Colouring on Hereditary Graph Classes

by

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Abstract

The graph colouring problems ask if one can assign a colour from a palette of colour to every vertex of a graph so that any two adjacent vertices receive different colours. We call the resulting problem k-Colourability if the palette is of fixed size k, and Chromatic Number if the goal is to minimize the size of the palette. One of the earliest NP-completeness results states that 3-Colourability is NP-complete. Thereafter, numerous studies have been devoted to the graph colouring problems on special graph classes. For a fixed set of graphs $\mathcal{H}$ we denote $Forb(\mathcal{H})$ by the set of graphs that exclude any graph $F \in \mathcal{H}$ as an induced subgraph. In this thesis, we explore the computational complexity of graph colouring problems on $Forb(\mathcal{H})$ for different sets of $\mathcal{H}$.

In the first part of this thesis, we study k-Colourability on classes $Forb(\mathcal{H})$ when $\mathcal{H}$ contains at most two graphs. We show that 4-Colourability and 5-Colourability are NP-complete on $Forb(\{P_7\})$ and $Forb(\{P_6\})$, respectively, where P_t denotes a path of order t. These results leave open, for $k \geq 4$, only the complexity of k-Colourability on $Forb(\{P_t\})$ for $k = 4$ and $t = 6$. Secondly, we refine our NP-completeness results on k-Colourability to classes $Forb(\{C_s, P_t\})$, where C_s denotes a cycle of length s. We prove new NP-completeness results for different combinations of values of k, s and t. Furthermore, we consider two common variants of the k-colouring problem, namely the *list k-colouring* problem and the *pre-colouring extension of k-colouring* problem. We show that in most cases these problems are also NP-complete on the class $Forb(\{C_s, P_t\})$. Thirdly, we prove that the set of forbidden induced subgraph that characterizes the class of k-colourable (C_4, P_6)-free graphs is of finite size. For $k \in \{3, 4\}$, we obtain an explicit list of forbidden induced subgraphs and the first polynomial *certifying algorithms* for k-Colourability on $Forb(\{C_4, P_6\})$.

We also discuss one particular class $Forb(\mathcal{H})$ when the size of $\mathcal{H}$ is infinite. We consider the intersection class of $Forb(\{C_4, C_6, \ldots\})$ and $Forb(caps)$, where a *cap* is a graph obtained from an induced cycle by adding an additional vertex and making it adjacent to two adjacent vertices on the cycle. Our main result is a polynomial time $3/2$-approximation algorithm for Chromatic Number on this class.

Keywords: Colouring; hereditary class; forbidden induced subgraphs; NP-complete; polynomial time algorithms; approximation algorithms

I dedicate this work
to my parents, wife and son
for their unconditional love.

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Declaration

The new results in this work are presented in Chapters 3–6. The results in Sections 4.1, 5.1 and 5.2 are joint work with my supervisor Pavol Hell, the results in Sections 4.2–4.4 are joint work with Matthew Johnson and Daniël Paulusma, and the results in Chapter 6 are joint work with Murilo V. G. da Silva.

1
Introduction

1.1 Prelude

My first encounter with graph theory was in my senior year during my undergraduate study at which time I took a course in discrete mathematics. In the study of the graph theory part in the course, I was amazed by the elegant half-page proof of the five colour theorem: every planar graph can be properly coloured using five colours. The proof reveals the unique beauty of the graph theory language that is entirely different from languages in such traditional areas of mathematics as analysis, algebra and geometry. When I saw it, I loved it! The experience opens a fascinating world of the subject, particularly of graph colouring, which has now become my topic in this dissertation.

The problem of ordinary graph colouring can be formally stated as follows. Given an undirected graph and a *palette* of available colours, is there a way to assign a colour from the palette to each vertex of the graph so that no two adjacent vertices receive the same colour? Five colour theorem says that as long as the palette consists of five colours or more, such a task is always achievable for planar graphs. (A graph is *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints)

The interest in colouring planar graphs stems from the map colouring problem around the middle of the nineteenth century. At that time, map makers observed that apparently every planar map can be coloured using four colours in such a way that countries sharing a boundary have distinct colours. In modern language of graph theory, to find a colouring of countries in a planar map is essentially to find a *face-colouring* of the planar graph M that represents the map, and this is equivalent to the ordinary graph colouring problem on the *dual graph* of M which is a planar graph as well. The seemingly overwhelmingly true observation of map makers, however, was not mathematically proven by anybody. This led the mathematician Guthrie to formulate this phenomenon in 1852 as the *Four Colour Conjecture*. Since then, a number of false proofs of the conjecture have appeared, among which is a proof given by Kempe in 1879. Kempe's proof was perceived to be correct until 1890 when Heawood found that there was a flaw in the proof. In addition to exposing the flaw, Heawood proved the five colour theorem, the one that leads this author to the study of graph colouring.

As close to a proof of the Four Colour Conjecture as the five colour theorem sounds, it remained the state-of-art for almost another century despite the tremendous efforts of many great minds. It was not until 1976 that Appel and Haken **[3, 4]** announced a computer-aided proof of the Four Colour Conjecture and hence the triumph over this old problem. The result is now known as the *Four Colour Theorem* and is probably the most celebrated graph-theoretic result within as well as outside graph theory community.

Theorem 1.1.1 (Four Colour Theorem) *Every planar graph can be coloured using only four colours.*

The announcement of Apple and Haken, nevertheless, has attracted much controversy due to the fact that the proofs rely largely on computer programs to check a considerable part of the proofs that cannot be done by hand. Twenty years after the initial announcement, Robertson, Sanders, Seymour, and Thomas **[95]** found a new proof of the Four Colour Theorem which is a significant simplification of the original proof of Apple and Haken. Although their new proof still appealed to a computer, Robertson, Sanders, Seymour and Thomas stated in their paper **[95]** that "we are making it possible for other scientists to verify all steps in our proof, including the computer programs and data". With their proofs verified by peer scientists and computer programs tested in different languages and on different platforms, Theorem 1.1.1 is nowadays widely accepted among graph theorists and mathematicians.

1.2 NP-completeness theory

The confirmation of the Four Colour Conjecture is far from being the end of the story. Instead, countless questions about graph colouring problems on planar graphs and other types of graphs with special structures are raised every year and the area has grown at a rapid rate in the past few decades, see **[10, 94]** for some recent surveys, and monograph of Jensen and Toft **[72]**.

A dynamic area in the study of the graph colouring problems is the design of efficient algorithms. The motivation for designing algorithms for graph colouring problems originates partly from the fact that graph colouring and its variants model a large number of real-life problems, ranging from scheduling committees, aircraft assignment, frequency assignment, physical layout segmentation to optimizing register allocations and reassembling DNA fragments. In his dissertation, Marx **[86]** gave an excellent and detailed account of how graph colouring problems provide a natural mathematical model for many of these problems. As such, efficient algorithms for finding optimal solutions to colouring problems that model real-life situations are highly desirable. Unfortunately, due to the intrinsic complexity of the problems, 'efficient' algorithms may be out of reach. To capture the notion of the intractability of the problems, a whole new paradigm, the so-called *NP-completeness theory*, has come into play since the beginning of the 1970s. The birth of the theory is usually attributed to Cook who published in 1971 his seminal work **[34]** in which the very first NP-complete problem (i.e., the satisfiability problem) was shown. Roughly speaking, a NP-complete problem is the most difficult problem in a large class (called *NP*) of decision problems. If an efficient (namely polynomial time) algorithm exists for any of the NP-complete problems, all problems in NP would be efficiently solvable. The general belief is that this is not the case: once shown to be NP-compete, a (NP) problem is unlikely to enjoy an efficient algorithm. We refer to the monograph of Garey and Johnson **[48]** for a rigorous treatment of the theory including the notoriously hard conjecture of $P \neq NP$ that formulates the intuition of 'a problem is unlikely to enjoy an efficient algorithm'.

Since the satisfiability problem was shown to be NP-complete, many important problems from areas like graph theory, computational geometry, game theory, linear programming, etc. were shown to be NP-complete as well using the method of polynomial time reduction. The fundamental work that lays the foundation for the fast development of the NP-completeness theory in subsequent years is due to Karp **[75]** who listed another 21 problems to the list of NP-completeness problems only one year after Cook's work. Among the 21 problems in Karp's paper is the graph 3-colouring problem, i.e., the ordinary graph colouring problem when the palette consists of exactly three elements. In complexity theory, it is desirable to formulate the problem at hand with the following form.

k-COLOURABILITY
Instance: An undirected graph $G = (V, E)$.
Question: Is G k-colourable?

Although listed in Karp's paper, 3-COLOURABILITY was first shown to be NP-complete by Lovász **[82]**. Other principal NP-complete problems in Karp's paper include 3-DIMENSIONAL MATCHING, VERTEX COVER, MAX CUT, UNDIRECTED HAMILTON CYCLE, KNAPSACK, INTEGER PROGRAMMING, JOB SEQUENCING among others (see **[75]** for formal definitions). From then on, a considerable number of problems in mathematics and computer science have been proven to be NP-complete and added to the already-large list of NP-complete problems annually. As we pointed out earlier, the NP-completeness of those problems indicates that it is unlikely to have a polynomial time algorithm for solving theses problems.

1.3 Handling NP-complete problems

As tough as NP-complete problems are, there are always situations when solutions to those problems are needed. Sometimes an approximate solution suffices. Other times one may not specify deterministically what each step does in an algorithm; random choices turn out to make algorithms fast and at the same time yield a good solution with high probability. What we just described are two popular ways to attack NP-complete problems: approximation and randomization. These two schemes have greatly advanced research on many central combinatorial problems that are NP-complete, among which are SATISFIABILITY, KNAPSACK, MAX CUT, VERTEX COVER and beyond. Not only have both of them proven to be productive and powerful as to solving NP-complete problems, they are now mature research fields on their own. We refer to Vazirani **[104]** for approximation algorithms and Mitzenmacher and Upfal **[88]** for randomized algorithms.

Besides approximation and randomization, there is a third common approach: that is, to restrict the instances of the problem under consideration. The k-colouring problem, for instance, may be asked for some particular graph class whose members possess certain special structure. Theorem 1.1.1 implies that 4-COLOURABILITY can be trivially answered 'yes' for planar graphs. One may feel that 3-COLOURABILITY is not more difficult than 4-COLOURABILITY. Yet, somewhat surprisingly, 3-COLOURABILITY remains NP-completeness for planar graphs. This is a result due to Stockmeyer.

Theorem 1.3.1 **[100]** 3-COLOURABILITY *is NP-complete on the class of planar graphs.*

Nevertheless, if we further forbid the occurrence of a triangle in a planar graph, then Grötzsch proved in 1959 the beautiful theorem stating that every such graph is 3-colourable, making the answer of 3-COLOURABILITY trivially 'yes' on this class of graphs.

Theorem 1.3.2 (Grötzsch's theorem **[58]**) *Every triangle-free planar graph is 3-colourable.*

Theorem 1.3.1 and Theorem 1.3.2 illustrate the typical situation in the study of computational complexity for combinatorial problems: adding constraints may or may not eliminate the intractability of the problem. In this thesis, we would like to continue this line of research and explore the complexity of graph colouring problems when additional structures are present in the instances. On one hand, the aim is to identify the boundary between the NP-completeness and the polynomial time solvability. We believe that this will enhance our understanding of the nature of colouring problems. On the other hand, graphs that arise from real applications normally have nice structural properties. In order to map the human genome, for instance, biologists use *interval graphs*, to model the overlaps of DNA clones (segments of a genome) **[87]**. The class of interval graphs forms a well-studied and well-understood class of graphs that admits very efficient (linear time) algorithm for many combinatorial problems like determining chromatic number, clique number and independence number. We refer to Golumbic **[54]** for efficient algorithms on interval graphs as well as further real-life applications. Our new results in this thesis that adopt the methodology of 'restricting the instances', we hope, do not rule out the possibility of (distant if not near) future applications.

1.4 Definitions and notation

Basic graph notions. All graphs in this thesis are finite, undirected and simple, i.e., they do not have parallel edges or loops. A graph is a pair $G = (V(G), E(G))$ where $V(G)$ is a set of *vertices* and $E(G)$ is a set of *edges*. The elements of $E(G)$ are 2-element subsets of $V(G)$. Always it is convenient to write uv instead of $\{u, v\}$ for an edge. We say that $V(G)$ and $E(G)$ are the *vertex set* and the *edge set* of G, respectively. The quantity $|V(G)|$ is the *order* of G.

The two vertices $u, v \in V(G)$ are *adjacent* (or *neighbours*), respectively, *non-adjacent* (or *non-neighbours*) if $uv \in E(G)$, respectively, $uv \notin E(G)$. The *open neighbourhood* of a vertex v, denoted by $N_G(v)$, is the set of neighbours of v. The *closed neighbourhood* of v is $N_G[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, let $N_G(X) = \bigcup_{v\in X} N_G(v) \setminus X$ and $N_G[X] = N_G(X) \cup X$. The *degree* of v, denoted by $d_G(v)$, is equal to $|N_G(v)|$. We shall omit the subscript G if the context is clear, i.e., we will write $N(v)$ instead of $N_G(v)$, etc. The *minimum degree* (respectively *maximum degree*) of G is equal to $\min\{d_G(v) : v \in V(G)\}$ (respectively $\max\{d_G(v) : v \in V(G)\}$), and is denoted by $\delta(G)$ (respectively $\Delta(G)$). Two edges $e, f \in E(G)$ are *adjacent* if $e \cap f \neq \emptyset$; otherwise e and f are *independent*. A vertex v is *incident* with an edge e if $v \in e$. The two vertices incident with an edge are its *end-vertices* or *ends*. We say that a vertex v *distinguishes* an edge $e = xy$ if v is adjacent to exactly one of x and y.

Subsets in graphs. A set of pairwise non-adjacent vertices is called an *independent* (or *stable*) set; and a set of pairwise independent edges is called a *matching*. A *clique* is a set of pairwise adjacent vertices. The *clique number* (respectively *independence number*) of G, denoted by $\omega(G)$ (respectively $\alpha(G)$), is the size of an independent set (respectively a clique) with maximum cardinality. For two subsets $X, Y \subseteq V(G)$, we say that X is *complete* (respectively *anti-complete*) to Y if every vertex in X is adjacent (respectively non-adjacent) to every vertex in Y. Clearly, X is complete (anti-complete) to Y if and only if Y is complete (anti-complete) to X, and hence we may say that X and Y are complete (anti-complete) to each other. If X consists of only a single element x, we simply say x, rather than $\{x\}$, is complete (anti-complete) to Y. For a set $S \subseteq V(G)$ and a vertex $x \in V(G)$, we write $N_S(x) := N_G(x) \cap S$. A set $D \subseteq V$ is *dominating* if $N_D(v) \neq \emptyset$ for each $v \in V \setminus D$. We say that D is a dominating set of G.

Relations between graphs. For a graph G, the *complement* of G, denoted by $\overline{G}$, is the graph with the vertex set $V(\overline{G}) = V(G)$ and the edge set $E(\overline{G}) = \{uv : uv \notin E(G)\}$. Let

$G = (V, E)$ and $G' = (V', E')$ be two graphs. We say that G and G' are *isomorphic* if there exists a bijection $\phi : V \to V'$ such that $xy \in E$ if and only if $\phi(x)\phi(y) \in E'$. Such a mapping is called an *isomorphism*. If $G' = G$, then it is called an *automorphism*. We say that G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If G' is a subgraph of G and $E' = \{xy \in E : x, y \in V'\}$, then G' is an *induced subgraph* of G, and we say that G' is an induced subgraph *on* V'. For any $S \subseteq V$, we denote the subgraph induced on S by $G[S]$.

Paths and cycles. Let $t \geq 1$ be a positive integer. A *path* P is a sequence of vertices $P = v_0 - v_1 - v_2 - \ldots - v_t$ such that $v_i v_{i+1}$ is an edge of G for each $0 \leq i < t$. The vertices v_0 and v_t are *ends* of the path and $v_1, \ldots, v_{t-1}$ are *internal vertices* of the path. We say that P *connects* v_0 and v_t. Sometimes we say that P is a $v_0 v_t$-path. The quantity t is the *length* of P. For any $0 \leq i < j \leq t$, $v_i - \ldots - v_j$ is a *sub-path* of P. If in addition $t \geq 2$ and v_0 and v_t are also adjacent, then the sequence is called a *cycle* and we write $C = v_0 - v_1 - v_2 - \ldots - v_t - v_0$. The *length* of C is the quantity $t+1$. A *chord* of a path or a cycle is an edge $v_i v_j$ with $j \neq i+1$ (in case we talk about cycles, the indices are modulo the length of the cycle). A path or a cycle is *chordless* if it does not contain any chord. We use P_t and C_t to denote a (chordless) path and cycle containing t vertices, respectively. A chordless cycle of length at least four is called a *hole*. A *k-hole* is a hole of length k. A hole is said to be *even* respectively *odd* if it is of even respectively odd length. They are referred to as *odd hole* and *even hole*, respectively. An *antihole* is the complement of a hole and k-antihole and odd antihole are defined analogously to k-hole and odd hole. The *girth* of G is the length of a shortest cycle and is denoted by $g(G)$. Let $P = v_0 - \ldots - v_t$ and $P' = u_0 - \ldots - u_s$ be two vertex-disjoint paths so that v_t and u_0 are adjacent. The *concatenation* of P and P', denoted by $P - P'$, is the path $v_0 - \ldots - v_t - u_0 - \ldots - u_s$. Note that any path can be viewed as the concatenation of its sub-paths.

Connectivity. We say that G is *connected* if for any two vertices of G there exists a path connecting them, and that G is *disconnected* if it is not connected. A *connected component* of G is a maximal connected subgraph. We stipulate that graphs with at most one vertex are connected. A vertex subset S of V is a *cutset* if $G - S$ has more connected components than G. Therefore, a cutset in a connected graph is a vertex subset whose removal results in a disconnected graph. If a cutset S is also a clique, it is called a *clique cutset*. If a cutset S consists of a single element v, then v is said to be a *cut-vertex*. A connected graph with no cut-vertices is 2-*connected*. A *block* of G is a maximal 2-connected subgraph. For a set $S \subseteq V$, we say that S is *connected* if $G[S]$ is connected.

Graph operations. Let $G = (V, E)$ be a graph. For any subset $E' \subseteq E$, the *edge deletion* of E' is the operation of removing every edge in E' from G. We denote the resulting graph $(V, E \setminus E')$ by $G - E'$. If E' consists of a single element e, we write $G - e$ instead of $G - \{e\}$. Similarly, for any subset $S \subseteq V$, the *vertex deletion* of S is the operation of removing every vertex in S and all edges that are incident with some vertex in S. We denote the resulting graph by $G - S$.

If S consists of a single element v, we write $G - v$ instead of $G - \{v\}$. We say that a vertex is *pendent* if it has exactly one neighbour in G and *universal* if it is adjacent to all other vertices in G. We say that a vertex v is *pendent to* u if u is the only neighbour of v. The operation of *adding a pendent* (respectively *universal*) vertex to G is to create a new vertex $u \notin V(G)$ and to make it adjacent to exactly one (respectively all) vertex in $V(G)$. Let $e = uv \in E$. The *contraction* of $e = uv$ is the operation of deleting vertices u and v, creating a new vertex $w \notin V(G)$ and making w adjacent to those vertices in G that were previously adjacent to u or v. The *subdivision* of e is the operation of removing e and creating a new vertex w and making it adjacent to u and v. Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ with $V(G) \cap V(H) = \emptyset$, the *disjoint union* of G and H, denoted by $G + H$, is the graph $(V(G) \cup V(H), E(G) \cup E(H))$. The disjoint union of $r \geq 2$ copies of G is written as rG. We note that any graph is the disjoint union of its connected components.

Special graphs. The graph $(\emptyset, \emptyset)$ is called the *null graph*. An *empty graph* is a graph with no edges (note that the null graph is a special empty graph). A graph $G = (V, E)$ is said to be a *complete multipartite graph* if V can be partitioned into $k \geq 2$ independent sets $V_1, \ldots, V_k$ so that V_i and V_j are complete to each other for $1 \leq i \neq j \leq k$. These sets $V_1, \ldots, V_k$ are *partites* of G. In case that $k = 2$, the graph is called a *complete bipartite graph*. We use $K_{r,s}$ to denote the complete bipartite graph with one partite having size r and the other having size s. The graph $K_{1,s}$ is called a *star* and the only vertex in V_1 is the *center* of the star. A *subdivided star* is a graph obtained from a star by subdividing each edge of the star exactly once. The graph $K_{1,3}$ is also known as a *claw*. In case that each V_i consists of a single vertex, the graph is a *complete graph* and is denoted by K_k. The graph K_3 is also called a *triangle*. We say that G is *bipartite* if V can be partitioned into two subsets V_1 and V_2 (that are not necessarily complete to each other); and is *chordal bipartite* if it is bipartite and every cycle of length at least 6 has a chord. A graph is *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. A *forest* is a graph with no cycles at all. A *tree* is a connected forest. A forest is said to be *linear* if it has maximum degree at most two. Equivalently, a linear forest is a disjoint union of chordless paths. The *line graph* of a graph H, denoted by $L(H)$, is a graph with vertex set $E(H)$ and edge set $\{ef : e \text{ and } f \text{ are incident edges in } H\}$. We say that G is a *line graph* if $G = L(H)$ for some graph H.

Colouring. A (*proper*) k*-colouring* of a graph $G = (V, E)$ is a mapping $\phi : V \to \{1, 2, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ whenever $uv \in E$. The value $\phi(u)$ is usually referred to as the *colour* of u under ϕ. We say that G is k*-colourable* if G admits a k-colouring. Observe that $\phi^{-1}(i)$ (for each $1 \leq i \leq k$) is an independent set in G. Therefore, a k-colouring of G can be equivalently described as a partition of V into k independent sets $V_1, \ldots, V_k$. We shall switch between the two terminology whenever convenient. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest positive integer k such that G is k-colourable. Note that G is 1-colourable if and only if G is an empty graph and G is 2-colourable if and only if G is bipartite. A k*-list assignment* of

G is a mapping $L : V(G) \to 2^{\{1,2,\ldots,k\}}$. A graph G with a k-list assignment L is said to be *L-colourable* if there exists a k-colouring $\phi : V \to \{1, 2, \ldots, k\}$ that *respects* L, i.e., $\phi(v) \in L(v)$ for each $v \in V$. Such a colouring is also called a *L-colouring* of G. A *pre-colouring* of G is a k-colouring $\phi_W : W \to \{1, 2, \ldots, k\}$ for some (possibly empty) subset W of V. We say that G is *ϕ_W-extendable* if there exists a k-colouring ϕ of G such that $\phi|_W = \phi_W$.

Decision problems. We list all decision problems that are relevant in this thesis. In the following $k \geq 2$ is a fixed integer.

k-COLOURABILITY
Instance: An undirected graph $G = (V, E)$.
Question: Is G k-colourable?

LIST k-COLOURABILITY
Instance: An undirected graph G and a k-list assignment L of G.
Question: Is G L-colourable?

k-PREXT
Instance: An undirected graph G and a pre-colouring ϕ_W for some subset $W \subseteq V$.
Question: Is G ϕ_W-extendable?

CHROMATIC NUMBER
Instance: A graph G and a positive integer q.
Question: Is $\chi(G) \leq q$?

3-SAT
Instance: A set $X = \{x_1, \ldots, x_n\}$ of variables and a set $\mathcal{C}$ of 3-literal clauses over X.
Question: Is $\mathcal{C}$ satisfiable, i.e, is there a truth assignment such that every clause contains at least one true literal?

NOT-ALL-EQUAL 3-SAT
Instance: A set $X = \{x_1, \ldots, x_n\}$ of variables and a set $\mathcal{C}$ of clauses each of which is a disjunction of three positive literals over X.
Question: Is $\mathcal{C}$ satisfiable, i.e, is there a truth assignment such that every clause contains at least one true literal and one false literal?

MAXIMUM WEIGHTED INDEPENDENT SET
Instance: A graph G, a weight function $w : V(G) \to \mathbb{R}^+$ and a positive real number r.
Question: Is there an independent set S of G such that $w(S) := \sum_{v \in S} w(v) \geq r$?

A note on the notation. We will not distinguish a vertex subset S and the subgraph induced on it. Namely, we use S to mean $G[S]$ whenever it causes no confusion. Similarly, we often identify the vertex set of a graph with the graph itself. We always let $n := |V(G)|$ and $m := |E(G)|$ unless otherwise told. For a graph G, we write $|G|$ instead of $|V(G)|$ to mean the order of G. Throughout the thesis, 'linear time' algorithm means an algorithm with running time $O(m+n)$ rather than $O(n)$.

2

Hereditary Graph Classes

In the study of graph problems, from both graph-theoretic and algorithmic view, restricting the problems to special graph classes turns out to be an effective and productive approach. The 'speciality' naturally depends on what constraints are added to the input instances. One could ask, for instance, the k-colouring problem with the constraints 'the input graph has at most 100 vertices'. This, however, does not yield an interesting problem: there are at most k^{100} (this number is a constant since k is fixed) possibly distinct k-colouring of the graph and checking if any of them is indeed a k-colouring of the graph one by one gives a constant algorithm. Instead of restricting the instances in an arbitrary way that may lead to trivial problems like the above example, we would like to take a systematic approach. A fairly successful approach taken in the literature is to forbid the appearance of certain structures in the input graphs. In other words, the input graphs are required to contain certain fixed graphs nowhere locally. The statement that 'a graph contains another graph' leaves some space for interpretation. The precise interpretation rests on a containment relationship between graphs. In this chapter, we survey several common graph containment relationships and the theories that arise from their corresponding containment relationships. The emphasis is given to 'induced subgraph' relationship from which the theory of hereditary graph classes results.

2.1 Subgraph relation

The most natural containment relationship one may come up with is the *subgraph relationship*. Recall that H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. One alternative way to define subgraph relation can be achieved through *graph operations*.

Definition (Subgraph) *A graph H is a* subgraph *of G if H can be obtained from G via repeatedly performing vertex deletions and edge deletions.*

Ramsey theory, one of the most exciting branches in combinatorics, deals with order in structures. Problems in Ramsey theory typically ask a question of the form: how big must a mathematical structure be in order to ensure that no matter how the original structure is cut into pieces at least one of the pieces has a given interesting property? In the context of graphs, the question may read as: given a fixed graph H does there exist a positive integer n so that for every graph G of order n either G or $\overline{G}$ contains H as a subgraph. Here the underlying structure is the complete graph K_n with G and $\overline{G}$ being its two pieces while the property that is of interest is 'contains H as a subgraph'. One of the most beautiful theorems in graph theory is due to Ramsey who gave a positive answer to the question.

Theorem 2.1.1 (Ramsey's theorem **[92]**) *For every fixed graph H, there exists a positive integer n such that either G or $\overline{G}$ contains H as a subgraph provided that G has at least n vertices.*

The original Ramsey's theorem **[92]** was stated solely for $H = K_r$ where r is a positive integer. Theorem 2.1.1 is a direct consequence of the original theorem, simply because H is a subgraph of K_h for $h := |H|$. Roughly speaking, Ramsey's theorem says that total chaos is impossible. Knowing the existence of such an integer in Theorem 2.1.1 does not fully satisfy mathematicians. Instead, they want to decide the smallest positive integer that makes the property in Theorem 2.1.1 hold. We denote this number by $R(H)$ and call it *Ramsey number*. Ramsey number is usually quite large. A constructive proof of the original Ramsey's theorem shows that $R(K_r) \leq 2^{2r-3}$. On the other hand, $R(K_r)$ is not 'too far' from the upper bound: $R(K_r) \geq 2^{r/2}$ which was shown via probabilistic method, see **[2]** for a proof. This means that the Ramsey number $R(K_r)$ grows exponentially in r. Nonetheless, the upper bound can be significantly reduced to be linear if the target graph H has its maximum degree bounded. This is a result due to Chvátal, V. Rödl, Szemerédi and Trotter **[29]**.

Theorem 2.1.2 **[29]** *For every positive integer Δ there is a constant c such that*

$$R(H) \leq c|H|$$

for all graphs H with $\Delta(H) \leq \Delta$.

As nice as the linear upper bound in Theorem 2.1.2 looks, determining the exact value of Ramsey number is counted as one of the most difficult problems in combinatorics. Even in the case that $H = K_r$ very little is known in this respect. The largest r for which $R(K_r)$ is known is $r = 4$ and the corresponding Ramsey number is 18. For $r = 5$ it is known that $43 \leq R(K_5) \leq 49$ but no one has succeeded in finding the exact value. As to $R(K_6)$, the difficulty can be well illustrated by a quote of Erdős from Spencer's book **[99]**, " Erdős asks us to imagine an alien force, vastly more powerful than us, landing on the earth and demanding the value of $R(K_5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(K_6)$. In that case, he believes, we should attempt to destroy the aliens." We refer the readers to **[55]** for more topics on Ramsey theory.

2.2 Minor relation

A different containment relation is the *minor relation* which is more subtle than the subgraph relation. Let us begin with the definition.

Definition (Minor) *We say that H is a minor of G if H can be obtained from G via repeatedly performing vertex deletions, edge deletions and edge contractions. In this case, we say that G contains H as a minor. If G does not contain H as a minor, G is said to be H-minor-free.*

By definition, any graph is a minor of itself. The minor relation gives rise to the study of so-called minor-closed classes. A graph class $\mathcal{G}$ is called *minor-closed* if it is closed under taking minors, i.e., for every member $G \in \mathcal{G}$ it holds that any minor of G also belongs to $\mathcal{G}$. Every minor-closed class $\mathcal{G}$ can be characterized by a set $\mathcal{M}(\mathcal{G})$ of *excluded minors* in the sense that a graph G belongs to $\mathcal{G}$ if and only if G is M-minor-free for every $M \in \mathcal{M}(\mathcal{G})$. This is true since $\mathcal{M}(\mathcal{G})$ can be taken as the set of graphs that are not in $\mathcal{G}$. Conversely, given a set $\mathcal{M}$ of graphs, the set of graphs that are M-minor-free for every $M \in \mathcal{M}$ form a minor-closed class. In other words, a graph class is minor-closed if and only if it can be characterized by a set of excluded minors. However, much more can be said about the set of excluded minors for a minor-closed class. The class of planar graphs, for example, can be characterized by exactly two excluded minors, namely K_5 and $K_{3,3}$ due to a celebrated theorem of Kuratowski **[78]**. In particular, the set of excluded minor is finite in this case. This motivated Wagner in 1970 to conjecture:

Conjecture 2.2.1 (Wagner's conjecture **[105]**) *Every minor-closed class can be characterized by a finite set of excluded minors.*

In the early 1980s, Robertson and Seymour started their monumental project on graph minors and proved Conjecture 2.2.1 in 2003 with a series of 23 papers. In fact, they offered a stronger theorem: the minor relations is *well-quasi-ordered*, i.e., for any infinite sequence of graphs $G_1, G_2, \dots,$ there exist two indices i and j with $i < j$ such that G_i is a minor of G_j. This became known as the *Graph Minor Theorem*.

Theorem 2.2.2 (Graph Minor Theorem) *The minor relation on the set of all finite graphs is well-quasi-ordered.*

The fact that Theorem 2.2.2 proves Conjecture 2.2.1 can be seen as follows. Take any minor-closed class $\mathcal{G}$ and suppose that $\mathcal{M}(\mathcal{G})$ is the set of excluded minors that characterizes $\mathcal{G}$. By definition no graph in $\mathcal{M}(\mathcal{G})$ belongs to $\mathcal{G}$. We may further assume that each graph $M \in \mathcal{M}(\mathcal{G})$ is minimal in the sense that any minor of M other than M itself lies in $\mathcal{G}$. Suppose by contradiction that $\mathcal{M}(\mathcal{G})$ is infinite. It follows then from Theorem 2.2.2 that there exist two graphs F and H in $\mathcal{M}(\mathcal{G})$ such that F is a minor of H. The minimality of H and the fact that $F \in \mathcal{M}(\mathcal{G})$ thus imply that $F = H$ and this is a contradiction. Over the course of the graph minor project, Robertsen and Seymour **[96]** also developed a cubic time algorithm to decide if a graph G is H-minor-free, provided that H is a fixed graph. This together with the truth of Wagner's conjecture implies the following.

Theorem 2.2.3 [96] *For any minor-closed class of graphs, the membership can be tested in cubic time.*

2.3 Induced subgraph relation

For a graph H to be a subgraph of G, we allow vertex deletions and edge deletions. In contrast, if we allow merely vertex deletions from the graph G, the subgraph H we end up with is said to be *induced*. The formal definition of induced subgraph relation is as follows.

Definition (Induced subgraph) *A graph H is an induced subgraph of G if H can be obtained from G via repeatedly performing the vertex deletion operation. In this case, we say that G contains H as an induced subgraph. If G does not contain H as an induced subgraph, G is said to be H-free. For a family $\mathcal{H}$ of graphs, G is $\mathcal{H}$-free if it is H-free for every $H \in \mathcal{H}$.*

The induced subgraph relation results in the study of hereditary classes. A graph class $\mathcal{G}$ is *hereditary* if it is closed under taking induced subgraphs, i.e., for every member $G \in \mathcal{G}$ it holds that every induced subgraph of G also belongs to $\mathcal{G}$. Hereditary classes can be characterized by *forbidden induced subgraphs* in the same way minor-closed classes are characterized by excluded minors.

Observation 2.3.1 *A family $\mathcal{G}$ of graphs is hereditary if and only if there exists a set $\mathcal{H}$ of graphs such that a graph $G \in \mathcal{G}$ if and only if G is $\mathcal{H}$-free.*

Definition (Characterization set) *The set $\mathcal{H}$ in Observation 2.3.1 is called the characterization set of the class $\mathcal{G}$. We also denote by $Forb(\mathcal{H})$ the class of $\mathcal{H}$-free graphs and write $\mathcal{G} = Forb(\mathcal{H})$.*

Unlike minor relation, however, the set of all finite graphs is not well-quasi-ordered under the induced subgraph relation, since $C_3, C_4, \ldots,$ is an infinite sequence of graphs so that no C_i is an induced subgraph of C_j for $i \neq j$. Therefore, Theorem 2.2.2 fails to hold for the induced subgraph relation. Nor does Wanger's conjecture hold in the context of hereditary classes:

$\mathcal{F}(\mathcal{G})$ is not always finite for any hereditary class $\mathcal{G}$. The class of bipartite graphs is apparently hereditary. It is well-known, see **[42]** among others, that the characterization set for this class is the set of odd cycles, which is of infinite size. Despite the lack of a general theorem analogous to Theorem 2.2.2, hereditary classes have generated a large body of research in the past decades. Two common approaches towards the investigation of hereditary classes are taken in the literature: (1) define a hereditary class by certain graph properties and then try to determine the characterization set or (2) define a hereditary class by specifying its characterization set and then seek graph properties it exhibits. Both approaches yield non-trivial and challenging questions that are not only interesting on their own but also linked to other areas of graph theory and computer science. We now survey a number of fundamental hereditary classes that illustrate the two approaches.

2.3.1 Perfect graphs

The chromatic number of a graph G is at least as big as its clique number, simply because any pair of vertices in a clique must not be coloured alike. Therefore, the clique number $\omega(G)$ of G provides a lower bound for $\chi(G)$. Nevertheless, this lower bound is 'bad' in the sense that the difference $\chi(G) - \omega(G)$ can be arbitrarily large. This fact was first shown by Mycielski **[90]** who constructed in 1955 a family of triangle-free graphs with arbitrarily large chromatic number. With the aid of probabilistic method, Erdős **[45]** generalized Mycielski's 'large chromatic number' result on triangle-free graphs to graphs with arbitrarily high girth four years later.

Theorem 2.3.2 [45] *For every positive integer t there exists a graph F with girth $g(F) > t$ and chromatic number $\chi(F) > t$.*

Erdős's result is quite discouraging: even if the local structure around every vertex is as simple as trees the chromatic number of which is merely two, properly colouring the whole graph still requires a significant number of colours. Berge, probably not disappointed by Theorem 2.3.2, took a different approach to study chromatic number of a graph. Instead of trying to bound the chromatic number from above by the clique number (Mycielski's and Erdős's result say that this is impossible but it becomes possible if further constraints are added to the graphs), Berge wondered whether or not those graphs G that do satisfy $\chi(G) = \omega(G)$ possess interesting properties. One may soon realize that requiring only $\chi(G) = \omega(G)$ might not be the 'right' definition. For instance, take any graph of order 10000 and take the disjoint union of the graph with the complete graph K_{10000}. The resulting graph thus has the property that its chromatic number equals its clique number. This is, however, not so interesting since nothing can be deduced about the graph picked in the first place. To overcome this difficulty, Berge made the property 'chromatic number equals clique number' hereditary. Here is Berge's definition of perfect graphs.

Definition (Perfect graph) *A graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G.*

The definition of perfect graph implies that any induced subgraph of a perfect graph is again perfect. In other words, the class of perfect graphs is a hereditary class. Thus, the class admits a forbidden induced subgraph characterization. For any $q \geq 2$, the odd hole C_{2q+1} has $\chi(C_{2q+1}) = 3$ and $\omega(C_{2q+1}) = 2$. This means that any odd hole is not perfect. Neither is the complement of such a graph. Berge made in 1962 the far-reaching conjecture stating that these are the only forbidden induced subgraphs for the class of perfect graphs. The conjecture was confirmed in 2002 by Chudnovsky, Robertson, Seymour, Thomas.

Theorem 2.3.3 (Strong Perfect Graph Theorem **[27]**) *A graph is perfect if and only if it does not contain any odd hole or odd antihole as an induced subgraph.*

The proof of Strong Perfect Graph Theorem falls into the paradigm of 'structural decomposition'. Results on structural decomposition for a graph class $\mathcal{C}$ are generally of the form: every member in $\mathcal{C}$ is either of a certain (usually well-understood) *basic type*, or can be decomposed via certain *cutsets* into smaller parts. As observed earlier, a perfect graph cannot contain any odd hole or its complement as an induced subgraph. These graphs are called *Berge* graphs. To prove Theorem 2.3.3, Chudnovsky, Robertson, Seymour, Thomas proved a decomposition theorem for the class of Berge graphs: every Berge graph G is either of five basic types each of which is known to be perfect, or G admits one of the three cutsets, proper 2-join, complement proper 2-join and balanced skew partition. This decomposition theorem implies Theorem 2.3.3: suppose that the theorem is false. Then there exists a Berge graph that is not perfect. Choose such a graph G with minimum number of vertices. It is known that any minimal counterexample does not admit any of the three cutsets and therefore G has to be one of five basic types. This, however, contradicts the assumption that G is imperfect. The complete proof spans over almost 200 pages and we refer to **[27]**.

One year after the announcement of Chudnovsky, Robertson, Seymour, Thomas's proof of Theorem 2.3.3, the first polynomial time algorithm for recognizing perfect graphs was also found by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković. Somewhat strangely, their algorithm is independent of the proof of Theorem 2.3.3.

Theorem 2.3.4 [21] *There exists a $O(n^9)$ algorithm to recognize if a graph of order n is perfect.*

Not only does the recognition of perfect graphs admit a polynomial time algorithm, a number of combinatorial problems that are NP-complete in general can be solved in polynomial time on the class of perfect graphs. Of particular interest to our work is a result of Grötschel, Lovász and Schrijver.

Theorem 2.3.5 [57] CHROMATIC NUMBER *is polynomial time solvable for perfect graphs.*

In addition to CHROMATIC NUMBER, polynomial solvability of the maximum independent set, the maximum clique and the clique covering problem were shown in the same paper **[57]**. We remark that those algorithms exist long before the validation of Strong Perfect Graph Theorem, and moreover, they make use of the ellipsoid method from linear programming and hence are not combinatorial. It is still a challenging open problem whether or not a combinatorial algorithm exists for optimally colouring perfect graphs. With the availability of Theorem 2.3.3, the quest we feel has a greater chance to reach a positive answer, see for example a partial result **[28]** in this respect.

2.3.2 Chordal graphs

The first approach of studying hereditary class we mentioned above is perfectly illustrated by the class of perfect graphs while the class of chordal graphs provides a representative example of the second approach. A graph is *chordal* if it does not contain any hole. In other words, the characterization set for the class of chordal graphs is $\{C_4, C_5, \ldots\}$. Historically, the study of chordal graphs dates back to the beginning of 1960s when Dirac investigated them under the name of 'rigid circuit graphs'. The principal result in Dirac's paper **[43]** is a decomposition theorem for the class of chordal graphs:

Theorem 2.3.6 (Dirac's theorem **[43]**) *Every connected chordal graph is either a complete graph or admits a clique cutset.*

As a matter of fact, every chordal graph can be built from smaller chordal graphs by pasting them together on clique cutsets. A direct consequence of Dirac's decomposition theorem is the fact that every chordal graph that is not a complete graph contains two non-adjacent *simplicial* vertices, i.e., a vertex whose neighbourhood induces a clique. This, together with the obvious fact that any complete graph contains a simplicial vertex, implies that every chordal graph has a simplicial vertex. The removal of a simplicial vertex in a chordal graph G results in a smaller chordal graph to which the fact can be again applied. One may repeat this process until the graph becomes the null graph. Putting the vertices of G in the order they were removed during the process forms a liner ordering $v_1, v_2, \ldots, v_n$ of $V(G)$. The ordering has the special property that for each i the vertex v_i is a simplicial vertex of the subgraph of G induced on $\{v_i, v_{i+1}, \ldots, v_n\}$. Such an ordering is referred to as a *perfect elimination ordering* the concept of which is originally due to Fulkerson and Gross **[47]**. Without too much difficulty it can be seen that the existence of a perfect elimination ordering prevents the occurrence of any hole in the graph.

Another profound result in the study of chordal graphs says that chordal graphs can be characterized as intersection graphs. Let $\mathcal{S}$ be a family of non-empty sets. The *intersection graph* of $\mathcal{S}$ is obtained by representing each set in $\mathcal{S}$ by a vertex and connecting two vertices by an edge if and only if their corresponding sets intersect. In 1974, Gavril **[49]** discovered that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree. The following theorem summarizes equivalent characterizations of chordal graphs that come from distinct perspectives.

Theorem 2.3.7 (Characterization of Chordal Graphs **[43, 47, 49]**) *For a graph G, the following statements are equivalent: (1) G is chordal; (2) G can be built from smaller chordal graphs by pasting them on clique cutsets. (3) G admits a perfect elimination ordering; (4) G is the intersection graph of a family of subtrees of a tree.*

The proof of Theorem 2.3.7 can be found in **[54]**. The characterizations in Theorem 2.3.7, on one hand, certify that the class of chordal graphs is a natural and beautiful class worth studying. They, on the other hand, serve as tools in the research of chordal graphs. Dirac's decomposition theorem, for instance, allows one to deduce the perfectness of chordal graphs.

Proposition 2.3.8 [7] *Any chordal graph is perfect.*

Proof. Let G be a chordal graph. We may assume that G is connected. It suffices to show that $\chi(G) \leq \omega(G)$. We prove this by induction on $|G|$. Since any complete graph is perfect, we may assume that G is not complete. It then follows from Theorem 2.3.7 that G admits a clique cutset K such that $G - K$ is a disjoint union of two subgraphs H_1 and H_2 of G. Let $G_i = H_i \cup K$ for $i = 1, 2$. Each G_i has strictly fewer vertices than G and by the inductive hypothesis we deduce that $\chi(G_i) \leq \omega(G_i)$. Since K is a clique, it follows that $\chi(G) = \max\{\chi(G_1), \chi(G_2)\} \leq \max\{\omega(G_1), \omega(G_2)\} \leq \omega(G)$. ∎

Proposition 2.3.8 and Theorem 2.3.4 imply that a chordal graph of order n can be recognized in $O(n^9)$ time. Nevertheless, a much faster algorithm due to Rose, Tarjan and Leuker **[97]** recognizes chordal graphs in linear time. Their algorithm uses *lexicographic breadth-first search* and outputs a linear ordering of the vertices of the input graph. The analysis of the correctness of the algorithm is based on perfect elimination ordering: if the input graph is chordal, then the ordering obtained at the end of the algorithm is a perfect elimination ordering (this is the heaviest work); and the other direction follows readily from Theorem 2.3.7.

Theorem 2.3.9 [97] *There exists an $O(m + n)$ time algorithm to decide if a graph with n vertices and m edges is chordal. Furthermore, the algorithm finds an induced cycle C_s for some integer $s \geq 4$ if the input graph is not chordal.*

By slightly modifying the algorithm of Rose, Tarjan and Leuker **[97]**, it is also possible to find the chromatic number ($=$ clique number) of a chordal graph in linear time.

Theorem 2.3.10 **[47]** *If G is chordal, then one can determine $\chi(G)$ ($= \omega(G)$) in linear time.*

2.3.3 Cographs

In this subsection we present another extensively studied hereditary class, i.e., the class of cographs. We begin with the definition.

Definition (Cographs) *A graph G is called a cograph if for every induced subgraph H of G with at least two vertices either H or $\overline{H}$ is disconnected.*

The definition of cographs implies that the class of cographs is hereditary and therefore can be characterization by forbidden induced subgraphs. In the 1970s, Seinsche **[98]** proved that the forbidden induced subgraph for a cograph is a single graph, namely the chordless path P_4. Later on, Corneil, Lerchs and Burlingham discovered an equivalent way of defining cographs. They investigated cographs under the name 'complement reducible graphs'.

Definition (Complement reducible graphs **[35]**) *A complement reducible graph is defined recursively as follows:*

(i) A graph on a single vertex is a complement reducible graph.

(ii) If $G_1, G_2, \ldots, G_k$ are complement reducible graphs, then so is their disjoint union $G_1 + G_2 + \ldots + G_k$.

(iii) If G is a complement reducible graph, then so is its complement $\overline{G}$.

The fundamental result in **[35]** is that complement reducible graphs are precisely cographs. The following theorem shows that P_4-free graphs and complement reducible graphs are the same as cographs.

Theorem 2.3.11 (Characterization of Cographs **[35, 98]**) *For a graph G the following statements are equivalent:*

(a) G is a cograph.

(b) G is a complement reducible graph.

(c) G is P_4-free.

Proof. We prove the theorem by proving the following three implications.

(a) $\Rightarrow$ (b). Let G be a cograph and we use induction on $|G|$. We may assume that $|G| \geq 2$. Since G is a cograph, either G or $\overline{G}$ is disconnected. Without loss of generality, we assume that G is disconnected. Thus, G is the disjoint union of its connected components G_i for $1 \leq i \leq k$. Applying the inductive hypothesis on each G_i we conclude that G_i is a complement reducible graph. Therefore, the G is also a complement reducible graph. $\diamond$

(b) $\Rightarrow$ (c). Let G be a complement reducible graph. We use induction on the number r of complementation and disjoint union operations that are needed to construct G. If $r = 0$, then G is a single vertex and thus is P_4-free. Now suppose that $r \geq 1$. If the last operation applied is the complementation, then $G = \overline{F}$, where F is the graph obtained using the first $r - 1$ operations. It follows from the inductive hypothesis that F is P_4-free. Since $\overline{P_4} = P_4$, G is also P_4-free. If the last operation applied is the disjoint union, then $G = G_1 + G_2 + \ldots + G_k$, where each G_i for $1 \leq i \leq k$ is constructed using less than r operations. Thus, G_i is P_4-free by the inductive hypothesis. Clearly, the disjoint union cannot create an induced P_4 and so G is P_4-free. $\diamond$

(c) $\Rightarrow$ (a). Let G be a P_4-free graphs. It suffices to show that either G or $\overline{G}$ is disconnected. We use induction on $|G|$. Pick any vertex $a \in V(G)$ and let $G' = G - a$. Then G' or $\overline{G'}$ is disconnected by the inductive hypothesis, since G' is P_4-free and has fewer vertices than G. Without loss of generality, we assume that G' is disconnected. Therefore, G' is the disjoint union of its connected components $F_1, F_2, \ldots, F_k$ for some integer $k \geq 2$. If a has no neighbour in F_i $(i = 1, 2, \ldots, k)$, then G is disconnected. On the other hand, if a is adjacent to each vertex in G', then $\overline{G}$ is disconnected. Therefore, we assume that a has a neighbour $b_i \in F_i$ for each $i = 1, 2, \ldots, k$ and a has a non-neighbour n in F_1. Since F_1 induces a connected subgraph of G', we may assume that b_1 and n are adjacent. But then $n - b_1 - a - b_2$ induces a P_4 in G. $\diamond$

The theorem follows from three implications. ■

The equivalence between (a) and (c) and between (a) and (b) are originally proven in **[98]** and **[35]**, respectively. Our proofs are slightly different from those in the literature. The *join* operation is a binary operation that takes two vertex-disjoint graphs G_1 and G_2 and adds an edge between any vertex in G_1 and any vertex in G_2. It is not difficult to see (by induction) that the complementation operation in the definition of complement reducible graphs can be replaced by the join operation. Theorem 2.3.11 thus implies that any cograph can be recursively constructed via the disjoint union and join operations. This leads to a canonical representation for cographs, i.e., the co-tree representation. The *co-tree* $C(G)$ of a cograph G is a tree in which leaves are vertices of G while the internal nodes of the tree are labeled as 0 and 1 which correspond to the join and disjoint union operations that reflect the construction of G. The co-tree of a cograph can be constructed in linear time **[36]**. This is due to Corneil, Pearl and Stewart who offered in the same paper the first linear time algorithm for recognizing cographs.

With the availability of the co-tree representation, the chromatic number of any cograph can be found in linear time in a bottom-up fashion on the co-tree.

Theorem 2.3.12 [36] *If G is a cograph, then one can determine $\chi(G)$ in linear time.*

The algorithm that finds $\chi(G)$ for a cograph in fact outputs a $\chi(G)$-colouring. This colouring turns out to be a $\omega(G)$-colouring: cographs are perfect. This can be easily seen from Theorem 2.3.11 (b): the complementation and the disjoint union operations preserve the perfection of a graph. Another important fact about cographs is due to Damaschke **[41]** who proved that the class of cographs is well-quasi-ordered under the induced subgraph relation.

In addition to its own development, the study of cographs also motivates additional related research. Here we briefly mention some that follows the line of 'generalizing cographs'. The *clique-width* of a graph G is defined to be the minimum number of labels that are needed in order to construct G under certain rules. The disjoint union and join operations are two of those rules. It can be easily deduced from Theorem 2.3.11 (b) that cographs are exactly the graphs that have clique-width at most two. Hence, graphs of clique-width at most some constant $k \geq 2$ generalize cographs. The concept of clique-width was introduced in the early 1990s and its importance is manifested in an algorithmic meta-theorem on polynomial solvability of a large class of problems proved by Courcelle, Makowsky and Rotics **[37]**. We refer to a recent survey **[74]** for formal definition and results on clique-width as well as its relation to tree-width.

Another way of generalizing cographs is achieved through the lens of the forbidden induced subgraph characterization (c) in Theorem 2.3.11. Relaxing the requirement 'nowhere containing induced P_4' to 'containing locally not too many induced P_4's' leads to so-called P_4-reducible and P_4-sparse graphs. A graph G is P_4-*reducible* if no vertex of G belongs to more than one induced P_4, and G is P_4-*sparse* if every set of five vertices contains at most one induced P_4. Clearly, any P_4-reducible graph is P_4-sparse. The classes of P_4-reducible and P_4-sparse graphs were introduced in 1980s by Jamison and Olariu **[70]** and Hoàng **[61]**, respectively. Both classes admit a linear time recognition algorithm **[70, 71]**. The class of P_4-sparse graph is hereditary and the forbidden induced subgraphs for this class are those graphs on five vertices that contain more than one induced P_4. Hoàng **[61, 62]** also gave a number of characterizations for P_4-sparse graphs and showed that these graphs are perfect.

More recently, there has been significant focus on the class of P_t-free graphs for fixed integer $t \geq 1$. When $t \geq 5$, the class generalizes the class of cographs in the most straightforward way. The first non-trivial generalized class of this kind is the class of P_5-free graphs. Bácso and Tuza **[6]** revealed that every connected P_5-free graphs has a dominating set that induces either a P_3 or a clique. Later on, Liu and Zhou **[81]** provided a characterization of P_5-free graphs: a graph G is P_5-free if and only if each connected induced subgraph H of G admits a dominating set D (of H) such that D induces either a 5-hole or a clique. Similar characterization was then obtained for the class of P_6-free graphs by van't Hof and Paulusma **[103]**. Such a theorem for general P_t-free graphs was not available until last year. In particular, the following theorem is due to Camby and Schaudt **[17]**.

Theorem 2.3.13 (Characterization of P_t-free graphs **[17]**) *Let G be a graph and $t \geq 4$ be an integer. Then G is P_t-free if and only if, for every connected induced subgraph H of G, H admits a connected dominating set D such that D induces either a C_t or a P_{t-2}-free graph.*

Note that the result of Liu and Zhou is a special case of Theorem 2.3.13, namely when $t = 5$. For $t = 4$, the theorem gives a characterization of cographs: a graph G is a cograph if and only if every connected induced subgraph of H has a dominating set that induces either a C_4 or P_1 (this was also previously proved). To prove the characterization theorem, Camby and Schaudt first demonstrated a critical property of the connected dominating sets in a connected P_t-free graph: any minimum connected dominating set is either P_{t-2}-free or isomorphic to P_{t-2}. They showed in addition such a dominating set can be found in polynomial time.

The hereditary classes presented here form just the tip of the iceberg. There are numerous graph classes that are of special structural properties. The monograph of Brandstädt, Le and Spinrad **[13]** provides a comprehensive survey on graph classes and their relations. Despite the fact that Theorem 2.3.13 is quite recent, the study of combinatorial problems on P_t-free graphs has existed long before that. The colouring problems, for example, has received much attention in the past decade. In the next three chapters, we devote ourselves to the computational complexity of ordinary colouring problems on the class of P_t-free graphs and related classes. New results in this thesis will be presented.

3

k-Colourability of P_t-Free Graphs

Theorem 2.3.11 and Theorem 2.3.12 imply that the chromatic number of a P_4-free graph can be determined in linear time. This is no longer true if we enlarge our graph under consideration from being P_4-free to being P_t-free for any fixed $t \geq 5$. Král, Kratochvíl, Tuza and Woeginger **[77]** showed that it is NP-complete to determine the chromatic number of a P_t-free graph whenever $t \geq 5$. This fact is a corollary of the main result in **[77]**, which is a dichotomy of computational complexity of Chromatic Number for H-free graphs.

Theorem 3.0.1 **[77]** Chromatic Number *is polynomial time solvable for the class of H-free graphs if H is an induced subgraph of P_4 or $P_3 + P_1$; and NP-complete otherwise.*

Roughly speaking, Theorem 3.0.1 says that for most graphs H, forbidding H in the input graph does not make computing chromatic number easier. In the same paper, Král, Kratochvíl, Tuza and Woeginger initiated a study of the computational complexity of Chromatic Number for graphs that do not contain two specified graphs H_1 and H_2 as induced subgraphs. Unlike forbidding a single graph, this problem seems to be much harder and no dichotomy is known so far for (H_1, H_2)-free graphs. We refer to **[40, 53, 84]** for some partial results and a recent survey by Golovach, Johnson, Paulusma and Song **[50]** for a summary in this respect.

In this chapter, we devote ourselves to k-Colourability of H-free graphs, with emphasis on the case $H = P_t$. The chapter is organized as follows. In Section 3.1 we validate the interest of k-Colourability of P_t-free graphs by surveying known results on k-Colourability of H-free graphs for various graphs H. We then present, in Section 3.2, a novel framework that allows us to deduce new NP-completeness results. Our results improve upon all previous NP-completeness results on k-Colourability of P_t-free graphs and narrow the attention to two major open problems in this area. In Section 3.3, we demonstrate a partial result towards one of these problems.

3.1 Problem history

Computing the chromatic number of H-free graphs enjoys a dichotomy of computational complexity as shown by Theorem 3.0.1. The situation is, however, more elusive when it comes to k-Colourability. Nevertheless, the problem has received much attention in the past decade and much progress has been made. Kamiński and Lozin **[73]** showed that, for any fixed $k \geq 3$ and $g \geq 3$, k-Colourability is NP-complete for the class of graphs of girth at least g. Their result has the following immediate consequence.

Theorem 3.1.1 **[73]** *For any fixed $k \geq 3$, k-Colourability is NP-complete for the class of H-free graphs whenever H contains a cycle.*

In the early 1980s, Holyer **[65]** proved that 3-Colourability is NP-complete for line graphs. Later, Leven and Galil **[80]** extended this result by showing that k-Colourability is also NP-complete for line graphs for $k \geq 4$. These two results, together with the fact that line graphs are claw-free, have the following consequence.

Theorem 3.1.2 **[65, 80]** *For any $k \geq 3$, k-Colourability is NP-complete for the class of H-free graphs if H is a forest with a vertex of degree at least three.*

Due to Theorem 3.1.1 and Theorem 3.1.2, only the case that H is a linear forest remains. Clearly, a path is the simplest linear forest and hence much study has been done on k-Colourability of P_t-free graphs. The problem is trivial for P_t-free graphs when $t \leq 3$. The first non-trivial case is the class of P_4-free graphs, namely cographs. Theorem 2.3.12 says that one can even find the chromatic number of a cograph and hence solve k-Colourability in linear time. This result was superseded by a breakthrough work due to Hoàng , Kamiński, Lozin, Sawada, Shu **[63]** who exhibited an elegant recursive algorithm confirming that k-Colourability can be solved in polynomial time for P_5-free graphs for any fixed $k \geq 1$. In addition, their algorithm applies to List k-Colourability of P_5-free graphs.

Theorem 3.1.3 **[63]** List k-Colourability *is polynomial time solvable for P_5-free graphs.*

When it comes to $t = 6$, only 3-Colourability is known. The polynomial solvability of 3-Colourability of P_6-free graphs was first shown by Randerath and Schiermeyer **[93]**. Their result was generalized by Broersma, Fomin, Golovach, Paulusma **[14]** to List 3-Colourability of P_6-free graphs and by Chudnovsky, Maceli and Zhong **[25, 26]** to 3-Colourability of P_7-free graphs.

On the negative side, Woeginger and Sgall **[106]** demonstrated in 2001 the NP-completeness of 5-Colourability of P_8-free graphs and 4-Colourability of P_{12}-free graphs. Their paper **[106]** was the very first one to study the colouring problem on P_t-free graphs, although the result of Kamiński and Lozin **[73]** that made the story more compelling was published 6 year later. Later on, Le, Randerath and Schiermeyer **[79]** improved 4-Colourability result of Woeginger and Sgall from P_{12}-free to P_9-free graphs. This was further improved to P_8-free graphs by Broersma, Golovach, Paulusma and Song **[15]**.

As much effort as researchers spent, no NP-completeness result was known for P_6-free graphs and P_7-free graphs around 2010. Therefore, List k-Colourability and k-Prext were also considered. Both problems generalize, and hence are harder than, k-Colourability. The hardness of these problems in fact makes the NP-completeness proof easier. Not surprisingly, the NP-completeness on P_6-free graphs and P_7-free graphs was soon established in the context of these problems. It was shown by Broersma, Golovach, Paulusma and Song **[15]** that 4-Prext is NP-complete for the class of P_7-free graphs and by Broersma, Fomin, Golovach, Paulusma **[14]** that 5-Prext is NP-complete for the class of P_6-free graphs. The latter result still holds for P_6-free graphs as far as List 4-Colourability is concerned **[52]**.

We use the following table to summarize all results in the literature. Here P and NP-c denote polynomial solvability and NP-completeness, respectively, while "?" means that the complexity status is still open. The literature corresponding to the maximal polynomial and minimal NP-complete results is listed.

	k-Colourability				k-Prext				List k-Colourability			
t	$k=3$	$k=4$	$k=5$	$k\geq 6$	$k=3$	$k=4$	$k=5$	$k\geq 6$	$k=3$	$k=4$	$k=5$	$k\geq 6$
$t\leq 5$	P	P	P	P	P	P	P	P	P	P	P	P
$t=6$	P	?	?	?	P	?	NP-c [14]	NP-c	P [14]	NP-c [52]	NP-c	NP-c
$t=7$	P [25, 26]	?	?	?	?	NP-c [15]	NP-c	NP-c	?	NP-c	NP-c	NP-c
$t\geq 8$	?	NP-c [15]	NP-c	NP-c	?	NP-c	NP-c	NP-c	?	NP-c	NP-c	NP-c

Table 3.1: The complexity status of k-Colourability, k-Prext and List k-Colourability of P_t-free graphs.

3.2 A new framework

As we mentioned earlier, all NP-complete results of k-Colourability of P_t-free graphs stop at $t = 8$, and no progress was made on P_7-free graphs. In this section, we derive new NP-complete results that break this barrier. Our main result here (cf. also **[67]**) is the following theorem.

Theorem 3.2.1 *4-Colourability of P_7-free graphs and 5-Colourability of P_6-free graphs are NP-complete.*

In order to prove Theorem 3.2.1, we provide a novel reduction framework that allows us to derive both results simultaneously. Before presenting the framework, we briefly explain why previous methods fail. In their groundbreaking work on k-Colourability of P_t-free graphs, Woeginger and Sgall **[106]** demonstrated the NP-completeness of the problem when $k = 5$ and $t = 8$ via a reduction from 3-SAT: for any 3-SAT instance I they constructed a graph G_I by introducing the gadget H^* (see Figure 3.1) for each clause of I. All subsequent improvements **[14, 15, 93]** are based on the same method by either choosing a different graph as the gadget or considering k-PREXT. The graph constructed this way, however, inherently contains an induced P_7. Specifically, regardless of the choice of the gadget, the construction as given in these papers always contains a copy of P_7. Therefore, all existing methods fail to work for P_7-free graphs. This issue is underscored by a subtle error in the proof of the claim that 6-COLOURABILITY is NP-complete for P_7-free graphs in **[14]**. This error was first noted in **[66]**.

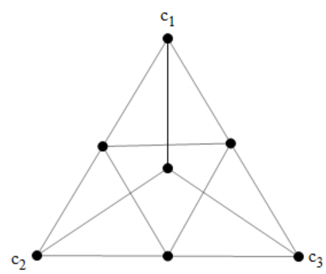


Figure 3.1: A nice 4-critical graph H^*: $\chi(H^*)$=4 and $\omega(H^*) = \omega(H^* - \{c_1, c_2, c_3\}) = 3$.

We now present our novel framework that not only fixes the error in **[14]** but also allows us to prove Theorem 3.2.1. A graph G is called *k-critical* if $\chi(G) = k$ and $\chi(G - v) < k$ for any vertex $v \in V(G)$.

Definition (Nice critical graphs) *A k-critical graph G is nice if $\omega(G) = k-1$ and moreover there exist three pairwise non-adjacent vertices c_1, c_2, c_3 in G such that $\omega(G - \{c_1, c_2, c_3\}) = \omega(G)$.*

As strange as the definition sounds, nice critical graph do exist. For instance, any odd hole of length at least 7 with any its three pairwise non-adjacent vertices is a nice 3-critical graph. The graph H^* in Figure 3.1 with vertices c_1, c_2 and c_3 is a nice 4-critical graph. Nice critical graphs hold the key to our new reduction.

3.2.1 Construction

Let I be a 3-SAT instance with variable set $X = \{x_1, x_2, \ldots, x_n\}$ and clause set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$. Let H be a nice k-critical graph with three specified independent vertices c_1, c_2, c_3. We now construct a graph $G_{H,I}$ that corresponds to H and I as follows.

Construction: $G_{H,I}$.

• For each variable x_i, introduce a *variable component* T_i that consists of two adjacent vertices x_i and $\overline{x_i}$. These vertices are said to be *X-type*.

• For each variable x_i, introduce a vertex d_i. These vertices are said to be *D-type*.

• For each clause $C_j = y_{i_1} \vee y_{i_2} \vee y_{i_3}$, introduce a *clause component* H_j that is isomorphic to H, where y_{i_t} is either x_{i_t} or $\overline{x_{i_t}}$ for $1 \leq t \leq 3$. The three specified independent vertices in H_j are then denoted by $c_{i_t j}$ for $1 \leq t \leq 3$. Those vertices $c_{i_t j}$ are said to be *C-type* while the remaining vertices in H_j are said to be *U-type*.

Without confusion we also denote X, D, C and U by the set of X-type, D-type, C-type and, respectively, U-type vertices. In other words, $V(G_{H,I}) = X \cup D \cup C \cup U$. For a C-type vertex c_{ij} in $G_{H,I}$, the vertex x_i or $\overline{x_i}$ is said to be its *corresponding literal vertex*, depending on whether $x_i \in C_j$ or $\overline{x_i} \in C_j$. We now describe the edges of $G_{H,I}$.

• Add an edge between each U-type vertex and each D-type vertex.

• Add an edge between each U-type vertex and each X-type vertex.

• For each C-type vertex c_{ij}, add an edge between c_{ij} and its corresponding literal vertex and add an edge between c_{ij} and d_i.

This completes the construction of the graph $G_{H,I}$. We note that clause components are pairwise disjoint in $G_{H,I}$. We shall show that the colourability of $G_{H,I}$ encodes the satisfiability of I. Our main contribution is the following theorem that makes the connection between NP-completeness and nice critical graphs.

Theorem 3.2.2 *Let $t \geq 6$ and $k \geq 3$ be fixed integers. Then k-*Colourability *of P_t-free graphs remains NP-complete whenever there exists a P_t-free nice $(k-1)$-critical graph.*

The proof of Theorem 3.2.2 will follow from two lemmas presented in the next subsection.

3.2.2 Proof of Theorem 3.2.2

Throughout the subsection, we assume that I is a 3-Sat instance, H is a nice k-critical graph, and $G_{H,I}$ is the graph constructed the way we present in subsection 3.2.1.

Lemma 3.2.3 *I is satisfiable if and only if $G_{H,I}$ is $(k+1)$-colourable.*

Proof. We assume that I is satisfiable. Let σ be a truth assignment satisfying each clause C_j. We define a mapping $\phi : V(G_{H,I}) \to \{1, 2, \ldots, k+1\}$ as follows.

• Set $\phi(d_i) := k+1$ for $1 \leq i \leq n$.

• For $1 \leq i \leq n$, if $\sigma(x_i) = TRUE$, set $\phi(x_i) := k+1$ and $\phi(\overline{x_i}) := k$. Otherwise, set $\phi(x_i) := k$ and $\phi(\overline{x_i}) := k+1$.

• Let $C_j = y_{i_1} \vee y_{i_2} \vee y_{i_3}$ be the jth clause of I. Since σ satisfies C_j at least one literal in C_j is $TRUE$, say y_{i_t} for some $t \in \{1, 2, 3\}$. The corresponding literal vertex of c_{i_tj} is therefore assigned colour $k+1$ under ϕ (note that this colour is the same colour that d_{i_t} receives under ϕ). We set $\phi(c_{i_tj}) := k$ so that the colour of c_{i_tj} conflicts with neither its corresponding literal vertex nor d_{i_t}.

• Since H_j is isomorphic to H and H is k-critical, it follows that $H_j - c_{i_tj}$ admits a $(k-1)$-colouring $\phi_j : V(H_j - c_{i_tj}) \to \{1, 2, \ldots, k-1\}$. We set $\phi := \phi_j$ on $V(H_j - c_{i_tj})$.

This completes the definition of ϕ and it can be readily checked that ϕ is indeed a mapping from $V(G_{H,I})$ to $\{1, 2, \ldots, k+1\}$. Note that all vertices in $X \cup D$ are assigned colours either k or $k+1$ while all vertices in U is never assigned colour k or $k+1$. This fact, together with the definition of ϕ, implies that ϕ is a $(k+1)$-colouring of $G_{H,I}$.

Conversely, suppose that $G_{H,I}$ is $(k+1)$-colourable. Let $\phi : V(G_{H,I}) \to \{1, 2, \ldots, k+1\}$ be a $(k+1)$-colouring of $G_{H,I}$. For each $1 \leq j \leq m$, H_j is a nice k-critical graph. It follows from the definition of nice critical graphs that $\omega(H_j \cap U) = k-1$. Let $R_j \subseteq V(H_j \cap U)$ be a clique of size $k-1$. Observe that $R_j \cup T_1$ is a largest clique in G_{H_I} and $|R_j \cup T_1| = k+1$. Since ϕ is a $(k+1)$-colouring of G_{H_I}, it follows that $\phi(u) \neq \phi(v)$ for any two distinct vertices u and v in $R_j \cup T_1$. We may assume, without loss of generality, that $\{\phi(x_1), \phi(\overline{x_1})\} = \{k, k+1\}$. Since U is complete to $X \cup D$, the following properties of ϕ hold.

(P1) For each $1 \leq i \leq n$, $\{\phi(x_i), \phi(\overline{x_i})\} = \{k, k+1\}$.

(P2) For each $1 \leq i \leq n$, $\phi(d_i) \in \{k, k+1\}$.

(P3) For each $u \in U$, $\phi(u) \in \{1, 2, \ldots, k-1\}$.

We then construct a truth assignment $\sigma : \{x_1, x_2, \ldots, x_n\} \to \{TRUE, FALSE\}$ as follows.

$$\sigma(x_i) := \begin{cases} TRUE & \text{if } \phi(x_i) = \phi(d_i). \\ FALSE & \text{if } \phi(x_i) \neq \phi(d_i). \end{cases}$$

It follows from **(P1)** and **(P2)** that σ is indeed a truth assignment. It remains to show that σ satisfies each clause in I. Suppose that some clause $C_j = y_{i_1} \vee y_{i_2} \vee y_{i_3}$ is not satisfied under σ. This means that $\sigma(y_{i_t}) = FALSE$ for $1 \leq t \leq 3$. It then follows from our definition of σ that the corresponding literal vertex of c_{i_tj} receives a colour different from $\phi(d_{i_t})$. This, together with **(P1)** and **(P2)** , implies that $\phi(c_{i_tj}) \notin \{k, k+1\}$ for $1 \leq t \leq 3$. Therefore, $\phi_j := \phi|_{V(H_j)}$ is a $(k-1)$-colouring of H_j due to **(P3)**. This, however, contradicts the fact that $\chi(H) = k$. ■

Lemma 3.2.4 *Let $t \geq 6$ be a fixed integer. If H is P_t-free, then $G_{H,I}$ is P_t-free.*

Proof. Assume that H is P_t-free. We prove the lemma by contradiction. Let $P = P_t$ be an induced path in $G_{H,I}$ with $t \geq 6$. We show the following properties of P.

(i) *P contains at least one vertex in $X \cup D$.*

Note that $G - (X \cup D)$ is the disjoint union of m copies of H. If P contains no vertex in $X \cup D$, then $V(P) \subseteq V(H_j)$ for some $1 \leq j \leq m$. This contradicts the assumption that H is P_t-free. This proves **(i)**. $\diamond$

(ii) *P contains at least one vertex in U.*

Let C_i and $\overline{C_i}$ denote the set of C-type vertices that are neighbours of x_i, and respectively, $\overline{x_i}$. Observe that every connected component of $G - U$ has a specific structure, namely it is the result of substituting independent sets into a 5-hole (and possibly removing some vertices). Specifically, the five independent sets are, in the cyclical order, $X_0 = \{x_i\}$, $X_1 = C_i$, $X_2 = \{d_i\}$, $X_3 = \overline{C_i}$, and finally $X_4 = \{\overline{x_i}\}$. This subgraph of $G_{H,I}$ does not contain an induced P_t, since the 5-hole does not and substituting independent sets cannot create a P_t. This proves **(ii)**. $\diamond$

(iii) *P contains at most three vertices in $X \cup D \cup U$ and these vertices form a sub-path of P.*

We recall that U and $X \cup D$ are complete to each other. By **(i)** and **(ii)**, P contains at least one vertex $u \in U$ and one vertex $x \in X \cup D$. This implies that u and x are consecutive on P. In addition, the other neighbour x^+ of x on P (if it exists) is not from $X \cup D$ (since it would be adjacent to u but P is induced), and similarly the other neighbour u^+ of u on P (if it exists) is not from U. By the same token, all other vertices on P (apart from x, u and their neighbours on P) are from C. Furthermore, at least one of x^+ and u^+ is also from C, for otherwise x^+ is in U, and u^+ is in $X \cup D$ implying that x^+ and u^+ are adjacent, but this is impossible since P is induced. Consequently, we conclude that apart from x, u and exactly one of x^+ and u^+, there are no other vertices on P from $X \cup D \cup U$, and clearly those that are from $X \cup D \cup U$ are consecutive on P. This proves **(iii)**. $\diamond$

It follows from **(iii)** that all but at most three consecutive vertices on P are from C. This implies $P \cap C$ contains adjacent vertices due to our assumption that P has at least $t \geq 6$ vertices. This is impossible since C is an independent set. This completes the proof. ∎

We are ready to prove our main result in this section.

Proof of Theorem 3.2.1. As we noticed before that H^* is a nice 4-critical graph and C_7 is a nice 3-critical graph. Moreover, it is routine to verify that the longest induced path in H^*

has five vertices. In other words, H^* is P_6-free. Clearly, C_7 is P_7-free. Applying Theorem 3.2.2 with H^* and C_7 completes the proof. ■

3.2.3 Consequences

Theorem 3.2.1 demonstrates for the first time the existence of an integer k such that k-Colourability of P_7-free and P_6-free graphs is NP-complete. It improves upon the NP-complete results from **[14, 15]** including those on k-PrExt. The following table is an updated complexity status with our new results.

	k-Colourability			
t	$k=3$	$k=4$	$k=5$	$k\geq 6$
$t\leq 5$	P	P	P	P
$t=6$	P	?	**NP-c**	NP-c
$t=7$	P	**NP-c**	NP-c	NP-c
$t\geq 8$	?	NP-c	NP-c	NP-c

Table 3.2: Updated complexity status of k-Colourability of P_t-free graphs. Our new results are indicated in bold cells.

As easily seen from Table 3.2, our result almost completes the classification of the complexity of k-Colourability of P_t-free graphs when $k\geq 4$, leaving as the only missing case 4-Colourability of P_6-free graphs. We conjecture that this problem can be solved in polynomial time.

Conjecture 3.2.5 *4-Colourability of P_6-free graphs admits a polynomial time algorithm.*

The conjecture is supported in part by the following observation.

Observation 3.2.6 *There exists no P_6-free nice 3-critical graphs.*

Proof. Suppose that H is a P_6-free nice 3-critical graphs where c_1,c_2 and c_3 are pairwise non-adjacent vertices in H. It is well-known that odd holes and the triangle are the only 3-critical graphs. Therefore, $H=C_{2s+1}$ for some integer $s\geq 1$. Since H is not a clique and does not contain any induced path with six vertices or more, it follows that $s=2$, i.e., H is isomorphic to a 5-hole. This contradicts the fact that the 5-hole has independence number two. ■

Observation 3.2.6 preludes the possibility of applying Theorem 3.2.2 to showing the NP-completeness of 4-Colourability of P_6-free graphs. This is some evidence that Conjecture 3.2.5 should hold. Moreover, Golovach, Paulusma and Song **[51]** completed the classification of 4-Colourability of H-free graphs when H has at most five vertices. The classification states that 4-Colourability is polynomial time solvable on the class of H-free graphs when H is a linear forest, and NP-complete otherwise. We note that linear forests with at most five vertices and no more than two connected components are induced subgraphs of P_6. Therefore, most polynomial cases in the classification are subclasses of the class of P_6-free graphs.

Besides Conjecture 3.2.5, the remaining question marks in Table 3.2 are the ones that correspond to 3-COLOURABILITY. This problem seems to be notoriously hard: it is not even known whether or not there exists an integer $t \geq 8$ such that 3-COLOURABILITY becomes NP-complete on the class of P_t-free graphs. It is not difficult to see that the existence of such an integer is equivalent to the existence of an integer r so that LIST 3-COLOURABILITY is NP-complete for P_r-free graphs.

Problem 3.2.7 *Is there an integer $t \geq 8$ such that (*LIST*)* 3-COLOURABILITY *is NP-complete for P_t-free graphs?*

3.3 A polynomial case

In this section we prove that Conjecture 3.2.5 is true for a subclass of $Forb(\{P_6\})$. As we indicated before, Conjecture 3.2.5 seems plausible due to Observation 3.2.6. Nevertheless, the problem resists an answer so far. This is largely because the usual techniques for 3-COLOURABILITY, see **[14, 15, 93, 106]** for example, do not apply. It turns out that the problem becomes easier if an additional induced subgraph is forbidden. For instance, it was shown that every (P_6, C_3)-free graph is 4-colourable, see **[94]**. A *paw* is a graph obtained from a triangle by adding a pendent vertex. A classical result of Olariu **[91]** says that every connected component of a paw-free graph is either triangle-free or a complete multipartite graph. This implies that the result of 4-COLOURABILITY of (P_6, C_3)-free graphs extends to (P_6, paw)-free graphs. Another interesting result in this direction was due to Golovach, Paulusma and Song **[53]**.

Theorem 3.3.1 **[53]** *For any fixed positive integers k, r, s and t,* LIST k-COLOURABILITY *of $(K_{r,s}, P_t)$-free graphs is polynomial time solvable.*

Taking $k = 4$, $r = s = 2$ and $t = 6$, it follows that 4-COLOURABILITY of (P_6, C_4)-free graphs is polynomial time solvable. The *banner* is the graph obtained from a 4-hole by adding a pendent vertex. See Figure 3.2.

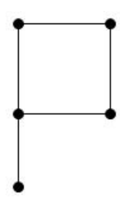


Figure 3.2: The banner graph.

The class of banner-free graphs generalizes the class of C_4-free graphs the same way the class of paw-free graphs does for the class of C_3-free graphs. It has been studied for MAXIMUM WEIGHTED INDEPENDENT SET in several papers in the context of P_t-free graphs, say **[11, 89]**. Here we show that Conjecture 3.2.5 is true for banner-free graphs. Before we present our algorithm, the following tools are needed.

Lemma 3.3.2 [44] *Let $G = (V, E)$ be a graph with a list $L(v)$ of admissible colours such that $|L(v)| \leq 2$ for each vertex $v \in V$. Deciding whether or not G admits a colouring that respects the lists can be solved in polynomial time.*

Lemma 3.3.3 [14] LIST 3-COLOURABILITY *can be solved in polynomial time for P_6-free graphs.*

We are now ready to prove our main result in this section.

Theorem 3.3.4 4-COLOURABILITY *is polynomial time solvable for $(P_6, banner)$-free graphs.*

Proof. Let $G = (V, E)$ be a $(P_6, banner)$-free graph with n vertices and m edges. It is folklore that G is k-colourable if and only if each block of G is k-colourable. In addition, all blocks of G can be found in $O(n + m)$ time using depth-first search **[101]**. This allows us to assume that G is 2-connected. We may also assume that G does not contain as an induced subgraph a K_5, a $\overline{C_7}$ with an additional vertex that is complete to the $\overline{C_7}$ or a C_5 with two additional adjacent vertices that are complete to the C_5, for otherwise we immediately conclude that G is not 4-colourable, and it takes $O(n^8)$ time to detect the presence of such an induced subgraph. We proceed by appealing to Theorem 2.3.4 to distinguish two cases according to whether or not G is perfect. If G is perfect, then we apply Theorem 2.3.5 to optimally colour G. From now on we assume that G is not perfect. We then test if G contains a C_5 or a $\overline{C_7}$ in $O(n^7)$ time. If the answer is no, then by the Strong Perfect Graph Theorem and the fact that G is P_6-free, G must contains an odd antihole of length at least 9 whose chromatic number is at least 5. Hence, G is not 4-colourable. So, we assume in the following that G contains either a C_5 or a $\overline{C_7}$ as an induced subgraph. We suppose first that G contains an induced $C = \overline{C_7} = v_0 - v_1 - \ldots - v_6 - v_0$ where $v_i v_j \in E$ if and only if $2 \leq |i - j| \leq 5$ (the indices are modulo 7).

(i) *Every vertex in $V \setminus C$ with at least one neighbour on C has at least four neighbours on C.*

Consider a vertex $x \in V \setminus C$ with at least one but at most three neighbours on C. We may assume, without loss of generality, that x is adjacent to v_0 and non-adjacent to v_1 and v_2. If x is non-adjacent to both v_4 and v_5, then $\{x, v_0, v_1, v_4, v_5\}$ induces a banner in G. If x is adjacent to exactly one of v_4 and v_5, then $\{x, v_1, v_2, v_4, v_5\}$ induces a banner in G. Therefore, x is adjacent to both v_4 and v_5. Since x has at most three neighbours on C, it follows that $N_C(x) = \{v_0, v_4, v_5\}$. This implies that $\{x, v_1, v_2, v_5, v_6\}$ induces a banner in G. This proves **(i)**. $\diamond$

(ii) *Every vertex in $V \setminus C$ has at least one neighbour on C.*

Consider a vertex $y \in V \setminus C$ with no neighbour on C. Since G is connected, y can be chosen with the property that it has a neighbour $x \in V \setminus C$ such that x has a neighbour on C. It follows from **(i)** that x has at least four neighbours on C. This implies that there exists some index $0 \leq i \leq 6$ such that both v_i and v_{i+1} are neighbours of x. We assume without loss of generality that $i = 0$. If x is non-adjacent to v_i where $i \in \{3, 4, 5\}$, then the set $\{y, x, v_0, v_1, v_i\}$ induces a banner in G. We therefore conclude that x is adjacent to v_i for each $3 \leq i \leq 5$. This implies that x is also adjacent to v_2 and v_6, for otherwise either $\{y, x, v_2, v_4, v_5\}$ or $\{y, x, v_3, v_4, v_6\}$ induces a banner in G. This contradicts our assumption that G does not contain $C \cup \{x\}$ as an induced subgraph. This proves **(ii)**. $\diamond$

Apparently, there are at most 4^7 different 4-colouring of the 7-antihole C, and G is 4-colourable if and only if there exists at least one such colouring of C that can be extended to G. Therefore, it suffices to explain how to decide if a given pre-colouring $\phi_C : C \to \{1, 2, 3, 4\}$ of C can be extended to a 4-colouring of G in polynomial time. This amounts to deciding in polynomial time whether or not G admits a 4-colouring that respects the following 4-list assignment.

$$L(v) = \begin{cases} \{1, 2, 3, 4\} & \text{if } v \notin C \\ \{\phi_C(v)\} & \text{otherwise} \end{cases}$$

We say that vertices with $|L(v)| = 1$ have been *pre-coloured*. We then reduce the lists of possible colours for other vertices in G using the following procedure.

Update the lists. For any pre-coloured vertex v and any $x \in N(v)$ we remove the colour $\phi(v)$ from the list of x, i.e., we let $L(x) := L(x) \setminus \{\phi(v)\}$.

Since every vertex $x \in V \setminus C$ has at least four neighbours on C (due to **(i)** and **(ii)** above), it follows that the list for x contains at most two distinct colours after updating the lists. Therefore, Lemma 3.3.2 allows us to decide in polynomial time if ϕ_C can be extended to a 4-colouring of G.

This shows that testing whether or not G is 4-colourable can be done in polynomial time if G contains an induced $\overline{C_7}$. Therefore, we assume in the following that G contains a 5-hole $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ where $ij \in E$ if and only if $|i - j| \in \{1, 4\}$ (indices are modulo 5). For $0 \leq i \leq 5$, a vertex $v \in V \setminus C$ is said to be an *i-vertex* if $|N_C(v)| = i$, and let S_i denote the set of i-vertices. Clearly, $V(G) = V(C) \cup \bigcup_{i=0}^{5} S_i$. For each $0 \le i \le 4$, we define

$$S_1(v_i) = \{x \in S_1 : N_C(x) = \{v_i\}\}.$$
$$S_2(v_i) = \{x \in S_2 : N_C(x) = \{v_{i-2}, v_{i+2}\}\}.$$
$$S_3(v_i) = \{x \in S_3 : N_C(x) = \{v_{i-1}, v_i, v_{i+1}\}\}.$$
$$S_4(v_i) = \{x \in S_4 : N_C(x) = V(C) \setminus \{v_i\}\}.$$

(a) *The set S_5 is an independent set and moreover all vertices in S_5 are coloured alike in any 4-colouring ϕ of G.*

Let $\phi : V(G) \to \{1, 2, 3, 4\}$ be a 4-colouring of G. Since $\chi(C_5) = 3$, it follows that at least three colours appear on C, say colours 1, 2 and 3. For any vertex $u \in S_5$, it follows that $\phi(u) = 4$ since u is complete to C. This and the fact that ϕ is a proper colouring imply that S_5 is an independent set. This proves **(a)**. $\diamond$

(b) *For each $1 \le j \le 4$, $S_j = \bigcup_{i=0}^{4} S_j(v_i)$.*

It suffices to prove **(b)** for $j = 2$ and $j = 3$, since it is trivially true for $j = 1$ and $j = 4$. This is equivalent to prove that for each $x \in S_2 \cup S_3$, the neighbours of x on C are consecutive along the cycle. Suppose that the neighbours of x on C are not consecutive. We assume without loss of generality that x is adjacent to v_0 and v_2 but non-adjacent to v_1 and v_4. This implies that $\{x, v_0, v_1, v_2, v_4\}$ induces a banner in G. This proves **(b)**. $\diamond$

(c) *For each $3 \le j \le 4$ and $0 \le i \le 4$, $S_j(v_i)$ is a clique of size at most two.*

Suppose by contradiction that for each $3 \le j \le 4$ there exists some i such that $S_j(v_i)$ is not a clique. We assume without loss of generality that $i = 0$ when $j = 3$ and $i = 3$ when $j = 4$. Since $S_j(v_i)$ is not a clique, it contains two non-adjacent vertices x and y. Thus, $\{x, y, v_1, v_3, v_4\}$ induces a banner in G, since neither x nor y is adjacent to v_3 (in both cases). This proves that $S_j(v_i)$ is a clique. Since G is K_5-free, it follows that each set has size at most two. This proves **(c)**. $\diamond$

It follows from **(b)** and **(c)** that $|S_j| \le 10$ for $j = 3, 4$. This implies that $|C \cup S_3 \cup S_4| \le 25$. Therefore, there are at most 4^{25} different 4-colourings of $C \cup S_3 \cup S_4$, and G is 4-colourable if and only if there exists at least one such colouring of $C \cup S_3 \cup S_4$ that can be extended to G. It remains to explain how to decide in polynomial time whether or not a given 4-colouring $\phi : C \cup S_3 \cup S_4 \to \{1, 2, 3, 4\}$ can be extended to a 4-colouring of G. This amounts to deciding in polynomial time whether or not G admits a 4-colouring that respects the following 4-list assignment.

$$L(v) = \begin{cases} \{1,2,3,4\} & \text{if } v \notin C \cup S_3 \cup S_4 \\ \{\phi(v)\} & \text{otherwise} \end{cases}$$

Just like before, starting from the pre-colouring ϕ of $C \cup S_3 \cup S_4$, we iteratively update the lists of uncoloured vertices by removing a colour i from $L(v)$ if v has a neighbour u with $L(u) = \{i\}$. Any time during updating the lists if the list of some vertex becomes empty, it means that there is no 4-colouring of G that extends ϕ . We then reject ϕ right away (this also applies to **Step 1** and **Step 2** in the algorithm below). After updating the lists it follows from **(a)** that $|L(x)| = 1$ for each vertex $x \in S_5$. Thus, all vertices in $V \setminus (S_0 \cup S_1 \cup S_2)$ have been pre-coloured at this point. Moreover, $|L(x)| \leq 2$ for each vertex $x \in S_2$. It remains to list-colour vertices in $S_0 \cup S_1 \cup S_2$. It clearly suffices to explain how to individually list-colour each connected component of these vertices, since different components can be coloured independently of each other. In the following algorithm we refer to U as the set of currently uncoloured vertices, namely those whose lists contain more than one colour. Initially we have $U = S_0 \cup S_1 \cup S_2$.

Algorithm: whether or not ϕ extends to $S_0 \cup S_1 \cup S_2$.

Step 1 For each $0 \leq i \leq 4$, test if $|S_1(v_i)| \leq 3$, and if so, then pre-colour all vertices in $S_1(v_i)$ and update the lists of remaining vertices. Note that this changes the set U.

Step 2 If S_2 is not anti-complete to S_1, pick a vertex $w \in S_2$ that has a neighbour in S_1, pre-colour w and update the lists. This again changes the set U. Note that in this step we pre-colour at most one vertex.

Step 3 For each connected component K of U, do the following:

- **(i)** if every vertex $v \in K$ satisfies that $|L(v)| \leq 2$, Lemma 3.3.2 allows us to find a L-colouring of K or conclude that none exits;
- **(ii)** if some colour $i \in \{1, 2, 3, 4\}$ is absent from the lists of all vertices in K, Lemma 3.3.3 allows to find a L-colouring of K or conclude that none exists.

Step 4 If all required colourings in **Step 3** exist, return Yes; otherwise return No.

Note that the algorithm pre-colours at most additional 16 vertices. Branching on these vertices, i.e., considering all possible 4-colouring of these vertices, leads to a constant number of sub-problems. Furthermore, each subproblem amounts to calling the polynomial time algorithms in Lemma 3.3.2 and Lemma 3.3.3 in **Step 3 (i)** and **Step 3 (ii)** at most n times, for U has at most n connected components. Therefore, the algorithm runs in polynomial time. We are left to prove the correctness of the algorithm, i.e., to prove that for any connected component K of U, either **(i)** or **(ii)** applies in **Step 3**. To this end, we first discuss the vertices in S_0.

Claim A *The following properties hold for S_0.*

(A1) *S_0 is anti-complete to $S_1 \cup S_2$.*

(A2) *Any two vertices in S_0 that lie in the same connected component of S_0 have exactly the same neighbours in S_3.*

(A3) *Any vertex in S_0 is either complete or anti-complete to S_5.*

(A4) *If two vertices in S_5 have neighbours in a connected component A of S_0, A is complete to S_5.*

To prove (A1), take any vertex $y \in S_0$ and suppose that y is adjacent to some $x \in S_1 \cup S_2$. We assume without loss of generality that x is adjacent to v_0 but non-adjacent to v_2, v_3 and v_4. Now $y - x - v_0 - v_4 - v_3 - v_2$ induces a P_6 in G. This proves (A1).

To prove (A2), it suffices to show that for any edge of S_0, its ends have the same neighbours in S_3. Suppose that this is not the case, i.e., there exist an edge $e = xy$ of S_0 and a vertex $z \in S_3$ such that z distinguishes x and y, say z is adjacent to x but not to y. Moreover, we assume by symmetry that $z \in S_3(v_0)$. Now $y - x - z - v_1 - v_2 - v_3$ induces a P_6 in G. This proves (A2).

To prove (A3), let x be an arbitrary vertex in S_0. Suppose by contradiction that x is neither complete nor anti-complete to S_5. This means that there exist two vertices y and z in S_5 such that x distinguishes y and z, say x is adjacent to y but not to z. Now $\{v_0, v_2, x, y, z\}$ induces a banner in G. This proves (A3).

To prove (A4), let x and y be two vertices in S_5 such that both of them have a neighbour in some connected component A of S_0. It follows from (A3) that every vertex in A is either complete or anti-complete to S_5. We define $A' = \{a \in A : a \text{ is complete to } S_5\}$. Note that $A' \neq \emptyset$ due to (A3). Take any vertex $t \in A \setminus A'$. We may assume that t has a neighbour $t' \in A'$ due to the connectivity of A. Thus, $\{v_0, x, y, t, t'\}$ induces a banner, since t is anti-complete to S_5. This shows that the vertex t cannot exist. This prove (A4).

This completes the proof of Claim A. $\diamond$

Let K be an arbitrary connected component of U that contains a vertex from S_0. It follows from (A1) that $K \subseteq S_0$. If K is anti-complete to S_3, then there exist at least two vertices in S_5 that have a neighbour in K due to the assumption that G is 2-connected. Hence, K is complete to S_5 by (A4). Recall that all vertices in S_5 have been pre-coloured and have the same colour, say $i \in \{1, 2, 3, 4\}$, due to **(a)**. This colour does not appear on the list of any vertex of K because we already updated the lists. If K has a neighbour in $x \in S_3$, it follows from (A2) that $\phi(x)$ does not appear on the list of any vertex of K. In both cases, **Step 3 (ii)** applies to K. In the following we assume that $K \cap S_0 = \emptyset$. If $K \subseteq S_2$, then **Step 3 (i)** applies to

K. Thus, let K contain a vertex u from S_1. We assume that without loss of generality that $u \in S_1(v_0)$.

Claim B *For each $0 \leq i \leq 4$, $S_1(v_i)$ is complete to $S_1(v_{i+2})$ and anti-complete to $S_1(v_{i+1})$. Furthermore, if both $S_1(v_i)$ and $S_1(v_{i+2})$ are not empty, then $|S_1(v_i)| \leq 3$ and $|S_1(v_{i+2})| \leq 3$.*

> It suffices to prove Lemma 5.2.3 for $i = 0$. Let $x \in S_1(v_0)$. If x is non-adjacent to a vertex $z \in S_1(v_2)$, then $x - v_0 - v_4 - v_3 - v_2 - z$ is an induced P_6 in G. If x is adjacent to a vertex $y \in S_1(v_1)$, then $\{v_0, v_1, v_2, x, y\}$ induces a banner. This implies the first part of the claim. Suppose that $|S_1(v_0)| \geq 4$. Since G is K_5-free, $S_1(v_0)$ contains two non-adjacent vertices x and x'. But then $\{v_0, v_4, x, x', z\}$ (where $z \in S_1(v_2)$) induces a banner. $\diamond$

We now consider the case $K \subseteq S_1$. Note that the vertex u is not pre-coloured since u lies in U. This implies that K does not contain a vertex from $S_1(v_j)$ for $j = 2, 3$ by **Step 1** and Lemma 5.2.3. In addition, $S_1(v_0)$ is anti-complete to $S_1(v_1) \cup S_1(v_4)$ by Lemma 5.2.3. The connectivity of K then implies that $K \subseteq S_1(v_0)$. Consequently, the colour $\phi(v_0)$ is absent on the list of any vertex in K. In other words, **(ii)** applies to K in **Step 3**. Finally, we consider the case that K contains also a vertex $v \in S_2$. Since K is connected, the vertices u and v can be chosen to be adjacent. This means that the algorithm pre-coloured a vertex $w \in S_2$ in **Step 2**.

Claim C *For each $0 \leq i \leq 4$, $S_1(v_i)$ is anti-complete to $S_2(v_j)$ for each $j \neq i$; moreover, if for some i and j with $i \neq j$ both $S_1(v_i)$ and $S_1(v_j)$ are not empty, then $S_1(v_i)$ is also anti-complete to $S_2(v_i)$.*

> It suffices to prove Claim C for $i = 0$. Let $x \in S_1(v_0)$, $y \in S_2(v_1)$ and $z \in S_2(v_2)$. If x is adjacent to y, then $\{v_0, v_1, v_4, x, y\}$ induces a banner. If x is adjacent to z, then $v_1 - v_2 - v_3 - v_4 - z - x$ induces a P_6. This implies the first part of the claim by symmetry. For the second part, suppose that $S_1(v_1) \cup S_1(v_2)$ contains a vertex, say y'. If x has a neighbour $z' \in S_2(v_0)$, then $y' - v_1 - v_0 - x - z' - v_3$ induces a P_6 if $y' \in S_1(v_1)$, and $v_4 - v_0 - x - z' - v_2 - y'$ induces a P_6 in G if $y' \in S_1(v_2)$. This proves the second part of the claim. $\diamond$

Recall that $u \in S_1(v_0)$ is adjacent to v. It follows from Claim C that v lies in $S_2(v_0)$.

Claim D *$S_2(v_0)$ is a star.*

> We show that v is adjacent to any other vertex in $S_2(v_0)$. Suppose by contradiction that $S_2(v_0)$ contains a vertex y that is non-adjacent to v. Since $y - v_2 - v - u - v_0 - v_4$ does not induce a P_6, it follows that u and y are adjacent. This implies that $\{v_0, v_2, u, v, y\}$ induces a banner. Therefore, $S_2(v_0)$ is a star centered at v, since G is K_5-free. $\diamond$

Note that the list of any vertex in $S_2(v_0)$ is a subset of $\{1, 2, 3, 4\} \setminus \{\phi(v_2), \phi(v_3)\}$. It follows from Lemma 5.2.4 that pre-colouring any vertex in $S_2(v_0)$ forces the colour of all remaining vertices in $S_2(v_0)$ (after iteratively updating the lists). Since $v \in S_2(v_0) \cap U$ is not pre-coloured, we conclude that no vertex of $S_2(v_0)$ is pre-coloured in **Step 2**. This implies that the vertex w (pre-coloured in **Step 2**) is in $S_2(v_i)$ for some $i \neq 0$. The choice of w implies that w has a neighbour in S_1 and this neighbour must lie in $S_1(v_i)$ by Claim C. In particular, $S_1(v_i)$ is not empty, and therefore $S_1(v_0)$ is anti-complete to $S_2(v_0)$ by Claim C. This contradicts the fact that $u \in S_1(v_0)$ and $v \in S_2(v_0)$ are adjacent. Therefore, no connected component of U contains a vertex from both S_1 and S_2. We have shown that in **Step 3** of the algorithm, either **(i)** or **(ii)** applies to each connected component K of U. This proves the correctness of the algorithm. ■

4

k-Colourability of (C_s, P_t)-Free Graphs

Table 3.2 shows that k-Colourability of P_t-free graphs remains NP-complete for most values of k and t. In this chapter, we refine our NP-complete results on P_t-free graphs to the class $Forb(\{C_s, P_t\})$. The graphs in $Forb(\{C_s, P_t\})$ are called (C_s, P_t)-free graphs. The study of k-Colourability of (C_s, P_t)-free graphs dates back to 2001. In addition to demonstrating the NP-completeness of 5-Colourability of P_8-free graphs, Woeginger and Sgall **[106]** showed that every (C_3, P_5)-free graph is 3-colourable. Later on, Randerath and Schiermeyer **[79]** proved that 4-Colourability can be solved in polynomial time for (C_5, P_5)-free graphs. These results exist before the polynomial solvability of k-Colourability of P_5-free graph was shown (Theorem 3.1.3). When it comes to P_6-free graphs, we already mentioned that every (C_3, P_6)-free graph is 4-colourable. Lemma 3.3.3 thus completes the study of k-Colourability of (C_3, P_6)-free graph. Up to date, $t = 7$ is the largest integer such that forbidding triangles makes k-Colourability polynomial time solvable for P_t-free graphs **[25]**. As to forbidding cycles of length at least four, the only systematic result is Theorem 3.3.1.

In this chapter, we undertake a systematic study of the complexity of k-Colourability of (C_s, P_t)-free graphs. Throughout the chapter, we assume that (1) $t \geq s$: for otherwise the class of (C_s, P_t)-free graphs coincides with the class of P_t-free graphs; (2) $s \neq 4$: this is due to Theorem 3.3.1; (3) $t \geq 6$: this is due to **[63]**. The chapter is organized as follows. In Section 4.1, we investigate the problem when $s \geq 5$. The triangle-free case (namely $s = 3$) is studied in Section 4.2. Our results find that for most values of k, s and t, the problem remains NP-complete. In Section 4.3, a number of NP-complete results are shown in the context of List k-Colourability and k-Prext. We then provide a summary of the complexity of all three problems in Section 4.4.

4.1 Forbidding large cycles

In this section, we forbid cycles of length at least five. Unlike colouring P_t-free graphs, k-Colourability of (C_s, P_t)-free graphs does not enjoy the monotonicity with respect to s. As such, individual treatment for different values of s is needed.

4.1.1 The case $s \geq 6$

It turns out that our framework (Theorem 3.2.2) can be generalized to (C_s, P_t)-free graphs (Theorem 4.1.2 below). We follow the notation from Section 3.2. Specifically, suppose that I is a 3-Sat instance, H is a nice k-critical graph, and $G_{H,I}$ is the graph constructed the way we present in subsection 3.2.1.

Lemma 4.1.1 *Let $s \geq 6$ be a fixed integer. If H is C_s-free, then $G_{H,I}$ is C_s-free.*

Proof. Assume that H is C_s-free. We prove the lemma by contradiction. Let $Q = C_s$ be an induced cycle in $G_{H,I}$ with $s \geq 6$. Due to our assumption that H is C_s-free and $s \geq 6$, the proof of Lemma 3.2.4 **(i)** and **(ii)** applies to Q: Q contains a vertex $u \in U$ and a vertex $x \in X \cup D$. We recall that U and $X \cup D$ are complete to each other. This implies that u and x are consecutive on Q (since Q is induced). Let x^+ and u^+ be the other neighbour of x and u on Q, respectively. Thus, x^+, x, u, u^+ are four consecutive vertices on Q. Furthermore, all vertices in $Q \setminus \{x^+, x, u, u^+\}$ are from C. Since $s \geq 6$, $Q \setminus \{x^+, x, u, u^+\}$ contains an edge. This contradicts the fact that C is an independent set. ■

Lemma 3.2.4 and Lemma 3.2.3 have the following consequence.

Theorem 4.1.2 *Let $s, t \geq 6$ and $k \geq 3$ be fixed integers. Then k-Colourability of (C_s, P_t)-free graphs remains NP-complete whenever there exists a (C_s, P_t)-free nice $(k-1)$-critical graph.*

We now use Theorem 4.1.2 to deduce our main results in this subsection.

Corollary 4.1.3 *For any fixed integers $s, t \geq 6$ and $k \geq 5$, k-Colourability of (C_s, P_t)-free graphs is NP-complete.*

Proof. For fixed $s, t \geq 6$, it suffices to prove the corollary for $k = 5$. The longest induced cycle in H^* (Figure 3.1) has lenght five and thus H^* is C_s-free. Hence, H^* is a nice 4-critical (C_s, P_t)-free graph, for any fixed $s, t \geq 6$. The corollary then follows from Theorem 4.1.2. ■

Corollary 4.1.4 *For fixed $s, t \geq 6$, 4-Colourability of (C_s, P_t)-free graphs is NP-complete if*

(1) $s \geq 6$ but $s \neq 7$, and $t \geq 7$; or

(2) $s \geq 6$ but $s \neq 9$, and $t \geq 9$.

Proof. It suffices to demonstrate the existence of a nice 3-critical (C_s, P_t)-free graph in each case: C_7 is a nice 3-critical (C_s, P_t)-free graph when $t \geq 7$ and $s \geq 6$ but $s \neq 7$, and C_9 is a nice 3-critical (C_s, P_t)-free graph when $t \geq 9$ and $s \geq 6$ but $s \neq 9$. ■

4.1.2 The case $s = 5$

The main result in this subsection is the following.

Theorem 4.1.5 4-COLOURABILITY *of (C_5, P_7)-free graphs is NP-complete.*

Construction

To prove Theorem 4.1.5, we offer a reduction from NOT-ALL-EQUAL 3-SAT. Suppose that I is an instance of NOT-ALL-EQUAL 3-SAT with variable set $X = \{x_1, x_2, \ldots, x_n\}$ and clause set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$. We construct a graph G_I out of I as follows.

• For each $1 \leq i \leq n$, introduce a single vertex x_i. These vertices are said to be *X-type*.

• For each$1 \leq i \leq n$, introduce an induced path $F_i = d_i - e'_i - e_i - d'_i$ (in this order).

• For each clause $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$, introduce clause components H_j and H'_j that are isomorphic to a 7-hole so that clause components are pairwise disjoint. Pick three pairwise non-adjacent vertices in H_j, and respectively in H'_j, and denote them by c_{i_1j}, c_{i_2j} and c_{i_3j}, and by c'_{i_1j}, c'_{i_2j} and c'_{i_3j}, respectively. For each $1 \leq t \leq 3$, c_{i_tj} is said to be *C-type* while c'_{i_tj} is said to be *C'-type*. The remaining vertices in $H_j \cup H'_j$ are said to be *U-type*.

We also denote X, U, C and C' by the set of X-type, U-type, C-type and, respectively, C'-type vertices. In addition, we denote by F the union of all vertices in F_i. In other words, $V(G_I) = X \cup U \cup C \cup C' \cup F$. We now describe the edges of G_I.

• Add an edge between each vertex in U and each vertex in $X \cup F$.

• For each C-type vertex c_{ij}, add an edge between c_{ij} and x_i, and between c_{ij} and d_i.

• For each C'-type vertex c'_{ij}, add an edge between c'_{ij} and x_i, and between c'_{ij} and d'_i.

This completes the construction of G_I. We shall show that the colourability of G_I encodes the satisfiability of I. Furthermore, G_I is (C_5, P_7)-free.

Proof of Theorem 4.1.5

Lemma 4.1.6 *I is satisfiable if and only if G_I is 4-colourable.*

Proof. We assume that I is satisfiable. Let σ be a truth assignment satisfying each clause C_j, i.e., C_j contains one true literal and one false literal under σ. We define a mapping $\phi : V(G_I) \to \{1,2,3,4\}$ as follows.

- For each $1 \le i \le n$, set $\phi(x_i) := 3$ if $\sigma(x_i) = TRUE$; otherwise set $\phi(x_i) := 4$.

- For each $1 \le i \le n$, set $\phi(d_i) := \phi(e'_i) := 3$, and set $\phi(d'_i) := \phi(e_i) := 4$.

- Let $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ be the jth clause. Since σ satisfies C_j, it contains a true as well as a false literal. We assume without loss of generality that $\sigma(x_{i_1}) = TRUE$ and $\sigma(x_{i_2}) = FALSE$. Thus, both x_{i_1} and d_{i_1} are assigned colour 3 under ϕ. We set $\phi(c_{i_1 j}) := 4$ so that the colour of $c_{i_1 j}$ conflicts with neither x_{i_1} nor d_{i_1}. Since H_j is 3-critical, there exists a 2-colouring $\phi_j : V(H_j - c_{i_1 j}) \to \{1,2\}$ of $H_j - c_{i_1 j}$. By the same token, we set $\phi(c'_{i_2 j}) := 3$ so that the colour of $c'_{i_2 j}$ is different from $\phi(x_{i_2})$ $(= \phi(d'_{i_2}) = 4)$. Moreover, there exists a 2-colouring $\phi'_j : V(H'_j - c'_{i_2 j}) \to \{1,2\}$ of $H'_j - c'_{i_2 j}$.

- For each $1 \le j \le m$, set $\phi := \phi_j$ on $V(H_j - c_{i_1 j})$, and set $\phi := \phi'_j$ on $V(H'_j - c'_{i_2 j})$.

This completes the definition of ϕ and it can be readily checked that ϕ is indeed a mapping from $V(G_I)$ to $\{1,2,3,4\}$. Note that all vertices in $X \cup F$ are assigned colours either 3 or 4 while all vertices in U are assigned colours 1 or 2. This fact, together with the definition of ϕ, implies that ϕ is a 4-colouring of G_I.

Conversely, suppose that G_I is 4-colourable. Let $\phi : V(G_I) \to \{1,2,3,4\}$ be a 4-colouring of G_I. For each $1 \le j \le m$, H_j (as well as H'_j) contains two U-type vertices that are adjacent. We assume that two such vertices in H_1 are coloured with colour 1 and 2 under ϕ, respectively. Since U is complete to $X \cup F$, the following properties of ϕ hold.

(P1) For each $1 \le i \le n$, $\phi(x_i) \in \{3,4\}$.

(P2) For each $1 \le i \le n$, $\{\phi(d_i), \phi(d'_i)\} = \{3,4\}$.

(P3) For each $u \in U$, $\phi(u) \in \{1,2\}$.

We then construct a truth assignment $\sigma : \{x_1, x_2, \ldots, x_n\} \to \{TRUE, FALSE\}$ as follows.

$$\sigma(x_i) := \begin{cases} TRUE & \text{if } \phi(x_i) = \phi(d_i). \\ FALSE & \text{if } \phi(x_i) = \phi(d'_i). \end{cases}$$

It follows from **(P1)** and **(P2)** that σ is indeed a truth assignment. It remains to show that σ satisfies each clause in I. Suppose that some clause $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ is not satisfied under σ.

This means that $\sigma(x_{i_t}) = FALSE$ for $1 \le t \le 3$ or $\sigma(x_{i_t}) = TRUE$ for $1 \le t \le 3$. Suppose that we are in the first case. It then follows from our definition of σ that d_{i_t} receives a colour different from $\phi(x_{i_t})$ for each $1 \le t \le 3$. This, together with **(P1)** and **(P2)** , implies that $\phi(c_{i_t j}) \notin \{3, 4\}$ for $1 \le t \le 3$. Therefore, $\phi_j := \phi|_{V(H_j)}$ is a 2-colouring of H_j due to **(P3)**. This, however, contradicts the fact that $\chi(H_j) = 3$. The case that all three literals are $TRUE$ is similar. ■

Lemma 4.1.7 *G_I is (C_5, P_7)-free.*

Proof. We prove the lemma by contradiction. Let Q be an induced subgraph of G_I that is isomorphic to either a P_7 or a C_5. We show the following properties of Q.

(i) *Q contains at least one vertex in $X \cup F$.*

Note that $G - (X \cup F)$ is the disjoint union of $2m$ copies of 7-holes. If Q contains no vertex in $X \cup F$, then $V(Q) \subseteq V(H_j)$ or $V(Q) \subseteq V(H'_j)$ for some $1 \le j \le m$. This contradicts the fact that C_7 is (C_5, P_7)-free. This proves **(i)**. ◇

(ii) *Q contains at least one vertex in U.*

Let C_i and C'_i denote the set of C-type and C'-type vertices that are neighbours of d_i and d'_i, respectively. Observe that every connected component of $G - U$ has a specific structure, namely it is the result of substituting independent sets into a 7-hole (and possibly removing some vertices). Specifically, the seven independent sets are, in the cyclical order, $X_0 = \{x_i\}$, $X_1 = C_i$, $X_2 = \{d_i\}$, $X_3 = \{e'_i\}$, $X_4 = \{e_i\}$, $X_5 = \{d'_i\}$, and finally $X_6 = C'_i$. This subgraph of G_I does not contain an induced Q, since the 7-hole does not, and substituting independent sets cannot create a Q. This proves **(ii)**. ◇

By **(i)** and **(ii)**, Q contains at least one vertex $u \in U$ and one vertex $x \in X \cup F$. This implies that u and x are consecutive on Q. Following the proof of Lemma 3.2.4 **(iii)** (with P and D replaced by Q and F respectively) we conclude that Q contains at most three vertices in $X \cup F \cup U$ and these vertices are consecutive on Q. Since Q is isomorphic to either a C_5 or a P_7, $V(Q) \setminus (X \cup F \cup U)$ $(\subseteq C \cup C')$ contains an edge. This contradicts the fact that $C \cup C'$ is an independent set. ■

Since k-COLOURABILITY of (C_s, P_t)-free graphs is monotone with respect to both k and t, the following result follows.

Corollary 4.1.8 *For any fixed integers $k \ge 4$ and $t \ge 7$, k-COLOURABILITY of (C_5, P_t)-free graphs is NP-complete.*

We remark that Theorem 4.1.5 does not apply to P_6-free graphs. Chudnovsky, Maceli, Stacho, and Zhong **[24]** recently developed a polynomial time algorithm deciding whether or not a (C_5, P_6)-free graph is 4-colourable. However, k-COLOURABILITY of (C_5, P_6)-free graphs remains unknown when $k \geq 5$. For $k = 5$, we observe that the framework Theorem 4.1.2 is not sufficient to prove the NP-completeness.

Observation 4.1.9 *There exists no nice* 4*-critical* (C_5, P_6)*-free graphs.*

Proof. Suppose not. Let H be a nice 4-critical (C_5, P_6)-free graphs. Since $\chi(H) = 4$ and $\omega(H) = 3$, H is imperfect. By Theorem 2.3.3 and the fact H is (C_5, P_6)-free, we conclude that H contains an induced $\overline{C_{2s+1}}$ for some $s \geq 3$. Since H is 4-critical, $H = \overline{C_7}$. This contradicts the fact that $\alpha(\overline{C_7}) = 2$. ∎

4.2 Forbidding Triangles

In this section we investigate the case $s = 3$, namely forbidding triangles. The existing NP-hardness reductions for k-COLOURABILITY of (C_s, P_t)-free graphs, say Theorem 3.2.2 and Theorem 4.1.2 are based on the presence of triangles in the gadgets and therefore do not apply for (C_3, P_t)-free graphs. In order to make a NP-completeness proof for triangle-free graphs, the main task is to design a triangle-free gadget that can replace a number of edges of a graph G without long induced paths in such a way that the resulting graph becomes triangle-free, and at the same time keep the length of the longest induced path relatively small. The first NP-completeness result for (C_3, P_t)-free graphs was obtained this way. This is a result due to Golovach, Paulusma and Song.

Theorem 4.2.1 **[53]** 4-COLOURABILITY *is NP-complete for* (C_3, P_{164})*-free graphs*

The edge-replacement gadget designed in **[53]** is fairly sophisticated. Nevertheless, the gadget itself contains 'long' induced paths so that the replacement of it results in a large increase on the length of the longest induced path in the resulting graph. Our main result here is an improvement on Theorem 4.2.1 (see also **[69]**).

Theorem 4.2.2 4-COLOURABILITY *is NP-complete for* (C_3, P_{22})*-free graphs*

The proof of Theorem 4.2.2 follows the scheme we just described. The main technical contribution here is that we make use of the well-known Mycielski graphs. By carefully exploring colouring properties and induced paths in (nearly) Mycielski graphs we are able to bring the constant down from 164 to 22.

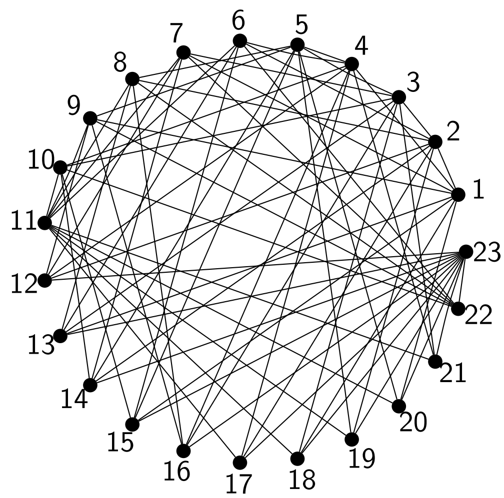


Figure 4.1: The Mycielski graph M_5. The vertex subsets $\{1, 2\}$, $\{i : 1 \le i \le 5\}$ and $\{i : 1 \le i \le 11\}$ induce M_2, M_3 and M_4, respectively.

4.2.1 Mycielski graphs

Definition (Mycielski construction) *The Mycielski construction of a graph G is created by adding, for each vertex v of G, a new vertex that is adjacent to each vertex in $N_G(v)$, and then adding a further vertex that is adjacent to each of the other new vertices.*

The Mycielski construction of K_2, for instance, is a 5-hole, and the Mycielski construction of a 5-hole is the well-known Grötzsch graph. These examples are the first ones in an infinite sequence of graphs $M_2, M_3, M_4 \ldots$ where $M_2 = K_2$ and M_r, $r \ge 3$, is the Mycielski construction of M_{r-1}. The graphs M_r ($r \ge 2$) are called *Mycielski graphs*. Mycielski **[90]** showed that each M_r is C_3-free and has chromatic number r. The proof of this fact may be found in **[8]**.

Theorem 4.2.3 [90] *For any fixed $r \ge 2$, M_r is C_3-free. Moreover, $\chi(M_r) = r$ and for any edge $e = uv \in E(M_r)$, $\chi(M_r - e) = r - 1$.*

Of particular importance to our proof is the graph M_5, see Figure 4.1. Since we make considerable use of this graph below, let us explain its construction carefully. For clarity, in the following text we denote an edge between two vertices u and v by $\{u, v\}$ instead of uv. Suppose that we start with M_3 where $V(M_3) = \{1, 2, 3, 4, 5\}$ and $E(M_3) = \{\{1, 2\}, \{2, 3\}, \{3, 5\}, \{5, 4\}, \{4, 1\}\}$. Then M_4 is obtained by adding each vertex i, $6 \le i \le 10$, and making it adjacent to the neighbours of vertex the vertex $i - 5$ in M_3 and to a further vertex 11. Finally, M_5 is obtained by adding a vertex i, $12 \le i \le 22$, making it adjacent to the neighbours of the vertex $i - 11$ in M_4 and to a further vertex 23.

We note that $N_{M_5}(17) = \{2, 4, 11, 23\}$. Let M' be the graph obtained from M_5 by removing the edge $\{17, 23\}$. It follows from Theorem 4.2.3 that M' is 4-colourable. The following simple observation is crucial to our NP-complete reduction.

Lemma 4.2.4 *Let $\phi : V(M') \to \{1, 2, 3, 4\}$ be any 4-colouring of M'. The following holds.*

(1) $\phi(17) = \phi(23)$;

(2) $\{\phi(2), \phi(4), \phi(11), \phi(17)\} = \{\phi(2), \phi(4), \phi(11), \phi(23)\} = \{1, 2, 3, 4\}$.

Proof. Since $\chi(M_5) = 5$, ϕ is not a proper colouring of M_5 and this implies the first item. We may assume that $\phi(17) = \phi(23) = 4$. Suppose that $\{\phi(2), \phi(4), \phi(11), \phi(17)\} \neq \{1, 2, 3, 4\}$. This means that some colour $i \in \{1, 2, 3\}$ is assigned to none of $\{2, 4, 11\}$ under ϕ. Therefore, we can define a new colouring $\phi' : V(M) \to \{1, 2, 3, 4\}$:

$$\phi'(j) := \begin{cases} i & \text{if } j = 17. \\ \phi(j) & \text{otherwise.} \end{cases}$$

Clearly, ϕ' is a proper 4-colouring of M_5 and this contradicts that $\chi(M_5) = 5$. Therefore, $\{\phi(2), \phi(4), \phi(11), \phi(17)\} = \{1, 2, 3, 4\}$. ■

We denote by T the subset $\{2, 4, 11, 23\}$ of $V(M')$. We shall use M' as our edge-replacement gadget. To this end, we need to bound the length of induced paths that have one end in T.

Lemma 4.2.5 *The following holds for paths in M'.*

(M1) *Every induced path connecting two vertices in T contains as most 7 vertices.*

(M2) *Every induced path with one end in T contains at most 8 vertices.*

(M3) *No disjoint union of two induced paths such that each of them has one end in T is isomorphic to a $P_8 + P_1$ or a $2P_7$.*

Proof. First let us make a few observations and definitions. Note that M' has many automorphisms. This means that, for instance, finding a path from 2 to 23 is equivalent to finding a path from 4 to 23 as there is an automorphism that maps 2 to 4 and fixes 23. In our proofs, we will often state that the number of cases can be reduced 'by symmetry' without explicitly describing any automorphisms. We introduce the notation $V_i^j = \{i, i+1, \ldots, j\}$ and also let ${}^+V_i^j$ and ${}^-V_i^j$ denote $V_i^j \cup \{17\}$ and $V_i^j \setminus \{17\}$, respectively. We note that V_1^{11} is isomorphic to M_4 and also note that M_4 is known to be P_6-free (checking this is an easy exercise). We now prove the three properties one by one.

(M1). Suppose by contradiction that $P = v_1 - \cdots - v_p$ is an induced path of M' with $p \geq 8$. We must consider the cases $(v_1, v_p) \in \{(2, 4), (2, 11), (2, 23), (11, 23)\}$ (the other cases are covered by symmetry). Here, and in later proofs, we use P_i^j to denote $\{v_i, v_{i+1}, \ldots, v_j\}$.

Case 1. $v_1 = 2$ and $v_p = 4$.

We can make much use of the symmetry of M' in this case: for example, if there is no such P with $v_2 = 8$, then we can also assume that $v_{p-1} \neq 10$. By considering which vertices neighbour v_1 and v_p but not both we find that $v_2 \in A = \{3, 8, 14, 19\}$ and $v_{p-1} \in B = \{5, 10, 16, 21\}$. We have immediately that $v_2 \neq 3$ as 3 is adjacent to every vertex in B, and, by symmetry, that $v_{p-1} \neq 5$.

If $v_2 = 8$, then $v_3 \in \{11, 13, 22\}$ since the other neighbours of 8 are in B. If $v_3 = 11$, then v_{p-1} is adjacent to v_2 or v_3, but this contradicts the assumption that P is induced. If $v_3 = 13$, then $v_4 = 23$ is the only possibility. This further implies that $v_5 \in C = \{15, 18, 20\}$, and v_6 is a neighbour of a vertex in C that is not adjacent to any vertex in $P_1^4 \cup \{v_p\} = \{2, 4, 8, 13, 23\}$ nor a member of either this set or B. It is easy to check that there is no such vertex. If $v_3 = 22$, then v_{p-1} must be 21, but then v_{p-2} has a neighbour in P_1^3. Thus $v_2 \neq 8$ and we can also conclude that $v_{p-1} \neq 10$. Hence, v_{p-1} is either 16 or 21.

If $v_2 = 14$, then $v_3 = 7$ since the other neighbours of 14 are adjacent to v_p or v_{p-1}. If $v_4 = 11$, then $v_{p-1} = 16$ and neither v_5 nor v_6 belongs to V_1^{11} as each of its vertices is in or adjacent to $P_1^4 \cup \{v_p\}$ or, in the case of 9, is adjacent to both v_4 and v_{p-1}. So, as v_5 and v_6 are in V_{12}^{23}, one of them must be 23 for otherwise they would not be adjacent. But 23 is a neighbour of v_2.

Therefore, we must have $v_2 = 19$ and $v_{p-1} = 21$. But this implies that v_3 is either 11 or 23 which are both also adjacent to 21.

Case 2. $v_1 = 2$ and $v_p = 11$.

If $p = 8$, then P_3^{p-2} must induce a path on at least four vertices. These vertices all belong to $A = \{4, 5, 13, 15, 16, 23\}$, the set of vertices adjacent to neither 2 nor 11. Note that $P_4^6 \subseteq A \setminus N_{M'}(v_2)$. We know that $v_2 \in \{1, 3, 12, 14\}$, and it is easy to check that, in each case, $A \setminus N_{M'}(v_2)$ does not contain a subset that induces a path on three vertices.

Case 3. $v_1 = 2$ and $v_p = 23$.

If P contains 8 vertices, then P_3^{p-2} must induce a path on at least four vertices. These vertices all belong to $\{4, 5, 7, 9, 10\}$. Since the vertex 7 is not adjacent to the other vertices, it is not on P which must therefore contain the P_4 induced on $\{4, 5, 9, 10\}$ with ends 9 and 10. Therefore v_{p-1} must be adjacent to 23, exactly one of 9 or 10 and neither of 4 and 5. It is easy to check that there is no such vertex.

Case 4. $v_1 = 11$ and $v_p = 23$.

We can see that $P_3^6 \subseteq V_1^5$ since every vertex in V_6^{23} is a neighbour of either 11 or 23. So, v_5 and v_6 are the two vertices of V_1^5 that are not neighbours of v_3. But v_2 is a neighbour of 11 but not 23 and thus belongs to ${}^+V_6^{10}$, and, as each of these vertices is adjacent to two non-adjacent

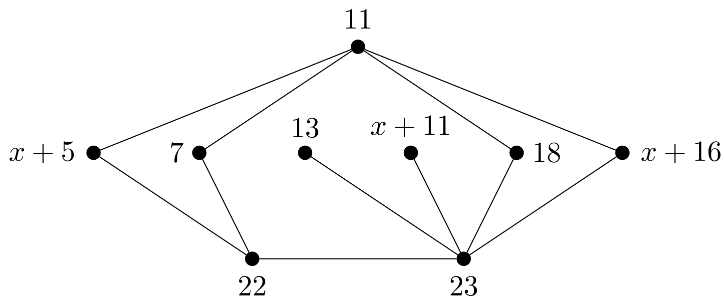


Figure 4.2: The graph C from Case 1 of the proof of (M2) where $x \in \{4, 5\}$.

vertices of V_1^5, it is adjacent to either v_5 or v_6 as well as v_3. This final contradiction completes the proof of (M1).

(M2). We will prove by contradiction. Let $P = v_1 - \cdots - v_9$ be an induced path of M' with $v_1 \in \{2, 11, 23\}$.

Case 1. $v_1 = 2$.

Let $A = \{4, 5\}$. If both vertices of A are on P, then $A = P_3^4$ since every neighbour of 2 is adjacent to a vertex in A. Then $V_6^9 \subset \{7, 11, 13, 18, 22, 23\} = B$, the set of vertices that do not neighbour 2, 4 or 5. Then $v_2 \in N_{M'}(2) = \{1, 3, 6, 8, 12, 14, 17, 19\}$ cannot be adjacent to more than two vertices in B. A quick check reveals that either $v_2 \in \{12, 14\}$ and $P_6^9 = \{11, 13, 18, 22\}$, or $v_2 \in \{17, 19\}$ and $P_6^9 = \{7, 13, 18, 22\}$, but neither of these sets induces a P_4.

If neither of the vertices in A is on P, then P_3^9 contains vertices of V_6^{23} that induce a P_7. If 23 is in P_3^9, then $P_3^9 \setminus^- V_{12}^{23}$ is a subset of ${}^+V_6^{11}$ and contains four vertices that induce one or two paths, a contradiction. So, P_3^9 contains only vertices of V_6^{22}. If 11 is in P_3^9, then $P_3^9 \setminus^- V_6^{11}$ is a subset of ${}^+V_{12}^{22}$ and contains four vertices that induce one or two paths, a contradiction. Thus, remembering now it contains no neighbour of 2, we have that P_3^9 contains only vertices in $\{7, 9, 10, 13, 15, 16, 18, 20, 21, 22\}$, and the graph induced on these vertices contains only three vertices of degree more than one and so is P_7-free.

Thus we know that P contains exactly one vertex of A. Let this vertex be x. We have that $x = v_i$ for $i \in \{3, \ldots, 9\}$. The vertices of P apart from 2, x and their neighbours form a subset of $\{7, x+5, 11, 13, x+11, 18, x+16, 22, 23\}$. The subgraph of M' that they induce is displayed in Figure 4.2; we will use C to denote both the vertex set and the graph.

If $i = 3$ or $i = 9$, then, respectively, v_4 or v_8 is a neighbour of x — and therefore also a neighbour either of 23 or of both 11 and 22 — and adjacent to the end of a P_5 that is a induced subgraph of C. But every induced P_5 in C contains 23 and either 11 or 22 as an internal vertex. If $i = 4$, then one of v_2 and v_3 is in V_6^{10} and the other is in V_{12}^{21}, and so neither 22 nor 23 is on P. This contradicts that P_6^9 is a subgraph of C.

So, $i \in \{5, 6, 7\}$ and P contains four vertices of C that induce either a $2P_2$ or a $P_1 + P_3$. In the former case, one P_2 contains 11 and the other contains 23, but each vertex in $D = \{v_2, v_{i-1}, v_{i+1}\}$ must be adjacent to either 11 or 23, a contradiction. In the following we assume that the vertices of C on P induce a $P_1 + P_3$. Then the P_3 can only include 23 if it is the vertex of degree two. Then $D \subset V_6^{10}$. Thus each vertex in D is adjacent to 11 which cannot therefore be on P. Thus, the P_1 is either 7 or $x + 5$ and therefore not adjacent to any vertex in D, a contradiction. If the P_1 is 23, then the P_3 has ends 7 and $x + 5$ which implies that each member of D is a neighbour of 23. So, we must have that 23 is not on P, and then the P_1 is either 13 or $x + 11$ and adjacent to at least one vertex of D which must be in V_6^{10}. But this implies it is also adjacent to both 11 and 22, one of which must also be on P.

Case 2. $v_1 = 11$.

Let $A = V_1^5 \cup V_{12}^{16} \cup \{23\}$ and note that this set induces a graph that is isomorphic to M_4 and so is P_6-free. The six vertices in P_4^9 are not neighbours of 11 and so belong to $A \cup \{22\}$, and must include 22 as they induce a P_6. But then no vertex in V_6^{10} is on P as each is adjacent to both 11 and 22. The only other neighbour of 22 is 23, but we must have $v_3 = 23$ as it is adjacent to v_2. Then $v_4 = 22$ and there is no vertex that can be v_5.

Case 3. $v_1 = 23$.

The only vertex of ${}^{-}V_{12}^{22}$ on P is v_2. Thus $P_3^9 \subset^{+} V_1^{11}$. Note that in the subgraph of M' induced on ${}^{+}V_1^{11}$, 6 and 17 have the same neighbours. This and the fact that M_4 is P_6-free imply that ${}^{+}V_1^{11}$ is also P_6-free. This contradicts the fact that P_3^9 has 7 vertices.

Thus, the proof of (M2) is complete.

(M3). We will prove by contradiction. Let $P = v_1 - \cdots - v_p$ and $Q = w_1 - \cdots - w_q$ be paths such that $P + Q$ is an induced subgraph of M' where $(p, q) \in \{(8, 1), (7, 7)\}$ and v_1 and w_1 belong to T. We can assume, by symmetry, that v_1 is in $\{2, 11, 23\}$.

Case 1. $v_1 = 2$.

Suppose that 23 is on Q. Then $P_3^p \subseteq \{4, 5, 7, 9, 10, 11\}$. The only induced P_5 on these vertices contains 9, 10 and 11 as internal vertices, but every neighbour of 2 is adjacent to one of these vertices.

Suppose instead that 23 is not on Q. Then 17 is in neither P nor Q since it is adjacent to v_1 and w_1. If 23 is on P, then no vertex of V_{12}^{22} is in Q. Thus Q is a subgraph of M_4, and so we must have $p = 8$ and $q = 1$. By (M1), 23 has two neighbours on P, say x and y.

If $w_1 = 4$, then only the nine vertices in the set $A = \{2, 3, 7, 8, 9, 11, 23, x, y\}$ can be on P, and as 3 and 8 are neighbours of 2 only one is on P (and is v_2). Thus, all the other vertices in A must be on P. As only two of 7, 8 and 9 can be on P (since all are neighbours of 11), we must have $v_2 = 3$. Then $v_3 = 7$, $v_4 = 11$ and $v_5 = 9$ and the remaining three vertices on P are x, 23 and y. So x (or, equivalently, y) is a common neighbour of 9 and 23 and so one of $\{12, 16, 22\}$, but these vertices are each adjacent to other vertices in the path.

If $w_1 = 11$, then the vertices of P must be in $V_1^5 \cup \{23, x, y\}$, but the graph they induce does not contain an induced path on 8 vertices.

Finally suppose that 23 is in neither P nor Q. If $w_1 = 4$, the vertices of V_1^{11} that might be on P are in $B = \{2, 3, 7, 8, 9, 11\}$. These vertices induce a 5-hole with a pendant edge. As P contains at least 7 vertices, if it contains r vertices of V_{12}^{22}, it must contain $7 - r$ vertices of B and these vertices must induce a graph having at least r components (as V_{12}^{22} is an independent set). It is easy to check that this is impossible for all values of r. If $w_1 = 11$, the vertices of V_1 that might be on P are in V_1^5 which induce a 5-hole and a similar argument can be used.

Case 2. $v_1 = 11$.

If 23 is on Q, then P_3^p is a subset of V_1^5. This implies $p < 7$.

If 23 is not on Q, we can assume that $w_1 = 2$. Suppose that 23 is on P. Then the vertices of Q must all be in V_1^{11} and so it has fewer than 6 vertices and $p = 8$ and $q = 1$. Let the neighbours of 23 on P be x and y. Note that at most one vertex of V_6^{11} is on P since they are neighbours of 11. The other vertices that might be on P are in $\{4, 5, 23, x, y\}$ and so P cannot have 8 vertices.

Finally suppose that 23 is in neither P nor Q. Thus the vertices of V_1^{11} that might be on P are 11, at most one of its neighbours, and the adjacent vertices 4 and 5. As these vertices induce a graph with at most two components, at most two vertices of V_{12}^{22} are on P and it cannot have more than 6 vertices.

Case 3. $v_1 = 23$.

The only vertex of $N_{M'}(23) =^- V_{12}^{22}$ on P is v_2, and 17 is not on P as it is adjacent to w_1. Thus $P_3^p \subset V_1^{11}$ and cannot induce a P_6. Therefore $p = q = 7$.

If $w_1 = 11$, then P contains no vertex in V_6^{10} which implies $P_3^7 = V_1^5$, a contradiction. So we can assume $w_1 = 2$. As Q has seven vertices it must contain a vertex not in V_1^{11} and this vertex must be 17 since 23 is on P. So, $w_2 = 17$ and w_3 is either 4 or 11. In either case the number of vertices in V_1^{11} that are not in or adjacent to Q is less than 5, contradicting that each vertex of P_3^7 must belong to this set.

This completes the proof of (M3) and thus the lemma. ■

4.2.2 Intermediate construction

To prove Theorem 4.2.2, we reduce Not-All-Equal 3-Sat to 4-Colourability of (C_3, P_{22})-free graphs. We shall construct a graph (denoted by J_I^* below) out of an instance I of Not-All-Equal 3-Sat. To describe our reduction we need a number of intermediate steps. Throughout this subsection, we assume that I is an instance of Not-All-Equal 3-Sat with variable set $X = \{x_1, x_2, \ldots, x_n\}$ and clause set $\mathcal{C} = \{D_1, D_2, \ldots, D_m\}$.

The graph J_I

In their paper **[52]**, Golovach, Paulusma and Song constructed, out of I, a graph J_I with an admissible list $L(v) \subseteq \{1,2,3,4\}$ for each vertex $v \in J_I$ as follows.

Construction: J_I.

- x-type vertices: for each variable x_i, J_I contains a vertex x_i with $L(x_i) = \{1,2\}$.
- a-type and b-type vertices: for each clause D_j, J_I contains two *clause components* D_j and D'_j each isomorphic to a P_5. Considered along the paths the vertices in D_j are $a_{j,1}$, $b_{j,1}$, $a_{j,2}$, $b_{j,2}$ and $a_{j,3}$ with lists of admissible colours $\{2,4\}$, $\{3,4\}$, $\{2,3,4\}$, $\{3,4\}$ and $\{2,3\}$, respectively, and the vertices in D'_j are $a'_{j,1}$, $b'_{j,1}$, $a'_{j,2}$, $b'_{j,2}$ and $a'_{j,3}$ with lists of admissible colours $\{1,4\}$, $\{3,4\}$, $\{1,3,4\}$, $\{3,4\}$ and $\{1,3\}$, respectively.

This completes the description of the vertices of J_I and their lists. We use X, A and B to denote the set of x-type, a-type and b-type vertices, respectively. We now describe the edges of J_I.

- Add an edge between each x-type vertex and each b-type vertex.
- For every clause D_j, its variables $x_{i_1}, x_{i_2}, x_{i_3}$ are ordered in an arbitrary (but fixed) way. For each $1 \leq h \leq 3$, add an edge between x_{i_h} and $a_{j,h}$, and between x_{i_h} and $a'_{j,h}$.

This completes the construction of J_I. See Figure 4.3 for an example of the graph J_I. In this figure, D_j is a clause with ordered variables $x_{i_1}, x_{i_2}, x_{i_3}$. The thick edges indicate the connection between these vertices and the a-type vertices of the two copies of the clause component. Indices of vertices in D_j and D'_j have been omitted to aid clarity.

The graphs J'_I

Definition (The graph J'_I) *Subdividing every edge between an a-type vertex and a x-type vertex in J_I results in a new graph J'_I. Every newly added vertex is assigned list $\{1,2\}$ and is said to*

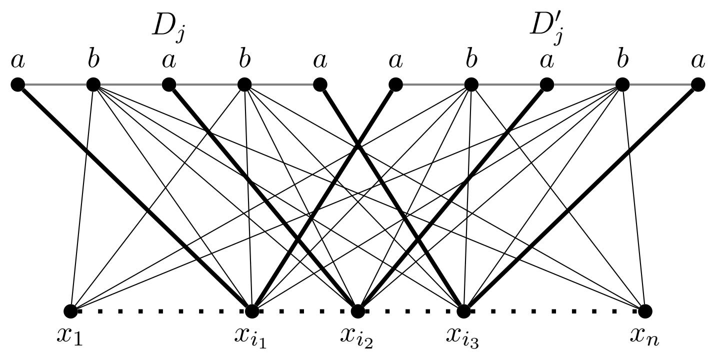


Figure 4.3: An example of a graph J_I as shown in **[69]**. Only the clause $D_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ is displayed.

be of c-type. The set of c-type vertices is denoted by C. The 4-list assignment of vertices in J'_I is denote by L'. Thus, (J'_I, L') is an instance of List 4-Colourability.

Note that each c-type vertex has exactly two neighbours in J'_I, one a-type and the other x-type.

Lemma 4.2.6 *I is satisfiable if and only if J'_I has a 4-colouring that respects L'.*

Proof. We assume that I is satisfiable. Let σ be a truth assignment satisfying each clause C_j, i.e., C_j contains one true literal and one false literal under σ. We define a mapping $\phi : V(G_I) \to \{1, 2, 3, 4\}$ as follows.

- For each variable x_i, set $\phi(x_i) := 1$ if $\sigma(x_i) = TRUE$; otherwise set $\phi(x_i) := 2$.
- For each clause $D_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$, assume without loss of generality that $\sigma(x_{i_1}) = TRUE$, $\sigma(x_{i_2}) = FALSE$ and $\sigma(x_{i_3}) = FALSE$. Set $\phi(a_{j,1}) := 1$ $(= \phi(x_{i_1}))$ and alternatively assign colours 3 and 4 to $b_{j,1}$, $a_{j,2}$, $b_{j,2}$ and $a_{j,3}$ (in this order). Similarly, set $\phi(a'_{j,3}) := 2$ so that it is the same as $\phi(x_{i_3})$. Alternatively assign colours 3 and 4 to $a'_{j,1}$, $b'_{j,1}$, $a'_{j,2}$ and $b'_{j,2}$ (in this order).
- For each c-type vertex, either its two neighbours have the same colour $i \in \{1, 2\}$ under ϕ or the a-type neighbour of it has been assigned colour 3 or 4. In both case, it is possible to assign a colour from $\{1, 2\}$ to this vertex.

It can be readily checked that ϕ is a 4-colouring of J'_I such that $\phi(v) \in L'(v)$ for each $v \in J'_I$.

Conversely, suppose that J'_I has a 4-colouring ϕ that respects L'. We construct a truth assignment $\sigma : \{x_1, x_2, \ldots, x_n\} \to \{TRUE, FALSE\}$ as follows.

$$\sigma(x_i) := \begin{cases} TRUE & \text{if } \phi(x_i) = 1. \\ FALSE & \text{if } \phi(x_i) = 2. \end{cases}$$

Since the list of each x-type vertex is $\{1, 2\}$, σ is a truth assignment. Consider a clause $D_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ (in this order). Suppose that $\sigma(x_{i_t}) = FALSE$ for each $1 \leq t \leq 3$. This means that each x_{i_t} is coloured with colour 2 under ϕ. Since ϕ is a proper colouring, all their c-type neighbours have colour 1. This implies that all a-type vertices in D_j are not coloured with colour 1, and so $\phi(a_{j,1}) = 4$ and $\phi(a_{j,3}) = 3$. This contradicts the fact that in any 2-colouring of P_5, the two ends must be coloured alike. Therefore, not all literals in D_j are $FALSE$. Similarly, not all literals in D_j are $TRUE$. ■

Although J_I is not triangle-free in general, the graph J'_I is. We prove a stronger property of J'_I.

Lemma 4.2.7 *J'_I is a P_8-free chordal bipartite graph.*

Proof. Note that both $A \cup X$ and $B \cup C$ are independent sets in J'_I and they form a bipartition of $V(J'_I)$. Therefore, J'_I is bipartite. It remains to show that J'_I is (C_6, C_8, P_8)-free. We prove this by contradiction. Suppose that Q is an induced subgraph of J'_I that is isomorphic to a C_6, a C_8 or a P_8.

(i) *Q contains at least one vertex in X.*

Note that $J'_I - X$ has a special structure, namely it is obtained from $J_I - X$ (which is the disjoint union of $2m$ copies of P_5) by adding vertices pendent to the vertices in A. Clearly, this subgraph contains no cycle. In addition, the longest induced path in $J_I - X$ has five vertices and adding pendent vertices increases this number by 2. Thus, Q must be a path containing at most 7 vertices if it does not contain a vertex from X. This is a contradiction to our assumption. ◇

(ii) *Q contains at least one vertex in B.*

Note that $J'_I - B$ has a special structure, namely it is a disjoint union of subdivided stars. Therefore, it contains no cycles or paths with six or more vertices. ◇

By **(i)** and **(ii)**, Q contains a vertex $x \in X$ and a vertex $b \in B$. Since $\Delta(Q) \leq 2$, Q is induced and $X \cup B$ is a complete bipartite graph, Q contains at most three vertices from $B \cup X$ and there vertices form a sub-path of Q. This implies that $Q - (B \cup X)$ contains an edge as Q has at least 6 vertices. This is, however, impossible, since $A \cup X$ is independent. ■

The next lemma (which we need later in our proofs of Theorem 4.2.2) bounds the length of the longest induced path in certain subgraph of J'_I.

Lemma 4.2.8 *$J'_I - C$ is P_6-free.*

Proof. We prove the lemma by contradiction. Suppose that $P = P_t$ is an induced path such that $t \geq 6$.

(i) *P contains at least one vertex in X.*

Note that $J'_I - (C \cup X)$ is the disjoint union of $2m$ copies of the clause components each of which is isomorphic to an induced P_5. The claim thus follows from our assumption that $t \geq 6$. $\diamond$

(ii) *P contains at least one vertex in B.*

This follows from the fact that $A \cup X$ is an independent set in J'_I. $\diamond$

By **(i)** and **(ii)**, P contains at least one vertex $x \in X$ and one vertex $b \in B$. Similar to the argument in Lemma 4.2.7, we conclude that P contains at most three vertices from $X \cup B$ and these vertices form a sub-path of P. Consequently, $P - (X \cup B)$ $(\subseteq A)$ contains an edge due to the assumption that $t \geq 6$. This contradicts the fact A is an independent set. ■

The graph J_I^k

Definition (The graph J_I^k) *For fixed integer $k \geq 4$, the graph J_I^k is obtained by adding, for each vertex $u \in V(J'_I)$, $k - |L'(u)|$ pendant vertices to u and pre-colouring these vertices with different colours from $\{1, \ldots, k\} \setminus L'(u)$. Denote the set of pendent vertices added by W_k. Thus, J_I^k with the pre-colouring $\phi_{W_k} : W_k \to \{1, 2, \ldots, k\}$ is an instance of k-*PREXT.

It is clear from the definition of J_I^k that it admits a k-colouring that extends the pre-colouring ϕ_{W_k} if and only if J'_I has a 4-colouring that respects L'. Moreover, adding pendent vertices increases the length of the longest induced paths by 2. This fact, Lemma 4.2.6 and Lemma 4.2.7 immediately imply the following.

Lemma 4.2.9 *For any fixed integer $k \geq 4$, I is satisfiable if and only if J_I^k admits a k-colouring that extends the pre-colouring ϕ_{W_k}. Moreover, J_I^k is P_{10}-free and chordal bipartite.*

The next lemma says that certain induced subgraph of J_I^k does not contain long induced paths.

Lemma 4.2.10 *Every induced path in $J_I^4 - B$ has*

(i) *at most 7 vertices;*

(ii) *at most 6 vertices if it contains only one pendant vertex of $J_I^4 - B$;*

(iii) *at most 5 vertices if it contains no pendant vertex of $J_I^4 - B$.*

Proof. We observe that $J_I^4 - B$ is a forest each connected component of which has a special structure, namely it is obtained from a subdivided star by adding one or more pendent vertices to each vertex. Since the longest induced path in any subdivided star contains five vertices and adding pendent vertices increase this length by at most 2, the lemma follows. ■

4.2.3 The final graph J_I^*

Having defined the graphs J_I' and J_I^k, we are ready to describe the final graph J_I^* we work with. The graph J_I^* is obtained from J_I^4 with pre-colouring ϕ_{W_4} in the following way.

Construction: J_I^*.

- Remove all vertices that are pendant to vertices in $B \cup C$.
- Add a copy of M'. Denote $t_1 = 2$, $t_2 = 4$, $t_3 = 11$ and $t_4 = 23$
- Let S be the set of vertices pendant to vertices in $A \cup X$. For each $v \in S$ do as follows: if $\phi_{W_4}(v) = i$ then add an edge between v and t_j for each $j \in \{1, 2, 3, 4\} \setminus \{i\}$.
- Add an edge between every vertex in B and t_i for $i = 1, 2$.
- Add an edge between every vertex in C and t_i for $i = 3, 4$.

We say that the vertices that are pendent to vertices in $A \cup X$ are p-*type* and that the vertices in T are t-*type*. Note that $V(J_I^*) = V(J_I') \cup S \cup V(M')$. We remark that J_I^* is not P_{21}-free in general. In order to see this take a chordless path with vertices of type

$$a - c - x - c - a - p - t - p - a - c - x - c - a - p - t - p - a - c - x - c - a.$$

The path contains 21 vertices. Note that such a path merely uses two vertices of M' and this two vertices must be t_1 and t_2 (since t_3 and t_4 are adjacent to all c-type vertices). As such, trying to optimize bounds in Lemma 4.2.5 (which we believe is possible) does not help us with improving Theorem 4.2.2.

4.2.4 Proof of Theorem 4.2.2

Lemma 4.2.11 *I is satisfiable if and only if J_I^* admits a 4-colouring.*

Proof. It suffices to show that J_I^* admits a 4-colouring if and only if J_I^4 has a 4-colouring that is an extension of c_{W_4} due to Lemma 4.2.9.

Suppose first that J_I^4 has a 4-colouring ϕ that extends ϕ_{W_4}. Note that every vertex $v \in V(J_I^*) \setminus V(M')$ is assigned a colour $\phi(v)$ under ϕ. We now extend ϕ to $V(M')$: set $\phi(t_j) := j$ for each $1 \leq j \leq 4$. Moreover, by Lemma 4.2.4 there exists a 4-colouring ϕ' of M' such that $\phi'(t_j) = \phi(t_j)$. It is straightforward to verify that $\phi \cup \phi'$ is a 4-colouring of J_I^*.

Conversely, suppose that J_I^* has a 4-colouring ϕ. Recall that $t_1 = 2$, $t_2 = 4$, $t_3 = 11$ and $t_4 = 23$. By Lemma 4.2.4 we assume without loss of generality that $\phi(t_j) = j$ for each $1 \leq j \leq 4$. Due to our construction it follows that every vertex $u \in S$ (the set of vertices pendant to vertices of $A \cup X$) has the colour $\phi(u) = \phi_{W_4}(u)$, and that every vertex of B has a colour from $\{3, 4\}$, and every vertex of C has a colour from $\{1, 2\}$. Therefore, ϕ can be extended to a colouring of J_I^4 in such a way that any vertex v pendent to $B \cup C$ receives the colour $\phi_{W_4}(v)$. In other words, this colouring is a 4-colouring that extends ϕ_{W_4}. ■

It remains to show that J_I^* is (C_3, P_{22})-free.

Lemma 4.2.12 *J_I^* is C_3-free.*

Proof. Suppose by contradiction that Z is a triangle in J_I^*. We note first that for any vertex $m \in V(M') \setminus T$, $N_{J_I^*}(m) \subseteq V(M')$, and therefore $m \notin Z$ due to the fact that M' is C_3-free. This means that $Z \subseteq A \cup X \cup T \cup B \cup C \cup S$. This is impossible, since $A \cup X \cup T$ and $B \cup C \cup S$ are two independent sets. ■

Lemma 4.2.13 *J_I^* is P_{22}-free.*

Proof. Let P be an induced path in J_I^*. Let α be the number of vertices on P that are from $T = \{t_1, t_2, t_3, t_4\}$, i.e., $\alpha = |T \cap V(P)|$. We distinguish five cases according to the value of α.

Case 1. $\alpha = 0$.

Either $P \subseteq J_I^4$ in which case $|P| \leq 9$ by Lemma 4.2.9 or $P \subseteq M' - T$ in which case $|P| \leq |V(M') \setminus T| = 19$.

Case 2. $\alpha = 1$.

It follows that P can be written as $P = P_L - t_j - P_R$ for some $1 \leq j \leq 4$, where each of the sub-paths P_L and P_R is fully contained in either J_I^4 or M'. If both P_L and P_R are contained in M', then $P \subseteq M'$ in which case $|P| \leq |V(M') \setminus T \cup \{t_j\}| = 20$; if one of P_L and P_R is contained in M' and the other is contained in J_I^4, then $|P| \leq 8 + 9 = 17$ by Lemma 4.2.5 and Lemma 4.2.9; otherwise both sub-paths are contained in J_I^4 in which case $|P_L|, |P_R| \leq 9$ by Lemma 4.2.5, and therefore $|P| \leq 9 + 1 + 9 = 19$.

Case 3. $\alpha = 2$.

In other words, P contains two vertices from T, say t_i and t_j for some $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$. It follows that P can be written as $P = P_L - t_i - P_M - t_j - P_R$, where each of the sub-paths P_L, P_M and P_R is fully contained in either M' or J_I^4. We bound the length of P_L and P_R first.

Claim E *Each of the sub-paths P_L and P_R has at most 6 vertices.*

It suffices to prove the claim for P_L due to symmetry. Suppose that $P_L \subseteq M'$. This means that $P_L - t_i$ and t_j are two disjoint paths in M', and therefore $|P_L| \leq 6$ by

Lemma 4.2.5. It remains to consider the case $P_L \subseteq J_I^4$. Since t_1 and t_2, and t_3 and t_4 are in symmetric position, we have two subcases.

(i) $t_i = t_1$.

Let t_1^- denote the right end of P_L. Since t_1 is anti-complete to $A \cup X \cup C$, $t_1^- \in B \cup S$. Recall that in J_I^* each vertex in S is adjacent to three vertices in T and t_1 is complete to B. This implies that $P_L \cap (B \cup S) = \{t_1^-\}$, since P is induced. In other words, $P_L \setminus \{t_1^-\} \subseteq J_I^4 - B$. If $t_1^- \in B$, then $P_L \setminus \{t_1^-\}$ does not contain any vertex from S, and so $|P_L| \leq 5 + 1 = 6$ by Lemma 4.2.10; otherwise $t_1^- \in S$ and P_L contains exactly one vertex (namely t_1^-) in S, implying that $|P_L| \leq 6$ by Lemma 4.2.10.

(ii) $t_i = t_3$.

Let t_3^- denote the right end of P_L. Since t_3 is anti-complete to $A \cup X \cup B$, $t_3^- \in C \cup S$. Recall that in J_I^* each vertex in S is adjacent to three vertices in T and t_3 is complete to C. This implies that $P_L \cap (C \cup S) = \{t_3^-\}$, since P is induced. In other words, $P_L \setminus \{t_3^-\} \subseteq J_I^4 - (C \cup S) = J_I' - C$. By Lemma 4.2.8, $P_L \setminus \{t_3^-\}$ contains at most five vertices, and this proves that $|P_L| \leq 6$.

This completes the proof of Claim E.

Claim F $|P_M| \leq 7$.

Suppose first that $P_M \subseteq M'$. This means that $t_i - P_M - t_j$ is a path in M'. By Lemma 4.2.5, it follows that $|P_M| \leq 5$. In the following, we assume that $P_M \subseteq J_I^4$. We have three cases to consider up to symmetry.

(i) $\{t_i, t_j\} = \{t_1, t_2\}$.

If P_M contains a vertex from B, then P_M is this b-type vertex, since $\{t_1, t_2\}$ is complete to B. Thus, $|P_M| = 1$. If P_M contains no vertex from B, then $P_M \subseteq J_I^4 - B$, and so $|P_M| \leq 7$ by Lemma 4.2.10.

(ii) $\{t_i, t_j\} = \{t_3, t_4\}$.

If P_M contains a vertex from C, then P_M is this c-type vertex, since $\{t_3, t_4\}$ is complete to C. Thus, $|P_M| = 1$. If P_M contains no vertex from C, then both ends of P_M are from S and all internal vertices of P_M are contained in $J_I' - C$. By Lemma 4.2.8, it follows that $|P_M| \leq 1 + 5 + 1 = 7$.

(iii) $\{t_i, t_j\} = \{t_1, t_3\}$. Without loss of generality, assume that $P = P_L - t_1 - P_M - t_3 - P_R$.

Let $P_M = t_1^+ - \ldots - t_3^-$. As before, $P_M \cap B \subseteq \{t_1^+\}$ and $P_M \cap C \subseteq \{t_3^-\}$. If $t_1^+ \notin B$, then $P_M \subseteq J_I^4 - B$, and so $|P_M| \leq 7$ by Lemma 4.2.10. Therefore, we assume that $t_1^+ \in B$. Note that $t_3^- \in C \cup S$ and in particular $t_3^- \neq t_1^+$. In addition, $P_M \cap (C \cup S) \subseteq \{t_3^-\}$. This implies that no internal vertex of P_M is from $C \cup S$, i.e., $P_M \setminus \{t_1^+, t_3^-\} \subseteq J_I^4 - (C \cup S) = J_I' - C$. It then follows from Lemma 4.2.8 that $|P_M| \leq 1 + 5 + 1 = 7$.

This completes the proof of Claim F. Finally, we conclude from Claim E and Claim F that

$$|P| = |P_L| + 1 + |P_M| + 1 + |P_R| \leq 6 + 1 + 7 + 1 + 6 = 21.$$

Case 4. $\alpha = 3$.

We write P as $P = P_L - t_h - P_M^1 - t_i - P_M^2 - t_j - P_R$ for some $h, i, j \in \{1, 2, 3, 4\}$, where each of the sub-paths P_L, P_M^1, P_M^2 and P_R is fully contained in either M' or J_I^4. Since P is induced and each vertex in S is adjacent to three vertices in T, we conclude that, for each $i = 1, 2$, $|P_M^i| = 1$ if P_M^i contains a vertex from S. In addition, $|P_L| \leq 6$ and $|P_R| \leq 6$ by the same arguments in **Case 3.** By the aforementioned symmetry between vertices and vertex pairs of $\{t_1, t_2, t_3, t_4\}$, we assume without loss of generality that $t_1, t_3, t_4 \in P$. We consider the following subcases according to the relative positions among these three vertices on P.

Case 4.1 $P = P_L - t_1 - P_M^1 - t_3 - P_M^2 - t_4 - P_R$.

We first prove the following claim.

Claim G *For each $1 \leq i \leq 2$, P_M^i is either contained in M' or consists of a single vertex.*

For $i = 1$, suppose that P_M^1 is not contained in M'. In other words, $P_M^1 \subseteq J_I^4$. We note that the right end of P_M^1 cannot be in C due to the presence of t_4, and hence it is in S, since t_3 is anti-complete to $A \cup X \cup B$. This implies that $|P_M^1| = 1$.

For $i = 2$, suppose that P_M^2 is not contained in M'. Note that P_M^2 contains a vertex from $C \cup S$. If P_M^2 contains a vertex from C, then P_M^2 is this c-type vertex, since it is adjacent to t_3 and t_4 and P is induced. Otherwise, P_M^2 contains a vertex from S and thus the claim follows from our earlier observation. This completes the proof of Claim G $\diamond$

If P_M^1 and P_M^2 are contained in M',

$$|P| = |P_L| + |t_1 - P_M^1 - t_3 - P_M^2 - t_4| + |P_R| \leq 6 + 7 + 6 = 19,$$

by Lemma 4.2.5 and the fact that $|P_L| \leq 6$ and $|P_R| \leq 6$. Therefore, one of P_M^1 and P_M^2 is not contained in M'. By symmetry, assume that $P_M^1 \subseteq J_I^4$. So, $|P_M^1| = 1$ due to Claim G. Moreover, again by Claim G, P_M^2 is either contained in M' or consists of a single vertex. In the former case, the sub-path $t_3 - P_M^2 - t_4$ of P contains at most 7 vertices by Lemma 4.2.5, and this implies that $|P| \leq 6+1+1+7+6 = 21$. In the latter case, $|P| = |P_L|+|t_1 - P_M^1 - t_3 - P_M^2 - t_4|+|P_R| \leq 17$. This completes the proof of **Case 4.1**.

Case 4.2 $P = P_L - t_3 - P_M^1 - t_1 - P_M^2 - t_4 - P_R$.

Since C is complete to $\{t_3, t_4\}$ and P is induced, P contains no vertices from C. Recall that for each $i = 1, 2$, $|P_M^i| = 1$ if P_M^i contains a vertex from S. These two facts imply that Claim G still holds, since $\{t_3, t_4\}$ is anti-complete to $A \cup X \cup B$. Repeating the argument in **Case 4.1** we find that $|P| \leq 21$.

Case 5. $\alpha = 4$.

In other words, all vertices t_j are on P. We write P as $P = P_L - t_h - P_M^1 - t_i - P_M^2 - t_j - P_M^3 - t_k - P_R$ for $\{h, i, j, k\} = \{1, 2, 3, 4\}$, where each of the sub-paths P_L, P_M^1, P_M^2, P_M^3 and P_R is fully contained in either M' or J_I^4. Observe that P does not contain any vertex from S, since any such vertex has three neighbours (namely those from T) on P. We consider the following subcases up to symmetry.

Case 5.1 $P = P_L - t_1 - P_M^1 - t_3 - P_M^2 - t_2 - P_M^3 - t_4 - P_R$.

Since B is complete to $\{t_1, t_2\}$ and C is complete to $\{t_3, t_4\}$, P contains no vertex from $B \cup C$, for otherwise such a vertex makes P not induced. Recall that $A \cup X$ is anti-complete to $V(M')$. Therefore, $P \subseteq M'$ and so $|P| \leq 21$.

Case 5.2 $P = P_L - t_1 - P_M^1 - t_3 - P_M^2 - t_4 - P_M^3 - t_2 - P_R$.

Since B is complete to $\{t_1, t_2\}$, P contains no vertex from B. By the same token, all sub-paths of P other than P_M^2 are contained in M' and moreover $|P_M^2| = 1$ in case $P \cap C \neq \emptyset$. This implies that $P_L - t_1 - P_M^1 - t_3$ and $t_4 - P_M^3 - t_2 - P_R$ are paths in M', and so each contains at most 8 vertices by Lemma 4.2.5. If $P \cap C = \emptyset$, then $P \subseteq M'$ and so $|P| \leq 21$. Otherwise $P \cap C = P_M^2$ and so $|P| \leq 8 + 1 + 8 = 17$.

Case 5.3 $P = P_L - t_1 - P_M^1 - t_2 - P_M^2 - t_3 - P_M^3 - t_4 - P_R$.

Since P is induced and contains no vertex from S, we conclude that P_L, P_M^2 and P_R are contained in M', and moreover each of P_M^1 and P_M^3 is either a single vertex (in $B \cup C$) or contained in M'. If both P_M^1 and P_M^3 are contained in M', then $P \subseteq M'$ and so $|P| \leq 21$; if exactly one of P_M^1 and P_M^3, say P_M^3, is contained in

M', then $|P| = |P_L - t_1| + |P_M^1| + |t_2 - P_M^2 - t_3 - P_M^3 - t_4 - P_R| \leq 8+1+8 = 17$ by Lemma 4.2.5; otherwise $|P_M^1| = |P_M^3| = 1$. Since P is induced, $P_L - t_1$, $t_2 - P_M^2$, and $t_4 - P_R$ are pairwise disjoint paths in M. By Lemma 4.2.5, the sum of the order of any two such paths is at most 13, and so $|P| \leq 3 \times 13/2 + 2 = 21.5$, implying that $|P| \leq 21$.

We have considered all the cases and this completes the proof. ■

Theorem 4.2.2 follows then from the above three lemmas. We now prove a final result for (C_3, P_t)-free graphs. This result extends a result of Golovach, Paulusma and Song **[53]** who proved that for all $s \geq 6$, there exists a constant t^s such that 4-COLOURABILITY is NP-complete for $(C_5, \ldots, C_{s-1}, P_{t^s})$-free graphs. Recall that Theorem 3.3.1 says that forbidding C_4 makes k-COLOURABILITY polynomial time solvable for P_t-free graphs. Therefore, we must exclude C_4 from the list of forbidden induced subgraphs in order to get a NP-complete result. In this sense, our new hardness result can be seen as best possible.

Theorem 4.2.14 *For all fixed integers $k \geq 4$ and $s \geq 6$, there exists a constant t_k^s such that k-COLOURABILITY is NP-complete for $(C_3, C_5, \ldots, C_{s-1}, P_{t_k^s})$-free graphs.*

Proof. Let $k \geq 4$ and $s \geq 6$ be fixed integers. By Theorem 2.3.2, there exists an edge-minimal graph F with $\chi(F) = k+1$ and $g(F) = s$. Pick any edge $e = pq \in E(F)$. By the definition of F, $F - e$ is k-colourable. Moreover, in every k-colouring of $F - e$, p and q must receive the same colour. Let F' be the graph obtained from F by adding a new vertex q^* and adding an edge between $q*$ and q. In other words, q^* is a vertex pendent to q in F'. It follows from the colouring property of F that p and q^* must receive a different colour in every k-colouring of F'. An F'*-identification* of an edge uv in a graph G is the following operation: delete the edge uv and add a copy of F' between u and v by identifying vertices u and v with p and q^*, respectively (we call these two new vertices u and v again). We now construct a graph from J_I^k as follows.

- Take a complete graph on k new vertices $r_1, \ldots, r_k$. Recall that there is a pre-colouring ϕ_{W_k} on the subset $W_k \subseteq V(J_I^k)$. We add an edge between a vertex r_i and a vertex $u \in W_k$ if and only if $\phi_{W_k}(u) \neq i$, for each $1 \leq i \leq r$.

- Perform an F'-identification of every edge between two vertices r_i and r_j and of every edge between a vertex r_i and a vertex in W_k.

Let G_I^k be the resulting graph (without any pre-colouring). It is clear from our construction that G_I^k admits a k-colouring if and only if J_I^k has a 4-colouring that extends ϕ_{W_k}. It then follows from Lemma 4.2.9 that I is satisfiable if and only if G_I^k is k-colourable. We observe that G_I^k is not C_4-free. However, since J_I^k is chordal bipartite (by Lemma 4.2.9) and we performed appropriate F'-identifications, G_I^k is $(C_3, C_5, \ldots, C_{s-1})$-free. It remains to bound the length of the longest induce path in G_I^k. Take any induced path Q in G_I^k. Let $h = |V(Q)| \cap \{r_1, \ldots, r_k\}$. It can be seen that Q can be written as $Q = Q_1 - r_{i_1} - Q_2 - r_{i_2} - \cdots Q_h - r_{i_h} - Q_{h+1}$, where $V(Q_i)$ is fully contained in either an F'-copy or J_I^k for each $1 \le i \le h+1$. Since $|F'|$ is a constant that depends only on k and s, say $|F'| = \alpha_s^k$, and J_I^k is P_{10}-free (by Lemma 4.2.9), we find that there exists a constant $t_k^s \le \max\{|F'|, 10\} \cdot (h+1) + h \le \max\{\alpha_s^k, 10\} \cdot (k+1) + k$ such that Q has length at most t_k^s. In other words, G_I^k is $P_{t_k^s}$-free. Clearly, t_k^s depends solely on k and s and this completes the proof. ■

Corollary 4.2.15 *For any integer $k \ge 5$, there exists a constant t_k (depending only on k) such that k-*Colourability *is NP-complete for (C_3, P_{t_k})-free graphs.*

Proof. Take $t_k := t_k^6$ where t_k^6 is the constant from Theorem 4.2.14. ■

We remark that a slight modification of the construction used in the proof of Theorem 4.2.14 gives us a better upper bound for t_k. Instead of using an edge-minimal graph F with chromatic number $k+1$ and girth s, we take the Mycielski graph M_{k+1}. Following the proof of Theorem 4.2.14 we pick an edge pq of M_{k+1} and obtain a modified graph M'_{k+1}. Since $|V(M_k)| = 3 \cdot 2^{k-2} - 1$ for all $k \ge 2$, replacing α_k^s in the computation for t_k^s with $|V(M_{k+1})|$ gives that $t_k \le \max\{|V(M_{k+1})|, 10\} \cdot (h+1) + h \le (3 \cdot 2^{k-1} - 1)(k+1) + k$.

4.3 Two variants of k-Colourability

In this section, we use the graphs J_I, J'_I and J_I^k introduced in subsection 4.2.2 to prove some new NP-complete results on list colouring and pre-colouring extension problems for the class of (C_s, P_t)-free graphs.

4.3.1 List k-colourability

Theorem 4.3.1 List 4-Colourability *is NP-complete for $(C_5, C_6, P_6, K_4, \overline{P_1 + 2P_2}, \overline{P_1 + P_4})$-free graphs.*

Proof. It was shown that in **[52]** that J_I is P_6-free, and that it has 4-colouring that respects L if and only if I is satisfiable. Note that J_I is 3-colourable since it can be partitioned into three independent sets A, B and X. Thus, J_I is K_4-free. It remains to show that no induced subgraph H of J_I is isomorphic to a graph in $\mathcal{F} = \{C_5, C_6, \overline{P_1 + 2P_2}, \overline{P_1 + P_4}\}$. Suppose not.

Case 1. $H \in \{C_5, C_6\}$.

Note that both $J_I - X$ and $J_I - B$ are forests and hence contain no cycles. This implies that H contains at least one vertex from each of the sets X and B. Since X and B are complete to each other and H is C_4-free, it follows that H contains at most three vertices from $X \cup B$ and these vertices form an induced path in H. Thus, $H-(X \cup B)$ contains an edge, since H is either a C_5 or a C_6. This, however, contradicts the fact that the remaining vertices in H are from A which is an independent set.

Case 2. $H \in \{\overline{P_1 + 2P_2}, \overline{P_1 + P_4}\}$.

Let u be the vertex that has degree 4 in H. Since any a-type vertex has degree at most three in J_I, $u \in B \cup X$. Since u is universal in H, it follows that $H - u$ is completely contained in $J_I - X$ or $J_I - B$ depending on whether $u \in X$ or $u \in B$. As we noted before, both subgraphs $J_I - X$ and $J_I - B$ are forests. This immediately implies that H cannot be a $\overline{P_1 + 2P_2}$, for otherwise $H - u$ induces a C_4. Thus, $H = \overline{P_1 + P_4}$ and $H - u$ is an induced P_4. This implies that $u \in X$, for otherwise $u \in B$ and $H - u$ is contained in $J_I - B$ (which is a disjoint union of stars), but $J_I - B$ is P_4-free. Therefore, $H - u \subseteq J_I - X$ and this subgraph is the disjoint union of $2m$ copies of the clause components. Hence, $H - u$ is contained in some clause component, say D_j for some $1 \leq j \leq m$. As a result, two vertices in $H - u$ are from A. This is impossible, as each x-type vertex has at most one a-type neighbour in a clause component.

This completes our proof. ■

The next result follows immediately from Lemma 4.2.6 and Lemma 4.2.7.

Theorem 4.3.2 List 4-Colouring *is NP-complete for P_8-free chordal bipartite graphs (and hence for (C_3, P_8)-free graphs).*

4.3.2 Pre-colouring extension

The first result follows immediately from Lemma 4.2.9.

Theorem 4.3.3 *For all $k \geq 4$, k-*PrExt *is NP-complete for the class of P_{10}-free chordal bipartite graphs (and hence for (C_3, P_{10})-free graphs).*

Our second result is slightly more complicated.

Theorem 4.3.4 4-PrExt *is NP-complete for the class of $(C_5, C_6, C_7, C_8, P_8)$-free graphs.*

Proof. Recall that J_I is the graph with list assignment L as constructed at the start of subsection 4.2.2. We now add new vertices to J_I and pre-colour these vertices (the pre-colouring of any vertex in J_I is therefore removed).

- For each clause component D_j, add five new vertices, s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$, and pre-colour s_j, t_j, $u_{j,1}$, $u_{j,2}$, $u_{j,3}$ with colours 3, 4, 1, 1, 1, respectively.

• For each clause component D'_j, add five new vertices, s'_j, t'_j, $u'_{j,1}$, $u'_{j,2}$, $u'_{j,3}$, and pre-colour s'_j, t'_j, $u'_{j,1}$, $u'_{j,2}$, $u'_{j,3}$ with colours 3, 4, 2, 2, 2, respectively.

• Add edges $a_{j,1}s_j$, $a_{j,3}t_j$ and $a_{j,h}u_{j,h}$, and add edges $a'_{j,1}s'_j$, $a'_{j,3}t'_j$ and $a'_{j,h}u'_{j,h}$ for each $h = 1, 2, 3$.

• Add two new vertices c_1, c_2 and for each $1 = 1, 2$ and each x-type vertex u, add an edge between c_i and u; similarly, add two new vertices y_1, y_2 and for each $1 = 1, 2$ and each c-type vertex u, add an edge between y_i and u.

• Pre-colour c_1, c_2, y_1, y_2 with colours 3, 4, 1, 2, respectively.

This results in a new graph J''_I. It is desirable to view y_1 and y_2 as x-type vertices, and similarly view c_1 and c_2 as b-type vertices. Except c_1, c_2, y_1, y_2 , any other new vertex is a pendant vertex in J''_I. Since J_I is (C_5, C_6, P_6)-free (by Theorem 4.3.1), we find that J''_I is $(C_5, C_6, C_7, C_8, P_8)$-free. Moreover, the pre-colouring in J''_I forces the list $L(v)$ on every vertex v of J_I. This means that J''_I admits a 4-colouring extending this pre-colouring if and only if J_I has a 4-colouring that respects L. This completes our proof by a result in **[52]** that J_I has 4-colouring that respects L if and only if I is satisfiable. ■

Here is the final result in this subsection.

Theorem 4.3.5 *For all $k \geq 5$, k-*Prext *is NP-complete for (C_5, C_6, P_6)-free graphs.*

Proof. In **[14]** the authors constructed a graph G_I with some vertices being pre-coloured (out of I) in such a way that G_I is P_6-free and G_I has 5-colouring that extends the pre-colouring if and only if I is satisfiable. To prove the theorem, it suffices to show that G_I is (C_5, C_6)-free. We refer to the concrete construction of G_I in **[14]**. Here we sketch the structure of G_I that are helpful for our purpose. $V(G_I)$ is the union of 8 subsets, A, B, C, X, P_1, $\overline{P_1}$, Q_1 and $\overline{Q_1}$. We let $V_1 = C \cup P_1 \cup Q_1$, $V_2 = X \cup \overline{P_1} \cup \overline{Q_1}$ and $V_3 = A \cup B$. It can be seen from the construction that each V_i is an independent set and $V_1 \cup V_2$ induces a complete bipartite graph. Suppose that Q is an induced subgraph of G_I that is isomorphic to a C_5 or a C_6. The graph G_I has the property that neither of $G_I - V_1$ and $G_I - V_2$ contains a C_5 or a C_6. As a result, Q contains a vertex from both V_1 and V_2. This implies that Q contains at most three vertices from $V_1 \cup V_2$, for otherwise Q would contain an induced C_4 (since $V_1 \cup V_2$ induces a complete bipartite graph). Moreover, these vertices form an induced path on Q since Q is an induced cycle. So, $Q - (V_1 \cup V_2)$ contains an edge. However, this is impossible as all remaining vertices on Q are from V_3 which contains no edge. ■

4.4 A summary

In this section we give a summary of the complexity of k-Colourability, k-Prext and List k-Colourability for (C_s, P_t)-free graphs. Before that we need the follow result due to Gravier, Hoàng and Maffray.

Theorem 4.4.1 [56] *For fixed integers $s, t \geq 1$, every (K_s, P_t)-free graph G has $\chi(G) \leq (t-2)^{s-2}$.*

Here is our summary.

Theorem 4.4.2 *Let k, s and t be three (fixed) positive integers. The following statements hold for (C_s, P_t)-free graphs.*

i List k-Colourability *is NP-complete if*

- ***(i.1)*** $k \geq 4$, $s = 3$ *and* $t \geq 8$.
- ***(i.2)*** $k \geq 4$, $s \geq 5$ *and* $t \geq 6$.

List k-Colourability *is polynomial time solvable if*

- ***(i.3)*** $k \leq 2$, $s \geq 3$ *and* $t \geq 1$.
- ***(i.4)*** $k = 3$, $s = 3$ *and* $t \leq 7$.
- ***(i.5)*** $k = 3$, $s = 4$ *and* $t \geq 1$.
- ***(i.6)*** $k = 3$, $s \geq 5$ *and* $t \leq 6$.
- ***(i.7)*** $k \geq 4$, $s = 3$ *and* $t \leq 6$.
- ***(i.8)*** $k \geq 4$, $s = 4$ *and* $t \geq 1$.
- ***(i.9)*** $k \geq 4$, $s \geq 5$ *and* $t \leq 5$.

ii k-Prext *is NP-complete if*

- ***(ii.1)*** $k = 4$, $s = 3$ *and* $t \geq 10$.
- ***(ii.2)*** $k = 4$, $s = 5$ *and* $t \geq 7$.
- ***(ii.3)*** $k = 4$, $s = 6$ *and* $t \geq 7$.
- ***(ii.4)*** $k = 4$, $s = 7$ *and* $t \geq 8$.
- ***(ii.5)*** $k = 4$, $s \geq 8$ *and* $t \geq 7$.
- ***(ii.6)*** $k \geq 5$, $s = 3$ *and* $t \geq 10$.
- ***(ii.7)*** $k \geq 5$, $s \geq 5$ *and* $t \geq 6$.

k-Prext *is polynomial time solvable if*

- ***(ii.8)*** $k \leq 2$, $s \geq 3$ *and* $t \geq 1$.
- ***(ii.9)*** $k = 3$, $s = 3$ *and* $t \leq 6$.
- ***(ii.10)*** $k = 3$, $s = 4$ *and* $t \geq 1$.
- ***(ii.11)*** $k = 3$, $s \geq 5$ *and* $t \leq 6$.
- ***(ii.12)*** $k \geq 4$, $s = 3$ *and* $t \leq 6$.

(ii.13) *$k \geq 4$, $s = 4$ and $t \geq 1$.*

(ii.14) *$k \geq 4$, $s \geq 5$ and $t \leq 5$.*

iii k-COLOURABILITY is NP-complete if

(iii.1) *$k = 4$, $s = 3$ and $t \geq 22$.*

(iii.2) *$k = 4$, $s = 5$ and $t \geq 7$.*

(iii.3) *$k = 4$, $s = 6$ and $t \geq 7$.*

(iii.4) *$k = 4$, $s = 7$ and $t \geq 9$.*

(iii.5) *$k = 4$, $s \geq 8$ and $t \geq 7$.*

(iii.6) *$k \geq 5$, $s = 3$ and $t \geq t_k$ where t_k is a constant that only depends on k.*

(iii.7) *$k \geq 5$, $s = 5$ and $t \geq 7$.*

(iii.8) *$k \geq 5$, $s \geq 6$ and $t \geq 6$.*

k-COLOURABILITY is polynomial time solvable if

(iii.9) *$k \leq 2$, $s \geq 3$ and $t \geq 1$.*

(iii.10) *$k = 3$, $s = 3$ and $t \leq 7$.*

(iii.11) *$k = 3$, $s = 4$ and $t \geq 1$.*

(iii.12) *$k = 3$, $s \geq 5$ and $t \leq 7$.*

(iii.13) *$k = 4$, $s = 3$ and $t \leq 6$.*

(iii.14) *$k = 4$, $s = 4$ and $t \geq 1$.*

(iii.15) *$k = 4$, $s = 5$ and $t \leq 6$.*

(iii.16) *$k = 4$, $s \geq 6$ and $t \leq 5$.*

(iii.17) *$k \geq 5$, $s = 3$ and $t \leq k + 2$.*

(iii.18) *$k \geq 5$, $s = 4$ and $t \geq 1$.*

(iii.19) *$k \geq 5$, $s \geq 5$ and $t \leq 5$.*

Proof. We first consider the intractable cases of LIST k-COLOURABILITY. Note that **(i.1)** follows from Theorem 4.3.2, and Theorem 4.3.1 implies that LIST 4-COLOURABILITY is NP-complete for the class of (C_5, C_6, P_6)-free graphs which proves **(i.2)**. We now consider the tractable cases. Lemma 3.3.2 implies that LIST 2-COLOURABILITY is polynomial time solvable on general graphs implying **(i.3)**. Bonomo, Chudnovsky, Maceli, Schaudt, Stein and Zhong **[9]** showed that LIST 3-COLOURABILITY is polynomial time solvable for (C_3, P_7)-free graphs. This result and Lemma 3.3.3 imply **(i.4)** and **(i.6)**. Theorem 3.3.1 implies **(i.5)** and **(i.8)**. The class of (C_3, P_6)-free graphs was shown to have bounded clique-width by Brandstädt, Klembt and Mahfud **[12]**. This implies that for any $k \geq 1$, LIST k-COLOURABILITY is polynomial time solvable on (C_3, P_6)-free graphs **[37]**. This proves **(i.7)**. Finally, **(i.9)** follows from Theorem 3.1.3.

We now consider k-PREXT. All the tractable cases follow from the results of LIST k-COLOURABILITY. We are left to consider the NP-complete cases. Theorem 4.3.3 implies **(ii.1)** and **(ii.6)**, and

Theorem 4.3.4 and Theorem 4.3.5 imply **(ii.4)** and **(ii.7)**, respectively. Finally, **(iii.15)**, **(ii.3)** and **(ii.5)** follow immediately from Corollary 4.1.4 and Corollary 4.1.8.

We finally consider k-COLOURABILITY. Theorem 4.2.2 and Corollary 4.2.15 imply **(iii.1)** and **(iii.6)**, respectively. All the other NP-completeness subcases follow from our results in Section 4.1. For the polynomial cases, Chudnovsky, Maceli and Zhong **[25, 26]** proved that 3-COLOURABILITY is polynomial time solvable for P_7-free graphs, which gives us **(iii.10)** and **(iii.12)**. Chudnovsky, Maceli, Stacho and Zhong **[24]** proved that 4-COLOURABILITY is polynomial time solvable for (C_5, P_6)-free graphs, implying **(i.2)**. Theorem 4.4.1 gives us **(iii.17)**. All other tractable cases follow from the corresponding tractable cases in LIST k-COLOURABILITY. This completes the proof. ■

Theorem 4.4.2 leaves the following cases open in the classification of the complexity of graph colouring problems for (C_s, P_t)-free graphs (recall that t_k is a constant only depending on k).

(i) For LIST k-COLOURABILITY the following cases are open:
- $k = 3$, $s = 3$ and $t \geq 8$.
- $k = 3$, $s \geq 5$ and $t \geq 7$.
- $k \geq 4$, $s = 3$ and $t = 7$.

(ii) For k-PREXT the following cases are open:
- $k = 3$, $s = 3$ and $t \geq 7$.
- $k = 3$, $s \geq 5$ and $t \geq 7$.
- $k = 4$, $s = 3$ and $7 \leq t \leq 9$.
- $k = 4$, $s \geq 5$ and $t = 6$.
- $k = 4$, $s = 7$ and $t = 7$.
- $k \geq 5$, $s = 3$ and $7 \leq t \leq 9$.

(iii) For k-COLOURABILITY the following cases are open:
- $k = 3$, $s = 3$ and $t \geq 8$.
- $k = 3$, $s \geq 5$ and $t \geq 8$.
- $k = 4$, $s = 3$ and $7 \leq t \leq 21$.
- $k = 4$, $s \geq 6$ and $t = 6$.
- $k = 4$, $s = 7$ and $7 \leq t \leq 8$.
- $k \geq 5$, $s = 3$ and $k + 3 \leq t \leq t_k - 1$.
- $k \geq 5$, $s = 5$ and $t = 6$.

The class of (C_5, P_6)-free graphs enjoys an interesting behavior with respect to these colouring problems. Specifically, the complexity of LIST k-COLOURABILITY and k-COLOURABILITY are in sharp contrast: the former being NP-complete and the latter being polynomial time solvable, for $k = 4$. For this reason, we feel it worths a closer study.

Problem 4.4.3 *For $k \geq 5$, is there a polynomial time algorithm for k-*Colourability *of (C_5, P_6)-free graphs?*

Problem 4.4.4 *What is the complexity of* 4-Prext *of (C_5, P_6)-free graphs?*

From the list above these two problems are the only open problems left for (C_5, P_6)-free graphs. Another interesting class is the class of (C_3, P_7)-free graphs. The polynomial algorithm from **[9]** is quite involved and it seems difficult to generalize to $k \geq 4$. The class of bipartite P_7-free graphs is, however, more promising.

Problem 4.4.5 *Is there a polynomial time algorithm for* List k-Colourability *(in particular $k = 4$) of bipartite P_7-free graphs?*

For any fixed $k \geq 4$, $t = 7$ is the only problem left for List k-Colourability of bipartite P_t-free graphs due to Theorem 4.3.2 and Theorem 4.4.2.

5

k-Colourability of (C_4, P_6)-Free Graphs

In this chapter we focus on the class $Forb(\{C_4, P_6\})$. By Theorem 3.3.1, List k-Colourability (and hence k-Colourability) is known to be polynomial for this class. Here we study the class through the lens of obstructions.

Definition (Obstructions) *A graph G is said to be an obstruction to k-*Colourability *if it is not k-colourable; and a minimal obstruction if it is an obstruction but no proper induced subgraph of G is an obstruction.*

Equivalently, a minimal obstruction to k-Colourability is simply a $(k+1)$-critical graph. In the following we always talk about the k-colouring problem, and so we simply say that a graph is an (minimal) obstruction without explicitly mentioning k-Colourability whenever convenient. The reason for studying minimal obstructions is that any obstruction must contain a minimal obstruction (that can be obtained by removing vertices). In this sense, the occurrence of minimal obstructions is the essential cause for a graph not to be k-colourable. Given a hereditary class $\mathcal{G}$ and a fixed integer $k \geq 1$, let $\mathcal{G}_k$ be the set of all k-colourable graphs that are in $\mathcal{G}$, and let $\mathcal{F}(\mathcal{G}_k)$ denote the set of minimal obstructions to k-Colourability that are also in $\mathcal{G}$. The typical question in the study of obstructions takes the follow form:

- *the characterization problem*: determine the set $\mathcal{F}(\mathcal{G}_k)$.

It turns out that this is a difficult question in general: the answer for general graphs is known only for $k \leq 2$. When restricted to special graph classes, very few cases can be completely answered. A weaker version of the problem asks

- *the finiteness problem*: whether or not $\mathcal{F}(\mathcal{G}_k)$ has finite size.

The finiteness problem is a meaningful because the characterization set for a hereditary class is not necessarily of finite size. More importantly, the finiteness of $\mathcal{F}(\mathcal{G}_k)$ has a fundamental implication.

Theorem 5.0.1 (Finiteness theorem) *Let $\mathcal{G}$ be a hereditary graph class and $k \geq 1$ be an integer. If $|\mathcal{F}(\mathcal{G}_k)|$ is finite, then k-*Colourability *is polynomial time solvable for $\mathcal{G}$.*

Proof. Suppose that $\mathcal{F}(\mathcal{G}_k) = \{F_1, \ldots, F_r\}$ for some finite integer $r \geq 1$ and $|V(F_i)| = n_i$ for $1 \leq i \leq r$. Let $G \in \mathcal{G}_k$ be a graph with n vertices. Then G is k-colourable if and only if G is F_i-free for each $1 \leq i \leq r$. We now brute-force on all n_i-tuples of $V(G)$ and check whether or not the given tuple induces a subgraph isomorphic to F_i. Since F_i has n_i vertices, there are at most n^{n_i} such tuples, and this implies that it takes $O(n^{n_i})$ time to decide if G is F_i-free. Therefore, it takes $\sum_i O(n^{n_i}) \leq O(n^{\max\{n_1,\ldots,n_r\}})$ time to determine if G is k-colourable. Since r is finite, the running time is a polynomial function in n. ■

Theorem 5.0.1 says that the finiteness of $\mathcal{F}(\mathcal{G}_k)$ generalizes the polynomial time solvability. In this case, we say that $\mathcal{G}$ admits a *finite characterization* to k-Colourability or k-Colourability of $\mathcal{G}$ admits a *finite characterization*. We remark that the converse of the theorem is not true. The class of all graphs does not admit a finite characterization to 2-Colourability (since odd cycles are the minimal obstructions) but 2-Colourability can still be solved in polynomial time for general graphs.

Beside the case that the characterization set for 2-Colourability of general graphs is known, the set of obstructions is also known for perfect graphs: for each $k \geq 1$ the complete graph K_{k+1} is the only obstruction (by definition). Other than that, not much can be said for the characterization problem. Not much is known for the finiteness problem, either. This is not surprising, because k-Colourability is NP-complete for many restricted classes and this implies by Theorem 5.0.1 that these classes do not admit a finite characterization (assuming that $P \neq NP$). Despite that, the research on obstructions for P_t-free graphs has been active in recent years. It was shown by Bruce, Hoàng and Sawada **[16]**, and Maffray and Morel **[83]**, that the class of P_5-free graph admits a finite characterization to 3-Colourability, while Hoàng, Moore, Recoskie, Sawada, and Vatshelle **[64]** proved that the class does not when $k = 4$. The latter result implies that for fixed integers $k \geq 4$ and $t \geq 5$, k-Colourability of P_t-free graphs does not admit a finite characterization. Recall that the class of P_4-free graphs is perfect and thus admits a finite characterization. This leaves the finiteness problem open only for 3-Colourability of P_t-free graphs when $t \geq 6$. Very recently, Chudnovski, Goedgebeury, Schaudt and Zhong **[22]** gave a complete answer to this problem: the class of P_t-free graph admits a finite characterization to 3-Colourability if $t = 6$, and it does not when $t \geq 7$.

We follow the line of a result in **[64]** where the authors have shown, aided by a computer search, that 4-Colourability of (P_5, C_5)-free graphs admits a finite characterization. We consider the obstructions for (C_s, P_t)-free graphs, mainly for (C_4, P_6)-free graphs. Since every (C_3, P_6)-free graph is 4-colourable, there is no obstruction for k-COLOURABILITY when $k \geq 4$. This, together with the result on 3-COLOURABILITY **[22]**, completes the study of (C_3, P_6)-free graphs (indeed there is a single obstruction M_4 for 3-COLOURABILITY, see for example **[94]**). For (C_5, P_6)-free graphs, it is not even known whether or not k-COLOURABILITY is polynomial time solvable when $k \geq 5$ (see the summary at the end of Chapter 4). For (C_6, P_6)-free graphs, we have shown that k-COLOURABILITY is NP-complete on this class when $k \geq 5$. We therefore focus on (C_4, P_6)-free graphs. After exploring structural properties of (C_4, P_6)-free graphs in Section 5.1, we show in Section 5.2 that for any $k \geq 1$, the class of (C_4, P_6)-free graphs does admit a finite characterization. We prove this by showing that any (C_4, P_6)-free minimal obstruction has at most $O(k^6)$ vertices. For $k = 3$ and $k = 4$, the exact minimal obstructions are found and these results are presented in Section 5.3 and Section 5.4, respectively. In Section 5.5, we turn our proofs into polynomial time certifying algorithms for k-COLOURABILITY $(k = 3, 4)$ of (C_4, P_6)-free graphs.

5.1 Structure around a hole

Let $G = (V, E)$ be a connected (C_4, P_6)-free graph. Since the only perfect obstruction is K_{k+1}, we assume that G is imperfect. By the Strong Perfect Graph Theorem (Theorem 2.3.3), G must contain a 5-hole $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ with ij being an edge if and only if $|i - j| = 1$ (indices are modulo 5).

Definition (C_5-structure) *A vertex $v \in V \setminus C$ is said to be a p-vertex with respect to C if v has exactly p neighbours on C. The set of p-vertices, for each $0 \leq p \leq 5$, is denoted by S_p. Moreover, we define*

$$S_1(v_i) = \{x \in S_1 : N_C(x) = \{v_i\}\}.$$
$$S_2(v_i, v_{i+1}) = \{x \in S_2 : N_C(x) = \{v_i, v_{i+1}\}\}.$$
$$S_3(v_i) = \{x \in S_3 : N_C(x) = \{v_{i-1}, v_i, v_{i+1}\}\}.$$

It follows immediately from the C_4-freeness of G that $S_4 = \emptyset$ and $S_p = \bigcup_{i=0}^{4} S_p(v_i)$ for each $1 \leq p \leq 3$. In the remaining of this chapter, we shall use S_p, $S_1(v_i)$, $S_2(v_i, v_{i+1})$ and $S_3(v_i)$ whenever we talk about a 5-hole C. Brandstädt and Hoàng **[11]** discovered an important property about 5-holes in (C_4, P_6)-free graphs that do not have clique cutsets.

Lemma 5.1.1 **[11]** *Suppose that G is a (C_4, P_6)-free graph containing an induced five-cycle C defined above. If G does not contain any clique cutset, C is a dominating set of G, i.e., $S_0 = \emptyset$.*

The following lemma lists a number of properties we find regarding the adjacency among vertices in different subsets. All indices are modulo 5.

Lemma 5.1.2 *Suppose that G is a (C_4, P_6)-free graph containing an induced five-cycle C defined above. Then the following holds for G.*

(P1) *For each $0 \le i \le 4$, $S_5 \cup S_3(v_i)$ is a clique.*

(P2) *For each $0 \le i \le 4$, $S_1(v_i)$ is complete to $S_1(v_{i+2})$ and anti-complete to $S_1(v_{i+1})$; moreover, if both $S_1(v_i)$ and $S_1(v_{i+2})$ are non-empty, then both sets are cliques.*

(P3) *$S_2(v_i, v_{i+1})$ is complete to $S_2(v_{i+1}, v_{i+2})$ and anti-complete to $S_2(v_{i+2}, v_{i+3})$; moreover, if both $S_2(v_i, v_{i+1})$ and $S_2(v_{i+1}, v_{i+2})$ are non-empty, then both sets are cliques.*

(P4) *$S_3(v_i)$ is anti-complete to $S_3(v_{i+2})$.*

(P5) *$S_1(v_i)$ is anti-complete to $S_2(v_j, v_{j+1})$ if $j \neq i+2$; moreover, if $y \in S_2(v_{i+2}, v_{i+3})$ is not anti-complete to $S_1(v_i)$, then y is adjacent to any other vertex in $S_2(v_{i+2}, v_{i+3})$.*

(P6) *$S_1(v_i)$ is anti-complete to $S_3(v_{i+2})$.*

(P7) *$S_2(v_{i+2}, v_{i+3})$ is anti-complete to $S_3(v_i)$.*

(P8) *One of $S_1(v_i)$ and $S_2(v_{i+3}, v_{i+4})$ is empty, and one of $S_1(v_i)$ and $S_2(v_{i+1}, v_{i+2})$ is empty.*

(P9) *One of $S_2(v_{i-1}, v_i)$, $S_2(v_i, v_{i+1})$ and $S_2(v_{i+2}, v_{i+3})$ is empty.*

(P10) *If both $S_1(v_{i-1})$ and $S_1(v_{i+1})$ are non-empty, then $S_2 = \emptyset$; if both $S_1(v_i)$ and $S_1(v_{i+1})$ are non-empty, then $S_2 = S_2(v_i, v_{i+1})$; if both $S_2(v_i, v_{i+1})$ and $S_2(v_{i+1}, v_{i+2})$ are non-empty, then $S_1 = S_1(v_{i+1})$.*

(P11) *Let $x \in S_3(v_i)$. If both $S_2(v_{i+1}, v_{i+2})$ and $S_2(v_{i+3}, v_{i+4})$ are non-empty, then x is either complete or anti-complete to $S_2(v_{i+1}, v_{i+2}) \cup S_2(v_{i+3}, v_{i+4})$. In the former case, both $S_2(v_{i+1}, v_{i+2})$ and $S_2(v_{i+3}, v_{i+4})$ are cliques. Moreover, if $S_2(v_{i+2}, v_{i+3})$ is also non-empty, then x is anti-complete to $S_2(v_{i+1}, v_{i+2}) \cup S_2(v_{i+3}, v_{i+4})$.*

(P12) *If $S_1(v_i)$ is not anti-complete to $S_2(v_{i+2}, v_{i+3})$, then $S_1 = S_1(v_i)$.*

(P13) *If G has no clique cutsets, then $S_1(v_i)$ is complete to $S_3(v_i)$.*

Proof. We prove these properties one by one.

(P1). Suppose by contradiction that $S_5 \cup S_3(v_i)$ contains two non-adjacent vertices x and y. Then $\{v_{i-1}, x, v_{i+1}, y\}$ induces a C_4 in G.

(P2). Let $x \in S_1(v_0)$, $y \in S_1(v_1)$ and $z \in S_1(v_2)$. Then $xy \notin E$, for otherwise $x - v_0 - v_1 - y - x$ induces a C_4, and $xz \in E$ for otherwise $x - v_0 - v_4 - v_3 - v_2 - z$ induce a P_6. Thus, the first part follows. Suppose further that $S_1(v_2)$ is not a clique. Then $S_1(v_2)$ contains two non-adjacent vertices z', z''. Now $x - z' - v_2 - z'' - x$ induces a C_4.

(P3). Let $x \in S_2(v_0, v_1)$, $y \in S_2(v_1, v_2)$ and $z \in S_2(v_2, v_3)$. If $xy \notin E$, then $x - v_0 - v_4 - v_3 - v_2 - y$ induces a P_6. If $xz \in E$, then $x - v_1 - v_2 - z - x$ induces a C_4. This proves the first part. Suppose that $S_2(v_1, v_2)$ contains two non-adjacent vertices y' and y''. Then $x - y' - v_2 - y'' - x$ induces a C_4.

(P4). Let $x \in S_3(v_0)$ and $y \in S_3(v_2)$. If $xy \in E$, then $x - v_4 - v_3 - y - x$ induces a C_4.

(P5). The first statement follows from the fact G is (C_4, P_6)-free. For the second part, it suffices to prove for $i = 0$. Let $x \in S_1(v_0)$ and $y \in S_2(v_2, v_3)$ with $xy \in E$. Suppose that y is not adjacent some vertex $y' \in S_2(v_2, v_3)$ with $y' \neq y$. Since $x - y - v_2 - y' - x$ (in this order) does not induce a C_4, it follows that $xy' \notin E$. But then $y' - v_2 - y - x - v_0 - v_4$ induces a P_6 in G, a contradiction.

(P6). Let $x \in S_1(v_0)$ and $y \in S_3(v_2)$. If $xy \in E$, then $x - v_0 - v_1 - y - x$ induces a C_4.

(P7). Let $x \in S_3(v_0)$ and $y \in S_2(v_2, v_3)$. If $xy \in E$, then $x - v_4 - v_3 - y - x$ induces a C_4.

(P8). It suffices to prove the property for $S_1(v_0)$ and $S_2(v_1, v_2)$. Let $x \in S_1(v_0)$ and $y \in S_2(v_1, v_2)$. Since $v_0 - v_1 - y - x - v_0$ does not induce a C_4, we conclude that $xy \notin E$. But then $x - v_0 - v_4 - v_3 - v_2 - y$ induces a P_6, a contradiction. Therefore, one of $S_1(v_0)$ and $S_2(v_1, v_2)$ is empty.

(P9). It suffices to prove the property for $i = 0$. Suppose that $x \in S_2(v_0, v_1)$, $y \in S_2(v_0, v_4)$ and $z \in S_2(v_2, v_3)$. It follows from **(P3)** that $xy \in E$, $xz \notin E$ and $yz \notin E$. Now $z - v_2 - v_1 - x - y - v_4$ induces a P_6.

(P10). Suppose that $S_1(v_0)$ and $S_1(v_2)$ are non-empty. Let $x \in S_1(v_0)$ and $y \in S_1(v_2)$. Note that $xy \in E$ by **(P2)**. Moreover, $S_2 = S_2(v_0, v_4) \cup S_2(v_2, v_3)$ by **(P8)**. Suppose that $S_2(v_0, v_4)$ contains a vertex z. Then z is not adjacent to x by **(P5)**. Since $z - y - x - v_0 - z$ does not induce a C_4, $zy \notin E$. But now $z - v_0 - x - y - v_2 - v_3$ induces a P_6. This shows that $S_2(v_0, v_4) = \emptyset$. Similarly, $S_2(v_2, v_3) = \emptyset$. This proves the first statement. The second and third statement follow directly from **(P8)**.

(P11). It suffices to prove the property for $i = 0$. Let $x \in S_3(v_0)$, $y \in S_2(v_1, v_2)$ and $z \in S_2(v_3, v_4)$. It follows from **(P3)** that $yz \notin E$. If x is adjacent to exactly one vertex in $\{y, z\}$, say $xy \in E$ but $xz \notin E$, then $v_0 - x - y - v_2 - v_3 - z$ induces a P_6 in G, a contradiction. This shows that x is either completer or anti-complete $\{y, z\}$. If x is anti-complete to $S_2(v_1, v_2) \cup S_2(v_3, v_4)$, then the property holds. In the following, we assume that x is complete to $\{y, z\}$. Applying the above argument for $\{y, z\}$ to $\{y, z'\}$ where z' is an arbitrary vertex in $S_2(v_3, v_4)$, we conclude that x is complete to $S_2(v_3, v_4)$, which in turn implies that x is also complete to $S_2(v_1, v_2)$. Since G is C_4-free, both $S_2(v_3, v_4)$ and $S_2(v_3, v_4)$ are cliques. This proves the first statement. Moreover, if $S_2(v_2, v_3)$ contains a vertex w, then x is anti-complete to $S_2(v_1, v_2) \cup S_2(v_3, v_4)$, for otherwise $x - z - w - y - x$ induces a C_4. This completes the proof.

(P12). Let $x \in S_1(v_0)$ and $y \in S_2(v_2, v_3)$ with $xy \in E$. Then $S_1(v_1) = S_1(v_4) = \emptyset$ by **(P8)**. Suppose that $S_1(v_3)$ contains a vertex z. Note that $zx \in E$ and $zy \notin E$ by **(P2)** and **(P5)**, and this implies that $z - x - y - v_3 - z$ induces a C_4. This shows that $S_1(v_3) = \emptyset$. Similarly, $S_1(v_2) = \emptyset$, and the property follows.

(P13). It follows directly from Lemma 5.1.1. ■

Let P be the graph obtained from the Petersen graph (see Figure 5.1) by adding a universal vertex. A graph is called *specific* if it results from replacing each vertex of P by a clique of arbitrary size (including possibly size 0, resulting in the deletion of the vertex). The next lemma explores the structure around a 6-hole in a (C_4, P_6)-free graph and will play an important role in subsequent proofs. The result is again due to Brandstädt and Hoàng.

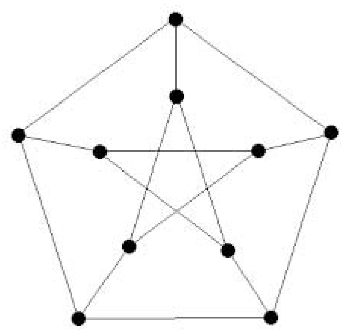


Figure 5.1: The Petersen graph.

Lemma 5.1.3 **[11]** *Let G be a (C_4, P_6)-free graph with no clique cutset. One of the following holds.*

1. *either G is C_6-free;*

2. *or G is not C_6-free but every 6-hole is dominating.*

3. *or G is not C_6-free and G is specific.*

5.2 Obstructions for general k

In this section our main result (cf. also **[60]**) is the following.

Theorem 5.2.1 *For any fixed positive integer k, any (C_4, P_6)-free minimal obstruction G has $O(k^6)$ vertices. In particular, the class of (C_4, P_6)-free graphs admits a finite characterization.*

Before proving the theorem, we need a preliminary result whose proof is a folklore.

Lemma 5.2.2 (Folklore) *Any minimal obstruction G to k-*COLOURABILITY *has $\delta(G) \geq k$ and no clique cutset.*

Moreover, our proof is based on the following two lemmas.

Lemma 5.2.3 *Let G be a (C_4, C_6, P_6, K_{k+1})-free graph so tha G contains an induced five-cycle $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ and has no clique cutset. For each $0 \leq i \leq 4$, $|S_1(v_i)| = O(k^6)$.*

Lemma 5.2.4 *Let G be a (C_4, C_6, P_6, K_{k+1})-free graph so tha G contains an induced five-cycle $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ and has no clique cutset. For each $0 \leq i \leq 4$, $|S_1(v_i, v_{i+1})| = O(k^6)$.*

We leave the proofs of Lemma 5.2.3 and Lemma 5.2.4 later and now prove the main result using these lemmas.

Proof of Theorem 5.2.1. Let G be a (C_4, P_6)-free minimal obstruction. By Lemma 5.2.2, G has $\delta(G) \geq k$ and no clique cutset. If G contains K_{k+1}, then G is isomorphic to K_{k+1}. From now on, we assume that G is K_{k+1}-free. In particular, since G is not k-colourable, we conclude that G is imperfect. This implies that G is not chordal, since any chordal graph is perfect by Proposition 2.3.8. Therefore, G contains an induced cycle C of length 5 or 6, since G is (C_4, P_6)-free.

Suppose first that C is an induced 6-cycle. It follows from Lemma 5.1.3 that either G is specific or C is dominating. If G is specific, then $|G| \leq 11k$ by the definition of specific graph and the fact that G is K_{k+1}-free, and the theorem holds. We thus assume that C is dominating, and we analyze the remaining vertices as to their connection to C, analogously to what we did in the previous section for C being a five-cycle. We define, for any $X \subseteq C$, the set $S(X)$ to consist of all vertices not in C that have X as their neighbourhood on C. Note that C being a dominating set in G means that $S(\emptyset) = \emptyset$. Let $X \subseteq C$. If X is not a clique, then $S(X)$ is a clique due to the C_4-freeness of G. If X is a clique, then $1 \leq |X| \leq 2$ and we note that $S(X) = \emptyset$, for otherwise a vertex in $S(X)$ and C contain an induced P_6. Therefore, $S(X)$ is a clique and hence $|S(X)| \leq k$ for any $X \subseteq C$. Since there are at most 2^6 subsets of C, it follows that $|G| = \sum_{X \subseteq C} |S(X)| + |C| \leq 64k + 6$.

We now assume that G is C_6-free and $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ is an induced five-cycle. We use the notations S_p, $S_3(v_i)$, $S_2(v_i, v_{i+1})$, and $S_1(v_i)$ from Section 5.1. Recall first that each $S_3(v_i) \cup S_5$ is a clique by **(P1)**. Since G is K_{k+1}-free, $|S_5| \leq k - 2$ and $|S_3(v_i)| \leq k - 2$ for each i. Moreover, $S_0 = \emptyset$ by Lemma 5.1.1. Now it follows that

$$|G| = |S_5| + |S_3| + |S_2| + |S_1| = O(k^6)$$

by Lemma 5.2.3 and Lemma 5.2.4, and this completes the proof of the theorem. ■

We remark that Theorem 5.2.1 is best possible in the sense that there are infinitely many P_6-free minimal obstruction and infinitely many C_4-free minimal obstructions. The former fact follows from **[64]** where it is shown that there are infinitely many P_5-free minimal obstruction, and the latter fact follows from Theorem 2.3.2.

5.2.1 Proof of Lemma 5.2.3

Let G be a (C_4, C_6, P_6, K_{k+1})-free graph so tha G contains an induced five-cycle $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ and has no clique cutset. It suffices to prove the claim for $i = 0$. Assume that $S_1(v_0)$ is not an empty set, for otherwise there is nothing to prove. If $S_1(v_2) \cup S_1(v_3) \neq \emptyset$, then $S_1(v_0)$ is a clique by **(P2)** and thus of size at most k, which proves the lemma. We thus assume that $S_1(v_j) = \emptyset$ for $j = 2, 3$ and then $S_1(v_0)$ is anti-complete to $S_1 \setminus S_1(v_0)$ by **(P2)**.

Dealing with vertices in $S_1(v_0)$ that have a neighbour in S_2

Let $X = \{x \in S_1(v_0) : N(x) \cap S_2(v_2, v_3) \neq \emptyset\}$ and $Y = \{y \in S_2(v_2, v_3) : N(y) \cap S_1(v_0) \neq \emptyset\}$. We recall first that each $y \in Y$ is universal in $S_2(v_2, v_3)$ by **(P5)**. This implies that $|Y| \leq k - 2$ since G is K_{k+1}-free. Moreover, $N(y) \cap S_1(v_0)$, for any $y \in Y$, is a clique since G is C_4-free, and so has size at most k. Clearly, $X \subseteq \bigcup_{y \in Y} N(y) \cap S_1(v_0)$ and so

$$|X| \leq (k-2)k = O(k^2). \tag{5.1}$$

Let $S_1'(v_0) = S_1(v_0) \setminus X$. It follows from the definition and **(P5)** that $S_1'(v_0)$ is anti-complete to S_2. The goal now is to consider each connected component of $S_1'(v_0)$, and we shall argue that there are not too many connected components and each such component has bounded size. To this end we fix a connected component A of $S_1'(v_0)$.

Claim H *Each $x \in X \cup S_3(v_1) \cup S_3(v_4)$ is either complete or anti-complete to A.*

> Let x be any vertex in $X \cup S_3(v_1) \cup S_3(v_4)$, and yz be any edge in A. In case when $x \in X$, we also let $p \in S_2(v_2, v_3)$ be a neighbour of x. If x is adjacent to exactly one vertex in $\{y, z\}$, say $xy \in E$ but $xz \notin E$, then $z - y - x - v_2 - v_3 - v_4$ or $z - y - x - v_4 - v_3 - v_2$ or $z - y - x - p - v_3 - v_4$ induces a P_6 in G, depending on which set x lies in. ◇

We now distinguish two types of connected components. We say that A is of *type* 1 if A has a neighbour in both $S_3(v_1)$ and $S_3(v_4)$, and of *type* 2 otherwise. Let n_i be the number of vertices in components of type i, $i = 1, 2$.

Dealing with components of type 1

Let A be of type 1 and assume that A has a neighbour $u \in S_3(v_1)$ and a neighbour $w \in S_3(v_4)$. By Claim H, u and w are complete to A. Moreover, u and w are not adjacent by **(P4)**, and this further implies that A is a clique since G is C_4-free. Hence, $|A| = O(k)$. On the other hand, each such component corresponds to a pair of neighbours (u, w) with $u \in S_3(v_1)$ and $w \in S_3(v_4)$ by definition. Since each $S_3(v_j)$ $(j = 1, 4)$ has at most $k - 2$ vertices, the number of such pairs (u, w) is less than k^2. This implies that there are at most k^2 connected components of type 1, for otherwise there exist two such components A and B corresponding to the same pair (u, w) of neighbours by the pigeonhole principle, and then the vertices u, w, a vertex from A and a vertex from B induce a C_4. This shows that

$$n_1 = O(k^3). \tag{5.2}$$

This takes care of the components of type 1.

Dealing with components of type 2

Let A be a connected component of type 2. We assume by symmetry and Claim H that A is anti-complete to $S_3(v_4)$. Let $X' = \{x \in X : x \text{ is complete to } A\}$ and $S_3'(v_1) = \{v \in S_3(v_1) : v \text{ is complete to } A\}$. Namely, X' and $S_3'(v_1)$ are the subsets of X and $S_3(v_1)$, respectively, that consist of precisely those vertices that are complete to A. Let $x_i \in X'$ $(i = 1, 2)$ be two arbitrary vertices in X, and let a be a vertex in A. Recall that a is anti-complete to S_2.

(i) *X' is a clique.*

We show that $x_1x_2 \in E$. Suppose not. If x_1 and x_2 have a common neighbour $y \in S_2(v_2, v_3)$, then $x_1 - a - x_2 - y - x_1$ induces a C_4. If there exist $y_i \in S_2(v_2, v_3)$ $(i = 1, 2)$ such that $x_iy_i \in E$ but $x_iy_j \notin E$ for $i \neq j$. Then y_1 and y_2 are adjacent by **(P5)** and thus $x_1 - y_1 - y_2 - x_2 - a - x_1$ is a 5-hole that is anti-complete to v_1, which contradicts Lemma 5.1.1. This shows that X' is a clique. $\diamond$

(ii) *$S_3'(v_1)$ is complete to X'.*

Let $x \in X'$ with a neighbour $y \in S_2(v_2, v_3)$, and let $z \in S_3'(v_1)$. If $zx \notin E$, then either $z - v_0 - x - y - z$ induces a C_4 or $z - a - x - y - v_3 - v_4$ induces a P_6, depending on whether or not $zy \in E$. Thus, $zx \in E$ and this proves **(ii)**. $\diamond$

Let $V' = \{v_0\} \cup S_3(v_0) \cup S_3'(v_1)$. If some $x \in S_3(v_0)$ is not adjacent to some $y \in S_3'(v_1)$, then $x - v_1 - y - a - x$ induces a C_4. Thus, V' is a clique. Moreover, $V' \cup S_5$ is a clique by **(P1)**, and $V' \cup X'$ is clique by **(ii)** and **(P13)**. Since G contains no clique cutset, A has a neighbour $x \in X'$ and a neighbour $u \in S_5$ with $ux \notin E$. In other words, each connected component

A of type 2 corresponds to a pair of non-adjacent neighbours (x, u) with $x \in X' \subseteq X$ and $u \in S_5$. Since $|X| = O(k^2)$ and $|S_5| \leq k$, the number of such pairs (x, u) is at most k^3. This implies that there are at most $O(k^3)$ components of type 2, for otherwise there exist two such components A and B corresponding to the same pair (x, u) of neighbours by the pigeonhole principle, and then the vertices x, u, a vertex from A and a vertex from B induce a C_4.

It remains to bound the size of a component A of type 2. We define $S_5^* = \{u \in S_5 : N(u) \cap A \neq \emptyset\}$, i.e., S_5^* is the set of vertices in S_5 that have a neighbour in A. Let $R \subseteq S_5^*$ be the set of vertices that have a non-neighbour in X' and let $S_5' = S_5^* \setminus R$. Note that $R \neq \emptyset$. By definition, S_5' is complete to X'. We then define $T = \{a \in A : N(a) \cap R \neq \emptyset\}$. Clearly, $T \subseteq \bigcup_{r \in R} N_A(r)$. By definition of R, any $r \in R$ is not adjacent some $x \in X'$. Recall that x is complete to A. This implies that $N_A(r)$ is a clique, for otherwise any two non-adjacent neighbours of r in A, x, and r induce a C_4. Therefore $|N_A(r)| \leq k$ for any $r \in R$ and so $|T| \leq k^2$. We now claim that $A = T$. By contradiction, suppose that $A \setminus T$ is not empty. Take any (non-empty) connected component B of $A \setminus T$. We shall find a clique cutset in G that separates B from the rest of G in the following way.

Let $S_5^B \subseteq S_5'$ be the set of vertices that have a neighbour in B. Clearly, $V^* = V' \cup X' \cup S_5^B$ is a clique.

(iii) *Each vertex $t \in T$ is either complete or anti-complete to B.*

Let $t \in T$ be any vertex and bb' be any edge in B. Then t has a neighbour $r \in R$, and r is not adjacent to some vertex $x \in X'$, which has a neighbour $y \in S_2(v_2, v_3)$. The existence of r, x and y follows from the definition of the sets T, R and X. If t is adjacent to exactly one vertex in $\{b, b'\}$, say $tb \in E$ but $tb' \notin E$, then $ry \notin E$ or $t - r - y - x - t$ induces a C_4, and this implies that $b' - b - t - r - v_2 - y$ induces a P_6. This contradiction shows that each vertex $t \in T$ is either complete or anti-complete to B. ◇

Let $T' \subseteq T$ be the set of vertices that are complete to B. We now show:

(iv) *$V^* \cup T'$ is a clique.*

It is clear that T' is complete to $V' \cup X'$. It then remains to show T' is a clique and it is complete to S_5^B. Let $b \in B$ be a vertex, and let t_1 and t_2 be two arbitrary vertices in T. Suppose that $t_1t_2 \notin E$. Then t_1 and t_2 do not have a common neighbour in R, for otherwise this common neighbour, t_1, t_2 and b induces a C_4. Thus, we assume that t_i has a neighbour $r_i \in R$ for $i = 1, 2$ such that $t_1r_2, t_2r_1 \notin E$. Observe that now $Q = b - t_1 - r_2 - r_1 - t_2 - b$ is a 5-hole. Take a vertex $x_i \in X'$ be a non-neighbour of r_i and $y_i \in S_2(v_2, v_3)$ be a neighbour of x_i, $i = 1, 2$. Since $r_i - v_0 - x_i - y_i - r_i$ does not induce a C_4, r_i is not adjacent to y_i. If $y_1 = y_2$ or $x_1 = x_2$ (then also $y_1 = y_2$), then Q does not dominate y_1, which contradicts Lemma 5.1.1. This implies that $x_1 \neq x_2$,

$y_1 \neq y_2$ and $y_i x_j \notin E$ for $i \neq j$. But then $x_1 - y_1 - y_2 - x_2 - x_1$ induces a C_4. This proves that $t_1 t_2 \in E$ and so T' is a clique.

Let $q \in S_5^B$. Note that we may choose $b \in B$ to be a neighbour of q. Since $q - b - t_1 - r_1 - q$ does not induce a C_4, we conclude that q is adjacent to t_1, and this proves that T' is complete to S_5^B. $\diamond$

By **(iv)**, $V^* \cup T'$ is the desired clique cutset separating B from G, which contradicts the fact that G has no clique cutset. We have indeed showed that $A = T$ and so $|A| = O(k^2)$. Recall that there are at most k^3 connected components of type 2, and this implies that

$$n_2 \leq O(k^2) \times O(k^3) = O(k^6). \tag{5.3}$$

This takes care of the components of type 2.

Completing the proof

Finally, it follows from (5.1), (5.2) and (5.3) that

$$|S_1(v_0)| = |X| + n_1 + n_2 \leq O(k^2) + O(k^3) + O(k^6) = O(k^6)$$

and this completes the proof of Lemma 5.2.3.

5.2.2 Proof of Lemma 5.2.4

It suffices to prove the claim for $S_2(v_2, v_3)$. Assume that $S_2(v_2, v_3)$ is not an empty set, for otherwise there is nothing to prove. If $S_2(v_1, v_2) \cup S_2(v_3, v_4) \neq \emptyset$, then $S_2(v_2, v_3)$ is a clique and by **(P3)** has at most k vertices, which proves the lemma. In the following we assume that $S_2(v_1, v_2) \cup S_2(v_3, v_4) = \emptyset$, and this implies that $S_2(v_2, v_3)$ is anti-complete to $S_2 \setminus S_2(v_2, v_3)$ by **(P3)**.

Dealing with vertices in $S_2(v_0)$ that have a neighbour in S_1

Let $Y \subseteq S_2(v_2, v_3)$ be the set of vertices that have a neighbour in $S_1(v_0) \cup S_3(v_1) \cup S_3(v_4)$. Note that any vertex $x \in S_3(v_1) \cup S_3(v_4)$ satisfies that $|N_{S_2(v_2,v_3)}(x)| \leq k$ since G is (C_4, K_{k+1})-free. This, together with **(P5)**, implies that

$$|Y| \leq 2k^2 + k = O(k^2). \tag{5.4}$$

Let $S_2'(v_2, v_3) = S_2(v_2, v_3) \backslash Y$. By definition, $S_2'(v_2, v_3)$ is anti-complete to $S_2 \cup S_3(v_1) \cup S_3(v_4)$. Take an arbitrary connected component A of $S_2'(v_2, v_3)$. Let $a \in A$ be any vertex.

Bounding the number of components of $S_2'(v_2, v_3)$

We begin with a few claims.

(i) *Any vertex $y \in Y \cup S_3(v_2) \cup S_3(v_3)$ is either complete or anti-complete to A.*

By definition, y has a neighbour $x \in S_1(v_0) \cup S_3(v_1) \cup S_3(v_4)$ if $y \in Y$. If y is adjacent to exactly one vertex of some edge aa' in A, say $ya \in E$ but $ya' \notin E$, then $a' - a - y - v_1 - v_0 - v_4$, $a' - a - y - v_4 - v_0 - v_1$, $a' - a - y - x - v_0 - v_4$ or $a' - a - y - x - v_0 - v_1$ induces a P_6, depending on where y and x lie in. This, together with the connectivity of A, proves the observation. $\diamond$

Let Y', $S_3'(v_3)$ and $S_3'(v_2)$ be the subsets of Y, $S_3(v_3)$ and $S_3(v_2)$, respectively that are complete to A. Let $V' = \{v_3, v_2\} \cup S_3'(v_2) \cup S_3'(v_3)$. If $p \in S_3'(v_2)$ is not adjacent to $q \in S_3'(v_3)$, then $p - a - q - v_4 - v_0 - v_1 - p$ induces a C_6 in G, which is impossible. This shows that V' is a clique.

(ii) *Y' is a clique.*

By contradiction, suppose that Y' contains two non-adjacent vertices y_1 and y_2. Let x_i be a neighbour of y_i in $S_1(v_0) \cup S_3(v_1) \cup S_3(v_4)$, $i = 1, 2$. Then $x_i \notin S_1(v_0)$ by **(P5)**. If $x_1 = x_2$, then $x_1 - y_1 - v_3 - y_2 - x_1$ or $x_1 - y_1 - v_2 - y_2 - x_1$ induces a C_4. This means that $x_1y_2, x_2y_1 \notin E$. If both x_1 and x_2 are in some $S_3(v_j)$, say in $S_3(v_4)$, then $x_1 - y_1 - a - y_2 - x_2 - x_1$ is a 5-hole in G and it does not dominate v_1. This contradicts Lemma 5.1.1. If $x_1 \in S_3(v_4)$ and $x_2 \in S_3(v_1)$, then $x_1 - y_1 - a - y_2 - x_2 - v_0 - x_1$ induces a C_6. This proves that Y' is a clique. $\diamond$

(iii) *Y' is complete to $S_3'(v_2) \cup S_3'(v_3)$.*

Suppose not, let $y \in Y'$ be non-adjacent to some $x \in S_3'(v_3)$, say. Then y has a neighbour $z \in S_1(v_0) \cup S_3(v_1) \cup S_3(v_4)$. Since $z - y - a - x - z$ does not induce a C_4, it follows that $xz \notin E$. This implies that $x - a - y - z - v_0 - v_1$ induces a P_6. $\diamond$

We have so far proved that V' is a clique, Y' is a clique and Y' is complete to V'. This means that $V' \cup Y'$ is a clique. Clearly, $V' \cup S_5$ is a clique. Since G contains no clique cutset separating A, A corresponds to a pair of non-adjacent neighbours (y, u) with $y \in Y' \subseteq Y$ and $u \in S_5$ such that y is complete to A. Since $|Y| = O(k^2)$ and $|S_5| \leq k$, the number of such pairs (y, u) is at most $O(k^3)$. This implies that

$$\text{the number of connected components of } S_2'(v_2, v_3) = O(k^3), \tag{5.5}$$

for otherwise there exist two components A and B corresponding to the same pair (y, u) of neighbours by the pigeonhole principle, and then the vertices y, u, a vertex from A and a vertex from B induce a C_4.

Bounding the size of components of $S_2'(v_2, v_3)$

It remains to bound the size of A. We define S_5^*, R, S_5', and T as in Lemma 5.2.3, except that X' is replacd by Y' in the definition. Similar to the case in Lemma 5.2.3, it can be shown that $|T| \le k^2$. We now claim that $A = T$. By contradiction, suppose that $A \setminus T$ is not empty. Take any connected component B of $A \setminus T$. We shall find a clique cutset in G that separates B from the graph G in the following way.

Let $S_5^B \subseteq S_5'$ be the set of vertices that have a neighbour in B. Then $V^* = V' \cup Y' \cup S_5^B$ is a clique. Using the same argument in Lemma 5.2.3, it can be shown that each vertex $t \in T$ is either complete or anti-complete to B. Let $T' \subseteq T$ be the set of vertices that are complete to B. Recall that $V^* = V' \cup Y' \cup S_5^B$ is a clique. We now show that $V^* \cup T'$ is a clique. It is clear that T' is complete to $V^* \setminus S_5^B$. It then remains to show T' is a clique and it is complete to S_5^B.

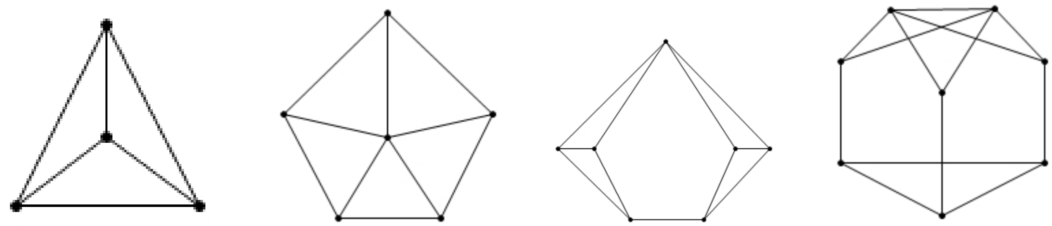
(iv) *T' is a clique and it is complete to S_5^B.*

Let $b \in B$ be a vertex, and let t_1 and t_2 be two arbitrary vertices in T. Suppose that $t_1t_2 \notin E$. Then t_1 and t_2 do not have a common neighbour in R, for otherwise this common neighbour, t_1, t_2 and b induce a C_4. Thus, we assume that t_i has a neighbour $r_i \in R$ for $i = 1, 2$ such that $t_1r_2, t_2r_1 \notin E$. Observe that now $Q = t_1 - r_1 - r_2 - t_2 - b - t_1$ induces a 5-hole. Take a vertex $y_i \in Y'$ be a non-neighbour of r_i and $x_i \in S_1(v_0) \cup S_3(v_1) \cup S_3(v_4)$ be a neighbour of y_i, $i = 1, 2$. Since $r_i - v_0 - x_i - y_i - r_i$ does not induce a C_4, r_i is not adjacent to x_i. If $x_1 = x_2$ or $y_1 = y_2$ (then also $x_1 = x_2$), then Q does not dominate x_1, which contradicts Lemma 5.1.1. This implies that $x_1 \neq x_2$, $y_1 \neq y_2$ and $y_1x_2, y_2x_1 \notin E$. Then $x_1x_2 \notin E$ or $x_1 - y_1 - y_2 - x_2 - x_1$ induces a C_4. Moreover, $x_1r_2, x_2r_1 \in E$, for otherwise x_1 (or x_2) is anti-complete to Q, and this contradicts Lemma 5.1.1. Finally, since $y_1 - t_1 - r_1 - r_2 - y_1$ does not induce a C_4, $y_1r_2 \notin E$. Similarly, $y_2r_1 \notin E$. But now $\{y_1, y_2, x_1, x_2, r_1, r_2\}$ induces a C_6. This proves that T' is a clique.

Now let $q \in S_5^B$ and we may choose b to be a neighbour of q. Since $b - t_1 - r_1 - q - b$ does not induce a C_4 and $br_1 \notin E$, t_1 is adjacent to q. This shows that T' is complete to S_5^B. This means that $V^* \cup T'$ is a clique cutset in G, a contradiction. $\diamond$

Therefore, each connected component A of $S_2'(v_2, v_3)$ has

$$|A| = O(k^2). \tag{5.6}$$



(a) K_4. (b) W_5. (c) The Hajós graph. (d) F.

Figure 5.2: All (C_4, P_6)-free minimal obstructions to 3-COLOURABILITY.

Completing the proof

It follows from (5.5) and (5.6) that $|S_2'(v_2, v_3)| \leq O(k^2) \times O(k^3) = O(k^6)$. Finally, it follows from (5.4) that

$$|S_2(v_2, v_3)| = |Y| + |S_2'(v_2, v_3)| \leq O(k^2) + O(k^6) = O(k^6),$$

and this completes the proof of Lemma 5.2.4.

5.3 Exact obstructions for 3-Colourability

In this section we explicitly describe (C_4, P_6)-free minimal obstructions to 3-COLOURABILITY. We note that **[83]**, in conjunction with **[16]**, describe all P_5-free minimal obstructions to 3-COLOURABILITY, and that **[64]** describes all (P_5, C_5)-free minimal obstructions to 4-COLOURABILITY.

Theorem 5.3.1 *There are exactly four (C_4, P_6)-free minimal obstructions, given in Figure 5.2.*

Proof. Let G be a (C_4, P_6)-free minimal obstruction to 3-COLOURABILITY. From the first few lines of the proof of Theorem 5.2.1, we know that G has $\delta(G) \geq 3$, contains no clique cutset, is K_4-free, and contains an induced five-cycle $C = v_0 - v_1 - \ldots - v_4 - v_0$. By Lemma 5.1.1 it follows that $S_0 = \emptyset$. Moreover, $|S_5| \leq 1$, for otherwise G contains a K_4. If $|S_5| = 1$, then G is isomorphic to W_5. If there exists an index $0 \leq i \leq 4$ such that $S_3(v_i) \neq \emptyset$ and $S_3(v_{i+2}) \neq \emptyset$, then G is the Hajós graph. From now on, we assume that $S_5 = \emptyset$ and at most two $S_3(v_i)$'s are not empty. Furthermore, each $S_3(v_i)$ is clique by **(P1)**, and so contains at most one vertex or G contain a K_4. It follows that $|S_3| \leq 2$. We distinguish three cases.

Case 1 $|S_3| = 2$. Without loss of generality, assume that $S_3(v_0) = \{x\}$ and $S_3(v_1) = \{y\}$.

Note first that $xy \notin E$ or G contains a K_4. Moreover, if $S_1(v_3)$ contains a vertex t, then $t - v_3 - v_2 - y - v_0 - x$ is a P_6. This shows that $S_1(v_3) = \emptyset$. Thirdly, if x is non-adjacent to some vertex $d \in S_2(v_3, v_4)$, then $d - v_3 - v_2 - y - v_0 - x$ is a P_6. This means that x is complete to $S_2(v_3, v_4)$. By symmetry, y is complete to $S_2(v_2, v_3)$. This implies that $S_2(v_3, v_4)$ and $S_2(v_3, v_2)$ are cliques of size at most one, or G contains a K_4. Since $d(v_3) \geq 3$ and $S_1(v_3) = \emptyset$, at least one of $S_2(v_3, v_4)$ and $S_2(v_3, v_2)$ is non-empty. By symmetry, we assume that $S_2(v_2, v_3) = \{z\}$. If $S_2(v_3, v_4)$ is also non-empty, say $S_2(v_3, v_4) = \{z'\}$, then $S_1 = \emptyset$ by **(P8)** and the fact that $S_1(v_3) = \emptyset$, and $S_2 = \{z, z'\}$ by **(P9)** and **(P11)**. Now G admits a 3-colouring, namely $\{v_3, x, y\}$, $\{v_0, v_2, z'\}$, $\{v_4, v_1, z\}$, contradicting the fact that G is a minimal obstruction.

In the following, we assume that $S_2(v_3, v_4) = \emptyset$. Note first that $S_1(v_1) = S_1(v_4) = \emptyset$ by **(P8)**. If $S_2(v_0, v_4)$ contains a vertex z', then $yz' \in E$ by **(P11)** and now $C \setminus \{v_1\} \cup \{y, z, z'\}$ induces the Hajós graph. This shows that $S_2(v_0, v_4) = \emptyset$. If $S_1(v_2) \neq \emptyset$, then $S_1(v_0) = \emptyset$ due to **(P10)** and the existence of z. This implies that $\{v_2, y\}$ is a clique cutset separating $S_1(v_2)$ from G, a contradiction. Therefore, $S_1(v_2) = \emptyset$. Next we show that $S_1(v_0)$ is also empty. If not, let $u_0 \in S_1(v_0)$. Note that x is complete to $S_1(v_0)$. Since $\{u_0, x, v_1, y\}$ does not induce a C_4. $u_0 y \notin E$. This proves that y is anti-complete to $S_1(v_0)$. Since $\{v_0, x\}$ is not a clique cutset separating $S_1(v_0)$, u_0 has a neighbour in $S_2(v_2, v_3)$, i.e., $u_0 z \in E$. Now $C \cup \{x, u_0, z\}$ induces the graph F. Therefore, $S_1(v_0) = \emptyset$. So far, we have proved that $S_1 = \emptyset$ and $S_2 = \{z\} \cup S_2(v_0, v_1) \cup S_2(v_1, v_2)$. If $S_2(v_1, v_2)$ contains a vertex p, then $yp \in E$ by **(P3)** and so either $\{y, v_1, p, z\}$ induces a C_4 or $\{v_1, v_2, y, p\}$ induces a K_4, depending on whether $yp \in E$. This shows that $S_2(v_1, v_2) = \emptyset$. Finally, we prove that $S_2(v_0, v_1) = \emptyset$. Suppose by contradiction that $S_2(v_0, v_1)$ contains a vertex p. Note that $S_2(v_0, v_1)$ is an independent set or G contains a K_4. Since G has no clique cutset, p is adjacent to both x and y. But then $\{p, v_0, v_1, x\}$ induces a K_4. This contradiction proves that $S_2(v_0, v_1)$ is indeed empty. Consequently, $V(G) = C \cup \{x, y, z\}$ and G admits a 3-colouring, namely $\{x, y, v_3\}$, $\{z, v_4, v_1\}$, $\{v_0, v_2\}$.

Case 2 $|S_3| = 0$, i.e., $V = C \cup S_1 \cup S_2$.

It follows from **(P9)** that there exists some index i such that $S_2(v_{i-1}, v_i) \cup S_2(v_i, v_{i+1}) = \emptyset$. Since $d(v_i) \geq 3$, it follows that $S_1(v_i) \neq \emptyset$. By symmetry, assume that $i = 0$. Note that $S_2(v_3, v_4) = S_2(v_1, v_2) = \emptyset$ by **(P8)**. We claim that $S_1(v_3) \neq \emptyset$ and $S_1(v_4) \neq \emptyset$. If not, since $d(v_3) \geq 3$ and $d(v_4) \geq 3$, we conclude that $S_2(v_2, v_3) \neq \emptyset$ and $S_2(v_0, v_4) \neq \emptyset$, and $S_1(v_3) = S_1(v_4) = \emptyset$. By **(P8)** and **(P9)**, we have that $S_2(v_0, v_1) = S_1(v_1) = \emptyset$. This implies that $d(v_1) = 2$, which is impossible. This indeed shows that both $S_1(v_3)$ and $S_1(v_4)$ are not empty. By symmetry, $S_1(v_1) \neq \emptyset$ and $S_1(v_2) \neq \emptyset$. We have shown that $S_1(v_i)$ is non-empty for each i. Let $u_i \in S_1(v_i)$ for $0 \leq i \leq 5$. Note that each of them is a clique, and $G = C \cup S_1$. Also, $|S_1| \leq 7$ or G contains a K_4. It is easy to check that G is 3-colourable if $|S_1| \leq 6$. Finally, assume that $|S_1| = 7$ and $|S_1(v_0)| = |S_1(v_1)| = 2$. Let $u_i' \in S_1(v_i)$ with $u_i' \neq u_i$ for $i = 0, 1$. Now the subgraph induced on $\{u_3, u_1, v_1, v_0, u_0, u_0', u_1'\}$ is isomorphic to the Hajós graph.

Case 3 $|S_3| = 1$. Without loss of generality, assume that $S_3 = S_3(v_0) = \{x\}$. We note that each $S_2(v_i, v_{i+1})$ is independent, for otherwise G contains a K_4. We consider two subcases. Suppose first that $S_1(v_0) = \emptyset$. We first prove a claim.

Claim I $S_2(v_2, v_3) = \emptyset$.

We prove the claim by contradiction. Assume that $S_2(v_2, v_3)$ contains a vertex z. Note that $S_2(v_2, v_3)$ is independent and anti-complete to x by **(P7)**. By **(P5)** and our assumption, the set $S_2(v_2, v_3)$ is anti-complete to S_1. Since $\{v_2, v_3\}$ is not a clique cutset separating $S_2(v_2, v_3)$, one of $S_2(v_3, v_4)$ and $S_2(v_1, v_2)$ is non-empty. We assume by symmetry that $S_2(v_3, v_4) \neq \emptyset$ and let $w \in S_2(v_3, v_4)$. By **(P8)**, $S_1 = S_1(v_3)$. Moreover, x is anti-complete to $S_2(v_1, v_2) \cup S_2(v_3, v_4)$. Otherwise we consider the 5-hole $C' = x - v_1 - v_2 - v_3 - v_4 - x$, and define S_3' with respect to C' in the same way as we define S_3. It is easy to check that $|S_3'| \geq 2$ and we are now in **Case 1**. Furthermore, $S_2(v_0, v_4) = \emptyset$. If not, let $t \in S_2(v_0, v_4)$. Since $x - v_0 - t - w - z - v_2$ does not induce a P_6, x must be adjacent to t, and thus $\{x, v_0, v_4, t\}$ induces a K_4. Note also that $S_2(v_0, v_1) = \emptyset$ by **(P9)**. This implies that $S_1(v_3) = \emptyset$, for otherwise $\{v_3\}$ is a clique cutset of G. This shows that $S_1 = \emptyset$. Now G admits a 3-colouring, namely $\{v_1, v_4, z\}$, $\{v_2, w, v_0\}$, $\{x, v_3\} \cup S_2(v_1, v_2)$. This proves the claim. ◇

By **(P8)**, one of $S_2(v_3, v_4)$ and $S_1(v_2)$ is empty, and one of $S_2(v_1, v_2)$ and $S_1(v_3)$ is empty. On the other hand, $S_2(v_3, v_4) \cup S_1(v_3) \neq \emptyset$ and $S_2(v_1, v_2) \cup S_1(v_2) \neq \emptyset$, by Claim I and $\delta(G) \geq 3$. This leads to the following two subcases.

(i) $S_1(v_2) \neq \emptyset$ *and* $S_1(v_3) \neq \emptyset$ *while* $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$.

By **(P8)**, we conclude that $S_2(v_0, v_1) = S_2(v_0, v_4) = \emptyset$, and so $S_2 = \emptyset$. Since $\{v_3\}$ is not a clique cutset separating $S_1(v_3)$, we have $S_1(v_1) \neq \emptyset$. Similarly, $S_1(v_4) \neq \emptyset$. Let $u_i \in S_1(v_i)$ for $i \neq 0$. By **(P2)**, each $S_1(v_i)$ is a clique, for $i \neq 0$. Moreover, $|S_1(v_1)| + |S_1(v_3)| = |S_1(v_2)| + |S_1(v_4)| = 3$ as $\delta(G) \geq 3$ and G contains no K_4. If $|S_1(v_1)| = 2$, then $|S_1(v_4)| = 1$ and this further implies that $|S_1(v_2)| = 2$. Now $\{u_4, v_1, v_2\} \cup S_1(v_1) \cup S_1(v_2)$ induces the Hajós graph. Therefore, $|S_1(v_1)| = |S_1(v_4)| = 1$ and $|S_1(v_2)| = |S_1(v_3)| = 2$. Note that x is anti-complete to $\{u_1, u_4\}$ or G would contain a W_5, since G is C_4-free. Now G admits a 3-colouring, namely $\{v_1, u_3, u_2, v_4\}$, $\{v_0, v_3, u_1, u_2'\}$, $\{x, u_4, u_3', v_2\}$ where $u_j' \in S_1(v_j)$ (if exists) with $u_j' \neq u_j$, $j = 2, 3$. ◇

(ii) $S_2(v_1, v_2) \neq \emptyset$ *and* $S_2(v_3, v_4) \neq \emptyset$ *while* $S_1(v_2) = S_1(v_3) = \emptyset$.

Let $y \in S_2(v_3, v_4)$ and $z \in S_2(v_1, v_2)$. If x is not anti-complete to $S_2(v_1, v_2) \cup S_2(v_3, v_4)$, then x is complete to the set by **(P11)** and so $C \setminus \{v_0\} \cup \{y, z\}$ induces the Hajós graph. Hence, x is anti-complete to $S_2(v_1, v_2) \cup S_2(v_3, v_4)$ By **(P9)**, we have $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$. Since $\{v_3, v_4\}$ is not a clique cutset, $S_2(v_3, v_4)$ has a neighbour in $S_1(v_1)$. Similarly, $S_2(v_1, v_2)$ has a neighbour in $S_1(v_4)$. However, this contradicts **(P12)**. ◇

This completes the proof of the case that $S_1(v_0) = \emptyset$. Now assume that $S_1(v_0) \neq \emptyset$. Let $y \in S_1(v_0)$. Note first that $xy \in E$ by **(P13)**. Furthermore, it follows from **(P8)** that $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$. If $S_1(v_0)$ is not anti-complete to $S_2(v_2, v_3)$, G contains the graph F. Therefore, we assume that $S_1(v_0)$ is anti-complete to $S_2(v_2, v_3)$. This implies that $S_2(v_2, v_3) = \emptyset$, for otherwise $\{v_2, v_3\}$ is a clique cutset of G. Since $\delta(G) \geq 3$, we have that $S_1(v_2) \neq \emptyset$ and $S_1(v_3) \neq \emptyset$. By **(P10)**, $S_2 = \emptyset$. Let $u_2 \in S_1(v_2)$ and $u_3 \in S_1(v_3)$. Note that $u_2u_3 \notin E$, $u_2y \in E$ and $u_3y \in E$ by **(P2)**. Consider the 5-hole $C' = y - u_2 - v_2 - v_3 - u_3 - y$. We define p-vertices, S'_3 and $S'_p(v_0)$ with respect to C' in the same way we define S_3 and $S_p(v_0)$ for $0 \leq p \leq 5$. If $S_1(v_2)$ contains a vertex u'_2 different from u_2, then $C' \cup \{x, u'_2\}$ induces the Hajós graph. Therefore, $S_1(v_2) = \{u_2\}$. Similarly, $S_1(v_3) = \{u_3\}$. Moreover, $S_1(v_0)$ is a clique by **(P2)**, and hence $S_1(v_0) = \{y\}$. This implies that $S'_3 = \emptyset$ and now we are in **Case 2**.

We have considered all the cases and hence our proof is complete. ■

5.4 Exact obstructions for 4-Colourability

It is clear that the graph obtained from any (C_4, P_6)-free minimal obstruction to 3-COLOURABILITY by adding a universal vertex is a (C_4, P_6)-free minimal obstruction to 4-COLOURABILITY. In this way, we already identify four minimal obstructions to 4-COLOURABILITY which arise from graphs in Figure 5.2. We call those obstructions *trivial*. The main result in this section is that there are 9 additional non-trivial minimal obstructions to 4-COLOURABILITY.

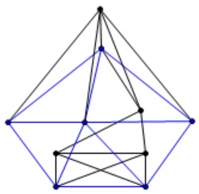
Theorem 5.4.1 *There are exactly 9 non-trivial (C_4, P_6)-free minimal obstructions to 4-COLOURABILITY, given in Figure 5.3.*

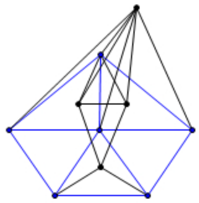
Our proof for Theorem 5.4.1 consists of two parts. The first part deals with the case when G contains an induced W_5 (see Figure 5.2). In the second part of the proof, we handle the case when G has no induced W_5. The technique we use here is to choose a 5-hole with certain minimality conditions and derive some additional properties, valid for graphs without induced W_5.

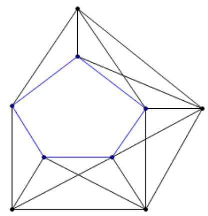
5.4.1 The case G contains a W_5

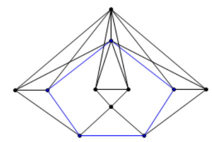
Lemma 5.4.2 *Let G be a (C_4, P_6)-free minimal obstruction that contains an induced W_5. Then either G is a trivial obstruction or G is isomorphic to F_1 or F_2, see Figure 5.3.*

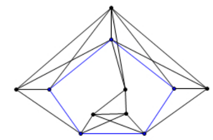
Proof. Let $W = \{v_0, v_1, v_2, v_3, v_4, w\}$ be an induced W_5 in G, where $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ is a 5-hole and w is complete to C. We define S_p, $S_3(v_i)$, $S_2(v, v_{i+1})$ and $S_1(v_i)$ with respect to C as before. Clearly, $|S_5| \geq 1$ since $w \in S_5$. Moreover, G has no clique cutset and

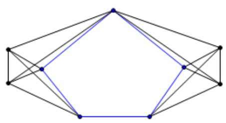

(a) F_1.

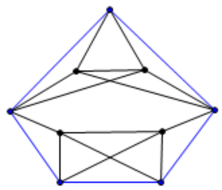

(b) F_2.

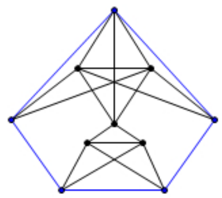

(c) G_{P_4}.


(d) H_1.


(e) H_2.


(f) $G_{3,1}$.


(g) $G_{2,2}$.


(h) $G_{2,1,1}$.

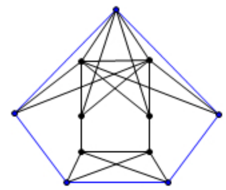

(i) $G_{1,1,1,1}$.

Figure 5.3: 9 non-trivial (C_4, P_6)-free minimal obstructions to 4-COLOURABILITY. In all figures, the blue part is the underlying induced C_5 or W_5. It is not difficult to verify that each graph is a minimal obstruction.

has $\delta(G) \geq 4$ by Lemma 5.2.2. If $|S_5| \geq 2$, then G is a trivial obstruction arising from W_5. Thus, we assume from now on that $S_5 = \{w\}$. Recall that S_5 is complete to S_3 by **(P1)**. If there exists i such that $S_3(v_i) \neq \emptyset$ and $S_3(v_{i+2}) \neq \emptyset$, then G is a trivial obstruction arising from the Hajós graph. This implies that there are at most two non-empty $S_3(v_i)$. Furthermore, $|S_3(v_i)| \leq 1$ or G contains a trivial obstruction K_5. So, $|S_3| \leq 2$.

Case 1 $|S_3| = 2$. We may assume that $S_3(v_0) = \{x\}$ and $S_3(v_1) = \{y\}$.

Note first that $xy \notin E$ or G contains a trivial obstruction K_5. If $S_1(v_3)$ contains a vertex t, then $t - v_3 - v_4 - x - v_1 - y$ induces a P_6. This shows that $S_1(v_3) = \emptyset$. If x is not adjacent to some vertex $d \in S_2(v_3, v_4)$, then $d - v_3 - v_2 - y - v_0 - x$ is a P_6. This shows that x is complete to $S_2(v_3, v_4)$, and y is complete to $S_2(v_2, v_3)$ (by symmetry). Consequently, $S_2(v_3, v_j)$ for $j = 2, 4$ is a clique since G is C_4-free. Moreover, w is complete to $S_2(v_2, v_3) \cup S_2(v_3, v_4)$, for otherwise a non-neighbour of w in $S_2(v_2, v_3) \cup S_2(v_3, v_4)$, v_3, w and y (or x) induce a C_4. Hence, $S_2(v_3, v_j)$ (for $j = 2, 4$) contains of a single element, or G contains a trivial obstruction K_5. Since $d(v_3) \geq 4$, $N_{S_2}(v_3)$ is non-empty. By symmetry, we assume that $S_2(v_2, v_3) = \{z\}$. If $S_2(v_3, v_4)$ is also non-empty, say $S_2(v_3, v_4) = \{z'\}$, then $S_1 = \emptyset$ by **(P8)** and the fact that $S_1(v_3) = \emptyset$, and $S_2 = \{z, z'\}$ by **(P9)** and **(P11)**. Now G admits a 4-colouring, namely $\{v_3, x, y\}$, $\{v_0, v_2, z'\}$, $\{v_4, v_1, z\}$, $\{w\}$.

In the following, we assume that $S_2(v_3, v_4) = \emptyset$. Note first that $S_1(v_1) = S_1(v_4) = \emptyset$ by **(P8)**. If $S_2(v_0, v_4)$ contains a vertex z', then $yz' \in E$ by **(P11)** and thus $wz' \in E$. Now $C \setminus \{v_1\} \cup \{y, z, z', w\}$ induces a trivial obstruction arising from the Hajós graph. This shows that $S_2(v_0, v_4) = \emptyset$. If $S_1(v_2) \neq \emptyset$, then $S_1(v_0) = \emptyset$ due to **(P10)** and the existence of z. This implies that $\{v_2, y, w\}$ is a clique cutset separating $S_1(v_2)$ from G, a contradiction. Therefore, $S_1(v_2) = \emptyset$. Next we show that $S_1(v_0)$ is also empty. If not, let $u_0 \in S_1(v_0)$. Note that x is complete to $S_1(v_0)$. Since $\{u_0, x, v_1, y\}$ does not induce a C_4. $u_0y \notin E$. This proves that y is anti-complete to $S_1(v_0)$. Since $\{v_0, x, w\}$ is not a clique cutset separating $S_1(v_0)$, u_0 has a neighbour in $S_2(v_2, v_3)$, i.e., $u_0z \in E$. Recall that $wz \in E$ and this implies that $wu_0 \in E$. Now $C \cup \{x, u_0, z, w\}$ induces a trivial obstruction arising from the graph F. Therefore, $S_1(v_0) = \emptyset$. So far, we have proved that $S_1 = \emptyset$ and $S_2 = \{z\} \cup S_2(v_0, v_1) \cup S_2(v_1, v_2)$. If w is non-adjacent to a vertex $q \in S_2(v_1, v_2)$, then $w - v_1 - q - z - w$ induces a C_4. This shows that w is complete to $S_2(v_1, v_2)$ and hence y is anti-complete to $S_2(v_1, v_2)$, for otherwise G contains a trivial obstruction K_5. If $S_2(v_1, v_2)$ contains a vertex q, then $yq \notin E$ and thus $y - v_1 - q - z - y$ induces a C_4. Therefore, $S_2(v_1, v_2) = \emptyset$. Finally, assume that $S_2(v_0, v_1)$ contains a vertex p. Take a connected component A of $S_2(v_0, v_1)$. Since G is P_6-free, each of x and y is either complete or anti-complete to A. Then x and y are complete to A, since neither $\{v_0, v_1, w, x\}$ nor $\{v_0, v_1, w, y\}$ is a clique cutset. This shows that $\{x, y\}$ are complete to $S_2(v_0, v_1)$. As a result, w is adjacent to p or $\{w, x, p, y\}$ induces a C_4. So, w is a universal vertex in G and so G is a trivial obstruction and the lemma holds. This shows that $S_2 = \{z\}$ and clearly G admits a 4-colouring.

Case 2 $|S_3| = 0$.

It follows from **(P9)** that there exists some index i such that $S_2(v_{i-1}, v_i) \cup S_2(v_i, v_{i+1}) = \emptyset$. Since $d(v_i) \geq 4$, it follows that $S_1(v_i) \neq \emptyset$. By symmetry, assume that $i = 0$. Applying the argument for **Case 2** in Theorem 5.3.1, we conclude that $S_1(v_i) \neq \emptyset$ for each $0 \leq i \leq 4$, and so $G = W \cup S_1$ by **(P10)**. Let $u_i \in S_1(v_i)$, $i = 0, 1, 2, 3, 4$. Suppose that w is adjacent to u_0. Then w is adjacent to u_2, since $w - u_0 - u_2 - v_2 - w$ does not induce a C_4. This together with **(P2)** implies that w is either complete or anti-complete to S_1. If w is anti-complete to S_1, then $w - v_2 - u_2 - u_4 - u_1 - u_3 -$ induce a P_6. Therefore, w is complete to S_1 and so w is a universal vertex in G. This implies that G is a trivial obstruction and the lemma holds.

Case 3 $|S_3| = 1$. We assume that $S_3 = S_3(v_0) = \{x\}$. We consider two cases. Suppose first that $S_1(v_0) = \emptyset$. We first claim that $S_2(v_2, v_3) = \emptyset$. By contradiction, suppose that $S_2(v_2, v_3)$ contains a vertex z. Then $S_1(v_1) = S_1(v_4) = \emptyset$ by **(P8)**. Moreover, x is anti-complete to $S_2(v_2, v_3)$ by **(P7)**. Since $\{v_2, v_3, w\}$ is not a clique cutset separating $S_2(v_2, v_3)$, one of $S_2(v_3, v_4)$ and $S_2(v_1, v_2)$ is not empty. By symmetry, we assume that $S_2(v_3, v_4)$ contains a vertex p. Then $S_2(v_0, v_1) = \emptyset$ by **(P9)**. Also, $S_1 = S_1(v_3)$ by **(P8)**. If $S_1(v_3) \neq \emptyset$, then $\{v_3, w\}$ is a clique cutset separating $S_1(v_3)$ from G. This shows that $S_1 = \emptyset$. By **(P9)**, one of $S_2(v_4, v_0)$ and $S_2(v_1, v_2)$ is empty. In either case, there are at most $S_2(v_i, v_{i+1})$ are non-empty and each of them is a clique (by **(P3)**) and has at most two vertices (or G contains a trivial obstruction K_5). This implies that G has at most 13 vertices. A straightforward (but tedious) case-by-case analysis according to how many vertices each non-empty $S_2(v_i, v_{i+1})$

contains shows that G is either 4-colourable (this is a contradiction) or a trivial obstruction and the lemma holds. This proves that $S_2(v_2, v_3) = \emptyset$.

Now similar to **Case 3** in Theorem 5.3.1, the following two cases are possible.

(i) $S_1(v_2) \neq \emptyset$ *and* $S_1(v_3) \neq \emptyset$ *while* $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$.

Then $S_2 = \emptyset$ by **(P8)**. If $S_1(v_1) = \emptyset$, then $\{v_3, w\}$ is a clique cutset separating $S_1(v_3)$ from G. This shows that $S_1(v_1) \neq \emptyset$. Similarly, $S_1(v_4) \neq \emptyset$. Let $u_i \in S_1(v_i)$ for $i = 1, 2, 3, 4$. Note that w is either complete or anti-complete to S_1. If w is anti-complete to S_1, then $w - v_2 - u_2 - u_4 - u_1 - u_3$ induces a P_6. Therefore, w is complete to S_1, and thus w is a universal vertex in G. This implies that G is a trivial obstruction and the lemma holds. $\diamond$

(ii) $S_2(v_1, v_2) \neq \emptyset$ *and* $S_2(v_3, v_4) \neq \emptyset$ *while* $S_1(v_2) = S_1(v_3) = \emptyset$.

It then follows from **(P9)** that $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$. If x has a neighbour $q \in S_2(v_3, v_4)$, then $C' = C \setminus \{v_0\} \cup \{x\}$ induces a 5-hole and its corresponding set S_3' of 3-vertices contains v_0 and q. In other words, we are in **Case 1**. This shows that x is anti-complete to S_2. Since $\{v_3, v_4, w\}$ is not a clique cutset separating $S_2(v_3, v_4)$, $S_2(v_3, v_4)$ has a neighbour in $S_1(v_1)$. Similarly, $S_2(v_1, v_2)$ has a neighbour in $S_1(v_4)$. However, this contradicts **(P12)**. $\diamond$

Now assume that $S_1(v_0) \neq \emptyset$. Let y be a vertex in $S_1(v_0)$. Then $S_2(v_1, v_2) \cup S_2(v_3, v_4) = \emptyset$ by **(P8)**. Suppose first that $S_2(v_2, v_3) = \emptyset$. Since $d(v_2) \geq 4$ and $d(v_3) \geq 4$, both $S_1(v_2)$ and $S_1(v_3)$ are not empty. This implies that $S_2 = \emptyset$ by **(P10)**, and thus $G = W \cup \{x\} \cup S_1$. Let $p \in S_1(v_3)$ and $q \in S_1(v_2)$. Note that w is either complete or anti-complete to S_1. If w is complete to S_1, then w is a universal vertex in G. So, G is a trivial obstruction and the lemma holds. From now on we assume that w is anti-complete to S_1. If $S_1(v_1)$ (or $S_1(v_4)$) contains a vertex t, then $w - v_1 - t - p - y - q$ (or $w - v_4 - t - q - y - p$) induces a P_6 in G, a contradiction. This shows that $S_1 = S_1(v_0) \cup S_1(v_2) \cup S_1(v_3)$. Moreover, $S_1(v_j) (j = 0, 2, 3)$ is a clique (by **(P3)**) and contains at most two vertices (or G contains a trivial obstruction K_5). Moreover, $|S_1(v_0)| \leq 2$ since x is complete to $S_1(v_0)$ by **(P13)**. If $S_1(v_0) = \{y\}$, then both $S_1(v_3)$ and $S_1(v_2)$ have exactly 3 vertices, since $d(p), d(q) \geq 4$. Now $S_1(v_3) \cup S_1(v_2) \cup \{y, v_2, v_3\}$ induces a $G_{3,1}$. Since G is a minimal obstruction, $G = G_{3,1}$ but this contradicts the assumption that G contains a W_5. Therefore, $S_1(v_0)$ contains a vertex y' different from y, and $S_1(v_0) = \{y, y'\}$. Again, since $d(p) \geq 4$, $S_1(v_3)$ contains a vertex p' other than p. But then $\{v_0, y, p, v_3, v_4, x, y', p', w\}$ induces a G_{P_4} with respect to the 5-hole $v_0 - y - p - v_3 - v_4 - v_0$. So, $G = G_{P_4}$ but this contradicts the assumption that G contains a W_5.

Now assume that $S_2(v_2, v_3)$ contains a vertex z. Since $\{v_2, v_3, w\}$ is not a clique cutset separating $S_2(v_2, v_3)$, $S_2(v_2, v_3)$ is not anti-complete to $S_1(v_0)$. Without loss of generality, we may assume that $yz \in E$. If $wy \in E$, then $wz \in E$ since $w - y - z - v_3 - w$ is not a C_4. This implies that G is a trivial obstruction arising from the graph F (see Figure 5.2). We assume, therefore, in the following that w is anti-complete to $\{y, z\}$, or more generally, to any edge between $S_1(v_0)$ and $S_2(v_2, v_3)$. On the other hand, $S_1 = S_1(v_0)$ by **(P12)**. Furthermore, one of $S_2(v_0, v_1)$ and $S_2(v_0, v_4)$ is empty by **(P9)**, say $S_2(v_0, v_1) = \emptyset$. This further implies that $S_2(v_0, v_4) = \emptyset$ or $\{w, x, v_0, v_4\}$ is a clique cutset. Thus, $G = W \cup \{x\} \cup S_1(v_0) \cup S_2(v_2, v_3)$. Note that neither of $S_1(v_0)$ and $S_2(v_2, v_3)$ contains a triangle or G contains a trivial obstruction K_5. This implies that any triangle $T \subseteq S_1(v_0) \cup S_2(v_2, v_3)$ contains a vertex from both $S_1(v_0)$ and $S_2(v_2, v_3)$, and so $T \cup W \cup \{x\}$ is isomorphic to F_1 or F_2, i.e., G is isomorphic to either F_1 or F_2 if such triangle exists. Therefore, no such triangle exists and the edges between $S_1(v_0)$ and $S_2(v_2, v_3)$ form a matching (since G is C_4-free). Since $d(y) \geq 4$ and $d(z) \geq 4$, we conclude that y and z have a neighbour $y' \in S_1(v_0)$ and $z' \in S_2(v_2, v_3)$, respectively. Then $y'z, yz' \notin E$, and this implies that $y'z' \notin E$ or $z' - y' - y - z - z'$ induces a C_4. If w is complete to $\{y', z'\}$, then $\{w, y, y', z, z'x, v_0, v_2, v_3\}$ induces a $G_{3,1}$ with respect to the 5-hole $w - y -' y - z - z' - w$. So, G is isomorphic to $G_{3,1}$ but this contradicts that G contains a W_5 since $G_{3,1}$ is W_5-free.

If $wy' \in E$, then $wz' \notin E$ and so $v_1 - w - y' - y - z - z'$ induces a P_6 in G. This shows that $wy' \notin E$, and (by symmetry) that $wz' \notin E$. On the other hand, z is a universal vertex in $S_2(v_2, v_3)$ by **(P5)**, and this implies that z' cannot have a neighbour in $S_2(v_2, v_3)$ different from z, for otherwise G contains a trivial obstruction K_5. Again, since $d(z') \geq 4$, z' has a neighbour y'' in $S_1(v_0)$. Clearly, $y'' \notin \{y, y'\}$. Recall that w is anti-complete to $\{z', y''\}$. Then y'' is not adjacent to y or y', for otherwise either $y'' - y - z - z' - y''$ induces a C_4 or $y'' - y' - y - z - v_3 - w$ induces a P_6. Now y'' must have a neighbour $y''' \in S_1(v_0)$ since $d(y'') \geq 4$. Clearly, y, y', y'', y''' are pairwise distinct. If y''' is adjacent to both y and y'', then $\{y, y', y''', v_0, x\}$ induces a trivial obstruction K_5. On the other hand, y''' is adjacent to at least one of y and y' since $y''' - y'' - z' - z - y - y'$ does not induce a P_6. This implies that y''' is adjacent to exactly one vertex in $\{y, y'\}$. But then $\{y, y', y'', y''', z', v_2\}$ induces a P_6. This final contradiction completes the proof of this case.

We have considered all the cases and hence our proof is complete. ■

5.4.2 The case G is W_5-free

We now investigate minimal obstructions with no induced W_5.

Observation 5.4.3 *Suppose that G is a (C_4, P_6, W_5, K_5)-free graph. Let $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ be an induced five-cycle in G so that $|S_3|$ is minimum among all induced five-cycles. Suppose that $t_i \in S_3(v_i)$ for $i = 2, 3$ and t_2 and t_3 are non-adjacent, then the the following properties hold.*

(O1) *If both $S_1(v_{i-1})$ and $S_1(v_{i+1})$ are non-empty, then $S_3(v_i)$ is anti-complete to $S_1(v_{i-1}) \cup S_1(v_{i+1})$.*

(O2) *If $S_2(v_{i-1}, v_i)$ and $S_2(v_i, v_{i+1})$ are non-empty, then $S_3(v_i)$ is complete to $S_2(v_{i-1}, v_i) \cup S_2(v_i, v_{i+1})$.*

(O3) *Let $x \in S_3(v_{i-1}) \cup S_3(v_{i+1})$ and pq be any edge with $p \in S_1(v_i)$ and $q \in S_2(v_{i+2}, v_{i+3})$. Then x is anti-complete to $\{p, q\}$.*

(The above three properties hold for any induced five-cycle.)

(O4) *$S_2(v_0, v_1)$ contains a neighbour of t_2 and $S_2(v_0, v_4)$ contains a neighbour of t_3.* In particular, $S_2(v_0, v_j)$ is non-empty for $j = 0, 4$.

(O5) *$S_2 = N_{S_2}(v_0)$.*

(O6) *$S_2(v_0, v_j)$ is a clique of size at most two for $j = 0, 4$, and S_2 contains at most three vertices.*

(O7) *S_1 is an empty set.*

(O8) *$S_3(v_j)$ and $S_3(v_{j+1})$ are complete to each other for any $j \neq 2$.*

(O9) *Either $S_3(v_2)$ or $S_3(v_3)$ consists of a single element.*

Proof. We prove them one by one. For **(O1)-(O3)**, it suffices to prove for $i = 0$.

(O1). Let x, y and z be arbitrary vertices in $S_1(v_4)$, $S_1(v_1)$ and $S_3(v_0)$, respectively. Then $xy \in E$ by **(P2)** and this implies that $v_4 - v_0 - v_1 - y - x - v_4$ induces a 5-hole. Since $\{v_4, v_0, v_1, x, y, z\}$ does not induce a W_5, it follows that z is not adjacent to x or y, say $zx \notin E$. Then it also implies that $zy \notin E$ or $z - y - x - v_4 - z$ induces a C_4. This proves that z is anti-complete to $\{x, y\}$ and hence the claim (by **(P2)**).

(O2). Let x, y and z be arbitrary vertices in $S_2(v_4, v_0)$, $S_2(v_0, v_1)$ and $S_3(v_0)$, respectively. Then $xy \in E$ by **(P3)**. Since $\{v_4, v_0, v_1, x, y, z\}$ does not induce a W_5, it follows that z is adjacent to x or y, say $zx \in E$. Then it also implies that $zy \in E$, for otherwise $z - x - y - v_1 - z$ induces a C_4. So, the claim follows.

(O3). Without loss of generality, assume that $x \in S_3(v_1)$. Note that $v_0 - v_1 - v_2 - q - p - v_0$ induces a 5-hole in G and x is adjacent to v_0, v_1 and v_2. Then the same argument used in **(O1)** shows that x is adjacent to neither p nor q. This proves **(O3)**.

To prove **(O4)-(O9)**, we consider the induced five-cycles $C' = C \backslash \{v_2\} \cup \{t_2\}$. Note that $t_3 \notin S'_3$ since $t_3 t_2 \notin E$. It then follows from the minimality of C that $S'_3 \cap (S_1 \cup S_2)$ contains a vertex, say p. It is easy to see by the definition of 3-vertex and **(P7)** that $p \in S_2(v_3, v_4) \cup S_2(v_0, v_1)$. In addition, p is adjacent to t_2. If $p \in S_2(v_3, v_4)$, then p is non-adjacent to t_3 since $\{p, t_3, t_2, v_2\}$ does not induce a C_4. But now the induced five-cycle $p - v_4 - t_3 - v_2 - t_2 - p$ and the vertex v_3 induce a W_5. This shows that p lies in $S_2(v_0, v_1)$, i.e., $S_2(v_0, v_1)$ contains a neighbour of t_2.

By symmetrically considering the induced five-cycles $C \setminus \{v_3\} \cup \{t_3\}$, it follows that $S_2(v_0, v_4)$ contains a neighbour of t_3, say q. This proves **(O4)** (see Figure 5.4 below).

Since $S_2(v_0, v_j)$ is non-empty for $j = 0, 4$, it follows by **(P9)** that $S_2(v_2, v_3) = \emptyset$. On the other hand, since $pt_2 \in E$ and $qt_3 \in E$, it follows by **(P11)** that $S_2(v_1, v_2) = S_2(v_3, v_4) = \emptyset$. This proves **(O5)**. **(O6)** follows directly from **(P3)** and the fact that G is K_5-free. Since $S_2(v_0, v_j)$ is non-empty, we derive from **(P8)** that $S_1 = S_1(v_0)$. Moreover, if $S_1(v_0)$ contains a vertex, say u_0, then $t_3 - v_3 - t_2 - v_1 - v_0 - u_0$ induces a P_6. This shows that $S_1 = \emptyset$ and hence proves **(O7)**. **(O8)** follows immediately from **(O4)** and **(O5)**.

Finally, suppose that both $S_3(v_2)$ and $S_3(v_3)$ have at least two vertices. Since G is K_5-free, it follows that $|S_3(v_j)| = 2$ for $j = 2, 3$. We assume that $S_3(v_2) = \{t_2, t_2'\}$ and $S_3(v_3) = \{t_3, t_3'\}$. We claim that $S_3(v_2) \cup S_3(v_3)$ contains at most one edge. To this end, assume without loss of generality that t_2' and t_3' are adjacent. Since G is K_5-free, t_2' and t_3' are non-adjacent to t_3 and t_2, respectively. This further implies that t_2 and t_3 are non-adjacent, for otherwise $S_3(v_2) \cup S_3(v_3)$ induces a C_4. It follows from the proof of **(O4)** that $S_2(v_0, v_1)$ and $S_2(v_0, v_4)$ contain two neighbours of t_2 and t_3, respectively. But then $\{v_0\} \cup S_2$ contains a K_5. This contradiction proves **(O9)**.

We have considered all items and hence our proof is complete. ∎

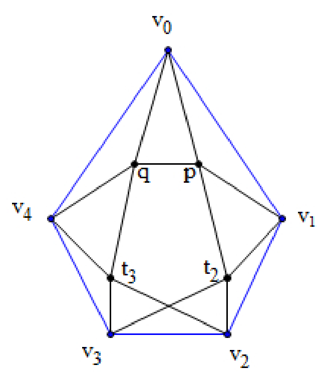


Figure 5.4: Illustration of **(O4)-(O9)**. The blue part is the underlying 5-hole C. The fact that t_2 and t_3 are non-adjacent implies that $S_2(v_0, v_1)$ contains a vertex p and $S_2(v_0, v_4)$ contains a vertex q. Moreover, $S_2 = S_2(v_0, v_1) \cup S_2(v_0, v_4)$, $2 \leq |S_2| \leq 3$ and $S_1 = \emptyset$.

We now prove the main result in this subsection.

Lemma 5.4.4 *Suppose that G is a (C_4, P_6, W_5, K_5)-free minimal obstruction to 4-COLOURABILITY. Then G is isomorphic to one of the graphs in $\{G_{3,1}, G_{2,2}, G_{2,1,1}, G_{1,1,1,1}, H_1, H_2, G_{P_4}\}$ (see Figure 5.3).*

Proof. Let G be a (C_4, P_6, W_5, K_5)-free minimal obstruction. By Lemma 5.2.2, G has $\delta(G) \geq 4$ and no clique cutset. Since G is an obstruction, G is not 4-colourable. We conclude that G is imperfect, and thus G contains a 5-hole by the Strong Perfect Graph Theorem. We choose a 5-hole $C = v_0 - v_1 - v_2 - v_3 - v_4 - v_0$ such that

$$|S_3| \text{ is as small as possible.}$$

If both $S_3(v_i)$ and $S_3(v_{i+2})$ contains at least two vertices for some i, then $C \cup S_3(v_i) \cup S_3(v_{i+2})$ induces a subgraph that is isomorphic to $G_{3,1}$, and since G is a minimal obstruction, we conclude that $G = G_{3,1}$. Moreover, since G is K_5-free, each $S_3(v_i)$ contains at most two vertices. This shows that $|S_3| \leq 7$. In the remaining of the proof, we shall frequently consider some induced five-cycle C' different from C. We define p-vertices with respect to C' the same way we define these notions for C and adapt those definitions by employing the notation S'_p. For instance, S'_1 is the set of vertices in $V \setminus C'$ that are adjacent to exactly one vertex on C'. Let $s = (s_0, \ldots, s_4)$ be an integer vector. We say that C is of *type* s if $S_3(v_i)$ has size s_i for each $0 \leq i \leq 4$.

We proceed by considering different possibilities of types of C. It follows from **(O9)** that at most one $S_3(v_i)$ contains two vertices. Therefore, $|S_3| \leq 6$. If $|S_3| = 6$, then there is some i such that $|S_3(v_i)| = 2$ and $|S_3(v_j)| = 1$ for any $j \neq i$. Since G is K_5-free, both $S_3(v_{i-1})$ and $S_3(v_{i+1})$ are not complete to $S_3(v_i)$. This, however, contradicts **(O8)**. By the same token, there are only two distinct configurations among S_3 when $|S_3| = 5$: C is of type either $(1, 1, 1, 2, 0)$ or $(1, 1, 1, 1, 1)$. If C is of type $(1, 1, 1, 1, 1)$, then S_3 contains at most one non-edge due to **(O8)**. But then G contains G_{P_4} as an induced subgraph. We now assume that C is of type $(1, 1, 1, 2, 0)$. Let $t_i \in S_3(v_i)$ for $i \neq 4$, and $t'_3 \in S_3(v_3)$. Since G is K_5-free, we assume that t_2 and t_3 are non-adjacent. This implies by **(O8)** that $t_0t_1, t_1t_2 \in E$, and therefore t_2 and t'_3 are non-adjacent, for otherwise $C \cup S_3 \setminus \{t_3\}$ induces a G_{P_4}. It then follows from the proof of **(O4)** that $S_2(v_0, v_1)$ contains two neighbours of t_2, say q and q', and that $S_2(v_0, v_4)$ contains a neighbour p of t_3, and $S_2 = \{p, q, q'\}$. In addition, t_0 is complete to S_2 by **(O2)** and **(O5)**. But now $\{t_0, v_0, p, q, q'\}$ induces a K_5, a contradiction. This proves that $|S_3| \leq 4$.

Case 1: $2 \leq |S_3| \leq 4$ and there is some i such that $S_3(v_i)$ and $S_3(v_{i+1})$ are not complete to each other.

Whenever possible, we choose $i = 2$. In other words, we always assume that some vertex $t_2 \in S_3(v_2)$ is non-adjacent to a vertex $t_3 \in S_3(v_3)$.

Case 1.1. $|S_3| = 4$ and $S_3(v_3) = \{t_3, t'_3\}$. It follows from **(O8)** and K_5-freeness of G that one of $S_3(v_2)$ and $S_3(v_4)$ is empty. We may assume that $S_3(v_4) = \emptyset$.

Case 1.1.a. $S_3(v_2) = \emptyset$. This means that $|S_3(v_i)| = 1$ for $i = 0, 1$. Let $t_i \in S_3(v_i)$ for $i = 0, 1$.

Then $t_0t_1 \notin E$ by **(P4)** and our assumption. It then follows from **(O4)** that $S_2(v_2, v_3)$ and $S_2(v_3, v_4)$ are non-empty and $S_2 = S_2(v_2, v_3) \cup S_2(v_3, v_4)$. This implies that $S_3(v_3)$ is complete to S_2 by **(O2)**. But now $\{v_3\} \cup S_3(v_3) \cup S_2$ contains a K_5, a contradiction.

Case 1.1.b. $S_3(v_2) \neq \emptyset$ and $S_3(v_0) \neq \emptyset$.

Let $t_i \in S_3(v_i)$ for $i = 0, 2$. We may assume that $t_3t_2 \notin E$ (since G has no K_5). Let $p \in S_2(v_0, v_4)$ and $q \in S_2(v_0, v_1)$ be a neighbour of t_3 and t_2, respectively, as shown in **(O4)**. Note that t_0 is complete to $N_{S_2}(v_0)$ by **(O2)**. Since G is K_5-free, we conclude that $S_2 = \{p, q\}$. This further implies that $t_2t_3' \in E$ (for otherwise the minimality of C forces S_2 to have at least three vertices). Note that p and t_3' may or may not be adjacent. In either case, G admits a 4-colouring, namely $\{v_2, v_4, q\}$, $\{v_1, v_3, p\}$, $\{t_0, t_2, t_3\}$, $\{v_0, t_3'\}$.

Case 1.1.c. $S_3(v_2) \neq \emptyset$ and $S_3(v_1) \neq \emptyset$.

Let $t_i \in S_3(v_i)$ for $i = 1, 2$. We assume that $t_3t_2 \notin E$. Moreover, t_1 and t_2 are adjacent by **(O8)**. Let $p \in S_2(v_0, v_4)$ and $q \in S_2(v_0, v_1)$ be a neighbour of t_3 and t_2, respectively, as shown in **(O4)**. Note that $pt_1 \notin E$ since $\{p, t_3, v_2, t_1\}$ does not induce a C_4. We now claim that t_2 is complete to $S_2(v_0, v_1)$. If $t_2q \notin E$, then either $\{t_2, v_1, q, p\}$ induces a C_4 or $t_2 - v_1 - q - p - v_4 - v_3$ induces a P_6, depending on whether $t_2p \in E$. This proves the claim and therefore $S_2(v_0, v_1) = \{q\}$. This, together with the proof of **(O4)**, implies that $t_2t_3' \in E$. Then $pt_3' \notin E$ since $\{p, q, t_2, t_3'\}$ does not induce a C_4. Now G admits a 4-colouring, namely $\{v_1, v_3\} \cup S_2(v_0, v_4) \setminus \{p\}$, $\{v_2, v_4, q\}$, $\{v_0, t_2, t_3\}$, $\{t_1, t_3', p\}$.

Case 1.2. $|S_3| = 4$ and every $S_3(v_i)$ contains at most one vertex.

By symmetry, let $t_i \in S_3(v_i)$ for $i \neq 0$. Then, there is exactly one index $i \in \{1, 2, 3\}$ such that $t_it_{i+1} \notin E$.

Case 1.2.a. $t_2t_3 \notin E$, and $t_1t_2, t_3t_4 \in E$.

Let $p \in S_2(v_0, v_4)$ and $q \in S_2(v_0, v_1)$ be a neighbour of t_3 and t_2, respectively, as shown in **(O4)**. Applying the same argument in **Case 1.1.c**, we conclude that t_1 is complete to $S_2(v_0, v_1)$ and therefore $S_2(v_0, v_1) = \{q\}$. Symmetrically, $S_2(v_0, v_4) = \{p\}$. Now G admits a 4-colouring, namely $\{v_2, v_4, q\}$, $\{v_1, v_3, p\}$, $\{v_0, t_2, t_3\}$, $\{t_4, t_1\}$.

Case 1.2.b. $t_3t_4 \notin E$, and $t_1t_2, t_2t_3 \in E$.

Let $p \in S_2(v_0, v_1)$ and $q \in S_2(v_1, v_2)$ be a neighbour of t_4 and t_3, respectively, as shown in **(O4)**. Note that t_1 is complete to S_2 by **(O2)**. This implies that $S_2 = \{p, q\}$ since G is K_5-free. Moreover, $qt_2 \notin E$ since $\{t_1, t_2, v_1, v_2, q\}$ is not a K_5. Now G admits a 4-colouring, namely $\{v_0, q, t_2\}$, $\{v_2, v_4, p\}$, $\{t_1, t_3, t_4\}$, $\{v_1, v_3\}$.

Case 1.3. $2 \leq |S_3| \leq 3$, and $S_3(v_i) = \{t_i\}$ for $i = 2, 3$ with $t_2t_3 \notin E$.

It follows from **(O4)** that t_3 and t_2 have a neighbour $p \in S_2(v_0, v_4)$ and $q \in S_2(v_0, v_1)$, respectively.

Case 1.3.a. $S_3(v_0) = \{t_0\}$.

It follows from **(O2)** that t_0 is complete to S_2. Consequently, $S_2 = \{p, q\}$. Now G admits a 4-colouring: $\{v_0, t_2, t_3\}$, $\{v_1, v_3, p\}$, $\{v_2, v_4, q\}$, $\{t_0\}$.

Case 1.3.b. $S_3(v_1) = \{t_1\}$.

Note first that $t_1 p \notin E$ since $\{t_1, p, t_3, v_2\}$ does not induce a C_4. This implies that $t_1 q \in E$, for otherwise $t_1 - v_1 - q - p - t_3 - v_3$ induces a P_6. This shows that t_1 is complete to $S_2(v_0, v_1)$ and thus $S_2(v_0, v_1) = \{q\}$. Let $p' \in S_2(v_0, v_4)$ be a second (possible) vertex. Now G admits a 4-colouring, namely $\{v_0, t_2, t_3\}$, $\{v_1, v_3\} \cup S_2(v_0, v_4) \setminus \{p\}$, $\{v_2, v_4, q\}$, $\{t_1, p\}$.

Case 1.3.c. $S_3(v_3) \setminus \{t_3\}$ contains at most one vertex, say t_3'.

Note that $2 \leq |S_2| \leq 3$ by **(O5)** and **(O6)**. Let r be a possible third vertex in S_2. We claim that $t_3' r \notin E$. If t_3' does not exist, the claim is trivially true. Now assume that t_3' exits. If $t_3' t_2 \notin E$, then the proof of **(O4)** shows that $r \in S_2(v_0, v_1)$ and this implies that $r t_3' \notin E$ by **(P7)**. So, $t_3' t_2 \in E$ and $r \in S_2(v_0, v_4)$. This implies that $r t_3' \notin E$, for otherwise $\{r, t_3', t_2, q\}$ induces a C_4. This proves the claim. Now G admits a 4-colouring, namely $\{v_0, t_2, t_3\}$, $\{v_1, v_3, p\}$, $\{v_2, v_4, q\}$, $\{t_3', r\}$.

This completes the proof of **Case 1**. In the following, we assume that $|S_3| \leq 4$ and $S_3(v_i)$ and $S_3(v_{i+1})$ are complete to each other for each i. If $|S_3| = 4$ and there is some i such that $|S_3(v_i)| = 2$, say $i = 3$, then it must be the case that $S_3(v_0) \neq \emptyset$ and $S_3(v_1) \neq \emptyset$. But then G is $G_{2,2}$. If $|S_3| = 4$ and each $S_3(v_i)$ has at most one vertex, then G is G_{P_4}. Therefore, $|S_3| \leq 3$. We distinguish four cases.

Case 2: $|S_3| \leq 2$ **and** $S_3 = S_3(v_0) \subseteq \{t_0, t_0'\}$**.**

It follows from **(P9)** that there exists some index i such that $N_{S_2}(v_i) = \emptyset$. We consider two subcases.

Case 2.1. $S_1 = \emptyset$.

Since $\delta(G) \geq 4$, it follows that $S_3(v_0) = \{t_0, t_0'\}$. In addition, $i \notin \{2, 3\}$. By symmetry, there are two subcases.

Case 2.1.a. $i = 1$, i.e., $S_2(v_0, v_1) \cup S_2(v_1, v_2) = \emptyset$.

Since $d(v_2) \geq 4$, it follows that $S_2(v_2, v_3)$ contains at least two vertices, say d_{23} and d_{23}'. Recall that $S_3(v_0)$ is anti-complete to $S_2(v_2, v_3)$ by **(P7)**. This implies that $S_2(v_3, v_4)$

contains a vertex, say d_{34}, for otherwise $\{v_2, v_3\}$ is a clique cutset separating $S_2(v_2, v_3)$ from G. It then follows from **(P3)** and K_5-freeness of G that $S_2(v_2, v_3) = \{d_{23}, d'_{23}\}$ and $S_2(v_3, v_4) = \{d_{34}\}$. Moreover, $S_2(v_0, v_4)$ is a clique of size at most two, and let $S_2(v_0, v_4) \subseteq \{d_{04}, d'_{04}\}$. Since G is (K_5, C_4)-free, $S_2(v_0, v_4) \cup S_3(v_0)$ contains at most one edge. We may assume by symmetry that $\{d_{04}, t'_0\}$ is the only (possible) edge in $S_2(v_0, v_4) \cup S_3(v_0)$. Now G admits a 4-colouring, namely $\{v_0, v_2, d_{34}\}$, $\{v_1, v_4, d_{23}\}$, $\{v_3, d_{04}, t_0\}$, $\{t'_0, d'_{23}, d'_{04}\}$.

Case 2.1.b. $i = 0$, i.e., $S_2(v_0, v_1) \cup S_2(v_0, v_4) = \emptyset$.

Suppose first that $S_2(v_2, v_3) = \emptyset$. Since $d(v_j) \geq 4$ for $j = 2, 3$, it follows that $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ contain at least two vertices. Recall that $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ are anti-complete to each other by **(P3)**. Since $\{v_1, v_2\}$ does not separate $S_2(v_1, v_2)$, we may assume that t_0 has a neighbour in $S_2(v_1, v_2)$, and therefore t_0 is complete to S_2 by **(P11)**. This implies that $S_2(v_i, v_{i+1})$ is a clique of size two for $i = 1, 3$. Now $C \setminus \{v_0\} \cup \{t_0\} \cup S_2$ contains $G_{3,1}$ as an induced subgraph. This settles the case that $S_2(v_2, v_3) = \emptyset$. In the following we assume that $S_2(v_2, v_3) \neq \emptyset$. Since $\{v_2, v_3\}$ does not separate $S_2(v_2, v_3)$, $S_2(v_1, v_2) \cup S_2(v_3, v_4) \neq \emptyset$, say $S_2(v_1, v_2)$ is non-empty. If $S_2(v_3, v_4)$ is also non-empty, then S_3 is anti-complete to S_2 by **(P11)**, and it is easy to check that G admits a 4-colouring. This shows that $S_2 = S_2(v_1, v_2) \cup S_2(v_2, v_3)$. Since $d(v_3) \geq 4$, it follows that $S_2(v_2, v_3) = \{d_{23}, d'_{23}\}$ and $S_2(v_1, v_2) = \{d_{12}\}$. Now G admits a 4-colouring, namely $\{v_1, v_4, d_{23}\}$, $\{v_3, v_0, d_{12}\}$, $\{t_0, v_2\}$, $\{t'_0, d'_{23}\}$.

Case 2.2. $S_1(v_0) \neq \emptyset$.

Then $S_2(v_1, v_2) \cup S_2(v_3, v_4) = \emptyset$ by **(P8)**. Suppose first that $S_3(v_0) \subseteq \{t_0\}$. Since $d(v_j) \geq 4$ for $j = 3, 4$, we conclude that either $S_2(v_2, v_3)$ and $S_2(v_0, v_4)$ are non-empty or $S_1(v_j) \neq \emptyset$ for $j = 3, 4$ and $S_2(v_2, v_3) \cup S_2(v_0, v_4) = \emptyset$. In the former case, it follows from **(P8)** and **(P9)** that $S_2(v_0, v_1) = \emptyset$ and $S_1(v_1) = \emptyset$. Consequently, v_1 has degree at most three in G and this is a contradiction. Therefore, both $S_1(v_3)$ and $S_1(v_4)$ are non-empty. Symmetrically, it follows that $S_1(v_1)$ and $S_1(v_2)$ are non-empty. In other words, $S_1(v_i) \neq \emptyset$ for each $i \in \{0, 1, 2, 3, 4\}$. This implies that $S_2 = \emptyset$ by **(P10)**. Let $u_i \in S_1(v_i)$ for each i. If $S_3 = \emptyset$, then since $\delta(G) \geq 4$, each set $S_1(v_i)$ contains at least two vertices, and then S_1 contains G_{P_4} as an induced subgraph. Therefore, let $S_3(v_0) = \{t_0\}$. Note that $|S_1(v_j)| \geq 2$ for $j = 2, 3$, and $|S_1(v_0)| \leq 2$ since G is K_5-free. Furthermore, each $S_1(v_i)$ is a clique of size at most three and $|S_1(v_i) \cup S_1(v_{i+2})| \leq 4$ by **(P2)** and the fact that G is K_5-free. In the following we shall use u'_i and u''_i to denote the second and the third vertex in $S_1(v_i)$, respectively (if they exist).

Case 2.2.a. $|S_1(v_0)| = 2$.

This implies that $|S_1(v_j)| = 2$ for $j = 2, 3$. If $S_1(v_1)$ or $S_1(v_4)$ contains two vertices, then S_1 is G_{P_4}. Therefore, both $S_1(v_1)$ and $S_1(v_4)$ are singletons. Now G admits a 4-colouring, namely $\{v_2, v_4, u_0, u_1\}$, $\{v_1, v_3, u'_0, u_4\}$, $\{v_0, u_2, u_3\}$, $\{t_0, u'_2, u'_3\}$.

Case 2.2.b. $|S_1(v_0)| = 1$.

We may assume that $|S_1(v_1)| = 1$, for otherwise S_1 contains G_{P_4}. If $|S_1(v_2)| = |S_1(v_3)| = 3$, then $\{u_0, v_2, v_3\} \cup S_1(v_2) \cup S_1(v_3)$ induces a $G_{3,1}$. Suppose first that $|S_1(v_3)| = 3$. Then $|S_1(v_2)| = 2$. If $|S_1(v_4)| = 2$, then S_1 induces a $G_{2,2}$. Otherwise, $|S_1(v_4)| = 1$ and G admits a 4-colouring, namely $\{v_1, v_4, u_2, u_3\}$, $\{v_0, v_2, u_3'', u_4\}$, $\{v_3, u_0, u_1\}$, $\{t_0, u_2', u_3'\}$. Now assume that $|S_1(v_3)| = 2$. Note that $|S_1(v_4)| \leq 2$. Since $|S_1(v_2) \cup S_1(v_4)| \leq 4$, either $|S_1(v_2)| = 2$ and $|S_1(v_4)| = 2$ or $|S_1(v_2)| = 3$ and $|S_1(v_4)| = 1$. Let r be either the third vertex in $S_1(v_2)$ or the second vertex in $S_1(v_4)$. Now G admits a 4-colouring, namely $\{v_1, v_4, u_2, u_3\}$, $\{v_0, v_3, u_1, u_2'\}$, $\{v_2, u_0, u_4\}$, $\{t_0, u_3', r\}$.

This completes the proof for the case $|S_3| \leq 1$. In the following we let $S_3(v_0) = \{t_0, t_0'\}$. Recall that our assumption is that $S_1(v_0) \neq \emptyset$. Let u_0 be a vertex in $S_1(v_0)$. Since $\{v_0, t_0, t_0'\}$ is not a clique cutset separating $S_1(v_0)$ from G, u_0 has a neighbour in $S_1 \cup S_2$, and the neighbour lies in $S_1(v_2) \cup S_1(v_3) \cup S_2(v_2, v_3)$ by **(P2)** and **(P3)**.

Case 2.2.c. u_0 has a neighbour in $S_1(v_2)$.

In particular, $S_1(v_2) \neq \emptyset$. It then follows from **(P10)** that $S_2 = \emptyset$. Since $d(v_j) \geq 4$, we conclude that $S_1(v_j)$ contains at least two vertices for $j = 2, 3$. Moreover, $S_1(v_j)$ is a clique by **(P2)** for $j = 0, 2, 3$, and hence $S_1(v_0) = \{u_0\}$ (recall that G is K_5-free). If $|S_1(v_2)| \geq 3$, then $\{v_0, t_0, t_0', v_1, u_0, v_2\} \cup S_1(v_2)$ induces a $G_{3,1}$. This shows that $|S_1(v_2)| = 2$. By symmetry, $|S_1(v_3)| = 2$. On the other hand, if $|S_1(v_1)| \geq 2$, then $\{v_0, t_0, t_0', v_1, u_0\} \cup S_1(v_1) \cup S_1(v_3)$ induces a $G_{2,2}$. This shows that $|S_1(v_1)| \leq 1$, and likewise $|S_1(v_4)| \leq 1$. Now it is not difficult to see that G admits a 4-colouring.

Case 2.2.d. u_0 has a neighbour d_{23} in $S_2(v_2, v_3)$.

Recall that $S_3(v_0)$ is complete to $S_1(v_0)$ by **(P13)**, and $S_1 = S_1(v_0)$ by **(P12)**. Moreover, $S_2(v_1, v_2)$, $S_2(v_3, v_4)$, and one of $S_2(v_0, v_1)$ and $S_2(v_0, v_4)$ are empty by **(P8)** and **(P9)**. By symmetry, we assume that $S_2(v_0, v_1) = \emptyset$. This implies that $S_2(v_0, v_4)$ is also empty, for otherwise $\{v_0, v_4, x, x'\}$ is a clique cutset. This shows that $S_2 = S_2(v_2, v_3)$. If u_0 has two neighbours in $S_2(v_2, v_3)$, then the two neighbours, u_0 and $C \cup S_3(v_0)$ induce a $G_{2,1,1}$. If d_{23} has two neighbours in $S_1(v_0)$, then the two neighbours and $\{v_0, t_0, t_0'\}$ induce a K_5. This proves that the edges between $S_1(v_0)$ and $S_2(v_2, v_3)$ form a matching. If $u_0'd_{23}'$ is an edge between $S_1(v_0)$ and $S_2(v_2, v_3)$ different from $u_0 d_{23}$, then d_{23} and d_{23}' are adjacent to by **(P5)**, and therefore u_0 and u_0' are not adjacent due to the C_4-freeness of G. Recall that u_0 and u_0' are adjacent to t_0 and t_0'. But then $C \cup \{t_0, t_0', u_0, u_0', d_{23}, d_{23}'\}$ induces a $G_{1,1,1,1}$. Therefore, $\{u_0, d_{23}\}$ is the only edge between $S_1(v_0)$ and $S_2(v_2, v_3)$. Since G has no clique cutset, we conclude that $S_1(v_0) = \{u_0\}$ and $S_2(v_2, v_3) = \{d_{23}\}$. Now $G = C \cup \{t_0, t_0', u_0, d_{23}\}$ and admits a 4-colouring.

Note that the proof of **Case 2.2** shows that $S_1(v_i) = \emptyset$ for each i if $S_3 = \emptyset$. Hence, **Case 2.1** and **Case 2.2** reach a contradiction if $S_3 = \emptyset$. In the following, we assume that $\{t_0\} \subseteq S_3(v_0) \subseteq \{t_0, t_0'\}$. Moreover, $S_1(v_0) = \emptyset$.

Case 2.3. $S_1(v_2) \neq \emptyset$.

Since $\{v_2\}$ does not separate $S_1(v_2)$ from G, $S_1(v_2)$ has a neighbour in either $S_2(v_0, v_4)$ or $S_1(v_4)$. If $S_1(v_2)$ has a neighbour in $S_2(v_0, v_4)$, then $S_1 = S_1(v_2)$ by **(P12)**. Furthermore, $S_2(v_3, v_4) = \emptyset$ by **(P8)**. Since $d(v_3) \geq 4$, it follows that $S_2(v_2, v_3) \neq \emptyset$. This implies that $S_2(v_1, v_2) = \emptyset$ by **(P9)**. Recall that $S_1(v_0) = \emptyset$ and $S_2(v_2, v_3)$ is anti-complete to $S_2(v_0, v_4)$ by **(P3)**. Now $\{v_2, v_3\}$ is a clique cutset separating $S_2(v_2, v_3)$ from G. Therefore, $S_1(v_2)$ has a neighbour in $S_1(v_4)$, and consequently $S_2 = \emptyset$ by **(P10)**. Since $d(v_j) \geq 4$, it follows that $|S_1(v_j)| \geq 2$ for $j = 2, 3$. In particular, $S_1(v_3) \neq \emptyset$ and this implies that $S_1(v_1) \neq \emptyset$, for otherwise $\{v_3\}$ is a clique cutset. Note that $S_3(v_0)$ is anti-complete to S_1 by **(O1)** and **(P6)**. If $S_3(v_0) = \{t_0\}$, this implies that $d(t_0) = 3$, a contradiction. Therefore, $S_3(v_0) = \{t_0, t_0'\}$. If $|S_1(v_1) \cup S_1(v_4)| \geq 4$, then $\{v_0, t_0, t_0', v_1, v_4\} \cup S_1(v_1) \cup S_1(v_4)$ induces a $G_{3,1}$ or $G_{2,2}$. This shows that $2 \leq |S_1(v_1) \cup S_1(v_4)| \leq 3$. Without loss of generality, assume that $|S_1(v_1)| = 1$. If $|S_1(v_2)| = 3$, then $|S_1(v_4)| = 1$; otherwise $|S_1(v_2)| = 2$ and $|S_1(v_4)| \leq 2$. In either case, $|S_1(v_3)| \leq 3$. Let u_i, u_i', u_i'' be the first, second, and third vertex (if exist) in $S_1(v_i)$, respectively, for $i \neq 0$. For example, u_2, u_2' and u_4 exist but u_2'' or u_4' may not exist. In fact, we denote x a possible third vertex in $S_1(v_2)$ or second vertex in $S_1(v_4)$. Now G admits a 4-colouring, namely $\{v_0, v_3, u_1, u_2\}$, $\{v_1, v_4, u_2', u_3'\}$, $\{v_2, u_3, u_4, t_0\}$, $\{t_0', u_3'', x\}$. This completes the proof of this subcase.

Therefore, $S_1(v_2) = \emptyset$. By symmetry, $S_1(v_3) = \emptyset$. Due to **Case 2.1** and **Case 2.2**, we assume by symmetry that $S_1(v_1) \neq \emptyset$. Then $S_2(v_2, v_3) = \emptyset$ by **(P8)**. Since $\{v_1\} \cup S_3(v_0)$ does not separate $S_1(v_1)$ from G, $S_1(v_1)$ has a neighbour in either $S_1(v_4)$ or $S_2(v_3, v_4)$. If $S_1(v_1)$ has a neighbour in $S_1(v_4)$, then $S_2 = \emptyset$ by **(P10)** and therefore v_3 has degree two in G. This contradicts the fact that $\delta(G) \geq 4$. Therefore, $S_1(v_1)$ has a neighbour in $S_2(v_3, v_4)$. This implies that $S_1 = S_1(v_1)$ by **(P12)**. Since $d(v_2) \geq 4$ and $d(v_3) \geq 4$, it follows that both $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ have at least two vertices. Thus, $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$ by **(P9)**. Since $\{v_1, v_2\}$ does not separate $S_2(v_1, v_2)$ from G, $S_2(v_1, v_2)$ is not anti-complete to $S_3(v_0)$, say t_0 has a neighbour in $S_2(v_1, v_2)$. This implies that t_0 is complete to S_2 by **(P11)**. But now $C \setminus \{v_0\} \cup \{t_0\} \cup S_2$ induces a $G_{3,1}$. This completes the proof of this case.

Case 3: Let $t_i \in S_3(v_i)$ for $i = 0, 1, 4$.

By our assumption, $t_0 t_j \in E$ for $j = 1, 4$. We consider two subcases.

Case 3.1. $S_2(v_2, v_3)$ is empty.

Since $\delta(G) \geq 4$, it follows from **(P8)** that either both $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ are non-empty but $S_1(v_2)$ and $S_1(v_3)$ are empty or both $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ are empty but $S_1(v_2)$ and $S_1(v_3)$ are non-empty.

Case 3.1.a. $S_2(v_1, v_2) \neq \emptyset$, $S_2(v_3, v_4) \neq \emptyset$, and $S_1(v_2) = S_1(v_3) = \emptyset$.

Then $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$ by **(P9)** and our assumption that $S_2(v_2, v_3)$ is empty. Moreover, $S_1 = S_1(v_1) \cup S_1(v_4)$ by **(P8)**. Since $S_2 \neq \emptyset$, we may assume by **(P10)** that $S_1(v_4) = \emptyset$. If t_0 has no neighbour in $S_2(v_1, v_2)$, then $\{v_1, v_2, t_1\}$ is a clique cutset of G separating $S_2(v_1, v_2)$ from G. This shows that t_0 has a neighbour in $S_2(v_1, v_2)$ and therefore is complete to S_2 by **(P11)**. Recall that t_0 is adjacent to both t_1 and t_4. Let d_{12} and d_{34} be a vertex in $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$, respectively. Now the induced five-cycle $C \setminus \{v_0\} \cup \{t_0\}$ together with $\{d_{12}, t_1, d_{34}, t_4\}$ induces a $G_{3,1}$. This completes the proof of this subcases.

Case 3.1.b. $S_1(v_j) \neq \emptyset$ for $j = 2, 3$, and $S_2(v_1, v_2) \cup S_2(v_3, v_4) = \emptyset$.

Then $S_2 = \emptyset$ by our assumption and **(P8)**. Note that $|S_1(v_0)| \leq 2$. Let $u_i \in S_1(v_i)$ for $i = 2, 3$. We first claim that $S_1(v_0) = \emptyset$. Suppose not, let $u_0 \in S_1(v_0)$. Then u_0 is adjacent to u_2 and u_3 by **(P2)**. Furthermore, t_1 and t_4 are anti-complete to $\{u_0, u_2\}$ and $\{u_0, u_3\}$, respectively by **(O3)**. Consider the induced five-cycle $C' = u_3 - v_3 - v_2 - u_2 - u_0 - u_3$. Note that $t_j, v_j \notin S'_3$ for $j = 0, 1, 4$. It follows from minimality of C that $S'_3 \cap S_1$ contains at least three vertices. Clearly, these vertices are in $S_1(v_0) \cup S_1(v_2) \cup S_1(v_3)$. If $S_1(v_2)$ contains two of them, then $\{v_2, u_0\} \cup S_1(v_2) \cup \{v_0, t_0, v_1, t_1\}$ induces a $G_{2,2}$. This shows that $|S'_3 \cap S_1(v_2)| \leq 1$. Similarly, $|S'_3 \cap S_1(v_3)| \leq 1$. Moreover, $|S'_3 \cap S_1(v_0)| \leq 1$. This implies that $|S'_3 \cap S_1(v_j)| = 1$ for $j = 0, 2, 3$. But now $\{v_0, t_0, v_1, t_1\} \cup S_1(v_0) \cup S_1(v_2)$ induces a G_{P_4}. This shows that $S_1(v_0)$ is indeed empty. Since G has no clique cutset, it follows that $S_1(v_j)$ contains a vertex, say u_j, for $j = 1, 4$. If $S_1(v_1)$ or $S_1(v_4)$ contains two vertices, then $\{v_0, v_1, v_4, t_0, t_1, t_4\} \cup S_1(v_1) \cup S_1(v_4)$ contains a G_{P_4}. Since $d(u_2) \geq 4$, it follows that $|S_1(v_2)| = 3$. Similarly, $|S_1(v_3)| = 3$. Now it is easy to check that G admits a 4-colouring.

Case 3.2. $S_2(v_2, v_3) \neq \emptyset$.

Let d_{23} be a vertex in $S_2(v_2, v_3)$. It follows from **(P8)** to **(P10)** that there are at most two non-empty $S_1(v_i)$. If there exists i such that $S_1(v_i) \neq \emptyset$ and $S_1(v_{i+1}) \neq \emptyset$, then $i = 2$ due to **(P8)** and the fact that $S_2(v_2, v_3) \neq \emptyset$. But then $\{v_2, t_1\}$ is a clique cutset separating $S_1(v_2)$. In other words, $S_1 = S_1(v_i)$ for some i. Note that $i \neq 1, 4$ by **(P8)**. If $S_2(v_1, v_2)$ contains two vertices d_{12} and d'_{12}, then t_1 is non-adjacent to one of them, say d_{12}. Since $\{t_1, v_1, d_{12}, d_{23}\}$ does not induce a C_4, t_1 is not adjacent to d_{23}. But then $t_1 - v_1 - d_{12} - d_{23} - v_3 - v_4$ induces a P_6. This proves that $|S_2(v_1, v_2)| \leq 1$. Similarly, $|S_2(v_3, v_4)| \leq 1$. Furthermore, if t_1 has two neighbours in $S_2(v_2, v_3)$, say r and r', then the induced five-cycle $v_0 - t_1 - r - v_3 - v_4 - v_0$

and its four 3-vertices $\{v_2, r', t_0, t_4\}$ induce a $G_{2,2}$. This shows that t_j $(j = 1, 4)$ has at most one neighbour in $S_2(v_2, v_3)$. Thirdly, no vertex in $S_2(v_2, v_3)$ is adjacent to both t_1 and t_4, for otherwise the vertex, v_0, v_1 and v_4 induce a C_4.

Case 3.2.a. $S_1 = \emptyset$.

If $S_2(v_1, v_2) \cup S_2(v_3, v_4) \neq \emptyset$, then $S_2(v_2, v_3)$ is a clique of size at most two by **(P3)**. Assume that $S_2(v_2, v_3) \subseteq \{d_{23}, d'_{23}\}$. By symmetry, assume that $S_2(v_1, v_2) = \{d_{12}\}$. Consider first that $S_2(v_0, v_1)$ is empty. Recall that $|S_2(v_3, v_4)| \leq 1$ and we let $S_2(v_3, v_4) \subseteq \{d_{34}\}$. Note that t_0 is anti-complete to S_2 by **(P11)** if $S_2(v_3, v_4) \neq \emptyset$. If $S_2(v_2, v_3) = \{d_{23}, d'_{23}\}$, then t_j has a non-neighbour $n_j \in S_2(v_2, v_3)$ for $j = 1, 4$. Now G admits a 4-colouring, namely $\{v_2, t_0\} \cup S_2(v_3, v_4)$, $\{v_3, v_0, d_{12}\}$, $\{v_4, t_1, n_1\}$, $\{v_1, t_4, n_4\}$ or $\{v_2, t_0\} \cup S_2(v_3, v_4)$, $\{v_3, v_0, d_{12}\}$, $\{t_1, t_4, n_1\}$, $\{v_4, v_1\} \cup S_2(v_2, v_3) \setminus \{n_1\}$ depending on whether or not $n_1 = n_4$. If $S_2(v_2, v_3) = \{d_{23}\}$, then G admits a 4-colouring, namely $\{v_2, t_0, d_{34}\}$, $\{v_3, v_0, d_{12}\}$, $\{v_4, v_1, d_{23}\}$, $\{t_1, t_4\}$. This shows that $S_2(v_0, v_1) \neq \emptyset$. It follows from **(P11)** and **(O2)** that t_4 is anti-complete to S_2 and that t_1 is complete to $N_{S_2}(v_1)$. This implies that $S_2(v_0, v_1) = \{d_{01}\}$ and t_0 and d_{01} are non-adjacent since G is K_5-free. Recall that t_1 has at most one neighbour in $S_2(v_2, v_3)$, say d_{23} is the only possible neighbour of t_1. Now G admits a 4-colouring, namely $\{v_2, t_0, d_{01}\}$, $\{v_0, v_3, d_{12}\}$, $\{v_1, v_4, d_{23}\}$, $\{t_1, t_4, d'_{23}\}$ (the vertex d'_{23} may or may not exist).

We have shown that $S_2(v_1, v_2) \cup S_2(v_3, v_4) = \emptyset$. Suppose now that $S_2(v_0, v_4)$ contains a vertex d_{04}. Since $\{v_0, v_4, t_0, t_4\}$ does not separate $S_2(v_0, v_4)$ from G, t_1 has a neighbour in $S_2(v_0, v_4)$ and this implies that t_1 is complete to S_2 by **(P11)**, and each non-empty $S_2(v_j, v_{j+1})$ is a clique of size at most two. In addition, d_{04} and t_0 are adjacent, since $\{v_4, d_{04}, t_1, t_0\}$ does not induce a C_4. This implies that t_0 is complete to $S_2(v_0, v_4)$ and therefore $S_2(v_0, v_4) = \{d_{04}\}$. On the other hand, if $S_2(v_2, v_3)$ contains a vertex different from d_{23}, say d'_{23}, then $C' \cup \{v_2, d'_{23}, t_0, t_4\}$ induces a $G_{2,2}$. This shows that $S_2(v_2, v_3) = \{d_{23}\}$. Note also that $t_4 d_{04} \notin E$ or $\{v_0, v_4, t_0, t_4, d_{04}\}$ induces a K_5. Recall that $S_2 = \{d_{23}, d_{04}\}$ by **(P9)** and **(P11)**. Now G admits a 4-colouring, namely $\{v_3, t_0\}$, $\{v_0, v_2\}$, $\{v_1, d_{23}, d_{04}, t_4\}$, $\{v_4, t_1\}$. This proves that $S_2(v_0, v_4) = \emptyset$ and similarly $S_2(v_0, v_1) = \emptyset$. Now $G = C \cup \{t_0, t_1, t_4\} \cup S_2(v_2, v_3)$. Note that $S_2(v_2, v_3)$ is a bipartite graph. Let $S_2(v_2, v_3) = X \cup Y$ be a bipartition of $S_2(v_2, v_3)$. Recall that t_j has at most one neighbour in $S_2(v_2, v_3)$ and we call this neighbour n_j for $j = 1, 4$. If n_1 and n_4 lie in the same partite of $S_2(v_2, v_3)$, say in X, then G admits a 4-colouring, namely $\{v_2, t_0\}$, $\{v_3, v_0\}$, $\{t_4, t_1\} \cup Y$, $\{v_1, v_4\} \cup X$. Otherwise n_1 and n_4 are in different partite, say $n_1 \in X$ and $n_4 \in Y$, then G admits a 4-colouring, namely $\{v_2, t_0\}$, $\{v_3, v_0\}$, $\{v_4, t_1\} \cup Y$, $\{v_1, t_4\} \cup X$.

Now we can assume that $S_1 = S_1(v_i) \neq \emptyset$ for some i. Recall that $i \in \{0, 2, 3\}$.

Case 3.2.b. $i = 2$, i.e., $S_1 = S_1(v_2) \neq \emptyset$.

Let $u_2 \in S_1(v_2)$. Since $\{v_2, t_1\}$ is not a clique cutset separating $S_1(v_2)$ from G, $S_1(v_2)$ is not anti-complete to $S_2(v_0, v_4)$. We may assume that u_2 has a neighbour $d_{04} \in S_2(v_0, v_4)$. Then $S_2 = S_2(v_2, v_3) \cup S_2(v_0, v_4)$ by **(P8)** and **(P9)**. It follows from **(O3)** that t_1 is anti-complete to $\{u_2, d_{04}\}$. This implies that t_1 is anti-complete to S_2 by **(P11)**. Consider the induced five-cycle $C' = d_{04} - u_2 - v_2 - v_3 - v_4 - d_{04}$. Note that $t_0, t_1, v_0, v_1 \notin S_3'$. This implies that $S_3' \cap (C \cup S_3) \subseteq \{t_4\}$. It follows from the minimality of C that $S_3' \cap (S_1 \cup S_2)$ contains at least two vertices, say r and r'. Clearly, $r, r' \notin S_2(v_2, v_3)$. Since $d_{23} - v_2 - u_2 - d_{04} - v_0 - t_0$ does not induce a P_6, t_0 is adjacent to d_{04}. This shows that d_{04} is the only neighbour of u_2 in $S_2(v_0, v_4)$, for otherwise $\{t_0, v_0, v_4\} \cup S_2(v_0, v_4)$ contains a K_5. Consequently, $r, r' \notin S_2(v_0, v_4)$. Therefore, $r, r' \in S_1(v_2)$. By the definition of S_3', r and r' are adjacent and they are also adjacent to u_2. But then $\{u_2, r, r', v_2, d_{04}\}$ induces a K_5 minus an edge and $\{t_0, v_0, t_1, v_1\}$ induces a K_4, and the union induces a $G_{2,2}$.

Case 3.2.c. $i = 0$, i.e., $S_1 = S_1(v_0) \neq \emptyset$.

Let $u_0 \in S_1(v_0)$. Then $S_2(v_1, v_2) \cup S_2(v_3, v_4) = \emptyset$ by **(P8)**. Suppose first that $d_{23}u_0 \in E$. Consider the induced five-cycle $C' = u_0 - v_0 - v_1 - v_2 - d_{23} - u_0$. Clearly, $v_3, v_4 \notin S_3'$. In addition, t_4 is anti-complete to $\{u_0, d_{23}\}$ by **(O3)**. This implies that $t_4 \notin S_3'$. It follows from the minimality of C that $S_3' \cap (S_1 \cup S_2)$ contains a vertex, say r. Note that $r \notin S_2(v_0, v_4) \cup S_2(v_0, v_1)$, and this means that $r \in S_1(v_0) \cup S_2(v_2, v_3)$. By the definition of S_3', r is adjacent to u_0 and d_{23}. Now G is isomorphic to either H_1 or H_2 depending on whether $r \in S_1(v_0)$ or $r \in S_2(v_2, v_3)$. We have therefore shown that $S_1(v_0)$ and $S_2(v_2, v_3)$ are anti-complete to each other. This implies that $\{t_1, t_4\}$ is complete to $S_1(v_0)$ since G has no clique cutset. On the other hand, since $\{v_2, v_3\}$ is not a clique cutset separating $S_2(v_2, v_3)$ from G, we may assume that $d_{23}t_1 \in E$. Then $d_{23}t_4 \notin E$ since G is C_4-free. Since d_{23} has at least four neighbours in G, it follows that there exists a vertex, say d_{23}', in $S_2(v_2, v_3)$ with $d_{23}'d_{23} \in E$. If $d_{23}'t_1 \in E$, then $\{d_{23}, d_{23}', v_2, v_3, t_1\}$ and $\{v_4, t_4, v_0, t_0\}$ induce a $G_{2,2}$. If $d_{23}'t_4 \in E$, then $\{u_0, v_0, t_0, t_1, t_4\}$ and $\{v_2, v_3, d_{23}, d_{23}'\}$ induce a $G_{2,2}$. This shows that d_{23}' is anti-complete to $\{t_1, t_4\}$. Since d_{23}' has at least four neighbours in G, it follows that there exists a vertex $d_{23}'' \in S_2(v_2, v_3)$ with $d_{23}'' \neq d_{23}$ and $d_{23}''d_{23}' \in E$. Moreover, $d_{23}d_{23}'' \notin E$ since $\{d_{23}, d_{23}', d_{23}'', v_2, v_3\}$ is not a K_5, and $d_{23}''t_1 \notin E$, since $d_{23}'' - t_1 - d_{23} - d_{23}' - d_{23}''$ is not a C_4. Note that $P = d_{23}'' - d_{23}' - d_{23} - t_1$ now induces a P_4. Since $P - u_0 - t_4$ does not induce a P_6, d_{23}'' is adjacent to t_4. But now $v_4 - t_4 - P$ induces a P_6. This completes the proof of the case.

Case 4: Let $\{t_0\} \subseteq S_3(v_0) \subseteq \{t_0, t_0'\}$ and $S_3(v_2) = \{t_2\}$.

Case 4.1. $S_1(v_3)$ contains a vertex u_3.

Since $\{v_3, t_2\}$ is not a clique cutset separating $S_1(v_3)$ from G, we may assume that u_3 has a neighbour, say n, in $S_2(v_0, v_1)$ or in $S_1(v_0) \cup S_1(v_1)$. If $n \in S_2(v_0, v_1)$, then $S_1 = S_1(v_3)$ and $S_2(v_0, v_4) \cup S_2(v_1, v_2) = \emptyset$ by **(P8)** and **(P12)**. If $S_3 = \{t_0, t_2\}$, then since $d(v_j) \geq 4$ for $j = 2, 4$, it follows that both $S_2(v_2, v_3)$ and $S_2(v_3, v_4)$ are non-empty. This, however, contradicts **(P9)**. If $S_3 = \{t_0, t_0', t_2\}$, it then follows from **(O3)** that t_2 is anti-complete to $\{n, u_3\}$. Since $t_2 - v_3 - u_3 - n - v_0 - t_0$ does not induce a P_6, n is adjacent to t_0. Similarly, $nt_0' \in E$. But then $\{n, v_0, v_1, t_0, t_0'\}$ induces a K_5. This proves that $S_1(v_3)$ is anti-complete to $S_2(v_0, v_1)$, and therefore $n \in S_1(v_0) \cup S_1(v_1)$. This implies that $S_2 = \emptyset$ by **(P10)**. Since $d(v_2) \geq 4$, it follows that $S_1(v_2)$ contains a vertex, say u_2. In the remaining proof of this case, for each i we shall use u_i, u_i' and u_i'' to denote the first, second and the third vertex in $S_1(v_i)$, respectively (if they exist).

Case 4.1.a. $S_3 = \{t_0, t_2\}$.

Note first that $|S_1(v_j)| \leq 2$ for $j = 0, 2$. In addition, if both $S_1(v_0)$ and $S_1(v_2)$ have size two, then $\{v_0, v_1, v_2, t_0, t_2\} \cup S_1(v_0) \cup S_1(v_2)$ induces a G_{P_4}. Thus, one of $S_1(v_0)$ and $S_1(v_2)$ has size one. Since $\delta(G) \geq 4$ and $S_2 = \emptyset$, it follows that $S_1(v_i)$ contains a vertex u_i for $i \neq 1$. Let u_1 be a vertex in $S_1(v_i)$ if $S_1(v_i)$ is not empty. Note that $C' = v_3 - u_3 - u_1 - u_4 - v_4 - v_3$ is an induced five-cycle. Clearly, $S_3' \subseteq S_1(v_1) \cup S_1(v_3) \cup S_1(v_4)$. We consider the following subcases in terms of the size of $S_1(v_1)$. If $|S_1(v_1)| = 3$, then $|S_1(v_3)| = |S_1(v_3)| = 1$, since G is K_5-free. This implies that $|S_3'| = 2$ and we are in **Case 2**. Now assume that $|S_1(v_1)| = 2$. Then $|S_1(v_3)| \leq 2$ and $|S_1(v_4)| \leq 2$. If both $S_1(v_3)$ and $S_1(v_4)$ have size two, then $|S_1(v_0)| = |S_1(v_2)| = 1$, for otherwise S_1 contains a G_{P_4}. Thus, G admits a 4-colouring, namely $\{v_0, v_2, u_3, u_4\}$, $\{v_1, u_3', u_4'\}$, $\{v_4, u_0, u_1, t_2\}$, $\{v_3, u_2, u_1', t_0\}$. So, we may assume that $|S_1(v_4)| = 1$. It then follows from the minimality of C that $|S_1(v_3)| = 2$. Moreover, if $|S_1(v_0)| = 2$, then $S_1(v_3) \cup S_1(v_1) \cup S_1(v_0) \cup \{t_0, v_0, v_1\}$ induces a G_{P_4}. Hence, $S_1(v_0) = \{u_0\}$. Now G admits a 4-colouring, namely $\{v_0, u_3, u_4, t_2\}$, $\{v_1, v_4, u_3', u_2\}$, $\{v_2, u_0, u_1\}$, $\{v_3, u_1', u_2', t_0\}$.

Next, assume that $|S_1(v_1)| = 1$. Suppose first that $|S_1(v_3)| = 3$. Then $|S_1(v_0)| = 1$. If $|S_1(v_4)| > 2$, then $S_1(v_3) \cup S_1(v_4) \cup \{u_1, v_3, v_4\}$ induces a $G_{3,1}$. On the other hand, if $|S_1(v_4)| < 2$, then C' has $S_3' = S_3'(u_3)$ and $|S_3'| = 2$, and thus we are in **Case 2**. This shows that $|S_1(v_4)| = 2$. This implies that $|S_1(v_2)| = 1$, for otherwise S_1 induces a $G_{2,2}$. Now G admit a 4-colouring: $\{v_1, v_4, u_2, u_3\}$, $\{v_0, u_4, u_3', t_2\}$, $\{v_2, u_3'', u_4', t_0\}$, $\{v_3, u_0, u_1\}$. Thus, we have $|S_1(v_3)| \leq 2$. By symmetry, $|S_1(v_4)| \leq 2$. It then follows from the minimality of C that $|S_1(v_3)| = |S_1(v_4)| = 2$. Recall that $|S_1(v_j)| \leq 2$ for $j = 0, 2$, and one of $S_1(v_0)$ and $S_1(v_2)$ have size one. Without loss of generality, assume that $S_1(v_0) = \{u_0\}$. Now G admit a 4-colouring: $\{v_2, u_0, u_4\}$, $\{u_3, u_4', t_0, t_2\}$, $\{v_0, v_3, u_1, u_2\}$, $\{v_1, v_4, u_2', u_3'\}$.

Finally, assume that $|S_1(v_1)| = 0$. If t_0 is adjacent to u_4, then $t_0 - u_0 - u_2 - u_4 - t_0$ induces a C_4. This shows that t_0 is anti-complete to $S_1(v_4)$. Similarly, t_2 is anti-complete to $S_1(v_3)$.

Suppose first that $|S_1(v_4)| = 3$. Then $|S_1(v_2)| = 1$. If $|S_1(v_0)| = 2$, then $\{v_0, v_4, t_0, u_2\} \cup S_1(v_0) \cup S_1(v_4)$ induces a $G_{2,2}$. Hence, $|S_1(v_0)| = 1$. Let $S_1(v_3) \subseteq \{u_3, u_3', u_3''\}$. Then G admits a 4-colouring, namely $\{v_3, u_0, u_4\}$, $\{v_0, u_3, u_4', t_2\}$, $\{v_2, u_4'', u_3', t_0\}$, $\{v_1, v_4, u_2, u_3''\}$. So, $|S_1(v_4)| \leq 2$. Symmetrically, $|S_1(v_3)| \leq 2$. We may assume that $S_1(v_0) = \{u_0\}$. Now G is an induced subgraph of the graph in the case where $|S_1(v_1)| = 1$ and $|S_1(v_j)| \leq 2$ for $j = 3, 4$. Hence, G is 4-colourable. This completes the proof of this subcase.

Case 4.1.b. $S_3 = \{t_0, t_0', t_2\}$.

We first show that $S_1(v_0) = \emptyset$. Suppose not, let $u_0 \in S_1(v_0)$. Since G is K_5-free, we have $S_1(v_0) = \{u_0\}$. If $|S_1(v_3)| = 3$, then $\{v_0, t_0, t_0', v_3, v_4, u_0\} \cup S_1(v_3)$ induces a $G_{3,1}$. If $|S_1(v_2)| \geq 2$, then $\{v_0, t_0, t_0', v_1, v_2, u_0, t_2\} \cup S_1(v_2)$ induces a $G_{2,2}$. Therefore, $|S_1(v_3)| \leq 2$ and $|S_1(v_2)| = 1$. Now consider the induced five-cycle $C' = v_3 - u_3 - u_0 - u_2 - v_2 - v_3$. It is routine to check that $S_3' \subseteq \{t_2\} \cup S_1(v_3) \setminus \{u_3\}$. In particular, $|S_3'| < 3$ and this contradicts the minimality of C. This proves the claim that $S_1(v_0) = \emptyset$. Therefore, u_3 has a neighbour in $S_1(v_1)$, say u_1. Since $\{v_2, t_2\}$ is not a clique cutset separating $S_1(v_2)$, we have $S_1(v_4) \neq \emptyset$. By **(O1)**, $\{t_0, t_0'\}$ is anti-complete to S_1 and $\{t_2\}$ is anti-complete to $S_1 \setminus S_1(v_2)$. If $|S_1(v_1)| + |S_1(v_4)| \geq 4$, then $\{v_0, v_1, v_4, t_0, t_0'\} \cup S_1(v_1) \cup S_1(v_4)$ contains either a $G_{3,1}$ or a $G_{2,2}$. Thus, $|S_1(v_1)| + |S_1(v_4)| \leq 3$. Now consider the induced five-cycle $C' = v_3 - u_3 - u_1 - u_4 - v_4 - v_3$. Clearly, $t_0, t_0', t_2 \notin S_3'$ and $S_3' \subseteq S_1(v_1) \cup S_1(v_3) \cup S_1(v_4)$. The fact that $|S_1(v_1)| + |S_1(v_4)| \leq 3$ means that $|S_3' \cap (S_1(v_1) \cup S_1(v_4))| \leq 1$. Also, $|S_3' \cap S_1(v_3)| \leq 2$. It then follows from the minimality of C that $|S_1(v_3)| = 3$. Thus, $|S_1(v_1)| = 1$ and $|S_1(v_4)| = 2$. Moreover, $|S_1(v_2)| \leq 2$. Now G admits a 4-colouring, namely $\{v_1, v_4, u_2, u_3\}$, $\{v_0, v_2, u_3', u_4'\}$, $\{t_0, t_2, u_3'', u_4\}$, $\{v_3, u_1, t_0', u_2'\}$.

In the following we assume that $S_1(v_3) = \emptyset$.

Case 4.2. $S_2(v_2, v_3) = \emptyset$.

Since $d(v_3) \geq 4$, it follows that $S_2(v_3, v_4)$ contains at least one vertex, say d_{34}. This implies that $S_1(v_2) = \emptyset$ by **(P8)** and then $S_1(v_1, v_2) \neq \emptyset$ since $d(v_2) \geq 4$. Let $d_{12} \in S_2(v_1, v_2)$. It then follows from **(P8)** and **(P9)** that $S_2 = S_2(v_1, v_2) \cup S_2(v_3, v_4)$ and $S_1 = S_1(v_1) \cup S_1(v_4)$. If $S_3 = \{t_0, t_2\}$, this means that $d(v_0) = 3$, a contradiction. In the following we let $S_3 = \{t_0, t_0', t_2\}$. We first claim that $S_1(v_4) = \emptyset$. Suppose not, let $u_4 \in S_1(v_4)$. Then $S_1(v_1) = \emptyset$ by **(P10)** and the fact that $S_2 \neq \emptyset$. Since $\{v_4, t_0, t_0'\}$ is not a clique cutset of G, it follows that $S_1(v_4)$ is not anti-complete to $S_2(v_1, v_2)$. Without loss of generality, assume that $u_4 d_{12} \in E$. Note that $\{t_0, t_0'\}$ is anti-complete to $\{u_4, d_{12}\}$ by **(O3)**. Consider the induced five-cycle $C' = u_4 - v_4 - v_3 - v_2 - d_{12} - u_4$. Note that $t_0, t_0' \notin S_3'$. It then follows from the minimality of C that $S_3' \cap (S_1 \cup S_2)$ contains at least two vertices, say r and s. Clearly, both r and s are in $S_2(v_1, v_2) \cup S_1(v_4)$ and complete to $\{u_4, d_{12}\}$ by definition. If $r, s \in S_1(v_4)$, then $\{t_0, t_0', v_0, v_1, v_4\} \cup S_1(v_4) \cup S_2(v_1, v_2)$ induces a $G_{3,1}$. Therefore, we may assume that

$s \in S_2(v_1, v_2)$. Since G is K_5-free, we may assume that $t_2 s \notin E$. But then $t_2 - v_2 - s - u_4 - v_4 - v_0$ induces a P_6.

Therefore, $S_1(v_4) = \emptyset$. Since $\{v_1, v_2, y\}$ is not a clique cutset separating $S_2(v_1, v_2)$ from G, $S_2(v_1, v_2)$ has a neighbour in $S_3(v_0)$, say t_0'. It then follows from **(P11)** that t_0' is complete to S_2. This implies that $S_2(v_1, v_2)$ and $S_2(v_3, v_4)$ are cliques of size at most two. Moreover, $S_2(v_3, v_4)$ and $S_1(v_1)$ are anti-complete to each other by **(O3)**. Note also that $t_2 d_{34} \notin E$ or $t_0' - d_{34} - t_2 - v_1 - t_0'$ induces a C_4. We now show that $S_1(v_1) = \emptyset$. Suppose not, let A be any connected component of $S_1(v_1)$. Since $\{v_1, t_0, t_0'\}$ is not a clique cutset separating A, t_2 has a neighbour $a \in A$. Furthermore, since $v_0 - t_0' - d_{34} - v_3 - t_2 - a$ does not induce a P_6, a is adjacent to t_0'. But now either the 5-hole $a - t_0' - d_{12} - v_2 - t_2 - a$ and v_1 induce a W_5 or $\{t_0', d_{12}, t_2, a\}$ induces a C_4 depending on whether $t_2 d_{12} \in E$.

Now $G = C \cup S_2 \cup S_3$. Note also that if $d_{34} t_2 \in E$, then $\{d_{34}, t_2, v_1, t_0'\}$ induces a C_4. This shows that t_2 is anti-complete to $S_2(v_3, v_4)$. Since $d(t_2) \geq 4$, it follows that $t_2 d_{12} \in E$. If $S_2(v_3, v_4) = \{d_{34}, d_{34}'\}$, then $\{t_0', v_4, d_{34}, d_{34}', v_3\}$ and $\{v_1, d_{12}, v_2, t_2\}$ induce a $G_{2,2}$. Finally, let $S_2(v_3, v_4) = \{d_{34}\}$. Then d_{34} is adjacent to t_0, since d_{34} has at least four neighbours in G. This implies that t_0 is complete to S_2 by **(P11)**. Now the induced five-cycle $t_0' - d_{12} - v_2 - v_3 - d_{34} - t_0'$ and its four 3-vertices $\{t_2, v_1, t_0, v_4\}$ induce a G_{P_4}.

Case 4.3. Let d_{23} be a vertex in $S_2(v_2, v_3)$.

Case 4.3.a. $S_3 = \{t_0, t_2\}$.

By symmetry, **Case 4.2** shows that $S_2(v_0, v_4) \neq \emptyset$. Let d_{04} be a vertex in $S_2(v_0, v_4)$. If $S_2(v_3, v_4) = \emptyset$, it follows from **(P8)** and **(P9)** that $S_1 = S_1(v_0) \cup S_1(v_2)$ and $S_2 = S_2(v_0, v_4) \cup S_2(v_2, v_3)$. Since $\{v_2, v_3, t_2\}$ does not separate $S_2(v_2, v_3)$, $S_2(v_2, v_3)$ is not anti-complete to $S_1(v_0)$. Similarly, $S_2(v_0, v_4)$ is not anti-complete to $S_1(v_2)$. But this contradicts **(P12)**. This proves that $S_2(v_3, v_4)$ contains a vertex d_{34}. So, $S_1 = \emptyset$ by **(P10)**. If t_2 is not adjacent to d_{23}, then either $\{t_2, v_2, d_{23}, d_{34}\}$ induces a C_4 or $t_2 - v_2 - d_{23} - d_{34} - d_{04} - v_0$ induces a P_6. This shows that t_2 is complete to $S_2(v_2, v_3)$ and thus $S_2(v_2, v_3) = \{d_{23}\}$. Similarly, $S_2(v_0, v_4) = \{d_{04}\}$. Note that any vertex in $S_2(v_3, v_4)$ is non-adjacent to either t_0 or t_2. We assume that $d_{34} t_0 \notin E$. If $S_2(v_3, v_4) = \{d_{34}\}$, then G admits a 4-colouring, namely $\{v_1, v_4, d_{23}\}$, $\{v_3, d_{01}\}$, $\{v_2, t_0, d_{34}\}$, $\{v_0, t_2\}$. Otherwise let d_{34}' be the second vertex in $S_2(v_3, v_4)$. Then G admits a 4-colouring, namely $\{v_1, v_4, d_{23}\}$, $\{v_3, d_{01}\}$, $\{v_2, t_0, d_{34}\}$, $\{v_0, t_2, d_{34}'\}$ or $\{v_1, v_4, d_{23}\}$, $\{v_3, d_{01}\}$, $\{t_2, t_0, d_{34}\}$, $\{v_0, v_2, d_{34}'\}$ depending on whether or not $t_2 d_{34} \in E$ (in which case $t_2 d_{34}' \notin E$).

Case 4.3.b. $S_3 = \{t_0, t_0', t_2\}$.

Suppose first that $S_2(v_1, v_2) \cup S_2(v_3, v_4)$ is empty. Since $\{v_2, v_3, t_2\}$ is not a clique cutset separating $S_2(v_2, v_3)$ from G, we assume that d_{23} has a neighbour $u_0 \in S_1(v_0)$. Note that u_0 is complete to $S_3(v_0)$ by **(P13)**. Moreover, $S_1 = S_1(v_0)$ by **(P12)**. Consider the induced five-cycle $C' = u_0 - v_0 - v_4 - v_3 - d_{23} - u_0$. Note that $t_2 \notin S_3'$. It then follows from the minimality of C that $S_3' \cap (S_1 \cup S_2)$ contains a vertex, say r. By the definition of S_3', $r \in S_1(v_0) \cup S_2(v_2, v_3)$, and is adjacent to u_1 and d_{23}. If $r \in S_1(v_0)$, then $\{t_0, t_0', v_0, u_1, r\}$ induces a K_5. If $r \in S_2(v_2, v_3)$, then $\{t_0, t_0', v_0, v_1, v_4, u_1, v_3, v_2, d_{23}, r\}$ induces a $G_{2,1,1}$.

Now let d_{12} be a vertex in $S_2(v_1, v_2)$. Then $S_1 = S_1(v_2)$ by **(P8)** and this implies that $S_1 = \emptyset$, for otherwise $\{v_2, t_2\}$ is a clique cutset of G. Furthermore, it follows from **(O2)** that t_2 is complete to $S_2(v_1, v_2) \cup S_2(v_2, v_3)$. This implies that $S_2(v_2, v_3) = \{d_{23}\}$ and $S_2(v_1, v_2) = \{d_{12}\}$. We claim that $S_2(v_0, v_1) = \emptyset$. Suppose not, let $d_{01} \in S_2(v_0, v_1)$. Since $v_3 - v_2 - d_{12} - d_{01} - v_0 - t_0$ does not induce a P_6, t_0 is adjacent to either d_{01} or d_{12}. If t_0 is adjacent to d_{12}, then t_0 is also adjacent to d_{01}, since $\{t_0, v_0, d_{01}, d_{12}\}$ does not induce a C_4. This shows that t_0 is adjacent to d_{01}. Similarly, t_0' is also adjacent to d_{01} and then $\{v_0, v_1, d_{01}, t_0, t_0'\}$ is a K_5. Now let $S_2(v_3, v_4) \subseteq \{r, r'\}$. If $S_2(v_3, v_4) \neq \emptyset$, then $S_3(v_0)$ is anti-complete to S_2 by **(P11)**. Note that we may assume $t_2 r' \notin E$ if $S_2(v_3, v_4) = \{r, r'\}$, since $\{r, r', d_{23}, v_3, t_2\}$ is not a K_5. Now G admits a 4-colouring, namely $\{v_1, v_4, d_{23}\}$, $\{v_0, v_3, d_{12}\}$, $\{v_2, t_0, r\}$, $\{t_0', t_2, r'\}$.

Finally, assume that $S_2(v_1, v_2) = \emptyset$ and let d_{34} be a vertex in $S_2(v_3, v_4)$. Then $S_1 = \emptyset$ by **(P8)** and **Case 4.1**. If $S_2(v_0, v_4)$ contains a vertex r, then we may assume that $r t_0 \notin E$ but then either $v_2 - d_{23} - d_{34} - r - v_0 - t_0$ induces a P_6 or $\{t_0, v_0, r, d_{34}\}$ induces a C_4, depending on whether $t_0 d_{34} \in E$. This shows that $S_2(v_0, v_4) = \emptyset$. If $t_2 d_{23} \notin E$, then either $t_2 - v_2 - d_{23} - d_{34} - v_4 - v_0$ induces a P_6 or $\{t_2, v_2, d_{23}, d_{34}\}$ induces a C_4. This proves that t_2 is complete to $S_2(v_2, v_3)$ and so $S_2(v_2, v_3) = \{d_{23}\}$. If $S_2(v_3, v_4) = \{d_{34}\}$, then G admits a 4-colouring, namely $\{v_0, v_2, d_{34}\}$, $\{v_1, v_4, d_{23}\}$, $\{v_3, t_0\}$, $\{t_2, t_0'\}$. Otherwise $S_2(v_3, v_4) = \{d_{34}, d_{34}'\}$. We may assume that $t_2 d_{34} \notin E$. If $t_2 d_{34}' \in E$, then $d_{34}' t_0, d_{34}' t_0' \notin E$ since G is C_4-free. Otherwise, $t_2 d_{34}' \notin E$ and we may assume without loss of generality that $d_{34}' t_0 \notin E$. In either case G admits a 4-colouring, namely $\{v_1, v_4, p\}$, $\{v_0, t_2, d_{34}\}$, $\{v_2, t_0, d_{34}'\}$, $\{t_0', v_3\}$.

Case 5: Let $S_3(v_i) = \{t_i\}$ for $i = 2, 3$ with $t_2 t_3 \in E$ and $S_3(v_0) \subseteq \{t_0\}$.

Case 5.1. $S_2(v_2, v_3) \neq \emptyset$.

Let d_{23} be a vertex in $S_2(v_2, v_3)$. Recall that $S_2(v_2, v_3)$ and $S_3(v_0)$ are anti-complete to each other by **(P7)**. Since $\{v_2, v_3, y, z\}$ is not a clique cutset separating $S_2(v_2, v_3)$, we assume that d_{23} has a neighbour n in $S_1(v_0)$ or $S_2(v_1, v_2) \cup S_2(v_3, v_4)$. If $n \in S_1(v_0)$, then since $t_2 - v_2 - d_{23} - n - v_0 - v_4$ does not induce a P_6, t_2 is adjacent to d_{23}. Symmetrically, t_3 is adjacent to d_{23}. But then $\{v_2, v_3, t_2, t_3, d_{23}\}$ induces a K_5. Therefore, $n \in S_2(v_1, v_2) \cup S_2(v_3, v_4)$. By symmetry, we assume that $n \in S_2(v_1, v_2)$ and in particular $S_2(v_1, v_2) \neq \emptyset$. Then $S_2(v_0, v_4) = \emptyset$ and $S_1 = S_1(v_2)$ by **(P8)** and **(P9)**. Since $d(v_0) \geq 4$, it follows that $S_2(v_0, v_1)$ contains a vertex, say d_{01}. On the other hand, t_2 is adjacent to d_{23} and n by **(O2)**. This implies that $t_3d_{23} \notin E$. Since $\{t_3, v_3, d_{23}, n\}$ is not a C_4, t_3 is non-adjacent to n. But now $t_3 - v_3 - d_{23} - n - d_{01} - v_0$ induces a P_6.

In the following we assume that $S_2(v_2, v_3) = \emptyset$.

Case 5.2. $S_1(v_0) \neq \emptyset$ and let $u_0 \in S_1(v_0)$.

Note first that $S_1(v_0)$ is anti-complete to S_2 by **(P5)**. Since $\{v_0\} \cup S_3(v_0)$ is not a clique cutset, it follows that u_0 has a neighbour in $S_1(v_2) \cup S_1(v_3)$. Without loss of generality, assume that $u_2 \in S_1(v_2)$ is a neighbour of u_0. This implies that $S_2 = \emptyset$ by **(P10)**. If $|S_1(v_j)| \geq 2$ for $j = 2, 3$, then $\{u_0, v_2, v_3, t_2, t_3\} \cup S_1(v_2) \cup S_1(v_3)$ contains a G_{P_4}. This shows that one of $S_1(v_2)$ and $S_1(v_3)$ contains at most one vertex.

Case 5.2.a. $S_3 = \{t_2, t_3\}$.

Since $d(v_j) \geq 4$ for $j = 0, 1, 4$, it follows that $|S_1(v_0)| \geq 2$, $|S_1(v_1)| \geq 1$, and $|S_1(v_4)| \geq 1$. The fact that $|S_1(v_0)| \geq 2$ implies that $S_1(v_j)$ contains at most one vertex for $j = 2, 3$, for otherwise $S_1(v_0) \cup S_1(v_2) \cup S_1(v_3) \cup \{v_2, v_3, t_2, t_3\}$ induces a G_{P_4}. Consequently, $S_1(v_2) = \{u_2\}$. Let $u_j \in S_1(v_j)$ for $j = 1, 4$. Consider the induced five-cycle $C' = u_1 - v_1 - v_0 - v_4 - u_4 - u_1$. Note that $v_2, v_3, t_2, t_3 \notin S_3'$. It then follows from the minimality of C that $S_3' \cap S_1$ contains at least two vertices r and r'. Clearly, $r, r' \in S_1(v_1) \cup S_1(v_4)$. This implies that $|S_1(v_1)| + |S_1(v_4)| \geq 4$ and therefore $|S_1(v_1)| + |S_1(v_4)| = 4$ by **(P2)** and the fact that G is K_5-free. If both r and r' are in $S_1(v_j)$ for some $j = 1, 4$, then $S_3' = S_3'(v_j) = \{r, r'\}$ and we are in **Case 2**. This shows that $|S_1(v_1)| = |S_1(v_4)| = 2$. If $|S_1(v_0)| = 3$, then $S_1(v_3) = \emptyset$, for otherwise $S_1(v_0) \cup S_1(v_2) \cup S_1(v_3) \cup \{v_2, v_3, t_2, t_3\}$ induces a $G_{2,2}$; otherwise $|S_1(v_0)| = 2$. Recall that $|S_1(v_3)| \leq 1$. Let u_i, u_i', u_i'' be the first, second, and third vertex in $S_1(v_i)$ for each i. We denote x either u_0'' or u_3 depending on the size of $S_1(v_0)$. Now G admits a 4-colouring, namely $\{v_1, t_3, u_4, u_0\}$, $\{v_2, u_4', x\}$, $\{v_0, v_3, u_1, u_2\}$, $\{v_4, t_2, u_0', u_1'\}$.

Case 5.2.b. $S_3 = \{t_0, t_2, t_3\}$.

Note first that $S_1(v_j)$ is a clique of size at most two, for $j = 0, 2, 3$. Let $\{u_0\} \subseteq S_1(v_0) \subseteq \{u_0, u_0'\}$. If both $S_1(v_0)$ and $S_1(v_2)$ contain two vertices, then $\{v_0, v_1, v_2, t_0, t_2\} \cup S_1(v_0) \cup S_1(v_2)$ contains a G_{P_4}. This show that one of $S_1(v_0)$ and $S_1(v_j)$ is of size at most one, for $j = 2, 3$. Recall that one of $S_1(v_2)$ and $S_1(v_3)$ contains at most one vertex. We first consider the case where $S_1(v_3)$ contains a vertex u_3. Thus, two of $S_1(v_0)$, $S_1(v_2)$ and $S_1(v_3)$ are single-element sets. Consider the induced five-cycle $C' = v_3 - u_3 - u_0 - u_2 - v_2 - v_3$. Clearly, $S_3' \subseteq S_1(v_0) \cup S_1(v_2) \cup S_1(v_3) \cup \{t_2, t_3\}$. Thus, the minimality of C implies that one of $S_1(v_0)$, $S_1(v_2)$ and $S_1(v_3)$ has size two. Therefore, we conclude that exactly one of $S_1(v_0)$, $S_1(v_2)$ and $S_1(v_3)$ has size two and the other two sets have size one. If $|S_1(v_2)| = 2$ or $|S_1(v_3)| = 2$, then we are in **Case 3**. Hence, we assume that $|S_1(v_0)| = 2$ and $|S_1(v_2)| = |S_1(v_3)| = 1$. Moreover, if $|S_1(v_4)| = 3$, then $S_1(v_4) \cup \{u_2, v_4, v_2, t_2, v_3, t_3\}$

induces a $G_{2,2}$. This shows that $|S_1(v_4)| \leq 2$. Similarly, $|S_1(v_1)| \leq 2$. Let $S_1(v_j) \subseteq \{u_j, u'_j\}$ for $j = 1, 4$. Now G admits a 4-colouring, namely $\{v_0, u_3, u_4, t_2\}$, $\{v_1, v_3, u_0, u'_4\}$, $\{u_1, u_2, t_0, t_3\}$, $\{v_2, v_4, u'_0, u'_1\}$.

Now we assume that $S_1(v_3) = \emptyset$. If $|S_1(v_4)| = 3$, then $S_1(v_4) \cup \{u_2, v_4, v_2, t_2, v_3, t_3\}$ induces a $G_{2,2}$. Thus, $|S_1(v_4)| \leq 2$. Moreover, if both $S_1(v_2)$ and $S_1(v_4)$ contain two vertices, then $S_1(v_4) \cup S_1(v_2) \cup \{v_4, v_2, t_2, v_3, t_3\}$ induces a G_{P_4}. This shows that either $S_1(v_2)$ or $S_1(v_4)$ contains at most one vertex. Next consider the induced five-cycle $C' = u_2 - u_0 - v_0 - v_1 - v_2 - u_2$. Clearly, $t_0 \in S'_3(v_0)$ and $t_2 \in S'_3(v_2)$. The minimality of C implies that one of $S_1(v_0)$ and $S_1(v_2)$ has size two. On the other hand, if both of them have size two, then $C' \cup \{t_0, t_2, u'_0, u'_2\}$ induces a G_{P_4}, where $u'_2 \in S_1(v_2)$. Therefore, one of $S_1(v_0)$ and $S_1(v_2)$ has size two and the other has size one. We now show that $|S_1(v_1)| \leq 2$. If $S_1(v_4)$ contains a vertex u_4, then $C'' = v_4 - u_4 - u_2 - u_0 - v_0 - v_4$ is a 5-hole. Clearly, $t_0 \in S''_3$ but $t_2, t_3 \notin S''_3$. Moreover, S''_3 contains at most one vertex from $S_1(v_0) \cup S_1(v_2)$. It then follows from the minimality of C that S''_3 contains a vertex from $S_1(v_4)$. In particular, $|S_1(v_4)| \geq 2$ and this implies that $|S_1(v_1)| \leq 2$, since $S_1(v_1) \cup S_1(v_4)$ is a clique of size at most four. If $S_1(v_4) = \emptyset$, then since G has no clique cutset, $\{t_0, t_2\}$ is complete to any connected component of $S_1(v_1)$. This implies that $S_1(v_1)$ is a clique of size at most two, since G is (C_4, K_5)-free.

We have thus proved that $|S_1(v_1)| \leq 2$. Recall that $|S_1(v_4)| \leq 2$. Let $S_1(v_j) \subseteq \{u_j, u'_j\}$ for $j = 1, 4$. Note that t_0 and t_3 are anti-complete to $S_1(v_4)$. If $|S_1(v_2)| = 1$, then $S_1(v_0) = \{u_0, u'_0\}$, and G admits a 4-colouring, namely $\{v_1, u_0, u_4, t_3\}$, $\{t_0, t_2, u'_4\}$, $\{v_0, v_3, u_1, u_2\}$, $\{v_2, v_4, u'_0, u'_1\}$. Otherwise, $|S_1(v_2)| = 2$, and therefore $S_1(v_0) = \{u_0\}$ and $S_1(v_4) \subseteq \{u_4\}$. Let $S_1(v_2) = \{u_2, u'_2\}$. Since $\{v_2, t_2, u_2, u'_2, t_3\}$ is not a K_5, t_3 is not adjacent to one of u_2 and u'_2, say u'_2. Now G admits a 4-colouring, namely $\{v_0, v_3, u_1, u_2\}$, $\{v_1, u'_2, t_3\}$, $\{v_2, v_4, u_0, u'_1\}$, $\{t_0, t_2, u'_4\}$. This completes the proof of this subcase.

In the following we assume that $S_2(v_2, v_3) = S_1(v_0) = \emptyset$. Since $d(v_0) \geq 4$, it follows that $N_{S_2}(v_0) \neq \emptyset$. Assume by symmetry that $S_2(v_0, v_1)$ contains a vertex d_{01}.

Case 5.3. d_{01} has no neighbour in S_2 or equivalently $S_2(v_0, v_4) = S_2(v_1, v_2) = \emptyset$.

Note that $S_2 = S_2(v_0, v_1) \cup S_2(v_3, v_4)$ and $S_1 = S_1(v_3) \cup S_1(v_1)$ by **(P8)**. In case of $S_3 = \{t_2, t_3\}$, $S_2(v_3, v_4)$ contains a vertex d_{34} since $d(v_4) \geq 4$. Since $\{v_0, v_1\} \cup S_3(v_0)$ is not a clique cutset separating $S_2(v_0, v_1)$ from G, either d_{01} has a neighbour $u_3 \in S_1(v_3)$ or d_{01} is adjacent to t_2.

Case 5.3.a. d_{01} has a neighbour $u_3 \in S_1(v_3)$.

Then $S_1 = S_1(v_3)$ by **(P12)**. Suppose first that $S_3 = \{t_2, t_3\}$. Recall that $S_2(v_3, v_4) \neq \emptyset$. Since $\{v_3, v_4, t_3\}$ does not separate $S_2(v_3, v_4)$ from G, we may assume that d_{34} is adjacent to t_2. This implies that t_2 is complete to S_2 by **(P11)** and in particular is adjacent to d_{01}.

This, however, contradicts **(O3)**. Now let $S_3 = \{t_0, t_2, t_3\}$. Since $v_2-v_3-u_3-d_{01}-v_0-t_0$ does not induce a P_6, t_0 and d_{01} are adjacent. This implies that d_{01} is the only neighbour of u_3 in $S_2(v_0, v_1)$, for otherwise a second such neighbour and $\{v_0, v_1, d_{01}, t_0\}$ induce a K_5. Consider the induced five-cycle $C' = d_{01} - v_1 - v_2 - v_3 - u_3$. Clearly, $v_0, v_4, t_0 \notin S'_3$. It then follows from the minimality of C that $S'_3 \cap (S_1 \cup S_2)$ contains a vertex r. Note that $r \in S_1(v_3) \cup S_2(v_0, v_1)$. Furthermore, r is complete to $\{d_{01}, u_3\}$ by the definition of S'_3. Since d_{01} is the only neighbour of u_3 in $S_2(v_0, v_1)$, it follows that $r \in S_1(v_3)$. Recall that t_3 is complete to $\{u_3, r\}$ by **(P13)**. Now G is isomorphic to H_2 (with respect to C': $t_3 \in S'_3(v_3)$, $t_2 \in S'_3(v_2)$, $r \in S'_3(u_3)$, $v_4 \in S'_1(v_3)$, $t_0, v_0 \in S'_2(d_{01}, v_1)$).

Case 5.3.b. d_{01} is adjacent to t_2.

We also assume by **Case 5.3.a** that $S_2(v_0, v_1)$ is anti-complete to $S_1(v_3)$. Suppose first that $S_3 = \{t_2, t_3\}$. Recall that $S_2(v_3, v_4)$ contains a vertex d_{34}. Since $d(v_0) \geq 4$, it follows that $|S_2(v_0, v_1)| = 2$. On the other hand, t_2 is complete to S_2 by **(P11)**. Now $(C \setminus \{v_2\}) \cup \{t_2\} \cup S_2 \cup \{t_3\}$ induces a $G_{3,1}$.

Now let $S_3 = \{t_0, t_2, t_3\}$. If $S_1(v_3) \neq \emptyset$, then $S_1(v_1) = \emptyset$ due to **(P10)** and the fact that $S_2 \neq \emptyset$. But then $\{v_3, t_3, t_2\}$ is a clique cutset separating $S_1(v_3)$ from G. This proves that $S_1(v_3) = \emptyset$ and hence $S_1 = S_1(v_1)$. If some vertex $s \in S_1(v_1)$ is adjacent to a vertex $t \in S_2(v_3, v_4)$, then t_2 is adjacent to t by **(P11)**. This, however, contradicts **(O3)**. This shows that $S_1(v_1)$ is anti-complete to $S_2(v_3, v_4)$. We now claim that $S_1(v_1)$ is a clique. Let A be an arbitrary connected component of $S_1(v_1)$. Note first that t_j $(j = 0, 2)$ is either complete or anti-complete to A since G is P_6-free. Since neither $\{v_1, t_0\}$ nor $\{v_1, t_2\}$ is a clique cutset separating A from G, we conclude that t_j is complete to A, for $j = 0, 2$. Since A is arbitrary, it follows that t_0 and t_2 are complete to $S_1(v_1)$. The (C_4, K_5)-freeness of G, therefore, implies that $S_1(v_1)$ is a clique of size at most two. Moreover, since $\delta(G) \geq 4$, $S_1(v_1)$ contains either no vertex or two vertices.

We next claim that $S_2(v_3, v_4) = \emptyset$. Suppose not. Let $d_{34} \in S_2(v_3, v_4)$. It follows from **(P11)** that t_2 is complete to S_2. This implies that t_3 is complete to $S_2(v_3, v_4)$ or $v_4 - t_3 - t_2 - d_{34} - v_4$ is a C_4. Since G is K_5-free, it follows that $S_2(v_3, v_4) = \{d_{34}\}$. Now $S_1(v_1) = \emptyset$, for otherwise $\{v_1, t_0, t_2, v_3, v_4, t_3\} \cup S_1(v_1) \cup S_2(v_3, v_4)$ induces a $G_{3,1}$. If $|S_2(v_0, v_1)| \geq 2$, then $(C \setminus \{v_2\}) \cup \{t_2\} \cup S_2 \cup \{t_3\}$ induces a $G_{3,1}$. Therefore, $S_2(v_0, v_1) = \{d_{01}\}$. Since $\{t_0, d_{34}, t_2, v_1\}$ does not induce a C_4, t_0 is non-adjacent to d_{34}. This implies that t_0 is adjacent to d_{01} since $d(t_0) \geq 4$. But now $G - v_2$ induce a $G_{2,2}$. Therefore, $S_2(v_3, v_4) = \emptyset$.

Next we show that $t_0 d_{01} \in E$. If not, consider the induced five-cycle $C' = t_3 - v_4 - v_0 - d_{01} - t_2 - t_3$. Note that $v_1, v_3 \in S'_3$ and $t_0, v_2 \notin S'_3$. It follows from the minimality of C that $S'_3 \cap (S_1 \cup S_2)$ contains a vertex, say s. Clearly, $s \in S_2(v_0, v_1)$ and thus is adjacent to v_0, d_{01} and t_2 by definition of S'_3. Moreover, since G is K_5-free, it follows that $S'_3 \cap (S_1 \cup S_2) = \{s\}$ and therefore $S'_3 = \{v_3, v_1, s\}$. Now it is easy to see that we are in **Case 4** $(s, v_1 \in S'_3(d_{01}))$.

This proves that t_0 is adjacent to d_{01}. In fact, the argument shows that t_0 is adjacent to any neighbour of t_2 in $S_2(v_0, v_1)$. It implies that $X = S_2(v_0, v_1) \setminus \{d_{01}\}$ is anti-complete to t_2, and thus $X = \emptyset$, for otherwise $\{v_0, v_1, t_0, d_{01}\}$ is clique cutset separating X from G. It is easy to see that G admits a 4-colouring.

Case 5.4. d_{01} has a neighbour in S_2.

Clearly, the neighbour is in $S_2(v_0, v_4)$ or in $S_2(v_1, v_2)$ by **(P3)**. We first claim that t_2 is complete to $S_2(v_1, v_2)$. If not, let $d_{12} \in S_2(v_1, v_2)$ be non-adjacent to t_2. Then either $\{d_{01}, d_{12}, v_2, t_2\}$ induces a C_4 or $t_2 - v_2 - d_{12} - d_{01} - v_0 - v_4$ induces a P_6, depending on whether or not $t_2 d_{01} \in E$. This proves the claim and this implies that $S_2(v_1, v_2) \subseteq \{d_{12}\}$ (since G is K_5-free).

Case 5.4.a. The neighbour is in $S_2(v_0, v_4)$. Let $d_{04} \in S_2(v_0, v_4)$.

Note that $S_1 = S_1(v_0) = \emptyset$ by **(P8)** and **Case 5.2**. Moreover, we may assume that $S_2(v_3, v_4) = \emptyset$ by **(P9)**. Note that if $S_2(v_1, v_2) \neq \emptyset$, then t_3 is anti-complete to S_2 by **(P11)**. Suppose first that $S_3 = \{t_0, t_2, t_3\}$. It then follows from **(O2)** that t_0 is complete to $S_2(v_0, v_1) \cup S_2(v_0, v_4)$. This implies that $S_2(v_0, v_1) = \{d_{01}\}$ and $S_2(v_0, v_4) = \{d_{04}\}$, since G is K_5-free. Now G admits a 4-colouring ϕ: $\{v_1, v_3, d_{04}\}$, $\{v_2, v_4, d_{01}\}$, $\{t_0, t_2\}$, $\{v_0, t_3\} \cup S_2(v_1, v_2)$.

Now let $S_3 = \{t_2, t_3\}$. If $S_2(v_0, v_1) = \{d_{01}\}$, then G admits a 4-colouring, namely $\{v_1, v_3, d_{04}\}$, $\{v_2, v_4, d_{01}\}$, $\{v_0, t_3\} \cup S_2(v_1, v_2)$, $\{t_2\} \cup S_2(v_0, v_4) \setminus \{d_{04}\}$. This shows that $S_2(v_0, v_1) = \{d_{01}, d'_{01}\}$, and then $S_2(v_0, v_4) = \{d_{04}\}$. If $S_2(v_1, v_2) = \emptyset$, then G admits a 4-colouring, namely $\{v_1, v_3, d_{04}\}$, $\{v_2, v_4, d_{01}\}$, $\{v_0, t_2\}$, $\{t_3, d'_{01}\}$. Finally, let $S_2(v_1, v_2) = \{d_{12}\}$. We may assume that $t_2 d_{01} \notin E$ since $\{t_2, d_{12}, v_1, d_{01}, d'_{01}\}$ is not a K_5. Note also that $t_3 d_{04} \notin E$ by **(P11)**. Now G admits a 4-colouring, namely $\{v_4, t_2, d_{01}\}$, $\{v_1, t_3, d_{04}\}$, $\{v_3, v_0, d_{12}\}$, $\{v_2, d'_{01}\}$.

Case 5.4.b. The neighbour is in $S_2(v_1, v_2)$.

Recall that t_2 is complete to $S_2(v_1, v_2)$. Consequently, $S_2(v_1, v_2) = \{d_{12}\}$ and $t_2 d_{12} \in E$. In addition, $S_2(v_0, v_4) = \emptyset$ due to **Case 5.4.a**. Note that $S_2(v_3, v_4) = \emptyset$ by **(P9)** and $S_1 = S_1(v_1)$ by **(P8)**. Since $d(v_4) \geq 4$, it follows that t_0 exits, i.e., $S_3 = \{t_0, t_2, t_3\}$. Applying the same argument in **Case 5.3.b**, it follows that both t_0 and t_2 are complete to $S_1(v_1)$. We claim that t_0 is complete to $S_2(v_0, v_1)$. If not, suppose that $t_0 d_{01} \notin E$. Then either $v_3 - v_2 - d_{12} - d_{01} - v_0 - t_0$ induces a P_6 or $\{t_0, v_0, d_{01}, d_{12}\}$ induces a C_4, depending on whether $t_0 d_{12} \in E$. Therefore, $S_2(v_0, v_1) = \{d_{01}\}$. If $u_1 \in S_1(v_1)$, then either $u_1 - t_2 - t_3 - v_4 - v_0 - d_{01}$ induces a P_6 or $\{d_{01}, u_1, t_0, t_2\}$ induces a C_4, depending on whether t_2 is adjacent to d_{01}. This proves that $S_1 = S_1(v_1) = \emptyset$. Now $S_2 = \{d_{01}, d_{12}\}$ and then G admits a 4-colouring, namely $\{v_0, v_3, d_{12}\}$, $\{v_2, v_4, d_{01}\}$, $\{t_0, t_2\}$, $\{t_3, v_1\}$.

Our proof of Lemma 5.4.4 is now complete. ■

5.5 Certifying algorithms

We now apply our results from previous sections to the questions of complexity of k-Colourability of (C_4, P_6)-free graphs. Reference **[53]** gives a linear time algorithm for k-Colourability of (C_4, P_t)-free graphs for any fixed integers $k \geq 1$ and $t \geq 1$. However, that algorithm depends on Ramsey-type results, and ends up using tree-decompositions with very high widths. We offer more practical algorithms for 3-Colourability and 4-Colourability of (C_4, P_6)-free graphs. Moreover, our algorithms are *certifying algorithms*. Indeed, they are based on our characterizations of (C_4, P_6)-free minimal obstructions, and when no colouring is found, they exhibit a forbidden induced subgraph from Theorem 5.3.1 and Theorem 5.4.1.

We first consider 3-Colourability. The proof of Theorem 5.3.1 in fact outputs a 3-colouring of the input graph G if it exists and a minimal obstruction if G is not 3-colourable. The proof involves the operations of partitioning $V(G)$ into subsets, determining the size of certain sets and deciding if there are edges between certain subsets. All these steps can be implemented in linear time. To obtain a certifying algorithm, we still need to deal with (C_4, P_6)-free graphs containing a 6-hole.

Lemma 5.5.1 *Suppose that G is a (C_4, P_6)-free graph containing a 6-hole $C = v_0 - \ldots - v_5 - v_0$. If G has $\delta(G) \geq 3$ and no clique cutsets, then there is a linear time algorithm that either finds a 3-colouring of G or an induced subgraph isomorphic to a graph in Figure 5.2.*

Proof. For any $X \subseteq V$, we define

$$S(X) = \{x \in V \setminus C : N_C(x) = X\}.$$

If $S(\emptyset) \neq \emptyset$, i.e., there is a vertex in $V \setminus C$ that is anti-complete to C, then G is a specific by Lemma 5.1.3. Moreover, by slightly modifying the proof of Theorem 7 in **[11]** one can find in linear time a 3-colouring of G if it exists or an induced subgraph isomorphic to a graph in Figure 5.2.

So, assume that $S(\emptyset) = \emptyset$. It is straightforward to verify (by the fact G is (C_4, P_6)-free) that

$$V(G) = C \cup \bigcup_i S(v_i, v_{i+3}) \cup \bigcup_i S(v_{i-1}, v_i, v_{i+1}) \cup \bigcup_i S(v_{i-1}, v_i, v_{i+1}, v_{i+2}) \cup S(C).$$

Moreover, each vertex in $S(C)$ is a universal vertex in G, each such set $S(X)$ is a clique, since G is C_4-free. If some vertex $x \in S(v_5, v_0, v_1)$ is non-adjacent to a vertex $y \in S(v_0, v_1, v_2)$, then $y - v_2 - v_3 - v_4 - v_5 - x$ is a P_6; if some vertex $x \in S(v_5, v_0, v_1)$ is adjacent to a vertex $y \in S(v_2, v_3, v_4)$, then $x - v_1 - v_2 - y - x$ is a 4-hole. So, we conclude that $S(v_{i-1}, v_i, v_{i+1})$ is anti-complete to $S(v_{i+2}, v_{i+3}, v_{i+4})$ and is complete to $S(v_i, v_{i+1}, v_{i+2})$.

If $S(C)$ contains a vertex, say u, then G is 3-colourable if and only if $G-u$ is bipartite. Thus, the lemma holds since testing bipartiteness can be done in linear time. Hence, we assume that $S(C)$ is also empty. If $S(v_i, v_{i+3})$ contains a vertex, say d, then $v_i - v_{i+1} - v_{i+2} - v_{i+3} - d - v_i$ is a 5-hole; if $S(v_{i-1}, v_i, v_{i+1}, v_{i+2})$ contains a vertex, say q, then $v_{i+2} - v_{i+3} - v_{i+4} - v_{i-1} - q - v_{i+2}$ is a 5-hole. Also, if some vertex $x \in S(v_{i-1}, v_i, v_{i+1})$ is adjacent to a vertex $y \in S(v_{i+1}, v_{i+2}, v_{i+3})$, then $x - v_{i-1} - v_{i-2} - v_{i-3} - y - x$ is a 5-hole. In all such cases, we follow the proof of Theorem 5.3.1. Since testing these conditions can clearly be done in linear time, the algorithm runs in linear time if such a 5-hole occurs.

We therefore may further assume that $V(G) = C \cup \bigcup_i S(v_{i-1}, v_i, v_{i+1})$. Since $S(v_{i-1}, v_i, v_{i+1})$ is complete to $S(v_i, v_{i+1}, v_{i+2})$, at most one of them is non-empty, for otherwise G contains a K_4. Therefore, at most three such sets are non-empty. If any of such sets contains two vertices, then we find a K_4 in G. So, each set contains at most one vertex. Now $|G| \leq 9$ and it is easy to see that G is 3-colourable. This completes our proof. ■

We now present an algorithm for 3-Colourability of (C_4, P_6)-free graphs with no clique cutsets or vertices of degree at most two.

Algorithm 1: A certifying algorithm for 3-Colourability of (C_4, P_6)-free graphs

Input: A connected (C_4, P_6)-free graph G with $\delta(G) \geq 3$ and no clique cutsets.

Output: A 3-colouring of G if it exists or one of the minimal obstructions in Figure 5.2.

1 **if** *G is chordal* **then**

2   **if** $|G| \geq 4$ **then** G contains a K_4;

3   **else** G is a clique of size at most 3;

4 **end**

// Now G is not chordal and contains a hole C of length 5 or 6

5 **if** *C is a 5-hole* **then** follow the proof of Theorem 5.3.1;

// Now C is a 6-hole

6 **else**

7   Apply Lemma 5.5.1

8 **end**

Theorem 5.5.2 *Algorithm 1 correctly decides if G is 3-colourable and runs in linear time.*

Proof. Suppose that G is chordal. Then since G has no clique cutsets, G must be a clique by Theorem 2.3.6. Therefore, line 2 and 3 correctly determine whether or not G is 3-colourable. If G is not chordal, the correctness follows directly from Theorem 5.3.1 and Lemma 5.5.1.

For the running time, we first test if G is chordal and in case that G is not chordal we find an induced cycle of length 5 or 6. All these steps can be done in linear time by Theorem 2.3.9. Therefore, the total running time is $O(m+n)$ by Theorem 5.3.1 and Lemma 5.5.1. ■

For 4-Colourability, the situation is similar but needs some clarification. The proof of Lemma 5.4.2 finds in linear time either a 4-colouring of G or a minimal obstruction or a universal vertex. In the last case, the problem reduces to 3-Colourability and so the proof is a linear time certifying algorithm. For Lemma 5.4.4 we first point out that the proof is a linear time certifying algorithm if the 5-hole C satisfies the minimality condition (those arguments that use the absence of W_5 reduce to Lemma 5.4.2). To find such a 5-hole, we could try all 5-holes but it can in worst case takes $O(n^5)$ and thus the algorithm would not be linear. Indeed, those parts of the proof that use the minimality condition of C could fail if C is not minimum. We note, however, that in these cases the proof actually finds another 5-hole C' (the red parts in the proof) with $|S_3'| < |S_3|$. So, we can apply the proof to C'. Since $|S_3| \leq 7$ (as we showed), this can happen at most 7 times. In other words, it takes $O(m+n)$ time to turn the initial 5-hole to the one that satisfies the minimality condition. Therefore, the total running time of Lemma 5.4.4 is $O(m+n)$.

Similarly, if G contains a 6-hole, we follow the idea in the proof of Lemma 5.5.1 to show that in linear time we either reduce to Lemma 5.4.2 and Lemma 5.4.4 or finds a 4-colouring of G or a minimal obstruction. So, the following holds.

Theorem 5.5.3 *For $k = 3$ and $k = 4$, there exists an $O(m+n)$ certifying algorithm for deciding if a (C_4, P_6)-free graph with minimum degree at least k and no clique cutsets is k-colourable.*

Colouring general (C_4, P_6)-free graphs

We now show that for general (C_4, P_6)-free graphs there is a polynomial time certifying algorithm. Clearly, we want to make use of Theorem 5.5.3. Hence, we need to decompose a general (C_4, P_6)-free graph G into subgraphs that have no clique cutsets or vertex of degree smaller than k.

Let G be a connected (C_4, P_6)-free graph. If G has a clique cutset K, then $G - K$ is a disjoint union of two subgraphs H_1 and H_2 of G. We let $G_i = H_i \cup K$ for $i = 1, 2$ and decompose G into G_1 and G_2. If we recursively decompose G_i via clique cutsets, then it was shown by Tarjan **[102]** that one can decompose G into at most n subgraphs (which are called *atoms*) in $O(mn)$ time such that each of the subgraphs has no clique cutsets and G is k-colourable if and only if each atom is k-colourable. However, we cannot directly use Tarjan's decomposition because our proofs rely heavily on the fact the graph has $\delta(G) \geq k$. Moreover, deleting a vertex of degree less than k in an atom may create a clique cutset (consider any odd hole for $k = 3$). So, removing vertices of small degree in an atom may end up with a graph that is not an atom.

We now present a combination of decomposition via clique cutsets and removing vertices of small degree. Again, if G has a clique cutset K we decompose G into G_1 and G_2 as above. On the other hand, if G contains a vertex v with $d_G(v) < k$ then we replace G by $G - v$. We then further decompose G_1 and G_2 or $G - v$ in the same way until either the graph has no clique cutsets and no vertex of degree less than k or the graph has at most k vertices. We refer to these subgraphs that are not further decomposed as *k-atoms*. The decomposition procedure can be represented by a binary tree $T(G)$ whose root is G, and G may have two children G_1 and G_2 or only one child $G - v$, depending on the way G is decomposed. Each leaf in $T(G)$ corresponds to a k-atom. We prove that there are only polynomially many nodes in $T(G)$. The proof idea is similar to that in **[11]**.

Observation 5.5.4 *$T(G)$ has $O(n^2)$ nodes.*

Proof. To see this, it is enough to prove that there are $O(n^2)$ internal nodes in $T(G)$. We label each internal node X of $T(G)$ with a pair of ordered vertices as follows.

- If X is decomposed via a clique cutset K into two subgraphs X_1 and X_2, then we choose a vertex $a \in X_1 - K$ and a vertex $b \in X_2 - K$, and label X with (a, b).
- If X is decomposed via removing a vertex v with $d_X(v) < k$, then we choose a non-neighbour u of v in X (note that this is always possible because X is not a k-atom and so has at least $k + 1$ vertices) and label X with (v, u).

Due to our choice of labeling, if X is labeled with (x, y), then $x, y \in X$ and $xy \notin E$. Now we show that no two internal nodes have the same label. Suppose not, let A and B be two internal nodes of $T(G)$ that have the same label, say (x, y). Suppose first that B is a descendant of A. If A is decomposed via a clique cutset into A_1 and A_2, then the unique path connecting A and B in the subtree rooted at A goes through either A_1 or A_2, say A_1. Then the fact that A has label (x, y) implies that $y \notin A_1$. On the other hand, the fact that B has label (x, y) implies that $y \in B$. But this is a contradiction, since B is an induced subgraph of A_1. So, A is decomposed via removing a vertex of degree less than k. This means that $A - x$ is the only child of A in $T(G)$ and so the unique path connecting A and B in the subtree rooted at A goes through $A - x$. Again, the label of B implies that $x \in B$ but this is a contradiction, since $B \subseteq A - x$.

So, we assume that B is not a descendant of A. Similarly, A is not a descendant of B. Let X be the lowest common ancestor of A and B. Then X must be decomposed via a clique cutset K into two subgraphs X_1 and X_2, for otherwise $X - v$ for some $v \in X$ with $d_X(v) < k$ would have been a common ancestor that is lower than X. Similarly, A and B lie in the subtree rooted at X_1 and X_2, respectively. As we observed earlier, $x, y \in A \subseteq X_1$ and $x, y \in B \subseteq X_2$. This implies that $x, y \in X_1 \cap X_2 = K$. Since K is a clique, $xy \in E$ but this is a contradiction.

Since there are at most n^2 distinct pairs of vertices, the number of internal nodes is $O(n^2)$. ■

We note that if G is connected, then every k-atom is also connected. This can be seen as follows. If G is decomposed via a clique cutset K, then both G_1 and G_2 are clearly connected.

If not, this means that G contains no clique cutset. In particular, G has no cut-vertex. Thus, deleting a vertex preserves the connectivity. Tarjan **[102]** shows that one can find a clique cutset of G in $O(m)$ time if one exists. Moreover, detecting a vertex of degree less than k can be done in $O(n)$ time. Hence, a single decomposition step takes $O(m + n)$ time. This and Observation 5.5.4 imply the following.

Lemma 5.5.5 *For any graph G, $T(G)$ can be found in $O(mn^2)$ time.*

Now we prove the final result.

Theorem 5.5.6 *For $k = 3$ and $k = 4$, there exists an $O(mn^2)$ time certifying algorithm for deciding if a (C_4, P_6)-free graph is k-colourable.*

Proof. We may assume that G is connected, for we can apply the algorithm to each connected component of G. By Lemma 5.5.5 we build $T(G)$ in $O(mn^2)$ time. Moreover, during the decomposition of G, we output a K_{k+1} whenever we encounter a clique cutset K with size $k + 1$. This shows that G is not k-colourable and we stop.

So, assume that the above situation never happens. Suppose that $A_1, \ldots, A_r$ are k-atoms that have at least $k + 1$ vertices. Let n_i and m_i be the number of vertices and edges in A_i, respectively. We now apply Theorem 5.5.3 to each A_i. If some A_i is not k-colourable, then the algorithm outputs a minimal obstruction and we stop. If each A_i is k-colourable, then the algorithm returns a k-colouring ϕ_i of A_i. Since any k-atom that has at most k vertices has a k-colouring (simply assigning pairwise distinct colours to the vertices), we conclude that G is k-colourable. Moreover, combining these colouring and ϕ_i we obtain a k-colouring of G. Therefore, the algorithm is indeed a certifying algorithm. Moreover, the running time of colouring all A_i's is

$$\sum_{i=1}^{r} O(n_i + m_i) = O(m + n) + (k^2 + k)O(n^2) = O(n^2),$$

where the first equality comes from the fact each decomposition step can increase the size of G by the size of the clique cutset K. Therefore, the total running time is $O(mn^2)$. ■

Note that the most expensive part of the algorithm is to find $T(G)$. Once $T(G)$ is given, the algorithm runs in $O(n^2)$ time. We finally notice that for general k, a polynomial time certifying algorithm for k-COLOURABILITY of (C_4, P_6)-free graphs can be obtained using Theorem 5.2.1 and the observation that a k-colourable (C_4, P_6)-free graph must have bounded tree-width (by a result of Atminas, Lozin and Razgon **[5]**): by brute force, we check all minimal obstructions to see if G contains any of them. If so, we output such an obstruction and G is not k-colourable. If this never happens, then we know that G is k-colourable and therefore has bounded tree-width. A standard dynamic programming algorithm thus finds a k-colouring of G in linear time.

6

Colouring Even-Hole-Free Graphs

In previous three chapters, we investigated the colouring problems on classes $Forb(\mathcal{H})$ when $\mathcal{H}$ is of finite size. We now turn our focus to a class with $\mathcal{H}$ being infinite size, namely $\mathcal{H} = \{C_4, C_6, \ldots\}$. In this case, the graphs in $Forb(\mathcal{H})$ are referred to as *even-hole-free graphs*. In this chapter, we study a subclass of even-hole-free graphs, namely (even-hole,cap)-free graphs. A *cap* is a graph induced by a hole with an additional vertex that is adjacent to exactly two adjacent vertices on the hole. If the hole has length k, then the cap is called a k-*cap* (see Figure 6.1). A graph is *cap-free* if it is k-cap-free for any $k \geq 4$. Our main result (cf. also **[68]**) in this chapter is the following.

Theorem 6.0.1 *For any (even-hole,cap)-free graph G, $\chi(G) \leq \lfloor \frac{3}{2}\omega(G) \rfloor$.*

We also develop a $3/2$-approximation algorithm for colouring (even-hole,cap)-free graphs. We shall give some background and recent work on even-hole-free graphs in Section 6.1 and then proceed to the main results.

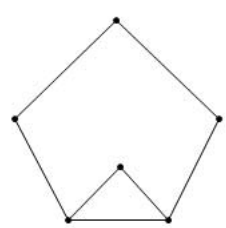


Figure 6.1: The 5-cap.

6.1 Background

The structure of even-hole-free graphs was first studied by Conforti, Cornuéjols, Kapoor and Vušković in **[31, 32]**. They focused on showing that even-hole-free graphs can be recognized in polynomial time (a problem that at that time was not even known to be in NP), and their primary motivation was to develop techniques that can then be used in the study of perfect graphs.

A vertex set $S \subseteq V(G)$ is a k-*star cutset* of G if S is a cutset and S contains a clique of size k that dominates S. A 1-star, 2-star and 3-star are referred to as *star*, *double star* and *triple star*, respectively. A graph G has a 2-*join* $V_1|V_2$, with special sets (A_1, A_2, B_1, B_2), if the vertices of G can be partitioned into sets V_1 and V_2 so that the following holds.

(J1) For each $1 \leq i \leq 2$, A_i and B_i are non-empty and disjoint with $A_i \cup B_i \subseteq V_i$.

(J2) A_1 and B_1 are complete to A_2 and B_2, respectively, and these are the only edges between V_1 and V_2.

(J3) For each $1 \leq i \leq 2$, the graph $G[V_i]$ induced on V_i contains a path with one end in A_i and the other in B_i but $G[V_i]$ is not a chordless path.

In 2002, Conforti, Cornuéjols, Kapoor and Vušković **[31, 32]** obtained the first decomposition theorem for even-hole-free graphs that uses 2-joins and star, double star and triple star cutsets. Using this decomposition theorem, they then developed the first polynomial time recognition algorithm for even-hole-free graphs. Since the main motivation was to show the existence of such an algorithm, they did not intend to optimize the running time which is $O(n^{40})$. Soon after, Chudnovsky, Kawarabayashi and Seymour **[23]** developed an $O(n^{31})$ recognition algorithm. Their algorithm is not based on decomposition theorems but on directly finding even holes in graphs and a technique called cleaning. Later on, Silva and Vušković obtained a new decomposition theorem which avoids double star and triple star cutsets.

Theorem 6.1.1 **[39]** *Every connected even-hole-free graph is either basic or admits a star cutset or a* 2*-join.*

Here the description of 'basic' graphs is somewhat technical and we refer to **[39]** for formal definitions. Taking advantage of this strengthened decomposition, Silva and Vušković **[39]** were able to obtain an $O(n^{19})$ algorithm to recognize even-hole-free graphs which is a significant improvement over the ones from **[23, 32]**. Very recently, Chang and Lu **[19]** showed that the $O(n^{19})$ algorithm does not fully exploit the power of Theorem 6.1.1 and they developed the best known recognition algorithm so far.

Theorem 6.1.2 [19] *For a graph G with n vertices and m edges, there exists an algorithm that runs in $O(m^3n^5)$ time to decide if G is even-hole-free. Moreover, the algorithm outputs an even hole if one exists.*

χ-boundedness and β-perfectness

Note that by excluding a 4-hole, one also excludes all antiholes of length at least 6. If we switch parity, the analogous class to even-hole-free graphs is the class of perfect graphs rather than just the class of the odd-hole-free graphs, i.e. graphs that do not contain any odd hole as an induced subgraph. Theorem 2.3.5 says that CHROMATIC NUMBER can be solved in polynomial time for perfect graphs. In contrast, it remains open whether one can optimally colour an even-hole-free graph (this is also the case for k-COLOURABILITY).

Problem 6.1.3 *What is the complexity of* CHROMATIC NUMBER *for even-hole-free graphs?*

Despite the unknown status of the complexity of determining $\chi(G)$ for even-hole-free graphs, an approximate version does exist. In 2008, Addario-Berry, Chudnovsky, Havet, Reed and Seymour **[1]** settled a conjecture of Reed by proving that every even-hole-free graph contains a *bisimplicial vertex* (a vertex whose neighbourhood induces a graph that is a union of two cliques). Since the degree of a bisimplicial vertex is at most $2\omega(G) - 2$, this has the following immediately consequence.

Theorem 6.1.4 [1] *If G is an even-hole-free graph, then $\chi(G) \leq 2\omega(G) - 1$.*

Gyárfás **[59]** introduced the concept of χ-bounded graphs as a natural extension of perfect graphs. A hereditary class $\mathcal{G}$ is called *χ-bounded* with *χ-binding function f* if for every induced subgraph G' of G it holds that $\chi(G') \leq f(\omega(G'))$. The class of perfect graphs is a χ-bounded family with identity function $f(x) = x$ being its χ-binding function. Translating Theorem 6.1.4 into this language, it says that the class of even-hole-free graphs belongs to the family of χ-bounded graphs with χ-binding function $f(x) = 2x - 1$. On the other hand, it is well-known that finding a maximum clique in C_4-free graphs (hence even-hole-free graphs) can be achieved in polynomial time. It was first observed by Farber **[46]** that 4-hole-free graphs have $O(n^2)$ maximal cliques and all of them can be listed in polynomial time. For even-hole-free graphs, Theorem 6.1.4 implies that the neighbourhood of a bisimplicial vertex is a chordal graph. The existence of a vertex whose neighbourhood induces a chordal graph in even-hole-free graphs was first proved by Silva and Vušković **[38]**. Since it takes linear time to find the clique number of a chordal graph (Theorem 2.3.10), this fact implies that $\omega(G)$ can be computed in $O(mn)$ for even-hole-free graphs. This and Theorem 6.1.4 imply:

Theorem 6.1.5 *There exists an $O(mn)$ 2-approximation algorithm for computing the chromatic number of even-hole-free graphs. Moreover, the algorithm outputs a (proper) colouring of G that uses at most $2\omega(G)-1$ colours.*

Another motivation for the study of even-hole-free graphs is their connection to β-perfect graphs introduced by Markossian, Gasparian and Reed **[85]**. For a graph G, consider the following linear ordering on $V(G)$: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Colouring greedily on this order gives the upper bound $\chi(G) \leq \beta(G)$, where

$$\beta(G) = \max\{\delta(G') + 1 : G' \text{ is an induced subgraph of } G\}.$$

A graph is *β-perfect* if for each induced subgraph H of G, $\chi(H) = \beta(H)$. Clearly, $\beta(C_{2s}) = 3$ and $\chi(C_{2s}) = 2$ for any $s \geq 2$. This means that any β-perfect graph must be even-hole-free. The converse of the statement is not necessarily true (replacing each vertex of a 5-hole by a clique of size two gives a counter-example). Nevertheless, if we forbid an additional graph in addition to even holes, it is possible to obtain β-perfect graphs. A recent result of Kloks, Müller and Vušković **[76]** showed that if the additional forbidden graph is the diamond, then this is indeed the case. A *diamond* is the graph obtained from K_4 by removing an edge.

Theorem 6.1.6 **[76]** *Every (even-hole,diamond)-free graph is β-perfect.*

The β-perfectness of (even-hole,diamond)-free graphs is a consequence of the fact that every such graph contains a *simplicial extreme*, namely a vertex that is either simplicial or of degree two, which in turn follows from a decomposition theorem for (even-hole,diamond)-free graphs that uses 2-joins, clique cutsets and bisimplicial cutsets (a special type of a star cutset). The β-perfectness of (even-hole,diamond)-free graphs implies that $\chi(G)$ can be computed in polynomial time by colouring greedily on the particular ordering of vertices we described above.

Corollary 6.1.7 $\textsc{Chromatic Number}$ *can be solved in $O(n^2)$ time for (even-hole,diamond)-free graphs.*

Proof. Let G be a (even-hole,diamond)-free graph. By Theorem 6.1.6, G is β-perfect. This implies that $\chi(G) = \beta(G)$. More accurately, let $v_1, v_2, \ldots, v_n$ be the linear ordering obtained from the procedure we described above, i.e., v_i is a vertex of minimum degree in $G_i = G[\{v_1, \ldots, v_i\}]$. Then $\chi(G) \leq \max\{\delta(G_i)+1 : 1 \leq i \leq n\} \leq \beta(G) = \chi(G)$. This means that $\chi(G) = \max\{\delta(G_i) + 1 : 1 \leq i \leq n\}$. Clearly, it takes $O(i)$ time to find v_i in G_i for each i. Thus, finding such a linear ordering can be done in $O(n^2)$ time. Moreover, greedily colouring G on $v_1, \ldots, v_n$ can be done in $O(m+n)$ time. Therefore, the corollary holds ■

In addition, the existence of a simplicial extreme immediately implies that the class of (even-hole, diamond)-free graphs is a χ-bounded family with χ-binding function $f(x) = x + 1$.

Corollary 6.1.8 **[76]** *For any (even-hole, diamond)-free graph G, $\chi(G) \leq \omega(G) + 1$.*

Very recently, efforts are made on subclasses of even-hole-free graphs by forbidding additional graphs besides even holes. The result of diamond-free graphs **[76]** already demonstrates the richness of this approach.

A *pan* is a graph induced by a hole with an additional vertex pendent to some vertex on the hole. Note that every pan contains a claw. Cameron, Chaplick and Hoàng **[18]** investigated (even-hole,pan)-free graphs which contains the class of (even-hole,claw)-free graphs. They first obtained a decomposition theorem for (even-hole,pan)-free graphs: every such graph can be decomposed via clique cutsets into (essentially) unit circular-arc graphs. The decomposition allows them to obtain an $O(mn)$ recognition algorithm and a polynomial time colouring algorithm. Although the class of (even-hole,pan)-free graphs is not β-perfect, it was shown to be χ-bounded with χ-binding function $f(x) = \frac{3}{2}x$.

Recall that a cap is a graph induced by a hole with an additional vertex that is adjacent to exactly two adjacent vertices on the hole. It was shown by Conforti, Gerards and Pashkovich **[33]** that the problem of maximum weighted independent set can be solved in polynomial time for (even-hole,cap)-free graphs. We study $\textsc{Chromatic Number}$ for (even-hole,cap)-free graphs below. Like the pan-free case, (even-hole,cap)-free graphs need not to be β-perfect. We show that the class of (even-hole,cap)-free graphs is a χ-bounded family with the same χ-binding function $f(x) = \frac{3}{2}x$.

6.2 Decomposition of cap-free graphs

In 1999, Conforti, Cornuéjols, Kapoor and Vušković **[30]** proved a decomposition theorem for cap-free graphs. To state their decomposition, we first define a special kind of 'cutset'. Let $X = (V_1, A_1, V_2, A_2, K)$ be an array of disjoint sets with union $V(G)$. We say that X is an *amalgam* of G if the following properties hold:

- A_1 and A_2 are complete to each other and both are non-empty.
- K is a clique (possibly empty) and K is complete to $A_1 \cup A_2$.
- V_1 is anti-complete to $A_2 \cup V_2$ and V_2 is anti-complete to $A_1 \cup V_1$.
- $|V_1 \cup A_1| \geq 2$ and $|V_2 \cup A_2| \geq 2$.

Note that possibly K may have neighbours in $V_1 \cup V_2$.

Theorem 6.2.1 **[30]** *Every cap-free graph with a triangle either admits an amalgam or a clique cutset or contains a universal vertex.*

Therefore, cap-free graphs can be built from triangle-free graphs. We say that two vertices u and v are *twins* in G if $N[u] = N[v]$, and that G contains twin vertices if there are vertices that are twins in G. We notice in the following that if we forbid even holes in cap-free graphs, then an amalgam of G gives rise to twin vertices.

Lemma 6.2.2 *Suppose that G is an (even-hole, cap)-free graph containing no clique cutset. If G contains an amalgam $X = (V_1, A_1, V_2, A_2, K)$, then G contain a pair of twin vertices.*

Proof. Suppose that both A_1 and A_2 are not cliques. Then A_1 (respectively A_2) contains two non-adjacent vertices, say, u, u' (respectively v, v'). But then $\{u, u', v, v'\}$ induces a 4-hole. So at least one of A_1 and A_2 induces a clique. By symmetry, we assume that A_1 induces a clique.

If $V_1 \neq \emptyset$, then $A_1 \cup K$ is a clique cutset separating V_1 from $V_2 \cup A_2$. So $V_1 = \emptyset$, and therefore $|A_1| \geq 2$. But then any two vertices of A_1 are twins in G. ■

Note that the proof of Lemma 6.2.2 makes use of merely the absence of 4-holes. The following decomposition of (even-hole,cap)-free graphs is an immediate consequence of Theorem 6.2.1 and Lemma 6.2.2.

Theorem 6.2.3 *Suppose that G is (even-hole,cap)-free graph that contains no universal vertices, no twin vertices, and no clique cusets. Then G is triangle-free.*

6.3 Colouring (even-hole, cap)-free graphs

In this section, we prove our main result in this chapter. Then we turn our proof into a polynomial time approximation algorithm. First we note that 'two vertices being twin vertices' in fact defines an equivalence relation $\sim_T$. Moreover, each equivalence class is a clique and for any two equivalence classes X and Y, X and Y are either complete or anti-complete to each other.

We are now ready to prove Theorem 6.0.1.

Proof of Theorem 6.0.1. We prove the theorem by induction on $|G|$. We may assume that G is connected, for otherwise applying the inductive hypothesis to each connected component of G completes the proof. If G contains a universal vertex u, then $G - u$ has $\chi(G-u) \leq \frac{3}{2}\omega(G-u)$. Clearly, $\chi(G) = \chi(G-u) + 1$ and $\omega(G) = \omega(G-u) + 1$. It follows that

$$\chi(G) = \chi(G-u) + 1 \leq \frac{3}{2}\omega(G-u) + 1 = \frac{3}{2}(\omega(G) - 1) + 1 \leq \frac{3}{2}\omega(G).$$

If G contains a clique cutset K, then $G - K$ is a disjoint union of two subgraphs H_1 and H_2. Let $G_i = H_i \cup K$ for $i = 1, 2$. Then $\chi(G) = \max\{\chi(G_1), \chi(G_2)\}$. Thus,

$$\chi(G) = \max\{\chi(G_1), \chi(G_2)\} \leq \max\{\frac{3}{2}\omega(G_1), \frac{3}{2}\omega(G_2)\} \leq \frac{3}{2}\omega(G).$$

Therefore, G has no universal vertices or clique cutsets. Now we partition $V(G)$ into equivalence classes $T_1, T_2, \ldots, T_r$ under $\sim_T$. Take an arbitrary vertex $t_i \in T$ for $1 \leq i \leq r$ and let $G' = G[\{t_1, \ldots, t_r\}]$. Note that G' is obtained from G by successively removing twin vertices. We claim that removing twin vertices does not create a clique cuset or a universal vertex.

Claim J *Suppose that u and v are twin in G. Then $G - u$ does not contain any universal vertex or clique cutset.*

> Suppose not. If $G - u$ contains a universal vertex, say x. Then x is adjacent to each vertex in $G - u$, in particular to v. This implies that x is also adjacent to u, since u and v are twins in G. Now x is a universal vertex in G, contradicting to our assumption. So, $G - u$ contains no universal vertices.
>
> Suppose that $G - u$ contains a clique cutset K. Now $G - K$ is disjoint union of two vertex-disjoint subgraphs H_1 and H_2. Let G_i be the subgraph of G induced by $V(H_i) \cup K$ for $i = 1, 2$. If v is in H_1 or H_2, then K is still a clique cutset in G. So, $v \in K$. But then $K \cup \{u\}$ is a clique cutset of G, a contradiction. $\diamond$

Hence, by Claim J and Theorem 6.2.3 we conclude that G' is triangle-free, and so $\chi(G') \leq 3$ by Corollary 6.1.8. On the other hand, note that G' is connected (since G is connected). In particular, each vertex of G' lies in an edge of G'. Therefore, any maximal clique in G' is an edge. This means that any maximal clique of G is a union of two T_j's, which implies that $\omega(G - G') = \omega(G) - 2$. By the inductive hypothesis, $\chi(G - G') \leq \frac{3}{2}\omega(G - G')$. Then

$$\chi(G) \leq \chi(G - G') + \chi(G') \leq \frac{3}{2}\omega(G - G') + 3 = \frac{3}{2}(\omega(G) - 2) + 3 = \frac{3}{2}\omega(G).$$

Since $\chi(G)$ is an integer, the theorem follows. ■

The bound in Theorem 6.0.1 is attained by odd holes and the Hajós graph (see Figure 5.2). Note that these graphs have clique number at most 3. For graphs with large clique number, we do not have an example showing that the bound is tight. Nevertheless, the optimal constant is at least $5/4$. For any integer $k \geq 1$, let G_k be the graph obtained from a 5-hole by replacing each vertex of the 5-hole with a clique of size $2k$ and making two cliques complete (respectively anti-complete) if the two original vertices are adjacent (respectively non-adjacent) on the 5-hole. Clearly, $|G_k| = 10k$, $\alpha(G_k) = 2$ and $\omega(G_k) = 4k$. Hence, $\chi(G_k) \geq \frac{|G_k|}{\alpha(G_k)} = 5k$. Moreover, it is easy to see that G_k does admit a $5k$-colouring. So, $\chi(G_k) = 5k = \frac{5}{4}\omega(G_k)$. A natural question is that whether or not one can reduce $\lfloor 3/2 \rfloor$ to $\lceil 5/4 \rceil$.

Problem 6.3.1 *Is it true that $\chi(G) \leq \lceil \frac{5}{4}\omega(G) \rceil$ for every (even-hole,cap)-free graph G?*

It was shown in **[20]** that this is true for the class of (C_4, P_5)-free graphs which is a subclass of (even-hole,cap)-free graphs.

A $3/2$- approximation algorithm

We now turn our proof of Theorem 6.0.1 into a $3/2$-approximation algorithm for colouring (even-hole,cap)-free graphs. The algorithm outputs a $\frac{3}{2}\omega(G)$-colouring of G in polynomial time. We need one more observation.

Observation 6.3.2 *Suppose that G is a graph without clique cutsets. If $u \in V(G)$ is a universal vertex, then $G - u$ contains no clique cutsets.*

Proof. If K is a clique cutset in $G - u$, then $K \cup \{u\}$ is a clique cutset in G. ■

The proof of Theorem 6.0.1 is almost algorithmic except for the last step where we deal with G with no clique cutsets or universal vertices. Essentially we want to successively remove a triangle-free subgraph, one vertex from each equivalence class, from G so that the removal of it reduces the clique number of the graph exactly by 2. During the removal process, however, if the graph becomes disconnected, a maximal clique could just be one equivalence class, say T_i (that forms a connected component of the graph). If T_i happens to be a maximum clique of the current graph, then removing a single vertex from T_i may reduce the clique number by at most 1. This happens when either T_i has at least two vertices or the current graph is just an independent set. But both cases have an easy fix. In the former case, we simply remove two vertices from T_i, and in the latter case we colour the independent set with a new colour that has not been used (at this point all vertices of G have been coloured). Clearly, the number of subgraphs we removed is at most $\omega(G)/2 = O(n)$. Moreover, each time it takes $O(m+n)$ time (determining the connected components) to find such a subgraph. Therefore, it takes $O(mn)$ time in total for finding subgraphs. On the other hand, by Corollary 6.1.7 we can colour all subgraphs in $O(n^2)$ time.

Lemma 6.3.3 *Suppose that G is a (even-hole,cap)-free graph without universal vertices or clique cutsets. If the equivalence classes T_i's under $\sim_T$ are given, one can find a $\frac{3}{2}\omega(G)$-colouring for G in $O(mn)$ time.*

We now present the algorithm for colouring general (even-hole,cap)-free graphs.

Algorithm 2: A $3/2$-approximation algorithm for CHROMATIC NUMBER

Input: A (even-hole,cap)-free graph G.

Output: A $\frac{3}{2}\omega(G)$-colouring of G.

1 Do clique cutset decomposition of G **[102]** and obtain a binary decomposition tree $T(G)$.
2 **for** *each atom* A **do**
3     $A' := A$;
4   **for** *each* $a \in V(A)$ **do**
5   **if** $|N_A(a)| = |A| - 1$ **then** `//` a `is a universal vertex in` A
6             $A' := A - a$;
7   **end**
8   **end**
9   Partition A' into equivalence classes $T_1, \ldots, T_r$ under $\sim_T$;
10   Obtain a $\frac{3}{2}\omega(A')$-colouring $\phi_{A'}$ of A' by Lemma 6.3.3;
11   Extend $\phi_{A'}$ to a colouring ϕ_A of A by colouring each vertex in $A \setminus A'$ with a new colour;
12 **end**
13 Combine colouring ϕ_A of the atoms along $T(G)$ and obtain a colouring ϕ of G.

We show that the algorithm is correct.

Theorem 6.3.4 *Algorithm 2 is correct and runs in $O(mn^2)$ time.*

Proof. We first discuss the running time. The clique cutset decomposition can be found in $O(mn)$ time and there are at most n atoms, see **[102]**. The **for** loop from line 4 to line 8 and line 11 apparently take $O(n)$ time. To partition A' into $T_1, \ldots, T_r$, we test for each edge $e = xy \in E(A')$ whether or not $N[x] = N[y]$. For each edge it takes $O(n)$ time and so line 9 takes $O(mn)$ time. Line 10 takes $O(mn)$ time by Lemma 6.3.3. In a word, the colouring ϕ_A, for each atom A, can be found in $O(mn)$ time. Since there are $O(n)$ atoms, the total running time is $O(mn^2)$.

To prove the correctness, we first note that A' (after line 8) contains no universal vertices. Suppose not, let $b \in A'$ be a universal vertex in A'. Since all vertices $A \setminus A'$ are universal vertices in A, they are all adjacent to b. This implies that b is a universal vertex in A and so it would have been removed during the **for** loop from line 4 to line 8, a contradiction. Furthermore, A' contain no clique cutsets by Observation 6.3.2. Therefore, the correctness follows from Lemma 6.3.3 and the fact that universal vertices and clique cutsets preserve the χ-binding function. ■

Bibliography

[1] L. Addario-Berry, M. Chudnovsky, F. Havet, B. Reed, and P. Seymour. Bisimplicial vertices in even-hole-free graphs. *J. Combin. Theory Ser. B*, 98:1119–1164, 2008.

[2] N. Alon and J.H. Spencer. *The Probabilistic Method (3rd edition)*. Wiley, 2008.

[3] K. Appel and W. Haken. Every planar map is four colorable. Part I: Discharging. *Illinois J. Math.*, 21:429–490, 1977.

[4] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. Part II: Reducibility. *Illinois J. Math.*, 21:491–567, 1977.

[5] A. Atminas, V.V. Lozin, and I. Razgon. Linear time algorithm for computing a small biclique in graphs without long induced path. In *Proc. SWAT 2012*, Lecture Notes in Computer Science 7357, pages 142–152, 2012.

[6] G. Bácso and Z. Tuza. Dominating cliques in P_5-free graphs. *Period. Math. Hungar.*, 21:303–308, 1990.

[7] C. Berge. Les problémes de coloration en théorie des graphes. *Publ. Inst. Stat. Univ. Paris*, 9:123–160, 1960.

[8] J.A. Bondy and U.S.R. Murty. *Graph Theory*, volume 244 of *Springer Graduate Texts in Mathematics*. Springer, 2008.

[9] F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein, and M. Zhong. Three-coloring graphs without triangles or induced paths on seven vertices. Submitted for publication, 2015.

[10] O.V. Borodin. Colorings of plane graphs: A survey. *Discrete Math.*, 313:517–539, 2013.

[11] A. Brandstädt and C.T. Hoàng. On clique separators, nearly chordal graphs, and the maximum weight stable set problem. *Theoretical Computer Science*, 389:295–306, 2007.

[12] A. Brandstädt, T. Klembt, and S. Mahfud. P_6- and triangle-free graphs revisited: structure and bounded clique-width. *Discrete Mathematics and Theoretical Computer Science*, 8:173–188, 2006.

[13] A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph Classes: A Survey*, volume 3 of *SIAM Monographs on Discrete Math. Appl.* Philadelphia, 1999.

[14] H.J. Broersma, F.V. Fomin, P.A. Golovach, and D. Paulusma. Three complexity results on coloring P_k-free graphs. *European J. Combin.*, 1:92–110, 2012.

[15] H.J. Broersma, P.A. Golovach, D. Paulusma, and J. Song. Updating the complexity status of coloring graphs without a fixed induced learn forest. *Theoret. Comput. Sci.*, 414:9–19, 2012.

[16] D. Bruce, C.T. Hoàng, and J. Sawada. A certifying algorithm for 3-colorability of P_5-free graphs. In *Proc. ISAAC 2009*, Lecture Notes in Computer Science 5878, pages 594–604, 2009.

[17] E. Camby and O. Schaudt. A new characterization of P_k-free graphs. arXiv:1402.7213v1, 2014.

[18] K. Cameron, S. Chaplick, and C.T. Hoàng. On the structure of (pan, even hole)-free graphs. arXiv:1508.03062v1, 2015.

[19] H.C. Chang and H.I. Lu. A faster algorithm to recognize even-hole-free graphs. *J.Combin. Theory Ser. B*, 113:141–161, 2015.

[20] S.A. Choudum, T. Karthick, and M.A. Shalu. Perfectly coloring and linearly χ-bound P_6-free graphs. *Journal of Graph Theory*, 54:293–306, 2007.

[21] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković. Recognizing Berge graphs. *Combinatorica*, 25:143–186, 2005.

[22] M. Chudnovsky, J. Goedgebeury, O. Schaudt, and M. Zhong. Obstructions for three-coloring graphs without induced paths on six vertices. arXiv:1504.06979v1, 2015.

[23] M. Chudnovsky, K. Kawarabayashi, and P. Seymour. Detecting even holes. *Journal of Graph Theory*, 48:85–111, 2005.

[24] M. Chudnovsky, P. Maceli, J. Stacho, and M. Zhong. 4-coloring P_6-free graphs with no induced 5-cycles. arXiv:1407.2487v1, 2014.

[25] M. Chudnovsky, P. Maceli, and M. Zhong. Three-coloring graphs with no induced seven-vertex path I: the triangle-free case. arXiv:1409.5164v1, 2014.

[26] M. Chudnovsky, P. Maceli, and M. Zhong. Three-coloring graphs with no induced seven-vertex path II: using a triangle. arXiv:1503.03573v2, 2015.

[27] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 64:51–229, 2006.

[28] M. Chudnovsky, N. Trotignon, T. Trunck, and K. Vušković. Coloring perfect graphs with no balanced skew-partitions. *J. Combin. Theory B*, 115:26–65, 2015.

[29] V. Chvátal, V. Rödl, E. Szemerédi, and W.T. Trotter. The ramsey number of a graph with bounded maximum degree. *J. Combin. Theory B*, 34:239–243, 1983.

[30] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. Even and odd holes in cap-free graphs. *Journal of Graph Theory*, 30:289–308, 1999.

[31] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. Even-hole-free graphs. Part I: Decomposition theorem. *Journal of Graph Theory*, 39:6–49, 2002.

[32] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. Even-hole-free graphs. Part II: Recognition algorithm. *Journal of Graph Theory*, 40:238–266, 2002.

[33] M. Conforti, B. Gerards, and K. Pashkovich. Stable sets and graphs with no even holes. *Math. Program. Ser. B*, 153:13–39, 2015.

[34] S. Cook. The complexity of theorem-proving procedures. In *Proc. 3rd Ann. ACM symp. on Theory of Computing*, pages 151–158, 1971.

[35] D.G. Corneil, H. Lerchs, and S. Burlingham. Complement reducible graphs. *Discrete Appl. Math.*, 3:163–174, 1981.

[36] D.G. Corneil, Y. Pearl, and L. Stewart. A linear recognition algorithm for cograph. *SIAM J. Comput.*, 14:926–934, 1985.

[37] B. Courcell, J.A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs on bounded clique width. *Theory of Computing Systems*, 33:125–150, 2000.

[38] M.V.G. da Silva and K. Vušković. Triangulated neighborhoods in even-hole-free graphs. *Discrete Mathematics*, 307:1065–1073, 2007.

[39] M.V.G. da Silva and K. Vušković. Decomposition of even-hole-free graphs with star cutsets and 2-joins. *J.Combin. Theory Ser. B*, 103:144–183, 2013.

[40] K. Dabrowski, V.V. Lozin, R. Raman, and B. Ries. Colouring vertices of triangle-free graphs without forests. *Discrete Mathematics*, 312:1372–1385, 2012.

[41] P. Damaschke. Induced subgraphs and well-quasi-ordering. *J. Graph Theory*, 14:427–435, 1990.

[42] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, 1997.

[43] G.A. Dirac. On rigid circuit graphs. *Abhandlungen aus dem Mathematischen Seminar der Universitat Hamburg*, 25:71–76, 1961.

[44] K. Edwards. The complexity of coloring problems on dense graphs. *Theoret. Comput. Sci.*, 43:337–343, 1986.

[45] P. Erdös. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.

[46] M. Farber. On diameters and radii of bridged graphs. *Discrete Mathematics*, 73:249–260, 1989.

[47] D.R. Fulkerson and O.A. Gross. Incidence matrices and interval graphs. *Pacific J. Math.*, 15:835–855, 1965.

[48] M.R. Garey and D.S. Johnson. *Computers and intractability: a guide to the theory of NP-completeness*. W. H. Freeman, 1979.

[49] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *J. Combin. Theory B*, 16:47–56, 1974.

[50] P.A. Golovach, M. Johnson, D. Paulusma, and J. Song. A survey on the computational complexity of colouring graphs with forbidden subgraphs. arXiv:1407.1482v6, 2015.

[51] P.A. Golovach, D. Paulusma, and J. Song. 4-coloring H-free graphs when H is small. *Discrete Applied Mathematics*, 161:140–150, 2010.

[52] P.A. Golovach, D. Paulusma, and J. Song. Closing complexity gaps for coloring problems on H-free graphs. *Information and Computation*, 237:204–214, 2014.

[53] P.A. Golovach, D. Paulusma, and J. Song. Coloring graphs without short cycles and long induced paths. *Discrete Appl. Math.*, 167:107–120, 2014.

[54] M.C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*, volume 57 of *Annals of Discrete Mathematics*. North-Holland Publishing Co., 2004.

[55] R.L. Graham, B.L. Rothschild, and J.H. Spencer. *Ramsey theory (2nd edition)*. John Wiley and Sons, 1990.

[56] S. Gravier, C.T. Hoàng, and F. Maffray. Coloring the hypergraph of maximal cliques of a graph with no long path. *Discrete Math.*, 272:285–290, 2003.

[57] A. Grötschel, M. Lovász, and L. Schrijver. Polynomial algorithms for perfect graphs. *Ann. Discrete Math.*, 21:325–356, 1984.

[58] H. Grötzsch. Ein dreifarbensatz für dreikreisfreie netze auf der kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 8:109–120, 1959.

[59] A. Gyárfás. Problems from the world surrounding perfect graphs. *Zastos. Mat.*, 19:413–431, 1987.

[60] P. Hell and S. Huang. Complexity of coloring graphs without paths and cycles. *Discrete Applied Mathematics*, http://dx.doi.org/10.1016/j.dam.2015.10.024, 2015.

[61] C.T. Hoàng. A class of perfect graphs. Master's thesis, McGill University, 1983.

[62] C.T. Hoàng. *Perfect graphs*. PhD thesis, McGill University, 1985.

[63] C.T. Hoàng, M. Kamiński, V.V. Lozin, J. Sawada, and X. Shu. Deciding k-colorability of P_5-free graphs in polynomial time. *Algorithmica*, 57:74–81, 2010.

[64] C.T. Hoàng, B. Moore, D. Recoskiez, J. Sawada, and M. Vatshelle. Constructions of k-critical P_5-free graphs. *Discrete Applied Mathematics*, 182:91–98, 2015.

[65] I. Holyer. The NP-completeness of edge coloring. *SIAM J. Comput.*, 10:718–720, 1981.

[66] S. Huang. Improved complexity results on k-coloring P_t-free graphs. In *Proc. MFCS 2013*, Lecture Notes in Computer Science 8087, pages 551–558, 2013.

[67] S. Huang. Improved complexity results on k-coloring P_t-free graphs. *European Journal of Combinatorics*, 51:336–346, 2016.

[68] S. Huang and M.V.G. da Silva. A note on coloring (even-hole,cap)-free graphs. arXiv:1510.09192, 2015.

[69] S. Huang, M. Johnson, and D. Paulusma. Narrowing the complexity gap for colouring (C_s, P_t)-free graphs. *The Computer Journal*, doi:10.1093/comjnl/bxv039, 2015.

[70] B. Jamison and S. Olariu. P_4-reducible graphs: a class of uniquely tree representable graphs. *Studies in Appl. Math.*, 81:79–87, 1989.

[71] B. Jamison and S. Olariu. A tree representation for P_4-sparse graphs. *Discrete Applied Mathematics*, 35:115–129, 1992.

[72] T.R. Jensen and B. Toft. *Graph Coloring Problems*. Wiley Interscience, 1995.

[73] M. Kamiński and V.V. Lozin. Coloring edges and vertices of graphs without short or long cycles. *Contrib. Discrete. Mah.*, 2:61–66, 2007.

[74] M. Kamiński, V.V. Lozin, and M. Milanič. Recent developments on graphs of bounded clique-width. *Discrete Applied Mathematics*, 157:2747–2761, 2009.

[75] R. Karp. Reducibility among combinatorial problems. *Complexity of Computer Computations The IBM Research Symposia Series*, 1:85–103, 1972.

[76] T. Kloks, H. Müller, and K. Vušković. Even-hole-graphs that do not contain diamonds: a structure theorem and its consequences. *Journal of Combinatorial Theory B*, 99:733–800, 2009.

[77] D. Král, J. Kratochvíl, Z. Tuza, and G.J. Woeginger. Complexity of coloring graphs without forbidden induced subgraphs. In *Proc. WG 2001*, Lecture Notes in Computer Science 2204, pages 254–262, 2001.

[78] K. Kuratowski. Sur le probleme des courbes gauches en topologie. *Fund. Math.*, 16:271–283, 1930.

[79] V.B. Le, B. Randerath, and I. Schiermeyer. On the complexity of 4-coloring graphs without long induced paths. *Theoret. Comput. Sci.*, 39:330–335, 2007.

[80] D. Leven and Z. Galil. NP-completeness of finding the chromatic index of regular graphs. *J. Algorithm*, 4:35–44, 1983.

[81] J. Liu and H. Zhou. Dominating subgraphs in graphs with some forbidden structures. *Discrete Mathematics*, 135:163–168, 1994.

[82] L. Lovász. Coverings and coloring of hypergraphs. In *Proceedings of the Fourth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Utilitas Math., Winnipeg, Man.,*, pages 3–12, 1973.

[83] F. Maffray and G. Morel. On 3-colorable P_5-free graphs. *SIAM J. Discrete Math.*, 26:1682–1708, 2012.

[84] D.S. Malyshev. The coloring problem for classes with two small obstructions. *Optimization Letters*, 8:2261–2270, 2014.

[85] S.E. Markossian, G.S. Gasparian, and B. Reed. β-perfect graphs. *Journal of Combinatorial Theory B*, 67:1–11, 1996.

[86] D. Marx. *Graph coloring with local and global constraints*. PhD thesis, Budapest University of Technology and Economics, 2004.

[87] B.G. Mirkin and S.N. Rodin. *Graphs and genes*. Springer, 1984.

[88] M. Mitzenmacher and E. Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*. Cambridge University Press, 2005.

[89] R. Mosca. Stable sets of maximum weight in $(P_7, banner)$-free graphs. *Discrete Mathematics*, 308:20–33, 2008.

[90] J. Mycielski. Sur le coloriage des graphes. *Colloq. Math.*, 3:161–162, 1955.

[91] S. Olariu. Paw-free graphs. *Inform. Process. Lett.*, 28:53–54, 1988.

[92] F.P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30:264–286, 1930.

[93] B. Randerath and I. Schiermeyer. 3-colorability $\in \mathcal{P}$ for P_6-free graphs. *Discrete Applied Mathematics*, 136:299–313, 2004.

[94] B. Randerath and I. Schiermeyer. Vertex colouring and forbidden subgraphs-a survey. *Graphs Combin.*, 20:1–40, 2004.

[95] N. Robertson, D.P. Sanders, P. Seymour, and R. Thomas. A new proof of the four-colour theorem. *Electronic Research Announcements Of The American Mathematical Society*, 2:17–25, 1996.

[96] N. Robertson and P. Seymour. Graph minors. XIII. the disjoint paths problem. *J. Combin. Theory B*, 63:65–110, 1995.

[97] D.J. Rose, R.E. Tarjan, and G.S. Leuker. Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput.*, 5:266–283, 1976.

[98] D. Seinsche. On a property of n-colorable graphs. *J. Comb. Theory, Ser. B*, 16:191–193, 1974.

[99] J.H. Spencer. *Ten Lectures on the Probabilistic Method (2nd edition)*. Society for Industrial and Applied Mathematics, 1994.

[100] L.J. Stockmeyer. Planar 3-colorability is NP-complete. *SIGACT news*, 5:19–25, 1973.

[101] R.E. Tarjan. Depth-first search and linear graph algorithms. *SIAM J. Comput.*, 1:146–160, 1972.

[102] R.E. Tarjan. Decomposition by clique separators. *Discrete Math.*, 55:221–232, 1985.

[103] P. van't Hof and D. Paulusma. A new characterization of P_6-free graphs. *Disc. App. Math.*, 158:731–740, 2010.

[104] V.V. Vazirani. *Approximation Algorithms*. Springer, 2013.

[105] K. Wagner. *Graphentheorie*, volume 61 of *B.J. Hochschultaschenbucher 248/248a*. Mannheim, 1970.

[106] G.J. Woeginger and J. Sgall. The complexity of coloring graphs without long induced paths. *Acta Cybernet.*, 15:107–117, 2001.