Unfolding of Diagramming and Gesturing between Mathematics Graduate Student and Supervisor during Research Meetings

by

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Abstract

This study has two main purposes. The first is to confirm and advance Gilles Châtelet's account of the role of diagramming in mathematics invention by extending his results, which were based solely on historical, mathematical manuscripts, to the context of live mathematical activity. The second purpose is to elucidate the enculturation process of a graduate student into mathematical research, which has hitherto received limited research attention. These two purposes are related through the virtual and physical gestures that the graduate student engages in during diagramming thereby providing insights into mathematical invention and the enculturation process. This study adopts a qualitative methodology based on field notes, video-recordings and digital images from nine research meetings, which were held weekly over a period of three months.

Current research theorises the role of mathematical diagrams in diametrically opposed ways: diagrams are either a visual representation of already existing mathematical objects and relations, or they are the means through which mathematical objects and relations emerge. The latter is due to Châtelet, who regards the diagram as a material site of engaging with and mobilizing the mathematics. His approach is employed in this thesis to create a window into the realm of mathematical thinking and invention by examining how a graduate student (as the less-expert mathematician) and his supervisor and two research colleagues (as the expert mathematicians) interact with diagrams. An embodied lens, based on the work of de Freitas, Roth, Rotman, Sinclair and Streeck, exposes the similarities and differences in the way that each class of mathematician gestures and diagrams. The analysis in this thesis reveals that gesturing and diagramming support and advance mathematical communication throughout the graduate student's enculturation process. Furthermore, a collective study of the abundant diagrams produced during research meetings leads to a life-cycle of diagrams, whose phases disclose a variety of distinct relationships between mathematician and diagram. Lastly, a detailed examination of the evolution of a particular diagram uncovers how mathematical invention emerges through gesturing and diagramming. These findings have implications for the teaching and learning of mathematics at all levels.

Keywords: diagram, diagramming, embodiment, enculturation, gestures, graduate student, materialism, mathematical research, mathematizing, virtuality.

For my family, because they are my roots that give me strength and let me blossom.

"But without my family, nothing – not even the beauty of mathematics – would have any meaning at all." (Zeitz, 1998, p.xii)

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I want to take this opportunity to voice my gratefulness and appreciation to the people who supported and guided my doctoral studies.

Foremost, I want to thank my son, Eli, who only had a partial mother during this personal journey. May he find his own journey in life, equally enriching and empowering. I thank and acknowledge my husband, Dr. John Stockie, for supporting my research endeavours, for squeezing my edits into his prolific research life, and for being willing to take a dive into the world of research in mathematics education. Meine Schwester Manuela und ihre Familie, meine Mutti und mein Papa haben mich die Jahre hindurch mit ihrem Verständnis und Gesprächen von nah und fern mit Liebe unterstützt. Dafür bin ich von Herzen dankbar.

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Cindy and Mary, thank you for the smiles, hugs and encouragements, and the many dinners to nourish not only my body but also my soul.

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Lastly, I am grateful for the yoga practices lead by Alix and Delia, who restored my spirit and my body on countless occasion with their technical expertise and wisdom. Lokāḥ samastāḥ sukhino bhavanthu – may we all be happy, healthy and at peace.

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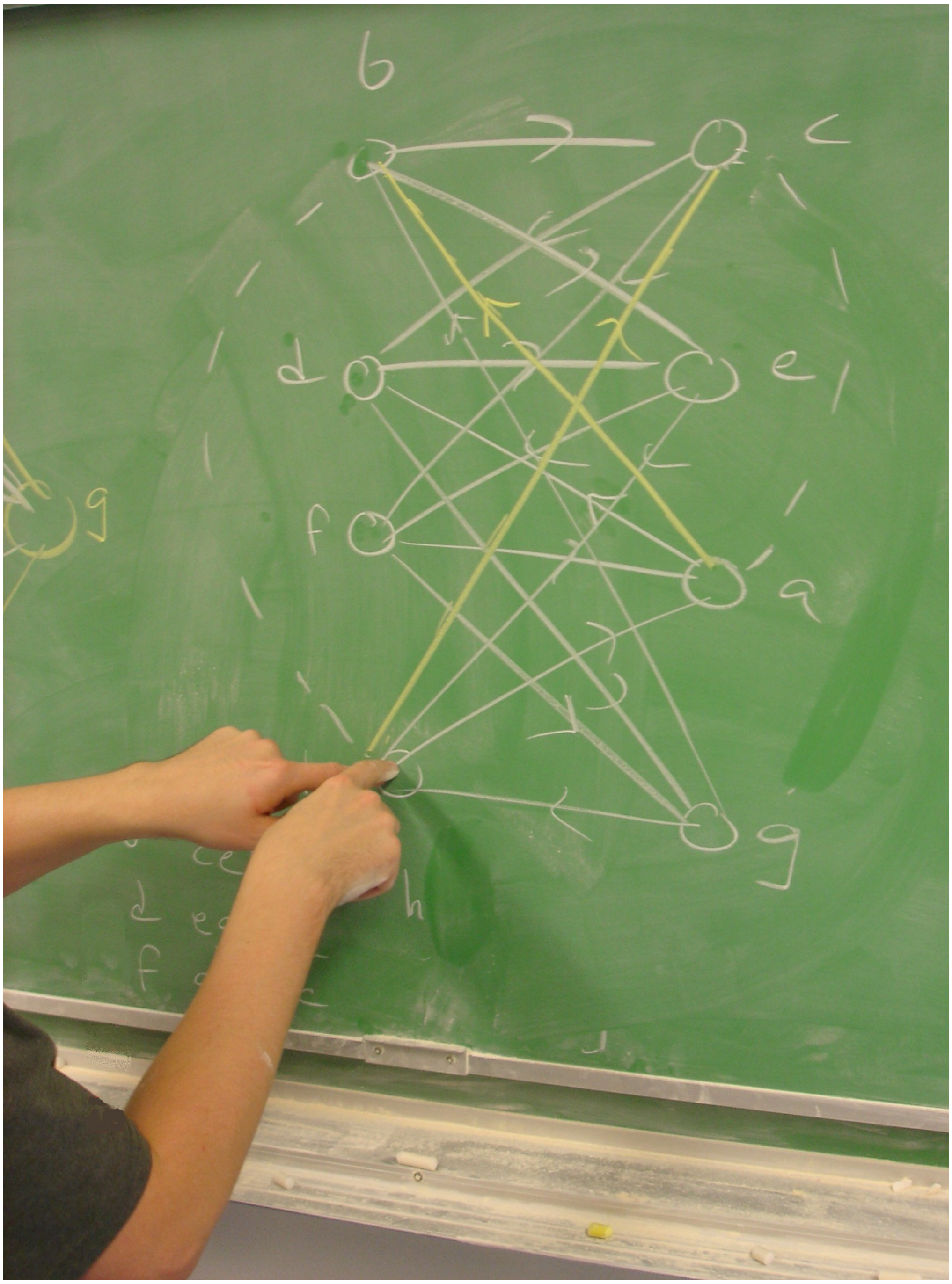
Transcription Glossary

.	falling intonation followed by noticeable pause
?	rising intonation
!	animated tone
,	continuing intonation
bold	used to highlight a phrase being discussed in the text
italic	indicates emphasis by the speaker within the utterance
(.)	indicates a brief interval within or between utterances
(2.3)	example of timed pause
[	indicates the point of overlap onset
]	indicates the point at which two overlapping utterances end
=	a pair of equal signs, one at the end of one line and one at the beginning of the next, indicate no break between the two lines
( )	indicates that the transcriber was unable to understand what was said
(word)	indicates a guess by the transcriber at what might have been said if unclear
(())	contains transcriber's descriptions
ah	interjection expressing understanding
er	interjection expressing hesitation
mhm	interjection expressing agreement
oh	interjection expressing surprise, understanding or amazement
uh	interjection expressing uncertainty or confusion
um	interjection as space filler
FG, TG	stands for graduate student in the foursome and twosome respectively
FS, TS	stands for supervisor in the foursome and twosome respectively
C	stands for colleague in the foursome
V	stands for visiting colleague in the foursome

Gesturing and Diagramming



Chapter 1.

Introduction

1.1. Math and Me

I love math! Why do I feel compelled to begin the introduction to my thesis with this sentence d'amour? Because this statement encompasses at once the foundation of my very being and the basis of my academic and career decisions that have brought me to this study. I am even married to a mathematician because we see the world through similar eyes. The sentiment of this little phrase also reverberates in the actions of the people whom I have studied for this research.

Let me begin by elaborating on the foundations of my connections with mathematics. I inherited my love for mathematics from my Papa, who taught me so much more than any school ever did about the beauty of numbers, their relationships, applications and problem solving in general. I recall fondly our many family hikes, which never failed to inspire my father to pose some mathematical problems that engaged me and helped me to forget completely how many kilometres we had put behind us. When my father led me to the solution of a problem, he was often visual with his explanations using sticks and stones and other natural features around us. In some sense, a seed was planted in me not just for mathematics but also for teaching.

I am deeply rooted in family and have a strong sense of belonging. This support allowed me to venture out to Canada from Germany immediately after completing high school in the mid-1980s. I eventually pursued a bachelor's degree in mathematics and computing sciences at the University of Toronto, studying in a language that at the time I was functional in but certainly had not yet mastered.

Perhaps because English is my second language, I have depended on more than words in communication. I would study the people in front of me, mostly mathematicians, and how they communicated with their faces, arms and hands, and sometimes even their entire body. I fondly recall my calculus professor, who would energetically walk up and down in front of the blackboard waving his arms around while pointing here and there. But rather than distracting me, his antics pulled me into his lectures. The spine of my calculus textbook still cracks because these lectures sufficed to teach me all I needed to know about calculus at that time.

During my undergraduate studies, two professors in particular took me under their wings and mentored me during my mathematical journey. I was invited to undergraduate seminars and was given the opportunity to run tutorials for mathematics courses while in my third and fourth years of undergraduate study. For the first time, my eyes were opened to a possible academic career in mathematics. Not only was I the first in my family to graduate with a university degree, I was also encouraged to continue graduate school, which I chose to pursue at the University of British Columbia.

I encountered one chauvinistic professor in my fourth year of undergraduate studies, who blatantly and repeatedly told us three females in the course that women belong in the kitchen. None of us took him seriously, but we realised that some professors held a very negative view of women in mathematics. Nonetheless, this encounter did not prepare me for the words that greeted me in the early 1990s as I introduced myself to the supervisor that was assigned to me at the beginning of my graduate studies: “Oh, Petra is a female name. Well, I will see you again if and when you will graduate.” To put it mildly, this greeting put the brakes on my academic mathematical career, and having barely begun I already prepared to abandon my dreams. In the second last semester of my M.Sc., I finally switched supervisors and successfully attained my Master’s degree along with an NSERC award to continue with my Ph.D. studies. But graduate school had been a difficult journey for me, working without the benefit of a supervisor-mentor, and so I turned my back on graduate school and pursued the other passion of my life: teaching.

I enjoyed the teaching program at the University of British Columbia tremendously. There are two events that stand out in my mind from this time, the first being courses taken with Dr. Brent Davis. I was fortunate enough to have met Dr. Davis, who is a simply inspirational educator and who provoked me to rethink teaching and learning as I had understood it at the time. He also encouraged me to continue to learn and to keep an open mind to pursue graduate school in education once I had gained more experience as a teacher. The result was that from early on I became an even more avid observer and critic of activity in the educational sphere. The second event that had a profound impact on my time in the teaching program was the teaching practicum report I received from a school advisor after completing one of my practicum lessons. This report was pointedly critical. The issue was my repeated use of "flailing hands" and "facial expressions" during teaching, which in her mind detracted from what I had to say. She encouraged me to repress my bodily gestures while teaching and to instead use my voice for emphasis as needed. Of course, I tried this for a while – a long while – until I came into my own as a teacher. Now, I have come to firmly believe that gesturing is part of human nature and is also a powerful non-verbal communication tool that is most useful especially in the learning-teaching context.

Almost twenty years later, with extensive teaching experience in secondary and post-secondary education behind me, I still have more questions than answers when it comes to teaching and learning. What I have discovered along the way is that teaching and learning are an interactive process with many social and cultural aspects and also deeply personal, bodily, and tangible experiences by each participant. So, why did I end up here at Simon Fraser University, pursuing my doctoral degree and embarking on this particular study? Because I still wonder:

- What can improve the teaching of mathematics?
- How can we better attend to the learning of mathematics?

About four years ago, I had a decisive experience that drew my attention to diagrams in a fundamental way. During one of my calculus lectures with 500 students, I was working through a classic related rates problem, which requires minimizing the time it takes to reach a particular point on a beach from some distance at sea by a combination of rowing and running as shown in Figure 1-1. Students had been asked to come prepared for this lecture by viewing an animation of the problem posted on their learning management system. As a first task in the lecture hall, I asked students to draw a diagram to represent all the relevant physical quantities and their relationships. Walking through the aisles, I was flabbergasted at what I saw, or rather what I didn't see. I ended up collecting all drawings from the students and investigated their diagrams further after class. Students had indicated whether they actually watched the animation or not, but it became clear to me that, even if they had, the animation that was provided (based on Figure 1-1) did not make sense to many students. This raised further questions:

- When I draw a diagram for my students, what does the diagram mean to me versus to my students?
- Similarly, what does the diagram that my student draws mean to this student versus to me?
- Are students and teachers equally engaged in the diagramming process?
- Why is there a disconnect between the diagram and the learner?
- How can mathematical understanding be made explicit through a diagram, if at all?

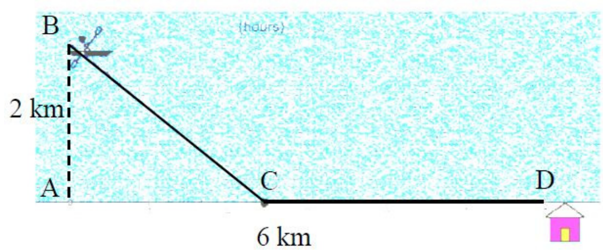


Figure 1-1. Diagram for related rates problem (Mulholland, n.d.)

As a mathematics educator, the wife of a research mathematician and mother to a child in elementary school, I cannot imagine the formal, symbolic world of mathematics without diagrams. I have observed several distinct purposes for diagrams as used by learners of mathematics, teachers of mathematics and mathematicians. Here are my proposed roles of diagrams, which are also illustrated with carefully selected diagrams as shown in Figure 1-2:

a. Diagrams illustrate concrete mathematical concepts, such as the Pythagorean theorem or the area between two curves, that we often find in a textbook.
b. Diagrams are simplified representations of otherwise unvisualizable abstract mathematical concepts such as the hypercube.

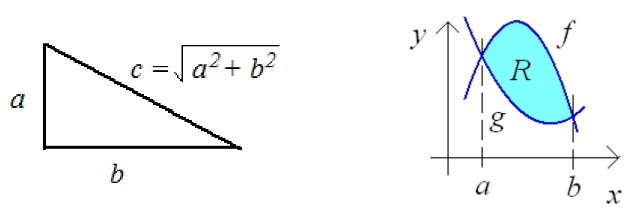
c. Diagrams are idealized representations of physical objects in mathematical modelling such as the stereotypical spherical cow (Brown, 2010, para. 3).

d. Diagrams are tools for constructing solutions to problems or mathematical proofs such as a proof of the Pythagorean theorem.

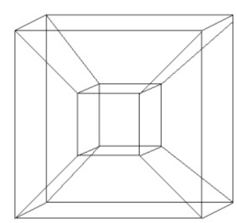
e. Diagrams are spontaneous and intuitive outbursts that capture new and as yet unformalized concepts for the person doing the mathematics, such as this drawing from my then eight-year-old son Eli about the addition of fractions. My son knew how to draw fractions but I had never before asked him to add fractions nor shown him how to add them. Furthermore, fraction addition was a subject he had not covered in school yet. So when he divided the one-third sections into quarters to be able to see them, it was a spontaneous and intuitive act that captured fraction addition for him.

f. Diagrams beautify mathematical objects such as a fraction flag from a fraction kit or the Sierpinski carpet.

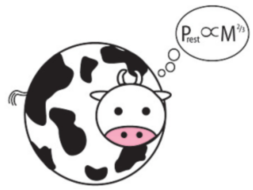
g. Diagrams are used as pedagogical tools for explaining mathematical procedures such as the multiplication of two whole numbers.



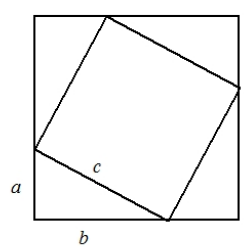
a. Pythagoras theorem and area between two curves



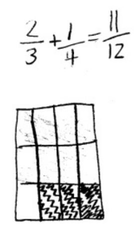
b. Hypercube



c. Spherical cow

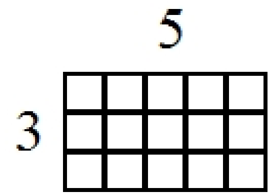


d. Proof of Pythagoras theorem



e. Addition of fractions

f. Fraction flag and Sierpinski carpet



g. Multiplication of two whole numbers

Figure 1-2. Sample of mathematical diagrams: The letters labelling each diagram correspond to the itemization of diagrammatic functions in the text.

From this list, one can readily see that diagrams themselves are concrete, but that the ideas they codify may not be so concrete to the viewer. Moreover, all that can be represented on the printed page is a static image of a diagram, while in fact the very act of drawing the diagram or how it emerges during the mathematizing is also very important. For example, the illustration of integration in Figure 1-2a presents the end result of drawing the area between two curves; however, it may not be so obvious while drawing the two curves where the actual area between them lies, as I point out over and over again to students of calculus. Another example is Figure 1-2g, which again is just the end product of a process for illustrating visually how two whole numbers multiply to produce an area.

The literature review has shown me that there is plenty of research that looks at the diagram as a product in mathematical problem solving or during mathematical proof construction, a literature which I expand upon in Sub-Section 3.3.1. However, these research results that view the diagram as a static representation are unsatisfying in light of my observations, because this research neglects to investigate how the diagram arises and forgets about the hand that draws the diagram and all the experiences that reside in the person drawing the diagram.

1.2. Pivotal Moment

In the fall of 2013 I was deeply immersed in the book *Figuring Space – Philosophy, Mathematics, and Physics* (2000, translation of 1993) while trying to understand Châtelet's ideas presented there on virtuality and reality in the emergence of mathematical discoveries. His ideas provoked and also resonated with me. Through my experiences as a mathematics teacher, I have come to believe that mathematical invention and intuition emerge long before a symbolic representation of the mathematics is encountered. I find it intriguing that a diagram could hold more meaning than the few lines that are needed to draw it; and that the very act of drawing the diagram and engaging with it can possibly provide keys about one's understanding of the mathematics that led to and from the diagram.

Since Châtelet based his analysis on manuscripts left behind by famous mathematicians and physicists without access to live observations of or interviews with them about their work, I became fascinated by how one can *see* mathematical creation through diagramming and tried to pay particular attention to it during the teaching of my courses. During this time, I undertook a little mathematical investigation with my then ten-year-old son, when I had my aha-moment. I made notes afterwards, but these notes were subjective as I did not make a video-recording of our exploration. However, there is still value in this anecdotal account of the mathematizing that unfolded, as it was a pivotal moment for me in making a connection with the meaning behind Châtelet's work.

My son and I were working through some math contest questions when something interesting happened during the problem solving of this scenario: "You have six sticks of lengths 1 cm, 2 cm, 3 cm, 2001 cm, 2002 cm and 2003 cm. You have to choose three of these sticks and form a [non-degenerate] triangle. How many different choices of three sticks are there which work?" (European Contest, 2003, p. 4).

Eli's first response to the problem question went something like, "this is easy, the 1 cm, 2 cm and 3 cm sticks form a triangle and the 2001 cm, 2002 cm and 2003 cm sticks form another triangle, that's it." When I asked Eli to draw the first triangle by measuring out the lengths with a ruler, he recognized after drawing the 1 cm and 2 cm lengths that the construction of his first triangle was impossible, and so he did not even draw the 3 cm stick. I then posed the following question: "Can you replace one of those sticks (pointing to the 1 cm, 2 cm and imaginary 3 cm sticks) with any of the other sticks in the problem?" Now, the moment came that transformed my thinking around diagrams: Eli reached out as if to grab the 2 cm stick and his hand traced a motion as if the 2 cm stick were pivoting around its connection with the 1 cm stick. With this action he said something like: “If I move this stick and put it up (now the two sticks formed a straight line), then those two are as long as this stick (meaning the 3 cm stick). But then I don’t have a triangle. And if I move it (meaning decreasing the angle between the 1 cm and 2 cm sticks), then I don’t have a stick that fits in between.” In this moment, Eli mathematized through the diagram. The diagram was created through my urging, but with the ‘grabbing’ and ‘moving’ that Eli was compelled to do, he now owned the diagram. The diagram was no longer static and silent. It was as if the diagram were suspended in space, a physical three-dimensional space held in a plane only because the diagram was drawn on paper, and Eli’s gesture of attempting to physically move the drawn stick actualized the virtual in him. This engagement helped Eli to discover the triangles that are possible with the given sticks, which in Châteletan terms is the possible that has been realized. Indeed, Eli continued to find all correct triangles for the posed problem.

1.3. Thesis Outline

Eventually, my interest in diagrams and my experiences as a graduate student and teacher, taken together with the ideas of Châtelet (2000), allowed me to find a fertile research area and formulate my research questions. Little research has been done on analyzing the mathematizing that arises during research meetings between graduate student and supervisor, and what there is focuses either on theorem and proof construction or the social and cultural aspects of such interaction. Furthermore, Châtelet’s work itself is based on the surviving written manuscripts of mathematical figureheads without access to sound- or video-recordings of these mathematicians during their research, as might be produced today and that could provide further data to illuminate or clarify creative moments. Therefore, as my title indicates, I decided to study the unfolding of diagramming and gesturing between mathematics graduate student and supervisor during research meetings.

My main interest is the emergence of the diagram as well as the manner in which the graduate student as the less-expert mathematician and the supervisor as the expert mathematician interact with the diagram. I suspect that the act of drawing and engaging with diagrams can capture mathematical creation or indicate aspects of mathematical intuition. In particular, I aim to demonstrate that the diagram is more than a visual product in that it has a voice of its own and is not neutral during mathematical engagement; that it provides a quintessential playground hovering near virtuality in the process of mathematizing; and that the diagram is not just a representation that leads to mathematical formalization.

In Chapter 2, I clarify what I mean by *mathematizing* and what implications it has for the players that are involved in research, namely the research mathematician-to-be and the research mathematician. Furthermore, I discuss the literature pertaining to the graduate student/supervisor relationship and the limited research that has been done on this relationship in the area of mathematics. In Chapter 3, I present a review of gesturing in general and how diagramming in connection with intuition, invention and abstract thought is emerging as a research area in its own right. Furthermore, I examine the diametrically opposed frameworks developed around diagrams in the research community, that view the diagram as either a visual product or the virtual that is being actualized. It is the latter which provides the main analytical framework for this study. Therefore, I exemplify the analytical framework with my examination of Oresme's work by Châtelet, which I express in modern mathematical terms. I conclude this chapter with a discussion of my research questions. The methods that have been employed in conducting my research are explained in Chapter 4, such as the concept of a *talking stick* and the characterization of gestures, actions and postures using *units of activity* for transcription purposes. Chapter 4 also includes a description of the participants as well as the data-gathering/viewing/selecting/transcribing processes, and the identification of the limits of the chosen research methodology. In Chapter 5, I summarize the mathematics of the research meetings and describe the evolution of the graduate student in terms of diagramming and gesturing during mathematizing, while in Chapter 6 I discuss the units of activity from the viewpoint of the diagram, present the analysis of diagrams in general that led me to a life-cycle of diagrams, and provide a detailed look at the evolution of a particular diagram that provides insights into mathematical thinking and invention. This is followed in Chapter 7 with the results of my study that provide responses to my research questions, overall concluding remarks pertaining to diagramming and gesturing, a re-definition of mathematizing, and a discussion on the implication of my study and directions for future work.

Chapter 2.

Three Key Elements of a Mathematical Research Meeting

2.1. Introduction

In this chapter, I provide a literature review regarding what I see as the three key elements of a mathematical research meeting: namely, the *graduate student* and the *supervisor* as the people involved in the meeting, and *doing mathematics* as the act they are engaged in. The graduate student is viewed as a research mathematician-to-be, which encompasses the roles of less-expert mathematician and novice researcher in which the graduate student is learning not only how to conduct mathematical research but also how to become a researcher. The former speaks to the skills and tools one must build in order to perform research, and the latter relates to the process of enculturation into the research community. In both of these processes, the supervisor as the research mathematician has the dual role of the expert mathematician who mentors the student in her mathematical journey, and the established researcher who demonstrates what being a researcher entails, both locally in the department and globally in the wider research community. The assumption here is that a research mathematician works at a university mathematics department. This literature review elaborates on these points of view in Sub-Sections 2.3.1 and 2.3.2, while also addressing the graduate student/supervisor relationship and the limited research that has been done in this area in the context of mathematics. First however, I expand broadly in Section 2.2 on what the literature says about what it means to *do mathematics*, and how this is demonstrated by the less-expert and expert mathematicians.

2.2. Mathematization

I am approaching this section rather cautiously, since the phrase *doing mathematics* incorporates the word mathematics, which potentially opens up broad discussions of both the ontology and the epistemology of mathematics. While it is necessary to review some ontological and epistemological aspects, since my framework rests on the research of the mathematician-philosopher Châtelet (2000), it is not my intention to place this study squarely into the philosophical realm. Being grounded in the practice of mathematics, I am instead motivated to shed light on the connection between diagrams and the ensuing invention of mathematics. This might also lead me to redefine mathematizing at the expert and near-expert levels, which should serve as guide for the graduate student in her journey from mathematics student to research mathematician. Moreover, this study should also provide pedagogical insight into how educators can facilitate the learning of mathematics even at the more novice level.

I began my literature search with what I thought was a straightforward question in mind: *what does a mathematician do?* What follows is a compilation of the prevalent answers that I encountered in my readings, which I am purposely highlighting to demonstrate the surprising complexity behind my question and the multitude of viewpoints on mathematics held by mathematicians and researchers. There is a quick answer to my question, namely, that what a mathematician does is to solve problems:

> [A] problem is a question that cannot be answered immediately. Problems are often open-ended, paradoxical, and sometimes unsolvable, and require investigation before one can come close to a solution. Problems and problem solving are at the heart of mathematics. Research mathematicians do nothing but open-ended problem solving. (Zeitz, 1998, p. ix)

There is an expanded answer to my question that derives from the personal point of view of the mathematician Paul Halmos during an interview:

> At that minute, for what it is worth, I became a mathematician. It is routine for every grown mathematician to have a big idea and to formulate it, to formalize it, to express it, to verbalize it precisely, and that is something I had never known how to do until that moment. (Csicsery, 2009)

In other words, not only does a mathematician problem-solve, but he also concerns himself with discovering the problems and shaping them before actually engaging with the problems and seeking satisfying solutions. There is a concrete answer to my question that hints at the abstract nature of mathematics, given by the British mathematician and Fields Medalist Sir Timothy Gowers in his keynote address *The Importance of Mathematics* (1999) at a millennium meeting in Paris:

> Rather than studying the world directly, mathematicians create so-called models of the world, and study them. This applies even to the simplest mathematics. After the age of four or five we do not study addition by actually combining groups of objects and counting them. Instead we use an abstract mathematical construction, or model, known as the positive integers (that is, the numbers 1,2,3,4,5 and so on). (pp. 2-3)

There is yet another answer that is a blatantly abstract one, stated by Davis and Hersh towards the end of their highly referenced work *The Mathematical Experience* (1980): a mathematician studies “mental objects with reproducible properties” (p. 399). There is also a more critical answer to my question provided by Thurston (1994) in response to the article *Theoretical Mathematics: Toward a cultural synthesis of mathematics and theoretical physics* by Jaffe and Quinn (1993). Thurston objects their claim that a mathematician’s work is restricted to the so-called definition-speculation-theorem-proof model. Thurston argues that mathematicians do not just churn out theorem after theorem, but more importantly, mathematicians “enable *people* to understand and think more clearly and effectively about mathematics” (p. 3, *emphasis in original*). Speaking for mathematicians in general, Thurston states that in order to facilitate the transmission of mathematical ideas, “we need to pay much more attention to communicating not just our definitions, theorems, and proofs, but also our ways of thinking” (p. 8).

This last phrase ‘ways of thinking’ indicates that I need to ask a more specific question, namely, *how can mathematical thinking be observed or accessed?* This question lies at the heart of my research and is not at all easy to answer in light of the current literature. While there are some theories on how a mathematician problem-solves, I was unable to find anything near a universally-accepted answer. My literature review offers a summary of pertinent work on this topic and the emerging themes that spoke to me as the instructor, practitioner and researcher of mathematics.

I begin with the term *mathematization*, which best describes the process of doing mathematics. In his article *Mathematization matters* (1982), the renowned mathematics education researcher David Wheeler elaborates on the term mathematization and how his definition of what it means to mathematize has evolved over the last decade. In some privately circulated notes in 1978, Wheeler attempts to define mathematization as “the phenomenology of the awareness and convictions that we experience when we are doing mathematics and which power the movements of our mathematical thought" (quoted in Wheeler, 1982, p. 46). In an earlier article published in *Mathematics Teaching* in 1975, he offers a more extensive definition of mathematization, which I am quoting in its entirety here, because this definition identifies various abilities that my data analysis needs to take into account:

> [Mathematization is] the ability to perceive relationships, to idealise them into purely mental material, and to operate on them mentally to produce new relationships. It is the capacity to internalise, or to virtualise, actions or perceptions so as to ask oneself the question, "What would happen if…?"; the ability to make transformations – from actions to perceptions, from perceptions to images, from images to concepts, as well as within each category – to alter frames of references, to refocus on neglected attributes of a situation, to recast problems; the capacity to coordinate and contrast the real and the ideal and to synthesise the systems of perception, imagery, language and symbolism. When these functionings are applied to pure relationships, detached from specific exemplars, the products will then be mathematics. (quoted in Wheeler, 1982, p. 47)

Wheeler's intricate statement engenders many questions, but I focus on only two here that are pertinent to this study:

- How can the mental activities described above be observed in the person doing mathematics?
- Who is the person mathematizing?

By this last question I mean to ask to whom mathematizing can be attributed: the expert mathematician only, or anyone engaged in the process of doing mathematics? In order to define mathematizing further and to understand how to engage in mathematical thinking, I reviewed literature from education researchers who study mathematicians, as well as the writings of mathematicians, who have not only published research articles but have also offered insights into the modus operandi of mathematicians. Most of the existing educational literature on mathematicians and what they do is based on the theme that proofs are a fundamental component of professional mathematical practice, and therefore research is driven by investigating various aspects that have to do with proof construction. I am not as interested in proofs as my study is concerned with the pre-proof stage, when mathematicians get the ideas that eventually enable them to write the proof. Therefore, my literature review is not a detailed investigation of the theme of proofs but, rather, a broad account of decisive developments during the last half-century on how to investigate mathematical thinking.

In the highly cited work *An essay on the psychology of invention in the mathematical field* (1954), the mathematician Jacques Hadamard investigates the role of the unconscious in inventive thought by surveying contemporary mathematicians including himself. Based on these introspective accounts, Hadamard argues that the unconscious is a meaningful part of cognition and is at work during mathematical engagement. By extending the three-step sequence of mathematical thinking given by mathematician von Helmholtz with a fourth and last step, Hadamard theorizes that mathematical thinking can be described as a process of preparation, incubation, illumination and verification.

Hadamard's mathematical thinking sequence is criticized by the popular British mathematician Ian Stewart (2006) as being too basic a process. Stewart argues that when mathematicians solve problems, the individual steps are applied at many levels, and each level may be broken down into further steps, "so instead of a single three-stage process, we get a complicated network of such processes" making "research [...] more like a strategic battle" (pp. 60-61). How can this intricate network of processes be observed in a mathematician? Stewart has this advice to offer without going into details: "Study the tactics and strategies of the great practitioners of the past and present. Observe, analyze, learn, and internalize" (p. 61).

Burton (1998) concludes from her study *The Practice of Mathematicians* that professional mathematicians have a rich sense of pleasure "from achieving an Aha!, even one which is later followed by an Oops!" (p. 135). This joy of mathematizing that is mingled with uncertainties, excitement and frustration is echoed by many mathematicians who describe the mathematical journey (e.g., Davis & Hersh, 1980; Hadamard, 1954; Halmos, 1985; Poincaré, 1963; Stewart, 2006). These emotional experiences are a result of an aesthetical engagement with mathematics. This role of aesthetics during mathematizing has long been emphasized by mathematicians such as Hadamard (1954), Halmos (1985) and Poincaré (1963). Having worked in the mathematics department of two universities and being married to a mathematician, I have observed mathematicians from afar as well close up for over a decade. The beauty, eloquence and significance of a theorem and proof are not just appreciated but very much at the heart of doing, writing and judging mathematics. However, these claims of aesthetics by mathematicians have not been adequately analyzed until recently from a pedagogical viewpoint in order to enrich students' learning of mathematics by promoting aesthetical engagement. In her pragmatic analysis of the roles of the aesthetic, Sinclair (2004) identifies three groups of aesthetic responses in mathematical inquiry: the *evaluative* is the most recognized and public response and concerns itself with judgement of mathematical products; the *generative* response guides the mathematician and is "responsible for generating new ideas and insights that could not be derived by logical steps alone" (p. 264); and the *motivational* response speaks to the attractions mathematicians find with certain problems. Sinclair argues that these aesthetical roles are intertwined with elements of the affective domain such as emotions, attitudes, beliefs and values. She demonstrates that "the links [...] between the affective and aesthetic domains reveal some of the beliefs and values of mathematicians that are, along with their knowledge and experience, central to their successes at learning and doing mathematics" (p. 269).

In the classical work *Mathematical Problem Solving* (1985), the mathematics education researcher Alan Schoenfeld characterizes the intellectual activities of expert and novice problem-solving behaviour in order to address two fundamental questions: "What does it mean to 'think mathematically'? and How can we help students to do it?" (p. xi). It is important to note that Schoenfeld does not specify what mathematical thinking is, but rather takes the stance that it occurs in problem-solving, and he is therefore interested in problem-solving behaviour. His framework consists of what he calls four complex intellectual activities: resources, heuristics, control and beliefs. Resources refer to the mathematical knowledge that the problem solver possesses. Heuristics are the broad range of general strategies that the problem solver applies, such as drawing a diagram or reformulating the problem. Control pertains to the global decisions the problem solver makes such as planning, monitoring, decision-making and other conscious meta-cognitive acts. Beliefs refer to how the self, the environment and the mathematics influence cognition and determine the behaviour of the problem solver. In the succeeding decades, Schoenfeld's framework has had a tremendous effect on shaping mathematics education research, work that has shed light on mathematical thinking and has influenced the learning of mathematics at a national level in the USA. Schoenfeld also encourages researchers to be “prudent to examine particular instances of problem-solving behaviour from as many perspectives as possible” (p. 316).

Wilkerson-Jerde and Wilensky (2011) studied the acquisition and application of the above-introduced resources by mathematicians, using a coding framework based on discourse to investigate the sense-making that expert mathematicians employ versus that employed by mathematics graduate students when confronted with unfamiliar mathematical ideas in a novel proof. They discuss the concept of “mathematical knowledge as a network of relations between different properties, objects and procedures that come to bear on a given mathematical idea” (p. 24). While Wilkerson-Jerde and Wilensky concede that their sample of ten novice and expert mathematicians is not large enough to generalize their findings, their results do nonetheless clearly indicate the strategic use of different types of resources by expert mathematicians:

> We found that rather than using examples as illustrative tools, experts used them in combination with other available resources in order to identify the units from which the object of study can be flexibly constructed and tested. These deconstructed components were then employed by experts to understand the individual relationships and substructures that defined the mathematical object of interest and the behaviors that underlied its formal definitions. (p. 41)

Although I am critical of Wilkerson-Jerde and Wilensky’s research, in that it is based solely on coding words that reference identified mathematical knowledge resources but leaves out any diagramming on the part of the mathematician, their results indicate places in mathematical discourse where new mathematics is developed. These may also be places to look out for in my analysis of diagramming.

The constructivist frameworks adopted by Schoenfeld, and also by Wilkerson-Jerde and Wilensky, do not take into account embodied activities such as gestures, nor do they allow for materials like a blackboard or calculator to play an integral part in mathematical thinking. My study takes an embodied point of view when analyzing the diagramming aspects during mathematizing in an attempt to uncover inventive mathematical thinking. The embodied theories of Roth (2011), as well as those of de Freitas and Sinclair (2013), are expanded on in the next few paragraphs since they offer insights into a contemporary view of mathematical work that provides a new approach to study mathematical thinking.

Supporting the theory that mathematizing is embodied, the cognitive scientist Wolff-Michael Roth (2011) draws attention to what he terms the living/lived work of mathematics versus the accounts of that mathematical work. He observes that “[the] lived praxis (labor) within which this written account *counts* as the proof, however, is not contained in the written account” (pp. 9-10, *emphasis in original*). By analyzing two proofs that he offers to the reader in order to experience the living work of mathematics, Roth concludes that “[t]he living/lived work has nothing to do with a mental construction, as the movements underlying the (intentional) drawing of a line emerge from experiences that have nothing at all to do with intentions” (p. 13). Accessing the living/lived work “requires [...] learning to ‘see the invisible’, or, more aptly, learning to experience what can only be obliquely pointed to by means of language” (p. 17). In other words, the *doing* of mathematics not only concerns itself with the final product, but also encompasses everything that leads up to this final product. Roth broadens the window through which the workings of a mathematician should be observed, but he does not elaborate on what it is in the living work of mathematics that should be observed, nor how to observe it.

Roth’s research on ‘learning to experience in order to see’ is consistent with the work by education researcher Keith Weber (2008; 2010; Weber et al., 2014), who studies how novice and expert mathematicians read proofs and determine their validity under the framework of epistemic cognition. Weber and his colleagues Matthew Inglis and Juan Pablo Mejia-Ramos (2014) point out that until recently the actual practice of mathematicians has been largely ignored by philosophers of mathematics, but that the philosophers’ attention has finally shifted to “how mathematics actually develops in the mathematical community” (p. 42), and the outcome of this research “illustrates how many widespread beliefs about mathematical practice are inaccurate” (p. 42). This research team concludes the following: there are times when mathematicians are convinced of the legitimacy of a proof either from authoritative or empirical sources; the sources that are considered by mathematicians are dependent on the epistemic aims of the mathematician; the higher the quality of the authoritative or empirical sources, the greater the persuasive power of these sources is to the mathematician; and there is substantial heterogeneity in the persuasiveness of different types of sources as perceived by the mathematicians. Furthermore, Weber and his colleagues provide instructional recommendations based on their arguments to aid the novice mathematician that "are not only more consistent with mathematicians' practice but also more realistic and provide greater learning opportunities for students" (p. 54).

In their theoretical article *New Materialist Ontologies in Mathematics Education: the Body in/of Mathematics* (2013), de Freitas and Sinclair argue that embodiment in mathematics education has not adequately described "what it is to be a 'body' and how mathematics itself partakes of a body" (p. 454). They extend the contemporary theory of embodiment by offering *new materialism* as a novel philosophical framework for studies in mathematics education:

> The new materialism we propose aims to embrace the 'body' of mathematics as that which forms an assemblage with the body of the mathematician, as well as the body of her tools/symbols/diagrams. [...] This new materialism sets itself apart from prior empiricist philosophies of mathematics because it posits a *virtuality* at the heart of mathematical concepts and locates this virtuality in the physical world. (p. 454)

In other words, the boundary of the physical human body is no longer considered to be merely the skin but extends to any item that the body is engaged with, whether it be physical material, like a chalk stick, a diagram on the blackboard or a cube constructed out of straws, or virtual material, such as a mathematical concept. de Freitas and Sinclair deliberately push this boundary and concept of materiality to make explicit the meaning of mathematical embodiment in terms of new materialism, which opposes the inert, inanimate and immaterial viewpoint of mathematical concepts that prevails in the current mathematics education literature, and thereby "to rethink the nature of learning and doing mathematics" (p. 461). They apply their theory to the teaching and learning of mathematics and claim that "tools become parts of the learner, forever changing the very constitution of their bodies" (2013, p. 458). The question for me is whether this new materiality can also be evidenced during the research meetings between a mathematics graduate student and the supervisor, and to what extent new materiality plays a role in creative mathematical thought. Leaning on the theories of Châtelet (2000), de Freitas and Sinclair hint at an answer to these questions, which will be elaborated on in Chapter 3:

> Mathematical entities are thus material objects with *virtual* and *actual* dimensions. Mathematicians engage these entities in thought experiments, but these are not the disembodied mental ruminations with which we typically associate mathematical thinking but, rather, gestural choreographies and acts of exploratory diagramming. (2013, p. 454, *emphasis in original*)

I began this literature review with the question *What does a mathematician do?* and extended it to ask *How can mathematical thinking be observed or accessed?* My aim was to provide a description of what it means to *do mathematics*, since this is one of the three key elements in a mathematical research meeting. The literature review has demonstrated that doing mathematics is entwined with mathematical thinking and is a process that is neither readily describable nor easily observable. It involves both non-cognitive abilities, such as belief and attitude, as well as cognitive abilities, such as control decisions, perception, imagery, language and symbolism. The term *mathematization* as used by Wheeler (1982) best describes this entwinement of doing mathematics and mathematical thinking. An etymological search on *mathematization* or *mathematize* using the *Online Etymology Dictionary* did not produce an origin for either word, nor did it provide a description of their meaning, even when using the alternative spelling with the 'z' replaced with an 's'. A general dictionary search finds that *mathematize* stands for *regard in mathematical terms*, which is too general a description for the purposes of this thesis. Throughout my thesis, I use the term *mathematization* to refer to an embodied engagement with mathematics based on Wheeler's extended description "to get as close as we can to the phenomenology of the awarenesses and convictions that we experience when we are doing mathematics and which power the movements of our mathematical thought" (Wheeler, 1982, p. 46).

In the next section, I review the literature on the identity of the graduate student as both researcher and mathematician, and how this is linked to that of the supervisor. I further investigate what the literature has to offer on how mathematization is demonstrated by the less-expert and expert mathematicians.

2.3. Graduate Student and Supervisor: The Research Mathematician-to-be and the Research Mathematician

I now provide a review of the literature from the past two decades on the graduate student and the supervisor as the last two key elements of a mathematical research meeting. There is an extensive body of research surrounding the graduate student, supervisor and graduate studies in general, but few of these studies are related to research in the sciences, let alone in mathematics. Most of the research topics fall into one of the following broad categories: graduate student attrition and success; graduate studies policies; structure of graduate degree programs; gender, ethical and racial issues; attitudes and beliefs on the part of either the graduate student or supervisor; process of enculturation into the research community; graduate student and supervisor/mentor relationship; teaching assistantship as formation of teacher identity; availability of resources; and writing and publication processes.

Of these categories, only the two topics related to mentorship and enculturation are relevant to this study – mentorship because it speaks to the relationship between the graduate student and supervisor, and the enculturation process because it describes how the graduate student becomes a researcher, although not necessarily a mathematician. I need to point out here that the term 'researcher' is not synonymous with the term 'mathematician' in my thesis. By researcher I refer to the person who has identified some research area of study that is not necessarily an area in mathematics, and who secures research funding, supervises graduate students, publishes articles, etc., whereas by mathematician I refer to the person who mathematizes and conducts mathematical research. Therefore, a further category of interest specific to mathematics graduate studies is the process by which the graduate student becomes a mathematician. The following two sections discuss the literature related to the two processes of a graduate student *becoming a researcher* as well as *becoming a mathematician*, and the role that the supervisor plays in these processes.

2.3.1. Becoming a Researcher

The anthropologist Tomas Gerholm (1990) studied knowledge and behaviour in academia, and points out that when a person wants to become a member of a new group (for example, a graduate student wants to become a researcher), then, besides having to attain explicit knowledge, that person also has to acquire implicit knowledge, which he terms *tacit knowledge*. Explicit knowledge stands for the group expertise related to theories and practices, which in the case of the graduate student in my study refers to the body of knowledge and related methods for a specific mathematical area of research. Tacit knowledge, on the other hand, refers to the fundamental cultural rules that are needed for one to be able to function within the group. Gerholm identifies six types of tacit knowledge: institutional norms; savoir-faire on how to handle conflicting rules; “special folklore thriving in most departments” (p. 265) such as the image of a mathematician as being nerdy and shy; intuitive feeling for the essence of one’s discipline; understanding of how one’s research relates to other research within the department; and lastly, the most significant type of tacit knowledge, “the one concerning scientific discourses, their characteristics and their uses” (p. 266). Gerholm claims “that failure to acquire this implicit knowledge is often taken as a sign of failure to have acquired the explicit knowledge itself” (p. 263).

In addition to Gerholm’s tacit knowledge, my literature review covering the past three decades indicates that factors such as socialization, mentorship and power struggles also play a role in the enculturation process of the graduate student on her way to becoming a researcher. The remaining section offers a summary of these findings.

In agreement with the research literature in the 1990s, the education research team of Donald et al. (1995) ascertain in their quantitative study that overall knowledge of the research field and availability of the supervisor are the most important factors in graduate supervision. In an effort to facilitate the establishment of policies and procedures, their study also identifies the responsibilities of both the graduate student and supervisor. For example, the supervisor needs to be knowledgeable about graduate studies policies; offer financial support in the form of research assistantships; meet regularly with the student; provide timely feedback about all aspects of research work; and introduce the student to the wider research community. The graduate student, on the other hand, needs to understand the scope of her graduate work; be knowledgeable about research methods and the supervisor's expectations; work within deadlines; "communicate directly with the supervisor (particularly if misunderstanding arises); and [...] submit a comprehensive annual progress report to the supervisor and the department" (p. 90). The support provided by the supervisor along with the graduate student's own work and understanding of graduate studies develops the student's research skills, shapes her values, and accustoms her to the rituals of the research community, all of which are signs of successful enculturation into the research group.

At the begin of the new millennium, the Carnegie Foundation for the Advancement of Teaching commissioned a five-year research activity called the Carnegie Initiative of the Doctorate in order to align the purpose and practice of doctoral education across six disciplines, including mathematics. Out of this study a collection of sixteen critical essays was published about the future of doctoral education (Golde & Walker, 2006). One of the editors of this essay collection is the education researcher Chris Golde, who elaborates on the themes of these essays in her introduction *Preparing Stewards of the Discipline* (2006): doctoral student development within a discipline, teaching preparedness, challenges, policies, responsibilities and career paths. Golde also proposes a definition for the doctoral student:

> This person is a scholar first and foremost, in the fullest sense of the term – someone who will creatively generate new knowledge, critically conserve valuable and useful ideas, and responsibly transform those understandings through writing, teaching, and applications. We call such a person a "steward of the discipline." (p. 5)

Furthermore, Golde posits that the doctoral degree signifies three competencies on the part of the graduate student, which she terms *generation*, *conservation* and *transformation*. Generation implies that the graduate student "is able to ask interesting and important questions, to formulate appropriate strategies for investigating these questions, to conduct investigations with a high degree of competence, to analyze and evaluate the results of the investigations, and to communicate the results to others to advance the field" (p. 10). Conservation refers to the ability of the graduate student to identify and protect the most important ideas within her discipline. Transformation means that the graduate student is able to explain and connect existing ideas from within and without the area of study. Golde ends her discussion of the doctoral student with the unanswered question: “How are stewards of the discipline best developed?” (p. 14). The fact that Golde, a highly referenced and published expert within the field of graduate studies, does not answer her own question indicates that much remains to be understood in this area of research.

Perhaps the largest body of research related to graduate studies that has emerged over the last two decades is in the area of mentorship (Hall & Burns, 2009; Fedynich & Bain, 2011; Herzig, 2002; Lechuga, 2011). The following are brief snapshots from the literature related to mentorship in graduate studies, many of which are based on the findings of the anthropologist Ian Lave and computer scientist Etienne Wenger (1991). These researchers propose that “learning is an integral and inseparable aspect of social practice” (p. 31), because social and cultural norms of a group are passed from seasoned practitioners to newcomers. They term this process *legitimate peripheral participation*, which has become the reference for much subsequent research in the area of socialization such as the small-scale study by Belcher (1994). Belcher notes the differences in supervisor relationship as hierarchical or dialogic, and recommends close graduate student-supervisor collaboration on a research project in order to “promote growth and independence” (p. 31).

Also based on the legitimate peripheral participation framework, the case study of one mathematics department by education researcher Abbe Herzig (2002) indicates that “a doctoral student needs to do more than just learn the content of the mathematics taught in classes; he needs to learn to participate in social and cultural practices” (p. 201), a sentiment that is echoed among many education researchers (Austin, 2002; Austin & McDaniels, 2006; Fedynich & Bain, 2011; Lechuga, 2011). Herzig also raises an interesting question: “are mathematicians independent because that is a style that is best suited to the nature of mathematics, or, since the social structure is one that requires independence, is it only the independent learners who succeed in mathematics?” (p. 202). Her study claims that neither of these may be the case, since some independent graduate students do not just rely on the relationship and collaboration with their supervisor, but also with their fellow graduate students. In support of this finding, the higher education researcher Ann Austin (2002) examines the socialization process of doctoral students and recommends that graduate schools “make deliberate use of informal peer relationships to foster socialization” (p. 115). Furthermore, Herzig encourages mathematics supervisors not to wait until graduate students have proven themselves through course work but instead to take an active mentoring role from the outset, as this engenders a better doctoral success rate and a reduction in attrition among graduate students. The higher education researcher Vicente Lechuga (2011) echoes this conclusion in a similar study on culturally and socially underrepresented faculty and graduate students: “because the faculty-graduate student relationship is one that develops over a period of many years in many cases, it would be wise for both the mentors and mentees to focus on the nuances of these relationships from the start” (p. 769).

The education researchers Leigh Hall and Leslie Burns (2009) use theories of identity as well as the legitimate peripheral participation framework to study doctoral students in education. They offer the following summary of current findings and understanding of mentorship in graduate studies:

> The personalized interactions that occur between mentors and their students have powerful effects, whether positive or negative, on students' understandings, motivations, willingness to engage, and access to opportunities. [...] (p. 58)
>
> Mentors maintain a great deal of power and control in their relationships with students. They often determine what it means to be identified as a researcher and decide what skills and experiences students need to become one. When mentors maintain power over the researcher identity, students have limited opportunities to participate in their own development.
>
> Mentoring relationships, then, are not inherently reciprocal and may in fact be coercive. (p. 60)

Based on their analysis, Hall and Burns propose several recommendations that aim to improve the mentoring relationship between graduate student and supervisor. In addition to graduate students learning the skills needed to perform research, they should also be given the opportunity by their mentor to openly discuss “what it means to be a researcher and from where those understandings come” (p. 61) so that they can develop a research identity that is based on shaping and challenging their current beliefs and understanding, rather than confusing them or making them vulnerable. Furthermore, Hall and Burns recommend that “[m]entoring relationships should be actively treated as social processes in which students develop and refine skills while also learning how different people enact a range of possible researcher identities" (p. 62). Similarly, the higher education scholars Ann Austin and Melissa McDaniels (2006) recommend that "[t]he graduate socialization experience should begin with clearly stated program expectations coupled with opportunities for students to articulate their own goals and interests" (pp. 439-440). The claim of both Hall and Burns as well as Austin and McDaniels is that an open process of communication and negotiation between the mentoring supervisor and the graduate student will increase the success rate for completion of graduate studies, and thereby produce more satisfied professionals who will in turn enrich the discipline.

The cognitive psychologist Tim Mainhard and his colleagues (2009) summarize the skills and attitudes of a supervisor, that have been shown by research to be effective in supervising a graduate student: "Supervisors should have listening skills, encourage argument and debate, provide continuous feedback and support, be enthusiastic, and show warmth and understanding" (p. 360). It has become apparent among the group of researchers focussing on the graduate student-supervisor relationship that the interpersonal styles of both people involved need to align in order for the supervisory relationship to be productive rather than lead to tension (Donald et al., 1995; Golde, 2006; Herzig, 2002; Lechuga, 2011). Mainhard's research team point out that there is currently no appropriate instrument for measuring interpersonal skills, behaviours, and attitudes of a graduate student and supervisor. Therefore, they undertook a study and developed an instrument specific to the graduate student experience in the supervisory relationship that is based on the interpersonal styles of the graduate student as well as the supervisor. This instrument allows for feedback to the supervisor "about their supervisory style with the aim to improve the quality of their supervision" (p. 369) and "maps the relationship between the doctoral student and his or her supervisor from the perspective of the student" (p. 370).

A study conducted in New Zealand specifically about experiences of mathematics doctoral students by mathematician and education researcher Morton and mathematics lecturer Thornley (2010) points out that "[i]nitially students expect a high degree of involvement from their supervisors, but then move through a more detached phase when they take increasing responsibility for their own work, before returning to greater involvement with their supervisors" (p. 119) close to the submission of their thesis. Morton and Thornley emphasize that even though the majority of graduate students in their study were satisfied with the supervision they receive, there is still a considerable number of students whose wants are not met by their supervisor. They recommend locally that mentors pay “attention to the everyday experiences of [the graduate students’] lives” (p. 123), and globally that “[g]raduate schools should foster a university-wide community by ensuring that opportunities occur for students to meet, both socially and for scholarly purposes”, because encouragement from mentors and fellow graduate students contributes to the overall success of the graduate student.

The book *A Mathematician’s Survival Guide: Graduate School and Early Career Development* (Krantz, 2003) provides a comprehensive and insightful manual for the graduate student intending to pursue a career in pure mathematics. Krantz addresses many identities of the graduate student such as being or becoming a mathematician, researcher, teacher, student and department member. In addition to his tongue-in-cheek view of mathematics graduate school expressed by the statement that “[i]n many instances the journey through [mathematical] graduate school is one of the blind being led by nobody” (p. xii), he offers this advice about the relationship between graduate student and supervisor:

> Your relationship with your thesis advisor really is a personal one. It is up to you, as much as it is up to the advisor, to make it a relationship that works for both of you. […] If your advisor is going to help you, then you must say directly where you are, where you are coming from, and where you are going. (p. 63)

The implication of this advice is that the onus is on the graduate student to understand what it is they need and want and approach the supervisor. Burton (2004) feels very strongly about the responsibility of supervisors as mentors in that they are “in a special position to be able to make their students aware of strategies that are productive for dealing with, amongst others, a stuck state. I believe that it is unprofessional to abrogate that responsibility” (p. 191). A decade later, the mathematician Sara Billey (2012) offers to mathematics graduate students very different advice from that which Krantz did: “Along the way, it will be very helpful to have a friend along to show you the gems and help you avoid disaster. The role of ‘your friend’ is played by your advisor” (p. 2). I have not found anywhere else in the graduate studies literature review where advisorship is considered as friendship.

So far, this literature review has provided many recommendations on how to improve the graduate student-supervisor mentoring relationship, in an effort to support the socialization aspects of graduate studies and the graduate student's journey to becoming a researcher. Austin (2002) points out that "[t]hese suggestions may sound fairly straight-forward and relatively easy to implement"; however, the longstanding call to make these changes, as well as the current culture, values and structures of North American research universities underline the difficulties in implementing the recommendations.

I would be remiss to ignore the other side of the coin representing the graduate student-supervisor relationship, which in the research literature is referred to as the power relationship. There is a hint of the power held by supervisors as mentors in the study by Hall and Burns (2009) that I summarized earlier. The study by Aguinis et al. (1996) shows that the perception of the graduate student about the supervisor's power is related to the perceived quality of supervision; the trustworthiness and credibility of the supervisor; and the extent of the student's compliance with requests from the supervisor. The research team concludes "that the power bases are important antecedents [in how] professors educate, shape, influence, and direct graduate students" (p. 292). The case study of one science department by education researchers Dorit Maor and Barry Fraser (1995) "suggests that clear expectations and procedures should be established in order to provide supervisors and graduate students with better opportunities to establish a collaborative working relationship" (p. 15). Nyquist et al. (1999) followed graduate students through their four-year journey in graduate studies and queried them on their personal experiences. Several themes emerge from the recounting of these experiences: "the tensions that graduate students experience in adapting to the values embodied in higher education; the mixed (or ambiguous) messages they receive about priorities in the academy; and the pleas for support" (p. 20). When these issues are unresolved, they can lead to disillusionment, isolation, resignation and disappointment on the part of the graduate student.

In summary, the supervisor in the role of advisor or mentor is the main person that enculturates the graduate student into the research community by demonstrating how to perform research; how to write articles and publish them; how to present work at conferences and academic institutions; how to communicate and network with colleagues; how to secure research grants; how to mentor graduate students; how to interact generally with students; and how to teach. This mentoring role comes with the great responsibility of guiding graduate students wisely and thoughtfully, in a way that permits them to grow into the role of a researcher. In the next section, I present the findings of the literature review on how the graduate student becomes a mathematician and what the role of the supervisor is in this process.

2.3.2. Becoming a Mathematician

My literature search has led me to believe that there is a dearth of research on the process by which a graduate student becomes a mathematician. Indeed, Burton (2004) notes that "there is a myth associated with mathematicians that they are born not made" (p. 34), and Beisiegel (2007) states that "[many] dichotomies exist in mathematics and, in the community, you either are or you are not a mathematician. There is little focus on the process" (p. 23). Most of the literature addressing the process of becoming a mathematician consists of handbooks or accounts written by mathematicians such as *The Mathematical Experience* (1980) by Davis and Hersh, *A Mathematician's Survival Guide: Graduate School and Early Career Development* (2003) by Krantz, or *I Want to Be a Mathematician* (1985) by Halmos. What I was able to find in research articles is mostly a by-product of a different research question such as *The Practices of Mathematicians: What Do They Tell Us About Coming to Know Mathematics?* (1998) or *Mathematicians as Enquirers* (2004) by Burton, or *Being (Almost) a Mathematician: Teacher Identity Formation in Post-Secondary Mathematics* (2009) by Beisiegel. The remainder of this section is a summary of my findings from this literature review about the process of becoming a mathematician. This is also the place where I begin to take a closer look at how mathematization is demonstrated by the research mathematician-to-be and the research mathematician.

In the article *A Process of Becoming* (2007), Beisiegel asks pertinent questions about the mathematics graduate student, even if she does not provide concrete answers:

> [W]hat are their experiences, what is it that they interpret or understand their lives to be like in mathematics? What has meaning for them in the process of becoming? What do graduate students in mathematics interpret as giving meaning for who they should be and how they should be as mathematicians? (p. 23)

Perhaps the best answers to these questions derive from the following two studies: one study on seven female mathematics graduate students who have chosen to leave the mathematics ‘pipeline’ (Stage & Maple, 1996), and the other a study of 34 mathematics graduate students, with a closer look at the behaviour of six successful students among them (Carlson, 1999). The higher education researchers Frances Stage and Sue Maple (1996) report that every female mathematics graduate student in their study experiences mathematics research as involving extensive time commitments that allow for few interests outside of research. Furthermore, these women are faced with “competitiveness or failure to be taken seriously by male classmates” (p. 38). Stage and Maple conclude that the process of becoming a mathematician was interrupted for these seven graduate students through their perception of isolation and negative experiences, and they recommend that more women mathematicians are needed as role models and also that more acts of understanding are required on the part of the mathematics community. Based on Schoenfeld’s (1985) framework of mathematical behaviour, the study by Carlson (1999) determines that incorrect attempts, mathematical enjoyment, individual effort, independence and persistence are contributing factors in the mathematizing of graduate students that are noticeably absent in those graduate students that ultimately drop out from graduate studies in mathematics. Furthermore, “these graduate students’ views about the methods of mathematics appear to be closely aligned with those of mathematicians, while their views about the nature of mathematics appear to be in transition” (p. 251). Indeed, Carlson points out that although graduate students’ beliefs are more expert than those of undergraduate students, they still pursue unfamiliar problems “with little reflection and monitoring during their solution attempts” (p. 256).

The mathematician Ian Stewart (2006) explains to his fictitious mathematics graduate student how he views a doctorate in mathematics and how it differs from a doctorate in sciences in general:

> I don't advise being that ambitious when you are working for a PhD! Big problems, like big mountains, are dangerous. [...] Math differs from the other sciences [...], you normally can't write a thesis saying, "Here's how I tried to solve the problem, and here's why it didn't work." (pp. 95-96)

Furthermore, Stewart suggests that a research mathematician-to-be has to deal with two competing realities: the normally solitary or even lonely work of a mathematician and the ability to negotiate her mathematical journey among people, such as the supervisor, supervisory committee members, other faculty and fellow graduate students. In the same vein, Burton (1998) argues that a mathematics student's "[l]earning is neither wholly individual nor wholly social" (p. 139); but that the mathematician as instructor generally does not seem to bring his experiences to his teachings, making communication with students one-way rather than two-way. The importance of communication is also echoed by Krantz in *A Mathematician's Survival Guide: Graduate School and Early Career Development* (2003), who gives this advice to the graduate student:

> It is also an important part of being a mathematician to be able to communicate – not just technical mathematics but also information *about* mathematics, about teaching, and about the profession. You are now not simply *learning* mathematics – you are learning to *create* it. (p. 36, *emphasis in original*)

A decade earlier, the mathematician William Thurston (1994) chides his profession about their selective communication ability:

> The transfer of understanding from one person to another is not automatic. It is hard and tricky. Therefore, to analyze human understanding of mathematics, it is important to consider **who** understands **what**, and **when**. Mathematicians have developed habits of communication that are often dysfunctional." (p. 5, **emphasis in original**)

However, Thurston blames the culture of mathematics and its division into numerous subfields for this dysfunctional manner of communication. He emphasizes that within subfields there is no problem in communicating ideas and results, but from one subfield to another it is like dealing with a foreign language.

There is also a huge discrepancy between the informal discourse between mathematicians and the presentation of work at a talk or in paper form. For example, Davis and Hersh (1980) state that "[t]o his fellow experts, he communicates these results in a casual shorthand" (p. 36), and Thurston (1994) elaborates that "[o]ne-on-one, people use wide channels of communication that go far beyond formal mathematical language. They use gestures, draw pictures and diagrams, make sound effects and use body language. Communication is more likely to be two-way, so that people can concentrate on what needs the most attention" (p. 6). The social scientists Christian Greiffenhagen and Wes Sharrock (2011) find that the discourse between mathematics graduate student and supervisor can be characterized as "fragmentary, informal, intuitive, tentative" (p. 854). In the analysis of my data, I pay attention to how mathematizing is enabled between graduate student and supervisor, especially since the graduate student as the less-expert mathematician is the one initiated into their mathematical subfield by the supervisor as the expert mathematician.

In continuation of the mathematical communication devices, talks, however, are more formal and mostly a one-way channel of communication, and papers are the ultimate product of formalization with most of the work translated into symbols and logic that demands the reader understand the formalism or exercise an ability to translate back to the informal structure. This indicates that in addition to the verbal communication there is also the written communication skill that the research mathematician-to-be has to learn, which is wittily captured by Davis and Hersh (1980) in their description of the 'ideal mathematician':

> His writing follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being. [...] To read his proofs, one must be privy to a whole subculture of motivations, standard arguments and examples, habits of thought and agreed-upon modes of reasoning. (pp. 36-37)

Moreover, Burton and Morgan (2000) conclude from their study on the writing of mathematicians that the mathematician-author is conforming to cultural norms rather than making a deliberate decision about the written form. In some sense, "the mathematician's role is subordinate to that of the mathematics itself" (p. 435). Despite the difficulties with mathematical writing and reading at all levels of mathematics, Burton and Morgan also observe that "the training of mathematicians does not appear to include any systematic attention to the development of writing skills" (p. 448).

In a recent study about the presentation of mathematics in research articles versus that in research meetings between graduate student and supervisor, Greiffenhagen and Sharrock (2011) point out that "[s]tandard mathematical publications have scarcely any historical elements and, unlike scientific papers, do not even feature 'methods' statements" (p. 842). They conclude that this absence of prior history in the written form of mathematics does not misrepresent the nature of mathematics but rather emphasizes the mathematical interest in the technical details of theorems and proofs. On the other hand, the written formalism in mathematics creates in some sense a barrier for the graduate student. Beisiegel and Simmt (2012) conclude from their study on the mathematics graduate students' formation of teacher identity that:

> [t]he graduate students understood mathematics to have a particular form that consisted of axioms, definitions, theorems and their proofs. They seemed to view mathematics as set in stone, inflexible, unbending. (p. 36)

The experiences in graduate studies affect the formation of the graduate student's identity as researcher, academic, professor, instructor and above all mathematician. In her thesis *Being (Almost) a Mathematician: Teacher Identity Formation in Post-Secondary Mathematics* (2009), Beisiegel argues that the training and performance of the graduate student as teaching assistant as well as student participation in mathematics courses will have some bearing on her identity as mathematics teacher. Similarly, the experiences of the graduate student as research mathematician-to-be in research meetings under the guidance of the mentoring supervisor influence how she interacts with other mathematicians, provide a mirror on how to mathematize, and even shape how she views her mentoring role as a young professor. Beisiegel's analysis of her conversations with mathematics graduate students demonstrates the struggle that mathematics graduate students must undergo in the process of becoming a mathematician:

> One theme that came through in the conversations was the sense that being a mathematician meant being somehow different in and to the world. Being a mathematician was a complicated and conflicted space of being set apart and unusual, of belonging, to some extent, to a distanced group

> of brainy eccentrics who, despite their mathematical abilities, were unable to function normally in the world. [...]
>
> To the participants in the study, being a mathematician was sometimes experienced as something that made them better, smarter, and set apart from others in a superior sense. At other times, though, some of the participants were pained by the otherness they felt and distanced themselves from mathematics and their particular interests in the discipline. (pp. 181-182)

Beisiegel's view of mathematician identity is a more holistic one than that described by most mathematicians, as she considers not only the operational aspects of being a mathematician, but also the social, cultural and emotional aspects. For example, Krantz (2003) states that "[t]he process of becoming a mathematician is a *synthesis*, both of ideas and technique" (p. xii, *emphasis in original*), which describes the identity of the mathematician only in terms of his ability to do mathematics. Similarly, for Stewart (2006) a key ingredient of becoming a mathematician is to be an original thinker: "Originality is one of those things that you either have or you don't: it can't be taught" (pp. 147-148). Davis and Hersh (1980) claim that if an outsider "accepts our discipline, and goes through two or three years of graduate study in mathematics, he absorbs our way of thinking" (p. 44), but I could not find a place where they explain what they mean by *absorbing* and one is left with the impression that indeed they refer to a process of osmosis.

In contrast to these authors, Beisiegel (2009) finds that the social and cultural norms that exist both implicitly and explicitly in a mathematics department put subtle pressure on the graduate student to make particular choices and to perform in particular ways in order to be "primed for a particular way of being" (p. 255). In the sense of Lave and Wenger's (1991) legitimate peripheral participation, the graduate student forms an identity that becomes ever more closely aligned to that of a mathematician. As Beisiegel concludes, this identity formation also includes avoidance of behaviours that the mathematics department deems illegitimate such as helping undergraduate students who have not demonstrated sufficient mathematical skills.

To summarize, in terms of legitimate peripheral participation, the development of the graduate student as a mathematician is not just about learning more mathematics, but also about learning the social and cultural practices within mathematics. The supervisor is in a unique position to actively and openly support those social and cultural processes. In this manner, the graduate student can be enabled to participate in their own development, and gather not only the knowledge of the field, but also the skills and experiences to become a research mathematician. Becoming a mathematician does not just entail the "ability to think like a mathematician" (Krantz, 2003, p. 10) and to have a shared experience and a shared language, as argued by Burton (1998). As Beisiegel (2009) puts it, "[b]eing in mathematics is a profound (deep, weighty, intense) and complicated part of who the participants are and how they are in the world" (p. 214).

Overall, this literature review has demonstrated that while several themes in graduate studies have been investigated, and that there are some findings suggesting that aspects of socialization, power bases and mentorship contribute to the process of becoming a researcher, there are nonetheless very limited results on the process of becoming a mathematician. However, supervisors as "[m]athematicians are significant figures in the lives of mathematics graduate students" (Beisiegel, 2009, p. 47). The literature review has also indicated that becoming a mathematician requires solid knowledge and command of mathematical discourse. It is important that a mathematician is able to communicate verbally to a wide variety of people about mathematics and to communicate in a specific manner of writing to the mathematical audience in particular.

The literature review has also shown that there is precious little guidance provided in how the mentoring process should work in aiding the graduate student to becoming a researcher and a mathematician: what specific or general help the supervisor should give the graduate student; who should come up with the mathematical problem, the graduate student or the supervisor; how much the researcher should be involved in the mathematical problem-solving of the graduate student; at which points during the problem-solving the researcher should provide hints, if at all; how the mathematical writing should be fostered; and when and how the graduate student should talk about and present the research accomplished thus far. In summary, there is little known about what goes on in the closed research meetings between graduate student and supervisor.

Since I want to study the unfolding of diagramming and gestures during research meetings between graduate student and supervisor, I need to be mindful that a research meeting serves the purpose of the graduate student's becoming both a mathematician and a researcher. In order to analyze mathematizing through the diagrams, I focus on those places in my analysis that show evidence of the process of becoming a mathematician. In the next chapter, I present a review of gesturing in general and how diagramming in connection with intuition, invention and abstract thought is emerging as a research area in its own right, which provides the main analytical framework for this study.

Chapter 3.

Gestures and Diagrams

3.1. Introduction

The analytical framework for my study is rooted in the research areas of gesturing and visualizing with a focus on diagramming. There has been quite a bit of research on visualization in mathematics as a helpful aid for doing mathematics, particularly proofs and problem solving (e.g., Nelson, 1993; Samkoff et al., 2012; Stylianou, 2002), and for the teaching and learning of mathematics (e.g., Bishop, 1989; Dreyfus, 1991; Fischbein, 1987; Zimmermann & Cunningham, 1991). However, diagrams are often regarded as a visual product that leads ultimately to some mathematical result with the connotation that the diagram is only a tool or crutch, while the *real* mathematics is synonymous with formal written mathematical statements.

Over the past decade, research has emerged to suggest that the key to understanding mathematical invention is the diagram as the virtual that is being actualized rather than the diagram as a product (e.g., Châtelet, 2000; de Freitas & Sinclair, 2012; Greiffenhagen, 2014; Sinclair, de Freitas & Ferrara, 2013). The current study is germane in that it aims to identify how mathematics is being invented through diagramming during research meetings between graduate student and supervisor. In particular, I seek to explore the concept of virtuality as something that is pushing the material aspects of mathematics, and how mathematics comes into being for either the graduate student or the mathematician supervisor through this conceptualization. This is particularly important in the evolution of the graduate student into the researcher-to-be, while also being relevant to the learning of mathematics at any age or level.

This section begins with a literature review within those two research areas, of gesturing and visualizing, that is critical in unraveling how diagramming is interwoven with mathematical invention and includes related studies covering mostly the last 30 years. The first topic is a review of gesturing in general. The second and third topics – aside from introducing the term diagram – make a distinction between how diagrams have come to be seen as either product or process in the research community. Throughout the discussion of these three topics, the points of view of diagramming and mathematizing as embodied acts are drawn out. Out of the literature review the analytical framework for this study emerges and, as a concluding part for this section, is exemplified through Oresme's study on acceleration along with the analysis by Châtelet.

3.2. Gesturing

This literature review begins on purpose with the study of gestures because this is the area that envelops diagramming, and furthermore because my research seeks the connection between diagramming and mathematical invention in the researcher and the researcher-to-be. Châtelet put it quite aptly when he wrote, "we pay particular attention to the dynasties of gestures of cutting out, to diagrams that capture them mid-flight, to thought experiments" (2000, p. 10). The term *cutting out*, translated from Châtelet's original French term *dispositif*, is used in the sense of *carving*, *creasing* or *excavating* in order to make the accompanying mathematical object, depicted through the diagram, material. Châtelet has a rather particular point of view on gestures and diagrams, and the ideas that grew out of his studies comprise one of the frameworks for my research analysis. Châtelet's key ideas are laid out and put into perspective after first providing a brief historical synopsis of gesture studies and then elaborating on the relevant themes that have developed in the study of gestures over the past decade.

The next few paragraphs serve as a historical summary of the study of gesture prior to the 1970s that is based on Kendon's (1983), McNeill's (2008, pp. 13-15) and Müller's (1998, 2000) respective overviews of gestures. This summary helps to position the current status of the study of gesture and its rise to an autonomous field of research.

Gestures have been studied throughout the centuries as recorded in these earliest works: *Institutio Oratoria* (Quintilian, 95 BC), *Chirologia: or the naturall language of the hand. Composed of the speaking motions, and discoursing gestures thereof. Whereunto is added Chironomia: or, the art of manuall rhetoricke. Consisting of the naturall expressions, digested by art in the hand, as the chiefest instrument of eloquence.* (Bulwer, 1644), *Traité de l'action de l'orateur, ou de la prononciation et du geste* (le Fauchert, 1657), or *Chironomia* (Austin, 1802). It is evident in these works that the main interest in gestures from classical antiquity to the Renaissance was in the way gestures do or should accompany rhetoric. However, as identified by Müller and elaborated on by McNeill, there are five themes that have gradually developed throughout the two thousand or so years of studying gestures which are reviewed next.

Until recently the viewpoint had been taken that gesture and language compete. Consequently, gestures had to be controlled, which is the first theme, and gestures had to be prescribed and regulated in public speech, which is the second theme. Recent evidence suggests that instead of competing, gesture and language rise and fall in complexity together. This framework is relevant to this study, insofar as gestures are yet another communication device that can offer insight into the mathematizing that unfolds during research meetings between graduate student and supervisor.

The third theme arises out of the study of manners, which emphasizes that a restrained use of gestures in private and dialogic conversation can be used to indicate a higher social status. This is still being practiced, for example in the preparation of many presidential speeches. The fourth theme is the idea that gestures can be considered as the basis of a universal language, which comes out of the gestural studies of the deaf. As Rotman (2005) summarizes, "various gestural systems – generically *Sign* – used throughout the world by the deaf [… are] full-blown (visual) languages, on a semantic, syntactic and pragmatic par with the (auditory) languages of human speech, and in some respects […] superior to them (p. 6, *emphasis in original*). This is yet another indication that gestures seem to embody more than the eye perceives, and perhaps this study allows a window into the realm of mathematical thinking and invention. The fifth and last theme, which emerged only in the eighteenth century, is the idea that perhaps gestures were the medium from which language developed. This fundamental hypothesis serves to emphasize that gestures can no longer be overlooked: they are an important, essential communication tool, and as such may provide insights into human thought.

Most of the twentieth century saw a stark decline in the interest and development of gesture studies, until in the 1970s a revival flourished under the umbrella field of semiotics of human action, and increased attention was given to the phenomena of gestures throughout diverse academic and creative disciplines, such as anthropology, linguistics, cognitive science and performance studies, to name but a few. At the beginning of the new millennium, the International Society for Gesture Studies (ISGS) was founded and instituted its own journal *Gesture*, with the aim of institutionalizing the study of gestures as a field in its own right.

For the purposes of this study, the literature review is now limited to the study of gestures as they pertain to mathematics, which is of fairly recent interest to the research community. In the words of Netz (1998), "there is a growing interest in the role of the visual in the creation and the teaching of mathematics, partly as a backlash against the more verbally-oriented mathematical ideals of the twentieth century and partly as a reflection of the new potentialities opened up by computers" (p. 33). Perhaps for these reasons, the last two decades have been fertile ground for the study of gestures in mathematics from the anthropological (Rotman, 2012), cognitive scientific (Lakoff & Núñez, 2000), educational (Krummheuer, 2013; Radford, 2001), psycholinguistic (Levelt, 1989; McNeill, 1992 and 2008), sociological (Greiffenhagen, 2014), and philosophical (Châtelet, 1993) points of view.

So, what is gesture? The origin of the word *gesture* derives from medieval Latin *gestura* meaning "bearing, behavior", which derives from Latin *gestus,* past participle of the root *gerere* meaning "carriage, posture" (www.etymonline.com, 2014). A recent point of view in gesture study is that a gesture is a movement with the hand or other body part to convey non-verbal messages that may or may not accompany speech. In other words, in this model thought is translated through the gesture. The term gesture encompasses more than just a bodily movement, according to the experimental psychologist Kendon (2004):

> [T]here is a wide range of ways in which visible bodily actions are employed in the accomplishment of expressions that, from a functional point of view, are similar to, or even the same as expressions in spoken language. At times they are used in conjunction with spoken expressions, at other times as complements, supplements, substitutes or as alternatives to them. These are the utterance uses of visible action and it is these uses that constitute the domain of 'gesture'. (pp. 1-2)

One of the groundbreaking studies in gesture theory appeared in the book, *Hand and Mind: What Gestures Reveal about Thought* (1992), that describes the theory of psycholinguist David McNeill developed over a ten-year period. This theory is "in a nutshell, […] that *gestures are an integral part of language as much as are words, phrases, and sentences – gesture and language are one system* (p. 2, *emphasis in original*). McNeill elaborates on this (at that time) revolutionary statement with three themes: images are included in language in contrast to the existing thinking that language is solely the spoken word; gestures are an integral part of discourse; and thought is impacted by gestures. This communication model of gesture is contemporary amongst philosophers such as Châtelet and Vygotsky as well as linguists such as McNeill and Kita. Contrary to the aforementioned conveying model, in the communication model the gesture *is* the thought rather than its translator.

McNeill's key theoretical concept is the so-called *growth point*, which unifies gestures and speech into an inseparable and simultaneous system (p. 220). This is in opposition to the *modular* cognitive theory developed by the contemporary psycholinguist Willem Levelt (1989):

> Rather than finding an impenetrable cut-off between conceptualization and formulation, as in modular information-processing theories (Levelt 1989), we see an interpenetration of thought and language: there is speaking in thought and thinking in speech. In this dialectic the formulation process changes thought from holistic with analytic elements to analytic with holistic elements. Gestures add individually conceived distinctiveness to the socially regulated linguistic structures. (McNeill, 1992, p. 247)

Echoed in this quotation is McNeill's (1992, 2008) utilization of Vygotsky's (1986) concept of material carrier of meaning in his theoretic construction of the growth point, wherein gesture itself is an element of thought and thus embodies meaning. In this sense, the gesture, however it is materialized, brings thought into existence. Kita (2000) extends McNeill's growth point "idea that two different structures of information interact to constitute the speaker's thought" (p. 165) by arguing that the two structures, analytic thinking and spatio-motoric thinking, act upon each other collaboratively in a virtual space out of which the gestures emerge for the purpose of speaking. For Kita, this virtual environment is "internally created as imagery" and "not an accurate reproduction of the physical environment that has been experienced" (p. 165). It seems as if Châtelet is echoed here with the attention to the virtual space of creation; however, Châtelet is concerned with the inventive nature of mathematics and how it emerges, while Kita focuses on how thoughts are linked to language via gestures.

McNeill also tries to apply his theories to mathematical concepts. His studies demonstrate that during mathematizing, gestures are tightly linked to language as measured by the time that elapses between the gesture and the words, and furthermore that the words that are linked to gestures are mostly nouns. McNeill goes on to hypothesize that "it may be appropriate to think of these gestures as imagistic words that take the place of linguistic words. When Hadamard insists that mathematicians do not think with linguistic words, we should ask if instead they are thinking with gestural words" (1992, p. 168).

The problem with trying to analyze *thinking* is that traditionally, thinking is considered a purely mental activity that is intangible and immaterial, and as such separate from the body. However, a decade after McNeill, the mathematics education researcher Luis Radford (2001, 2008, 2009) began to bring forth his idea of *sensuous cognition* based on Gehlen's (1988) notion of multimodal cognition – perceptual, auditory, tactile and kinesthetic sensorial channels – and merges this with the meaning of gestures seen from a cultural perspective. Radford (2009) emphasizes "that gestures, considered in isolation, have a very limited cognitive scope and that the cognitive possibilities of gestures can only be understood in the broader context of the interplay of the various sensuous aspects of cognition as they unfold against the background of social praxes" (p. 112). It is in this sense that Radford claims that gestures "are rather *genuine constituents* of thinking" (p. 113, *emphasis in original*) which therefore allow insight into abstract mathematical thinking.

Similar to how McNeill (1992) employed Vygotsky's mental carriers, Radford "see[s] the potential of semiotics in a rather Vygotskian perspective, that is, signs as psychological tools, or as prostheses of the mind, or even (but this is no longer Vygotsky) as the external locus where the individual's mind works" (2001). In this manner, the bodily sign of a gesture is objectifying the mental process that leads to mathematical understanding.

While McNeill dramatically extended the thinking around gestures during the 1990s, and while Radford brings light to gestures by looking at semiotics through lenses of cultural meaning and sensuous cognition, the problem remains, according to de Freitas and Sinclair (2011), that:

> [m]uch of this work conceives of diagrams and gestures as 'external' representations of abstract mathematical concepts or cognitive schemas. According to this approach, the diagram is assigned a static completeness, while the gestures – and the hands – that the diagram mobilized are forgotten. The diagram is then demoted to merely an illustration or representation of some other more fundamental or prior concept, while the gestures through which it emerged are erased from the text. (p. 4)

Châtelet does not forget about the gesture and its link to the diagram. On the contrary, he interprets gesture as even more than a visible, non-verbal, bodily action that carries meaning; indeed, a gesture is the articulation between the virtual and the actual and as such is immediate and embodied. In this sense, a gesture is inseparable from where it came from and what it creates. As a philosopher, Châtelet questions the boundary between physics and mathematics that Aristotle created with his interpretation that physics represents applications "that exist in Nature [… and are] mobile" whereas mathematics represents abstraction "which exists only by proxy through the wit" and is "immobile" (Châtelet, 2000, p. 17). Furthermore, Châtelet rejects Aristotle's reconciliation between the two rivals, physics and mathematics, "by subordinating them to metaphysics […] whose objective is the theory of immobile and real being, immutable substance" (p. 17).

Instead, Châtelet introduces his physico-mathematical space, which essentially envelops the thought experiments of mathematics – the place of intuition and premonition – as well as "mathematical abstraction [that] cannot be divorced from 'sensible matter', from the movement and agency of bodies" (Rotman, 2005, p. 18, 'emphasis in original'). Châtelet claims that "this concept of *gesture* seems to us crucial in our approach to the amplifying abstraction of mathematics" (Châtelet, 2000, p. 9, *emphasis in original*). Gestures are the virtual cut-outs that bring mathematics into being; they push the material aspects of mathematics. "In other words, Châtelet is asking that we imagine the inventive gesture as an action that literally breaks down previously taken-for-granted determinations of what is sensible or intelligible, and actually carves up space in new, unscripted ways" (de Freitas & Sinclair, 2012, p. 138).

The physicist Francis Bailly and his philosopher of mathematics colleague Giuseppe Longo (2011) develop an epistemological framework for cognitive processes in natural sciences wherein “mathematics helps [...] to constitute the very objects and objectivity of the exact sciences, because it is within mathematics that “thought stabilizes itself” (p. 1). Bailly and Longo explore forms of knowledge “via action within physical time and space and via living phenomena” and by “grasping the role of action in time and space, as well as the organization of these by means of [Châteletan] ‘gestures’” (p. 2). Indeed, Bailly and Longo expand Châtelet’s gestural thinking even further by acknowledging the animal life experience that precedes the intellectual experience. Their *naturalized* gesture “makes us conscious of our body and, through action, places this body in space [and] which would have, as a consequence, at once the thought itself, the thinking of self, and the consciousness of being alive within an environment” (p. 82). This framework derives from an earlier work by Longo (2005), who argues that “gesture of imagination should be included in this physical and mathematical intuition as ‘sense of construction’” (p. 8) and uses it to recap some of Châtelet’s case studies:

> By doing this kind of gesture, a human performs a conceptual experiment: “Archimedes, in his bathtub, imagines that his body is nothing but a gourd of water…Einstein takes himself for a photon and positions himself on the horizon of velocities” [Châtelet, 1993; p. 36; transl. 2000, p. 12]. “Gauss and Riemann…[conceive]…a theory on the way of habiting the surfaces” [Châtelet, 1993; p. 26]. (p. 8)

Châtelet restricts his research to historical, mathematical figureheads such as Oresme, Leibniz, Cauchy and Poisson, and their surviving written manuscripts. He needed to delve deep into the research of these mathematicians and trace their thoughts and actions purely from their writing and their diagramming. In Châtelet’s observational journey from mathematical research to the construction of a mathematically sound theoretical framework he was searching for “strategies of cognitive intuition” (Knoespel, 2000, p. xii). The crux of Châtelet’s study is his demonstration of how the virtual is evoked in historical mathematical inventions “through diagramming experiments whose sources Châtelet can trace to mobile gestural acts” (Sinclair & de Freitas, 2014, p. 563). It is this trace of the gesture lingering in the creation of new mathematics that is provocative and challenging to the abstract nature that has come to be associated with mathematics. In this way, Châtelet is pointing to the physical nature of mathematics that is often associated with its own creation. Knoespel concludes that through Châtelet's research "mathematics is historicized at the same time that it is recognized to contain patterns of thought. Châtelet (2000) opens a new science […] for a new survey of mathematical topoi and the way that they provoke thought" (p. xii).

In contrast, the cognitive linguist George Lakoff and cognitive scientist Rafael Núñez (2000) establish a conceptual system for mathematizing in their influential book, *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*, that is at odds with Châtelet's research on cognitive intuition. Lakoff and Núñez theorize that mathematics is conceptualized and understood based on normal cognitive mechanisms such as schemas and metaphors together with bodily activity. For example, they argue that elementary arithmetic reasoning is constituted through the 'stopping', 'starting' and 'continuing motion' schemas, and that such schemas can be combined by 'linking' metaphors to create more complex mathematical structures. This theory leaves no room for the possibility that conceptual mathematics can be realized in other modalities because it looks upon thinking processes as static structures, and therefore it cannot cope with analyzing the crucial connection between gestures and mathematical diagrams.

Nemirovsky and Ferrara (2009) agree that this "characterization of embodied cognition [is] illuminating, but also lacking […] imaginary activity" (p. 162). Rather than speaking of a sensory-motor system they speak of a perceptuo-motor activity, which "is inscribed in a realm of possibilities encompassing all those for which the subject achieves a certain state of readiness" (p. 162). Their research is based on two main perspectives: a multimodal interpretation of utterance that encompasses gestures, eye motion, body poise, sound production, and much more; as well as the inclusion of the imaginary in the scope of embodied cognition. Through these lenses their research reveals that, in the learning of mathematics, attention can be shifted "'from 'what *is*' to 'what *could be*'" (p. 173). It is in this sense that Châtelet's notion of hinge-horizon is echoed. To Châtelet, the horizon is the place of intuition, possibilities, and making sense of relationships, actions and objects that must occur as gestures give birth to diagrams:

> [It is] the hinge-horizons where inchoate systems begin to unfold, in all those places where orientation can't be had for free and where the true is not synonymous with the verifiable. […] Gestures and problems mark an

> epoch and unknown to geometers and philosophers guide the eye and hand. (p. 3)

It is apparent that Châtelet takes an embodied stance in the exploration and invention of mathematics. While I have demonstrated earlier that the understanding of gestures and diagrams by some of Chatelet's contemporaries are at odds with his, their gesture studies – like Châtelet's – support the view that an embodied approach is needed to understand abstract thinking because gestures play a significant role in mathematical thinking and learning, and that there is a close linkage between gestures and speech as well as gesture and metaphorical thinking (Radford, 2009; Núñez, 2004).

Various attempts have been made to extend Châtelet's framework to study the learning of mathematics among children, most notably the research of Sinclair and her colleagues Elisabeth de Freitas and Francesca Ferrara (de Freitas & Sinclair, 2012 and 2013; Sinclair, de Freitas & Ferrara, 2013; Sinclair & de Freitas, 2014). From these studies, it is apparent that gestures emerge as a very important resource when learners speak about mathematics, and furthermore that diagramming itself functions as a non-linguistic form of mathematical communication. The interplay between diagramming and gesturing brings virtuality into actuality through creative acts in learners in the context of school mathematics, which suggests that further investigation is needed at the level of mathematician (Sinclair, de Freitas & Ferrara, 2013). In the following section, the literature review expands on the extensions of Châtelet's frameworks by Sinclair and her colleagues in order to prepare the framework for my analysis.

3.3. Diagramming

The origin of the word *diagram* derives from the French *diagramme,* which in turn comes from the Latin *diagramma* and Greek *diagramma* meaning "geometric figure, that which is marked out by lines," and depends also on the word *diagraphein* meaning "mark out by lines, delineate" (www.etymonline.com, 2014). The word diagram first appeared in 1840 as a verb, from which the gerund diagramming is formed, which refers to the act of producing a diagram. The adjective diagrammatic has become a technical term in mathematics education studies such as in diagrammatic argumentation (Krummheuer,

2013) and diagrammatic thinking or reasoning (Radford, 2014), which are elaborated on in Sub-Section 3.3.1.

Diagrams play an important role in mathematics. One of the earliest fields of mathematics is the highly visual field of geometry, which is concerned with spatial figures, their properties, and their relationships, and in which succinct and expressive diagrams are essential. “Indeed, the abstract arguments presented in Euclid’s *Elements* rely heavily on the use of diagrams, and this use of visual representations remained an acceptable practice in mathematics well into the eighteenth century” (Stylianou & Silver, 2004, p. 354).

In his research paper “Greek Mathematical Diagrams: Their Use and their Meaning” (1998), Netz points out that in the treatises of the ancient Greek mathematician Pappus and philosopher Aristotle, their use of the term *diagramma* embodied much more than the meaning of diagram as we know it today, in that to them the word *diagramma* captures proofs and mathematics. Through the ensuing millennia, mathematics has branched into many fields and moved away from the notion that diagrams are synonymous with mathematics and perhaps even useful. There are various reasons why the subject of mathematics has come to be connected with abstraction and formalism, notably the following three influential events: (1) the invention of non-intuitive hyperbolic geometry in the nineteenth century that questioned Euclid’s intuitive visual arguments; (2) the twentieth-century Bourbaki movement whose goal was to write all of existing mathematics based on axiomatic set theory – thereby representing mathematics through formalism and symbolism; and (3) the ever-increasing complexity of mathematical fields. This diametric development between formalism and visualization within mathematics is captured well by Netz: “The metonym of modern science is a formula, a symbolic text. The metonym of ancient science is a diagram, a visual representation” (p. 37).

My main interest is the emergence of the diagram, as well as the manner in which the less-expert and expert mathematician interact with the diagram, and how these aspects capture mathematical inventiveness. In other words, I aim to demonstrate that the diagram is more than a visual product in that it has a voice of its own and is not neutral during mathematical engagement, that it provides a quintessential playground hovering near virtuality in the process of mathematizing, and that the diagram is not just a representation on the way to formalization. But first, I examine the diametrically opposed frameworks developed around diagrams in the research community, that view the diagram as either a visual product or the virtual that is being actualized. It is the latter which provides one of the analytical frameworks for this study.

3.3.1. The Diagram as a Visual Product

The late twentieth century saw the rise of visualization studies in the context of mathematical problem solving at both novice and expert levels. This is perhaps not a surprise, as this was also the era when computers became readily available, and funding agencies such as the US National Science Foundation were urged "to get visualization tools into the 'hands and minds' of scientists" (McCormick, DeFanti & Brown, quoted in Zimmermann & Cunningham, 1991, p. 2). Subsequently, there was a push by education ministries, at least in the Western world, to move computers and their graphics capabilities along with visualization programs such as *The Geometer's Sketchpad* into the school arena in general and mathematics education specifically to "restore geometry[,] restore intuitive and experimental mathematics" (Davis & Anderson, quoted in Zimmermann & Cunningham, 1991, p. 2).

Aside from the technological advancements to aid mathematicians visually, and the potential of visual representations to provide useful learning and teaching tools for mathematics, the rise of cognitive studies during this time also helped visualization research to gain popularity. As Stylianou and Silver put it, "the growth of [the cognitive] perspective on complex performance and problem solving rescued visualization from the brink of oblivion, where it had resided during the reign of behaviorism in psychology" (2004, p. 355).

Since one of the obvious visual representations of mathematics is in the form of a diagram, this literature review needs to investigate the viewpoint of diagrams that is undertaken in visualization research. I start by stating the etymology of the word *visualization* and providing a description of visualization as it is used in the research community. The noun visualization first appeared in 1881 and derived from *visualize*, which is originally attested in Coleridge's "Biographa literaria, 1817" (1971) and leans on the adjective *visual*. The origin of the word visual derives from the Late Latin *visualis* meaning “from sight”, which derives from Latin *visus* meaning “sight”, past participle of the root *videre* meaning “to see” (www.etymonline.com, 2014). Today, research studies in the context of mathematics employ “the term visualization to describe the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated” or achieved mentally, as opposed to referring to the pure, direct meaning of mental image in the everyday use of the term visualization (Zimmermann & Cunningham, 1991, p. 1).

Some of the early studies of visualization demonstrate that visualization is seen as a helpful aid to learning mathematics but that it is nonetheless considered as a crutch to reach the formal, finished product, which is understood to be the so-called *real* mathematics (e.g., Fischbein, 1987; Bishop, 1989; Blum and Kirsch, 1991). In some later research circles, visualization is upgraded to a recognized tool for mathematical reasoning and offers a more balanced view of mathematics that includes the visual and intuitive rather than just the symbolic, verbal and analytical elements (e.g., Arcavi, 2003; Dreyfus, 1991; Zimmermann & Cunningham, 1991; Stylianou & Silver, 2004). One of the visionary researchers in visualization studies, Presmeg (1986a, 1986b), who considers visualization as an epistemological learning tool, establishes her theories by studying high school students and their teachers. She distinguishes two types of learners as well as two types of teachers, namely those that prefer a visual versus non-visual problem-solving heuristic/teaching method. Her theory introduces the *mathematical visuality* to indicate the “extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods” (1986b, p. 42) and points out that “textbooks and current teaching methods favour the nonvisualiser insofar as most teachers are unaware of difficulties inherent in visual cognition and of ways of overcoming these difficulties” (1986a, p. 309). Presmeg and contemporary scholars list the following difficulties with visual methods: production takes time; usage is not validated in problem solving; thought is tied to irrelevant details in the visualization; thought becomes inflexible moving from standard to non-standard diagrams; imagery is uncontrollable, vague, or hindering. These findings promote further research in how visualization can become a pedagogical tool in mathematics. Further support for a closer look at visualization is provided by mathematicians such as Hadamard (1954), Poincaré

(1963) and Halmos (1985), who have analyzed the reasoning processes of themselves or other mathematicians, and conclude that the ability to visualize is an important one for a mathematician.

I suggest that whether visualization is determined to be a crutch, a tool or an ability makes no difference; the problem is that these views hinder the analysis of the creative mathematical moments during diagramming, because the mathematical objects and relations that are under consideration are reduced to a static external representation. “Zazkis et al. emphasizes that the act of visualization is a *translation* from external to mental (or vice versa) and, particularly, the connection made by the individual between the image and the mental” (Stylianou, 2002, p. 306, *emphasis in original*). My study questions that a translation occurs and attempts to analyze diagramming during mathematizing under a different framework as stated in Section 3.2 and further elaborated on in Sub-Section 3.3.2.

The research community that is grounded in the framework of visualization relies on the underlying principle that visualization implies understanding (Alcock & Simpson, 2004; Dreyfus, 1991; Gibson, 1998; Zimmermann & Cunningham, 1991). Often the diagram is cited as the visualizing means for understanding:

> In mathematics, visualization is not an end in itself but a means toward an end, which is understanding. Notice that, typically, one does not speak about visualizing a *diagram* but visualizing a *concept or problem*. To visualize a diagram means simply to form a mental image of the diagram, but to visualize a problem means to understand the problem in terms of a diagram or a visual image. (Zimmermann & Cunningham, 1991, p. 3, *emphasis in original*)

In this quotation, it becomes apparent that the mathematical diagram is still referred to only as a static, external product rather than a creative embodied act that gives birth to a material and movable mathematics.

But what the literature review of mathematical visualization reveals are some of the following insights into mathematical reasoning by novice, less-expert and expert mathematicians. Visualization methods can support thought experiments through valuable cognitive ideas and provide insights into sophisticated mathematics (Tall, 1994). The drawing of diagrams helps post-secondary students to absorb and critique information as well as to uncover and deliberate hypotheses (Gibson, 1998). A visualizer/analyzer model is developed to support the interdependence of visual and analytical thinking as opposed to the earlier orthogonal view of these axes of thinking (Zazkis et al., 1996). The drawing of a diagram by a mathematician during problem solving stimulates four types of actions under the extended visualizer/analyzer model, namely inferring consequences, elaborating on new information, creating sub-goals and engaging in meta-reasoning (Stylianou, 2002). Visualization is not only a means to illustrate symbolic results in mathematics, but it is also a key component for all kinds of mathematical activities such as reasoning, proving and problem solving (Arcavi, 2003). Under the external/internal authority point of view, students link visual representations to symbolic representations only if they have an internal sense of authority about their role as learner, i.e. their understanding of mathematical definitions is consistent and appropriate (Alcock & Simpson, 2004).

Scholars in visualization studies seem to agree "that experts have an extensive and deeply organized knowledge base with respect to visual representation use as a heuristic" (Stylianou & Silver, 2004, p. 381), while novices may not have developed such a rich visual representation toolkit depending on their sociological and cultural background (Alcock & Simpson, 2004; Arcavi, 2003).

To summarize, mathematical visualization theory emphasizes cognition over perception, supports thought experiments (Tall, 1994), and "gives depth and meaning to understanding, serves as a reliable guide to problem solving, and inspires creative discoveries" (Zimmermann & Cunningham, 1991, p. 4). However, in the literature review I conducted on mathematical visualization coupled with invention or creation of mathematics at the novice or expert levels, I was unable to find evidence of visualization studies shedding light on the mathematical creative process. While anecdotal accounts from mathematicians support the strong connection between visual representations and mathematics,

> research suggests that the relation is actually not so clear, or at least that there may be a need for greater clarity about the nature of the relation. […] For example, it would be useful to know how experienced mathematicians

> make use of visual representations and how they overcome the difficulties associated with their use. (Stylianou & Silver, 2004, p. 357)

In the last decade, most visualization studies in mathematics, particularly those adopting a semiotic approach, concentrate directly on the diagram and diagrammatic reasoning. Philosophers of mathematics education such as Dörfler (2005) and Radford (2008) have expanded on Peirce's epistemology in order to expose the limits and possibilities of diagrammatic thinking. Dörfler concludes that a Peircean perspective on diagrams "shifts mathematics from an esoteric, abstract, or purely mental activity 'down' to a material activity on perceivable and therefore also communicable objects, that is, the diagrammatic inscriptions" (p. 66). But Radford criticizes Peirce's view for its exclusion of the subject: "in so far as the reality of mathematical relations (Peirce) or the transcendental ontological status of mathematical objects (Kant) precedes all semiotic experience, Kant and Peirce coincide in leaving the actions of the subject out of the ontological realm" (p. 13). Radford argues that diagrammatic thinking is a deeply subjective activity embedded in the socio-cultural experiences of the individual, which constructs a reality in accordance with semiotic systems of signification out of which creation emerges.

Some of the latest research demonstrates how diagrammatic argumentation is employed in children aged 3-10 years during mathematical activities (e.g., Krummheuer, 2013), as well as in mathematicians during lectures and proof constructions (e.g., Samkoff et al., 2012). Diagrammatic reasoning is seen as or sometimes made equivalent to informal reasoning, and is basically a heuristic approach to problem solving. Krummheuer builds on his earlier constructed conceptual framework termed "interactional niche in the development of mathematical thinking" (p. 249), which states that as children age their reasoning shifts from a diagrammatic to a narrative argumentation and that this shift is not linear. Krummheuer concludes in his latest work that the age range from 5 to 7 years is a "crucial developmental phase in which the children acquire narrative language skills" (p. 250) and that this shift towards the narrative needs to be attended to pedagogically. Samkoff et al. extend previous studies on the use of diagrams during proof construction and conclude in their qualitative and interpretive study that:

> undergraduates use diagrams primarily to try to understand a mathematical claim, and sometimes to generate an explanation of why a theorem is true (see Alcock and Weber 2010a, 2010b). However, these students generally

> did not use diagrams for the same reasons that some of our participants did, such as verifying that their logical deductions were true, uncovering errors in their logic, or identifying false mathematical assertions. (p. 64)

Nonetheless, Krummheuer and Samkoff et al. base their analysis on viewing the diagram as a finished object that must adhere to mathematical conventions, and which is produced by the individual in a structured manner.

To summarize, current visualization studies in mathematics concern themselves with the production, interpretation and utilization of visual representations during mathematical problem solving or proof production from the novice to the expert level mathematician; however, these studies take the approach that the visual representations are evoked from the invisible representations that existed a priori for the person engaged in mathematical activities. In contrast to this point of view that treats the diagram as a product in mathematizing, the next sub-section discusses the literature review of diagrams as the virtual that is being actualized coupled with the mathematical invention as the possible that is being realized and shows how this view helps shed light on the creative processes in mathematizing. As Knoespel points out, "[t]o activate our understanding of intuition, invention and discovery in mathematics we need to become cognisant that the space of mathematics has been mediated not only through theorems but also through diagrams" (Châtelet, 2000, p. x).

3.3.2. The Diagram as the Virtual Being Actualized

Figure 1-2d from the *Introduction* (see Chapter 1) depicts a diagram that is often used to illustrate a proof of the Pythagorean theorem: a square of side length $a+b$ is drawn with congruent right triangles placed in each corner that have as their legs the lengths a and b and hypotenuse of length c, where the four hypotenuses form an inner rotated square. When a diagram is interpreted as a visual representation of mathematics, i.e. as a product and not as a process, then it can be easy to overlook how the diagram is produced and how it connects the mathematician to the mathematics and to the experiences in the material and virtual worlds. Seen as a representation of a proof, Figure 1-2d shows the mathematician that the area of the larger square can be calculated in two different ways and then equated: $(a+b)^2=c^2+4((ab)/2)$. The manipulation of this equation yields the Pythagorean theorem. But as a representation, Figure 1-2d has little chance to tell its story. Why is this diagram effective, and where does the mathematician *see* the Pythagorean theorem? “Diagrams are more than depictions or pictures or metaphors, more than representations of existing knowledge; they are kinematic capturing devices, mechanisms for direct sampling that cut up space and allude to new dimensions and new structures” (de Freitas & Sinclair, 2011, p. 138).

As I have shown earlier in Section 3.2, even when the diagram is taken as a process, it is often interpreted as a bodily action that merely aids the knowing mind, as in the case of Núñez’s (2004) study. In contrast to Núñez, Châtelet (2000) sees the diagram as the source of intuition, invention and discovery of mathematics through the body: diagrams “illustrate the urgency of an authentic way of conceiving information which would not be committed solely to communication, but would aim at a rational grasp of allusion and of the learning of learning” (p. 14).

In this sub-section, my literature review presents recent findings in philosophy, mathematics education and the social sciences that demonstrate a new orientation towards gestures, diagrams and mathematics that lean on Châteletan thinking. Châtelet aptly states that “[a] diagram can transfix a gesture, bring it to rest, long before it curls up into a sign, […] But unlike the metaphor the diagram is not exhausted: if it mobilizes a gesture in order to set down an operation, it does so by sketching a gesture that then cuts out another” (p. 10). This emergence of diagramming as the written form of gesture to analyze the mathematizing of teachers, learners and mathematicians alike has intensified research activity on gestures. Before I elaborate on the latest findings, it is time to establish Châtelet’s framework fully.

During Châtelet’s time as Program Director at the International College of Philosophy from 1989 to 1995, his provocative work *Les enjeux du mobile: mathématique, physique, philosophie* was published in 1993. Just after Châtelet’s death, the English translation *Figuring Space – Philosophy, Mathematics, and Physics* was published in 2000. *Figuring Space* is a study of the research notes of several famous mathematicians, such as Oresme, Leibniz, Maxwell and Hamilton. In this book in particular, Châtelet lays the foundation of the concept of virtuality as something that pushes the material aspects of mathematics, and where the diagram is the connection between the virtual and actual. In his foreword to *Figuring Space*, Knoespel writes that

> in contemporary Greek the verb διαγραμμα means [...] a word that was used in Greek to refer to writing on a wax note pad in which the mark of a stylus would simultaneously cross over the marks that had been drawn previously. Διαγραμμα in effect embodies a practice of figuring and defiguring. (p. xvi)

It is this flexible viewpoint of *figuring* and *defiguring* through which Châtelet brings new thoughts to the study of gestures, diagrams and mathematics, and the underlying key ideas are summarized here:

- The diagram is never really fixed – it is erased, drawn over, redrawn, or reassembled. Not only does the mathematician interact with the diagram, the diagram also interacts with the mathematician. This alludes to the diagram having a life on its own.
- It is through interaction with the diagram, which is a material process, that a person invents or discovers mathematics. This pushing of the material boundary is in strong opposition to the long-prevailing Aristotelian belief that the doing of mathematics is an abstract process, disconnected from the mathematician and the material world.

The connection between these two key ideas is what Châtelet terms *virtuality*. Contrary to its name, virtuality is a tangible concept for Châtelet in that it indicates the place of creative play for a mathematician, and through the diagram and the mathematical gestures that the diagram provokes and captures this virtuality is being actualized. According to de Freitas and Sinclair (2012), "[m]athematical intuition, according to this approach, is less about mystical insight into an ideal realm and more about the pre-linguistic apprehension of embodiment itself" (p. 138).

I use the analysis of Oresme's work by Châtelet to exemplify the above ideas regarding virtuality and the diagram. By way of background, Oresme lived during the 14th century in France and was one of the principal founders of modern science and a leading original thinker of his time. Oresme was among a group of kinematician-philosophers of the School of Paris that studied the motion of motion (acceleration), which was considered a difficult problem at the time, partly due to the prevailing Aristotelian view of mathematics and physics as an unbridgeable divide between abstraction and application.

Working from the relation $L=VT$, where L represents length, V constant velocity and T time, these scholars reconsidered how the relationship between velocity, time and length can be thought of and visualized, which ultimately allowed them to tackle the problem of the motion of motion. To begin with, Oresme and his colleagues were constrained by the theory of Aristotle in that they had to distinguish between the extensive and intensive measurements in their problem. In modern terms, extension refers to quantitative measurements, while intensity refers to ontological measurements. In the relation $L=VT$, length and time are the measurements that can be quantified, while velocity was not considered directly measureable, but instead a description of a quality that gauges how fast or slow the motion is, i.e. how intense it is. The conflict for scholars of this era was that extensions belong to the application because they are ‘real’ and can be measured with devices such as a ruler or a clock, whereas intensity is considered an abstract concept. Oresme’s diagrams in particular were able to reconcile this divide, while at the same time, capturing a new way of understanding motion that led to the analysis of acceleration. Figure 3-1 depicts Oresme’s diagrams from his work in *De configuratione qualitatum* as presented in *Figuring Space* (p. 39). Châtelet guides the reader through these diagrams:

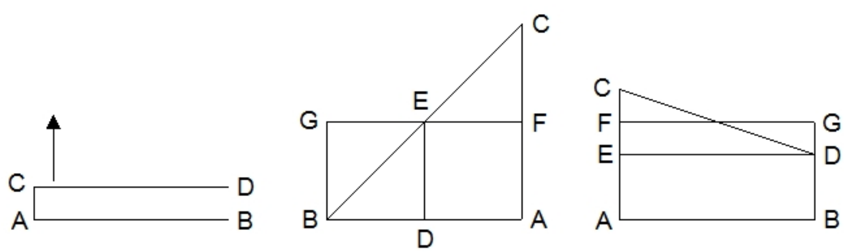


Figure 3-1. Diagrams given by Oresme in his *De configuratione qualitatum*

> The idea is very simple: a mobile subject manages to give extension to the quality of which it partakes (here velocity represented vertically) by travelling a certain distance during the unit of time. We see how clearly the diagram underlines the distinction between extensive subject (horizontal) and intensive subject (vertical).
>
> The length is calculated as the *area* comprised between the line of the degrees [velocity] and the line of the extended subject [time], and the different motions appear therefore as *deformations* of the standard rectangle [see Figure 3-2]. (p. 39, *emphasis in original*)

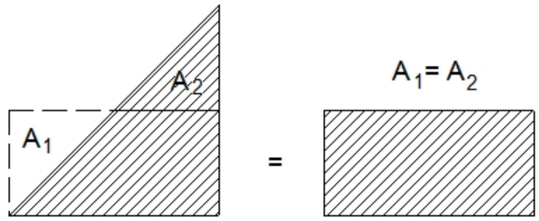


Figure 3-2. Uniformly deformed motion represented as area is simply a deformation of the standard rectangle, where areas are shown as presented in *Figuring Space* (p. 41)

This means that Oresme went from a constant velocity diagram (Figure 3-2 right) to a constant acceleration diagram (Figure 3-2 left) by mobilizing the velocity line in the diagram. Despite the prevalent belief that velocity is not directly measureable, and thereby hindering access to the motion of motion, these diagrams are evidence that Oresme and his colleagues found a way out of this constraint.

Oresme's diagrams contrast sharply with my own experience of graphs during my schooling and in textbooks, and how I have drawn the graph in my calculus classes, namely by letting the horizontal axis represent time T and the vertical axis length L (see Figure 3-3). With this choice, the axes become immediately immobile in that the relation $L=VT$ is reduced to input (time) corresponding to output (distance) and "the line can only represent a *transit* of forms" (p. 39, *emphasis in original*). In other words, the traditional approach to graphing the velocity-time-length relation is to plot both extensive measurements (time and distance), leaving the velocity to be calculated from the resulting curve as a rate of change or slope while the rate of change of velocity (acceleration) is visualized as curvature. Oresme's diagram is the flip-side of this approach, because an extensive measurement (time) and intensive measurement (velocity) are graphed together giving length as area, or in other words the integral of the resulting curve. When viewed in this manner, Oresme and his colleagues can be thought of as having anticipated the Fundamental Theorem of Calculus 300 years before Leibniz and Newton. Figure 3-1 has the added benefit of illustrating both sides of the calculus coin because the slope of the velocity-time graph is acceleration. But I am getting ahead of myself.

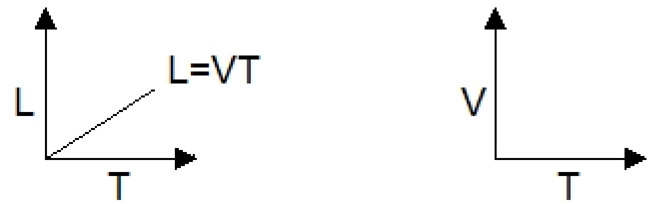


Figure 3-3. Modern (left) and medieval (right) versions of the graph of *L=VT*

It is important to point out again that Oresme's manner of diagramming (Figures 3-1 and 3-2) frees up velocity and allows it to become mobile during thought experiments, which Oresme explains in his own words:

> The quantity of any linear quality at all is to be imagined by a surface whose longitude or base is a line protracted in some way in the subject [...] And I understand by 'linear quality' the quality of some line in the subject 'informed' with that quality. For that the quantity of such a quality can be imagined by a surface of this kind is obvious, because there can be found a surface relatable to that quantity, a surface equivalent in length or extension and whose altitude is similar to the intension. (Oresme, quoted in Châtelet, 2000, p. 41)

It is within this physico-mathematical space that Châtelet speaks of *virtuality* which is being *actualized* through Oresme's diagrams, and of the *possibility* of satisfying both the extensive and intensive points of view thereby gaining an understanding of not only motion but also motion of motion, which is *realized* in portraying the length as an area. Châtelet emphasizes Oresme's achievement not only in his sublime invention depicting motion and opening up a way to comprehend acceleration, but also in how his intuition achieves freedom from the constraint of the Aristotelian standpoint:

> The kinematician-philosopher can thus cut out forms of motion obtained through continuous variations of the standard unit, that is, by the rectangle associated with uniform motion [...]. Such *deformations* provide access to the motion in motion, which is however strictly prohibited in Aristotle's physics. We already know that the question of motion as such is a delicate one for a metaphysic of substance and we see all the advantages gained by these diagrams, which are capable of presenting motion as a regulated unfolding of velocity, as a plastic and undivided unit through which a subject appropriates space. (pp. 41-42)

So how do Oresme's diagrams provide access to the understanding of acceleration? Figure 3-4 depicts how the degrees of velocity – their intensity – can be virtually grasped as they unfold from maximal degree, called degree 1, to the zero degree: "The velocities should not be added up like rulers placed end to end, but should be grasped in the way that a single subject would unfold them in thought in order to inspect a space more or less promptly" (Châtelet, 2000, p. 47).

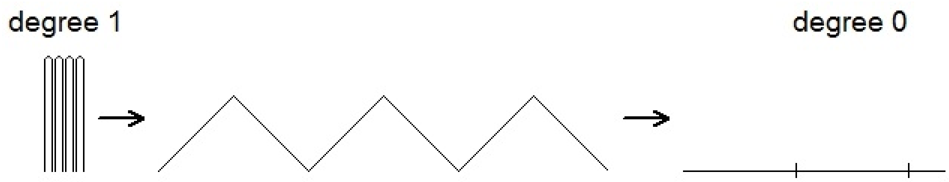


Figure 3-4. Degrees of velocity

The extensive and intensive measurements are ingeniously wrapped up in this representation without being fused, and as Châtelet points out "Oresme takes care to distinguish between two terms for 'to measure', *mensuratur* and *attenditur*," which correspond respectively to the spatial or temporal extension of motion (its quantitative measurement) and the change in velocity (its ontological measurement) (p. 43, *emphasis in original*). In this manner of measuring and diagramming, Oresme's thought experiments have given rise to a virtual space that allows motion itself to become mobile:

> [B]y considering, in defiance of naive intuition, the length as an area, Oresme means to make it clear that it comprehends the parameter. It is not only that which is already dissipated in the extension, but above all that which makes it possible to give out the 'ontological' measurement by degrees and the 'quantitative' measurement by standard units by embracing in a single gesture the spreading out of the discrete units by consecution and the selection of a degree of the continuous spectrum of the velocities. To consider the length as an area is to make clear that the cooperation of the two measurements involves the invention of a continuum capable of presenting as contemporaneous that which appears as already divided and that which asserts itself as an undivided entity. (p. 44)

Châtelet is astute in how he chooses to discuss Oresme's investigation of motion. Not only does this study exemplify Châtelet's diagrammatical framework, but the virtual structure established by Oresme also foreshadows how Einstein's relativity theory emerges: this continuum of space and time described above is controlled by the degrees of velocity and necessitates that the velocities are thought of as a spectrum rather than discrete units.

> To render the coalition of the velocities diagrammatically, it is not enough just to associate a rectangle that is simply 'bigger' than the others to the first degree [Figure 3-5a], rather it is necessary to make it an *operator* of perspective projection [Figure 3-5b].
>
> If we now want the unfolding of the spectrum of the velocities and the coalition of the degrees to *jump out at us*, we must make ourselves capable of a *contemplation* that takes in all the diagrams in one go and makes the subordination of the imperfect degrees to the degree 1 patent and creative. [...]
>
> The degree 1 of the velocities must therefore function as a *horizon* in Oresme's diagrams: this is probably the central intuition of relativity. (pp. 48-49, *emphasis in original*)

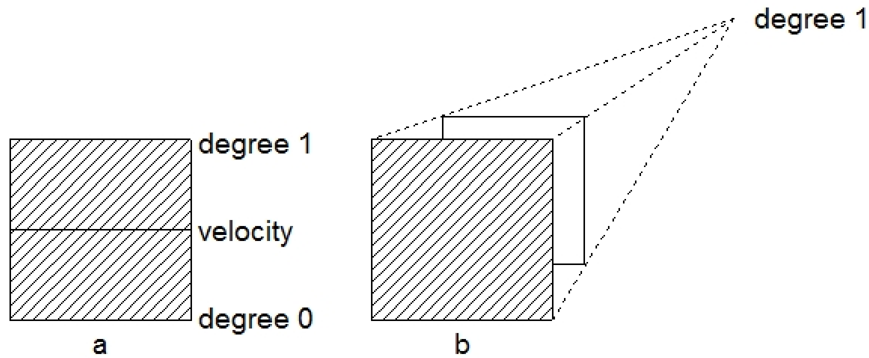


Figure 3-5. Rectangle as *operator* of perspective projection to render the coalition of the velocities diagrammatically (Châtelet, 2000, p. 48, *emphasis in original*)

When Châtelet speaks of the horizon or hinge-horizon, he refers in effect to the place of intuition, where sense is made of objects, their relationships and their actions, and where – like a hinge – possibilities also arise in the *figuring* and *defiguring* of these objects, their relationships and their actions that bring about the creation of something new. In the case of Oresme, the hinge-horizon is the contemplation of a maximal degree velocity, which enacts itself as depth in the diagram, and thereby encompasses the continuum of velocities. However, once the horizon appears, it cements again what was briefly in flux:

> The horizon is neither a boundary marker that prohibits or solicits transgression, nor a barrier drawn in a dotted line across the sky. Once it has been decided, one always carries one's horizon away with one. This is the exasperating side of the horizon: corrosive like the visible, tenacious like a smell, compromising like touch, it does not dress things up with

> appearances, but impregnates everything that we are resolved to grasp. (p. 54)

By rewriting Oresme's study of the motion of motion through the lens of Châtelet, I have laid out Châtelet's diagrammatic framework. In summary, to use Châtelet's own words: "The thought experiment taken to its conclusion is a diagrammatic experiment in which it becomes clear that a diagram is for itself its own experiment" (p. 12).

I end this section with the latest findings of contemporary scholars, whose research leans on Châteletan thinking. I start with the mathematician-philosopher Rotman (2012), whose exposé on categories versus sets as a means to conceptualize mathematics explores how diagrammatic language supports an embodied understanding of mathematics and its setting in social and cultural dynamics.

> Diagrams/gestures operate in a pivotal space: they are embodied acts that bridge the gulf between thought and the sign. On one side: intuition, a-articulated [sic] images, and rumination; on the other: syntax, representation, symbols. [...]
>
> Châtelet's work suggests the possibility of accessing the intrinsic corporeality of topological ideas through a kind of reverse engineering: retrieve the gestures that have been operationalized/internalized into symbols, make the problems they respond to and the intuitions guiding them physically explicit within a material context. (pp. 256-257)

In his article *The Materiality of Mathematics: Presenting Mathematics at the Blackboard* (2014), the sociologist Christian Greiffenhagen "explores how writing mathematics (on paper, blackboards or even in the air) is indispensable for doing and thinking mathematics" (p. 1). His research is based on data collected from graduate mathematics lectures and research meetings between a supervisor and his graduate students. While Greiffenhagen gives no indication that he has read Châtelet's work, his findings are nonetheless reminiscent of Châteletan thinking in their interplay between virtual and material substances through gesturing and diagramming, and furthermore extend it to the *act of writing*:

> What is specific to mathematics (and other conceptual sciences) is the absence of inscription devices that transform materials *into* inscriptions, i.e., inscriptions on the blackboard have not been obtained through

> instruments out of material substances. In a sense, the material substances of mathematics *are* the inscriptions (and *vice versa*). (p. 23)

I end this sub-section on the subject of diagrams as the virtual that is being actualized by describing the research findings of the mathematics education scholars Nathalie Sinclair and Elisabeth de Freitas as well as their colleagues. Sinclair and Armstrong (2011) construct a classroom task teaching algebra via geometry involving *The Geometer's Sketchpad* with the goal of furthering students' spatial and kinetic reasoning skills, and report on their results. They conclude that, "[d]rawing diagrams can be an important mediator between embodied experiences and mathematical abstraction" (p. 348), and that their "highly successful" (p. 353) approach allows students to have the spatial and kinetic experiences in order to "have the images and drawings in mind that they need" when moving from geometric to algebraic lessons.

de Freitas (2012) claims that the works of Healy and Sinclair (2007), Hofstadter (1997) and Sfard (1994) indicate that the purpose of narrative is to access the logical structure of mathematics but do not show how this is to be achieved. Her study attempts to demonstrate how narrative performs this function by analyzing the video data of one mathematician who was explicitly asked to tell the story that each of three diagrams reveals to him. Employing the analytical tools of free indirect narration and virtual voicing, and following the philosophical ideas of Rotman (2008) and Châtelet (2000) that the diagram mobilizes mathematics, de Freitas concludes that "treating the diagram as story allows us to think about the event-structure of the mathematical diagram, and to explore the ways a diagram is a material site of engagement" (p. 33). She further suggests that more research is needed on how the learner can utilize the roles of narrative and voice to enhance engagement with diagrams.

Sinclair and de Freitas (de Freitas & Sinclair, 2012 and 2013; Sinclair & de Freitas, 2014) collaborate extensively to further Châtelet's work and "argue for a fundamental philosophical shift to better conceptualize the relationship between gesture and diagram, and suggest that such an approach might open up new ways of conceptualizing the very idea of mathematical embodiment" (2012, p. 133). The research setting of one of their studies is an undergraduate geometry class, where a black and white stop-action film entitled *Families of circles in the plane* is shown to "evoke ideas related to projective geometry, namely, the notion of a point at infinity" (2012, p. 144). Their analysis is largely framed in Châtelet's work and how it fits into or intersects with contemporary social, cultural, pedagogical and philosophical perspectives. Focusing on the diagrams that the students drew, de Freitas and Sinclair deduce from their empirically-based pedagogical study that they:

> can see in some of these diagrams precisely what Châtelet found in significant historical developments in mathematics: inventive 'cutting out' gestures that interfere and trouble assumed spatial principles, new and radical 'symmetrizing devices' and the emergence of new perspectival dissymmetries within the given work surface. (p. 150)

de Freitas and Sinclair conclude this particular study by raising several pedagogical questions based on their firm belief that "the work of Châtelet challenges educators to reconsider the power of student diagramming as a disruptive and innovative practice that sheds light on the very nature of mathematical agency" (p. 151).

Sinclair, de Freitas and Ferrara (2013) analyze the data from a computer-based lesson held in a grade 1 classroom that "focused on conceptualizing intersecting and parallel lines, which the students had never formally encountered" (p. 243). Châtelet's viewpoint allows the researchers "to shift [their] attention away from the doer and focus on the doing" (p. 241) in their analysis of the lesson. With this altered perspective, they define four essential qualities for their so-called *creative act* that conceptualizes mathematical inventiveness and "point[s] to the centrality of the body and its movement (actions) – rather than internal mental disposition" (p. 242).

> A creative act:
>
> 1. introduces or catalyzes the new – quite literally, it brings forth or makes visible what was not present before,
>
> 2. is unusual in the sense that it must not align with current habits and norms of behavior,
>
> 3. is unexpected or unscripted, in other words, without prior determination or direct cause,
>
> 4. is without given content in that its meaning cannot be exhausted by existent meanings. (pp. 241-242)

Sinclair et al. emphasize that by separating the two processes – the actualization of the virtual and the realization of the possible – mathematical creativeness can be analyzed and ultimately furthered from a new perspective. They conclude that these characteristics of the creative act "should open up new areas of research" (p. 252).

The Virtual Curriculum: New Ontologies for a Mobile Mathematics (Sinclair & de Freitas, 2014) is an analysis of the mathematics curriculum in the US from the philosophical perspective of Châtelet with the aim of stimulating educators "to re-think questions and assumptions about the school mathematics curriculum" (p. 575). The authors "believe that the concrete-abstract binary imposes a straightjacket on mathematics and misrepresents the actual activity of doing mathematics" (p. 560). They argue that *virtuality* (as defined by Châtelet) allows the bridging between the concrete and the abstract, and necessitates a re-examination of the curricular sequencing of mathematical concepts.

> When the concept is used only as a logical tool, while the ontological aspect (actualizing the virtual) is abandoned, the activity reduces to adhering to logical constraints. In such cases, the ossified concept doesn't sustain the mobility from which it came forth. (p. 566)

Sinclair and de Freitas provide several primary school examples that are related to number sense from Châtelet's perspective on mathematical mobility in order to support a change from the current sequencing of mathematical ideas:

> The history of mathematics points to several encounters with the virtual that the small set of counting numbers could entail: the concept of zero and its bringing into existence of nothing; the flight into integers; the excavation into rational numbers. A curriculum that was less concerned with operations and more with creativity might delay the upward extension of the counting numbers and the early practicing of addition-as-grouping in favour of providing opportunities for children to engage with the new virtual spaces created by the concepts mentioned above. (p. 571)

I conclude this section of the literature review by listing the six major features that Châtelet ascribes to diagrams and which Knoespel summarizes in his foreword to *Figuring Space*:

- *Diagrams constitute technologies that mediate between other technologies of writing*. […]

- *Diagrams create space for mathematical intuition*. […]
- *Diagrams are not static but project virtuality onto the space which they seek to represent*. […]
- *Diagrams represent a visual strategy for entailment*. […]
- *Diagrams are mediating vehicles which means that they cannot only be recovered but rediscovered*. […]
- *Diagrams have a pedagogical force that could be integrated into mathematical education*. […] (Châtelet, 2000, p. xviii, emphasis in original)

My study is an attempt to answer the following questions through the analytical frameworks on gestures and diagrams established in this Chapter: how is the diagram situated as the place of mathematical invention and discovery? How and when does Châteletan diagramming emerge during research meetings between a graduate student and supervisor? Whom does the diagram engage among the less-expert and expert mathematicians? What is the interplay between the diagrams and the gestures during these research meetings? Attention needs to be paid to speaking, diagramming, gesturing and virtual gesturing in the sense of Châtelet when analyzing the data for the multifarious ways in which the body is involved in communicating mathematical thinking. As a recognized integral part of language, gestures may hold the key to providing immediate insights into thought and how thought is expressed. Concrete gestures such as pointing at the diagram, caressing the diagram, drawing over the diagram, erasing the diagram, redrawing the diagram, or starting a new diagram need to be investigated for evidence of the underlying mathematical thought. By examining the material processes of diagramming and interacting with the diagram, on the part of both the less-expert and expert mathematician, this study aims to examine whether the diagram has a voice in mathematizing and whether, as Châtelet asserts for historical cases of mathematical invention, it is an essential medium for mathematical thinking, discovery and intuition.

3.4. Research Questions

The literature review in Chapter 2 on the key elements of a research meeting – namely the supervisor, the graduate student, and the mathematization – provided the backdrop for two main aims of this study. The first aim is to investigate diagramming and gesturing during mathematizing from a Châteletan viewpoint. The second aim is to gain an understanding of how the graduate student goes through the process of becoming a researcher and a mathematician. Research has shown that the supervisor in the role of advisor or mentor is the main person who enculturates the graduate student into the research community as well as being the main person who demonstrates mathematizing. The literature review has also demonstrated that there are two major concerns: there is precious little guidance provided in the literature in how the mentoring process should guide the graduate student in becoming a researcher and a mathematician, and furthermore, the literature contains very limited results on the process of becoming a mathematician. A by-product of my two aims is to further refine the term *mathematization* based on its current use by education researchers and mathematicians. For the purpose of my study and as I have argued in Chapter 2, the term *mathematization* refers to an embodied engagement with mathematics in order to get as close as possible to the emergence of mathematical creativity.

In Chapter 3, I have presented the work on diagrams and gestures by Châtelet as the analytical framework for my study. In order to fine-tune my analysis of diagrams, I am guided by the philosophical framework of new materiality as an extension of embodiment as explicated by de Freitas and Sinclair (2013). From this perspective, mathematical entities have virtual and actual dimensions and are considered material objects with which the mathematician is engaging through thought experiments. This leads naturally to my first research question:

1. *How is new materiality evidenced in the expert and less-expert mathematicians during research meetings?*

My hypothesis is that the mathematical diagram is a catalyst that initiates a transformation of mathematical entities into material objects. This is not just to say that the diagram is a material object, but that the mathematical concept under consideration also emerges as a material object through the diagram. This leads me to my second set of research questions:

2. *When does the diagram talk and to whom? What story does the diagram tell?*

If I consider the diagram to be an actual, material object, one that the mathematician engages with, then I also wonder if there is a duality in the conversation between the diagram and the mathematician. In other words, I hypothesize that the direction of conversation is not just *from* the mathematician *to* the diagram, but also *from* the diagram *to* the mathematician. With this research question, I seek not only to find evidence in support of my hypothesis, but also to explore how the diagram itself converses with the mathematician. At the same time, I must investigate the mathematician in this conversation, and how he maintains the conversation with the diagram. My literature review on gestures concluded that, as a recognized integral part of language, gestures may hold the key to providing immediate insights into thought and how thought is expressed. Therefore, I need to investigate concrete gestures such as pointing at the diagram, caressing the diagram, drawing over the diagram, erasing the diagram, blocking the diagram, redrawing the diagram, or starting a new diagram when looking for evidence of the underlying mathematical thought. In particular, I need to determine whether there are any differences in these regards between the less-expert and expert mathematicians. This brings me to my third set of research questions:

3. *What role do gestures play in the culture of mathematical research? In particular, what differences are there between the way expert and less-expert mathematicians interact with the diagram?*

The guiding principle of the analytical framework of my study is Châtelet's definition of diagramming as the virtual that is being actualized, thereby realizing the possible mathematics that resides within the mathematician. Since Châtelet asserts his analytical framework in historical cases of mathematical invention, this prompts the questions as to whether Châteletan diagramming can be evidenced *in vivo* and to what extent such diagramming is observable in either or both the expert mathematician and the less-expert mathematician. These hypotheses are reflected in my fourth and final research question:

4. *How does mathematical invention emerge through diagramming carried out by expert and less-expert mathematicians during research meetings?*

An answer to this question could provide better guidance to the supervisor as the mathematician who is modelling mathematizing for the graduate student and who is also mentoring the graduate student in the process of becoming a mathematician.

In summary, if the diagram is indeed a birth-place where mathematical entities are actualized, and perhaps even where some mathematical creations are realized, then the philosophical movement of new materialism (that opposes the inert, inanimate and immaterial viewpoint of mathematical concepts prevailing in the current mathematics education literature) is strengthened. Furthermore, the answers to my research questions could allow a redefinition of mathematization, one that necessitates pushing the boundaries between the virtual and actual not just in the entwinement of doing mathematics and mathematical thinking but also in the objects of mathematization themselves, namely the mathematical entities. These results are of interest to educators like myself, in order to facilitate diagramming when students engage with mathematical concepts. For example, it may guide me in new ways to model diagramming purposely for the students, or to design new tasks for students that necessitate diagramming. Furthermore, having an understanding of the gestures that are involved in diagramming, I can emphasize them during my teaching and also look for them in my students as evidence of mathematical engagement.

Chapter 4.

Methodology

The previous chapter, *Gestures and Diagrams,* described my analytical framework and also included my research questions, a discussion of what motivated me to ask these particular research questions, and what I hope to gain from them. Before I address these research questions, I first want to speak to the tension between the identities and expertise of both the graduate student and the supervisor that led me to choose names carefully that distinguish their various roles.

In Chapter 2, I presented my literature review on the research surrounding the graduate student, supervisor and graduate studies in general and the limited research that has been done on both the supervisor-graduate student relationship in the area of mathematics as well as the process of becoming a mathematician. I introduced the graduate student as the research mathematician-to-be with the dual roles of less-expert mathematician and novice researcher, and the supervisor as the research mathematician having the dual identities of expert mathematician and research mathematician. I have chosen these names carefully since I want to distinguish between the process of becoming a researcher and the process of becoming a mathematician. The graduate student, having already been accepted into the graduate program, has proven skills in mathematics, and therefore cannot be called a novice mathematician anymore. However, the graduate student has not yet developed full expertise in a mathematical field like their supervisor has. I therefore chose to call the graduate student a *less-expert* mathematician and the supervisor an *expert* mathematician. With regard to the role of researcher, the graduate student is in the graduate program precisely in order to test the waters about becoming a researcher, a role that until now has only been experienced by proxy during undergraduate studies, and hence I identify the graduate student as a *novice* researcher. I could have called the supervisor an *expert* researcher, but I settled on identifying this role as *research mathematician* or alternatively *mathematics researcher,* because this best describes where their expertise lies.

While I do care about how the graduate student develops the skills necessary to move from novice researcher to research mathematician, my research area is about the graduate student and supervisor as mathematicians. I am ultimately interested in how the diagramming and gesturing unfolds in the mathematical research meetings and how the graduate student and supervisor as the less-expert and expert mathematicians participate in the understanding or creation of new mathematics. I now confront my research questions, which are restated below for convenience:

1. *How is new materiality evidenced in the expert and less-expert mathematicians during research meetings?*
2. *When does the diagram talk and to whom? What story does the diagram tell?*
3. *What role do gestures play in the culture of mathematical research? In particular, what differences are there between the way expert and less-expert mathematicians interact with the diagram?*
4. *How does mathematical invention emerge through diagramming carried out by expert and less-expert mathematicians during research meetings?*

In order to address these questions, I decided to follow the research meetings between graduate student and supervisor in the field of mathematics. This is an appropriate approach, because during such meetings the supervisor as the expert mathematician and the graduate student as the less-expert mathematician at the very least converse about their research mathematics; and perhaps even do mathematics together, which creates an ideal opportunity to observe their mathematizing. I discounted meetings where no actual mathematical exchanges take place. I also decided to look for graduate students who were just entering their graduate program. This choice was made for two reasons, the first being that the difference between research mathematician and research mathematician-to-be is more clearly evident. The second reason is that the research meetings would more likely be about finding ways to answer the student's research question(s) rather than about the late writing stage of the established answer (in which case I would not be able to observe the emergence of newly invented mathematics even though that may yet occur).

In this chapter, I describe the participants in the study as well as the data-gathering, data-viewing, data-selection and data-transcription processes. I conclude this chapter by listing possible limitations of the research method.

4.1. Participants

The participants in this study are two supervisor-graduate student pairs in the Department of Mathematics at a prominent Canadian university. One of the pairs was joined by the research colleague of the supervisor from the same department and a visiting professor from another university, which I refer to as the 'foursome'. The remaining supervisor-graduate student pair is referred to as the 'twosome'. These two groups have been chosen for several reasons: (1) their respective mathematical fields lend themselves to the study of diagramming and gestures, (2) the research topics are accessible to myself as the investigator, (3) I have no conflict of interest with the research participants, and (4) these two graduate students were at the beginning of their respective graduate programs.

Through no design of my own, all participants in this study are Caucasian, and the supervisors are at similarly advanced stages in their research career. I was careful to choose one male and one female supervisor, not because I intended to study differences in gender but because I am looking for as much diversity as possible and I am mindful to include at least one female in the field of mathematics, which is known as a male-dominated subject. Each participant in this study is referred to by a pseudonym, which I chose.

First I describe the twosome. The supervisor Tracy is an associate professor, who has experience supervising undergraduate students in NSERC (Natural Sciences and Engineering Research Council) summer research projects and Master's graduate students, as well as working with postdoctoral fellows. She is currently supervising her first two doctoral graduate students, one of whom is the graduate student Tim of this study. At the time of data gathering, Tim was in the second year of his studies. The research of the twosome is situated in the field of Enumerative Combinatorics, which can be roughly described as involving the counting of objects subject to constraints. The graduate student is using a combinatorial approach to asymptotic growth of restricted lattice paths. His research question is: given an integer grid and constraints on vectors (size and direction), how many paths are there of length *n* within this region that can be taken using these vectors to construct the path? The research question was already established at the time when the data collection started.

Next, I describe the foursome. The supervisor Fred is an associate professor whose direct supervisory experience at the time of data gathering is similar to Tracy's, with some undergraduate students in NSERC summer research projects, two graduate students each in the Master's and PhD programs, and a few postdoctoral students. Fred's graduate student Finn is in the Master's program and is participating in this study. Finn has just begun his studies in the area of Topological Graph Theory. His research area is the study of embedding graphs in surfaces, where graphs are considered as topological spaces. Finn's research question is: what are the minor, minimal, non-planar, 2-regular, directed graphs? This research question was established by the supervisor Fred. Initially, the research meetings were planned to include just the supervisor Fred and his graduate student Finn. However, the research topic attracted Fred's research colleague Colin from the same department, who is a prominent mathematician. Since Colin was sponsoring the visit of his research colleague Victor from another university at that time, this person was also invited along to the research meetings. Consequently, the research meetings ended up consisting of four people: the supervisor Fred, the graduate student Finn, the research colleague Colin, and the visiting researcher Victor. Fred tells everyone that he is very excited about this research question, because he believes that the proof is at their fingertips.

Aside from the research team, there were two other people who attended the research meetings on four occasions, namely a research colleague from the computing department, who attended once, and another graduate student, who attended three times. I made the decision to ignore these visitors, partly because they were not in regular attendance, and mostly because they contributed little to the research meetings.

4.2. Data-gathering Process

Video-recordings and camera images are quite commonly employed as data sources in qualitative research, and both are regarded as advantageous in a variety of fields such as education and sociology (Erickson, 2006; Heath & Luff, 2000; Jordan & Henderson, 1995; Silverman, 2006). As Silverman points out, recordings and transcripts can offer a highly reliable record compared with field notes of observational data and moreover, researchers can return to these recordings as they develop new hypotheses. In particular, Heath and Luff (2000) argue that:

> [w]ithout video-recordings of the 'naturally occurring' events in the various settings, it would be difficult, if not impossible, to undertake analysis which examines the interactional production and coordination of workspace activities, and the ways in which personnel use tools, artefacts and various features of the local environment to accomplish the actions in which they engage. (p. 21)

I attended the research meetings of both the twosome and the foursome, from September to December 2013, which is considered one research term at the university where the data-gathering took place. I considered this time to be sufficient to gather evidence to answer my research questions based on conversations I had with supervisors about progress during graduate studies in mathematical fields. I attended meetings as an observer and recorder, so that discussions as befitted the research meeting can be considered to have taken place 'normally'. However, I was also present as a semi-participant. In other words, I attempted to follow the mathematics that was being developed during the research meetings, and at the same time took research notes that attended to changes in themes and mood. This was necessary in order to capture whether new understanding or creation of mathematics took place. However, I never interfered with the interaction between the participants nor purposely interrupted the research meetings in general. If I had been absent during the recording of the research meetings, then this would have been less than desirable for several reasons. For one, I have only a graduate-level knowledge of graph theory and topological spaces, which was last used twenty years ago. However, being an avid learner, I analyzed my learning style a long time ago. I have come to understand that I am very skilled at absorbing the major mathematical aspects of a lesson or mathematics presentation when I am present in the room with the mathematician. While I am still able to learn from a taped lecture, I require more effort to follow along, because the sound is somehow flat, the blackboard/screen is too small, and, as odd as it might sound, there is no atmosphere of learning. Therefore, I chose to learn the mathematics of the research meeting by being present. This means that when I view the video-recordings, I already have a base-knowledge of the mathematics of the research meetings and can use this knowledge to notice when mathematics changes, becomes understood by the participants, and is discovered. Furthermore, I am not distracted by having to learn the mathematics, and as the researcher of diagramming and gesturing I can pay attention to the interactions between the mathematicians and the diagram. Without my presence at the research meetings, I can only venture to guess which would have had the stronger influence during data-viewing: the lens of the mathematician needing to understand or the lens of the researcher wanting to analyze the data. Consequently, my presence in dual roles allowed me to make field notes during the meeting that attended both to the mathematics that developed during the meeting and to the observations. These observations concern the emotional mood that was being created by any of the participants, and any diagramming or gesturing that stood out for me during the time of recording.

All data that I analyze here were collected during these research meetings. Digital videos captured the diagramming and gesturing involved during the research meeting. Digital still images were taken using a second camera that was trained on either the blackboard, computer screen or paper used by any one of the participants to capture the diagramming. In the exceptional cases when I was not present during a research meeting, the supervisors captured the aforementioned digital images and videos using the two cameras.

The twosome initially planned to meet once per week for one hour, but two factors conspired against this plan: the supervisor, Tracy, had just returned from a one-year, out-of-town sabbatical and the return did not go a there are three sets of adjacent blackboards, each set comprising one fixed wall-mounted board at a lower level and two moveable boards on vertical tracks that can be independently moved between an upper or a lower level, so that six of the nine boards are visible at any time s smoothly as planned; and Tracy learned quickly that the graduate student was not able to produce sufficient data and analytical work in a one-week period. Therefore, Tracy rescheduled the research meetings to take place every third week in Tracy's office. The digital camera was positioned in front of the twosome for the duration of the meeting, in order to capture both the supervisor and the graduate student as well as the computer in the background. I was present only as observer and sat next to the video camera so I could read the recording time for the purpose of making field notes. I collected a total of six video-recordings, all but one of which were thirty to sixty minutes in length as shown in Table 4-1. In addition, I captured 38 digital images of notes and drawings made by either the supervisor Tracy or the graduate student Tim as well as screen shots from the computer that were referenced by Tracy or Tim (see Table 4-1).

Table 4-1. Video-recordings and Images from the Research Meetings of the Twosome

Date (yr-m-d)	Data description	Recording length	Reference #
2013-09-23	video-recording	59:59	T1-v1
2013-09-30	video-recording	45:26	T2-v1
	digital images of computer		T2-i1
	digital images of grad student notes		T2-ng1
	digital images of supervisor notes		T2-ns1 to 4
2013-10-21	video-recording	30:14	T3-v1
	digital images of computer		T3-i1 to 4
	digital images of supervisor notes		T3-ns1
2013-11-17	video-recording	42:57	T4-v1
	digital images of computer		T4-i1
	digital images of grad student notes		T4-ng1 to 4
	digital images of supervisor notes		T4-ns1 to 4
2013-12-02	video-recording	58:52	T5-v1
	video-recording	06:22	T5-v2
	digital images of computer		T5-i1 to 13
	digital images of grad student notes		T5-ng1 to 3
	digital images of supervisor notes		T5-ns1 to 4

The supervisor of the foursome chose to conduct the research meetings in the departmental seminar room, which is formatted as a regular classroom with blackboards and rectangular tables. Each rectangular table seats two people comfortably, and there are five rows of three tables each. There are three sets of adjacent blackboards, each set comprising one fixed wall-mounted board at a lower level and two moveable boards on vertical tracks that can be independently moved between an upper or a lower level, so that six of the nine boards are visible at any time. The four people involved worked almost solely in front of the blackboards with the supervisor being the typical person to work on the blackboards, unless any one of the other mathematicians felt compelled to come up to the blackboard as well for a short period of time; the most common scenario involved the graduate student, research colleague, and visitor sitting at tables spread throughout the room. I therefore had to make a choice of where to place the video camera. During the first meeting I placed the camera to capture just the blackboard that was being worked on, and elected to swivel the camera whenever another blackboard was being used. I found out quickly that almost all blackboards were employed during the meetings, and moreover that this method of data collecting was not effective, as I would capture only the supervisor and whomever happened to be in the camera view, but not the rest of the people involved in the dialogue. As an unobtrusive observer, I also did not want to influence the way the foursome placed themselves in the room. Therefore, the digital camera was positioned at the back of the room near the left end on the second to last table row for the rest of the meetings, in order to capture the full expanse of the six visible blackboards along with the person that was working on any one of the blackboards, and include at least the graduate student as well. The downside of this video camera placement is that the blackboard images were too far removed and so were not clearly visible on the videos. To compensate, I took digital images of the blackboard. During the first meeting, I was hesitant to walk up to the blackboard to capture the images, but I learned quickly that if I did not do so, then the diagram might be erased before I had a chance to capture it. So, as of the second research meeting, I simply went up to the blackboard to take a digital image whenever I saw an opportune moment, hoping not to be too intrusive. I assumed that this interfered very little, if at all, with the research meeting because conversation went on ‘through my body’, so to speak. Another downside of the video camera’s placement towards the back of the room is that voices are at times not clearly audible when the participants were huddling over one of the desks for example, but this did not occur often. The meetings were scheduled to take place for one to two hours once per week, and were cancelled only if the supervisor was away at a conference.

Given that each of the batteries for my video camera lasted for exactly one hour, I had to switch batteries after one hour during research meetings that lasted for more than one hour. I collected a total of nine video-recordings, each mostly between one to two hours in length (see Table 4-2). In addition, I captured 156 digital images of notes and drawings made by the four participants during their combined research meetings as shown in Table 4-2.

Table 4-2. Video-recordings and Images from the Research Meetings of the Foursome

Date (yr-m-d)	Data description	Recording length	Reference #
2013-09-24	video-recording	59:59	F1-v1
	video-recording	04:58	F1-v2
	digital images of blackboard		F1-i1 to 3
2013-10-08	video-recording	59:59	F2-v1
	video-recording	00:31	F2-v2
	digital images of blackboard		F2-i01 to 15
	digital images of grad student notes		F2-ng1 to 5
	digital images of colleague notes		F2-nc1 to 5
2013-10-15	video-recording	59:59	F3-v1
	video-recording	36:43	F3-v2
	digital images of blackboard		F3-i01 to 19
	digital images of grad student notes		F3-nc1 to 3
2013-10-22	video-recording	59:59	F4-v1
	digital images of blackboard		F4-i01 to 12
2013-10-29	video-recording	59:59	F5-v1
	video-recording	43:29	F5-v2
	digital images of blackboard		F5-i01 to 26
	digital images of grad student notes		F5-ng1 to 4
	digital images of colleague notes		F5-nc1
2013-11-05	video-recording	59:59	F6-v1
	digital images of blackboard		F6-i1 to 9
2013-11-12	video-recording	59:59	F7-v1
	video-recording	34:05	F7-v2
	digital images of blackboard		F7-i01 to 23
	digital images of grad student notes		F7-ng1

	digital images of colleague notes		F7-nc1 to 2
2013-11-19	video-recording	59:59	F8-v1
	video-recording	07:11	F8-v2
	digital images of blackboard		F8-i01 to 10
	digital images of grad student notes		F8-ng1 to 4
2013-11-26	video-recording	59:59	F9-v1
	video-recording	54:51	F9-v2
	digital images of blackboard		F9-i01 to 14

4.3. Data-viewing Process

When I video-taped a research meeting, that video-recording became data. As a researcher I need to be mindful that the recording is a layering of events that may all influence each other, such as the gestures that are being made, the words that are being uttered, where and how the diagram is drawn, the physical layout of the room and the position of the participants relative to each other. Therefore, I should not treat data as fixed records and flat images. I need to be able to “see the invisible” (Roth, 2011, p. 17) and in particular, I need to see the *virtual gestures*. In light of the research by McNeill (1992 and 2008), Rotman (2005) and Châtelet (2000) discussed in Chapter 2, I was greatly interested in both the *actual gestures* that were being made by the participants and also the *virtual gestures* that reveal themselves through the diagram, other material and even the gestures themselves.

Roth warns that when “persons retrospectively talk about their living/lived mathematical experiences, [...] these accounts inherently involve *re*presentations of the experience, that is, means of making some past experience present again” (p. 15, *emphasis in original*). Especially as the viewer of the video-recordings, I have to be mindful that I am not only watching past experiences that are being made present again, but that these experiences are not my own. Furthermore, there is a tension between the two roles that I play: the analyst of the data, and the mathematician attempting to follow along with the mathematics that is being unravelled. However, this tension is necessary, because as an analyst I am able to reveal the actual and virtual gestures that are being made in connection with the diagram, and as the mathematician I am able to reveal mathematical understanding and creation. Thereby, I am able to *unfold* the diagramming and gesturing and expose their connection to newly created mathematics.

Roth broadens the window through which the workings of a mathematician should be observed using the idea of 'learning to experience in order to see'. This takes away the focus on the final mathematical product and emphasizes the whole journey of mathematical discovery. Nonetheless, he does not elaborate on what it is in the living work of mathematics that should be observed, nor how to observe it. The interaction analysts Brigitte Jordan and Austin Henderson (1995) make the practical suggestion that when video-recordings are first viewed, the researcher should provide a content listing as a first step in the data analysis:

> The level of detail is determined by the interests of the researcher and the available time. [...] Content listings are useful for providing a quick overview of the data corpus, for locating particular sequences and issues, and as a basis for doing full transcripts of particularly interesting segments. (p. 43)

The following section outlines which questions I asked myself and which actions I looked for while watching the video-recordings for the first time, in order to 'see the invisible' in the words of Roth (2011) as well as to locate particular sequences of interest.

4.4. Data-selection Process

During the data-viewing process, I was able to perform a preliminary analysis of the amount and type of data for the twosome and foursome respectively. As mentioned in Section 4.1, supervisor Tracy rescheduled the research meetings to take place every third week during the data gathering period, since her transition from sabbatical did not go as planned and her graduate student Tim did not perform to her expectation. It turned out that Tim had had a change of heart and dropped out of the PhD program during the subsequent term. While I was able to collect approximately four hours of recorded research meetings, the graduate student was still stuck in the phase of data production rather than graphing the data and analyzing the various components for possible relationships. Most of the research meetings were about Tracy attempting to guide Tim in the direction of producing diagrams, but very little was achieved. Therefore, these data do not provide sufficient material to examine diagramming as well as the mathematical enculturation of the graduate student. On the other hand, the data that I collected from the foursome, which amounts to a little over 12 hours of recorded research meetings, provided ample and rich data regarding graduate student involvement and diagramming to pursue my research. While I hope, nonetheless, to analyze the data of the twosome at a future date, I decided to perform the data analysis for my doctoral research solely on the data of the foursome.

Having collected the data and constructed my analytical framework leaning on Châteletan thinking about gestures, diagrams and mathematics, I was faced with decisions about how to apply this knowledge to my data analysis. During the attendance at the actual research meetings and later during data viewing, I was often drawn to how some participants commanded the conversation and also startled to find the diagram figuring dominantly in this conversation. During a discussion with research colleagues about my observations, I was made aware of the function of the indigenous talking stick, which immediately struck a chord with me as I had come across the talking stick as an art object during my visit at the Vancouver Museum of Anthropology. Upon further investigation, I realized that the concept of talking stick is what helps me to apply Châtelet's ideas to my data. Before I further expand on how I am accessing the Châteletan framework, I first want to explain the relevance of the talking stick.

4.4.1. Talking Stick

The talking stick originates from the indigenous people of the Northwest Coast of North America. Coastal native tribes have a hereditary head and political spokesman, who possesses a talking stick or speaker's staff that is typically "painted and carved with the supernatural patrons and guardians of his family" (Wade, 1986, p. 31), in order to indicate the lineage represented in this leading tribal elder. Locust (1997) describes the proceedings of a tribal meeting in the following way. The leading elder is in possession of the talking stick and initiates a discussion about some matter of concern. After he has concluded his talk, he holds out the talking stick and offer it to the council members to ensure an equal opportunity of speech: "In this manner, the stick would be passed from one individual to another until all who wanted to speak had done so. The stick was then passed back to the elder for safe keeping" (para. 1). Because the talking stick indicates authority and a right to speak, it is both an instrument and a symbol of aboriginal democracy.

In discourse analysis, authority for speaking is typically studied using participation frameworks such as those outlined by mathematics education researchers Hegedus and William Penuel (2008). I am employing the talking stick instead, because it serves my study in both a literal and metaphorical sense. As a metaphor, the talking stick indicates authority of speech (Locust, 1997). By pointedly asking questions that are listed below on the theme of the talking stick, I can extract from my data how the participants interact with or around the diagram, and also what role the diagram itself plays in the interaction. Furthermore, the participants of the study were often holding a writing device in their hand such as a pen, pencil or chalk stick. These concrete devices may indicate that the talking stick is literally in the hand of the participant, for example, by tapping on the blackboard with the chalk stick or pointing with the pen at the computer screen. These may also be places where new materiality is evidenced. I have therefore listed questions that I asked myself while viewing the video-recordings, in order to reveal how the diagram communicates with the mathematician and conversely, how the mathematician communicates with the diagram:

A. *Who holds the talking stick among the expert mathematician and less-expert mathematician? Is it possible for the diagram on the blackboard/paper/computer screen to hold the talking stick?*
B. *Who holds the talking stick most often?*
C. *How does the talking stick get passed among participants?*
D. *Is the talking stick fair, in the sense of allowing for equal rights to be heard?*
E. *During diagramming, how is the diagram blocking understanding (in the sense of a communication breakdown)? Or how is it supporting understanding (in the sense of a communication breakthrough)?*
F. *During diagramming, what are the interactions between the diagram and its creator? If the diagram is drawn by one participant, then how is the diagram communicating with other participants who did not draw it?*

Jordan and Henderson (1995) point out that "[a]s particular tape segments emerge as significant, content [listings] are expanded into transcriptions. These may be more or less elaborate and detailed, depending on the nature of the researcher's analytic interests" (p. 47). Using the data-viewing process that is described in Section 4.3 and the above guiding questions centred on the talking stick, I watched the nine one-to-two-hour video-recordings of the foursome. Based on this viewing, I selected those episodes that involved a diagram for further examination. From each of the nine research meetings I ended up with 9, 11, 13, 20, 19, 11, 14, 14 and 11 episodes respectively varying in length from approximately thirty seconds to approximately six minutes, for a total of 122 episodes. I viewed these episodes several more times and made brief notes describing the mathematics that occurs, tracking who is holding the talking stick mostly, illuminating the role of the diagram, identifying the overall body language associated with the diagram, recording spoken phrases that caught my attention, and labelling the apparent mood (see Appendix A for some sample pages). By this time, I had a more holistic notion of the various roles that the diagram plays during the mathematizing. Based on this understanding, I then transcribed one to three time intervals per research meeting for a detailed examination in my data analysis. The data-transcription process and the structure of the transcription tables are discussed in the next section.

4.5. Data-transcription Process and Tables

Jordan and Henderson (1995) argue that "there is no ideal or complete transcript according to any abstract standard. Rather, the question that guides the transcription of the research data must be: How adequate is this transcript for purposes of the analysis to be performed?" (p. 48). My research interests demand that I pay attention to both gestural and verbal communication, and that I express the relationship between these two aspects as accurately as possible. However, "transcripts should be relatively neutral" as Ochs (1979, p. 47) points out, in that the researcher should not impose conventions on the transcript based on cultural expectations that may not necessarily be there. For example, the researcher should not impose a sentence structure on an utterance, simply because texts are written in sentences. Furthermore, Ochs argues that there is an inherent contradiction between detailed transcription layout of the verbal and non-verbal communication and its readability for analysis. The researcher has to make decisions about keeping the transcript readable within the framework of the study. I adopted the column layout suggested by Ochs (1979). Therefore, the transcriptions of my study contain the following four columns, which are tailored to my data analysis:

1. Time in minutes from the beginning of the research meeting
2. Participants' non-verbal activity
3. Participants' utterances
4. Images (still images from the video-recordings or still images from camera of blackboard, computer screen or paper used by any one of the participants)

Of these four columns, the *time* column is the only truly factual and objective column, because the recording device time-stamps the video-recordings. One can argue that the *images* column is also factual and objective in the sense that it is taken from what the camera sees; however, the still images are chosen based on a selection of non-verbal activities described in the second column, and are also cropped to narrow in on the non-verbal activity. Therefore, the *images* column really displays factual but subjective research information. In the following sub-sections I elaborate on the second and third columns.

4.5.1. Participants' Non-verbal Activity

The second column consists of short descriptions of pertinent gestures, actions and postures made by participants that I collected during multiple data-viewing. As I pointed out in Chapter 3, *Gestures and Diagrams*, Radford (2008) reviews the classical mind-oriented view of cognition and makes a case for what he terms *sensuous cognition*, which is based on Gehlen's (1988) notion of multimodal cognition. Under the sense-oriented view, Radford argues that gestures must be considered within the social praxes of a culture, so that gestures can provide insight into abstract mathematical thinking. Therefore, I am taking a closer look at body language in the culture of mathematical research. Body language encompasses various forms of non-verbal communication, such as gestures, actions, postures, facial expressions, gaze and tone of voice. Non-verbal communication provides clues to the attitude or state of mind of a person and is part of what creates a certain mood in conversations. While I paid attention to facial expressions, gaze and tone of voice during data viewing, I did so for the purpose of determining certain moods and places of interest that helped me with selecting intervals in the recordings that are relevant to diagramming. Here is the list of moods that I employed: aha-moment, excitement, confidence, joy, laughter, evenness, silence, hesitation, doubt, puzzlement. Once these short intervals were selected, I did not find it necessary to transcribe facial expressions, gaze and tone of voice and concentrated instead on gestures, actions and postures.

In order to bring the data closer to the reader, I chose to observe the gestures, actions and postures of the mathematician who is doing the drawing and talking in relationship to the diagram and also to the other participants of the research meeting. I first explain how I have come to think of and term the other participants. Initially I called the other participants the *audience* because at first glance the research meetings give the impression of a typical mathematics lecture at a post-secondary institution; that is, the mathematician at the blackboard and his seated colleagues and graduate student are in similar positions as the instructor and the listening and observing students. However, viewing the video-recordings fully and attentively, I have come to the conclusion that *audience* is an unsuitable term to choose for two reasons. First, the origin of the word audience derives from the Latin *audentia* meaning “hearing, listening”, which derives from Latin *audientum* (nominative *audiens*), past participle of the root *audire* meaning “to hear” (www.etymonline.com, 2015). The use of the term audience today may refer not only to listeners but spectators as well, but in the end both listening and observing are one-way skills and do not entail participating in the event or even being on par with the performer.

Second, having been in the room with the mathematicians during recording and also viewing the recording afterwards, I never had the sense that I was sitting in a lecture. In fact, I recall being astonished at the time how readily any of the mathematicians voiced that something was puzzling them or that they were not understanding something. I have never observed this in any mathematics lecture that I have sat in on. Certainly, Fred typically started conversation and the diagram drawing on the blackboard, but he is also the graduate student’s supervisor and so he had the role of providing guidance during the research meetings. However, every mathematician, at times, did not hesitate to take charge (although the relative proportions of times doing so were not necessarily equal).

Finn's participation in and contribution to the research discussions were not on par with that of the expert mathematicians, but I will discuss the evolution of the graduate student as mathematician in Section 5.5.

The important point is that this was truly a research meeting, where every mathematician participated and contributed to the learning/creating of the material under discussion. I therefore decided to term the participants *mathematikoi*, after the name of the members belonging to the inner circle in the community that Pythagoras founded for the study of knowledge. Pythagoras made the distinction of an outer and inner circle within the community. The members of the outer circle were called *akousmatics* and "did not study mathematics or philosophy" but instead listened to the teachings and received spiritual guidance (Wertheim, 1996, p. 7). "The mathematikoi, however, lived inside the community and dedicated themselves to a Pythagorean life" (p. 7), essentially being philosopher-mathematicians who fully participated in the acquisition of knowledge.

I now proceed to explain how the participants' non-verbal activities are transcribed for the second column of the transcription tables. Just as the utterances are transcribed for the purpose of closer analysis, the body language undergoes a similar process. While it was at times challenging to transcribe everything that participants uttered, I had the aid of technology for slowing down the recording; and after all, utterances consist of words and interjections that have been firmly established not only as the units of the English language but also by the research community in discourse analysis. However, transcribing non-verbal activities is more difficult because there are no universally established units for the body language of the participants. It is therefore necessary to outline my approach for this type of transcription.

Similarly to sociologist Hugh Mehan's topically related sets, where "the beginning of each topically related set is signaled by a combination of kinesic [sic], verbal, and paralinguistic behavior" (Cazden, 1983, p. 4), I created *units of activities* for non-verbal communication. Topically related sets are used to sequence the verbal and non-verbal communications that occur with the participants of a lesson in order to further analyze the data (Mehan, 1978; Cazden, 1983). Likewise, I use these activity units to sequence the non-verbal communication that occurs among the participants of the research meeting.

These activity units are based on my interest in diagramming and gesturing during mathematizing and are derived from multiple data viewing, where I noted recurring gestures, actions and postures that seem to be characteristic of what the mathematicians do.

However, it took several attempts of defining and refining these activity units before I achieved a suitable collection that then allowed me to reasonably transcribe the non-verbal communication. I began by describing anything that the mathematician did which had to do with the diagram: e.g., draws, erases, adds to, points to, touches, traces. Then I realized that I also needed to describe what the mathematician did otherwise which had to do with mathematizing or else I would not see a fuller picture of the mathematics unfolding in front of me: e.g., faces mathematikoi versus faces diagram, turns to mathematikoi versus turns to diagram. I started to notice that the physical gap between the blackboard and the tables at which the mathematikoi were mostly seated created a space for the acting mathematician to reside in. The data viewing made me realize that this space is fully used by the mathematician, and that seated mathematicians at times either claim this space by walking into it or reach out into this space by means of gesturing from their seated position. It was therefore necessary that the activity units should not be tied to one particular person because any of the participants performed non-verbal communication with each other and also with the diagram. Furthermore, when the mathematician walks to the blackboard, turns to the diagram or faces the mathematikoi, then these non-verbal activities are noteworthy.

As I was studying the various activity units that I had already gathered, a pattern of activities emerged, namely a pattern of engagement/disengagement and of creation. The creation was solely about the diagram and the engagement could be clustered into engagement with mathematikoi, the blackboard and the diagram. This clustering also helped me to determine if any activity units were missing that would help in refining transcription of the non-verbal activities. For example, I already noted the activity unit (mathematician) *steps away* (from blackboard), when upon closer examination of the proximity between the mathematician and the blackboard I realized that it was necessary to have the activity unit (mathematician) *steps far away* (from blackboard). Before I proceed, I list the activity units for the clusters of creating diagram, engaging with blackboard and engaging with mathematikoi:

- Creating diagram: draws diagram, stops drawing, continues drawing, duo-draw (two or more mathematicians draw diagrams side-by-side at the same time).
- Engaging with blackboard: walks to blackboard, steps away, steps far away, returns to seat, turns away.
- Engaging with mathematikoi: faces mathematikoi, turns to mathematikoi, gestures a mathematical object/action.

Most research meetings centred around digraphs and mathematical operations on them, which caused the drawing of many diagrams; so naturally, the cluster of activity units for the engagement with the diagram was the longest. I then examined these activity units more closely and readily saw that they needed to be separated into two clusters differentiating between *direct* and *material* engagements with the diagram. The activity units that express a direct engagement with the diagram also physically alter the diagram:

- Engaging with diagram directly: erases object, draws over object, adds to diagram, redraws diagram, starts over.

The activity units that express a material engagement with the diagram do so at several levels of intimacy covering distant, close and contact engagements.

- Engaging with diagram in a material sense (distant): turns to diagram, stares at diagram, ignores diagram, discards diagram, points to diagram, hand-points to diagram, tandem-points to object (two or more mathematicians point to the same object at the same time).
- Engaging with diagram in a material sense (close): sweeps object, traces object, caresses diagram, covers up object.
- Engaging with diagram in a material sense (contact): touch-points object, holds object.

Unlike direct engagement which physically alters the diagram, engagement in the material sense may or may not change the diagram physically or virtually. I provide three examples to clarify each case: (1) The tracing of the edges in the diagram makes the mathematician realize that an edge is missing and so he now adds an edge to the diagram. This demonstrates that the material engagement with the diagram can cause a physical alteration of the diagram. (2) The mathematician traces the edges in the diagram while verbalizing how these edges can be projected onto the torus. This demonstrates that the material engagement with the diagram can cause a virtual alteration of the diagram. (3) The mathematician traces the edges in the diagram and the utterances indicate neither a physical nor a virtual change of the diagram.

Table 4.3 contains the complete collection of activity units clustered around creation and engagement. When I transcribed the gestures, actions and postures of the mathematicians using the collection of activity units, I chose that activity unit which best fitted an accompanying utterance (not necessarily the same two participants), so that the transcription tables can be read horizontally as well. My focus during data viewing shifted among participants, which may not be in parallel with the turn taking of spoken words (see transcription tables in Chapter 5).

Table 4-3. Activity units for gestures, actions and postures clustered around engagement with and creation of a diagram

Theme	Activity unit (*object* refers to a mathematical part of the diagram)	Explanation or example when activity unit is not obvious
engaging with blackboard	walks to blackboard	
	steps away	steps away still facing blackboard
	steps far away	steps far away still facing blackboard
	turns away	turns away from blackboard
	returns to seat	
engaging with mathematikoi	faces mathematikoi	
	turns to mathematikoi	
	gestures *a mathematical object/action*	e.g., gestures vertex facing mathematikoi
creating diagram	draws diagram	
	stops drawing	
	continues drawing	
	duo-draw, trio-draw, …	two (three …) mathematicians draw side-by-side either on paper or blackboard
engaging with diagram directly	erases *object*	e.g., erases edge
	draws over *object*	e.g., draws over edge, draws over diagram
	adds to diagram	
	redraws diagram	diagram is drawn again or slightly altered
	starts over	
	turns to diagram	

engaging with diagram in a material sense at several levels of intimacy	stares at diagram	
	ignores diagram	e.g., during discussion, turns to new topic
	discards diagram	mathematician works with diagram, then realizes diagram does not hold needed information, so he turns away from diagram
	points to diagram	index finger is used to point to diagram without touching it
	hand-points to diagram	open hand is used to point to diagram without touching it
	tandem-point to diagram	two mathematicians point to the same *object* at the same time
	sweeps *object*	mathematician performs a broad, dynamic hand-motion over an *object*
	traces *object*	mathematician performs a precise, dynamic hand-motion along an *object*
	covers up *object*	hand(s) and/or arm(s) are used to cover up an *object* in the diagram
	caresses diagram	multiple sweeps
	touch-points *object*	index finger is used to briefly touch an *object* in the diagram
	holds *object*	index finger or hand is used to touch an *object* in the diagram and to remain there for several seconds

The following is a list of facts about the participants that I made use of when recording the activity units, and also of conventions that I followed in the non-verbal activity column:

- In the foursome, Fred (FS), Finn (FG) and Colin (C) are right-handed, while Victor (V) is left-handed.
- Unless otherwise indicated, the activity unit is made with the natural writing arm/hand, if applicable to the gesture, action or posture.
- The capital letter *L* preceding the activity unit for a gesture indicates that the gesture was made with the left hand or left arm. Similarly, the capital letter *R* indicates that the gesture was made with the right hand or right arm, but *L* and *R* are used only if it is unclear which hand or arm is meant.
- When an activity unit is in bold, then it is accompanied by an image shown in the fourth column. Every effort has been made to make these line up horizontally.

4.5.2. Participants' Utterances

For the transcriptions of the participants' utterances, which are shown in the third column of the transcription tables, I applied a mix of Schiffrin's transcription key (1987, p. ix) and Jefferson's transcription system (2004). I chose this mix because the full spectrum of the Jefferson system is not needed in my analysis of the data in the realm of discourse, and would only add unnecessary complexity and reading difficulty. Schiffrin's convention of intonation and stress indicators are chosen over Jefferson's due to their clarity of use and interpretation. The transcription keys and their meanings can be found on the page titled *Transcription Glossary* (see p. xiv).

My data analysis pays particular attention to how language is changing with each non-verbal communication, i.e. talking *at* versus talking *with* the diagram, but also what words and phrases are being said when the diagram clashes with the user or supports the user. I resorted to spatial, temporal and personal deictic words that have already been identified with the basic gesture of pointing as a determinant of speech (Bühler et al., 2011). Educational psychologist Greg Kearsley (1976) points out that question asking plays an important role in the acquisition of knowledge and offers a taxonomy of question words. Since I am interested in how mathematics is understood and invented by mathematicians, I also identified question words. Lastly, I paid attention to modal verbs as a special class of auxiliary verbs. Mathematics education researchers Beth Herbel-Eisenmann and David Wagner (2010, p. 49) emphasize that modal verbs are one of the strongest indicators of personal feelings, attitudes and values of the speaker, and as such contribute to the positioning of the speaker within mathematical discussions. I colour coded these words in the utterance column in order to analyze whether and how diagramming becomes visible through the spoken words. The following list identifies the colour codes that I used and for what purpose:

- Spatial deictic words are identified with a green underline: here, there, this, that, these, those.
- Temporal deictic words are identified with an orange underline: now, then, when, always, first, just
- Personal deictic words are identified with a red underline: I, you, we, it, they, your, them, their

- Question words are identified with a purple underline: what, where, which, who, why, how
- Modal verbs are identified with a blue underline: can, could, will, would, should, may

I employed the software NVivo® and ZoomBrowser EX® to transcribe the verbal and non-verbal communication by the participants. I also used ZoomBrowser EX® to cut the video-recordings to create shorter segments, to create still images from the video-recordings, and to trim and resize images.

4.6. Limitations of Research Method

My study can be considered small-scale because my sample of convenience is only a quartet of one mathematics graduate student and three expert mathematicians. The reason for this limitation is that it is difficult to obtain the consent of graduate students in such a critical part of their career journey. Nonetheless, I was able to follow the graduate student's research meetings with the three expert mathematicians for over three months, which provided me with sufficient data for analysis.

The talking stick metaphor assumes that a conversation is a sequence of main speakers who are separated by the passing of the right to speak. However, there could be instances of no main speaker or multiple main speakers, in which case the talking stick metaphor does not help in drawing information from the data. There could also be an instance of no identifiable *passing* of the talking stick, in which case it may be unclear if the speaker is indeed a main speaker. My data analysis shows that these instances are rare, and that overall, the talking stick operates as a superb organizing metaphor.

Another limitation arises from the fixed position of the video camera, since it was located on a stationary tripod during research meetings. Because the work of the foursome was spread over so many blackboards, I was forced to locate the camera quite far away in the room to capture all six blackboards and four participants. As a result, voices are occasionally not clearly audible in the video-recording, and images of the blackboards are not as sharp as they could otherwise be. Hence there is a risk that I may have missed some significant phrases uttered by one of the participants, and some of the diagrams that are included in my data analysis are quite blurred.

A further limitation of my study is the specificity of the area of mathematics that the graduate student Finn is working in. This area has been chosen because it is accessible to me and my supervisors, and the likelihood of diagrams being drawn is very high. I therefore need to be cautious in generalizing any of my research findings to other areas of mathematics.

Chapter 5.

Revealing the Mathematics and the Graduate Student's Development

My analysis pertains to the following three elements of the research meetings: the mathematics, the enculturation process of the graduate student, and the diagrams and gestures that are actualized. This chapter reveals the first two of these elements: the mathematics of the research meeting, and how the graduate student changes from being a less-expert mathematician to more-expert-like one. The third element, diagramming and gesturing, is examined in Chapter 6. In this chapter, I begin my analysis of the research meetings by providing an overview of the graduate student's area of research, which is followed by a summary of the mathematics covered during each research meeting along with the prevalent moods. Lastly, the evolution of the graduate student is elucidated by describing what, when and how the graduate student contributes to the research meetings and interacts with the expert mathematicians.

5.1. The Mathematics of the Research Meetings

The mathematics of the research meetings is situated in the area of Topological Graph Theory. I first give an overview of the area of mathematics pertaining to the graduate student's research, followed by brief descriptions of the mathematics discussed during each research meeting. These descriptions are based on my own understanding of the mathematics that I have come to know through two consultations with the supervisor prior to the research meetings, background participation during the research meetings (see Section 4.2), as well as multiple data viewing, and verifications of some terminology and diagrams via emails with the graduate student after all research meetings were recorded.

In order to answer Finn's research question (What are the minor, minimal, non-planar, 2-regular, directed graphs?), the research team studied the class of 2-regular directed graphs and how they embed in different surfaces such as the projective plane or torus. The term *directed graph* or *digraph* refers to a graph for which every edge has been given a direction, either entering or leaving a vertex. In a *2-regular* directed graph, every vertex has exactly two edges entering it and exactly two edges leaving it. Therefore, a 2-regular digraph is a graph for which the in-degree and out-degree of every vertex equals two.

Fred was predominantly interested in an operation called *vertex split*, which is simply a deletion of a vertex followed by pairing up the four edges that were incident with the vertex (see Figure 5-1). Through this operation, the initial graph can be reduced to a smaller graph, which is then referred to as a *graph minor* of the initial graph. A graph that cannot be reduced to a smaller graph through a vertex split is called a *minimal* graph. Much of the first two research meetings was about applying vertex split(s) to known graphs and exploring the embedding of the smaller graphs thereby obtained.

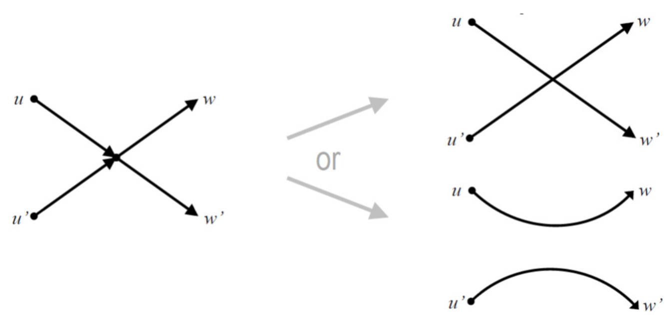


Figure 5-1. Illustration of a vertex split based on supervisor's publication

Furthermore, a smaller graph is called an *immersion* of a larger graph if the smaller graph can be obtained from the larger graph by a sequence of vertex splits. If a class of graphs is *closed* under the operation immersion, i.e. the application of immersion on any member of this class of graphs always produces a member of that same class of graphs, then there is a finite list of *excluded minors* according to the Graph Minor Theorem: a graph can be embedded in a surface if and only if it does not contain an excluded minor as an immersion. In this way, an excluded minor can be viewed as an obstruction for a graph which hinders its embedding in a surface. The Graph Minor Theorem, also known as the Robertson-Seymour theorem, was finally proven in 2004 after a span of 21 years and generalizes Wagner's theorem, which was formulated by the German mathematician Klaus Wagner (1937) and determines the excluded minors (namely the graphs K_5 and $K_{3,3}$) in the special case when the surface for the graph embedding is the plane.

The remainder of the research meetings was about compiling a list of obstructions for the projective plane in particular, although other surfaces such as the torus and Klein bottle were also explored at times. The research goal was to classify these obstructions for 2-regular directed graphs, for various surfaces. Below, I have compiled a list consisting of nine brief synopses of the mathematical explorations during each of the research meetings to provide a general overview. Included in the summaries is a one-sentence description of the dominant moods that I encountered while viewing the video-recordings of the research meetings. These moods show where the mathematics is going and also how the mathematicians themselves are feeling about the progress they are making. I continue to use the reference numbers for the research meetings that I have introduced in Table 4.2 from Section 4.2:

F1: Fred provides a general overview of how to go about researching the minor, minimal, non-planar 2-regular digraphs. It is the general consensus to review the vertex split operation via some familiar examples, to go over the Robertson-Seymour proof of the Graph Minor Theorem and understand how Euler's tour is used there, and to explore the embedding of the half-octahedron in the plane and in the projective plane. This research meeting is marked by alternating moods of excitement and puzzlement interjected with many silences, sometimes several minutes long.

F2: During this research meeting, the foursome come across a graph, which they term *funny octahedron*. In a later correspondence with the graduate student, I found out that they called this graph funny because the way they drew the octahedron in the plane was not in its obvious orientation. Their funny octahedron keeps the foursome busy during the entire research meeting as they apply Euler tours and explore its embedding. It is unclear to them whether the interesting embedding is in the torus, Klein bottle or projective plane. They finally learn that this particular graph is a minimal obstruction for the projective plane. Throughout this research meeting, there is a lot of laughter and excitement, but also doubt and puzzlement, and two moments when an obvious aha-moment occurs.

F3: The theme for this research meeting is to explore embeddings in the projective plane by hand in order to understand how obstructions arise well enough to produce a computer code that will generate the finite list of obstructions. This is the first time that they mention the need to organize their findings. They also introduce new terminology in order to communicate about some of the objects they have discovered and their relationships, which is necessary for describing the embedding process. A considerable amount of their time is taken up by something they refer to as *crossings* of edges, and how these crossings influence the faces they must get in the embedding.

Comments are made several times about the need to be able to describe in words what they are doing. Yet, they diagram heavily and there is a sense of urgency to see what goes on using diagrams. The moods that stand out in this research meeting are evenness, excitement, puzzlement and several long stretches of silence up to six minutes long.

F4: Fred starts this meeting with the words “everything’s actually looking extremely good now, er, but it’s a question of somehow navigating a lot of cases”. Throughout the remainder of this meeting they use the idea of conflict graphs and crossings to explore these cases. Towards the end of the meeting they come to understand that any 2-regular digraph either embeds in the plane or produces a double C_3, which will lead to one of three types of conflicts. They also count the number of crossings possible in a pentagon and a triangle to get a firm grip on the number of cases. For the first time, formal writing about their material and full labelling of diagrams appear intermittently throughout their meeting. The dominant moods of this research meeting are evenness and confidence.

F5: Fred excitedly reports that Finn has worked through all the unsolved cases from the previous research meeting and confirms that all cases behave as predicted. This leads to their first theorem, which is based on four cases of obstruction, two of which they have explored previously and understand fully. Victor confirms that their research is on solid grounds, notices “some kind of duality”, and suggests using “cross caps”. Colin changes topic by raising the question of whether the fourth case of their theorem, the funny octahedron, embeds in the torus. During the majority of the remainder of the research meeting, the mathematicians explore the funny octahedron’s embedding in the torus, and whether the embedding is unique. Towards the end, they begin exploring C_4, but leave a full analysis for the next meeting. This research meeting starts with an even mood that turns to much puzzlement followed by long silences interjected with aha-moments and confidence.

F6: Colin suggests they study one case of their theorem in detail so that they are able to set up a programming procedure. They settle on the funny octahedron’s embedding in the Klein bottle, which Fred draws based on Victor’s previous idea of cross caps, and then they start analyzing its immersions. A discussion between Fred and Finn shifts the focus to the embedding of the funny octahedron in the torus. During this exploration that lasts half the research meeting, first Fred and then Victor claim to have found a theorem regarding their obstructions; however, both of their arguments lead to a disproof. Evenness, silence, puzzlement and hesitation are the dominant moods of this research meeting, interspersed with a few moments of excitement.

F7: This research meeting begins with Victor instead of Fred guiding the mathematikoi. Contrary to all previous research meetings, Victor uses formal mathematical writing on the blackboard to introduce new terminology, notation, graphical objects and one more operation to describe their findings and generalizations thereof. At first, Victor shows all the obstructions that have anti-digons for the Klein bottle, and then he guides the mathematikoi through four propositions and two theorems along with their proofs. Victor concludes with the drawing of one new obstruction from a nonagon for the Klein bottle. This leads to a thirty-minute discussion and exploration of vertex splits and four-edge cuts on the nonagon as well as the orientation of the resulting face-cycles in order to explore the nonagon's embedding in the torus. The research meeting starts with moods of confidence and excitement, and concludes with puzzlement and many stretches of silence.

F8: Only Fred and Finn are present during this research meeting, which begins with Fred asking if Finn has anything to share. Finn has found no new graphs since the last research meeting, but he has thought about Colin's proof and Victor's arguments and asks Fred some questions about them. This leads to Fred's consideration of a 4-regular graph on eight vertices, which he suggests they explore. Most of the time is spent first drawing the complement of this graph and then attempting to find its embedding in the torus. While Fred and Finn work together during the first half of the meeting, Fred becomes absorbed in this exploration on his own. Towards the end of the meeting, Fred explains the method he found to Finn and a theorem that Finn should know about. The meeting ends with Fred outlining how he wants to move forward. The dominant moods of this research meeting are evenness, puzzlement and silence.

F9: Fred's aim for the research meeting is the study of the embeddings in the torus by starting from the double C_3, C_4, K_5 and the funny octahedron. The mathematikoi explore what the obstruction graph looks like in general. They still search for connections among the graphs by considering vertex splits and edge jumps. Towards the end of the meeting, Fred conjectures a hierarchical relationship among the four graphs they have studied in this research meeting. The primary mood is evenness interjected with puzzlement and some long silences, especially in the latter half of the research meeting.

5.2. Evolution of Graduate Student as Mathematician

Since Finn had only arrived about a month prior to the commencement of the research meetings, it is important to investigate how Finn contributes to the research meetings, and how the expert mathematicians are interacting with him, in order to shed light on Finn's journey from less-expert mathematician to expert mathematician. Because Finn had just completed undergraduate studies, he is familiar with mathematical lectures, but not with mathematical research meetings. It is worthwhile to point out the differences between these two venues of learning, in order to highlight that Finn has to reorient his modus operandi in mathematics. I argue that a lecture is mostly scripted, because an organized and effective lecturer prepares the lesson beforehand and maybe even rehearses the lecture before delivering it. Research, on the other hand, is by definition an investigation of a hitherto unsolved problem, and while there is typically a starting point, how to move forward from such a starting point is driven by improvisations. Furthermore, the lecturer draws the students into the world of the lesson, while the participants of a research meeting share the common world of explorations. Since the mathematical explorations of the research meeting are new to all participants, the graduate student's trajectory cannot be viewed from less-expert to expert in terms of mathematical knowledge, but rather in terms of enculturation into mathematical research. This mathematical research culture encompasses, for example, how to participate, when and how to speak, when and what to gesture, how to act, and what body language to use.

The following sections outline chronologically the various roles that the graduate student Finn takes on during the research meetings. While the focus is on Finn, I sometimes present the expert mathematicians as well, in order to bring out similarities and differences between the mathematizing and diagramming of the less-expert and expert mathematicians. I highlight Finn's unsuccessful as well as successful attempts at joining the discussions and also draw attention to the manner in which the expert mathematicians interact with Finn. During these exchanges, I pay particular attention to the gestures made by the less-expert and the expert mathematicians in order to tease out what role these gestures play in the culture of mathematical research, and what differences there are between the way less-expert and expert mathematicians interact with the diagram. These close-up examinations provide insights into how mathematical invention emerges through diagramming within the less-expert and expert mathematicians.

Prior to the research meetings, Finn's supervisor Fred appoints Finn to the role of note taker. During the first two research meetings Fred and the research colleague Colin occasionally turn to Finn asking him if he has taken down a particular diagram; however, it becomes apparent that Finn is a very good note taker since his notes are sometimes referenced and so these questions subside.

5.2.1. Research Meeting 1 (1:04:57): No Participation

During this first research meeting, the mathematikoi create a common research ground for their study of the minor, minimal, non-planar 2-regular digraphs, by working through applications of the operation vertex split and reviewing the Graph Minor Theorem and its proof. Supervisor Fred quickly establishes himself as the writer and drawer on the blackboard, while Finn, Colin and Victor are seated facing the blackboard, as shown in Figure 5-2. None of the seated mathematikoi come up to the blackboard during this first research meeting. The talking stick is shared between Fred and Colin with an occasional interjection by Victor, but Finn never contributes verbally or non-verbally. There is one further colleague in attendance at just this research meeting. Throughout the meeting, he asks only two questions to seek clarification in notation and application of the vertex split, and otherwise stays in the background.

Figure 5-2. F1-v1 28:03, Fred is at the blackboard, Finn (middle) and Colin (right) look at blackboard, Victor is out of camera view

5.2.2. Research Meeting 2 (1:00:30): First Attempts at Participation

For the first time, Finn points a half-raised hand towards the diagram and holds his pointing gesture for 3 seconds before lowering his hand (see Figure 5-3). However, Finn does not accompany his pointing gesture with any verbal utterance indicating that he has something to offer. All the while, Victor, who sits next to him, bends sideways – most likely to get a better look at the diagram on the blackboard – and Fred and Colin are engaged with the diagram on the blackboard facing away from both Finn and Victor (see Figure 5-3). Finn's attempt at joining the verbal communication is meek and not acknowledged by any of the expert mathematicians.

Figure 5-3. F2-v1 35:45, Finn's unacknowledged pointing gesture

A few minutes after Finn's unsuccessful attempt at joining the discussion, it is clear from the conversation between the three expert mathematicians that they have arrived at an impasse (see Table 5-1 for transcription).

Table 5-1. Transcription of Foursome F2-v1, 36:14-36:32

Time	Participants' non-verbal activity	Participants' utterances	Images
36:14	**FS, C, V, FG stare at diagram** C steps away C turns to V C turns to diagram FS steps away **C turns away from blackboard** FS turns away from blackboard	FS: Something is funny! No, I don't think (4.4) C: It does not look right, right? ((V shakes head as if to indicate 'no')) FS: No idea. (3.9) C: But maybe it's a different one. So (2.8) let's see what can be a different one?	 

Finn bends over his own paper and engages with the material there (see first image in Table 5-2). Even though the mathematicians are attempting another approach to the problem, Finn is ignoring them for 37 seconds before he raises his head to look at what Fred does on the blackboard. In the meantime, Colin has taken his seat next to Finn again and is now voicing the suggestion to perform a face count, which Fred is eagerly taking up on the blackboard. Finn continues to bend over his own paper and is clearly working on something there. After 5 seconds Colin says "let me draw it on paper" and also bends over his paper. After a further 13 seconds Colin murmurs "where to begin?" and leaning over to Finn looks at Finn's paper. Finn now turns to Colin, shows him his work and both engage in a short, barely audible side conversation (see Table 5-2 for transcription).

Table 5-2. Transcription of Foursome F2-v1, 37:00-37:45

Time	Participants' non-verbal activity	Participants' utterances	Images
37:00	**FG ignores mathematikoi** FS draws diagram V draws dia. on paper **C turns to FG's diagram** FG points to diagram FG points to diagram FG points to diagram C turns to FS	C: (5.0) Let me draw it on paper. V: Yah! FS: So, I'm just going to draw it in a big way here. (8.0) C: Where to begin? (3.5) You found, you found a shorter cycle in this? FG: There (.) C: Oh, there *are* triangles= FG: =there is the yellow triangle. C: And then *this* is a triangle. Isn't it? Right. FG: Yes! Mhm (.) And you have these triangles. C: And these are the only triangles? And then the others are some four cycles. Right? FS: So? Sorry, what's happening?	 

At last, Fred becomes aware of some part of the side conversation between Colin and Finn and wants to know what is going on. Colin then faces Fred and argues as before through the face count. Colin conjectures that there need to be four triangles and wonders why he sees only two of them. This then spurs another attempt at finding an embedding with four directed 3-cycles and two directed 4-cycles.

During the remainder of the research meeting, there are no more side conversations but Finn, Colin and Victor trio-draw side by side on paper. While Fred is not captured on the video-recording in the beginning, the chalk-scratching sounds lead me to conclude that even Finn draws – albeit on the blackboard. This mixed-media quartet-drawing continues in silence and lasts for over four minutes, during which time I adjusted the view window of the camera to alternately capture Fred working on the blackboard (see Figure 5-4, left) and the trio working on paper (see Figure 5-4, right).

Figure 5-4. F2-v1 42:29, Fred draws diagram on blackboard (left); F2-v1 40:17, Victor, Finn and Colin trio-draw diagram on paper (right)

5.2.3. Research Meeting 3 (1:36:42): Intermittent Participation

In the previous two meetings, occasional side remarks were made to Finn by either Fred or Colin along the lines of taking notes to capture their findings, but mostly the acting mathematician directed his talk at everyone. This meeting starts out with Colin diagramming and providing an approximately two-minute summary of their understanding so far, which is almost entirely directed at Finn (see Figure 5-5, left). Then Fred interjects and emphasizes for about thirty seconds the need to come up with a sensible way of keeping track of their findings. Still directing his talk at Finn, Colin continues summarizing for about another four minutes, describing the process of embedding in the projective plane and what an obstruction may look like, before he is joined by both Fred and Victor discussing how best to construct the faces of the embedding.

Figure 5-5. F3-v1 01:27, Colin faces Finn while he sweeps diagram (left); F3-v1 08:35, Fred reports findings prior to research meeting (right)

The summaries are followed by Fred reporting on a discussion Fred and Finn had in their office prior to this research meeting about these obstructions and the newly created terminology *face sets* and *neighbourly* to talk about aspects of the embedding (see Figure 5-5, right). Yet, Finn never offers anything towards this report nor is he encouraged to contribute.

Figure 5-6. F3-v1 14:17, Finn points upwards similar to a student wanting to contribute (left); F3-v1 14:18, Finn points to diagram (right)

About 14 minutes into the research meeting, during a heated discussion between Fred and Colin, Finn first points upward like a student wanting to contribute (see Figure 5-6, left) and then extends his arm pointing at a diagram (see Figure 5-6, right). Finn points for 3 seconds before retrieving the pointing gesture as he is not acknowledged by any of the other three mathematikoi. Finn gets another opportunity to contribute when a pause occurs in the debate about 38 seconds later. This time Finn verbally interjects and gets the attention of Fred and Colin, who erupt in loud, overlapping talk (see Table 5-3 for transcription).

Table 5-3. Transcription of Foursome F3-v1, 14:54-15:21

Time	Participants' non-verbal activity	Participants' utterances	Images
14:54	**FG points to diagram** FS steps away **FS, C turn to FG** **FS, C turn to diagram** FS turns to FG FS turns to diagram FS attempts to move to blackboard FS walks to blackboard **FS, C tandem-point to diagram** FS stares at diagram C points at diagram FS turns away, *L* points at diagram FS walks far away	FG: I er, I I don't think the middle drawing goes to c four. (.) I, I think it's= FS: =Uh? FG: I think it's ((almost immediately both mathematicians erupt in speech simultaneously)) C: [No, no, the middle drawing has a local edge.] FS: [No! This is fine, this is fine.] C: [( ) That's the] local edge. (.) FS: [Yah! Yah!] C: [That's the] (.) FS: [Yah!] C: That's the local edge (.) in below. So, yah, in below. Yah! FS: Yah, this one, we understand. C: That's fine!	  

	V walks to blackboard FG returns to seat	FS: That one's totally cool. You can embed that one. Right! We all agree with that one, yah. (3.0)	
15:17	**V *R* touch-points *edge*** **FS walks to blackboard** C points to diagram V returns to seat	V: That one and that one= FS: =Oh, it does say it goes to c-four. Sorry?= C: =We all agree whether () the third one= V: =( ) the drawing changed.	

During the loud eruption of talk by Fred and Colin, the graduate student's voice is no longer heard. Although Finn moves out of the camera's view when he gets up, he never makes it up to the blackboard (the diagram under consideration is in the view window of the camera). After Fred finishes uttering "we all agree with that one, yah", silence ensues during which Victor walks to the blackboard. Victor first points to the label above the diagram and then touch-points the diagram while he says "that one and that one […] the drawing changed" (see Table 5-3 for transcription). This shows everyone that the label of the diagram is no longer correct, and all four of them end up in agreement with what the diagram now represents. As a last action, Fred corrects the diagram's label.

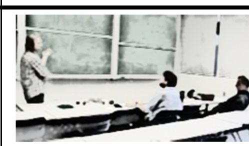
This is a first attempt by Finn to get up and walk to the blackboard but he never makes it to the blackboard when Fred and Colin burst out talking. It is clear from the conversation that, although Fred and Colin are stumped in understanding what goes on mathematically, they are in agreement with what the diagram represents. In other words, their mathematizing is not hindered by the sketchily-drawn diagram, which initially represents one mathematical object with a correct label and then by an added stroke represents another mathematical object where the label becomes incorrect. This is not the case for the graduate student. Finn's attention is still on the mislabelled diagram, and he is not following the mathematical arguments as his verbal interjection is clearly disruptive to the mathematical conversation that is in progress.

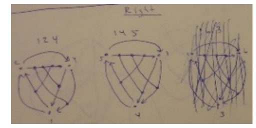
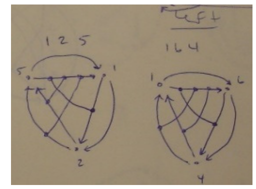
In a later discussion, Fred addresses Finn directly by asking if the diagram on the blackboard is what they call the funny octahedron. Finn points to the diagram and says "yah, and there are others, I have some others", to which Fred responds "oh, really? Okay! ((chuckles)) Hm, maybe saying in words how you ((chuckles))" and moves on to something else, ignoring for the second time what Finn has to offer.

After about an hour of working together on finding obstructions, the mathematikoi get stuck about how to see these obstructions in the embedding. This is followed by mostly silent attempts to draw topological graphs and their embeddings that may lead to an obstruction. The few utterances that occur never draw in the entire group and are mostly brief exchanges in pairs or mutterings to themselves. Both Fred and Victor work on the blackboard either alternately or duo-drawing. Neither of these actions attracts the interest of the other mathematicians. Finn mostly observes the mathematician at the blackboard or seems to follow what little exchange there is, but every now and then he bends over his paper and seems to be drawing there as well. Colin is out of the camera's view window.

After about 23 minutes of this non-cohesive work among the mathematikoi, Colin speaks into the silence asking Finn: "So, [Finn], you have uh (.) if you have this things? So this and the, the, the (.) you get the funny octahedron in there? Or?" To which Finn replies "yes, so, with one, with one jump the funny octahedron." This starts a two-minute discussion among all four mathematikoi about which one among their drawings leads to an obstruction. During this group talk the view window of the camera is adjusted so that Colin is now in view as well. Finn's utterances do not seem to offer anything insightful to the mathematicians and Colin soon turns to Victor instead and ignores Finn. Finn tries again by pointing to the diagram and saying "if we could do the same argument and get down to one jump". Victor gets up, moves to the blackboard, points to one diagram, then sweeps with the other hand over to another diagram and responds with "you can provided things are sufficiently (.) *distinct*". He then talks about one, two and three crossings and ends with "I don't know how to make that more precise" and at the same time makes a dismissive gesture with both arms. After a bit of silence Fred starts up the discussion again. During this time, Finn contributes to the discussion and offers his findings of immersing the funny octahedron (see Table 5-4 for transcription).

Table 5-4. Transcription of Foursome F3-v2, 24:38-27:28

Time	Participants' non-verbal activity	Participants' utterances	Images
24:38	**FS faces mathematikoi** FS turns to FG diagram	FS: "Maybe if, I mean, if crossings help to nail things down, maybe we should try to *get* to crossings". FG: So these are the ones here= FS: =Oh, yah.=	

	C turns to FG V walks to FG **FS, C, V, FS stare at diagram** V turns away C turns away V, C ignore FS, FG FG gestures *left turn* V walks to FG diagram FG traces diagram V walks away FG sweeps diagram FG *L* gestures *rotates* ((C, V have a side conversation)) ((FS had stared at FG diagram all this time)) FS walks to blackboard FS walks to FG diagram FS walks to blackboard	FG: =besides the jumps. (3.1) FS: So these, these are all the ones, so you took funny octahedron and you found all the immersions= FG: =Mhm.= FS: =of c three in it. FG: Yes! FS: And these are all the bad things that you can get. FG: Yah. FS: That's good. FG: So, it's sort of the same picture, I mean. FS: What's the right and left here? FG: Oh, I was just, uh, I was just assuming two of the edges were right= FS: =Oh, I see.= FG: =in the case when you want a cross (3.5) and now it just sort of rotates. (3.1) It seems to be even. I guess for the even there is only one, cause (3.2) if you look here, everything seems to be while in these pictures that's the same. (7.1) FS: I see c three one here. (3.2) Oh! Those are cool! (3.9) I see! Those are very nice. Yah, yah, I know, I, I, er. (.) Yah, so, these guys sort of tell us (.) um. (10.5) Right, so. (1.8) What's happening there, yah, I just wanna try to interpret this one. So, (3.0) what's happening here is (.) you've got (.) uh (1.2)	  
26:28	C turns to FG diagram V walks to FG diagram C, V, FS stare at diagram V walks away	C: Can I see this? So you started with splitting this ( ) in all possible ways? FG: Yes! I split, I split in most possible ways. And then I was, I was getting a bunch of repeats. C: And then you say these are the ( )= FG: =Yah. But I think, I think it is even fewer than these.= FS: =Yah, it's (.) this is the (2.8) FG: Yah. Like so, if you take these two (.) I was saying these two are distinct. But there are actually if you interchange them here (.) however many ( ). You just switch them. You get this picture here. ( ) These are actually the same. (.) And these are the same, (2.4) yah.	

	C *L* points to diagram on blackboard	C: So my, my question is, do you always get two answers like (.) so? (2.3) That would be (2.4) probably helping in the proof because we can then try to find this with two edges and subdivide them and then this definitely would be the number of cases.	

All of the above episodes demonstrate that during the 1:36:42 hour long research meeting, it takes Finn six attempts to draw attention to his work before his diagrams are acknowledged as contributing to the mathematikoi's understanding of obstructions. Finn exhibits misunderstanding of the diagrams on the blackboard and he is hesitant in explaining his diagrams, which are drawn differently from the expert mathematicians' diagrams. For the first time, Fred and Colin are interested in Finn's work, spend time in understanding what he has drawn, and offer praise: Fred exclaims "that's good" and "those are cool", while Colin even conjectures "that would be (2.4) probably helping in the proof".

5.2.4. Research Meeting 4 (0:59:50): Supervisory Influence

In addition to the research team, Fred's other graduate student is in attendance, but he mostly stays quietly in the background. About 20 minutes into this research meeting, Fred starts another discussion with the phrase "so what I'm claiming now is, if you take this picture" and then proceeds to point to existing diagrams on the blackboard and draws a new diagram while making utterances. However, none of these utterances refers to what exactly it is he is claiming. During the drawing Fred hesitates and Colin points out that there are actually two ways of drawing this, which Fred readily agrees with. Fred then traces over the diagram and verbally walks through the two cases muttering "I have forgotten the details". When he is finished, Colin, who is seated next to Finn, turns to Finn, points to Fred's diagram and asks "[Finn], you, you looked at this too?" (see Figure 5-7). Without letting Finn speak, Fred immediately responds to this question with "Well, I didn't tell him. I told him to look at other stuff. This part was my homework. […] We can look at that stuff later." To which Colin replies "Okay. (.) Yah, yah. Okay." Fred then completes the drawing and again argues verbally through the two cases; however, he does so in a hand-waving manner that relies on his experience in the field but not on hard facts that he has researched and prepared. Then, 1:24 minutes after his first utterance about making a claim, Fred says, “let me leave this as a claim for now,” and finally puts the claim in writing on the blackboard for all to see.

Figure 5-7. F4-v1 21:17, Colin points at diagram and turns to Finn

This exchange between Fred and Colin makes visible the underlying hierarchy among the mathematicians. Fred as the supervisor controls what tasks are given to Finn, but he also controls Finn’s contributions to the research meetings by allowing or blocking his speech. Colin also has a vested interest in the mathematics they develop, and when Fred could not deliver, Colin wants to find out what the graduate student Finn has been working on, just as he asked him in the previous meeting (see Section 5.2.3) and found out that Finn had an interesting perspective on working through the different cases of obstructions. However, this time, Finn was not able to respond, because Fred’s immediate interjection shuts down both Finn and Colin. Up until now, I observed only a few hesitant attempts from Finn to claim the talking stick and not all of these attempts were successful. However, I know from the remarks that his supervisor Fred has been making during the research meetings that Finn works quite diligently and successfully at drawing some of the obstructions they are looking for, as well as the embeddings they are studying, as noted in Sub-Section 5.4.3. I am therefore unsure whether Finn has perhaps worked on this material on his own, but does not want to speak up against his supervisor, or if Finn really has not worked on these cases.

A few minutes after the above incident, Fred suggests that these two cases “actually maybe should be our first, our first sort of sanity check”, meaning an initial approach for the programming scheme. Victor, who sits farthest back right behind Finn and Colin, then grabs the talking stick. When he refers to the chords in the diagram, he gestures with both arms extended upwards (see Figure 5-8, left). This is a huge gesture and immediately draws Fred’s attention and stops him from talking. Not only does Victor verbally point out that “this little loopy guy” – meaning an edge – in between these two chords creates conflict with embedded faces, but he also points out this conflict physically with his left arm. During this 19-second long massive gesturing by Victor, Fred faces Victor and makes frequent interjections “mhm” and once even bursts out “ah” and “that’s right”, Colin turns around to face Victor and once nods in agreement, but Finn never once turns around to look at Victor (see Figure 5-8, right). This provides an example of an opportunity lost for the graduate student to receive gesturing clues that accompany verbal explanations. Throughout the research meetings, all three expert mathematicians habitually use bodily enactments of mathematical objects during discussions. Thus far, Finn has not exhibited such gesturing, which may be because he has not held the talking stick many times and when he has been talking, it was always for very brief moments or with just one of the three expert mathematicians as described in the previous research meetings. Furthermore, his closed body posture is in stark contrast to the relaxed and open body postures of the three expert mathematicians.

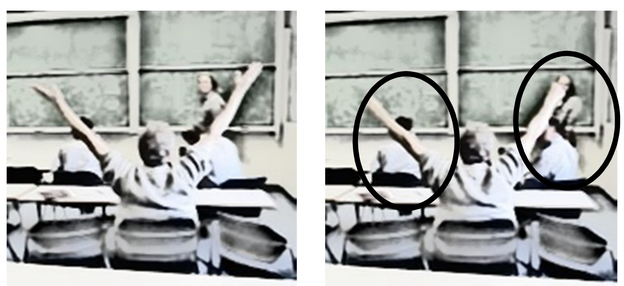


Figure 5-8. F4-v1 24:00, Victor gestures *two chords* (left); F4-v1 24:11, Fred and Colin face Victor, while Finn never does (right)

Half-way through the research meeting, Fred is reminded through his diagramming that this is something that Finn had researched and he reports on Finn’s findings. Victor wants to hear more, and so Finn gets a brief moment to contribute, but immediately Fred steps in again and claims the talking stick back to direct the discussion towards the next item (see Table 5-5 for a transcription). Here again, Fred exhibits his supervisory control over Finn and shuts down Finn’s chance at joining the discussion. This takes away an opportunity for Finn to practice talking among expert mathematicians and to learn how to communicate with them.

Table 5-5. Transcription of Foursome F4-v1, 26:23-27:03

Time	Participants' non-verbal activity	Participants' utterances	Images
26:23	FS*L* points FS walks to diagram **FS *R,L* holds two edges** FS steps away FS sweeps chord FS faces mathematikoi **FG turns to C** FG turns to FS FS gestures *those*	FS: Oh, uh, anyway. So that's, that's the first thing that [Finn] tested. So [Finn] looked at all (.) any two edges that are not on the same face. So if you have two edges that are not on the same face, you add a, a, a, you add a chord between them. What happens? And, er= V: =And what happened? FG: [c-four] C: [Funny] octahedron? FG: Yah. C: four-c-four, four-c-four.= FG: Yah. FS: =So.= V: =So, four-c-four. [Right.] FS: [So] every single time those give us= V: =Yah.= FS: =one of our, one of our two. (.) And, um (.) yah, and the good news here is that I (.) think that, er, well. Right! I mean anyway if you have, if you have this kind of configuration, then we can reroute to get that chord.	 

Just a few minutes later, Finn claims the talking stick, when there is a pause in Fred's summary. At first it seems that he is being shut down by the visiting colleague Victor this time; however, supervisor Fred speaks up in support of Finn's claim and explains what it means. Finn then continues in the discussion and elaborates on what he has found out. This prompts Victor to offer a guess as to what these findings mean, which is again accompanied by extensive gesturing. This exchange is transcribed in Table 5-6 and marks the first time that Finn converses on equal terms with all of the expert mathematicians for nearly half a minute: he points at the diagram; his body posture opens up to both Fred and Victor when they are talking; he faces Victor and observes the elaborate bodily enactments of mathematics objects; and he vigorously nods his head in agreement along with Colin. However, being mathematically enculturated also entails speaking like a mathematician. Of the 21 spatial deictic terms (underlined in green in Table 5-6) and 16 mathematical nouns (written in turquoise in Table 5-6) that are being used in this exchange, the ratio of spatial deictic terms to mathematical nouns is 2:4 for Finn, 14:6 for

Fred and 5:6 for Victor. Through the employment of deictic terms, Fred and Victor are able to converse informally and to some extent vaguely about the mathematics. Their exploration is not hindered by using precise terms that may not even have meaning yet in the realm of exploration. By contrast, Finn's talk is still specific, marked by twice as many mathematical nouns as spatial deictic terms.

Table 5-6. Transcription of Foursome F4-v1, 31:12-32:24

Time	Participants' non-verbal activity	Participants' utterances	Images
31:12	FS faces mathematikoi FS points to diagram FS sweeps diagram	FS: This, this thing doesn't embed. Right, I mean? I already know this is *the* obstruction.	
		V: Yah.	
	FS gestures *family* FS sweeps diagram FS gestures *plus* FS traces *chords* FS gestures *not contain* FS steps far away	FS: So, so in fact, we, we know at this point that all of our, we, we know that this whole family of excluded immersions are the ones we know, plus possibly anything that you get by adding exactly two chords like this, that don't, that doesn't contain one of the ones we've got. Right?	
		V: Yah.	
	FG *L* points to diagram	FS: Uh, I mean, that's er that's the whole list and it's only seven vertices. (.) Uh. (.)	
	FS faces FG	FG: That one (.) is octahedron.	
	FS turns to diagram	FS: Oh! ((chuckles)) very good. So ((chuckles))	
	FS turns to V	V: Uh, no it isn't.	
	FS hand-points to dia.	FG: No?	
	FG turns to V	FS: Well, well no, I think it's gonna *lead* [up to it.]	
		V: [Oh, it's] gonna lead up to it.	
	FS turns to diagram	FS: It's gonna be *fine*. [We split one of these.]	
		FG: [Yah, you split] one of the vertices.=	
	FS turns to FG	V: =One of the vertices. Yah. You are gonna get that funny octahedron.	
	FG faces FS,C,V	FS: Um. (.)	
	FG points to diagram V *R* hand-points to dia.	FG: But, what I'm sort of (.) sometimes it goes to funny octahedron and sometimes it goes to c four? Of anyone of [these guys?]	
	V gestures C_3^2 V gestures *put* **FG faces V**	V: [Yah.] That's where your math there is a whole bunch of those c three squares sittin in here. And any time you put	
	V gestures *edge*	FS: [That's right]	
	V gestures *parallel* FG faces FS	V: [an edge] between two parallel guys in the c three three they give c four one.	
		FS: *Oh*, maybe? That's a (.) yah, that's a good comment.	

At 33:40 minutes of this meeting, Fred once more summarizes their findings, namely that immersing 2-regular digraphs is reduced to three main cases, of which two have now been explored and the last case “is really the only one to test”. During this summary, Fred turns to Finn to let him successfully fill in what one of the sub-cases of their last case leads to (see time interval 34:12-34:18 of Table 5-7 for a transcription), and Finn also attempts a second contribution a few seconds later, but is unable to find the place in his notes and is ignored by the three expert mathematicians (see time interval 35:05-35:18 of Table 5-7 for a transcription).

Table 5-7. Transcription of Foursome F4-v1, four brief time intervals within 33:40-44:26

Time	Participants’ non-verbal activity	Participants’ utterances	Images
34:12 34:18	FS turns to FG FS turns to diagram	FS: If they do cross that was something (.) FG: That is funny octahedron. FS: That’s funny octahedron. FG: Yah. FS: Okay.	
35:05 35:18	FS faces mathematikoi FS turns to FG FS turns to diagram	FS: I think it’s actually an octahedron again. (3) FG: I think I did that (.) but what did I do with that? (3.5) V: It embeds. FS: Yah. It does, it does embed.	
36:59 37:27	FS faces mathematikoi V *R* points to diagram FS turns to diagram **FG *L* points to dia.** FS walks to blackboard FG still *L* points FG still *L* points FS touch-points vertices	FS: Yah, this is actually the planar one. (.) V: Yah, but it *can’t* be, because that is the immersion of the= C: =something is wrong! FS: Oh. (.) FG: This little, this little (.) a does not go to (.) z. FS: Oh. FG: z goes to a. FS: Oh. FG: The bottom triangle azc. FS: a z c, so you are telling me (.) oh! z goes to a. Ah!	
43:13 43:29	FS faces diagram FS turns to FG FS turns to blackboard	FS: This goes to a, and now what? FG: We are jumping three ahead instead of jumping two ahead. FS: Yah! So this is a slightly different, I mean (.) this might be (.) C: a goes to (.) oh FS: I suggest we take the other pentagon. (.) I want to try the other pentagon. C: Yes, yes.	

During the next 10 minutes, the mathematikoi explore this last case, which marks one of the longest time periods spent on one particular diagram. However, as the expert mathematicians repeatedly remark throughout this meeting, they “don’t quite see it” (Fred, 42:03 minutes) and they need to “make a new picture” (Colin, 42:11 minutes) before employing a computing program. During the 10-minute time interval, Fred draws the immersion of their case three times, each time using labels and multiple gestures of pointing, touch-pointing, holding and tracing. The first time Fred says “let me just draw it”, and his diagramming, directed by both Colin and Victor, takes 1:30 minutes while Finn observes. When the immersion is complete, Fred remarks that this is the planar immersion, but Victor and Colin reject this statement and Finn points out an error in the diagram (see time interval 36:59-37:27 of Table 5-7 for a transcription). Fred then says “shall we just try this again?”, and Fred’s second diagramming lasts 1:22 minutes with Colin and Victor even more carefully directing Fred. However, this time Finn ignores them and is clearly engaged with his own diagramming (see Figure 5-9). When the mathematikoi try to understand what the new diagram tells them about the immersion, Colin succeeds in seeing C5 in the immersion, but Fred says “I don’t quite see it”. Colin encourages Fred to “make a new picture,” so Fred begins a third time drawing the immersion but even Finn’s comment about the jumps that occur does not help Fred understand any better, and Fred gives up on this diagram (see time interval 43:13-43:29 of Table 5-7 for a transcription). Fred then proceeds diagramming completely on his own for 0:46 minutes. This fourth time, the diagramming is successful and they all agree on what their last case says about immersion.

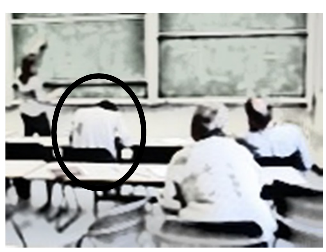


Figure 5-9. F4-v1 39:17, Fred draws while Colin and Victor direct the drawing, but Finn ignores them and draws on his own

During the 11-minute time interval from Fred’s summary to the end of the fourth diagramming, Finn openly participates four times. While his gestures are still restricted to pointing, compared to the previous three research meetings Finn is more frequently speaking up on his own throughout this research meeting, even though not all of his contributions are yet successful. Towards the end of the research meeting, Finn chimes in once more with an answer during Fred's counting of possible crossings in a pentagon and a triangle respectively. At times the supervisor Fred controls Finn's contribution by either not acknowledging what Finn utters or by encouraging Finn to provide an answer to a question. It is noticeable that Fred does most of the diagramming; however, the other two expert mathematicians freely come up to the blackboard to join in on the diagramming or to seize the talking stick from Fred, while Finn either observes the diagramming or withdraws from the expert mathematicians' diagramming for a small period of time (37 seconds is the largest measured interval) and draws on his own on the paper in front of him. These acts are not note-taking and are therefore significant as actions that the graduate student is performing, which will be highlighted in the next section when it is particularly relevant.

5.2.5. Research Meeting 5 (1:43:28): Notebook Reveals Diagramming

Again Fred starts the research meeting by reporting on Finn's work based on the results from the previous research meeting. Apparently their findings have now resulted in a theorem, which Fred spends two minutes writing up on the blackboard providing small diagrams for the cases (see Figure 5-10). Since the supervisor is steadfastly holding on to the role of reporter, opportunities are taken away from Finn to speak in front of the mathematikoi. As pointed out in Section 2.2, mathematizing is entwined in the lived experiences of reading, writing, speaking and doing mathematics. Finn's chances of speaking are increasing over the course of the research meetings; however, the supervisor is not paying specific attention to and fostering Finn's development in this regard. Finn's notebook brings to light that when he copies work from the blackboard the copies are faithfully rendered aside from vertical or horizontal rearrangements. So the role of note-taker befits Finn. Interestingly, Finn's notebook also reveals his own, private diagramming and questions he raises about the material of the research meetings, which I highlight in this section.

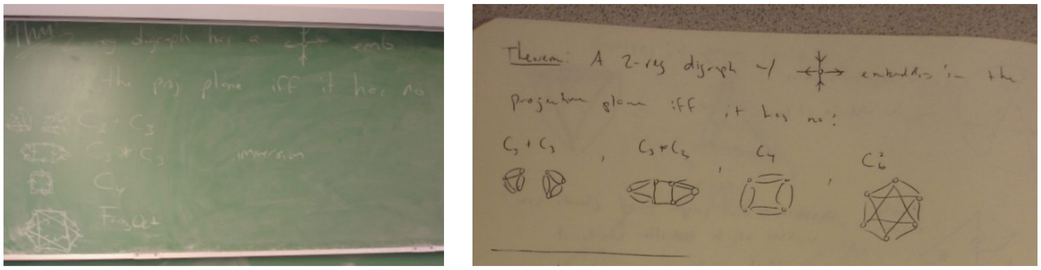


Figure 5-10. F5-i02, Fred’s theorem on the blackboard (left); excerpt from F5-ng1, Finn’s copy of Fred’s theorem in his notebook (right)

After writing the theorem on the blackboard, Fred pauses, says “proof”, pauses again, but instead of proceeding to the proof, he opens up a discussion about automation. As soon as Fred hesitates, Finn points to the theorem and offers what he knows about automatic generation of graphs (see Table 5-8 for a transcription). When Finn says the word “circular”, his outstretched hand that is still holding his pen is making a small circular gesture. This marks the first time that I observe Finn making a mathematical gesture. When Finn finishes his contribution, Fred tries to find words but pauses many times, so Victor grabs the talking stick to report on his own work thereby ignoring Finn’s contribution. Victor spends a little more than a minute recounting that he has gone through all of their arguments and that he is convinced that these arguments are mathematically sound, but the different use of vertices is awkward: “there is nothing *wrong* with that, but doesn’t somehow quite seem balanced”. During this narration, Finn chimes in “there is one weird thing,” but he is ignored once more.

Table 5-8. Transcription of Foursome F5-v1, 02:47-03:22

Time	Participants’ non-verbal activity	Participants’ utterances	Images
02:47	FS faces mathematikoi FS gestures *automate* FS turns to FG **FG points to theorem and gestures *circular*** FS turns to blackboard	FS: I’m kind of intrigued about the possibility of trying to automate this (.) and see what happens. V: Yah. FS: Um (.) FG: Er, I, I’ve heard of a program (.) so far it, it just generates these circular graphs= FS: =Oh.= FG: =it jumps to a different size. FS: Okay. Mhm.	

	FS turns to V	FG: So I don’t know the particulars and how it solves, it just jumps to a different size. FS: Um. I, I mean, er= V: =Er, sorry. To continue with the er summary. (.) Er, er, I went back last week, and er (.) looked sort of over the er whole structure.	

At 8:57-minutes into the research meeting, Fred addresses Finn directly with “so, you know this, this is about sort of adding a cross cap?” and proceeds to verbalize, draw (see Figure 5-11, left) and gesture this type of mathematics for Finn, all the while ignoring the other mathematicians in the room. There are two statements that Fred makes to Finn that are illuminating about how the supervisor is directing the graduate student’s mathematizing: “what [Victor] is observing is, another way, like, there’s a lot of ways to think about these funny objects”, and “more generally, that’s an operation you can do to fix a triangle”. Other than “mhm”, Finn does not utter any other words during Fred’s 4:27-minutes long explanations, but faces him observantly or makes notes. His notebook reveals that he is deeply engaged with the material presented by Fred, because he draws his own diagrams, writes down a brief note, and raises one question to himself “3 edges though?” (see Figure 5-11, right).

Figure 5-11. F5-i01, Fred’s diagrams (left) in order they were drawn: cross cap (blackboard right), embedding that goes wrong at the inner six vertices (blackboard left), usual octahedron with two opposite faces in red reversed to produce funny octahedron (blackboard middle); excerpt from F5-ng1, Finn’s version of Fred’s diagrams and a question to himself (right)

During the next 17 or so minutes of the research meeting, the mathematikoi explore whether the funny octahedron embeds in the torus, which then leads to analyzing the symmetries of the embedding. Most of the diagramming on the blackboard is done by

Fred (see Figure 5-12, left), but both Colin and Victor alternately or together get up from their seats and join Fred at the blackboard. Once again, Finn does not leave his seat to join any of the expert mathematicians at the blackboard; instead, Finn silently observes and also attempts the embedding on his own, as his notes disclose (see Figure 5-12, right). None of the 3-, 4- and 6-cycles was graphed by any of the other mathematikoi on the blackboard, but Finn graphed them for his embedding. Finn's diagram on the left in the top notebook excerpt sets the 6-cycle in the centre rather than the 3-cycle. It would be interesting to know if Finn's diagram on the bottom notebook excerpt is a copy of Fred's diagram from the blackboard or his own embedding. Unfortunately, when I inquired a year later, Finn could not remember. The agreement in directionality and labelling between Finn's and Fred's diagrams makes me believe that Finn simply copied Fred's diagram, but I hesitate to claim this outright, because of the asymmetric rendering without paying attention to the square-ness of three of the faces in light of Finn's usually faithful note taking.

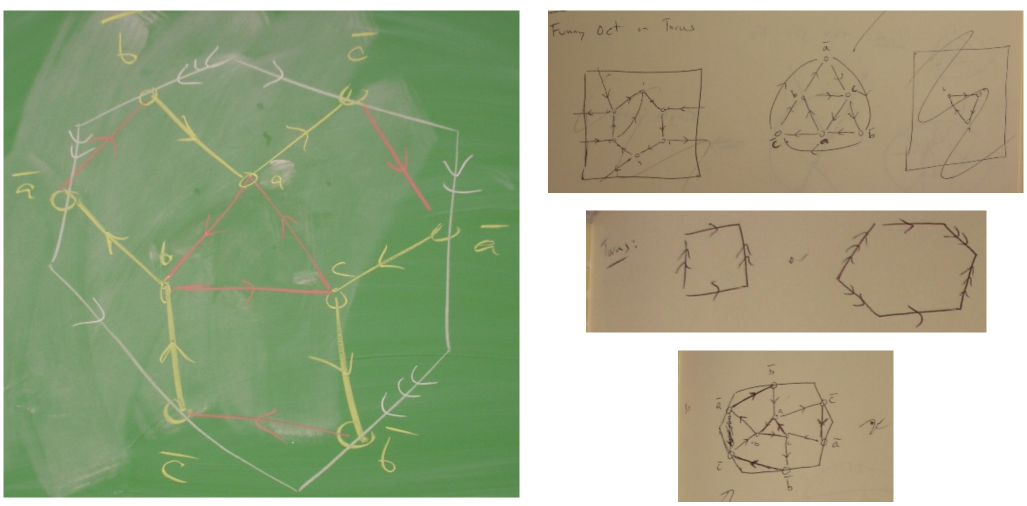


Figure 5-12. F5-i03, Fred's embedding of the funny octahedron in the torus (left); three excerpts from F5-ng1 and F5-ng2, Finn's embedding of the funny octahedron in the torus (right)

Forty-three minutes into the research meeting, Fred is asking Colin where his drawing of the embedding in the Klein bottle is. Colin hesitates and Finn points to a diagram on the blackboard right next to Colin (see Figure 5-13). Even though Colin faces Finn, he ignores Finn's pointing, actually turns away from the blackboard, and explains verbally what such an embedding would look like, so Finn retrieves his pointing almost immediately. It is remarkable how Finn continues, meeting after meeting, with his pointing efforts and only every now and then employs his voice in addition to his pointing. This speaks to Finn's tenaciousness to contribute and his willingness to be part of the mathematikoi, but these recurring episodes also indicate that Finn is not yet enculturated into how to mathematize among these three expert mathematicians. Instead, Finn's pointing is reminiscent of gestures by students in a classroom setting.

Figure 5-13. F5-v1 43:08, Finn points to diagram

During the next nine minutes or so, the mathematikoi work on the idea of *forcing* the embedding through the two types of 4-cycles they identified earlier. Fred and Colin are up at the blackboard while Finn and Victor are seated. There is a time interval of 2:35 minutes when the mathematikoi quartet-draw in silence. Finn intermittently pays attention to Fred and Colin but none of the expert mathematicians acknowledges any of the others (see Figure 5-14).

Figure 5-14. F5-v1 48:18, Mathematikoi quartet-draw

At 49:24 minutes, Colin interrupts the silence by uttering "yah, I think it is forced and then it doesn't work", which marks the time that Fred and Colin start working together on the blackboard, but are still being ignored by Finn and Victor (see Figure 5-15, left). At 51:59 minutes, Fred says "*Oh*! But, but, then, then you are just screwed right off the bat! Right? So you can't, I mean you can't, um (.), you can't pick up, I mean, there is, there is an instant trouble." This statement draws Finn's attention and he looks up at Fred and Colin following them from this moment onwards (see Figure 5-15, middle). Fred and Colin shift to the middle blackboard with the first diagram of the funny octahedron on it. Fred excitedly proceeds to draw over this diagram to demonstrate the understanding that they just gained about their forced approach to embed the funny octahedron. When Fred keeps touch-pointing on the diagram and repeating phrases similar to *can't use this, can't use that* (52:23-36 minutes), Victor finally pays attention to them (see Figure 5-15, right). At the end, Colin asks "is this clear? So (.) [Victor]?" (52:49 minutes) to which Victor responds by nodding his head. Finn is not addressed and he interjects neither verbally nor non-verbally.

Figure 5-15. Fred and Colin work on the blackboard: F5-v1 51:10, Finn and Victor ignore them and work on their own (left); F5-v1 52:21, Finn starts giving his attention to Fred and Colin (middle); F5-v1 52:32, Victor starts giving his attention to Fred and Colin (right)

The above episode is one of many during which some or all of the expert mathematicians discuss and diagram together, but Finn is not openly part of their journey through the mathematics. Finn is nonetheless attentive and keeping up with the mathematical discoveries as the next three minutes reveal, in which at first Fred and then also Colin address Finn directly to reiterate their discoveries. Finn points to Fred's diagram three times, two of which are accompanied by the respective utterances "there is one of them right here" and "it's all of this", both of which are acknowledged by Fred in the affirmative. Moreover, Finn's notebook reveals once more that he is not just taking notes but also participating in the explorations. Finn's diagram (see Figure 5-16, left) is quite different from Fred's diagram (see Figure 5-16, right) in directionality, use of colour and types of lines drawn.

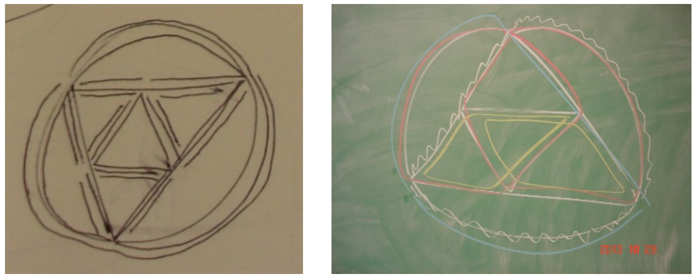


Figure 5-16. Exploring *forced* embedding of funny octahedron: excerpt from F5-ng2, Finn's diagram in his notebook (left); F5-i10, Fred's diagram on the blackboard (right)

At 57:40 minutes, Fred wants to know if these are the only 4-cycles they can use, to which Colin replies "so, by Euler's formula, what er, what's the face size?" and then Fred and Colin work on the blackboard arguing through Euler's formula. However, their bodies cover up some of the diagrams from view, and at 59:38 minutes Victor gets up to get a closer look at the diagrams. For the first time, Finn follows suit at 60:00 minutes to leave his seat and walk to the blackboard (see Figure 5-17). At 60:15 minutes, Victor is back in his seat, and at 60:17 minutes, Finn has returned to his seat. Even though Victor and Colin have left their seats many times before, Finn finally follows their examples to position himself better when a diagram is blocked from view.

Figure 5-17. F5-v2 00:00, Finn walks to blackboard for the first time

During the next 12:35 minutes, the mathematikoi explore the embedding of the funny octahedron using the face sizes. First, they exhaust the use of 4-cycles before trying out 3-cycles. Fred does most of the diagramming, Colin is alternately up at the blackboard or seated, and Victor and Finn remain seated except for one other instance of walking to the blackboard, which is almost a repeat of the previous time when they consecutively got up to have a closer look at the diagram on the blackboard. During this exploration, Finn also offers two contributions, both of which are acknowledged by the supervisor, but one of them is corrected by Colin as being a wrong approach. When they have completed their exploration, Fred exclaims: "So, that's *done*! So, that's *fantastic*! So, so, this tells us that funny octahedron graph has a unique embedding in the torus and a unique embedding in the Klein bottle" (69:50 minutes). During the next 5 minutes, the three expert mathematicians debrief their findings in relation to the theorem that Fred put up on the blackboard at the beginning of the research meeting. Finn is neither addressed, nor does he offer any contributions.

When Fred gets stuck around 75:20 minutes, Victor offers a way of thinking about the obstructions, which Fred immediately picks up on and even records formally on the blackboard. This leads to a further 4 minutes reiterating their findings on the first two cases of their theorem. Then, at 79:30 minutes, the mathematikoi start exploring C_4, which is the third case of their theorem. Fred diagrams on the blackboard, and initially the other mathematikoi observe him with Colin even making some small comments, but very soon, one by one. the other mathematikoi also explore on their own piece of paper. When Fred gets stuck, Colin walks up to the blackboard and shares his diagram, which Fred comments on as follows, "oh, nice, nice" (82:13 minutes), "yes, that is much prettier" (82:32 minutes) and "yah, nice" (82:44 minute). Shortly afterwards, Victor comments by saying "here is my picture, yours is prettier" (83:04 minutes). Figure 5-18 shows C_4's embedding from Fred on the blackboard, Finn on paper, and Colin both on paper and the blackboard. Finn's diagram is the only one that has the label C_4 and numbered vertices. This diagram also lacks identification of all faces, which Fred unsuccessfully attempted and Colin achieved with his two u-shaped lines.

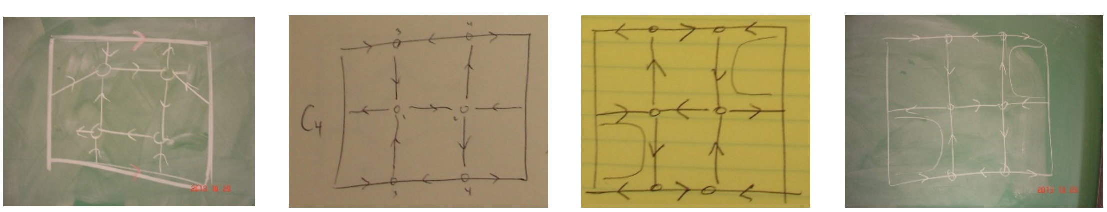


Figure 5-18. Embedding C_4 (from left to right): F5-i18, Fred's diagram on blackboard; excerpt from F5-ng3, Finn's diagram in his notes; excerpt from F5-nc1, Colin's diagram in his notes; F5-i19, Colin's diagram on blackboard

At 84:04 minutes, Fred introduces binary labelling to C_4 to get a sense of the structure through 4-digit binary words, some of which lead to legitimate objects. This sparks an almost six-minute discussion between Fred and Colin. At 91:04 minutes, Colin suggests checking that the embedding of the funny octahedron does not have the same pair of edges appearing on two faces, to which Fred replies "that was *really* significant for our earlier analysis", and so they proceed with the suggestion. However, only Fred and Colin work together, being ignored by both Finn and Victor, and when they conclude their exploration Fred addresses Finn directly and provides a 1:21-minute synopsis of their findings. During the remaining seven or so minutes of the research meeting, the three expert mathematicians discuss C_4's embedding, and alternately Colin and then Fred add coloured chords to their previous embedding of C_4 to gain an understanding of the mechanism of embedding. During their conversation, Colin remarks firmly "we have the lemma already for the projective plane" and Fred replies "yah, right, in the sense that it's in our heads". These utterances demonstrate a high level of confidence in understanding the mathematics that the mathematikoi have informally explored so far, and also direct how they proceed in their investigation. So far, Finn has not contributed in these overarching mathematical discussions, although the expert mathematicians often include him at least by making eye contact, and forming an open posture circle facing each other, and Finn reacts with head nodding (see Figure 5-19).

Figure 5-19. F5-v2 36:19, expert mathematicians discuss, but Finn is included with eye contact and facing one another

5.2.6. Research Meeting 6 (0:59:50): Asking Questions

I was not present for this research meeting, so Fred, Colin and Finn set up the camera to video-record. While the view window is focused on the spread of blackboards, it is angled in such a way that only Fred and Finn are in view, as well as Fred's other graduate student, who is again in attendance. However, Colin and Victor, who sit on the sides are not in view unless they come up to the blackboard. This limits the actions, postures and gestures I can describe for Colin and Victor and additionally the still images that I am able to make from the video-recording to supplement the narrative below.

This meeting starts with Colin's request to go over one of their cases of the theorem from the last research meeting (see Section 5.2.5) in detail, so that they are able to set up a programming procedure. During the ensuing discussion between Colin and Fred about which is the best case to start with, Victor claims the talking stick when he interjects "can we back up for just a second". His question is responded to by Fred, but Fred points twice at Finn to pass on the talking stick to him. During Fred's first pointing gesture, Finn is bent over his notes and does not react. Fred's second pointing gesture is accompanied by a question, which offers a tentative invitation to speak: "do, do you wanna (.) say er or draw a picture or something?" This time Finn responds, albeit in the negative, because he had not attempted to draw this embedding. Fred begins to draw the funny octahedron accompanied by a verbal response to Finn, but he is interrupted by Colin, who maintains the focus on exploring the embedding of the funny octahedron in the Klein bottle. Table 5-9 contains the transcription of this episode, which marks one of the rare times that the talking stick is offered to Finn.

Table 5-9. Transcription of Foursome F6-v1, 01:57-03:06

Time	Participants' non-verbal activity	Participants' utterances	Images
01:57	FS faces mathematikoi	FS: Maybe we should just start from the start. We er let's try and get to the funny octahedron. We know funny octahedron is good. (2.7) Or do you rather try (.) yes [Victor].	
		V: Can we back up for just a second?	
	FG turns to V	FS: Sure.	
		V: Um, what did those other embeddings of the funny octahedron on the Klein bottle look like?	
	FS points to FG	(0.6) Were they isomorphic under=	
	FG looks through notes	FS: =Yes=	
		V: =Yah=	
		FS: Yah.	
		V: So they are the same as unlabeled	
	FS steps back	embeddings, right?	
	FS gestures *unlabeled*	FS: Right! So, unlabeled there is only one.	
	FS walks to blackboard	V: Mhm.	
		FS: So, it was uh=	

	FS faces mathematikoi	V: I was just wondering what kind of switches went on.	
	FS faces FG	FS: Do you= (barely audible)	
		V: So that I have some concept of how the embeddings work.	
	FS points to FG	FS: Mhm. (1.5) Do, do you wanna (.) say er or draw a picture or something?	
	FS walks to blackboard	FG: Uh, I never actually drew it on the Klein bottle. I just sort of counted the (1.8) cycles.	
	FS draws diagram	FS: And you can verify without (.) sort of getting=	
	C points to blackboard C gestures *torus* C gestures *third*	C: So the Klein (.) you know, if we sort of want to get the results for a surface, that's (.) the Klein bottle would be the first one to start because we *know* one of those must be there, so, but for the torus we have no idea. So (1.0) uh, we would need to prove that one of those would be there. Or maybe a third one, a third graph (.) need to be there. (3.3)	

The next four minutes are occupied with Fred drawing the three embeddings of the funny octahedron in the Klein bottle, for which he references the colour scheme that they used in the previous research meeting. When the diagram is completed, Fred comments "everything is well-behaved transition-wise" (07:02 minutes) and proceeds to verbally argue that the diagram must represent the three embeddings in the Klein bottle, all the while tracing, touch-pointing and holding objects in the diagram. At 8:46 minutes, Victor asks "there were three additional ones?" and indicates that he is not following the argument by Fred. Both Fred and Colin try to convince Victor that the count is correct, but Victor is still puzzled. When Victor asks another question, it is Finn's response that correctly orients Victor towards the counting argument (see Table 5-10 for transcription).

Table 5-10. Transcription of Foursome F6-v1, 08:46-09:19

Time	Participants' non-verbal activity	Participants' utterances
08:46		V: Yes, but I remember you saying there were three additional ones?=
	FS *L* holds diagram	FS: =Oh, that's what I meant by three additional.
		C: *Two* additional, right?
		V: *Oh*, two additional, three total? (1.7)
	FS *L* touch-points *cycle*	FS: I'm, I'm adding *three*. For each (1.0) there, there are three cycles of this type, and you get to pick up two of them, you missed one. (1.4) So that gives three (.) three new ones. Right?
		V: What was the old one? (1.3)
	FG turns to V	FG: All quadrilateral.

	FS draws diagram	V: Oh, all quadrilaterals. FS: Oh, the old one= V: Yah, I remember. FS: Oh, the old one was all fours. (3.7) V: Ah, okay, okay.

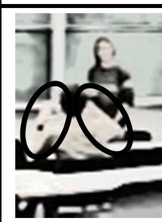
After another three minutes of discussing the uniform four embedding (face count consists of six 4-cycles) of the funny octahedron in the Klein bottle, Fred remarks “did we figure out a nice way of drawing it?” (12:26 minutes). Both Finn and Victor rustle through notes but none of the mathematikoi responds, and so after twenty seconds Fred faces the backboard and states “I guess I can just start drawing it”, but he does not proceed to do so. At 14:19 minutes Fred exclaims “man, I don’t know if this is the best way to draw it. Yah? Let me draw it. Okay, so, okay” and finally begins to draw the embedding. Over two minutes later, Fred steps back from the blackboard and says “so there is the graph” (see Figure 5-20). During this episode, Finn is witness to one expert mathematician’s struggles with how to represent the embedding. However, Finn also observes how Fred resolves his struggles, because Fred not only verbalizes the idea of the cross caps that they explored in the previous meeting, he also employs dashed lines and dots (see Figure 5-20) to indicate where edges and vertices go in and out on the Klein bottle.

Figure 5-20. F6-i02, Fred’s diagram of the 444444 embedding of the funny octahedron in the Klein bottle

During the next two minutes, Fred, Colin and Victor discuss what Figure 5-20 tells them about obstructions and how to continue. At 18:20 minutes, Fred turns to Finn and asks him “so, [Finn], did you (.) did you do any, any of these?” This starts the longest stretch of conversation that Finn has so far been involved in, and also marks the first time that Finn is questioning a decision made by the expert mathematicians. Table 5-11 displays the transcription of the 1:54-minute discussion separated into three sections.

Each section begins with a question asked by Fred that also serves to indicate the subject for this part of the exchange. The first section opens with Fred asking Finn whether he has looked at any of the material that they are exploring right now, and contains Finn's response as well as Fred's reaction; the second section begins with Fred's probing question "what did you find there?" which is followed by Finn's elaborate response that prompts Fred to draw a diagram; and the third section starts with Fred questioning Finn's claim, which sets off two further questions, one by Victor and one by Finn.

Table 5-11. Transcription of Foursome F6-v1, 18:20-20:14

Time	Participants' non-verbal activity	Participants' utterances	Images
18:20 18:45	FS erases *blackboard* **FS faces FG** **FS *L* points to blackboard** FS gestures *torus* FS gestures *unique* FS gestures *adding*	FS: So, [Finn], did you (.) did you do any, any of these? You, you were looking at some? FG: I, I was looking at the (.) I was looking at this on the torus? FS: *Ah*! Oh, okay. So you were doing torus. FG: Yah. FS: *Oh*! Actually torus should be better. At least in the sense that it is unique. FG: Right. FS: So every time you added something, you, you should see some, I mean, you've seen *something* that either *is* a forbidden (.) is one of the minimal obstructions or at least contains one. FG: Yah.	
18:46 19:31	FG gestures *adding* **FG gestures *chord*** FG *L* points to diagram FG gestures *vertices* FS walks to blackboard FS draws diagram FS stops drawing	FS: What did you find there? FG: I, I found a really nice, um (1.2) by adding one chord (1.5) we can see (1.3) the sort of (3.1) octahedron (.) but on seven vertices? But we get two jumps? FS: *Oh*! *Oh*! That's a happy thing. Oh, okay, *oh*, okay, so this is now on a slightly different track. But so you are telling me if you take seven vertices arranged in a cycle= FG: =Mhm. FS: And you (.) uh (2.1) and you give me a directed cycle like this, and then the jumps of length two= FG: =Correct. (1.2) That, that can be obtained (1.4) by adding a chord (.) to the (.) octahedron (.) in the torus. (2.1)	
19:32	FS steps away	FS: And, and so you are saying this is a minimal obstruction for the torus? (1.2)	

	FG gestures *yah*	FG: Yah? (1.7)	
		V: Is it minimal?	
	FS walks to blackboard	FS: Yah, okay! So, so we are guessing.	
		FG: Well, er, er that's what I was wondering? [Um]	
	FS faces FG	FS: [So?]	
	FG turns to C and V	FG: So, can we say that we would (2.1) we sort of	
	FG *L* points to dia.	(.) we're really surprised that we found	
	FS turns to diagram	anything smaller than *that*. (1.9) As an	
	FS turns to diagram	obstruction.	
		FS: Um. I bet. Yah.	
		FG: So (1.6) we, we've sort of been thinking of a	
	FS walks to blackboard	way to split two vertices here. (1.1)	
		FS: Mhm. (0.8) But I mean, the thing is (.) ah (0.5)	
20:14		like in this one (1.1) if you split the correct	
		way (0.7) then you see an octahedron.	

In order to bring attention to the differences and similarities in Fred's and Finn's manner of communication, the transcription in Table 5-11 has the modal verbs, the question markers as well as the spatial, temporal and personal deictic words underlined in the colour scheme presented in Section 4.5.2. Mathematical terms are written in turquoise, while the remaining words are written in black. Of the 44 mathematical terms, Fred speaks 25, while Finn speaks 19. Considering that Fred says a total of 183 words and Finn 121 words in this episode, this means that Fred's utterances consist of 13.7 percent mathematical terms versus 15.7 percent in Finn's utterances. In addition, this entire conversation contains only 11 spatial and three temporal deictic words in a ratio of 5:6 and 3:0 respectively, comparing Fred's usage to Finn's. In other words, this conversation is marked by a formal manner of talking about mathematics, with the graduate student and supervisor exhibiting a similar level of formality.

Fred utters the personal deictic word 'you' nine, five and three times respectively in the first, second and third section of the dialogue. I discount the usage of 'I' in the two times that Fred utters the discourse marker 'I mean', and so Fred uses 'we' and 'I' only once each. Interestingly, Fred uses 'you' to address Finn directly, but when he starts wondering about Finn's claim in the third section, which is responded to by Finn with "yah" in rising intonation, and followed by Victor's equal questioning of Finn's claim, Fred's next statements "yah, okay! So, so we are guessing" does not contain a 'you' but rather a 'we'. This use of 'we' functions to both soften the accusation of cheating and to speak to the general nature of their research approach, namely that the mathematikoi collectively are guessing which graphs will cause an obstruction in a certain surface. Finn never says the word ‘you’, but utters ‘we’ nine times and ‘I’ six times, where the ‘we’ indicates the non-personal use of the pronoun ‘we’ that is typically encountered in mathematical textbooks and used by teachers of mathematics.

Fred’s only use of modal verbs is ‘should’, namely two times in the first section, whereas Finn uses only ‘can’ and ‘would’, three times and once respectively in the second and third sections. This usage is indicative of the roles that Fred and Finn have towards each other: Fred is the mentor who provides guidance to Finn through the mathematics, and Finn is the graduate student who is trying out various mathematical solution processes.

The transcription also indicates rising, falling, animated and continuing intonation according to the *Transcription Glossary* (p. xv). Unlike the expert mathematicians, Finn states four of his findings and responses to Fred with a rising intonation, which are highlighted in pink colour, while the five questions that were asked by various mathematicians are highlighted in grey colour. In addition, Finn’s speech is marked by many small pauses up to three seconds long, as can be seen especially in sections two and three, whereas Fred’s speech contains fewer pauses and these are of much shorter duration. The pauses and rising intonation employed by Finn indicate that, while both Fred and Finn talk in a similar formal manner about mathematics, Finn is either less sure about his mathematics or reluctant to appear to be showing up his supervisor. Furthermore, unlike in previous discussions when Fred took the talking stick away from Finn during Finn’s pauses, here, Finn continues to hold the talking stick, even throughout his many pauses in speech. What the transcription does not capture is that Fred is a fast and Finn a slow speaker, but Fred slows down just like Finn by the third section.

Lastly, I want to draw attention to the postures, actions and gestures by the speakers. While Fred is standing and Finn is sitting along with Colin and Victor, Fred specifically faces Finn unless he is up at the blackboard drawing, and Finn orients his body towards Fred (see Table 5-11, images column). Both Fred and Finn point at the diagram on the blackboard, and moreover, Finn accompanies most of his speech with many gestures just like Fred does. Until now, this was the most animated that I had observed Finn to be while talking.

Over the next five minutes, the mathematikoi continue exploring the funny octahedron, until Fred says at 25:13 minutes: “Actually, I am a little unclear (.) how unique (.) this (.) how close to unique this embedding is. I think that’s a pertinent question here.” This sets off a drawing-and-staring episode by Fred on the blackboard that lasts for 3 minutes, for 2:42 minutes of which all mathematikoi are silent. At 28:21 minutes, Fred excitedly exclaims: “Wait, I (.) sorry. I mean, I think, I have a proof that k five doesn’t embed on the torus. (.) Which seems shocking. (.) But let me try.” For the next 10 minutes, Fred works close to the blackboard verbally and through many tracing, pointing, touch-pointing and holding gestures guiding the other mathematikoi through his claimed proof. However, at 37:59 minutes, he shouts: “Bloody hell. It works. (.) One, two, three. Okay! Bloody hell. There is one way to do it. It’s three, three, four (.) three, seven.” Fred chuckles, shakes both his arms, and everyone is silent for 11 seconds. During this episode, Finn is witness to the formulation of a conjecture, the argumentation through the proof-attempt, and the failure of the conjecture, all of which are fundamental aspects of the nature of mathematics.

At 38:37 minutes, a short exchange begins between Fred and Finn: staring at the K_5-embedding (see Figure 5-21, left), Fred states “that is a bit of a strange creature”, to which Finn responds with “yah.” When Fred says that “it would be lovely to have a small obstruction ‘cause you bump into it all the time”, Finn responds “but, but it would be sort of weird to (.) maybe”, which Fred supports with “the fact that we are trying to go on an orientable surface and, and then you can split these two faces (touch-pointing two faces on the diagram) does seem to have some pretty strong (.) force”. From 39:05 minutes to 45:30 minutes, Fred draws the embedding of K_5 in the torus (see Figure 5-21, right), carefully referencing the labels in Figure 5-21 (left), tracing along the edges, and lastly adding the colour scheme of red and yellow to indicate the faces on the torus.

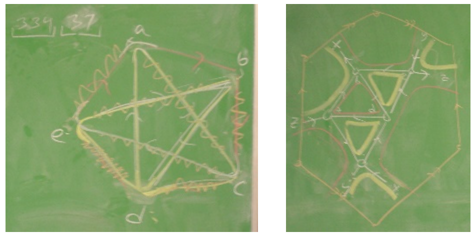


Figure 5-21. Excerpts from F6-i04: Fred's face count of K_5 to analyze embedding in torus (left); Fred's embedding of K_5 in torus (right)

At 45:48 minutes, Victor walks to the blackboard, touch-points and holds his earlier graph and discusses with Fred what the various splits of this graph achieve. Then Victor claims that he can prove that the graph in his diagram does not embed in the torus, which would then indicate a new obstruction. Victor begins drawing while saying "I think the graph looks like this. I haven't double-checked anything, so excuse me if I make a mistake" (46:46 minutes). About two-and-a-half minutes later, the proof fails, and during the next three minutes the expert mathematicians debrief on why the conjecture is false. This episode is another instance that allows Finn to observe an attempt at formalizing mathematics.

During the remaining seven or so minutes of the research meeting, the mathematikoi are engaged in finding an obstruction. All three expert mathematicians are alternately or in pairs up at the blackboard either duo-drawing or tracing their diagram. During this time, there is a 1:46-minute stretch of silence that is briefly interrupted by a meta-statement by Fred, who after prolonged staring at a graph begins to diagram and mutters "this is a nice graph though, I, I, this is much better", which speaks to the relationship that Fred has with the graph. There are also two instances of Finn pointing to Fred's diagram: the first time Finn silently gestures *cycles* with his outstretched hand for five seconds, and the second time Finn gestures *edge-tracing* for four seconds while whispering "across, top, down".

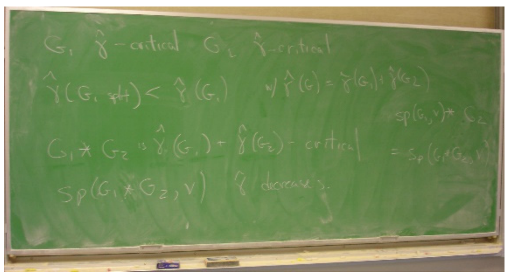
5.2.7. Research Meeting 7 (1:34:04): Informal versus Formal Mathematizing

For the first time, formal mathematical writing and proving is introduced to the research meeting lasting for about one hour with the remaining thirty minutes of the research meeting spent exploring. Unlike all the other research meetings, this one begins with Victor holding the talking stick. He stands near the blackboard, with Finn, Fred and Colin seated in front of him, and announces that he will show the mathematikoi “all the obstructions that have, um, anti-digons for the Klein bottle (.) as well as a (.) few others” (0:14 minutes), which is met with “mhm” from Colin and “cool” from Fred. During the next seven-and-a-half minutes, Victor introduces a *four-some* graph as opposed to the *two-some* graph they have been working with until now, writes an inequality about the face count on the blackboard, proves the inequality, and deals with the questions from Colin and Fred about this work until they are satisfied with the notation and the key idea behind Victor’s work. During this time, Finn does not participate either verbally or non-verbally.

At 7:39 minutes, without being asked to, Fred turns to Finn and spends 1:15-minutes explaining the concept of face boundaries underlying Victor’s work, while Victor first erases the blackboard and then paces back and forth in front of them, and Colin sits behind them staring at the blackboard (see Figure 5-22, left). During the explanation, Fred draws two diagrams on paper, as shown in Figure 5-22 (right); touch-points, holds and traces the diagram; and gestures the uttered words “faces” and “embedding”. Finn does not utter anything and his posture never changes.

Figure 5-22. F7-v1 08:09, from left to right Victor at the blackboard, Finn listening and Fred explaining, Colin staring at the blackboard (left); F7-v1 i07, Fred’s diagrams to aid explanation (right)

When Fred finishes his monologue, Colin walks to the blackboard, draws a diagram, and offers an alternative way of elucidating Victor’s key concept to Fred and Finn. At 10:39 minutes, Victor continues on the blackboard by formalizing the relationship between a graph and its immersion (see Figure 5-23, left) along with two diagrams (see Figure 5-23, right), which he uses to explain his new vertex operation termed *blowing up* *a vertex*. At 18:02 minutes, Victor summarizes: “If I had any obstruction and I blow up a vertex to an anti-digon, it’s an obstruction for the next higher up genus surface (2.8) that’s pretty cool”. He then proceeds to draw the basic obstructions they have collected so far for the Klein bottle, and introduces a new obstruction he found from blowing up the octahedron. This prompts Fred to turn to Finn asking whether they had already discovered this one themselves when they continued working on the funny octahedron after their last research meeting. After a 1:33-minute discussion between Fred, Finn and Victor it turns out that none of their graphs is this new obstruction that Victor found, and Fred concludes with “our vertex in the middle is not present […] it looked visually similar” (22:33 minutes).



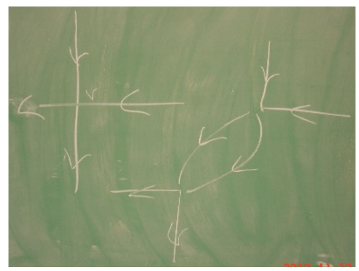


Figure 5-23. **F7-v1 i09, Victor’s formalization of relationship between a graph and its immersion (left); F7-v1 i11, Victor’s diagram depicting a vertex blow-up (right)**

At 23:25 minutes, Victor poses the question whether all Klein obstructions arise through the digons, which sparks a short discussion among the expert mathematicians. During this time, Fred remarks that Finn “has a new one too” (26-11 minutes), referring to an obstruction for the Klein bottle. From 26:23 minutes until 35:04 minutes, Victor writes a three-part proposition on the blackboard and then proceeds to clarify each proposal with a diagram that is accompanied with sweeps, traces and holds that often involve both hands and arms (see Figure 5-24, top left and middle, bottom left).

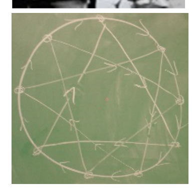
Figure 5-24. Gestures by expert mathematicians from F7-v1: Victor 30:10 (top left), 30:11 (top middle), 34:13 (bottom left); Colin 35:11 (top right); Victor mimics Colin 35:16 (bottom middle); Fred 35:23 (bottom right)

At the end of this period, Colin gestures with both hands and arms and provides a short synopsis of how he understands Victor's work (see Figure 5-24, top right). During Colin's gesturing, Victor mimics Colin's gesture, albeit in the vertical direction (see Figure 5-24, bottom middle). Neither Fred nor Finn turn around to observe Colin's gesturing, but both face Victor. Immediately following Colin's summary, Fred leans over to Finn and spends over one minute explaining Victor's argument to Finn with the same gesturing that Colin used and Victor mimicked (see Figure 5-24, bottom right). Finn does not exhibit any of these gestures that each of the expert mathematician uses to express their understanding of the relationships within a dense graph on the torus.

During the next 10 minutes, Victor writes another proposition and two theorems on the blackboard along with their proofs. Before he continues with the proofs, he is interrupted by Colin who is "thinking differently" (40:52 minutes) and who uses the earlier terminology to paraphrase Victor's work. After about one-and-a-half minutes of discussion between Colin and Victor, a common understanding is reached and Victor resumes. During the entire time of formal mathematics, no diagrams are drawn. Finn is seated either hunched forward, leaning back, or resting his chin on his fist, but since the start of the research meeting he has neither gestured nor volunteered any verbal statements.

At 49:17 minutes, Victor announces that he will now show the other obstruction he found and draws a nonagon, to which Fred responds with a chuckle and the statement "I guess this is the same as the one that [Finn] got" (49:22 minutes). Finn does not speak up. After the obstruction is completely drawn and discussed by the three expert mathematicians, Victor points to Finn at 52:33 minutes (see first image in Table 5-12) and now asks whether this is the obstruction that Finn analyzed as well. Finn responds in the affirmative, but also admits to having found only one of the splits (see Table 5-12 for transcription). After Victor has shown everyone how to do the other split, Fred wonders if a vertex split would produce an embedding in the torus, to which Finn responds "that's what I was trying for" (57:11 minutes). This marks Finn's first voluntary, unprompted contribution during this research meeting.

Table 5-12. Transcription of Foursome F7-v1, 52:33-53:05

Time	Participants' non-verbal activity	Participants' utterances	Images
52:33	V faces FG **V points to FG** V *R* points to diagram V walks to notes **V touch-points *edges*** FG turns to C	V: So you checked this out, right? FG: Well= V: ((chuckles)) And there is only two ways to split it? FG: I er I did one of the splits, uh suggesting the other one I couldn't quite find uh= V: =Oh I got it! (4.5) But it took me a long while. (.) The split you probably did was the *easy* one. You pulled *these* two edges off.= FG: =And you get the digon? V: And you get the= C: =And then you get the normal digon which you can then leave out. V: [Yah.] FG: [Mhm.] FS: [Yah.]	 

At 57:33 minutes, Colin claims the talking stick, walks to the blackboard, and begins to alter the nonagon shown in the image column of Table 5-12. This brings both Fred and Victor close to the blackboard, while Finn remains seated (see Figure 5-25, left). One minute later, Victor engages Finn in a conversation about the embedding of the nonagon in the torus, and twice points to the nonagon drawn on the blackboard (see Figure 5-25, middle). At 8:49 minutes, Victor claims the talking stick and explains to the mathematikoi why the nonagon does not embed in the torus, while he holds and touch-points the existing diagram, and then draws a new diagram (see Figure 5-25, right). This is followed by the three expert mathematicians debriefing their findings for about two minutes, which in the words of Victor is "a hell of a powerful piece of information (.)

whipping these chords in we can't make a digraph (.) digon (.) anti-digon, sorry" (62:56 minutes).

Figure 5-25. F7-v1 57:45, Colin diagrams on the blackboard (left); F7-v1 58:40, Victor engages Finn in conversation about diagram on blackboard (middle); F7-v1 59:30, Victor diagrams on the blackboard (right)

During the two-minute break that ensues, Finn asks Fred "so, so this (.) ((points at diagram on blackboard)) we took one octahedron and turned ((gestures by opening and closing his hand)) one of those vertices into an anti-digon?" (65:26 minutes), to which Fred responds in the affirmative. This marks Finn's first time gesturing during this research meeting. Then Finn says "I'm trying to draw it" (65:41 minutes), and proceeds to do so. One minute later, Finn is interrupted by Colin with "so, Finn, the conclusion was that one, one of the splittings didn't go, one of the splitting didn't go in the torus, right?". Fred responds with "yah" and Finn nods his head. Colin continues with "so it's not, this is not a minimal toroidal either?", and again Fred responds with "yah" and Finn nods his head. The discussion proceeds without Finn contributing verbally or non-verbally. From 68:04 minutes until 72:10 minutes, the three expert mathematicians discuss the orientation of the cycles in the embedding at the blackboard (see Figure 5-26, left), while Finn is hunched over his notebook (see Figure 5-26, right).

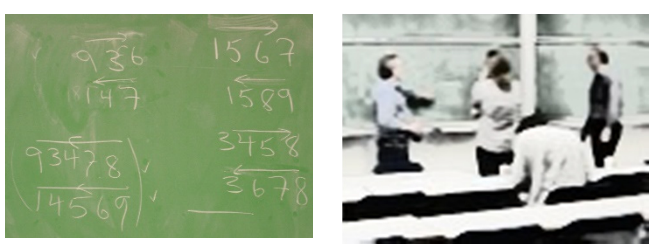


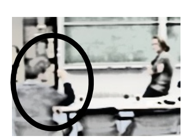
Figure 5-26. F7-i22, orientation of cycles on blackboard (left); F7-v2 11:29, Fred, Colin and Victor discuss at the blackboard, while Finn is hunched over his notebook (right)

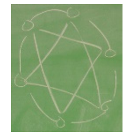
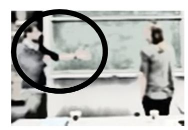
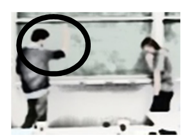
After 1:33 minutes of silence, Fred starts up a discussion again about what other cycles there could be. For the remaining 22 minutes of the research meeting, the three expert mathematicians discuss Victor's new obstruction for the Klein bottle and whether there is any vertex split or four-edge cut that produces obstructions for any other surface such as the plane, projective plane, sphere or torus. They are also wondering what the theorems do not cover in terms of obstructions and how to proceed from here. Fred diagrams once and Victor twice on the blackboard, but each diagram is discarded. There are many intervals of silence. Finn sits hunched forward, leaning back or resting his chin on his fist, but otherwise he does not contribute verbally or non-verbally, and he is also not addressed by any of the three expert mathematicians.

5.2.8. Research Meeting 8 (0:67:10): Almost like an Expert

This is the only research meeting where both Colin and Victor are absent. Since there are no other visitors, the meeting is just between the supervisor and the graduate student. Fred begins the meeting by asking Finn "uh, so, well, I guess I should ask if you have anything to tell me cause I certainly don't have anything", which begins an approximately two-minute exchange about a proof that Finn found interesting. During this time, Finn gestures mathematical terms, walks to the blackboard, draws a diagram, and engages with the diagram during his speech. Table 5-13 displays the transcription of the 1:44-minute discussion separated into two sections. In the first section, Finn works alone on the blackboard, and in the second section, Fred joins Finn at the blackboard.

Table 5-13. Transcription of Foursome F8-v1, 00:19-02:03

Time	Participants' non-verbal activity	Participants' utterances	Images
00:19	FS faces FG FG faces notebook FG faces FS FG gestures *join* FG gestures *infinite* **FG gestures *cycles***	FS: Uh, so, well, I guess I should ask if you have anything to tell me cause I certainly don't have anything. FG: Um (1.2) not really. Um (2.6) I sort of liked, um (.) [Colin]'s proof about the (0.8) join of the, the (heinous) (1.1) um (.) yah the infinite group of graphs we can think of now (1.3) as the directed cycles sort of moving (.)= FS: =Mhm= FG: =around= FS: =Right.	

	FG still gestures *cycles* FG walks to blackboard	FG: Um (.) I thought about that a little bit (0.9) cause I was, I was thinking about (.) could it still be uniform case (.) where we had (1.7) where the cycles were of the same length.	
		FS: Mhm.	
	FG draws diagram 1 **FG draws diagram 2**	FG: Um (6.2) or (.) for the six (7.6) but it doesn't even have to be uniform because we can have the (.)	
		FS: That same argument works.	
	FG sweeps diagram	FG: Right. (1.9) I was sort of looking at the .() different ways (1.8)	
	FG draws diagram 3	FS: Sort of wrap two cycles around?	
		FG: Yah. (.) or (1.1) I, I guess any minimum number of cycles?	
	FS faces blackboard	FS: Um, right. So that in particular (.) is another one. (.)	
01:31	FG points to diagram	FG: [Yah, but this is one] that [Victor], [Victor]	
01:31	FS walks to blackboard FG steps away FS sweeps diagrams FS points to diagram	FS: [But in this guy] oh, oh this is basically below or above (.) oh and this (.) so you basically got the same argument. Oh, I didn't realize that. So [Victor]'s got the same argument why this one, um=	
	FG gestures *blown up*	FG: =this, this is, this is octahedron blown up. [Yah.]	
	FG walks to blackboard	FS: [Yah] that's right. So you, so the same argument tells=	
		FG: =yah=	
	FS steps away	FS: =explains why this guy doesn't embed.	
	FG adds to diagram	FG: Yah. Everything, everything is going this way? (1.6)	
		FS: I see, that's *very* nice.	
	FG sweeps diagram	FG: And, and (.) every edge-cut uses three edges? So we can get the Euler characteristic argument in.	
02:03	FG steps away		
		FS: Right.	

Similar to the analysis of the transcription in Table 5-11, the transcription in Table 5-13 has the modal verbs, the question markers as well as the spatial, temporal and personal deictic words underlined in the colour scheme presented in Section 4.5.2, in order to bring attention to the differences and similarities in Fred's and Finn's manner of communicating. Mathematical terms are written in turquoise, while the remaining words are written in black. Of the 54 mathematical terms, Fred speaks 20, while Finn speaks 34. Considering that Fred says a total of 106 words and Finn 173 words in this episode, this means that Fred's utterances consist of 18.9 percent mathematical terms versus 19.7 percent in Finn's utterances. In addition, this entire conversation contains 17 spatial and one temporal deictic words in a ratio of 10:7 and 0:1 respectively, comparing Fred's usage to Finn's. In other words, this conversation is again marked by a formal manner of talking about mathematics with the graduate student and supervisor exhibiting a similar level of formality.

The personal deictic word 'I' is spoken five times by Fred and seven times by Finn, which makes it about evenly used. Fred utters the personal deictic word 'you' three times, while Finn does not use it; and similarly but in reverse, Finn utters the personal deictic word 'we' four times, while Fred does not use it. Here again, Fred uses 'you' to address Finn directly, and Finn's use of 'we' indicates the non-personal use of the pronoun 'we' that is typically encountered in mathematical textbooks and used by teachers of mathematics. 'Now' is the only temporal deictic word in this dialogue, and it is spoken by Finn. Fred's only use of modal verbs is 'should', namely once, whereas Finn uses only 'can' and 'could', four times. The usage of personal deictic words and modal verbs is again indicative of the supervising role that Fred has versus the questioning and trying graduate student role that Finn has.

Similar to the previous analysis of intonation in Sub-Section 5.2.6, Finn states three of his findings and responses to Fred with a rising intonation, which are highlighted in pink colour, while the one question that was asked by Fred is highlighted in grey colour. It is also noticeable that Finn's speech is disrupted by many pauses up to 7.6 seconds in duration. Despite these pauses, Finn continues to hold the talking stick. Furthermore, in the second section, Fred speaks more slowly and Finn uses fewer pauses. Although all three of Finn's responses in rising intonation occur towards the end of the second section, the exchange between the two speakers sounds more fluid.

Lastly, I draw attention to the speakers' postures, actions and gestures. Not only does this segment mark the first time that Finn is up at the blackboard, but Finn is also up at the blackboard on his own. His actions are reminiscent of the actions of the expert mathematicians displayed in all other research meetings, such as creating and engaging with diagrams through a variety of gestures during speech. Furthermore, Finn is joined by Fred at the blackboard, and both exhibit similar gestures of pointing and sweeping as well as use of mathematical terms, which makes this the first time that Finn is participating like an expert mathematician.

During the next 2:33-minutes, Fred generalizes from the argument that Finn brought forth, and verbally guides Finn on how they can go about proving their work, pointing out what is difficult and what is easy. Both Fred and Finn are still standing near the blackboard, although no diagrams are being drawn. During this monologue, Fred makes typical statements about mathematizing (summarized in Table 5-14).

Table 5-14. Transcription of Significant Utterances during Fred's Monologue F8-v1, 02:23-04:55

Time	Fred's utterances
02:37	I'm a little unclear where all of this is going to go.
03:35	It'd be nice if we could make some general arguments about other things that might be minimal uh uh not embeddable graphs.
03:46	Well this is a very robust argument for showing that something doesn't embed.
03:50	Showing minimality is tricky. Um (2.5) quite tricky. ((chuckles))
04:17	Showing something doesn't embed is usually hard. (.) Right? I mean that's showing that there does *not exist* a collection of cycles. Showing that it does embed that's just a matter of finding the right collection and verifying. That tends to be easy.

When Fred hesitates, Finn grabs the talking stick and asks Fred some questions again about Victor's argument, which he is attempting to answer himself. This takes about one minute, during which time Fred gives very brief answers or acknowledgements. This is followed by two minutes filled with many short silences, where Fred mentions three times that he wants to explore one example fully on either the torus or the Klein bottle until he makes the hedged statement "I think maybe what we should do (.) is sort of (.) get our fingers dirty ((chuckles)) so I can dig into one of those things" (7:43 minutes). At the end of this statement, Finn sits down. At 8:01 minutes, Fred whispers "What does [Victor]'s argument get us exactly?" and he mumbles, while Finn searches through his notebook. At 8:11 minutes, Finn replies "[Victor]'s argument gives us (.) all the (1.2) here, here are the (.) basically the nice (.)" and points to his notebook (see Figure 5-27). Fred responds with "oh well, those are starting from (.) you have to be a little bit careful here". Then Fred proceeds to explain the difference in starting their exploration on the torus versus on the Klein bottle.

Figure 5-27. F8-v1 08:17, Finn seated points to diagrams in his notebook

At 9:47 minutes, Finn interrupts Fred with “right, yah, we had this one”, again pointing to a diagram in his notebook. Fred reacts surprised: “oh, we had an example?” and Fred proceeds to draw this diagram on the blackboard. During the next five minutes, Finn and Fred discuss this graph, while Fred is at the blackboard all the time and Finn joins him at the blackboard only two minutes into the discussion. Fred verbally guides Finn through the diagramming process of splitting an edge, all the while engaging with the diagram through pointing, touch-pointing, holding, tracing, erasing, and adding (see Figure 5-28).

Figure 5-28. F8-v1 08:17, Fred diagrams and explains, while Finn observes

At 15:17 minutes, Finn questions the correctness of Fred’s diagram: “do we have this edge in here and here?” (see Figure 5-29, left). While Fred tries to find his mistake, Finn starts drawing as well, and they end up duo-drawing on the blackboard (see Figure 5-29, second from left). Fred finally finds his mistake, which prompts him to explain the concept of *complement graph* to Finn, and how this view sometimes aids the understanding of the graph at hand. Then, Fred finishes drawing his complement graph, while Finn attempts to draw the complement as well. However, Finn stops drawing, gets his notebook, looks at his diagram in the notebook, then looks at Fred’s diagram, and finally completes his graph by copying four edges from Fred’s graph. The two right-most diagrams in Figure 5-29 show Fred’s and Finn’s diagrams. It is noticeable that Fred worked on top of the original graph when producing the complement in yellow chalk, while Finn drew the vertices and edges of only the complement graph.

Figure 5-29. F8-v1 (from left to right): 15:17, Finn points to diagrams, while questioning its correctness; 16:11, Finn and Fred duo-draw; i01, Fred’s diagram; ng2, Finn’s diagram

At 20:17 minutes, Fred asks Finn “actually, the question I want to ask is, um (.) does this still ((points at Figure 29, second from the right)) uh, I mean, we have proofs that things don’t embed, right?” While Fred paraphrases what he means, Finn searches through his notes, and responds 30 seconds later “it should but then that means it shouldn’t embed on the Klein bottle either”. Fred likes Finn’s response, because he replies “oh, that’s an excellent (.) that’s a very good point”. Finn finally finds the place in his notebook he was looking for, and while Fred produces an alternative version of the complement graph and then erases the complement edges in his original diagram (see Figure 5-30, left and right), Finn copies the formulas from his notebook onto the blackboard (see Figure 5-30, left).

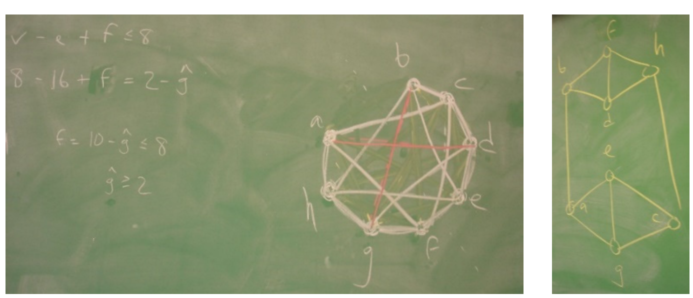


Figure 5-30. F8-i03, Finn’s formulas and Fred’s diagram (left); F8-i02, Fred’s new diagram (right)

After a two-minute discussion, Fred indicates that he wants to try to embed this graph in the torus because he believes "the set of graphs we are looking for is going to be smaller" (25:22 minutes). Fred begins to draw while saying "I'm gonna draw it a little bit bigger and then try and fill things in" (25:44 minutes). After about 2:30 minutes of Finn observing Fred drawing near the blackboard, Fred exclaims "yah, I've just got too many edges crossing, this is just gonna be a nightmare" (28:10 minutes), but he continues to draw. Finally, at 28:51 minutes, Fred steps back from the blackboard and states "oh, yah, well there, isn't that beautiful ((chuckles))" (see Figure 5-31).

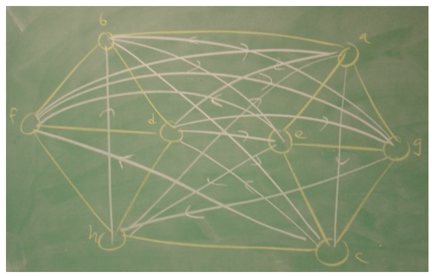


Figure 5-31. F8-i05, Fred's attempt at embedding the graph in the torus

At 29:01 minutes, Finn interrupts the silence with "so, do you, do you try to do (.)". After a 6-second long pause, Fred responds "*I* don't even know what I am trying to do. I, I guess I was hoping to see this thing a little bit better". Then Fred proceeds to explain to Finn why this graph is difficult to deal with and likens it to a reduced version of the graph $K_{4,4}$. This comparison gives Fred an idea of how to understand the graph; he turns towards the blackboard, talks to himself about the number of edges, and starts drawing a new diagram. After about 20 seconds of silence, Finn begins to draw on the blackboard as well, then 10 seconds later Fred asks Finn two questions about their 4-regular graph on eight vertices. Since Finn is closest to this graph on the blackboard, he checks it and answers Fred's questions. From 31:19 minutes to 33:26 minutes, Fred and Finn duo-draw in silence (see Figure 5-32). Then Fred stops drawing and starts up a discussion with Finn again. Fred corrects one of Finn's responses with "well that's not quite true" (34:43 minutes) and proceeds to explain which arguments hold.

Figure 5-32. F8-v1 32:16, Finn and Fred duo-draw in silence for 2:17 minutes

At 35:07 minutes, Finn walks to his desk and searches in his notebook, while Fred continues to talk about their graph and its symmetry. At 35:25 minutes, Fred begins to diagram silently on the blackboard (see Figure 5-33, left top), which lasts for 5:11 minutes. At 37:49 minutes, Finn seems to have found what he was looking for and walks with his notebook towards Fred (see Figure 5-33, left middle). After waiting for 27 seconds without gaining Fred's attention, Finn returns to his seat (see Figure 5-33, left bottom). Figure 5-33 (middle and left bottom) also shows how immersed Fred is in the diagramming. His upper body is close to the blackboard, and often both of his hands are on the diagram, while he holds, touch-points, traces and draws. At 40:13 minutes, Fred turns towards Finn and exclaims "oh, well, I think I found the best, or, or, I found *a* way of saying this or at least it permits me to describe it compactly". Finn slides the blackboard he was working on up, and the diagram in Figure 5-33 (middle) becomes hidden behind the top blackboard. For the next 4:30 minutes, Fred shares with Finn what he has found out about the 4-regular graph on eight vertices during his silent exploration, by redrawing his hidden graph from scratch accompanied by verbal explanations (see Figure 5-33, right). The redrawn graph has different and incomplete labels, which Fred says is done on purpose to bring out the graph's inherent symmetry. There is also an absence of dashed lines and edges drawn in colour in the new graph. Furthermore, while the total number of vertices matches in both graphs, the edge count does not match, yielding 14 and 12 edges respectively.

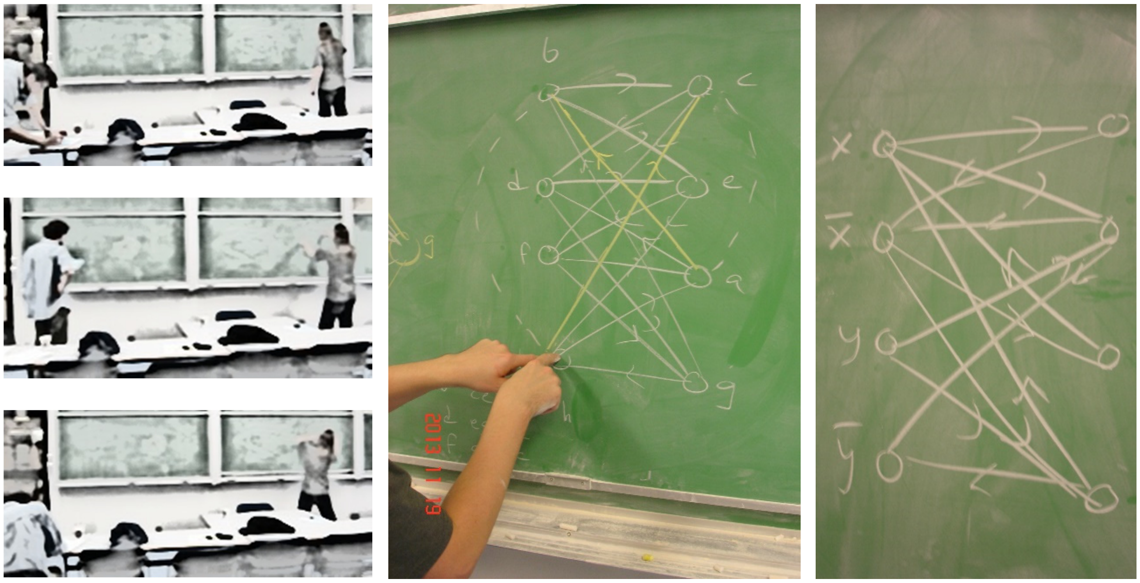


Figure 5-33. F8-v1 Fred diagrams in silence for 5:11 minutes: 37:39, Finn searches in his notebook (left top); 38:01, Finn oriented towards Fred (left middle); 38:48, Finn seated (left bottom); i06, Fred holds the diagram he is creating at 39:15 (middle); i08, Fred redraws the diagram for Finn

At 45:08 minutes, Fred states “I have the tendency to dwell on some of these examples too much” and that he wants to come back to the projective plane and the torus. Finn gets up with his notebook, walks towards Fred and replies “that’s sort of what I was going back to as well” (see Figure 5-34). The ensuing discussion lasts for about one minute. Then Fred takes Finn’s notebook, says “this is *very* nice” (46:59 minutes) referring to Finn’s embedding, and draws Finn’s diagram on the blackboard. This leads to a discussion of all of Fred’s jumps and splits with this graph that lasts for about three minutes.

Figure 5-34. F8-v1 45:39, Finn shares notebook with Fred

At 50:15 minutes, Fred turns to Finn and asks him “do you know the perfect graph theorem?” to which Finn neither gestures nor utters a sound, but steps closer. This prompts Fred to explain this theorem to Finn, which takes about 3:30 minutes. Then Fred summarizes briefly why this theorem is useful in the context of complement graphs, which in turn help analyzing whether graphs lead to obstructions. The meeting finishes with Fred outlining that he wants to go forward by taking the “paradigm that we have that worked for the projective plane” (55:38 minutes) and applying it to the torus. Fred spends the next 11 minutes going through this outline in rough detail, starting from the beginning, and accompanies his monologue with the familiar graphs from the first few research meetings (see Figure 5-35).

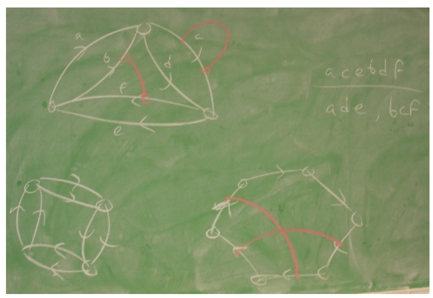


Figure 5-35. F8-i10, Fred’s diagrams accompanying talk on how to move forward

Throughout this research meeting, it is noticeable that Finn’s actions, postures and gestures have become much like those of his supervisor Fred in that he easily stood up to move to the blackboard, freely picked up chalk and began to draw on the blackboard, gestured during speech, asked questions, contributed on his own, had answers ready when he was addressed, and was also engaged in several mathematical conversations with Fred. There were only a few times when Finn sat down, but it was clear from his utterances afterwards that he was equally busy exploring and trying to understand their 4-regular graph on eight vertices.

5.2.9. Research Meeting 9 (1:54:50): Effortless Participation

Victor is absent from this research meeting, and Fred’s other graduate student is again in attendance. Fred begins the meeting by noting that K_5 has only one Eulerian orientation up to isomorphism, which Finn questions, so Fred draws the graph and explains. Then Fred asks Finn how many orientations the octahedron has, but Finn cannot recall and says he did not make notes about it. After a pause, Fred says that he wants to aim for the torus for today. He lists the doubled triangle, C_4 and K_5 as their starting points, while drawing the graphs on the blackboard. Colin and Fred begin exploring how K_5 embeds in the torus, with Fred diagramming and Colin directing Fred. At 8:10 minutes, Fred exclaims “how does this embed on the torus?” as both Colin and Fred are puzzled which faces would work. This is the same embedding that baffled the mathematikoi during the sixth research meeting, but which Fred managed to find (see Figure 5-21). For about four minutes, Colin and Fred discuss which embedding is possible by arguing through cycles of various lengths. At 13:03 minutes, Fred turns to Finn and explains Colin’s argument about forced cycles, while pointing at and tracing edges in the diagram (see Figure 5-36, left). At 14:36 minutes, Fred finds that the embedding consists of the [7,3][3,3,4] cycles. It takes another five minutes of Fred drawing and meta-talking until he has a diagram of the actual embedding in the torus. All the while Finn and Colin are hunched over their notebooks, drawing as well (see Figure 5-36, right).

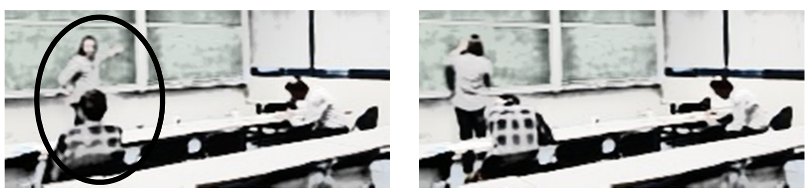


Figure 5-36. F9-v1 13:07, Fred points at diagram while explaining Colin’s argument to Finn (left); F9-v1 17:05, Fred, Finn and Colin trio-draw (right)

At 20:32 minutes, Fred redirects the mathematikoi to start over with the doubled triangle to find a better embedding. For three minutes, Fred addresses his subsequent plan to Finn, who silently faces Fred. At the end, Colin contributes with some embeddings they know already, which sparks another discussion about the doubled triangle, during which time Finn asks Fred a question, but then answers it himself. At 27:27 minutes, Fred asks Finn “you’ve looked at a couple of things of this?” referring to smaller obstructions. Finn points at Fred’s diagram and reports “yah, yah, I found (.) I was starting with this guy (.) and adding chords (.) and I found, uh, two excluded guys, one is the double C5 (.)”. Now, Finn gets up to walk to the blackboard while still explaining which obstructions he found. First, he adds two chords to Fred’s diagram to show the double C5 (see Figure 5-

37, left), and then erases the chords he just added and adds another chord to show the other excluded graph (see Figure 5-37, right). Fred observes Finn's diagramming and at the end says "I do want to get to that one, but I, I, I actually want to start back, back here right now ((pointing at the doubled triangle))" (28:31 minutes).

Figure 5-37. F9-v1 27:55, Finn adds chords to Fred's diagram (left); F9-i04, changed diagram (right)

During the next four minutes, Fred talks about the complexity of their problem and how they have to make choices regarding edges and faces. At 33:43 minutes, Finn points at a diagram and interrupts Fred: "So, so, I sort of did this" and continues to describe for thirty seconds how he investigated the embeddings by considering the effects of chord crossings and splits on the faces. All the while he sits upright and gestures mathematical terms that occur in his speech (see Figure 5-38). When Finn has finished his report, Fred responds "oh, oh, that's exactly what I want to do, yah, yah, yah, okay" (34:17 minutes). Then Colin asks "oh, so even, uh, if it wasn't really much looking, looking much better, you got to a better graph?" (34:22 minutes), which prompts Finn to share some of his diagrams in his notebook.

Figure 5-38. Finn gestures mathematical terms (from left to right) from F9-v1: 33:50, *embeddings*; 34:05, *faces*; 34:07, *chords*; 34:11, *splitting down*

At 35:17 minutes, Fred says "I would like to argue something in general about, um (.) general about this obstruction guy" and proceeds with his arguments. Fred is interrupted by a question from Finn, which he clarifies. For about 13 minutes, Fred explores what the obstruction graph looks like in general with regular contributions by Colin and one contribution by Finn. When Fred gets stuck with his argument, he guesses “maybe we picked this, sort of a lousy choice for our embedding” (48:15 minutes) and discards the diagram. At 49:02 minutes, Finn asks “what, what do you mean it kills this?” to which Colin replies: “This, this is, uh, uh shows that the red embedding doesn’t work. (1.1) So you have to go to a different one, right?” But Finn does not indicate that he understands this argument, so Fred tries to explain, and finally, Finn starts nodding his head. Then Fred suggests that perhaps they need to ask themselves, where the jumps occur in order to get a better grasp at the obstructions. Aside from one brief utterance by Fred, the mathematikoi work in silence from 53:00 minutes until 57:37 minutes with Fred at the blackboard and Colin and Finn seated. At 57:38 minutes, Fred breaks the silence and shares that he has an idea how to get rid of 4-edge cuts. For the next six minutes, Fred explores all possible jumps from an edge on the doubled triangle without contributions from Colin and Finn. This is followed by six minutes of exploration of jumps on C_4 that is started by Fred with intermittent contributions by Colin. Then Colin grabs the talking stick, draws a new diagram and points out what mathematics is needed to reduce this graph to a smaller one. At 73:31 minutes, Colin points at Finn and says “you remember we had this for the projective plane” to which Finn responds verbally while also gesturing mathematical terms.

At 76:48 minutes, Colin draws attention to the triangular faces in the embedding of C_4. When Finn asks “and you pick the two triangles?” (77:03 minutes), Colin walks to the blackboard and draws a new diagram. While facing Finn, Colin points, touch-points, traces, and sweeps his diagram to explain his argument to Finn (see Figure 5-39, left). At the end of his explanations, Colin adds dashed lines to the diagram (see Figure 5-39, right), accompanied by the utterance “you have the possibility to have this one or to have this one” (78:10 minutes). Fred and Colin continue to comment on C_4 for another five minutes, but are often silent for up to 30 seconds.

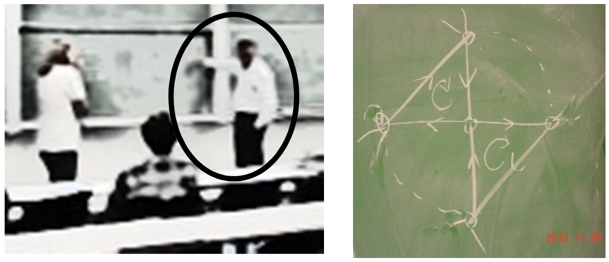


Figure 5-39. F9-v2 17:48, Colin faces Finn while sweeping his diagram (left); F9-i10, Colin's diagram (right)

At 83:51 minutes, Fred asks Finn "was funny octahedron uniquely embedded in the torus?" to which Finn responds "I don't remember" and begins searching through his notebook. A minute later, Finn says "I think, I think this one". Fred walks to Finn, bends over Finn's notebook, and both of them discuss Finn's diagram for about two minutes (see Figure 5-40, left). At 86:42 minutes, Fred exclaims "so the claim here is that for embedding funny octahedron there should be 3, 3, 6 and then opposite 4, 4, 4" and proceeds to write these numbers on the blackboard. The mathematikoi explore the splits of the funny octahedron for another five minutes, until silence ensues. Then Colin leaves the research meeting. Now Fred argues how the four graphs () are connected, labels them, and draws a hierarchical relationship of the graphs on the blackboard (see Figure 5-40, right).

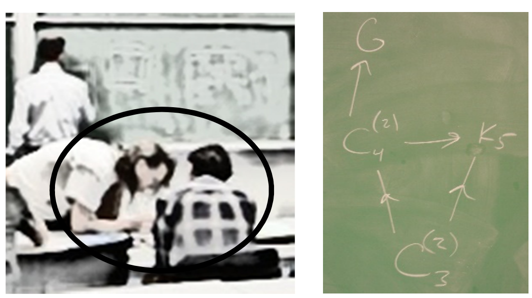


Figure 5-40. F9-v2 26:33, Fred bends over Finn's diagram (left); F9-i14, hierarchical diagram of obstructions (right)

At 94:12 minutes, silence ensues that lasts for almost 10 minutes. During this time, Fred walks to the blackboard twice and starts drawing, only to stop drawing about 20 seconds later each time. Otherwise, Fred either stands or sits facing the blackboard, while Finn remains seated and does not draw. At 104:10 minutes, Fred faces Finn and reasons: "I think what's really weird is that the argument (.) these *chord* diagrams are weird. They are hard to follow. […] Arguing about these things is weird and unnatural. […] It just seems (.) strange. Like it's hard to argue (.) what you (.) what you are getting. It seems like either we need some better, uh, better understanding of those, better (.) like lemmas and tools […] or we should just switch back to the more traditional way". During the remaining nine minutes of the research meeting, Fred offers other methods of exploring obstructions. During this time, Finn asks the following questions, and even walks to the blackboard but never draws a diagram: "You would never have a case like this ((points at a diagram))? Is that what you are talking about?" (106:14 minutes); "I, I don't think it's the same, but I was sort of looking for the smallest instances (.) that would get me in trouble? (.) Is that sort of wrong or something?"(109:19 minutes); "we can't have one going to the top ((gestures top)), on the other side? Or from the right to the left or something like that?" (114:27 minutes). Fred addresses all the questions before the research meeting comes to a close.

5.2.10. Summary

Sub-Sections 5.2.1 to 5.2.9 concerning the nine research meetings describe the actions, postures, gestures and interactions of the graduate student Finn during mathematizing and diagramming, and how his verbal and non-verbal communications are similar or different compared to those of the three expert mathematicians. I will now provide a summary of these verbal and non-verbal forms of communications. The following is a list of the typical gestures, actions and postures by Finn during the research meetings in general and predominantly during the first four research meetings, when he is still not taking on the role of main speaker: facing blackboard, facing speaker(s), attentive listening and note taking (see Figure 5-41). Finn's non-verbal communication is synchronous to the verbal communication by the other mathematikoi in the room because his face follows the speaker.

Figure 5-41. F1-v1 38:00, Finn takes notes (left); F1-v2 02:14, Finn faces speakers and blackboard (right)

It is worthwhile noting that when Finn speaks, his voice is low and does not carry well to the camera, which is situated in the back of the room, while the voices of the three expert mathematicians carry well throughout the room regardless of where they are situated: standing facing the blackboard, standing facing the mathematikoi, seated facing the blackboard or seated facing the room. There is an assuredness in the manner in which the three expert mathematicians (locals and visitor alike) occupy the room, carry themselves upright and naturally, and speak to make themselves heard. As an onlooker, I got the impression that this room is home turf, so to speak. Barany and MacKenzie (2014) point out that the blackboard "as a semiotic technology [is] as much a stage as a writing surface" and as such public and "ostentatious so much so that colleagues in shared offices expressed shyness about doing board work when office-mates are present" (p. 12). Clearly, the blackboard-as-a-stage syndrome does not affect the three expert mathematicians. Finn, on the other hand, often sits hunched over and seemingly occupies as little space as possible (see Figure 5-41). It takes five research meetings before Finn sits back with ease, stretches when he needs to, and gets up freely to the blackboard or another location in the room, just like any of the other three expert mathematicians have modelled from the first research meeting onward.

Moreover, during the first research meeting, Finn makes no attempt at joining the discussion and he is also not addressed by any of the expert mathematicians. In the second and third research meetings, Finn tries a number of times to gain the attention of the expert mathematicians, but only a few attempts are successful. During the fourth, fifth and sixth research meetings, Finn is successful at joining discussions and asking some questions, but those instances are infrequent and of short duration. It is not until the seventh research meeting that Finn is consistently making himself heard, which allows him to frequently ask questions and contribute to discussions, all the while employing gestures other than pointing, such as drawing on the blackboard, tracing diagrams, and gesturing mathematical objects during speech.

Throughout all research meetings, supervisor Fred and, at times, research colleague Colin often summarize for Finn major findings or new insights that the mathematikoi have gained in their diagramming explorations or discussions. Every now and then, each one of the three expert mathematicians has also addressed Finn, either asking about some information in his notebook or engaging him in a discussion about some graphs that he had explored. Table 5-15 provides a summary of Finn's successful and unsuccessful communication as well as communication directed at him, and also identifies the progression of his typical gestures, activities, postures and voice throughout the nine research meetings.

Table 5-15. Summary of graduate student's verbal and non-verbal communication

<table>
<tr><th>Progression of typical gestures, activities, postures and voice</th><th>Research Meeting</th><th>"-" indicates unsuccessful communication by graduate student,
"+" indicates successful communication by graduate student,
"=" indicates communication directed at graduate student</th></tr>
<tr><td rowspan="2">faces blackboard
faces speaker(s)
listens attentively
takes notes</td><td>1</td><td>- no interactions</td></tr>
<tr><td>2</td><td>- two instances: points to diagram for 3 s but doesn't gain attention
+ addressed by research colleague, side conversation ensues
+ participates in trio-drawing</td></tr>
<tr><td rowspan="4">occupies little space
sits hunched over
sits leaning back

speaks in low voice
speaks slowly
speaks with rising intonation

points to diagram</td><td>3</td><td>= summary directed at him by research colleague
- not part of reporting on supervisor-graduate student activity
- points to diagram but does not gain attention
+ verbally interjects and gains attention,
- but he misunderstood and never makes it to the blackboard
- addressed by supervisor, then dismissed
+ addressed by research colleague, all hover over his diagrams</td></tr>
<tr><td>4</td><td>- two instances: asked by research colleague but supervisor takes over
- unaware of extensive gesturing accompanying visitor's speech
- not part of reporting on supervisor-graduate student activity
+ points to diagram and contributes successfully to discussion
+ addressed by supervisor and contributes
- fails to produce diagram from his notes
+ points to diagram and identifies error in diagram
- unhelpful comment to supervisor about meaning of diagram</td></tr>
<tr><td>5</td><td>- not part of reporting on supervisor-graduate student activity
- points to diagram and gestures "circular" but is unsuccessful
- verbally interjects but does not gain attention
- three instances: points to diagram but does not gain attention
= explanations directed at him by supervisor or research colleague
+ three instances: points to diagram and verbally participates
+ two instances: walks to blackboard to get closer look at diagram</td></tr>
<tr><td>6</td><td>+ two instances: addressed by supervisor and contributes
+ resolves visiting researcher's misunderstanding</td></tr>
</table>

		+ questions expert mathematicians' decision, which leads to conversation + converses mathematically with supervisor - two instances: points and gestures but does not gain attention
walks to blackboard draws diagrams points to diagram traces diagram asks questions speaks confidently speaks audibly gestures math terms	7	= two instances: explanation directed at him by supervisor + addressed by supervisor and contributes = two instances: supervisor points out his work + two instances: addressed by visiting researcher and contributes + contributes on his own + asks supervisor questions = asked by research colleague but supervisor takes over
	8	+ four instances: addressed by supervisor and contributes + four instances: asks supervisor questions + two instances: contributes on his own + questions supervisor's diagram, which leads to conversation + two instances: duo-draws with supervisor = two instances: summary directed at him by supervisor = theorem explained to him by supervisor = directions given to him by supervisor
	9	+ eight instances: asks supervisor or research colleague questions = addressed by supervisor but cannot contribute = four instances: summary directed at him by supervisor + three instances: addressed by supervisor or colleague and contributes + two instances: contributes on his own

The accounts about Finn in Sub-Sections 5.2.1 to 5.2.9 provide insights into his journey as a mathematics graduate student. Of the three phases of graduate studies that Thornley (2010) identifies, Finn is clearly in the first phase of high-degree involvement from his supervisor, since Finn has only just arrived to begin his graduate studies. Indeed, the research meetings demonstrate that Finn has not yet taken sole responsibility for the research but is still reliant on the contributions of and guidance from the expert mathematicians. However, the research meetings have also shown that Finn explores the 2-regular digraphs outside the research meetings, and that he is developing some insights of his own into the behaviour of graphs. Two of his principal difficulties are gaining the attention of the expert mathematicians and being able to communicate like them.

The research meetings offer a multitude of opportunities for Finn "to ask interesting and important questions, to formulate appropriate strategies for investigating these questions, to conduct investigations with a high degree of competence, to analyze and evaluate the results of the investigations, and to communicate the results to others to advance the field" (Golde, 2006, p. 10), which Golde identifies as the competency called *generation*. Moreover, the research meetings allow Finn to observe how the expert mathematicians enact generation. Each of the expert mathematicians models for Finn how to propel research by asking pertinent questions about obstructions and embeddings; applying such tools as the Euler characteristic from graph theory or cross caps from topology; inventing new tools such as vertex splitting and edge jumping; devising strategies to deal with a particular graph or a particular surface but also generalizing their research ideas to discover relationships among graphs or common behaviours among surfaces; and most importantly, analyzing and evaluating their findings to set a further course of action. Lastly, because the mathematicians often continue working on the mathematics of the research meeting between their weekly meetings, they also model how to communicate their individual findings to each other during the next research meeting.

This scientific discourse is the most significant type among the six types of tacit knowledge a graduate student has to acquire to become a researcher, according to anthropologist Tomas Gerholm (1990, p. 266). Indeed, the research meetings show that the speech of the expert mathematicians is tightly linked with the bodily enactments of mathematical objects, and that there is a flow of communication among the expert mathematicians. Furthermore, the discourse of the expert mathematicians often centres on diagrams and is heavily saturated with gestures of pointing, hand-pointing, touch-pointing, holding, tracing, sweeping and covering up. Finn's utterances, on the other hand, are initially tentative, short and unaccompanied by gestures other than pointing. Finn also is not often given the opportunity to create diagrams on the blackboard and engage with them. Lastly, the units of discourse in terms of deictic and mathematical words provide another source of comparison between Fred and Finn. Fred hardly ever uses the impersonal "we", whereas most of Finn's initial speech is based on this mathematical "we". In Finn's earlier attempts, he also tries to speak in perfect sentences using indicative mathematical terms, while Fred and the other expert mathematicians freely employ spatial and temporal deictic words. Indeed, the manner in which Fred speaks is best described in the words of Davis and Hersh (1980), who state that "[t]o his fellow experts, he communicates these results in a casual shorthand" (p. 36). This points to the huge discrepancy between the informal discourse among mathematicians and the formal discourse of the mathematician's presentation of work at a talk or in paper form that Finn as a mathematics graduate student encounters. As part of the enculturation process, Finn has to learn to differentiate between these two types of mathematical discourse, and be able to accomplish both himself.

Mathematicians like Thurston (1994) have pointed out this ambivalent state of mathematical discourse. While there has been much focus in K-12 education on how to build bridges between everyday language and the formal language of mathematics over the past two decades, as my literature review shows, few studies address how less-expert mathematicians at the level of graduate student learn to navigate between formal and informal mathematical discourse. Finn, as the less-expert mathematician, is the one initiated into the mathematical subfield by Fred as the expert mathematician. While the mathematizing between graduate student and supervisor is “fragmentary, informal, intuitive, tentative” (Greiffenhagen & Sharrock, 2011, p. 854), it is also enabled through the diagramming and gesturing. To summarize, Finn has to learn the characteristics and uses of mathematical discourse, and Table 5-15 shows that it takes many research meetings for Finn to learn how to join in the diagramming, gesturing and mathematizing, and how to hold his own during mathematical discussions. In the sense of Lave and Wenger's (1991) legitimate peripheral participation, through these research meetings Finn learns about the social and cultural practices in mathematics that go beyond learning how to talk and present mathematics both formally and informally. From this point of view, Finn also acquires attributes such as communication skills, willingness to engage on the blackboard, motivation and resourcefulness at trying out ideas, and perseverance and flexibility when a solution does not readily present itself. Thereby, Finn forms an identity that becomes ever more closely aligned to that of a mathematician.

I now address the moods that I was drawn to during the viewing of the research meetings in order to elucidate another way in which Finn gets enculturated to mathematics. First of all, the wide spectrum of moods that I observed during the research meetings, ranging from aha-moment, excitement, confidence, joy, laughter, evenness, silence, hesitation, doubt to puzzlement point to the human nature that resides in mathematizing which just does not exist in the mathematics of textbooks and lectures.

Although disappointment occurs during the research meetings, I do not identify it as a mood, because it was extremely short-lived, often dismissed with a shrug or laughter. Indeed, it does not serve a mathematician, or any other researcher for that matter, well, if he succumbs to disappointment because an idea does not work out. However, I do identify silence as a mood, not only because I was surprised at finding silence so prevalent during the research meetings, but also because silence is often the place of substantial diagramming. Supervisors as “[m]athematicians are significant figures in the lives of mathematics graduate students” according to Beisiegel (2009, p. 47). In this sense, the many silences that occur during the research meetings are indicative of the space and time that the expert mathematicians need in order to mathematize. In terms of Roth’s “living/lived mathematical experiences” (2011, p. 15), the shared silences provide powerful experiences for Finn not only in terms of Carlson’s (1999) contributing factors in mathematizing such as incorrect attempts, mathematical enjoyment, individual effort, independence and persistence, but also what it means to *be* a mathematician, namely an ultimately solitary endeavour.

Lastly, I speak to the mentorship and power struggles between the graduate student and the supervisor, which also play a role in the enculturation process of the graduate student. Advice on mentorship particular to mathematics graduate students has been published by Krantz (2003), Burton (2004) and Billey (2012) in varying degrees of relationships (close friend to distant observer) between the supervisor and graduate student. However, all researchers agree that mentorship is important to facilitate two-way communication in the graduate student’s journey in becoming a mathematics researcher. The interactions between Fred and Finn demonstrate that their supervisor/graduate-student relationship is healthy. Finn is an eager participant in the research meetings, who keeps up with the work that is expected of him between meetings. More than once the expert mathematicians ask him about the findings of a particular graph, and Finn is often able to shed light on these findings. While in the first few research meetings, Fred at times cuts Finn off from contributing, he more often draws Finn into the mathematizing, and, above all, ensures that Finn keeps up with the development of the mathematics by frequently summarizing for him or asking questions.

There is no denying that Finn is in an unusual research meeting setting. Typically, graduate-student/supervisor meetings are held in the supervisor's office (Herzig, 2002; Stewart, 2006), which, from my experience, have little board-space for writing, whereas Finn's research meetings are run like informal seminars with ample blackboard space. Nonetheless, the data analysis of the nine research meetings supports the findings in the research area of graduate studies that socialization, mentorship and power struggles contribute to the process of becoming a researcher. Furthermore, the data analysis supports that all three expert mathematicians exhibit similar verbal and non-verbal communication patterns during mathematizing. The expert mathematicians' discourse during explorations is laden with deictic words, and mathematical terms are often loosely or imprecisely used. For example, in topology, the technical term 'embed in' is used when there is a homeomorphism between two topological spaces *X* and *Y*. The term 'embed on' is incorrect. However, all three expert mathematicians use the terms 'embed in' and 'embed on' interchangeably in speech. Table 5-16 provides some examples of Fred using either 'embed in' or 'embed on'. Perhaps the term 'embed on' is used so prolifically because the mathematicians explore the topological spaces 'on' the blackboard.

Table 5-16. Transcription of Utterances by Fred Using the Term 'Embed'

Research Meeting	Time	Fred's utterances
F1-v2	01:19	If you could embed this one in the plane!
F2-v1	09:33	Even testing if this thing embeds in the projective plane is not so easy, right?
F5-v2	09:53	So, so, this tells us that funny octahedron graph has a unique embedding in the torus and a unique embedding in the Klein bottle.
F6-v1	28:21	Wait, I (.) sorry. I mean, I think, I have a proof that k five doesn't embed on the torus. (.) Which seems shocking. (.) But let me try.
F8-v1	20:47	It should but then that means it shouldn't embed on the Klein bottle either.
F9-v1	08:10	How does this embed on the torus?
F9-v2	23:51	Was funny octahedron uniquely embedded in the torus?

Furthermore, the experts' informal discourse is accompanied by an abundance of gestures that attempt to express the mathematics that is being explored with their hands and arms, whether it is in the air or on the blackboard during diagramming, and also specific gestures of pointing at, hand-pointing at, touch-pointing, tandem pointing at, holding, sweeping, tracing, staring at, and covering up a diagram. Because my analysis rests on three expert mathematicians, these acts of diagramming and gesturing during mathematizing offer profound insights into the culture of doing mathematics at the research level. As supported in the literature review provided in Chapter 2, in order for the mathematics graduate student to grow into an expert mathematician, he must not only know the mathematics of his field, but he must also become accustomed to principles and conventions of mathematical communication so that he becomes a more effective mathematician.

> Mathematics is a staggeringly fragmented discipline whose practitioners must master the art of communicating without co-understanding. Indeed, mathematicians seem persistently preoccupied with sharing their work with each other, boldly blinding themselves to the petty incommensurabilities of their studies in order to join, on scales ranging from meetings with collaborators to international congresses, in mutual mathematical activity. (Barany & MacKenzie, 2014, p. 5)

This implies that learning how to deal with other mathematicians is part of learning how to become an expert mathematician. While my data are insufficient to conclude that Finn's personal growth during these nine meetings is representative of the enculturation process of mathematics graduate students in general; I, nonetheless, posit that diagramming and gesturing during mathematizing are important aspects in the enculturation process of a graduate student into research mathematics.

Chapter 6.

Unfolding Diagramming and Gesturing

I begin my data analysis of diagramming and gesturing by first explaining what I mean by the term *unfold*. Because the word unfold is composed of the prefix *un-* meaning "opposite of" and the verb *fold*, I performed an etymological search on both *fold* and *unfold* (www.etymonline.com, 2015). The origin of the word fold can be traced as far back as Proto-Indo-European (*pel-to-* meaning "fold") from which it entered the Proto-Germanic languages (*falthan, faldan*) and then Old English (*faldan*). The word fold is transitive and means "to bend (cloth) back over itself, wrap up, furl". The word unfold has developed to mean "to open or unwrap the folds of", and also figuratively to mean "to disclose, reveal, explain". The double meaning of the word unfold is the reason why I have chosen to use it for the title of my thesis. Not only is the data analysis revealing the diagramming and gesturing that occurs during the research meetings as already shown in Chapter 5, but the data analysis is also providing evidence that there is indeed a layering of mathematics which can be unfolded from the virtual to the actual gestures. However, before I present the unfolding of diagramming and gesturing in terms of Châtelet, I first revisit the activity units that I designed for data transcription, as these are used to analyze the gestures of the mathematicians centred on the creation of and engagement with a diagram.

6.1. Activity Units Revisited

Table 4.3 from Chapter 4, *Methodology*, lists the activity units for gestures, actions and postures clustered around an engagement and creation of diagram that are designed from the viewpoint of the mathematician. I revisit these activity units that I designed for data transcription in order to critique my choices, and also to juxtapose the activity units from the viewpoint of the mathematician with those from the viewpoint of the diagram.

6.1.1. Critique on Activity Units

My critique of the activity units is based on their usefulness in both viewing and transcribing the video-recording. Even though I could have counted the number of times I have used each activity unit in the transcriptions and descriptions presented in Chapter 5, this count would leave out the multiple viewings of the video-recordings on which the data-viewing notes are based (see Appendix A for some sample pages), which contribute to my sense of the appropriateness and significance of the activity units in capturing the non-verbal communication of the mathematicians. Based on the five clusters of activity units that I created, I discuss which of the activity units are the most fruitful and which are less essential.

The activity units for the cluster *engaging with blackboard* (walks to blackboard, steps away, steps far away, returns to seat, turns away) are all essential in describing how the mathematician deals with the space between himself and the blackboard. In Chapter 5, I pointed out that the graduate student has to learn to cross this space, while the expert mathematicians are completely at ease in this space and in command of their proximity to the diagram on the blackboard. Furthermore, the activity units *walks to*, *steps away* and *steps far away* are necessary to describe the dance-like character of the steps that the expert mathematician performs in getting closer or further away from a diagram on the blackboard. If anything, the additional activity unit *steps closer* would have fine-tuned the movement rather than just using *walks to blackboard*.

In the cluster *engaging with mathematikoi*, all three activity units (faces mathematikoi, turns to mathematikoi, gestures a mathematical object/action) are vital. This comes as no surprise, as the fundamental component of a research meeting is communication. However, during the data-viewing, it struck me how the gestures of mathematical objects and actions by the expert mathematicians that accompany their speech stood out in the absence of these gestures in the graduate student. I created one additional activity unit, namely *ignores mathematikoi*, because this is needed to describe either a mathematician's deep engagement with the mathematics, ignoring all outside input, or to expose that a mathematikoi's attention-seeking efforts are ignored by the other mathematicians.

I generated the cluster *creating diagram* (draws diagram, stops drawing, continues drawing, duo-draw) based on the actions that are necessary to create a diagram and in case of the last activity unit *duo-draw* (or trio-draw, quartet-draw) what I observed during the data-viewing. Obviously, when the mathematician draws, he will eventually stop drawing and sometimes continue drawing. While it is essential that I note the drawing, there are only a few instances in the data viewing when it is noteworthy to point out that the mathematician stops drawing or continues to draw, such as after prolonged drawing or the interruption of drawing by another mathematician that leads to new insights and an alteration of the diagram.

There are at least two intervals of high diagramming activity during each research meeting. Especially during these times, the activity units in the cluster *engaging with diagram directly* (erases object, draws over object, adds to diagram, redraws diagram, starts over) are indispensable in describing the direct engagement of the mathematician with the diagram. In particular, the activity unit *starts over* is a significant action by the mathematician that is often accompanied by meta-statements: e.g., “shall we just try this again?”, “make a new picture”, “something is funny”, “it doesn’t look right, right?” or “what happened to the diagram here?”. Furthermore, I was surprised to observe the action of redrawing a diagram multiple times by the expert mathematicians. Not only were there redrawings of diagrams that accentuated details in the diagram or stripped the diagram down to its essential parts, but often there were redrawings that did not change the diagram at all or caused only a minimal change such as a rotation, scaling or relabelling of the original diagram. For example, compared to the original diagram (Figure 6-1, left), the redrawn diagram (Figure 6-1, right) is drawn to a smaller scale and the direction of the interior edges is the reverse of those on the left. The latter is an important change in the redrawing, since the various orientations baffled the mathematikoi (see Section 6.3).

Figure 6-1. F2-i01, Fred’s original (left) and, F2-i15, redrawn diagrams (right)

By far, the most frequently-occurring activity units – with the exception of *ignores diagram* and *caresses diagram* – are the ones in the cluster *engaging with diagram in a material sense* (distant: turns to diagram, stares at diagram, ignores diagram, discards diagram, points to diagram, hand-points to diagram, tandem-point to diagram; close: sweeps object, traces object, caresses diagram, covers up object; contact: touch-points object, holds object). First, I comment on the exceptions. Both *ignores diagram* and *caresses diagram* are difficult activities to assess. While an absence in speech, or a posture turned away from a diagram, can be taken to imply that a mathematician is not acknowledging a diagram, it is often unclear in such instances: sometimes the mathematician comes back to the diagram later in the research meeting or even during another research meeting. Strictly speaking, the term *caress* indicates a gentle or loving touch. While I frequently observed a multiple tracing or sweeping by a mathematician that made me *feel* like the mathematician was caressing the diagram, I hesitated to use this activity unit.

The remaining activity units listed in this cluster are crucial in describing how the mathematician interacts with the diagram during diagramming and mathematizing. For example, the actions of redrawing and starting over often necessitate a material engagement with the original diagram. The mathematician *points*, *touch-points*, *holds, traces* and *covers up* the original diagram when creating a new diagram. This multitude of senses that the mathematician evokes – sight, touch, kinesthetic – allows the mathematician to access more information in the original diagram and to virtually reassemble the original diagram to become the new diagram. In addition, these particular actions are constantly performed when the mathematician explains, elaborates on or explores a mathematical concept. The activity units *turns to diagram*, *stares at diagram* and *discards diagram* are necessary to describe how the mathematician acknowledges the diagram, is captivated by the diagram, and finds the current diagram unsatisfactory. *Points to diagram* and *hand-points to diagram* are typically used by the mathematicians from a distance, while *sweeps* is usually employed when close to the diagram. These three activity units are the most commonly utilized gestures and the first observable gestures by the graduate student. Indeed, Kendon and Versante (2003, p. 239) observe that three types of pointing are typically used for individuating a concrete object (such as the graph in the diagram) while three types of hand-pointing generally refer to the symbolic or conceptual status of the object (such as a particular class of graphs represented by the diagram). Lastly, *tandem-pointing* is a rare but significant action as it shows up in situations when mathematicians are explaining some concept to each other or are coming to an agreement using a diagram: just like their utterances merge, so do their actions.

6.1.2. Activity Units from the Diagram's Viewpoint

The activity units outlined in Section 4.5 describe the gestures, actions and postures by observing what the mathematician does. This was a choice I made in order to bring the data closer to the reader. As the researcher, I chose to interpret what the diagram does, and not just what the mathematician does. Therefore, I could have described the gestures, actions and postures by choosing the viewpoint of the diagram. For example, the activity unit *stares at diagram* names the action by observing what the mathematician does. That same activity unit could be changed to *transfixes mathematician*, and now the action of the diagram is described rather than that of the mathematician. A more balanced activity unit would be *mathematician communes with diagram* as this expresses that both parties involved – the mathematician and the diagram – act on each other. By pretending to be situated just behind the blackboard upon which the diagram resides and looking out towards the mathematikoi, I have gone through the exercise of re-labelling the activity units from the viewpoint of the diagram (see Table 6-1). Following Châtelet, I want to draw attention to the diagram as a material object that can not only be acted upon, but that can also act upon the mathematician and the mathematikoi.

Under the theme of *engaging with mathematikoi*, none of the four activity units can be described from the viewpoint of the diagram, as these activity units are about the engagement between mathematicians. Therefore, these four activity units are not applicable to the diagram. This made me wonder if there are some activity units from the viewpoint of the diagram that are not applicable to the mathematician. Certainly, the diagram on the blackboard is always facing the mathematikoi, but whether the diagram is acknowledged by the mathematician or not depends on the need of the mathematician. I have demonstrated in Chapter 5 that the diagram can pull the mathematician, which is evidenced in action when the mathematician turns towards the diagram in order to complete his utterance. However, the diagram can also simply be facing the mathematician without acting on the mathematician, and therefore, this activity unit is not applicable to the mathematician. In addition to the newly proposed activity units *steps closer* and *ignores mathematician* from Sub-Section 6.1.1, both activity units *faces mathematikoi* and *pulls mathematician* are added to Table 6-1 compared to Table 4-3. All four of these activity units are shaded in gray in Table 6-1 since these are activity units that are new compared to Table 4-3.

Table 6-1. Activity units from the viewpoints of mathematician and diagram

Theme	Activity unit from the viewpoint of mathematician (*object* refers to a mathematical part of the diagram)	Activity unit from the viewpoint of diagram
engaging with blackboard/ mathematician	walks to blackboard	attracts mathematician
	steps away from blackboard	pushes mathematician away
	steps far away from blackboard	pushes mathematician further away
	steps closer to blackboard	attracts mathematician closer
	turns away from blackboard	deflects mathematician
	returns to seat	disengages mathematician
engaging with mathematikoi	faces mathematikoi	n/a
	turns to mathematikoi	n/a
	gestures *a mathematical object/action*	n/a
	ignores mathematician	n/a
	n/a	faces mathematikoi
	turns to diagram	pulls mathematician
creating diagram	draws diagram	emerges
	stops drawing	pauses
	continues drawing	continues to emerge
	duo-draw	duo-emerge
engaging with diagram directly	erases *object* / erases diagram	loses *object* / ceases to exist
	draws over *object*	engorges *object*
	adds *object* to diagram	gains *object*
	redraws diagram	re-emerges
	starts over	has nothing more to offer
	turns to diagram	draws mathematician’s attention
	stares at diagram	transfixes mathematician

engaging with diagram in a material sense at several levels of intimacy	ignores diagram	draws no attention
	discards diagram	repels mathematician
	points to diagram	attracts mathematician's finger
	hand-points to diagram	attracts mathematician's hand
	tandem-point to diagram	attracts mathematicians' fingers
	sweeps *object*	*object* attracts mathematician's hand
	traces *object*	*object* attracts mathematician's finger
	caresses diagram	caresses mathematician
	covers up *object*	*object* withdraws from view
	touch-points *object*	*object* touches mathematician's finger/hand
	holds *object*	*object* holds mathematician's touch

6.2. Phases and Episodes in the Life-cycle of a Diagram

Throughout the research meetings, the mathematicians created an abundance of diagrams. This was expected since their research is in the area of Topological Graph Theory, and the premise of their research meetings is one of exploration. Almost all diagrams in the data are created on the blackboard except for: those that the graduate student draws in his notebook for record keeping; the few diagrams that the research colleague draws in his notebook; and those created in the two instances when the supervisor sits next to the graduate student diagramming on a piece of paper. My references are therefore to diagrams on the blackboard. Interestingly, the word "diagram" was uttered only once in all nine research meetings. The mathematicians referred to the diagram by mathematical words such as "crossing land", "anti-digon", "surface" or "graph" and even more often by deictic words such as "this", "this guy", "this creature", "this one", "this thing", "here" or "it" to name but a few; and the closest they came to calling their drawings a diagram was with the occasionally-used words "picture" and "drawings". This is evidence in itself that, for the mathematician, the diagram is not just a visual product, but also a mathematical object in its own right.

During the data-viewing, I noticed that not only does the diagram undergo transformations through the creation and engagement by the mathematicians, but also the relationship that exists between the diagram and the mathematicians, which reminded me of a life-cycle. Therefore, I became interested if there are characteristics that allow me to set one type of diagram engagement apart from other types. I analyzed each of the 122 time intervals selected for viewing and identified distinct characteristics surrounding the creation and engagement with the diagram.

Before I elaborate on the process that led me to identify these distinct characteristics, I explain why I refer to the time interval that depicts a particular type of diagram engagement as a *diagram episode*, follow this with a summary of my findings, and conclude by detailing the life-cycle of a diagram. I chose the term *episode* because it alludes to both a period of time and a side-story being told. An etymological search shows that the word episode can be traced back to the 1670s, where it was used as "commentary between two choric songs in a Greek tragedy" and also "an incidental narrative or digression within a story, poem, etc." (www.etymonline.com, 2015). The origin of the word is derived from the French *episode* literally meaning “an addition”, which is constructed from the two Greek words *epi* meaning “in addition” and *eisodos* meaning “a coming in, entrance”. A general dictionary search finds that episode stands for *an event or a group of events occurring as part of a larger sequence; an incident or period considered in isolation*. The word *diagram* in the phrase *diagram episode* acts as an adverb, and I use the terms *diagram episode* and *episode of a diagram* interchangeably.

My analysis shows that throughout the life-cycle of a diagram, from its creation to its ‘establishment’ or ‘obliteration’, there are eleven distinct diagram episodes: *emerging*, *present*, *unsupportive*, *disruptive*, *supportive*, *pulling*, *central*, *discarded*, *obliterated*, *absent* or *established* (see Figure 6-2). I have deliberately not used mathematical labels in this figure of the life-cycle, even though everything in this thesis pertains to mathematical enquiry. The reason is that I want to emphasize the relationship between the diagram and the people interacting with it; indeed, I believe that the diagram life-cycle presented here might be more widely applicable outside a strict mathematical context.

There are perhaps other diagram episodes that I have not captured here, because my data did not contain them. For example, the mathematikoi did not work with any textbook and so did not use published diagrams from such a source. Maybe such diagrams, which are not created out of the exploration by the mathematikoi, allow for a different type of engagement that does not fall into any of the eleven diagram episodes that I have identified.

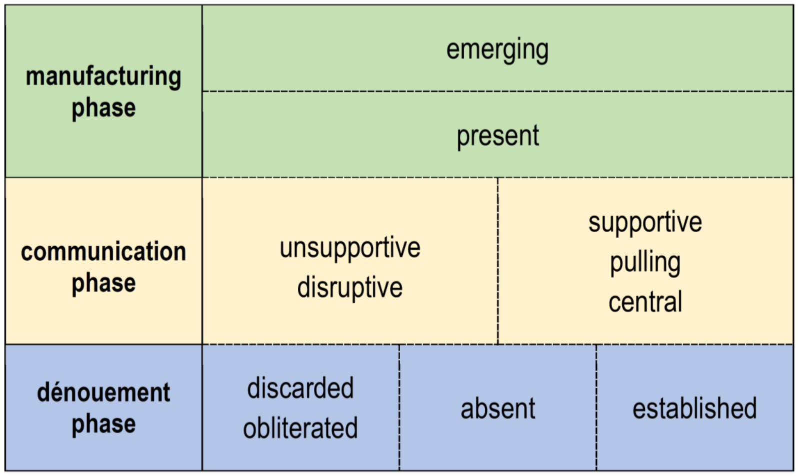


Figure 6-2. Phases and Episodes of a Diagram

I now describe the process that led me to identify distinct diagram episodes. During the data-viewing of all these diagrams, I realized that I needed a succinct label in my data-viewing notes to describe what I called "illuminating the role of the diagram" in Section 4.4 (see Appendix A for some sample pages). I needed to manage the amount of data surrounding the approximately 200 diagrams, and more importantly, identify differences and similarities surrounding the creation of as well as engagement with diagrams. For example, from early on, I observed that there were particular settings when a diagram is created such as when a mathematician sketches a diagram from scratch or a mathematician uses an already existing diagram in order to draft a new diagram. I refer to all different manners of creating a diagram as *diagram-is-emerging* (see Section *Diagram-is-Emerging*), which is similar to Châtelet's "diagrams that capture [gestures] mid-flight" (2000, p. 10). However, my analysis is based on both the physical and virtual gestures that take place in a research meeting and are recorded on video, which narrows the period of emergence; whereas Châtelet had access only to manuscripts, and therefore could not address the period of emergence. Furthermore, I noticed that once a diagram is drawn on a blackboard, it is *present* on the blackboard, regardless whether any of the mathematikoi engages with it or refers to it in speech (see Section *Diagram-is-Present*). Therefore, a diagram's presence made me pay particular attention to when and how the diagram is interacted with from that moment onwards. Through this process, I identified a variety of different characteristics which led to further diagram episodes. For example, a diagram can also *support* a mathematician among other possible episodes of a diagram (see Section *Diagram-is-Supportive*). Since I repeatedly came across several different types of diagram episode during data-viewing, I was convinced that these diagram episodes are legitimate stages in the life-cycle of a diagram. Therefore, I no longer identified time intervals that contain particular types of diagram episode for further analysis during the complete data-viewing of the approximately 12 hours of research meetings, unless a diagram episode was coupled with some new interaction, which became exceedingly rare. Instead, I focused on fine-tuning how, when and why diagrams are created and engaged with in order to detect any new occurrences of diagram episode.

Lastly, similar to the life-cycle of a product in economics or the life-cycle of a being in biology, the life-cycle of a diagram can be divided into phases: (1) the mathematician brings the diagram into being; (2) the diagram communicates – positively, neutrally or negatively – with the mathematician during mathematizing; and (3) the mathematician resolves, in simplistic terms, whether the diagram stays on the blackboard or not. I termed these phases *manufacturing* (emerging, present), *communication* (unsupportive, disruptive, supportive, pulling, central) and *dénouement* (discarded, obliterated, absent, established) respectively (see Figure 6-2), which are expounded in the next three subsections along with their respective episodes. I end this section with a quantitative analysis and a discussion of the relationships among the episodes, which is hinted at in the layout of the terms in Figure 6-2.

6.2.1. Manufacturing Phase

The mathematician may introduce the diagram that he is about to draw with gestures that accompany his utterances and that are indicative of the diagram to be drawn (for an example, see *Diagram-is-Absent* in Section 6.2.3); or the mathematician does not know how to begin the drawing process, but he has the need for drawing, which is exquisitely expressed in the following utterance by Fred: "I would like to get better at *seeing* this sort of thing. ((laughs)) It take (.) er, I mean, it, it vaguely makes sense to me, I mean, it takes me a long time to verify this sort of thing. […] I just have to draw (.) um. Yah (.) just draw" (F4-v1 27:04 minutes). Regardless of the manner in which a diagram comes about, its emergence is a crucial episode in its life-cycle, because it is an attempt by the mathematician to actualize the mathematics that is being explored. While the drawing may not yet define mathematical objects, and the relationships among the objects may be tentatively placed in the diagram, the mathematician is intent on making his intuitive knowledge appear on the blackboard. In other words, during this episode, the diagram not only takes on its physical chalk mark form, but the diagram is also a material expression of the mathematics that is being investigated or understood. Furthermore, the emergence of a diagram has as a consequence that this diagram is given a presence on the blackboard no matter for how short or long a period of time. Thereafter, an engagement with the diagram may or may not occur, but because the diagram now exists in chalk marks on the blackboard, this engagement brings about different types of diagram episodes as explicated in Sub-Sections 6.2.2, *Communication Phase*, and 6.2.3, *Dénouement Phase*. Hence, the *manufacturing phase* of a diagram encompasses the two episodes *diagram-is-emerging* and *diagram-is-present*, which are elaborated on after I explain the choice of the word *manufacturing*.

I have chosen the term *manufacturing* for this phase in the life-cycle of the diagram to echo the attention to the *hands* as the bodily part that executes drawing and gesturing, which Châtelet (2000) draws attention to with the phrase “talking *in* the hands” (p. 11, *emphasis in original*), and distinguished communication researcher Jürgen Streeck exposes in his book *Gesturecraft: The Manu-facture of Meaning* (2009). In order to bring the reader closer to his embodied view, Streeck likens the hand/drawing-gesturing connection with the vocal-organ/speaking connection. He emphasizes “that the structure of (spoken) human languages is to some extent shaped - or constrained - by the properties of our vocal organs”; and therefore, “[t]he ability to process language sounds, to produce and recognize very small articulation differences, is a contingent and convergent by-product of other developments, but it is nevertheless essential to the structures of spoken human languages which, in all their diversity, make systematic and economic use of these possibilities that the human body provides” (p. 39). Similarly, drawing and gesturing depend on the hands’ kinesthetic and haptic abilities alongside the experiences that reside in the person who manufactures. Streeck pointedly concludes: “hands know best” (p. 58), and therefore, the term *manufacturing phase* befits the initial phase in the life-cycle of a diagram.

Diagram-is-Emerging

A diagram is emerging every time a mathematician makes chalk marks on the blackboard that are not writings. The scope of the final diagram can encompass a wide range of detail, from just a few strokes to elaborate, colourful drawings. The period of emergence lasts from the beginning of the chalk markings until the mathematician steps or turns away from the blackboard, in which case the diagram is now *present*. Any of the mathematicians may add to the original diagram or erase parts of the original diagram; this signals the emergence of another diagram based on the original diagram. I identified five settings when the episode diagram-is-emerging comes about: (1) the mathematician explores some mathematics and diagrams from scratch (see Appendix B, Table B-2, row four: FS,C duo-draw), which occurs at a minimum at the beginning of the first six research meetings and also during sustained high diagramming activity such as in research meetings 3, 5 and 9; (2) the mathematician adds to an existing diagram, thereby altering it to a new diagram (see Table 5-13, row two: FG adds to diagram; Appendix B, Table B-2, row three: C continues drawing; and Appendix B, Table B-3, row three: FS adds to diagram); (3) the mathematician produces a new diagram based on the information from another diagram, which occurs at the very least every time the mathematikoi explore if a given graph on the blackboard embeds by tracing the edges and vertices of the given graph in order to draw the embedded faces; (4) the mathematician retrieves a known diagram from memory (see Table 5-10: FS draws diagram; and Table 5-13, row one: FG draws diagram 1 and 2); and (5) the mathematician is directed by another mathematician how to draw a diagram (see Table 5-11, row two).

Diagram-is-Present

As soon as the diagram has emerged, the drawn diagram is present on the blackboard no matter for how short or long a time period. This is similar to a puddle of water that has emerged in my garden after some rainfall: regardless of my seeing the puddle or not, the puddle is there. Therefore, I call such an episode diagram-is-present, and this episode can lead to any of the other nine episodes, similarly to the presence of the puddle possibly leading to my child playing in the puddle, a bird taking a sip of its water, more water being added to the puddle due to more rainfall, or the puddle just being there to name but a few examples. While the puddle eventually dries up, and therefore ceases to exist, a diagram, on the other hand, either stays present, returns to being present after some virtual or physical engagement by the mathematician, or is erased from the blackboard. All the data provided so far containing a diagram provide examples for the episode diagram-is-present; still, I now offer two specific examples of diagram-is-present:

- Appendix B, Table B-1, row two: In this episode, Fred draws a diagram, then stops drawing. As he turns to face the mathematikoi, his body covers up the diagram. He proceeds to verbalize the behaviour of the octahedron as a planar graph accompanied by gestures when he utters the words “c three”, “antipodal” and “projective plane”. In this example, the diagram emerges, and then is present, although not visible. More significantly though, Fred’s utterances and gestures are still engaged with the diagram and already project the completed graph before it is even completely drawn. This is a *live* example that demonstrates what Châtelet coins “actualizing the virtual”. While Fred physically gestures during speech, these gestures also virtually mould the diagram into being until the actual chalk marks create the diagram on the blackboard.
- Appendix B, Table B-2, row two: While Colin attempts to verbally explain the relationship among immersions in different surfaces, Fred is not convinced that the splitting process leads to this relationship. During this exchange, Fred turns away from the diagram, then stares at the diagram, and finally faces the mathematikoi. In this episode, the diagram is present and only engaged with through Fred’s staring.

6.2.2. Communication Phase

When a diagram is created by any of the mathematicians, then, there is usually a subsequent engagement with the diagram, because of the information that the diagram holds for the mathematician. As explained in Sub-Section 4.5.1, this engagement with the diagram can be direct or in a material sense at several levels of intimacy through the mathematician’s gestures, actions and postures. My data analysis shows that, through this engagement, the diagram itself – or what it stands for – affects a mathematician’s exploration and understanding of the mathematics in a positive, neutral or negative way. These are the episodes that I labelled *disruptive*, *unsupportive*, *supportive*, *pulling* and *central*, and which I elucidate shortly. These labels do not refer to properties of the diagram and do not evaluate the diagram’s mathematical correctness but, rather, these labels indicate how the information that the diagram holds relates to the knowledge of the mathematician. This suggests that there is a two-way communication between the diagram and the mathematician about the mathematics under exploration. Therefore, I grouped these diagram episodes under the term *communication phase*.

There are episodes when a diagram no longer has any such effect as just described on the mathematician; instead, the mathematician has come to an understanding about the mathematics and is now making a decision regarding the status of the diagram: whether it continues to exist or not. These are diagram episodes that belong to the dénouement phase (see Section 6.2.3). While it is possible that a diagram goes through all three phases in its life-cycle, there are diagrams that skip the communication phase. For example, those diagrams that emerge and then stay present, because they are never engaged with; or those diagrams that emerge, are present, and are then erased from the blackboard. In other words, the life-cycle of these diagrams skip the communication phase, and move from the manufacturing phase directly to the dénouement phase.

Diagram-is-Disruptive

When the diagram reveals something to the mathematician that is unexpected, or when the information in the diagram is in discord with the knowledge of the mathematician, then the thinking track that the mathematician is on is interrupted. Therefore, I termed this kind of episode *diagram-is-disruptive*. Such episodes are usually quite tumultuous and often end with laughter. The onsets of this kind of episode are typically speech repairs or exclamations such as "oh" and "ah" in the utterances of the mathematician who is engaged with the diagram. Furthermore, sometimes the mathematicians erupt almost simultaneously in speech. During the ensuing discussion, there is an equal use of spatial deictic words as well as mathematical terms, and often a mathematical object is coupled with a spatial deictic word such as "this edge" or "those vertices".

The transcriptions in Table 5-3, the third row in Table 5-7, and the third row of Table B-1 (see Appendix B) are examples of episodes where the diagram does not show to the mathematician what it is supposed to show, thereby creating episodes of diagram-is-disruptive. In the first and second examples all mathematikoi are affected by the diagram's disruption, while in the third example only Fred is interrupted by the diagram. The following is a detailed example of diagram-is-disruptive that has the element of unexpectedness. About three-quarters of the way through research meeting 2, the mathematikoi have agreed that the faces of the embedding could be four 3-cycles and two 4-cycles, and everyone is attempting to create this embedding. Fred is the only one working on the blackboard. He has just spent four minutes repeatedly walking to the blackboard, tracing edges, erasing edges, drawing over edges, and stepping away from the blackboard (see Appendix B, Table B-3, row three). Now he stares at the resulting diagram and then exclaims "*oh*! It's not three, three, three, three, four, four. It's *five* threes and a five" (see Appendix B, Table B-3, row four). Because the premise of the exploration was to look for four 3-cycles and two 4-cycles, the information that the diagram reveals to Fred after his diagramming exploration comes as a surprise.

Diagram-is-Unsupportive

The episode *diagram-is-unsupportive* turns out to be rare within the 122 selected time intervals, but still I identified two settings of this type of diagram episode. The first setting is when a mathematician uses a diagram to explain some mathematics to another mathematician, and this other mathematician does not follow. Neither the verbal utterances nor the diagram and accompanying gestures help with understanding. For example, the time interval from 08:46 minutes to 09:19 minutes during research meeting 6 (see Table 5-10 for a transcription) shows that the diagram is first disruptive and then unsupportive; only the newly drawn diagram provides supportive. The second setting is when a mathematician interacts with a diagram to further his own understanding; however, the diagram is not supporting the proposed mathematics (see Appendix B, Table B-3, row two). More details are provided in the next section, when I elaborate on a time interval that exhibits episodes of both diagram-is-unsupportive and diagram-is-supportive.

Diagram-is-Supportive

In contrast to the episode diagram-is-unsupportive, the episode *diagram-is-supportive* is the second most frequent episode (after *diagram-is-central* – see below). This comes as no surprise as the mathematicians choose to resort to diagramming over computing in order to understand the underlying principles of obstructions, which is captured when the supervisor utters: "So, I guess, I wanna know what that looks like? How did that happen?" (F3-v1, 32:39 minutes). Diagram-is-supportive refers specifically to any support that the diagram lends to understanding during mathematizing. During this type of episode, utterances are marked by mathematical terms (e.g., “edge”, “triangle”, “funny octahedron”, “split”) rather than spatial deictic words. Furthermore, the mathematicians involved often exhibit some direct engagement with the diagram by drawing over the diagram, adding to the diagram or erasing parts of it; but more typically, the mathematicians interact with the diagram in a material sense by staring at, pointing to, touch-pointing or holding the diagram or parts of it during speech. Through this amalgamation of speech and gestures by the ‘explaining’ mathematician, a three-way communication is opened between the diagram, the ‘explaining’ mathematician and the ‘understanding-seeking’ mathematician. The following are three examples of episodes diagram-is-supportive:

- Table 5-2: When Finn points at his diagram and utters “there”, Colin, who has been staring at Finn’s diagram, immediately picks up that the embedding contains triangles even before Finn is able to verbalize this fact. Finn continues to point repeatedly at his diagram, and Colin deduces that the remaining faces of the embedding must be quadrilateral in nature.
- Table 5-12: In this episode, Victor helps Finn understand which one of the two possible immersions Finn has found by touch-pointing the edges in the diagram while saying "you pulled these two edges off". This leads Finn to see the regular digon.
- Appendix B, Table B-3, row five: Fred has just discovered that the embedding consists of five 3-cycles and one 5-cycle. Colin now wants to know “so where is the fifth triangle” which Fred responds with “the outer directed triangle”. While Fred explains more, first Colin questioningly interjects with “two?” and then Victor with “yes?”. Then Fred makes his longest utterance (shown in the bottom of row five), which no longer provides any explanations; however, during this time, Fred adds to his diagram by colouring this outer directed triangle in red chalk. Now, Colin responds with “yah, this is good. Yah, definitely” and Victor with “there’s your five-cycle, the red, right?” to which Fred replies in the affirmative. This demonstrates that the verbal instruction was not enough for either Colin or Victor to follow Fred; however, through the addition of colour, the outer directed triangle materializes for both Colin and Victor, who now agree with Fred’s finding.

Lastly, I want to single out the transcription in Table B-2 (see Appendix B) as an example with episodes of diagram-is-supportive and diagram-is-unsupportive that showcases how a diagram keeps getting moulded by an expert mathematician until it supports the understanding of another expert mathematician.

- Row one (diagram-is-emerging; diagram-is-supportive to Colin and Victor; diagram is unsupportive to Fred): Fred draws a diagram to get the mathematikoi started with exploring the immersion of a 2-regular digraph with a 2-edge cut in different surfaces. While making several utterances, Fred repeatedly holds, touch-points and sweeps his diagram until he just stares at his diagram. The statements by Colin as well as Victor indicate that they understand that planar immersion leads to immersion in other surfaces; however, Fred's last statement indicates that he is not following.

- Row two (diagram-is-present): Colin attempts to verbally explain the relationship between planar immersion and immersion in any other surface, but he neither gestures nor engages with Fred's diagram.
- Row three (diagram-is-emerging; diagram-is-unsupportive to Fred): Now, Colin draws a new diagram while providing explanations, and then repeatedly either touch-points his diagram or continues to draw. Colin even makes a gesture of holding Fred's diagram; yet, Fred's utterances indicate that he is still not following the argument, and Colin starts over, drawing yet another diagram.
- Row four (diagram-is-emerging; diagram-is-supportive to Fred): Then Fred starts diagramming for himself while voicing Colin's explanation, whereas Colin continues drawing his new diagram. When Fred utters "*oh*! Okay" he stares at Colin's diagram, and then stresses "*I* see". From this time forward, his utterances indicate understanding of how the immersions work.

Diagram-is-Pulling

There are many instances during the research meetings when the mathematician, who holds the talking stick, is turned away from the blackboard and faces the mathematikoi while elaborating on their mathematical investigation. Among these scenarios, there is a high number of occurrences when the diagram that is left behind on the blackboard is not left behind for this mathematician. This is evidenced through the interruption of speech, which only continues once the mathematician has made often direct physical contact again with the diagram through gestures of pointing, hand-pointing, holding, touch-pointing or sweeping. Such an episode is often accompanied with energetic body movements such as walking back and forth between diagram and mathematikoi or sharp turns of the body alternately facing the diagram or the mathematikoi, which looks as if the mathematician is connected to the diagram by a *tether*. I term such an episode *diagram-is-pulling*, as elaborated on in a paper I presented at the *Mathematics Education Doctoral Students Conference* (Menz, 2014). There is one other setting that I also identify with diagram-is-pulling, namely when the mathematician stares at a diagram for a prolonged time interval neither moving nor speaking such as Fred exhibits in research meeting 2 (see Appendix B, Table B-3, row one). Such an episode reminds me of my cat Freckles, who would sit enraptured on the windowsill with body frozen but ready to pounce and with only the eyes following a bird outside.

Now, I present an example of the *tether*-like connection as evidence for the episode diagram-is-pulling (see Appendix B, Table B-4): Fred is explaining behaviour of faces based on the edges in a graph, when his speech becomes broken and his demeanour is that of someone searching. It is evident that he is looking for a diagram, which he announces himself: “what happened to the diagram over here?” Fred finally locates the diagram, touch-points its edges several times, and resumes with his explanations. When Fred moves the blackboard with the diagram on it up, he indicates with this act that he does not need the diagram any longer. The content of his speech supports this conclusion, because he is attempting to generalize. However, a short while later, his speech is once more interrupted by himself and resumes only after he again pulls down the blackboard and this time holds the diagram.

Diagram-is-Central

From early on, I noticed that there are times when either a single mathematician or all mathematikoi are so deeply engaged with diagramming and mathematizing, that these two activities seem almost inseparable. These occurrences stand out because of the high engagement with the diagram, regardless of whether the engagement is direct (e.g., drawing, redrawing, erasing, adding or drawing over) or in a material sense (e.g., pointing, hand-pointing, touch-pointing, holding, tracing or sweeping). Almost always, all levels of intimacy from distant, close to contact engagements are covered. Often, such an episode begins with one mathematician working on the blackboard, who is then joined by one or two more mathematicians at the blackboard, so that there is more than one mathematician working with the diagram on the blackboard. Sometimes the mathematikoi come to an agreement and I observe tandem-pointing. At other times, each of the mathematikoi needs to work out the mathematics for themselves, which is observable as duo-, trio- or even quartet-drawing. Utterances are heavily laden with modal verbs and spatial, personal and temporal deictic words; and despite the many uses of “this” and “that” the mathematikoi are able to communicate their ideas and understanding to each other. Every so often, the discourse breaks away from the use of spatial deictic words and mathematical terms dominate, which are accompanied by gestures often involving both arms that echo the features of the diagram that the mathematikoi are involved with.

Throughout my data-viewing notes, I labelled these episodes *necessary*, but a more appropriate description is *diagram-is-central*. The reason for the change in label is that central not only has the connotation that the diagram is necessary, but central also identifies that the diagram takes the centre stage. In other words, if I imagine that the diagram is taken out of the episodes diagram-is-central, then there will be little meaning left in the mathematizing, because the referencing and engagement rely heavily on the diagram. This is similar to Châtelet's *thought experiment* elaborated on in Chapter 3, which is the place of intuition and premonition where new objects are created and novel relationships are discovered. Châtelet's description that "[t]he thought experiment taken to its conclusion is a diagrammatic experiment in which it becomes clear that a diagram is for itself its own experiment" (2000, p. 12) is precisely what can be observed during episodes when the diagram is central. In the hands of a mathematician, the diagram becomes organic and is persistently moulded in the quest to uncover relationships. During this type of episode, there is a duality between diagram-is-central and diagram-is-emerging, because of the dynamic nature of the central diagram in the hands of the mathematican, who adds, erases and draws over the diagram, which is thereby constantly emerging anew.

The third row of Table B-3 (see Appendix B) is an episode of diagram-is-central, where Fred, as the single mathematician working on the blackboard, repeatedly traces edges, erases edges, draws over edges, steps away from the blackboard, stares at the diagram, and walks back to the blackboard during a time period of over four minutes. During this time, no utterances are being made except for some brief mumbling by Fred at the beginning; and while the other mathematikoi are not visible in the camera view, my field notes indicate that they are silently working on paper.

Table 5-6 is an episode of diagram-is-central that involves Fred, Finn and Victor with Fred standing near the blackboard and everyone else seated facing him. This episode begins with Fred pointing at a diagram and talking about the obstruction they know, which Victor audibly agrees with. This is followed with a short monologue by Fred, in which he aptly summarizes: "We know that this whole family of excluded immersions are the ones we know". During this time, Fred points, sweeps and traces the diagram, which is interspersed with many gestures of mathematical terms such as “family” and “not contain” that he voices during his speech. These gestures echo the diagram that represents the family of immersions. Then Finn points out that the diagram is the funny octahedron, which prompts a brief discussion about the diagram among the mathematicians. At the end, Fred agrees that the diagram leads to the octahedron and is a representation of the family of immersions they have found. The episode ends with Victor grabbing the talking stick and generalizing their finding, all the while facing Finn. Now, Victor points to the diagram and makes big, bold gestures using both of his arms while he utters the mathematical terms “c three squared, “put”, “edges” and “parallel” that are again reminiscent of how the diagram came about.

6.2.3. Dénouement Phase

The third phase in the life-cycle of a diagram concerns whether the diagram continues to exist or not. This is the phase when the mathematician has come to an understanding about the mathematics that the diagram holds and is now making a decision regarding the status of the diagram. During this phase, the diagram can be obliterated, discarded, or established, but the diagram can also simply be absent from the mathematizing of the mathematicians, both in speech and gestures. My data support that once a diagram is obliterated it does not appear again on the blackboard, and once it is discarded it is not interacted with again. However, when a diagram is absent or established, then my data also show that it is possible that the diagram can return to the communication phase. Next, I expand on each of the four diagram episodes that belong to the dénouement phase.

Diagram-is-Discarded

The episode *diagram-is-discarded* does not happen very often, but it took place in the following way: a diagram arises; the mathematicians discuss mathematics pertaining to the diagram; then the mathematicians realize that the diagram does not hold mathematics that is possible; and the mathematicians subsequently dismiss the diagram. The typical action is that the mathematicians turn their back on the diagram and walk away from it. There was never an instance when a discarded diagram is also immediately erased. It may later be erased to make room on the blackboard for some new material, but otherwise the diagram also stays present.

An episode of diagram-is-discarded occured during research meeting 5 in the time interval from 31:55 minutes to 35:06 minutes. Fred just finished up their discussion by concluding that their graph is actually a tiling of the plane, when Victor walks to the blackboard and begins drawing a diagram in silence on the blackboard. At first, Fred observes Victor, but then Fred points to their established diagram and discusses how “the fact that this guy really *does* embed could be, could be really good” (32:30 minutes) for finding other obstructions on the torus. At 33:04 minutes, Victor stops drawing, turns to the other mathematikoi, points at his diagram, and says “here is a nice drawing” (see Figure 6-3, left). Fred walks over to the diagram, while Victor claims “all quadrilateral pieces are on the torus” to which Fred responds: “What? (.) No! It is?”. Then Victor traces all the faces of the diagram while stating that “all faces are directed”, which brings Fred even closer, but Colin points out that some of the edges “should be shifted”. Now, all three expert mathematicians are positioned in a semi-circle around the diagram on the very left blackboard with Colin pointing at the diagram and Victor gesturing two edges of the diagram (see Figure 6-3, middle), while Finn sits far off to the right side of the room. At 34:15 minutes, Colin says that “this could be one of the double cycles”, walks to the blackboard, and traces through the edges while counting them out loud. Colin then turns to Victor and makes a shrugging gesture with his hand, while Fred turns away from the blackboard and a 7-second silence ensues. Colin now voices that there is perhaps a 6-cycle rather than a 4-cycle, turns to the diagram, traces through its edges and even labels them with numbers. Fred, who watched what Colin was doing, now turns away again, while Victor also turns away from the diagram and returns to his seat (see Figure 6-3, right). About 25 seconds later, Colin exclaims “oh, no!”, flings the chalk in the tray, turns away from the diagram, and returns to his seat. The mathematicians all laugh and move on, never to include this diagram in their discussion again.

Figure 6-3. F5-i01, Victor's diagram (left); F5-v1 33:55, Victor gesturing edges, Colin pointing, and Fred facing the diagram (middle); F5-v1 34:56, Fred and Victor turn away from the diagram (right)

Diagram-is-Obliterated

I want to point out that the action *erases object*, recorded in any of the transcription tables, needs to be carefully read: *erases blackboard* refers to a blackboard that has not yet been used by the mathematikoi during that research meeting and is now being cleaned before drawing or writing on it; *erases edges/vertex/face/etc.* alters the diagram by deleting the indicated object from the diagram; and *erases diagram* terminates the existence of the diagram on the blackboard and indicates an episode of *diagram-is-obliterated*.

I identified two settings of the episode diagram-is-obliterated. The first setting is when the mathematician obliterates a diagram when he deems it incorrect almost immediately after he has drawn it. I observed this only a handful of times during the nine research meetings such as in research meeting 5, when at 63:28 minutes, Fred states the following while adding to a diagram: "Suppose I use this one and this one. (2.1)". Then he steps back from the blackboard and exclaims: "*Oh*! Shoot! These are the ones that are forbidden from using". First, Fred chuckles, then utters "I just get rid of that picture" while erasing the diagram. There are many diagrams in the research meetings that never lead anywhere during explorations or turn out to misrepresent information, but these diagrams are not obliterated when the exploration ends or the error is discovered. Instead, the mathematician rather performs one of the following actions: ignores diagram (diagram-is-discarded), erases *object in diagram*, or starts over.

The second setting is when the mathematician wants to draw a new diagram but there is no more room on the blackboard he is working on. Typically, the mathematician searches for a suitable blackboard among the other eight blackboards by scrolling the blackboards up or down until an unused blackboard is found. If there is no unused blackboard, then the mathematician erases some or all of their former work on one of the blackboards, thereby obliterating some diagrams. The following provides an example of two diagrams that are obliterated during research meeting 4. At 15:28 minutes, Fred says "we can actually see this obstruction in just *one* step looking like that" and proceeds to add two yellow lines to an existing diagram on the blackboard (see Figure 6-4, middle). Shortly afterwards, Fred utters "this guy is, uh (.) my new favourite (1.3) this guy is (2.1) this k five (4.2) so this k five is nice because it's (1.1) it's unique" (15:36 minutes). During this time, Fred draws K_5 on the blackboard, and the postures of the sitting mathematicians indicate a leaning in, conveying the idea that they are paying attention (see Figure 6-4, left and middle). The mathematikoi continue to explore obstructions by drawing on all available blackboards, pushing them up or down to find free space. At 22:43 minutes, Fred pushes the middle blackboard up, and the blackboard behind it reveals C_3 and K_5 along with another diagram. While erasing C_3 but not K_5, Fred utters: "*Oh* (.) this was very nice. ((chuckles)) There is the cyclic k five." At 27:04 minutes, Fred states: "I would like to get better at *seeing* this sort of thing. ((chuckles)) It takes, er, I mean, it, it vaguely makes sense, to me, I mean, it takes me a long time to verify this sort of thing." During this time, Fred first points at another diagram, and then erases K_5 (see Figure 6-4, right). This time, the postures of Finn and Victor are relaxed, conveyed by the leaning back in their chairs, and Colin does not even observe Fred. This indicates the mathematicians are not concerned that the diagram is being erased.

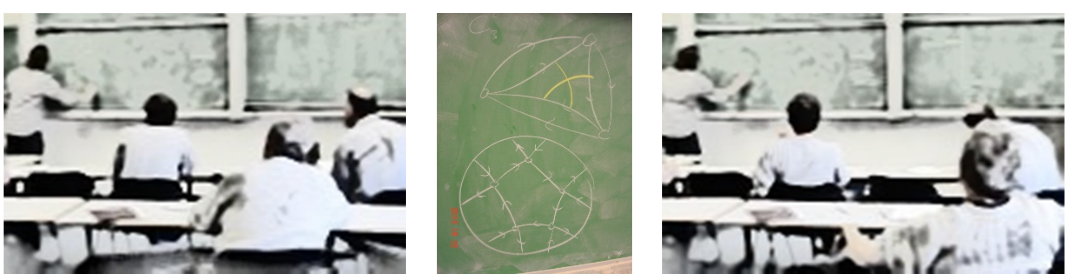


Figure 6-4. F4-v1 15:46, Fred draws K_5 (left); F4-i03, C_3 above and K_5 below (middle); F4-v1 27:15, Fred erases K_5 (right)

Diagram-is-Absent

An episode of *diagram-is-absent* occured in two different settings. The first setting requires that none of the diagrams present on the blackboard is referenced by the mathematicians: not in speech (e.g., by deictic words “this” or “that”, by naming the diagram “embedding” or “funny octahedron”), nor through gestures (e.g., pointing, holding or sweeping), nor even by looking at or turning to the diagrams. This setting occured every time the mathematikoi discuss either established mathematics that has not made it onto the blackboard or some new turn in their research. The second setting occurs when the utterances of the mathematician foreshadow the arrival of a diagram, but the diagram is as yet absent from the blackboard. It is noteworthy that during the absence of a diagram, the mathematician sometimes gestures mathematical terms (e.g., *edges, vertices, face, parallel, chords, family,* C_3*, automate, circular, torus, third, unique, adding,* etc.) during utterances, which are indicative of a diagram that has already been investigated or is about to be drawn and explored. These gestures actualize the virtual structure of the graphs that the mathematicians are engaged with. While I settled on calling such an episode diagram-is-absent, I might equally well have considered this a third setting for diagram-is-pulling, in the sense that the diagram is pulling on the mathematician to be brought into existence on the blackboard.

The time interval from 49:15 minutes to 49:36 minutes during research meeting 1 (see Appendix B, Table B-1, row one) provided evidence for both settings of diagram-is-absent: Colin and Fred discuss which graph they should explore next. While there are diagrams present on all blackboards, these diagrams are also absent for Fred and Colin, because neither of them acknowledges these diagrams in speech, looks or gestures; and furthermore, their discourse content indicates that they want to explore some particular embedding that they have not explored so far. It is clear that Fred is about to draw the graph of C_3; however, he is torn between facing the blank blackboard and facing the mathematikoi as he repeatedly turns from one to the other. Furthermore, before Fred draws, his statements are accompanied by his gestures of the mathematical terms *dual*, C_3 (twice) and *half-octahedron* (see Figure 6-5).

Figure 6-5. F1-v1 49:31, Fred gestures half-octahedron

Diagram-is-Established

The goal of the research meetings is to compile a list of obstructions and to classify them, so that the graduate student Finn can eventually write programming code in order to answer his research question: what are the minor, minimal, non-planar, 2-regular, directed graphs? Therefore, a diagram becomes established either if it is an ascertained obstruction or if it identifies pertinent relationships or operations that become a tool for doing mathematics in this field. For example, the operation of splitting vertices as shown in Figure 5-1 was introduced in the very first research meeting and became an established tool for finding immersions in subsequent research meetings. The funny octahedron (see Figure 6-6) is a graph that engaged the mathematikoi from research meeting 2 onwards, and which became established as a legitimate obstruction for the torus in research meeting 5. During subsequent research meetings, this diagram was recalled from memory, and these recollections are instances when diagram-is-emerging immediately becomes diagram-is-present and diagram-is-established.

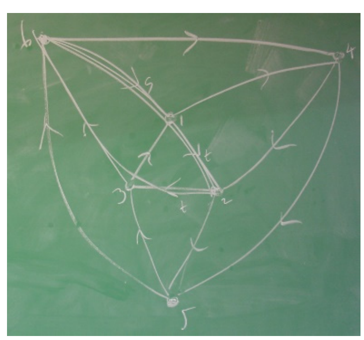


Figure 6-6. F2-i01, C_6^2 embedded in projective plane

6.2.4. Quantitative Analysis and Relationships of Episodes

While I did not perform an exact count of the number of diagrams that were drawn on the blackboard during the combined times of almost twelve hours of research meetings, I estimated this number to be approximately 200. I reached this estimate by viewing the 131 digital images that I managed to take with the picture camera during the meetings, and counting the number of diagrams on them as 179. However, this number does not include the erased and drawn-over diagrams that I could not capture but that are recorded on the videos of the research meetings. On rare occasions, a mathematician would place just a few chalk marks on the blackboard and immediately erase them, which occurred at most once in each research meeting. Furthermore, during sustained high diagramming activity such as in research meetings 3, 5 and 9, for every two diagrams that I was able to take a digital photo of, another diagram was either drawn over or partly erased. Since there is no clear demarcation between the beginning and ending of diagramming, there was a fine balance between getting my data and not disturbing the meeting. Based on my viewing, I estimated there to be about twenty diagrams that are recorded only on video and not as a digital photo, which led to my estimate of 200 diagrams drawn during the nine research meetings of the foursome.

As indicated in Chapter 4, I selected 122 time intervals of interest for further analysis, which contain a total of 128 diagrams. I counted only first occurrences of a diagram. In other words, if a diagram in a particular time interval has already been counted in any of the previous time intervals, then this diagram is not counted again. This means that of the estimated total of 200 diagrams, my data analysis contains 64 percent of the diagrams. In each of the selected 122 time intervals, I identified to which episode each diagram belongs. The following is a quantitative analysis of the eleven diagram episodes, the result of which is shown in Figure 6-7. Even though only 94 out of the 128 diagrams are shown to emerge in my selection of 122 time intervals, I still regard the episode diagram-is-emerging to be 100 percent of the total number of diagrams, because every one of these diagrams was created by the mathematicians at some point during the nine research meetings. Once a diagram has emerged, however sketchily or incompletely drawn, it is present on the blackboard, no matter for how short or long a time period. Therefore, the episode diagram-is-present also occurs as 100 percent of the total number of diagrams. I counted the remaining nine types of diagram episodes and calculated their percentage compared to the 128 total diagrams. Every so often, I identified more than one episode in the life-span of a diagram, and therefore, the percentages in Figure 6-7 for the bottom nine types of episodes do not add up to 100 percent.

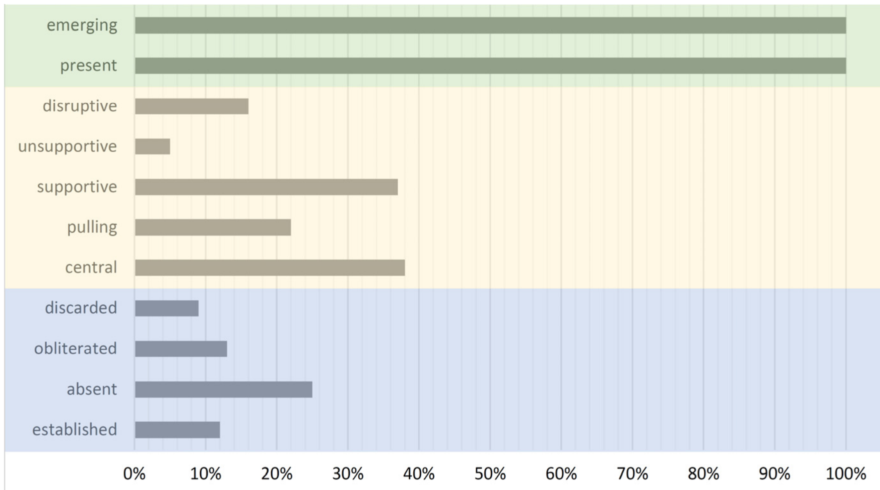


Figure 6-7. Quantitative Analysis of Diagram Episodes

Of the nine episode types from the communication and dénouement phases, the two most common are diagram-is-supportive and diagram-is-central at 37 percent and 38 percent respectively. This does not come as a surprise, because the mathematicians choose to diagram in order to explore the mathematics of their research. In other words, a diagram is the quintessential playground, where a mathematician's intuition and experience shape the mathematics that is under exploration. Moreover, as Châtelet aptly puts it, "one is infused with the gesture before knowing it" (2000, p. 10), which is evident in the mathematicians, who gestured a diagram during speech before the chalk marks created it on the blackboard. Similarly, de Freitas and Sinclair point out that "[the potential] marks that which is latent or ready in a body. In the case of the diagram, the potential is the virtual motion or mobility that is presupposed in an apparently static figure—and that was central to its creation in the first place" (2011, p. 139). Perhaps this also explains why the episode diagram-is-unsupportive is the least common type of episode with only a five percent occurrence. The mathematician's physical and mathematical intuition shapes his gestures that actualize the diagram, thereby realizing the mathematics that is possible rather than the mathematics that is not.

While I identified the episode diagram-is-absent in about 20 percent of the 122 time intervals of interest, I suspect the true proportion taken over all the research meetings to be much higher. The selected time intervals were chosen for their diagramming content; however, in about half of the time intervals between those of interest, the mathematikoi discussed theorems, made generalizations or summarized their findings without creating or engaging with diagrams. Hence, these are also settings where the diagram is absent. My estimate for the episode diagram-is-absent was a conservative 35 percent of about 240 time intervals covering all the research meetings.

I now discuss the relationships that I observed among the eleven diagram episodes and among the three diagram phases. Uncovering these relationships was rather a complex task, because I had to go back many times to view the data again to ensure that a diagram episode could indeed lead to another type of diagram episode. During this process, I went through my own diagramming experiments in order to present the phases and episodes coherently, the result of which is shown in Figure 6-8.

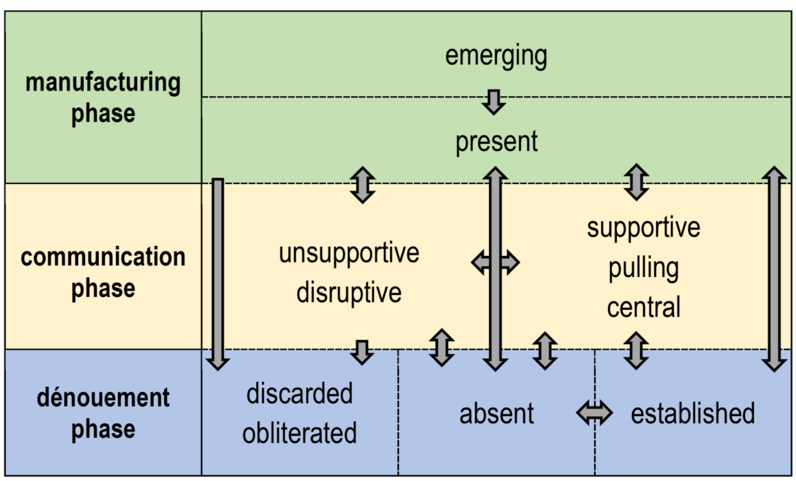


Figure 6-8. Relationships among Diagram Episodes

It is important to point out that these diagram episodes are not mutually exclusive in two ways. First, since the relationship which exists between the diagram and the various mathematicians who are engaged with it dictates which diagram episode applies, there are times when the diagram is one type of episode to one mathematician, yet another type of episode to another mathematician. For example, a diagram may be supportive to the thinking of one mathematician but unsupportive to another mathematician's understanding. Second, more than one diagram episode can occur to the same diagram at the same time with the same participants. For example, especially during an episode of diagram-is-central, the diagram may be in flux and constantly changing in the hands of the mathematicians. This implies that while this is an episode of diagram-is-central, there could be many instances of diagram-is-emerging. As mentioned earlier, not all of these episodes need to occur in the life-cycle of a diagram, except for diagram-is-emerging followed by diagram-is-present, and a diagram does not necessarily move through the three phases in a linear fashion.

6.3. Evolution of a Particular Diagram

The majority of time in the research meetings is spent on understanding how various vertex splittings change a 2-regular digraph, and whether or not the resulting immersions lead to obstructions or not. During the mathematikoi's exploration in the second research meeting, they discover a planar embedding of the octahedron, C_6^2, with an unfamiliar orientation. This discovery raises their interest because an immersion of C_6^2 leads to the graph K_5, which is a well-known obstruction in the plane. The vertex split that produces K_5 occurs at vertex 1 (top vertex of centre triangle) of the graph of C_6^2 (see Figure 6-9). Figure 6-9 (right) is a different rendering of C_6^2 and K_5, which shows the similarity between the two graphs in that the edges of both graphs can be partitioned into cycles whose orientation is always clockwise. Fred calls the new graph *funny octahedron*, which is a name that sticks throughout their meetings. In a later correspondence with the graduate student, I found out that the research group eventually called this graph the *reversed-planar octahedron*. As a side note, at the time of the video-recorded research meetings, the foursome knew of only two 2-regular orientations of the octahedron, while they later discovered that there are four 2-regular orientations.

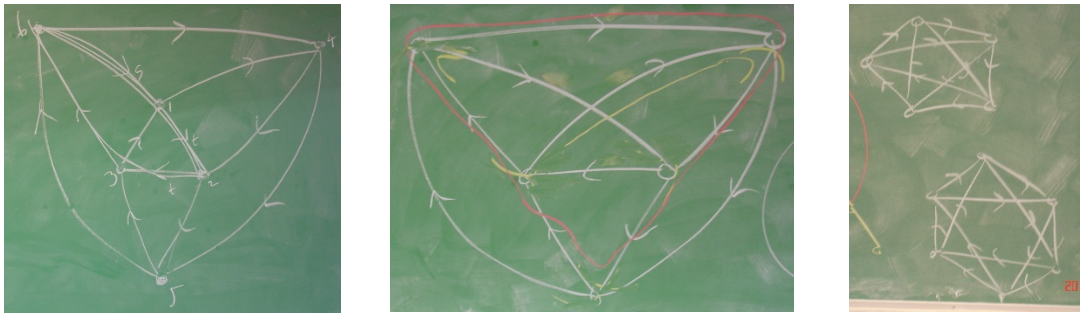


Figure 6-9. F2-i01, C_6^2 (left); F2-i05, immersion of C_6^2 leads to the graph K_5 (middle); F4-i11, alternative drawing of K_5 (right, top) and C_6^2 (right, bottom) in the plane showing their similar structures

The mathematikoi were intrigued by the funny octahedron, making many attempts in several research meetings to understand its structure and repeatedly exploring its embedding in the projective plane, torus and Klein bottle. Furthermore, in an email discussion with the graduate student a year after the nine research meetings took place, I found out that this particular diagram now appears in a stylized, printed form in a joint paper in preparation. Therefore, I have chosen some of the diagrams associated with the funny octahedron to analyze how these diagrams were drawn and interacted with by the mathematicians in order to determine their evolution.

About twenty minutes into the fifth research meeting, the mathematikoi begin to explore whether the funny octahedron embeds in the torus. In the second research meeting they had already explored its embedding in the projective plane, which is shown in Figure 6-9 (left). Fred draws the funny octahedron in yellow chalk on the blackboard, but seems to be stuck finding the faces of the embedding. Then Colin joins Fred at the blackboard and asks, “may I please destroy this picture?” to which Fred replies “oh, sure”. Colin picks up a piece of red chalk and draws over Fred’s diagram, all the while explaining which pairing of edges must yield the faces in the torus (see Figure 6-10, left). When Fred understands Colin’s strategy, he makes the following observation: “In the future, I mean, in the future, all the faces of the other, all the faces of the other colour class have to alternate, red, yellow, red, yellow. Right?”, to which Colin responds “yes”. Fred continues: “So, if you look at a face, so this goes (.) red (.) yellow (.) red (.) yellow. *Ah*! So that’s one.” Then he stares at the diagram for two seconds and says, “*oh!* And you got three of that sort. Agreed?”. Colin replies “that’s a four cycle, right?” and Fred echoes “that’s a four cycle (.) but that’s just right”. Colin agrees, saying “yah”, puts his chalk in the tray and turns away from the diagram, while Fred utters “so that’s six faces” and steps away from the diagram.

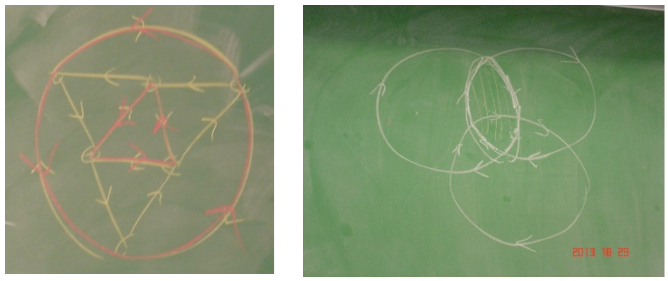


Figure 6-10. Embedding of C_6^2 in torus: excerpt from F5-i01, by Fred and Colin (left); F5-i04, by Victor (right)

At the beginning of these statements, Fred cups his right hand and places it over each of the faces in the diagram as if his hand is mimicking the perimeter of the faces. During the times when he utters “red, yellow”, he draws his fingers into a fist except for the index and middle fingers, which he alternately moves up and down while at the same time moving his hand successively from face to face. Both of these are interesting gestures, as they not only embody the alternation between edge colourings and thereby virtually assemble the faces that the diagram holds; but the gestures also underline the content of Fred’s utterances, which is a generalization of the process of finding the faces.

In the meantime, at the very beginning of Colin and Fred’s exploration at the blackboard, Victor gets up saying “I can’t see it”, but instead of joining Fred and Colin, he quietly draws his own version of the embedding on the blackboard (see Figure 6-10, right) until both Colin and Fred have stepped far away from their diagram. Now, Fred utters “so that’s (.) what am I doing? How did I get such funny numbers? (.) No, but that’s *perfect*. That’s, that’s an embedding on the torus”, during which time he steps closer to the diagram and touch-points it. This statement raises Victor’s interest, who stops drawing and turns to Colin and Fred’s diagram and steps closer. Now, all mathematikoi stare at this diagram for several seconds (see Figure 6-11). Fred emphatically repeats “that’s an embedding on the torus”. Colin replies “uh, so you have six vertices, twelve” and pauses, so Fred finishes with “yah! Uh, uh, six vertices, twelve edges and six faces.” Then Colin responds “six faces is the torus, yah, yah", and Victor chimes in with "yah, yah, that's what I was doing" and walks to his diagram. Fred turns to Victor's diagram and exclaims "that's a nice, uh, so, so, yah", but immediately turns back to his diagram and continues saying "let's draw, somebody draw that in a nicer way, uh". Colin laughingly points at Victor's diagram and states, "[Victor]'s is good right there", to which both Victor and Fred respond with laughter as well and a six second pause ensues. The silence is interrupted when Victor and Fred speak up simultaneously. Fred says, "so, what's happening", while Victor utters "what? Are you drawing all quadrangular faces by the way?" to which Fred responds "*no*! No, *no*! I've got, It's all gons ((polygons)). *Silly*! There are (.)" and touch-points a face. Victor turns away from Fred, while Colin says "there are two triangles, they, they" and Fred follows with "yah, we are using the two opposite triangles ((gestures with both hands high up in the air, opposite each other)), so, so one, one of the (.) so you, uh the faces are two coloured, right? ((gestures with both hands still high up in the air but now alternately moving up and down))". Now, Victor is facing Fred again and replies "yah". Fred continues with his explanation: "One colour class has the two opposite triangles and the six cycle, and the other class has three squares", which is accompanied by touch-pointing the opposite triangles in the diagram, tracing the six cycle and sweeping the diagram. Victor interjects with "*oh*" and head-nodding.

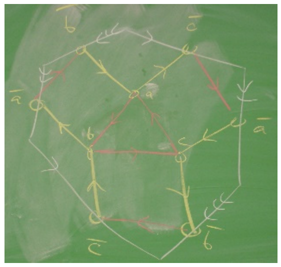
During this episode of diagram-is-central, the diagramming, gesturing, positioning of the bodies, and verbalizing are necessary and entwined acts, which culminate in the three expert mathematicians forming a shared understanding of their discovery of the funny octahedron's embedding in the torus. Finn's body posture indicates that he is an observant listener; however, he does not participate in the diagramming, gesturing and verbalizing.

Figure 6-11. F5-v1 22:50, mathematikoi stare at diagram for several seconds

Now, I analyze the similarities and differences between the diagram drawn by Victor and that by Fred and Colin. I notice that the diagram in Figure 6-10 (left) is in keeping with the symmetry of the diagram in Figure 6-9 (left), while the diagram in Figure 6-10 (right) is an upside-down version. The mathematikoi render either form of the funny octahedron throughout the research meetings, always drawing it so that the line of reflectional symmetry is vertical to the onlooker. However, they never draw the funny octahedron so that the line of symmetry would be rotated away from the vertical centre line. Perhaps these two forms of the funny octahedron are visually pleasing to the mathematikoi; perhaps there is a strategy based on intuition that makes the mathematician draw the funny octahedron in this form; or perhaps the established cultural norms in Topological Graph Theory demand that graphs are drawn in such a fashion. Regardless, the manner in which the diagrams are drawn speaks to the roles of the aesthetic in mathematical inquiry to which Sinclair (2004) draws attention.

During the drawing, Victor as well as Fred and Colin attend to the direction in which the edges travel. Their diagrams also display rotational symmetry with a 120-degree angle of rotation. Furthermore, I noticed that Fred and Colin's diagram is based on an existing representation of the funny octahedron, albeit in the projective plane. The difference here is that the colouring of red and yellow edges is used to dictate how the faces are made up when embedded in the torus. Victor's diagram, on the other hand, does not resemble any previous representation of the funny octahedron and employs only one colour, although two edges stand out more because he draws over them so often when he is explaining his method to the mathematikoi. At a fleeting glance, one can argue that these are still the same diagrams, by just morphing the curved edges of Figure 6-10 (right) into straight lines and rotating the entire diagram by a 180-degree angle of rotation through the centre. Yet, Victor's explanations, which he offers a little while later, elucidate that he is using the idea of Venn diagrams to represent the inner triangle, outer triangle, hexagon and the three quadrilaterals of the embedding. When Fred and Colin are curious how the faces align, they almost immediately discover that Victor's diagram is actually an embedding of the funny octahedron in the Klein bottle, which Colin explains to Victor as follows: "You see, the, the three, if you look at these three quadrilaterals, so this one and this one have an edge in common. And similarly the others. So this should be different colours." Victor agrees and the diagram is discarded because at this point they are interested in embeddings only in the torus (see *Diagram-is-Discarded* for a fuller description). The two diagrams in Figure 6-10 (left and right) are not only differently drawn, they also demonstrate how uniquely virtual gestures are actualized; and furthermore, that not every actualization leads to a realization of the possible mathematics.

At 27:07 minutes, Fred states “I am curious how much symmetry of this you can *keep* when you take this funny embedding of the thing on the torus” and begins to redraw the embedding of the funny octahedron in the torus, which takes him about two minutes to do. He uses Figure 6-10 (left) to trace along the edges, which guides him drawing the faces in his new version of the torus embedding using the same yellow and red chalk colouring (see Figure 6-12, left). In the meantime, Finn occasionally observes Fred but is also bent over his notes, which later reveal Finn’s drawing of the funny octahedron’s embedding in the torus (see Figure 6-12, right). I elaborated on the differences between Fred’s and Finn’s drawings in Section 5.2.5, where I pointed out that the asymmetric rendering without paying attention to the square-ness of three of the faces in light of Finn’s usually faithful note-taking makes me believe that this is Finn’s own diagram rather than a copy of Fred’s diagram. This is important in two ways. First, Fred set out to redraw in order to reveal further symmetries, and during the drawing process Fred took his time meticulously placing every edge and vertex. Finn’s diagram, on the other hand, although symmetric in nature, does not visually display this symmetry. While Fred, as the expert mathematician, is able to tend to the symmetry while building up the face-structure of the embedding, Finn does not demonstrate the same ability. Second, these diagrams are drawn by the mathematicians in order to understand the mathematical structure of the funny octahedron on the torus. It was necessary for Fred to trace along the edges and pause at the vertices of the diagram shown in Figure 6-10 (left) in order to produce a new version of the same graph. In Châtelet’s terms, the constant back and forth between Figure 6-10 (left) and Figure 6-12 (left) helps actualize the virtual structure.



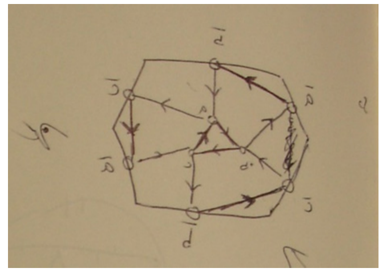


Figure 6-12. Embedding of C_6^2 in torus: F5-i03, Fred (left); excerpt from F5-ng2, Finn (right)

Now that the mathematikoi have discovered the embedding of C_6^2 in the torus, they want to know if this embedding is unique and how C_6^2 can embed generally in a surface of Euler characteristic zero. For this, they need to understand how the faces of the embedding are connected to the edges of the original graph, which they refer to as *closed walks*. Fred notices that "also around each, around each *red* edge you see a directed four cycle just taking those two faces" (39:08 minutes), and half a minute later he states "so if we say we have only quadrangles, maybe we can work out, uh, what the possibilities are". Colin responds with "so we can try using no triangles, uh, uh, and then there would be only squares". During this utterance, Victor walks to the blackboard and joins Fred there tracing and touch-pointing at the diagram in Figure 6-10 (left) while exploring the connection between the edges and faces. After about a minute, Fred exclaims: "Wait! I *don't* understand. Are you taking, are you taking *these* two?" A silence of 11 seconds ensues, then Colin walks to the blackboard, utters "look at, uh, look at" and is about to begin drawing a new diagram (see Figure 6-13, left). During the drawing, he explains: "You have ((draws centre triangle)), you have the (1.4) ((draws circle)). Every quadrangle it's either of these types, (.) ((starts drawing squiggly lines on top of edges)) uses two of this (2.3) ((adds more squiggly lines on top of edges)) and then two of this. That is one type" (see Figure 6-13, left, bottom). Before Colin finishes his explanation, Fred interjects with "that's right (.) well, that's right". Then Colin draws the other type accompanied by verbal explanations (see Figure 6-13, left, top). During this drawing, Fred interjects again "red, red, yellow, yellow", while pointing at Colin's first diagram, which is drawn with white chalk (see Figure 6-13, right). When Colin is completely finished drawing and explaining, Fred responds “so you can say, the directed four cycles, relative to this colouring have two types” and he draws the two diagrams shown in Figure 6-13 (middle).

Figure 6-13. Identifying the two types of 4-cycles: F5-i07, Colin’s diagrams (left); F5-i08, Fred’s diagrams (middle); F5-v1 41:49, Colin draws and Fred points

During this episode of diagram-is-supportive, Colin’s diagram helps Fred identify the necessary structure of the two types of 4-cycles. Moreover, Fred strips Colin’s diagram down to its essence, when he renders his version of the two types, by focusing not only on the four edges, but also on the colour scheme from their earlier exploration (see Figure 6-10, left). For the next five minutes or so, the mathematikoi discuss the implications of these two types in how the faces of the embedding are drawn, and they refer to their discovery as the “parity argument”. At the end, Fred is summarizing their findings. During this time, Fred is leaning against the blackboard constantly pointing, touch-pointing, holding, and tracing the edges of the diagrams and also gesturing mathematical terms during his speech (see Figure 6-14).

Figure 6-14. F5-v1, Fred gestures: 44:41, leans against blackboard and *L*-holds an edge; 44:53 still leans against blackboard and *L*-touch-points an edge; 45:02 gestures *inner edges* with his right hand

The mathematikoi continue exploring the embedding through forced closed walks using the two types of 4-cycles. Colin and Fred are still at the blackboard, while Finn and Victor remain seated. However, it is clear from the conversation that they have reached an impasse. At 46:47 minutes, a 2:35-minutes long silence ensues, during which time the mathematikoi quartet-draw. Colin uses his diagram (see Figure 6-13, left) to count through the closed walks by adding dashed and squiggly lines with white chalk (see Figure 6-15, left). Fred bases his exploration on the diagram that he and Colin produced earlier, which is shown here again for convenience (see Figure 6-15, middle). Fred keeps drawing over this diagram, also using white chalk, but to contrast the red and yellow edges (see Figure 6-15, right).

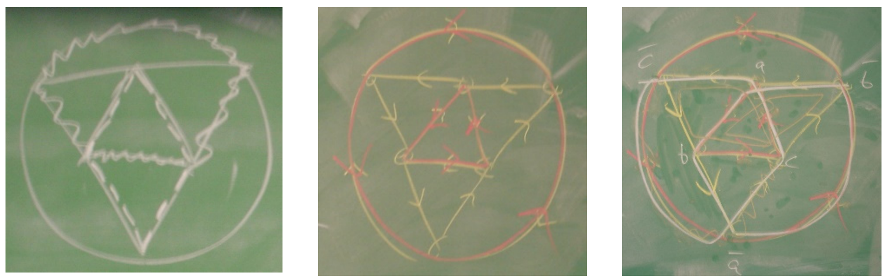


Figure 6-15. Exploration of closed walks in C_6^2: excerpt from F5-i07, Colin's diagram (left); excerpt from F5-i01, earlier embedding of C_6^2 in torus by Fred and Colin (middle); F5-i09, Fred's diagram (right)

During this time, which I label as episodes of diagram-is-central, diagram-is-pulling, and diagram-is-emerging, I could not keep up taking digital images as edges were quickly drawn over, and I did not want to be obtrusive. Therefore, I captured only the end-product of Colin's and Fred's continuously changing diagram. Still, the diagrams in Figure 6-15 demonstrate the dynamic nature of diagramming in the hands of the mathematician. By drawing various types of lines (dashed, squiggly, coloured) on top of each other, the diagrams by Colin and Fred become layered, which creates a sense of plasticity. The smudges of chalk and finger prints on the blackboard within Fred's diagram (see Figure 6-15, right) are evidence of the erasing and touching of edges and faces, thereby speaking of the intimate engagement with the diagram. Diagramming "in effect embodies a practice of figuring and defiguring" (Châtelet, 2000, p. xvi), as Knoespel writes in his foreword to Châtelet's work; and it is this *defiguring* which reorients the mathematician anew to the information that the diagram holds. Interestingly, both diagrams are drawn with the same orientation, which is not the case in the previous exploration (see Figure 6-10, middle) and a subsequent exploration (see Figure 6-16, left), which I elucidate shortly. Furthermore, Fred's diagram has vertices labelled and edges are oriented, while Colin's diagram is void of any labels and orientation. Therefore, the diagrams by Colin and Fred provide further evidence how uniquely virtual gestures are actualized. However, these personal actualizations also provide a common ground among the mathematicians to realize the possible mathematics as is seen next.

After the silent exploration, Colin steps away from the blackboard and states: “Yah, I think it is forced, and then it doesn’t work, so there is some parity issue”. Fred replies “yah I can see that things are forced but I can’t quite tell if they work or not”. Colin steps close to the blackboard again and responds “so, let me, let me show you”. First, he asks if Fred used the one type of quadrilateral by pointing at Figure 6-13 (middle, rightmost quadrilateral). Fred denies, and then Colin says: “Let me, let me try to convince you. Take two of these, okay? ((again pointing at the one type of quadrilateral)) By symmetry there is only one”. Fred replies with a chuckle, “anyone I like, oh yah”. Now, Colin directs Fred how to draw the diagram by telling him which colour of chalk to use, where to begin and which edges to follow (see Figure 6-16, left). During his explanations, Colin keeps walking between the two types of quadrangles on the centre blackboard and Fred’s new drawing on the right blackboard, using gestures of pointing, hand-pointing, touch-pointing, holding, tracing and sweeping any of the three diagrams. Then he draws Fred’s attention by hand-pointing at the centre triangle and saying “now you have to look at this triangle” (see Figure 6-16, middle). Fred understands Colin’s strategy and completes the diagram. Now, both Fred and Colin discuss how the closed walks are forced by the two types of 4-cycles. Fred states that “there is an instant trouble” (52:08 minutes), walks to his diagram, draws over his diagram holding the chalk sideways to create thick lines (see Figure 6-16, right), and explains what he means. During this time Finn and Victor observe Colin and Fred. At the end of Fred’s explanation, Colin agrees, turns to Victor and asks, “is this clear? So (.) [Victor]?” (52:49 minutes) to which Victor responds by nodding his head. Now, Fred turns to Finn and spends the next five minutes summarizing their findings.

Figure 6-16. Fred is directed by Colin to use forced closed walks to draw the embedding of C_6^2 in the torus: F5-i10, Fred's diagram (left); F5 50:19, Colin hand-points at centre triangle (middle); F5-i11, Fred's diagram (right)

During these episodes of diagram-is-unsupportive and diagram-is-supportive, I am struck by the great care with which Fred's diagram in Figure 6-16 (left) was drawn in order to separate the closed walks and to ensure that no more than two walks appeared at each edge, as otherwise the faces of the embedding are not created correctly. More importantly, even though Fred held the chalk, it was Colin who guided Fred's drawing through words and gestures. This provides evidence that actualization does not need to be a single-person process, but that it can occur through a shared process of gesturing and diagramming. When Fred reaches the same level of understanding that Colin has, Fred's singles out the top-most 4-cycle in Figure 6-16 (right), by thickly drawing over the edges, to voice back the argumentation for forced closed walks, which in Châtelet's terms is the realization of the possible. Through this act of drawing, Fred's diagram echoes Colin's diagram in Figure 6-15 (left), which uses squiggly lines for that particular 4-cycle. The fact that Fred's new diagram is not in the same orientation as Fred's old diagram, neither comes up during their engagement with the diagram nor in speech. This shows that the information that the diagram holds is the important aspect when drawing it.

During the next 15 minutes or so, the mathematikoi collectively explore the embedding of the funny octahedron using the face sizes and extend the strategy of forced closed walks to 3-cycles in order to show the uniqueness of embedding C_6^2 in a surface of Euler characteristic zero. Fred does most of the diagramming, Colin is alternately up at the blackboard or seated, and Victor and Finn get up twice to have a closer look at the diagram on the blackboard, but each time they return to their seats. When they have completed their argumentation, Fred exclaims: “So, that’s *done*! So, that’s *fantastic*! So, so, this tells us that funny octahedron graph has a unique embedding in the torus and a unique embedding in the Klein bottle” (69:50 minutes). During this time of high diagramming, I managed to take a digital photo only of the final diagram through which Fred argues the uniqueness of funny octahedron’s embedding in the torus and the Klein bottle (see Figure 6-17). The cloud of erased chalk on the blackboard behind the diagram no longer reveals the diagrams that were drawn, each in turn emphasizing the start of the closed walks from a different 3-cycle. The final diagram emphasizes the outer 3-cycle using thick chalk lines, which again provides evidence for the reconfiguration and moulding in the hands of the mathematician in order to prove uniqueness.

Figure 6-17. F5-i12, Fred’s diagram to prove uniqueness of funny octahedron’s embedding in torus

In this section, I recounted four mathematical inventions that occurred during the fifth research meeting, namely the discovery of funny octahedron’s embedding in the torus through edge colouring, the drawing of the actual embedding in the torus, the parity argument, and the uniqueness proof of this embedding through forced closed walks. There are two fundamental aspects at work here: the embodied acts that actualize the diagram and realize these inventions, and the material dependence on the diagram in all four instances of inventions. Anthropologist Ingold points out this embodiment during gesturing and diagramming: “Every hand-drawn line, then, is the trace of a gesture […]. Drawings comprised of gestural lines are, by the same token, non-propositional. They issue *from* things (including bodies) rather than making statements *about* things” (2013, p. 126 *emphasis in original*). In a similar way, but more specifically in relation to mathematics, sociologists Michael Barany and Donald MacKenzie (2014) take a detailed look at the practice of blackboard writing in mathematics, and call attention to the material aspect in mathematics:

> There is, we contend, an essential relationship between the supposedly abstract concepts and methods of advanced mathematics and the material substituents and practices that constitute them. This process operates even in the rarefied realm of mathematical research, where the pretense of dealing purely in abstract, ideal, logical entities does not liberate mathematicians from their dependence on materially circumscribed forms of representation. (p. 2)

The detailed recounts from this section demonstrate that the three expert mathematicians treat the diagram as a material site through their gestures, actions and postures during their explorations. Furthermore, through these embodied engagements, the diagram is constantly reconfigured and moulded, and the mathematician reoriented to the information that the diagram holds, which allows mathematical invention to emerge through diagramming.

I close this section with a brief look at the stylized renderings of C_6^2 (see Figure 6-18), which graduate student Finn shared with me close to the completion of my studies. Finn produced two of the four 2-regular orientations of C_6^2 ready for publication of his work. Gone is the hand that holds the chalk to draw the lines of these two diagrams; and instead, graphic software is employed to produce the two diagrams. Of course, the hand is still there to direct the software what to draw; nonetheless, the connection between hand and diagram is not an obvious one anymore. Analyzing the two diagrams, I see evenly drawn edges with perfectly aligned arrows at the end of each edge. Additionally, the three outermost edges form a perfect circle, and the inner edges form equilateral triangles. Both the circle and the equilateral triangles are highly symmetric objects, which are not necessary to show the orientation and connection of edges. Furthermore, these objects were never discussed during any of the research meetings, but seem to work further in the direction of an idealized object that hides the traces of its human making. The black and white diagram has also lost all its colour and uneven thickness of lines compared to the diagrams drawn with white, yellow and red chalk on the blackboard. The small curved arrows were never drawn during the research meetings and replace the messier, more colourful and thick chalk marks of before. Furthermore, the clockwise-pointing set of arrows is rendered solidly, while the counter-clockwise-pointing set of arrows is rendered in dashed lines. The placing of these arrows and the manner in which they are drawn put a heavy emphasis on the orientation of each face. This emphasis is an echo that remains of the struggles that the mathematikoi faced in understanding the possible orientations of this particular graph, and whether each of the orientations leads to an embedding or an obstruction. This brings to question how diagrams are being used in teaching and learning of mathematics. The diagrams in textbooks look pretty, but they neither tell the story of how they came about and the places of struggle, nor do they hint at the time and effort it takes to get to this form. In short, these diagrams are end products and not the playground for mathematical invention that the hand-drawn ones were.

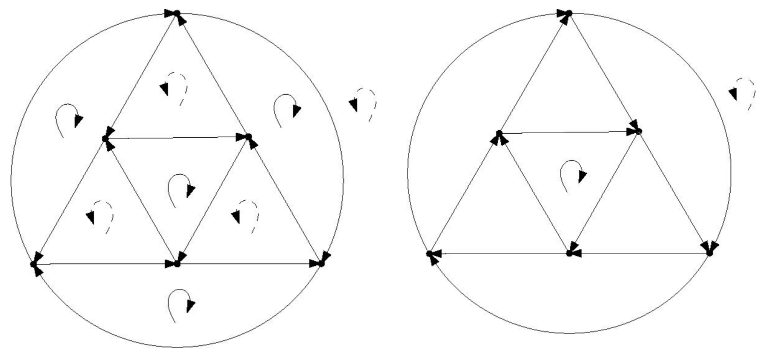


Figure 6-18. Two 2-regular orientations of C_6^2 in their stylized form produced by the graduate student Finn ready for publishing

Chapter 7.

Conclusion and Discussion

In this final chapter of my thesis, I provide individual answers to my research questions, discuss briefly how the particular mathematical topic of the research meetings and the fact that I analyzed only one mathematics graduate student pertains to my study, and conclude with a holistic summary of my findings on diagramming and gesturing. I follow my conclusion by offering a modified definition of mathematizing; contributions to the teaching and learning of mathematics in light of my findings; and suggestions for future research based on my work.

7.1. Answers to Research Questions

My thesis addresses two themes: the enculturation process of the graduate student from being a less-expert mathematician to a more-expert one, and diagramming and gesturing during mathematizing. My overarching ambition is to connect these two themes, which I voiced through the four research questions in Section 3.4. Through my data analysis, which is expounded in Chapter 5 and Chapter 6, I am now ready to provide answers to my four research questions:

1. *How is new materiality evidenced in the expert and less-expert mathematicians during research meetings?*

One might see each mathematical result in the nine research meetings as an abstraction that does not partake of the physical world, but is instead some kind of mental process. In contrast to this point of view, my data provide evidence that the mathematical result derives from an embodied engagement with diagrams, where the diagrams are material entities of mathematical objects and relationships that are engaged with virtually as well as physically. The three expert mathematicians' abundant creation of diagrams, the subsequent direct, intimate and material engagement with the diagrams, and the prolific gestures that accompany speech and hint at the mathematical entities that are being explored through the diagram are evidence that the diagram is indeed a material object for the three expert mathematicians. Finn, as the less-expert mathematician, is initially hesitant but increasingly participatory in these acts of creating, engaging and gesturing, which shows that he is drawn into this manner of exploration and communication alongside the expert mathematicians. While Finn was not yet able to offer as much insight, direction, and purposefulness as the expert mathematicians during their investigations, he nonetheless made tentative efforts during diagramming that were attempts at reorienting the information contained in the diagram. The constant reconfiguration and moulding of the diagram in the hands of the expert mathematicians allow the mathematical concept under consideration to emerge as a material object through the diagram. Therefore, new materiality is evidenced virtually and physically through the mathematical diagram, which acts as a catalyst that initiates a transformation of mathematical entities into material objects.

Before I answer the second research question about the diagram, I come back to the talking stick, which I employed as a means to reveal how the diagram communicates with the mathematician and, conversely, how the mathematician communicates with the diagram. The latter has been addressed in my answer to research question 1 above. While there is nothing that I was able to ascertain as a talking stick per se during the viewing of the research meetings, the metaphor of the talking stick and the six questions I generated (restated below for convenience) were crucial in allowing me to identify the speaker among the four participants and the diagrams (in particular, see Sub-Sections 5.2.1 through 5.2.9); expose the interactions between mathematician and diagram (in particular, see Section 6.3); and notice and classify the relationships between mathematician and diagram (see Section 6.2). These findings contribute holistically in answering each of the four research questions. However, some of the talking-stick-questions have, so far, not been addressed, and I do so now, in order to bring closure to the talking stick as a means, and also, to lead into my response to research question 2. I provide brief answers to each of the six talking-stick-questions when possible, and otherwise, refer to places in the thesis that delivers detailed evidence and answers:

A. *Who holds the talking stick among the expert mathematician and less-expert mathematician? Is it possible for the diagram on the blackboard/paper/computer screen to hold the talking stick?*

   As the detailed transcriptions and analytic discussions in Chapter 5 and Chapter 6 attest, there is no doubt that the talking stick is held by the supervisor, the research colleague, the visiting colleague, the graduate student and even the diagram, which is evidenced by episodes such as diagram-is-supportive, diagram-is-pulling or diagram-is-disruptive.

B. *Who holds the talking stick most often?*

The supervisor is the self-elected main speaker, mostly standing at the blackboard, and he therefore holds the talking stick most often.

C. *How does the talking stick get passed among participants?*

Among the three expert mathematicians, the talking stick gets passed using either a verbal interjection, a deictic gesture or a change in body posture, such as standing up or walking to the blackboard. These three mathematicians freely grab the talking stick but also respect when someone else takes it over. As pointed out in the summary of Chapter 5 (see Sub-Section 5.2.10), the graduate student is not yet skilled in this manner of communication, but throughout the nine research meetings, he becomes increasingly adept at employing those same strategies as the expert mathematicians for procuring the talking stick. The diagram is afforded the talking stick through its relationship with the mathematician. However, it is not just the mathematician who *lets* the diagram speak or not in such episodes as diagram-is-central and diagram-is-obliterated respectively: the diagram also *demands* the talking stick through such episodes as diagram-is-pulling or diagram-is-disruptive.

D. *Is the talking stick fair, in the sense of allowing for equal rights to be heard?*

Of all six questions, this is the most difficult question to answer, because my data consist of just three expert mathematicians and only one graduate student. From these data, I deduce that the talking stick is fair among the three expert mathematicians; however, the talking stick does not provide an equal right to be heard for the graduate student, since he still needs to be enculturated to the communication processes during research meetings. Since diagrams are both the mathematician's thinking made visible and a playground for exploration, the fairness for diagrams to talk or rather, be heard, depends on the mathematician.

E. *During diagramming, how is the diagram blocking understanding (in the sense of a communication breakdown)? Or how is it supporting understanding (in the sense of a communication breakthrough)?*

Detailed answers are provided in Section 6.2 through the discussion about the phases and episodes in the life-cycle of the diagram.

F. *During diagramming, what are the interactions between the diagram and its creator? If the diagram is drawn by one participant, then how is the diagram communicating with other participants who did not draw it?*

> Detailed answers are provided in Sub-Sections 5.2.1 through 5.2.9, which analyze, in particular, the graduate student's interactions with diagrams, and juxtapose these interactions with those of the expert mathematicians. Section 6.3 provides further evidence about the expert mathematicians' interactions with diagrams based on the funny octahedron and its exploration.

Now, I am in a position to answer the second research question and argue that the diagram talks and tells a story.

> 2. *When does the diagram talk and to whom? What story does the diagram tell?*

As the diagram episodes from the appearance, communication and dénouement phases confirm (reproduced here for convenience, see Figure 7-1), it is through the virtual and physical gestures of the mathematician that the diagram is given a voice, which speaks not only to the gesturing mathematician but all of its participating onlookers.

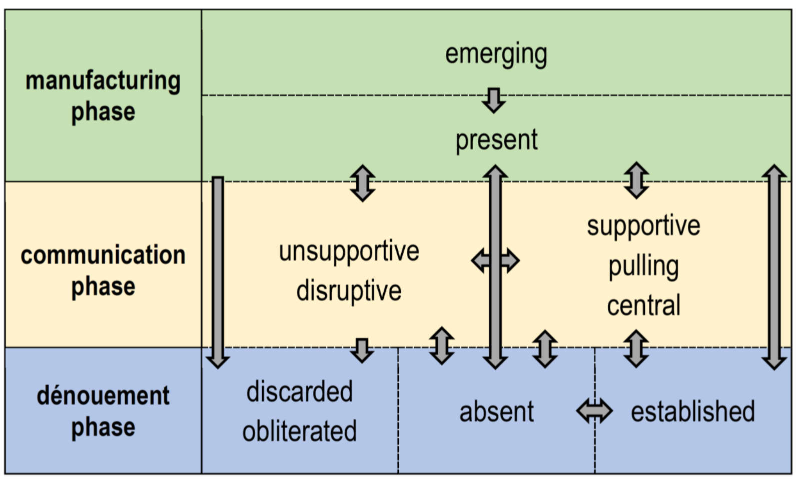


Figure 7-1. Diagram Episodes and their Relationships

Thereby, the diagram comes into existence (diagram-is-emerging), laboriously and furtively reveals its information (diagram-is-central), teasingly shouts the blatantly obvious (diagram-is-pulling), forcefully announces information thus far unknown (diagram-is-disruptive), steadfastly reinforces the mathematician's knowledge (diagram-is-supportive), explains, alas, without making sense (diagram-is-unsupportive), proclaims untruthfully (diagram-is-discarded), has its voice taken away because either it was assembled erroneously or another diagram needs to be heard (diagram-is-obliterated), is no longer audible (diagram-is-absent), or firmly and unequivocally articulates its truths (diagram-is-established). Therefore, the diagram is more than what it depicts (for example, the funny octahedron exemplified in Section 6.3): it constantly reassembles itself for the mathematician as he engages with the diagram virtually through gestures or physically through the hand that adds, erases, draws over or otherwise alters it. In his book *The Brain That Changes Itself* (2007), psychiatrist and psychoanalyst Norman Doidge recounts case studies in support of the adaptability of the brain, which is constantly changing because neural pathways are newly assembled, reorganized, damaged or destroyed. This behaviour of the brain is referred to as *neuroplasticity*. The term *plasticity* equally befits the diagram, and Sinclair and de Freitas (in press) write about this plasticity of the diagram:

> Châtelet suggests that we must distinguish between the figure of the parallelogram and its diagram: while the former operates in terms of a relationship of illustration or resemblance, the diagram exceeds similarity and even analogy because it invites new indexations, and mobilises both image and calculation. A diagram is not a pure icon because in the process of reasoning, the diagram is the thing in and of itself—with its past successive acts of inscribing, as well as its future ones. A diagram vibrates with the labour that produced it, inviting the hand in to engage and alter it. A diagram thus remains plastic, never exposing itself fully, and proceeding always by indices and indexations. (pp. 9-10)

The distinguished anthropologist Tim Ingold offers the following insight from his study of drawings through the lenses of anthropology, archaeology, art and architecture: "The drawing that tells is not an image, nor is it the expression of an image; it is the trace of a gesture. […] Thus the drawing is not the visible shadow of a mental event; *it is a process of thinking, not the projection of thought*" (2013, p. 128, *emphasis in original*). Ingold's conclusion equally serves my study of mathematical drawings, because ultimately, the diagram tells the story of how innovative mathematics comes into being, which is elaborated on when I answer Research Question 4, but first I will attend to Research Question 3.

3. *What role do gestures play in the culture of mathematical research? In particular, what differences are there between the way expert and less-expert mathematicians interact with the diagram?*

There is a noticeable difference between how the less-expert mathematician interacts with the diagram and how the three experts do. While the expert mathematicians freely and abundantly create and engage with diagrams either directly (erasing, adding, drawing over, redrawing and starting over) or in a material sense at several levels of intimacy (pointing, hand-pointing, touch-pointing, tandem pointing, holding, sweeping, tracing, staring, covering) during their explorations, the less-expert mathematician initially neither creates nor engages with diagrams at the blackboard. Furthermore, the experts' discourse is accompanied by an abundance of gestures through which they attempt to express with their hands and arms the mathematical entities that are being explored. As Châtelet posits, gestures give rise to diagrams, and through the diagram new gestures are evoked. However, throughout the nine research meetings, the less-expert mathematician increasingly exhibits similar gestures to those of the expert mathematicians, both during engagement with diagrams and during discourse, and even starts participating in the creation of diagrams on the blackboard during the last two research meetings.

In Section 2.2, I summarized and critiqued Wilkerson-Jerde and Wilensky's (2011) study about the sense-making that less-expert and expert mathematicians employ when confronted with unfamiliar mathematical ideas in a novel proof. Now that I have completed my analysis of the less-expert and expert mathematicians during mathematizing in my study, I concur with Wilkerson-Jerde and Wilensky "that experts do not necessarily rely on one method or adopt a consistent preference for what resources they rely on to comprehend proof but instead leverage a wide collection of resources when needed" (p. 33). However, I maintain my criticism of their research for using only a coding framework based on discourse to pursue their investigation. My research demonstrates that gesturing and diagramming are two further resources that mathematicians employ in order to make sense of mathematical objects and their relationships during mathematizing.

As the literature review from Section 3.2 reveals, researchers agree that gestures make thought visible. What is new in recent literature "is the effort to identify causal and measureable relations and interactions between bodily behavior and hypothesized internal processes and to explain these within embracing and detailed theories of kinetic, communicative, cognitive, and symbolic systems" (Streeck, 2009, p. 172). At least in the culture of mathematical research in Topological Graph Theory, gestures play a vital role in that they afford the mathematician not only a means to support communication with his other fellow mathematicians, but also kinesthetic and haptic experience with the diagram and the mathematical meaning that the diagram holds; and thereby support the mathematician in his ideational realm.

> Gestures thus occupy a unique position in human behaviour: they are bodily actions, but they are also cognitive actions. [...] Often it is not the visual image that communicates, but the fact that the action indexes a familiar kinesthetic (haptic) experience, and it is this kinesthetic experience that aids the gesturer's thought process. By way of gesture, relationships cannot only be seen, but also *felt*. (Streeck, 2009, p. 171, *emphasis in original*)

Thus, learning to gesture is an essential component in the enculturation process of the graduate student in becoming a more effective mathematician.

4. *How does mathematical invention emerge through diagramming carried out by expert and less-expert mathematicians during research meetings?*

This question is about the invention of mathematics, which necessitates that I consider the mathematician as the inventor and consider the trials and tribulations of the undertaken explorations that may lead to the invention. I find the following famous quotation helpful in echoing my observations about the mathematical explorations of the research group during their meetings. In his exposé *De la Métaphysique aux Mathématiques* (1960), renowned French mathematician André Weil writes about analogies in mathematics, and thereby beautifully captures a mathematician's journey of discovery and triumph at succeeding:

> Rien n'est plus fécond, tous les mathématiciens le savent, que ces obscures analogies, ces troubles reflets d'une théorie à une autre, ces furtives caresses, ces brouilleries inexplicables; rien aussi ne donne plus de plaisir au chercheur. Un jour vient où l'illusion se dissipe; le pressentiment se change en certitude; les théories jumelles révèlent leur source commune avant de disparaître; comme l'enseigne le *Gītā* on atteint à la connaissance et à l'indifférence en même temps. La métaphysique est devenue mathématique, prête à former la matière d'un traité dont la beauté froide ne saurait plus nous émouvoir. (p. 52, *emphasis in original*)

The mathematical physicist Michael Laurent Lapidus (2008) provides his own translation of this quotation:

> As all mathematicians know, nothing is more fruitful than those obscure analogies, these disturbing reflections of one theory on another; these fleeting caresses, this inexplicable blurriness. Nothing gives more pleasure to the researcher. One day comes where the illusion dissipates, the premonition turns into certainty. Twin theories reveal their common source before vanishing. As is taught by the Gītā, we reach knowledge and indifference at the same time. Metaphysics has turned into mathematics, ready to form the contents of a treatise whose cold beauty could no longer move us. (2008, p. 215)

Weil's expressive words underline the two global observations that struck me, as I followed this research group for over two months. The first observation concerns their steadfast pursuit of obstructions: they persevered with a resourcefulness based on their experiences and instincts in the subject matter; they laughingly manoeuvered out of culs-de-sac; and they uncompromisingly tested their findings until they were assured of their truth. My second observation is about their attitude once they found an obstruction: they were initially elated at having made a break-through; but what astounded me was that by the next research meeting this particular discovery was treated as old news and they simply went on about their business of exploring immersions and perhaps establishing further obstructions.

So, how does mathematical invention emerge through diagramming carried out by expert and less-expert mathematicians during research meetings? The answer lies in the engagement with the diagram through the mathematician's virtual and physical gestures, which keep moulding the diagram and reorienting the mathematician until the potential is actualized. Through this entwinement of bodily and cognitive actions, mathematics is mobilized. When this process is complete, then the possible that resides within the mathematician through his understanding and experience can be realized, which results in mathematical invention. In other words, my data of live mathematizing by expert mathematicians support Châtelet's findings from his analysis of expert mathematicians' and physicists' manuscripts that a diagram demands a virtual presence within the mathematician; provides a material site of engaging with mathematics; and far from being a static representation, it mobilizes mathematics. Moreover, my data provide evidence that not only the virtual but also the physical gestures are the principal mechanism through which these findings are realized. This is a mechanism which could not, in fact, have been directly observed by Châtelet, as he was working from only the diagrams, and which is summarized next.

While it seems second nature for the expert mathematicians to constantly reconfigure and mould the diagram and to reorient themselves to the information that the diagram holds for them based on their understanding, the graduate student as the less-expert mathematician initially does not exhibit such engagement. However, throughout the nine research meetings, his interactions with the diagram became increasingly more involved: first through the note-taking, where he redrew the diagrams from the blackboard, but also explored some of the concepts for himself; then, increasingly more, through the pointing at and gesturing of the mathematical entities held in the diagram while speaking; until in the eighth research meeting, he participated in the gesturing and diagramming at the blackboard in a fashion very close to that of the expert mathematician.

An organic, material view of the diagram sheds light on the creative processes in mathematizing, in that this view draws attention to the virtual and physical gestures that are at play during diagramming through which the expert mathematicians mathematize. Thereby, this view makes visible a neglected connection between the abstract and the physical worlds, namely materialism. Therefore, in answer to my final research question, mathematical invention emerges through diagramming, in that the three expert mathematicians engaged with the diagram directly or in a material sense at several levels of intimacy through their gestures, actions and postures during their explorations, which constantly reconfigured and moulded the diagram and reoriented the mathematician to the information that the diagram holds. At this stage in the less-expert mathematician's journey towards becoming an expert mathematician, the less-expert mathematician partakes in the invention of mathematics through shared experience and joint interaction with the expert mathematicians during mathematizing.

Before I provide an overarching summary of diagramming and gesturing, I discuss how the particular mathematical topic of the research meetings impacts my study and address the fact that I analyzed only one mathematics graduate student. As is evident in my data analysis in Chapter 5, the topic of Topological Graph Theory lends itself well to diagramming. This does not come as a surprise, because in this particular field of mathematics, the fundamental object is a graph. Not every field in mathematics relies on objects that are geometric in nature such as a graph that can be readily drawn. However, as the research meetings have shown, the significant aspect about diagramming is not necessarily the graph itself, but relationships that the mathematician is investigating and generalizations that he can determine from these relationships. Surely, the principal nature of mathematics is its study of objects, their operations and their relationships. I indicated at the beginning of Section 2.2 that I have no intention to discuss philosophically what mathematics is. Instead, I offer the following observations from the perspective of being grounded in the practice of mathematics: As I walk the hallway of my department, which is populated with researchers from a variety of mathematical fields, I catch glimpses of my colleagues mathematizing with their fellow researchers or graduate students on the whiteboards in their offices. I am struck by the informal manner of writing, the juxtaposition of objects and symbols, the mobility that is indicated through sketchily drawn lines that connect symbols and objects, and the gestures that are reminiscent of the gestures that I observed in the data-viewing of the research meetings and described in detail in Chapter 5.

While my study rests on just one mathematical field, there are other studies that similarly investigate diagramming and gesturing. First and foremost, Châtelet based his analysis on manuscripts of figureheads in mathematics and physics, which cover Algebraic Manifolds, Calculus, Complex Numbers, Coordinate Geometry, Geometry, Infinite Sequences and Series, Ordinary Differential Equations, Partial Differential Equations, Symmetry, Classical Mechanics, Electricity, Magnetism, Quantum Mechanics and Relativity Theory. The sociologist Christian Greiffenhagen (2013) uses one mathematical lecture in Mathematical Logic as a paradigmatic example “to extend the ‘material turn’ to an instance of abstract thought” (p. 21). In his analysis, he similarly observes gestures of pointing and of mathematical entities during utterances, as I have described, and notes that while logicians are “predominately [sic] concerned with symbolic writing [diagrams] are nevertheless a crucial aspect of some parts of their practice” (p. 23). In their publication *Chalk: Materials and Concepts in Mathematics Research* (2014), Barany and MacKenzie summarize their observations of mathematicians, who study partial differential equations and related topics, working on the blackboard:

> It matters little that the full measure of the blackboard's glory is confined to the narrow environs that lend profound influence to it. In the pregnant space between chalk and slate there reposes a germ of the bursts of inspiration, triumphs of logic, and leaps of intuition that dominate mindcentered accounts of mathematics. (pp. 10-11)

I cannot say with certainty that every mathematical field is prone to so much diagramming as I witnessed in Topological Graph Theory through the research meetings. Nonetheless, my findings suggest that awareness needs to be raised to the acts of both diagramming and gesturing as places of mathematizing. This is particularly important for both the supervisor, who is the expert mathematician and mentor, and the mathematics graduate student, whose cognitive as well as affective activities are being developed during the enculturation process of becoming an expert mathematician. Critiquing the limitations of the cognitive apprenticeship model, Belcher (1994) points out that there is an assumption that supervisors automatically model the role of researcher and naturally take on the role of mentor for the graduate student, “yet these roles seem not to be intuitive to some of the advisors of my students” (p. 24).

Lastly, the fact that I have data relating to only one mathematics graduate student allows me to analyze my data deeply rather than widely. This means that generalizability is not evident in terms of the enculturation process of the graduate student in becoming an expert mathematician. Still, my findings might inform other mathematics graduate students, as well as mathematics education researchers and mathematicians themselves, as to how diagramming and gesturing in addition to verbalizing and writing provide a means to think. By drawing attention to these embodied and material acts during mathematizing, I hope to raise the awareness of future mathematics graduate students, so that they may, perhaps, have a richer mathematical experience in their journey to becoming a mathematician; and also point a way for mathematics education research to illustrate further the practices of expert mathematicians in order to shape the learning and teaching of mathematics at all levels.

Overarching Conclusion

The diagramming that I observed in the research meetings of the foursome is in stark contrast with the diagramming experience that I have had in mathematics courses as an undergraduate or graduate student in the early 1990s, as well as now, and that I have offered until recently to students in my mathematics classroom. When the expert mathematicians are diagramming, they do not open textbooks or browse the internet in search of the picture-perfect image about some mathematical objects under discussion and their relationship. Their premise is that what they are researching does not yet exist. Certainly, there are existing definitions, constructs, operations and theorems, and they are used as a springboard, as is seen in the first research meeting when the research mathematics is reviewed and considered. However, from that point onwards, diagrams are created by the mathematicians without reluctance (as to how to get started and what to do), without confinement (any writing surface and any writing material is used) and without inhibition (driven by curiosity).

Moreover, the mathematicians are deeply involved with the created diagrams at several levels of intimacy covering distant, close and contact engagements as elaborated on in Sub-Section 4.5.1. The diagrams come alive through the mathematician’s pointing, hand-pointing, touch-pointing, holding, tracing, covering up and sweeping gestures; the leaning in of the mathematician’s body towards the diagram – often bending sideways; and the almost dance-like stepping back and forth either with just one diagram or between diagrams that are spread far apart between several blackboards, yet hold meaning together. As the mathematician draws over the diagram, adds some parts to the diagram, erases other parts of the diagram or re-draws the diagram, the diagram orients itself newly to the mathematician and reveals information that has the possibility of being realized.

The meta-talking of the expert mathematicians during diagramming reveals the intricate relationship that the mathematician has with diagramming. Meta-statements such as “there is like a fifty-fifty chance” and “I see all these parity things, I basically have a fifty percent chance of getting it right” are typically accompanied by laughter and indicate that the mathematician is aware of having a choice in drawing the diagram; but ultimately he does not take care to make the correct choice because the essence of the mathematics is still held in the diagram, and that is what he wants to bring to light. Other statements, such as “I shouldn’t be trying to remember, at this point I should just do” and “I wanna know what that looks like?”, refer to the need to diagram in order to expose mathematical structures or relationships, which is what Châtelet describes as actualizing the virtual. The following meta-statement by the supervisor is particularly illuminating in that it describes the mathematician's complex relationship with diagramming: "I would like to get better at *seeing* this sort of thing. ((laughs)) It take (.) er, I mean, it, it vaguely makes sense to me, I mean, it takes me a long time to verify this sort of thing. […] I just have to draw (.) um. Yah (.) just draw." At times, the diagram helps with the mathematical understanding: "we are learning good things here", "I am getting slightly better at this, I am not sure if I am good yet", and "it's taken me a long time where I can see this"; but at other times, the diagram is also disruptive: "I don't know why I draw it this way?", "I don't know why I insist on drawing it the wrong way", "something is funny", "it doesn't look right, right?", "if I would know what I'd be doing, it wouldn't be research", "I don't even know what I'm looking for", and "super simple […] did I get that wrong?". The meta-statements also reveal that there is a fundamental respect for each other's diagramming and that the diagram does not just belong to the creator: "may I please destroy this picture?" to which is replied "oh, sure". During the formalizing of their findings, the mathematicians no longer diagram anymore in the sense that they are exploring the mathematics; instead, diagrams are regurgitated from earlier explorations or not employed at all: "there is probably some nice little picture that does that but I started relying too much on formulas because I couldn't remember what picture meant what".

Diagramming is not just a tool, it is not just used for communication, and it is not just the representation of some particular set of mathematical objects and relationships. A tool by its very definition carries out a particular function, can only be applied to certain material, and has a predictable outcome if applied correctly. Diagramming, on the other hand, produces neither fixed nor predictable representations of mathematics. The diagrams drawn by the mathematicians are organic, because they transform and hint at more than the mathematics that is visible. The diagram is not silent, and it is connected not only to its creator but to the other mathematicians in the room. While the diagram exposes mathematical objects and their relationships, the diagram also hints at how these objects can transform or how the relationships can alter. In this sense, the diagram holds more information than an outside onlooker can see. The very experience of the mathematicians and the act of drawing are vital to the existence of the diagram and what it can reveal.

As a result of my research, the metaphor of a diagram having an existence of its own has shaped the way I think about a diagram now: this metaphor allows me to perceive that the diagram is far from static in its creation and in the manner that the mathematician works with the diagram; it encompasses the abstract notion that a diagram can pull, support, disrupt, and otherwise influence a mathematician during mathematizing; it positions the diagram as a material site of engagement; and it draws attention to the diagram in its role of mobilizing mathematics.

To summarize, diagramming is messy (lines may change quickly and often, the diagram is not perfectly drawn, some lines are not connected, etc.) but also robust (it endures the mathematician's pondering and probing); natural (it befits the mathematician and is in character with the mathematics, and does not appear out of place) but also creative (structures and relationships are positioned in novel ways, similar to the images that a kaleidoscope produces); shared (just like a language that is shared among speakers with the same mother tongue), but also single-minded (driven by the need to see for oneself, as evidenced by duo/trio-drawing and failing to acknowledge another's diagram); disruptive (when the information revealed by the diagram jars with the mathematician's current understanding), but also essential (innovative mathematical understanding is realized out of the diagramming).

7.2. Re-defining Mathematizing

At the end of the literature review and extensive discussion surrounding the term *mathematization* in Section 2.2, I concluded that I would use the term mathematization to refer to an embodied engagement with mathematics based on Wheeler's extended description of mathematization published in *Mathematics Teaching* in 1975, which is restated below for convenience:

> [Mathematization is] the ability to perceive relationships, to idealise them into purely mental material, and to operate on them mentally to produce new relationships. It is the capacity to internalise, or to virtualise, actions or perceptions so as to ask oneself the question, "What would happen if…?"; the ability to make transformations – from actions to perceptions, from perceptions to images, from images to concepts, as well as within each category – to alter frames of references, to refocus on neglected attributes

> of a situation, to recast problems; the capacity to coordinate and contrast the real and the ideal and to synthesise the systems of perception, imagery, language and symbolism. When these functionings are applied to pure relationships, detached from specific exemplars, the products will then be mathematics. (quoted in Wheeler, 1982, p. 47)

Now that I have concluded my research, I come back to this definition for two reasons: to evaluate critically the definition based on the insights I have gained, and to offer a re-definition based on my findings.

Based on my combined experience as a learner, teacher and researcher of mathematics, I am struck by the depth, breadth, insightfulness and complexity that this definition entails even though it was created forty years ago. However, I am also drawn to the words “idealise”, “purely”, “mental”, “mentally”, “ideal”, “pure” and “detached”, which are woven together with the words “material”, “virtualise”, “actions”, “images”, “real” and “imagery”. The latter set of words has gained a new meaning for me while working on this thesis, one that is in disagreement, I believe, with Wheeler’s intended meaning of the words. The two sets of terms create a juxtaposition that brings me back to Châtelet’s critique of the Aristotelian view positing that the doing of mathematics is an abstract process that is disconnected from the mathematician and the material world. While Wheeler does point to virtuality and images as important aspects of mathematizing, his use of the term “material” is not in the sense of *new materialism*, because he combines “material” with the words “purely mental”.

The clause “[mathematization is] the ability to make transformations – from actions to perceptions, from perceptions to images, from images to concepts” echoes the two Châteletan processes that lead to the invention of mathematics, namely the actualization of the virtual (potential) and the realization of the possible. Unlike Châtelet, Wheeler does not indicate how the transformation is to be accomplished. With the clause “[mathematization is] the capacity to coordinate and contrast the real and the ideal”, Wheeler maintains that mathematics is not just abstraction, it is a bridge between application and abstraction. The clause that follows next, “[mathematization is] the capacity […] to synthesise the systems of perception, imagery, language and symbolism”, alludes to the complex process of utilizing the various communication systems during mathematizing, which intrigues researchers from a variety of backgrounds including anthropology, linguistics, mathematics education, sociology (e.g., Burton & Morgan, 2000; Greiffenhagen, 2014; Ingold, 2013; Netz, 1998). However, the element that is missing in the list of communication systems is the human body itself as a communication device through physical gestures, actions and postures; and the body which has experiences that influence the virtual gestures. The last sentence: “When these functionings are applied to pure relationships, detached from specific exemplars, the products will then be mathematics”, points to the generalization of mathematical inventions, but does not do justice to the humble beginnings of many generalizations that root in the specifics. My data do not support that the generalization occurs “detached from specific exemplars”. On the contrary, the expert mathematicians that I studied gained understanding of general relationships by moulding concrete graphs and orienting themselves anew to the diagram through their virtual and physical engagement with the diagram. Therefore, the diagram is not just the specific mathematical object that it depicts, but it often also stands for a general class to which the object belongs.

Lastly, Wheeler’s entire definition does not reference any direct involvement of the body through gestures, actions and postures, nor does it mention any material engagement during mathematizing. I claim that this definition still stands with one leg in the Aristotelian realm and neglects the material engagement during mathematizing through gesturing and diagramming. Rather than providing an entirely new definition for mathematization, I offer here a re-definition based on Wheeler’s extended description, because, fundamentally, I still support his definition over any of the others that I drew attention to in Section 2.2. I have replaced “material” with “entities” and “mentally” with “virtually and physically”, inserted some extensions and also deleted one phrase. I indicate my changes using italic font and strike-through text:

> Mathematization is the ability to perceive relationships, *to create and engage with them in a material sense,* to idealise them into purely mental ~~material~~ *entities*, and to operate on them ~~mentally~~ *virtually and physically* to produce new relationships. It is the capacity to internalise, or to virtualise, actions or perceptions so as to ask oneself the question, “What would happen if…?”; the ability to make transformations – from actions to perceptions, from perceptions to *gestures, from gestures to* images, from images to concepts, as well as within each category – to alter frames of references, to refocus on neglected attributes of a situation, to recast problems; the capacity to coordinate and contrast the real and the ideal and to synthesise the systems of perception, imagery, language and

> symbolism. When these functionings are applied to pure relationships, ~~detached from specific exemplars,~~ *then the possible has been realized and* the products will then be mathematics.

7.3. Contributions to Teaching and Learning Mathematics

From the outset, I wanted to elicit from my thesis some insights and directions for my personal teaching of mathematics. As I mentioned in the *Introduction*, Chapter 1, I have been interested in diagrams from an instructor's point of view for a long while now. Through this study, I have developed a new orientation related to diagramming in mathematics teaching and learning. Although I have long desired to bring diagrams closer to my students, I now realize that I have so far approached this desire in an unsatisfactory way by offering textbook-ready, stylized reproductions of mathematical objects and their relationships. While I did emphasize the act of diagramming to my students, I did not provide sufficiently scaffolded approaches to diagramming, nor did I allow students the extensive time that is necessary to engage with their own diagrams and truly mathematize. In the remainder of this section, I address the question: what does my study imply for teaching and learning?

My analysis shows that in their creative phase the expert mathematicians needed to *see* and *feel* objects and their relationships in order to gain an understanding about them or to explore how they could be altered or newly positioned to extract new insights in their field. This brings into question how we, as teachers in general and at all levels, allow our students to *see* and *feel* in the classroom or the office. As I glance at my bookshelf lined with mathematics textbooks, I wonder how prevalent is the use of regurgitated visual products in order to explain, for example, what a parabola looks like instead of what a parabola is. How often do we create an environment that places each student in the position of a mathematician, where the student explores the relationship between input x and output $x^2 + c$, perhaps dynamically using geometry software, or drawing by hand the locus of points such that the distance to a given focus equals the distance to a given directrix? In any of these scenarios, the student is bound to point, touch, hold, trace, add and delete, which may lead to further material engagement with the curve and the curving that *is* the parabola.

I was surprised to observe the action of redrawing a diagram multiple times by expert mathematicians, especially when there were no changes in the final version of the diagram or a minimal change such as a rotation of the original diagram. This reminded my instructor-self how I ask my students to redraw my diagrams. Perhaps this action is not good enough for the student, as the redrawing is really the first instance of the student *drawing* the diagram, and therefore the student may not be able to own the diagram yet. Furthermore, the process of drawing in and of itself provides insights into the information that the diagram holds, as Nunokawa (1994) points out:

> some changes in the diagrams cannot be noticed from their outer appearance. Recognizing these changes requires taking account of the way of drawing the diagrams. Taking the way of drawing as an important aspect of a diagram, one which is closely related to the sense of the diagram, we can analyze the detailed relationship between diagrams and the problem-solving process and understand the roles diagrams can play in that process. Such understanding will lead to the construction of appropriate instructional suggestions concerning diagrams. (p. 37)

My study shows how prolifically the expert mathematician gestures both during the engagement with a diagram and also in speech when mathematical terms are uttered. Through the gesturing and diagramming the potential can be actualized, which then provides a gateway for the possible, that resides within the expert mathematician, to be realized, thereby creating mathematics. Perhaps more attention needs to be focused on gesturing during teaching and learning of mathematics. The units of activity that I devised and used for the purpose of transcribing the gestures, actions and postures by each research participant during their research (see Table 4.3) offer a useful framework for capturing these acts by students and teachers during mathematics lessons. Such a viewpoint brings attention to how much virtual and physical engagement is supported through a particular mathematics lesson or exploration. Furthermore, the diagram episodes resulting from my data analysis (see Figure 7-1) can be used as lenses through which to identify the type of gesturing and diagramming that students exhibit during mathematics lessons. This may inform the teacher of the students' mathematizing and also provide insights into the structure of the mathematics lesson.

As my parting answer to the *so-what* question that permeates this section, I want to draw attention to one aspect of teaching and learning that this study confirms I have instinctively done right in my own teaching: the pen that draws and writes during mathematizing needs to be held in the hand of the learner.

7.4. Future Research

One obvious direction for future research is a study involving more mathematics graduate students and a broader range of mathematical fields of study. As my literature review demonstrates, there is very little existing research regarding the enculturation process that a mathematics graduate student has to undergo. In order to generalize the results of my study regarding Finn's enculturation process, other graduate students in mathematics need to be studied during their research meetings. This is a place where the data from the twosome are still useful for analysis, because they provide me with another graduate student, whose studies are in a different field from Topological Graph Theory.

The findings in this thesis regarding gesturing and diagramming pertain to the mathematization of the less-expert mathematician compared with that of an expert. In an effort to gain a clearer understanding of gesturing and diagramming at all levels of mathematizing, two extensions of this study can be explored. The first extension is to involve a more varied participant group as noted above, but also in terms of gender and cultural background. Whereas the less-expert and three expert participants in this study were white male and from North America and central Europe, a gender-balanced study of gesturing and diagramming during mathematizing across the world might give rise to a more global view of materialism and mobility in mathematics. The second extension pertains to studying the mathematizing of mathematicians ranging from novice to expert, which should include students and teachers from pre-K, grades K-12, as well as undergraduate studies. Such a study would not only raise awareness of an embodied view of mathematics, but might also bring to light a scaffolded approach to gesturing and diagramming that could provide new insights into the teaching and learning of mathematics.

Furthermore, a broader study of gestures employed in a variety of mathematical fields, ranging from the applied to the pure, could not only provide unequivocal support for Châtelet's notion of *virtuality* in mathematics, but might also reveal parallels between gestures during diagramming and gestures during symbolic manipulation while mathematizing. This idea came to me while observing research meetings 7 and 9, where the mathematicians formalized their mathematics and worked with text and symbols written on the blackboard.

The notion of doodling in mathematics as a teaching and learning aid has been popularized by the recreational mathematician Veronica Hart (en.wikipedia.org, 2015), also known as Vi Hart. By posting YouTube videos showing only her hand drawing and writing while talking about a particular mathematical concept such as fractals, the Golden Ratio and binary trees, Vi Hart introduced a new form of mathematical story-telling. Her doodling method has spawned other mathematics video series such as the similarly produced *MinutePhysics* videos on mathematics and physics by Henry Reich (perimeterinstitute.ca, 2015) or the documentary films *Numberphile* produced by Brady Haran (en.wikipedia.org, 2015). In a related vein, the algebraic geometer Ravi Vakil presents a variety of doodles in his article *The Mathematics of Doodling* (2011) in order to guide the reader playfully to ask questions: "[m]athematics is about asking the right questions, and that's what we will do here, finding interesting questions, and then finding questions behind the questions" (p. 116). This in turn raises the questions: How is doodling defined? Does doodling provide the same material site as diagramming? And does this playful notion of drawing and writing in mathematics mobilize mathematics thereby leading to mathematical invention (Vakil claims that the mathematical doodling "journey is a metaphor for mathematical exploration in general" (p. 129))?

Lastly, while I generally observed the gestures, actions and postures of the research group and created units of action in order to record these mobile acts, I nonetheless concentrated throughout my analysis on the gestures pertaining to diagramming. I noted the dance-like footsteps by mathematicians within the space in front of the blackboard, which brings them alternately closer to the blackboard and further away from it. Moreover, the apparent constant need of the expert mathematician to get up and not only diagram at the blackboard, but also gesture at close range with the diagram, makes me wonder if there is a difference in these acts according to whether diagramming occurs on paper while sitting at a desk or whether it occurs while standing at a blackboard and being able to move the entire body. Interestingly, the ergonomic standing desk has become a popular feature of offices in my department, and I regularly see at least ten mathematicians working while standing rather than sitting. A natural question to ask here is: does the difference in postural dynamics between *sitting* and *standing* affect the doing of mathematics?

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Appendix A.

Sample Pages of Data-viewing Notes

Page 1 of 2 of Notes about Video-recording from 2013-09-24

2013-09-24 F1-v1, F1-v2

- introduction of new operation "splitting" and understanding it via examples
- looking at embeddings in the plane
- use of Euler tour

1/2

	TIME	DESCRIPTION	NEW MATH	TALKING STICK	DIAGRAM	POSTURE	GESTURE	LANGUAGE	MOOD
	6:08	splitting a vertex preserves embedding	understanding	FS, dia	ever present	facing / away	pointing < for / close	MATH TERMS + this / this one	
	-7:33		operation		connecting to existing knowledge, abrupt turn to diagram (0:41)				
	10:18	fine-tuning operations:	"	conflict	tether to MS	facing blackboard		this < edge / face / graph	
		deleting / splitting edge / vertex		dia. → FS		facing audience	pointing at diagram		
	-14:05			FS / C	connecting to just established new knowledge				
	37:10	detail looking at splitting	examples of	FS	is pulling,	facing, half, away	pointing, covering new gesture for splitting	MATH + this	
			splitting	dia.	when explaining the operation to audience head keeps				
	-39:01				getting pulled back to diagram (5 TIMES)				
	43:54	excluded minor proof	"so little	FS	MS draws, ~~steps back~~	facing / away	head looks	this, read	JOY
		summary "just understanding the	associated	dia.	(3 TIMES)		back (3 TIMES)	off diagram	
	-45:09	nature of this operation"	pain"						
T*	49:15	ex, half octahedron in projected plane,		FS / C	of C3 (not) wrong	covering dia.	pointing with	this < edge / vertex	
		C3		dia.	→ disruptive	getting close	movement	(very specific)	LAUGH
	-51:26				i↑ legs		touch-point		
T*	52:22	diagram contains this thing as immersion,		FS		walks back	holds object	this < much / unique / argument	
		argument for double cycle in greater		dia.	is pulling	to blackboard	in hand but doesn't get anywhere	MATH	
	-52:59	generality							
*	54:37	"what does it correspond to in 'crossing land'"	"understanding	FS, V	helpful and	stares	pointing, walking	new terms	PUZZLEMENT
		"read that off" over here"	how to see	through diagram,	necessary		recalling		SILENCE
	-55:49	"not remember ... do"		which is communicated					

Page 2 of 2 of Notes about Video-recording from 2013-09-24

2013-09-24 FI-v1, FI-v2 2/2

TIME	DESCRIPTION	NEW MATH	TALKING STICK	DIAGRAM	POSTURE	GESTURE	LANGUAGE	MOOD
57:01	"Why does Eulertour argument work so	why it works?	FS	established	facing / away	pointing	MATH	PUZZLEMENT
	well?" "stick to embedding	what can go wrong?	V, dia.	helpful		sweeping		SILENCE
	connected things"		C, dia.	expanded on				
	"there is an IF there"		FS/V/C	rescue BUT	stares			PAUSE
				i1 right				EXCITEMENT
-59:59								
T* 00:55	~~what does a~~ two-edge cut do to graph?		FS, dia.	helpful				
	which graph can be embedded		FS/V/C					PUZZLEMENT
	in the plane? what are its		C, dia.	not helpful	C edge			SILENCE
	features?			(3 attempts)				
			FS/C			duo-drawing		
			C	4th attempt				AHA
-3:41				helpful				

Appendix B.

Transcriptions of Foursome

Table B-1. Transcription of Foursome F1-v1, 49:15-50:54

Time	Participants' non-verbal activity	Participants' utterances	Images	Diagram Episode
49:15 49:36	C faces blackboard C faces FG FS walks to blackb. C faces FS FS turns to C FS erases blackboard FS turns to C FS gestures *dual* **FS gestures *C_3* x 2** FS gestures *half-octahedron*	C: So maybe today let's go just over some examples for, for projective plane c four, right? FS: Yah. [So, so here's] what I am C: [Yah, yah] FS: Yah, actually, [I mean I think we can do?] C: [Yah, I think c three?] FS: Sorry? (.) Oh, yah, yah! So. Um, uh, yah so c three = C: = Yah, let's do [c three in the projective plane.] FS: [So c three embeds the dual of the (.) I mean c three is the, er is the, is the half-octahedron ((chuckle)) so if you [take] C: [Yah!]		diagram is absent
49:37 49:59	FS draws diagram FS stops drawing FS gestures *C_3* **FS gestures *antipodal*** **FS gestures projective plane** FS continues drawing	FS: If you take the projective plane (.) right? Then um you have this um (1.4) um (1.5) uh (1.1) like octahedron is a nice four-regular planar graph but it has antipodal edges? Right each vertex has an opposite vertex, each edge has uh has an opposite edge. So you can think of it on the um projective plane ((laughs)) right? And then it just gets cut in half, what it would give you is, is exactly this guy.	 	diagram is emerging diagram is present

50:00 50:54	 FS turns to mathem. FS touch-points *edge* FS *L* points to dia. FS continues drawing **FS *L* holds *edge*** FS traces diagram FS *L* holds diagram FS touch-points *edge* FS continues drawing FS steps away FS continues drawing FS *L* traces *edge* **FS erases *edge*** FS continues drawing **FS steps away**	FS: So now uh this goes maybe like that (1.7) so just, so those are the only three vertices. Er er what I keep track of who, which, which edge goes to where, right? So this goes to a, b, c, (1.1) a (.) b, c. And then uh, uh then I will orient them like this, so that this, this edge (2.3) a is going out to here (.) that's, uh (.) oh, sorry. (1.7) Have I done this one? (3.2) This edge (.) should be going to here. C: [Oh, right! It is, it is.] FS: [Yah! ((laughs))] (3.9) ((laughs)) Yah, there is nothing wrong, I just er ((laughs)), yah, so, so the the, this edge is going out of this [vertex] C: [Uh, no! In! Yah.] FS: Oh, shoot! = V: = In. = FS: = By which I meant in (.) yah (.) there is like a [fifty-fifty] chance ((laughs)). V: [Yah.]	  	diagram is disruptive

Table B-2. Transcription of Foursome F1-v2, 00:55-03:41

Time	Participants' non-verbal activity	Participants' utterances	Images	Diagram Episode

00:55	FS draws diagram	FS: If you *had* a 2-regular digraph that		diagram is emerging diagram is supportive to C and V diagram is unsupportive to FS
	FS turns to mathem.	*has* a 2-edge cut (.)		
	FS steps away	V: Mhm.=		
	FS *L* holds diagram	FS: =I can *immerse* in it, like er you		
		could just run wild splitting over		
		here until you got rid of		
		everything. ((laughs)) Right? You		
	FS continues drawing	can, that will give you an, that		
	FS turns to mathem.	would give you this guy.		
		((laughs))=		
	FS sweeps	V: =Yah!=		
	FS turns to mathem.	FS: =Right! And you can do the same		
	FS continues drawing	over here. So you can run wild		
	FS touchpoints *vertex*	splitting and, and, and, and get to		
	FS steps away	this.=		
	FS *L* holds diagram	V: =Mhm.		
		FS: And now (.) if (1.2) The thing is? If		
		you could embed this one in the		
	FS still *L* holds dia.	plane! (.) If this one is actually		
		planar, then you're, you're then,		
	FS *L* sweeps dia.	then, er,		
		V: You are done!		
		FS: Er,		
		V: Oh, yah.		
	FS *L* sweeps diagram	FS: Then,		
	FS turns to mathem.	C: Because the genus is just the, the		
	C gestures *sum*	sum of the two. So, that's easy to		
		see.=		
	FS steps far away	FS: =Oh, yah.=		
		C: =Right?		
	FS stares at diagram	FS: Yah.		
		V: Yah.		
01:38		C: Yah.		
	FS turns away	FS: Uh, (6.8) I ss, wait!		
01:39		C: You just take a er (5.3)		diagram is present
		FS: ((mumbles))		
		C: [Yah.]		
	FS stares at diagram	V: [Yah.]		
		C: You just face together, I mean it's		
		er		
	FS faces mathem.	FS: Yah, I mean there is a little, I		
	C gestures *connected*	C: [You take the connected sum of		
		the *surfaces*.]		
01:56	**C walks to blackb.**	FS: [There is a little check here.]		

01:57	**C draws diagram**	C:	So it's er, so this edge is some- where in the first surface, so (1.3)		
		FS:	Mhm.		
	C touch-points face C continues drawing C touch-points face	C:	So like this (1.1) so there is a face here right? And then (.) the other one has a, has a face (1.6) face here. So (2.4)		
	C continues drawing	FS:	Mhm.		
	 C erases *vertex* C continues drawing C faces mathematikoi C steps away C holds Fred's dia.	C:	So there is a (.) you think of, you *cut* this face together and then just join them (.) here so (1.1) you will delete this but cut this edge out, so there will be the appropriate rotation here, you see? And then (1.5) you delete this edge, so (.) this is coming in and this is going out (.) so, it's (1.5) just taking the connected sum. So the rotation would be (.) appropriate, right? (2.6)		
	FS turns away	FS:	I (1.2) believe you. But I am a little uncomfortable with the operation you did in the surface. So [on the surface, what have you done?]		
	C starts over C sweeps *face*	C:	[() Surface? Well, you take a] you take a face, you cut out a		
		FS:	[You remove all this?]		
02:57	C gestures *disk* C continues drawing	C:	[()] Er, well? Er, or, or a little disk in it and then take the other surface so this is one surface (1.8) okay, so (1.3)		diagram is emerging diagram is unsupportive to FS

02:58 03:41	FS walks to blackb. **FS,C duo-draw** FS *L* hand-points to his diagram C stops drawing **FS continues draw.** **C steps away** **C draws over his diagram** FS steps away **FS walks to blackb.** **FS stares at C's dia.** FS turns away C hand-points to dia. C stops drawing FS faces mathem. **FS gestures walking**	FS: Oh, [I see! Oh, oh, oh! I see! Yah!] C: [Here is the graph and then you take the other one.] FS: Oh, okay! So like I can think of this, this edge as being, I can think, think of this edge living in this face and then you're gonna choose, you are going to take a little disk that, that looks like this? C: Yah. [Yah. I, er, or just take] FS: [On both sides?] C: Or just take the () neighborhood. I mean, they will change. There was (.) this one was (.) was in a face here (.) right? This face will disappear, so it's er= FS: =*Oh*! Okay.= C: =Er, so and then there was a face here, at this edge= FS: =*I* see! C: So this is [now a new face here] FS: [Oh that's right. So, so, yah] yah yah. So= C: =So then you just do a face count. FS: Yes. I mean, you can always figure out your genus by just er V: [Euler's formula, yah.] FS: [walking around counting the] counting the faces and using Euler's formula here.	    	diagram is emerging diagram is supportive to FS

Table B-3. Transcription of Foursome F2-v1, 39:47-45:45

Time	Participants' non-verbal activity	Participants' utterances	Images	Diagram Episode
39:47 39:58	**FS stares at diagram**	FS: (12.1)		diagram is pulling

39:59 40:28	FS walks to blackboard **FS *L* holds *vertex, R* traces vertices** FS steps away **FG, C, V trio-draw**	FS: Oh, yah. And I can go (.) here, here, here? (8.2) No! (18.1) ((mumbles)) No, no, that is not what I said.	 	diagram is unsupportive to FS
40:29 44:32	FS repeatedly walks to blackboard **traces *edges***, erases *edges*, **draws over *edges***, steps away from blackboard, stares at diagram	FS: (4:03.1)	 	diagram is emerging diagram is central
44:33 44:38	FS steps away FS stares at diagram	FS: (.) *Oh*! It's not three, three, three, three, four, four. It's *five* threes and a five. (.)		diagram is disruptive

44:39	FS walks to blackb. **FS adds to diagram** FS steps away	C: Oh, you found one? FS: Yah! C: So where is the fifth triangle? FS: The outer directed triangle. (2.5) C: ((mumbles)) Two?		
	FS walks to blackb.	FS: Yah! It's all the triangles and a five-cycle. (2.2) And I believe it works. (1.9)		
	FS steps away FS adds to diagram	C: Yes? (2.9) FS: Um! I don't know what the easy way, I, I mean, I am actually having a hard time (8.1) I may not have much of Paul but I may have a memory of Paul. ((chuckles))		
		C: Yah, this is good. Yah, definitely.		
	FS steps away	FS: So, um= V: =There's your five-cycle, the red, right?		
	FS turns to mathem. FS turns to diagram FS walks to blackb.	FS: Yah, the red cycle is my five-cycle and then I am taking *all* the triangles. (4.6) It's (.) uh, it's hard to, uh=		
	FS steps away	C: =It's nice! (2.5)		
45:45	FS stares at diagram FS walks to blackb. **FS erases *edges***	FS: Yah, like almost it would be best to (.) I don't know what the right way of colouring this is, like er it seems like it would be better if we just would have a, a whole tub of colours or something, because it is hard to, I mean it is hard to check if you used this rotation scheme properly.		diagram is supportive

Table B-4. Transcription of Foursome F3-v1, 10:03-11:10

Time	FS' non-verbal activity	Participants' utterances	Images	Diagram Episode

10:03	touch-points *edges* **turns and walks from blackboard to blackboard** pulls down blackboard ***touch-points edge*** touch-points *edges* turns to mathematikoi	FS: This figure with these extra yellow edges in (.) and it is clear where almost all the yellow edges have to go, right? Cause if you have a yellow edge, uh (1.1) wow (.) we said (.) sorry. I don't think I know what I'm looking for. Ah, I'm looking for the sub-divided (2.1) what happened to the diagram over here? Oh, oh, I know. It's this one. So, if you have (1.4) you know, if I have an edge like this, that edge just has to go in *the* unique face that connects these two direct edges. And then (.) it has no other choice.	 	
	turns to diagram turns to C	C: Yah, except this one, the local ones [are a problem.]		
11:10	turns to diagram touch-points *edge* gestures *excluded* touch-points *edges* turns to mathematikoi **moves blackboard up** and steps away turns to mathematikoi gestures *obstruction* **hand-points to dia.** touch-points other dia. pulls blackboard down holds *edges*	FS: [Right!] You, you don't have edges from here to here cause that gives you one of our, our, our excluded things. So the *only* thing that is up in the air are *these* guys. And so the, the question is (.) um (.) like in some sense (2.0) uh, in some sense one of our minimal obstructions is gonna have the property that, um, sort of whichever face, whichever the two face sets you pick, you can't, you can't sort of choose faces for *these* types of edges without leading to crosses. ((chuckles))	 	diagram is pulling

Table B-5. Transcription of Foursome F5-v1, 21:39-23:38