Arithmetic aspects of the Burkhardt quartic threefold

by

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Abstract

The Burkhardt quartic is a 3-dimensional projective hypersurface defined over the rational numbers. It is known that sufficiently general points on the Burkhardt quartic parameterize abelian surfaces with a full level 3 structure. Furthermore, it is classical that the Burkhardt quartic is birational to 3-dimensional projective space after adjoining a cube root of unity. In this thesis we will show that the Burkhardt quartic is birational to 3-dimensional projective space over the rational numbers, and describe a geometric method of constructing a generic family of hyperelliptic curves corresponding to points on the Burkhardt quartic, whose Jacobians have a full level 3 structure. Specifically, we give an explicit family of hyperelliptic curves which contain almost all complex genus 2 curves with a full level 3 structure.

Keywords: level three structure; genus 2 curves; moduli of curves; arithmetic geometry; Burkhardt quartic; Abelian surfaces

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Introduction and Outline of Results

The Burkhardt quartic threefold is a hypersurface in $\mathbb{P}^4_{\mathbb{Q}}$ defined by the equation

$$\mathcal{B}\colon y_0^4 + 8y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 48y_1y_2y_3y_4 = 0. \tag{1}$$

Our motivation for studying $\mathcal{B}$ is that it has a moduli interpretation. In particular, sufficiently general points on $\mathcal{B}$ correspond to abelian surfaces with a full level three structure. For this reason, the Burkhardt quartic has been studied extensively. The original work by Burkhardt himself can be found in [4] and [5]. Further classical work by Coble is found in [8]. Van der Geer studied $\mathcal{B}$ and the moduli interpretation with modern language in [34]. More recent work involving the moduli interpretation and abelian surfaces can be found in [10]. For a connection to toric/tropical geometry see [31]. In [13], Elkies discusses the moduli interpretation from an arithmetic minded point of view. For connections to modular forms see [15], [16], and [22].

It is classical that $\mathcal{B}$ is birationally equivalent to $\mathbb{P}^3_{\mathbb{Q}(\zeta_3)}$ (see [25, 5.2.7]). According to Hunt in [25, 5.2.7] the following rationalization was given by Baker in 1942, and verified by Finkelnberg in [14]:

$$\begin{aligned}
\xi_0 &= t_3\left(t_0(t_1^2 + t_1t_2 + t_2^2) + t_1t_2(t_1 + t_2) - t_3^3\right),\\
\xi_1 &= -t_3\left((1-\zeta_3)t_0^2(t_1 - \zeta_3^2t_2) - \zeta_3t_0(t_1 - \zeta_3^2t_2)^2 - \zeta_3t_1t_2(t_1 + \zeta_3t_2) + t_3^3\right),\\
\xi_2 &= -t_1^2t_2^2 - t_0t_1t_2(t_1 + t_2) + (t_0 + t_1 + t_2)t_3^3,\\
\xi_3 &= -\zeta_3t_0^2(t_1 - \zeta_3^2t_2)^2 + (1-\zeta_3)t_0t_1t_2(t_1 - \zeta_3^2t_2) + t_1^2t_2^2 - \zeta_3(t_1 + \zeta_3t_2)t_3^3,\\
\xi_4 &= t_0\left(t_0(t_1^2 + t_1t_2 + t_2^2) + t_1t_2(t_1 + t_2) - t_3^3\right).
\end{aligned}$$

We will remove the dependency on a cube root of unity and prove the following.

Theorem A. *The following 4 forms define a birational map* $\mathbb{P}^3_{\mathbb{Q}} \dashrightarrow \mathcal{B}$.

$$\begin{aligned}
\mathcal{Q}_0 &= -x^4 + 4x^3y - 6x^2y^2 + 5xy^3 - 2y^4 - xyz^2 + y^2z^2 \\
&\quad + \frac{1}{3}xz^3 - \frac{1}{3}yz^3 - xyzw + y^2zw + \frac{2}{3}xz^2w - \frac{2}{3}yz^2w \\
&\quad - xyw^2 + y^2w^2 + \frac{2}{3}xzw^2 - \frac{2}{3}yzw^2 + \frac{1}{3}xw^3 - \frac{1}{3}yw^3, \\
\mathcal{Q}_1 &= -x^3y + 3x^2y^2 - 3xy^3 + 2y^4 + x^3z - 3x^2yz \\
&\quad + 3xy^2z - 2y^3z + \frac{1}{3}yz^3 - \frac{1}{9}z^4 + x^3w - 3x^2yw \\
&\quad + 3xy^2w - 2y^3w + \frac{2}{3}yz^2w - \frac{2}{9}z^3w + \frac{2}{3}yzw^2 - \frac{1}{3}z^2w^2 \\
&\quad + \frac{1}{3}yw^3 - \frac{2}{9}zw^3 - \frac{1}{9}w^4, \\
\mathcal{Q}_2 &= -x^4 + 4x^3y - 6x^2y^2 + 5xy^3 - 2y^4 - \frac{1}{3}xz^2w \\
&\quad + \frac{1}{3}yz^2w - \frac{1}{3}xzw^2 + \frac{1}{3}yzw^2 - \frac{1}{3}xw^3 + \frac{1}{3}yw^3, \\
\mathcal{Q}_3 &= -x^3y + 3x^2y^2 - 3xy^3 + 2y^4 - \frac{1}{3}yz^3 + \frac{1}{9}z^4 \\
&\quad + x^3w - 3x^2yw + 3xy^2w - 2y^3w - \frac{1}{3}yz^2w \\
&\quad + \frac{2}{9}z^3w - \frac{1}{3}yzw^2 + \frac{1}{3}z^2w^2 + \frac{2}{9}zw^3 + \frac{1}{9}w^4, \\
\mathcal{Q}_4 &= -x^3y + 3x^2y^2 - 3xy^3 + 2y^4 - y^2z^2 + \frac{1}{3}yz^3 \\
&\quad - y^2zw + \frac{2}{3}yz^2w - y^2w^2 + \frac{2}{3}yzw^2 + \frac{1}{3}yw^3.
\end{aligned}$$

It is classical that for each sufficiently general $\mathbb{C}$-valued point $\alpha \in \mathcal{B}$ there is a smooth genus 2 curve $\mathcal{C}_\alpha$, and an isomorphism

$$\psi_\alpha \colon \mathrm{Jac}(\mathcal{C}_\alpha)[3](\mathbb{C}) \to (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$$

that respects the Weil pairing on $\mathrm{Jac}(\mathcal{C})[3](\mathbb{C})$ in a suitable sense. Furthermore, any curve that admits such an isomorphism lies in the isomorphism class of $\mathcal{C}_\alpha$ for some $\alpha \in \mathcal{B}$. The curve and the isomorphism can be recovered from α and $\mathcal{B}$ in the form of a conic with 6 marked points. Over an algebraically closed field, a conic is isomorphic to $\mathbb{P}^1$, so the 6 marked points can be used to construct a double cover of $\mathbb{P}^1$ ramified at 6 points. Over an algebraically closed field this determines the isomorphism class of a genus 2 curve. Over a non-algebraically closed field, such as $\mathbb{Q}$, the situation is more complicated; a conic may fail to be isomorphic to $\mathbb{P}^1$. In this case, the Galois-stable isomorphism class of genus 2 curves over $\overline{\mathbb{Q}}$ determined by the 6 points does not contain a genus 2 curve defined over $\mathbb{Q}$ (see [29]). Thus, to find a model of $\mathcal{C}_\alpha$ we must find an isomorphism between the conic and

$\mathbb{P}^1$. This is accomplished by constructing an isomorphism between the conic and a suitable twisted space cubic. Furthermore, over fields where not all non-zero elements are squares, there are multiple non-isomorphic genus 2 curves that are geometrically isomorphic. Thus we also need to determine an appropriate twist of the curve recovered from the conic. We prove the following.

Theorem B. *Let $\alpha = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) \in \mathcal{B}$ be a sufficiently general point. The curve $\mathcal{C}_\alpha$ is given by the model*

$$y^2 = G^2 + \lambda H^3,$$

where

$$\begin{aligned} G &= \left(-\frac{1}{2}\alpha_0^3\alpha_2^3 - \frac{1}{2}\alpha_0^3\alpha_4^3 - 8\alpha_1^3\alpha_4^3\right)\alpha_0^{-1}x^3 + \left(\frac{3}{2}\alpha_0^2\alpha_2^2\alpha_3 - 6\alpha_1^2\alpha_2\alpha_4^2\right)x^2 \\ &\quad - \left(\frac{3}{2}\alpha_0^2\alpha_2\alpha_3^2 - 6\alpha_1^2\alpha_3\alpha_4^2\right)x + \frac{1}{2}\alpha_0^2\alpha_3^3 - \frac{1}{2}\alpha_0^2\alpha_4^3, \\ H &= x^2 + \frac{1}{2}\frac{\alpha_0\alpha_2}{(\alpha_1\alpha_4)}x - (\frac{1}{2}\frac{\alpha_0\alpha_3}{(\alpha_1\alpha_4)}, \\ \lambda &= (-8\alpha_0^3\alpha_1^3\alpha_4^6 - 64\alpha_1^6\alpha_4^6)\alpha_0^{-2}. \end{aligned}$$

We see that

$$\frac{(y-G)(y+G)}{\lambda} = H^3.$$

The existence of such an equation means that there is a divisor T on $\mathcal{C}_\alpha$ such that

$$\operatorname{div}(y - G) = 3T.$$

Thus our equation specifies two 3-torsion classes on $\mathcal{C}$: the classes of T and $-T$. Our computations give enough information to compute the remaining 3-torsion classes and write down an explicit isomorphism

$$\operatorname{Jac}(\mathcal{C}_\alpha)[3](\mathbb{C}) \to (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2.$$

Chapter 1

Preliminaries

Given a field k, we will use the notation $\overline{k}$ to denote an algebraic closure. Unless otherwise stated, we will take ζ_3 to be a cube root of unity. Define

$$B = y_0^4 + 8y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 48y_1y_2y_3y_4.$$

Unless otherwise specified the statement $\alpha \in \mathcal{B}$ will mean that $\alpha \in \mathcal{B}(K)$ for some field extension $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$. If $\{F_i\}_{i\in I}$ is a collection of polynomials we will denote the subscheme cut out by these polynomials by $V(\{F_i\}_{i\in I})$. The context will make clear the ambient space of $V(\{F_i\}_{i\in I})$. If X is a projective variety with a k-rational point x, we we will let $\mathbb{T}_x(X)$ be the tangent space of X at x.

1.1 Algebraic Geometry

We assume familiarity with modern algebraic geometry at the level of Hartshorne [20] or Liu [27]. When possible we will use classical language at the level of Hindry-Silverman [21].

1.1.1 Hyperelliptic Curves

In this section we will review some of the theory of hyperelliptic curves. Fix a field k of characteristic 0 and take an algebraic closure $\overline{k}$. We will follow [6],[27] and [3] here. The curves of interest are those that have a model of the form

$$y^2 = f(x),$$

where $f(x) \in k[x]$ is of degree 6 and has non-zero discriminant. From [6], such curves are of genus 2, and every genus 2 curve is birationally equivalent to such a curve. Following [6] we will denote such a curve $\mathcal{C}$. There is an alternative description of the curve $\mathcal{C}$ as given in [27].

Definition 1.1. Let X be a smooth geometrically connected curve over a field k, of genus $g \geq 1$. We say that X is *hyperelliptic* if there is a separable degree 2 morphism $\phi \colon X \to \mathbb{P}^1_k$.

If $\mathcal{C}$ is a hyperelliptic curve, one can show (see [27, 4.27]) that there is an automorphism $\iota \colon \mathcal{C} \to \mathcal{C}$ of degree 2 called the *hyperelliptic* involution. We say that a point $x \in \mathcal{C}(k)$ is ramified if $\iota(x) = x$. One can show that the descriptions are equivalent, and we have the following.

Theorem 1.2 ([27, 4.29])**.** *Let $\mathcal{C}$ be a hyperelliptic curve over k of genus $g \geq 2$. Then $\mathcal{C}$ has a unique separable degree 2 map $\mathcal{C} \to \mathbb{P}^1_k$. In other words, $\mathcal{C}$ is a hyperelliptic curve in a unique way.*

Given a genus 2 curve $\mathcal{C}$, we write $\mathrm{Div}(\mathcal{C})$ to denote the group of divisors of $\mathcal{C}$ over $\overline{k}$ and $\mathrm{Pic}^j(\mathcal{C})$ for the group of degree j divisor classes over $\overline{k}$. We will write $\mathrm{Div}_k(\mathcal{C})$ for the divisors of $\mathrm{Div}(\mathcal{C})$ that are $\mathrm{Gal}(\overline{k}/k)$ invariant and $\mathrm{Pic}^j_k(\mathcal{C})$ for the divisor classes of degree j that are $\mathrm{Gal}(\overline{k}/k)$ invariant. We will mostly be concerned with the divisors of degree 0, degree 1, degree 2, and degree 3. Another object of interest is

$$\mathrm{Pic}^0_k(\mathcal{C})[3] = \{[D] \in \mathrm{Pic}^0_k(\mathcal{C}) : [3D] = 0\}.$$

That is to say, we will be interested in the three torsion of Pic^0_k. There is a good way of representing a 3-torsion element in terms of a hyperelliptic equation for $\mathcal{C}$.

Lemma 1.3 (see [3] Lemma 3)**.** *Let $\mathcal{C}$ be a curve of genus 2, given by an equation $y^2 = f(x)$. Then a decomposition*

$$f(x) = G(x)^2 + \lambda H(x)^3,$$

with $\deg G(x) \leq 3$ and $H(x)$ quadratic, corresponds to a 3-torsion point of $\mathrm{Pic}^0_k(\mathcal{C})[3]$. Specifically, let x, x' be the roots of H. Then the divisor

$$[(x, G(x) + (x', G(x')) - \kappa]$$

where κ is a canonical divisor, is a three-torsion class of $\mathrm{Pic}^0_k(\mathcal{C})[3]$. Furthermore, one can choose f in such a way as every non-zero point of $\mathrm{Pic}^0_k(\mathcal{C})[3]$ arises precisely in this way.

One can in fact say more.

Lemma 1.4 (see [3] Lemma 3)**.** *Let $\mathcal{C}$ be a curve of genus 2. Then $\mathrm{Pic}^0_k(\mathcal{C})$ has a subgroup Σ of size 9 if and only if $\mathcal{C}$ admits a model*

$$\mathcal{C} \colon y^2 = f(x) = G_1(x)^2 + \lambda_1 H_1(x)^3 = G_2(x)^2 + \lambda_2 H_2(x)^3$$

for $G_i, H_i \in k[x]$ with H_i monic and, quadratic, $\lambda_i \in k^\times$ and $\mathrm{GCD}(H_1, H_2) = 1$.

1.1.2 Projective Duality

Fix a vector space V over a field k. We let $\mathbb{P}(V)$ be the collection of 1-dimensional subspaces of V. We see at once that $\mathbb{P}^n_k = \mathbb{P}(k^{n+1})$. We can now define the dual projective space $(\mathbb{P}^n_k)^* = \mathbb{P}((k^{n+1})^*)$ where $(k^{n+1})^*$ is the dual vector space of k^{n+1}. Let X be a projective variety in $\mathbb{P}^n_{\mathbb{C}}$ and let $x \in X$ be a non-singular point. Then the tangent plane $\mathbb{T}_x(X)$ gives a point in $(\mathbb{P}^n_k)^*$.

Definition 1.5. Let X be a projective hyperplane in $\mathbb{P}^n_k$. The dual projective variety X^* is defined to be the closure in $(\mathbb{P}^n_k)^*$ of the tangent planes $\mathbb{T}_x(X)$ for non-singular $x \in X$.

From the above definition one can prove the following.

Theorem 1.6 (Reflexivity theorem)**.** *Let X be an irreducible projective hypersurface. Then $(X^*)^* = X$.*

For a proof see [17]. For some of the properties of the dual variety and how it relates to classical geometry see [11].

1.2 Abelian Varieties

Our main source will be Milne [30].

Definition 1.7. A group variety G over a field k consists of the following data: a variety X with a k-rational point $e \in X(k)$ and morphisms $m\colon X \times_k X \to X$ and $i\colon X \to X$ such that the above maps induce a group structure on $X(\bar{k})$ with identity e.

We will often say that X is a group variety leaving m, i and e implicit.

Definition 1.8. An abelian variety X is a complete connected group variety.

It is a standard fact that the group structure on an abelian variety is in fact abelian, and so the name is deserving.

Definition 1.9. Fix an integer $n \in \mathbb{Z}$. The multiplication map $m\colon A \times A \to A$ allows us to define a morphism

$$n_A\colon A \to A$$

which is called multiplication by n. The fiber above the k-rational point e will be denoted $A[n]$.

The subscheme $A[n]$ is thought of as the kernel of multiplication by n.

We are mostly concerned with a special type of abelian variety that can be obtained from an algebraic curve, called the *Jacobian* of the curve. The following result characterizes these varieties.

Theorem 1.10 (See [30]). *Let C be a smooth projective curve over a field k. Then there is an abelian variety* $\mathrm{Jac}(C)$*, such that for any field extension L of k we have*

$$\mathrm{Jac}(C)(L) \cong \mathrm{Pic}^0_L(C).$$

In our case, we will be interested in Jacobians of non-singular curves given by an affine equation of the form $y^2 = f(x)$, where $f(x)$ has non-zero discriminant and is of degree 6. In this case, there is an explicit description of the Jacobian variety given by [6].

1.2.1 Kummer Surfaces

Let A be an abelian variety A. Multiplication by -1 gives a morphism $-1\colon A \to A$. We define the *Kummer variety* of A to be $A/\pm 1$. We will often be given a curve described by an equation $\mathcal{C}\colon y^2 = f(x)$, where f is a degree 6 polynomial over a field of characteristic not 2 or 3 with non-zero discriminant. In this situation, we refer to [6, Ch.3] for the description of the Kummer surface of $\mathrm{Jac}(\mathcal{C})$.

Theorem 1.11 (see [6]). *Let $\mathcal{C}$ be the curve given by the equation $y^2 = f(x)$ as above. Then the Kummer surface of the Jacobian variety* $\mathrm{Jac}(\mathcal{C})$ *is a projective degree 4 surface in $\mathbb{P}^3$, that is non-singular except for 16 nodes. Furthermore, the Kummer surface has a rational node.*

The Kummer surface of a genus 2 curve will mean the Kummer surface of its Jacobian.

Definition 1.12. Let k be a field of characteristic 0. A Kummer surface over k, is a degree 4 surface that is non-singular except at 16 nodes, one of which is defined over k.

It is classical that a quartic surface K that is non-singular except at 16 nodes in $\mathbb{P}^3_{\mathbb{C}}$ has 16 associated planes. These planes are called the *tropes* of the surface. Each trope intersects K in a double counting conic that contains precisely 6 of the nodes. On the other hand, every node lies in precisely 6 tropes. More generally, we refer to a quartic surface with 16 nodes as a *geometric Kummer surface*, since over the algebraic closure, one is free to choose one of the nodes to make it a Kummer surface.

1.2.2 The Dual Kummer Surface

It is classical that over an algebraically closed field a Kummer surface and its dual variety as defined in Section 1.1.2 are projectively equivalent. However, this is no longer the case over a non-algebraically closed field. The nodes on a Kummer surface correspond under projective duality to tropes on the dual Kummer surface. If we have a specified rational node on a Kummer surface, then the dual Kummer surface has a rational trope. For a more complete discussion see [6, Ch.3-5]. The important fact is that if $\mathcal{K}$ is a Kummer surface defined over a field k, then after a suitable field extension $\mathcal{K}$ and $\mathcal{K}^*$ are projectively isomorphic.

The construction of the Kummer surface of the curve $\mathcal{C}\colon y^2 = f(x)$ in [6] chooses coordinates such that $(0:0:0:1)$ is a node of the Kummer surface. However, the dual Kummer surface may not have a rational node. On the other hand, the dual Kummer surface does have a rational trope. If we choose coordinates dual to the choice above, a rational trope of the dual Kummer surface is given by $V(t_3)$, where $t_0, ..., t_3$ are taken as the dual coordinates in the dual projective space. The dual Kummer surface also has a useful interpretation in terms of divisors.

Theorem 1.13 ([6, 4.2.1])**.** *Let $\mathcal{K}$ be the Kummer surface of a genus 2 curve $\mathcal{C}$. Then the points of the dual Kummer surface $\mathcal{K}^*$ classify effective degree 3 divisors on $\mathcal{C}$ up to linear equivalence and the natural hyperelliptic involution on the curve.*

1.2.3 Weil Pairings

It is well known (see [30, Section 16]) that for $n > 1$ we may equip Jacobian varieties with a bilinear, non-degenerate, alternating, Galois covariant pairing

$$e_n\colon \mathrm{Jac}(C)[n](\overline{k}) \times \mathrm{Jac}(C)[n](\overline{k}) \to \mu_n(\overline{k}),$$

where μ_n is the group scheme that represents the n^{th} roots of unity. This bilinear form is called the *Weil pairing.* For a genus 2 curve we have the following formula for the Weil pairing on the 3-torsion.

Lemma 1.14 (see [3], Lemma 4)**.** *Let $\mathcal{C}$ be a curve of genus 2. Suppose that we have*

$$\mathcal{C}\colon y^2 = f(x) = G_1(x)^2 + \lambda_1 H_1(x)^3 = G_2(x)^2 + \lambda_2 H_2(x)^3$$

for $G_i, H_i \in k[x]$ with H_i monic and quadratic, $\lambda_i \in k^\times$ and $\mathrm{GCD}(H_1, H_2) = 1$. Define $D_i = \{y - G_i(x), H_i(x)\}$ and let $T_i = [D_i - \kappa] \in \mathrm{Pic}^0_k(\mathcal{C})$ for κ an effective canonical divisor. Then

$$e_3(T_1, T_2) = \frac{\lambda_2}{\lambda_1}\frac{\mathrm{Resultant}(G_2 - G_1, H_2)}{\mathrm{Resultant}(G_1 - G_2, H_1)}.$$

The previous result gives a convenient way to compute the Weil Pairing on the three torsion if we can get decompositions in the correct form. Using Lemma 1.4 we can always find generators of this form, and then use bilinearity to compute the other pairings.

1.2.4 Finite étale Group Schemes

In this section we recall some facts about finite étale group schemes following [35]. Let G be a finite group scheme over a field k of characteristic 0. We write $k[G]$ for the algebra that represents G.

Definition 1.15. A finite group scheme G is said to be étale if $k[G]$ is a reduced algebra.

Finite étale group schemes are relevant for us because $A_\alpha[3]$ is a finite étale group scheme (see [27, 4.38]) and so are $(\mathbb{Z}/3\mathbb{Z})^2$ and $(\mu_3)^2$ (their representing algebras are reduced).

We will also want to consider the Cartier dual of a finite abelian group scheme. If B is a finite rank free k-module then so is $B^\vee = \hom_k(B, k)$.

Theorem 1.16 ([35, 2.4])**.** *Let G be a finite abelian group scheme represented by an algebra B. Then $B^\vee$ represents a finite abelian group scheme $G^\vee$. Furthermore, the action of taking duals preserves finite products and $(G^\vee)^\vee \cong G$.*

There is another characterization.

Lemma 1.17 ([35, p. 18])**.** *Let G be a finite abelian group scheme. Then*

$$G^\vee \cong \hom(G, \mathbb{G}_m).$$

Where $\hom(G, \mathbb{G}_m)(R) = \hom(G(R), \mathbb{G}_m(R))$.

We are working with finite étale group schemes because we wish to study the group scheme $(\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$. We will also be interested in the dual group scheme. Because of this, we begin studying the relationship between $\mathbb{Z}/3\mathbb{Z}$ and μ_3.

Lemma 1.18 ([35, p. 18])**.** *Consider the constant group scheme $\mathbb{Z}/n\mathbb{Z}$. Then $(\mathbb{Z}/n\mathbb{Z})^\vee \cong \mu_n$.*

As we have remarked, taking duals preserves products, we obtain the following.

Lemma 1.19. *Let $\Sigma = (\mathbb{Z}/n\mathbb{Z})^g$. Then $\Sigma^\vee \cong (\mu_n)^g$.*

Put $\Sigma = (\mathbb{Z}/3\mathbb{Z})^2$. The finite étale group scheme

$$\Sigma \times \Sigma^\vee$$

will frequently be of interest to us. Since taking the Cartier dual preserves products we obtain the following lemma.

Lemma 1.20. *Let $\Sigma_{n,g} = (\mathbb{Z}/n\mathbb{Z})^g$. Then*

$$(\Sigma_{n,g} \times \Sigma_{n,g}^\vee)^\vee \cong \Sigma_{n,g} \times \Sigma_{n,g}^\vee$$

.

The previous result allows us to define a pairing on $\Sigma_{n,g} \times \Sigma_{n,g}^\vee$.

Definition 1.21. The pairing

$$(\Sigma_{n,g} \times \Sigma_{n,g}^{\vee}) \times (\Sigma_{n,g} \times \Sigma_{n,g}^{\vee}) \to \mathbb{G}_m,$$

obtained by identifying the second factor of $(\Sigma_{n,g} \times \Sigma_{n,g}^{\vee}) \times (\Sigma_{n,g} \times \Sigma_{n,g}^{\vee})$ with $(\Sigma_{n,g} \times \Sigma_{n,g}^{\vee})^{\vee}$ will be called the *standard symplectic pairing.*

We will be interested in the case that $n = 3$ and $g = 2$. Our final tool in this regard is the following.

Theorem 1.22 ([35, p. 49])**.** *Finite étale group schemes are equivalent to finite groups where* $\mathrm{Gal}(\overline{k}/k)$ *is acting continuously via group automorphisms. The equivalence is given by* $G \mapsto G(\overline{k})$*.*

We can thus consider the 3-torsion of abelian varieties as $\mathrm{Gal}(\overline{k}/k)$-modules. We will now give a criterion for identifying a subgroup of $\mathrm{Jac}(\mathcal{C})[3]$. Note that $\mathrm{Jac}(\mathcal{C})[3](\overline{k})$ has the group structure of a 4-dimensional vector space over $\mathbb{F}_3$. The Weil pairing equips this space with a non-degenerate alternating pairing. It is straightforward to show that such a vector space has 40 two-dimensional subspaces on which the pairing restricts to the trivial one. Furthermore, these subspaces are maximal with respect to the property that the Weil pairing is trivial. We will call these subspaces *maximal isotropic.*

Fix a field k in characteristic 0. Let $\Sigma \cong (\mathbb{Z}/3\mathbb{Z})^2$ be a maximal isotropic subgroup of $\mathrm{Jac}(\mathcal{C})[3](\overline{k})$ (regarded as a $\mathrm{Gal}(\overline{k}/k)$-modules).

Lemma 1.23. *Suppose that there is a Galois sub-module* Σ' *of* $\mathrm{Jac}(\mathcal{C})[3](\overline{k})$ *that intersects trivially with* Σ *and is maximal isotropic. Then* $\mathrm{Jac}(\mathcal{C})[3](\overline{k})/\Sigma \cong \Sigma' \cong \mu_3 \times \mu_3$*.*

Proof. As Σ' intersects Σ trivially we have an injection $\Sigma' \hookrightarrow \mathrm{Jac}(\mathcal{C})[3](\overline{k})/\Sigma$. Since we are working with finite objects of the same cardinality we have that the injection is an isomorphism. On the other hand, for each $u \neq 0 \in \Sigma'$ we obtain a homomorphism

$$\theta_u \colon \Sigma \to \mu_3, v \mapsto e_3(v, u).$$

As Σ is maximal isotropic, θ_u is non-trivial for $u \neq 0$. We obtain an injection

$$\Sigma' \hookrightarrow \mathrm{hom}(\Sigma, \mu_3).$$

On the other hand, we have that $\mathrm{hom}(\Sigma, \mu_3) \cong \mu_3 \times \mu_3$. Since we have an injection $\Sigma' \hookrightarrow \mu_3 \times \mu_3$ we see that it must be an isomorphism, as both sets have the same cardinality. □

1.3 Moduli Spaces

In this thesis we will study a particular family of abelian surfaces. The family we are interested in can be parametrized in the following sense. For each sufficiently general point $\alpha \in \mathcal{B}$ there is an isomorphism class of genus 2 curves $\mathcal{C}_\alpha$, that is equipped with some extra data. For a given α, we would like to find equations for $\mathcal{C}_\alpha$ in terms of α. This situation is an example of what is called a *moduli problem*. The typical formalism to describe such problems is that of schemes and functors. We will introduce this notion for completeness, and because it is the framework used by our main source [25, 1.2]. A moduli problem in algebraic geometry is often described in the following way. We take a category of schemes over a field k and define a contravariant functor $F\colon \mathrm{Sch}/k \to \mathrm{Sets}$. We consider the elements of $F(S)$ for S a scheme to be families of objects over S and we also require some sort of notion of equivalence of families. We denote this equivalence relation $\sim$. We are interested in describing $F(S)/\sim$. For example, this is the framework used by Morrison and Harris in their book *The Geometry of Curves* [19]. In our case we can be more specific about the notion of family and equivalence that is of interest. Following Hunt in [25], we define a moduli problem for abelian varieties as follows.

Definition 1.24. A moduli problem consists of the following information. Fix a base field k. We require the following data.

- A collection of polarized abelian varieties (and possibly other data) denoted by $\mathfrak{F}(k)$.
- A family of objects over a scheme S will be the following datum: a proper and flat morphism $\pi\colon \mathcal{X} \to S$ with an invertible sheaf $\mathcal{C}$ on $\mathcal{X}$ such that for all $s \in S$ we have $(\mathcal{X}_s, \mathcal{C}\mid_{\mathcal{X}_s}) \in \mathfrak{F}(k)$.
- A notion $\sim$ of isomorphism of families.
- A functor $F\colon \mathrm{Sch}/k \to \mathrm{Sets}$, given by

$$S \mapsto \{\text{families of objects over S}, \quad \pi\colon \mathcal{X} \to S\}/\text{isomorphism}$$

We say that the functor F is the moduli functor of the problem.

We say that a scheme M is a fine moduli space for a moduli functor F if F is representable by M. That is, F is naturally isomorphic to the functor

$$\hom_{\mathrm{Sch}/k}(-, M)\colon \mathrm{Sch}/k \to \mathrm{Sets}, S \mapsto \hom_{\mathrm{Sch}/k}(S, M). \tag{1.1}$$

For details on the above definitions see [28].

An abelian A variety equipped with a *principal polarization* λ (see [9, p. 126]) gives rise to a Weil pairing

$$\overline{e}_n^\lambda\colon A[n] \times A[n] \to \mu_n.$$

See [9, pp. 131-132] for more details on the construction.

Definition 1.25. Let (A, λ) be a g-dimensional abelian variety with a principal polarization. A *full level n structure* on (A, λ) is a pairing-preserving isomorphism of group schemes

$$\alpha\colon A[n] \to (\mathbb{Z}/n\mathbb{Z})^g \times (\mu_n)^g,$$

where $(\mathbb{Z}/n\mathbb{Z})^g \times (\mu_n)^g$ is given the standard symplectic pairing (see Definition 1.21).

The relevant moduli problem for our purposes is the following, as described in [25, 1.2.1].

Definition 1.26. Let $\boldsymbol{A}^*_{g,n}\colon \mathrm{Sch}/k \to \mathrm{Sets}$ be defined as follows. $\boldsymbol{A}^*_{g,n}(S) = \{$isomorphism classes of principally polarized abelian schemes over S of relative dimension g with a level n structure$\}$. We set $\mathfrak{F}^*_{g,n}(k)$ to be isomorphism classes of triples (X, λ, α) where X is a g-dimensional abelian variety over k, λ a principal polarization, and $\alpha\colon X[n] \to (\mathbb{Z}/n\mathbb{Z})^g \times \mu_n^g$ is a full level n structure on (X, λ).

We will be interested in the case $n = 3$ and $g = 2$. It is a theorem that $\boldsymbol{A}^*_{g,n}$ is representable [25, 1.2.5]. Again following [25, 1.2-1.3], we will denote $\mathcal{A}^*_{2,3}$ the fine moduli scheme for $\boldsymbol{A}^*_{2,3}$. The reason the Burkhardt quartic is relevant to the above definitions is that an open subset of the Burkhardt quartic is isomorphic to an open subset of $\mathcal{A}^*_{2,3}$ (See Section 2.4).

1.4 Classical Geometry

1.4.1 The Enveloping Cone

Let $X = V(f) \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ be a irreducible hypersurface. Fix a point $\alpha \in \mathbb{P}^n_{\mathbb{C}}$. For each $x \in X$ let l_x be the line through x and α. The *enveloping cone* of X with respect to α is the closure of the union of all lines l_x such that $x \neq \alpha$ and the intersection multiplicity of l_x with X at x is greater than 1. The enveloping cone is denoted $\mathrm{EC}_\alpha(X)$. For more details see [11, p. 9].

1.4.2 Polars

We will give an economical description of the polar hypersurfaces related to a hypersurface f taken from [25]. Let f be a degree d homogeneous polynomial in variables $x_0, ..., x_n$. Let $Q = (Q_0 : ... : Q_n)$ be a $\mathbb{C}$-valued point of $\mathbb{P}^n_{\mathbb{C}}$. We define the the first polar of f at the point Q as

$$P_Q(f) = \frac{1}{d}\sum_{j=0}^{n} Q_j \frac{\partial f}{\partial x_j}. \tag{1.2}$$

We then define the k^{th} polar inductively as

$$P_{Q^k}(f) = P_Q(P_{Q^{k-1}}(f)). \tag{1.3}$$

The polars of a hypersurface $X = V(f)$ will be denoted $P_Q(X)$ and taken to mean $V(P_Q(f))$. In Dolgachev [11, Ch.1] the properties of the polars are described in detail. Some of the properties of polars are listed below.

Proposition 1.27 ([11], 1.1.5)**.** *Let $Q \in \mathbb{P}^n_{\mathbb{C}}(\mathbb{C})$ and let f be a degree d irreducible homogeneous form. If Q is a smooth point of $V(f)$ then $P_{Q^{d-1}}(V(f)) = \mathbb{T}_Q(V(f))$. If Q is a singular point of $V(f)$ then*
$P_{Q^{d-1}}(V(f)) = \mathbb{P}^n_{\mathbb{C}}$. Finally, for arbitrary Q we have

$$P_Q(f) \cap V(f) = \{P \in V(f) \colon Q \in \mathbb{T}_P(V(f))\}, \tag{1.4}$$

where $\mathbb{T}_P(V(f))$ is the projective tangent space at P of $V(f)$.

Example 1.28. The Burkhardt quartic has linear,quadric, and cubic polars. Let
$\alpha = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4) \in \mathbb{P}^4(\mathbb{C})$. Then the equations of the polars of $\mathcal{B}$ with respect to the point α are given below.

- $P_\alpha(\mathcal{B}) = \alpha_0 y_0^3 + 2\alpha_0 y_1^3 + 2\alpha_0 y_2^3 + 2\alpha_0 y_3^3 + 2\alpha_0 y_4^3 + 6\alpha_1 y_0 y_1^2 + 6\alpha_2 y_0 y_2^2 + 6\alpha_3 y_0 y_3^2 + 6\alpha_4 y_0 y_4^2 + 12\alpha_4 y_1 y_2 y_3 + 12\alpha_3 y_1 y_2 y_4 + 12\alpha_2 y_1 y_3 y_4 + 12\alpha_1 y_2 y_3 y_4$.

- $P_{\alpha^2}(\mathcal{B}) = \alpha_0^2 y_0^2 + 4\alpha_1^2 y_0 y_1 + 4\alpha_2^2 y_0 y_2 + 4\alpha_3^2 y_0 y_3 + 4\alpha_4^2 y_0 y_4 + 4\alpha_0\alpha_1 y_1^2 + 8\alpha_3\alpha_4 y_1 y_2 + 8\alpha_2\alpha_4 y_1 y_3 + 8\alpha_2\alpha_3 y_1 y_4 + 4\alpha_0\alpha_2 y_2^2 + 8\alpha_1\alpha_4 y_2 y_3 + 8\alpha_1\alpha_3 y_2 y_4 + 4\alpha_0\alpha_3 y_3^2 + 8\alpha_1\alpha_2 y_3 y_4 + 4\alpha_0\alpha_4 y_4^2$.

- $P_{\alpha^3}(\mathcal{B}) = (\alpha_0^3 + 2\alpha_1^3 + 2\alpha_2^3 + 2\alpha_3^3 + 2\alpha_4^3) y_0 + (6\alpha_0\alpha_1^2 + 12\alpha_2\alpha_3\alpha_4) y_1 + (6\alpha_0\alpha_2^2 + 12\alpha_1\alpha_3\alpha_4) y_2 + (6\alpha_0\alpha_3^2 + 12\alpha_1\alpha_2\alpha_4) y_3 + (6\alpha_0\alpha_4^2 + 12\alpha_1\alpha_2\alpha_3) y_4$.

1.4.3 Webs of Quadrics

Let k be a field of characteristic not 2. Let S_2 be the vector space of homogeneous degree 2 polynomials in $k[x_0, x_1, x_2, x_3]$.

Definition 1.29. Let W be a 4 dimensional subspace of S_2. The *web of quadrics* determined by W is $\mathbb{P}(W) \subseteq \mathbb{P}(S_2)$. The base points of the web of quadrics are those points common to every quadric in the web.

Example 1.30. The quadrics $t_0^2, t_1^2, t_2^2, t_3^2$ are linear independent and so their span form a web. The web consists of all quadrics

$$at_0^2 + bt_1^2 + ct_2^2 + dt_3^2$$

for $(a : b : c : d) \in \mathbb{P}^3$.

Definition 1.31. Let $\mathcal{W}$ be a web of quadrics. The *singular locus of* $\mathcal{W}$ is the subset of $\mathcal{W}$ consisting of the singular quadrics in $\mathcal{W}$. If $\mathcal{W}$ has a basis $Q_0, ..., Q_3$ then the locus of singular quadrics is given by the equation

$$\det(\sum_{i=0}^{3} t_i Q_i) = 0, \tag{1.5}$$

where the Q_i are regarded as symmetric matrices. In this case if $\mathcal{W}$ contains any singular quadrics then the locus of singular quadrics is a hypersurface in $\mathbb{P}^3 = \mathcal{W}$.

Example 1.32. In this example we will illustrate a method of constructing a web of quadrics, and compute the singular locus. Consider the quadrics

$$\begin{aligned} Q_1 &= t_0 t_2 - t_1^2, \\ Q_2 &= t_0 t_3 - t_1 t_2, \\ Q_3 &= t_1 t_3 - t_2^2. \end{aligned}$$

Note that Q_1, Q_2, Q_3 cut out a twisted space cubic curve with parametrization $\psi\colon \mathbb{P}^1 \to V(Q_1, Q_2, Q_3)$ given by $(x : 1) \mapsto (1 : x : x^2 : x^3)$. Let $f = x^6 + x + 1$. We can define a 4th quadric by demanding $f = \psi^* Q_4$. The polynomial

$$Q_4 = t_0^2 + t_0 t_1 + t_3^2$$

suffices. We let $\mathcal{W}$ be the web of quadrics generated by the Q_i. Consider the symmetric matrices M_i corresponding to Q_i. Explicitly we have

$$M_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{-1}{2} & 0 \\ 0 & \frac{-1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$
$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \quad M_4 = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$\begin{aligned} M(t_0, t_1, t_2, t_3) &= t_0 M_1 + t_1 M_2 + t_2 M_3 + t_4 M_4 \\ &= \begin{pmatrix} t_3 & \frac{1}{2}t_3 & \frac{1}{2}t_0 & \frac{1}{2}t_1 \\ \frac{1}{2}t_3 & -t_0 & -\frac{1}{2}t_1 & \frac{1}{2}t_2 \\ \frac{1}{2}t_0 & -\frac{1}{2}t_1 & -t_2 & 0 \\ \frac{1}{2}t_1 & \frac{1}{2}t_2 & 0 & t_3 \end{pmatrix}. \end{aligned}$$

The singular locus of web of quadrics is then given by $\det M(t_0, t_1, t_2, t_3)$ which is

$$4t_0^3t_3 + t_0^2t_2^2 - 2t_0t_1^2t_2 - 4t_0t_1t_3^2 + t_0t_2t_3^2 + t_1^4 - 4t_1^2t_3^2 - 4t_1t_2^2t_3 + 4t_2^3t_3 + 4t_2t_3^3.$$

Remark 1.33. The singular locus of a web of quadrics is classically known as the *Symmetroid* of the web. This is how it is referred to in the literature such as [25]. For a classical discussion see [26, Ch.IX]

There is a related construction.

Definition 1.34. Let $Q_0, ..., Q_3$ be a basis for a web of quadrics $\mathcal{W}$. The Jacobian locus of the four quadrics will be denoted $\mathcal{J}(Q_0...Q_3)$. Precisely we have

$$\mathcal{J}(Q_0...Q_3) = V(\det(\frac{\partial Q_i}{\partial t_j})_{ij}). \tag{1.6}$$

If the four quadrics have 6 common base points we will call the Jacobian locus of the web a *Weddle* surface.

Example 1.35. Consider the web $\mathcal{W}$ of Example 1.32. The Jacobian matrix of the quadrics $Q_1, ..., Q_4$ is given by

$$M_{\mathcal{J}} = \begin{pmatrix} t_2 & -2t_1 & t_0 & 0 \\ t_3 & -t_2 & -t_1 & t_0 \\ 0 & t_3 & -2t_2 & t_1 \\ 2t_0 + t_1 & t_0 & 0 & 2t_3 \end{pmatrix}.$$

The Jacobian locus is then cut out by the polynomial

$$\det M_{\mathcal{J}} = -2t_0^3t_3 + 6t_0^2t_1t_2 - 2t_0^2t_1t_3 + 2t_0^2t_2^2 - 4t_0t_1^3 + 2t_0t_1^2t_2 + 2t_0t_3^3 - 2t_1^4 - 6t_1t_2t_3^2 + 4t_2^3t_3.$$

One reason we will be interested in the singular locus and Jacobian locus of a web of quadrics is the following.

Theorem 1.36 ([25, B.5.15])**.** *Let $Q_1, Q_2, Q_3,$ and Q_4 be linearly independent quadrics defined over $\mathbb{C}$. Let $\mathcal{W}$ be the web they span. If the quadrics have d points in common for $d \leq 6$ then the Jacobian locus of $\mathcal{W}$ has d nodes and $\binom{d}{2}$ additional lines, so $10+\binom{d}{2}$ lines total. The singular locus of $\mathcal{W}$ has d additional nodes so $10 + d$ nodes and $\binom{d}{2}$ tropes. If the quadrics have 6 points in common then the singular locus of $\mathcal{W}$ is a geometric Kummer surface.*

Example 1.37. Let $\mathcal{W}$ be the web of Example 1.32. Let $\text{Sing}(\mathcal{W})$ be the singular locus of the web $\mathcal{W}$ and $\mathcal{J}(Q_1, ..., Q_4)$ the Jacobian locus of the web $\mathcal{W}$. One can check that the quadrics $Q_1, ..., Q_4$ are linearly independent and have 6 base points. Furthermore, computation shows that $\mathrm{Sing}(\mathcal{W})$ is non-singular outside of 16 singularities, and that $\mathcal{J}(Q_1, ..., Q_4)$ is non-singular outside of 6 singularities.

The singular locus and Jacobian locus of a web of quadrics with 6 points in common were studied classically. For example see [26] and [24]. More recent accounts can be found in [6, Ch.5] and [25, B.5.11].

Relations between the Jacobian and Singular Locus of a Web of Quadrics

Here we follow [6, Ch.5]. Let $\mathcal{W}$ be a web of quadrics with a given basis $Q_0, ..., Q_3$. The Jacobian locus of $\mathcal{W}$ and the singular locus are intimately related. Let P be a k-valued point of $\mathbb{P}^3_k$ and let $\mathcal{J}_P$ be the Jacobian matrix $(\frac{\partial Q_i}{\partial x_j} |_P)_{i,j}$. If a point P lies on $\mathcal{J}(Q_0, ..., Q_3)$ then there is a k-valued point Z such that

$$Z \cdot \mathcal{J}_P = 0$$

where $\mathcal{J}_P$ is the Jacobian matrix defined above. That is to say, for each $0 \leq j \leq 3$ we have

$$\sum_{i=0}^{3} Z_i \frac{\partial Q_i}{\partial t_j}(P) = 0.$$

This means that the quadric $\sum_{i=0}^{3} Z_i Q_i$ is singular, with a singularity at P. Conversely, suppose that $Q = \sum_{i=0}^{3} Z_i' Q_i$ lies in the singular locus of $\mathcal{W}$. Then there is a point P' such that for each $1 \leq j \leq 3$

$$\sum_{i=0}^{3} Z_i' \frac{\partial Q_i}{\partial t_j}(P') = 0.$$

We see that points on the Jacobian locus of the web can be thought of as the singular points of quadrics through the base points of the web. We also see that there is a correspondence between the Jacobian locus and the singular locus.

Lemma 1.38 ([25, Ch.5])**.** *Let $\mathcal{W}$ be a web of quadrics with a given basis $Q_1, ..., Q_4$. The map $P \mapsto Z$ in the notation above defines a birational map from the Jacobian locus of the web of quadrics to the singular locus of the web of quadrics with inverse given by $Z' \mapsto P'$.*

Example 1.39. As above we work with the web $\mathcal{W}$ from Example 1.32. Consider the matrix

$$M_{\mathcal{J}} = \begin{pmatrix} t_2 & -2t_1 & t_0 & 0 \\ t_3 & -t_2 & -t_1 & t_0 \\ 0 & t_3 & -2t_2 & t_1 \\ 2t_0 + t_1 & t_0 & 0 & 2t_3 \end{pmatrix}.$$

Using a computer algebra system we can show that if $\det M_{\mathcal{J}} = 0$ then $M_{\mathcal{J}}$ has a 1-dimensional row kernel. The resulting map

$$\phi\colon \mathcal{J}(Q_1, ..., Q_4) \to \mathrm{Sing}(\mathcal{W})$$

is given by

$$\begin{aligned}
\phi_0 &= t_1^4t_3 + 2t_1^3t_2^2 + 2t_1^3t_2t_3 + 2t_1^2t_2^3 + 4t_1^2t_3^3 + 16t_1t_2^2t_3^2 + 8t_1t_2t_3^3 + 16t_2^4t_3 + 16t_2^3t_3^2 + 4t_2^2t_3^3,\\
\phi_1 &= 2t_0^2t_1t_2t_3 + t_0t_1^3t_3 - 6t_0t_1^2t_2^2 + 2t_0t_1^2t_2t_3 - 2t_0t_1t_2^3 + 4t_0t_1t_3^3 + 8t_0t_2^2t_3^2\\
&\quad + 4t_0t_2t_3^3 - 4t_1^3t_2^2 - 16t_1^2t_2t_3^2 - 32t_1t_2^3t_3 - 20t_1t_2^2t_3^2 - 8t_2^4t_3 - 4t_2^3t_3^2,\\
\phi_2 &= -t_0^2t_1^2t_3 + 4t_0t_1^3t_2 + 2t_0t_1^2t_2^2 + 4t_0t_1t_2t_3^2 + 2t_0t_1t_3^3 + 8t_0t_2^3t_3 + 8t_0t_2^2t_3^2\\
&\quad + 2t_0t_2t_3^3 + 2t_1^4t_2 + 8t_1^3t_3^2 + 16t_1^2t_2^2t_3 + 10t_1^2t_2t_3^2 + 4t_1t_2^3t_3 + 2t_1t_2^2t_3^2,\\
\phi_3 &= -2t_0^2t_1t_3^2 - 4t_0^2t_2^2t_3 - 2t_0^2t_2t_3^2 + 6t_0t_1^2t_2t_3 - t_0t_1^2t_3^2 + 12t_0t_1t_2^3 + 6t_0t_1t_2^2t_3\\
&\quad - 2t_0t_1t_2t_3^2 + 4t_0t_2^4 + 2t_0t_2^3t_3 - 4t_1^4t_3 - 8t_1^3t_2^2 - 5t_1^3t_2t_3 - 2t_1^2t_2^3 + 2t_1^2t_2^2t_3 - 2t_1t_2^4.
\end{aligned}$$

Let us pause and take stock of what we have done with this example. From a sextic $f(x) = x^6 + x + 1$ we constructed a web of quadrics $\mathcal{W}$. The singular locus of $\mathcal{W}$ is a quartic surface with 16 nodes. Using the methods of [6] one can show that the singular locus of $\mathcal{W}$ is the dual Kummer surface of the genus 2 hyperelliptic curve given by $y^2 = x^6 + x + 1$. Recall that the dual Kummer surface of $y^2 = x^6+x+1$ comes equipped with a marked trope $V(t_3)$. The intersection $V(t_3) \cap \mathrm{Sing}(\mathcal{W})$ contains precisely 6 nodes which lie on the double counting conic $(t_1^2 - t_0t_2)^2 = 0$ in the plane $t_3 = 0$. On the other hand, Q_1, Q_2, Q_3 cut out the standard twisted cubic curve in $\mathbb{P}^3$. One can also show that $V(Q_1, Q_2, Q_3) \subseteq \mathcal{J}(Q_1, ..., Q_4)$, and that $V(Q_1, Q_2, Q_3)$ contains the 6 base points of $\mathcal{W}$. Finally, ϕ maps $V(Q_1, Q_2, Q_3)$ isomorphically onto the conic $t_1^2 - t_0t_2 = 0$ in the plane $t_3 = 0$. Furthermore, ϕ sends the 6 base points of $\mathcal{W}$ to the 6 nodes in the conic. In conclusion, from a model of a hyperelliptic curve, we constructed a web of quadrics $\mathcal{W}$. From $\mathcal{W}$ we constructed 2 surfaces, $\mathrm{Sing}(W)$ and $\mathcal{J}(Q_1...Q_4)$, and a birational map ϕ between them. We then constructed a conic on $\mathrm{Sing}(\mathcal{W})$ with 6 marked points and a twisted cubic curve on $\mathcal{J}(Q_1, ..., Q_4)$ with 6 marked points. Finally, ϕ provides an isomorphism between the conic and the twisted space cubic that preserved the marking each of the curves.

Remark 1.40. The above map can be described intrinsically. A point on the Jacobian locus of the quadrics is the singular point of a quadric in the singular locus of the web, and the birational map associates the singular point to the coordinates of the quadric in the singular locus. Conversely, a singular quadric in the web is mapped to its singularity where possible.

The following result is classical.

Lemma 1.41 ([26, Section 106]). *Suppose 4 linearly independent quadrics have 6 base points. Then the Jacobian locus of the quadrics $Q_1, ..., Q_4$ consists of the singular points of the singular quadrics in the web they generate.*

Remark 1.42. When doing computations, it can be useful to write down equations for the birational map. One way that works in practice is as follows. Given the quadrics $Q_1, ..., Q_4$, for formal P write down the matrix $(\frac{\partial Q_i}{\partial x_j}|_P)_{i,j}$. If we assume that the determinant is zero, then using the determinant relation we can formally construct (usually with the help of a computer algebra system) the row kernel of the matrix $(\frac{\partial Q_i}{\partial x_j}|_P)_{i,j}$. If this is one dimensional, then the generator of the row kernel gives polynomial equations for Z in terms of P, thus describing a rational map.

Chapter 2

The Burkhardt Quartic

In this chapter we will describe some geometric aspects of the Burkhardt quartic threefold. My exposition will mostly follow Hunt in [25, Ch.5] and Finkelnberg in [14].

2.1 The Nodes of the Burkhardt Quartic and their Properties

Define

$$\mathcal{B} = V(y_0^4 + 8y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 48y_1y_2y_3y_4) \hookrightarrow \mathbb{P}^4_{\mathbb{Q}}. \tag{2.1}$$

Recall that $\mathcal{B}$ is rational with rationalization (see [25, 5.2.7]) given by

$$\begin{aligned}
\xi_0 &= t_3\left(t_0(t_1^2 + t_1t_2 + t_2^2) + t_1t_2(t_1 + t_2) - t_3^3\right),\\
\xi_1 &= -t_3\left((1-\zeta_3)t_0^2(t_1 - \zeta_3^2 t_2) - \zeta_3 t_0(t_1 - \zeta_3^2 t_2)^2 - \zeta_3 t_1 t_2(t_1 + \zeta_3 t_2) + t_3^3\right),\\
\xi_2 &= -t_1^2t_2^2 - t_0t_1t_2(t_1 + t_2) + (t_0 + t_1 + t_2)t_3^3,\\
\xi_3 &= -\zeta_3 t_0^2(t_1 - \zeta_3^2 t_2)^2 + (1-\zeta_3)t_0t_1t_2(t_1 - \zeta_3^2 t_2) + t_1^2t_2^2 - \zeta_3(t_1 + \zeta_3 t_2)t_3^3,\\
\xi_4 &= t_0\left(t_0(t_1^2 + t_1t_2 + t_2^2) + t_1t_2(t_1 + t_2) - t_3^3\right).
\end{aligned}$$

More precisely, we have a birational map

$$\Xi\colon \mathbb{P}^3_{\mathbb{Q}(\zeta_3)} \dashrightarrow \mathcal{B} \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta_3),\ t \mapsto ((\xi_0(t)) : \xi_1(t) : \xi_2(t) : \xi_3(t) : \xi_4(t)).$$

In [14] Finkelnberg verified that the rational map ξ has a birational inverse. Later, we will show that $\mathcal{B}$ is birational to $\mathbb{P}^3_{\mathbb{Q}}$.

We now turn to the singular part of $\mathcal{B}$. It is known that $\mathcal{B}$ contains 45 nodal singularities, when regarded as a scheme in $\mathbb{P}^4_{\mathbb{Q}(\zeta_3)}$ and that this is the maximum number of nodal singularities that a quartic hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ can achieve (the so-called *Varchenko bound*).

Furthermore, it has been proved that any 3-dimensional hypersurface with these properties is projectively equivalent to $\mathcal{B} \times_{\mathbb{Q}} \mathbb{C}$ (see[14, p. 1]). Many of the properties of $\mathcal{B} \times_{\mathbb{Q}} \mathbb{C}$ can be studied by considering the interaction of $\mathcal{B} \times_{\mathbb{Q}} \mathbb{C}$ with a certain collection of linear subspaces. We consider a collection of linear subspaces of $\mathbb{P}^4_{\mathbb{Q}(\zeta_3)}$ of dimension d that contain at least $d-1$ nodes. The particular numbers that are relevant for the Burkhardt quartic are given below. The left column of the following table describes the possible dimensions of the linear subspaces, and the right column tells us the number of nodes the linear spaces of interest can contain.

Space	Incidence
Lines	2 nodes or 3 nodes
Planes	3 nodes, 4 nodes, 6 nodes, or 9 nodes
Hyperplanes	7 nodes, 10 nodes, 12 nodes, 18 nodes

I will call a plane with 9 nodes a *distinguished plane* and a hyperplane with 18 nodes a *distinguished hyperplane*. We will mostly be interested in the planes and hyperplanes that contain the maximal number of nodes, that is, the distinguished planes and hyperplanes. In the literature (especially in [25], one of our major sources), distinguished planes are called *j-planes* (or Jacobi planes) and distinguished hyperplanes are called *Steiner primes*. In the classical literature (see [25, Chapter 5] for a modern account), the properties of the above arrangement are discussed in detail.

The singular subscheme in $\mathbb{P}^4_{\mathbb{Q}}$ of $\mathcal{B}$ will be denoted $\mathrm{Sing}(\mathcal{B})$ and is cut out by the following polynomials,

$$y_0y_4^2 + 2y_1y_2y_3,\ y_0y_3^2 + 2y_1y_2y_4,\ y_0y_2^2 + 2y_1y_3y_4,\ y_0y_1^2 + 2y_2y_3y_4,$$
$$4y_0^3 + 8y_1^3 + 8y_2^3 + 8y_3^3 + 8y_4^3,\ y_0^4 + 8y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 48y_1y_2y_3y_4.$$

Proposition 2.1. $\mathrm{Sing}(\mathcal{B})$ *is a reduced zero dimensional degree 45 scheme.*

The proof can be done by direct computation and is omitted; it is certainly classical. One can show that the singular points are rational over $\mathbb{Q}(\zeta_3)$.

2.2 The Automorphism Group of the Burkhardt Quartic

Another important aspect of the Burkhardt quartic is its automorphism group, which is the unique simple group of order 25920. For classical studies involving this group see [1],[32] and [33]. For more recent studies involving the automorphism group see [15],[16], and [13]. I will generally write $\mathrm{Aut}(\mathcal{B})$ to denote the automorphism group. One of the interesting aspects of $\mathrm{Aut}(\mathcal{B})$ is that it is isomorphic as a group to $\mathrm{PSp}_4(\mathbb{F}_3)$. Furthermore, it is known that $\mathrm{Aut}(\mathcal{B})$ acts linearly on both $\mathbb{P}^3$ and $\mathbb{P}^4$. Furthermore, the linear action on $\mathbb{P}^4$ induces the natural action of $\mathrm{Aut}(\mathcal{B})$ on $\mathcal{B}$. See [25, Ch.4] for a discussion of these actions. The automorphism group is a very convenient tool for proving statements about the distinguished planes and distinguished hyperplanes because $\mathrm{Aut}(\mathcal{B})$ acts transitively on them. It is often sufficient to prove the statement for one of them and then invoke symmetry to conclude it holds for all. Using the table in [25, p. 153] we can give generators for $\mathrm{Aut}(\mathcal{B})$. The generators over $\mathbb{Q}(\zeta_3)$ are as follows.

$$\frac{1}{2\zeta_3 - 1}\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \zeta_3 & \zeta_3^2 \\ 0 & 0 & 1 & \zeta_3^2 & \zeta_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \zeta_3^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \zeta_3^2 & 0 \\ 0 & 0 & 0 & 0 & \zeta_3^2 \end{bmatrix}.$$

2.3 The Configuration of Distinguished Planes and Hyperplanes in $\mathbb{P}^4$

The configuration of linear subspaces through the nodes of the Burkhardt quartic threefold has many interesting properties. For example see [25] or [31]. There are 40 distinguished planes in $\mathbb{P}^4_{\mathbb{Q}(\zeta_3)}$. Each distinguished plane contains 9 of the nodes on the Burkhardt quartic. In fact, the distinguished planes lie on the Burkhardt quartic.

Proposition 2.2 ([25, Ch. 5])**.** *Let L be a distinguished plane. Then $L \subseteq \mathcal{B} \times_{\mathbb{Q}} \mathbb{Q}(\zeta_3)$.*

Of the 40 distinguished planes, 8 are defined over $\mathbb{Q}$ and the rest are defined over $\mathbb{Q}(\zeta_3)$. The rational distinguished planes are given by

$$\begin{gathered} V(y_0, y_1), V(y_0, y_2), V(y_0, y_3), V(y_0, y_4) \\ V(y_0 + 2y_1, y_2 + y_3 + y_4), V(y_0 + 2y_3, y_1 + y_2 + y_4) \\ V(y_0 + 2y_4, y_1 + y_2 + y_3), V(y_0 + 2y_2, y_1 + y_3 + y_4). \end{gathered}$$

See A.1 for the complete list.

On the other hand, there are 40 distinguished hyperplanes, each containing 18 of the nodes. Of the 40 distinguished hyperplanes there are 6 rational distinguished hyperplanes. The 6 rational distinguished hyperplanes are

$$V(y_0), V(y_0 + 2y_3), V(y_0 + 2y_1), V(y_0 + y_2)$$
$$V(y_0 + 2y_4), V(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4).$$

For the complete set of distinguished hyperplanes see A.1.

We will also be interested in the incidence properties of the distinguished planes and distinguished hyperplanes (see [25, Ch. 5] for an extensive discussion of the incidence properties of the configuration of linear subspaces). Each distinguished plane lies in exactly 4 distinguished hyperplanes, and each distinguished hyperplane contains precisely 4 distinguished planes. In fact, the rational distinguished planes are all contained in two distinguished hyperplanes, namely

$$V(y_0), V(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4).$$

The distinguished hyperplanes intersect $\mathcal{B}$ in 4 distinguished planes.

Proposition 2.3 ([25, Ch. 5])**.** *Let S be a distinguished hyperplane that contains distinguished planes j_1, j_2, j_3, j_4. Then*

$$\mathcal{B} \times_{\mathbb{Q}} \mathbb{Q}(\zeta_3) \cap S = j_1 \cup j_2 \cup j_3 \cup j_4.$$

That is to say, each distinguished hyperplane meets the Burkhardt quartic threefold in the 4 distinguished planes in that hyperplane.

2.4 The Moduli Interpretation

There is an open subset of the Burkhardt quartic that is isomorphic to an open subset of the moduli space of principally polarized abelian varieties with a full level three structure. Let $\mathfrak{J}$ be the set of distinguished planes of $\mathcal{B}$.

Theorem 2.4 ([25, 5.3.4])**.** *$\mathcal{B} \times_{\mathbb{Q}} \mathbb{C} - \{\bigcup_{j \in \mathfrak{J}} j\}$ is isomorphic to an open subset of the moduli space of principally polarized abelian varieties with a full level three structure.*

We will consider points $\alpha \in \mathcal{B}$ not lying on any distinguished plane and find a genus 2 curve $\mathcal{C}_\alpha$ whose Jacobian is the abelian surface corresponding to α. We now outline the procedure. Hunt's transcendental work will give us an embedding of the abelian variety in $\mathbb{P}^8_{\mathbb{C}}$, which will have the full level 3 structure and required principal polarization. We will then eschew the transcendental methods, and work in a geometric/arithmetic context. This will have the following benefits.

1. We will be able to obtain concrete formulas for the curve whose Jacobian realizes the abelian surface corresponding to sufficiently general points $\alpha \in \mathcal{B}$.

2. By paying attention to the base field we will obtain a independent proof that the abelian surface is defined over the field of definition of the point $\alpha \in \mathcal{B}$. Thus, there is no difference between the field of moduli and the field of definition of $\mathcal{C}_\alpha$.

2.5 Some Projective Geometry of the Burkhardt Quartic Relating to the Moduli Interpretation

In this section we will discuss some of the geometry of the Burkhardt quartic threefold and how it relates to the moduli interpretation. Because of Theorem 2.4 we are interested in the union of the distinguished planes. There is another way to describe this locus of points. Recall that the *Hessian* of a hypersurface $X = V(f)$ is defined as

$$\mathrm{He}(X) = V(\det(\frac{\partial f}{\partial x_i x_j})_{i,j}).$$

Proposition 2.5 ([25, 5.3.2])**.** *Let* $\mathrm{He}(\mathcal{B})$ *be the Hessian variety of* $\mathcal{B}$*. Then the union of the 40 distinguished planes is equal to* $\mathcal{B} \cap \mathrm{He}(\mathcal{B})$*.*

There is a property of $\mathcal{B}$ that we have not discussed, that $\mathcal{B}$ is self-Steinerian.

Definition 2.6. Let $X = V(f)$ be a degree $d > 2$ irreducible hypersurface in $\mathbb{P}^n_{\mathbb{C}}$. The *Steinerian* hypersurface of X is defined as

$$\mathrm{St}(X) = \bigcup_{\alpha \in \mathrm{He}(X)} \mathrm{Sing}(P_{\alpha^{d-2}}(X)). \tag{2.2}$$

That is to say, the Steinerian hypersurface of X consists of those points that are singular points of the polar quadrics of X. The name is fitting as we have.

Theorem 2.7 ([11, 1.1.6])**.** *Let* $X = V(f)$ *be an degree* $d > 2$ *irreducible hypersurface in* $\mathbb{P}^n_{\mathbb{C}}$*. Then* $\mathrm{St}(X)$ *is a hypersurface.*

We can in fact define $\mathrm{St}(X)$ in another way.

Lemma 2.8 ([11, 1.1.6])**.**

$$\mathrm{St}(X) = \{\alpha \in \mathbb{P}^n \colon P_\alpha(X) \text{ is singular}\}.$$

The above definitions make clear that there is a relationship between the Steinerian hypersurface and the Hessian hypersurface of X. Precisely we have the following result.

Theorem 2.9 ([11, p. 20])**.** $\mathrm{St}(X)$ *are* $\mathrm{He}(X)$ *are birationally equivalent. A birational map between them is given by*

$$\mathrm{st} \colon \mathrm{He}(X) \dashrightarrow \mathrm{St}(X), \ a \mapsto \mathrm{Sing}(P_{a^{d-2}(X)})$$

and

$$\mathrm{st}^{-1}\colon \mathrm{St}(X) \dashrightarrow \mathrm{He}(X),\ a \mapsto \mathrm{Sing}(P_a(X)).$$

The reason that we discuss this is because of the following.

Theorem 2.10 ([25, 5.3])**.** *$\mathcal{B}$ is self-Steinerian. That is, $\mathcal{B} = \mathrm{St}(\mathcal{B})$.*

Because of the above we can say that the points of $\mathcal{B}(\mathbb{C})$ are those points $\alpha \in \mathbb{P}^4_{\mathbb{C}}(\mathbb{C})$ such that the hypersurface defined by

$$\begin{aligned} \alpha_0 y_0^3 + 2\alpha_0 y_1^3 + 2\alpha_0 y_2^3 + 2\alpha_0 y_3^3 + 2\alpha_0 y_4^3 \\ +6\alpha_1 y_0 y_1^2 + 6\alpha_2 y_0 y_2^2 + 6\alpha_3 y_0 y_3^2 + 6\alpha_4 y_0 y_4^2 \\ +12\alpha_4 y_1 y_2 y_3 + 12\alpha_3 y_1 y_2 y_4 + 12\alpha_2 y_1 y_3 y_4 + 12\alpha_1 y_2 y_3 y_4 \end{aligned}$$

is singular. On the other hand, from Section 2.4 and Theorem 2.4 the points on $\mathcal{B}$ that correspond to a smooth genus 2 curve with a full level three structure are those that do not lie on $\mathrm{He}(\mathcal{B}) \cap \mathcal{B}$. This is of interest because of the following.

Theorem 2.11 ([25, 1.1.17])**.** *Let $X = V(f)$ be an degree d irreducible hypersurface in $\mathbb{P}^n_{\mathbb{C}}$. Then*

$$\mathrm{He}(X) = \bigcup_{\alpha \in \mathrm{St}(X)} \mathrm{Sing}(P_\alpha(X)). \tag{2.3}$$

Because $\mathrm{St}(\mathcal{B}) = \mathcal{B}$, we have that

$$\mathrm{He}(\mathcal{B}) = \bigcup_{\alpha \in \mathcal{B}} \mathrm{Sing}(P_\alpha(\mathcal{B})).$$

We have the alternative characterization of points $\alpha \in \mathbb{P}^4_{\mathbb{C}}(\mathbb{C})$ that correspond to smooth genus 2 curves with a full level three structure.

1. $P_\alpha(\mathcal{B})$ is singular.
2. There is no point $\alpha' \in \mathcal{B}$ such that $\alpha \in \mathrm{Sing}(P_{\alpha'}(\mathcal{B}))$.

That is to say, the obstruction to a point $\alpha \in \mathcal{B}(\mathbb{C})$ corresponding to a smooth genus 2 curve with a full level three structure is that α is a singular point of some other first polar of $\mathcal{B}$.

2.6 Moduli points on $\mathcal{B}$

Let $\alpha \in \mathcal{B}$ and write $\alpha = (\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \alpha_4)$. Put $k_\alpha = \mathbb{Q}(\alpha)$ and fix coordinates $(y_0 : y_1 : y_2 : y_3 : y_4)$ on $\mathbb{P}^4_{k_\alpha}$ and $(t_0 : t_1 : t_2 : t_3)$ on $\mathbb{P}^3_{k_\alpha}$. We are working with the equation

$$\mathcal{B} = V(y_0^4 + 8y_0(y_1^3 + y_2^3 + y_3^3 + y_4^3) + 48y_1y_2y_3y_4) \hookrightarrow \mathbb{P}^4_{k_\alpha}. \tag{2.4}$$

In light of Theorem 2.4 we assume that α does not lie in any of the distinguished planes. Let A_α be the corresponding abelian surface. We wish to recognize A_α as the Jacobian of a hyperelliptic curve $y^2 = f(x)$ for f a sextic with distinct roots. In fact, we would like $f \in k_\alpha[x]$. From [25, 5.7.1] we know that A_α is in fact a Jacobian of a genus 2 curve. Fix coordinates on $\mathbb{P}^8_{k_\alpha}$ given by $y_0, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4$. Using the analytic theory of theta functions, one is able to obtain an embedding of the abelian surface in $\mathbb{P}^8_{k_\alpha}$.

Theorem 2.12 ([25, pp. 188-189])**.** *Let α be a point on the Burkhardt quartic that does not lie on any distinguished plane. Then the abelian surface corresponding to α can be cut out in $\mathbb{P}^8_{k_\alpha}$ in the following way. Fix coordinates $y_0, ..., y_4, z_1, ..., z_4$ on $\mathbb{P}^8_{k_\alpha}$. Then A_α is given by the equations*

$$\begin{aligned}
\alpha_0 y_0^2 + 2\alpha_1 y_1^2 + 2\alpha_2 y_2^2 + 2\alpha_3 y_3^2 + \alpha_4 y_4^2 - 2\alpha_1 z_1^2 - 2\alpha_2 z_2^2 - 2\alpha_3 z_3^2 - 2\alpha_4 z_4^2,\\
\alpha_0 y_1^2 + 2\alpha_1 y_0 y_1 + 2\alpha_2 y_3 y_4 + 2\alpha_3 y_2 y_4 + 2\alpha_4 y_2 y_3 + \alpha_0 z_1^2 - 2\alpha_2 z_3 z_4 - 2\alpha_3 z_2 z_4 - 2\alpha_4 z_2 z_3,\\
\alpha_0 y_2^2 + 2\alpha_1 y_3 y_4 + 2\alpha_2 y_0 y_2 + 2\alpha_3 y_1 y_4 + 2\alpha_4 y_1 y_3 + \alpha_0 z_2^2 + 2\alpha_1 z_3 z_4 + 2\alpha_3 z_1 z_4 - 2\alpha_4 z_1 z_3,\\
\alpha_0 y_3^2 + 2\alpha_1 y_2 y_4 + 2\alpha_2 y_1 y_4 + 2\alpha_3 y_0 y_3 + 2\alpha_4 y_1 y_2 + \alpha_0 z_3^2 + 2\alpha_1 z_2 z_4 - 2\alpha_2 z_1 z_4 + 2\alpha_4 z_1 z_2,\\
\alpha_0 y_4^2 + 2\alpha_1 y_2 y_3 + 2\alpha_2 y_1 y_3 + 2\alpha_3 y_1 y_2 + 2\alpha_4 y_0 y_4 + \alpha_0 z_0^2 + 2\alpha_1 z_2 z_3 + 2\alpha_2 z_1 z_3 - 2\alpha_3 z_1 z_2,\\
z_1\pi_{01} + z_2\pi_{43} + z_3\pi_{24} + z_4\pi_{32},\\
z_1\pi_{43} + z_2\pi_{02} + z_3\pi_{14} + z_4\pi_{13},\\
z_1\pi_{24} + z_2\pi_{14} + z_3\pi_{03} + z_4\pi_{12},\\
z_1\pi_{32} + z_2\pi_{13} + z_3\pi_{12} + z_4\pi_{04},
\end{aligned}$$

where $\pi_{ij} = \alpha_i y_j - \alpha_j y_i$.

The above notation is suggestive. We have two linear projective subspaces, where the $y_i = 0$ and the $z_i = 0$. If we set the $y_i = 0$ then we obtain 4 quadrics in the corresponding $\mathbb{P}^3_{k_\alpha}$ cut out by the equations

$$\begin{aligned}
Q_1 &= \alpha_0 z_0^2 - 2\alpha_2 z_2 z_3 - 2\alpha_3 z_1 z_3 - 2\alpha_4 z_2 z_1.\\
Q_2 &= \alpha_0 z_1^2 + 2\alpha_1 z_2 z_3 + 2\alpha_3 z_0 z_3 - 2\alpha_4 z_2 z_0.\\
Q_3 &= \alpha_0 z_2^2 + 2\alpha_1 z_1 z_3 - 2\alpha_2 z_0 z_3 + 2\alpha_4 z_0 z_1.\\
Q_4 &= \alpha_0 z_3^2 + 2\alpha_1 z_2 z_1 + 2\alpha_2 z_0 z_2 - 2\alpha_3 z_0 z_1.
\end{aligned} \tag{2.5}$$

These 4 quadrics will be fundamental for what follows.

Lemma 2.13 ([25, 5.3.5])**.** *The four quadrics Q_1, Q_2, Q_3, Q_4 have 6 common base points, and the singular locus is a geometric Kummer surface.*

Let $\mathcal{W}_\alpha$ denote projective web spanned by the Q_i. By the above lemma, we have produced a geometric Kummer surface. On the other hand, $\mathrm{Km}(A_\alpha) = A_\alpha / \pm 1$ is also a Kummer surface over $\mathbb{C}$. Using different methods (that we will review in Section 5.1) Hunt is able to reconstruct the Kummer surface corresponding to α. We will later show that the singular loci of $\mathcal{W}_\alpha$ and $\mathrm{Km}(A_\alpha)$ coincide. When working over $\mathbb{C}$, we can classically recover the curve whose Jacobian is isomorphic to A_α. However, we are also interested in questions relating to the field of definition, and the explicit computation of the curve whose Jacobian is A_α. To this end, we will employ methods described by Cassels and Flynn in [6] which will be developed in Chapter 4. We will now begin the original work of the thesis. We will first discuss twists of the Burkhardt quartic and address the rationality question. We will then delve into the moduli interpretation.

Chapter 3

Birational Geometry and Twists of the Burkhardt Quartic

We will first discuss some of the birational geometry of the Burkhardt quartic and how this relates to twists of the Burkhardt quartic.

3.1 Twists of the Burkhardt Quartic

In the literature, there is an alternative description of $\mathcal{B} \times_{\mathbb{Q}} \mathbb{C}$ that sacrifices the property of being a hypersurface in exchange for more symmetric equations. We will denote by $\mathcal{B}'$ the variety in $\mathbb{P}^5_{\mathbb{Q}}$ given by the polynomials

$$\sigma_1(x_0, ..., x_5) = \sum_{i=0}^{5} x_i, \tag{3.1}$$
$$\sigma_4(x_0, ..., x_5) = \sum_{i,j,k,l} x_i x_j x_k x_l.$$

Consider the following linear equations,

$$X_0 = y_0 + y_1 + y_4,\ X_1 = y_0 + \zeta_3 y_1 + \zeta_3^2 y_4,\ X_2 = y_0 + \zeta_3^2 y_1 + \zeta_3 y_4,$$
$$X_3 = -(y_0 + y_2 + y_3),\ X_4 = -(y_0 + \zeta_3 y_2 + \zeta_3^2 y_3),\ X_5 = -(y_0 + \zeta_3^2 y_2 + \zeta_3 y_3).$$

Lemma 3.1 ([25, p. 170])**.** *The map given by*

$$(y_0 : y_1 : y_2 : y_3 : y_4) \mapsto (X_0 : X_1 : X_2 : X_3 : X_4 : X_5)$$

gives an isomorphism $\mathcal{B} \times_{\mathbb{Q}} \mathbb{Q}(\zeta_3) \xrightarrow{\sim} \mathcal{B}' \times_{\mathbb{Q}} \mathbb{Q}(\zeta_3)$.

However, as we are interested in arithmetic questions, we will work with $\mathcal{B}$. This is due to the following easy result.

Proposition 3.2. *$\mathcal{B}$ is not isomorphic to $\mathcal{B}'$ over $\mathbb{Q}$.*

Proof. Calculation shows that $\mathcal{B}$ has 7 rational nodes. On the other hand, $\mathcal{B}'$ has 15 rational nodes. A isomorphism defined over $\mathbb{Q}$ would preserve the number of rational nodes. □

In Finkelnberg's thesis, he verifies that the rational map

$$\Xi\colon \mathbb{P}^3_{\mathbb{Q}(\zeta_3)} \to \mathcal{B} \times_{\mathbb{Q}} \mathbb{Q}(\zeta_3)$$

given by the quartic forms

$$\begin{aligned}
\xi_0 &= t_3\left(t_0(t_1^2+t_1t_2+t_2^2)+t_1t_2(t_1+t_2)-t_3^3\right),\\
\xi_1 &= -t_3\left((1-\zeta_3)t_0^2(t_1-\zeta_3^2t_2)-\zeta_3t_0(t_1-\zeta_3^2t_2)^2-\zeta_3t_1t_2(t_1+\zeta_3t_2)+t_3^3\right),\\
\xi_2 &= -t_1^2t_2^2-t_0t_1t_2(t_1+t_2)+(t_0+t_1+t_2)t_3^3,\\
\xi_3 &= -\zeta_3t_0^2(t_1-\zeta_3^2t_2)^2+(1-\zeta_3)t_0t_1t_2(t_1-\zeta_3^2t_2)+t_1^2t_2^2-\zeta_3(t_1+\zeta_3t_2)t_3^3,\\
\xi_4 &= t_0\left(t_0(t_1^2+t_1t_2+t_2^2)+t_1t_2(t_1+t_2)-t_3^3\right).
\end{aligned}$$

is birational. In the process of doing so he describes some of the combinatorics of the base locus of the map Ξ.

Proposition 3.3 ([14, p. 4])**.** *Define*

$$\begin{aligned}
M_0 &= V(t_1-\zeta_3^2t_2, t_2^3+t_3^3) \hookrightarrow \mathbb{P}^3_{\mathbb{Q}(\zeta_3)}\\
M_1 &= V(t_0+t_1, t_1^3+t_3^3) \hookrightarrow \mathbb{P}^3_{\mathbb{Q}(\zeta_3)}\\
M_2 &= V(t_0+t_2, t_2^3+t_3^3) \hookrightarrow \mathbb{P}^3_{\mathbb{Q}(\zeta_3)}.
\end{aligned}$$

Then the reduced subscheme of the base scheme of Ξ is equal to $M = M_0 \cup M_1 \cup M_2$.

It is elementary that each of the M_i split into three lines. Specifically we have that

$$\begin{aligned}
M_0 &= V(t_1+\zeta_3t_3, t_2+\zeta_3^2t_3)\cup V(t_1+\zeta_3^2t_3, t_2+t_3)\cup V(t_1+t_3, t_2+\zeta_3t_3),\\
M_1 &= V(t_0-\zeta_3^2t_3, t_1+\zeta_3^2t_3)\cup V(t_1+\zeta_3^2t_3, t_2+t_3)\cup V(t_1+t_3, t_2+\zeta_3t_3),\\
M_2 &= V(t_0-\zeta_3^2t_3, t_2+\zeta_3^2t_3)\cup V(t_0-\zeta_3t_3, t_2+\zeta_3t_3)\cup V(t_0-t_3, t_2+t_3).
\end{aligned}$$

That is to say, the reduced subscheme of the base locus of Ξ splits into 9 lines. The lines contained in M_i intersect in a single point Q_i, where $Q_0 = (1:0:0:0), Q_1 = (0:0:1:0), Q_2 = (0:1:0:0)$. Note that each component of M_i meets exactly one component of M_j with $j \neq i$ at exactly 1 point. Furthermore, the 9 lines form a closed polygon. This can be verified by direct computation.

We will be interested the following data, which generalizes the above situation: we consider three algebraic sets T_1, T_2, T_3 where T_i is the union of three lines that intersect at a single common point P_i, subject to the following restrictions.

1. There are planes W_1, W_2, W_3 with $T_i \subseteq W_i$.

2. The 9 lines that compose the T_i form a closed polygon.

We have generalized the data that described the base locus of Ξ. However, all such collections of 9 lines are projectively equivalent to the base locus of Ξ.

Lemma 3.4 ([14, I.1.1]). *Let N be a collection of 9 lines satisfying the above requirements. Then N is projectively equivalent to M.*

3.2 Proof of Theorem A; Rationality over $\mathbb{Q}$

We will now use the ideas introduced above to construct a birational map $\mathbb{P}^3_{\mathbb{Q}} \dashrightarrow \mathcal{B}$ defined over $\mathbb{Q}$. Our first goal is to construct a configuration that satisfies the above properties and is Galois invariant. Take $P_0 = (0 : 0 : 1 : \zeta_3), P_1 = (0 : 0 : 1 : \zeta_3^2), P_2 = (1 : 1 : 1 : 1)$. These will be the common points of our unions of lines. For the origin, (that is where the three planes intersect) we will take $P = (1 : 0 : 0 : 0)$. Take projective coordinates x, y, z, w on $\mathbb{P}^3_{\mathbb{Q}(\zeta_3)}$. Consider the planes

$$W_1 = \zeta_3 z - w,$$
$$W_2 = \zeta_3^2 z - w,$$
$$W_3 = y + Tz - (1 + T)w,$$

where T is to be determined. Formally, we work over the function field of $\mathbb{Q}(\zeta_3)(T)$. The planes W_1, W_2, W_3 will be the planes of the desired configuration. Let

$$l_1 = W_1 \cap W_2,$$
$$l_2 = W_2 \cap W_3,$$
$$l_3 = W_1 \cap W_3.$$

We think of l_1, l_2, l_3 as the coordinate axes of a tetrahedron we are forming. To form the closed polygon, we start with the point $C_0 = (0 : 1 : 0 : 0)$. This will be one of the vertices of our closed polygon, that lies on the one of the axes. Now consider the line $L_1 = \overrightarrow{C_0 P_0}$. It intersects l_2 at a point C_1. Now consider the line $L_2 = \overrightarrow{C_1 P_1}$, which intersects l_3 at a point C_2. We now take the line $L_3 = \overrightarrow{C_2 P_2}$ and the intersection with l_1 to be C_3. We have thus created three lines. Now continue this process two more times to obtain points $C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9$ and lines $L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9$. The collection of lines $\{L_1, L_4, L_7\}$ and $\{L_2, L_5, L_8\}$ and $\{L_3, L_6, L_9\}$ are all through common points. The planes they describe are also distinct up to a suitable choice of parameter T. We almost have the desired configuration, but we want the polygon formed by the lines to be closed, so we require $C_9 = C_0 = (0 : 1 : 0 : 0)$. We have that

$$C_9 = (-2\zeta_3 - 1)T^2 + (-2\zeta_3 - 1)T + \frac{2}{9} \cdot (-2\zeta_3 - 1) : T^3 + (-\zeta_3 + 1)T^2 - \zeta_3 T + \frac{1}{9}(-2\zeta_3 - 1) : 0 : 0).$$

We wish to solve $(-2\zeta_3 - 1)T^2 + (-2\zeta_3 - 1)T + \frac{2}{9} \cdot (-2\zeta_3 - 1) = 0$ for rational T. Multiplying by a constant gives the equation

$$T^2 + T + \frac{2}{9} = (T + \frac{1}{3})(T + \frac{2}{3}) = 0.$$

So we have a choice of $T = -\frac{1}{3}, -\frac{2}{3}$. Furthermore, these points are not zeros of

$$T^3 + (-\zeta_3 + 1)T^2 - \zeta_3 T + \frac{1}{9}(-2\zeta_3 - 1),$$

so a choice of $T = -\frac{1}{3}, -\frac{2}{3}$ will give a closed polygon. If we take $T = -\frac{1}{3}$ then the third plane is

$$y - \frac{1}{3}z - \frac{2}{3}w.$$

We have constructed a Galois stable configuration. It remains now to find a suitable projective transformation from our configuration to the base locus M of Ξ. To do so, we could start at the point $\tilde{Q}_0 = (\zeta_3 + 1 : -\zeta_3 - 1 : 1 : 1)$. Now take the line $\overrightarrow{\tilde{Q}_0 Q_2}$ and let $\tilde{Q}_1$ be the intersection with $V_2 \cap V_3$ and continue as above to construct 9 points $\tilde{Q}_i$ for $0 \leq i \leq 8$. Now take the projective transformation that maps

$$\begin{aligned} P_0 \mapsto Q_0, P_1 \mapsto Q_1, P_2 \mapsto Q_2 \\ C_0 \mapsto \tilde{Q}_7, P \mapsto Q. \end{aligned}$$

If we let φ be the projective automorphism described above, direct computation shows that $\Xi \circ \varphi$ is given by the following equations $\mathcal{Q}_i$ for $i = 0, 1, 2, 3, 4$,

$$
\begin{aligned}
\mathcal{Q}_0 &= -x^4 + 4x^3y - 6x^2y^2 + 5xy^3 - 2y^4 - xyz^2 + y^2z^2 \\
&\quad + \frac{1}{3}xz^3 - \frac{1}{3}yz^3 - xyzw + y^2zw + \frac{2}{3}xz^2w - \frac{2}{3}yz^2w \\
&\quad - xyw^2 + y^2w^2 + \frac{2}{3}xzw^2 - \frac{2}{3}yzw^2 + \frac{1}{3}xw^3 - \frac{1}{3}yw^3, \\
\mathcal{Q}_1 &= -x^3y + 3x^2y^2 - 3xy^3 + 2y^4 + x^3z - 3x^2yz \\
&\quad + 3xy^2z - 2y^3z + \frac{1}{3}yz^3 - \frac{1}{9}z^4 + x^3w - 3x^2yw \\
&\quad + 3xy^2w - 2y^3w + \frac{2}{3}yz^2w - \frac{2}{9}z^3w + \frac{2}{3}yzw^2 - \frac{1}{3}z^2w^2 \\
&\quad + \frac{1}{3}yw^3 - \frac{2}{9}zw^3 - \frac{1}{9}w^4, \\
\mathcal{Q}_2 &= -x^4 + 4x^3y - 6x^2y^2 + 5xy^3 - 2y^4 - \frac{1}{3}xz^2w \\
&\quad + \frac{1}{3}yz^2w - \frac{1}{3}xzw^2 + \frac{1}{3}yzw^2 - \frac{1}{3}xw^3 + \frac{1}{3}yw^3, \\
\mathcal{Q}_3 &= -x^3y + 3x^2y^2 - 3xy^3 + 2y^4 - \frac{1}{3}yz^3 + \frac{1}{9}z^4 \\
&\quad + x^3w - 3x^2yw + 3xy^2w - 2y^3w - \frac{1}{3}yz^2w \\
&\quad + \frac{2}{9}z^3w - \frac{1}{3}yzw^2 + \frac{1}{3}z^2w^2 + \frac{2}{9}zw^3 + \frac{1}{9}w^4, \\
\mathcal{Q}_4 &= -x^3y + 3x^2y^2 - 3xy^3 + 2y^4 - y^2z^2 + \frac{1}{3}yz^3 \\
&\quad - y^2zw + \frac{2}{3}yz^2w - y^2w^2 + \frac{2}{3}yzw^2 + \frac{1}{3}yw^3.
\end{aligned}
$$

In conclusion we have proven the following result.

Theorem 3.5. *The Burkhardt quartic $\mathcal{B}$ is birational to $\mathbb{P}^3_{\mathbb{Q}}$ via the map given by $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4$. The reduced subscheme is given by the Galois stable configuration described above.*

Remark 3.6. The automorphism that maps the configuration we constructed to the base locus of Ξ is not unique. Only a suitable choice of projective automorphism will result in a map that is defined over $\mathbb{Q}$.

We close the section with the following question. As $\mathcal{B}$ is rational over $\mathbb{Q}$, it is natural ask the following.

Question. Is the variety $\mathcal{B}'$ defined by the equations (3.1) rational over $\mathbb{Q}$?

Chapter 4

Reconstructing a Curve from a Web of Quadrics

In this chapter we review some classical results on the relationship between genus 2 curves and webs of quadrics in $\mathbb{P}^3$. Any genus 2 curve gives rise to a web of quadrics in $\mathbb{P}^3$ with 6 base points. We will see that we can recover the curve (up to a twist) from the web. Furthermore, we will make this construction explicit. Here is some motivation for the topics in this chapter. Let $\mathcal{C}$ be a smooth genus 2 curve defined over a field k of characteristic not 2 or 3. It is well known that $\mathcal{C}$ can be described as a double cover of $\mathbb{P}^1$ branched along 6 points in essentially one way. Specifically, if $y^2 = f(x)$ and $v^2 = g(u)$ are two models of $\mathcal{C}$ then there is a birational map of the form

$$(x, y) \mapsto (\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^3})$$

with $ad-bc \neq 0$ and $e \neq 0$ (see [6, Ch.1] or [27, 4.33] for a version that allows higher genera). Thus, the data of a genus 2 curve is intimately related to that of 6 marked points on a $\mathbb{P}^1$. Given 6 points in $\mathbb{P}^1$, we can construct the genus 2 curve branched over these 6 points. Place these 6 points on a twisted space cubic in $\mathbb{P}^3$. The resulting 6 points in $\mathbb{P}^3$ lie in general position, and so the space of quadrics in $\mathbb{P}^3$ through the points is four-dimensional. The corresponding web has a singular locus which is isomorphic to the dual Kummer surface of the genus 2 curve branched over the original 6 points [6, Ch.5]. Conversely, suppose that we are given a web spanned by 4 linearly independent quadrics with 6 base points in general position. The singular locus of the web is a geometric Kummer surface of a genus 2 curve by Theorem 1.36. As the base points are in general position, there is a unique twisted space cubic curve through the base points. We can recover a twist of the curve in the following way.

- Recognize the twisted space cubic through the base locus of the web.

- Parametrize the twisted space cubic through the base locus of the web.
- Pull back the base locus along the parametrization to obtain 6 points in $\mathbb{P}^1$.

To perform this construction explicitly, we will need to know the following.

1. Given a web of quadrics with 6 base points, how does one find a twisted space cubic $\mathcal{T}$ through the 6 base points.
2. How does one parametrize a twisted space cubic $\mathcal{T}$.

4.1 The Web of Quadrics associated to a Curve of Genus 2

In this section we will describe a heuristic method of finding a twisted cubic curve in a web of quadrics. It will turn out that this will be enough for us in the future. Suppose we are given an affine model of a curve

$$\mathcal{C} : y^2 = f(x) = \sum_{i=0}^{6} f_i x^i \in k[x]$$

with $\mathrm{Disc}(f) \neq 0$ and $f_6 \neq 0$. We denote the Jacobian of the curve $\mathcal{C}$ by $\mathrm{Jac}(\mathcal{C})$, and the Kummer surface $\mathcal{K}_{\mathcal{C}}$. We will denote by $\mathcal{K}_{\mathcal{C}}^*$ the dual Kummer surface as described in Section 1.2.2. A natural problem is to find a projective model for $\mathcal{C}$. One way to obtain a projective model in $\mathbb{P}_k^4$ can be found in [6]. Let $\mathcal{T}$ be the twisted space cubic in the hyperplane $x_4 = 0$ given by

$$V(x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2) \hookrightarrow V(x_4) \cong \mathbb{P}_k^3.$$

For simplicity set

$$\begin{aligned} T_1 &= x_0x_2 - x_1^2 \\ T_2 &= x_0x_3 - x_1x_2 \\ T_3 &= x_1x_3 - x_2^2 \\ T_4 &= x_4^2 - (f_0x_0^2 + f_1x_0x_1 + f_2x_1^2 + f_3x_1x_2 + f_4x_2^2 + f_5x_2x_3 + f_6x_3^2) \end{aligned} \tag{4.1}$$

in $\mathbb{P}_k^4$.

Proposition 4.1 ([6, p. 2]). *$V(T_1, T_2, T_3, T_4) \hookrightarrow \mathbb{P}_k^4$ is a projective model for $\mathcal{C}$.*

Notice that if we are given a quadric

$$x_4^2 - (f_0x_0^2 + f_1x_0x_1 + f_2x_1^2 + f_3x_1x_2 + f_4x_2^2 + f_5x_2x_3 + f_6x_3^2)$$

with the $f_i \in k$ we can construct a sextic polynomial f as above. Namely, consider the map that sends $x_i \mapsto x^i$. If $\mathrm{Disc}(f)$ and f_6 are non-zero then we have constructed a sextic equation from 4 quadrics that corresponds to a curve. Notice that we obtain 4 quadrics in the projective space given by $V(x_4)$, by setting $x_4 = 0$ in (4.1).

Let $\mathcal{W}$ be the web of quadrics with generators

$$\begin{aligned} Q_1 &= x_0x_2 - x_1^2, \\ Q_2 &= x_0x_3 - x_1x_2, \\ Q_3 &= x_1x_3 - x_2^2, \\ Q_4 &= f_0x_0^2 + f_1x_0x_1 + f_2x_1^2 + f_3x_1x_2 + f_4x_2^2 + f_5x_2x_3 + f_6x_3^2, \end{aligned} \tag{4.2}$$

in $\mathbb{P}^3_k$. We are interested in $\mathcal{W}$ because of the following.

Lemma 4.2 ([6, 5.2])**.** *The singular locus $\mathcal{S}$ of $\mathcal{W}$ given by the equation*

$$\det(\sum_{i=1}^{4} t_iQ_i) = 0$$

is the dual Kummer surface of $\mathcal{C}$. Furthermore, the surface $V(t_4)$ is a trope of $\mathcal{S}$.

The web $\mathcal{W}$ has another property; $\mathcal{W}$ contains generators of the ideal of a twisted space cubic through the base locus of $\mathcal{W}$, namely Q_1, Q_2, Q_3. The twisted space cubic allows us to recognize the 6 points common to $\mathcal{W}$. Let $\psi\colon \mathbb{P}^1 \dashrightarrow V(Q_1, Q_2, Q_3)$ be the rational map $x \mapsto (1 : x : x^2 : x^3)$. As ψ maps onto $V(Q_1, Q_2, Q_3)$ the base locus is given by $Q_4(1, x, x^2, x^3) = f(x) = 0$. That is to say, the roots of $f(x)$ determine the 6 base points of the web $\mathcal{W}$.

It is natural to ask if there a geometric way to recognize the twisted space cubic in the web. In this case there is an easy method. The Kummer surface of $\mathcal{C}$ has a marked node at $(0 : 0 : 0 : 1)$. It follows that the dual Kummer surface (which is the singular locus of the web $\mathcal{W}$) has a marked trope, given by $V(t_4)$. A basis for the linear space given by the trope is $(1 : 0 : 0 : 0), (0 : 1 : 0 : 0), (0 : 0 : 1 : 0)$. If we consider these points as coordinates on the web $\mathcal{W}$ then we obtain the coordinates of quadrics that cut out a twisted space cubic, namely $V(Q_1, Q_2, Q_3)$.

We will be concerned with webs of quadrics with 6 base points, such that the singular locus is the dual Kummer surface of a genus 2 curve. The dual Kummer surface will be equipped with a marked trope. In all cases of interest we will be able to verify that the ideal generated by the three dimensional system described by the trope is the ideal of a twisted space cubic.

4.2 Parameterizing a Twisted Cubic Curve

In this section we will describe how to explicitly parametrize a twisted space cubic. We first review some of the classical results involving space cubics.

Definition 4.3. A twisted space cubic in $\mathbb{P}^3_k$ for some field $\mathbb{Q} \subseteq k \subseteq \mathbb{C}$ is the image of a map $\psi\colon \mathbb{P}^1_k \to \mathbb{P}^3_k$ where ψ is the map defined by 4 linearly independent cubic forms.

The following is well known.

Lemma 4.4 ([18, p. 10]). *Any twisted space cubic is projectively equivalent to*

$$\mathcal{T}_0 = V(x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2,\ x_1x_2 - x_2^2).$$

Corollary 4.5. *Let $\mathcal{T}$ be a twisted space cubic. Then $\mathcal{T}$ is the intersection of 3 quadrics.*

Lemma 4.6 ([18, p. 14]). *Let $p_1, ..., p_6$ be points in $\mathbb{P}^3$ with no four of them dependent. Then there is a unique twisted space cubic through $p_1, ..., p_6$.*

We will be interested in the following question. Suppose that we know (or suspect) that a variety X is a twisted cubic curve, but we do not have a parametrization $\mathbb{P}^1 \to X$. How does one find a rationalization of X over the field of definition of X? Recall that a chord of a variety X is a line through precisely 2 points of X. That is, a line that intersects X in a degree 2 zero dimensional subscheme. The following result is classical.

Lemma 4.7 ([6, p. 42]). *Let $\mathcal{T}$ be a twisted cubic curve. Suppose that P is a point not on $\mathcal{T}$. Then $\mathcal{T}$ has a unique chord through P.*

Returning to the above situation with X we see that we have a general method of constructing a rationalization of X defined over the required basefield. Choose a point $\alpha \notin X$ that is defined over the field of definition of X. Construct the chord l_α of X with respect to α. A general plane will intersect X in three points as X has degree 3. This means that a general plane containing l_α will intersect X in a unique third point. Thus, rotating a $\mathbb{P}^1$ of planes about l_α will provide a rationalization. Thus, to construct the rationalization we must construct the chord corresponding to the point α.

Lemma 4.8. *Let*

$$\mathcal{T} = V(T_1, T_2, T_3) \hookrightarrow \mathbb{P}^3.$$

be a twisted cubic curve and let T_i be quadrics. Then for a general point $\alpha \notin \mathcal{T}$ we have that

$$P_\alpha(T_1) \cap P_\alpha(T_2) \cap P_\alpha(T_3)$$

consists of a single point γ and the line through γ and α is a chord of $\mathcal{T}$.

Proof. By Lemma 4.4 it suffices to prove this for the standard twisted cubic curve $\mathcal{T}_0$. In this case, one can verify by direct computation, that for $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, the intersection of the first polars is given by

$$(-\alpha_0^2\alpha_3+3\alpha_0\alpha_1\alpha_2-2\alpha_1^3 : -\alpha_0\alpha_1\alpha_3+2\alpha_0\alpha_2^2-\alpha_1^2\alpha_2 : \alpha_0\alpha_2\alpha_3-2\alpha_1^2\alpha_3+\alpha_1\alpha_2^2 : \alpha_0\alpha_3^2-3\alpha_1\alpha_2\alpha_3+2\alpha_2^3).$$

If we denote by γ the point above and $l_{\alpha\gamma}$ the line through α, γ then direct computation shows for general α that $l_{\alpha\gamma}$ intersects $\mathcal{T}_0$ in a zero dimensional degree 2 reduced scheme. □

Remark. The above computation can be done with a computer algebra system. We used Magma (see [2]).

In general, to parametrize X we choose a point α over the field of definition of X that is not on X, compute the first polars of X with respect to that point, and find with linear algebra their common intersection, which is a single point γ. Now choose a line S that is skew to the line through α and γ. Choose an isomorphism $\mathbb{P}^1 \to S$. For each λ in the base field let s_λ be the image of λ in S (so the map $\mathbb{P}^1 \to S$ is $\lambda \mapsto s_\lambda$). Let L_λ be the plane spanned by

$$\alpha, \gamma, s_\lambda. \tag{4.3}$$

For almost all λ, the plane L_λ intersects X at two points of $l_{\alpha\gamma}$, and a third point p_λ. Thus the rational map $\mathbb{P}^1 \dashrightarrow X$ is given by $\lambda \mapsto p_\lambda$. Let us finish this part with an example of the method.

Example. Fix coordinates t_0, t_1, t_2, t_3 on $\mathbb{P}^3_{\mathbb{Q}}$ and consider the quadrics

$$\begin{aligned}
T_1 &= 74t_0^2 - 100t_0t_1 + 120t_0t_2 - 137t_1t_2 - 185t_1t_3 - 222t_2t_3 + 40t_3^2 \\
T_2 &= 235t_0t_1 - 296t_1^2 - 134t_0t_2 + 235t_1t_2 - 740t_0t_3 + 740t_2t_3 - 94t_3^2 \\
T_3 &= 12t_0t_1 + 30t_0t_2 - 25t_1t_2 + 74t_2^2 - 222t_0t_3 - 185t_1t_3 + 10t_3^2.
\end{aligned}$$

I claim that T_1, T_2, T_3 cut out a twisted cubic. To prove this, I will use the methods outlined above. Take $\alpha = (1 : 0 : 0 : 0)$. Notice that $\alpha \notin V(T_1, T_2, T_3)$. We have that

$$\begin{aligned}
P_\alpha(T_1) &= V(74t_0 - 50t_1 + 60t_2) \\
P_\alpha(T_2) &= V(\frac{235}{2}t_1 - 67t_2 - 370t_3) \\
P_\alpha(T_3) &= V(6t_1 + 15t_2 - 111t_3).
\end{aligned}$$

Solving this system of linear equations gives the point $\gamma = (0 : 6 : 5 : 1)$. Finally we take $\rho_\lambda = (0 : 0 : 1 : \lambda)$ for $\lambda \in \mathbb{Q}$. The plane generated by $\alpha, \gamma, \rho_\lambda$ is the image of of the map

$$\Psi : \mathbb{P}^2 \to \mathbb{P}^3, \ (u : v : w) \mapsto u\alpha + v\gamma + w\rho_\lambda$$

which is given by

$$L_\lambda \colon (\lambda-5)t_1+5t_2+(5-6\lambda)t_3=0.$$

The intersection $L_\lambda \cap V(T_1,T_2,T_3)$ has a single point that depends on λ and is given by $(L_{\lambda,0} : L_{\lambda,1} : L_{\lambda,2} : L_{\lambda,3})$ with

$$\begin{aligned}
L_{\lambda,0} &= \frac{1075}{67}\lambda^3 - \frac{7365}{134}\lambda^2 + \frac{32175}{268}\lambda - \frac{20125}{268}\\
L_{\lambda,1} &= 6\lambda^3 + \frac{1595}{67}\lambda^2 - \frac{16325}{134}\lambda + \frac{6125}{67}\\
L_{\lambda,2} &= -\frac{635}{67}\lambda^3 + \frac{8425}{134}\lambda^2 - \frac{8575}{268}\lambda - \frac{875}{268}\\
L_{\lambda,3} &= \lambda^3 - \frac{1085}{134}\lambda^2 + \frac{1475}{268}\lambda + \frac{25375}{268}.
\end{aligned}$$

One can now check that this in fact gives a birational map $\mathbb{P}^1 \dashrightarrow V(T_1,T_2,T_3)$. Thus, we have proven the claim.

4.2.1 Classical Geometry of Kummer and Weddle Surfaces

In this section we will review some classical geometry involving twisted cubics and webs of quadrics. For a complete discussion in a classical language see [24, p. 167]. Let X be a hyperelliptic genus 2 curve with hyperelliptic equation $y^2=f(x)$. Let K be the Kummer surface of $\mathrm{Jac}(X)$ as constructed in [6, Ch.2] and let K^* be the dual Kummer surface. Let C be the conic in the marked trope of K^* containing 6 nodes of the dual Kummer surface. Let W be the Weddle surface of the Kummer surface, that is the Jacobian determinant of the quadrics given by (4.2).

Lemma 4.9 ([6, p. 43])**.** *The nodes of the Weddle surface W are give by $(1:\theta_i:\theta_i^2:\theta_i^3)$ where the θ_i are the roots of f.*

Lemma 4.10. *The standard twisted cubic given by Section 4.1 lies on the Weddle surface W.*

Proof. This is a formal identity that can be verified in a computer algebra system. □

Recall that the dual Kummer surface can be given by the singular locus of the standard web of quadrics described in (4.2) (see Lemma 4.2). In this case, we can easily identify the conic in the marked trope through the nodes.

Lemma 4.11. *The conic in the marked trope of the dual Kummer surface K^* can be described as $t_0t_2-t_1^2$.*

Proof. From [6, p. 25] the 6 nodes in the marked trope are $(\theta_i^2 : -\theta_i : 1 : 0)$ where the θ_i are the roots of f. On the other hand, as the dual Kummer surface is given as the singular locus of an explicit web of quadrics, we can check that the intersection with the marked trope $V(t_3)$ is equal to $(t_0t_2 - t_1^2)^2$. Thus the conic in the trope through the nodes is $t_0t_2 - t_1^2$. □

We can now show that the birational map of Lemma 1.38 maps the conic in the dual Kummer surface to the twisted cubic on the Weddle surface.

Proposition 4.12. *Let C be the conic in the marked trope of K^* containing 6 nodes of the dual Kummer surface. Let W be the Weddle surface of the Kummer surface. Then the birational map given by Lemma 1.38 maps the twisted cubic unto the conic. Furthermore, the 6 nodes of the Weddle surface map unto the 6 nodes in the trope.*

Proof. It is easier to consider the map from the conic to the twisted cubic. A point on the conic corresponds to a point in space $(t_0 : t_1 : t_2 : 0)$ such that $t_0t_2 - t_1^2 = 0$. If we take $t_2 = 1$ then the corresponding quadric in the web is

$$-t_1^2x_1^2 + t_1^2x_0x_2 - t_1x_1x_2 - x_2^2 + t_1x_0x_3 + x_1x_3$$

which has a singularity at $(1 : -t_1 : (-t_1)^2 : (-t_1)^3)$. Since the birational map of Lemma 1.38 takes a singular quadric and maps it to its singularity (see Remark 1.40) we see that the image of a point on the Weddle surface is $(1 : -t_1 : (-t_1)^2 : (-t_1)^3)$ which lies on the twisted cubic. Furthermore, the above calculation shows that the nodes $(\theta_i^2 : -\theta_i : 1 : 0)$ in the trope are mapped to

$$(1 : \theta_i : \theta_i^2 : \theta_i^3)$$

which are the nodes of the Weddle surface.

□

4.3 Recovering the Curve from a Dual Kummer Surface

Let X be a genus 2 curve with model $y^2 = f(x)$ with f of degree 6. Let $f \in k[x]$ with k of characteristic 0. Let $\mathcal{K}$ be the Kummer surface of X as described in [6] and let $\mathcal{K}^*$ be the corresponding dual Kummer surface and T be the marked trope of $\mathcal{K}^*$. We seek a method of recovering X from $\mathcal{K}^*$. The trope T contains 6 of the nodes of $\mathcal{K}^*$. We have the following classical fact.

Lemma 4.13 ([7, p. 169])**.** *There is a plane conic C in the trope T through 6 of the nodes of K^*. Furthermore there is a hyperelliptic curve X' geometrically isomorphic to X with branch points at the 6 nodes in the trope.*

In Chapter 5 we will attempt to recover the geometric isomorphism class of a smooth genus 2 curve from an associated web of quadrics. The following lemma is the tool that allows us to do so.

Lemma 4.14. *Suppose that a dual Kummer surface $\mathcal{K}^*$ of a smooth genus 2 curve X is given as the singular locus of a web of quadrics $\mathcal{W}$ with 6 base points. Let T be the marked trope of $\mathcal{K}^*$ and let C be the conic through the 6 nodes contained in T. Suppose that we have the following.*

1. *There is a basis T_1, T_2, T_3, Q of $\mathcal{W}$ such that $V(T_1, T_2, T_3)$ cut out a twisted cubic curve.*
2. *There is a birational map $\Psi\colon \mathbb{P}^1_k \to V(T_1, T_2, T_3)$*
3. *There is an isomorphism $\varphi\colon V(T_1, T_2, T_3) \to C$.*

Then X can can be described as $y^2 = d\Psi^* Q$ for some constant d.

Proof. Let $\psi\colon X \to C$ be the map described above in Lemma 4.13. Composing with $\Psi^{-1} \circ \varphi$ gives a degree 2 map $X \to \mathbb{P}^1_k$ branched along $\Psi^* Q$. Because the geometric isomorphism class of the curve X is determined by the branch locus, the result follows. □

Chapter 5

The Genus 2 Curve corresponding to a Moduli point

We now return to the moduli interpretation of $\mathcal{B}$. Let $\alpha \in \mathcal{B}$ be a point that does not lie on any distinguished plane. Let k_α be a field that α is defined over. In this chapter rationality will mean rationality over k_α. From Section 2.6 there is a genus 2 curve $\mathcal{C}_\alpha$ such that $\mathrm{Jac}(\mathcal{C}_\alpha)[3]$ has a full level three structure. We will use the notation $A_\alpha = \mathrm{Jac}(\mathcal{C}_\alpha)$. We will now review Hunt's construction of a genus 2 curve whose Jacobian has a full level three structure and point out some complications that arise when dealing with a non-algebraically closed base field.

5.1 Hunt's Geometric Construction

Associated to the point α are the polar hypersurfaces $P_\alpha(\mathcal{B}), P_{\alpha^2}(\mathcal{B}), P_{\alpha^3}(\mathcal{B})$. See Example 1.28. Consider the first polar $P_\alpha(\mathcal{B})$ given by the equation

$$\begin{aligned}\alpha_0 y_0^3 + 2\alpha_0 y_1^3 + 2\alpha_0 y_2^3 + 2\alpha_0 y_3^3 \\ +2\alpha_0 y_4^3 + 6\alpha_1 y_0 y_1^2 + 6\alpha_2 y_0 y_2^2 + 6\alpha_3 y_0 y_3^2 + 6\alpha_4 y_0 y_4^2 \\ +12\alpha_4 y_1 y_2 y_3 + 12\alpha_3 y_1 y_2 y_4 + 12\alpha_2 y_1 y_3 y_4 + 12\alpha_1 y_2 y_3 y_4 = 0.\end{aligned}$$

The geometric isomorphism class of the first polar is well known, and is constant for the points on $\mathcal{B}$ lying outside the distinguished planes. To understand the geometric isomorphism class of the first polar we introduce the following cubic threefold.

Definition 5.1. The Segre cubic threefold is the hypersurface $\mathcal{S}_3$ in $\mathbb{P}^4_{\mathbb{Q}}$ defined by the equation

$$y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3 - (y_0 + y_1 + y_2 + y_3 + y_4)^3 = (\sum_{i=0}^{4} y_i^3) - (\sum_{i=0}^{4} y_i)^3 = 0.$$

We are interested in the Segre cubic because of the following.

Lemma 5.2 ([25, 5.3.8])**.** *Suppose that $\alpha \in \mathcal{B}(\mathbb{C})$ and α does not lie on any distinguished plane. Then $P_\alpha(\mathcal{B})$ is geometrically isomorphic to $\mathcal{S}_3$.*

The Segre cubic is classified up to projective isomorphism by the property of being a degree 3 hypersurface with the maximum number of nodal singularities, which is 10. The Segre cubic is also the moduli space of 6 ordered points in $\mathbb{P}^1$, modulo projective equivalence For more on this see [12] or [23]. The dual variety of $P_\alpha(\mathcal{B})$ will also be of interest.

Definition 5.3. The Igusa quartic threefold is the hypersurface $\mathcal{I}_4$ in $\mathbb{P}^4_{\mathbb{Q}}$ defined by the equation

$$(y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4y_0y_1y_2(\sum_{i=0}^{4} y_i) = 0.$$

The Igusa quartic is of interest because of the following classical result.

Theorem 5.4 ([25, 3.3.8])**.** *For any sufficiently general point $x \in \mathcal{I}_4(\mathbb{C})$ the tangent hyperplane section of x with $\mathcal{I}_4$ is a Kummer surface.*

In fact, $\mathcal{I}_4$ has a another moduli interpretation. The Igusa quartic can be regarded as the moduli space of 6 marked points in a $\mathbb{P}^3$. See [23] for more details. The connection between the Igusa quartic, the Segre cubic, and our current situation is the following classical result.

Theorem 5.5 ([25, 3.3.1])**.** *The Segre cubic and Igusa quartic are projectively dual varieties.*

We now have an immediate way of constructing a Kummer surface corresponding to the point α. Because α does not lie on any distinguished plane, α does not lie on the Hessian variety of $\mathcal{B}$, so by Theorem 2.11 the point α is not a singular point of $P_\alpha(\mathcal{B})$. We may now consider $\mathbb{T}_\alpha(P_\alpha(\mathcal{B}))$ (the tangent plane of $P_\alpha(\mathcal{B})$ at α) as a point in the dual projective space $(\mathbb{P}^4)^*$. It is immediate that $\mathbb{T}_\alpha(P_\alpha(\mathcal{B}))$ lies on the projective dual of $P_\alpha(\mathcal{B})$ which by Lemma 5.2 and Theorem 5.5 is isomorphic to $\mathcal{I}_4$ over $\mathbb{C}$. By Theorem 5.4 the tangent hyperplane section of $\mathbb{T}_\alpha(P_\alpha(\mathcal{B}))$ at the dual variety of $P_\alpha(\mathcal{B})$ is a Kummer surface. In conclusion, we have the following result.

Theorem 5.6 ([25, p. 192])**.** *Suppose that α is a sufficiently general point on $\mathcal{B}$. Write A_α for the corresponding abelian surface. Consider $\mathbb{T}_\alpha(P_\alpha(\mathcal{B}))$ as a point in the dual projective space. Then the Kummer surface of A_α is the tangent hyperplane section of $P_\alpha(\mathcal{B})^*$ at $\mathbb{T}_\alpha(P_\alpha(\mathcal{B}))$.*

With the Kummer surface in hand, one is then (in theory) able to reconstruct the curve using the methods in [6]. While the method outlined above is theoretically satisfying, it is also not explicit in the sense that it seems difficult to compute the twist of the Igusa quartic that is used in the above construction. We will now describe an alternative method.

5.2 Arithmetic Recovery of the Curve; Proof of Theorem B

Let us sketch the proof of Theorem B. The work of Hunt provides a web of quadrics $\mathcal{W}_\alpha$ with 6 points in common (see Lemma 5.10 and preceding discussion). From $\mathcal{W}_\alpha$ and the polar hypersurfaces of $\mathcal{B}$ with respect to α, we can construct a conic with 6 marked points (see Proposition 5.17) and a twisted space cubic with 6 marked points (see Lemma 5.16 and preceding discussion). Furthermore, there is an isomorphism between the conic and twisted space cubic that respects the markings on both curves (see Corollary 5.22). Using work of Hunt we are then able to apply Lemma 4.14 and recover the geometric isomorphism class of the curve. In Section 5.3 we describe how to find the correct twist of the curve.

5.2.1 Proof of Theorem B

In [25, Ch. 5] Hunt does more than the above construction. Because Hunt is working geometrically, he does not distinguish between a Kummer surface and its dual. This is evident from the fact that Hunt constructs a Kummer surface in a dual projective space. On the other hand, Hunt provides an alternative and explicit way of computing a quartic surface that he views as the Kummer surface of A_α. However, for arithmetic purposes the quartic surface is naturally a *dual Kummer surface*. We will now review the construction and point out how the quartic is naturally a dual Kummer surface. The construction in [25] relies on the enveloping cone of the first polar $\mathrm{EC}_\alpha(P_\alpha(\mathcal{B}))$ as defined in Section 1.4.1. We have a good description from [25].

Proposition 5.7 ([25, 5.23]). *The enveloping cone* $\mathrm{EC}_\alpha(P_\alpha(\mathcal{B}))$ *is the determinantal hypersurface cut out by*

$$\det \begin{bmatrix} \pi_{01} & \pi_{43} & \pi_{24} & \pi_{32} \\ \pi_{43} & \pi_{02} & \pi_{14} & \pi_{13} \\ \pi_{24} & \pi_{14} & \pi_{03} & \pi_{12} \\ \pi_{32} & \pi_{13} & \pi_{12} & \pi_{04} \end{bmatrix}, \tag{5.1}$$

where $\pi_{ij} = \alpha_i y_j - \alpha_j y_i$.

We can now describe the geometric Kummer surface.

Proposition 5.8 ([25, 5.3.3]). *The variety* $\mathrm{EC}_\alpha(P_\alpha(\mathcal{B}))$ *is a cone over a geometric Kummer surface. Over an algebraically closed field, the projection of* $\mathrm{EC}_\alpha(P_\alpha(\mathcal{B}))$ *from* α *to a hyperplane not containing* α *is isomorphic to the Kummer surface of* A_α.

Equivalently, we could intersect with a hyperplane not containing α. As α does not lie in any distinguished plane, we see that $\alpha \notin V(y_0)$. We let K_α be the intersection $V(y_0) \cap \mathrm{EC}_\alpha(P_\alpha(\mathcal{B}))$. We can now say why K_α is arithmetically a dual Kummer surface. In general, K_α does not have a k_α-rational node. On the other hand, K_α does have a

k_α-rational trope given by $T = \mathbb{T}_\alpha(\mathcal{B}) \cap V(y_0)$. Explicitly we have that T is cut out by

$$(6\alpha_0\alpha_1^2+12\alpha_2\alpha_3\alpha_4)y_1+(6\alpha_0\alpha_2^2+12\alpha_1\alpha_3\alpha_4)y_2+(6\alpha_0\alpha_3^2+12\alpha_1\alpha_2\alpha_4)y_3+(6\alpha_0\alpha_4^2+12\alpha_1\alpha_2\alpha_3)y_4$$

in $V(y_0)$. The fact that T is a trope means the following.

Proposition 5.9 (see 5.3.3 and 5.4 in [25])**.** *T contains precisely 6 of the nodes of K_α.*

We see that the projective dual K_α^* is a geometric Kummer surface with a rational node, and so is a Kummer surface in the arithmetic sense. There is another method of recognizing K_α as a dual Kummer surface. Consider two copies of $\mathbb{P}^3_{k_\alpha}$ with coordinates $t_0, ..., t_3$ and $z_0, ..., z_3$. Recall that we defined quadrics

$$\begin{aligned}
Q_1 &= \alpha_0 z_0^2 - 2\alpha_2 z_2 z_3 - 2\alpha_3 z_1 z_3 - 2\alpha_4 z_2 z_1.\\
Q_2 &= \alpha_0 z_1^2 + 2\alpha_1 z_2 z_3 + 2\alpha_3 z_0 z_3 - 2\alpha_4 z_2 z_0.\\
Q_3 &= \alpha_0 z_2^2 + 2\alpha_1 z_1 z_3 - 2\alpha_2 z_0 z_3 + 2\alpha_4 z_0 z_1.\\
Q_4 &= \alpha_0 z_3^2 + 2\alpha_1 z_2 z_1 + 2\alpha_2 z_0 z_2 - 2\alpha_3 z_0 z_1.
\end{aligned}$$

in Equation 2.5.

The Q_i arise from a geometric projection $\mathbb{P}^8_{k_\alpha} \to \mathbb{P}^3_{k_\alpha}$ described after Theorem 2.12.

From Hunt we have the following.

Lemma 5.10 ([25, 5.3.5])**.** *The Q_i span a web of quadrics $\mathcal{W}_\alpha$ with 6 base points.*

Let $\mathcal{W}_\alpha$ be the web of quadrics spanned by $Q_1, ..., Q_4$ in the coordinates $(t_0 : t_1 : t_2 : t_3)$ and let $\mathcal{S}_\alpha$ be the singular locus. Because of Theorem 1.36 we have the following.

Corollary 5.11. *$\mathcal{S}_\alpha$ is geometrically a Kummer surface.*

The connection between $\mathcal{S}_\alpha$ and K_α is given by the following.

Lemma 5.12. *$\mathcal{S}_\alpha$ is isomorphic to K_α via the map $(t_0 : t_1 : t_2 : t_3) \mapsto (0 : t_0 : t_1 : t_2 : t_3)$.*

Proof. This is done by direct computation. Inspect the equations of the two varieties in question. K_α is a hyperplane section of the plane $V(y_0)$ in $\mathbb{P}^4$ while $\mathcal{S}_\alpha$ is defined in the $\mathbb{P}^3$ with coordinates $t_0, ..., t_3$. □

The above result tells us that K_α is given by the singular locus of a web of quadrics. We are now in a position to use the methods outlined in Chapter 4. Specifically, we will work to apply Lemma 4.14. In light of the discussion after Lemma 4.2, we will show that the coordinates of a basis of the trope T will give a twisted cubic in $\mathcal{W}_\alpha$. We will make some assumptions to simplify our calculations. Since α is non-singular, one of the partial derivatives of B (the Burkhardt relation) must not vanish. Direct computation verifies the following.

Lemma 5.13. *Suppose that we have a point $\alpha \in \mathcal{B}(\mathbb{C})$ that does not lie on any distinguished plane. Then*

$$\frac{\partial B}{\partial y_i}(\alpha) \neq 0$$

for some $1 \leq i \leq 4$.

So we may assume that

$$\frac{\partial B}{\partial y_i}(\alpha) \neq 0$$

for some $1 \leq i \leq 4$. We will assume that $\dfrac{\partial B}{\partial y_4}(\alpha) \neq 0$. As the automorphism group of the Burkhardt contains all permutations of $y_1, ..., y_4$ we do not lose generality by this assumption. We can now compute a basis for the trope T.

Proposition 5.14. *Suppose that $\dfrac{\partial B}{\partial y_4}(\alpha) \neq 0$. Let e_1, e_2, e_3 and e_4 be the standard basis vectors for k_α^4. Then the vectors*

$$v_i = \frac{\partial B}{\partial y_4}(\alpha) e_i - \frac{\partial B}{\partial y_i}(\alpha) e_4$$

for $1 \leq i \leq 3$ are a basis for the trope T.

Proof. It is easy to check that the v_i lie in T. The v_i are clearly independent. □

Definition 5.15. Define $T_i := \dfrac{\partial B}{\partial y_4}(\alpha) Q_i - \dfrac{\partial B}{\partial y_i}(\alpha) Q_4$.

Our methodology in Section 4.1 suggests that we should attempt to prove that $V(T_1, T_2, T_3)$ is a twisted space cubic. We have outlined a strategy to do so in Section 4.2. Take $p_0 = (1 : 0 : 0 : 0)$. An easy calculation with a computer algebra system gives the following.

$$P_{Q_0}(T_1) \cap P_{Q_0}(T_2) \cap P_{Q_0}(T_3) = (0 : \alpha_2 : \alpha_3 : \alpha_4)$$

where $P_{Q_i}(T_j)$ is the first polar of T_j at Q_i (see Section 1.4.2).

Let q_0 be the point $(0 : \alpha_2 : \alpha_3 : \alpha_4)$ computed above. For each point $\lambda \in k_\alpha$ we let $\rho_\lambda = (0 : 1 : \lambda : 0)$ and L_λ be the plane spanned by p_0, q_0, ρ_λ. Explicitly we have that L_λ is given by the equation

$$\lambda \alpha_4 z_1 - \alpha_4 z_2 + (\alpha_3 - \alpha_2 \lambda) z_3 = 0$$

On the other hand, a parametric description of L_λ is given by

$$L_\lambda = \{(x : y\alpha_2 + z : y\alpha_3 + z\lambda : y\alpha_4) \mid (x : y : z) \in \mathbb{P}^2_{k_\alpha}\}.$$

We wish to find the intersection of $L_\lambda \cap V(T_1, T_2, T_3)$. We can proceed in an elementary fashion (using a computer algebra system) and find an element of the intersection. The element is given by,

$$X(\lambda) = (\frac{1}{2}\alpha_0^2\alpha_1\alpha_2^3 - \frac{1}{2}\alpha_0^2\alpha_1\alpha_4^3)\lambda^3 + (-\frac{1}{2}\alpha_0^3\alpha_2\alpha_4^2 - \frac{3}{2}\alpha_0^2\alpha_1\alpha_2^2\alpha_3 + 2\alpha_1^3\alpha_2\alpha_4^2)\lambda^2 +$$
$$(\frac{1}{2}\alpha_0^3\alpha_3\alpha_4^2 + \frac{3}{2}\alpha_0^2\alpha_1\alpha_2\alpha_3^2 - 2\alpha_1^3\alpha_3\alpha_4^2)\lambda - \frac{1}{2}\alpha_0^2\alpha_1\alpha_3^3 + \frac{1}{2}\alpha_0^2\alpha_1\alpha_4^3$$
$$Y(\lambda) = (-\frac{1}{2}\alpha_0^2\alpha_2^3 - \frac{1}{2}\alpha_0^2\alpha_4^3 - 2\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4)\lambda^3 + (\frac{1}{2}\alpha_0^2\alpha_2^2\alpha_3 + 2\alpha_0\alpha_1\alpha_3^2\alpha_4 + 2\alpha_1^2\alpha_2\alpha_4^2)\lambda^2 +$$
$$(\frac{1}{2}\alpha_0^2\alpha_2\alpha_3^2 + 2\alpha_0\alpha_1\alpha_2^2\alpha_4 + 2\alpha_1^2\alpha_3\alpha_4^2)\lambda - \frac{1}{2}\alpha_0^2\alpha_3^3 - \frac{1}{2}\alpha_0^2\alpha_4^3 - 2\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4$$
$$Z(\lambda) = (\alpha_0^2\alpha_2^3\alpha_3 + \alpha_0^2\alpha_3\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4 - 4\alpha_1^2\alpha_2^2\alpha_4^2)\lambda^2 +$$
$$(-2\alpha_0^2\alpha_2^2\alpha_3^2 - 2\alpha_0\alpha_1\alpha_2^3\alpha_4 - 2\alpha_0\alpha_1\alpha_3^3\alpha_4 + 2\alpha_0\alpha_1\alpha_4^4 + 4\alpha_1^2\alpha_2\alpha_3\alpha_4^2)\lambda$$
$$+\alpha_0^2\alpha_2\alpha_3^3 + \alpha_0^2\alpha_2\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2^2\alpha_3\alpha_4 - 4\alpha_1^2\alpha_3^2\alpha_4^2.$$

The solution holds whenever the following polynomials in λ are not identically zero.

$$(-48\alpha_0\alpha_2\alpha_3^2 - 96\alpha_1\alpha_2^2\alpha_4)\lambda + (48\alpha_0\alpha_3^3 + 48\alpha_0\alpha_4^3 + 192\alpha_1\alpha_2\alpha_3\alpha_4) \neq 0$$

$$D = (\alpha_0^2\alpha_2^3\alpha_3 + \alpha_0^2\alpha_3\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4 - 4\alpha_1^2\alpha_2^2\alpha_4^2)\lambda^2$$
$$+(-2\alpha_0^2\alpha_2^2\alpha_3^2 - 2\alpha_0\alpha_1\alpha_3^3\alpha_4 + 2\alpha_0\alpha_1\alpha_4^4 + 4\alpha_1^2\alpha_2\alpha_3\alpha_4^2)\lambda$$
$$+\alpha_0^2\alpha_2\alpha_3^3 + \alpha_0^2\alpha_2\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2^2\alpha_3\alpha_4 - 4\alpha_1^2\alpha_3^2\alpha_4^2 \neq 0$$

Now define

$$E_1 = -48\alpha_0\alpha_2\alpha_3^2 - 96\alpha_1\alpha_2^2\alpha_4$$
$$E_2 = 48\alpha_0\alpha_3^3 + 48\alpha_0\alpha_4^3 + 192\alpha_1\alpha_2\alpha_3\alpha_4$$
$$F1 = \alpha_0^2\alpha_2^3\alpha_3 + \alpha_0^2\alpha_3\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4 - 4\alpha_1^2\alpha_2^2\alpha_4^2$$
$$F2 = -2\alpha_0^2\alpha_2^2\alpha_3^2 - 2\alpha_0\alpha_1\alpha_3^3\alpha_4 + 2\alpha_0\alpha_1\alpha_4^4 + 4\alpha_1^2\alpha_2\alpha_3\alpha_4^2$$
$$F3 = \alpha_0^2\alpha_2\alpha_3^3 + \alpha_0^2\alpha_2\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2^2\alpha_3\alpha_4 - 4\alpha_1^2\alpha_3^2\alpha_4^2.$$

To have a solution we assume that at least one of the E_i are non-zero and at least one F_i is non-zero. We will return to these forms later.

The above forms $X(\lambda), Y(\lambda), Z(\lambda)$ give a point on $V(T_1, T_2, T_3)$ namely,

$$\begin{aligned}
\mathcal{T}_0(\lambda) = &(\frac{1}{2}\alpha_0^2\alpha_1\alpha_2^3 - \frac{1}{2}\alpha_0^2\alpha_1\alpha_4^3)\lambda^3\\
&+ (-\frac{1}{2}\alpha_0^3\alpha_2\alpha_4^2 - \frac{3}{2}\alpha_0^2\alpha_1\alpha_2^2\alpha_3 + 2\alpha_1^3\alpha_2\alpha_4^2)\lambda^2\\
&+ (\frac{1}{2}\alpha_0^3\alpha_3\alpha_4^2 + \frac{3}{2}\alpha_0^2\alpha_1\alpha_2\alpha_3^2 - 2\alpha_1^3\alpha_3\alpha_4^2)\lambda\\
&- \frac{1}{2}\alpha_0^2\alpha_1\alpha_3^3 + \frac{1}{2}\alpha_0^2\alpha_1\alpha_4^3,\\
\mathcal{T}_1(\lambda) = &(-\frac{1}{2}\alpha_0^2\alpha_2^4 - \frac{1}{2}\alpha_0^2\alpha_2\alpha_4^3 - 2\alpha_0\alpha_1\alpha_2^2\alpha_3\alpha_4)\lambda^3\\
&+ (\frac{3}{2}\alpha_0^2\alpha_2^3\alpha_3 + \alpha_0^2\alpha_3\alpha_4^3 + 4\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4 - 2\alpha_1^2\alpha_2^2\alpha_4^2)\lambda^2\\
&+ (-\frac{3}{2}\alpha_0^2\alpha_2^2\alpha_3^2 - 2\alpha_0\alpha_1\alpha_3^3\alpha_4 + 2\alpha_0\alpha_1\alpha_4^4 + 6\alpha_1^2\alpha_2\alpha_3\alpha_4^2)\lambda\\
&+ \frac{1}{2}\alpha_0^2\alpha_2\alpha_3^3 + \frac{1}{2}\alpha_0^2\alpha_2\alpha_4^3 - 4\alpha_1^2\alpha_3^2\alpha_4^2,\\
\mathcal{T}_2(\lambda) = &(\frac{1}{2}\alpha_0^2\alpha_2^3\alpha_3 + \frac{1}{2}\alpha_0^2\alpha_3\alpha_4^3 - 4\alpha_1^2\alpha_2^2\alpha_4^2)\lambda^3\\
&+ (-\frac{3}{2}\alpha_0^2\alpha_2^2\alpha_3^2 - 2\alpha_0\alpha_1\alpha_2^3\alpha_4 + 2\alpha_0\alpha_1\alpha_4^4 + 6\alpha_1^2\alpha_2\alpha_3\alpha_4^2)\lambda^2\\
&+ (\frac{3}{2}\alpha_0^2\alpha_2\alpha_3^3 + \alpha_0^2\alpha_2\alpha_4^3 + 4\alpha_0\alpha_1\alpha_2^2\alpha_3\alpha_4 - 2\alpha_1^2\alpha_3^2\alpha_4^2)\lambda\\
&- \frac{1}{2}\alpha_0^2\alpha_3^4 - \frac{1}{2}\alpha_0^2\alpha_3\alpha_4^3 - 2\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4,\\
\mathcal{T}_3(\lambda) = &(-\frac{1}{2}\alpha_0^2\alpha_2^3\alpha_4 - \frac{1}{2}\alpha_0^2\alpha_4^4 - 2\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4^2)\lambda^3\\
&+ (\frac{1}{2}\alpha_0^2\alpha_2^2\alpha_3\alpha_4 + 2\alpha_0\alpha_1\alpha_3^2\alpha_4^2 + 2\alpha_1^2\alpha_2\alpha_4^3)\lambda^2\\
&+ (\frac{1}{2}\alpha_0^2\alpha_2\alpha_3^2\alpha_4 + 2\alpha_0\alpha_1\alpha_2^2\alpha_4^2 + 2\alpha_1^2\alpha_3\alpha_4^3)\lambda - \frac{1}{2}\alpha_0^2\alpha_3^3\alpha_4 - \frac{1}{2}\alpha_0^2\alpha_4^4 - 2\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4^2.
\end{aligned}$$

We obtain the following.

Lemma 5.16. *The map* $\psi\colon \mathbb{P}^1_{k_\alpha} \to \mathcal{T}_\alpha = V(T_1, T_2, T_3)$ *given by*

$$\lambda \mapsto (\mathcal{T}_0(\lambda) : \mathcal{T}_1(\lambda) : \mathcal{T}_2(\lambda) : \mathcal{T}_3(\lambda))$$

defines a birational map and thus $V(T_1, T_2, T_3)$ *is a twisted cubic curve.*

We are now close to being able to apply Lemma 4.14. We have the web of quadrics, and the twisted cubic. Recall that there is a conic C_α in the marked trope through the 6 nodes in the trope. To that end we have the following result.

Proposition 5.17 ([25, 5.4.2])**.** *The conic in the trope of* K_α *through the 6 nodes is given by* $K_\alpha \cap P_{\alpha^2}(\mathcal{B}) \cap \mathbb{T}_\alpha(\mathcal{B})$*, where* $\mathbb{T}_\alpha(\mathcal{B})$ *denotes the tangent plane at* α *of* $\mathcal{B}$*.*

To apply the results from Lemma 4.14 we must now show that there is an isomorphism between twisted cubic and the conic in the trope. Let $\mathcal{W}_\alpha$ be the web of quadrics spanned by the Q_i. Take $\mathcal{J}_{\mathcal{W}_\alpha}$ to be the Jacobian locus of $Q_1, ..., Q_4$. Take Sing$\mathcal{W}_\alpha$ be the singular locus of $\mathcal{W}$. Recall that there is a birational map $\varphi\colon \mathcal{J}_{\mathcal{W}_\alpha} \to \mathrm{Sing}\mathcal{W}_\alpha$ that takes a singular quadric through the 6 base points of $\mathcal{W}_\alpha$ to the singular point of that quadric (see Lemma 1.41). Let $b_1, ..., b_6$ be the base points of the web $\mathcal{W}$. Choose a projective automorphism $\Theta\colon \mathcal{T}_\alpha \to \mathcal{T}_0$ where $\mathcal{T}_0$ is the standard twisted cubic given by (4.1). Let $\mathcal{W}_0$ be the linear system of quadrics through the 6 points $d_i = \Theta(b_i)$. As the space of quadrics in 4 variables is 10-dimensional, the linear system of quadrics through $d_1, ..., d_6$ is 4-dimensional. In fact, there is a convenient choice of basis. Let $g(x) = \sum_{i=0}^{6} g_i x^i$ be the sextic polynomial with roots d_i. Consider the quadrics

$$S_1 = x_0x_2 - x_1^2,\ S_2 = x_0x_3 - x_1x_2,\ S_3 = x_1x_2 - x_2^2$$
$$S_4 = g_0x_0^2 + g_1x_0x_1 + g_2x_1^2 + g_3x_1x_2 + g_4x_2^2 + g_5x_2x_3 + g_6x_3^2.$$

These quadrics are linearly independent as $g_6 \neq 0$ and contain the points d_i by construction. Let $\mathcal{W}_0$ be the web spanned by $S_1, ..., S_4$. From [6, 5.2] the singular locus of $\mathcal{W}_0$ is a dual Kummer surface with marked trope given by $V(t_3)$. Note that here we are using $t_0, ..., t_3$ as coordinates for the space that contains the singular locus.

Lemma 5.18. *The map Θ induces a isomorphism of vector spaces between $\mathcal{W}_0$ and $\mathcal{W}$.*

Proof. The equation of $\Theta(Q)$ is $Q \circ (\Theta^{-1})$. Since Q is a quadric, and Θ a projective linear transformation, Θ can be represented by an invertible matrix M_Θ and

$$Q = (x_0, x_1, x_2, x_3)M_Q(x_0, x_1, x_2, x_3)^T$$

for some symmetric matrix M_Q. So the action of Θ is

$$M_Q \mapsto M_\Theta^{-1} M_Q M_\Theta^{-T}.$$

It is immediate that the action preserves addition and scalar multiplication. □

We also have that for each quadric $Q \in \mathcal{W}_0$ there is a unique quadric $\tilde{Q} \in \mathcal{W}$ such that $\Theta(\tilde{Q}) = Q$. Furthermore,the above correspondence between $\mathcal{W}$ and $\mathcal{W}_0$ respects singular quadrics. Putting these facts together we can construct a projective automorphism that maps $\mathrm{Sing}\mathcal{W}_0 \to \mathrm{Sing}\mathcal{W}$.

Lemma 5.19. *The singular locus of $\mathcal{W}_0$ with basis $\Theta(Q_1), \Theta(Q_2), \Theta(Q_3), \Theta(Q_4)$ is the singular locus of the web Q_1, Q_2, Q_3, Q_4.*

Proof. Let $x = (x_0, x_1, x_2, x_3)$. The claim follows from

$$\det(\sum_{i=0}^{3} t_i Q_{i+1} \circ \Theta) =$$
$$\det(\sum_{i=0}^{3} t_i M_\Theta^{-1} M_{Q_{i+1}} {M_\theta}^{-T}) =$$
$$\det(\sum_{i=0}^{3} t_i M_{Q_{i+1}}).$$

□

We can now construct the projective automorphism. We define F_Θ to be the projective automorphism given by the change of basis matrix from $\Theta(Q_1), ..., \Theta(Q_4)$ to $S_1, ..., S_4$. As this matrix is invertible, we get a linear transformation of $\mathbb{P}^3$ that takes

$$\mathrm{Sing}\mathcal{W} \mapsto \mathrm{Sing}\mathcal{W}_0.$$

On the other hand, we can reverse the construction using Θ^{-1} to construct a linear automorphism that maps $\mathrm{Sing}\mathcal{W}_0 \to \mathrm{Sing}\mathcal{W}$. Explicitly this is given by the change of basis matrix $\Theta^{-1}(S_1), ..., \Theta^{-1}(S_4) \mapsto Q_1, ..., Q_4$. Now let P be a point on the the Jacobian locus of $Q_1, ..., Q_4$. The map φ of Lemma 1.38 takes the point P and associates a quadric Q_P with singularity at P.

Lemma 5.20. *Let Θ, F_Θ, $F_{\Theta^{-1}}$, and φ be as above. Take φ_0 to be the map from the Jacobian locus of $S_1, ..., S_4$ to the singular locus of the web spanned by $S_1, ..., S_4$. Then the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Sing}W_0 & \xrightarrow{F_{\Theta^{-1}}} & \mathrm{Sing}W \\ \varphi_0 \uparrow & & \uparrow \varphi \\ \mathcal{J}_{\mathcal{W}_0} & \xleftarrow{\Theta} & \mathcal{J}_{\mathcal{W}} \end{array}$$

Proof. Let $P \in \mathcal{J}_{\mathcal{W}}$ and let Θ be a point where the map φ is defined and suppose that $\Theta(P)$ is in the domain of φ_0. Let $(t_0, ..., t_3) = \varphi(P)$. Then $\Theta(P)$ lies on the twisted cubic in $\mathcal{J}_{\mathcal{W}_0}$ by construction. Take $\varphi_0(\Theta(P)) = (t'_0, ..., t'_3)$. Recall that $F_{\Theta^{-1}}(t'_0, ..., t'_3)$ are the coordinates of

$$\sum_{i=0}^{3} t'_i \Theta^{-1}(S_{i+1}).$$

Since Θ^{-1} acts as a linear transformation we have

$$\sum_{i=0}^{3} t'_i \Theta^{-1}(S_{i+1}) =$$
$$\Theta^{-1}(\sum_{i=0}^{3} t'_i(S_{i+1})) =$$
$$\Theta^{-1}(\sum_{i=0}^{3} t_i \Theta(Q_{i+1})) =$$
$$\sum_{i=0}^{3} t_i Q_{i+1}.$$

This is precisely the statement that the diagram commutes. □

We can now ask about the image of the marked trope of $\mathrm{Sing}\mathcal{W}_0$ in $\mathrm{Sing}\mathcal{W}$.

Lemma 5.21. *Let $V(t_3)$ be the marked trope of* $\mathrm{Sing}W_0$. *Then the image of $V(t_3)$ (with respect to $F_{\Theta^{-1}}$) is the marked trope of* $\mathrm{Sing}W$.

Proof. The marked trope of $\mathrm{Sing}(\mathcal{W}_0)$ is the linear span of S_1, S_2, S_3 which are the generators of the ideal of a twisted space cubic. As $F_{\Theta^{-1}}$ is is a projective automorphism, the same holds for the image of $V(t_3)$. Let X be the twisted space cubic generated by the image of $V(t_3)$. By construction, X lies in the web $\mathcal{W}$ and so contains the points $b_1, ..., b_6$. As there is a unique twisted space cubic through 6 points, we have that $X = \mathcal{T}_\alpha$. Then $\Theta^{-1}(S_1), \Theta^{-1}(S_2), \Theta^{-1}(S_3)$ lie in the ideal of $\mathcal{T}_\alpha$. Since we are working with a homogeneous ideal, and all the elements are quadrics it must be the case that $\Theta^{-1}(S_1), \Theta^{-1}(S_2), \Theta^{-1}(S_3)$ lie in the linear span of T_1, T_2, T_3 the generators of the ideal of $\mathcal{T}_\alpha$. By construction, the elements T_1, T_2, T_3 generate the marked trope of $\mathrm{Sing}\mathcal{W}$. □

As a corollary we obtain the an isomorphism between the twisted cubic and the conic in the trope.

Corollary 5.22. *The map φ provides an isomorphism between the twisted cubic $\mathcal{T}_\alpha$ and the conic in the marked trope of K_α. Furthermore, the 6 nodes on the conic are mapped to by the 6 base points of the web $\mathcal{W}$.*

Proof. By Proposition 4.12 the map φ_0 of Lemma 5.20 maps the twisted cubic to the conic. By Lemma 5.21 the marked trope of $V(y_0)$ is mapped to the marked trope of $\mathrm{Sing}(\mathcal{W}) = K_\alpha$. Thus the $F_{\Theta^{-1}}$ takes the conic in the trope $V(y_0)$ to the conic in the trope of $\mathrm{Sing}\mathcal{W}$ and the result follows. □

We can now apply Lemma 4.14.

Theorem 5.23. *Let $\alpha \in \mathcal{B}$. Define*

$$f_\alpha(\lambda) = \psi^* Q_4 = Q_4(\mathcal{T}_0(\lambda), \mathcal{T}_1(\lambda), \mathcal{T}_2(\lambda), \mathcal{T}_3(\lambda)).$$

If f_α has degree 6 and $\mathrm{Disc}(f_\alpha)$ is non-zero and $\frac{\partial B}{\partial y_4}(\alpha) \neq 0$ and E_1, E_2 are not both zero and F_1, F_2, F_3 are not all zero then $y^2 = f_\alpha$ is a twist of the curve $\mathcal{C}$ corresponding to α.

Proof. A direct computation shows that the discriminant locus of f_α contains all of the distinguished planes. Thus if the discriminant of f_α is non-zero then α does not lie on any distinguished plane. We have seen K_α is the singular locus of a web of quadrics with 6 points in common (see Lemma 5.12 and Lemma 5.10). Furthermore, we have found a basis T_1, T_2, T_3, Q_4 of W_α where T_1, T_2, T_3 cut out a twisted cubic curve. (see Lemma 5.16) Finally, we have seen that the conic in the trope of the dual Kummer surface through the 6 nodes is isomorphic to the twisted cubic curve (see Corollary 5.22). By Lemma 4.14, $\psi^* Q_4 = Q_4(\mathcal{T}_0(\lambda), \mathcal{T}_1(\lambda), \mathcal{T}_2(\lambda), \mathcal{T}_3(\lambda))$ represents a twist of the curve $\mathcal{C}$. □

We will let $\hat{f}_\alpha$ denote sextic that corresponds to a twist of $\mathcal{C}_\alpha$ found above.

5.2.2 The Discriminant Locus of f_α

The discriminant of the polynomial $\hat{f}_\alpha$ is given by

$$\alpha_4^{30} \alpha_0^3 (\alpha_0 + 2\alpha_1)^3 D_1 (D_2 D_3 D_4 D_5)^3,$$

where

$$D_1 = \alpha_0\alpha_2^3 + \alpha_0\alpha_3^3 + \alpha_0\alpha_4^3 + 6\alpha_1\alpha_2\alpha_3\alpha_4,$$
$$D_2 = (\alpha_0\alpha_3 + \alpha_0\alpha_4 - 2\alpha_1\alpha_2)(\alpha_0\alpha_2 + \alpha_0\alpha_4 - 2\alpha_1\alpha_3)(\alpha_0\alpha_2 + \alpha_0\alpha_3 - 2\alpha_1\alpha_4)(\alpha_0^2 - 2\alpha_0\alpha_1 + 4\alpha_1^2),$$
$$D_3 = \alpha_0^2\alpha_3^2 - \alpha_0^2\alpha_3\alpha_4 + \alpha_0^2\alpha_4^2 + 2\alpha_0\alpha_1\alpha_2\alpha_3 + 2\alpha_0\alpha_1\alpha_2\alpha_4 + 4\alpha_1^2\alpha_2^2,$$
$$D_4 = \alpha_0^2\alpha_2^2 - \alpha_0^2\alpha_2\alpha_4 + \alpha_0^2\alpha_4^2 + 2\alpha_0\alpha_1\alpha_2\alpha_3 + 2\alpha_0\alpha_1\alpha_3\alpha_4 + 4\alpha_1^2\alpha_3^2,$$
$$D_5 = \alpha_0^2\alpha_2^2 - \alpha_0^2\alpha_2\alpha_3 + \alpha_0^2\alpha_3^2 + 2\alpha_0\alpha_1\alpha_2\alpha_4 + 2\alpha_0\alpha_1\alpha_3\alpha_4 + 4\alpha_1^2\alpha_4^2.$$

In addition to the above, we have the forms

$$E_1 = -48\alpha_0\alpha_2\alpha_3^2 - 96\alpha_1\alpha_2^2\alpha_4$$
$$E_2 = 48\alpha_0\alpha_3^3 + 48\alpha_0\alpha_4^3 + 192\alpha_1\alpha_2\alpha_3\alpha_4$$
$$F1 = \alpha_0^2\alpha_2^3\alpha_3 + \alpha_0^2\alpha_3\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4 - 4\alpha_1^2\alpha_2^2\alpha_4^2$$
$$F2 = -2\alpha_0^2\alpha_2^2\alpha_3^2 - 2\alpha_0\alpha_1\alpha_3^3\alpha_4 + 2\alpha_0\alpha_1\alpha_4^4 + 4\alpha_1^2\alpha_2\alpha_3\alpha_4^2$$
$$F3 = \alpha_0^2\alpha_2\alpha_3^3 + \alpha_0^2\alpha_2\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2^2\alpha_3\alpha_4 - 4\alpha_1^2\alpha_3^2\alpha_4^2.$$

The sextic $\hat{f}_\alpha$ defines a twist of the genus 2 curve corresponding to the point α if the discriminant does not vanish, and if at least one of E_i and F_i are non-zero.

5.3 Finding the correct twists of f_α

In Theorem 5.23 we found an explicit expression for a polynomial

$$\hat{f}_\alpha(x) = \sum_{i=0}^{6} f_i(\alpha_0, ..., \alpha_4)x^i$$

where the f_i are polynomials in $\mathbb{Q}[y_0, ..., y_4]$. Consider

$$\hat{f}_\alpha(x) = \sum_{i=0}^{6} f_i(\alpha_0, ..., \alpha_4)x^i \in (\mathbb{Q}[\alpha_0, ..., \alpha_4])[x].$$

The terms of the polynomial $\hat{f}_\alpha(x)$ all have a common factor, namely $\frac{\partial B}{\partial y_4}(y_0, ..., y_4)$. This suggests that correct twist could be $\frac{\partial B}{\partial y_4}(\alpha_0, ..., \alpha_4)^{-1}$. Recall that a smooth genus 2 curve $\mathcal{C}$ with a full level 3 structure is one with an isomorphism of group schemes $\varphi\colon \mathrm{Jac}(\mathcal{C})[3] \to (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$ that is compatible with the Weil pairing. We have described a family of curves $\mathcal{C}'_\alpha\colon y^2 = \hat{f}_\alpha(x)$. For these curves to correspond to the moduli point α, their Jacobians must have the required torsion structure. We can now check for explicit α that satisfy the conditions of Theorem 5.23 that $y^2 = \frac{\partial B}{\partial y_4}(\alpha_0, ..., \alpha_4)^{-1}\hat{f}_\alpha(x)$ has the required torsion structure. We will see later in Chapter 6 that our choice of twist allows us to write down an isomorphism of Galois modules $\mathrm{Jac}(\mathcal{C}_\alpha)[3] \to (\mathbb{Z}/3\mathbb{Z})^2 \times \mu_3^2$ (see Theorem 6.13).

Lemma 5.24. *Let $\mathcal{C}\colon y^2 = f(x)$ be a hyperelliptic genus 2 curve over a field k of characteristic 0. Suppose that* $\mathrm{Jac}(\mathcal{C})$ *has a full level three structure. Let $\mathcal{C}'$ be another genus 2 hyperelliptic curve over k with a full level three structure. If $\mathcal{C} \cong \mathcal{C}'$ over $\overline{k}$ then $\mathcal{C}'$ can be represented by $y^2 = df(x)$ where $d \in \{1, -3\}$.*

Proof. As $\mathcal{C}'$ lies in the same geometric isomorphism class of $\mathcal{C}$ we may assume that $\mathcal{C}'$ is of the form $y^2 = df(x)$. First suppose that $\zeta_3 \in k$. This means that $\mu_3(k)$ has three elements. As $\mathcal{C}$ has a full level three structure, the 80 non-zero three torsion elements of $\mathrm{Jac}(\mathcal{C})$ are defined over k. Combining the existence of a full level three structure with Lemma 1.3 tells us that there are exactly 80 triples (G_i, H_i, λ_i) such that

$$f = G_i^2 + \lambda_i H_i^3$$

with $G_i, H_i \in k[x]$ and H_i is monic of degree at most 2. Since $\mathcal{C}'$ also admits a full level three structure, there is a decomposition

$$df = (G')^2 + \lambda'(H')^3.$$

Over $k(\sqrt{d})$ we also have a decomposition

$$f = \left(\frac{G'}{\sqrt{d}}\right)^2 + \frac{\lambda'}{d}(H')^3$$

for some G', H', λ'. However, this forces that $(\frac{G'}{\sqrt{d}}, H', \frac{\lambda'}{d})$ is equal to some triple (G_i, H_i, λ_i), which implies that d must be a square in k and so $\mathcal{C} \cong \mathcal{C}'$.

Now suppose that $\zeta_3 \notin k$. Then over the quadratic extension $k(\zeta_3)$ the above argument shows that

$$\mathcal{C} \times_k k(\zeta_3) \cong \mathcal{C}' \times_k k(\zeta_3).$$

However, this means that d is a square in $k(\zeta_3)$ and so is either in a square in k or -3 times a square in k. In either case the result follows. $\square$

Thus if we can show that $y^2 = \dfrac{\partial B}{\partial y_4}(\alpha_0, ..., \alpha_4)^{-1}\hat{f}_\alpha(x)$ has a full level three 3 structure, then we have identified the k-isomorphism classes of $y^2 = \hat{f}_\alpha$ which have a full level three structure.

We now set

$$\frac{\partial B}{\partial y_4}(\alpha_0, ..., \alpha_4)^{-1}\hat{f}_\alpha = f_\alpha. \tag{5.2}$$

We will later see a compact way of writing down f_α. The equation can be found in computer readable format in A.2.2.

Chapter 6

Marking the Three Torsion

We have described how to find a twist of the genus 2 curve associated to a sufficiently general point on $\mathcal{B}$. In this chapter we will verify that we found the correct twist of the curve in Section 5.3 by exhibiting a full level 3 structure, and give some applications of our explicit formulas. We let $\alpha \in \mathcal{B}$ satisfy the hypothesis of Theorem 5.23, and let f_α be the sextic given in Section 5.3. We let $\mathcal{C}_\alpha$ be the curve described by $y^2 = f_\alpha$ and A_α the Jacobian of $\mathcal{C}_\alpha$.

6.1 The Relationship between the Distinguished Planes and the 3-Torsion

It is well known that A_α contains 81 three-torsion points. The 3-torsion can be grouped into 40 pairs of points: the identity, and 40 points along with their inverses. Furthermore, $A_\alpha[3]$ has the group structure of a 4-dimensional vector space over $\mathbb{F}_3$. The Weil pairing equips this space with a non-degenerate alternating pairing. It is straightforward to show that such a vector space has 40 two-dimensional subspaces to which the pairing restricts to the trivial one. Furthermore, these subspaces are maximal with respect to the property that the Weil pairing is trivial. We will call these subspaces maximal isotropic. The numerology suggests a possible connection between these objects. We will show that a distinguished plane corresponds to a a non-trivial 3-torsion point and its inverse, while a distinguished hyperplane corresponds to a maximal isotropic subspace of $A_\alpha[3]$.

6.1.1 The Distinguished Planes

Fix α that satisfies the hypothesis of Theorem 5.23 and take coordinates $y_0, ..., y_4$ on $\mathbb{P}^4_{k_\alpha}$ (where k_α is a field that α is defined over). In the following computations it will be useful to introduce the following notation. We will put $\mathcal{T}_\alpha, \psi_\alpha$ for the twisted space cubic and the corresponding rationalization found in Lemma 5.16. Let $j_i = V(y_0, y_i)$ and $S = V(y_0)$. Note that the j_i are the distinguished planes in the distinguished hyperplane S. Recall that in Section 5.2 we saw that the dual Kummer surface of A_α is isomorphic to the the projection of $\mathcal{K} = \mathrm{EC}_\alpha(P_\alpha(\mathcal{B}))$ away from α to any hypersurface not containing α. Fix a hyperplane H_α not containing α and let

$$\mathrm{Pr}_\alpha \colon \mathbb{P}^4_{k_\alpha} \dashrightarrow H_\alpha$$

be the rational map defined by projecting away from α to H_α. Put $\mathcal{K}_\alpha = \mathrm{Pr}_\alpha(\mathcal{K})$.

Remark 6.1. Recall that $\mathcal{K}$ is a cone over a dual Kummer surface (see Proposition 5.8). Note that the projection of a cone over α to the hypersurface H_α is projectively equivalent to the intersection of the cone with any other hyperplane H not containing α. For computation, instead of projecting to a hyperplane we will intersect with the hyperplane $V(y_0)$. In this case, the intersection $\mathcal{K} \cap V(y_0)$ will be denoted K_α as in Section 5.2.

Given a distinguished plane J we may ask about the image $J_\alpha = \mathrm{Pr}_\alpha(J)$.

Lemma 6.2. *Let J_1 and J_2 be distinct distinguished planes. Suppose that $J_1, J_2 \subseteq H$ where H is a hyperplane. Then H is a distinguished hyperplane.*

Proof. By using the automorphism group we may assume that $J_1 = j_1$. If j_1 and J_2 lie in H, then they span H. We can now check the linear span of j_1 and J_2 for all distinguished planes. We find that the only hyperplanes spanned by j_1 and another distinguished plane are distinguished hyperplanes. □

Lemma 6.3. *Let $\alpha \in \mathcal{B}$ not lie on any distinguished plane. If J_1 and J_2 are distinct distinguished planes then $\mathrm{Pr}_\alpha(J_1)$ and $\mathrm{Pr}_\alpha(J_2)$ are distinct in H_α.*

Proof. Let J_1 and J_2 be distinguished planes. The projections of J_1 and J_2 to H_α from α coincide if and only if J_1, J_2 and α lie in a single hyperplane. By Lemma 6.2 the hyperplane must be a distinguished hyperplane. However, then α lies on the Burkhardt quartic and on a distinguished hyperplane. As $\mathcal{B}$ meets a distinguished hyperplane in 4 distinguished planes we obtain a contradiction, as α does not lie on any distinguished plane by assumption.

□

It natural to ask if the 40 planes in H_α as described above have a relation to $\mathcal{K}_\alpha$.

Proposition 6.4. *Let J be a distinguished plane. Then J_α is tangent to $\mathcal{K}_\alpha$.*

Proof. It suffices to prove the result for $J = j_1$. As in Remark 6.1 we can also assume that $\mathcal{K}_\alpha = K_\alpha$. We can now directly check that the point $P = (0 : \alpha_2 : \alpha_3 : \alpha_4)$ lies on K_α and that the tangent space of K_α at P is j_1. □

From Lemma 6.3 and Proposition 6.4 we see that each distinguished plane J corresponds to a plane J_{H_α} in H_α that is tangent to $\mathcal{K}_\alpha$. As $\mathcal{K}_\alpha$ is a dual Kummer surface, projective duality tells us that each distinguished plane corresponds to a point on the Kummer surface of A_α. By pulling back to A_α we obtain a pair of points on that consist of a point and its inverse. Thus we have obtained 80 points on A_α grouped into 40 pairs of points that consist of a point and its inverse. Our goal is to show that these points are 3-torsion. Let Y_α be the conic in the marked trope of $\mathcal{K}_\alpha$ (see Proposition 5.17). The projection of each distinguished plane J intersects Y_α in a degree 2 subscheme. Because $\mathcal{K}_\alpha$ is projective to K_α we obtain an isomorphism $Y_\alpha \to \mathcal{T}_\alpha$. Thus, each distinguished plane marks two points on $\mathcal{T}_\alpha$. Pulling back by ψ_α gives a degree two subscheme in $\mathbb{P}^1_{k_\alpha}$ and it is natural to ask about the fiber of this subscheme under the hyperelliptic double cover $x_\alpha \colon \mathcal{C}_\alpha \to \mathbb{P}^1_{k_\alpha}$. The computation is simplified by the following observations. The four distinguished planes $j_1, j_2, j_3, j_4 \subseteq \mathbb{P}^4_{k_\alpha}$ determine four points.

$$\begin{aligned}
\tilde{p}_1 &= j_2 \cap j_3 \cap j_4 = (0:1:0:0:0)\\
\tilde{p}_2 &= j_1 \cap j_3 \cap j_4 = (0:0:1:0:0)\\
\tilde{p}_3 &= j_1 \cap j_2 \cap j_4 = (0:0:0:1:0)\\
\tilde{p}_4 &= j_1 \cap j_2 \cap j_3 = (0:0:0:0:1).
\end{aligned}$$

By forgetting the first coordinate we obtain four points in the $\mathbb{P}^3_{k_\alpha}$ containing $\mathcal{T}_\alpha$ which we identify with $V(y_0)$. Recall that we parametrized $\mathcal{T}_\alpha$ by finding a chord through p_1 and then rotating a $\mathbb{P}^1$ of planes about that chord (see the discussion following Definition 5.2.1). The construction specifies two points on the twisted cubic curve, and so under the isomorphism of Lemma 5.16 specifies two points in $\mathbb{P}^1_{k_\alpha}$. As $\mathcal{C}_\alpha$ is a double cover of $\mathbb{P}^1_{k_\alpha}$ we obtain two points on $\mathcal{C}_\alpha$ up to the action of the hyperelliptic involution.

Lemma 6.5. *The points on the chord of $\mathcal{T}$ through p_1 are given by the vanishing of the 3×3 minors of*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_2 & \alpha_3 & \alpha_4 \\ \mathcal{T}_0(x) & \mathcal{T}_1(x) & \mathcal{T}_2(x) & \mathcal{T}_3(x) \end{bmatrix}. \tag{6.1}$$

Proof. Recall that the chord is the line through p_1 and $(0 : \alpha_2 : \alpha_3 : \alpha_4)$. This line contains a point on the twisted cubic precisely when the 3 rows of the above matrix are dependent and thus when the 3×3 minors vanish. □

Explicitly, the two points on the chord of $\mathcal{T}$ through p_1 are the roots of the polynomial

$$H_1 = x^2 + \frac{-2\alpha_0^2\alpha_2^2\alpha_3^2 - 2\alpha_0\alpha_1\alpha_2^3\alpha_4 - 2\alpha_0\alpha_1\alpha_3^3\alpha_4 + 2\alpha_0\alpha_1\alpha_4^4 + 4\alpha_1^2\alpha_2\alpha_3\alpha_4^2}{\alpha_0^2\alpha_2^3\alpha_3 + \alpha_0^2\alpha_3\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4 - 4\alpha_1^2\alpha_2^2\alpha_4^2}x + \frac{\alpha_0^2\alpha_2\alpha_3^3 + \alpha_0^2\alpha_2\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2^2\alpha_3\alpha_4 - 4\alpha_1^2\alpha_3^2\alpha_4^2}{\alpha_0^2\alpha_2^3\alpha_3 + \alpha_0^2\alpha_3\alpha_4^3 + 2\alpha_0\alpha_1\alpha_2\alpha_3^2\alpha_4 - 4\alpha_1^2\alpha_2^2\alpha_4^2}.$$

Temporarily, regard the α_i as formal variables. Computation with a computer algebra system shows that the principal divisor of H_1 in the function field of the curve $y^2 = f_\alpha(x)$ is supported by the divisor at infinity and 2 other points $\mathcal{P}_i$ together with their hyperelliptic involutes.

Remark. We used Magma to perform the above calculation (see [2]).

Let κ_∞ be the fiber of the hyperelliptic double cover above the point at $x = \infty$. Direct computation shows that $3 \cdot (\mathcal{P}_i - \kappa_\infty)$ is the divisor of a function $y - G_1$. In fact, we can say more. By examining $\frac{(y-G_1)(y+G_1)}{H_1^3}$ we obtain a formal identity $f_\alpha(x) = G_1(x)^2 + u_1 H_1(x)^3$ with $u_1 = \frac{w_1}{r_1}$ for some polynomials w_1 and r_1. Explicitly we have,

$$\begin{aligned} w_1 = \alpha_0^6\alpha_2^9\alpha_3^3 + 3\alpha_0^6\alpha_2^6\alpha_3^3\alpha_4^3 + 3\alpha_0^6\alpha_2^3\alpha_3^3\alpha_4^6 + \alpha_0^6\alpha_3^3\alpha_4^9 + 6\alpha_0^5\alpha_1\alpha_2^7\alpha_3^4\alpha_4 \\ +12\alpha_0^5\alpha_1\alpha_2^4\alpha_3^4\alpha_4^4 + 6\alpha_0^5\alpha_1\alpha_2\alpha_3^4\alpha_4^7 - 12\alpha_0^4\alpha_1^2\alpha_2^8\alpha_3^2\alpha_4^2 \\ +12\alpha_0^4 a\alpha_1^2\alpha_2^5\alpha_3^5\alpha_4^2 - 24\alpha_0^4\alpha_1^2\alpha_2^5\alpha_3^2\alpha_4^5 + 12\alpha_0^4\alpha_1^2\alpha_2^2\alpha_3^5\alpha_4^5 - 12\alpha_0^4\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^8 \\ -48\alpha_0^3\alpha_1^3\alpha_2^6\alpha_3^3\alpha_4^3 + 8\alpha_0^3\alpha_1^3\alpha_2^3\alpha_3^6\alpha_4^3 - 48\alpha_0^3\alpha_1^3\alpha_2^3\alpha_3^3\alpha_4^6 + 48\alpha_0^2\alpha_1^4\alpha_2^7\alpha_3\alpha_4^4 - 48\alpha_0^2\alpha_1^4\alpha_2^4\alpha_3^4\alpha_4^4 \\ +48\alpha_0^2\alpha_1^4\alpha_2^4\alpha_3\alpha_4^7 + 96\alpha_0\alpha_1^5\alpha_2^5\alpha_3^2\alpha_4^5 - 64\alpha_1^6\alpha_2^6\alpha_4^6 \end{aligned}$$

$$\begin{aligned} r_1 = \alpha_0^2\alpha_2^6 + 2\alpha_0^2\alpha_2^3 a3^3 + 2\alpha_0^2\alpha_2^3\alpha_4^3 + \alpha_0^2\alpha_3^6 \\ +2\alpha_0^2\alpha_3^3\alpha_4^3 + \alpha_0^2\alpha_4^6 + 12\alpha_0\alpha_1\alpha_2^4\alpha_3\alpha_4 + 12\alpha_0\alpha_1\alpha_2\alpha_3^4\alpha_4 \\ +12\alpha_0\alpha_1\alpha_2\alpha_3\alpha_4^4 + 36\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2 \end{aligned}$$

$$\begin{aligned} G_1 = {} & (\alpha_0\alpha_2^3 + \alpha_0\alpha_3^3 + \alpha_0\alpha_4^3 + 6\alpha_1\alpha_2\alpha_3\alpha_4)^{-1}((\frac{1}{2}\alpha_0^3\alpha_2^6 - \frac{1}{2}\alpha_0^3\alpha_2^3\alpha_3^3 \\ & + \alpha_0^3\alpha_2^3\alpha_4^3 - \frac{1}{2}\alpha_0^3\alpha_3^3\alpha_4^3 + \frac{1}{2}\alpha_0^3\alpha_4^6 + 3\alpha_0^2\alpha_1\alpha_2^4\alpha_3\alpha_4 + 3\alpha_0^2\alpha_1\alpha_2\alpha_3\alpha_4^4 + 12\alpha_0\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2 + 8\alpha_1^3\alpha_2^3\alpha_4^3)x^3 \\ & + (-\frac{3}{2}\alpha_0^3\alpha_2^5\alpha_3 + \frac{3}{2}\alpha_0^3\alpha_2^2\alpha_3^4 - \frac{3}{2}\alpha_0^3\alpha_2^2\alpha_3\alpha_4^3 - 3\alpha_0^2\alpha_1\alpha_2^3\alpha_3^2\alpha_4 - 6\alpha_0^2\alpha_1\alpha_3^2\alpha_4^4 \\ & + 6\alpha_0\alpha_1^2\alpha_2^4\alpha_4^2 - 18\alpha_0\alpha_1^2\alpha_2\alpha_3^3\alpha_4^2 - 6\alpha_0\alpha_1^2\alpha_2\alpha_4^5 - 12\alpha_1^3\alpha_2^2\alpha_3\alpha_4^3)x^2 \\ & + (\frac{3}{2}\alpha_0^3\alpha_2^4\alpha_3^2 - \frac{3}{2}\alpha_0^3\alpha_2\alpha_3^5 - \frac{3}{2}\alpha_0^3\alpha_2\alpha_3^2\alpha_4^3 - 3\alpha_0^2\alpha_1\alpha_2^2\alpha_3^3\alpha_4 - 6\alpha_0^2\alpha_1\alpha_2^2\alpha_4^4 \\ & - 18\alpha_0\alpha_1^2\alpha_2^3\alpha_3\alpha_4^2 + 6\alpha_0\alpha_1^2\alpha_3^4\alpha_4^2 - 6\alpha_0\alpha_1^2\alpha_3\alpha_4^5 - 12\alpha_1^3\alpha_2\alpha_3^2\alpha_4^3)x \\ & - \frac{1}{2}\alpha_0^3\alpha_2^3\alpha_3^3 - \frac{1}{2}\alpha_0^3\alpha_2^3\alpha_4^3 + \frac{1}{2}\alpha_0^3\alpha_3^6 + \alpha_0^3\alpha_3^3\alpha_4^3 + \frac{1}{2}\alpha_0^3\alpha_4^6 \\ & + 3\alpha_0^2\alpha_1\alpha_2\alpha_3^4\alpha_4 + 3\alpha_0^2\alpha_1\alpha_2\alpha_3\alpha_4^4 + 12\alpha_0\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2 + 8\alpha_1^3\alpha_3^3\alpha_4^3). \end{aligned}$$

Returning now to the situation with $\alpha \in \mathbb{P}^4_{k_\alpha}$ (and satisfying the usual assumptions of this section) we see that if r_1 is non-zero, and the denominators in H_1 and G_1 do not vanish then we obtain a decomposition of

$$f_\alpha = G_1^2 + u_1 H_1^3.$$

By Lemma 1.3 we obtain a 3-torsion element of $\mathrm{Pic}^0_{k_\alpha}(\mathcal{C}_\alpha)$ (and thus of the Jacobian) given by $[\{y - G_1, H_1\} - \kappa]$ where κ is an effective canonical divisor on $\mathcal{C}_\alpha$. We can now ask about the other p_i. One checks that for $1 \le i \le 3$ that the non-vanishing of $\alpha_0 \dfrac{\partial B}{\partial y_4}(\alpha)$ implies that $T_i(p_i) \neq 0$ (with T_i defined as Definition 5.15) which means that p_i does not lie on the twisted cubic $\mathcal{T}$. For p_2, p_3 we obtain two points on the twisted cubic curve, and thus two points on $\mathbb{P}^1_{k_\alpha}$. The situation with p_4 is only slightly different. One computes that $T_i(p_4) = -\alpha_0 \dfrac{\partial B}{\partial y_i}(\alpha)$. Thus p_4 lies on the twisted cubic when all the partial derivatives of B vanish except $\dfrac{\partial B}{\partial y_4}(\alpha)$. In any case, we can compute the intersection $P_{p_i}(T_1) \cap P_{p_i}(T_2) \cap P_{p_i}(T_3) = q_i$ and find the intersection of the line through p_i and q_i and $\mathcal{T}$. That is to say, consider the matrix with rows

$$p_i, q_i, [\mathcal{T}_0(x), \mathcal{T}_1(x), \mathcal{T}_2(x), \mathcal{T}_3(x)].$$

Compute the 3×3 minors and look at the greatest common denominator, and then work formally as above.

Corresponding to the points $p_2, q_2 = (\alpha_1 : 0 : \alpha_3 : -\alpha_4)$ we obtain.

$$\begin{aligned} G_2 = \frac{(-\frac{1}{2}\alpha_0^3\alpha_2^3 - \frac{1}{2}\alpha_0^3\alpha_4^3 - 8\alpha_1^3\alpha_4^3)}{\alpha_0} x^3 \\ +(\frac{3}{2}\alpha_0^2\alpha_2^2\alpha_3 - 6\alpha_1^2\alpha_2\alpha_4^2)x^2 \\ +(-\frac{3}{2}\alpha_0^2\alpha_2\alpha_3^2 + 6\alpha_1^2\alpha_3\alpha_4^2)x \\ +\frac{1}{2}\alpha_0^2\alpha_3^3 - \frac{1}{2}\alpha_0^2\alpha_4^3, \\ H_2 = x^2 + \frac{1}{2}\frac{\alpha_0\alpha_2}{(\alpha_1\alpha_4)}x - \frac{1}{2}\frac{\alpha_0\alpha_3}{(\alpha_1\alpha_4)}, \\ u_2 = \frac{-8\alpha_0^3\alpha_1^3\alpha_4^6 - 64\alpha_1^6\alpha_4^6}{\alpha_0^2}. \end{aligned}$$

Corresponding to the points $p_3, q_3 = (\alpha_1 : -\alpha_2 : 0 : \alpha_4)$ we obtain.

$$\begin{aligned} G_3 &:= (-\frac{1}{2}\alpha_0^2\alpha_2^3 + \frac{1}{2}\alpha_0^2\alpha_4^3)x^3 \\ &+(\frac{3}{2}\alpha_0^2\alpha_2^2\alpha_3 - 6\alpha_1^2\alpha_2\alpha_4^2)x^2 \\ &+(-\frac{3}{2}\alpha_0^2\alpha_2\alpha_3^2 + 6\alpha_1^2\alpha_3\alpha_4^2)x \\ &+\frac{(\frac{1}{2}\alpha_0^3\alpha_3^3 + \frac{1}{2}\alpha_0^3\alpha_4^3 + 8\alpha_1^3\alpha_4^3)}{\alpha_0}, \\ H_3 &= x^2 - \frac{\alpha_3}{\alpha_2}x - 2\frac{\alpha_1\alpha_4}{(\alpha_0\alpha_2)}, \\ u_3 &= \alpha_0^4\alpha_2^3\alpha_4^3 + 8\alpha_0\alpha_1^3\alpha_2^3\alpha_4^3. \end{aligned}$$

Corresponding to the points $p_4, q_4 = (\alpha_1 : \alpha_2 : -\alpha_3 : 0)$ we obtain.

$$\begin{aligned} G_4 &= \frac{(-\frac{1}{2}\alpha_0^3\alpha_2^3 - \frac{1}{2}\alpha_0^3\alpha_4^3 - 8\alpha_1^3\alpha_2^3)}{\alpha_0}x^3 \\ &+\frac{(\frac{3}{2}\alpha_0^3\alpha_2^2\alpha_3 + 6\alpha_0\alpha_1^2\alpha_2\alpha_4^2 + 24\alpha_1^3\alpha_2^2\alpha_3)}{\alpha_0}x^2 \\ &+\frac{(-\frac{3}{2}\alpha_0^3\alpha_2\alpha_3^2 - 6\alpha_0\alpha_1^2\alpha_3\alpha_4^2 - 24\alpha_1^3\alpha_2\alpha_3^2)}{\alpha_0}x \\ &+(\frac{1}{2}\alpha_0^3\alpha_3^3 + \frac{1}{2}\alpha_0^3\alpha_4^3 + 8\alpha_1^3\alpha_3^3)\alpha_0, \\ H_4 &= x^2 + \frac{(-\frac{1}{2}\alpha_0\alpha_4^2 - 2\alpha_1\alpha_2\alpha_3)}{(\alpha_1\alpha_2^2)}x + \frac{\alpha_3^2}{\alpha_2^2}, \\ u_4 &= \frac{-8\alpha_0^3\alpha_1^3\alpha_2^6 - 64\alpha_1^6\alpha_2^6}{\alpha_0^2}. \end{aligned}$$

For each i we have that

$$f_\alpha(x) = G_i(x)^2 + u_i H_i(x)^3. \tag{6.2}$$

If all of the above expressions are well defined, (that is, the denominators do not vanish) and each of the u_i is non-zero, and at least two of the expressions are distinct, then by Lemma 1.3 we have obtained 4 pairs of 3-torsion points. An immediate benefit to our computations is the following. Let κ be an effective canonical divisor and set $\mathcal{C} : y^2 = G_i(x)^2 + u_i H_i(x)^3$. Now put

$$D_i = \{y - G_i(x), H_i(x)\}$$

and let

$$T_i = [D_i - \kappa] \in \mathrm{Pic}^0(\mathcal{C}/k_\alpha).$$

We can now compute Weil Pairing of the points using the criterion given by Lemma 1.14. We have

$$e_3(T_i, T_j) = \frac{u_j}{u_i} \frac{\text{Resultant}(G_j - G_i, H_j)}{\text{Resultant}(G_i - G_j, H_i)}. \tag{6.3}$$

Direct computation shows that the Weil pairing of all the points T_i with the representation given above is 1. We see that all the 3-torsion points that we have given pair trivially. We obtain the following.

Lemma 6.6. *The divisors T_i defined above generate an isotropic subgroup of size 9 in* $\mathrm{Pic}^0_{k_\alpha}(\mathcal{C})$

Proof. By [3, Lemma 3] and the decomposition of f_α given above we see that for each choice of G_i, H_i and G_j, H_j with $i \neq j$ we obtain a subgroup of order 9 in $\mathrm{Pic}^0_{k_\alpha}(\mathcal{C})[3]$ which means that the divisors in question are distinct. By (6.3) and Lemma 1.14 we see that the subgroup generated by all T_i is isotropic. Since the maximal size of an isotropic subgroup of the three torsion is at most size 9 we obtain the result. □

Before continuing, we now recap and update where our construction works.

Definition 6.7. Let $\tilde{Z}_{\text{Discriminant}}$ be the points on $\mathcal{B}$ where the discriminant of f_α vanishes. Define $\tilde{Z}_{\text{parametric}}$ to be the locus of points where $E_i = E_2 = 0$ or $F_1 = F_2 = F_3 = 0$ (where the E_i, F_i are defined in Section 5.2.2). Let $\tilde{Z}_{3-torsion}$ be the locus of points where the denominators of H_i, G_i vanish, or one of the u_i is zero. Set

$$\tilde{Z} = \tilde{Z}_{3-torsion} \cup \tilde{Z}_{\text{parametric}} \cup \tilde{Z}_{\text{Discriminant}}.$$

Define Z be the full orbit of $\tilde{Z}$ under the automorphism group of $\mathcal{B}$.

We see that $\mathcal{B} - Z$ is a non-empty open subset of $\mathcal{B}$ where our construction so far is guaranteed to work.

We can now return to the situation described after Proposition 6.4. We briefly review the situation for the convenience of the reader. Recall that we have the projection of the Kummer cone $\mathcal{K}_\alpha$. For each distinguished plane J we have the projection J_α. We also have the projection of the marked trope, which is $T_\alpha = \mathrm{Pr}_\alpha(\mathbb{T}_\alpha(\mathcal{B}))$. There is a conic Y_α in the marked trope T_α that contains 6 singularities of $\mathcal{K}_\alpha$. We also have an isomorphism $\theta_\alpha \colon Y_\alpha \to \mathcal{T}_\alpha$, where $\mathcal{T}_\alpha$ is a twisted space cubic with rationalization $\psi_\alpha \colon \mathbb{P}^1_{k_\alpha} \to \mathcal{T}_\alpha$. Set $\Phi_\alpha = \theta_\alpha \circ \psi_\alpha$. Finally, let $x \colon \mathcal{C}_\alpha \to \mathbb{P}^1_{k_\alpha}$ be the hyperelliptic double cover. For suitably general α we have the following.

Theorem 6.8. *Work over $\overline{k}$. Let J be a distinguished plane, and let*

$$X_J = \Phi_\alpha(J_\alpha \cap T_\alpha \cap Y_\alpha).$$

Let F_J be the quadratic form such that $X_J = V(F_{J_\alpha})$. Then

$$\operatorname{div}(x^* F_{J_\alpha}) = P_{J_\alpha} + Q_{J_\alpha} - 2\kappa_\infty$$

where $[P_{J_\alpha} - \kappa_\infty]$ and $[Q_{J_\alpha} - \kappa_\infty]$ are 3-torsion in $\operatorname{Pic}^0_{k_\alpha}(\mathcal{C}_\alpha)$ and κ_∞ denotes the fiber above $x = \infty$.

Proof. First consider the distinguished plane j_2. Reduce to the situation in the way described in Remark 6.1. One checks with a computer algebra system over the function field of $\mathcal{B}$ that $\theta_\alpha(X_{j_2})$ is the intersection of the line through $(0 : 1 : 0 : 0)$ and $(\alpha_1 : 0 : \alpha_3 : -\alpha_4)$. Let J be any other distinguished plane. As the automorphism group of $\mathcal{B}$ acts transitively on the distinguished planes we may choose a projective isomorphism Θ such that $\Theta(J) = j_2$. On the other hand, Θ induces a projective isomorphism $K_\alpha \to K_{\Theta(\alpha)}$. We see that J satisfies the theorem. □

Remark 6.9. The sufficiently general part of Theorem 6.8 is needed in two parts. We need that the divisors described by the decompositions given by (6.2) are well-defined and that we lie in the open subset where the above calculation applies. To compute with a computer algebra system, we need equations for the birational map between the twisted cubic and the conic in the trope. These are obtained by the procedure outlined in Remark 1.42. The equations can be found in Section A.2.1.

6.1.2 The Distinguished Hyperplanes

Proposition 6.10. *There is a one-to-one correspondence between distinguished hyperplanes and maximal isotropic subgroups of* $\mathrm{Pic}^0_{k_\alpha(\zeta_3)}(\mathcal{C}_\alpha)[3]$. *Specifically, the divisors represented by the distinguished planes in a distinguished hyperplane constitute a maximal isotropic subgroup with respect to the Weil pairing.*

Proof. The automorphism group of the Burkhardt quartic acts transitively on the distinguished hyperplanes. We have shown that the distinguished planes in $V(y_0)$ correspond to divisor classes of $\mathrm{Pic}^0_{k_\alpha(\zeta_3)}(\mathcal{C}_\alpha)[3]$ that generate a maximal isotropic subspace. As automorphisms of $\mathcal{B}$ correspond to Weil pairing preserving automorphisms of $A_\alpha[3]$ we see that for each distinguished hyperplane we obtain a maximal isotropic subgroup. As there are precisely 40 such subgroups, we get the result. □

Corollary 6.11. *Let* $P_J, P_{J'} \in \mathrm{Pic}^0_{k_\alpha(\zeta_3)}(\mathcal{C}_\alpha)[3]$ *be points that correspond to distinguished planes* J *and* J'. *Then* $e_3(P_j, P_{J'}) = 1$ *if and only if* J *and* J' *lie in a common distinguished hyperplane.*

Proof. If J and J' lie in a common distinguished hyperplane the above proposition says that J and J' lie in a maximal isotropic subgroup with respect to the Weil pairing and thus pair trivially. Conversely, if J and J' pair trivially, then they lie in a maximal isotropic subgroup. However by the above proposition each maximal isotropic subgroup contains divisors represented by distinguished planes that lie in a common distinguished hyperplane. □

6.1.3 Extra Structure

Our original computations assumed that $\alpha \in \mathcal{B}$. The formulas lift to an open subset of $\mathbb{Q}(\alpha_0, ..., \alpha_4)$. In particular, the formulas

$$f_\alpha(x) = G_i(x)^2 + u_i H_i(x)^3. \tag{6.4}$$

hold on an open subset of $\mathbb{P}^4$. Thus by [3] we see that the hyperelliptic curve given by f_α has k_α-rational subgroup of order 9. It would be interesting to have a geometric explanation for this fact.

6.2 Marking the Level Structure

We still not have not discussed the marking of the level structure on the Jacobian. Our earlier results can now be applied. Let $\alpha \in \mathcal{B}$ be a point that satisfies Theorem 6.8 and let f_α be the sextic from the previous sections. Suppose that α is defined over a field k. Let A_α

be the Jacobian of $y^2 = f_\alpha(x)$. The level structure is marked by an isomorphism of group schemes over k,

$$\gamma\colon A_\alpha[3] \to (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2.$$

By Theorem 1.22 this amounts to finding an isomorphism of $\mathrm{Gal}(\overline{k}/k)$ modules. Recall that there are 8 rational distinguished planes. The formulas are given below.

$$\begin{aligned} V(y_0, y_1), V(y_0, y_2), V(y_0, y_3), V(y_0, y_4) \\ V(y_0 + 2y_1, y_2 + y_3 + y_4), V(y_0 + 2y_3, y_1 + y_2 + y_4) \\ V(y_0 + 2y_4, y_1 + y_2 + y_3), V(y_0 + 2y_2, y_1 + y_3 + y_4). \end{aligned}$$

Note that we have inclusions

$$V(y_0, y_1), V(y_0, y_2), V(y_0, y_3), V(y_0, y_4) \subseteq V(y_0)$$

and

$$\begin{aligned} V(y_0 + 2y_1, y_2 + y_3 + y_4), V(y_0 + 2y_3, y_1 + y_2 + y_4) \subseteq V(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4) \\ V(y_0 + 2y_4, y_1 + y_2 + y_3), V(y_0 + 2y_2, y_1 + y_3 + y_4) \subseteq V(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4). \end{aligned}$$

There are two distinguished hyperplanes that are unique with respect to the property that they contain 4 rational distinguished planes. The hyperplanes are given by the equations

$$\begin{aligned} y_0 &= 0 \\ y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 &= 0. \end{aligned}$$

Direct computation shows that the rational distinguished planes are contained in the following distinguished hyperplanes. Each distinguished plane is contained in the distinguished hyperplanes in the right hand column.

Distinguished Plane	Incidence
$V(y_0, y_1)$	$V(y_0)$, $V(y_0+2y_1)$, $V(y_0+2\zeta_3^2 y_1)$, $V(y_0+2\zeta_3 y_1)$
$V(y_0, y_2)$	$V(y_0)$, $V(y_0+2y_2)$, $V(y_0+2\zeta_3^2 y_2)$, $V(y_0+2\zeta_3 y_2)$
$V(y_0, y_3)$	$V(y_0)$, $V(y_0+2y_3)$, $V(y_0+2\zeta_3^2 y_3)$, $V(y_0+2\zeta_3 y_3)$
$V(y_0, y_4)$	$V(y_0)$, $V(y_0+2y_4)$, $V(y_0+2\zeta_3^2 y_4)$, $V(y_0+2\zeta_3 y_4)$
$V(y_0+2y_1, y_2+y_3+y_4)$	$V(y_0+2y_1+2\zeta_3 y_2+2\zeta_3 y_3+2\zeta_3 y_4)$ $V(y_0+2y_1+2\zeta_3^2 y_2+2\zeta_3^2 y_3+2\zeta_3^2 y_4)$ $V(y_0+2y_1), V(y_0+2y_1+2y_2+2y_3+2y_4)$
$V(y_0+2y_2, y_1+y_3+y_4)$	$V(y_0+2\zeta_3 y_1+2y_2+2\zeta_3 y_3+2\zeta_3 y_4)$ $V(y_0+2\zeta_3^2 y_1+2y_2+2\zeta_3^2 y_3+2\zeta_3^2 y_4)$ $V(y_0+2y_1+2y_2+2y_3+2y_4), V(y_0+2y_2)$
$V(y_0+2y_3, y_1+y_2+y_4)$	$V(y_0+2\zeta_3 y_1+2\zeta_3 y_2+2y_3+2\zeta_3 y_4)$ $V(y_0+2\zeta_3^2 y_1+2\zeta_3^2 y_2+2y_3+2\zeta_3^2 y_4)$ $V(y_0+2y_1+2y_2+2y_3+2y_4), V(y_0+2y_3)$
$V(y_0+2y_4, y_1+y_2+y_3)$	$V(y_0+2\zeta_3 y_1+2\zeta_3 y_2+2\zeta_3 y_3+2y_4)$ $V(y_0+2\zeta_3^2 y_1+2\zeta_3^2 y_2+2\zeta_3^2 y_3+2y_4)$ $V(y_0+2y_1+2y_2+2y_3+2y_4), V(y_0+2y_4)$

Let $\pm P_i$ be the 3-torsion points on the Jacobian that correspond to the distinguished planes $V(y_0, y_i)$. Similarly let $\pm Q_i$ be the 3-torsion points corresponding to $V(y_0 + 2y_i, \sum_{j\neq i} y_j)$.

From the above table we have the following.

Lemma 6.12. *Let e_3 be the Weil pairing on $A_\alpha[3](\overline{k})$. Then P_1, P_2, Q_1, Q_2 are a basis for $A_\alpha[3](\overline{k})$.*

Proof. Suppose not. Then there is a $\mathbb{Z}$-linear combination such that

$$P = aP_1 + bP_2 + cQ_1 + dQ_2 = 0$$

in $A_\alpha[3](\overline{k})$. We can take $a, b, c, d \in 0, 1, 2$. Note that $e_3(P_1, Q_1) = 0$ by the above table. So we have that $e_3(P, P_1) = e_3(P_1, Q_2)^d = 1$. As $e_3(P_1, Q_2) \neq 1$ by the above table we see that $d = 0$. Similar arguments for a, b, c show that they too must be zero. □

The subgroup generated by P_1, P_2 is isomorphic to $\Sigma = (\mathbb{Z}/3\mathbb{Z})^2$. On the other hand, we have a second maximal isotropic subgroup Σ' corresponding to $V(y_0+2y_1+2y_2+2y_3+2y_4)$. The above computation shows that

$$\Sigma \times \Sigma' \cong A_\alpha[3](\overline{k}). \qquad (6.5)$$

On the other hand, $\Sigma \cap \Sigma' = \{0\}$. Thus by Lemma 1.23 we have the following.

Theorem 6.13. *Let α be sufficiently general. Then $A_\alpha[3] \cong (\mathbb{Z}/3\mathbb{Z})^2 \times (\mu_3)^2$.*

We see that our choice of twist in Section 5.3 has been validated. Furthermore, we see that the computation in Theorem 6.8 may be done over $k_\alpha(\zeta_3)$.

6.2.1 A Basis for the Three Torsion

Using the methods of Section 5.2 we can compute a basis for the 3-torsion of $\mathrm{Pic}(\mathcal{C}_\alpha)^0_{k_\alpha}(\mathbb{Q}(\zeta_3))$ or equivalently, the Jacobian of the curve $\mathcal{C}_\alpha$. Specifically, we take the distinguished plane given by the equations $y_0 + 2y_2 = y_1 + y_3 + y_4 = 0$ and $y_0 + 2y_1 = y_2 + y_3 + y_4 = 0$ and compute the x-coordinate of the corresponding point on $\mathcal{C}$. In practice, one can then compute corresponding point using a computer algebra system. Unfortunately, the formulas are larger than the formulas we obtained for the rational 3-torsion. The formulas can be found in A.3.

Bibliography

[1] H. F. Baker, *A Locus with* 25920 *Linear Self-Transformations*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 39, Cambridge, at the University Press; New York, The Macmillan Company, 1946.

[2] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993).

[3] Nils Bruin, E. Victor Flynn, and Damiano Testa, *Descent via* $(3, 3)$*-isogeny on Jacobians of genus 2 curves*, Acta Arith. **165** (2014), no. 3, 201–223.

[4] Heinrich Burkhardt, *Untersuchungen aus dem Gebiete der hyperelliptischen Modulfunctionen*, Math. Ann. **38** (1891), no. 2, 161–224.

[5] ———, *Untersuchungen aus dem Gebiete der hyperelliptischen Modulfunctionen*, Math. Ann. **41** (1892), no. 3, 313–343.

[6] J. W. S. Cassels and E. V. Flynn, *Prolegomena to a middlebrow arithmetic of curves of genus* 2, London Mathematical Society Lecture Note Series, vol. 230, Cambridge University Press, Cambridge, 1996.

[7] Ciro Ciliberto and Thomas Dedieu, *Limits of pluri-tangent planes to quartic surfaces*, Algebraic and complex geometry, Springer Proc. Math. Stat., vol. 71, Springer, Cham, 2014, pp. 123–199.

[8] Arthur B. Coble, *Point sets and allied Cremona groups. III*, Trans. Amer. Math. Soc. **18** (1917), no. 3, 331–372.

[9] Gary Cornell and Joseph H. Silverman (eds.), *Arithmetic geometry*, Springer-Verlag, New York, 1986, Papers from the conference held at the University of Connecticut, Storrs, Connecticut, July 30–August 10, 1984.

[10] I. Dolgachev and D. Lehavi, *On isogenous principally polarized abelian surfaces*, Curves and abelian varieties, Contemp. Math., vol. 465, Amer. Math. Soc., Providence, RI, 2008, pp. 51–69.

[11] Igor V. Dolgachev, *Classical algebraic geometry*, Cambridge University Press, Cambridge, 2012, A modern view.

[12] ———, *Corrado segre and nodal cubic threefolds*, 2015, arXiv:1501.06432.

[13] Noam D. Elkies, *The identification of three moduli spaces*, (1999), arXiv:math/9905195.

[14] Hans Finkelnberg, *On the Geometry of the Burkhardt Quartic*, Ph.D. thesis, Leiden, 1989.

[15] Eberhard Freitag and Riccardo Salvati Manni, *The Burkhardt group and modular forms*, Transform. Groups **9** (2004), no. 1, 25–45.

[16] ———, *The Burkhardt group and modular forms. II*, Transform. Groups **9** (2004), no. 3, 237–256.

[17] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008, Reprint of the 1994 edition.

[18] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992, A first course.

[19] Joe Harris and Ian Morrison, *Moduli of curves*, Graduate Texts in Mathematics, vol. 187, Springer-Verlag, New York, 1998.

[20] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52.

[21] Marc Hindry and Joseph H. Silverman, *Diophantine geometry*, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000.

[22] J. William Hoffman and Steven H. Weintraub, *The Siegel modular variety of degree two and level three*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3267–3305 (electronic).

[23] Ben Howard, John Millson, Andrew Snowden, and Ravi Vakil, *A description of the outer automorphism of* S_6*, and the invariants of six points in projective space*, J. Combin. Theory Ser. A **115** (2008), no. 7, 1296–1303.

[24] R. W. H. T. Hudson, *Kummer's quartic surface*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1990, With a foreword by W. Barth, Revised reprint of the 1905 original.

[25] Bruce Hunt, *The geometry of some special arithmetic quotients*, Lecture Notes in Mathematics, vol. 1637, Springer-Verlag, Berlin, 1996.

[26] Charles Minshall Jessop, *Quartic surfaces with singular points*, Cambridge University Press, 2016.

[27] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications.

[28] Saunders Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[29] Jean-François Mestre, *Construction de courbes de genre* 2 *à partir de leurs modules*, Effective methods in algebraic geometry (Castiglioncello, 1990), Progr. Math., vol. 94, Birkhäuser Boston, Boston, MA, 1991, pp. 313–334.

[30] J. S. Milne, *Abelian varieties*, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 103–150.

[31] Qingchun Ren, Steven V. Sam, and Bernd Sturmfels, *Tropicalization of classical moduli spaces*, Math. Comput. Sci. **8** (2014), no. 2, 119–145.

[32] J. A. Todd, *On the simple group of order* 25920, Proc. Roy. Soc. London. Ser. A. **189** (1947), 326–358.

[33] ______, *The invariants of a finite collineation group in five dimensions*, Proc. Cambridge Philos. Soc. **46** (1950), 73–90.

[34] Gerard van der Geer, *Note on abelian schemes of level three*, Math. Ann. **278** (1987), no. 1-4, 401–408.

[35] William C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York-Berlin, 1979.

Appendix A

A.1 Chapter 2

Below are the equations of the distinguished planes.

```
[y0,y1],[y0 + 2*y3,w*y1 + (-w - 1)*y2 + y4],
[y0 + 2*y1 + 2*w*y2 + 2*w*y3 + 2*w*y4,
y0 - y1 - w*y2 + 2*w*y3 - w*y4],
[y0 + 2*y1,y2 + (-w - 1)*y3 + w*y4],
[y0 + (-2*w - 2)*y4,y1 + y2 + w*y3],
[y0 + 2*w*y3,y1 + y2 + (-w - 1)*y4],
[y0 + 2*y1 + (-2*w - 2)*y2 + (-2*w - 2)*y3 + (-2*w - 2)*y4,
(-w - 1)*y0 + (w + 1)*y1 - w*y2 - w*y3 + 2*w*y4],
[y0 + 2*y4,(-w - 1)*y1 + y2 + w*y3],
[y0 + 2*y4,w*y1 + y2 + (-w - 1)*y3],
[y0 + (-2*w - 2)*y4,y1 + w*y2 + y3],
[y0,y3],[y0,y2],[y0 + 2*y1,y2 + w*y3 + (-w - 1)*y4],
[y0 + 2*w*y2,(-w - 1)*y1 + y3 + y4],
[y0 + 2*y1 + 2*y2 + (-2*w - 2)*y3 + 2*w*y4,
w*y0 - w*y1 - w*y2 + 2*y3 + (w + 1)*y4],
[y0 + 2*y2,y1 + y3 + y4],[y0 + 2*w*y2,y1 + (-w - 1)*y3 + y4],
[y0 + (-2*w - 2)*y1,y2 + w*y3 + y4],[y0 + (-2*w - 2)*y2,
w*y1 + y3 + y4],
[y0 + 2*w*y1,(-w - 1)*y2 + y3 + y4],
[y0 + 2*y4,y1 + y2 + y3],
[3*y0 + 6*y3,3*w*y1 + 3*y2 + (-3*w - 3)*y4],
[y0 + 2*y3,y1 + y2 + y4],[y0,y4],
[y0 + 2*y2,y1 + (-w - 1)*y3 + w*y4],
[y0 + (-2*w - 2)*y1,(-w - 1)*y2 + (-w - 1)*y3 + y4],
[y0 + 2*w*y2,w*y1 + w*y3 + y4],
[y0 + 2*y2,y1 + w*y3 + (-w - 1)*y4],
[y0 + 2*w*y1,y2 + (-w - 1)*y3 + y4],
[y0 + 2*y1,y2 + y3 + y4],
[y0 + 2*y1 + 2*y2 + (-2*w - 2)*y3 + 2*w*y4,
(-w - 1)*y0 + (w + 1)*y1 + (w + 1)*y2 - w*y3 + 2*y4],
```

```
[y0 + 2*w*y1 + 2*y2 + 2*w*y3 + 2*w*y4,
y0 - w*y1 - y2 + 2*w*y3 - w*y4],
[y0 + 2*y1 + 2*w*y2 + 2*w*y3 + 2*w*y4,
y0 - y1 - w*y2 - w*y3 + 2*w*y4],
[y0 + (-2*w - 2)*y1,w*y2 + y3 + y4],
[y0 + (-2*w - 2)*y3,y1 + w*y2 + y4],
[y0 + (-2*w - 2)*y2,y1 + w*y3 + y4],
[y0 + (-2*w - 2)*y2,(-w - 1)*y1 + (-w - 1)*y3 + y4],
[y0 + 2*w*y4,y1 + (-w - 1)*y2 + y3],
[(-2*w - 1)*y0 + (2*w + 4)*y1,
(-2*w - 1)*y2 + (-2*w - 1)*y3 + (w - 1)*y4],
[y0 + (-2*w - 2)*y3,w*y1 + y2 + y4]
```

The equations of the distinguished hyperplanes are given below.

```
[-1/2*y0],[1/2*(-2*w - 1)*y0 + (w + 2)*y1
+ (-2*w - 1)*y2 + (-2*w - 1)*y3 + (w -1)*y4],
[-1/2*y0 - y1 + (w + 1)*y2 - w*y3 - y4],
[1/2*(2*w + 1)*y0 + (-w - 2)*y1
+ (-w - 2)*y2 + (2*w + 1)*y3 + (-w -2)*y4],
[1/2*y0 - y1 - w*y2 - w*y3 - w*y4],
[1/2*(-2*w - 1)*y0 + (w - 1)*y1
+ (w + 2)*y2 + (w - 1)*y3 + (w + 2)*y4],
[-1/2*y0 - y1 + (w + 1)*y2 + (w + 1)*y3 + (w + 1)*y4],
[-1/2*y0 - y3],[-1/2*y0 + (w + 1)*y4],
[1/2*(-2*w - 1)*y0 + (w - 1)*y1
+ (w + 2)*y2 + (-2*w - 1)*y3 + (-2*w -1)*y4],[-1/2*y0 - y1],
[-1/2*y0 + (w + 1)*y1],[-1/2*y0 + (w + 1)*y2],
[-1/2*y0 - w*y1 - y2 + (w + 1)*y3 - y4],
[1/2*(2*w + 1)*y0 + (2*w + 1)*y1
+ (-w - 2)*y2 + (2*w + 1)*y3
+ (-w +1)*y4],[-1/2*y0 - y1 - y2
+ (w + 1)*y3 - w*y4],
[1/2*(2*w + 1)*y0 + (-w - 2)*y1
+ (-w - 2)*y2 + (-w - 2)*y3 + (2*w +1)*y4],
[-1/2*y0 - w*y1 - w*y2 + (w + 1)*y3 + (w + 1)*y4],
[-1/2*y0 - y1 - y2 - w*y3 + (w + 1)*y4],
[3/2*y0 + (-3*w - 3)*y1 + (-3*w - 3)*y2 + 3*w*y3 + 3*w*y4],
[-1/2*y0 + (w + 1)*y3],
[-1/2*y0 + (w + 1)*y1 + (w + 1)*y2 - y3 + (w + 1)*y4],
[-1/2*y0 - y1 + (w + 1)*y2 - y3 - w*y4],
[-1/2*y0 - y2],[-1/2*y0 - y1 - y2 - y3 - y4],
[1/2*(2*w + 1)*y0 + (-w - 2)*y1
+ (2*w + 1)*y2 + (-w - 2)*y3 + (-w -2)*y4],
[-1/2*y0 - y1 - w*y2 + (w + 1)*y3 - y4],[-1/2*y0 - y4],
[1/2*(-2*w - 1)*y0 + (w - 1)*y1
```

```
 + (w + 2)*y2 + (w + 2)*y3 + (w - 1)*y4],
[-1/2*y0 - w*y3],[-1/2*y0 + (w + 1)*y1 - y2 - w*y3 - y4],
[1/2*(-2*w - 1)*y0 + (w + 2)*y1
 + (w - 1)*y2 + (-2*w - 1)*y3 + (-2*w -1)*y4],
[-1/2*y0 - w*y2],
[-1/2*y0 - w*y1 + (w + 1)*y2 - w*y3 + (w + 1)*y4],
[-1/2*y0 - w*y1 + (w + 1)*y2 + (w + 1)*y3 - w*y4],
[-1/2*y0 + (w + 1)*y1 - y2 + (w + 1)*y3 + (w + 1)*y4],
[-1/2*y0 + (w + 1)*y1 - y2 - y3 - w*y4],
[-1/2*y0 - w*y1],
[-1/2*y0 - w*y4],
[-1/2*y0 + (w + 1)*y1 + (w + 1)*y2 + (w + 1)*y3 - y4]
```

A.2 Chapter 5

A.2.1 The birational map between the Jacobian Variety and Singular Locus

Here we give the map between The **a**i here represent the coordinates of the point α and the **x**i are coordinates on a $\mathbb{P}^3$. The map here is the map between the Weddle surface and Singular locus of the web of quadrics.

```
[
a0*a1*a2*a3*a4*x1^4
+ (1/2*a0^2*a2*a3^2 - a0*a1*a2^2*a4)*x1^3*x2
+ (1/2*a0^2*a2*a4^2 - a0*a1*a2^2*a3)*x1^3*x3 +
(-a0^2*a2^2*a3 - 2*a1^2*a2*a4^2)*x1^2*x2^2 +
(a0*a1*a2^3 + 2*a1^2*a2*a3*a4)*x1^2*x2*x3 +
(-a0^2*a2^2*a4 - 2*a1^2*a2*a3^2)*x1^2*x3^2 +
(1/2*a0^2*a2^3 - a0*a1*a2*a3*a4)*x1*x2^3 +
(a0*a1*a2*a3^2 +
2*a1^2*a2^2*a4)*x1*x2^2*x3
+ (a0*a1*a2*a4^2 +
2*a1^2*a2^2*a3)*x1*x2*x3^2 +
(1/2*a0^2*a2^3 - a0*a1*a2*a3*a4)*x1*x3^3
+ a0*a1*a2^2*a4*x2^4 + (1/2*a0^2*a2*a4^2 -
a0*a1*a2^2*a3)*x2^3*x3 +
(-a0^2*a2*a3*a4 - 2*a1^2*a2^3)*x2^2*x3^2
+ (1/2*a0^2*a2*a3^2 - a0*a1*a2^2*a4)*x2*x3^3
+ a0*a1*a2^2*a3*x3^4,
a0*a1*a2*a3*a4*x0^2*x1^2 +
1/2*a0^2*a2*a3^2*x0^2*x1*x2
+ 1/2*a0^2*a2*a4^2*x0^2*x1*x3
- 1/2*a0^2*a2^2*a3*x0^2*x2^2 - a0*a1*a2^3*x0^2*x2*x3 -
1/2*a0^2*a2^2*a4*x0^2*x3^2 - a0*a1^2*a2*a4*x0*x1^2*x2 +
```

```
a0*a1^2*a2*a3*x0*x1^2*x3 - 1/2*a0^2*a1*a2*a3*x0*x1*x2^2
+ 1/2*a0^2*a1*a2*a4*x0*x1*x3^2 +
1/2*a0^2*a1*a2^2*x0*x2^3 + 1/2*a0^3*a2*a4*x0*x2^2*x3 -
1/2*a0^3*a2*a3*x0*x2*x3^2
 - 1/2*a0^2*a1*a2^2*x0*x3^3
- 2*a1*a2^2*a4^2*x1^2*x2^2 + (a0*a2*a3^3 +
a0*a2*a4^3 + 2*a1*a2^2*a3*a4)*x1^2*x2*x3
- 2*a1*a2^2*a3^2*x1^2*x3^2 - a0*a2^2*a3*a4*x1*x2^3
+ (-a0*a2^2*a3^2 + 2*a1*a2^3*a4)*x1*x2^2*x3
+ (-a0*a2^2*a4^2 +
2*a1*a2^3*a3)*x1*x2*x3^2
- a0*a2^2*a3*a4*x1*x3^3
+ a0*a2^3*a4*x2^4 - 2*a1*a2^4*x2^2*x3^2
 + a0*a2^3*a3*x3^4,
-1/2*a0^2*a2^2*a3*x0^2*x1^2
+ 1/2*a0^2*a2^3*x0^2*x1*x2
- a0*a1*a2*a3^2*x0^2*x1*x3 + a0*a1*a2^2*a4*x0^2*x2^2
+ 1/2*a0^2*a2*a4^2*x0^2*x2*x3 -
1/2*a0^2*a2*a3*a4*x0^2*x3^2
- 1/2*a0^2*a1*a2*a3*x0*x1^3 +
1/2*a0^2*a1*a2^2*x0*x1^2*x2
- 1/2*a0^3*a2*a4*x0*x1^2*x3
+ a0*a1^2*a2*a4*x0*x1*x2^2 +
1/2*a0^3*a2^2*x0*x1*x3^2
- a0*a1^2*a2^2*x0*x2^2*x3
- 1/2*a0^2*a1*a2*a4*x0*x2*x3^2 +
1/2*a0^2*a1*a2*a3*x0*x3^3 + a0*a2*a3^2*a4*x1^4
- a0*a2^2*a3*a4*x1^3*x2 -
2*a1*a2*a3*a4^2*x1^2*x2^2 + (-a0*a2^3*a3
+ 2*a1*a2*a3^2*a4)*x1^2*x2*x3 - 2*a1*a2*a3^3*x1^2*x3^2
+ (a0*a2^4 + a0*a2*a4^3
+ 2*a1*a2^2*a3*a4)*x1*x2^2*x3
+(-a0*a2*a3*a4^2 + 2*a1*a2^2*a3^2)*x1*x2*x3^2
- 2*a1*a2^3*a3*x2^2*x3^2
- a0*a2^2*a3*a4*x2*x3^3 + a0*a2^2*a3^2*x3^4,
-1/2*a0^2*a2^2*a4*x0^2*x1^2
- a0*a1*a2*a4^2*x0^2*x1*x2
+ 1/2*a0^2*a2^3*x0^2*x1*x3
- 1/2*a0^2*a2*a3*a4*x0^2*x2^2
+ 1/2*a0^2*a2*a3^2*x0^2*x2*x3 +
a0*a1*a2^2*a3*x0^2*x3^2 +
1/2*a0^2*a1*a2*a4*x0*x1^3
+ 1/2*a0^3*a2*a3*x0*x1^2*x2
- 1/2*a0^2*a1*a2^2*x0*x1^2*x3
 - 1/2*a0^3*a2^2*x0*x1*x2^2 -
a0*a1^2*a2*a3*x0*x1*x3^2 - 1/2*a0^2*a1*a2*a4*x0*x2^3
+ 1/2*a0^2*a1*a2*a3*x0*x2^2*x3
```

```
+ a0*a1^2*a2^2*x0*x2*x3^2
+ a0*a2*a3*a4^2*x1^4 - a0*a2^2*a3*a4*x1^3*x3 -
2*a1*a2*a4^3*x1^2*x2^2 + (-a0*a2^3*a4 + 2*a1*a2*a3*a4^2)*x1^2*x2*x3
- 2*a1*a2*a3^2*a4*x1^2*x3^2
+ (-a0*a2*a3^2*a4 + 2*a1*a2^2*a4^2)*x1*x2^2*x3
+ (a0*a2^4 +
a0*a2*a3^3 + 2*a1*a2^2*a3*a4)*x1*x2*x3^2 +
a0*a2^2*a4^2*x2^4 - a0*a2^2*a3*a4*x2^3*x3 - 2*a1*a2^3*a4*x2^2*x3^2]
```

A.2.2 The Sextic

Below the equation of the sextic $f_\alpha(q)$ after removing the required twist.

```
(1/4*a0^4*a2^6 + 1/2*a0^4*a2^3*a4^3 + 1/4*a0^4*a4^6
+ 8*a0*a1^3*a2^3*a4^3)*q^6 +
(-3/2*a0^4*a2^5*a3
- 3/2*a0^4*a2^2*a3*a4^3
+ 6*a0^2*a1^2*a2^4*a4^2
- 6*a0^2*a1^2*a2*a4^5
- 24*a0*a1^3*a2^2*a3*a4^3)*q^5
+ (15/4*a0^4*a2^4*a3^2 + 3/2*a0^4*a2*a3^2*a4^3
- 6*a0^3*a1*a2^2*a4^4 - 24*a0^2*a1^2*a2^3*a3*a4^2
+ 6*a0^2*a1^2*a3*a4^5 + 24*a0*a1^3*a2*a3^2*a4^3
- 12*a1^4*a2^2*a4^4)*q^4
+ (-5*a0^4*a2^3*a3^3 - 1/2*a0^4*a2^3*a4^3 - 1/2*a0^4*a3^3*a4^3
+ 1/2*a0^4*a4^6 + 12*a0^3*a1*a2*a3*a4^4
+ 36*a0^2*a1^2*a2^2*a3^2*a4^2
- 8*a0*a1^3*a2^3*a4^3 - 8*a0*a1^3*a3^3*a4^3
+ 8*a0*a1^3*a4^6 + 24*a1^4*a2*a3*a4^4)*q^3
+ (15/4*a0^4*a2^2*a3^4 + 3/2*a0^4*a2^2*a3*a4^3
- 6*a0^3*a1*a3^2*a4^4 - 24*a0^2*a1^2*a2*a3^3*a4^2
+ 6*a0^2*a1^2*a2*a4^5 + 24*a0*a1^3*a2^2*a3*a4^3
- 12*a1^4*a3^2*a4^4)*q^2
+ (-3/2*a0^4*a2*a3^5 - 3/2*a0^4*a2*a3^2*a4^3
+ 6*a0^2*a1^2*a3^4*a4^2
- 6*a0^2*a1^2*a3*a4^5 - 24*a0*a1^3*a2*a3^2*a4^3)*q
+ 1/4*a0^4*a3^6 + 1/2*a0^4*a3^3*a4^3
+ 1/4*a0^4*a4^6 + 8*a0*a1^3*a3^3*a4^3
```

A.3 Chapter 6

A.3.1 Formulas for the 3-torsion

Below are the q-coordinates of the Mumford representation (computed in magma) of the twisted curve $y^2 = -3f_\alpha(q)$. One can use the automorphism $(q, y) \mapsto (q, \frac{y}{\sqrt{(-3)}})$ to recover the corresponding point on $y^2 = f_\alpha(q)$. The y-coordinates are large and not suitable for print. The y-coordinate in Mumford form can be found using the points command in magma if needed. However, the computation is expensive to do generically.

```
q^2 +
((-4*a2^4*a4^2 +
 4*a2^3*a3*a4^2 + 4*a2^3*a4^3
 - 4*a2^2*a4^4
 - 4*a2*a3*a4^4
 + 4*a2*a4^5)/(8*a2^6*a4^3 - 24*a2^5*a3*a4^3
+ 8*a2^3*a3^3*a4^3 + 24*a2^4*a3*a4^4 -
24*a2^3*a3*a4^5
+ 8*a2^3*a4^6 - a2^6 + 3*a2^5*a4 - 6*a2^4*a4^2
 + 8*a2^3*a4^3 - 6*a2^2*a4^4
  + 3*a2*a4^5 - a4^6)*a1^2
  + (2*a2^5*a4 - 2*a2^4*a3*a4 - 4*a2^4*a4^2 +
2*a2^3*a3*a4^2 + 6*a2^3*a4^3 - 6*a2^2*a4^4
 - 2*a2*a3*a4^4 +
 4*a2*a4^5 + 2*a3*a4^5 - 2*a4^6)/(8*a2^6*a4^3
  - 24*a2^5*a3*a4^3 + 8*a2^3*a3^3*a4^3
 + 24*a2^4*a3*a4^4 -
24*a2^3*a3*a4^5 + 8*a2^3*a4^6 - a2^6 + 3*a2^5*a4
- 6*a2^4*a4^2 + 8*a2^3*a4^3 - 6*a2^2*a4^4
+ 3*a2*a4^5 - a4^6)*a1 + (-8*a2^6*a4^3
- 8*a2^5*a3*a4^3 +
48*a2^4*a3^2*a4^3 - 8*a2^3*a3^3*a4^3
 - 8*a2^2*a3^4*a4^3 + 8*a2^5*a4^4
 - 24*a2^4*a3*a4^4 -
 24*a2^3*a3^2*a4^4 + 8*a2^2*a3^3*a4^4
  + 24*a2^3*a3*a4^5 - 8*a2^3*a4^6 -
8*a2^2*a3*a4^6 + 8*a2^2*a4^7 + 2*a2^5*a3
- a2^5*a4 - 5*a2^4*a3*a4 + 3*a2^4*a4^2
+ 8*a2^3*a3*a4^2
- 6*a2^3*a4^3 - 8*a2^2*a3*a4^3
 + 6*a2^2*a4^4 + 4*a2*a3*a4^4 -
3*a2*a4^5 - a3*a4^5 + a4^6)/(8*a2^6*a4^3
 - 24*a2^5*a3*a4^3 + 8*a2^3*a3^3*a4^3
 + 24*a2^4*a3*a4^4 -
  24*a2^3*a3*a4^5 + 8*a2^3*a4^6
```

```
  - a2^6 + 3*a2^5*a4
 - 6*a2^4*a4^2 +
8*a2^3*a4^3 - 6*a2^2*a4^4
+ 3*a2*a4^5 - a4^6))*q + (4*a2^3*a3*a4^2
 - 4*a2^2*a3^2*a4^2 -
  4*a2^2*a4^4 + 4*a2*a3*a4^4)/(8*a2^6*a4^3
- 24*a2^5*a3*a4^3 +
8*a2^3*a3^3*a4^3 + 24*a2^4*a3*a4^4
- 24*a2^3*a3*a4^5 + 8*a2^3*a4^6 - a2^6 + 3*a2^5*a4
- 6*a2^4*a4^2 + 8*a2^3*a4^3
 - 6*a2^2*a4^4 + 3*a2*a4^5 - a4^6)*a1^2 +
(-2*a2^4*a3*a4 + 2*a2^3*a3^2*a4
+ 2*a2^3*a3*a4^2 - 2*a2^2*a3^2*a4^2 + 2*a2^3*a4^3
 - 4*a2^2*a3*a4^3 + 2*a2*a3^2*a4^3 - 2*a2^2*a4^4
 + 2*a2*a3*a4^4 + 2*a2*a4^5 -
2*a3*a4^5)/(8*a2^6*a4^3
 - 24*a2^5*a3*a4^3 + 8*a2^3*a3^3*a4^3
  + 24*a2^4*a3*a4^4 - 24*a2^3*a3*a4^5
  + 8*a2^3*a4^6
 - a2^6 + 3*a2^5*a4 - 6*a2^4*a4^2
  + 8*a2^3*a4^3 -
6*a2^2*a4^4 + 3*a2*a4^5 - a4^6)*a1
 + (8*a2^5*a3*a4^3
 - 24*a2^3*a3^3*a4^3 + 8*a2^2*a3^4*a4^3
+ 24*a2^3*a3^2*a4^4 - 24*a2^3*a3*a4^5
 + 8*a2^2*a3*a4^6 - a2^4*a3^2 +
a2^4*a3*a4 + 2*a2^3*a3^2*a4 - a2^4*a4^2
- 2*a2^3*a3*a4^2 - 3*a2^2*a3^2*a4^2
+ 2*a2^3*a4^3 + 4*a2^2*a3*a4^3 + 2*a2*a3^2*a4^3
- 3*a2^2*a4^4 - 2*a2*a3*a4^4
 - a3^2*a4^4
+ 2*a2*a4^5 + a3*a4^5 - a4^6)/(8*a2^6*a4^3
- 24*a2^5*a3*a4^3 + 8*a2^3*a3^3*a4^3
 + 24*a2^4*a3*a4^4 - 24*a2^3*a3*a4^5
  + 8*a2^3*a4^6 - a2^6
 + 3*a2^5*a4 - 6*a2^4*a4^2 +
8*a2^3*a4^3 - 6*a2^2*a4^4 + 3*a2*a4^5 - a4^6)
```

```
q^2 + ((-16*a2^6*a4^2 + 16*a2^3*a3^3*a4^2
+ 16*a2^5*a4^2 - 16*a2^2*a4^5 - 4*a2^4*a3*a4
 - 8*a2^4*a4^2
+ 4*a2*a3*a4^4 + 8*a2*a4^5)/(32*a2^8*a4^3
- 64*a2^5*a3^3*a4^3 +
32*a2^5*a4^6 - 48*a2^4*a3^4*a4^2
- 64*a2^7*a4^3
```

```
+ 80*a2^4*a3^3*a4^3 - 48*a2^4*a3^2*a4^4
- 64*a2^4*a4^6 - 24*a2^6*a3^2*a4
+ 24*a2^6*a3*a4^2 + 32*a2^6*a4^3 -
40*a2^3*a3^3*a4^3 + 48*a2^3*a3^2*a4^4
+ 32*a2^3*a4^6 - 4*a2^8 + 12*a2^5*a3^2*a4
- 24*a2^5*a3*a4^2 - 4*a2^5*a4^3 - 24*a2^2*a3^2*a4^4
+ 12*a2^2*a3*a4^5 - 4*a2^2*a4^6
+ 2*a2^7 + 6*a2^4*a3*a4^2 - 4*a2^4*a4^3
- 6*a2*a3*a4^5 + 2*a2*a4^6 - a2^6 + 2*a2^3*a4^3
- a4^6)*a1^2 +
(32*a2^7*a4^3 - 64*a2^4*a3^3*a4^3
+ 32*a2^4*a4^6 -
16*a2^6*a3*a4^2 - 16*a2^3*a3^4*a4^2
- 48*a2^6*a4^3 + 48*a2^3*a3^3*a4^3
- 48*a2^3*a3^2*a4^4 - 16*a2^3*a3*a4^5
- 48*a2^3*a4^6 - 8*a2^5*a3^2*a4 + 24*a2^5*a3*a4^2 +
24*a2^5*a4^3 - 24*a2^2*a3^3*a4^3
+ 32*a2^2*a3^2*a4^4 + 24*a2^2*a4^6
- 16*a2^4*a3*a4^2 - 4*a2^4*a4^3
- 12*a2*a3^2*a4^4 + 4*a2*a3*a4^5
- 8*a2*a4^6 + 2*a2^3*a3*a4^2 -
2*a2^3*a4^3
- 2*a3*a4^5 + 2*a4^6)/(32*a2^8*a4^3
- 64*a2^5*a3^3*a4^3 + 32*a2^5*a4^6
- 48*a2^4*a3^4*a4^2 - 64*a2^7*a4^3
+ 80*a2^4*a3^3*a4^3 - 48*a2^4*a3^2*a4^4 -
64*a2^4*a4^6
- 24*a2^6*a3^2*a4 + 24*a2^6*a3*a4^2
+ 32*a2^6*a4^3 - 40*a2^3*a3^3*a4^3
+ 48*a2^3*a3^2*a4^4 + 32*a2^3*a4^6 - 4*a2^8
+ 12*a2^5*a3^2*a4 - 24*a2^5*a3*a4^2
- 4*a2^5*a4^3 - 24*a2^2*a3^2*a4^4
+ 12*a2^2*a3*a4^5 - 4*a2^2*a4^6 + 2*a2^7
+ 6*a2^4*a3*a4^2 - 4*a2^4*a4^3 - 6*a2*a3*a4^5
+ 2*a2*a4^6 - a2^6 + 2*a2^3*a4^3 - a4^6)*a1
+ (-96*a2^7*a3*a4^3 + 96*a2^4*a3^4*a4^3
- 32*a2^7*a4^4 + 64*a2^4*a3^3*a4^4
- 32*a2^4*a4^7 - 32*a2^6*a3^2*a4^2
+ 64*a2^3*a3^5*a4^2 + 160*a2^6*a3*a4^3 -
80*a2^3*a3^4*a4^3
+ 48*a2^6*a4^4 - 48*a2^3*a3^3*a4^4
+ 16*a2^3*a3^2*a4^5 - 32*a2^3*a3*a4^6
+ 48*a2^3*a4^7
- 8*a2^8*a4 + 40*a2^5*a3^3*a4
- 24*a2^5*a3^2*a4^2 -
104*a2^5*a3*a4^3 + 40*a2^2*a3^4*a4^3
- 32*a2^5*a4^4 - 24*a2^2*a3^3*a4^4
```

```
- 24*a2^2*a3^2*a4^5 + 16*a2^2*a3*a4^6
- 24*a2^2*a4^7 + 8*a2^7*a3 + 8*a2^7*a4 -
20*a2^4*a3^3*a4 + 24*a2^4*a3^2*a4^2
+ 24*a2^4*a3*a4^3 + 12*a2^4*a4^4
+ 16*a2*a3^3*a4^4 - 8*a2*a3*a4^6
+ 8*a2*a4^7 - 4*a2^6*a3 - 4*a2^6*a4
- 8*a2^3*a3^2*a4^2 -
2*a2^3*a3*a4^3 + 2*a3^2*a4^5 - 2*a4^7
+ 2*a2^5*a3 + a2^5*a4 - 2*a2^2*a3*a4^3
- a2^2*a4^4)/(32*a2^8*a4^3
- 64*a2^5*a3^3*a4^3 + 32*a2^5*a4^6
- 48*a2^4*a3^4*a4^2 -
64*a2^7*a4^3 + 80*a2^4*a3^3*a4^3
- 48*a2^4*a3^2*a4^4 - 64*a2^4*a4^6
- 24*a2^6*a3^2*a4 + 24*a2^6*a3*a4^2
+ 32*a2^6*a4^3 - 40*a2^3*a3^3*a4^3
+ 48*a2^3*a3^2*a4^4 +
32*a2^3*a4^6 - 4*a2^8 + 12*a2^5*a3^2*a4
- 24*a2^5*a3*a4^2 - 4*a2^5*a4^3
- 24*a2^2*a3^2*a4^4 + 12*a2^2*a3*a4^5
- 4*a2^2*a4^6 + 2*a2^7 + 6*a2^4*a3*a4^2
- 4*a2^4*a4^3
- 6*a2*a3*a4^5 + 2*a2*a4^6 - a2^6
+ 2*a2^3*a4^3 - a4^6))*q + (16*a2^5*a3*a4^2
- 16*a2^2*a3^4*a4^2 + 16*a2^2*a3^3*a4^3
- 16*a2^2*a4^6 - 8*a2^4*a3*a4^2 - 8*a2^4*a4^3
- 8*a2*a3^3*a4^3 + 8*a2*a3*a4^5 + 16*a2*a4^6
+ 4*a2^3*a3^2*a4 + 4*a2^3*a3*a4^2
+ 4*a2^3*a4^3 - 4*a3^2*a4^4 - 4*a3*a4^5
- 4*a4^6)/(32*a2^8*a4^3 - 64*a2^5*a3^3*a4^3 +
32*a2^5*a4^6 - 48*a2^4*a3^4*a4^2
- 64*a2^7*a4^3 + 80*a2^4*a3^3*a4^3
- 48*a2^4*a3^2*a4^4 - 64*a2^4*a4^6
- 24*a2^6*a3^2*a4
+ 24*a2^6*a3*a4^2 + 32*a2^6*a4^3 -
40*a2^3*a3^3*a4^3 + 48*a2^3*a3^2*a4^4
+ 32*a2^3*a4^6 - 4*a2^8 + 12*a2^5*a3^2*a4
- 24*a2^5*a3*a4^2 - 4*a2^5*a4^3
- 24*a2^2*a3^2*a4^4 + 12*a2^2*a3*a4^5
- 4*a2^2*a4^6
+ 2*a2^7 + 6*a2^4*a3*a4^2 - 4*a2^4*a4^3
- 6*a2*a3*a4^5 + 2*a2*a4^6 - a2^6
+ 2*a2^3*a4^3 - a4^6)*a1^2
+ (-32*a2^6*a3*a4^3 + 64*a2^3*a3^4*a4^3
- 32*a2^3*a3*a4^6 +
16*a2^5*a3^2*a4^2 + 16*a2^2*a3^5*a4^2
+ 32*a2^5*a3*a4^3 - 16*a2^2*a3^4*a4^3
```

```
+ 16*a2^5*a4^4 + 16*a2^2*a3^3*a4^4
+ 16*a2^2*a3^2*a4^5 + 32*a2^2*a3*a4^6
+ 16*a2^2*a4^7
+ 8*a2^4*a3^3*a4 - 16*a2^4*a3^2*a4^2
- 16*a2^4*a3*a4^3 + 8*a2*a3^4*a4^3
- 16*a2^4*a4^4 - 8*a2*a3^2*a4^5
- 16*a2*a3*a4^6 - 16*a2*a4^7
+ 8*a2^3*a3^2*a4^2 +
8*a2^3*a3*a4^3 + 4*a2^3*a4^4 + 4*a3^3*a4^4
+ 4*a3^2*a4^5
+ 4*a3*a4^6 + 4*a4^7)/(32*a2^8*a4^3
- 64*a2^5*a3^3*a4^3 + 32*a2^5*a4^6
- 48*a2^4*a3^4*a4^2 - 64*a2^7*a4^3 +
80*a2^4*a3^3*a4^3 - 48*a2^4*a3^2*a4^4
- 64*a2^4*a4^6 - 24*a2^6*a3^2*a4
+ 24*a2^6*a3*a4^2 + 32*a2^6*a4^3
- 40*a2^3*a3^3*a4^3 + 48*a2^3*a3^2*a4^4
+ 32*a2^3*a4^6 -
4*a2^8 + 12*a2^5*a3^2*a4 - 24*a2^5*a3*a4^2
- 4*a2^5*a4^3 - 24*a2^2*a3^2*a4^4
+ 12*a2^2*a3*a4^5 - 4*a2^2*a4^6
+ 2*a2^7 + 6*a2^4*a3*a4^2
- 4*a2^4*a4^3 - 6*a2*a3*a4^5
+ 2*a2*a4^6 - a2^6 + 2*a2^3*a4^3 - a4^6)*a1
+ (64*a2^6*a3^2*a4^3 - 32*a2^3*a3^5*a4^3
+ 64*a2^6*a3*a4^4 - 32*a2^3*a3^4*a4^4
- 32*a2^3*a3^2*a4^6 - 32*a2^3*a3*a4^7 +
32*a2^5*a3^3*a4^2 - 16*a2^2*a3^6*a4^2
- 80*a2^5*a3^2*a4^3 + 16*a2^2*a3^5*a4^3
- 112*a2^5*a3*a4^4 + 32*a2^2*a3^4*a4^4
- 16*a2^5*a4^5 - 32*a2^2*a3^3*a4^5 +
16*a2^2*a3^2*a4^6 + 32*a2^2*a3*a4^7
- 16*a2^2*a4^8
+ 8*a2^7*a3*a4 - 16*a2^4*a3^4*a4
+ 64*a2^4*a3^2*a4^3
- 8*a2*a3^5*a4^3 + 56*a2^4*a3*a4^4
 + 16*a2^4*a4^5 +
16*a2*a3^3*a4^5 - 8*a2*a3^2*a4^6 + 16*a2*a4^8
- 4*a2^6*a3^2 - 8*a2^6*a3*a4 + 8*a2^3*a3^4*a4
- 4*a2^6*a4^2 - 4*a2^3*a3^3*a4^2
- 20*a2^3*a3^2*a4^3 - 16*a2^3*a3*a4^4 -
4*a3^4*a4^4 - 8*a2^3*a4^5
 - 4*a3^3*a4^5 - 4*a3*a4^7
- 4*a4^8 + 2*a2^5*a3^2 + 4*a2^5*a3*a4
+ 2*a2^5*a4^2 + 4*a2^2*a3^3*a4^2
+ 4*a2^2*a3^2*a4^3 + 2*a2^2*a3*a4^4 -
a2^4*a3^2 - a2^4*a3*a4
```

```
- a2^4*a4^2
+ a2*a3^2*a4^3 + a2*a3*a4^4 + a2*a4^5)/(32*a2^8*a4^3
- 64*a2^5*a3^3*a4^3 + 32*a2^5*a4^6
- 48*a2^4*a3^4*a4^2 - 64*a2^7*a4^3 +
80*a2^4*a3^3*a4^3 - 48*a2^4*a3^2*a4^4
- 64*a2^4*a4^6 - 24*a2^6*a3^2*a4
+ 24*a2^6*a3*a4^2 + 32*a2^6*a4^3
- 40*a2^3*a3^3*a4^3 + 48*a2^3*a3^2*a4^4
+ 32*a2^3*a4^6 -
4*a2^8 + 12*a2^5*a3^2*a4 - 24*a2^5*a3*a4^2
- 4*a2^5*a4^3 - 24*a2^2*a3^2*a4^4
+ 12*a2^2*a3*a4^5 - 4*a2^2*a4^6 + 2*a2^7
+ 6*a2^4*a3*a4^2 - 4*a2^4*a4^3 - 6*a2*a3*a4^5
+ 2*a2*a4^6 - a2^6 + 2*a2^3*a4^3 - a4^6)
```