PRICING VOLATILITY SWAP USING HESTON STOCHASTIC VOLATILITY MODEL

by

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Abstract

In this paper, we price the volatility swap as an OTC derivatives aimed for direct trading of volatility. Our pricing method is based on a PDE approach on Heston stochastic volatility model. Heston model has received the most attention since it can give a satisfactory description of the underlying asset dynamics. We follow the PDE approach suggested by Broadie and Jain (2008) to price volatility swap. In addition to their work, we also

- Use loss function minimization to calibrate the Heston parameters to the current data on S&P 500 index and construct implied volatility surface.
- Solve the PDE using numerical computation, Crank-Nicolson finite difference method.
- Price the volatility swap and compare our model expected volatility (fair volatility strike) with the realized volatility, in order to assess the accuracy of this approach.

Our result shows that the model fair volatility strike is close to the realized volatility for long maturity swaps.

Keywords: Volatility swap; Heston stochastic volatility model, Crank-Nicolson finite difference method

To my beloved mother and all my friends who support me during my study at SFU

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1: Introduction

The major purpose of this paper is to price volatility swap under stochastic volatility. Standard call and put options were traded traditionally to gain exposure to the market's volatility. However, these positions also depend on the price level of the underlying asset. Investors can take views on the future realized volatility directly, by trading volatility derivatives. The simplest such instruments is volatility swap. A volatility swap is a forward contract on future realized price volatility.

Volatility has several characteristics that make trading attractive. It is likely to grow when uncertainty and risk increase. As with interest rates, volatilities appear to revert to the mean: High volatilities will eventually decrease; low volatilities will likely rise. Finally, volatility is often negatively correlated with stock or index level, and tends to stay high after large downward moves in the market.

Different methods have been developed for pricing volatility swap, which can be classified in two main groups: Replication approach and Analytical approach. The Analytical approach is mostly based on considering stochastic processes for stock price and volatility. Discrete stochastic model and pure jump model have been studied previously.

Broadie and Jain (2008) suggested a PDE based on Heston stochastic volatility model. This PDE implies that the fair volatility strike is a function of three variables: time, stochastic variance and accumulated variance. In this paper, we follow their approach and solve the suggested PDE using numerical computation, Crank-Nicolson finite difference method. Since the PDE has three variables, we need to solve it through a three dimensional grid in the finite difference method. The details of solving the PDE is explained in the related section of this paper.

We consider the bivariate system of stochastic differential equations in Heston model for stock price process and volatility process. In order to make our calculation more accurate we estimate the Heston parameters on our current datasets and use the calibrated parameters to price volatility swap following Braodie and Jain approach. The parameter calibration is done through loss function minimization, trying to minimize implied volatility mean squared error. We use Lewis (2000) expansion of volatility of volatility to estimate the model implied volatility.

As an empirical evidence, we price the volatility swap on S&P 500 index and compare our model expected volatility to the realized volatility in different maturities. Our result shows that the model fair volatility strike is close to the realized volatility for long maturity swaps, which implies that this method can be a useful pricing tool for long maturity swaps. However, a risk of misspecification and miscalibration presents in any one model. While the stochastic volatility model fits the longer behaviour of the asset price, to adequately describe the short-term behaviour of the asset price it should be augmented with return jumps.

This paper is structured as follows: firstly, we give a brief background and literature review on volatility instruments. In the second part, the Heston parameters are calibrated to our datasets by minimizing the loss function of implied volatility, and Heston implied volatility surfaces are created for five data sets on S&P 500 index. And finally, we use the calculated Heston parameters to price volatility swap. We price the volatility swap by solving the PDE suggested by Broadie and Jain (2008).

1.1 Background

In order for investors and traders to manage their portfolios, they must have views on future volatility; otherwise, they must try to control and minimize their volatility exposure.

Delta-hedging was practiced as the first attempts to hedge against changes in the value of the underlying assets. The principle of delta-hedging was understood only after the Black and Scholes wrote their paper on option pricing in 1973. However, delta-hedging does not eliminate the volatility risk. This issue arises from the assumption of constant volatility in Black-Scholes model.

In further attempts to provide an accurate and flexible volatility hedge, the Log Contract was developed. The Log Contract is a futures style contract whose settlement price is equal to the logarithm of the price of the underlying asset. A Log Contract can be used to provide a payoff that depends only on the difference between the volatility expected at the time at which the contract was entered and the actual volatility that occurs over its duration (Neuberger, 1994).

Over the past two decades, the volatility of an underlying stock or an index has developed as an asset class in its own right. Just as stock investors predict the direction of the stock market, or bond investors predict the probable direction of interest rates, so one might want to predict the level of future volatility. The market for variance and volatility swaps has since then been growing, and many investment banks and other financial institutions are now actively quoting volatility swaps on various assets: stock indexes, currencies, as well as commodities.

Investors can trade volatility and regard it as an underlying asset in order to price other volatility sensitive instruments, including volatility swaps, variance options, VIX futures, etc (Broadie and Jain, 2008). Volatility swaps are forward contracts on future realized stock volatility. Variance swaps are similar contracts on variance, the square of future volatility. Both these instruments provide an easy way for investors to gain exposure to the future level of volatility.

Three applications are mentioned for trading variance and volatility swaps: *Directional Trading of Volatility Levels*- One can use these instruments to speculate on future volatility levels, *Trading the spread between realized and implied volatility levels*- By unwinding the swap before expiration, one can trade the spread between realized and implied volatility. *Hedging Implicit Volatiltiy Exposure*- To hedge the volatility exposure of other positions or businesses (Demeterfi, Derman, Kamal and Zou, 1999) The variance swap's replicating portfolio became in 2003 the basis for how Chicago Board Option Exchange (CBOE) calculates the VIX index.

The 1987 crash showed that the assumption of normal distribution for returns in equity markets is not correct and returns exhibit skewness and kurtosis, indicated by smiles and skews in the implied volatility surface. A stochastic volatility model is needed for both the variance and the volatility swap pricing. To overcome the shortcoming of constant volatility assumed for underlying asset in the Black-Scholes model, stochastic volatility (SV) models were developed. Two types of volatility models have been derived: continuous-time stochastic volatility models and discrete-time Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models. In the GARCH models, it is assumed that volatility is readily observable from the history of asset prices, however implied volatilities inferred from other contemporaneous options are used in continuous-time stochastic volatility models (Heston and Nandi, 2000). GARCH model parameters are calibrated from historical prices, and SV model parameters are calibrated from implied volatilities inferred from contemporaneous options. In these models, the volatility process is assumed time-homogeneous, so they can explain different Black-Scholes implied volatilities, known as volatility smile on options with different strikes and expirations.

Depending on the assumed process for volatility, different methods have been developed to price the volatility swap. In the literature review section, we summarize these methods.

Bank of America Securities LLC — Indicative Terms

(For Discussion Only) — October 8, 1999

S&P 500 Index Realized Variance Swap

Equity Payer:	Bank of America, N.A. ("BofA")
Equity Receiver:	Merrill Lynch International
Trade Date:	October 8, 1999
Maturity Date:	May 7, 2003
Underlying Index:	The Standard & Poor's 500 Composite Stock Price Index

Equity Calculations:

(a) "Initial Price" means 0.305

(b) "Final Price" means the actual realized index Variance defined in accordance with the following formula and definition:

$$\sqrt{\frac{\sum_{i=1}^{n-1}\left[\ln\left(\frac{P_{i+1}}{P_i}\right)\right]^2}{n-2}}\times\sqrt{52}$$

(c) "Natural Logarithm" means for any Daily Quotient, as determined by the Calculation Agent, the exponential number which equates 2.718281828 to such Daily Quotient;

(d) "n" means the total number of Valuation Dates;

(e) "P_i" means the closing level of the index on the ith valuation date (i.e.: P_1 is the closing level of the index on October 6, 1999, P_2 is the closing level of the index on the first Wednesday that is an Exchange Business Day following the Trade Date and P_n is the closing level of the index on the Final Valuation Date.

(f) "Valuation Dates" means, commencing on October 6, 1999, and each Wednesday thereafter up to and including the Final Valuation Date and if any such date is not an Exchange Business Day, the next following day that is an Exchange Business Day, subject to the Market Disruption Events as set forth in the 1996 ISDA Equity Derivatives Definitions.

(g) "$\sum_{i=1}^{n}$" means the summation from $i=1$ to $i=n$.

Notional:	111,230,666
Equity Payment:	Notional * [Final Price^2 – Initial Price^2] If the Equity Payment is a positive value, then the Equity Payer pays the Equity Receiver this value. If the Equity Payment is a negative value, then the Equity Receiver pays the Equity Payer the absolute value of this number.
Credit Terms:	n/a

Figure 1- An Example of Variance Swap Contract

1.2 Literature Review

Based on the demand from volatility traders, the market in volatility and variance swaps has developed rapidly over the last decade and is expected to grow more in the future. Hence several researchers in academia and industry have been working on pricing and hedging of these derivatives.

Carr and Madan (1998) reviewed three methods for trading volatility: First, *static positions in options alone*- The classic example is that of a long position in a straddle, since the value usually increases with a rise in volatility. Unfortunately once the underlying moves away from its initial level these contracts can take on significant price exposure. Second, *dynamic trading in futures* to create or hedge a volatility exposure, however they always have price exposure as well.; and finally they *combine static positions in options with dynamic trading in futures* to synthesize an over-the-counter volatility contracts as a further alternative for trading volatility which has no exposure to price. They assume that an investor follows the classic replication strategy specified by the Black model, with the delta calculated using a constant volatility. Since the volatility is actually stochastic, the replication will be imperfect and the error results in either a profit or a loss realized at the expiration of the hedge.

Demeterfi, Derman, Kamal and Zou (1999) showed how a hedged portfolio of standard options with suitably chosen strikes could theoretically replicate a variance swap, as long as stock prices evolve without jumps. The fair value of the variance swap is the cost of the replicating portfolio. They point out that volatility swaps can be replicated by dynamically trading the more straightforward variance swap. As a result, the value of the volatility swap depends on the volatility of volatility itself.

Although interesting and model-free forecasts of subsequent realized volatility, these last two approaches require that a continuum of options of every strike and maturity on the underlying asset be traded to span volatility. Since in practice there is only a limited number of market quotes to imply the volatility skew per single maturity, we need to introduce an interpolation and extrapolation method to get market implied volatilities at certain strikes, which is crucial for replicating strategy. As a result, we might ultimately end up introducing model-dependence in our replication method.

Heston and Nandi (2000) developed analytical solutions for volatility derivatives such as options and futures on volatility as well as volatility swaps. Their analytical solutions are based on a discrete-time GARCH volatility process.

Little and Pant (2001) developed a finite-difference approach for valuing variance swaps based on a discretely sampled variance. They showed that the price of a variance swap could be obtained as a sum of solutions to a set of degenerate two-dimensional parabolic partial differential equations (PDE). Their method is based on decomposing the problem of valuing a variance swap into a set of one-dimensional PDE problems, each of which is then solved using a finite difference method. They obtain a numerical algorithm for pricing variance swaps.

Javaheri, Wilmott and Haug (2002) estimated the expected variance in a discrete GARCH(1,1) model. They used a general and flexible PDE approach to determine the first two moments of the realized variance in a continuous or discrete context. Then they use this information to approximate the expected realized volatility via a convexity adjustment.

Carr, German, Madan and Yor (2005) derived a closed form solution for the prices of options on realized variance. They employ the models, which hypothesize that returns are pure jump processes with independent increments.

Carr and Lee (2009) price volatility derivatives by replication. Replicating portfolios trade the underlying asset and vanilla options, in quantities in terms of vanilla option prices. Their results are valid if volatility satisfies an independence condition, which means zero correlation between stock price process and variance process.

Broadie and Jain (2008) price variance and volatility swaps when the variance process is a continuous diffusion given by the Heston stochastic volatility model. They compute fair volatility strikes and price variance options by deriving a partial differential equation that must be satisfied by volatility derivatives.

Sepp (2008) developed analytical methodology for pricing and hedging options on the realized variance under Heston stochastic volatility model augmented with jumps in asset returns and variance. He employed generalized Fourier transform to obtain analytical solutions for volatility and variance swaps.

Swishchuk and Xu (2011) studied the valuation of the variance swaps under stochastic volatility with delay and jumps. In their model, the volatility of the underlying stock price process incorporates jumps and also exhibits past dependence.

In this paper we ignore the jumps and delay in the stochastic volatility process and assume Heston stochastic volatility model. To get the fair volatility strike, we solve the PDE suggested by Broadie and Jain (2008). We use finite difference approach to solve the PDE.

2: Heston Model

In an attempt to remedy the drawback of the constant volatility assumption in the Black-Scholes model, many models have been proposed to incorporate stochastic volatility. To describe the stochastic evolution of asset return variation we choose square-root diffusion, which is a central part of a few important financial models including CIR interest rate model (Cox, Ingersoll and Ross (1985)), the Heston stochastic volatility model (Heston (1993)), and the general affine model (Duffie-Pan-Singleton (2000)). This attractiveness of the square-root diffusion is motivated by several essential properties including positivity, mean-reversion, and closed-form solution for the transition density. Among all the stochastic volatility models in the literature, model proposed by Heston (1993) has received the most attention since it can give a satisfactory description of the underlying asset dynamics. The Heston model accommodates a volatility parameter that can increase with the level of kurtosis. Daniel, Joseph and Bree (2007) confirmed that the Heston model improve on the fit provided by the Gaussian model, particularly at higher frequency, since it is supposed to capture both the volatility of price fluctuations and skewness which are more pronounced in high frequency data. Some of the advantages of the model are as follow:

- The parameters in Heston model are able to generate skewness and kurtosis and respectively produce smile or skew in the implied volatilities extracted from option prices generated by the model.
- It allows for the inverse relationship between the stock price and volatility
- A closed form solution for the European option prices is available in the Heston model, which makes the calibration to market prices relatively quick and efficient.

In the Heston model, the stock price process is assumed as Black-Scholes type stochastic process. And the volatility process is based on the Cox, Ingersoll, and Ross (1985) process. The bivariate system of stochastic differential equations in Heston model is as follow:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{v_t}dW_{1,t} \\ dv_t = k(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t} \end{cases} \quad (1.1)$$

The parameters of the model are

μ the drift of the stock price process;

k>0 the mean reversion speed for the variance;

θ>0 the mean reversion level for the variance;

σ>0 the volatility of the variance;

v_0>0 the initial level of the variance;

$\rho \epsilon$ [-1,1] the correlation between the two Brownian motions the $W_{1,t}$ and $W_{2,t}$

$dW_{1,t}$ and $dW_{2,t}$ describe the random noise in asset and variance respectively.

An increase in ρ generates an asymmetry in the distribution while a change of volatility of variance σ results in a higher kurtosis.

2.1 Parameters Estimation

Lewis (2000) derives a volatility of volatility series expansion for the call price and for the implied volatility that are valid under a general class of stochastic volatility models. The general parametrized model for the risk-adjusted pricing process is:

$$\begin{cases} dS = (r-\delta)Sdt + \sigma S dW_{1,t} \\ dv = (\omega - \theta v)dt + \ \xi v^{\varphi} dW_{2,t} \end{cases}$$

Where $dW_{1,t}$ and $dW_{2,t}$ are correlated Brownian motions with correlation $\rho(v)$, r, δ and ξ are constants and $b(v)$ and $\eta(v)$ are independent of ξ.

According to Lewis there are two general steps in the ξ – expansion. The expansion starts from the fundamental solution for the option price. For a call option, and taking k-plane integration along Im k=1/2 for simplicity, this solution representation is

$$C(S,v,\tau) = Ae^{-\delta\tau} - \frac{Ke^{-r\tau}}{2\pi}\int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} \exp(-ikX)\frac{H(k,v,\tau)}{k^2 - ik}dk$$

where $X = \ln\left(\frac{S}{K}\right) + (r-\delta)\,\tau$

Lewis calculate the call option price from the equation (1.3). As it turns out, the k-plane integrals can all be done analytically because they reduce to derivatives of the B-S formula. Doing those integrals yield as result of the form

Series I:

$$C(S,v,\tau) = \ C^{(0)}(S,v,\tau) + \ \xi C^{(1)}(S,v,\tau) + \xi^2 C^{(2)}(S,v,\tau) + \cdots$$

Then implied volatility is calculated from $C(S,v,\tau) = c\big(S, v_{imp}, \tau\big)$, where $C(S,v,\tau)$ is th B-S formula.

Alternatively, one can expand

Series II:

$$v_{imp} = v_{imp}^{(0)} + \xi v_{imp}^{(1)} + \xi^2 v_{imp}^{(1)} + \cdots$$

Then option price is calculated from $C(S,v,\tau) = c\big(S, v_{imp}, \tau\big)$

We use the expansion for Heston model, where $\varphi = \frac{1}{2}$. It is very fast to obtain the Heston call price with this method, because numerical integration is not required. The name is derived from the fact that both series expansions are in terms of powers of the volatility of variance parameters, ξ. The first series is based on an expansion about the Black-Scholes price evaluated at the average variance $\overline{v}$.

Lewis use the B-S formula $c(S, v_0, t)$ and its first derivative with respect to v. That is using $\tau = T - t$

$$c(S, v_0, t) = Se^{-\delta\tau}\Phi(d_+) - Ke^{-r\tau}\Phi(d_-)$$

Where $d_\pm = \frac{1}{\sqrt{v_\tau}}\,[X \pm \frac{1}{2}v_\tau]$, $\quad X = \ln[\frac{Se^{-\delta\tau}}{Ke^{-r\tau}}]$,

And $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{z^2}{2}}\,dz$,

Then, the derivative we need is

$$c_V(S, v_0, \tau) = \frac{\partial}{\partial V}c(S, v_0, \tau) = \sqrt{\frac{\tau}{8\pi v}}\;Se^{-\delta\tau}\exp(-\frac{1}{2}d_+^2\,)$$

When we use these functions, they are not evaluated at the initial volatility v_0. Insteadv_0, is replaced by $\overline{v} = v(v_0, T)$. The quantity $\overline{v}$ denotes the average expected variance over (0,T)

$$\overline{v} = E[\frac{1}{\tau}\int_0^\tau v_t dt\,|v_0] = \frac{1}{\tau}\int_0^\tau E(v_t|v_0)dt$$

$$= \frac{1}{\tau}\int_0^\tau [\theta + (v_0 - \theta)e^{-kt}]dt = (v_0 - \theta)\left(\frac{1 - e^{-k\tau}}{k\tau}\right) + \theta$$

The average expected variance $\overline{v}$ is also the fair strike of a variance swap in the Heston model, as demonstrated by (Gatheral, 2006).

Lewis use 4 integrals, labeled $J_i = J_i(v_0, \tau)$, $i = 1,2,3,4$. It turns out that J_2 vanishes because the drift is linear. The non-vanishing integrals are

$$J_1(v, \tau) = \frac{\rho}{k}\int_0^\tau (1 - e^{-k(\tau - s)})\left(\theta + e^{-ks}(v - \theta)\right)ds$$

$$= \frac{\rho}{k}[\theta\tau + \left(1 - e^{-k\tau}\right)\left(\frac{v}{k} - \frac{2\theta}{k}\right) - e^{-k\tau}(v - \theta)\tau]$$

$$J_3(v, \tau) = \frac{1}{2k^2}\int_0^\tau \left(1 - e^{-k(\tau - s)}\right)^2\left(\theta + e^{-ks}(v - \theta)\right)ds$$

$$= \frac{\theta}{2k^2}\left[T + \frac{1}{2k}\left(1 - e^{-2k\tau}\right) - \frac{2}{k}\left(1 - e^{-k\tau}\right)\right]$$

$$+ \frac{(v - \theta)}{2k^2}\left[\frac{1}{k}\left(1 - e^{-2k\tau}\right) - 2Te^{-k\tau}\right]$$

$$J_4(v,\tau)=\frac{\rho^2}{k^2}\int_0^\tau(\theta+e^{-k(\tau-s)}(v-\theta))(\frac{1}{k}(1-e^{-ks})-se^{-ks})ds$$

$$=\frac{\rho^2\theta}{k^3}\left[\tau\left(1+e^{-k\tau}\right)-\frac{2}{k}\left(1-e^{-k\tau}\right)\right]$$

$$-\frac{\rho^2}{2k^2}\tau^2e^{-k\tau}(v-\theta)+\frac{\rho^2(v-\theta)}{k^3}\left[\frac{1}{k}\left(1-e^{-k\tau}\right)-\tau e^{-k\tau}\right]$$

Lewis also uses certain ratios of derivatives of the B-S formula, as $\tilde{R}^{(p,q)}$:

$$\tilde{R}^{(1,1)}=\left[\frac{1}{2}-W\right]$$

$$\tilde{R}^{(2,0)}=\tau\left[\frac{W^2}{2}-\frac{1}{2Z}-\frac{1}{8}\right]$$

$$\tilde{R}^{(1,2)}=\left[W^2-W-\frac{4-Z}{4Z}\right]$$

$$\tilde{R}^{(2,2)}=\tau\left[\frac{W^4}{2}-\frac{W^3}{2}-\frac{3X^2}{Z^3}+\frac{X(12+Z)}{8Z^2}+\frac{48-Z^2}{32Z^2}\right]$$

In these ratios, Lewis (2000) defines $W=\frac{X}{Z}, X=\log\left(\frac{S_0}{k}\right)+(r-q)\tau$ and $Z=\overline{v}\tau$.

In terms of these expressions, which are explained below, the two series are given. The first series expansion produces Heston call prices $C_I(S_0,v_0,T)$ directly:

$$C_I(S_0,v_0,\tau)\approx c(S_0,\overline{v},\tau)+\sigma\frac{J_1}{\tau}\tilde{R}^{(1,1)}C_v(S_0,\overline{v},\tau)$$
$$+\sigma^2C_v(S_0,\overline{v},\tau)\left[\frac{J_2}{\tau}+\frac{J_3\tilde{R}^{(2,0)}}{\tau^2}+\frac{J_4\tilde{R}^{(1,2)}}{\tau}+\frac{(J_1)^2\tilde{R}^{(2,2)}}{2\tau^2}\right]+O(\sigma^3)$$

The second series produces an implied variance the Heston call price $C_{II}(S_0,v_0,\tau)$.

$$V_{imp}\approx\overline{v}+\sigma\frac{J_1}{\tau}\tilde{R}^{(1,1)}+\sigma^2\left[\frac{J_2}{\tau}+\frac{J_3\tilde{R}^{(2,0)}}{\tau^2}+\frac{J_4\tilde{R}^{(1,2)}}{\tau}+\frac{(J_1)^2}{2\tau^2}(\tilde{R}^{(2,2)}-\left(\tilde{R}^{(1,1)}\right)^2\tilde{R}^{(2,0)}\right]+O(\sigma^3)$$

The implied variance is then fed into the Black-Scholes model to produce the Heston call price under Series II.

$$C_{II}(S_0,v_0,\tau)=c\left(S_0,v_{imp},\tau\right)$$

In other words, we use (1.4) to obtain the call price, but we replace $\overline{v}$ with v_{imp} from previous equation

Finally, in the expressions for the call price in equations (1.14) the integrals J_1,J_3,J_4 are all evaluated at the initial variance v_0, so that v is replaced by Heston parameter v_0 everywhere in (1.7), (1.8) and (1.9).

We used the model parameters as Table 3.1 of Lewis (2000) to reproduce the implied volatility graph. Model parameters are ρ=0.5, v_0=0.0225, k=0.04, $\theta = \frac{0.09}{4}$, $\sigma = 0.1$. We also used $\tau = 0.25$ years, $r = \delta = 0, S = 100$.

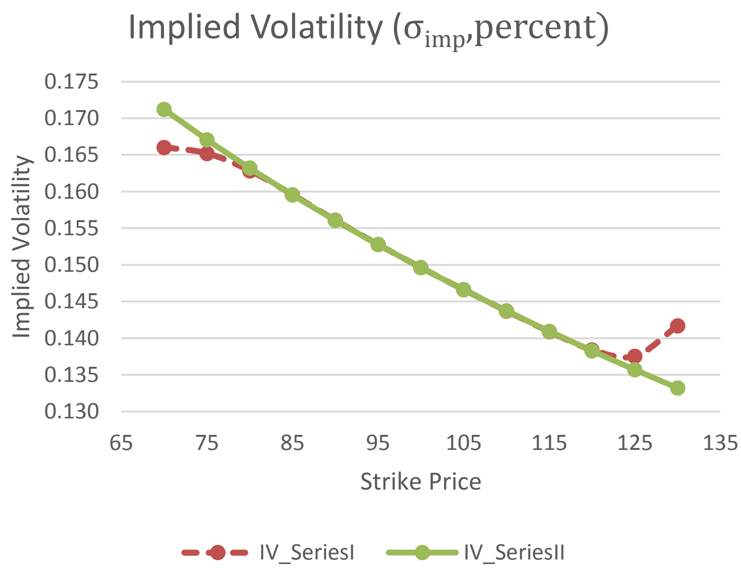


Figure 2- Reproduction of Figure 3.1 of Lewis (2000)

	Strike Price												
Model	**70**	**75**	**80**	**85**	**90**	**95**	**100**	**105**	**110**	**115**	**120**	**125**	**130**
CallPrice_Series I	30.000	25.000	20.007	15.056	10.303	6.116	2.984	1.137	0.324	0.067	0.010	0.001	0.000
IV_SeriesI	0.166	0.165	0.163	0.160	0.156	0.153	0.150	0.147	0.144	0.141	0.138	0.138	0.142
CallPrice_Series II	30.000	25.000	20.003	15.037	10.258	6.073	2.991	1.193	0.381	0.098	0.020	0.003	0.001
IV_SeriesII	0.171	0.167	0.163	0.160	0.156	0.153	0.150	0.147	0.144	0.141	0.138	0.136	0.133
CallPrice_BS	30.000	25.000	20.003	15.037	10.258	6.073	2.991	1.193	0.381	0.098	0.020	0.003	0.001
IV_BS	0.150	0.150	0.150	0.150	0.150	0.150	0.150	0.150	0.150	0.150	0.150	0.150	0.150

Table 1- Option Prices and Implied Volatility: Black-Scholes versus Volatility of Volatility Series Expansions

The results from the two methods are often very close. The two series differ the most from each other for relatively far out-of-the money options. Lewis also calculated the Heston call price and implied volatilities using the trapezoidal rule and called it the exact method. Numerical comparison with the exact result shows that Series II is usually but not always more accurate. So he recommends using Series II.

2.2 Estimating Heston parameters for S&P 500 index

As the most common estimation method, we try to select the parameters in a way that the quoted market implied volatilities are as close as possible to the model implied volatilities. This is called Loss Function approach. The market for S&P 500 index options is the second most active index options market in the United States and, in terms of open interest in options, it is the largest. We collected the Options data on S&P 500 index on different dates and used Lewis Series II expansion to estimate the Heston parameters using the loss function. The quoted (market) implied volatility is extracted from Bloomberg. Bloomberg uses the following method to calculate implied volatility:

European Options:

Value is derived using the Black-Scholes formula.

American Options:

Value is derived numerically using a finite-difference PDE (partial differential equation) price employing a constant volatility to option maturity and a term structure of interest rates.

We represent Heston parameters as the vector $\Theta=(k, \theta, \sigma, v_0, \rho)$, and their corresponding estimate as $\hat{\Theta}$. We use a set of N_τ maturities τ_t $(t=1, \ldots, N_T)$ and a set of N_K strikes k_k $(k=1, \ldots, N_K)$. For each pair of (τ_t, k_k), we have a market implied volatility $IV(\tau_t, k_k)$ and a corresponding model implied volatility $IV(\tau_t, k_k; \Theta) = IV_{tk}^{\Theta}$. The error is usually defined as the squared difference between the quoted and implied volatilities, or the absolute value of the difference, relative error can also be used.

Mean error sum of square (MSE) loss function: $IVMSE = \frac{1}{N}\sum_{tk}(IV_{tk} - IV_{tk}^{\Theta})^2$

We used a constrained minimization algorithm on the error between quoted and model implied volatilities subject to the constraints on the parameters as:

$k>0$, $\theta>0$, $\sigma>0$, $v_0>0$, $\rho \epsilon [-1,1]$

We use the approximated implied volatility from Lewis' (2000) Series II expansion and use that instead of IV_{tk}^{Θ}. As the initial parameters, for the first data set we used the Heston paramaters calculated by Rouah (2013) using the options collected on April 13, 2012, on the S&P500 index. The parameters are ρ=-0.7799, v_0=0.0344, k=1.9214, $\theta = 0.0904$, $\sigma = 1.0193$. And for the other data sets we used the parameters achieved on the first data set. At each date, we calibrate the model to S&P500 index options with 10 different maturities and 10 different strikes, so in each data set we used 100 options implied volatility.

Dataset	IVMSE	ρ	θ	v_0	κ	σ_v
30-Apr-15	7.71E-06	-0.6336	0.0463	0.0174	2.2614	0.8906
11-Sep-15	5.22E-06	-0.5178	0.0444	0.0308	10.8046	2.4443
01-Oct-15	7.33E-06	-0.4796	0.0453	0.0275	6.7932	1.6087
16-Oct-15	5.76E-04	-0.353	0.0429	0.0063	11.5455	1.7743
11-Nov-15	2.69E-05	-0.4829	0.0451	0.0276	6.7902	1.5658

Table 2- Result of Heston Parameter Estimation for Different Data Sets

Confirming Matytsin(1999) findings, among the stochastic volatility parameters, long run diffusion volatility $\sqrt{\theta}$ is relatively stable. The following figures show the market and Heston implied volatilities on the S&P 500 index, estimated using IVMSE on different data sets.

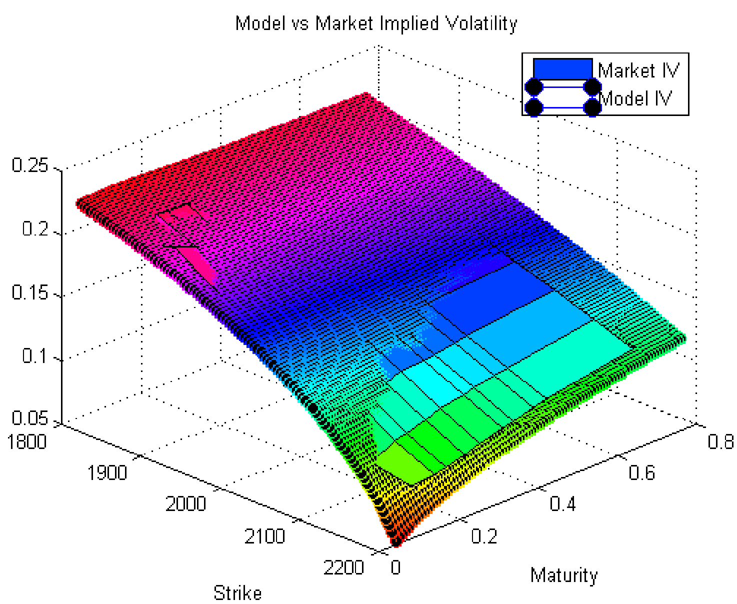


Figure 3- Volatility surface on dataset 1

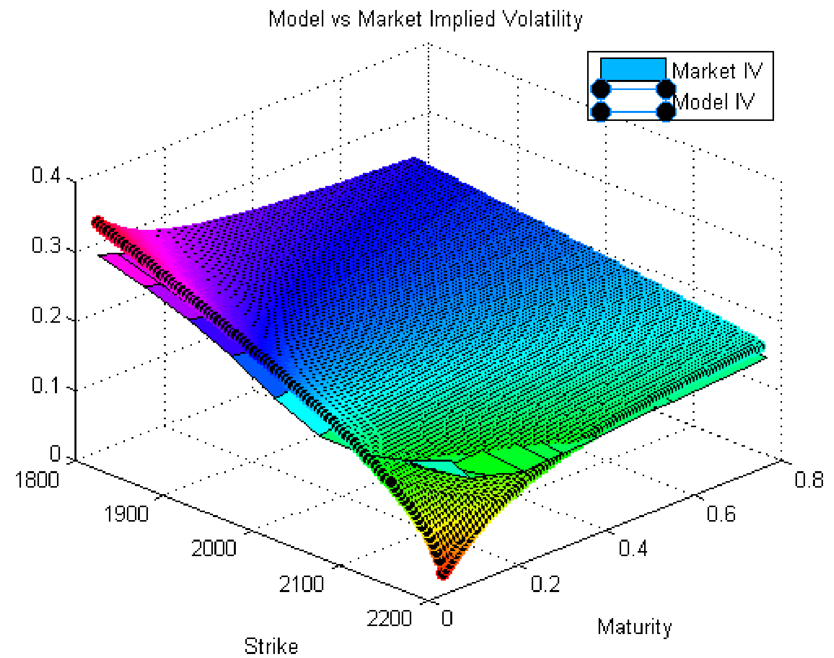


Figure 4- Volatility surface on dataset 2

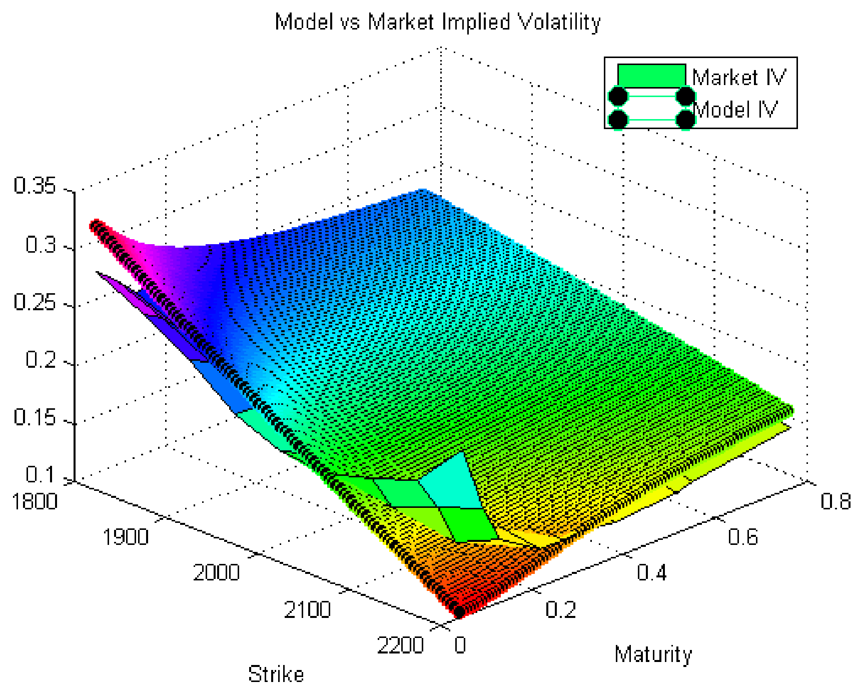


Figure 5- Volatility surface on dataset 3

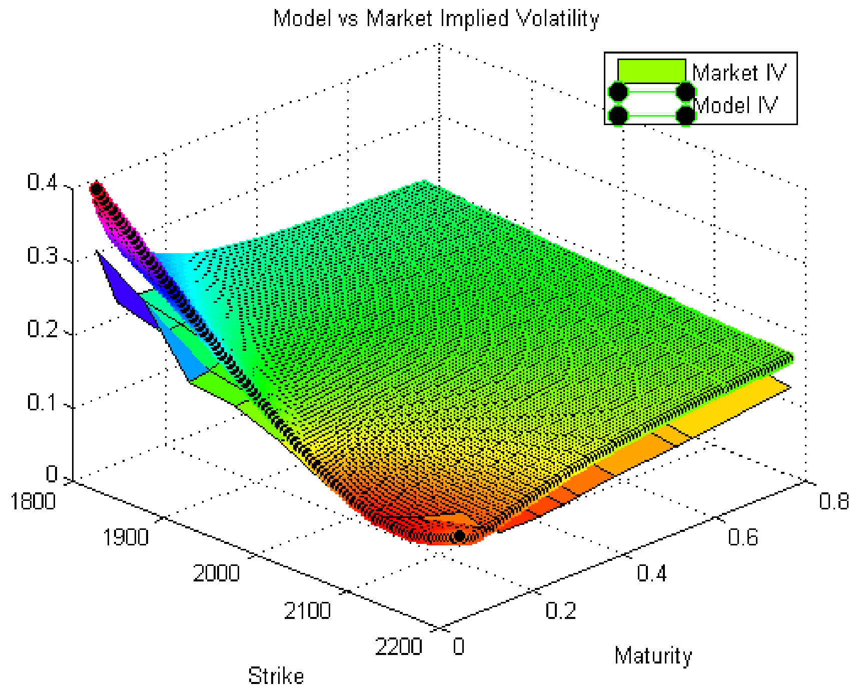


Figure 6- Volatility surface on dataset 4

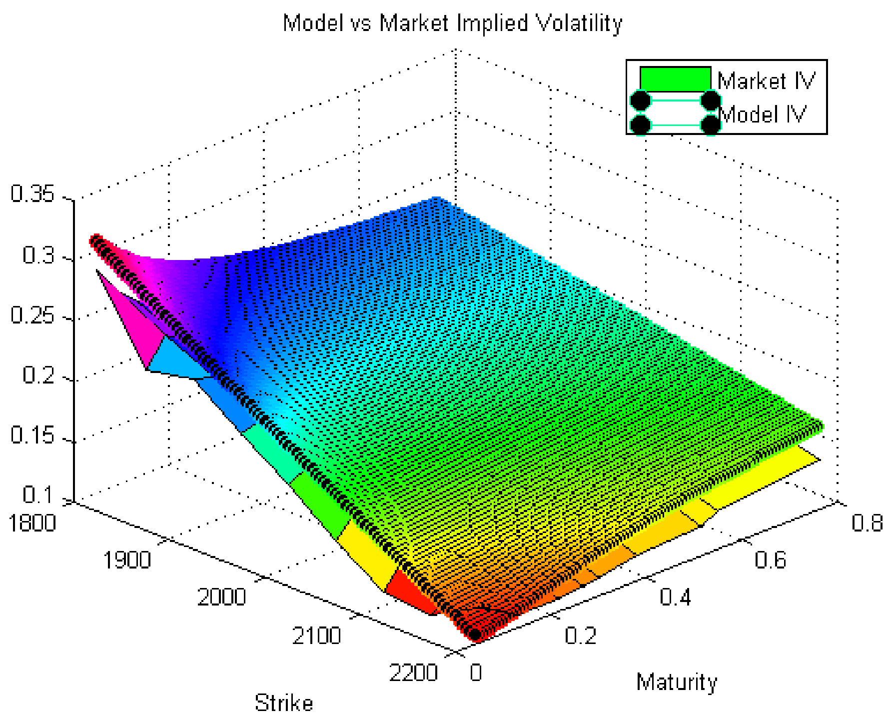


Figure 7- Volatility surface on dataset 5

Based on the parameters estimation for the five data series, it is indicated that ρ (correlation between the stock price and volatility Brownian motions) and θ (the mean reversion level of the variance) are stable; however the other parameters vary significantly between data sets. We are going to use the parameters from the first data set, to calculate the volatility swap price in the next part of the project.

As demonstrated in the graphs and indicated by the IVMSE, the model implied volatility based on the Heston parameters is very close to the market implied volatility. The major difference is in out-of-the money and short term maturity options.

3: Volatility Swap

The easy way to trade volatility is to use volatility swaps, because they provide pure exposure to volatility (and only to volatility). A volatility swap is an OTC contract, which similar to any swap contract has zero upfront premium. At the expiry date, the long side of the volatility swap pays a positive dollar amount, which was agreed upon at inception. In return for this fixed payment, the long side receives a positive dollar amount at expiry, called the realized volatility of the underlying index. The realized volatility, which is the floating leg of the contract, is a square root of annualized average of squared daily returns (Carr and Lee, 2009). In this part we calculate the fair volatility strike based on Heston stochastic volatility model.

3.1 Fair volatility strike

A stock volatility swap is a forward contract on annualized volatility. Its payoff at expiration is equal to

$$\left(\sqrt{V_c(0,T)} - K\right) \times N$$

where $\sqrt{V_c(0,T)}$ is the realized stock volatility (quoted in annual terms as defined below) over the life of the contract, K is the volatility strike, and N is the notional amount of the swap in dollars. $V_c(0,T)$ is the continuously sampled realized variance. However, it can be replaced with $V_d(0,n,T)$ which is discretely sampled realized variance, n is the number of sampling dates.

The procedure for calculating realized volatility and variance should be specified in the derivative contract with respect to the several aspects: detail about the source and observation frequency of the price of the underlying asset, the annualization factor to be used in moving to an annualized volatility and the method of calculating the variance. Most traded contracts define realized variance to be

$$V_d(0,n,T) = \frac{AF}{n-1} \sum_{i=0}^{n-1} \left(\ln\left(\frac{S_{i+1}}{S_i}\right)\right)^2$$

Here S_i is the price of the asset at the i^{th} observation time t_i and AF is the annualization factor, e.g., 252 ($= n/T$) if the maturity of the swap, T, is one year with daily sampling. In contrary to definition of realized variance, in the above formula, the sample average is not subtracted from each observation. Since the sample average is approximately zero, the realized variance is close to the sample variance.

In the stochastic volatility models, continuous realized variance is given by

$$V_c = \frac{1}{T}\int_0^T \upsilon_s ds$$

(2.1)

The fair variance strike, K_{var}^* is defined as the value which makes the contract's net present value equal to zero, i.e., it is the solution of

$$E_0^Q[e^{-r\tau}(V_c(0,T) - K_{var}^*)] = 0$$

(2.2)

where the superscript Q indicates the risk-neutral measure and the subscript 0 denotes expectation at time $t = 0$ (Broadie and Jain, 2008). In the SV model, the fair variance strike is given by

$$K_{var}^* = E[V_c(0,T)] = E(\frac{1}{T}\int_0^T \nu_s ds) = \theta + \frac{\nu_0 - \theta}{\kappa T}(1 - e^{-\kappa T})$$

(2.3)

where the last equality follows, e.g. from Broadie and Jain (2007). The fair volatility strike is defined as the value, which makes the contract net present value equal to zero, i.e., it solves the equation

$$E_0\left[e^{-rT}\left(\sqrt{V_c(0,T)} - K_{vol}^*\right)\right] = 0$$

(2.4)

Hence, the fair volatility strike can be expressed as

$$K_{vol}^* = E\left[\sqrt{\frac{1}{T}\int_0^T \nu_t dt}\right] = E[\sqrt{V_c(0,T)}]$$

(2.5)

Broadie and Jain (2008) define Y_t^T to be the price process of the floating leg of a volatility swap:

$$Y_t^T = E_t^Q\left[\sqrt{\frac{1}{T}\int_0^T \nu_s ds}\right]$$

This security has a payoff at time T which depends on the variance process from time $t = 0$ until maturity. At time T it represents the payoff of the floating leg of the volatility swap. At time 0 it gives the fair volatility strike:

$$K_{vol}^* = Y_0^T$$

Broadie and Jain (2008) define a state variable I_t to measure the accumulated variance so far:

$$I_t = \int_0^t \nu_s ds$$

I_t is the variance over the life of the contract, as opposed to ν the instantaneous variance at a point in time. This state variable is a known quantity at time t and satisfies the differential equation:

$$dI_t = \nu_t dt$$

The forward price process, Y_t^T, can be expressed as

$$Y_t^T = E_t\left[\sqrt{\frac{1}{T}(I_t + \int_t^T \nu_s ds)}\right] = F(t, \nu_t, I_t)$$

and is a function of time, the stochastic variance ν_t and a deterministic quantity I_t. Applying Ito's lemma to $F(.)$ we get

$$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial \nu}d\nu + \frac{\partial F}{\partial I}dI + \frac{1}{2}\frac{\partial^2 F}{\partial \nu^2}d\nu^2$$

which is simplified using equation (1.1) to

$$dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial \nu}\kappa(\theta - \nu_t) + \frac{\partial F}{\partial I}\nu_t + \frac{1}{2}\frac{\partial^2 F}{\partial \nu^2}\nu_t\sigma_\nu^2\right]dt + \frac{\partial F}{\partial \nu}\sigma_\nu\sqrt{\nu_t}dW_{2,t}$$

(2.6)

Since F is a forward price process, its drift under the risk-neutral measure must be zero. Hence,

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial \nu}\kappa(\theta - \nu_t) + \frac{\partial F}{\partial I}\nu_t + \frac{1}{2}\frac{\partial^2 F}{\partial \nu^2}\nu_t\sigma_\nu^2 = 0$$

(2.7)

Thus, the forward price process satisfies the partial differential equation (2.7) in the SV model (Broadie and Jain, 2008). We solve the partial differential equation in the region:

$0 \le t \le T, I_{min} \le I \le I_{max}, \nu_{min} \le \nu \le \nu_{max}$, with boundary condition

$$Y_T^T = F(T, \nu_T, I_T) = \sqrt{\frac{I_T}{T}}$$

(2.8)

At other boundaries (I and V) we set the second order variation of the price process to zero (Broadie and Jain, 2008). In particular we use the boundary conditions:

$$\left.\frac{\partial^2 F}{\partial \mathrm{I}^2}\right|_{(I=I_{max}, I_{min})} = 0 \qquad \left.\frac{\partial^2 F}{\partial \mathrm{v}^2}\right|_{(\nu=\nu_{max}, \nu_{min})} = 0$$

(2.9)

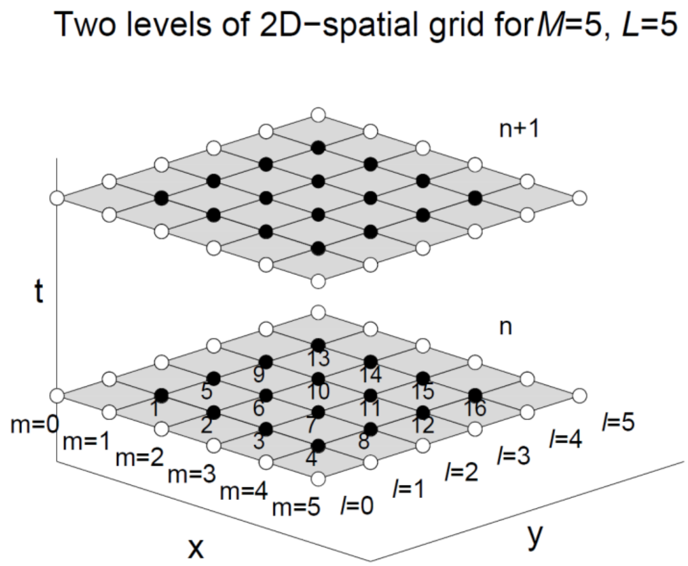
Thus by solving the partial differential equation (2.7) with boundary conditions (2.8) and (2.9) we can compute the fair volatility strike and get the price at all times until maturity.

Next we present numerical results to illustrate the computation of fair volatility strike. We use model parameters from what we calculated in part 1 for data set 1.

3.2 Solving the PDE

We solve the PDE (2.7) with the conditions (2.8) and (2.9). Regarding the Heston parameters, We use the parameters from the first part of this paper by minimizing mean-squared differences between model and market S&P500 index implied volatilities on April 30, 2015. Therefore we are calculating $F_{i-1,j,k}$ using numerical method, i.e. recurrence relation to $F_{i,j,k}$. It turns out when the ratio of time step to I- or V- step is bigger than 1.3% , the approximate solutions can contain (decaying) spurious oscillations. Therefore for example for time to maturity 125 days we used 7000 intervals in t-direction and 40 intervals in each V- and I- directions to meet the stability condition. For easier demonstration, We change the notation of $F_{i,j,k}$ to $U_{m,l}^{n+1}$ and $F_{i-1,j,k}$ to $U_{m,l}^{n}$, where $n \approx i-1, m \approx j, l \approx k$.

The solution vector can be set up on a 3-dimensional grid (two dimensions in V and I and one dimension in time). (Lakoba, lecture notes)



We assume that the step size on x and y directions are the same and equal to h. We also discretize the time variable with a step size k. Then the three-dimensional grid for the 2D equation consists of points $(x = mh, y = lh, t = nk)$. $0 \le m \le M = \frac{V_{max}}{h}$, $0 \le l \le L = \frac{I_{max}}{h}$ and $0 \le n \le N = \frac{T}{k}$.

We denote the solution on the above grid as

$U_{m,l}^{n} = u(mh, lh, nk)$, $\quad 0 \le m \le \frac{v_{max}}{h}$, $0 \le l \le \frac{I_{max}}{h}$ and $0 \le n \le \frac{T}{k}$

We expect that any numerical scheme that we design will give some recurrence relation between $U_{m,l}^{n+1}$

and $U_{m,l}^{n}$. As long as our grid are rectangular, the array of values $U_{m,l}^{n}$ at each given n can be conveniently represented as $(M-1) \times (L-1)$ matrix. To step from level $(n+1)$ to level n, we just apply the recurrence formula to each element of the matrix $U_{m,l}^{n+1}$. The first components $(M-1)$ of the solution vector $\bar{\mathrm{U}}$ will be the values $U_{m,1}^{n}$ with $m = 1,2, \dots, M-1$. The next $(M-1)$ components will be $U_{m,2}^{n}$ with $m = 1,2, \dots, M-1$, and so on. The resulting vector is:

$$\vec{\mathbf{U}} = \begin{pmatrix} U_{1,1} \\ U_{2,1} \\ \cdot \\ \cdot \\ U_{M-1,1} \\ U_{1,2} \\ U_{2,2} \\ \cdot \\ \cdot \\ U_{M-1,2} \\ \cdot \\ \cdot \\ \cdot \\ U_{1,L-1} \\ U_{2,L-1} \\ \cdot \\ \cdot \\ U_{M-1,L-1} \end{pmatrix} \quad \begin{array}{l} \left.\begin{array}{l} U_{1,1} \\ \vdots \\ U_{M-1,1} \end{array}\right\} \text{1st row of 2D level (along } y = 1 \cdot h \text{ (i.e. } l = 1)) \\ \left.\begin{array}{l} U_{1,2} \\ \vdots \\ U_{M-1,2} \end{array}\right\} \text{2nd row of 2D level (along } y = 2 \cdot h \text{ (i.e. } l = 2)) \\ \left.\begin{array}{l} U_{1,L-1} \\ \vdots \\ U_{M-1,L-1} \end{array}\right\} (L-1)\text{th row of 2D level (along } y = (L-1) \cdot h \text{ (i.e. } l = L-1)) \end{array}$$

We use the Crank-Nicolson finite difference model, so for each 2D grid along the x and y directions the coefficients calculation would be as follow:

Central difference for $\partial U/\partial t$

$$\frac{\partial U_{m,l}^{n+\frac{1}{2}}}{\partial t} = \frac{U_{m,l}^{n+1} - U_{m,l}^{n}}{\delta t} + O(\delta t^2)$$

Symmetric difference for $\partial U/\partial v$

$$\frac{\partial U_{m,l}^{n+\frac{1}{2}}}{\partial v} = \frac{1}{2}\left[\frac{\partial U_{m,l}^{n}}{\partial v} + \frac{\partial U_{m,l}^{n+1}}{\partial v}\right]$$

$$= \frac{1}{2}\left[\frac{U_{m+1,l}^{n} - U_{m-1,l}^{n}}{2\delta v} + \frac{U_{m+1,l}^{n+1} - U_{m-1,l}^{n+1}}{2\delta v}\right] + O(\delta v^2)$$

Symmetric difference for $\partial U/\partial I$

$$\frac{\partial U_{m,l}^{n+\frac{1}{2}}}{\partial I} = \frac{1}{2}\left[\frac{\partial U_{m,l}^{n}}{\partial I} + \frac{\partial U_{m,l}^{n+1}}{\partial I}\right]$$

$$= \frac{1}{2}\left[\frac{U_{m,l+1}^{n} - U_{m,l-1}^{n}}{2\delta I} + \frac{U_{m,l+1}^{n+1} - U_{m,l-1}^{n+1}}{2\delta I}\right] + O(\delta I^2)$$

Standard 2nd derivative difference

$$\frac{\partial^2 U_{m,l}^{n+\frac{1}{2}}}{\partial v^2} = \frac{1}{2}\left[\frac{\partial^2 U_{m,l}^{n}}{\partial v^2} + \frac{\partial^2 U_{m,l}^{n+1}}{\partial v^2}\right]$$

$$= \frac{1}{2}\left[\frac{U_{m+1,l}^{n} - 2U_{m,l}^{n} + U_{m-1,l}^{n}}{\delta v^2} + \frac{U_{m+1,l}^{n+1} - 2U_{m,l}^{n+1} + U_{m-1,l}^{n+1}}{\delta v^2}\right] + O(\delta v^2)$$

Using the above expressions in the PDE (2.7), result in:

$$a_j U_{m-1,l}^{n} - (1 + b_j) U_{m,l}^{n} + c_j U_{m+1,l}^{n} - e_j U_{m,l-1}^{n} + e_j U_{m,l+1}^{n} =$$

$$-a_j U_{m-1,l}^{n+1} - \left(1 - b_j\right) U_{m,l}^{n+1} - c_j U_{m+1,l}^{n+1} + e_j U_{m,l-1}^{n+1} - e_j U_{m,l+1}^{n+1}$$

(2.10)

where $v_t = j\delta v$, and

$$a_j = \frac{j\sigma_v^2 \delta t - \kappa(\theta - j\delta v)\delta t}{4\delta v}$$

(2.11)

$$b_j = \frac{j\sigma_v^2 \delta t}{2\delta v}$$

(2.12)

$$c_j = \frac{\kappa(\theta - j\delta v)\delta t + j\sigma_v^2 \delta t}{4\delta v}$$

(2.13)

$$e_j = \frac{j\delta v\delta t}{4\delta I}$$

(2.14)

So for the first three terms in equation (2.10), we define matrix A as:

$$A = \begin{bmatrix} -b_1 & c_1 & 0 & \cdots & 0 & 0 \\ a_2 & -b_2 & c_2 & \cdots & 0 & 0 \\ 0 & a_3 & -b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{M-1} & -b_{M-1} \end{bmatrix}$$

(2.15)

And for the last two terms in equation (2.10), we define matrix E as:

$$E = \begin{bmatrix} e_1 & 0 & \cdots & 0 \\ 0 & e_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{M-1} \end{bmatrix}$$

(2.16)

Then we build the coefficient of vector $U_{m,l}^{n+1}$, as a block-tridiagonal matrix with the size $[(M-1)\times(L-1)]\times[(M-1)\times(L-1)]$. Therefore in equation $CU_{m,l}^{n} = DU_{m,l}^{n+1}$, the C and D would be as follow:

$$C = (I - \mathcal{A})$$

(2.17)

$$D = (I + \mathcal{A})$$

(2.18)

where

$$\mathcal{A} = \begin{bmatrix} A & E & O & \cdots & O & O \\ -E & A & E & \cdots & O & O \\ O & -E & A & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & -E & A \end{bmatrix}$$

(2.19)

3.3 Boundary conditions

The boundary conditions are:

$$\left.\frac{\partial^2 F}{\partial \mathrm{I}^2}\right|_{(I=I_{max},I_{min})} = 0 \qquad \left.\frac{\partial^2 F}{\partial \mathrm{v}^2}\right|_{(\nu=\nu_{max},\nu_{min})} = 0$$

This kind of condition, which is a condition on derivative of U instead of U itself, is called a *Neumann boundary condition* (LeVeque, 2006). To solve this problem numerically, we need to introduce one more unknown than we previously had: U(0) at the point $\nu = 0$ or $I = 0$. I used the one sided expression for U″(0)=0 and U″(end)=0

So for ν_{min} and ν_{max} the equations would be:

$$U''(0) = \frac{\mathrm{U}^{\mathrm{n}}_{2,:} - 2\mathrm{U}^{\mathrm{n}}_{1,:} + \mathrm{U}^{\mathrm{n}}_{0,:}}{\partial \nu^2} = 0$$

$$U''(end) = \frac{\mathrm{U}^{\mathrm{n}}_{M,:} - 2\mathrm{U}^{\mathrm{n}}_{M-1,:} + \mathrm{U}^{\mathrm{n}}_{M-2,:}}{\partial \nu^2} = 0$$

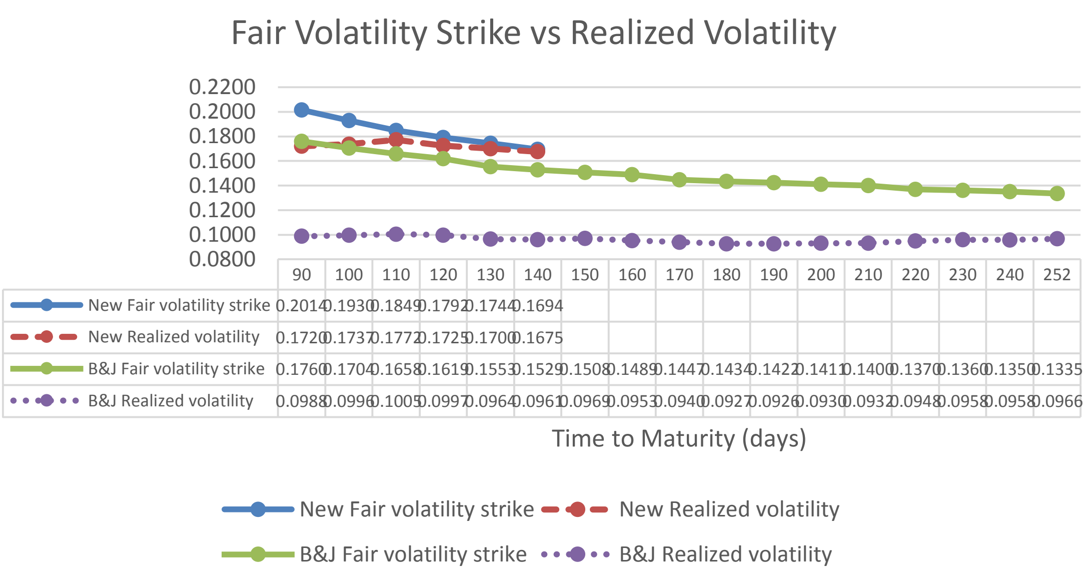
This means that in equation $CU^n = DU^{n+1}$, I need to change the coefficients of $\mathrm{U}^{\mathrm{n}}_{2,:}$, $\mathrm{U}^{\mathrm{n}}_{1,:}$, $\mathrm{U}^{\mathrm{n}}_{0,:}$, $\mathrm{U}^{\mathrm{n}}_{M,:}$, $\mathrm{U}^{\mathrm{n}}_{M-1,:}$ and $\mathrm{U}^{\mathrm{n}}_{M-2,:}$ based on the above conditions. This requires adding one row to the matrices A and E for elements $\mathrm{U}^{\mathrm{n}}_{0,:}$. Respectively the size of matrices $\mathcal{A}, C$ and D change to $(M \times L) \times (M \times L)$.

Similarly for I_{min} and I_{max} the equations would be:

$$U''(0) = \frac{\mathrm{U}^{\mathrm{n}}_{:,2} - 2\mathrm{U}^{\mathrm{n}}_{:,1} + \mathrm{U}^{\mathrm{n}}_{:,0}}{\partial \mathrm{I}^2} = 0$$

$$U''(end) = \frac{\mathrm{U}^{\mathrm{n}}_{:,L} - 2\mathrm{U}^{\mathrm{n}}_{:,L-1} + \mathrm{U}^{\mathrm{n}}_{:,L-2}}{\partial \mathrm{I}^2} = 0$$

We calculated the fair volatility strike based on the parameters used by Broadie and Jain (2008). The parameters were calibrated to the market data on November 02, 1993. The fair volatility strike of 13.35% for one year maturity swap matches with the result as 13.26% mentioned in their paper. We also compared fair volatility strike with the realized volatility for different maturities after November 02, 1993. (Indicated in figure 8 as B&J Fair volatility strike vs B&J Realized volatility)



	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	252
New Fair volatility strike	0.2014	0.1930	0.1849	0.1792	0.1744	0.1694											
New Realized volatility	0.1720	0.1737	0.1772	0.1725	0.1700	0.1675											
B&J Fair volatility strike	0.1760	0.1704	0.1658	0.1619	0.1553	0.1529	0.1508	0.1489	0.1447	0.1434	0.1422	0.1411	0.1400	0.1370	0.1360	0.1350	0.1335
B&J Realized volatility	0.0988	0.0996	0.1005	0.0997	0.0964	0.0961	0.0969	0.0953	0.0940	0.0927	0.0926	0.0930	0.0932	0.0948	0.0958	0.0958	0.0966

Figure 8- Fair volatility strike vs Realized volatility

As demonstrated in figure 8, B&J fair volatility strike is getting closer to the realized volatility for longer term maturities, however there is significant difference between them.

Furthermore using the Heston parameters as calibrated on the first dataset at April 30, 2015 in part 1 of this paper, we calculated the fair volatility strike on S&P500 index for different maturities and compared it with the realized volatilities after that date. (Indicated in figure 8 as New Fair volatility strike vs New Realized volatility). As we can see, the fair volatility strike is close to the realized volatility for longer time to maturity swaps.

4: Conclusion

In this paper, we priced volatility swap considering Heston stochastic volatility model. For calibrating Heston parameters, we minimized the loss function on implied volatility based on the Lewis (2000) expansion of the volatility of volatility. For pricing the volatility swap, we exploited the PDE introduced by Broadie and Jain (2008) based on the no-arbitrage strategy. We solved the three variable PDE using numerical computation by finite difference Crank-Nicolson method in a 3D grid. In order to assess Broadie and Jain approach, we price the volatility swap on S&P 500 index and compare our model expected volatility to the realized volatility. Our result shows that the model fair volatility strike is close to realized volatility for long maturity swaps.

5: Future works

It would be interesting to consider contracts, which pay nonlinear functions of realized or local volatility. One of the drawbacks of model based pricing, is that this approach is subject to significant model risk since one is unlikely to guess the correct volatility process. So a risk of misspecification and miscalibration presents in any one model. While the stochastic volatility model fits the longer behaviour of the asset price, to adequately describe the short-term behaviour of the asset price it should be augmented with return jumps; furthermore for the realistic modeling of volatility skew observed in market prices of VIX options, the jumps in the variance should also be included.

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