \cdot P \cdot Q_{x+k}. \qquad \text{(A.6)}
\end{aligned}
$$

On the other hand,

$$\begin{aligned} {}_{k+1}p_x &= kp_x \cdot p_{x+k} \\ &\le \Omega^T \cdot {}_kP_x \cdot \mathbf{1} \cdot p_{x+k} \\ &= \Omega^T \cdot {}_kP_x \cdot \mathbf{1} \cdot (1 - q_{x+k}) \\ &= \Omega^T \cdot {}_kP_x \cdot \mathbf{1} - \Omega^T \cdot {}_kP_x \cdot \mathbf{1} \cdot q_{x+k}. \end{aligned} \tag{A.7}$$

Comparing (A.6) and (A.7), if $\Omega^T \cdot {}_kP_x \cdot \mathbf{1} \cdot q_{x+k} \ge \Omega^T \cdot {}_kP_x \cdot P \cdot Q_{x+k}$, then the inequality holds for $t = k + 1$. Actually, we have

$$\begin{aligned} & \Omega^T \cdot {}_kP_x \cdot \mathbf{1} \cdot q_{x+k} - \Omega^T \cdot {}_kP_x \cdot P \cdot Q_{x+k} \\ = \;& \Omega^T \cdot {}_kP_x \cdot \mathbf{1} \cdot q_{x+k} - \Omega^T \cdot {}_kP_x \cdot P \cdot (q_{x+k} \cdot \mathbf{1} - q_{x+k} \cdot \delta_{x+k} \cdot \Theta) \qquad \text{(by (3.1))} \\ = \;& \Omega^T \cdot {}_kP_x \cdot P \cdot q_{x+k} \cdot \delta_{x+k} \cdot \Theta \\ \ge \;& 0. \qquad \text{(by Lemma A.0.1)} \end{aligned} \tag{A.8}$$

Hence, by (A.8),

$$\Omega^T \cdot {}_kP_x \cdot \mathbf{1} \cdot q_{x+k} \ge \Omega^T \cdot {}_kP_x \cdot P \cdot Q_{x+k},$$

implying

$${}_{k+1}p_x \le \sum_{s=0}^{L-1} \omega_s \sum_{i=0}^{L-1} {}_{k+1}p_x^{si}.$$

Therefore, by the mathematical induction, the lemma has been proved. □

Appendix B

Mortality Table

Table B.1: 1997-04 Canadian Institute of Actuaries (CIA) Mortality Table - Male.

x	q_x	x	q_x	x	q_x	x	q_x	x	q_x
15	0.00032	37	0.00090	59	0.00626	81	0.05796	103	0.45000
16	0.00038	38	0.00093	60	0.00705	82	0.06487	104	0.45000
17	0.00043	39	0.00096	61	0.00801	83	0.07305	105	0.45000
18	0.00049	40	0.00101	62	0.00908	84	0.08259	106	0.45000
19	0.00054	41	0.00108	63	0.01028	85	0.09372	107	0.45000
20	0.00059	42	0.00115	64	0.01160	86	0.10681	108	0.45000
21	0.00063	43	0.00125	65	0.01304	87	0.12232	109	0.45000
22	0.00067	44	0.00135	66	0.01462	88	0.14018	110	0.45000
23	0.00070	45	0.00148	67	0.01632	89	0.16018	111	0.45000
24	0.00073	46	0.00162	68	0.01816	90	0.18264	112	0.45000
25	0.00076	47	0.00178	69	0.02013	91	0.20160	113	0.45000
26	0.00078	48	0.00196	70	0.02225	92	0.22283	114	0.45000
27	0.00080	49	0.00216	71	0.02450	93	0.24583	115	0.45000
28	0.00081	50	0.00238	72	0.02691	94	0.26983	116	0.45000
29	0.00082	51	0.00263	73	0.02946	95	0.29383	117	0.45000
30	0.00083	52	0.00291	74	0.03216	96	0.31783	118	0.45000
31	0.00083	53	0.00323	75	0.03501	97	0.34183	119	0.45000
32	0.00084	54	0.00359	76	0.03802	98	0.36583	120	1.00000
33	0.00084	55	0.00400	77	0.04119	99	0.38983		
34	0.00085	56	0.00445	78	0.04453	100	0.41423		
35	0.00086	57	0.00498	79	0.04802	101	0.43973		
36	0.00087	58	0.00558	80	0.05169	102	0.45000		

Appendix C

The status distribution evolution through P

Table C.1: The status distribution evolution of the homogeneous Markov chain process

Year \ Status	Blue	Bronze	Silver	Gold	Diamond
1	30.00%	40.00%	15.00%	10.00%	5.00%
2	19.05%	26.25%	23.70%	19.25%	11.75%
3	14.42%	21.44%	23.34%	25.13%	15.68%
4	12.36%	19.09%	22.82%	28.07%	17.66%
5	11.39%	17.97%	22.51%	29.50%	18.63%
6	10.92%	17.43%	22.36%	30.19%	19.10%
7	10.70%	17.17%	22.29%	30.52%	19.32%
8	10.59%	17.05%	22.25%	30.68%	19.43%
9	10.54%	16.99%	22.23%	30.76%	19.48%
10	10.52%	16.96%	22.23%	30.79%	19.50%
11	10.50%	16.95%	22.22%	30.81%	19.52%
12	10.50%	16.94%	22.22%	30.82%	19.52%
13	10.50%	16.94%	22.22%	30.82%	19.52%
14	10.49%	16.93%	22.22%	30.83%	19.53%
15	10.49%	16.93%	22.22%	30.83%	19.53%
16	10.49%	16.93%	22.22%	30.83%	19.53%
17	10.49%	16.93%	22.22%	30.83%	19.53%
18	10.49%	16.93%	22.22%	30.83%	19.53%
19	10.49%	16.93%	22.22%	30.83%	19.53%
20	10.49%	16.93%	22.22%	30.83%	19.53%