Die Freude an der Gestalt: Methods, Figures, and Practices in Early Nineteenth Century Geometry

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ABSTRACT

As recounted by later historians, modern geometry began with Jean Victor Poncelet, whose contributions then spread to Germany alongside an opposition between geometric methods that came to be exemplified by the antagonism of Julius Plücker, an analytic geometer, and Jakob Steiner, a synthetic geometer. To determine the participants, arguments, and qualities of this perceived divide, we drew upon historical accounts from the late nineteenth and early twentieth centuries. Several themes emerged from the historical perspective, which we investigated within the original sources. Our questions centred on how geometers distinguished methods, when opposition arose, in what ways geometry disseminated from Poncelet to Plücker and Steiner, and whether this geometry was "modern" as claimed. Our search for methodological debates led to Poncelet's proposal that within pure geometry the figure was never lost from view, while it could be obscured by the calculations of algebra. We examined his argument through a case study that revealed visual attention within constructive problem solving, regardless of method. Further, geometers manipulated and represented figures through textual descriptions and coordinate equations. In these same texts, Poncelet and Joseph-Diez Gergonne instigated a debate on the principle of duality. Rather than dismiss their priority dispute as external to mathematics, we consider the texts involved as a medium for communicating geometry in which Poncelet and Gergonne developed strategies for introducing new geometry to a conservative audience. This conservative audience did not include Plücker and Steiner, who adapted new vocabulary, techniques and objects. Through comparing their common research, we found they differentiated methods based on personal considerations. Plücker practiced a "pure analytic geometry" that avoided calculation. Steiner admired "synthetic geometry" because of its organic unity. These qualities contradicted descriptions of analytic geometry as computational or synthetic geometry as ad-hoc. Finally, we turned to claims for novelty in the context of contemporary French books on geometry. Most of these books point to a pedagogical orientation, where the methodological divide was grounded in student prerequisites and "modern" implied the use of algebra in geometry. By contrast, research publications exhibited evolving forms of geometry that evaded dichotomous categorization.

Keywords: geometry, nineteenth century, analysis and synthesis, visualization, Jean Victor Poncelet, Joseph-Diez Gergonne, Julius Plücker, Jakob Steiner

RÉSUMÉ

Die Freude an der Gestalt: méthodes, figures et pratiques dans la géométrie au début du dix-neuvième siècle

L'histoire standard de la géométrie projective souligne une opposition entre les méthodes analytiques et synthétiques. Selon les historiens de la fin du dix-neuvième siècle, la géométrie moderne a commencé avec le *Traité des propriétés projectives* de Jean Victor Poncelet en 1817, puis, pendant le premier tiers du siècle, les contributions de Poncelet se répandirent en Allemagne, ainsi qu'une opposition entre différentes approches géométriques dont l'exemple toujours cité est l'antagonisme entre Julius Plücker, un géométre analytique, et Jakob Steiner, un géomètre synthétique. Ce n'est qu'à partir des années 1870, selon ces récits, que les géomètres mirent fin à une distinction qui avait cessé d'être pertinente.

Pour déterminer les participants, les arguments et les qualités de cette apparente division méthodologique, nous avons puisé dans les récits historiques écrits à la fin du dix-neuvième siècle et au début du vingtième siècles. Même s'ils insistent pour résumer la situation dans l'opposition globale et binaire que nous avons décrite, leurs écrits suggèrent déjà qu'il y avait plutôt une multitude d'oppositions à plus petite échelle, qui résultaient en particulier de ce que des découvertes multiples, presque simultanées, étaient faites par un petit groupe de géomètres.

Plusieurs thèmes principaux émergeaient de cette perspective historiographique, qui forme le premier chapitre de cette thèse, et nous avons décidé de les approfondir en les confrontant à une étude détaillée des textes originaux : des textes de Poncelet, Plücker, Steiner et Joseph-Diez Gergonne sur la géométrie et la méthodologie écrits pendant le premiers tiers du dix-neuvième siècle. Nos questions sont centrées sur la manière dont ces géomètres ont distingué leurs propres méthodes géométriques de celles des autres mathématiciens contemporains, quand une opposition surgissait en géométrie, sur la manière dont à la fois la géométrie et ces oppositions se sont transmises de Poncelet à Plücker et Steiner, et dans quelle mesure cette géométrie était “moderne” et nouvelle comme le clamaient ses praticiens, et plus tard les historiens de la géométrie.

Dans le deuxième chapitre, nous examinons donc en détail un problème de mathématiques concernant l'inscription de figures dans des coniques, problème qui a été discuté entre plusieurs mathématiciens (Gergonne, Poncelet, Plücker, etc.) et a déclenché des polémiques sur les méthodes de solution. J'ai étudié en détail les arguments, les techniques, etc. et les oppositions, j'ai aussi montré le rôle des nouveaux moyens de diffusion que sont les journaux mathématiques.

Notre recherche sur ce débat méthodologique nous a conduit à un échange épisté-

mologique entre Poncelet et Gergonne sur l'utilisation de l'analyse algébrique en géométrie. En distinguant ce qu'il appelle la pure géométrie de la géométrie analytique, Poncelet insistait sur le role central de la figure. Il suggérait que tant dans l'ancienne géométrie pure, celle héritée d'Euclide et de l'Antiquité, que dans la géométrie pure moderne qu'il incarnait, la figure n'était jamais perdue de vue, et qu'elle pourrait être obscurcie par les calculs de l'algèbre appliquée à la géométrie. En géométrie synthétique, les objets de la géométrie étaient representationnels et tangibles, ces qualités étant encore mises en avant même lorsque les géomètres introduisirent des points imaginaires ou à l'infini.

Nous examinons ici l'argument de Poncelet en action à travers l'étude de plusieurs solutions à un même problème de construction géométrique qui a été discuté et résolu dans un ensemble de publications étroitement connectées, mais écrites par plusieurs auteurs différents. Il s'agit de construire une courbe du second ordre ayant un contact d'ordre trois avec une courbe plane donnée, dont cinq solutions paraissent entre 1817 et 1826. En étudiant comment les auteurs invoquaient les figures et quelles formes alternatives leur servaient à représenter les objets géométriques, nous avons trouvé que l'attention visuelle est au coeur de la résolution de ces problèmes de construction, indépendamment de la méthode suivie, mais qu'elle n'est pas non plus réservée aux figures. Se fondant sur ce qu'il décrit comme le manque d'élégance des calculs analytiques, Poncelet perçoit cet exemple de construction comme particulièrement adapté à une approche de géométrie pure et développe en particulier le concept de corde idéale pour généraliser les cas où les points de tangence sont imaginaires.

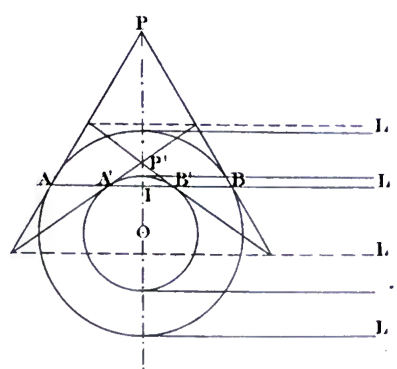

Figure 1: Illustration selon Poncelet de la corde idéale commune à l'infini entre deux cercles concentriques, *Traité des propriétés projectives*, 1822.

Mais Poncelet et Plücker ont aussi développé des stratégies pour manipuler et visualiser d'autres types de représentations du problème. Plücker par exemple a développé de nouvelles formes d'équations permettant de révéler des éléments clés des objets géométriques sans recourir à des calculs excessifs. Pour une courbe du deuxième degré

$$y^2 + 2\alpha xy + \beta x^2 + 2\delta x = 0$$

tangente à l'axe des y à l'origine, Plücker utilise un système de deux droites passant par l'origine, $y + mx = 0$, $y + nx = 0$ pour déterminer la corde commune entre la courbe donnée et celle cherchée, représentée par

$$(2\alpha - (m + n))y + (\beta - mn)x + 2\delta = 0.$$

Nous montrons donc comment la distinction fondée sur les modes de représentation visuelle se matérialise effectivement dans les pratiques géométriques de l'époque et ce qui constitue une évidence géométrique quand la figure ordinaire n'est plus le support de l'intuition.

Le même ensemble de textes a attiré notre attention sur des discussions de méthodologie pendant le premier tiers du dix-neuvième siècle entre Gergonne et Poncelet, s'achevant sur un accord entre les deux : le choix du problème devrait déterminer le choix de la méthode. Mais une décennie plus tard, Gergonne et Poncelet furent impliqués dans un débat bien plus violent entre plusieurs auteurs publiant des résultats très proches et témoignant d'un manque de reconnaissance mutuelle supposé ou affiché sur le principe de dualité. Plutôt que d'ignorer cet épisode comme une simple dispute de prioritié externe aux mathématiques, nous avons, dans le troisième chapitre, étudié les textes en question comme un moyen de communiquer de la géométrie en signalant à l'intention d'un public en formation de nouvelles zones de recherche et en créant même une audience intéressée par ces questions. Malgré leurs désaccords sur l'orgine et l'étendue du principe de dualité, tant Poncelet que Gergonne s'inquiétaient des difficultés à défendre la nouvelle géométrie face à un auditoire plus conservateur. Cependant, en quelques années, la publicité engendrée par la controverses avait réussi à engendrer de nouveaux travaux de recherche dans un groupe plus important de mathématiciens.

Cet auditoire conservateur n'incluait évidemment pas Plücker et Steiner, dont les premières publications témoignent de l'adoption et de l'adaptation du vocabulaire, des techniques et des objets de Poncelet et de Gergonne. Ces deux mathématiciens, pourtant souvent décrits comme "mathématiciens allemands" atteignaient souvent les mêmes résultats, et c'est en comparant en détail leurs recherches parallèles que nous avons trouvé comment se distinguaient leur travail et leurs méthodes. Des comptes rendus de leurs travaux et des correspondances nous ont permis aussi d'approfondir comment leur recherche était décrite et comprise par leurs contemporains. Le troisième chapitre est une étude comparative de deux mathématiciens Steiner et Plücker, que je suis dans leurs travaux de recherche, leurs cours, etc. L'un décrivant la géométrie synthétique comme celle où la figure n'est "jamais perdue de vue", nous avons examiné en particulier leurs usages des figures, et aussi au contraire comment les notations algébriques pouvaient apparaître comme des représentations

figurées, faisant appel à l'intuition. L'examen de leurs publications de recherche pendant cette période nous a permis de montrer que leurs différentes approches particulières à la géométrie contredisaient les descriptions grossières entre termes de méthodes analytiques ou synthétiques. Plücker voulait de fait pratiquer une "géométrie analytique pure" dans laquelle les équations représentaient en un sens fort les objets géométriques, y compris sur le plan de l'intuition visuelle et contrairement aux caractérisations trop sommaires de la géométrie analytique, il voulait au contraire éviter les calculs. Steiner justifiait son choix d'une approche synthétique par un désir d'unité organique au sein des mathématiques.

Nous nous sommes enfin tournée vers les affirmations faites par les géomètres sur la nouveauté et la modernité de leurs tentatives en examinant les livres français sur la géométrie publiés pendant le premier tiers du dix-neuvième siècle. La majorité de ces textes ont une orientation pédagogique et la division de méthode se fonde sur les connaissances antérieures des étudiants, l'usage du mot "moderne" impliquant dans ces manuels l'utilisation d'équations aux coordonnées en géométrie. Il ne s'agit donc pas de passer de l'ancienne géométrie synthétique, euclidienne, à une nouvelle géométrie sysnthétique, incorporant projection et points imaginaires, mais bien de contrer l'apport présenté comme simplificateur de l'analyse ; ceci explique aussi pourquoi les géomètres novateurs ont fait tant d'efforts pour présenter leurs recherches comme intuitives, accessibles visuellement et constructives. Par contraste avec le contenu et les pratiques visibles dans les manuels contemporains, tant Gergonne et Plücker que Steiner ont continué à développer des formes de géométrie qui ne se pliaient pas si facilement à une caractérisation dichotomique, mais répondaient de manière nuancée à l'état des connaissances, mais aussi des pratiques mathématiques et des modes d'interaction de leur temps.

Mots-Cléfs: géométrie, XIXe siècle, analyse et synthèse, visualisation, Jean Victor Poncelet, Joseph-Diez Gergonne, Julius Plücker, Jakob Steiner

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Thank you.

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Introduction

The story goes that in the early nineteenth century, the animosity between synthetic and analytic geometers had reached such an impasse that Jakob Steiner (a synthetic geometer) informed the editor, August Leopold Crelle, that he would no longer submit articles to his *Journal für die reine und angewandte Mathematik*, as long as Julius Plücker (an analytic geometer) continued to be published there.

This dramatic episode has resurfaced in the history of geometry since the 1870s and is reproduced to this day (Greenberg (2008), Gray (2010b), Eccarius (1980), Kline (1972), Boyer and Merzbach (1968), Boyer (1956)). One recent account can be found in Jeremy Gray's *Worlds Out of Nothing*.

> But relations between Plücker and Steiner were particularly bad, and Steiner was in the growing University of Berlin with ready access to Leopold Crelle, the successful editor of the first journal in Germany devoted exclusively to mathematics. Steiner identified strongly with synthetic geometry, and let it be known that he would not be willing to write for Crelle's journal if it continued to carry articles by Plücker. Plücker felt completely denigrated in Berlin, and switched fields.(Gray (2010b), 167)

Where did this anecdote come from? When we explore backwards the references mentioned, we find several other twentieth century historical sources. Morris Kline's *Mathematical Thought from Ancient to Modern Times* contained a version of this conflict, which placed greater emphasis on the methodological disparity. Steiner's actions only exemplified the most bitter effect.

> The upshot of the controversy is that the pure geometers reasserted their role in mathematics. As if to revenge themselves on Descartes because his creation of analytic geometry had caused the abandonment of pure geometry, the early nineteenth-century geometers made it their objective to beat Descartes at the game of geometry. The rivalry between analysts and geometers grew so bitter that Steiner, who was a pure geometer, threatened to quit writing for Crelle's

> *Journal für Mathematik* if Crelle continued to publish the analytical papers of Plücker. (Kline (1972), 836)

Kline's account appeared in his chapter on "The Revival of Projective Geometry," which included a brief bibliography with two corroborating sources.

Among these sources, the more recent text is Carl Boyer's *History of Analytic Geometry* (1956). Boyer repeated the story of Steiner's antagonism to "the analytic point of view" and gave an exact citation to Florian Cajori's *A History of Mathematics* (1931), while Kline had cited the 1919 edition. Cajori's version of the events originated with his first edition from 1893.

> But in Germany Plücker's researches met with no favour. His method was declared to be unproductive as compared with the synthetic method of Steiner and Poncelet! His relations with Jacobi were not altogether friendly. Steiner once declared that he would stop writing for Crelle's Journal if Plücker continued to contribute to it. [66] (Cajori (1893), 359)

Happily, here we find another precise bibliographic link back to Adolf Dronke's 1871 biography *Julius Plücker Professor der Mathematik und Physik an der Rhein.*

> Jacobi, dessen Grösse als Mathematiker zu schmälern wohl Niemand zu versuchen sich erlauben wird, handelte gegenüber Plücker, gelinde gesagt, eigentümlich. In einem Aufsätze schliesst er sich zwar an die Arbeiten des Letztern an, ignoriert aber deren Verfasser vollständig. Steiner erklärte, nicht mehr in Crelle's Journal schreiben zu wollen, falls noch Arbeiten Plücker's fernerhin Aufnahme fänden. Dadurch war ihm Berlin vollständig verleidet und ist es wohl begreiflich, wie er seine wissenschaftlichen Arbeiten meist in ausländischen Journalen niederlegte, wo er wusste, dass seine Leistungen wenigstens nicht verachtet wurden. (Dronke (1871), 11–12)[1]

In his foreword, Dronke advertised that he had been close to Plücker as a student and a friend, and with the permission of the family had consulted his teacher's unpublished papers in composing this account. Yet, another biography of Plücker, by Alfred Clebsch, also from 1871, contained no mention of Steiner's blackmail attempt, nor did Felix Klein's historical analyses, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (Clebsch (1872), Klein (1926a)).

[1] "Jacobi, whose greatness as a mathematician probably no one would allow himself to attempt to besmirch, acted strangely against Plücker, to say the least. In an article, in fact, he follows the works of the latter, but ignores the author completely. Steiner declared that he would no longer write in Crelle's Journal, if Plücker's work still continued to appear there. Berlin was thereby ruined for him and it is easy to grasp why he submitted his scientific works mostly to foreign journals, where he knew that his contributions were at least not despised."

Looking beyond this bibliographic chain, there is further evidence of Steiner's personal feelings towards Plücker. In *Die Mathematik und ihre Dozenten an der Berliner Universität* Kurt Biermann remarked that "Steiner hasste Plücker gründlich," citing the correspondence between Siegfried Aronhold and Otto Hesse in 1849 (Biermann (1973), 47). This correspondence was posthumously published in Crelle's *Journal* in 1902. In a letter from December 18th, Aronhold reassured Hesse, who had apparently received harsh criticism from Steiner regarding his analytic research.

> Ihren Gruss habe ich Steiner überbracht. Den kleinen Krieg mit ihm dürften Sie weniger ernst nehmen. Er ist zu sehr Hypochonder, als dass er von dem abgehen sollte, was er sich einmal in den Kopf gesetzt hat. Er scheint sich schon durch Ihre früheren Arbeiten verletzt zu fühlen, wie überhaupt durch die der Analytiker, und es ist besonders Plücker, den er gründlich hasst, und welcher wohl seine geringe Zärtlichkeit für die Analysis hervorgerufen hat. Uebrigens dürfte Steiner seine wichtigsten Entdeckungen immer publicirt haben und nur neue Methoden für die Behandlung bekannter Probleme in seinen Manuskripten zurückbehalten. Von dem Probleme der Wendepunkte für Curven vierten Grades hat er oft gesagt, dass er daraus "nichts Gescheites" machen kann. Es scheint überhaupt die Synthesis der Gränze nahe zu sein, wo sie ohne die Hülfe der Analysis nicht weiter kommen kann. Vielleicht irre ich mich, wenn ich aus dem Umstande, dass Steiner recht gerne analytische Resultate benutzt, einige Gründe für die Behauptung entnehmen kann. (Aronhold (1902), 64)[2]

While this exchange certainly corroborated Steiner's hatred of Plücker, the reason behind Steiner's hatred was left ambiguous.[3] Instead, Aronhold attributed Steiner's alleged (and perhaps even affected) distaste for analysis to his prior feelings against Plücker. Conversely, Steiner's threat to Crelle regarding Plücker has been explained as caused by Steiner's dedication to the purely geometric or synthetic method.

Dronke's account of Steiner's actions appears to be the first published source of this story. The origins in a biography of Plücker help to explain some of its literary tropes: the hostile and powerful Steiner (sometimes accompanied by the even more powerful Carl Gustav Jacob

[2] "I've delivered your greeting to Steiner. You should take the little war with him less seriously. He is such a hypochondriac, that he goes off on whatever once he has made up his mind. He seems to have felt injured by your previous work, as by those of all the analysts, and particularly Plücker whom he hates thoroughly, and which has probably caused his limited sympathy for Analysis. Besides, Steiner should have published all his most important discoveries and withheld only new methods for the treatment of known problems in his manuscripts. From the problems of the turning points for fourth degree curves, he has often said that he can make "nothing clever" of it. The limit of synthesis seems to be close where it can not get anywhere without the aid of analysis. Maybe I'm wrong, though I can see some reasons for the claim, from the fact that Steiner very gladly used analytical results."

[3] Plücker may have been difficult to get along with in general, as is suggested by Hermann von Helmholtz in a letter to Rudolf Lipschitz from 1856 (Lipschitz (1986), 113–116).

Jacobi), Plücker's subsequent exile from German mathematics to experimental physics at the University of Bonn, followed by his eventual international recognition, notably only returning to mathematics after Steiner's death.[4] The dramatic arc overshadows the lack of concrete details. To take the most striking anomaly, it is completely unclear when Steiner may have issued his ultimatum and whether it was indeed effective. Plücker continued publishing alongside Steiner in Crelle's *Journal* for twenty years, from 1827 until 1847.[5] After this date, Plücker stopped publishing on mathematics in any journals, in Germany or abroad, until 1865 when he finished his career in experimental physics and returned to geometry. However, by 1847 Steiner and Jacobi were no longer on cordial terms, so would not have been acting against Plücker collectively. The animosity between these former friends has been documented within letters between Jacobi and his brother in 1846, and was well known enough among contemporary geometers that Poncelet commented on the circumstances (Ahrens (1907), 132–148; Poncelet (1866), 411). Jacobi did remark on the advantages of geometry over analytic geometry in a paper given to the French *Académie des sciences* in 1844, but his description only referred to geometry without calculation or figures–a description that could apply to Steiner or Plücker, as we will see.[6] Further, published letters of recommendation from Crelle regarding Plücker's advances in analytic geometry, general merits, and positive comparison to Steiner dating from the early 1830s and through mid 1840s, raise doubt that Crelle would have rejected Plücker's work (Ernst (1933)). Finally, if Dronke learned of Steiner's actions through Plücker's papers, one must suppose that Plücker knew. Did Crelle inform him? These speculations convolute the

[4]Plücker's contributions to physics were briefly documented in a note by Johann Wilhelm Hittorf following Alfred Clebsch's mathematical biography (Hittorf (1872)). More recently, Roland Jackson's study of the early history of diamagnetism detailed Plücker's experimental contributions to that field (Jackson (2014)).

[5]Citing Dronke, Wolfgang Eccarius argued that Steiner's actions were motivated by limited professional opportunities, rather than methodological differences. In particular, Eccarius showed through letters between Crelle and the ministry that Steiner and Plücker were both being considered for the same available teaching positions at a proposed polytechnic institute in Berlin. This cause would then suppose that the episode occurred in the early 1830s when both geometers lived in that city and neither held a permanent post. By the mid-1830's the plans for the polytechnic institute were abandoned (Eccarius (1980)).

[6]This paper was published in the *Comptes rendus*, t. XIX, 1844, 1239–1261 and quoted in (Chasles (1870), 119).

> Si l'on considère que tant de résultats, dont chacun exigerait en Géométrie analytique une démonstration différente et peut-être parfois difficile, dérivent aisément ici d'un seul théorème primitif, dont ils ne sont, en quelque sorte, que des transformations qui se font par le seul raisonnement, sans exiger ni calcul ni figures, on verra, je crois, dans cette fécondité et cette facilité de démonstration, un nouvel exemple des ressources que pourraient offrir les méthodes géométriques, si cette partie si importante des Mathématiques était plus cultivée.
>
> "If we consider that so many results, of which each requires a different and perhaps sometimes difficult proof in analytic Geometry,here derive easily from a single primitive theorem, of which they are only, in some way, transformations made by reasoning alone, without requiring calculation or figures, we will see, I believe, in this fruitfulness and in this ease of proof, a new example of resources that geometric methods could provide, if this important part of Mathematics was cultivated more."

Michel Chasles remarked that this extract related to the "culture of methods of synthetic Geometry."

catchy narrative, and call into question the story's frequent repetition as historical fact.

Certainly, Plücker's abrupt transition from analytic geometry to experimental physics demands an explanation. However, as Boyer suggested, following his account of the role of Steiner, Plücker may have stopped publishing on geometry for an altogether less exciting reason.

> There is, however, another explanation available which would appear more plausible. From 1825 until 1846 Plücker had taught mathematics–first at Bonn, then at Berlin, and finally at Halle. In 1847 he became professor of physics at Bonn; and it is said that there was some criticism of the fact that a chair in physics should be held by a pure mathematician. (Boyer (1956), 255)

While this explanation may be more plausible, it is apparently less popular. Despite its vagaries, the story of Steiner thwarting Plücker has endured as emblematic of early nineteenth century geometry and corroborate a larger narrative: the methodological opposition between synthetic and analytic geometry. As we will see, this narrative of two competing methods has dominated histories of early nineteenth century geometries from biographies of individual geometers, such as the case of Plücker above, to general contemporary histories of mathematics.

Yet, when we examine the texts of the alleged analytic and synthetic geometers we will find remarkable fluidity between content, objects, and techniques. Regardless of purported method, geometric practices remained visually informed and dedicated toward reaching ever better solutions inspired by classic planar geometry problems. While these theorems, tools and principles also found generalizations and applications in increasingly complicated geometric domains and problem sets, the subject matter remained focused on conic sections and their three-dimensional analogues. We will see that this was a repetitive, competitive, and pedagogically oriented atmosphere, where authors distinguished their work by emphasizing the novelty of their methods. With innovative processes but familiar results, geometers directly contrasted the simplicity, elegance and clarity of their approaches against those of specific colleagues who had solved identical problems. The subtlety of these distinctions often led to public and private suspicions of plagiarism, which in turn fuelled sharper divisions. In efforts to recognize and communicate new areas of geometric understanding, geometers continually redefined the limits of methodological boundaries.

Hidden by apparently stagnant visual planar figures, invisible, infinite, imaginary, ideal, and generally inaccessible geometric objects were developed, investigated, and proliferated among researchers. Rather than a dichotomy between analytic and synthetic, we will find a continuum of geometries proliferating and evolving new proofs, objects, operations, and modes of representation. One distinction we will explore concerns how early nineteenth century geometers presented their methods as "modern" in contrast to the figure bound particularity characteristic of so-called "ancient geometry." While developing and adapting modern geometric objects and principles that promoted generality over individual cases for each possible configuration, the overarching connection to the figure ultimately demanded a commitment to visual representation. Even a general theorem was intended to be applicable within specific constructive problems. With this evolution of the domain of geometry, early nineteenth century researchers moved away from contemporary textbook and teaching practices.

Before approaching the complex relationship between analytic and synthetic geometries, we must first clarify what we intend by the two contrasting descriptions. The connotations of analytic and synthetic in mathematics have encompassed a wide range of conflicting meanings, and we are confronted with diverse usages among individuals in the early nineteenth century as well as later historical evaluations. In our case, some of the many uses of analysis and synthesis in mathematics have been approached and dissected in Michael Otte and Marco Panza's collection *Analysis and Synthesis in Mathematics: history and philosophy.* Among the numerous possible interpretations, analytic and synthetic geometry most resemble the 'historico-theoretical' interpretation,

> A mathematical theory is synthetic, if it refers to the classical geometrical objects or arguments or even to the classical theories of proportion, or numbers or magnitudes. It is analytic if it considers its objects as arguments of certain equations (rather than proportions) or operations, or even as functions. (Otte and Panza (2005), xi)

However, this neat either/or leaves ambiguous geometric practices adopted in the early nineteenth century, such as incorporating imaginary points of intersection and projecting lines to infinity.

Even within early nineteenth century geometry, the possible connotations of analytic and synthetic have been shown to be complicated. Massimo Mazzotti's study of analytic and synthetic geometry in early nineteenth century Italy focuses on an identically named and roughly contemporary opposition with vastly different motivations and outcomes than the one we just described (Mazzotti (1998)).[7] His contextualized reading of geometric texts in the Kingdom of Naples, reveals underlying political and religious divisions that we found had almost no correspondence in French and German publications. Analysis and synthesis have too many meanings unless we specify particular places and people shaping the terms of their debate, and we cannot simply transpose conclusions between disparate contexts.

As Giorgio Israel has shown in "The Analytical Method in Descartes' Géométrie" the term "analytic geometry" was first defined by Sylvestre François Lacroix in 1797, as

[7]The divide between Italian geometers was striking enough that Jacobi wrote to his brother, Moritz, of it after visiting Naples in 1844. "Die Mathematiker, die hier etwas zurück sind, theilen sich hier in zwei feindliche Parteien und Schulen [...]" (Ahrens (1907), 113)

“l’application de l’Algèbre à la Géométrie” (Lacroix (1797), xxvi quoted in Israel (2005), 9). Lacroix professed to be following Gaspard Monge in this description, and particularly his *Application de l’Analyse à la Géométrie* first published in 1795. As we will further show in Chapter V, nineteenth century geometers appear to have interpreted “analytic methods” as more or less corresponding to geometry utilizing coordinate equations, and interchangeably used the terms “the application of algebra to geometry” and “analytic geometry,” while “the application of analysis to geometry” often implied a use of differential or integral calculus. Synthetic geometry had no such simple corresponding translation, and was not a widespread term in the early nineteenth century. For example, the *Annales des mathématiques pures et appliquées* contained one subject heading or more for each article, and among these we find elementary geometry, geometry of position, pure geometry and geometry of the ruler, but no synthetic geometry. Even when geometers divided the subject in two, the opposition was not necessarily between synthesis and analysis. For instance, we will see how Jean Victor Poncelet contrasted his modern pure geometry with modern analytic geometry. In our account, we will be careful to maintain the authors’ original usages, which immediately suggest problems in the simple terms of the historical summary. Catherine Goldstein has observed the power of following the use of often cited key words describing kinds of mathematics. Although seemingly clichéd, terms such as “simplicité, clarté, fécondité”

> [...] jouent, tout comme dans l’entourage de Felix Klein le mot *Anschauung*, le rôle d’une bannière, ralliant des mathématiciens, des types d’explication, des méthodes. Employés ensemble ou non, assez rarement en association avec d’autres termes (“simple et général” , un peu plus souvent “précis”), ces mots viennent s’opposer de manière récurrente au couple “rigoureux” et “compliqué.” (Goldstein (2011), 14)

Similarly, “analytic and synthetic” provide a means of association as well as disassociation to existent results and techniques. Rather than precise adjectives, the terms analysis and synthesis in geometry often acted as substitutes among historians for a perceived methodological bifurcation that might emerge under different names when applied by earlier geometers. Our thesis thus does not simply argue against a clear division between analytic and synthetic geometry in early nineteenth century geometry, since this could be easily shown by pointing to the absence of “synthetic geometry” as a common category among the actors themselves. Rather we will show the absence of a uniform dichotomy under any pair of contrasting adjectives.

In his comparison with early and late nineteenth century methodological claims in geometry, Moritz Epple presented the specific qualities of the “pre-modern” early nineteenth century analysis and synthesis debate as between “not two different branches of mathematical knowledge but rather two different modes of presenting, acquiring and justifying this knowledge" (Epple (2005), 179). Epple continued by pointing to the methodological purism on both sides of the divide, and suggesting that this emphasis ultimately resulted in a "petrification" that was only resolved through later methodological diversity, as exemplified, for instance, in Felix Klein. One-hundred years earlier, Gino Fano similarly suggested that the methodological difference between analytic and synthetic geometries was a matter of "Sprache" (Fano (1907), 228). Yet, style cannot simply be divorced from substance, as has been recently reiterated in the numerous papers from the Oberwolfach Report on "Disciplines and Styles in Pure Mathematics" (Rowe, Volkert, Vullermot and Remmert (2010)). To take two succinct comments, Gray observed, "It is probably not possible to say the same thing in different ways" (Gray (2010a), 586). With similar doubt, Norbert Schappacher found "that different styles are rarely just varying ways to express an invariant content" (Schappacher (2010), 657). Thus, our study of early nineteenth century geometries considers not only how the authors described their work, but also what changed when the same "content" appeared in different styles. The early nineteenth century geometers themselves did not use the term style, but instead often referred to the "form" in juxtaposition to the "content" of an article or presented result. We will more closely examine the relationship between form and content as they were used to describe mathematics.

The historiography of early nineteenth century geometry is primarily limited to developments in France and Germany, as will be shown in greater detail in Chapter I. In particular, we will see Jean Victor Poncelet (1788–1867) and Michel Chasles (1793–1880) consistently cited as the major early nineteenth century French geometers, while August Ferdinand Möbius (1790–1868), Jakob Steiner (1793–1863) and Julius Plücker (1801–1868) represented the key figures of German geometry. In the historical literature, the latter two mathematicians would come to symbolize the oppositional nature of the synthetic and analytic method.

Reviewing the past 150 years of secondary literature we find varied perspectives on the lives and mathematics of these two geometers. Both have been the subjects of several biographical studies from their deaths up to the early 1930's. Swiss mathematicians and mathematical organizations have revisited Steiner's legacy on several occasions, beginning with Steiner's first biography by his nephew Karl Friedrich Geiser in 1872 (Geiser (1874)), which is still the most widely cited. At the turn of the century, Johann Heinrich Graf and Julius Lange respectively provided further biographic details of Steiner's life, emphasizing his origins in Switzerland and his life in Berlin (Graf (1897), Lange (1899)). Lange supported his narrative with many letters to, from, and about Steiner, which supplemented perspectives on his methodology drawn from published mathematics texts. Similarly, the most recent biography of Plücker was a thesis from Wilhelm Ernst, a student at Halle who had access to enough of Plücker's correspondences to shed new light on the 1872 accounts by Clebsch and Dronke (Ernst (1933), Clebsch (1872), Dronke (1871)). Plücker's contributions to analytic geometry were further explored in the mid-twentieth century, when Boyer featured Plücker as the protagonist of the “Golden Age of Analytic Geometry” (Boyer (1956)).

Having worked at the University of Berlin, both Plücker and Steiner appeared in Kurt Biermann’s comprehensive history of the institution, although Steiner, as a former professor there, received more attention (Biermann (1973)). Both geometers published in the same venues, and Wolfgang Eccarius has provided a direct comparison between their social situations and search for employment in Berlin (Eccarius (1980)). Drawing on previously unpublished letters from the editor of the *Journal für die reine und angewandte Mathematik*, August Leopold Crelle, who also lobbied for salaried positions for several of his contributors, Eccarius proposed that the methodological divide was greatly motivated by these economic concerns. With respect to their mathematical legacies, Gray recently underscored Plücker’s substantial contributions to the theory of duality, where “Plücker’s contribution has been rather marginalised” (Gray (2010b), vi).

After the nineteenth century, Steiner too became a marginal character, falling under the shadow of his so-called ”successor,” Christian von Staudt (Klein (1926a), Coolidge (1940)). Consequently Steiner’s synthetic geometry often appears as transitional, making steps towards, but never achieving, a completely non-metric geometry. For instance, David Rowe has described how imaginary objects, a consequence of algebraic analysis applied to geometry, maintained an uncertain status in the work of Steiner and synthetic geometry in general, until von Staudt’s successful interpretation in 1847 (Rowe (1997)). In Philippe Nabonnand’s history of points and lines at infinity, he examined Steiner’s specific contributions in the development of geometry from Poncelet to von Staudt. For Steiner, “l’objet central de la géométrie n’est plus la figure mais devient la notion de formes fondamentales” (Nabonnand (2011a), 167). This then points to a very different concept of pure geometry than the figure centred geometry professed by Poncelet, and we will later investigate this conclusion with respect to Steiner’s early publications. Yet as much as Steiner can be interpreted as breaking away from Euclidean geometry, his work remained committed to constructions and questions from ancient Greek geometry. Viktor Blåsjö has attempted to rescue Steiner’s *Systematische Entwicklung* from “a lasting and undeserved depreciation” by interpreting it as “a monumental unification of classical geometry” (Blasjö (2009), 21). From a more literary perspective, Anne Boyer has integrated Steiner within the Romantic movement of his time, drawing particularly on his Socratic style of pedagogy (Boyer (1999)).

Although Plücker and Steiner have often costarred in the drama of analysis versus synthesis, their specific mathematical contributions have yet to be directly compared, except in encyclopedic summaries (for example, Kötter (1901), Fano (1907), Schoenflies (1909)). Through several focused case studies, we will find that the early geometry of Steiner and Plücker (from their first publications in 1826 to Steiner’s first monograph in 1832) explicitly engaged with "modern" pure geometry as first proposed by Poncelet in 1817. The striking similarity between Steiner and Plücker's research subjects, at least initially, was due more to this common source than their mutual influence. Thus the history of both analytic and synthetic geometry must include Poncelet's contributions.

The historiography of Poncelet draws on his education at the École polytechnique, the development of projective geometry while a prisoner of war at Saratoff, the controversial principle of continuity that sparked antagonism with Augustin-Louis Cauchy, and the contested priority over the principle of duality with Joseph-Diez Gergonne. Poncelet's contributions to projective geometry constitute much of the impetus and subject matter of *Eléments d'une biographie de l'espace projectif* in articles by Philippe Lombard, Jean-Pierre Friedelmeyer, and Philippe Nabonnand (Bioesmat-Martagnon (2011b), Lombard (2011), Friedelmeyer (2011), Nabonnand (2011a)). These detailed textual analyses of Poncelet's writings show his development of a dynamic figurative approach, invention of new geometric objects, and complicated relationship to analytic geometry. Poncelet's emphasized generality, manifested in not separating solutions and proofs into multiple or exceptional cases. This generality underscored his specific form of pure geometry, in which the generality usually attributed to analytic geometry extended both to the figure and to the act of viewing. In particular, Jean-Pierre Friedelmeyer has contributed to understanding Poncelet's "pure geometry" by detailing what exactly was original in his work, and how Poncelet's attempt to replace analytic calculation with geometric reasoning, while still avoiding particular case studies, led to his invariant projective properties, points at infinity, ideal objects, and the principle of continuity.

This latter geometric principle had been imported from analytic geometry and algebraic analysis, as professed by Poncelet towards the end of his career and well documented by later historical studies of published and unpublished manuscripts (Poncelet (1864), Belhoste (1998), Gray (2005), Nabonnand (2011b), Lombard (2011), Friedelmeyer (2011)). As we will see in Chapters II, III and IV, the principle of continuity established which properties remained invariant as a primitive figure, such as a conic section, was deformed. Although many of Poncelet's contemporaries viewed this principle with skepticism, both Plücker and Steiner would find inspiration and further develop the results suggested by this form of continuity (Chasles (1837), Sturm (1826a), Plücker (1829b), Steiner (1828d)).

The rich available literature on Poncelet's biography situates our study of his relationship with the figure, which he believed was imperative to pure geometry. Considering Poncelet from the perspective of his "mathematical philosophy" on pure and analytic geometry, we will closely examine the ramifications in his problem solving and theorem proving practices in Chapters II and III. Recent historiography has established that Poncelet's form of generality revealed a tension between analytic and pure geometry. We intend to see how this tension manifested with respect to visual representations and other modes of communicating geometry.

Poncelet's original methodological statements, in which he put forward the distinction between the particularity of ancient pure geometry and the generality of modern pure geometry and analytic geometry, were directed toward Gergonne, the editor of the *Annales des mathématiques pures et appliquées*, the first journal not affiliated with any school or institution and devoted solely to mathematics in France (Verdier (2009b)). The early research and publications of Poncelet, Steiner and Plücker are inextricably bound up with the medium of the *Annales* and its editor, whose role in the development of projective geometry has been recognized since the mid-twentieth century (Struik (1970)). Most of Gergonne's own research was dedicated to analytic geometry and showing that coordinate equations could be used to arrive at solutions as elegant and simple as those from pure geometry. Later in life he became increasingly interested in what he described as positional or non-metric geometry, specifically through his principle of duality. As Karine Chemla has shown, Gergonne developed his research on spherical geometry to extend to duality between geometric figures in the plane and in space (Chemla (1989), Chemla and Pahaut (1988)). In later historical comparisons of Gergonne and Poncelet's respective principles of geometric duality, Mario Otero and Christian Gérini have independently described Gergonne's form of duality as more general because it relied upon a linguistic or philosophical relationship rather than mathematical techniques (Otero (1997), Gérini (2010a)). This generality translated into Gergonne's advocacy for analytic geometry, which avoided the particularity of the figure (Otero (1997), Gérini (2010a), Gérini (2010b), and Dahan Dalmedico (1986)). We will consider how Gergonne responded to contributors and contributions as a reflection of his own geometric research. Recent statistical analyses of the contents of the *Annales* has shown how the disproportionate emphasis on figure-based geometry in the *Annales* served as a fruitful environment for synthetic and analytic geometers alike (Otero (1997)). In particular, we will focus on the implications of Gergonne's editorial position on the research paths of Poncelet, Steiner, and Plücker and examine how he enabled new geometric practices to flourish and proliferate.

Our case studies of the research and publications of Poncelet, Steiner, Plücker and Gergonne will demonstrate numerous complexities challenging the traditional interpretation of two opposing methodologies that would later be unified as modern projective geometry in the 1870's. In focusing on the highly interconnected publications of the above four geometers, we leave aside the notable contributions of Michel Chasles and August Ferdinand Möbius. The story of Chasles' interpretation of Poncelet's geometry, beginning also in the late 1820's, has been addressed by Nabonnand and Chemla in their respective investigations of generality in pure geometry (Nabonnand (2011b), Chemla (1998)). Chasles argued against the use of coordinate equations in geometry, and his statements to this effect have served as evidence of the desire for methodological purity in Kline (1972) and Epple (2005).

Thus, Chasles' mathematical writings would serve as an interesting parallel to our analysis of Plücker and Steiner, but we will only elaborate particular moments of intersection between the German and French cases because Chasles' work appears to have had little influence on the early research of our primary actors. Chasles himself professed to not read German, and thus erected a barrier between his own studies and that of many of his contemporaries. On the other hand, we will utilize his historical texts as a counterpoint to the historical debate between analytic and synthetic (or pure) geometry.

Though classified as both an analytic and a synthetic geometer, Möbius' work (perhaps because written only in German) did not initially receive the publicity afforded to Poncelet, Steiner, or Plücker in the early nineteenth century (Loria (1887), Klein (1926a)). As an apparently methodologically neutral geometer, Möbius could also offer an interesting contrast, in particular because his lack of allegiance seemed to have no effect on the historical characterization of opposition. He is often cited as an exception, yet without any impact on modifying the general rule. We exclude him here because he was not widely read by French or German geometers until the early 1830's. At this time, Möbius began to gain recognition in terms that emphasized his outsider status. Responding to Möbius' professed independent research, Poncelet expressed suspicion that Möbius truly had no knowledge of either Steiner or Poncelet's research in 1827, due to his use of techniques remarkably similar to Poncelet's "polar reciprocity" (Poncelet (1866), 407). Yet, Steiner credited Möbius with clarifying the principle of duality in his *Der barycentrische Calcul* of 1827 (Steiner (1832), 5).[8] Later historians would take Möbius' geometry research as evidence that multiple geometers might independently arrive at the same discoveries at around the same time.

As our choice of primary actors suggests, our study geographically begins in France and extends gradually to German speaking regions with the advent of Crelle's *Journal*, although even then the prevalence of German geometers writing in French demonstrates a continued deference and desire to be read. Histories of early nineteenth century French mathematics often centre on Parisian mathematics, as the seat of both the École polytechnique and the *Académie des sciences* (Grattan-Guinness (1990), Grabiner (1981), Langins (1987), Dhombres and Dhombres (1989)). Certainly, both of these institutions inform our knowledge not only of Poncelet, who had studied at the École polytechnique, but also Gergonne and Plücker who considered their work as following in the Mongean analytic geometry tradition. Looming over early nineteenth century geometries, Gaspard Monge was invoked as a teacher

[8] In his introduction, Steiner elaborated Möbius' contribution, which must be interpreted broadly, since Möbius had apparently not known of the principle of duality until after publishing his monograph.

> Übrigens tritt die genannte Theorie durch die gegenwärtige Entwicklung in vollständigerer und allgemeinerer Gestalt hervor, als es in ihrer früheren Darstellungsweise geschehen konnte, wobei indessen nicht zu übersehen ist, dass der scharfsinnige Moebius zuerst eine freiere Auffassung dieser Theorie ans Licht gefördert hat (Barycentr. Calcül). (Steiner (1832), 5)

among both synthetic and analytic practitioners (Plücker (1828a), Dupin (1813), Steiner and Gergonne (1827)). Even Steiner, though not a devoted follower, responded specifically to Monge's results (Steiner (1826c)). In general, early nineteenth century geometers consistently attributed the reemergence of figurative geometry as a fruitful research area to Monge and the pedagogical tradition epitomized by the École polytechnique, and these claims have been borne out in rich historical studies (Taton (1951), Shinn (1980), Laurentin (2007), Sakarovitch (2005), Belhoste (2001)). Moreover, the archives of the École polytechnique contain Poncelet's archival documents, the contents of which more clearly illuminated Poncelet's use of the figure in research and publications. Drawing from these documents, the *Bulletin de la Sabix* devoted a special issue to Poncelet and his mathematical developments, which examined his particular relationship to the École polytechnique, the history of his experience as a prisoner of war, and his career as an engineer (Belhoste (1998), Billoux and Devilliers (1998), Gouzévitch and Gouzévitch (1998)).

But although both Plücker and Poncelet lived in Paris for intervals during the 1820's, the story of analytic and synthetic geometry is tangential to the Parisian social milieu during the first third of the century, as documented by Caroline Ehrhardt (Ehrhardt (2010)). The *Annales* appeared out of Montpellier, Poncelet primarily lived in Metz through the 1820's, and Plücker and Steiner remained in and around Berlin. Certainly, approval from the French *Académie des sciences* was deemed highly valuable to both French and German geometers, as will be seen with respect to Poncelet's publicity strategies and testy relationship to Augustin Louis Cauchy. However, a more accurate institutional focus for our study would be the provinces, and in this particular context one could even interpret geometry in Berlin as provincial with respect to Paris, which we will further explore in Chapter IV.

We will focus on the first third of the nineteenth century, and in particular the period between 1817 and 1832. The start date coincides with Poncelet's earliest publication advertising his new method in Gergonne's *Annales*, in which Poncelet introduced the concept of modern pure geometry. This interval offers a rich confluence of new journals on mathematics, including Crelle's *Journal*, the *Bulletin des sciences mathématiques, astronomiques, physiques et chimiques* of the Baron de Férussac, and the Belgian *Correspondance mathématique et physique* edited by Adolphe Quetelet and Jean Guillaume Garnier, leading up to the end of Gergonne's *Annales* in 1832. In this timespan we also find Poncelet's methodologically directed texts (Poncelet republished many of these works with further commentary in the 1860's) and Gergonne's responses, as well as the first publications of Plücker and Steiner, both in their earliest articles and books. These texts capture evolving geometric practices and methodological approaches. Our choice of dates also approximately follows the timeline presented in several historical accounts of nineteenth century analytic and synthetic geometry. In Ernst Kötter's *Die Entwickelung der synthetischen Geometrie von Monge bis auf von Staudt*, the beginning of projective geometry was traced back to Poncelet, and his second chronological section "Von Poncelet bis auf Steiner" begins in 1822 with Poncelet's *Traité des propriétés projectives* and ends in 1832 with Steiner's *Systematische Entwicklungen*. Historians have attributed events in this fifteen year period to the rise of projective geometry and the transition of geometrical research from France to Germany alongside the methodological opposition.

In 2010, Gray observed that

> We lack a recent history of projective geometry and of many, if not all, of its major protagonists. For too long now, readers have had to fall back on Coolidge's *A history of geometrical methods*, first published in 1940, supplemented by a very small number of more specialised studies. (Gray (2010b), vi)

This dissertation aspires to the latter description, thus adding to the very small number. Much like in the book edited by the collective author Lise Bioesmat, *Eléments d'une biographie de l'espace projectif*, we intend to show dimensions of the emerging practices that a history of projective geometry might consider. Moreover, as projective geometry did not yet exist as a well defined research area (Clebsch (1872)), we will demonstrate some of the ways in which geometers understood their contributions. These early nineteenth century comments and practices both confirm and contradict the simple dichotomies of analytic and synthetic, figurative and computational, abstract and concrete, French and German, urban and provincial, ancient and modern, or classic and romantic. Although evidence for each of these divisions can be found in the texts analyzed below, we will find this is not the whole story.

In Chapter I, we will begin by unravelling the narratives of synthetic versus analytic geometry as portrayed in historical accounts. Our historical survey covers the late nineteenth and early twentieth century. We will methodically build a corpus of thirteen authors, all of whom posed some distinction between synthetic (or pure) and analytic geometry and geometers in their analyses of the early nineteenth century. Collectively, these accounts emphasize a common set of features of geometry in this period, leading to an image that will be confronted and revised in subsequent chapters.

Although our thirteen authors represent a range of French, German, American, and Italian nationalities, the prevalent chronological narrative traced the origins of the synthetic-analytic debate to French mathematicians (usually Lazare Carnot or Poncelet), which transferred to Germany along with specific research in geometry. Following this geographic distribution, in Chapter II we will focus on methodological arguments in French publications, specifically in the writings of Poncelet, while in Chapter IV we will trace how Plücker and Steiner directed their research presentations to a French audience, and how these choices equally influenced the content of their German publications.

Synthetic geometry and geometers were portrayed as reactionary toward the use of coordinate equations in figure based geometry. This historical interpretation has been supported by direct quotes for methodological purity that emerge one-sidedly against analytic geometry from Carnot, Poncelet, Steiner, and Chasles. By examining some of these arguments with respect to the surrounding mathematical setting, we will attempt to determine what specific qualities within geometry were perceived as advantageous or undesirable. Particularly in Chapter III and IV, we look to methodological arguments both for and against the use of coordinate equations, thus bringing to light a more balanced picture of antagonism and defence.

Despite the alleged methodological opposition, historical surveys revealed how past geometers confronted practically identical problems and theorems. This is particularly striking in texts such as Max Simon's *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, where specific popular problems were documented in pages of citations spanning the whole nineteenth century. Juxtaposed to distinct and even antagonistic methods, the choice of content persisted. In Klein's *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, this constant recycling was perceived as stagnation. We will utilize this feature in our case studies in Chapters II through IV, examining instances where a set of problems or theorems attracted diverse geometers, who often used their solutions or proofs as evidence fro the necessity or superiority of their particular approach. In Chapter V, we will investigate the prevalence and significance of these reoccurring problems and theorems by considering the content of books on geometry, a medium largely neglected in later histories.

Perhaps in consequence of this repetition, the history of early nineteenth century geometry has portrayed a competitive atmosphere to the point of excessive drama. In biographies of Plücker and Steiner, the opposition between synthetic and analytic methods was exacerbated by conflicting attributions of credit and limited professional opportunities. This competition has been qualified as both productive and paralyzing, and we will examine how the medium of controversies shaped the community of geometers in Chapter III.

A final theme especially prominent in early twentieth century texts, is one of modern geometry and progress toward unification. From Klein's *Erlangen Programme* in 1872, practicing geometers (including those who wrote the histories reviewed here) called for an end to the no longer relevant methodological distinctions, instead considering the subject of synthetic and analytic geometry more generally as projective geometry (Klein (1872)). Thus a divided early nineteenth century served as a contrast and backdrop to contemporary progress.

Yet, progress and modernity also implied a break or move away from older geometric practices and their associated qualities. While recognizing the power and acclaim of the new projective geometry, some lamented the disappearance of synthetic geometry as an active research subject (Coolidge (1940), Darboux (1904)). The new projective geometry absorbed and dismissed certain qualities formerly attributed to analytic or synthetic methods. In Chapters II and IV, we will observe how the ideas concerning progress and modernity took shape in the synthetic and analytic geometries of Poncelet, Gergonne, Plücker, and Steiner. Then in Chapter V, we will observe how the concept of “modern” could describe very different styles and qualities.

These themes of the chronological and geographical path of geometry, the purported opposition of synthesis and analysis, the common geometric content, the competitive and controversial atmosphere, and the concepts of mathematical modernity and progress will direct the inquiries of our subsequent chapters, and provide points of comparison in examining the mathematical publications of so-called analysts and synthesists.

In 1817 Poncelet proposed that unlike analytic geometry, in pure geometry the figure is never lost from view.[9] Whether illustrated, described or constructed, Poncelet presented the figure as the primary form of geometrical evidence, a means of justification based in sensory perception. These qualities persisted as Poncelet extended results from particular figures to any possible deformation. By contrast, though classified as analytic geometry, Plücker’s contemporary research treated coordinate equations as visual geometric objects–evidence–by focusing on their form and endeavouring to avoid calculations. Plücker specifically formulated his work to counter Poncelet’s claims for the advantages of pure geometry by providing an alternative means of visualization. Moreover, the centrality of the figure was not always manifest in illustrated figures, but sometimes only in written descriptions. Following Dominique Tournès’ concept of a “virtual diagram” as a diagram “that one must have in mind, but that is no longer physically drawn on the paper, or at least which is left to the reader to draw,” we will employ the term “virtual figure” to stand for a description of a figure that is not actually provided with the text (Tournès (2012), 272). Especially in journal articles, we will find these virtual figures widely employed, and the fact that not every geometric object needed to be figured permitted including geometric objects that might not be illustratable. Thus we will consider how the virtual figure was both more and less than its actual counterpart.

Historically informed philosophical inquiries have recently focused on visualization in mathematics through the use of diagrams, such as in Mancosu, Jørgensen and Pedersen (2005) and Giaquinto (1992). In historical summaries, early nineteenth century geometry follows the decline of the figure in mathematics, often traced to the late eighteenth century and Joseph-Louis Lagrange, who boasted of his mechanics that it rested entirely on algebraic considerations. In the words of Jeanne Peiffer on “Rôles des figures dans la production et la transmission des mathématiques,”

> Chez Lagrange, le banissement des figures traduit un nouvel équilibre entre deux

[9]Poncelet may have been paraphrasing Carnot, who wrote, that “la synthèse [...] ne peut jamais perdre de vue son objet” as compared to analysis, in his *Géométrie de position* (1803) Carnot (1803).

> branches des mathématiques, la prédominance de l'analyse algébrique sur la géométrie. C'est dire que la question des figures se trouve au coeur de certaines représentations que l'on s'est faites des mathématiques dans l'histoire. (Peiffer (2006))

By the late nineteenth century, the figure no longer remained the focus of geometry, a development often linked to non-Euclidean and axiomatic geometries, most famously David Hilbert's *Grundlagen der Geometrie* in 1899. Drawing from Reviel Netz' work on diagrams and texts in ancient Greek mathematics as well as Michael Baxandall's findings on the textual explanation of pictures, we aim to elucidate the exchange of cognitive properties when an illustrated figure is replaced by words or equations (Netz (1999), Netz (2005), Baxandall (1985)). Our investigation nuances these broader themes of increased geometric abstraction and extension beyond the capabilities of two or three dimensional visualizations, by pointing to a persistent ebb and flow of the importance of the figure and its capabilities in conveying geometric results.

Following Poncelet's division between pure and analytic geometries, in Chapter II we focus on five versions by three different geometers, of a single conic section construction written between 1817 and 1826. Despite the similarity of their results, each geometer addressed the problem from contrasting methodological perspectives. We examine how the figure-based distinction materialized in contemporary geometric practices, and what constituted geometric evidence when the figure was lost from view.

The limited domain of pedagogically oriented problems and applications bred controversy alongside innovation. In Chapter III, we compare several interrelated controversies in early nineteenth century geometry concerning methodology, the value of generality, accusations of plagiarism, and contested priority. In particular, we will focus on the rhetoric surrounding the duality controversy and argue that this and other early nineteenth century controversies in geometry functioned as successful mediums of publicizing new principles and nuancing results. We will present mathematical duality insofar as it pertains to grasping the stakes of the associated controversy, thus complementing the more technical expositions of duality, which can found in a wide assortment of historical texts (Klein (1926a), Pedoe (1975), Chemla and Pahaut (1988), Otero (1997), and Gray (2010b)).

The central investigation in Chapter III will return to many of the texts introduced in Chapter II from a new perspective. Our initial analysis of the mathematical results will be reexamined in light of the surrounding controversy. We will see that the expanding polemic over the principle of duality shaped the choice and emphasis of geometrical content produced during that time period. From this broader outlook, we will gain a vantage on the wider textual exchange, which will provide additional contexts in which to understand Poncelet's, Gergonne's and Plücker's interdependent approaches to methodology and open areas of research.

Since the 1980's, historians and sociologists of science have increasingly advocated studying controversies and outlined possible categories and approaches as tools of research. Though controversies in mathematics have been studied by historians, they are generally not viewed as indispensable to mathematical theory formulation nor even as necessarily leading to epistemic gains. Further, the debate over what does and does not constitute a scientific controversy remains open among many historians and sociologists of science. We will adhere to H. Tristram Engelhardt Jr and Arthur L. Caplan's definition of a scientific controversy as "the existence of 'a' community of disputants who share common rules of evidence and reasoning with evidence" (Engelhardt Jr. and Caplan (1987), 12). Following this criterion, we will argue that the exchange concerning duality initiated by Poncelet and Gergonne constituted a true scientific controversy, in which mathematical practices (including deriving general principles, applying methods of proof, and composing original material) were questioned and supported through evidenced based reasoning (in the form of generating new mathematical problems, demonstrating scientific applications, and tracing historical chronologies). As a controversy, the textual exchange over duality was public, and the necessary collective endeavour of beginning and ending this controversy will serve to illustrate one venue in the circulation of geometry. In the midst of simultaneous less dramatic controversies over the choice of method in geometry, Poncelet's principle of continuity, and the originality of Steiner and Plücker, we display the relative normalcy of controversies in this area of geometry that also will serve to establish collective boundaries of the nebulous community of geometry researchers and an evolution of what was considered geometric practice. In turn, geometers demonstrated the strength and malleability of their research subject during the early nineteenth century by accepting the publicity of controversial exchanges.

In Chapter IV we we shift our attention to the early research of Steiner and Plücker to determine the ways in which their work and personalities became associated. As is well documented in the historical literature, Steiner and Plücker arrived at many of the same results by independent research paths. Alongside a detailed portrait of their mathematical processes through comparative case studies, we document the development of their personal methodologies and how they were received. Engaging with a primarily French audience, both became associated as "German mathematicians" and "modern" geometers. Drawing from reviews, private correspondence, and letters of recommendation, we witness how Steiner and Plücker were at once categorized into different methodological camps, but also refashioned the boundaries, qualities, and perception of synthetic and analytic geometry. The frequent repetition that enables our direct comparison of Steiner and Plücker also shaped the practice and presentation of geometry. Those who reiterated an extant construction, thus emphasized instead their contributions to distinct, new, and superior methods.

Both the methodological separatism, and the re-appropriated content caused competition between these close contemporaries.

To complement our emphasis on research publications, most often in article form, we conclude our study of early nineteenth century geometry by incorporating books and the broader overlapping audience of teaching and learning geometry. As Jean Dhombres has determined, most mathematics books published in France in the early nineteenth century were intended primarily as textbooks (Dhombres (1985)). This conclusion is confirmed in our survey of books on geometry published between 1800 and 1832. Using the Bibliothèque nationale de France holdings catalog, we searched for books during this time period classified or titled as geometry and intended to be read by students, teachers, and researchers in mathematics. In total, we included fifty-two titles, many of which ran to multiple editions. For each book we read the table of contents, introductions, available figures, and portions of text that pertained to any of the case studies or geometric objects discussed in the above four chapters. Almost all of these books were pedagogically oriented with introductions focused on best teaching practices, curation of content to maximize student interest, justification for omitting or maintaining high levels of rigour, and claims for the practical and even national importance of learning geometry. Relatively few texts (including those by our principal actors) included new research that did not fit neatly in these institutional boundaries. In general, these books offer no concept of modern pure geometry, much less an overarching projective geometry that would include both analytic and synthetic methods. Instead, modern geometry in books often denoted coordinate based geometry. Perspective and descriptive geometry books, though documenting a much newer discipline, were aimed at engineering applications and almost never described as modern. We thus encounter a further standard of division aligned between not only methods but also content and above all prerequisites. For the most part, the difference between elementary and analytic geometry was simply a matter of whether the student had successfully completed a course of study in analytic algebra or not.

As our first chapter historicized the opposition between "modern" synthetic and analytic geometry, the final chapter will emphasize how these new kinds of geometries fit into a broader group of contemporary practices. Our investigation of books in Chapter V will add weight to Poncelet's use of the term modern, to Gergonne's claim of a mathematical revolution, and to Plücker's emphatically "new principles" and "new coordinate systems." We will find that the conservative nature of these books and their pedagogical market reinforced repeating familiar problems and theorems, thus rationalizing why authors such as Poncelet, Plücker and Steiner went to such lengths to present new research and methods as intuitive, visually accessible, constructively malleable, and often based in figures. The repetition in teaching content will thus further illuminate potential motivation behind differentiating geometrical methods.

Our research into the features of a methodological division in early nineteenth century geometry consisted in a systematic reading of all geometric publications by Gergonne, Poncelet, Plücker and Steiner that appeared between 1817 and 1832. Within these texts we searched for descriptions, arguments, examples, and commentaries with respect to geometries and methods. We argue that these relatively rare instances are best understood in light of the particular geometric research involved. Thus we will pursue a deep analysis of specific practices and results that the actors and their contemporaries claimed as evidence in support of reading and writing about their individual forms of geometry. With these problems, theorems, and surrounding rhetoric we aim to present the methodology in action. Through Chapters II, III, and IV we will show how the figure was used, new geometric objects were introduced, and evolving geometries were communicated. These findings will be framed by the broader investigations of Chapters I and V, which motivate, situate, and add contrast to our specific case studies.

Chapter 1

Unravelling the methodological opposition: sources from histories of geometry (1872–1933)

1.1 Building a corpus

What are the origins of the perceived divide between analytic and synthetic methods in early nineteenth century geometry? Several English language general histories of mathematics reveal similar descriptions of a divided subject in the works of Dirk Struik, Carl Boyer, Uta Merzbach, Morris Kline, and Ivor Grattan-Guinness (Struik (1948), Boyer (1956), Boyer and Merzbach (1968), Kline (1972), Grattan-Guinness (1997)). Consulting bibliographies from these texts, we found a common set of historical and primary sources for this time period. The primary literature will be featured in subsequent chapters. Here we aim to understand if and how histories of geometry and historical narratives on nineteenth century geometry have contributed to a construction of the divide between synthetic and analytic methods. In particular, we will investigate historical texts to determine which were the components of the debate, where and how it emerged, and who were the main actors.

As illustrated in Appendix A, these initial bibliographic entries resulted in a finite list of historical sources on early nineteenth century geometry. Working backward, when a potential source contained a discussion of geometry in the early nineteenth century, we included it in our corpus and then consulted its bibliography. We limited our search to documents on the history of geometry or the history of particular methods or episodes (excluding the parallel postulate and resultant non-Euclidean geometries) in the first third of the nineteenth century geometry. Thus, we only consulted general histories of mathematics when they were cited as specific references to this research area and time period. With respect to publication dates, we restricted our corpus to texts published after the 1860's in order to emphasize the historic dimension since most early nineteenth century geometers were no longer publishing by this time. However, theories and results from early nineteenth century geometers, in particular those categorized as the new projective geometry, directly inspired new directions of research. As we will see, our corpus is situated in an interval when early nineteenth century geometry was considered a valuable resource by the next generation of working mathematicians who continued to publish until the early twentieth century.

Our corpus contains books and articles by thirteen authors published between 1872 and 1933. Although we initiated our corpus construction from contemporary English language histories, the bibliographies reflect a primary focus on Germany: eight of our authors are German, three are French, two are Italian (one writing in German), and two are American. These texts can be divided into three general sections, characterized by publication contexts: biographies of early nineteenth century mathematicians usually composed shortly after their deaths by their colleagues and successors; turn of the century texts on specific aspects of the history of geometry mostly by German and Italian mathematicians centred around the publication of Felix Klein's *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*; and general histories of mathematics relying more heavily on secondary sources. In each section, we will proceed chronologically following the author's first publication in the event of multiple texts or different language editions.

We bookend our corpus by considering a pair of texts on the history of methods in geometry, Michel Chasles' *Aperçu historique sur l'origine et le développement des méthodes en géométrie particulièrement de celles qui se rapportent à la géométrie moderne* (1837) and Julian Lowell Coolidge's *A History of Geometrical Methods* (1940) (Chasles (1837), Coolidge (1940)). The *Aperçu* served as a common historical source for all subsequent works in our corpus, although Chasles did not classify his contemporaries as analytic or synthetic geometers, and reserved the terms solely for historical texts and mathematicians. Likewise, Chasles did not consider his research as "projective geometry" although, as we will see, it would be later classified as such. In 1940, Coolidge presented his text as a contemporary equivalent to the *Aperçu*. However, Coolidge looked back to early nineteenth century projective geometry as an area of research that had already reached its full potential and was no longer active. Coolidge claimed that the text, *Projective Geometry*, written by Oswald Veblen and John Wesley Young in 1910, had been the "last great work dealing with this field" (Coolidge (1934), 227).[1]

Within our corpus, we will examine how historians divided and compared methods in geometry, most often as analytic and synthetic. In showing a division inside geometry,

[1]A much stronger assessment came from Bourbaki in 1960, who claimed that all of classical geometry–excluding differential and algebraic geometry–had been dead as a research field to professional mathematicians since the late nineteenth century (Bourbaki (1960), 140).

historians comparatively described each method.[2] These descriptions included qualities of both sides of the division, names of geometers who were identified as practicing or advocating one or the other method, the content of the mathematics addressed by both methods, as well as characteristics of the opposition itself. We will see how historians adopted and adapted the model of analysis versus synthesis over time, and used these descriptions to draw conclusions about how the geometry of the past related to the geometry of the present. We note that descriptions of historical methodology were often very brief, and our treatment of the following texts will reflect this brevity.

1.2 Michel Chasles (1837)

Chasles (1793–1880) wrote his *Aperçu Historique sur l'origine et le développement des méthodes en géométrie particulièrement de celles qui se rapportent à la géométrie moderne* as a response to a question posed by the *Académie des sciences* of Brussels in 1829 on the different methods of modern geometry. With the help of Adolphe Quetelet, Chasles expanded the work and it was printed by the *Académie* of Brussels in 1837. The work was well-received, Chasles became a professor at the École polytechnique in 1841 and the faculty of science at the University of Paris created a chair in "Géométrie supérieure" for him in 1846. By the time he issued a second edition, with almost no changes, in 1875, Chasles' name was followed by a paragraph of international scientific academy affiliations. The third and fourht editions, which appeared posthumously in 1889, were both published by Gauthier Villars.[3]

In his introduction, Chasles limited the scope of his historical research to "pure Geometry" and broadly referred to recent discoveries and doctrines by which geometry had achieved a generality and scope, heretofore limited to Analysis. He divided the history of geometry into five chronological *époques* of unequal length, beginning with Thales and ending with his own recent contributions. The geometric advances over the last thirty years, initiated by the descriptive geometry of Gaspard Monge, comprised Chasles' fifth *époque*. He claimed these new developments often equalled or rivalled those in analysis.

Chasles contrasted "two methods which divide the domain of mathematical sciences," analysis and geometry. Chasles assured the reader that he admired the analytic method and was not arguing that the geometric method could extend as far in all aspects. However, rather than apply the analytic method to geometry, Chasles described the advantages of

[2]We will refer to these authors as historians as their texts contain historical accounts. However, as we will see, most of the authors were mathematicians by profession, some with no other historical output than the texts considered here.

[3]When Chasles died in 1880 the New York Times reported he was "the most distinguished mathematician in France probably" (Anonymous (1880)). Bertrand wrote a more extensive biographical eulogy in 1892 for the *Académie des Sciences* (Bertrand (1892)). Recently, Nabonnand and Chemla have respectively analyzed particular aspects of Chasles' geometric contributions and interpretation of generality, (Chemla (1998), Nabonnand (2011b)).

pure Geometry.

> Mais, convaincu qu'on ne saurait avoir trop de moyens d'investigation dans la recherche des vérités mathématiques, qui toutes peuvent devenir également faciles et intuitives quand on a trouvé et suivi la voie étroite qui leur est propre et naturelle, nous avons pensé qu'il ne pouvait qu'être utile de montrer, autant que nos faibles moyens nous le permettaient, que les doctrines de la pure Géométrie offrent souvent, et dans une foule de questions, cette voie simple et naturelle qui pénétrant jusqu'à l'origine des vérités, met à nu la chaine mystérieuse qui les unit entre elles, et les fait connaître individuellement, de la manière la plus lumineuse et la plus complète. (Chasles (1837), 2–3)[4]

Chasles then classified geometry into three parts: ancient pure geometry, analytic geometry, and recent pure geometry. In general, he described pure geometry as more natural and simple precisely because it did not rely upon outside means of investigation. However, Chasles admired the "abstraction and universality" of analytic geometry that distinguished the work of Descartes from the so-called "Géométrie ancienne" (ibid, 94). While "la Géométrie des Anciens" could be "particular, incomplete, and unconnected," the third and newest kind of geometry, *Géométrie récente*, matched analytic geometry in its generality and fruitfulness but remained pure because independent of algebraic calculations.

> La troisième enfin est cette Géométrie pure, qui se distingue essentiellement par son abstraction et sa généralité; dont Pascal et Desargues ont donné les premiers exemples dans leurs traités des coniques, et dont nous verrons que Monge et Carnot, au commencement de ce siècle, ont assis les fondements sur des principes larges et féconds. (ibid, 117)[5]

By focusing on the differences between recent and ancient geometries, Chasles went beyond the separation between analytic and pure geometry.[6] Moreover, in his description of recent geometry, he listed four new "methods": the theory of transversals, the doctrine of the transformation of figures, the theory of polar reciprocity, and the doctrine of stereographic

[4]"However, convinced that one cannot have too many means of investigation in the search for mathematical truths, that all can be equally easy and intuitive when one has found and followed the narrow path which is proper and natural to them, we thought that it could be useful to show, as far as our feeble means permit us, that the doctrines of pure Geometry offer often, and in a great many questions, this simple and natural path that, penetrating to the origin of truths, strips bare the mysterious chain which links them, and makes them known individually, in the most luminous and complete manner."

[5]"The third finally is this pure Geometry, which is essentially distinguished by its abstraction and its generality; of which Pascal and Desargues have given the first examples in their treatises on conics, and of which we see that Monge and Carnot, at the beginning of the century, have seated the foundations on broad and fruitful principles."

[6]In 1817, Poncelet had introduced a very similar differentiation between modern and ancient pure geometries, which we will investigate in Chapter II (Poncelet (1817c)).

projections. He further listed nine "sub-methods," which were even more specific techniques such as the use of perspective and homological figures. In this level of analysis, the term "method" did not divide into analysis and geometry, but rather a particular set of practices and results that could be applied toward studying different aspects of geometry. In declaring the theory of transversals and the theory of polar reciprocity to be different methods, Chasles implied that a particular analytic or geometric method was a tool for addressing a specific set of problems. Different methods had different applications, so for Chasles, two methods were not just two ways of saying the same thing. Following his description of methods, Chasles advocated for including more geometry in the French mathematics curriculum, which he viewed as too heavily skewed toward analysis. In particular, he argued that pure geometry would be useful for students interested in applications and who could thus avoid unnecessary calculations. However, Chasles did not want geometry to replace analysis. To the contrary, he concluded that since all of mathematics was interconnected, neglecting geometry would impede scientific progress.

Chasles discussed "synthesis" by name only rarely and in historical contexts, not to describe the work of his contemporaries. His first reference was to Frans van Schooten's *Tractatus de concinnandis demonstrationibus geometricis ex calculo algebraico*. In this work from the mid-seventeenth century, Schooten showed that "the synthetic method can always be deduced from the analytic method" (99). In this context, the synthetic method stood for the methods of the "Ancients" and the analytic method was the use of coordinates applied to curves in space, which Chasles also denoted as the Geometry of Descartes. In a later footnote, Chasles classified Philippe de La Hire's proofs from later in the seventeenth century as purely synthetic and very simple, but capable of improvement through contemporary nineteenth century notation. Both of these references seem to imply a synonymy between synthetic methods and so-called ancient geometry. This interpretation is further confirmed in Chasles' quote from Joseph-Louis Lagrange, who had expressed similar sentiments on the nature of synthesis.

> "Quelques avantages que l'Analyse algébrique ait sur les méthodes géométriques des Anciens, qu'on appelle vulgairement, quoique fort improprement, *synthèse*, il est néanmoins des problèmes où celle-ci paraît préférable, tant par la clarté lumineuse qui l'accompagne que par l'élégance et la facilité des solutions qu'elle donne. Il en est même pour lesquels l'Analyse algébrique paraît en quelque sorte insuffisante, et où il semble que la méthode synthétique soit seule capable d'atteindre." (*Sur l'attraction des sphéroïdes elliptiques.* Nouveaux Mémoires de l'Académie de Berlin, année 1773.) (quoted in Chasles (1837), 252)[7]

[7]"'Whatever advantages algebraic Analysis has over the geometric methods of the Ancients, that one commonly calls, although very improperly, *synthesis*, there are nevertheless problems where the latter appears preferable, either because of the luminous clarity which accompanies it or of the elegance and ease of

By agreeing with this assessment, Chasles signalled that synthetic geometry possessed the elegance and simplicity characteristic of the geometric methods of the ancients.Yet, Lagrange's comment that the description "synthesis" was an improper name for ancient geometry reflected a common trope in the mathematical philosophy of Chasles's contemporaries, including Joseph-Diez Gergonne and Emmanuel Develey (Gergonne (1817c), Develey (1831)).[8] These geometers recognized that "synthetic" carried too many meanings to be used clearly. Chasles himself rarely used the term and did not describe his own geometry as synthetic, although other historians would. For Chasles, synthetic geometry contrasted both with analytic geometry and with modern pure geometry, the latter of which incorporated his own research.

Although Chasles did not present a bifurcation between analytic and synthetic methods or geometries, nearly every subsequent history of synthetic and analytic geometries referred back to the *Aperçu* as a foundational text. By cataloging how Chasles framed methodological differentiation, we provide a counterpoint to show how later historians drew from the same historical events while presenting new and even contradicting conclusions.

1.3 Biographies

Chasles lived until 1880, but many of his contemporaries in France and Germany passed away during the 1860s. The earliest references in our corpus relate to their subsequent biographies.

Although biographies of early nineteenth century French geometers exist and served as valuable resources in our broader research, the bibliographies consulted in our corpus construction only included those of Plücker and Steiner. The overwhelming emphasis on Plücker may be attributed to Boyer, Klein, and Plücker's first biographer, Alfred Clebsch (1833–1872), whose biography of Plücker was considered by the Italian geometer, Eugenio Beltrami, to be equally a history of all early nineteenth century geometry (Loria (1887)).

1.3.1 Alfred Clebsch (1872)

Plücker died in 1868, and his first biography was written and read by Clebsch before the Königlichen Gesellschaft der Wissenschaften in Göttingen in 1871. The text was published the following year, when a French translation by Paul Mansion also appeared in the Italian journal *Bollettino di bibliografia e storia delle scienze matematiche* (Clebsch (1872)). In Clebsch's own geometrical research he had further developed many of the results first

the solutions that it gives. There are even those for which algebraic Analysis appears somewhat insufficient, in which, it seems, only the synthetic method is capable of succeeding."'

[8] Dahan Dalmedico has described key features of the original 1813 text, which Gergonne's later publication on geometrical methods was based on in Dahan Dalmedico (1986).

attributed to Plücker. Clebsch began his biography by discussing the year 1826, in which Plücker published his first articles and Crelle's *Journal* was founded. In order to show how Plücker's work related to that of his predecessors, contemporaries, and successors, Clebsch proposed that his biography would at the same time be a history of the last 50 years of geometry, beginning with Gaspard Monge who marked the separation between new and ancient geometries. Thus, Clebsch summarized the personal events of Plücker's life in a paragraph, and dedicated the remainder of the text toward examining his scientific contributions.

Clebsch used the term "projective geometry" to describe the collective work of Poncelet, Gergonne, Möbius, Steiner and Plücker among other early nineteenth century geometers. He may have been the first to employ this designation, which he claimed better signified that synthetic and analytic geometry referred only to the form of presentation, and not to two disparate disciplines, as had once been thought.[9] He introduced the term in describing the non-metric geometry of Carnot.

> Bei ihm [Carnot] tritt das Bestreben, Lagenverhältnisse allein zu betrachten und alles Metrische auszuscheiden, noch nicht so rein hervor, wie später bei Poncelet und Andern, ein Bestreben, welches endlich zur Auflösung des Metrischen in projectivische Begriffe führen sollte; doch erkennt man leicht den halb unbewussten Zug, welcher demjenigen entgegentreibt, was wir heute unter projectivischer Geometrie verstehen; ein Name, der besser als die nur auf die Form der Darstellung bezüglichen Namen der synthetischen und der analytischen Geometrie das Wesen der Sache und den Gesichtspunkt bezeichnet, unter welchem tatsächlich diese beiden früher gesonderten Disciplinen sich vereinigt haben. (Clebsch (1872), 10)[10]

Continuing chronologically, Clebsch noted the profound impact of Monge on geometric research, but claimed that Poncelet had created projective geometry, as it was understood in its present form. From the work of Poncelet and Joseph-Diez Gergonne, Plücker began his own research in projective geometry.

During the first third of the nineteenth century, Clebsch included Carnot, Monge, Poncelet, Gergonne, Chasles, Gabriel Lamé, Étienne Bobillier, August Ferdinand Möbius, Ludwig Magnus, and Jakob Steiner as practicing projective geometry in connection to Plücker.

[9] Jeremy Gray traced the first use of the expression "projective geometry" back to Klein's Erlangen speech "Vergleichende Betrachtungen über neuere geometrische Forschungen," first published in 1873 (Klein (1872)). Clebsch's obituary marginally precedes this date (Gray (2010b)).

[10] "With him [Carnot] the effort to consider positional relationships alone and eliminate everything metric occurs, not yet so purely as later with Poncelet and others, an effort which should ultimately lead to the dissolution of the metric into projective concepts; indeed one easily recognizes the half-unconscious move, driving against what we today understand as projective geometry; a name that better designates the essence of things and point of view under which these two previously separate disciplines have effectively been unified, than the names synthetic and analytic geometry which only refer to the form of representation."

Clebsch did not divide these geometers into synthesists and analysts, but nevertheless observed particular tensions among geometers, often due to priority disputes or lack of proper citation. From the outset, Plücker had become involved in a dispute between Gergonne and Poncelet stemming from his first publication in Gergonne's *Annales* early in his career.[11] Then, on several occasions, both Steiner and Carl Gustav Jacob Jacobi had neglected to cite Plücker when employing his results. Both Plücker and Steiner had also been involved in separate discussions of priority with Magnus. Even Plücker, Clebsch noted, on occasion believed he had made discoveries, which in fact had been previously published by someone else (ibid, 8). From Clebsch's point of view, attention to priority was too prevalent among early nineteenth century geometers and propagated by histories of science. Ultimately, he concluded that these conflicts were "always sterile" since in natural scientific progress there were often coincidental simultaneous discoveries.

So while Clebsch pointed out disputes among projective geometers, they were not methodologically driven and after introducing the term "projective geometry" Clebsch did not segregate geometers or their publications as analytic or synthetic. He acknowledged that all of Plücker's geometrical research had been presented in an "analytic form," but found that Plücker's work in analytic geometry was neither abstract nor computational. Instead, Clebsch described Plücker's "Art der Forschung" or "Forschungsmethode" or "Denkweise" as drawn particularly from "die Freude an der Gestalt," driven by an appreciation for the aesthetic and intuition (ibid, 6). In their research and productivity, Clebsch compared Plücker to Steiner, describing the former as "richer." He did not suggest that either geometer carried any personal or professional animosity toward the other. However, in his conclusion Clebsch asserted that Plücker had not received the acknowledgment he deserved during his lifetime due to the partisanship [*Parteinahme*] of the epoch (ibid, 32).

Clebsch presented an informative image of a progressive early nineteenth century geometry including who practiced "projective geometry", the subject's chronological and geographical progression, and how numerous discoveries occurred among multiple geometers at once, often prompting public disputes. He minimized any methodological division within geometry, claiming that the names synthetic and analytic no longer designated separate disciplines. On the one hand, Clebsch admitted that what he called "projective geometry" was once divided in two. On the other hand, only in describing the "form" or "version" of Plücker's geometry did he use the adjective "analytic." Overall, Clebsch's description of Plücker's work is highly individualistic, and not necessarily indicative of analytic geometry in general.

[11] The events behind Plücker's charge of plagiarism and the surrounding controversy will be discussed in Chapter III. We will find numerous mentions of this episode in the following historical texts.

1.3.2 Adolf Dronke (1872)

The same year as Clebsch's biography appeared in print, Adolf Dronke (1837–1898) published his account of Plücker's life and work. Dronke had been an assistant to Plücker in his physical researches at the University of Bonn. In 1872, Dronke worked as the director of a gymnasium in Koblenz. His text, *Julius Plücker Professor der Mathematik und Physik an der Rhein*, was a biographical narrative of Plücker's life, in which his scientific contributions appeared chronologically (Dronke (1871)).

Unlike Clebsch, Dronke described Plücker's method as analytic geometry. Moreover, Plücker's analytic geometry had consistently been opposed by Steiner, a synthetic geometer. Dronke explained how Steiner had undertaken many of the same investigations through a synthetic method, which overshadowed Plücker's originality during his early career. This opposition continued, Dronke observed, as both Steiner and Jacobi later neglected to properly acknowledge Plücker's prior contributions. On top of this, Dronke lamented how Plücker's mathematical theories were attacked as infertile in Germany, and only received proper recognition abroad.

> Grossen Angriff erfuhren die Theorieen Plücker's in Deutschland und war der Hauptvorwurf, den man ihnen machte, der, dass die Theorien unfruchtbar seien namentlich gegenüber der damals vorzüglich von Steiner und Poncelet vertretenen synthetischen Methode. Ich glaube kaum, dass dieser Vorwurf einer Entgegnung bedarf für den, der sehen will. Die grossen Resultate, die Plücker selbst und die von seinen Anschauungen ausgehenden englischen und französischen Mathematiker erzielten, beweisen hinlänglich die Hinfälligkeit jener Behauptungen. Leider aber brachte der Umstand, dass Plücker und Steiner, wenn auch auf verschiedene Methode gestützt, dasselbe wissenschaftliche Feld bebauten und dieselben Probleme zu lösen suchten, grosse Misshelligkeiten. (Dronke (1871), 10)[12]

Dronke portrayed these disagreements as one sided: Plücker was a victim of unjust German criticisms and even attacks. This opposition was most strongly exemplified by Steiner's threat to Crelle regarding publishing Plücker's analytic work, a story that may have originated with Dronke, as we discussed in our Introduction. Jacobi also featured as an antagonist, although Dronke acknowledged that "the great mathematician" had no need to

[12]"Plücker's theories encountered great opposition in Germany and the main criticism made against them was that the theories were infertile, particularly at the time with respect to the superior synthetic method represented by Steiner and Poncelet. I hardly think that this allegation requires a response for those who accept to see for themselves. The major results achieved by Plücker and derived from his intuition by the English and French mathematicians, sufficiently prove the invalidity of those claims. Unfortunately, however, the circumstances that brought Plücker and Steiner to build the same scientific field and to search for solutions to the same problems, albeit based on different methods, also brought great disagreements."

claim the results of others. The synthesis versus analysis rivalry echoed Dronke's account of Plücker's first publication, which had been rewritten by Gergonne and led to an accusation of plagiarism from Poncelet. In Dronke's account, early nineteenth century geometry was rife with conflict that often negatively affected Plücker.

Dronke tangentially mentioned the geometric contributions of Monge, Carnot, Bobillier and Möbius as they intersected with Plücker's work. With respect to methodological divisions he only designated Plücker and Gergonne as analytic geometers and Steiner and Poncelet as synthetic geometers. Dronke argued that this methodological division underlay both the many disagreements that occurred within early nineteenth century geometry as well as the general under appreciation of Plücker in Germany. He defined analytic geometry as tracing back to Descartes and involving the use of coordinate points, but gave no definition for synthetic geometry. Dronke suggested that the two methods could be applied to the same research field, and only differentiated them in terms of their practitioners.

1.3.3 Wilhelm Ernst (1933)

The most recent biography of Plücker was written over sixty years later, in 1933, as an inaugural dissertation by Wilhelm Ernst (1889–??) for his post at the University of Bonn. The subject of his paper, *Julius Plücker eine zusammenfassende Darstellung seines Lebens und Wirkens als Mathematiker und Physiker auf Grund unveröffentlichter Briefe und Urkunden*, had been suggested by Dr. Konen, a physicist at Bonn who worked in spectroscopy (Ernst (1933)). Due to the relatively late date of this source, we begin to see new references and narratives, which will reappear in our subsequent investigations of earlier histories. Thus, in his bibliography, Ernst cited the biographies by Clebsch and Dronke and Felix Klein, but also drew upon archival material as revealed by long excerpts from unpublished letters written by Crelle, Plücker, and various officials regarding Plücker's persistent search for stable funding, mathematical posts and research travel. The text divided into four parts: youth and studies, Plücker's work as a mathematician, Plücker's work as a physicist, and Plücker's personality. In his discussion of Plücker's method, Ernst closely followed prior interpretations from secondary sources. He related how "Monge's school" naturally united spatial intuition with analytic operations, consequently the analytic formula was only a concise expression of spatial relationships. A student of Monge's school, Poncelet, had introduced projective geometry, thus marking the transition from "älteren zur neueren synthetischen Geometrie." Then the "sceptre" of geometry passed from France to Germany via Möbius, Steiner and Plücker (Ernst (1933), 10). The former two German geometers were assigned methodological categories–Möbius was both a synthesist and an analyst, Steiner was only a synthesist. Ernst succinctly summarized the opposition between synthesis and analysis in his description of Steiner's relationship with Plücker.

> Steiner stand in einem persönlichen und auch sachlichen Gegensatz zu Plücker, womit der alte bereits an der Ecole Polytechnique hervorgetretene Streit zwischen synthetischer und analytischer Geometrie von neuem auflebte. (Ernst (1933), 11)[13]

By the former "Streit," Ernst appeared to be referring back to the work of Poncelet, who was the only French geometer he described methodologically. As we will soon see, this neatly echoed Klein's earlier historical account. Ernst also may have been following Klein or Dronke, in commenting on the opposition Plücker faced from the Jacobi-Steiner circle.

> Um so mehr ist es zu bedauern, dass ihm während seines Lebens nicht immer, wenigstens nicht überall, die Anerkennung zuteil wurde, die seinen grossen Leistungen gebührte. Gerade in Deutschland, wo der parteiliche Einfluss widerstrebender Schulmeinungen, insbesondere des Jacobi-Steiner'schen Kreises, der von ihm vertretenen Richtung entgegenwirkte, fand er nicht die verdiente Anerkennung. (ibid, 85)[14]

We will find the Jacobi-Steiner circle invoked in several other texts, never with any mention of other members nor with any details of its collective efforts.

While Ernst appears to have drawn the Jacobi-Steiner circle from secondary literature, he provided direct evidence for a divide between analytic and synthetic geometers through Crelle's letters of recommendation for Plücker. In 1835, Crelle wrote that while Plücker's results had already been found in other analytic and especially synthetic ways, Plücker's specific treatment was new and would be particularly appreciated by "friends of analytic geometry" (Ernst (1933), 26). Crelle thus suggested that Plücker's audience might be based on methodological lines. In 1846, Crelle again acknowledged that many of Plücker's results had already been derived by the synthetic method that "Poncelet, Möbius, etc. and particularly Steiner" had undertaken. However, Crelle asserted that Plücker's methodological improvements would eventually lead to further results and thus were more valuable than specific new results. From Crelle's letters, Ernst found further evidence of the simultaneous derivation of the same results by multiple geometers. However, Crelle's accounts also contrast with Ernst's conclusions, as the former mentioned no opposition between Steiner and Plücker. Crelle instead deemphasized the methodological opposition, even correcting himself in clarifying the "analytischen gegenüberstehende, oder vielmehr neben ihr bestehende synthetische Methode" (Ernst (1933), 32).

[13] "Steiner stood in personal and professional opposition to Plücker, which revived anew the old division between synthetic and analytic geometry that had emerged at the Ecole Polytechnique."

[14] "All the more to be regretted that during his life, the recognition due to his great achievements was not always, at least not everywhere, given to him [Plücker]. Especially in Germany, where the partisan influence of oppositional schools of thought, in particular from the Jacobi-Steiner circle, opposed the directions represented by him, he did not find the recognition he deserved."

Ernst only briefly touched on potential differences between the two methods in describing synthetic geometry as intuitive and analytic geometry as operational, but he then claimed that Monge's students had unified these qualities in their geometric research. These adjectives also did not ally with Ernst's description of Plücker's specific research as intuitive, general, and elegant. So we find methodological differences were not defined by qualities, but by specific analytic and synthetic geometers who often arrived at the same results.

1.3.4 Karl Friedrich Geiser (1874)

In contrast to Plücker, who from his death onward had been consistently represented as undervalued and deserving of greater recognition, Steiner primarily appeared as having been once considered the greatest German geometer of the nineteenth century, but as no longer occupying such a superlative post. His death in 1863 was followed by a brief note in Crelle's *Journal für die reine und angewandte Mathematik* by Otto Hesse a few weeks later, who claimed that Steiner had been "the first geometer of his time" (Hesse (1863), 199).[15] Eleven years after this, and perhaps in response to Dronke's recent book, Steiner's nephew, Karl Friedrich Geiser (1843–1934), wrote "Zur Erinnerung an Jakob Steiner" (Geiser (1874)). A lecture on Steiner had also been read at the Swiss science society in Schaffhausen in 1873. Geiser had studied in Berlin with Leopold Kronecker and Karl Weierstrass, and by this time worked as a professor of higher mathematics and synthetic geometry at the Swiss Polytechnik. Geiser was also the editor of Steiner's unpublished works, and had already published *Vorlesungen über Synthetische Geometrie* in 1867. Other biographies of Steiner followed, but Geiser's text remained the exclusive resource for later detailed accounts of Steiner's personality and mathematics.[16]

Following a summary of Steiner's youth up to 1826, Geiser turned to his mathematical contributions. Geiser introduced the synthetic method within the context of eighteenth century mathematics, when, according to him, analytic methods had dominated.

> Durch Euler, Lagrange und Laplace schien, wie diess [sic] auch Gauss an verschiedenen Stellen sehr scharf ausspricht, die Superiorität der analytischen, rech-

[15] Hesse described Steiner's deep attachment toward synthesis as resulting in occasional bad feelings towards analysis.

> Steiners Wirken steht mit der synthetischen Geometrie in unauflöslicher Verbindung. Mit unermüdlicher und ausschliesslicher Thätigkeit widmete er sich ihr, bis zu dem Grade der Schwärmerei, dass er es wie eine Schmach der Synthesis aufnahm, wenn bisweilen die Analysis, deren Macht er nicht unterschätzte, gleiche oder gar weitergreifende Resultate brachte. (Hesse (1863), 199)
>
> Steiner's work is inextricably bound to synthetic geometry. With tireless and unwavering activity he devoted himself to it, with such a level of enthusiasm that he interpreted it as a reproach to synthesis when sometimes analysis, whose power he did not underestimate, reached the same or even more comprehensive results.

[16] For other turn of the century biographies of Steiner see Graf (1897) and Lange (1899).

> nenden Methoden gegenüber den synthetischen, anschauenden in einer Weise festgestellt, dass während der grössern Hälfte des achtzehnten Jahrhunderts die Geometrie beinahe stille stand. (Geiser (1874), 16)[17]

Geiser thus implied that the opposition initiated with the analysts who argued against synthetic methods. In describing this time period, Geiser associated synthetic methods with those of traditional geometry and analytic methods as those involving calculations. According to Geiser, the work of Monge temporarily and happily revealed a unified account of analysis and synthesis as collaborative and mutually beneficial methods in his application of analysis to geometry.

> Nicht minder bedeutsam wirkten die "Applications" [*Applications d'Analyse à la Géométrie* (1795)], indem sie zeigten, dass Analysis und Synthesis nicht als feindliche Mächte einander gegenüber stehen müssen, sondern erst in ihrer Vereinigung die tiefen Geheimnisse der Mathematik einschließen, zu denen sie vereinzelt nie gelangt wären. Lässt das Buch, was die Anordnung des Stoffes anbetrifft, Manches zu wünschen übrig und mögen auch die Beweismethoden nicht überall genügen, so wird ihm, als einer glücklichen Verbindung von analytischem Scharfsinn und geometrischem Erfindergeist doch eine dauernde Stelle in der Geschichte der Mathematik verbleiben. (ibid)[18]

Geiser proceeded by describing the "great geometric principles" of Poncelet and then detailed Steiner's many contributions to geometry, none of which he described as synthetic.

In fact, Geiser only described Steiner as a synthesist in defending him against unjust characterizations as being against analytic methods. Instead, Geiser pointed to specific examples where Steiner had positively assessed analytic methods.

> Steiner ist durchaus Synthetiker gewesen, so dass man ihn sehr oft, allerdings mit Unrecht, als Gegner der analytischen Methoden bezeichnete, denen er doch, freilich mit Vorbehalt, wie z. B. in der Vorrede zur "Systematischen Entwicklung" und in der Einleitung zu den "Maximum et Minimum" eine ehrenvolle Rolle zuertheilte. (ibid, 28)[19]

[17] "Through Euler, Lagrange and Laplace, as Gauss also sharply expressed it at various places, the superiority of the analytic, computational methods with respect to the synthetic, intuitive ones was established in such a way that during the greater part of the eighteenth century geometry almost stood still."

[18] "The "Applications" operated no less significantly by showing that Analysis and Synthesis did not have to remain as hostile forces one against the other, but comprise in their union the deep secrets of mathematics, which they would never reach in isolation from each other. If the book leaves something to be desired as regards the arrangement of the material, and may not completely satisfy the methods of proof, it will retain a permanent place in the history of mathematics as a fortunate combination of analytical acumen and geometric ingenuity."

[19] Steiner is thoroughly a synthesist, so that one very often describes him, although unjustly, as opposed to analytic methods, to which he actually, to be sure with reservation, ascribed an honourable role, for example in the preface to the "Systematischen Entwicklung" and in the introduction to the "Maximum et Minimum."

Geiser observed how Steiner had been attacked posthumously, both in Poncelet's last writings (Poncelet (1865), Poncelet (1866)) and in Dronke's recent publication. Geiser described the latter work as a sort of conspiracy theory derived from Plücker's "spectral-analytisch-geometrisch-entwickeltes Herz," and wondered whether his memories of Steiner would also be construed as engaging in this "zweite Auflage des Kampfes mit dem Drache."[20] Geiser thus suggested that a previous conflict between Plücker and Steiner had existed, not along the lines described by Dronke, but conversely perhaps initiated by Plücker. Plücker did not appear elsewhere in Steiner's biography.

Geiser proved the definitive source for biographical details of Steiner's life, but he situated the opposition between analytic and synthetic methods as preceding Monge and resolving with Monge's *Applications d'Analyse à la Géométrie*. He followed the geographic and chronological progression of "new geometrical methods" from Poncelet to Germany, and none of the geometers during this interval were described with methodological affiliation. When Geiser presented Steiner as a synthesist, he showed that Steiner had acknowledged the merits of analytic methods.

1.4 Turn of the century: Felix Klein and his circle

1.4.1 Felix Klein (1873, 1926)

Felix Klein (1849–1925) was Plücker's doctoral student at Bonn and consistently expressed admiration for the results in geometry and physics of his Doktor-Vater. He had also studied under Clebsch, and his inspiration from both teachers has been detailed in "Felix Klein and His 'Erlanger Programm"' by Garrett Birkhoff and M. K. Bennett (Birkhoff and Bennett (1988)). Klein first assessed developments in nineteenth century geometry as early as 1872 in a distributed paper to the faculty at the University of Erlangen, where he had just been appointed full professor. Although not widely read at the time, the contents of "Vergleichende Betrachtungen über neuere geometrische Forschungen" would become known as the Erlanger Programm, and the paper was republished in German in the 1890's along with Italian, French, and English translations (Klein (1890), Klein (1891), Klein (1893b), Klein (1893a)).[21] Our quote pagination and translations will refer to "A comparative review of recent researches in geometry" translated by M. W. Haskell in Klein (1893a), with comparative German from Klein (1872). Though professing to document the past fifty years of geometry, the majority of Klein's paper focused on research from the mid-century for-

[20] "[...] spectral-analytic-geometric-developed heart [...] second stage of battle with the dragon."

[21] The differences between and historical confusion over Klein's speech and distributed paper at Erlangen are documented by David Rowe in "A Forgotten Chapter in the History of Felix Klein's *Erlanger Programm*" (Rowe (1983)). The relative importance of Klein's Programm itself has been contested by Thomas Hawkins in "The *Erlanger Programm* of Felix Klein: Reflections on Its Place in the History of Mathematics" (Hawkins (1984)).

ward, as indicated by his many textual references to his contemporaries. His discussion of methodology was restricted to his introduction and endnotes and appears as much an assessment of his present view as his historical view.

Whereas Chasles had perceived progressive advantages in the many methods of geometry, Klein regretted the fragmentary nature of geometry which "has been only too much broken up in the course of its recent rapid development into a series of almost distinct theories, which are advancing in comparative independence of each other" (Klein (1893a), 216). Klein negatively assessed the division of modern geometry into theories and consequently described "the present state of mathematical knowledge as exceedingly incomplete and, it is to be hoped, as transitory." Instead, he argued for a more connected geometry, which he called *projective geometry* and contained both synthetic and analytic geometry.

> The distinction [*Unterschied*] between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form [*gestaltet*] in both. We choose therefore in the text as common designation of them both the term *projective geometry.* Although the synthetic method has more to do with space-perception [*räumlicher Anschauung*] and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulae of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations. On the other hand, the advantage to original research of a well formulated analysis should not be underestimated, – an advantage due to its moving, so to speak, in advance of the thought. But it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident [*begrifflich evident*], and the progress made by the aid of analysis is only a first, though a very important, step. (ibid, 243)[22]

Despite this assessment of similarity, Klein drew attention to the differentiating features between the two methods, belying a prescriptive rather than descriptive motivation in his proposal for unification. While the current state of geometry might be moving away from a division between synthetic and analytic geometries, Klein implied that this trend was at best recent. Indeed, Klein titled this note "On the Antithesis between the Synthetic and the Analytic Method in Modern Geometry," where "antithesis" [*Gegensatz*] connoted a much stronger opposition than "distinction" [*Unterschied*].[23] Klein himself assigned complementary properties to each method, suggesting the advantage of using both analysis to

[22]We include the original German for certain of Haskell's translated terms, in particular because *Anschauung* is usually translated as "intuition," while *begrifflich evident* is "conceptually evident."

[23]We follow Haskell's translation, the original note was titled "Ueber den Gegensatz der synthetischen und analytischen Richtung in der neueren Geometrie."

advance the thought and synthesis to make it evident when conducting projective geometric research.

Klein reiterated a similar, though more critical, comparison between the two methods in his *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert* (1926). He first described the qualities of analytic geometry,

> Die analytische Geometrie hat den bequemen Algorithmus für sich, der die höchsten Verallgemeinerungen ermöglicht, der aber auch leicht dazu verführt, das eigentliche Objekt der Geometrie : die Figur und die Konstruktion, aus dem Auge zu verlieren. (Klein (1926a), 115)[24]

Klein pointed to figures, constructions, and physical viewing, all qualities that emphasized the concrete and evident side of geometry. These contrasted with his characteristics of analytic geometry as algorithmic and aimed at generality. In his definition of synthetic geometry, the visual form and intuition played a more prominent role, but only at the expense of ad hoc algorithms and a consequent greater difficulty.

> Bei der synthetischen Geometrie wiederum droht die Gefahr, dass der Geist am einzelnen angeschauten Fall oder doch nur einer beschränkten Zahl von Möglichkeiten haften bleibt; die Lage wird wenig gebessert, wenn, um ihr zu entgehen, ein neuer Algorithmus ad hoc erfunden wird, der schwerfällig bleibt, solange er sich nicht in die einfachsten Ansätze der analytischen Geometrie verwandelt. Zu begrüßen ist bei der synthetischen Behandlung das deutliche Bewusstein der lebendigen Wurzel aller Geometrie, der Freude an der Gestalt. (ibid)[25]

In his section on French geometry and the École polytechnique, Klein traced the origins of the "modern" divide back to Carnot, who had opposed the "Hieroglyphenschrift der Analysis" in favour of a purely synthetic form at the turn of the century.

> Von geschichtlicher Bedeutung ist das Carnotsche Buch [*Géométrie de position*] durch seine Ablehnung der Analysis. Hier ist die Quelle für den nun bald hervortretenden Gegensatz zwischen analytischer und synthetischer neuerer Geometrie, der sich schliesslich zu einer Gegnerschaft von prinzipieller Bedeutung

[24] "Analytic geometry has the comfort of the algorithm going for it, which allows the highest generalizations, but which also easily misleads one into losing sight of the proper object of geometry: the figure and the construction."

[25] "In turn, with synthetic geometry there is a danger that the mind remains bound to the single intuited case or at least to a limited number of possibilities; the situation is little improved, when to avoid this a new ad hoc algorithm is invented, which remains cumbersome as long as it does not reduce to the simplest theorem of analytic geometry. The clear consciousness of the living root of all geometry, the joy of the figure is to be welcomed in the synthetic treatment."

> auswächst. (ibid, 80)[26]

By contrast, Klein explained how Monge had successfully combined analysis and geometry in his analytic geometry. He viewed Poncelet as combining the ideas of both Carnot and Monge, and described his particular theories as "projective"–not analytic or synthetic.

In his later section on early nineteenth century German geometry and Crelle's *Journal*, Klein claimed that the opposition between the two methods had spread from the École polytechnique to Germany along with new developments in French mathematics. Here, Klein did not perceive the methodological divide as solely negative, but suggested that it had been compounded in Germany by a second, more personal, and less "objective" antagonism based in personal cliques or schools of thought. Klein attributed the dispute between Jacobi and the synthesist Steiner against Plücker to this "less factual" disaccord.

> In unserem Falle handelt es sich um den Streit des von Jacobi und seinem Anhang gestützten Synthetikers Steiner gegen Plücker. Moebius steht in seiner stillen Art mehr ausserhalb dieser Kämpfe, die zudem auch durch den Gegensatz von Hauptstadt und Provinz verschärft werden. Noch heute sind ihre Spuren nicht selten zu entdecken, so etwa, wenn noch bis vor kurzem in gewissen Kreisen Steiner als der unvergleichliche, grösste Geometer der ersten Hälfte des 19. Jahrhunderts gefeiert wurde. (ibid, 116)[27]

Klein's assessment developed several features, independent of geometric methodology and suggestive of Klein's particular context. First, Klein considered Jacobi as an analyst, and not a geometer, yet in this version of events, Jacobi appeared as supporting if not instigating the feud between Steiner and Plücker. Secondly, Klein emphasized the importance of Berlin and the urban setting in exacerbating the divide.[28] Finally, Klein challenged Steiner's celebrated rank as the incomparable, greatest geometer of the first half of the nineteenth century, apparently contradicting "certain circles."

Rather than claiming one greatest geometer, Klein presented scientific biographies of the "three great geometers" published in Crelle's *Journal*: Möbius, Plücker, and Steiner.[29] In

[26]"Carnot's book [*Géométrie de position*] is of historical significance through its rejection of analysis. Here is the source for the coming prominent contrast between analytic and synthetic modern geometry, which eventually grows into an opposition of fundamental importance."

[27]"In our case, it concerns the conflict of the synthesist Steiner, supported by Jacobi and his followers, against Plücker. Moebius remains in his quiet way more outside of these struggles that are also exacerbated by the opposition between the capital and the provinces. Even today, it is not uncommon to discover traces, such as when until recently Steiner was celebrated in certain circles as the incomparable, greatest geometer in the first half of the 19th century."

[28]Klein would continue to describe Berlin mathematics as clique-ish when considering the history of later nineteenth century mathematical developments (ibid, 281–285).

[29]Möbius' work was presented independently of any methodological classifications, although Klein concluded by declaring the geometer as a "typical representative" of a "classical" rather than "romantic" mathematician.

his brief biography of Plücker, Klein surmised that the conflict between the "Jacobi-Steiner circle" and Plücker dated back to Plücker's time in Berlin between 1832 and 1834, but offered no details as to the substance of this conflict nor who else may have been involved. Klein provided a nuanced overview of Plücker's geometric method, noting the tradition of Monge's method in combining constructions and analytic forms coupled with Plücker's unique ability to "read equations." In general, Plücker appeared as visually focused on the "true geometric image of forms."

While Klein's biography of Plücker emphasized his scientific career, he introduced Steiner on more personal terms: underscoring his humble origins as a Swiss farmer who surprisingly entered academia despite his "primitive rudeness" and argumentative character.[30] Klein derided Steiner's early teacher, Pestalozzi, as advocating a completely unintuitive pedagogy, and determined that Steiner's intuitive strength did not come from this source. Klein also noted Steiner's systematic treatment of geometry and peculiar "art of instruction." As an example, Klein relayed how Steiner's devotion to the Socratic method extended so far as to not include figures in his geometry lessons. Instead,

> [...] das lebendige Mitdenken des Hörers sollte ein so deutliches Bild in seiner Vorstellung erzeugen, dass er das sinnlich Angeschaute entbehren könnte. (ibid, 128)[31]

Thus, Steiner's teaching style contradicted the connection between pure geometry and the visual form or figure. These additional details illustrate how Klein's account of Plücker and Steiner stood in opposition to his general descriptions of analytic and synthetic geometry. Plücker, the analytic geometer, emphasized the visual properties of geometry, while Steiner, the synthetic geometer, excelled at systematizing but often avoided physical representations. Even the most emblematic representatives of both methods created exceptions to the classification of analytic as uniform but abstract and synthetic as evident but ad hoc.

Klein stated that methodological distinctions were no longer relevant in what should be referred to as "projective geometry," but his historical exposition revealed an underlying propensity toward analytic practices. Klein pointed to the specific advances in geometry, the relationship of algebraic curves to higher function theory or to set theory as well as differential geometry, that could not have been addressed synthetically (ibid, 116). He then praised Plücker, and argued against Steiner's acclaim as the greatest geometer of his time. When explaining the technical work of both Poncelet and Steiner, Klein employed coordinate representations. Finally, in Klein's narrative all antagonism between methods resulted from the personal, professional, and mathematical aggression of synthetic geometers. By claiming

[30] Klein cited Geiser for further documentation of Steiner's personality and development.

[31] "[...] the active thinking of the listener should produce such a clear picture in his mind that he could dispense with the sensory evidence."

that the methodological opposition was over, Klein implicitly sided with the apparently neutral analytic geometers.

These two publications, Klein's "Erlanger Programm" from the very beginning of his career and his lectures on the history of nineteenth century mathematics published posthumously, are representative of the historical content in Klein's intermediary texts where he maintained a consistent position toward unifying the diverse methodological approaches of geometry to promote scientific progress (for example, in *Vorlesungen über höhere Geometrie* (Klein (1926b))). Klein reiterated this call for unification perhaps most iconically in his collaborative organization of the collective mathematical encyclopedia, *Encyklopädie der mathematischen Wissenschaften*, which began publication in 1898 and ended in 1933.[32] In the next texts, we examine several articles associated with this collection. This includes articles published directly as texts in the *Encyklopädie*, as well as those by other European geometers in the late nineteenth century, who followed a similar investigative structure in evaluating the recent past of specific branches of geometry. The German *Encyklopädie* and its partial French adaptation, are the most extensive examples of this trend. In both collections, the third volume was devoted to geometry and included articles originally written by many prominent working mathematicians including Gino Loria, Gino Fano, and Arthur Schoenflies.

1.4.2 Gino Loria (1887)

When Gino Loria (1862–1954) wrote his technical mathematical article "Spezielle ebene algebraische Kurven von höhere als 4. Ordnung" for the *Encyklopädie* in 1914, he had already established a reputation as an algebraic geometer and an historian of mathematics. While Loria's *Encyklopädie* article did not concern the history of geometry, this had been the subject of his first book: *Il passato ed il presente delle principale teorie geometriche* (Loria (1887)). This work had originally been published serially, and Loria later wrote two substantially revised editions in 1896 and 1907, as well as many other histories of geometry through the 1940s.[33] In each edition, he devoted the first forty pages to the history of geometry before 1850 and the remaining hundreds of pages covered the next half-century emphasizing Italian contributions. The majority of his text was organized with respect to specific areas of geometric study, such as algebraic planar curves and surfaces, differential geometry, or non-euclidean geometry. In these sections, traces of early nineteenth century

[32]The impetus, undertaking, and impact of Klein's *Encyklopädie* project have been studied in depth by both David Rowe and Renate Tobies (Rowe (1989), Tobies (1994)). Hélène Gispert has extended this research into a comparison of the nationalist differences between the original German articles and those that were subsequently adapted, with numerous differences, into French (Gispert (1999)). In particular we will look at both the German and French versions of two articles on geometry.

[33]Excerpts from work were often translated into English, and the originals and translations served as a frequent source for Boyer's *History of Analytic Geometry* (Boyer (1956)). Loria's *Il passato ed il presente delle principale teorie geometriche* was also translated into German by R. Sturm.

contributions would appear. By contrast, the first historical section progressed chronologically rather than thematically.[34]

Loria's history of geometry began in ancient Greece and proceeded rapidly to more recent developments, encompassing both analytic geometry and pure geometry (which he also called synthetic geometry or synthesis). In Loria's presentation, pure geometry stood opposed to algebraic analysis. Pure geometry had been revitalized in the early nineteenth century first by Monge and Carnot, but most significantly by Poncelet, whose *Traité des propriétés projectives des figures* heralded the "risorgimento della geometria pura" in 1822 (Loria (1887), 24). Loria enthusiastically digressed from his historical overview to detail some of Poncelet's achievements as they related to more contemporary researches. Continuing to document landmark years in French mathematics, Loria next focused on 1837, and Chasles' *Aperçu*. Here he presented his first comparison between pure geometry and analysis in describing Chasles' *Apercu* as a fascinating work.

> [...] opera affascinante in cui l'autore, dopo aver esposto con uno stile la cui bellezza potrà raggiungersi ma non superarsi tutto quanto costituiva ai suoi tempi il patrimonio della geometria pura, gagliardamente sostenne i diritti che questa aveva alla considerazione degli scienziati e che le venivano continuamente contrastati dai ciechi adorati dell'analisi. Non bisogna però credere che questa sia un'opera di sola polemica, e che quindi abbia, oggi che la lotta è finita, soltanto un valore storico. (ibid, 30–31)[35]

Loria declared that the polemic between geometry and analysis had ended. However, with his praise of pure geometers and their contributions, he seemed to appreciate the fruits of the past struggle.

From France, Loria proceeded to developments in Germany, and here presented Steiner and Möbius as two contrasting examples of German geometers. While Möbius engaged in geometry, analysis, mechanics and astronomy, Steiner was exclusively dedicated to geometry.

> [...] essi palesino una sostanziale differenze fra lui e l'altra delle stelle di prima grandezza che illuminavano in quell'epoca il cielo della matematica tedesca, cioè J. Steiner (1796–1863) (1), il quale fu così esclusivamente geometra, che coll'analisi non volle mai scendere palesemente a patti (2). (ibid, 33)[36]

[34] This particular chapter was translated by George Bruce Halsted into a two-part article "Sketch of the origin and development of geometry prior to 1850" published in *The Monist* in 1903.

[35] "'[... a] fascinating work in which the author, after having described with a style, whose beauty could be matched but not superseded, all that constituted the heritage of pure geometry in his time, vigorously supported the rights that it [pure geometry] had to the considerations of scientists, [rights] that were continually thwarted by the blind worship of analysis. But we must not believe that this was only a work of controversy, and is only of historical value, now that the fight is over."

[36] [...] they [Möbius' other research interests] revealed substantial differences between him and the other

With only two great stars in the sky of German mathematics, Loria appeared to be neglecting Plücker, who by order of birth, appeared last in his list of German geometers: Möbius, Steiner, Staudt, and Plücker. However, both in the introduction and throughout the body of the text, Loria declared that many of Plücker's achievements were decisive to the progress of analytic geometry. In support of this, Loria cited Clebsch's biography of Plücker, which Beltrami had claimed was "'il miglior elogio che si possa fare a Plücker, considerato come geometra, è questo, che Clebsch non ha potuto tessere il racconto dei suoi lavori, senza rifare in gran parte la storia della moderna geometria analitica"' (ibid, 35).[37] In Loria's specific examples, not only modern analytic geometry, but also modern synthetic geometry had benefitted from Plücker's research. In particular, Loria claimed that the principle of duality had doubled the domain of both geometries, thanks "in large part to Plücker" (ibid, 207).

Even though geometry was divided between methods, Loria contended that both methods could benefit from the same innovation. Conversely, Loria demonstrated that Poncelet, Steiner and Plücker working from different methods arrived at many of the same results. His portrayal of synthetic and analytic geometry thus illustrated two complementary methods, even if specific geometers practiced one method exclusively.

1.4.3 Ernst Kötter (1901)

Ernst Kötter (1859–1922) focused on documenting the history of synthetic geometry in "Die Entwickelung der synthetischen Geometrie von Monge bis auf Staudt (1847)" (Kötter (1901)). This 484-page report was published as part of the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, and thus was not part of the *Encyklopädie* although its structure and content were similar. The text was organized in three time periods, designated by the major publications of geometers: Monge to Poncelet (1822), Poncelet to Steiner (1822–1832), and Steiner to Staudt (1832–1847). These sections were further divided by results (Pascal's theorem), techniques (stereographic projection), or problem sets (construction of conic sections form five points and tangent lines). In his introduction, Kötter described the premise of his work.

> Der Bericht soll eine möglichst vollständige Darlegung aller derjenigen geometrischen Entwickelungen bieten, die sich mit rein geometrischen Mitteln auf den Funda-

star of first magnitude, which in that epoch illuminated the sky of German mathematics, J. Steiner (1796–1863) (1), who was so exclusively a geometer that he never deigned to come to terms openly with analysis." Intriguingly, the English translation of Loria's introductory chapter offers a subtle modification: "that he was never willing to come to terms openly with the analysts." (231) The change from subject matter to practitioners suggests a much more personal interpretation of Steiner's proclivities.

[37] "'[...] the best eulogy that could be made about Plücker, considered as a geometer, is this, that Clebsch could not succeed in recounting his works, without remaking in great part the history of modern analytic geometry."

> mentalsatz der Geometrie der Lage zurückführen lassen. (Kötter (1901), iii)[38]

The effect is expectedly encyclopedic and genetic, succinct listings of specific theories, problems, and theorems often moving from the seventeenth century to current research with limited narrative structure.

Like Steiner's nephew Geiser, Kötter had studied under Weierstrass and Kronecker, and lectured on synthetic geometry, at the University of Berlin and taught as a professor of descriptive geometry at the technical Hochschule in Aachen. The Steiner prize had been established by Steiner through the University of Berlin following his death to award 8000 Thaler once every two years for geometric research treated synthetically, and Kötter had received this award for his synthetic geometry research in 1886.[39] He was awarded the prize for solving a problem posed by Weierstrass on higher curves and surfaces (Biermann (1973), 108). Thus, in his time, Kötter was considered a successful synthetic geometer.

In his history, Kötter clarified that synthetic geometry included practices from pure or elementary geometry and geometry of position. Explaining the necessary length and detail of his presentation, Kötter pointed to the arduous calculations of analytic geometry and the particular cases necessary in the development of synthetic geometry. He described the gradual development of a synthetic method that could address geometric problems, which at first had been limited to analytic approaches.

> Griff sie zunächst nur gelegentlich und wenig methodisch ein, um Abkürzungen in der Herleitung von Resultaten zu erzielen, welche sich mit den noch wenig ausgebildeten Methoden der analytischen Geometrie nur nach beschwerlichen Rechnungen gewinnen liessen, so forderte sie schliesslich immer entschiedener die völlige Aufdeckung eines rein geometrischen Weges, der von den ersten Voraussetzungen zum Resultate führt. Vielfach wird dieses Ziel nur nach mehreren Versuchen erreicht. So erweisen sich öfters gerade solche Arbeiten für die Entwickelung der synthetischen Geometrie als sehr bedeutungsvoll, die an sich neue Resultate nur in geringem Masse bieten. (ibid)[40]

[38]"The report aims to provide the most complete exposition of all those geometric developments, which can be traced back to the fundamental theorem of geometry of position by purely geometric means."

This fundamental theorem, also known as the fundamental theorem of projective geometry, traced back to Von Staudt and states that a projective relationship is determined by three initial elements (often points or lines) and their corresponding three elements under that projection. Jean-Daniel Voelke details the evolving proof of the history of the fundamental theorem of projective geometry in Voelke (2008).

[39]In "A History of Prize Problems" Gray provides a partial list of recipients of the Steiner prize, the terms of which were changed in 1888 to be awarded less frequently under the impetus of Kronecker (Gray (2006), 17).

[40]"If it [synthetic geometry] stepped in at first only occasionally and not very methodically in order to arrive at abbreviations in the derivation of results, which were obtained with still underdeveloped methods of analytic geometry only after arduous calculations, then it more and more resolutely claimed the complete exposure of a purely geometric path, which leads from the first hypotheses to the results. Frequently this goal was achieved only after many attempts. Often exactly such works which provide new results only in limited quantity prove to be very meaningful for the development of synthetic geometry."

Thus Kötter distinguished early nineteenth century synthetic geometry as more intuitive than analytic geometry, but lacking innovation and uniformity, which only arrived later. Even after the transition to modern geometry, the primary difficulty of synthetic geometry, for Kötter, remained the particularity of the figure.

> Die einzige Schwierigkeit besteht darin, genau zu entscheiden, ob der Zustand einer Figur ein allgemeiner oder ein besonderer ist. (ibid, 121)[41]

Kötter suggested that modern synthetic geometry was only achieved after Poncelet, who still struggled to distinguish between general and particular figures. Consequently, Kötter dated the beginning of "synthetic geometry as a science" back to 1822 and Poncelet's *Traité*. He then chronicled how the work of Poncelet, Steiner, Chasles and others gradually had rendered the synthetic geometry increasingly general.

Kötter explained that analytic results had often preceded their later synthetic development. To show this "natural" progression, he therefore included the original analytic reference, which explained the preponderance of formulas in his text. Indeed, although the title claimed to be a report on synthetic geometry, Plücker appeared in the appendix over thirty times and Kötter also thoroughly described results from analytic geometry in the work of Möbius, Magnus, Bobillier, and Gergonne. In part because of this choice between methods, Kötter cited numerous instances of Plücker and Steiner arriving at the same result or proving the same theorem. While on occasion Steiner had not properly credited Plücker's prior publications, Kötter's text indicated no overt hostility between the two geometers (ibid, 272, 450).

This was not because Kötter avoided controversial subjects all together. Although he focused on explaining the technical geometry and surrounding publications, this included Poncelet's "unjustified attacks" on Plücker with respect to potential plagiarism in his debut publications in "modern analytic geometry."

> An die dargelegten Entwickelungen knüpft sich noch ein nicht sehr angenehmes Nachspiel; ich habe der in ihrer Masslosigkeit jedenfalls unberechtigten Angriffe Poncelet's auf zwei Arbeiten von Plücker zu gedenken. Die eine ist lediglich analytischer Natur und gehört zu Plücker's Erstlingsarbeiten auf dem Gebiete der modernen analytischen Geometrie. (ibid, 137)[42]

Despite the occasional misunderstandings and improper citations between analytic and synthetic geometers, Kötter observed how early nineteenth century geometers mutually developed results not limited by methodological boundaries. For Kötter, this resulted in a

[41] "The only difficulty remains to decide exactly whether the state of a figure is general or particular."

[42] "A not very pleasant sequel joined in the known developments; I have to mention Poncelet's attacks, altogether unjustified in their intemperance, against two works of Plücker. The one was exclusively analytic in nature and belonged to Plücker's first works in the field of modern analytic geometry."

deepening of both proof methods. Specifically, synthetic geometry had progressed from the studying particular figures toward studying geometric objects "in general."

1.4.4 Gaston Darboux (1904)

In a similar vein to Klein's Erlanger Programm, Gaston Darboux (1842–1917) gave an address before the section on Geometry at the International Congress of Arts and Science entitled "A Survey of the Development of Geometric Methods" in St. Louis in 1904. The Congress was intended for scholars from the United States, Canada and Europe to share talks on their research areas, and in order to encourage a larger audience there was no attendance fee. Presumably, Darboux originally spoke in French, as the published version, which appeared in the *Bulletin of the American Mathematical Society* had been translated by Henry Dallas Thompson. However, we were unable to locate an original French manuscript. In this talk, Darboux traced the trajectory of geometry from the late eighteenth century, and particularly the work of Monge, to the present. He classified this time period as constituting "modern geometry." Darboux hinted at a fissure between geometry and analysis in characterizing the unifying nature of Monge's contributions.

> He, the regenerator of modern geometry, pointed out from the beginning–though his successors may have forgotten it–that the alliance between geometry and analysis is useful and productive; and that perhaps this alliance is a condition for success to them both. (Darboux (1904), 519)

Darboux described Poncelet as one of these forgetful successors, who had neglected "everything in Monge's work that belongs to cartesian analysis or concerns infinitesimal geometry." Further, Darboux posited that Poncelet's methods "in opposition to the analytic geometry" met with no favour among French analysts, but were quickly adopted by geometers "stirring up in many directions the most profound researches." In particular, Darboux mentioned the articles in Gergonne's *Annales* as well as independent publications by Möbius, Plücker, Steiner and Chasles. The latter two geometers practised pure geometry or synthesis and defined their work in opposition to analysis or analytic geometry.

> Those who, like Chasles and Steiner, devoted their whole lives to inquiries in pure geometry, opposed to analysis that which they called synthesis; and adopting in the main rather than in detail the tendencies of Poncelet, they proposed to institute an independent theory, a rival to the cartesian analysis. (ibid, 522)

Darboux explained in broad detail how Poncelet, Chasles and Steiner had generalized and extended synthetic geometry to match analytic geometry. On the other side, was the perfection of analytic geometry at the hands of Gergonne, Bobillier, Sturm and Plücker.

> While Chasles, Steiner, and later, as we shall see, von Staudt, applied themselves to the task of constructing a rival doctrine to analysis, thus in a way setting up one altar against another, Gergonne, Bobillier, Sturm, and above all Plücker, were perfecting the cartesian geometry and developing an analytic system somewhat adequate to the discoveries of the geometricians. (ibid, 524)

Here also, Darboux emphasized opposition. The analytic geometers measured their system against geometry. In this rivalry, Darboux allotted greater success to analytic geometers who could "bring out the full meaning of those conceptions which the so-called synthetic geometry had not been able to master completely." However, both methods benefitted from the competitive atmosphere and Darboux celebrated a "brilliant period [...] for geometric research of every kind," with analysts stimulating geometricians and vice versa. Even if analytic geometry appeared as ultimately more general and more complete, synthetic geometry occupied an "honourable position."

> Such were the principal investigations which at that time reinstated synthetic geometry in its honorable position, and assured to it during the last century the place which belongs to it in mathematical research. Numerous and illustrious laborers took part in this great geometric movement, but it must be acknowledged that it had Chasles and Steiner as its leaders. (ibid, 529)

In fact, Darboux suggested that the "brilliancy displayed by their marvellous discoveries" threw the foundational work of von Staudt into the shade. Darboux summarized many more areas of study than those practiced by the geometers that he called analysts or synthesists, including infinitesimal geometry, intrinsic geometry, non-euclidean geometry, and analysis situs. Despite the cross-over between certain geometers, these various other "methods" were presented as independent of the ongoing rivalry.

In concluding, Darboux returned to analysis and pure geometry. He admired the fruitfulness and power of the former, but expressed concern about the limited number of pure geometers at the beginning of the twentieth century.

> Do not let us forget that while analysis has acquired means of investigation which formerly it lacked, nevertheless it owes those means largely to the concepts introduced by the geometricians. Geometry must not remain, as it were, shrouded in its own triumph. It was in the school of geometry that we have learned, and there our successors will have to learn it, never blindly to trust to too general methods, but to consider each question on its own merits, to find in the particular conditions of each problem either a direct way toward a simple solution or the means to apply in an appropriate manner those general methods which every science should collect. (ibid, 542)

However, instead of attempting to match geometry against analysis, Darboux proposed cultivating it for "its own advantages," because of and not despite its individual and visual nature. In this conclusion, Darboux used geometry and pure geometry interchangeably, and anticipated that the subject would be "revived" through future applications of mathematics.

1.4.5 Max Simon (1906)

Several of the geometry articles originally intended for the *Encyklopädie* resulted in separate monographs. Among these was a report on the development of elementary geometry by Max Simon (1844–1918), who taught mathematics in Strassburg. In his introduction, Simon described how Klein had first asked him to write *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*, which was then instead published as a book within the *Jahresbericht der Deutschen Mathematiker-Vereinigung* in 1906 (Simon (1906)).[43]

Simon referenced scores of histories, textbooks, and research publications in compiling this volume, which is less a historical exposition of the development of elementary geometry than a reference book. Following the table of contents or the name index, one could determine who contributed to what elementary geometry developments, including geometers preceding the nineteenth century (Archimedes, for example, had 31 entries).[44]

In his introduction, Simon explained the difficulty of delimiting elementary geometry, especially in light of nineteenth century developments. In order to not include too much or too little, Simon proposed to focus on problems determined "primarily by the needs of middle school teachers." He continued by acknowledging, through the example of Gergonne's solution to the problem of Apollonius, that many theorems of elementary geometry could be found analytically.[45] Simon never explained what constituted analytic geometry, although he used the term throughout the text to describe the contributions of Plücker, Gergonne, Carnot, Möbius, Karl Wilhelm Feuerbach, François-Joseph Servois, Charles Sturm, Christian Gudermann and numerous later geometers, including himself. On the other hand, he only labeled a proof, solution, or text as "synthetic geometry" when directly comparing it to an adjacent example of analytic geometry. Thus, the publications of Monge, Poncelet, Steiner, Charles Brianchon, and Michel Chasles were sometimes described as "synthetic geometry" and sometimes just as "geometry." His numerous bibliographies scattered

[43]Accounting for the limited readership of Simon's text, Jesper Lützen explained that Klein decided to omit Max Simon's contribution due to imprecise reference citations (Lützen (2009), 376).

[44]Coolidge conveyed the overwhelming breadth of Simon's text in his general description. "In Simon we have an attempt to make a catalogue of contributions in the nineteenth century alone. The writer covers ten broad topics: 1) History and methods, 2) Parallels, 3) The circle, 4) Areas, 5) The triangle, 6) Polygons, 7) Plane configurations, 8) General space relations, 9) Special space relations, 10) Plane and spherical trigonometry. He does not give titles; occasionally he gives a line or two of explanation, frequently merely the author's name and place of publication. But the book contains upwards of 250 pages, and the number of references seems to be in the general vicinity of ten thousand." (Coolidge (1940), 51)

[45]We will examine several solutions to this problem in Chapter IV.

throughout the text suggest that the term "synthetic geometry" became a more common description in German texts during the second half of the nineteenth century.[46]

As we have noted above, Steiner and Plücker often contributed to the same areas of research, and many of these were documented within dense chronological paragraphs of textual references. Simon even designated a result on conic sections and polygons as the *Steiner-Plückerschen Sätze* (Simon (1906), 185).[47] Overall, Simon's work fully displayed the consistent repetition of results among nineteenth century geometers that had only been suggested by other authors.

1.4.6 Gino Fano (1907)

Gino Fano (1871–1952) had been a student of Klein, and in 1871 he translated Klein's "Erlanger Programm" into Italian. In 1907, as a professor at the University in Turin, he wrote "Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im XIX Jahrhundert" for the volume on geometry in Klein's *Encyklopädie* (Fano (1907)). By its title alone, the text suggested the "historical" existence of a methodological opposition, an opposition that Fano would present as no longer active. Fano explicitly defined both methods, although rather circuitously. For him, analytic geometry was geometry with the help of analysis.

> Man unterscheidet allgemein zwei Arten von Geometrie: die *synthetische Geometrie*, welche die Figuren an sich betrachtet, und die *analytische Geometrie*, welche mit Hilfe der Analysis ihr Lehrgebäude aufstellt. Es liegt in der Natur der Sache, dass von diesen beiden Arten, geometrische Gebilde zu untersuchen, ausschließlich die erste in den älteren Zeiten angewandt wurde, während die zweite erst im 17. Jahrhundert, nach Entstehung der Algebra, als *Anwendung der Algebra auf die Kurvenlehre* zur Geltung kam. (Fano (1907), 223)[48]

[46] Simon cited three books with "synthetic" in their title, following his citation style: "*T. Geiser*, Einleitung in die synthetische Geometrie 1869;" "*Th. Reye*, Die synthetische Geometrie im Altertum und in der Neuzeit. Rektorrede, Strassburg, 1886, 2. Auth. 1899, auch im XI Band des Jahrbuchs des Deutsch. Math. Verein. 1901.;" "*A. Milinowski*, Elementar-synthetische Geometrie der Kegelschnitte. 1. Abschnitt Leipzig (1882). (2. wohlfeile Ausgabe 1896.)"

[47] Simon referenced "Note sur le theorème de Pascal" written by Plücker for Crelle's *Journal* in 1847. Here Plücker corrected a recent article by Cayley "Sur quelques théorèmes de la géométrie de position," in which Cayley had referenced and critiqued Steiner's research on the sixty hexagons formed by six points on a conic (Cayley (1845)).

> Je me propose de faire disparaître de ce qui précède tout ce qu'il renferme d'incorrect et d'hypothétique. (Plücker (1847), 337)

Plücker noted that Steiner had first published an incorrect theorem in 1828, which Plücker had corrected in 1829. Steiner accordingly modified his theorem in his 1832 book, and that book [*Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von Einander*] had been cited by Cayley. Here Plücker supplied the analytic proof, missing in Cayley's presentation, and referred to the theorem as belonging to both him and Steiner.

[48] "One generally distinguishes two types of geometry: *synthetic geometry*, which examines the figures in

Fano thus emphasized the figure as the central object of geometry, to which analysis could be applied. Generally, the difference between analytic and synthetic geometry hinged on the representation of geometric objects: by coordinates and equations or by points and figures. Later in the text, Fano provided a more detailed description of the process of analytic geometry, as the translation of geometric statements into equations, their solution via analysis, and a translation back into geometry.

> In der analytischen Behandlung lassen sich *drei* wesentliche Momente unterscheiden: 1) die analytische Übersetzung des Problems, in welcher es sich darum handelt, die vorliegenden Umstände durch *Gleichungen* auszudrücken, d.h. "in der Sprache der analytischen Geometrie zu schreiben"; 2) die Ableitung, von diesem System von Gleichungen ausgehend, der weiteren Gleichungen oder der Koordinatenwerte, welche die aufgesuchten Gebilde darstellen; eine rein analytische Aufgabe, in welcher die von der Analysis der Geometrie geleistete Hilfe besteht; 3) die geometrische Deutung der analytisch gewonnenen Resultate. Der in dem Einzelproblemen 1) und 3) verlangte Übergang von einer geometrischen Beziehung zu ihrem analytischen Bilde und umgekehrt ist oft ein unmittelbarer, fast unbewusster. (ibid, 228)[49]

Such a description emphasized the *application* of analysis to geometry.[50]

The nineteenth century participants in the development of both geometries remained familiar. Monge served as a forerunner for both geometrical methods, followed by Poncelet,

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themselves, and *analytic geometry*, which sets up its theories with the aid of analysis. It is in the nature of things that among these two ways of investigating geometric structures, exclusively the former was applied in ancient times, while the latter came into acceptance only in the 17th century, after the creation of algebra as the *application of algebra to the theory of curves*."

[49]"In the analytic approach one can distinguish *three* essential moments: 1) the analytic translation of the problem in which the question is to express the existing situation through *equations*, that is, "to write in the language of analytic geometry"; 2) the derivation, starting from this system of equations, of further equations or coordinate values, representing the desired structures; a purely analytic exercise in which consists the assistance provided to geometry by analysis; 3) the geometric interpretation of the analytically obtained results. The translation required in steps 1) and 3) from a geometric relationship to an analytic picture and vice versa is often direct and almost unconscious."

[50]Kline explored the question of whether analytic geometry was perceived as real geometry in the work of Chasles.

> The objections to analytic methods in geometry were based on more than a personal preference or taste. There was, first of all, a genuine question of whether analytic geometry was really geometry since algebra was the essence of the method and results, and the geometric significance of both were hidden. Moreover, as Chasles pointed out, analysis through its formal (836) processes neglects all the small steps which geometry continually marks. The quick and perhaps penetrating steps of analysis do not reveal the sense of what is accomplished. The connection between the starting point and the final result is not clear. Chasles asks, "Is it then sufficient in a philosophic and basic study of a science to know that something is true if one does not know why it is so and what place it should take in the series of truths to which it belongs?" The geometric method, on the other hand, permits simple and intuitively evident proofs and conclusions. (Kline (1972), 835)

Möbius, Steiner and Chasles on the side of synthetic geometry opposite Möbius (again) and Plücker on the side of analytic geometry. Later geometers including von Staudt, Grassmann, Cremona, Thieme, Kötter, De Paolis, and Clebsch were classified as neither synthetic nor analytic. Fano concluded with a chapter on differential geometry and the work of Monge, Dupin, Gauss, and Lie.

While appreciating the differences between both geometries, Fano concluded the synthetic and analytic methods could address the same geometric questions. Fano modified this statement with respect to specific areas of inquiry, such as general algebraic curves and surfaces of any degree, which he assessed as solely within the purvey of analytic geometry. With self-declared "historic hindsight" Fano thus assessed that the differences between the methods were superficial. In particular, he classified the distinction as merely a matter of expression or speech [*Sprache*].

> Trotz des scheinbaren Gegensatzes lässt sich in den Methoden der beiden Geometrieen [sic] vielfach derselbe Leitfaden erkennen. Wird eine Frage den beiden Behandlungsweisen unterworfen, so ist oft der Gedankengang nur einer; geändert ist bloss die Ausdrucksweise, die "Sprache". (ibid, 228) [51]

Nevertheless, he recognized the advantages accrued from the past divide, beginning with Poncelet's "neueren synthetischen Geometrie" and its progress through the nineteenth century. The two methods were portrayed as driving each other and developing new and more general areas and strategies of research.

> Durch die synthetische Entwicklung der projektiven Geometrie erhielt die analytische Geometrie einen bedeutenden Anstoss; neue Probleme boten sich ihrer Behandlung dar, und für diese Behandlung mussten auch neue Methoden geschaffen werden. (ibid, 238)[52]

Fano continued to describe some of these recent developments in more specific details, relying upon published texts. In the context of generating and propagating new mathematics, the "Gegensatz" ultimately appeared as positive and productive.

In the French version of the *Encyklopädie*, the Algerian mathematician Sauveur Carrus adapted Fano's article, which appeared in 1915 (Carrus and Fano (1915)).[53] Carrus added citations to Darboux's speech on geometrical methods (although with a French title) (Darboux (1904)), and placed greater emphasis on the work of French geometers, in particular

[51] "Despite the apparent opposition, the same guidelines can often be seen in the methods of both geometries. If a question is subject to the two modes of treatment, the course of thought is often unique; what is changed is only the expression, the 'language'."

[52] "Through the development of synthetic projective geometry, analytic geometry received an important impetus; new problems offered themselves to its treatment, and for this treatment new methods also had to be created."

[53] Hélène Gispert has compared and analyzed the differences between the French and German editions in Gispert (1999).

Chasles. The definitions of analysis and synthesis remained essentially the same. The three most significant differences between the German and French version are (1) the change in title to a much more neutral "Exposé parallèle du développement de la géométrie synthétique et de la géométrie analytique pendant le 19ième siècle;" (2) crediting Chasles with a successful synthetic treatment of imaginary points (Carrus and Fano (1915), 202); (3) the emphasis on contributions of French geometers and editing out Fano's suggestion that "the centre of geometric activity" moved to Germany.

> Mais il faut, des maintenant, noter une différence fondamentale entre les tendances des grands géomètres. Les uns, comme M. Chasles et J. Steiner, K. G. Chr. von Staudt plus tard, se consacrèrent uniquement aux recherches de pure géométrie et se proposèrent de constituer une doctrine autonome, rivale de l'analyse cartésienne. Les autres, J. D. Gergonne, E. Bobillier, J. Ch. F. Sturm, et en Allemagne A. F. Möbius et J. Plücker surtout, perfectionnaient l'instrument de R. Descartes et découvraient de nouvelles méthodes s'adaptant en quelque sorte aux travaux de J. V. Poncelet. (Carrus and Fano (1915), 197–198)[54]

By broadening the scope of early nineteenth century geometers, Carrus further moved analysis and synthesis away from Plücker and Steiner, and instead emphasized the parallel, mutually influential, and widespread development of both pure and analytic geometry, only subtly underscored by rivalry.

1.4.7 Arthur Schoenflies (1909)

Perhaps in accord with Klein's call for unifying the methods of geometry, the *Encyklopädie* article that followed Fano's historical treatment of the opposition between synthetic and analytic methods was on "Projektive Geometrie" written by Arthur Schoenflies (1853–1928) in 1909 (Schoenflies (1909)). At the time, Schoenflies held the chair in applied mathematics at the University of Königsberg. Schoenflies explained that he had first submitted the text several years earlier, in 1901, but the publication had been postponed "for editorial reasons." He divided the text into six parts addressing different defining features of projective geometry: a historical introduction, general terms and methods, particular problems, foundational questions, and projectivity as an operation. Schoenflies provided a list of textbooks on projective geometry, which included works by Carnot, Poncelet, and Steiner as

[54]But one must now note a fundamental difference between the tendencies of the great geometers. The ones, like M. Chasles and J. Steiner, K. G. Chr. von Staudt later, devoted themselves uniquely to research in pure geometry and proposed to constitute an autonomous doctrine, rival of Cartesian analysis. The others, J. D. Gergonne, E. Bobillier, J. Ch. F. Sturm, and in Germany A. F. Möbius and J. Plücker above all, perfected the instrument of R. Descartes and discovered new methods to adapt somewhat to the work of J. V. Poncelet.

representatives from the early nineteenth century. However, it was not until the mid-1870's that the term "projective geometry" began appearing in titles of books and articles.

Schoenflies cited Kötter's report on synthetic geometry as his primary historical source, and extended Kötter's research up to the end of the nineteenth century. However, Schoenflies only used the term "synthetic geometry" once, in a footnote, in order to describe a historical account written by Otto Ludwig in 1900. Thus he did not use synthetic to describe nineteenth century developments.

Schoenflies described Poncelet's work as the first exhibiting a "general projective way of thinking," by providing a geometric interpretation of infinite elements and imaginary quantities. Schoenflies noted that "analysts," including Gergonne and Plücker, had developed "analytische Darstellung" of these concepts. Similarly, the imaginary infinitely far tangent points between two circles that Poncelet had introduced geometrically were only later fully understood on the basis of their "analytic-metric nature" (Schoenflies (1909), 396). Throughout his text, although usually in footnotes, Schoenflies cited other contributions from "analysis" by contrast to those from geometry. For example, in describing Steiner's contributions to the development of quadratic relationships Schoenflies included a footnote to the earlier analytic treatment by Plücker and Magnus.

> Analytisch tritt die B_2 zuerst bei J. Plücker auf, J.f. Math. 5 (1829), p. 28, sowie bei A. J. Magnus, der sie freilich für die allgemeinste eineindeutige Punktverwandtschaft hielt, da er die Eineindeutigkeit von vornherein als Bestehen einer bilinearen Relation deutete, J. f. Math. 8 (1831), p. 51. (ibid, 477)[55]

Schoenflies provided a few other instances of simultaneous results by Steiner and Plücker, labelling the former as geometric and the latter as analytic.

Schoenflies had served as an editor for Plücker's mathematical works and wrote with great admiration for his contributions to geometry in his *Einführung in die analytische Geometrie der Ebene und des Raumes* (Schoenflies (1931)). Yet Plücker's work was largely excluded from this historical treatment. As a case in point, Schoenflies described how Poncelet and Gergonne defined the principle of duality and their resulting polemical exchange without mention of Plücker's involvement. When Schoenflies returned to the principle of duality in his 1931 monograph *Einführung in die analytische Geometrie der Ebene und des Raumes*, both Plücker and Möbius also received credit.

> Ein letztes Streben wissenschaftlichen Erkennens ist auf das *Einheitliche* in der Fülle der mannigfachen Gestalten gerichtet. In der Geometrie ist dies Streben

[55] "J. Plücker first addressed the B_2 analytically, J. F. Math. 5 (1829), p. 28, similarly A. J. Magnus, who certainly maintained the most general individual change of coordinates, here he interpreted the unity correspondence from the first as a bilinear relation, J. f. Math. 8 (1831), p. 51."

in den letzten hundert Jahren von grossem Erfolg gekrönt gewesen. *Dualität, projektive* Denkweise und *Übertragungsprinzipien, Dualisierung* und *Homogenisierung* des Koordinatenbegriffs sind seine Marksteine. In V. C. Poncelet [sic], J. D. Gergonne, A. F. Möbius und J. Plücker haben wir die Lehrmeister zu erblicken, die uns in erster Linie dahin führten. (Schoenflies (1931), 193)[56]

Schoenflies did not use the expressions "synthetic" or "pure" to describe geometry, and by differentiating the work of "analysts" he suggested that geometry itself was inherently not analytic. The method and results called projective geometry by Schoenflies aligned with those of the modern synthetic geometry described by Kötter. For Schoenflies, geometry possessed certain well-defined positive qualities, exemplified in the work of Steiner.

> Der mächtige Fortschritt, den das Studium der geometrischen Gebilde den Erzeugungsmethoden von J. Steiner verdankt, beruht wesentlich auf zwei Umständen. Erstens besitzen die durch ihn geschaffenen Begriffe und Methoden unmittelbare *konstruktive*, der Anschauung zugängliche und daher im besten Sinn geometrische Vorzüge, zweitens aber erwiesen sie sich als der weitesten Verallgemeinerung zugänglich. (ibid, 415–416)[57]

In particular, as exhibited by Steiner's method, Schoenflies emphasized the constructive nature of geometry, and thus, a non-constructive or indirectly constructive method was not geometric.[58]

Schoenflies did not describe the qualities or practices of analysis, although he observed that it was better suited to particular problems concerning higher degree curves or surfaces and imaginary points, lines, and planes. Yet even when geometry appeared more difficult than analysis, Schoenflies emphasized how recent developments had resulted in greater generalization and uniformity.

As both a sequel and counterpoint to Fano's article, Schoenflies presented a history of a single modern geometry with an autonomous set of geometric practices, problems, and principles that defined projective geometry. By separating his subject from analysis or analytic geometry, he offered an alternative and more limited interpretation of "projective geometry" as compared to the methodologically unified version proposed by Clebsch and Klein, which included analytic and geometric practices.

[56] A last pursuit of scientific knowledge is directed toward *unity* in the mass of multiple forms. In geometry, this quest has achieved great success in the last hundred years. *Duality, projective* ways of thinking and *transmission principles, dualization* and *homogenization* of coordinates are its landmarks. In V.C. Poncelet [sic], J.D. Gergonne, A.F. Mobius and J. Plücker we may see the teachers, who led us in the first place.

[57] "The mighty progress that the study of the geometric structure owes to the production methods of J. Steiner is based mainly on two circumstances. First the concepts and methods created by him presented the merits of being directly *constructive*, accessible to intuition, and therefore geometrical in the best sense, secondly, they also proved open to the broadest generalization."

[58] We will explore the constructive nature of geometry, with respect to figures, in Chapter II.

The German article on projective geometry was also adapted into French. Like Carrus, the French mathematician Arthur Tresse also emphasized the importance of Chasles in the 1913 "Géométrie projective," especially in footnotes (Tresse and Schoenflies (1913)). Tresse also cited analytic developments in the text with greater prominence than Schoenflies had. Writing on alternative versions of the principle of duality presented by Gergonne and Poncelet, Tresse noted that the principle was established by the analytic methods of Möbius, Plücker, and Chasles.

> Ce point devait d'ailleurs être établi quelques années plus tard, par des méthodes analytiques basées sur l'analogie des représentations analytiques du point et de la droite, par A. F. Möbius et par J. Plücker en Allemagne, par M. Chasles en France. (Tresse and Schoenflies (1913), 14–15)

This quote is somewhat surprising compared to our other sources, as Tresse seemed to suggest that Chasles had also engaged in analytic geometry.

Tresse also emphasized Chasles contributions to the methods of projective geometry, thus substantially modifying Schoenflies' praise of Steiner.

> La découverte par J. Steiner des méthodes projectives de génération des figures géométriques marque une date capitale dans le développement de la géométrie projective. Vulgarisées presque aussitôt, grâce aux travaux de M. Chasles, qui ne connaissait qu'incomplètement ceux de J. Steiner, ces méthodes prirent immédiatement une place prépondérante dans les recherches géométriques. (ibid, 41)

Tresse interjected the contributions of Chasles throughout the history of projective geometry, and so rendered the subject as an outgrowth of the single French geometer's studies. Thus Tresse's interpretation appeared more homogeneous with respect to participants and contributors than Schoenflies's original text.

1.5 Histories of mathematics

1.5.1 Florian Cajori (1896)

Florian Cajori (1859–1930) was an American professor of applied mathematics and physics, best know for his numerous texts on the history of mathematics. He wrote *A History of Mathematics* for an audience of "teachers and students" (Cajori (1893)). Beginning with Babylonian mathematics and extending up to the end of the nineteenth century and the research of his contemporaries, Cajori organized his text under broad chronological headings.

The final section, "Recent Times," included seven subject-based chapters, the first two of which were "Synthetic Geometry" and "Analytic Geometry." Cajori opened his discussion of synthetic geometry by declaring an end to the "conflict between geometry and analysis."

> The conflict between geometry and analysis which arose near the close of the last century and the beginning of the present has now come to an end. Neither side has come out victorious. The greatest strength is found to lie, not in the suppression of either, but in the friendly rivalry between the two, and in the stimulating influence of the one upon the other. (Cajori (1893), 341)

The rivalry Cajori described was not inside geometry, but within mathematics in general. In Cajori's assessment, the conflict was friendly and stimulating, and by the late nineteenth century had ended in a draw. Cajori seemed to suggest that the divide between geometry and analysis served as an impetus for the creation of "modern synthetic geometry" in this same time period.

Cajori characterized modern synthetic geometry by its investigative goal, "a desire for general methods which should serve as threads of Ariadne to guide the student throughout he labyrinth of theorems, corollaries, porisms, and problems." As developers and practitioners of synthetic geometry, he cited first Monge, Carnot, and Poncelet in France; then Möbius and Steiner in Germany; and finally Chasles, von Staudt, and Luigi Cremona in their respective countries. In a brief biography on Steiner, Cajori quoted his title as "'the greatest geometrician since the time of Euclid"' (ibid, 343). His account of Steiner's life accorded with this positive assessment: "in his hands synthetic geometry made prodigious progress." Cajori also emphasized Steiner's exclusivity: "Steiner's researches are confined to synthetic geometry. He hated analysis as thoroughly as Lagrange disliked geometry." (ibid, 345). Although, Cajori provided careful references in his summary of Steiner's mathematical contributions, this comment was left uncited. Cajori bestowed equal credit to Chasles and Steiner for the elaboration of "modern synthetic or projective geometry." Although Chasles was not categorized as a participant in the "conflict," Cajori continued to compare his work in synthetic geometry with analysis. "The labours of Chasles and Steiner raised synthetic geometry to an honoured and respected position by the side of analysis." (ibid 347). Cajori's description of synthetic geometry continued to the end of the nineteenth century and encompassed non-Euclidean geometry. Steiner served as the lone example of overtly expressing methodological preference.

Cajori directly spoke to the difference between modern synthetic and analytic geometry in his introduction to the following chapter on "Analytic Geometry."

> Modern synthetic and modern analytical geometry have much in common, and may be grouped together under the common name "projective geometry." Each has advantages over the other. The continual direct viewing of figures as existing

> in space adds exceptional charm to the study of the former, but the latter has the advantage in this, that a well-established routine in a certain degree may outrun thought itself, and thereby aid original research. While in Germany Steiner and von Staudt developed synthetic geometry, Plücker laid the foundation of modern analytic geometry. (ibid, 358)

Although Steiner and Plücker were positioned on either side, the methods they practiced were not presented as antagonistic. As noted in our introduction, Cajori, citing Dronke, repeated the allegations against Steiner with respect to Plücker and Crelle's *Journal.* Cajori situated this episode with respect to a general German preference for synthetic methods, "in Germany Plücker's research met with no favour. His method was declared to be unproductive as compared with the synthetic method of Steiner and Poncelet!" Cajori added succinctly that Plücker's "relations with Jacobi were not altogether friendly" (ibid, 359).

Cajori thus exhibited changing interpretations of a methodological divide within or involving geometry. Although he stated that both synthetic and analytic geometry could be commonly considered "projective," in describing the distinct advantages of each he emphasized their differences, which was reinforced by their separation into individual chapters. Cajori did not describe a conflict or rivalry within geometry, but instead described a conflict between geometry and analysis, citing the *Mécanique Analytique* of Lagrange, by way of introducing the subject of synthetic geometry. The inclusion of Lagrange within a chapter on early nineteenth century geometry is striking and perhaps points to a lack of consensus regarding the boundaries of analysis and geometry, or analytic geometry and synthetic geometry.

1.5.2 Johannes Tropfke (1903)

Johannes Tropfke (1866–1939), a gymnasium teacher in Berlin, began publishing his seven volume *Geschichte der Elementar-Mathematik in systematischer Darstellung* in 1902, which he continued to re-edit and republish until 1940 (Tropfke (1903)). Each volume was devoted to a different set of subjects, with the second, containing geometry, appearing in 1903.

This volume included a wide range of chapters on geometry, logarithms, planar trigonometry, spherical trigonometry, series, compound interest, combinatorics, calculating probability, continued fractions, stereotomy, analytic geometry, conic sections, and maximums and minimums–in that order. His discussion of "newer synthetic geometry" was in the first chapter on simply "Geometrie" in the section on teaching similarity. Parenthetically, Tropfke associated new synthetic geometry with projective geometry and geometry of position. Here he summarized the progressive and collective contributions of Monge, Carnot, Poncelet, Steiner, Plücker, von Staudt, and Möbius. Without methodological distinctions, both Steiner and Plücker appeared on the same side of this new geometry. According to

Tropfke, modern geometry began in France with the work of Monge, Carnot, and Poncelet. Referring to the latter, he noted:

> In eigenen Lande wenig anerkannt, waren seine Untersuchungen von grösstem Einfluss auf eine deutsche Schule, zu deren Vorkämpfter sich Jakob Steiner (1796–1863, Berlin) aufschwang, jener hochbegabte Mathematiker, der sich aus einfachsten Verhältnissen [...] zu den höchsten Gipfeln der Mathematik hindurchgerungen hatte. Ihm zur Seite stehen Plücker (1801–1868, Bonn), v. Staudt (1798–1867, Erlangen), Moebius (1790–1868, Leipzig) und viele andere. (Tropfke (1903), 94)[59]

Here, Tropfke draws a very harmonious image of geometric progress, without methodological distinction between geometers nor the content of synthetic and projective geometry.

In his later chapter on analytic geometry, Tropfke attributed the first "elementary-systematic treatment of analytic geometry" in modern textbooks to Meier Hirsch and pointed to the further work of August Crelle in this area. No other works of nineteenth century analytic geometry were mentioned in this chapter. As we can see from these cursory treatments. The subject of early nineteenth century geometry did not occupy a large portion of these two general histories of mathematics, which would perpetuate through later examples of this genre.

1.6 Julian Lowell Coolidge (1940)

Julian Lowell Coolidge (1873–1954) had obtained his PhD on the subject of projective geometry at the University of Bonn under Eduard Study. Through the early twentieth century he wrote several textbooks on geometry while teaching as a professor at Harvard University. Coolidge provided several interchangeable accounts of nineteenth century synthetic and analytic geometry in his books and articles, which covered various aspects of the history of geometry. Coolidge divided his first historical monograph, *A History of Geometrical Methods* into three "books": Synthetic Geometry, Algebraic Geometry, and Differential Geometry (Coolidge (1940)). Coolidge included his discussion of "analytic geometry" within the book on Algebraic Geometry and in general used the terms interchangeably, while "Differential Geometry" focused on the application of calculus to geometry.

Coolidge began the first book with a discussion of geometry "in the Animal Kingdom," and the first three chapters progressed chronologically from there. The latter three chap-

[59] "Little acknowledged in his own country, his [Poncelet's] investigations were of the greatest influence on a German school, of which Jakob Steiner (1796–1863, Berlin) rose to be the champion, a gifted mathematician who from humble circumstances struggled through [...] to the highest peaks of mathematics. At his side stood Plücker (1801–1868, Bonn), v. Staudt (1798–1867, Erlangen), Moebius (1790–1868, Leipzig) and many others."

ters were subject specific: non-euclidean geometries, projective geometry, and descriptive geometry. Coolidge designated the nineteenth century as the "great period" of projective geometry. In this section he praised the advances of Poncelet, Chasles, Steiner, and Von Staudt in formulating general, ingenious, and deep principles. However, when explaining the details of these geometers, he admittedly put "the thing analytically," by setting up coordinate axes and clarifying that "the geometric treatment is very different" (Coolidge (1940), 93, 95). As a further authorial interjection, Coolidge was judgmental of many of his historical subjects, denigrating Poncelet's definition of imaginary points and failure to find the invariant cross-ratio. Like Klein, Coolidge tempered his esteem for Steiner in light of Von Staudt's work.

> The most characteristic feature of Steiner's work is that given by the adjective in its title 'systematisch'. He has a consistent uniform method for treating a variety of figures, and he handles it beautifully. He is usually considered as the greatest of the German school of projective geometers, but this seems to me entirely incorrect. In originality and depth he falls far below his great successor, Johann Karl Christian Von Staudt. (ibid, 97)

Moreover, Coolidge concluded, even Von Staudt's "revolutionary" and "remarkable" contributions could not match the simplicity and adaptability of analytic geometry, and the section on projective geometry ended on a fairly final note that the subject has been exhausted.

Coolidge returned to these same developments in describing the "Extension of the system of linear coordinates," his second chapter on algebraic geometry. During the early nineteenth century, the progress of Poncelet, Chasles and Steiner had inspired "algebraic geometers" such as Lamé, Bobillier, Gergonne, and above all Plücker. Coolidge described Plücker's exclusive dedication to algebraic methods in geometry.

> He had an unshakable belief that for most purposes, algebraic methods were infinitely preferable to the purely geometric ones recently brought into fashion by Poncelet and Steiner. He went a good way towards proving the correctness of his belief. (ibid, 144)

Coolidge appeared to agree with Plücker's preference for "algebraic methods". By the 1940s, he saw little opportunity for new developments in the field of synthetic geometry, although he professed to admire the subject greatly.

> Until and unless, some totally new principle is discovered, the subject of synthetic projective geometry is not to-day a fruitful field for original research. (ibid, 105)

Coolidge argued that synthetic geometry should still be taught, for its "beauty" and "invaluable insight into the inner significance of geometrical science." Even this well-intentioned advocacy suggested a lost cause, which was reinforced by Coolidge's frequent use of coordinate equations in representing historical "synthetic" geometry.[60]

1.7 Conclusion

While each of the above texts inform a history of early nineteenth century geometry, they do not coincide on a single image or set of characteristics. Instead we find many different accounts of methodological opposition. Nineteenth century geometry has been classified as uniformly projective, as divided between analytic or synthetic, as transitioning from ancient to modern, and as an evolving subject seeking to rival analysis. The lines of conflict shift between narratives, as do the participants and qualities.

Even when early nineteenth century geometry was portrayed as divided between analytic and synthetic methods, historical accounts emphasized different attributes of each method. Texts dedicated to particular aspects of geometry sometimes provided definitions or descriptions of the actual difference between the two geometries. Klein, Kötter, Darboux, Fano, and Schoenflies point to the particular, directly constructive, figure-based, intuitive, or evident nature of synthetic geometry or projective geometry. Analytic geometry or analysis were described as the progressive, general, uniform, calculation-based, and fruitful. Some of these descriptions contradicted the biographies of geometers that emphasized the individual qualities of the mathematician, and not his method. For instance, the research of both Plücker and Steiner was seen as both intuitive and general, even though the former practiced analytic and the latter synthetic geometry. These geometers were thus both emblematic and problematic in portraying two sides of a geometry. A deeper inquiry into their mathematics and surrounding rhetoric would help explain why they became representative of two methods in mathematics.

In attempting to answer who were the main actors in early nineteenth century geometry, we discovered a varying list. Among numerous cited geometers, Poncelet, Plücker and Steiner were repeatedly listed as key figures in the development of geometry. Further, only Plücker and Steiner served as examples of explicit conflict between individuals, their alleged personal dispute sometimes overshadowing any more general methodological issues. The emphasis on Steiner and Plücker may be in part due to a national bias in our corpus. With the influence of Klein's *Encyklopädie* project that is only slightly mitigated by examining the

[60] Contemporary articles in the *American Mathematical Monthly*, echoed Coolidge's regret over the present condition of synthetic geometry. These include titles such as "The Rise and Fall of Projective Geometry" (Coolidge (1934)), "The Neglected Synthetic Approach" (Kelly (1948)), "A Note on Synthetic Projective Geometry" (Simons (1914)), and "Synthetic Projective Geometry as an Undergraduate Study" (Bussey (1913)).

French versions and the independent text of Darboux, the common chronological narrative generally related how modern geometry began in France and then migrated to Germany. We find this same story in the accounts of Clebsch, Dronke, Ernst, Klein, Loria, Kötter, Fano, Cajori, and Tropfke. The French version of the same events placed greater emphasis on the simultaneous and independent research of Chasles, thus rejecting the suggestion that geometry research had moved exclusively to Germany or that Chasles followed German geometers. Klein and Fano both suggested that the dissemination of French geometry also carried the methodological debate to Germany. However, none of these texts explained how this transfer of methodological preferences might have been communicated and adopted.

The methods of Poncelet, Steiner, Plücker and others were often described as modern, thus emphasizing that they broke with an older geometric tradition and initiated a set of practices that could be applied in contemporary research. Mitigating the innovations in methods and theories documented in the historiography of early nineteenth century geometry, we find geometers continually arrived at the same results. Each text in our corpus exhibits how geometers repeatedly republished new solutions to solved problems or new versions of theories that had already appeared elsewhere. This aspect of the historiography is perhaps most apparent amidst the repetition of Simon's text. In more narrative texts in our corpus, the lack of new results appeared to lead to priority disputes and unequal distribution of credit. This may be one possible source for the later reputation for methodological hostilities. In order to better understand what differentiated two methods, this repetition suggests that a direct comparison could be possible within the context of a single geometry problem. We turn to such an investigation in the following chapter.

Chapter 2

The role of the figure: a case study in methodological differences

2.1 Introduction

In 1817, Jean Victor Poncelet wrote a letter to the editor of the *Annales*, Joseph-Diez Gergonne, on the use of "algebraic analysis" in geometry (Poncelet (1817c)).[1] He reiterated these remarks when introducing his *Traité des propriétés projectives des figures* in 1822 and explaining the practice of ordinary geometry,

> la figure est décrite, jamais on ne la perd de vue, toujours on raisonne sur des grandeurs, des formes réelles et existantes, et jamais on ne tire de conséquences qui ne puissent se peindre, à l'imagination ou à la vue, par des objets sensibles; on s'arrête dès que ces objets cessent d'avoir une existence positive et absolue, une existence physique. (Poncelet (1822), xxi)[2]

In this geometry, the figure–a real and existent form composed by sensible objects with positive and absolute physical existence–was central. Moreover, this geometry was a visual, tangible practice in which the figure was never lost from view. Poncelet continued by criticizing the severe rigour of such a "restrained" geometry where one must repeat a proof based on whether a given point is to the right or left of a line, and proceeded to promote a new, modern geometry. However, the focus of study in Poncelet's geometry (as his title indicated) continued to be the figure.

[1]We simply quote Poncelet's use of "algebraic analysis" here. On the changing meanings of algebraic analysis from antiquity through the nineteenth century, see the articles in Jahnke (2003).

[2]"...the figure is described, one never loses it from view, one always reasons about magnitudes, real and existent forms, and never reaches consequences that cannot be painted in the imagination or in sight, by sensible objects; one stops when these objects no longer have a positive and absolute existence, a physical existence."

In the preceding chapter, we have seen that although the opposition between analytic and synthetic geometry was often emphasized by writers on the history of geometry, how to characterize this difference was inconsistent, especially with respect to particular geometers or general methods. Further, analytic and synthetic was not the only operational division applied to historical geometry, which was also differentiated as pure, projective, ancient or modern. One argument, put forward in the texts of Felix Klein, Ernst Kötter, and Gino Fano, rested on the centrality of the figure in synthetic geometry as opposed to analytic geometry. As Poncelet demonstrated, the use of figures to draw boundaries between methods was also used by early nineteenth century geometers.

The word *figure* itself, did not have a fixed meaning in Poncelet's *Traité*. On the most concrete level, the term *figure* signified the numbered and labelled two-dimensional illustrations often accompanying geometric constructions and definitions, but these particular figures occupied only a fraction of Poncelet's researches. Poncelet discussed "a certain general arrangement of the objects of a figure" as well as figures composed of graphic magnitudes [les grandeurs graphiques] (Poncelet (1822), xii)). The connection to objects and magnitudes suggest that figures were positional and drawable. However, figures extended well beyond the page. They could be three dimensional, could actively move or deform, could be projected (even projected to infinity), and could contain imaginary points. Figures could be indeterminate or of a particular type, could be primitive or correlative. Reasoning, quantities, expressions, and notions could be figured, as we will see below. Fundamentally, by characterizing ordinary or pure geometry as figure-based, Poncelet intended to contrast it with the equations and calculations of analytic geometry where the figure disappeared. This designation may have been drawn from Lazare Carnot (1753–1823), as Poncelet had read his 1803 *Géométrie de position*, where Carnot claimed that synthesis

> [...] est restreinte par la nature de ses procédés; elle ne peut jamais perdre de vue son objet: il faut que cet objet s'offre toujours à l'esprit réel et net, ainsi que tous les rapprochemens et combinaisons qu'on en fait. Elle ne peut donc employer des formules implicites, raisonner sur des quantités absurdes, sur des opérations non exécutables: elle peut bien faire usage des signes, pour aider l'imagination et la mémoire; mais ces signes ne peuvent jamais être pour elle que de simples abréviations. (Carnot (1803), 9)[3]

[3] "[...] is restrained by the nature of its procedures; it can never lose sight of its object: it is necessary that this object always presents itself truly and clearly to the mind, as well all the relationships and combinations made from it. It cannot therefore use implicit formulas, nor reason about absurd quantities, about non-executable operations: it may well make use of signs to aid imagination and memory, but these signs can never be more than simple abbreviations."

Carnot drew a contrast between synthesis and analysis, which Poncelet applied particularly to geometry. Thus the figure was the object of geometry.[4]

In all its guises, Poncelet presented the figure as the primary form of geometrical evidence. Following his use of the word *évidence* and *évident*, most prominently in his *Traité*, we take "evidence" as a means of justification based in sensory perception or as a description of clear mathematics. Poncelet used these terms complementarily, for instance the coincidence of two points was a constructive procedure that could be clearly seen (Poncelet (1822), 31). Even when this evidence was not supplied by concrete illustrations of figures, the objects of geometry were emphatically representational and tangible. Thus, according to Poncelet, the convolutions of computations found in analytic geometry were hardly bearers of evidence. Working from Poncelet's division between pure and analytic geometries, in this chapter we explore how the figure-based distinction between geometric methods materialized in contemporary geometric practices, enabling us to witness how methodological decisions effected the mathematical content and conversely. Further, we will observe other mediums through which geometers conveyed geometric evidence and under what circumstances the figure was lost from view. Finally, by focusing on the figure we obtain a perspective from which to examine a second division in geometry, between ancient and modern methods. As Kötter suggested in his historical analysis, the particular nature of traditional geometry proved a challenge to modern synthetic geometers who strove for greater generality. This chapter will then also consider the compatibility between figures and generality in geometry.

2.2 Five solutions to the problem of osculating curves: a case study in geometric methods

Early nineteenth century geometers professed that the choice of geometric method (and consequently the form of geometric evidence) should depend on the particular problem at hand (Dupin (1813), Poncelet (1817c), Gergonne (1817e)). In contrast to the opinion often expressed by historians in Chapter I, early nineteenth century French geometers often claimed that multiple methods should be cultivated to best answer a variety of geometry problems. We will spend more time on such "philosophical" arguments about geometric methodology in Chapters III and IV. In Chapter I, we found that geometers using new methods often revisited the same geometric problems. It is thus particularly interesting to focus on such a case in order to ascertain what was involved in the different methodological approaches, particularly with respect to a division of geometry. Here we will focus on a single geometric problem–to construct a second order curve sharing a third order contact with a

[4]The common aspirations and similar methods of Carnot and Poncelet have been described by Karine Chemla and Philippe Nabonnand respectively in Chemla (1998) and Nabonnand (2011b).

given planar curve–whose multiple solutions included conic sections, tangents, secants, and points of intersection constructed with the ruler alone. We were led to this choice of problem by Poncelet himself. In a methodologically driven 1817 article published in the *Annales des mathématiques pures et appliquées*, Poncelet included this problem and his solution without proof as an example of the simplicity and elegance of pure geometry (Poncelet (1817c)). His first proof appeared in his *Traité* five years later (Poncelet (1822)). Unaware of the contents of the latter publication, Plücker gave an analytic proof for the problem's solution in an article submitted to the *Annales* in 1826. The published version appeared in two articles under the name "Pluker", dramatically altered by the journal's editor, Gergonne, with a new non-analytic proof. The original remained unpublished until 1904 (Plücker (1826b), Plücker (1826a), Plücker and Schoenflies (1904)). These five texts intersected when Poncelet charged Plücker with plagiarism in 1827 (Poncelet (1827a), Poncelet (1827b), and Poncelet (1828c)). The resulting confusion and controversy over this editorial indiscretion will be discussed at length in Chapter III.

In situating the mathematics, the significance of the fact that the same publishing venue is shared by all but one of our texts can hardly be overestimated. As we indicated in our introduction, Mario Otero's recent biography of Gergonne provides an insightful view of the development of Gergonne's *Annales*, which has been further supplemented by the dissertation and articles of Christian Gérini (Otero (1997), Gérini (2010a), Gérini (2010b)). Not only did Gergonne enjoy broad editorial liberties as in the case of Plücker's article, he also commented frequently on many publications through the copious use of footnotes and subsequent critiques. His especial interference with the work of Plücker and Steiner will be further revealed in Chapter IV. Gergonne's own research included conic sections and he further promoted this area of study among contributors by the inclusion of posed problems. Finally, Gergonne gave every article in his journal at least one subject heading categorized by discipline, method, content, or all of the above. This subjective labelling could create new liaisons than those suggested by the article's particular results.

We will begin in Section 2.3 with Poncelet's article, "Réflexions sur l'usage de l'analise algébrique dans la géométrie; suivies de la solution de quelques problèmes dépendant de la géométrie de la règle" (Poncelet (1817b)). Here Poncelet advocated the use of what he called "modern pure geometry" as simpler and more elegant than "analytic geometry" and more general than "ancient pure geometry." This claim to modernity in pure geometry appears to have originated with Poncelet, and we will compare his connotation of modern with that of his contemporaries in Chapter V. He supported this argument with several unproved problem solutions and theorems, including the problem of third order contact. In his article, we will focus on Poncelet's designation of three geometries and use of the figure as the key to categorization.

Poncelet's proposed dichotomy between modern pure and analytic geometry motivated

Julius Plücker's 1826 text.[5] In Section 2.4, we turn to Plücker's original manuscript. Through examining Plücker's proofs and solutions we will argue that Plücker used coordinate equations as visual geometric objects–evidence–by focusing on their form and endeavouring to avoid calculations. This was Plücker's first geometric manuscript, and foreshadowed Plücker's consistent focus on developing innovative "forms" of geometry, often at the expense of generating new results. Though classified as analytic geometry, Plücker's work maintained a visual attention to geometric evidence. We will consider how the relationship between evidence and existence shifted between the contexts of proving theorems versus solving problems. In the latter situation, finding geometric solutions implied constructing geometric objects and the imaginary was dismissed out of hand. In the former, coordinate equations could serve as representation.

Though chronologically prior, Poncelet's *Traité* was unknown to Plücker. The text represents a dramatic shift in pure geometry that would eventually inspire Plücker to reshape analytic geometry. Poncelet claimed his method never lost sight of the figure, but there was no attempt to provide a literal representation of imaginary objects. Instead, we will witness how one could "view" imaginary points through constructing the real lines that contained them. Poncelet could then define and manipulate points and lines at infinity without providing a pictorial correspondence. Sometimes Poncelet visually indicated the points at infinity as the infinitely distant intersection points of parallel lines, shown in Figure 2.1.

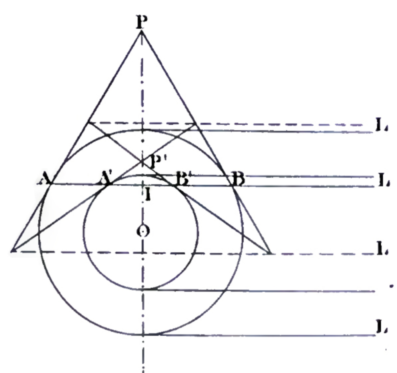

Figure 2.1: Points at infinity indicated off the page (Poncelet (1865), Planche VI)

In Section 2.6 we will review the two articles ascribed to "Pluker," which we consider as a unilateral collaboration between Plücker and Gergonne. In these texts, the results remained the same, but Plücker's attention to analytic form was completely lost. We will demonstrate

[5] In its published version, Schoenflies titles Plücker's work as "Die an Gergonne gesandte Abhandlung" (Plücker and Schoenflies (1904)). While the existence of this posthumously published text is well-documented, its contents have been little studied and even misinterpreted. For instance, Carl Boyer suggested that Plücker originally pursued pure geometry, only turning to analytic geometry as a result of Poncelet's negative assessment (Boyer (1956)).

how these changes had a profound effect on the mathematical content. Plücker's original treatment, conceived as providing evidence in favour of analytic geometry, was rewritten in a particular, figure-based style with the use of two parallel columns denoting corresponding left and right content. This latter feature was Gergonne's stylistic invention for portraying duality (Klein (1926a), Chemla (1989), Otero (1997), Gray (2010b)). We will consider the implications caused by Gergonne rewriting Plücker in this new form for the published versions of his articles.

While the form of evidence varied from text to text, we will see that Poncelet, Plücker, and Gergonne argued their results within a visually sensitive geometry. Poncelet's insistence on keeping the figure in view shows the significance of vision in geometry. However, Poncelet's 1817 text contained no illustrated figures. Nevertheless, in the geometric constructions of Poncelet, Plücker, and Gergonne a figure would be described and manipulated with enough unambiguous detail for the reader to construct their own illustration. As we explained in our general Introduction, Dominique Tournès has described a "virtual diagram" as a diagram "that one must have in mind, but that is no longer physically drawn on the paper, or at least which is left to the reader to draw" (Tournès (2012), 272). Further, Tournès has shown that the use of virtual diagrams "is frequent in the analytical period after 1750: mathematicians commonly evoke geometrical diagrams with words rather than drawing them." Similarly, we will denote these described but absent figures as virtual figures. We will see that Poncelet, Plücker, and Gergonne actively referred to virtual figures as if they were available to the reader. The difference is subtle. In the written text the use of actual versus virtual figures might only be distinguished by a parenthetical figure enumeration, e.g. *(fig 1)*, and the actual figure was never embedded in the body of the text. In the case of Gergonne's *Annales* any figures were presented at the end of the issue and in Poncelet's *Traité* the figures followed the entirety of the text. So the actual figure could only be viewed by actively flipping back and forth between the content and the illustration.

That actual accompanying figures were desirable is attested to by their proliferation in journals and monographs. For instance, Poncelet's *Traité* contains 104 figures, and the general prevalence of figures in books on geometry will be analyzed in Chapter V. Further, the number of figures included in each text was advertised in their catalog listings put out by publishers as well as subsequent text reviews (for example, in Férussac's *Bulletin*). On the other hand, as the articles considered here show, many mathematical texts contained no figures. Even for authors, such as Poncelet, who argued on behalf of the figure, the fact that not every geometric object needed to be figured permitted including geometric objects that might not be figurable. We will see how Poncelet, Gergonne, and Plücker used this allowance to speak of infinite and imaginary points and lines as if they were on the page. The virtual figure was both more and less than its actual counterpart.[6] Attention

[6]Our differentiation between virtual and actual figures, has occasionally been side-stepped in recent stud-

to the figure (both illustrated and virtual) reinforced different forms of visualization as a means towards securing evidence. These extended beyond geometric constructions to the appearance of the text itself as a visual medium.

We will also attend to different forms of visualization as a means towards supplying evidence. In particular, Poncelet described the use of a *tableau artificiel* in his 1817 article– an illustratable device for counting ordered points.[7] Further, Gergonne presented articles in dual columns to visually reinforce the correspondence between results on the left and right sides.

In Poncelet's *Traité*, he gave new meaning to common geometrical objects including points of intersection and common chords. Poncelet also referred to a single object by multiple names, emphasizing at once its myriad properties (Poncelet (1822), 155). Though not in the texts considered here, attention to proper vocabulary was important to Gergonne and Plücker when introducing new geometric methods (Gergonne (1826), Gergonne (1827f), Plücker (1828a)). Of course, a word is not a strictly visual object. Nevertheless, words comprising mathematical works were intended to be seen rather than heard, and word choice carried visual impact. In particular, we will see how Poncelet's use of synonyms reinforced the relationships between his figures. Though neither Poncelet's counting strategy and attention to language nor Gergonne's use of dual columns were strictly geometric, we contend that each of these factors contributed to the visual atmosphere and the mathematics conveyed.

Despite the time and geography separating the mathematical educations of Gergonne, Poncelet, and Plücker, they shared a background in constructive geometric practices. Because of their common research interests in geometry, they could be characterized as participants in what was known even at the time as *"l'École de Monge,"* though not even Poncelet, who was a student at the École polytechnique, actually attended Monge's lectures.[8] Though

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ies by denoting depicted figures as "diagrams" (Tournès (2012), Netz (1999), Mancosu *et al.* (2005)). Since this terminology is absent from early nineteenth century geometry texts, we will adhere to contemporary usage of "figure," following Chemla (2011), Peiffer (2006) and Decorps-Foulquier (1999). When necessary to distinguish illustrated figures, we will attempt to be as clear as possible through the use of adjectives. Finally, we will use the phrase "geometric object" to refer to the sorts of things that geometers studied, which in some contexts may be figures.

[7]This strange phrase occurs uniquely to describe this very particular problem solution in the whole of Poncelet's published geometric research. Furthermore, he did not use *tableau* in any other (real?) sense nor *artificiel* to describe any other objects. Contemporary uses of *artificiel* lead us to an 1810 article by Brianchon in the *Journal de l'École polytechnique* in which he considered *calcul* in geometry as *la méthode artificielle des coordonnées* (p. 6). Otherwise, the descriptor can be found rather frequently in chemistry or engineering articles within the same publication venues alluding to a crafted device.

[8]Karen Parshall provides a list of characteristics defining a mathematical research school in Parshall (2004).

> [...] Second, that leader advocates a fundamental idea or approach to some set of inherently related research interests or research interests that become related by virtue of the idea or approach. Third, the leader trains students and, in so doing, imbues them with a sense not only of the validity and fruitfulness of the approach but also of the "right" way to go

not even Poncelet, who was a student at the École polytechnique, actually attended Monge's lectures, all geometers situated themselves with respect to this research and practical tradition. To give one example, Plücker described his *Analytisch-geometrische Entwicklungen* (1828) as pure analytic geometry in the Mongean sense: "Die von mir aufgestellte und durchgeführte Behandlungsweise ist eine rein analytische, in demjenigen Sinne des Wortes, in welchem man dasselbe seit Monge nimmt" (Plücker (1828a), iii).[9] In Poncelet's association of evidence with geometry, he also followed Monge, who had accordingly differentiated the qualities of geometry and analysis in his *Géométrie Descriptive* (1798):

> Il serait à désirer que ces deux sciences fussent cultivées ensemble: la Géométrie descriptive porterait dans les opérations analytiques les plus compliquées l'évidence qui est son caractère, et, à son tour, l'Analyse porterait dans la Géométrie la généralité qui lui est propre. (Monge (1798), 18)[10]

As Monge had succeeded in descriptive geometry, so too Poncelet strived to incorporate greater generality without compromising the evidence of geometry.

The appropriation of Monge's geometrical teaching and reputation is well documented in the historical literature, see for instance Taton (1951), Belhoste and Taton (1992), Sakarovitch (2005), and Laurentin (2007). These studies show that Monge's school was both more and less than his actual former students. Above all the students of Monge were practicing geometers. For example, Belhoste describes the principal role of geometric intuition for Monge and his school.

> Le premier est le rôle primordial accordé à l'intuition géométrique. Monge dit-on, faisait de la géométrie avec les mains. Au contraire de la géométrie synthétique des Anciens, les démonstrations de "la géométrie générale et rationnelle" que pratiquait l'École de Monge s'appliquent en effet à des représentations sans figures d'objets fictifs, généralement dans l'espace, comme des lignes, des plans et des surfaces illimités, disposés de façon arbitraire. (Belhoste (1998), 6)[11]

> about asking and answering questions; explicit and tacit knowledge are conveyed through the education process. [...] Fourth, the publication of the research not only represents recognition of the research done but also comes to reflect the external validation of the approach. This external validation may result in the extension of the ideas and approach by other researchers nationally and internationally. (Parshall (2004), 274)

As the secondary literature indicates, the school of Monge fits these criteria (Taton (1951), Belhoste and Taton (1992), Sakarovitch (2005), and Laurentin (2007)). The notion of a mathematical research school has also been discussed by Parshall and David Rowe in Parshall and Rowe (1994).

[9] "The method of treatment that I have established and carried out is a purely analytical one, in that sense of the word that has been taken since Monge."

[10] "It is desirable that these two sciences be cultivated together: descriptive geometry brings its characteristic evidence to the most complicated analytic operations, and in turn, analysis brings to geometry the generality that is proper to it."

[11] "First is the primordial role accorded to geometric intuition. Monge they say, did geometry with his

We note in particular, that Gergonne, Poncelet, and Plücker were well versed in creating and reading geometric figures. Even so, the *épineuse* or *hérissée* nature of geometric problem solving was criticized by early nineteenth century geometers, including Poncelet, as necessitating ingenious approaches to address all possible exceptional cases (Lacroix (1805), Poncelet (1865), Otero (1997), Nabonnand (2011b)). In his widely disseminated textbook on teaching mathematics, Lacroix advised perfecting geometric practices through habituation to the techniques, rather than rote memorization.

> ...je dis autant qu'il est possible, car il serait ridicule, et quelquefois même dangereux, de vouloir rendre raison de tous les artifices que les géomètres ont employés dans leurs recherches. Des yeux dirigés par une longue habitude de ce genre de spéculations, aperçoivent dans une figure, dans un calcul, des circonstances, prévoient dans une opération à effectuer, des effets, qu'on ne peut jamais rendre sensibles à un commençant. (Lacroix (1805), 200)[12]

Later in the same text, Lacroix compared the mathematician's habituation to a mechanization of operations–for geometry, constructions.

> Il y a dans chaque science des choses qui ne peuvent s'enseigner, et que l'élève doit acquérir par lui-même; c'est l'habitude des procédés de la science, ou autrement le mécanisme des opérations qu'elle prescrit: en arithmétique et en algèbre ce sont des calculs, en géométrie, des constructions. (Lacroix (1805), 207)[13]

Habituation was obtained only through practice, which was further enforced in Lacroix's textbook on trigonometry and analytic geometry, where he recommended further literature to students in need of additional practice problems. These, and other elementary geometry texts, will form the subject of Chapter V.

Conic sections were defined by Poncelet in the sense of Apollonius as planar sections of a three-dimensional cone with a circular base (Poncelet (1822), 4). These circles, ellipses, parabolas and hyperbolas were also collectively referred to as conics, curves, and lines of

hands. Contrary to the synthetic geometry of the Ancients, the proofs of the "general and rational geometry" that the school of Monge do indeed apply to representations without figures of general fictive objects, generally in space, like unbounded lines, planes and surfaces positioned arbitrarily."

[12] "...I claim that though it is possible, it would be ridiculous, and sometimes even dangerous, to memorize all the artifices that geometers have employed in their researches. Eyes directed by a long habituation to this kind of speculation perceive in a figure, in a calculation, circumstances that foreshadow an operation to perform, effects that one can never render sensible to a beginner."

[13] "In each science there are things which cannot be taught, and that the student must acquire by himself; this is the habit of the procedures of science, or otherwise the mechanism of the operations it prescribes: in arithmetic and in algebra these are calculations, in geometry, constructions."

second order or degree.[14] We will employ the author's original usage in each case, but preference of term appears to have been independent of choice of method. Through proof or definition, early nineteenth century geometers showed that two conic sections had at most four points of intersection.

Two conics with a third order contact–sharing a common tangent at a real point with no other points of intersection–were defined as osculating curves and the study of contact points of higher order was thus called the theory of osculation (shown in Figure 2.2).

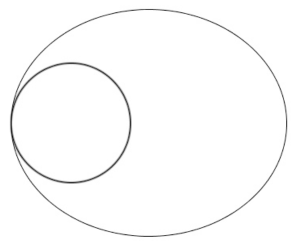

Figure 2.2: Two conics with a third order contact

From an analytic perspective, the order of contact between two second order curves could be found through successive differentiation at their intersection points.[15]

In this chapter, as well as our entire dissertation, we endeavour to clearly illustrate the geometric processes imbedded in early nineteenth century French mathematics. The reader is encouraged to practice this geometry through participation in constructions or sketches. However, arguments described as simple, easy, elegant, or evident may no longer seem as such from a contemporary perspective. We cannot fully bridge the distance separating our experiences from those of the original mathematicians, and so we must rely upon their own definitions, often implicit and varying from author to author.[16] With this in mind, we intend to consider ways of perceiving geometrical evidence and what kinds of geometrical evidence that could be perceived.

[14]We will see the importance of *order* for the geometric classifications of Poncelet, Plücker, and Gergonne. Simultaneously, a quite different concept of *order* with respect to position was developed and discussed most prominently by Louis Poinsot , see Jenny Boucard (Boucard (2011)).

[15]For details on the history of finding contacts between two oblique curves or two second order surfaces, J. Delcourt (2011a) compares the use of a polygon derived geometric method to a differential geometry method from the early eighteenth through early twentieth centuries.

[16]As a brief example, drawing from Poncelet's use of simple and simplicity in Poncelet (1817c), we begin to understand the cognitive properties associated with this description. Poncelet considered *simple* proofs as those with brief arguments with relatively few constructive steps or containing equations with relatively few unknowns. He noted that purely geometric considerations could *simplify* a problem in analysis by suggesting an appropriate choice of variables. With respect to geometric objects, Poncelet described straight lines as *simple*, that is, simpler than curved ones. Though he often equated elegance with simplicity, he considered generality as an independent quality. Turning by comparison to Plücker and Schoenflies (1904), we find that Plücker aligned simplicity with symmetry, but more often described a proof or theorem as *facile* if it contained few variables or followed immediately from an earlier result.

2.3 Poncelet's solutions without proofs, 1817

Poncelet framed his article, "Réflexions sur l'usage de l'analise algébrique dans la géométrie; suivies de la solution de quelques problèmes dépendant de la géométrie de la règle," as a letter to the editor of the *Annales.* Though clearly intended for publication, he directly addressed Gergonne and referred to the limited space available in his chosen medium. As the lengthy title suggests, Poncelet's article was two-fold consisting of a philosophical methodological argument followed by corroborating new results, his supporting evidence.

2.3.1 Poncelet's philosophical argument

Poncelet began by critiquing Gergonne's bipartite division of geometry into analytic and pure. Instead, Poncelet differentiated three types of geometry: analytic geometry or the method of coordinates, ancient pure geometry as represented by "Euclid, Apollonius, Viète, Fermat, Viviani, Halley, etc.", and modern pure geometry.[17] Modern pure geometry refrained from using the coordinate equations or any type of calculation which permitted losing view of the figure concerned. However, unlike ancient pure geometry, modern geometry included infinity and infinitesimals, invariable relations among variable figures, and three dimensions applied to planar geometry.

> ...cette géométrie, cultivée par les modernes, dans laquelle, au moyen des notions d'infiniment grands et d'infiniment petits, on parvient à découvrir les relations qui existent entre les diverses parties d'une figure supposée variable; [...] cette géométrie qui consiste à chercher, dans les propriétés de l'étendue à trois dimensions, la solution des problèmes de la géométrie plane [...] (Poncelet (1817c), 142-143)[18]

Following this distinction, Poncelet henceforth referred only to analytic geometry and pure (or "rational" or "ordinary") geometry–by which he presumably intended the modern variant. While depicting the latter geometry as the path of intuition (*la voie d'intuition*), he conceded that the choice of method should depend on the problem being solved. Citing

[17]Poncelet's admittedly partial list of ancient pure geometers does not give a very clear sense of what ancient pure geometry comprised, but it is his only positive description. Otherwise, one must infer what Poncelet thought constituted ancient pure geometry with respect to what it lacks. In the introduction to his *Traité*, Poncelet gave a similar, but not identical list of practitioners of ancient geometry including here Euclid, Archimedes, Apollonius, Viète, Fermat, Grégoire de St.-Vincent, Halley, Viviani, R. Simson, etc. (Poncelet (1822), xxviii). Once again we find a striking similarity to Carnot, this time in the latter's much longer list of elementary geometers: Archimedes, Hipparchus, Apollonius, Napier, Viète, Fermat, Descartes, Galileo, Pascal, Huygens, Roberval, Newton, Halley, and Maclaurin (Carnot (1803), xxx).

[18]"...this geometry, cultivated by the moderns, in which, by means of notions of infinitely great and infinitely small, it is possible to discover relations that exist between the different parts of a figure supposed variable; [...] this geometry which consists of finding, in the properties of the extension to three dimensions, the solution to planar geometric problems,..."

Charles Dupin's *Développemens de géométrie* (1813), Poncelet advocated that one should cultivate both *sciences* for their mutual advancement, since analysis provided generality to geometry, and reciprocally the particularity of geometry facilitated simplifying the equations of analysis by choosing convenient unknowns, interpreting and developing geometrical consequences from results of calculation.[19] However, since Gergonne had been advocating the superiority of analytic geometry, Poncelet offered his results as evidence for the simplicity and elegance of pure modern geometry, without the use of coordinates or calculation.[20]

2.3.2 Poncelet's evidence in favour of pure geometry

The first two problems were exactly those that Gergonne had proposed to prove by analytic geometry at the end of his "Solution et construction, par la géométrie analitique, de deux problèmes dépendant de la géométrie de la règle" published just a few months earlier (Gergonne (1817b)). These were: first, to inscribe with the ruler alone an m-sided polygon to a conic with sides passing through m given points and, second, to circumscribe an m-vertexed polygon to a conic with vertices lying on m given lines. Poncelet claimed that the solution of the second problem, the circumscription, would follow analogously from that of the first, the inscription.

The problem divided into two essentially distinct parts.[21] The first part reduced to determining how many different types of polygons could be formed with vertices on m given points. Poncelet solved this problem through what he called *géométrie de situation*. He proceeded by first considering three letters a, b, c arranged on the circumference of a circle. Since reading left to right or right to left on the circumference would be equivalent, there was only one unique arrangement. With an additional letter d, there would be three possible different arrangements by placing d between any two consecutive letters. To avoid confusion, Poncelet suggested putting each of these arrangements on three new circumferences. Then for each of these three, a fifth letter e could be placed in-between any two consecutive letters, so there would be four different arrangements per circumference, and thus $3*4 = 12$ in total. Again, these twelve different arrangements could be placed on twelve different circumferences. After completing these first few cases, Poncelet indicated that by continuing thus, one would arrive at $3*4*5*\cdots(m-1)$ possible arrangements of m given points.

[19] Dupin had explicitly proposed that,

En considérant ainsi l'analyse et la géométrie dans leurs rapports, ces deux sciences s'éclaireront mutuellement, et chacune d'elles s'accroîtra de tous les progrès de l'autre. (Dupin (1813), 238)

["In thus considering analysis and geometry in their relationships to each other, these two sciences mutually clarify and enhance all their progress."]

He followed by showing examples of analysis applied to geometry and geometry applied to analysis. We further discuss aspects of Dupin's book in Chapter V.

[20] We will delve further into Gergonne's provocative arguments in favour of analytic geometry in Chapter III.

[21] A more technically detailed exposition of the solutions to these problems can be found in Friedelmeyer (2011), (130–138).

Poncelet remarked that nothing would be easier than to imagine the use one could make of this type of *tableau artificiel*. For example, an ordered arrangement $abcd \ldots f$ on a circumference would signify that the polygon's first side passed through a, the second side through b, the third through c, and so on with the last side through f.

Intriguingly, Poncelet's first visualization in this article was thus not at all of the Euclidean variety. With the ambiguous term *tableau*, Poncelet emphasized the ordered organizational as well as pictorial aspect of these hypothetical circumferences. A nearly identical treatment of the same problem was also published in Poncelet's *Traité* five years later. In his manuscript notes to this section currently located in the archives of the École polytechnique, Poncelet included an illustration for the case of the twelve possible different polygons determined by five given points (Figure 2.3). By describing this form as *artificiel*, Poncelet seemed to emphasize that one need not truly construct any of these circumferences (which quickly becomes tedious), and the visualization served a very different purpose than that of ordinary geometrical figures.

The second part of the problem depended on *géométrie ordinaire*, to find an inscribed polygon with sides passing through m ordered points in the plane. Poncelet presented what he termed indirect and direct solutions to this more particular problem. The indirect solution treated the cases of polygons with two, three, or four vertices individually, and then showed a process toward reducing any polygon of more than four vertices to the two or three vertex case using the constructive procedure described in the four vertex case. The much briefer direct solution contained only one construction for any number of vertices, and relied upon the collinearity of intersections of opposite sides of an inscribed hexagon.[22] Poncelet qualified these findings as remarkably general and simple. He noted that the direct solution was superior with respect to generality and symmetry, though for a smaller number of given points the indirect solution could be preferable. Poncelet added a very short treatment for the case of the circumscribed polygon, "tout-à-fait remarquable par sa parfaite analogie" to the direct construction of the inscribed polygon. Poncelet then continued from these problems to two special cases and two fundamental theorems on inscribed and circumscribed polygons.

For Poncelet's final solved problem he switched emphasis from constructing polygons given a conic to constructing a conic satisfying given parameters. This problem type was also a popular theme for *Annales* contributors. One contributor, former École polytechnique student, and later collaborator with Poncelet, Charles Brianchon (1783–1864), had systematically solved the problem of finding a conic given any combination of five given points or tangent lines in his *Mémoire sur les lignes du second ordre* (Brianchon (1817)).

Poncelet introduced his version of this problem as "l'autre exemple que j'ai promis, au

[22] This result is today known as Pascal's theorem, and was often attributed to Pascal (without a specific citation) in early nineteenth century geometry texts, though not in this one.

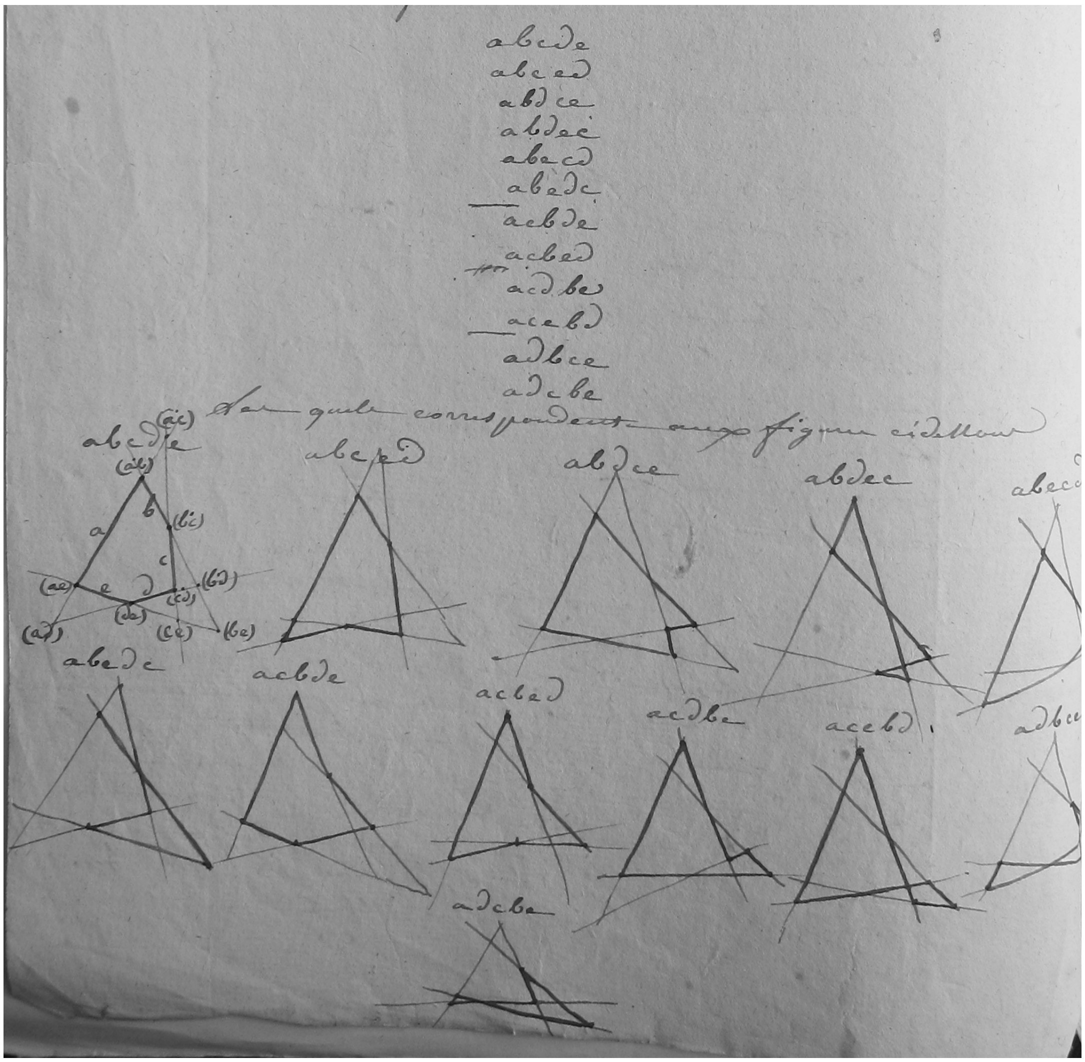

Figure 2.3: Tableau artificiel from Poncelet's "Notes de géométrie" in the École polytechnique archives (5175)

commencement de cette lettre, en faveur de la géométrie pure" (Poncelet (1817c), 153). The problem concerned a given conic, a point on the conic, and another point in the plane, and asked to find with the ruler alone arbitrarily many points of another conic with a third order contact at the first given point and passing through the second.[23]

All of Poncelet's constructions referenced but did not include any illustrated figures. As discussed in our introduction, we will use *virtual figure* to denote the invocation of geometric objects where points, lines, or faces are named with letters in order to direct the reader toward their precise construction, and there is no printed or drawn figure. While we have found the best way to understand this and other constructions (whether or not accompanied by an actual figure) is through sketching them, to assist the reader here, we provide a figure below illustrating the construction step-by-step in the case of a given circle. The points and lines are taken from Poncelet's description, and we also employ a variety of solid, dashed, and dotted lines (as was customary within Poncelet and his contemporaries' actual figures) to signify different phases in the construction process and aid in visibility. The actual figure, whether present in the original text or created by the reader through a described virtual figure, illustrated in Figure 2.4, could thus summarize the complete construction at a glance.

Poncelet introduced an unnamed conic in the plane, which we represent in the following figures with a circle, a point called P on its perimeter, and another planar point A. The desired conic would have a third order contact with the given conic at P and pass through A. Poncelet instructed the reader to draw an indefinite tangent line through P to the given conic, and then draw the secant line PA, whose second intersection with the conic would be the point Q (on the left).

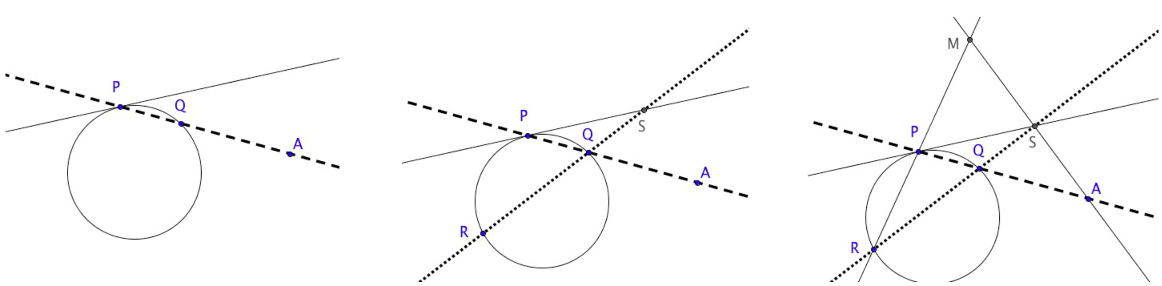

Figure 2.4: Virtual figures illustrating Poncelet's conic section construction

Then choosing any other point on the given curve, R, and drawing a secant QR, this new secant would meet the tangent line through P at S (centre figure). Finally, one should draw lines PR and SA concurring at a point M, which belonged to the curve sought (right figure). By varying the position of the point R one could obtain the locus of the points M.

[23]Poncelet did not directly define a third order contact in this text, though we shall see he went into ample detail on this topic in his *Traité*.

Poncelet next considered the case where the other given point A was infinitely distant [*infiniment éloigné*]. In that case, the conic with third order contact (which Poncelet now designated as the *osculatrice*) would be either a parabola or a hyperbola. He asserted that the same construction as above would still apply, but could no longer be executed with the ruler alone. Poncelet did not elaborate the procedure for the modified construction, implying that it was practically identical. However, the process may have been like that in the solution to a similar problem from Brianchon's *Mémoire*: to describe a conic when three or fewer of the five given points were inaccessible or situated at infinity [*inaccessibles ou situés à l'infini*].[24] In this case, one needed to know the direction of a line containing one of the two or more given finite points and the inaccessible infinite point. The same procedure could be applied to Poncelet's construction. Then the secant PA as well as the line SA would be parallel to the given direction, and hence intersecting it at a point at infinity, A.

Poncelet extended this problem by proposing that if the point A was replaced by a given tangent to the desired conic the construction would be little different in its first part and lose none of its simplicity. Further, Poncelet claimed it would not be difficult to deduce the entire theory of osculations among conic section from the above construction. However, he continued, this was not the place. Such a sweeping generalization laid claim to deeper researches for which this letter was only a small preview. Republishing this article in 1864, Poncelet reconstructed his earlier motivations, both philosophical and personal.

> Toutefois j'appréhende fort que le présent Article de *Philosophie mathématique* et peut-être quelques-uns des suivants, venus d'un jeune officier désireux de se préparer à l'avance un nom pour la publication du *Traité des Propriétés projectives*, n'aient indisposé, malgré ses formelles intentions, le savant rédacteur des *Annales* de Montpellier contre leur auteur qui, probablement à ses yeux, n'avait point encore acquis le droit de combattre, quoique avec circonspection et courtoisie, les idées philosophiques d'un ancien professeur, déjà justement estimé pour ses services scientifiques. (Poncelet (1864), 466)[25]

Thus, at least in retrospect, Poncelet viewed this article as a step toward securing his scientific reputation and drumming up interest for the publication of his first monograph.

[24]Brianchon did not clarify what he intended by inaccessible points. Poncelet used this term often in his *Traité*, even titling a chapter "Conséquences qui en résultent pour la détermination des droites, ou des points qui appartiennent à un point, ou à une droite, supposés tous deux inaccessibles, invisibles ou placés à l'infini." With respect to inaccessible objects, Poncelet referred both to Brianchon's *Mémoire* as well as problems treated by Frans van Schooten on this subject in 1656 (Poncelet (1822), 80).

[25]"However I strongly fear that the present article of *Mathematical Philosophy* and perhaps some of the following, from a young officer eager to advance a name for the publication of the *Treatise of Projective Properties*, despite its formal intentions, indisposed the learned editor of the *Annales* of Montpellier to their author who, probably in his eyes, had never acquired the right to combat, albeit with caution and courtesy, the philosophical ideas of a former professor, already justly esteemed for his scientific services."

Poncelet concluded his paper with the statement of two additional problems also possible to be treated by pure geometry, and doubted whether analytic geometry could reach such general, symmetric, and simple constructions due to the large number of unknown variables. However, he tactfully admitted that this could be the fault of his own analytic abilities, rather than the method itself.

> C'est la faute que j'avais moi-même commise avant de connaître vos solutions; et cela prouve de nouveau qu'on ne doit jamais se hâter d'imputer à l'analise des imperfections qui souvent sont uniquement le faut de ceux qui ne savent point en faire un usage convenable. (Poncelet (1817c), 155)[26]

The choice of variables and translation of equations in analytic geometry required a well-disciplined analytic practice, which Poncelet professed (perhaps with excessive modesty) to lack.

2.3.3 Poncelet's missing figures

Though Poncelet criticized analytic geometry for losing sight of the figure, there were no figures to be actually seen in his purely geometric paper. It is possible that Poncelet had originally included figures, which were not printed. Such a situation occurred within two other articles from the same *Annales* volume when the figures provided by Poncelet were not printed in their entirety (Poncelet (1817d), Poncelet (1817b)). In an earlier article, Poncelet's final figure was incomplete (Poncelet (1817d)). Gergonne apologized for its absence in a later footnote.

> Nous saisissons cette occasion pour demander pardon au lecteur de ce que, dans la figure 4 du mémoire cité, le tracé de la parabole a été oublié. L'erreur peut heureusement se réparer à la main avec beaucoup de facilité. (Poncelet (1817b), 71)[27]

Gergonne initialed "J.D.G. fecit." under all the figures in the *Annales*, thus signifying his role as the artist in the production process. In 1817, most French printmaking was done through engraving, a lengthy and expensive process usually involving a draftsman [*fecit*] and an engraver [*sculpt*]. With this in mind, it is possible that figure omissions might save time or money. Supporting this hypothesis, we will see that the same problems and constructions when self-published in Poncelet's *Traité* were accompanied by numerous illustrations. Further, when Poncelet reprinted this article in a supplement to a monograph

[26] "This is a fault that I committed before knowing your solutions; and that proves anew that one must never rush to impute to analysis the imperfections which often are uniquely the fault of those who do not know how to use it well."

[27] "We take this opportunity to beg the reader's pardon, for forgetting to trace the parabola in figure 4 of the referenced memoir. The error can happily be repaired with great ease by hand."

in 1865 he referred the reader to either his *Traité* or to the first volume of his *Applications* to find the figures discussed as well as the missing proofs.

> Les personnes qui n'ont pas entre les mains le *Traité des Propriétés projectives* pourront recourir au Ier Cahier de ce second volume des *Applications*, pour le tracé des figures et les démonstrations géométriques, (Poncelet (1866), 471)[28]

When given complete editorial power, Poncelet chose to include actual figures to illustrate his constructions.

However, without the presence of actual figures, Poncelet effectively created and actively manipulated a well coordinated ensemble of virtual figures. Moreover, these figures were constructible with a ruler alone for a given conic in the plane–barring noted exceptions of the intersection points of parallel lines. Returning to the language of his first stated theorem, we note the vivid description of the geometer animating the polygon, whose third side turns "constantly" while remaining "perpetually" tangent to another conic section. With his choice of vocabulary, Poncelet took advantage of the descriptive possibilities of the medium of pure geometry. By following Poncelet's written instructions, the well-versed reader could easily create sketches or constructions to suggest the narrated procedures. Even so, the description of an illustration, no matter how precise, was not the same as an illustration itself.

The problematic relationship between a description and a picture has been addressed within the field of art history in Baxandall (1985). Michael Baxandall cites a passage from the fourth-century Greek Libanius that described an absent picture and then asks what the description represents. He replies,

> It would not enable us to reproduce the picture. In spite of the lucidity with which Libanius progressively lays out its narrative elements, we could not reconstruct the picture from his description. [...] What happens as we read it is surely that out of our memories, our past experience of nature and of pictures, we construct something–it is hard to say what–in our minds, and this omitting he stimulates us to produce feels a little like having seen a picture consistent with his description.[...] In fact, language is not very well equipped to offer a notation of a particular picture. It is a generalizing tool. (Baxandall (1985), 3)

The great difference here is that for visual art the description of the figure is dependent on the existence of an illustrated figure, which is certainly not the case in geometry. Moreover, the language of geometry functions comparatively much better as pictorial notation. However, in both descriptions of art and descriptions of geometrical figures, language generalizes. For Poncelet's geometry, the generalization was advantageous. A construction about

[28]Those who do not have the *Treatise on Projective Properties* in hand can have recourse to the first book of the second volume of the *Applications*, in order to trace the figures and the geometric demonstrations.

conic sections could be about all conic sections in a description, while a single figure would show only one ellipse, hyperbola, or parabola. If his reader then constructed the figure, that particularization would be independent of Poncelet's original contribution and so had no effect on the generality of his method.

Yet, while language enables a myriad of simultaneous equally valid visualizations, it also imposes a linear temporal structure. We read mathematical writing from left to right, from top to bottom. Not so for mathematical pictures, which can be viewed in much the same way as any other picture.

> Within the first second or so of looking we have a sort of impression of the whole field of a picture. What follows is sharpening of detail, noting of relations, perception of orders, and so on, the sequence of optical scanning being influenced both by general scanning habits and by particular cues in the picture acting on our attention. (Baxandall (1985), 4)

An illustrated figure could make a construction evident at a glance, whereas the evidence supplied by only the description of a figure unfolded gradually and in a prescribed order. With respect to ancient Greek geometrical figures, Reviel Netz found that the diagrams were synoptic–summarizing the construction's content (Netz (1999), 29). This summary quality persisted into the early nineteenth century. In publications like the *Annales* and Poncelet's *Traité* actual figures accompanying geometrical texts most often illustrated the completed construction–only the text conveyed the progressive constructive steps. Though Poncelet seems to have preferred an illustrated figure when possible in his independent publications, he did not explicitly emphasize this distinction. For Poncelet's ideal objects, such an emphasis might have been detrimental to their acceptance. To stress the difference between figures illustrated and described would have also served to broaden the gulf between geometric objects that could be illustrated and those that could not.

As we saw in our introduction, Poncelet would describe ordinary geometry in his *Traité* by emphasizing the centrality of the figure. This figure could be made sensible, but did not need to be presented in a strictly sensible form. For Poncelet the description or painting of the figure through invoking imagined or visualized sensible objects, more so than any realized illustration, was the characteristic feature of a purely geometric approach. Furthermore, as we shall see in the *Traité*, imaginary, infinite, or ideal objects could become sensible through projection or Poncelet's principle of continuity.[29] Despite the abridged nature of Poncelet's preliminary article in 1817, we can observe the importance of these qualities here.

[29]In Chapter IV, we will extend our study of Poncelet's "modern geometry" by considering its adaptation in the work of Plücker and Steiner.

2.3.4 Reception of Poncelet's article

Poncelet's paper was met with responses to both its methodological arguments and its mathematical results. In the immediately following, "Réflexions sur l'article précédent" (a frequent editorial feature in the *Annales* accompanying philosophical articles or innovative mathematics) Gergonne seemed impressed by Poncelet's ingenious and elegant solutions but cautious about the lack of proof or underlying theory.

> On ne peut donc que faire des vœux pour que l'auteur, après avoir aussi vivement piqué la curiosité des lecteurs, veuille bien enfin la satisfaire complètement, en faisant connaître les théories sur lesquelles reposent ses ingénieuses et élégantes constructions. On doit désirer, en outre, que M. Poncelet ne borne point là ses recherches; [...] (Gergonne (1817e), 162)[30]

Gergonne would have to wait five years until the publication of Poncelet's over 400 page *Traité des propriétés projectives* unequivocally confirmed Poncelet's attestation that his theorems rested on "principes dont la développement excéderait nécessairement les bornes d'une simple lettre." (Poncelet (1817c), 144)[31] Meanwhile, geometers could either take for granted the veracity of Poncelet's results,[32] or work toward grounding these results in a new or existing theory. Plücker's 1826 article for the *Annales* was in direct reaction to Poncelet's results, though first appearing nine years later.

By that time, Poncelet's *Traité* was published, Plücker had completed his studies in Paris between 1822 and 1824.[33] However, he later claimed he had been ignorant of this text as well as all issues of the *Annales* published after 1817.

[30]"We can only wish that the author, after having so strongly piqued the readers' curiosity, will be good enough to satisfy it completely, in revealing the theories on which his ingenious and elegant constructions rest. We desire, moreover, that M. Poncelet does not end his researches here; [...]"

[31]Poncelet's new principles were first introduced to *Annales* readers through a review by Cauchy (on behalf of himself, Arago, and Poisson) of Poncelet's submission of preliminary research to the *Académie des sciences* in 1820, which we will review in more detail in Chapter III (Poncelet and Cauchy (1820)). Poncelet funded the publication of 800 copies of his *Traité* in Metz two years later (Gray (2005), 367). The slow reception of Poncelet's text may be attributed to its unusual classification, since most geometry books of this time period concerned either elementary, analytic or descriptive geometry, as we will show later.

[32]Most immediately displayed by Jean Baptiste Durrande in "Questions résolues. Solution des deux problèmes de géométrie proposés à la page 164 de ce volume" Durrande (1817), where he extended Poncelet's results to polyhedra in response to a problem posed by Gergonne on this theme.

[33]Plücker's biography by Wilhelm Ernst from 1933 is decidedly vague with respect to his professional interactions with Parisian scientists.

> Um seine Kenntnisse in der höheren Mathematik und der theoretischen Physik zu erweitern und zu vertiefen, reiste er daher Anfang März 1823 nach Paris, wo er die bedeutendsten Fachgelehrten, wie Biot, Cauchy, Lacroix, Poisson, Pouillet, Thenard sowie Clément, Dulong und Binet in ihren öffentlichen Vorlesungen hörte und auch zu einigen von ihnen in persönlichen Verkehr trat. (Ernst (1933), 7)
>
> "To expand and deepen his knowledge of higher mathematics and theoretical physics, he therefore went in early March 1823 to Paris, where he heard important scholars, such as Biot, Cauchy, Lacroix, Poisson, Pouillet, Thenard as well as Clément, Dulong and Binet in their public lectures and also personally met some of them."

> Je n'avais pas alors la facilité de me procurer *Traité des propriétés projectives*, que je connaissais seulement par le catalogue de M. Bachelier. (Plücker (1828c), 331)[34]

We look first to Plücker's original submission to see how Poncelet's results on osculating curves could be proved and extended through analytic geometry. In light of Poncelet and Plücker's later writings, we will note especially Plücker's differentiation between real and imaginary points of intersection and the resulting exceptions to the validity of his constructions.

2.4 Plücker's analytic geometry without calculation, 1826

Plücker framed his early research on osculating curves in reaction to Poncelet's 1817 article, addressing both his broad methodological claim for the superiority of pure geometric methods and his specific conic section examples.

> En faveur de la géométrie pure M. Poncelet donne dans le huitième volume de ce même recueil, de bien jolies constructions de problèmes de géométrie. [...] Voyons si ce problème se prête si difficilement aux méthodes de la géométrie analytique. (Plücker and Schoenflies (1904), 392)[35]

Thus Plücker emphasized that, at least initially, he was not using analytic geometry to find new solutions, but rather to prove the solutions Poncelet had already provided. That said, Plücker presented these researches within a new theory on osculating curves. Poncelet had suggested in 1817 that one could deduce the entire theory of osculating conic sections from his geometric construction of the third order contact. However, as we saw above, he refrained from providing any of the theory's details in his article. Moreover, Poncelet had formulated the problem of third order contact as an isolated example, neither following from his earlier results nor leading to new ones. By contrast, Plücker situated the problem firmly within the article's content, and actually gave multiple proofs.

We will see that Plücker's overarching theory relied upon perceiving coordinate equations as representing geometric objects. That is, Plücker intended for his equations to be read as clear visual means towards justifying geometric arguments like the figures of Poncelet's geometry. The equations were being used as evidence. Plücker introduced this notion

Even if Plücker did personally meet the mathematicians among this list, none of them were actively engaged in the geometric research he pursued at this time.

[34] "I was not then able to obtain *Treatise of Projective Properties*, which I knew of only through the catalog of M. Bachelier."

[35] "In favour of pure geometry, Poncelet gives in the eighth volume of this same issue, very pretty constructions of geometry problems. Let us see if this problem lends itself with such difficulty to methods of analytic geometry."

gradually, beginning with the standard procedures of analytic geometry: choosing axes and translating geometric objects into equations.

2.4.1 Plücker's theory of osculating curves

Plücker opened his article by providing a coordinate representation of intersection points between two conic sections in the plane. Assuming two curves intersect, Plücker assigned one of their points of intersection as the origin of the coordinate system. Then if this intersection was of "first order (contact of two points)," by choosing the y-axis as the common tangent to the origin, Plücker was able to write the *équations primitives* of the curves as

$$y^2 + 2Axy + Bx^2 + 2Dx = 0, \tag{2.1}$$

$$y^2 + 2axy + bx^2 + 2dx = 0, \tag{2.2}$$

two second degree equations of curves passing through the origin and tangent to the y-axis there. Taking the difference of these equations resulted in what Plücker described as a "third geometric locus passing through their points of intersection." That is,

$$2(A - a)xy + (B - b)x^2 + 2(D - d)x = 0, \tag{2.3}$$

which he resolved into the system of two straight lines,

$$x = 0, \tag{2.4}$$

$$2(A - a)y + (B - b)x + 2(D - d) = 0. \tag{2.5}$$

Plücker identified these lines as the y-axis and the common chord of the two conics, asserting that

> Dans la discussion de cette équation (3) est renfermé toute la théorie de l'osculation dans les courbes du second degré et cette discussion s'offrira d'elle même et sans aucun calcul. (ibid, 387)[36]

This was Plücker's first subtle reference to Poncelet's "Réflexions," where Poncelet had promised the entire theory of osculating conic sections could be deduced from a single given construction. Further, by explicitly pointing to the absence of calculation in his analysis Plücker proposed a correction to Poncelet's characterization of analytic geometry where any type of calculation whatever could allow the figure to be momentarily lost from view. As Poncelet had written in 1817, analytic geometry was

[36] "The entire theory of osculation of second degree curves is contained in the discussion of this equation (3), and this discussion presents itself without any calculation."

> [...] la méthode des coordonnées, ou même de toute espèce de calcul quelconque qui permettrait de perdre momentanément de vue la figure dont s'occupe. (Poncelet (1817c), 143)[37]

Plücker would return to the theory of osculations and the associated equations later in his paper.

Plücker systematically examined the coefficients of these equations with respect to the order of contact between the two curves. He positively summarized these findings as elegant and effortless.

> Ces divers résultats me paraissent élégants en vertu de leur symétrie et de leur extrême simplicité, et pour cette même raison, il est évident que les propriétés fondamentales de deux ou de plusieurs courbes, assujetties aux conditions que nous discutons, se déduiront sans aucun effort de la combinaison de leurs équations, en y introduisant les diverses relations entre les constantes. (Plücker and Schoenflies (1904), 389)[38]

Plücker was able to use coordinate equations as representational rather than computational, whereby a reader could immediately discern properties of the geometric objects by looking at the analytic form.

Like Poncelet, Plücker asserted the proper definition of the order of a contact point was fundamental to the theory of tangents and osculations.

> Toute théorie du contact et de l'osculation des divers ordres est fondée sur la supposition que deux ou plusieurs points d'intersection de deux courbes se réunissent en un seul, supposition qu'on peut se conscrire de plusieurs manières, mais que, selon moi, l'on ne peut pas éluder, ni même remplacer par d'autres considérations, qui en donnent une idée plus nette. (ibid, 389)[39]

From this notion of order, Plücker created a basis for the theory, not just of conic section osculations (as above), but osculations of curves of any order. He argued that the conception of contact order as the result of coinciding points was "inescapable and irreplaceable" in considering intersection and tangency between curves. In practice, Plücker

[37] "[...] the method of coordinates, or even of any type of calculation that would momentarily allow the figure concerned to be lost from view."

[38] "These diverse results appear elegant to me by virtue of their symmetry and their extreme simplicity, and for this reason it is evident that the fundamental properties of two or several curves, subject to the conditions which we are discussing, are deduced without any effort from the combination of their equations, by introducing there the diverse relations between the constants."

[39] "Every theory of contact and of osculation of different orders is based on the supposition that two or several points of intersection of two curves meet in one, a supposition that one can write up in several ways, but that, in my view, one cannot elude, nor even replace by other considerations, which give a clearer idea of it."

textually emphasized the relationship between order and coincidence by writing "un contact de deuxième ordre, (un contact de trois points)," or "un contact de troisième ordre, (un contact de quatre points)." Plücker could immediately apply his procedure of making the points of intersection of the two curves (2.1) and (2.2) coincide to show that the common chord (2.5) would contain a point of higher order. On a deeper level, Plücker's attention to the meaning of order in geometry emphasized the relationship between algebraic notions of lines and points of given order and geometric lines and points in the plane. Again this relationship required no intermediary figure. Plücker analytically expressed the coincidence of two or several points of intersection by eliminating variables between the equations of two curves.

Plücker next found the difference of the equations of two conic sections (2.1) and (2.2). The roots in x or y of this new equation would correspond to the common points of the two curves. Plücker noted that the computed number of real roots determined the number of intersections between the two curves.

> Statuer sur la réalité ou l'imaginarité, sur l'égalité ou l'inégalité des ces racines, c'est assujettir les intersections à certains conditions, indiquées par des équations entre les constantes. (389)[40]

Plücker explained that the correspondence from roots of the equation to intersection points of the two geometric curves included the possibility of imaginary roots found analytically, which corresponded geometrically to a lack of intersection in the plane. Imaginary intersections were geometrically invisible.[41]

Plücker drew attention to a flaw in the above form of solution in that two equal roots in x could represent either one unique point or two distinct points lying on a parallel to the y-axis. Thus there was a potential indeterminacy in this form of solution. Plücker proposed to show an alternative treatment, "*plus particulièrement encore par la géométrie analytique*" (emphasis in original) to reach the same results by a "faster" and "more precise" route. Just as Poncelet had made a nuanced distinction between ancient and modern pure geometry, Plücker compared the particular advantages of one analytic treatment over another. There was more than one method of analytic geometry.

This new route still relied upon the equation of the common chord (2.5). Plücker remarked that for two conics with a common tangent at the origin and a chord through two other common points, the angle formed between the common tangent and the common chord would be independent of the values of the linear coefficients, D and d. In varying D and d in the original curve equations (2.1) and (2.2) while keeping the other coefficients

[40] "To rule on the real or imaginary character, on the equality or inequality of the roots, is to subject the intersections to certain conditions, indicated by the equations between the constants."

[41] In his later monograph, Plücker would argue that the treatment of imaginary numbers in constructive (as opposed to analytic) geometry was indirect [*Umwegen*] (Plücker (1828a)).

constant, all common chords between these two equations would be parallel to each other. In particular, in the case where the second conic was a circle,

$$y^2 + x^2 - 2dx = 0, \tag{2.6}$$

varying d would only change the magnitude of the radius.

From these properties, Plücker deduced a construction, which he admitted was already known in "la synthèse," for an osculating circle at a given point of a conic section based on the common chord between any tangent circle and the given conic. This is Plücker's only use of "synthesis" in the paper, and we suggest that his peculiar language–rather than using the French phrase *la géométrie synthétique* or *la méthode synthétique*– can be attributed to a fault of translation. Nevertheless, the effect of contrasting to his "analytic" approach was successfully conveyed as he would do two years later, in Plücker's *Analytisch-geometrische Entwicklungen*, in describing a theorem of Poncelet as *synthetisch* (Plücker (1828a), 216). In this 1826 paper, though Plücker was generally very careful with citations, he did not provide a specific source. Thus it appears that Plücker intended that this construction had been deduced from non-analytic considerations, and was known well enough in the standard literature to not require specific citation. Intriguingly, in the Gergonne-edited version the reference would be changed to "une autre construction déjà connue" (Plücker (1826b), 72).

As noted above, Plücker argued that two given conics might not share a real common chord if they did not intersect in two distinct real points. In this case, he suggested that "on pourra alors construire une ligne droite exprimée par l'équation (3) qui jouira de toutes ces propriétés géométriques" (390).[42] Plücker cited the concept of radical axes as a precedent with respect to constructible intersections which might not appear as intersections when constructed. Thus Plücker used the coordinate representation to extend the notion of a chord beyond that of a real chord in the plane. Though the chord did not "exist as such," it was still "expressible" as an equation, which determined a straight line. Along with generalizing the concept of a chord as being contained within a conic section, Plücker appears to have ignored the issue of endpoints which are not at all indicated by the representation in equation (2.5). This notion of geometric existence motivated by coordinate representation seems to imply a lack of exception in procedure despite a difference in ontological status between different types of chords. There is also a distinction implied between the existence of an object as opposed to the representation and determination of an object. A *non-existent* object might still be representable or made evident, and conversely representation or evidence did not guarantee existence. Within this context of deducing properties and theorems in common to osculating curves, Plücker considered representability rather

[42]"...one could thus construct a straight line expressed by equation (3) which enjoys all these geometric properties."

than existence as a criterion for determining geometric objects.

2.4.2 Applications to geometric constructions

From observing the standard equation of a curve through the origin (2.1), Plücker determined limitations on the types [*espèce*] of curves that would satisfy having a third or second order contact with a given point on a given curve. For example, if and only if the coefficient $A = 0$, could an osculating circle have a third order contact with a conic unless the point of osculation was also a vertex of one of the conic's principal diameters. To find equations for the diameters Plücker referenced "les méthodes connues de la géométrie analytique" or successive differentiation with respect to x and y to obtain linear equations.[43] Using those methods, Plücker studied the loci of centres of osculating conics.

In the second half of his article, Plücker transitioned from finding necessary coordinate and coefficient conditions for osculating conic sections, towards deriving constructions for conic sections that met the desired criteria. He described this as a shift from theory to application. While Plücker had not referenced figures while introducing his theory of osculating curves, he would apply this theory toward finding figure constructions. Specifically, as an application of his osculation theory, Plücker proposed to derive Poncelet's construction of a curve sharing a third order contact with a given curve. In the same coordinate system as before, he designated the first curve as

$$y^2 + 2\alpha xy + \beta x^2 + 2\delta x = 0, \tag{2.7}$$

a curve tangent to the y-axis at the origin. Plücker then considered the system of two straight lines intersecting at the origin,

$$y + mx = 0 \tag{2.8}$$

$$y + nx = 0, \tag{2.9}$$

which could be multiplied to form the single second order equation,

$$y^2 + (m + n)xy + mnx^2 = 0, \tag{2.10}$$

containing the origin. Plücker proceeded by subtracting (2.10) from (2.7) and dividing by

[43]Plücker did not provide details as to the procedure of the known analytic geometry methods, but perhaps he was referring to Gergonne's article from the ninth volume of the *Annales* Gergonne (1818), in which Gergonne had proposed methods from analytic geometry to replace transcendental calculus in finding derivatives.

the common factor x,

$$(2\alpha - (m+n))y + (\beta - mn)x + 2\delta = 0, \tag{2.11}$$

obtaining "the equation of the chord through the points where each of our two lines respectively meet the curve for a second time."

Plücker focused on the relationship between the coefficients in this final equation. He noted that if any two of α, β, δ were held constant with the third coefficient remaining arbitrary, then the equation (2.7) still represented an infinite number of curves, but one could now deduce certain properties from (2.11). For the case of the third order contact, if α, δ remained constant, then (2.11) would meet the y-axis (that is, the shared tangent at the point of osculation) at the fixed point,

$$y = -\frac{2\delta}{2\alpha - (m+n)}. \tag{2.12}$$

Since both curves shared the same α and δ, this y value determined the intersection of the straight lines through the curves and the x-axis. Plücker concluded: "[...] le premier de ces trois cas conduit précisément à la construction de M. Poncelet du problème cité."[44] Plücker proved this result constructively, naming geometric objects without equations or calculation, thus creating a virtual figure shown below. Given two conics tangent in exactly one point, the point of osculation, O, draw two arbitrary lines OMR, OAQ meeting one conic at M and A and the other at R and Q. Then the straight lines MA and RQ were both represented by (2.11) with fixed α, δ and variable β. So both these lines would meet on the common tangent at O.

With this theorem Plücker had verified Poncelet's construction, shown with Plücker's notations in Figure 2.5. Given a conic section, a point on its perimeter O and another point on the plane A, one could find a new point M on a second conic section sharing a third order contact at O and passing through A. Then Plücker considered the special case where the arbitrary lines OMR and OAQ coincided. In this case, the coefficients m, n would be equal. Consequently, the chord represented by (2.11) would become tangent to the given conic, and still meet the tangent at O to the desired conic at a fixed point. If the given point A was replaced by a given tangent line, Plücker explained how one could determine the contact point on the given tangent.

Plücker continued to the second order tangency case, holding β, δ as invariable and allowing α to have any value. Solving (2.11) for y, he determined that the x-axis would be cut at the constant value $-\frac{2\delta}{\beta - mn}$. With this Plücker could prove the construction of a curve with a given second order contact at O, a given intersection point at P, and passing

[44] "[...] the first of these three cases leads precisely to M. Poncelet's construction of the cited problem."

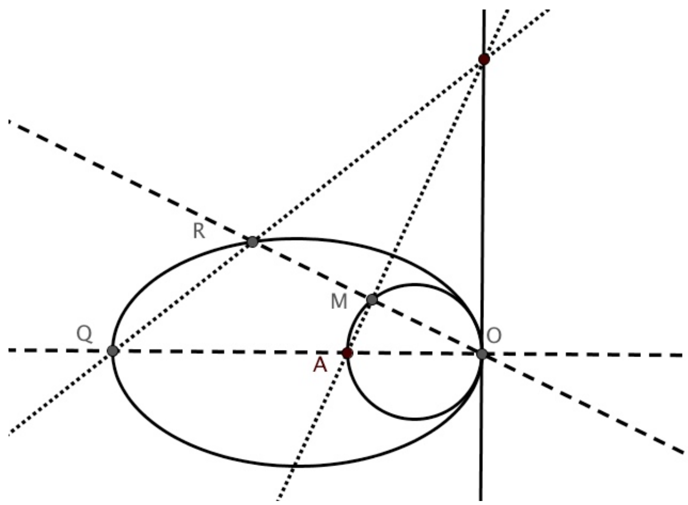

Figure 2.5: An interpretation of Plücker's virtual figure

through a given third coplanar point A – a solution Plücker described as "analogous" to the one given by Poncelet above. Plücker remarked that he had described the theorem and construction of only two conic sections for ease of presentation, but the same properties could be applied to several conics satisfying the coefficient constraints of fixed β and δ. In the case where the tangent point was replaced by a tangent line, "one could easily determine the contact point of the given straight line and the curve to construct" (399).

By substituting the given second order line with a system of two straight lines, Plücker stated two of many possible theorems, "which at first do not appear related to the announced theorem" (399). In this brief aside, Plücker provided evidence for the potential generality of analytic geometry in synonymously treating a conic section as a pair of straight lines.

As a final consequence of this theorem, Plücker "observed in passing" that the third order case could be derived "immediately" from the second order case if the point of intersection P "concurred" with the second order contact O, thereby replacing the secant line OP with a common tangent. Notably, though Plücker emphasized the definition of contact order as point coincidence in his theory of osculation, in this application he only referred to this aspect of the definition in retrospect, since he moved from higher to lower order contact.

In the case of a simple contact, for given values of the coefficients α, β all chords represented by (2.11) with varying δ would be parallel. Plücker emphasized the analogy of this result to the above and the potential application to any numbers of conic sections through suggestive notation.

> [...] si l'on mène par le point de contact O un couple quelconque de lignes droites Oq et Or, qui rencontrent les différentes courbes l'une aux points $Q, Q', Q'', ...$ et l'autre aux points $R, R', R'', ...$, les cordes [sic] qui passent respectivement par

Q et R, par Q' et R', par Q'' et R'',..., seront toutes parallèles entre elles.[45]

Plücker's derivation and resulting ruler construction for curves of simple contact only applied to finding points on a conic similar to the given conic. Once more, Plücker remarked that the given points could be replaced by tangents and a valid theorem would result, but did not elaborate the details of the procedure. The third order case again followed as a corollary, when the two simple points of contact coincided.

Plücker repeatedly described theorems, problems and corollaries as analogous, the use of only virtual figures in this process encouraged seeing similarities rather than differences. Another advantage to employing only virtual figures for Plücker's constructions was in the possibility of suggesting a large number of possible objects. His use of "..." in theorem statements had no figurative equivalent.

As in the case of the common chord to two non-intersecting curves, Plücker again faced existence considerations in examining conditions such that several second order lines shared two distinct points of tangency. As in his introduction, the equations of the conics were (1) and (2), tangent to the y-axis at the origin. Plücker combined the two equations by multiplying the first by d, the second by D and taking their difference. Plücker explained that the resulting equation,

$$(d - D)y^2 + 2(Ad - aD)xy + (Bd - bD)x^2 = 0. \tag{2.13}$$

contained the points of intersection between the two curves. Plücker noted, "at the same time," (2.13) was a system of two straight lines passing through the origin. Here, if the roots of the equation, and consequently the corresponding lines, were not imaginary then each line would intersect each curve at one other point besides the origin. In the case of exception, when the lines were imaginary, then the second intersection points "would no longer exist." Plücker provided no additional direction toward obtaining a construction in this case, which contrasted sharply with Plücker's treatment of a common chord containing imaginary points as expressible by an equation in his theory of osculating curves.

From Plücker's theory of osculation, the desired condition for the conics to share two points of contact would be obtained if for each pair the two intersection points coincided. That is, both straight lines passed through the real square root of (2.13), and the system of lines decomposed into two identical equations. However, Plücker recognized the potential radical calculation required to find that root, and instead chose a coordinate system that would modify (2.13) into a conveniently manipulable form. Within this context, Plücker designated the common chord between the two common tangent points as the x-axis. Then

[45] "[...] if one draws through the point of contact O any pair of straight lines Oq and Or that meet the different curves respectively at the points Q, Q', Q'', ... and R, R', R'', ..., the chords that pass respectively through Q and R, through Q' and R', by Q'' and R'',..., will be mutually parallel."

(2.13) would be of the form $y^2 = 0$, and consequently the coefficients would be in the following invariable ratios:

$$\frac{d}{a} = \frac{D}{A}, \frac{d}{b} = \frac{D}{B}, \frac{b}{a} = \frac{B}{A}. \tag{2.14}$$

In an accompanying footnote, Plücker justified his choice of coordinates, suggesting an alternative choice based on the conic section equations, but concluding that the calculation free approach was superior.

> Mais en partant de ces équations, la discussion du numéro suivant [section 10] ne gagnerait rien en breveté [sic] et perdrait en symétrie. Et, étant permis sans doute, de choisir les axes des coordonnées à volonté, il me semble cependant qu'une démonstration gagne en élégance si le calcul même nous détermine à ce choix. (397)[46]

Thus, for Plücker, calculation should be minimized by the choice of coordinates, in order to create more elegant and symmetrical proofs.

Within this new coordinate system, there would be an infinite number of conics of the form (2.7) that would be tangent to each other at the origin O and at a second point P given by $x = -\frac{2\delta}{\beta}$. Plücker found that any line through O could be represented as $x + my = c$ and any line through P as $\beta x + \beta m' y + 2\delta = 0$, where m, m' were indeterminate and variable quantities. By combining these three above equations, Plücker eventually determined that the straight line through the two points where the lines containing O and P respectively meet the curve for a second time was given by

$$(mm'\beta - 1)y + ((m + m')\beta - 2\alpha)x + 2\delta m = 0. \tag{2.15}$$

Then the intersection of this line with the x-axis would be at

$$x = -\frac{2m\frac{\delta}{\alpha}}{(m + m')\frac{\beta}{\alpha} - 2}. \tag{2.16}$$

Because Plücker had chosen his coordinate system so that $\frac{\delta}{\alpha}, \frac{\beta}{\alpha}$ were in constant ratios, this intersection would be constant for all curves satisfying (2.7). So, finally, given a conic containing points O and P Plücker could determine a conic of double contact at O and P and passing through a third planar point Q. As in all the prior cases, Plücker noted one could replace the given point Q with a given tangent line. One could also coincide the tangent points O and P in order to deduce the construction of Poncelet's problem as a corollary.

[46] "But in starting from these equations, the discussion of the following section gains nothing in brevity and loses symmetry. And, being undoubtedly permitted to choose the coordinate axes at will, it seems to me however that a proof gains in elegance if the calculation itself determines this choice for us."

Thus, we can see that in Plücker's systematic treatment of different types of curve tangency he presented four alternative variations on the derivation of Poncelet's construction. First, he directly determined a conic section of third order contact. Then, from the three cases of lower order contact, Plücker derived three additional constructions. He considered these derivations as more general than Poncelet's solution, which followed as a corollary to each of them.

Plücker concluded his paper by another application of this definition of order in which he considered two conics with four distinct points of intersection, "pour faire voir l'uniformité du mode de discussion" (400). Plücker thus treated every case of intersection between two conic sections.

2.4.3 The form of Plücker's geometry

The uniformity Plücker described in his final application reflected the structural symmetry between the successive problems related to conic section construction. Every tangent discussion opened by considering the coefficients and properties of the coordinate representation from which resulted a constructive geometric theorem. Every theorem led immediately to a constructive solution followed by suggested variations for the number of possible curves and replacing points with tangent lines. Finally, in all but the first problem, Plücker would return to Poncelet's construction, which could be derived as a corollary to each of the following more general cases. Plücker emphasized the generality of his results only with respect to a lower order contact. However, one could also interpret Plücker's claim to uniformity as one positive quality of a more general method. Gergonne had presented uniformity and generality as the twin benefits of analytic geometry in his "Réflexions" following Poncelet's 1817 article, which Plücker almost certainly would have read (Gergonne (1817e), 156). While achieving uniformity, Plücker was willing to sacrifice some initial simplicity or symmetry in order to use coordinate equations with minimal calculation. Where he succeeded, Plücker described his results and methods as elegant, simple, and effortless. Applying Poncelet's criteria, these results were thus *evident*.

Plücker's approach to analytic geometry contrasted with that of his predecessors. Plücker himself would describe his contributions as "eine neue Behandlungsweise der analytischen Geometrie" (Plücker (1828a), iii). In a later review of his *Analytisch-geometrische Entwicklungen* by August Cournot appearing in the *Bulletin des sciences*, Plücker's geometry was described as an analysis that "ressemble fort à la synthèse" (Cournot (1828), 178). Similarly, historian Philippe Lombard defined the synthetic approach, such as that practiced by Poncelet, as that "où elle s'efforçait d'éviter au maximum les calculs" (Lombard (2011), 33). Thus, by using fewer calculations, Plücker's analytic geometry appeared categorically less analytic. Plücker consistently identified his research as analytic geometry, or even pure analytic geometry, but his style could be perceived as a middle ground between the pure and analytic methods.

While Plücker countered Poncelet's description of analytic geometry as geometry with calculation, we should also consider Poncelet's criticism that through calculation one lost the figure from view. In Plücker's geometry, each geometric construction utilized only visualizable geometric objects, non-figurable results like imaginary points were dismissed. The separation into theory and application enabled Plücker to develop two different standards of geometrical existence. In Plücker's theory of osculating curves, the coordinate equation guaranteed representability even when the geometrical object did not "exist as such". In the applications to geometric constructions, non-existence seemed to be a constructive dead end. There was no place for imaginary points in deriving solutions and Plücker was able to ignore existence concerns in the statement of problems by assuming as given the number of points and their order of intersection. Imaginary objects could thus be avoided a priori. All of Plücker's constructions rested entirely on the use of virtual figures and followed from theorems stated in terms of figures. However, these theorems were proved through consideration of coefficients in coordinate equations designed in order to minimize calculation. In Plücker's theory of osculation, the corresponding visualization of these systems was neither obvious nor explained. The applications relied on figures and required visualizability to secure evidence, the theory did not.

For Plücker, figures did not play an intermediary role in theorems or proofs, they were only present in constructions and these two types of geometric practice were structurally segregated in the form of this article. Further, these constructions were not intended to be original, but rather followed from the pure geometry of Poncelet. While Poncelet referred to figures as ambiguously equivalent to geometric objects, in this paper Plücker referred specifically to geometric loci, curves, lines or points–never providing a general designator for the group as a whole. Overall, Plücker did not use the figure as a tool of proof in this paper, except in his frequent mention of deriving higher order contact as a corollary. Yet even in this case, the use of the figure was not the means of discovery, only an alternative route via a corollary to a theorem that had already been given an analytic proof. It seems inaccurate to say that the figure was "lost" from view, when Plücker never described a figure in the first place.[47] For Plücker the equation expressed, represented, indicated and gave the geometric results. In particular, coordinate equation representation was enough for Plücker to legitimate the use of imaginary objects in his theory of osculating curves. Consequently, Plücker's analytic geometry was not analysis applied *to* geometry, but a tool *within* geometry itself. In this role, the form of the equation–not the computations–was the

[47] As we will witness in Chapter IV, within the pedagogical context of his *Analytisch-geometrische Entwicklungen* Plücker relied heavily on illustrated figures as a means of explanation. Even so, his other research articles from the 1820s and 1830s continued to focus on the form of equations.

evidence in support of Plücker's geometric research.

2.5 Poncelet's secants without intersection points, 1822

Poncelet first published his proof of the third order contact construction in *Traité des propriétés projectives, ouvrage utile à ceux qui s'occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain*, which appeared four years before Plücker's article, in 1822. With this text, we will focus on Poncelet's new definitions and principles comprising his theory of osculating curves.[48] Poncelet's seemingly simple constructions from 1817 relied on a deep theoretical background, above all his *principle of continuity*.[49] As Poncelet explained in his introduction, the principle of continuity established the projectively invariant properties (which Poncelet would elaborate more precisely throughout his text) as a primitive figure was deformed.

> Considérons une figure quelconque, dans une position générale et en quelque sorte indéterminée, parmi toutes celles qu'elle peut prendre sans violer les lois, les conditions, la liaison qui subsistent entre les diverses parties du système; supposons que, d'après ces données, on ait trouvé une ou plusieurs relations ou propriétés, soit métriques, soit descriptives, appartenant à la figure, en s'appuyant sur le raisonnement explicite ordinaire, c'est-à-dire par cette marche que, dans certains cas, on regarde comme seule rigoureuse. (Poncelet (1822), xiii)[50]

The indeterminate figure was Poncelet's purely geometric equivalent to indeterminate coefficients in a coordinate equation. He admired the powerful generality of the method of coordinates and sought to establish that same generality in pure geometry directly through the figure. The generality then became evident.[51]

[48]For an in-depth historical analysis of Poncelet's new definitions and principles, the reader is further referred to Friedelmeyer (2011).

[49]Poncelet's controversial principle of continuity is well-known and studied, especially as it pertains to generality. See Chasles (1837), Belhoste (1998), Gray (2010b), Nabonnand (2011b), and Friedelmeyer (2011). In the context of the figure, we will suggest a supplementary interpretation of Poncelet's principle as a means for manipulating non-visualizable objects. In Chapter III we will describe how the principle became known as "controversial," and several attempts to remedy this.

[50]"Let us consider any figure in a general and in some sense indeterminate position, among all those that it can hold without violating the laws, the conditions, the relationships that exist between the different parts of the system; let us suppose that, according to these givens, we have found one or several relations or properties, either metric or descriptive, belonging to the figure, and resting on ordinary explicit reasoning, that is to say, by the path that, in certain cases, we regard as the only rigorous one."

[51]Nabonnand describes how Poncelet assumed objects as inherently general, and consequently the choice of method (analytic or pure) should not effect the particularity of the result.

> L'objectif que se fixe Poncelet est en partie identique à celui de Carnot: "faire passer dans la Géométrie ordinaire la généralité des conceptions de l'Analyse algébrique, généralité qui doit nécessairement appartenu à l'essence même de la grandeur figurée, indépendamment de toute manière de raisonner". La généralité est inhérente à l'objet étudié ce qui justifie le projet de

In his *Traité* Poncelet would explore how descriptive and metric properties changed or remained invariable for all possible states of the corresponding indeterminate figure. One result of this emphasis on invariance was an extension of geometric definitions to incorporate and geometrically explain results derived from analysis. Poncelet justified this step as an extension of the indeterminate generality already present in ordinary geometry. He pointed to the Euclidean use of proportions in geometry, which left the position of points and lines unspecified. Likewise, Poncelet proposed to prove geometric results that would apply without exception to real and imaginary figures.

> Or c'est précisément cette dernière dépendance, entre des figures qui paraissent, au premier abord, n'avoir rien de commun, qui peut exiger qu'on introduise, dans le langage et les conceptions de la Géométrie, les expressions et les notions abstraites de l'Analyse; elles seules, en effet, peuvent permettre d'établir un point de contact, sinon absolu, au moins fictif, entre certaines figures et certains résultats géométriques. (Poncelet (1822), xxv)[52]

2.5.1 Poncelet's extended definitions

Among these abstract notions and expressions imported from analysis, was the concept that any two conic sections shared a common secant, even when they did not intersect. The common secant could always be determined analytically as a linear equation the combination of two second order equations. Likewise, Poncelet asserted that any two coplanar conic sections shared a common secant. He defined a common secant with respect to the figure presented below through the use of harmonic proportions (Figure 6 in Poncelet's original text, as can be seen in the image; here it is Figure 2.6.)

From this definition, Poncelet introduced a second *secant* to C as the line parallel to MN and containing O', the line $M'N'$. The segment of the line parallel to MN with midpoint O', was then a *chord* $M'N'$ to C–even though it did not appear to be one.[53] In

> chercher des méthodes générales en géométrie pure et donc de ne pas se contenter de l'approche analytique. (Nabonnand (2011b), 29)
>
> "The objective that Poncelet sets is in part identical to that of Carnot: "to transport into ordinary Geometry the generality of the concepts of algebraic Analysis, generality which must necessarily belong to the same essence of the figured quantity, independent of all manner of reasoning." Generality is inherent to the object studied which justifies the project to find general methods in pure geometry and thus not be content with the analytic approach."

[52] "Thus it is precisely this latter dependence, between figures which appear, at first, to have nothing in common, which can exist when one introduces the expressions and abstract notions of Analysis into the language and the conceptions of Geometry; they alone, in effect, can help establish a point of contact, if not absolute, at least fictive, between certain geometric figures and results."

[53] Poncelet recognized that the relationship between O' and MN, or between O and $M'N'$ was that of pole and polar. However, he preferred to use the term secant in order to preserve the analogy between ideas and language corresponding to real chords (Poncelet (1822), 27).

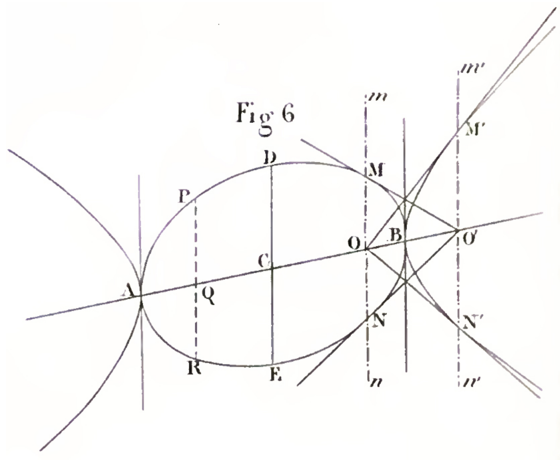

Figure 2.6: Poncelet (1865), Planche I

Poncelet's Figure 5, the chord $M'N'$ did not intersect the curve C at real points. Thus the points M' and N' were imaginary points of intersection, but the chord $M'N'$, which was also referred to as the secant $M'N'$, was termed "ideal". Poncelet explained the choice of the adjective ideal in order to distinguish between constructible objects containing imaginary points (the ideal chord and secant) and entirely non-constructible and impossible objects (the imaginary points of intersection).

Thus, since any point in the plane had a corresponding conjugate harmonic point with respect to a given curve, this harmonic relationship could be applied to find a common secant (real or ideal) between any two curves in the plane. Applying this to determine ideal common chords shared by two conic sections, one could find an infinite number of chord pairs satisfying the conjugate harmonic relationship through tangent construction. By choosing a pair where the chord of the first conic section was greater than the second and another pair with the inverse relationship, the law of continuity would then serve to prove the existence of a shared equal length chord common to both.

The generalized common secant was not necessarily evident as such in a two dimensional representation, it might not recognizably share points with either conic. Poncelet enhanced his proportion based definition with a simple planar visual interpretation of when the secant did not intersect two conics at real points, that is, the conics shared two imaginary points of intersection. By taking successive points along the diameter AB, one could derive a series of parallel *ideal* chords whose endpoints would lie on a *supplementary conic* to the curve C. Within this supplementary conic, the ideal chords of C would be real chords, and vice versa. In our Figure 2.6, the given ellipse C is supplementary to the hyperbola containing points A, B, M', and N'. The general common secant played an important role when applied to osculating curves, as we will see.

Poncelet rested his treatment of conic sections on a preliminary examination of circles. He then extended results derived for circles to general conics via perspective or other applications of the law of continuity. This approach was not at all original with Poncelet. By the early nineteenth century geometers commonly (and often without justification) extended circle results to any conic section.[54] By focusing on circles, Poncelet was able to exploit the unique property that any two circles were similar and similarly placed (Poncelet adopted the abbreviation s. and s.p., which we will also use here) in a plane.

Poncelet had first introduced the concept of s. and s.p. with respect to polygons. For example, two s. and s.p. coplanar triangles would have corresponding *homologous* proportional sides and be in the same orientation so that lines containing each pair of corresponding "homologous" vertices would intersect at the same point. Following contemporary usage, Poncelet called this point the "similitude centre."[55] Since any two circles are s. and s.p. figures (and so each point of one circle was homologous with a corresponding point of the other), Poncelet could determine their similitude centre. In general, a direct or opposed similitude centre of two circles could be found by the respective intersection of their internal or external common tangents.

Poncelet explained that when the common tangents were impossible or imaginary, such as when one circle lay within the other, one could still find a graphic and intuitive mode of representation by considering the "supplementary hyperbolas" of these circles. These supplementary curves would have the same similitude centre as the circles and in the case where the circles' common tangents were imaginary, those of the hyperbolas would be real. Since any circle or supplementary equilateral hyperbola has a centre of symmetry (which is the origin of a circle or the centre of the equilateral hyperbola), any two circles or equilateral hyperbolas would have two centres of similitude. Conversely, given a similitude centre, Poncelet demonstrated how to find corresponding homologous points on two given circles, and proposed that this argument was *evident* because clearly apparent in the figure.

Each pair of directly homologous lines would intersect in a point. In Figure 38 (our Figure 2.7), the line AB is parallel to its directly homologous line $A'B'$ so their point of intersection at infinity is not pictured. Directly homologous lines ED and $E'D'$ would also intersect in a point, here also at infinity. Because of the proportional invariance, the line containing these two intersection points would be a common secant to the two circles, here a line at infinity. Likewise, the line determined by the intersection of AB with $E'D'$–the

[54] Gergonne made frequent references to proofs or constructions for the circle being valid for any conic section through perspective. For instance, in a footnote to a posed problem response he extended the original solution, "La proposition étant ainsi démontrée pour le cercle , se trouve l'être aussi pour toute section conique, qui peut toujours être considérée comme la perspective d'un certain cercle" (Gergonne (1813a), 163). The prevalence of this practice among geometers in the early nineteenth century is noted in Friedelmeyer (2011).

[55] We will return to the similitude centre construction as described by Steiner and Gergonne in Chapter IV.

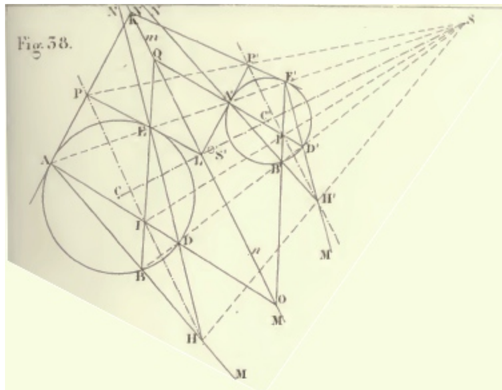

Figure 2.7: Poncelet (1865), Planche V

point M–and the intersection of ED with $A'B'$–the point N–as the intersections of inversely homologous lines would be another common secant. Note that in his Figure 38 (our Figure 2.7), Poncelet employed his extended definition of common secants, that is, the secants presented were not visually lines passing through either of the two circles. Moreover, as the construction showed, the common secant determined by intersecting directly homologous lines was at infinity since the lines determining this secant were parallel. From this construction, Poncelet observed that the finite common secant was also the *radical axis* between two circles, and thus, tangents at homologous points of the two circles would meet on this line.[56] This enabled a construction of a common secant given only one transversal through the similitude centre.

Poncelet offered two different three-dimensional interpretations of the above planar relationship, which would lend evidence to the not-figured metric properties. First, Poncelet noted that any two circles could be considered as the projection or perspective of one another with the similitude centre taken as the centre of projection. Correspondingly, one could manipulate the figures and examine their properties through changing the centre of projection. For example, projecting the infinite secant to the finite secant would result in a switch between inversely and directly homologous points and lines. In Poncelet's alternative visualization, the two circles could be pictured as planar sections of a conic surface with its vertex at the similitude centre. This viewpoint had the advantage of providing an intuitive notion of common secants at infinity as represented by parallel planar sections. The common finite secant of two circles was the line on which the two planar sections concurred.

Poncelet admitted that the construction of common secants would become impossible and illusory when the given similitude centre was at infinity. Because each common secant was also the polar to a similitude centre with respect to the given circles, if the similitude

[56]The concept of a radical axis was attributed to Louis Gaultier de Tours, whose definition is examined in Appendix F.

centre, say S was at infinity and the corresponding secant of contact MN would coincide with the line containing the circles' centres, CC'. The same situation arose in the case where the transversal through a similitude centre contained the circles' centres. In these cases, the corresponding tangents between the two circles ($AB, ED, A'B', E'D'$) would all be parallel. This interpretation, although imaginable, exceeded the bounds of the illustrated figure.

Poncelet summed up his findings: for two circles the similitude centre was also the concurrent point of common tangents, the convergence point of homologous lines, and the centre of projection between the two circles. Generalizing, Poncelet asserted that any conic section could be considered as the projection of another with the centre of projection at an intersection point between their common tangents. Further, he defined homologous points between the two conic sections as those lying on lines through the centre of projection and the homologous lines (straight or curved) containing these points would concur on the common secants (or axes of projection). For any two curves, when their common secant passed to infinity the centre of projection would become the similitude centre and the figures would become s. and s.p..

In a visualizable three-dimensional explanation, Poncelet proposed considering any two coplanar conic sections as the projection of two planar sections of a cone whose vertex projected to the point of concurrence of the common tangents to the original coplanar conic sections. To show the validity of this interpretation, he offered what he described as a simple, direct proof. From an examination of Poncelet's archival notes at the École polytechnique, we found that most of his working drawings were not formal constructions, but instead approximative sketches through which one could gain the sense of the construction. We present an illustrated sketch for the case of two ellipses based on the virtual figure in Poncelet's text (Figure 2.8).

Let C, C' be two conics with point S as any point where their common tangents concur. Consider the surface of a cone with C as its base and its vertex at any point in space. Project the cone such that its vertex goes to S. Then the cone's two extreme edges would project to the common tangents of C and C' passing through S. Next, take any point a on C' and project this point onto the conic surface from S as the centre of projection. Project the points of tangency of the common tangent on C' to the cone as well. These three points would define a plane cutting the cone in a conic section, whose projection onto the plane of the base C would coincide with C' because three points with a tangent at each of two points determined a unique conic section. Since the projection of a corresponded to two points on the conic surface, there are two such possible planes determined. Thus conic sections could be considered in two different ways as planar sections of the same cone with S as the vertex.

Following this proof, Poncelet pointed out an exception for when the point S did not belong to two real common tangents. To cover this case, he referred to the principle of

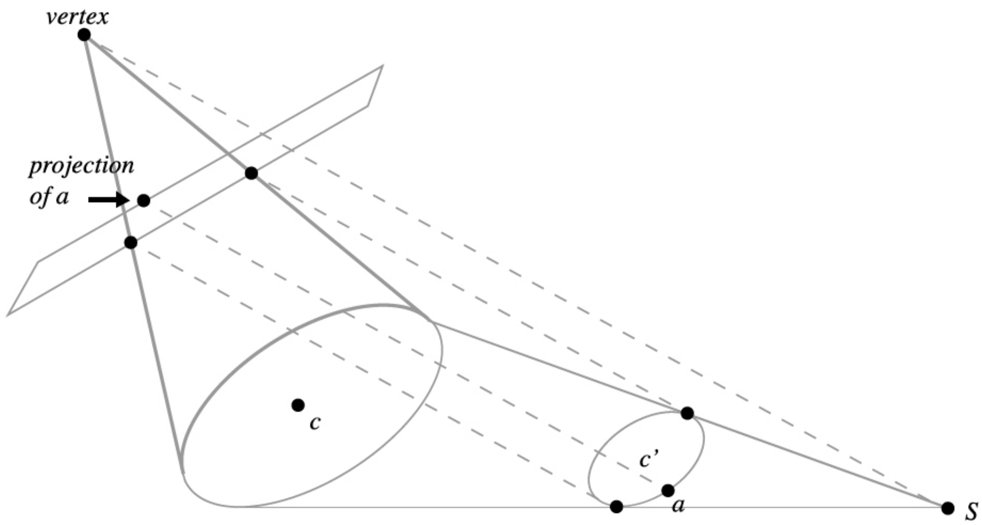

Figure 2.8: Sketch of projection to and from a cone

continuity which extended the result to the non-constructible case through deforming the tangent lines by insensible degrees until they intersected at a real point. Poncelet argued that common tangents of two curves uniquely determined two common conjugate secants and reciprocally, in the imaginary tangent case one could use constructible real or ideal common secants to obtain the necessary point of projection.

Having shown the intimate relationship between the particularity of two circles and the generality of two conic sections, Poncelet explained his choice of language and extension of definitions by projective invariance and analogy

> ...d'autant qu'il nous semble extrêmement avantageux, pour la langue géométrique, de pouvoir désigner un même objet par plusieurs mots, quand ces mots correspondent à des vues différentes de l'esprit, ou rappellent des propriétés distinctes de cet objet. (Poncelet (1822), 155)[57]

Poncelet argued that multiple designations of a single specified geometric object could emphasize the variety of properties inherent in that object. Further, his understanding of the role of language reinforced his position on the importance of the visual nature of geometry. Specifically, by labelling a point as a centre of projection, the name signified the projective relationship between that point and the other figures. However, the name gave no indication that the relationship of this point with respect to the projected conics was analogous to the similitude centre with respect to two circles. So Poncelet recommended

[57]"...insofar as it seems extremely advantageous, for geometric language, to be able to designate the same object by several words, when these words correspond to different mental views, or recall distinct properties of the object."

that geometric language should employ the use of synonyms in order to signify different properties and intentions inherent to the same object. In the situation of two conic sections that were not similar, the term "similitude centre" could no longer be applied in reference to the centre of projection without creating misunderstanding. Instead, Poncelet designated this point as the "centre of homology." Then the line on which homologous lines concurred would be called the "axis of homology." Finally, the lines converging toward the centre of homology would be "rays of homology." These new definitions indicated the "perfect analogy" with s. and s.p. figures that Poncelet intended to utilize in this context. Like figures, well chosen vocabulary provided immediate information. By changing the terms of reference, Poncelet could use fewer words to convey new and familiar objects and relations.

2.5.2 Poncelet's conic section constructions

Poncelet applied these definitions and properties to constructing conics of third order intersection. He introduced the chapter on "Applications à la théorie des contacts des sections coniques" by explaining a physical derivation of points of higher order by the coincidence of intersection points: a procedure he described as direct and simple.

> [...] quand deux points communs au système de deux sections coniques viennent par un mouvement continu, à se réunir en un seul, ces deux courbes se touchent nécessairement en ce point; [...] ces mêmes courbes deviennent osculatrice du second ordre et du troisième ordre, lorsqu'un ou deux nouveaux points communs à ces courbes viennent pareillement à se réunir en un seul avec les deux premiers. (Poncelet (1822), 166)[58]

Higher order contact points were essentially the unification of two or more intersection points into one. Poncelet actively described the process of two points continuously moving together and thus forming osculating curves from intersecting ones. Connecting this description with his newly defined terms and beginning with the scenario of two conics intersecting in four points, there would be in general six points of homology. As Poncelet had explained with respect to circles, common tangents to two curves always concurred on the curves' common secants. Thus if a common secant reduced to a point, that point would be the intersection of tangents and by definition a centre of homology for the two curves. Poncelet discussed the higher order tangent points each with reference to the modified figure of two conic sections with four points of intersection. This primitive configuration was the only figure neither pictured nor described, but we give an unlabelled figure with an ellipse and a circle for reference indicating where the points of homology would be (Figure 2.9).

[58] "[...] when two common points in the system of two conic sections move continuously to coincide in one point, these two curves must be tangent in this point; [...] these same curves become osculating curves of second and third order, when one or two new common points of these curves similarly coincide in one of the two first points."

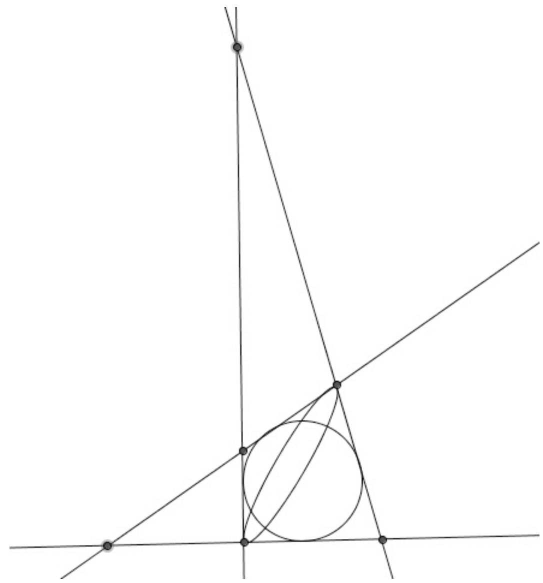

Figure 2.9: Six points of homology and four points of intersection between two conics

Poncelet began by describing how to construct a first order contact between two conics from a pair of conics intersecting in four distinct points. By a continuous movement, one common secant could reduce to a tangent point S along the common tangent line to both curves at S (Fig 41, our Figure 2.10). This point S, having been derived from a common secant, would be a centre of homology. The remaining three common tangents would intersect at the three other centres of homology (S', s, s') belonging to the two conics. The line MN containing the two intersection points of the conics would be a common secant with respect to the two conics, so an axis of homology with respect to the centre of homology S'. Any pair of homological lines $a'b'$ and ab would coincide on MN, as pictured in the point l. The other two centres of homology s, s' would lie on the common tangent line to the conics at S. Recall, from Poncelet's definition of common secants that he used the term conjugate point and line to describe the relationship also denoted as pole and polar. Here he used the terms interchangeably.

While Plücker's theorem concerning simple tangency applied only to similar conics, Poncelet had already established a means to translate properties from similar to non-similar figures through the use of projection to infinity. Thus we see that Poncelet's approach applied to any two conic sections.

To create a second order tangency, Poncelet continuously deformed Figure 41 (our Figure 2.10) until the point N coincided with the centre of homology S. In Figure 42 (our Figure 2.11) this point is represented by S. For this configuration, there would be two common tangents, $S'T$ and $S'S$, and one common secant SM. This secant thus contained the intersection of the homologous lines ab and $a'b'$.

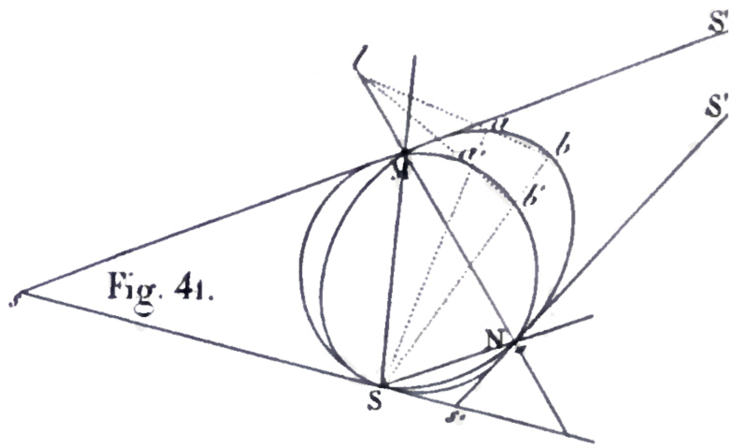

Figure 2.10: Poncelet's Figure 41 (Poncelet (1865), Planche V)

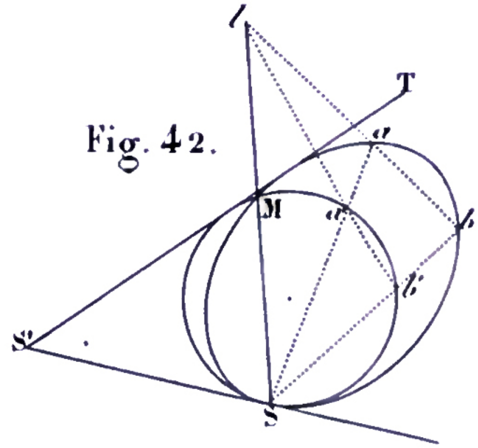

Figure 2.11: Poncelet's Figure 42 (Poncelet (1865), Planche V)

From Figure 42, Poncelet continuously moved point M until it met with point S. Since now all four former intersection points coincided at point S, this was a third order contact. As shown in Figure 43 (our Figure 2.12), only one centre of homology remained at point S. The former common secant SM was now the tangent line at point S. Thus this tangent line would also contain the intersections of all pairs of homologous lines.

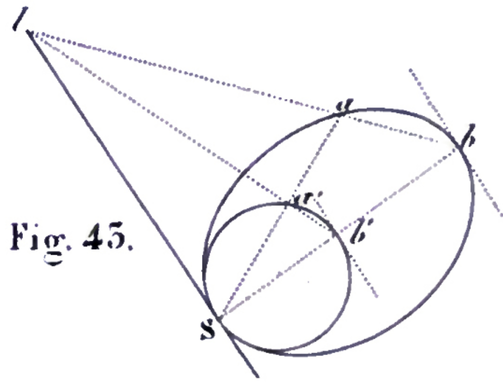

Figure 2.12: Poncelet's Figure 43 (Poncelet (1865), Planche V)

Finally, to obtain a double contact, between two conics, i.e. two points of simple contact, Poncelet returned to the configuration represented by Figure 41. If the two points of intersection M and N coincided in a single point, labeled in Figure 44 (our Figure 2.13) as S', then this point would be a centre of homology for the two conics. The common tangents to the curves at S and S', the lines SP and $S'P$, would be axes of homology. Their point of concurrence P would be the centre of homology, as well as the pole, conjugate to the finite common secant SS'.

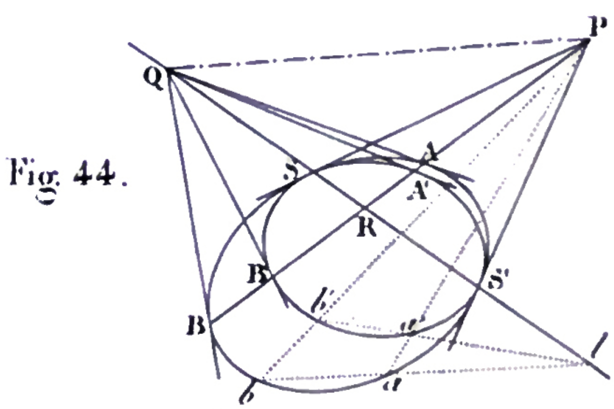

Figure 2.13: Poncelet's Figure 44 (Poncelet (1865), Planche V)

In passing, Poncelet noted that for the case where a centre of homology was at infinity, the corresponding tangents would become parallel and the homologous lines would meet on the axis of homology, the conjugate (or polar) of the point at infinity.

Poncelet asserted that the properties found in these four cases could be extended and used to determine a conic of first, second, third order single tangency or double tangency. In the case of a third order tangency–as presented in 1817–one would require only the third order contact (here, equivalent to the centre of homology) and either a second point on the other conic or a tangent to the other conic to derive a construction. Through his detailed and figure driven exposition, the description of such a curve was now evident.

> Il est évident que, dans ces diverses circonstances comme dans celles qui précèdent, l'on pourra toujours décrire l'une des deux courbes au moyen de l'autre et de certaines données, par quelqu'un des procédés généraux qui font le sujet des articles [...], le tout sans employer autre chose qu'une simple règle ou des jalons, si l'on opère sur le terrain. (174)[59]

Poncelet concluded by describing the necessary physical tools, ruler or ranging pole, and surfaces, the earth, to effect the above constructions. More than a result, Poncelet wanted to demonstrate an evident and thereby memorable procedure for practical derivation.

2.5.3 Generality and evidence

Poncelet strove to achieve geometric generality without losing the evidence of the figure, which led to two types of generalizations: a generalization of specific objects through the definition of ideal chords, secants, and tangents as well as a generalization of the process of gathering evidence.[60] Imaginary points were still not represented nor considered representable, but they could be manipulated and they possessed well-known properties derived from their real counterparts. Furthermore, ideal objects containing imaginary points only needed to be made evident through representation of their real points in the figures.[61]

> Pour concevoir l'objet de ces définitions, il suffit de supposer que la section conique que l'on considère ne soit pas décrite, mais seulement donnée par certaines conditions, et qu'alors on se propose de rechercher, soit les points où elle

[59] "It is evident that, in these different circumstances as in those which preceded, we could always describe one of two curves by means of the other and certain data, through one of the general methods which are the subject of these articles [...], all without using anything other than a simple ruler or ranging pole, if we use it on the terrain."

[60] Poncelet's attention to generality serves as a dominant theme in his historiography, in particular recent studies on how Poncelet' interpreted and manifested generality Friedelmeyer (2011), Nabonnand (2011b), Nabonnand (2011a), Gérini (2010b), Chemla (1998).

[61] Almost simultaneously, in the *Annales* mathematicians, including Gergonne, were discussing the representation of numbers as points in the complex plane. Although both called for greater generality in mathematics, we have not observed any specific overlap in reference between these two discussions. For more on the geometric representation of numbers in the complex plane see Lützen (2001).

> est rencontrée par la droite réelle tracée sur son plan, soit tout autre objet qui en dépende; car on ignore alors si les uns ou les autres sont ou non possibles, et il est naturel de persister, dans tous les cas, à regarder cette ligne droite comme une sécante véritable de la courbe, et par conséquent de la traiter comme telle dans le raisonnement géométrique qui sert à faire découvrir les objets qu'on cherche. (Poncelet (1822), 28)[62]

One consequence of Poncelet's proposed generality was that formerly finite and real objects–such as a pair of concentric circles in the plane–possessed infinite and imaginary points–two imaginary points of double contact on their common chord at infinity. Poncelet's new definitions weakened the boundary between real and imaginary objects, and as justification he pointed to the *diverses modes of existence* already present in geometry with respect to infinity and infinitesimals. Poncelet was able to manipulate infinite and imaginary points by a process of translation using perspective, cones in space, supplementary curves, and his principle of continuity.

Poncelet provided liberal illustrations of his geometric practices with figures depicting his numerous planar definitions, constructions, and proofs. However, Poncelet's so-called *modern* geometric techniques were not shown on paper. The effects appeared in figures, but points at infinity, imaginary lines, central projection, and even three dimensional objects were deliberately invisible. In this, Poncelet followed an established practice for three-dimensional geometry. As Michel Chasles (1793-1880) later described, Monge's lectures on the application of space to the plane were presented without physical representations either, simply through the evocative use of gestures and speech.

> C'est une tradition, dans l'École polytechnique, que Monge savait, à un degré inouï, faire concevoir dans l'espace toutes les formes les plus compliquées de l'étendue, et pénétrer dans leurs relations générales et leurs propriétés les plus cachées, sans autre secours que celui de ses mains, dont les mouvements secondaient admirablement sa parole, quelquefois difficile, mais toujours douée de la véritable éloquence du sujet: la netteté et la précision, la richesse et la profondeur d'idées. (Chasles quoted in Belhoste and Taton (1992), 290)[63]

[62] "To conceive the object of these definitions, it suffices to suppose that the conic section that we consider is not described, but only given by certain conditions, and that then we propose to find, either the points where it meets the real line traced on its plane, or any other object on which it depends; for we don't know then if the one or the other are possible or not, and it is natural to continue, in all cases, to regard this straight line as a true secant of the curve, and consequently to treat it as such in the geometric reasoning, which serves to discover the desired objects."

[63] "There is a tradition, in the École polytechnique, that Monge knew to an unheard of degree, how to render conceivable space forms of the most complicated extension, and how to penetrate their general relations and their most hidden properties, without any other recourse except that of his hands, whose movements admirably followed his words, sometimes difficult, but always endowed with true eloquence on the subject: the clarity and the precision, the richness and the depth of ideas."

Thus in the academic setting it was not always expected to produce three-dimensional renderings on paper in order to grasp enough evidence for geometric understanding. Similarly, Gergonne spoke to the possible confusion in accompanying three-dimensional descriptions with physical figures in a footnote to a construction on enveloping surfaces.

> Je sous-entends la figure, qu'il est plus aisé de concevoir que de représenter, sans confusion. (Gergonne (1813c), 365)[64]

Gergonne later argued that figures in the geometry of space were far more useful when conceived, or, if necessary, constructed by the reader. Including illustrations of such figures to accompany the text was likely to be a hindrance, enforcing the particularity of geometry.

> Nous croyons superflu d'accompagner ce mémoire de figures, souvent plus embarrassantes qu'utiles, dans la géométrie de l'espace; figures que nous ne pourrions d'ailleurs offrir que sous un aspect unique et individuel au lecteur qui pourra, au contraire, les construire et façonner à son gré, si toutefois il en juge le secours nécessaire. Il ne s'agit ici, en effet, que de déductions logiques, toujours faciles à suivre, lorsque les notations sont choisies d'une manière convenable. (Gergonne (1826), 212)[65]

Though one can find illustrations of three-dimensional objects in descriptive geometry texts, the use of three-dimensional description by Poncelet, Gergonne, or even Monge appears intended to create an impression rather than a technical construction.

Part of Poncelet's innovation in this respect was the use of projections to bring non-figured, and not obviously figurable, objects into the real plane where they could be manipulated, illustrated, and then become imaginary or infinite while maintaining certain specific properties. Poncelet provided criteria for these invisible objects to be made evident and thereby count as evidence in further applications.

However, Poncelet's constructions as they appeared in the eighth volume of the *Annales* were merely applications of elementary geometry (secants, conics, tangents, intersecting points) with no trace of the underlying theory, just as Plücker's constructions nine years later revealed none of the coordinate manipulation in their analytic proofs. In the construction the proof itself could be lost from view.[66]

[64] "I intend the figure, which is easier to imagine than represent, without confusion."

[65] "We believe it superfluous to accompany this memoir with figures, often more overwhelming than useful, in the geometry of space; figures that we could besides only present in a unique and individual aspect to the reader, who could, instead, construct and fashion them to his taste, if he judges the assistance at all necessary. We are concerned, indeed, only with logical deductions, always easy to follow when the notations are chosen in a convenient manner."

[66] In his study on geometry in ancient Greek mathematics, Netz classifies the construction as a description and the proof as a narrative (Netz (2005), 262). In this sense these articles present different stories with the same characters and in the same locations. However, as is so often true with stories, the characters and locations are not so similar as they first appear.

In their presentations, both Poncelet and Plücker centred their results around the relationship between coincidence of intersection points and order of tangent points. The procedure of deriving new results from gradual deformation of a figure was not unique to their writings, although their insistence on the importance of the procedure is marked. Further, Plücker's coordinate equations and Poncelet's ideal objects were both introduced as new forms of geometric evidence. From this perspective, the rather redundant constructions were merely a vehicle to show off their potential applications.

2.6 Gergonne's "Pluker" without Plücker, 1826

Poncelet never saw Plücker's original manuscript because the published version which appeared in the *Annales* had been edited beyond its author's recognition. As Plücker asserted four years later in the introduction to his *Analytisch-geometrische Entwicklung*,

> Die Veranlassung dazu war, dass ein von mir eingesandter Aufsatz in den Annalen (1826 *Août et Septembre*) abgedruckt wurde, nachdem er zuvor getheilt und in eine so ganz verschiedene Form gegossen worden war, dass ich den ersten Theil desselben später nur wiederkannte, weil mein Name an der Spitze desselben stand. (Plücker (1831), vi)[67]

Plücker emphasized that the form of his published article was completely different from his submission. As we shall see, this factor was decisive in shaping the article's immediate impact. To begin with, Gergonne divided the manuscript into two articles. The first was listed under the subjects "Géométrie des Courbes et Surfaces" and "Géométrie de Situation" entitled "Théorèmes et problèmes sur les contacts des sections coniques." This article built up lemmas and theorems based on Brianchon's *Mémoire sur les lignes du second ordre* (1817) to solve the problem of finding conics given points and tangents of first, second, or third order. There were no coordinate equations and the text was presented in dual columns. The second article was listed under the subjects "Géométrie analytique" and "Géométrie des Courbes et Surfaces" entitled "Recherche d'une construction graphique de cercle osculateur, pour les lignes du second ordre." Though only running four pages, this second article was more in tune with Plücker's manuscript as Gergonne included the coordinate expressions for second order equations and combined these equations to find a locus of intersection points.

Since third order contacts between two conics were addressed in both articles, we will consider the two quite different treatments. In referencing the author of these texts we will refer to "the author" or "he" because of the melange of contributions of Plücker and Gergonne.

[67] "The occasion of this was the submission of an essay that I sent to the Annales (1826 August and September), after which he [Gergonne] divided and moulded it into a so completely different form that I only recognized the first part afterwards because my name was at the top of it."

2.6.1 Pluker's first publication

The first article, "Théorèmes et problèmes," had a striking architecture. Not only was it written almost entirely in dual columns (barring a bibliographic introduction and a summary conclusion), but the material progressed following the gradual deformation of a complete quadrilateral into a triangle, then an angle, then a chord. Each deformation resulted in one or more theorems, which were then used in a subsequent problem to determine a conic with the ruler alone and a set of given conditions. The repetitive systematic style closely resembled L. M. P. Coste's recent contribution to the same subject, "Propriétés peu connues de la parabole, et construction de cette courbe, au moyen de quatre conditions données" written in 1817 and also inspired by Poncelet, in which Coste laboriously determined how to construct a parabola given any combination of four points or tangent lines (Coste (1817)). This required 18 different cases with 20 lemmas and 24 figures occupying an entire page with every figure a slight deformation of the previous one, shown in Figure 2.14.

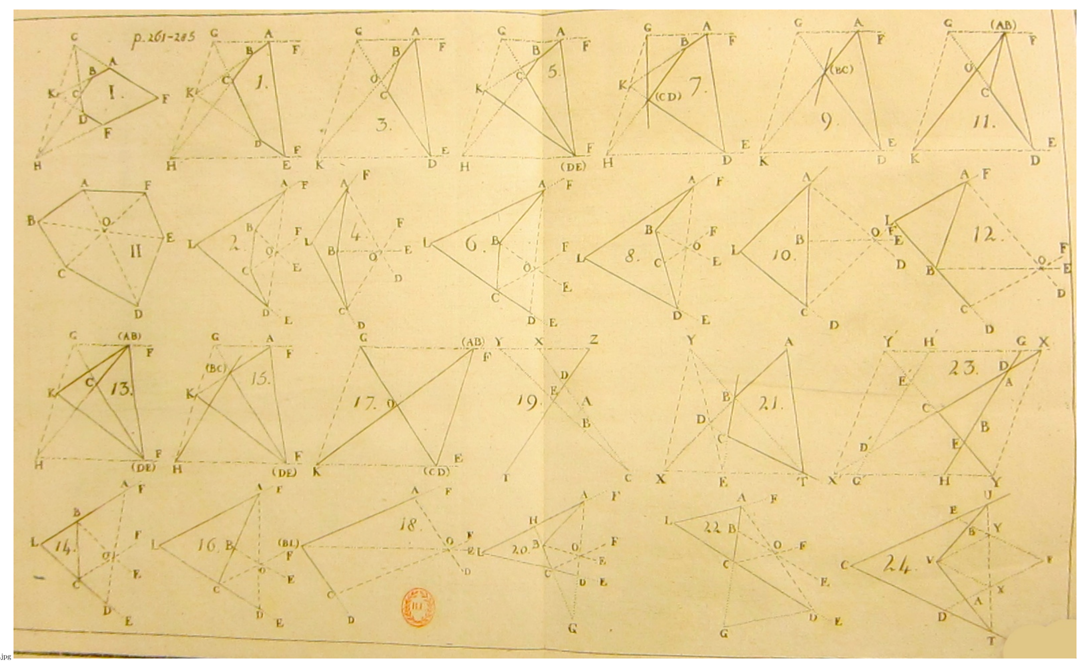

Figure 2.14: Coste's variations of parabolas (Coste (1817))

Though neither Pluker article contained any real figures, we will see that the invoked virtual figures were of the same type as Coste's parabolic variations.

The author (Gergonne's version of Plücker) opened the article by citing Brianchon's determination of any conic section given five conditions of passing through points or being tangent to lines. The author observed that a simple tangent point was equivalent to two given conditions, a second order tangent to three conditions, and a third order tangent to four conditions. Thus with a smaller number of points of higher order tangency one could still completely determine a conic. He cited Poncelet as having considered the research of this series of "curious" problems in his *Traité*. Without directly critiquing Poncelet's treatment, the author proposed to derive his results based on very simple considerations and constructions, deduced from two fundamental lemmas. Thus the author subtly seemed to imply that Poncelet's procedure was unnecessarily complicated. Since the problems, constructions, theorems and proofs corresponded in pairs, the author proposed employing two columns. He referred to three articles from the two most recent volumes of the *Annales* (the first two by Gergonne and the last by Vallès in response to a pair of problems posed by Gergonne (Gergonne (1825), Gergonne (1826), Vallès (1826)) as precedents in this formatting choice. The author did not use the term reciprocity or duality in this context, only *correspondence* to denote the relationship between the columns' contents.[68]

The style of two columns was a recent Gergonne innovation, introduced in 1825 within the context of corresponding polyhedral faces and vertices and soon to become a regular feature in the *Annales* (Gergonne (1825)). Gergonne was very enthusiastic about this particular style of presentation, and used it to rewrite his past results along with those from other *Annales* contributors.

In the side-by-side lemmas, circumscribing was symmetric to inscribing, sides to vertices, concurrence to collinearity, arbitrary secants through determined points and determined tangents through arbitrary points. On the left hand side the lemma was,

> Deux coniques étant circonscrites à un même quadrilatère ; si, par les deux extrémités d'un même côté de ce quadrilatère on mène aux deux courbes des sécantes arbitraires ; les cordes menées à ces courbes, par les points où elles seront respectivement coupées par ces sécantes, iront concourir toutes deux sur la direction du côté opposé du quadrilatère.[69]

The corresponding right hand side lemma was,

> Deux coniques étant inscrites à un même quadrilatère; si, sur les deux côtés d'un même sommet de ce quadrilatère on prend arbitrairement deux points par chacune desquels on mène des tangentes aux deux courbes; les points de concours des tangentes respectives à ces deux courbes seront en ligne droite avec le sommet opposé du quadrilatère. (Plücker (1826b), 39)[70]

[68] Karine Chemla has shown how Gergonne developed duality from spherical geometry, and the references he provided in this article reflect these origins, see Chemla and Pahaut (1988) and Chemla (1989).

[69] "Two conics are circumscribed to the same quadrilateral; if, by two vertices of the same side of this quadrilateral one draws arbitrary secants to two curves; then the chords drawn through the points where these curves will be respectively cut by these secants, will all concur in pairs on the the extension of the opposite side of the quadrilateral."

[70] "Two conics are inscribed to the same quadrilateral; if, on two sides of the same vertex of this quadrilateral one arbitrarily takes two points and draws tangents to the two curves through each of them; then the points of concurrence of the respective tangents to these two curves will be in a straight line with the opposite vertex of the quadrilateral."

The proofs of these lemmas preceded their statement and each side employed a carefully described virtual figure. The proof took the form of a construction respectively using Pascal's theorem and Brianchon's theorem on the left and right side for which we present an illustrated figure based on his description, Figure 2.15.

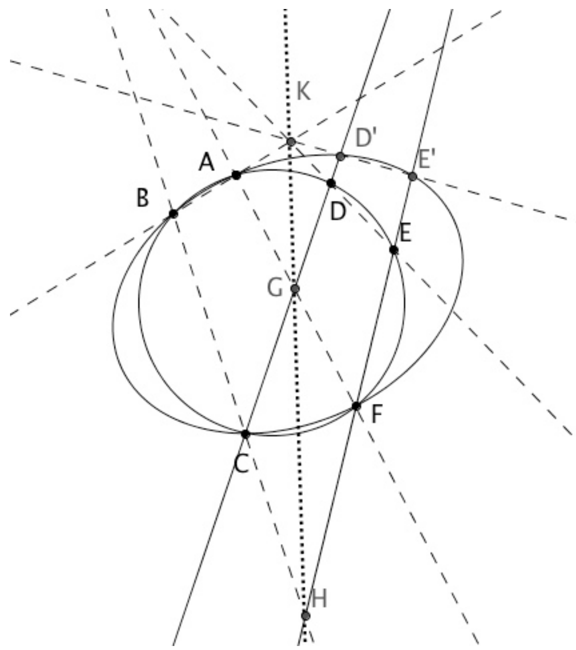

Figure 2.15: Illustrated figure for Pluker's lemma

As reference to Pascal's and Brianchon's theorems, the author cited five articles from the *Annales*, two by Gergonne, one by Jean Baptiste Durrande, and two by Germinal Dandelin–extracted from the *Mémoires de l'académie royale des sciences de Bruxelles* by Gergonne. While Pascal and Brianchon were cited by name, neither of their original works were referenced here. The author added in a footnote that this property applied equally to convex and concave quadrilaterals. From the lemmas, the author proceeded to ever fewer points, and correspondingly ever fewer tangent lines, of higher order contact. There was a large amount of repetition in the constructive steps and derived properties with twelve pairs of theorems each corresponding to a different deformation of the original quadrilateral. Every proof in this article similarly reduced the number of points in the prior polygon by one, without affecting the veracity of the lemma. So we may skip forward to the treatment of third order contact while still maintaining a sense of the style and content. We present only the left hand side.

> Lorsque deux coniques ont en un point commun un contact du troisième ordre, elles ne sauraient avoir alors aucun autre point commun, ni conséquemment aucune corde inscrite commune. On peut, dans ce cas, les considérer comme

étant toutes deux circonscrites à un même quadrilatère dont les côtés, d'une longueur nulle, sont dirigés suivant la tangente commune, et dont les sommets se confondent tous quatre avec le point de contact. Notre *Lemme* ne cesse pas pour cela d'être vrai. (56)[71]

From this informal proof, the author concluded that for two conics with a third order tangency and two arbitrary secants through that point, the chords drawn from the respective intersections concurred on the common tangent. This theorem enabled the solution to problem VII; with the ruler alone given a conic, a point on the conic, and a second point in the plane, to find all the necessary points to determine another conic with a third order contact at the first point and passing through the second. We present an illustration based on the virtual figure in Figure 2.16 where A is the point on the given conic, P is the coplanar point and Q is the new point on the desired conic section.

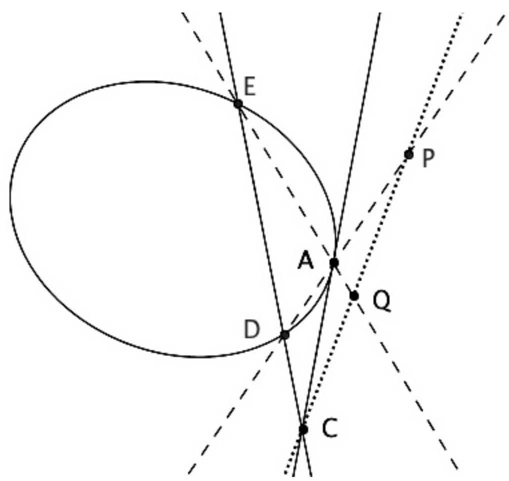

Figure 2.16: Third order tangency construction, 1826

The author continued by considering the case where the angle between two secants diminished to zero (or correspondingly the distance between two points diminished to zero), in order to derive a similar proof, then theorem, then problem and ultimately solution for the case of a third order tangent point and a given planar tangent line.

This final case completed the two column format. The author concluded the article by pointing to further generalizations by considering a system of two straight lines as a conic (which Plücker had also done) or extending to figures on a spherical surface (for which the author cited an article by L. Magnus, another Prussian mathematician, which had appeared in the preceding volume of the *Annales*).

[71] "When two conics have a common point of third order contact, they cannot have any other common point, nor consequently any common inscribed chord. We can, in this case, consider them as both circumscribed to the same quadrilateral where the sides, of no length, are directed following the common tangent, and where all four vertices coincide with the point of contact. Our *Lemma* does not stop being true."

Comparing the above construction and Figure 2.16 to Figure 2.4 (our illustration of Poncelet's original virtual figure) and Figure 43 (from Poncelet's *Traité*), we observe that except for a change in point names this 1826 solution is nearly identical to Poncelet's both in the 1817 article and in his 1822 *Traité*. Also, the emphasis on coinciding points closely mirrors Poncelet's in the *Traité*, here the process was explained in terms of the polynomial. In Pluker's "Théorèmes et problèmes sur les contacts des sections coniques," figures were manipulated by diminishing distances to zero or diminishing angles to zero, or considering figures with sides of no length or angles of no magnitude. In this way the principle of continuity was implicitly invoked, since for each of these deformations, the author insisted that "notre *Lemme* ne cesse pas pour cela d'être vrai et applicable." The focus on the physically realizable construction process in this article is further emphasized by the author's frequent *Remarks* which (out of multiple possible solutions) suggested the solution that did not require *drawing* a tangent line as the most advantageous.

Many of these features, from the introductory lemma to the concluding generalization for a spherical surface, were entirely absent in Plücker's original article. Perhaps most significantly, Plücker's evidence in support of analytic geometry had been reformulated as evidence in support of simple considerations and constructions of elementary geometry. Plücker's theory of osculations was based upon the coordinate representation of a common chord between two conic sections as well as a very explicit definition of tangency order with respect to coinciding points. The relationship between coinciding points and tangency order within the published article is employed repetitively, but never explained as a fundamental procedure or principle. Even the common chord had vanished.

Likewise, despite their similar appearance the proofs and constructions in "Théorèmes et problèmes sur les contacts des sections coniques" and Poncelet's *Traité* are based upon very different principles. In the former, the author used properties of the inscribed or circumscribed hexagon to derive each construction meticulously, while Poncelet's constructions rely on broader properties of similar figures, projection, and common secants or tangents. In this article, the author's particularity in examining each possible hypothesis contrasts strongly with Poncelet's much more general treatment in which replacing given points by given tangent lines is seen as analogous or reciprocal and not worth entering into detail.[72]

The role of the virtual figure within "Théorèmes et problèmes" was central to proving results, but emerged as specifically limited to the context of reiterative polygon deformation to fewer and fewer vertices. Special cases needed to be considered dependent on the number of intersections or contacts between each side of the derived polygon and the given conics. Thus this paper succumbed to Poncelet's 1822 criticism of pure geometry;

[72]Because of the striking similarity in results and lack of originality in method, Poncelet would accuse Plücker of the twin offences of plagiarism and loss of generality in Poncelet (1827a). This controversy forms the subject of our Chapter III.

> ...sa marche dépend tout-à-fait de la sagacité de celui qui l'emploie, et ses résultats sont, presque toujours, bornés à l'état particulier de la figure que l'on considère. (Poncelet (1822), xix)[73]

Gergonne's use of dual columns emphasized an algebra-like substitutability between words that suggested an association between elements in the figure that could not be represented by a constructive procedure: there was no ruler and compass manipulation that could translate a figure from the left hand to the right hand side. Historian Mario Otero argues that the metamathematical principle of duality marked the move from physical to abstract geometry (Otero (1997), 164). While we would counter that there were many contributions toward abstraction at this time, duality represented by double columns emphasized another step away from the descriptive figure and a physically manipulable object of geometry. But though the figure was obscured, dual columns enabled an immediate visual transfer of information. That is, Gergonne's dual columns, like Plücker's equations and Poncelet's ideal objects, were a form of new geometric evidence. Moreover, the use of dual columns interrupted the traditional direction of textual reading. With dual columns one could read one column and then the other, or switch continually between corresponding features in both columns, or even simply read one side of the column and thereby gain a sense of what the other column would say (admittedly, the latter option seems the most appealing). In Chapter III, we will witness Gergonne's additional experiments in lay-out to convey duality. Dual columns were not figures, but they were a striking visual vehicle for transferring geometric information.

With the use of dual columns "Théorèmes et problèmes sur les contacts des sections coniques" might be seen as innovative, but the painstaking process of determining individual particular cases was decidedly conservative. Despite the potential to include Poncelet's specific results, here the author assumed the desired number of intersections or contacts between two curves within the construction's hypothesis, and so issues of existence were entirely outside the scope of the paper's contents. Though Poncelet was cited in the introduction, the rest of the text contained no further references to the *Traité*. Moreover, by associating Poncelet's *Traité* with a variety of well-documented elementary geometry results (many by Gergonne), the author de-emphasized the originality of Poncelet's contributions. In "Théorèmes et problèmes", ideal objects were not mentioned and all references to chords, secants, and tangents were assumed real. However, Pluker would directly employ these terms in the following article.

[73]"...its progress depends completely on the wisdom of he who uses it, and its results are, almost always, limited to the particular state of the figure that he considers."

2.6.2 Pluker's second article

This second article opened with the author determining the coordinate plane: with the origin as any point on a second order line, the y-axis as the tangent at that point, and the x-axis as the normal. This led to two equations,

$$y^2 + 2axy + bx^2 + 2cx = 0, \tag{2.17}$$

$$y^2 + 2Axy + Bx^2 + 2Cx = 0, \tag{2.18}$$

for two second order lines intersecting with common tangents at the origin.

The author declared that two second order lines could "in general" intersect in four points. By giving equal status to real and imaginary intersections, the author's use of "in general" here seemed to signify a lack of exception. In the case of the two curves given by (2.17), (2.18), since the origin of the plane was the coincidence of two of their four points of intersection, there also existed two other real or imaginary intersection points between the curves. The author proposed to find the line joining these two points.

With slightly different variables from Plücker's original manuscript, the equation for the line joining the intersections of the two curves was found by subtracting (2.17) from (2.18). This yielded a system of two lines: the y-axis and

$$2(A - a)y + (B - b)x + 2(C - c) = 0. \tag{2.19}$$

As Plücker himself had pointed out, here the author noted that the two points of intersection between the common chord (2.19) and each of the curves could be imaginary. Plücker had relied upon the validity of the *equation* as securing a constructive representation for a straight line possessing common chord properties. However, in the published article the author stated that in the case of imaginary points the line (2.19) would "become what Poncelet had called an "ideal chord, but we can still construct it" (Plücker (1826a), 70). The use of *but* in this context suggests a still uncertain status for ideal objects. Apparently, the author chose to use ideal chords as evidence for a lack of exception to the construction, but cautiously had to remind the reader that ideal chords were still constructible objects.[74]

Significantly, this was Gergonne's (though under the guise of Pluker) first published use of ideal chords, which he had criticized strongly in 1820 in a footnote to Cauchy's report to the *Académie des sciences* on Poncelet's researches.

> [...] or, s'il est une définition de ces droites qui convienne également à tous les cas, ne faudrait il pas l'adopter de préférence à une autre définition sujette

[74] Plücker himself would begin to use the designation "ideal chords" in analytic geometry, as we will see in Chapter IV.

à des exceptions nombreuses, pour lesquelles il faut recourir à des conceptions ingénieuses, si l'on veut, mais qui tendent à faire perdre à la géométrie une partie des avantages et de la supériorité qu'on lui a toujours accordé sur toutes les autres sciences? Dans le cas de deux cercles, par exemple, ne vaut-il pas mieux définir l'axe radical, le lieu des points pour lesquels les tangentes aux deux cercles sont de même longueur, que de dire que c'est la corde commune à ces deux cercles? (Poncelet and Cauchy (1820), 80)[75]

Gergonne had expressed concern that Poncelet's alternative definitions would introduce unexpected exceptions and require great ingenuity in geometry–two features that Poncelet was directly attempting to avoid through generality and uniformity. By contrast, within "Recherche d'une construction graphique de cercle osculateur, pour les lignes du second ordre," the author referenced Poncelet's new definition of "ideal chords" casually as a means towards avoiding exception.

The author continued to examine the consequences of (2.19) with respect to the coefficients. This procedure followed that of the first half of Plücker's original article, his so-called theory of osculation, although in a more abbreviated format. If $C = c$ the lines (2.17, 2.18) would pass through the origin and the two curves would share a second order contact. Further if $A = a$ and $C = c$, the chord (2.19) would coincide with the y-axis and the two curves would share a third order contact at the origin.

The article ended with "une construction facile" for the centre of an osculating circle at a given point of a second order line. The derivation and exposition of this construction concur with those of Plücker's original article. By varying the coefficient C in $y^2 + x^2 + 2Cx = 0$ all osculating circles tangent at a given point to the curve would have parallel common chords. It was within this context that Plücker had suggested the extension of the notion of chord to include lines satisfying the chord equation and the chord properties. The author in "Recherche d'une construction graphique de cercle osculateur, pour les lignes du second ordre" had already referenced ideal chords, and the possible constructive procedure was not mentioned further.

This abrupt ending eliminated Plücker's study of diameters and his commentary on coinciding points of intersection as the basis for the theory of osculation. Though Plücker's specific proofs and solutions remained intact, the absence of theoretical and methodological context made the published version appear limited and particular. As the title of this published article suggested, the results were now framed as leading only to a graphic con-

[75]"[...] thus, if there is a definition of the lines which is suitable for all cases, we should not adopt a preference for another definition subject to numerous exceptions, for if we do then we must resort to ingenious designs, which tend to lose a part of the advantages and superiority that geometry gives to all the other sciences? In the case of two circles, for example, would it not be better to define the radical axis as the locus of points for which the tangents to two circles are of the same length, than to say that this is the common chord to two circles?"

struction of an osculating circle. The abbreviated treatment of coefficients deemphasized Plücker's attention to avoiding calculation and creating a direct correspondence between representational equations and geometric objects. The first section of Plücker's original text, though containing these same results as the second published article, were intended as exposition of his theory of osculation and particular method of analytic geometry. Finally, by placing these researches in a separate article following the work on tangent conics, the relationship between theory and application that Plücker strongly emphasized was almost entirely severed. The potential reader would not be introduced to Plücker's true style of analytic geometry until the following year, when it was met with general praise, as we shall see.

2.7 Conclusion

We return to Poncelet's 1817 article where he differentiated three methods in geometry: analytic geometry, pure modern geometry, and pure ancient geometry. The articles of Plücker, Poncelet, and Gergonne/Plücker respectively fulfill some of the characteristic features of Poncelet's trichotomy. Plücker used properties of coordinate representation, Poncelet eschewed calculation while developing general tools far beyond the scope of Gergonne's more elementary pure exposition. However, the three different presentations also illustrate the limitations of such a clear cut classification, as well as any of the classifications described in Chapter I. Plücker chose his coordinate axes to avoid calculation and emphasized the role of representation in analysis. Poncelet's treatment of imaginary and infinite points admitted objects into pure geometry that no longer had an immediate figurative correspondence, and his definitions of the ideal challenged the intuitive nature of geometry. Gergonne's *simple* figure deformations at least implicitly required some form of continuity, did not include any actual figures, and with the use of dual columns suggested a kind of symbolic substitution between words signifying geometric objects. Conversely, though Gergonne as Pluker cited Poncelet's *Traité*, he did not attempt to import the new *ideal* objects except in an analytic context. In Chapter IV, we will see how ideal objects spread through the pages of the *Annales* in both figure based and coordinate based representations.

Looking beyond these methodological boundaries, the greatest commonality between the five texts is apparent in the statement of the problems and constructive solutions. For these problems of planar geometry, there was an imperative to translate back to the vocabulary and situation of the problem's statement regardless of the method used to solve it.[76] Actually viewing or drawing the constructed figure makes the comparison between these different treatments even more striking. While the named points and constructive

[76]Friedelmeyer suggests that Gergonne's ability to translate between analytic calculation and simple geometric constructions was uncommon among his readers.

steps might vary in order, the completed solutions were visually more or less identical. In this respect, problems were conservative, while innovation lay in theorems and theory.[77] On the constructive level, the evidence for each author was the same.

In all the above presentations, the evidence provided by an actual or virtual figure determined the object's geometric existence. Though Plücker proposed coordinate equations as a representative geometric form, only geometric construction secured an object's reality. In particular, he noted that the common chord no longer existed as such when it was not bound by two real intersecting points, though it was still representable both as an equation and as a real line segment on the plane. Poncelet's introduction of ideal objects extended the concept of geometric existence, but not so much as to include the non-figurable imaginary and infinite. Within the context of ideal objects, Poncelet also suggested multiple kinds of existence.

> Son admission ouverte en Géométrie ne saurait donner lieu à aucune difficulté sérieuse; car, si la propriété qu'on examine et qui, par hypothèse, a été établie pour une situation non singulière, mais indéterminée, des parties de la figure, ne concerne que des objets actuellement réels et constructibles, elle aura lieu d'une manière entièrement absolue et géométrique; dans la supposition contraire, elle cessera d'être applicable à ces objets d'une manière absolue, sans pour cela devenir ni fausse ni absurde à l'égard des objets demeurés réels; en sorte que, si l'on conserve mentalement une existence de signe ou d'expression aux objets impossibles, la propriété devient purement idéale à l'égard de ces objets. (Poncelet (1822), 66)[78]

Thus an object could be neither real nor constructible, but also neither false nor absurd. This new kind of existence would seem to transcend figurative or coordinate forms of representation, and permit less tangible types of evidence.

> Ensuite, la solution analytique de Gergonne suppose de la part de son auteur une capacité peu commune d'interpréter et d'orienter les calculs analytiques pour en déduire une construction aussi simple. (Friedelmeyer (2011), 90)
>
> "Following the analytic solution of Gergonne supposes an uncommon capacity on the part of its author to interpret and orient analytic calculation in order to deduce a construction just as simple."

[77] We will further argue this claim in examining problem solving in Plücker and Steiner in Chapter IV, as well as the content of contemporary French geometry books in Chapter V.

[78] "Its [ideal object's] open admission in Geometry will not lead to any serious difficulty; because, while the property that we examine and which, by hypothesis, has been established for a non-singular, but indeterminate, situation of the parts of the figure, only concerns now real and constructible objects, the property will obtain in an entirely absolute and geometric manner; in the contrary supposition, the property will stop being applicable to these objects in an absolute manner, without becoming either false or absurd with respect to real objects; as a result, if we mentally conserve an existence of a sign or expression of impossible objects, the property becomes purely ideal with respect to these objects."

Despite the plethora of virtual figures in the constructive solutions we saw very few actual figures. Of the five texts considered, only Poncelet's lengthy monograph supplemented descriptions with an illustrated appendix. This technical observation merits consideration. With an imperative to accompany every proposition with a figure, as was common in ancient geometry, comes a limitation on the kinds of figures propositions may include (Decorps-Foulquier (1999), 64). On the contrary, when two dimensional representation is no longer an expectation in geometry, the figures can go to infinity, have imaginary components, be replaced by equations, substitute into other figures, or deform continuously. We saw the advantages of virtual figures in references to infinitely distant points, deforming planar objects, suggestions for application to any number of possible conic sections, and substitutions of points for lines or vice versa.

The geometric practices advertised by Poncelet, Plücker, and Gergonne not only extended the kinds of geometric figures, but also the ways of seeing them. In projective geometry, Poncelet could manipulate imaginary objects through their invariant relationships with real counterparts on the plane. Plücker shifted coordinate geometry away from computation and toward observation. Gergonne's dual columns encouraged viewing two figures in one. The figure could be kept in sight, if one was willing to broaden the scope of the subject and the action.

Plücker addressed this issue directly in the introduction to his *Analytisch-geometrische Entwicklungen*, where, without explicitly mentioning Poncelet, he returned to the critiques voiced in Poncelet's 1817 philosophical article (Plücker (1828a), Poncelet (1817c)).

> Man braucht nur zu erwägen, dass wir die gegebenen Figuren, und zwar zunächst die, sie vertretenden, Symbole, nie aus dem Auge verlieren, und, bei der Einfachheit der Verbindung, in jener Gleichung bis zur Endgleichung hin, die Beziehung zu den gegebenen Gleichungen wiedererkennen–während, von der einen Seite, in der alten Geometrie, wie sie z.B. Apollonius handhabt, das Hauptthema in den Schatten von Umschreibungen zurücktritt, eben so wie die Hauptconstructionen von Hilfslinien maskiert werden, und, von der andern Seite, die blosse Anwendung der Algebra auf Geometrie in Elimination sich verliert. (Plücker (1828a), iv)[79]

In adopting Poncelet's emphasis on keeping the figure in sight, Plücker differentiated his "pure analytic geometry" from both ancient geometry and the "mere" application of algebra

[79] "One only needs to consider that we never lose sight of the given figures, indeed first of all those symbols that represent them, and we recognize through the simplicity of combination, in each equation up to the final equation, the relationship to the given equations–however, on the other hand, in ancient Geometry, such as that which Apollonius employed, for example, the main theme recedes in the shadows of descriptions, just as the main constructions will be masked by auxiliary lines, and, on the other hand, the mere application of algebra to geometry loses itself in elimination."

to geometry. Plücker thus proposed an alternative three-part division to geometry, again mirroring Poncelet in 1817. However, Plücker's figures were now symbolic coordinate equations. For him, these new figures were more capable of conveying geometric relationships than cluttered illustrations.

The graphic sense of geometry was manifested by actual or virtual figures, real or imaginary objects, coordinate equations, the relationship between coefficients, or even the use of dual columns. Though stemming from visual considerations, not all of these geometric forms (for example, the common secant of two concentric circles) could even be visualized as geometric objects. By focusing on a constructive geometric problem, we can recognize the centrality of the figure. Concurrently, however, Poncelet, Plücker and Gergonne developed new means to obtain and present geometric evidence: through rendering the imaginary tangible, through transforming equations into representations, through effecting substitution with alignment.

In Chapter I, we found that early nineteenth century geometry was characterized by numerous disputes regarding method, proper allocation of credit, and even personality clashes. Having examined ways of differentiating geometrical methods, here we will shift our focus from methodology to opposition, considering how the early nineteenth century came to be characterized as divisive or antagonistic. Continuing to centre on the main protagonists of Chapter II, we will examine several interrelated controversies running parallel to the set of texts analyzed above. In revisiting the textual exchange between Gergonne, Poncelet and Plücker, we ask how these controversies served geometers and shaped the diffusion of mathematical content and the practicing community. In this pursuit, we will engage with the recent historical and sociological literature on controversies in science, considering how the study of controversies can be applied to the history of mathematics.

Chapter 3

Polemics in public: controversies around methods, priority, and principles in geometry.

3.1 Introduction

Early nineteenth century geometry resembles early modern geometry in both its emphasis on methods over results and the prevalence of controversies and priority disputes. From the Marin Mersenne correspondence, to Isaac Newton and Gottfried Wilhelm Leibniz, to Leonhard Euler and Jean d'Alembert, early modern mathematicians used results to fight for the merits and priority of their methods (as seen, for instance, in Goldstein (2013), Goldenbaum and Jesseph (2008), and Bradley and Sandifer (2010)). We will see similar arguments arise in the *Annales* and contemporary publications. Indeed, Gergonne modelled his journal off of early eighteenth century developments by including posed problems for his readers to solve and theorems to prove that he hoped would bring about scientific progress.

> Personne n'ignore d'ailleurs combien ces sortes de défis ont ajouté de perfectionnement à l'analise, au commencement du dernier siècle ; et il n'est point déraisonnable de penser qu'en les renouvelant, on peut, peut-être, lui préparer encore de nouveaux progrès. (Gergonne (1810a), iii)[1]

Finding and publishing the best solutions or proofs was but one facet of a competitive and driven atmosphere.

The best-documented controversy within early nineteenth century geometry, both with respect to textual evidence and later historical analyses, concerned the origins and applica-

[1] "No one is ignorant of how these sorts of challenges have added to the perfection of analysis, at the beginning of the last century; and it is not unreasonable to think that in renewing them, one can, perhaps, prepare again for new progress."

tions of the principle of duality (for instance, the duality controversy can be found in Clebsch (1872), Kötter (1901), Coolidge (1940), Boyer (1956), Otero (1997)). In this chapter, we will unpack how the duality controversy contained sentiments, arguments, and actors from earlier related controversies on the application of analysis and the use of the principle of continuity in geometry. In examining these sources, we will determine whether we are confronted with one general controversy or a series of them. The duality controversy revolved around Poncelet and Gergonne, but because Plücker's article was rewritten by Gergonne, the young German geometer became involved with his first publication. However, Plücker's understanding of his own participation was delayed due to limited availability of French journals outside of France.

In August of 1828, Plücker's received a letter in Bonn from Gergonne, responding to an apparent inquiry by Plücker on a matter of some gravity. Gergonne's tone was formal, but friendly as he told Plücker that the rumours were very true, Poncelet had openly accused Plücker of plagiarism. Gergonne explained the circumstances,

> Je n'avais pas voulu d'abord, dans son intérêt, publier ce qu'il m'avait écrit sur ce sujet; mais il m'a ensuite publiquement accusé d'une sorte de complicité avec vous; j'ai alors tout publié avec des réflexions convenables. Il est encore revenu dernièrement à la charge ; mais je lui dirai encore quelques mots en annonçant votre ouvrage. M Poncelet a beaucoup de talent ; mais il a mauvais estomac ; et quand au digère mal on a quelque fois de l'humeur. Il ne souffre qu'impatiemment qu'on s'occupe des choses sur lesquelles il s'exerce et surtout qu'on ne sache pas son livre par coeur. (Gergonne (1828a))[2]

With this brief reassurance, Gergonne changed the subject to geometry and Plücker's recent remarkable researches. He closed in again praising Plücker's "innovative" [*nouvelle*] and "fruitful" [*fécond*] use of analytic geometry.

Poncelet's accusations first appeared within the pages of the *Annales* in 1827, shortly before Gergonne's response (Poncelet (1827a)). Gergonne's promised defence of Plücker had little effect and Poncelet repeated his accusations twice more in the *Bulletin universel des sciences mathématiques, astronomiques, physiques et chimiques* in 1827 and 1828 (Poncelet (1827b), Poncelet (1828c)). Faced with these public statements, Plücker composed his own defence, which appeared in the *Bulletin* that year (Plücker (1828c)). His response was succinct: the objectionable features of his text had been added by the editor of the *Annales*,

[2]"At first I had not wanted, in his [Poncelet's] own interest, to publish what he had written to me on this subject; but he has subsequently accused me publicly of a kind of complicity with you; so I published everything with appropriate comments. He has once again returned to the charge; but I will again send him a few words announcing your work. M. Poncelet has much talent; but he has a bad stomach, and when one digests badly one is sometimes in a bad mood. He tolerates only with little patience that one looks at things that have occupied him and above all if one does not know his book by heart."

Gergonne himself. Moreover, Plücker declared the form of his work was so altered that he would not have recognized it if his name was not at the top. The article, "Théorèmes et problèmes sur les contacts des sections coniques," by Dr. Pluker [sic] thus was not the work of Plücker alone.

Plücker's original manuscript, as we saw above, aimed to promote the method of analytic geometry. However, we also demonstrated that the published version contained no coordinate equations or algebraic symbols typical of this method. Moreover, it was strikingly written in parallel columns of text, which had not been a feature of Plücker's original work. Every theorem, proof, problem, and solution aligned opposite a corresponding *dual*, where points became lines, inscription became circumscription, collinearity became concurrence and so on. This use of *dual columns* was especially objectionable to Poncelet because of an ongoing controversy between himself and Gergonne over the origins and scope of polar reciprocity or duality.[3]

Poncelet had introduced his *principle of polar reciprocity* in 1822,

> [...] en général, qu'il n'existe aucune relation descriptive d'une figure donnée sur un plan, qui n'ait sa réciproque dans une autre figure; car tout consiste à examiner ce qui se passe dans sa polaire réciproque par rapport à une section conique quelconque prise pour directrice [...] (Poncelet (1822), 120)[4]

In Poncelet's earliest published version, polar reciprocity concerned descriptive relations between two-dimensional figures and a fixed conic section. Four years later, Gergonne explained his own theory of duality in the plane,

> [...] dans la géométrie plane, à chaque théorème il en répond toujours nécessairement un autre qui s'en déduit en y échangeant simplement entre eux les deux mots *points* et *droites* ; (Gergonne (1826), 210)[5]

We see that though Poncelet's polar reciprocity chronologically preceded Gergonne's duality, the latter was considered by both geometers as more general, based on a linguistic translation without reference to a conic section or indeed *any* then recognized mathematical procedure.

At the time of Pluker's publication, Poncelet and Gergonne agreed that Poncelet had published first on what would soon be called duality, and that Gergonne's duality was

[3]In this chapter, we will present mathematical duality insofar as it pertains to grasping the stakes of the associated controversy. For a more technical exposition of the mathematics involved the reader is referred to (Klein (1926a), Chemla (1989), Otero (1997), Gray (2010b))

[4]"[...] in general, there exists no descriptive relation of a planar figure which does not have a reciprocal in another figure; which involves examining what happens in its polar reciprocal with respect to any conic section [...]"

[5]"[...] in planar geometry, to each theorem there always necessarily responds another which is deduced from it by simply exchanging between them the two words *points* and *lines*;"

more "philosophical." However, they disagreed as to proper attribution of authorship, the potential applications of the new principle(s), and the necessity of a fixed reference conic (Poncelet (1822), Poncelet (1826), Gergonne (1826), Gergonne (1827f)). We will see that relations between the two geometers remained professionally cordial, but the tension was such that any publication on duality could be perceived as allying with one side or the other. By presenting Plücker's content in double columns and his results without analytic geometry, Gergonne transformed an innocent contribution into a charged declaration.[6]

[6]Beginning with Chasles' *Aperçu*, we have benefitted from a diverse secondary literature of mathematical, historical and philosophical analyses on the duality controversy, see Chasles (1837), Clebsch (1872), Klein (1926a), Kötter (1901), Coolidge (1940), and Boyer (1956). One of the most amusing early histories of the duality controversy comes from Joseph Bertrand's biography of Julius Plücker, written for the *Journal des savants* in May 1867 following the latter's receipt of the Copley Medal. The piquant episode is worth quoting at length.

> Les premiers travaux de M. Plücker relatifs à la géométrie analytique, envoyés aux annales de mathématiques publiées par Gergonne, à Montpellier, donnèrent lieu à une discussion très-vive dont les incidents singuliers sont restés dans la mémoire des géomètres.
>
> M. Gergonne, homme de mérite d'ailleurs, et fort zélé pour la science, avait en ses propres lumières une confiance un peu exagérée; il annotait et transformait sans scrupule les articles destinés à son journal et y introduisait les réflexions qui lui semblaient utiles, sans prévenir l'auteur, auquel il croyait sincèrement rendre service, en lui prêtant son style et ses propres idées.
>
> Sa générosité envers M. Plücker dépassa toutes les bornes; il doubla le nombre de ses théorèmes en les disposant sur deux colonnes parallèles auxquelles l'auteur n'avait pas songé, mais en mêlant, malheureusement, à des résultats irréprochables plusieurs assertions erronées. Le mémoire ainsi défiguré attira des critiques et des réclamations fondées, et M. Poncelet, dans le bulletin de Ferussac, maltraita fort le jeune débutant, qui n'y comprenait rien, et n'avait pas même lu le traité des propriétés projectives, que M. Gergonne citait sous son nom. Tout s'expliqua, un peu lentement il est vrai, car les communications n'étaient pas rapides, et M. Gergonne, auquel les documents étaient naturellement adressés, ne se pressait pas de les publier. (Bertrand (1867), 269–270)
>
> "The first works of M. Plücker on analytic geometry, sent to the annales de mathématiques published by Gergonne in Montpellier, gave place to a very lively discussion whose singular incidents remain in the memory of geometers.
>
> M. Gergonne, otherwise a man of merit and full of zeal for science, had a slightly exaggerated confidence in his own enlightenment; he unscrupulously annotated and transformed the articles destined for his journal and introduced in them reflections that appeared useful to him, without warning the author, which he sincerely believed helpful, by giving the author his own style and ideas.
>
> His generosity towards M. Plücker exceeded all bounds; he doubled the number of his theorems and composed them in two parallel columns about which the author had never dreamed, but unfortunately mixing irreproachable results in with several erroneous assertions. The thus disfigured memoir attracted criticisms and justified complaints, and M. Poncelet, in the bulletin of Ferussac, greatly mistreated the young beginner, who had understood nothing, and had not even read the traité des propriétés projectives, that M. Gergonne had cited under his name. Everything was explained, a little slowly it's true, because the communications were not rapid, and M. Gergonne, to whom the documents were naturally addressed, was in no hurry to publish them."

Contemporary historians have also conducted more specific studies of the events preceding the controversy. In Belhoste (1998) and Friedelmeyer (2011), Bruno Belhoste and Jean-Pierre Friedelmeyer respectively offer explanations of Poncelet's theory of polar reciprocity in the context of his broader mathematical goals and detail the development of his theory before his 1822 publication. Karine Chemla reveals a history of

We intend to analyze the writings on duality of Poncelet and Gergonne (and related work of Plücker) specifically as texts in a scientific *controversy.* Since the 1980's, historians and sociologists of science have increasingly advocated studying controversies as tools of research. Perhaps most iconically exemplified in Steven Shapin and Simon Schaffer's *Leviathan and the Air Pump* and Martin Rudwick's *The Great Devonian Controversy*, the historical study of scientific controversies has been praised as a tool toward gaining deeper understanding of "knowledge in the making" (Shapin and Schaffer (1985), Rudwick (1985)). From a quite different point of view, it has also been suggested recently that controversies are essential to the very process of science.[7] Though controversies in mathematics are occasionally studied by historians, they are generally not viewed as indispensable to mathematical theory formulation, nor always concluding with compromise, nor even as necessarily leading to epistemic gains.[8]

Further, the debate over what does and does not constitute a scientific controversy remains open among many historians and sociologists of science. We will follow H. Tristram Engelhardt and Arthur Caplan's definition of a scientific controversy as "the existence of 'a' community of disputants who share common rules of evidence and reasoning with evidence" (Engelhardt Jr. and Caplan (1987), 12).[9] Following this criterion, we will argue that the exchange concerning duality initiated by Poncelet and Gergonne constituted a scientific controversy, in which mathematical practices (including deriving general principles, applying methods of proof, and composing original results) were questioned and supported through evidential reasoning (in the form of generating new mathematical problems, demonstrating scientific applications, and tracing historical chronologies).

While the historical literature offers valuable resources toward understanding the development, content, and effects of Poncelet's polar reciprocity and Gergonne's duality, the nature of the controversy itself seems to have deterred an in-depth textual study of the

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duality from Euler to Gergonne via spherical trigonometry, indicating inspiration outside of Poncelet's polar reciprocity (Chemla (1989)). Christian Gérini and Mario Otero each focus on the development of increasingly abstract geometry through Gergonne's linguistic duality (Gérini (2010a) and Otero (1997)). Gérini has also discussed other controversies within the *Annales*, including the 1817 debate between Poncelet and Gergonne over choice of method in geometry.

[7]For instance, Gideon Freudenthal has argued that the resolution of scientific controversies results in an "epistemic gain" for the scientific community (Freudenthal (1998), 158–159). Marcelo Dascal has asserted an even stronger claim for the "indispensable" nature of controversies within science because controversies are the means by which "theories are elaborated and their meaning progressively crystallizes" (Dascal (1998), 147–148). Other viewpoints have been expressed in recent controversy studies including Brante and Elzinga (1990), Engelhardt Jr. and Caplan (1987), and Prochasson and Rasmussen (2007).

[8]Within the history of mathematics, recent studied mathematical controversies include Catherine Goldstein's study of controversies as normal mathematics in the Mersenne correspondence (Goldstein (2013)), the algebraic controversy between Kronecker and Jordan in Brechenmacher (2008), the priority controversy between Liouville and Libri in Ehrhardt (2011), a controversy over impossible geometry problems between John Wallis and Thomas Hobbes in Jesseph (1999), a methodological controversy among Italian geometers in the early nineteenth century in Mazzotti (1998), the foundational controversy between Hilbert and Brouwer in Posy (1998), and various controversies involving Leibniz (Dascal (2007), Goldenbaum and Jesseph (2008)).

[9]The scare quotes around "a" speak to the difficulty of segregating a single kind of community.

explicitly *polemical* writings that we will consider here. Despite Poncelet's and Gergonne's purported mathematical motivations, in the scope of the controversy their written rancour strayed far from academic civility. The literature has described the events as an "unpleasant controversy" (Coolidge (1940)), a "regrettable polemic" (Friedelmeyer (2011)), a "rather violent quarrel" (Otero (1997)), and (despite the addresses of the participants) a "typically Parisian controversy" (Gray (2010b)). The historical consensus indicates that while Poncelet and Gergonne had different and valid viewpoints that ultimately required compromise to achieve resolution, the controversy digressed from mathematics into unproductive personal squabbling.

Despite the individual antagonism, the controversy brought the principle of duality to a widening community of geometers. As a public exchange the duality controversy extended beyond the personal agendas of Gergonne and Poncelet to become a collective scientific enterprise. By incorporating the "philosophical" exchanges between Gergonne and Poncelet from 1817 onward, we will find that what was at stake went beyond assigning priority to incorporate the status of generality and discovery, the proper form of presenting new principles, and the scope of a changing geometry. Beginning with Plücker's re-appropriated article, the publicity created by the dispute engaged and informed a participating public. In turn, this public could address unanswered questions and direct research in unanticipated directions, resulting in epistemic benefits to the mathematical community. In the texts of the duality controversy we are thus able to analyze the circulation of new geometry.

By using the term "publicity" we identify the geometers' rhetoric as a means of advertisement and persuasion. In publicizing their research, geometers aimed, in part, to be read, and accordingly adapted their presentations to suit a range of audiences through an assortment of publications. The public nature of scientific controversy has been discussed in detail for particular controversies. To take one well-known example, Martin Rudwick showed that, among the gentlemen specialists of the Devonian controversy, the most vicious critiques were communicated through private correspondence, while the geologists' printed publications remained cordial. The gain for preserving a unified scientific front was to show "natural science as straightforward and objective knowledge" (Rudwick (1985), 25). Similarly, Caroline Ehrhardt portrayed how an acrimonious controversy among Parisian mathematicians in the late 1830's was edited into a disciplined interchange within publications by the Parisian *Académie des sciences* (Ehrhardt (2011)). In both of the above instances, alternative media outlets popularized more sensational, argumentative versions of the events. Although the duality controversy was roughly contemporary, it involved neither a formal academic institution nor the popular press. We will see that the duality controversy was delocalized from any particular institution or journal with an audience mostly confined to teachers and researchers in mathematics. Perhaps as a result of this independence and more homogeneous public, eventually neither Gergonne nor Poncelet chose to restrain their insults and insinuations.

In fact, private correspondence between the mathematicians suggested a much more polite communicative atmosphere. Public animosity coupled with private amicability has been noted in Frédéric Brechenmacher's study of a late nineteenth century mathematical controversy between German and French mathematicians.

> L'opposition public / privé est un des moteurs de la controverse et la correspondance entre les deux savants se présente comme une tentative de ramener le débat de la scène publique (journaux et communications académiques) à une relation privée. Dans la sphère privée, les témoignages de sympathie de Kronecker visent à apaiser le sentiment de Jordan d'avoir été agressé « devant tout le monde » : il ne s'agit ni d'une attaque personnelle [...], ni d'une accusation de plagiat [...]. (Brechenmacher (2008), 195)[10]

Although Brechenmacher was discussing a later time period, we will similarly see that cordiality between Poncelet and Plücker was restored only in private. Moreover, when Poncelet privately accused the French mathematician Michel Chasles in 1828 of improper citation and credit, the same charges publicly levelled against Plücker, Chasles' response suggested no more than a friendly misunderstanding. On the one hand, these contrasts allow us to examine to what extent the public or private nature of the controversy determined the choice of rhetoric. On the other hand, we will consider how the intended public message decided the communicating medium. In particular, we will consider how Plücker's obscure background, foreign nationality, and misspelled name (Pluker) constrained him to undertake critiques in a very public arena.

We will see that Poncelet attempted to frame the duality controversy as a narrow polemic over priority between two persons, but his decision to publish these grievances speaks to the immediate importance of a reading public. The public of the duality controversy constituted a broader audience including other geometers who chose to adopt duality, the general readers of existent mathematical journals, and those who would later historicize the turn of events. To Poncelet's dismay, the polemical dialogue attracted new participants who instead focused on the concept and practice of duality itself. The resulting public dramatically changed the outcome and the nature of the controversy. Though Plücker only joined midway, the mathematical community agreed on his geometric competency, which enabled his role in shaping the controversy.[11] We will see that his ultimate resolution was found acceptable by later historical assessment and most of his contemporaries with noted exceptions–most relevantly,

[10]"The public/private opposition is one of the motors of the controversy and the correspondence between the two scholars presents itself as an attempt to bring the debate away from the public arena (journals and academic communications) toward a private relationship. In the private sphere, the testimony of sympathy from Kronecker seemed to appease Jordan's feeling of having been attacked "in front of everyone": it concerns neither a personal attack [...] nor an accusation of plagiarism [...]."

[11]In regard to the members of a controversy, Rudwick described a *gradient of attributed competence*,

Poncelet and Gergonne. Their enduring obstinacy demonstrates how the controversy could be resolved despite a lack of compromise among its original instigators.

The peculiar nature of the duality controversy was largely shaped by the mathematical journals of the early nineteenth century, and in particular Gergonne's *Annales*. To this end, we will consider what was (and was not) published and the dissemination of the published material. This line of inquiry will help to unravel some of the aspects of these publications, including the importance of venues and accessibility in the origins and growth of the controversy. The duality controversy was carried out employing a variety of publications, from monographs to articles to public letters to reviews to catalogue summaries to footnotes to footnotes of footnotes. We will observe how the medium of distribution informed the intended message.

We will begin our study in Section 3.2 with the first set of exchanges between Gergonne and Poncelet over the best choice of method in geometry. The initial debate between our two protagonists will serve to establish their respective positions as well as expected standards among *Annales* readers. In Section 3.3, we will follow the shifting dialogue between Gergonne and Poncelet, through the publicity and reception of Poncelet's *Traité des propriétés projectives* and in particular his principle of continuity. We then introduce Gergonne's principle of duality, and review Plücker's publication as Pluker as it pertained to this controversy in Section 3.4. These earlier publications will serve as preliminaries to Section 3.5, which discusses the duality controversy proper as it unfolded in texts from the *Annales*, the *Bulletin*, and existent archival letters between Poncelet, Gergonne, and Plücker between 1826 and 1829. Here we will examine the development of duality through the competing publicities that the controversy engendered. In Section 3.6, we will consider the end of the controversy and contrast its resolution with the final accounts of Gergonne and Poncelet.

3.2 On the application of algebraic analysis to geometry (1817)

In a series of papers beginning in 1810, Gergonne claimed that analytic geometry was superior in elegance and simplicity to pure geometry (Gergonne (1810b), Gergonne (1814a), Gergonne (1817a), Gergonne (1817b)). His arguments took the form of solving "difficult" geometric problems through the use of coordinate axes and equations, such as in a paper

> Whatever the power–then or now–of an official myth of democratic equality with "the scientific community," practitioners knew then–as they know now–that some scientists are more equal than others, and that the formal hierarchies of position and influence are by no means coincident with what will be termed the informal and tacit *gradient of attributed competence*. (Rudwick (1985), 419)

Rudwick found this hierarchy especially relevant in determining the close to the Devonian controversy, where only those with attributed competence could determine when the controversy was settled.

on the Apollonius problem.

> [...] j'essayais de prouver que cette géométrie analitique, convenablement maniée, offrait les solutions les plus directes, les plus élégantes et les plus simples de deux problèmes dès-long-temps célèbres, et qui passent pour difficiles. (Gergonne (1817a), 289)[12]

Poncelet responded to these methodological claims in "Réflexions sur l'usage de l'analise algébrique dans la géométrie; Suivies de la solution de quelques problèmes dépendant de la géométrie de la règle" published in the *Annales* under the subject heading *Philosophie mathématique* (Poncelet (1817b)). Gergonne categorized each article in the *Annales* under one or more such descriptors and mathematical philosophy was a small, but persistent category through the journal's existence.[13]

We studied the mathematical contents of this article in Chapter II, and here reiterate the key structural features to focus on Poncelet's broader arguments. "Réflexions" was a two-part paper: a self-described philosophical commentary followed by mathematical evidence in support of the arguments advanced. Poncelet formatted the first half as a letter to the editor, addressed in first person to Gergonne but also designed for the journal-reading public. This form of correspondence was fairly common in *Annales* publications, including two other letters by Poncelet (Poncelet (1817a), Poncelet (1827a)). He prefaced his critique with a compliment, praising the editorial open-mindedness of publishing divergent philosophical viewpoints on science. As a researcher, Poncelet noted Gergonne's advancement of the analytic method in geometry, and admired his recent successes in deriving elegant solutions to geometric problems. However, Poncelet contested Gergonne's claims for the *superiority* of analytic over pure methods in geometric problem solving.

Poncelet clarified what he intended by pure geometry. First, there was "ancient pure geometry", and Poncelet agreed with Gergonne that this kind of pure geometry lacked the generality of analytic geometry. Conversely, "modern pure geometry" was just as simple and elegant as any geometry using coordinates and "never lost view of the [geometric] figure," here referring to the illustrated or described labelled figures in geometric constructions. Since these two modern methods were so evenly matched, Poncelet suggested that the nature of the geometric problem should decide the choice of method. He continued even-handedly, "on ne pourrait, sans un grand préjudice pour l'avancement de la science, cultiver

[12] "[...] I will try to prove that this analytic geometry, appropriately handled, will offer the most direct, the most elegant, and the most simple solutions of two long-famous problems, which are thought of as difficult."

[13] In fact, Poncelet's article was categorized under three different headings. In the journal's table of contents, the second half of the title was listed under *Géométrie de la règle*, the first half of the title was listed under *Philosophie mathématique* and the same article with the title "Réflexions sur les méthodes de la géométrie analitique" was listed under *Géométrie analitique*. In the text itself, the subject heading appeared only as *Philosophie mathématique*. This kind of varied classification was not uncommon in *Annales* articles.

l'une ou l'autre d'une manière trop exclusive" (144).[14] To prove the first half of this point Poncelet presented new solutions to as yet unsolved problems. He omitted the proofs for these solutions, explaining that their length exceeded the bounds of a letter, but asserted that they had been derived through purely geometric means.

By no coincidence, the first of these problems had been advertised by Gergonne earlier in 1817, purportedly to be solved by analytic geometry: to construct an m-sided polygon circumscribed to a given conic section with vertices lying on m given lines or inscribed to a given conic section with sides passing through m given points (Gergonne (1817b)). For the general case, Poncelet contended that the large number of possible variables precluded an elegant analytic approach. This indicated a whole class of general geometric problems, which pure geometry was better equipped to solve. The other problems Poncelet addressed all concerned relationships between polygons and conic sections in the plane, including the problem later solved by Plücker and discussed at length in Chapter II. As we have seen, the solutions contained steps for constructing a labeled figure, although no illustrated figures accompanied the article.

Poncelet ended, as he had begun, on a modest note praising Gergonne's recent publications in showing the surprising capabilities of analytic geometry. He admitted that the flaws he perceived in analytic geometry might be more a result of his own inexperience than any inherent faults in the method.

> [...] on ne doit jamais se hâter d'imputer à l'analise des imperfections qui souvent sont uniquement le fait de qui ne savent point en faire un usage convenable. (ibid, 155)[15]

Though phrased as a confession of his own inexperience, Poncelet's conclusion could also be interpreted as a subtle challenge.

Gergonne responded in the article immediately following, "Réflexions sur l'article précédent" (Gergonne (1817e)). While Poncelet had addressed Gergonne specifically, Gergonne aimed his response at the general readership of the *Annales*, and so, indirectly, Poncelet. As Gergonne's intended audience, his readers were recast as direct participants in the public exchange and Gergonne took advantage of this with an opening anecdote about a recent discourse on exaggeration published by "a man of very distinguished merit." Then, in the most roundabout of concessions, Gergonne suggested that *if* Poncelet's description of Gergonne's methodological claims had been correct, then Gergonne would have been guilty of exaggeration. However, Gergonne claimed never to have intended to imply that analytic geometry was superior to pure geometry. Rather, because of its generality and uniformity,

[14] "[...] that we cannot, without a great prejudice against advancing science, cultivate one or the other [geometry or analysis] too exclusively."

[15] "[...] one must never hasten to impute to analysis any imperfections which are often uniquely the act of those who do not know how to use it properly."

analytic geometry could provide solutions *as* elegant and simple as those from pure geometry. Further, in several instances the analytic geometry solutions were "incomparably more elegant and simple than any yet known" (ibid, 157).

In surveying his publications on analytic geometry from 1810 to 1817, Gergonne deemphasized his earlier methodological claims. He had previously argued that analytic methods were superior and more convenient because they did not depend on the nature of the individual figures. It was rare that such demonstrations "were not subject to several exceptions or limitations" (Gergonne (1814a), 383). As an example, Gergonne showed that, when an argument is based on figures, one must consider as distinct cases when a given point was inside or outside of a closed curve, while an argument employing a single coordinate equation required no such distinction. Similarly, Gergonne had proposed that analytic geometry would lead to the most direct, the most elegant and the simplest solutions of contemporary and ancient constructive geometry problems (Gergonne (1817b)). In these earlier works, Gergonne's examples and descriptive superlatives were often directed against Simon Antoine Jean Lhuilier (1750–1840) who had defended classical geometry and criticized modern analytic geometry for a lack of simplicity and elegance (Lhuilier (1809), Gergonne (1810b)).[16] Here, with Poncelet's expressly methodological letter to the editor, Gergonne found a new, more modern opponent and adapted his position accordingly.

Gergonne appeared to agree with both of Poncelet's broader philosophical assertions: the importance of choosing a method based on the specific geometric problem at hand and the distinction between ancient and modern pure geometry (and preference for the latter over the former). He further nuanced the discussion by admitting that some problems "remained equally stubborn to all methods," and critiqued Carnot's so-called "mixed methods" for a lack of elegance and simplicity despite their advantageous application in certain "beautiful examples" (Gergonne (1817e), 160).

In conclusion, Gergonne restated his goal of multiplying the examples of analysis applied to geometry, while expressing confidence in the common goals of both pure and analytic geometry. "[...] j'ose croire que la diversité de nos méthodes ne fera jamais naître d'autre rivalité entre nous que celle du zèle pour l'avancement de la science" (ibid, 161).[17] In advocating diversification and specialization to promote scientific progress, Gergonne claimed to be merely elaborating and clarifying a consistent position. However, in comparison with

[16] Lhuilier set up a dichotomy between "la Géométrie ancienne et les méthodes modernes," the former of which he associated with purely geometric methods, while the latter involved the application of algebra and use of calculations to geometric problems. Although Lhuilier demonstrated facility with both methods in his *Élémens d'analyse géométrique et d'analyse algébrique appliquées à la recherche des lieux géométriques*, he had a reputation for preferring "l'ancienne géométrie" and using algebra only with parsimony (Trélis (1810), 170).

[17] "[...] I dare to believe that the diversity of our methods will give birth to no other rivalry between us than that of the zeal for advancing science."

his earlier statements, Gergonne was discreetly yielding to Poncelet's criticisms.[18] His decision reflected well on the *Annales* for Poncelet, who continued to publish articles there and indeed both pure and analytic geometry remained well represented in the journal's contents.

The exchange of articles had been publicized to *Annales* readers as "mathematical philosophy." Between 1811 and 1831, there were twenty-four *Annales* articles listed as "mathematical philosophy" most of which concerned methods of research and representation. The subject category suggested an appreciation of philosophical discourse among the mathematical community. Whatever the degree of enthusiasm of the audience for such discussions, Poncelet and Gergonne resolved that philosophy should be subsidiary to the mathematics: the mathematical problem dictated which method would be best employed.

That said, in practice both Gergonne and Poncelet saw many of the exact same problems as fitting research targets for their competing methods. The telling point in the methodological argument on both sides was the ability of the advocated method to solve problems, as well as features of the solution (simplicity, elegance, brevity, etc.). This would have the effect of demonstrating the effectiveness of their methods and thus publicizing them. For example, Gergonne and Poncelet independently proved solutions to the problem of finding a circle tangent to three given circles respectively employing analytic and pure geometry (Gergonne (1817a), Poncelet (1821a)). With this evidence, each method could thus be compared quantitatively, by the number of solved problems, and qualitatively, by the simplicity and elegance of the solutions in the context of problems known to many readers.

Poncelet found an additional use for geometric problems as a means of staking his claim for priority. Constructive solutions contained enough information to show his geometric abilities, while he promised the theoretic underpinnings in a later work. For example, Poncelet could give instructions on how to draw a circle tangent to three given circles without proving that his instructions would always be possible to carry out. Retrospectively, Poncelet would characterize himself as a young and ambitious officer wishing to promote his first book, and Gergonne as a learned and esteemed editor somewhat wary of the new upstart's philosophical ideas. Poncelet's first philosophical contribution to the *Annales* was an effort to "advance a name for the publication of the *Traité des propriétés projectives*" not merely an apology for pure geometry (Poncelet (1864), 466). Indeed, throughout his life Poncelet's geometric publications aimed at generating an audience for his *Traité*, and one that would read the text in its entirety. For Poncelet, a controversy over choice of method could be good publicity.

As articles in the *Annales*, both texts from 1817 courted the same potential public. Poncelet framed his article as a deferential letter to the editor and spoke specifically to Gergonne's recent publications and role in the *Annales*. Poncelet attributed any accidental

[18]Gergonne further showed his support of Poncelet's results, by publishing a posed problem generalizing Poncelet's solutions to three dimensions (Gergonne (1817d)).

lapses from the formality of a well-prepared article to the epistolary format and space restrictions. This form fused aspects of public and private, since the audience was foremost Gergonne but also the anonymous other readers. He showed caution in his flatteries and critiques, ending on a self-deprecating note. By contrast, Gergonne's response was entirely public and conformed to the standards of an article. He addressed Poncelet's mathematical concerns, but also added a lightness to the correspondence, both informing and entertaining his readers–a savvy strategy to capture a still emergent market.

Both the form and content of their textual exchange express a hierarchical relationship between the two mathematicians, who also filled the respective roles of editor and contributor. In his cultural references and expansive chronology of past publications, Gergonne contrasted to Poncelet as an experienced geometer who respected Poncelet's opinions and mathematical status enough to publish and comment on them. Poncelet wrote deferentially and directly to Gergonne, while Poncelet was not Gergonne's primary audience. Choosing to publish Poncelet's variant methodological perspective spoke well to the open-mindedness of the *Annales* editor, as Poncelet had noted. The methodological resolution also struck an attractive balance, encouraging contributions from future pure and analytic geometers. Gergonne had control of the message, and Poncelet's critique was addressed with gravity. Finally, the conclusion pointed to new and better geometry. Though Gergonne preferred analytic geometry, this did not prevent him from reading and publishing articles with opposing positions, which were then often modified with editorial footnotes and responses. Gergonne's almost unlimited editorial capacities ensured the *Annales* was not only a venue for individual articles, but also a compendium of conversations knitting together the mathematical content.

3.3 Generality and continuity: Poncelet's modern pure geometry and its reception (1820)

Poncelet continued to promote the principles and theories behind his modern pure geometry through journal articles, papers submitted to the *Académie royal des sciences*, and eventually his *Traité* in 1822. As we will see, Gergonne reacted with mixed approval. Their indirect correspondence in this period further cemented their different perspectives on generality, proof and discovery in geometry.[19] As Poncelet's scientific reputation grew, their former hierarchical relations shifted toward greater professional equality. All of these developments are instantiated in the 1820 publication in the *Annales* of a review by Augustin-Louis Cauchy

[19]The importance of generality in Poncelet's geometry has been discussed above, as well as in the historical analyses of Chemla, Nabonnand, Lombard, and Friedelmeyer (Chemla (1998), Nabonnand (2011b), Nabonnand (2011a), Lombard (2011), Friedelmeyer (2011)). Moreover, Gérini has compared Gergonne and Poncelet's conceptions of generality in mathematics (Gérini (2010b)). Here, we consider generality as another axis of potential contention between the two geometer's understandings of geometry.

(1789–1857) of a memoir by Poncelet on the principle of continuity, which he had submitted to the *Académie*. Gergonne took the opportunity to publish his own mathematical research and offer opinions on philosophical and scientific matters with extensive footnotes.

To establish the reputation of his forthcoming *Traité*, Poncelet submitted a memoir on the projective properties of conic sections to the *Académie royal des sciences* in 1820. François Arago, Siméon-Denis Poisson, and Cauchy reviewed this work, and their report was written up by Cauchy. Shortly after, the text appeared with Gergonne's footnotes in the *Annales*, thus serving as the first public exposure to the modern pure geometry that underlay Poncelet's results, and which we explored in Chapter II. Among the many novelties, the one controversial aspect was Poncelet's *principle of continuity*, the principle that certain properties of figures remain invariant under deformation, even when parts of the figure disappear.[20] Consider, for instance, the points of intersection of two concentric circles. Poncelet argued, contrary to perception, that the two circles intersected at infinity.

In his review, Cauchy warned against the principle of continuity.[21]

> Ce principe n'est, à proprement parler, qu'une forte induction, à l'aide de laquelle on étend des théorèmes établis, d'abord à la faveur de certaines restrictions, aux cas où ces mêmes restrictions n'existent plus. Etant appliqué aux courbes du second degré, il a conduit l'auteur à des résultats exacts. Néanmoins, nous pensons qu'il ne saurait être admis généralement et appliqué indistinctement à toutes sortes de questions en géométrie; ni même en analise : En lui accordant trop de confiance, on pourrait tomber quelque fois dans des erreurs manifestes. (Poncelet and Cauchy (1820), 73)[22]

As justification for this caution, he presented an example concerning definite integrals for measuring lengths, areas, and volumes, in which the principle of continuity could be used to derive untrue results.

In a mitigating footnote, Gergonne proposed applying the principle of continuity as a valuable instrument toward mathematical discovery if not proof.

> II faut donc employer le principe de M. Poncelet, ainsi que le tour de démonstra-

[20]The historical development and mathematical details of Poncelet's principle of continuity have been studied is described by Nabonnand in Nabonnand (2011b) and by Belhoste in Belhoste (1998).

[21]Cauchy's review of Poncelet and Poncelet's directed response (not published until the mid-1860's after Cauchy had died) has been described as an "intense polemic" by Otero (1997) and is treated in detail as a controversy by Claude Paul Bruter in Bruter (1987). However, compared to the duality controversy and given the extended time interval between the textual exchange over which one of the participants was no longer living, we suggest that it was of only minor importance to the actors involved at the time.

[22]"This principle is, properly speaking, no more than a strong induction, by aid of which one extends theorems established at first by favour of certain restrictions, to cases where these same restrictions no longer exist. If applied to second degree curves, it leads the author to exact results. Nevertheless, we think that it should not be admitted generally and applied indistinctly to all sorts of questions in geometry, nor even in analysis: in according in it too much confidence, one could at times fall into manifest errors."

> tion introduit par Monge, à peu près comme on employait le calcul différentiel lorsqu'on n'en voyait pas bien encore la métaphysique; c'est-à-dire, uniquement comme instrumens de découverte ; mais ce n'en seront pas moins des instrumens très-précieux ; car, le plus souvent, en mathématiques, découvrir est tout; et ce ne sont pas d'ordinaire les démonstrations qui embarrassent beaucoup. (ibid)[23]

Gergonne thus suggested that proofs would be fairly easy to obtain following a mathematical discovery, a perspective that went against Cauchy's viewpoint.

In his paper submitted to the *Académie*, Poncelet also proposed the adoption of new mathematical objects. In particular, he had defined ideal chords as the constructible line segments "in common" between any two conic sections that did not share two real points of intersection. Instead of representing chords visually by a line segment bounded by the interior of two conic sections, ideal chords could be recognized by a ratio relationship. Cauchy declared *ideal chords* as one of the most remarkable aspects of Poncelet's geometry and thus meriting further study. Gergonne objected, describing Poncelet's new definition as "subject to numerous exceptions" [*sujette à des exceptions nombreuses*] and requiring "ingenious concepts" [*des conceptions ingénieuses*] (80). Poncelet had argued that ideal chords followed from the coordinate representation of conic sections as second degree equations. In analytic geometry, one could derive an expression for non-intersecting conics through substitution. Gergonne did not share Poncelet's determination to import this generality from analysis to geometry, especially at the risk of destroying geometry's visual certainty. The use of ideal chords would compromise the advantages and superiority of geometry over other sciences. That is, a chord should look like a chord.

Unlike Poncelet, neither Gergonne nor Cauchy found any reason to privilege or modify pure geometry when analytic geometry would suffice. Consequently, the article and footnotes contained almost none of the modern pure geometry underlying Poncelet's derivations, and was instead replaced by coordinate equations in Cauchy's interpretation. Cauchy concluded by recommending Poncelet's memoir, but his summary of its contents was neither favourable nor representative of Poncelet's methodology.

Though a single critical review does not comprise a controversy, Cauchy's caution against Poncelet's principle of continuity spread through the geometrical community. On the one hand, ideal secants, chords, and points of intersection were soon adopted by several young geometers in the *Annales* (including Charles Sturm (Sturm (1826a), Sturm (1826b)), Étienne Bobillier (Bobillier (1827)), and Michel Chasles (Chasles (1828a))). In particular, we

[23] "One must thus use the principle of M. Poncelet, just as the manner of proof introduced by Monge, almost like how we used differential calculus when we hadn't yet understood the metaphysics; that is to say, only like instruments of discovery; but no less precious instruments; because, most often in mathematics discovery is everything; and what is troublesome is not the proofs." Based on an earlier reference to Jean Baptiste Durrande, Gergonne appeared to associate Monge's "manner of proof" with the application of three-dimensional geometry to planar figures, thus enabling more general principles (Durrande (1820), 1–5).

will see how the "German geometers," Plücker and Steiner, utilized Poncelet's ideal objects in Chapter IV. On the other hand, these same geometers avoided Poncelet's dubious principle of continuity.[24] Gergonne would later refer to the principle of continuity as "subject to controversy" [*sujettes à controverse*]. Consequently, he would attribute Poncelet's lack of popularity to the uncertain status of this fundamental principle, as we shall see (Gergonne (1827d)).

Following Cauchy's reception of the principle of continuity, Poncelet continued to submit memoirs for *Académie* review. Poncelet seemed to value the status associated with the *Académie* (where he became a member for his contributions to engineering in 1834) and the associated reviews contained enough positive publicity to merit wider publication in the *Annales.* As an introduction to his *Traité* in 1822, Poncelet republished the projective properties review without Gergonne's footnotes (and then again in 1864). Cauchy's review of Poncelet's memoir submitted to the *Académie* on harmonic ratios appeared in the *Annales* in 1826, where, as an anonymous reviewer in the *Bulletin de Férussac* remarked, Cauchy took "cette occasion de combattre de nouveau le *principe de continuité*" (Anonymous (1826c), 109).[25] By 1824, the *Académie* had received a third memoir by Poncelet, this one on polar reciprocity, but it was not to be reviewed until 1828. By this time, due to severed relations between Poncelet and Gergonne, the review appeared in the *Bulletin* and not the *Annales.* The memoirs on harmonic ratios and polar reciprocity that Poncelet submitted to the *Académie* were first published in Berlin within August Crelle's new journal, *Journal für die reine und angewandte Mathematik*, between 1827 and 1829, and again by Poncelet in the 1860's (Poncelet (1865), Poncelet (1866)).

Presentations to the *Académie* might secure priority, but did not guarantee a public. When Poncelet's work was publicized, Cauchy's review only gave an approximate picture of Poncelet's methods. The anonymous *Bulletin* summary of the *Annales* publication of Cauchy's review of Poncelet's memoir on harmonic ratios noted that "M. Cauchy ayant substitué des considérations de statique aux considérations de géométrie pure qui avaient guidé M. Poncelet, ce rapport paraît peu propre à donner une idée du mémoire de l'auteur" (Anonymous (1826c), 109).[26] As with Poncelet's 1817 article, the interested reader would

[24]For instance, Sturm was careful to disassociate his own general researches from those of Poncelet.

> On ne doit pas d'ailleurs les confondre avec les considérations de M. Poncelet sur la loi de continuité. La distinction en a été déjà faite, avec soin, par M. Cauchy, dans son rapport inséré au tome XI.e des *Annales* (pag. 69) et placé depuis en tête du *Traité des propriétés projectives des figures.* (Sturm (1826a), 279)
>
> "One must not also confuse them [theories of transversals] with the considerations of M. Poncelet on the law of continuity. The distinction between them has already been made, with care, by M. Cauchy in his report inserted in the IXth volume of the *Annales* (page 69) and later placed at the start of the *Traité des propriétés projectives des figures.*"

[25]"[...] this occasion to combat anew the *principle of continuity.*"

[26]"M. Cauchy having substituted considerations from statics for the considerations of pure geometry,

need to consult Poncelet's *Traité* to see both his results and his method. Though Poncelet was misrepresented by the *Académie* reports, his goal of publicity succeeded. Meanwhile others also achieved an audience from Poncelet's circuitous publicity. Cauchy had an opportunity to express hesitations about imprecise, unrestrained invocation of continuity.[27] Gergonne used footnotes to promote other *Annales* content, reassert methodological considerations, and offer alternative proofs. From 1824 onward, the *Bulletin* found content in summarizing these exchanges, and beginning in 1827, Crelle's *Journal* gained relevant, unpublished but anticipated new articles. The reception of Poncelet's memoir demonstrates that there existed a small but well-connected community of mathematical publications and readers. These would be utilized as the setting for documenting and disputing evidence in the ensuing controversy.

3.4 Duality and *Pluker*'s first publication

Poncelet's multifaceted efforts to publicize his *Traité des propriétés projectives* underscore some of the difficulties of disseminating new mathematics. Obtaining recent publications appeared to be difficult as well. Despite having studied in Paris between 1822 and 1824, Plücker claimed to not have had access to the *Annales* past the eighth volume, even though it was distributed in Paris by the publisher Courcier.[28] Similarly, while Poncelet had published his *Traité* in Metz in 1822, Plücker only knew of it through catalog advertisements.

By 1826, Plücker had returned to Bonn and was busy teaching. Throughout his mathematical career Plücker's interests lay exclusively in analytic geometry and, as we saw in Chapter II, his first article was no exception. Upon reading Poncelet's 1817 "Réflexions", Plücker determined to show that Poncelet's result on tangency between two conic sections could be derived as simply through analytic geometry. However, Gergonne took Plücker's article submission as a further opportunity to promote the principle of duality.

Gergonne initially propounded his concept of duality as *mathematical philosophy.* His 1826 article, "Considérations philosophiques sur les élémens de la science de l'étendue", was not the first research or even publication on what would soon be called duality (Gergonne (1826)). Even in this early publication (with no priority dispute underway) Gergonne claimed to have begun research in the field ten years prior. Here Gergonne first coined the term *duality*, giving an explicit, though sketchy, definition of duality as a simple exchange

which had guided M. Poncelet, this report appears scarcely appropriate to give an idea of the memoir of the author."

[27] Cauchy is well-known today for his work on continuity within Calculus. For the historical development of Cauchy's approach to continuity see Grabiner (1981), Gilain (1989), and particularly the bibliography in Gilain and Dhombres (1992). More recently Umberto Bottazzini and Jeremy Gray have analyzed Cauchy's mid-century work on continuity in Bottazzini and Gray (2013).

[28] In his biography of Plücker, Wilhelm Ernst attests that Plücker studied under Biot, Cauchy, Lacroix, Poisson, Pouillet, Thenard as well as Clément, Dulong and Binet (Ernst (1933), 7).

of the words *point* and *line* in the plane or *point* and *plane* in space to pass from a theorem to its correlative as illustrated with two facing columns (Figure 3.1).[29]

§. I.

Notions préliminaires.

1. Deux points, distincts l'un de l'autre, donnés dans l'espace, déterminent une droite indéfinie qui, lorsque ces deux points sont désignés par A et B, peut être elle-même désignée par AB.

2. Trois points donnés dans l'espace, ne se confondant pas deux à deux et n'appartenant pas à une même ligne droite, déterminent un plan indéfini qui, lorsque ces trois points sont respectivement désignés par A, B, C, peut être lui-même désigné par ABC.

1. Deux plans, non parallèles, donnés dans l'espace, déterminent une droite indéfinie qui, lorsque ces deux plans sont désignés par A et B, peut être elle-même désignée par AB.

2. Trois plans, non parallèles deux à deux dans l'espace, et ne passant pas par une même ligne droite, déterminent un point qui, lorsque ces trois plans sont respectivement désignés par A, B, C, peut être lui-même désigné par ABC.

Figure 3.1: Gergonne's adoption of two columns in Gergonne (1826)

Gergonne had first used a two column format the year before in an article on polyhedra, with corresponding *sommet* and *face* (Gergonne (1825)).[30] In his philosophical article, he explained the columns' advantage in making the corresponding content more visually striking.

> Nous aurons même soin, afin de rendre cette correspondance plus apparente, de présenter les théorèmes analogues dans deux colonnes, en regard l'une de l'autre, comme nous en avons déjà usé, dans l'article sur les polyèdres rappelé plus haut; de telle sorte que les démonstrations puissent se servir réciproquement de contrôle. (Gergonne (1826), 212)[31]

He further emphasized the novel nature of his research by offering an entirely elementary treatment grounded solely in concepts from Euclidean geometry, and by citing very few predecessors, among which Poncelet was notably absent. In this context, duality was not used as a means of proving theorems, but to show a correspondence between theorems that

[29]Gergonne extended this definition to spherical geometry as well. Karine Chemla has documented Gergonne's development of duality inspired from his work in spherical trigonometry (Chemla (1989)).

[30]We explored the visual aspects of a two column format in Chapter II.

[31]"We will likewise take care, in order to render this correspondence more apparent, to present the analogous theorems in two columns, one facing the other, as we have already been accustomed to do in the article on polyhedra cited above, in such a way that the proofs can serve reciprocally as checks."

had been proved previously. Along with his elementary exposition of planar, spatial, and spherical geometry in dual form, Gergonne also suggested that the work of others could be rewritten and expanded through duality. As an example, he pointed out how a recent theorem of the Belgian geometer Germinal Pierre Dandelin (1794–1847) could be expressed in dual form. Overall, Gergonne emphasized duality as a "philosophical fact," whose formal character was considered even more important than the new results it generated.

Gergonne's adoption of duality marked a transition in his research activities. He claimed that geometry, *la science de l'étendue*, divided into metric (concerning measurable quantities) and non-metric (situational or positional) relations, and proposed that any non-metric geometric property had a dual property. Gergonne presented duality as independent of calculation and measurement, since the dual of a theorem was found by the proper word substitution. Further, none of the articles he referenced employed coordinate equations to represent geometric objects. According to Gergonne's own subject headings, these were articles on elementary geometry, geometry of the ruler, or pure geometry. The properties and proofs within his "Considérations philosophiques" implied that Gergonne's interest in duality was at this point independent from, and had to some extent supplanted, his interest in analytic methods in geometry.

At least some readers were impressed by the apparent power of this method. From its inception in 1823, the *Bulletin de Férussac* summarized and reviewed almost every article published in the *Annales*. Intriguingly, the *Bulletin* review of "Considérations philosophiques" began by describing a constructive process using properties of perspective and black and red pencils to arrive at bicolored dual figures on a plane and a sphere. From here, the anonymous author proceeded to show how poles and polars could transform one planar figure into a corresponding figure where points of one colour replaced lines of the other, and the same positional properties held. Only after this introductory and highly visual constructive procedure was Gergonne's article discussed.

In this summary, the anonymous reviewer portrayed Gergonne's contribution as deep and reaching far beyond polar reciprocity.

> Mais si les propriétés des pôles, polaires et plans polaires mettent en évidence cette espèce de *dualité* d'une partie notable de la géométrie, ce n'est certainement pas en vertu de ces propriétés, mais bien en vertu de la nature même de l'étendue qu'elle a lieu ; et c'est principalement à mettre cette vérité dans le plus grand jour que M. Gergonne a consacré la livraison de son recueil qui vient de paraître. (Anonymous (1826b))[32]

[32]"But if the properties of poles, polars, and polar planes places this type of duality in evidence as a notable part of geometry, this certainly does not take place in virtue of these properties, but in virtue of the nature of extension; and M. Gergonne has devoted the most recent issue of his journal principally to place this truth in the clearest light."

To convey Gergonne's evidence, the *Bulletin* printed an example of Gergonne's two corresponding columns, and concluded by reiterating Gergonne's suggestion of how duality and the use of dual columns could be applied to certain recent *Annales* articles. Gergonne's enthusiasm for duality and dual columns would also manifest in future articles submitted to his journal.

Gergonne's form of duality was further affirmed in the seventeenth volume of the *Annales*, which contained two articles by a new German contributor, "M. Pluker, doctor from the University of Bonn." The first was classified under the subject headings "Géométrie des Courbes et Surfaces" and "Géométrie de Situation" entitled "Théorèmes et problèmes sur les contacts des sections coniques" (Plücker (1826b)). In the introduction, the author cited two texts, Brianchon's *Mémoire sur les lignes du second ordre* (1817) and Poncelet's *Traité* (1822), the latter containing research on "a series of curious problems." Without explicitly criticizing Poncelet's method, Pluker proposed to address these same problems through "very simple" considerations. Since the propositions corresponded two-by-two, the author would use the "form already adopted in different places in the present journal" of dual columns (Plücker (1826b), 38). An accompanying footnote cited several recent works by Gergonne where dual columns had been employed.

The body of the text contained twenty-four theorems and sixteen problems, among them the problem that had inspired Plücker's original article. As observed in Chapter II, the resulting constructive solution was nearly identical to that from Poncelet's 1817 paper, which received no mention.

Regular readers of the *Annales* may have been struck by the similarity of form and expression between Pluker's first article and Gergonne's article on duality from the previous volume. With its dual columns, elementary considerations, and particular applications of ruler-based planar geometry, the Pluker article almost functioned as a sequel to Gergonne's. Just as Gergonne had incorporated Dandelin's results into his duality paper, Plücker's research had been rewritten to serve as evidence in favour of Gergonne's version of duality. However, while Dandelin's original results had already been published in the *Annales* (Dandelin (1825)), there was no indication that the work of Pluker was not entirely that of Plücker. In this, Gergonne seemed to take advantage of Plücker's status as a young, foreign scholar. Gergonne had entirely abandoned Plücker's analytic geometry and with it his methodological argument and theory of tangent conics. Finally, by changing Plücker's citations, originally limited to the eighth volume of the *Annales*, Gergonne gave the impression that *Pluker* was well-versed in recent *Annales* publications as well as Poncelet's *Traité* and Brianchon's *Mémoire*. On the one hand, this brought Plücker's original paper into a more contemporary context, described in greater detail in Chapter II. On the other hand, the article's contents, limited to planar constructions and deformations, suggested a rejection of the modern principles and theories advocated particularly in the work of Poncelet.

With respect to duality, *Pluker* cited the exact same texts as those in Gergonne (1826), again slighting Poncelet's contributions. Poncelet appeared only as a source of "une série de problèmes curieux," whose proofs had been replaced by simpler ones.

Poncelet was completely absent from the subsequent *Bulletin* summaries of Pluker's articles, which appeared later in 1826. After a brief technical synopsis, the first article was described as "extremely simple" and the second as having been deduced from a "very short and very simple analysis" (Anonymous (1826a), 272-273).

The following year Gergonne continued to publish results presented in dual columns derived from the principle of reciprocity. While Gergonne had demonstrated that his results in both columns could be proved through Euclidean geometry in 1826, by 1827 he was confident enough to state theorems on polar reciprocity without proof pertaining to algebraic curves and surfaces of any order (Gergonne (1827b)). In particular, Gergonne suggested that a curve and its dual would have the same order, even though this had only be shown for curves of second order. This generalization combined with the recent *Bulletin* reviews would be the initial sparks to ignite the duality controversy.

3.5 The duality controversy erupts (1826–1832)

The attacks and rebuttals that we alluded to in the introduction of this chapter quickly escalated. Poncelet condemned Pluker's article in a letter to Gergonne (Poncelet (1826)), of which Gergonne published excerpts and responded with a public defence (Gergonne (1827f)). Following this, Poncelet wrote two more articles restating his argument for the *Bulletin* (Poncelet (1827b), Poncelet (1828c)). Gergonne then republished Poncelet's *Bulletin* content and the remainder of Poncelet's original letter, taking the opportunity to add footnoted commentary in the *Annales* (Gergonne (1827e), Poncelet (1827a)). In 1828 Plücker offered his own interpretation in the *Bulletin*, which elicited a reply from Poncelet (Plücker (1828c), Poncelet (1829)). That same year, Poncelet's original writings on polar reciprocity would finally be published not in the *Annales*, as had his previous work, but in Crelle's *Journal* (Poncelet (1828a)). We turn to the publicity in these texts, which together comprised the main drama of the duality controversy.

In 1824, Poncelet had read and submitted a memoir on polar reciprocity to the *Académie*, which had yet to be reviewed by 1826. Gergonne wished to share details of this work which he correctly surmised bore strong relations to his duality, and wrote to Poncelet requesting additional information. Poncelet's response included a lengthy preamble and postscript responding to Pluker's article and his own priority. However, Gergonne edited out the charges against Plücker and published only an excerpt, "Analyse d'un mémoire présenté à l'Académie royale des Sciences" (Poncelet (1826)).

As in 1817, Poncelet was limited in his exposition by the bounds of the letter, and often referred the reader to his *Traité* for a more in-depth treatment and further examples. Poncelet recognized his polar reciprocity as Gergonne's duality, and used the two terms more or less interchangeably. In this text, both terms referenced "a simple substitution of names and letters" and a process independent from calculation. Both geometers viewed the *mechanical* or *reason-free* derivation of dual propositions as its primary advantage and appeal. As Poncelet wrote,

> Le but principal que je me propose, dans ce mémoire, c'est d'examiner quelle espèce de modification éprouvent une figure donnée et les relations qui lui appartiennent, lorsque l'on passe à celle qui en est la polaire réciproque, et vice versa, et de réduire, en quelque sorte, à un pur mécanisme, à une simple substitution de noms et de lettres, écrites à la place les unes des autres, la traduction de toutes les affections, de toutes les propriétés tant soit peu générales qui appartiennent à une figure donnée et a sa réciproque ; enfin de montrer comment on peut, au simple énoncé d'une proposition qui se rapporte soit aux relations projectives, en général, soit aux relations d'angles des figures situées dans un plan ou dans l'espace, comment ou peut, dis-je, obtenir sur-le-champ et sans recourir à aucun calcul ou raisonnement, une, deux ou trois autres propositions, tout-à-fait distinctes de la première et néanmoins tout aussi générales. (Poncelet (1826), 266)[33]

Poncelet described Gergonne's treatment as "very philosophical" [*très-philosophique*] and perhaps subject to "too vague generalities" [*des généralités trop vagues*]. He attempted to show that his method did not possess these disadvantages by giving several concrete examples. Furthermore, Poncelet emphasized the role of a reference conic section in deriving dual properties, which Gergonne had mentioned but not employed in his presentation. While Gergonne stressed the application of duality to non-metric relations, Poncelet demonstrated that his principle could also be applied to metric properties. Finally, Poncelet hinted at even a three-fold [*trial*] or four-fold [*quadrial*] correspondence between propositions through the transformation of projective properties, such as those concerning angles. This multiplicity suggested that the term *dual* and the use of dual columns, would be insufficient for conveying the entire scope of the potential relationship.

[33]"The principal goal that I set myself in this memoir is to examine which type of modification a given figure, and the relations which belong to it, undergo when one passes to its polar reciprocal, and vice versa; and to reduce this, in some sense, to a pure mechanism, a simple substitution of names and letters, written one instead of the other, the translation of all the relations, of all the properties, no matter how slightly general though they may be, that belong to a given figure and to its reciprocal; finally to show how one can, by the simple statement of a proposition which concerns either projective relations, in general, or to relations of angles of figures situated in a plane or in space; how one can, I say, obtain immediately and without recourse to any calculation or reasoning, one, two or three other propositions, completely distinct from the first and nevertheless all also completely general."

Gergonne's response followed directly, and like in 1817 he addressed the broader readership of the *Annales*, not specifically Poncelet. He praised Poncelet's contributions, but suggested that Poncelet himself, who worked in applied mathematics and had just revealed his apprehension toward Gergonne's philosophical leanings, might not comprehend their full significance.

> Les esprits superficiels, ceux qui n'étudient les sciences que comme on apprend un métier, et qui n'en comptent pour rien la philosophie, pourront ne voir dans le beau travail de M. Poncelet, que quelques théorèmes nouveaux ajoutés à ceux dont nous sommes déjà en possession, et une manière nouvelle de démontrer des théorèmes. déjà connus. (Gergonne (1827f))[34]

Gergonne portrayed the new doctrine of duality as a "revolution for science." He explained that duality was "important and eminently philosophical" because it revealed part of the underlying structure of geometry, where perhaps every theorem had a "double face." This was the revolution. Here Gergonne referred to the recent article by "Pluker" where more general and complete solutions were already known "as people have remarked to us already several times." However, Gergonne argued, the *form* was almost everything, and this form was what principally recommended "Pluker's little memoir." Unknown to Gergonne's readers, he was the person responsible for the published *form* of Pluker's little memoir. Gergonne was not so much defending Plücker as engaging in self-promotion through the advertisement of his new technology.

After defending his philosophical approach, Gergonne took the offensive. He acknowledged Poncelet's early contributions to duality, but insinuated that Poncelet had at first regarded polar reciprocity as "very accessory" and did not realize its import. Gergonne suggested that by mixing the theory of polar reciprocity with the "controversial" principle of continuity, Poncelet had diluted the significance of the former and rendered it subject to attacks aimed at the latter. Finally, Gergonne criticized Poncelet's use of language:

> Il est au surplus un obstacle réel à la propagation facile des doctrines que M. Poncelet et nous cherchons à populariser, et cet obstacle, comme nous l'avons déjà insinué plusieurs fois, réside dans l'obligation où nous nous trouvons de parler la langue créée pour une géométrie bien plus restreinte que celle qui nous occupe. (275)[35]

[34] "Those who superficially study science like one learns a career, without taking account of philosophy will only appreciate Poncelet's beautiful work for its new theorems and as a new manner of demonstrating already known theorems."

[35] "It is moreover a real obstacle against easily propagating the doctrines that we and M. Poncelet are trying to popularize, and this obstacle, as we have already suggested several times, resides in the obligation to speak a language created for a much more restricted geometry than that with which we are occupied."

Gergonne proposed that the new theory would be the most advantageous if one could *first* create a language suited to it, but acknowledged that this would be difficult to make and that perhaps it would be difficult to obtain full acceptance for such a language. He implied that duality had not yet become widespread because of an inherent conservatism in the language of geometry. This sentiment coheres with Gergonne's overall efforts toward propagating duality to a wider audience.

The tone of this interchange between Poncelet and Gergonne contrasted sharply with that of 1817. Perhaps most strikingly, Gergonne now praised Poncelet's numerous contributions, while Poncelet appeared only concerned with Gergonne's research as far as it related to Poncelet's publications and legacy. This shift in their hierarchical dynamic was further reflected in the context of Poncelet's letter written only as a response to Gergonne's request. By 1826, Gergonne appeared to believe that even Poncelet's as yet unpublished work would be of interest to his readers. Though as an editor Gergonne still had full power to decide what of Poncelet's letter would be printed, Poncelet had since become a well-respected and established geometer whose research was an asset in popularizing duality. Poncelet's reputation beyond the *Annales* is evidenced by his reception in *Académie* reports and subsequent *Bulletin* summaries (Poncelet and Cauchy (1820), Poncelet and Cauchy (1825), Anonymous (1826c)).

The *Bulletin* only devoted a paragraph to the correspondence of Poncelet and Gergonne (Anonymous (1827b)). The paper submitted by Poncelet to the *Académie* was described as including more ample developments on the same subject. Poncelet's particular results were listed summarily:

> [...] il a trouvé qu'un grand nombre de théorèmes relatifs à des relations métriques, soit d'angles, soit de longueurs, étaient, comme les théorèmes de situation, susceptibles de cette sorte de *dualité* signalée par M. Gergonne, et a donné des règles pour reconnaître les théorèmes de cette classe et pour en déduire leurs corrélatifs par de simples mutations de mots et de symboles. (Anonymous (1827b), 274)[36]

By attributing *duality* to Gergonne, the article implied that Poncelet's results were derivative. Neither the article published in the *Annales* nor the *Bulletin* review mentioned the date of Poncelet's 1824 *Académie* submission, and thus the readership of the *Bulletin* might easily infer that Gergonne was the first to recognize duality. Poncelet sensed this possibility, and took action.

The following volume of the *Bulletin* contained a "Note" by Poncelet commenting on recent articles published in the *Bulletin* (Poncelet (1827b)). Taking advantage of the scientific

[36] "[...] he has found that a great number of theorems relative to metric relations either of angles or of lengths, are, like theorems of position, susceptible to this sort of *duality* introduced by M. Gergonne; and has given rules for recognizing the theorems of this class and for deducing their correlatives by simple mutations of words and symbols. "

hierarchy, Poncelet addressed his paper to those geometers "who did not have the advantage of attending the *Académie* of sciences," and thus had missed Poncelet's presentation on polar reciprocity on April 12, 1824. To prove his version of events as authentic, Poncelet invoked as witnesses the people who had been present, and recalled the reception of his work, "dont la singularité des vues a même fait dire plaisamment à d'illustres académiciens, que c'était de la *géométrie romantique*, de la *géométrie à quatre dimensions*" (Poncelet (1827b), 110).[37] Poncelet observed that the marked response of such a well-informed and important public had already been noted in the *Annales* the previous year (Poncelet (1826)), and verified the priority of Poncelet's results. His 1827 article in the *Annales* had been only an extract of this earlier presentation, sent to Gergonne upon request. Meanwhile, Poncelet had been occupied with the professional public obligations of military service and teaching mechanics applied to machines at the engineering school in Metz. Despite this lapse in mathematical activity, Poncelet desired to claim credit for his already semi-public earlier achievements.

In particular, Poncelet presented the Pluker article as an example of how little acknowledgement he had received for his new theories and principles. Poncelet explained that he had written about the matter in a letter to Gergonne, which had been "suppressed." Because of his "high esteem" for Gergonne, Poncelet had initially remained silent, hoping that Gergonne would eventually address the omission publicly. The *Bulletin* review had prompted Poncelet to seek a new outlet, in which he could establish proper priority.

Poncelet argued that he was seriously committed to research on polar reciprocity as evidenced by an entire memoir on the subject as well as treating it at length in his *Traité*. He contended that *Académie* reports had misrepresented his work as controversial by suggesting that Poncelet practiced a "espèce de géométrie où l'on remplace la rigueur du raisonnement par des inductions hasardées, des aperçus de pur sentiment" (115).[38] Poncelet attributed

[37] "[...] whose singularity of views has even pleased the illustrious academicians, that this was from *romantic geometry*, from *geometry of four dimensions*." This unusual descriptor offered an alternative delineation between classical mathematics and romantic mathematics. The descriptor *romantic geometry* and Poncelet's self-described marginalization may seem to serve as examples in favour of the "romantic narrative" and "romantic mathematics" described in Alexander (2006). However, Poncelet repeatedly emphasized that the value in his work was toward deriving the "useful new results" that Amir Alexander has classified as the province of eighteenth century mathematicians. Indeed, the term "romantic" carried assorted meanings, as Gergonne would later apply it to describe disputable new mathematics Gergonne (1827e). The adjective "romantic" might also describe qualities associated with novels and poems, as defined in the *Dictionnaire de l'Académie françoise* in 1798.

> Il se dit ordinairement Des lieux, des paysages, qui rappellent à l'imagination les descriptions des poèmes et des romans. (Anonymous (1798), 510)
>
> "One calls [romantic] usually places, landscapes, that recall in the imagination the descriptions of poems and novels."

[38] "[...] type of geometry where one replaces the rigour of reasoning by lucky inductions, perceived through pure sentiment."

this reproach to a lack of understanding of his *Traité* in its entirety, where his exposition was fully deductive.

> Nous aurions bien mal employé notre temps et nos peines, si nous n'avions pas réussi [...] à mettre le résultat de nos démonstrations à l'abri de toute attaque ; peut-être même n'en saurait-on dire autant de beaucoup d'écrits géométriques de notre époque, où la logique sévère des anciens est quelquefois négligée par suite de l'habitude acquise, assez généralement, d'accorder aux symboles et aux opérations de l'algèbre une rigueur mathématique presque indéfinie. (ibid, 116)[39]

Unlike Gergonne, Poncelet had not "rushed into a revolution." He proceeded "prudently" from "ordinary" geometry exemplified in his *Traité*, which contained over 400 pages of methodical exposition. Poncelet pointed specifically to how he had established his propositions through "synthetic geometry" just as in all the elementary geometry texts.

> Cette recommandation de la géométrie synthétique sera, si l'on veut, une concession faite aux idées du siècle, un moyen détourné de faire goûter les nouvelles doctrines et de ne point effrayer les géomètres qui tiennent à l'ancienne forme des élémens ; (ibid, 116)[40]

Poncelet reiterated his position on synthetic geometry, earlier stated in his *Traité*, as a restrained geometry comparable to "ancient pure geometry." Even so, he admitted its usefulness as a medium for introducing conservative geometers to new material, such as the principle of duality.

While Poncelet praised Gergonne as the "learned editor" of the *Annales*, his description appears more formal than heartfelt. Without any explicit charge of plagiarism, Poncelet set up a subtle contrast. He expressed his admiration for Brianchon as a modest geometer with enough of his own achievements not to envy those of others, while Gergonne's 1826 paper bore what Poncelet delicately termed an *analogy* with his prior researches. Yet, Poncelet's main concern in this article was not the fault of Gergonne, but rather of the subsequent *Bulletin* articles publicizing Gergonne's research at the expense of his own.

Poncelet concluded by correcting and cautioning against the overenthusiastic generality and scope of Gergonne's duality. Gergonne had "exaggerated" the importance of duality in order to reach uncertain or even controversial conclusions. While Poncelet did not think

[39] "We would have very badly employed our time and pains, if we had not succeeded [...] in sheltering the result of our demonstrations from all attack; perhaps even one could not say as much for many geometric writings of our time, where the severe logic of the ancients is sometimes neglected by following the rather generally acquired habit of according an almost indefinite amount of mathematical rigour to the symbols and operations of algebra."

[40] "This recommendation of synthetic geometry will be, if one likes, a concession made to the ideas of the time, a roundabout way to introduce new doctrines and not frighten the geometers who value the older form of the elements;"

anyone "could attack the exactitude of his results," he showed wariness toward publicizing duality in general. Instead of encouraging future research in the subject, Poncelet drew attention to his own past contributions.

Poncelet's article extended his audience to reach readers of the *Bulletin* who were neither members of the *Académie* nor avid readers of the *Annales*.[41] For Gergonne, Poncelet's demand for priority amounted to a public accusation, and so Gergonne would respond publicly as well.

However, Gergonne shifted the exchange back to the *Annales* by republishing Poncelet's *Bulletin* article accompanied by the *suppressed* preface and postscript from Poncelet's 1826 letter (Gergonne (1827e)). He placed this collection of articles and notes under the subject heading "Polémique mathématique"–the first and last use of this title in the *Annales*. In an introduction, Gergonne described Poncelet's article as "une sorte de note diplomatique, en forme de manifeste, où, à travers des expressions beaucoup trop flatteuses pour moi, on voit percer, de toutes parts, beaucoup d'amertume, des reproches et même des accusations assez graves" (125).[42] Gergonne explained to his readers that he had intended to convince Poncelet that his accusations were unfounded in private, and thus chose to leave these segments of the original letter unpublished. He phrased his motivation strategically, "finally, to add still more, if it is possible, to the publicity which he [Poncelet] desires to obtain" [*afin d'ajouter encore, s'il est possible, à la publicité qu'il a désiré d'obtenir*].

By organizing his response into footnotes, Gergonne implied a sort of editorial objectivity. Poncelet's text was presented as polemical and through editorial commentary his suggestions seemed more like accusations. By contrast, Gergonne's response was delivered less as a rebuttal than a rectification. Beneath this veneer of scholarly dispassion, Gergonne's language oozed with vicious sarcasm and subtle insults. Each of Poncelet's claims was interpreted as extreme then swiftly ripped apart, still Gergonne claimed neutrality.

> Le lecteur jugera. Je ne fais d'ailleurs aucun reproche, j'énonce simplement une opinion. (ibid, 136)[43]

In these frequent asides, Gergonne played with his readers by making fun of the debate itself and letting them in on his jokes, thus securing their support of his position. We will not attempt to simulate the peculiar format of footnotes, although traces remain with the numerous parenthetical asterisks (e.g. (*)), and consider Gergonne's response as a cohesive piece comprising three main themes. First, in response to Poncelet's reactive

[41] At the time Poncelet speculated that Gergonne himself was working as an editor for the *Bulletin* and had composed the anonymous summaries in his own favour. Gergonne heard of these insinuations and quickly denied them (Anonymous (1828b), 25).

[42] "[...] a sort of diplomatic note, in the form of a manifesto, where, in terms too flattering to me, one perceives, in all parts, much bitterness, many reproaches, and and even very serious accusations."

[43] "The reader will judge. I make no reproach, I simply announce an opinion."

accusations that Gergonne was coveting undue credit, Gergonne supplied evidence of his earlier and independent researches. Secondly, Gergonne portrayed Poncelet as melodramatic and overreacting. Finally, without suggesting that Poncelet was a plagiarist, Gergonne shifted the blame for Poncelet's lack of recognition to Poncelet himself.

With his 1826 paper, Gergonne explained that he had intended to provide an elementary introduction to duality following the shared views of Poncelet and himself. This equal division of credit implied that Gergonne's own research was independent, and Gergonne asserted that he had begun developing the idea in 1819. He referred to three of his earlier, duality-infused, publications on polyhedra and spherical trigonometry from 1825, which preceded his publications in planar geometry. The duality of spherical trigonometry, Gergonne explained, was well known, "there is not, I think, a single geometer for whom this duality is a mystery" [*il n'est, je pense, aucun géomètre pour qui cette dualité soit un mystère*] (126).

From a different angle, Gergonne insisted that he could not have stolen Poncelet's ideas because they were not public until 1824, and then only presented to the *Académie* in Paris, while Gergonne was in Nîmes. He scoffed at the possibility of co-opting Poncelet's private research: "Qui sait? J'écoutais peut-être aux portes" (130).[44] Gergonne followed up with a hypothetical reconstruction of the events that could have merited Poncelet's very serious accusation: Gergonne had hidden the date of Poncelet's memoir, continued to avoid citations cautiously, and finally written long memoirs on the subject of duality. Gergonne concluded that none of these events had actually occurred, and the accusation was "a pure phantom that the lightest breath could easily cause to vanish" [*un pur fantôme, que le souffle le plus léger pourrait faire aisément évanouir*] (131). In truth, Gergonne reminded the reader that he had published Poncelet's notes on polar reciprocity shortly after receiving them in 1826, and any delay in publicizing Poncelet's 1824 presentation was the fault of the *Académie*, whose members had waited three years before issuing a report.

Not all of Poncelet's criticisms were dismissed. Gergonne conceded that when he had failed to mention Poncelet's name with respect to polar reciprocity, this was because the association was so strongly identified as to be "superfluous"–just as we invoke the properties of right triangles, "without naming Pythagoras." With respect to the "suppressed" preamble and postscript, Gergonne suggested he had done so in Poncelet's best interest, and apologized for unintentionally obscuring the 1824 date of Poncelet's *Académie* submission. However, Gergonne continued, had Poncelet wished to gain greater publicity for his work, Gergonne would have happily printed Poncelet's 1824 memoir in its entirety when it was first written:

> [...] les *Annales* y auraient gagné un article fort piquant, et tous les miens n'eussent ainsi paru qu'après celui-là. (133)[45]

[44]"Who knows? Perhaps I listen at doors.

[45]"[...] the *Annales* would have gained an extremely stimulating article, and all mine [Gergonne's publi-

By choosing a closed venue for his research, Poncelet's obscurity was perhaps his own fault. And while Gergonne acknowledged and admired Poncelet's current engineering contributions, Poncelet could not expect "the whole world to cross their arms and wait" for his eventual return to mathematics.

Ultimately, Gergonne's footnotes depicted Poncelet increasingly frivolous in his insistence on priority and paranoid in his fear that Gergonne would receive all the accolades. "Who dreams to deny it?" Gergonne retorted in response to Poncelet's assertion of his independent results. Gergonne interpreted the *Bulletin*'s review as simply stating that Gergonne had *reported* on duality in 1826. However, Gergonne went on, "Je veux, bien d'ailleurs que le Bulletin se soit trompé, pour peu que cela puisse être agréable à M. Poncelet, car l'essentiel est ici que la vérité se manifeste, et il importe peu qu'elle emploie tel ou tel autre organe" (131).[46] That is, Poncelet and Gergonne were merely vessels for mathematical knowledge and any concern over priority was to ignore the collective scientific gain. Gergonne pointed to common and well-known sources available to both geometers, in particular research by the engineer Gaspard-Gustave Coriolis.[47]

Turning finally to the mathematical content, Gergonne reinforced his earlier criticisms that Poncelet had de-emphasized the significance of duality and rendered it vulnerable through association with Poncelet's *romantic* geometry, that the latter had highlighted in Poncelet (1827b).

> Je persiste à penser, comme je l'écrivais alors, que M. Poncelet a gravement compromis ses doctrines, en mêlant au *classique*, que tout le monde admet, le *romantique* que, pour ma part, je suis fort loin de repousser, mais sur lequel enfin on dispute encore, et que MM. les commissaires de l'Académie eux-mêmes, au jugement de qui M. Poncelet déclare attacher beaucoup de prix, ont traité assez sévèrement. (Gergonne (1827e), 135)[48]

Gergonne's version of duality showed correspondence between theorems that could be independently derived through Euclidean methods. Poncelet's *Traité* and associated memoirs were not only subject to dispute, they were also lengthy, "où les recherches sont assez difficiles à faire, à raison du grand nombre des résultats de détail qu'il embrasse;" (134).[49]

cations on duality] would thus not have appeared until afterward."

[46] "I would even want the *Bulletin* to be mistaken, however little that might be agreeable to M. Poncelet, because the essential thing here is that the truth appears, and it matters little whether it employs this or the other organ"

[47] Alexandre Moatti has addressed the contributions and life of Coriolis, including the relationship between Coriolis and Poncelet as engineers during this time period (Moatti (2011)).

[48] "I persist in thinking, as I wrote then, that M. Poncelet has gravely compromised his teachings, in mixing with the *classic*, which all the world admits, the *romantic* which, for my part, I am very far from rejecting, but about which however one still disputes, and which MM. the commissaries of the Académie themselves, whose judgment M. Poncelet declares to attach great importance, have treated rather severely."

[49] "[...] where research is rather difficult to do because of the great number of detailed results he includes;"

Gergonne professed to be more interested in developing new *methods* than new theorems, and so perhaps some of the theorems that he proved were also contained in Poncelet's *Traité*. He contrasted Poncelet's 400 pages with his simple announcement that theorems of non-metric geometry:

> 1. are double, and 2. could be established without any form of calculation. (139)

Moreover, a change in organization might have encouraged the reception of Poncelet's work.

> M. Poncelet aurait pu débuter par une géométrie dans le genre de celle dont j'ai ébauché les premières pages (tom. XVI, pag, 209) et contre laquelle aucune objection ne se serait élevée. Il aurait pu traiter ensuite de la théorie des transversales et des projections, dont les principes sont également admis, par tout le monde, et réserver, pour la fin de son livre, tout ce qui pouvait être controversé. (ibid)[50]

Then Poncelet would not have seemed to "rush into a revolution" [*brusquer une révolution*] and his work would have gained greater acceptance. Beginning with his principle of continuity and only treating polar reciprocity later suggested that Poncelet had not recognized the importance of duality at the time.

Poncelet had concluded his article by pointing to a potential flaw in Gergonne's applications, as Gergonne had incorrectly claimed that the dual of a curve of given order was another curve of the same order (Gergonne (1827b)). Poncelet had specifically cited an 1817 paper published in the *Annales*, where he had arrived at "very different results" (Poncelet (1817b)). These results also featured in Poncelet's submitted memoir to the *Académie* and accorded entirely with Gergonne's earlier research on algebraic lines and surfaces (Gergonne (1818)). Instead of addressing the error, Gergonne interpreted Poncelet's caution as a sign of a more particular version of duality. Poncelet linked polar reciprocity to duality, while Gergonne claimed duality applied to all non-metric geometry. On the other hand, Poncelet found that polar reciprocity also applied to some metric properties. Gergonne dismissed this as a separate domain of research. "Il est clair, en effet, que les relations *métriques* sont du domaine du *calcul*; mais il n'était nullement question de ces relations dans mes réflexions sur l'analyse du mémoire du M. Poncelet" (Gergonne (1827e), 140).[51] He proposed that finding the dual to a given statement was more a matter of constructing the correct language, but in this early stage of development the correct corresponding vocabulary was less

[50] "M. Poncelet could have begun with a geometry in the genre of that in which I have drafted the first pages (tom. XVI, pag. 209) and against which no objections would be raised. He could have then treated the theory of transversals and projections, whose principles are equally admitted by everyone, and reserved, for the end of his book, all that which might be controversial."

[51] "It is clear, in effect, that the *metric* relations which are from the domain of *calculation*; but there was no question of these relations in my reflections on the analysis of the memoir of M. Poncelet."

important than the assurance that the principle of duality itself “suffered neither exceptions nor any modifications” applying to all geometric objects with “absolute generality.”

This mock dialogue continued in Gergonne’s publication of Poncelet’s “suppressed” preamble and postscript. The communication was again one-sided: Poncelet addressed Gergonne directly and Gergonne spoke to his wider public. In fact, Gergonne repeatedly invoked “tout le monde” in his statements against Poncelet, thus suggesting he held a unanimous opinion. Gergonne was able to use the footnote medium to interject wherever he felt it appropriate, and his commentary formed a fragmented but consistent response to Poncelet’s complaints.

The postscript was entirely devoted to complaints against Plücker (who now was referred to as Plucker rather than Pluker, but we will employ the proper spelling) and his lack of sufficient citation in his 1826 articles. To this Gergonne declared that he did not see “another salvation for geometers except to learn this treatise by heart without omitting a single number.” The Pluker article had opened with a general citation of the *Traité*, but this was not scrupulous enough for Poncelet, who argued for references to specific results. Poncelet pointed in particular to the similarity between his printed figures and the figures suggested by Plücker’s constructions.

Gergonne questioned Poncelet’s assumption that Plücker had read his *Traité*. Instead he noted Plücker’s citation of the *Annales* with respect to Poncelet’s construction from 1817. However, as Gergonne might have recalled, this manuscript citation had been edited out of the published version.

To Poncelet, Plücker’s insufficient citation was further compounded by the use of dual columns.

> Quant à l’usage de mettre en deux colonnes les propositions de la géométrie de la règle (****) , il me semble que c’est un double emploi très-pénible, peu motivé quant aux démonstrations (*****) ; et qu’il suffira toujours d’indiquer, d’après les principes de la théorie des polaires réciproques, la manière de conclure les unes de ces propositions des autres déjà démontrées. (148)[52]

Gergonne, who we know was the author of this very feature, responded that “mes quelques articles à deux colonnes ont plus efficacement servi la cause de la dualité que ne l’ont fait les 400 pages de son ouvrage.”[53] Gergonne thus implied that new geometry might better be introduced in short articles, than in voluminous books.[54]

[52]“As for the use of putting propositions from ruler geometry into two columns, it seems to me that this is a very painful repetition of work (****), little motivated from the point of view of the demonstrations (*****); and that it suffices always to indicate, after the principle of the theory of polar reciprocity, the manner of concluding one of these propositions from the others already demonstrated.”

[53]“[...] my few articles in two columns have more efficiently served the cause of *duality* than have the 400 pages of his work [Poncelet’s *Traité*].”

[54]In Chapter V, we will examine Gergonne’s suggestion by considering the content of contemporary books

Whereas Poncelet's research articles had promised the appearance of new research in future publications, here he promised only more complaints.

> J'aurais, Monsieur, plusieurs autres réclamations à vous adresser pour mon propre compte, mais elle trouveront naturellement leur place ailleurs, et la tâche deviendra alors pour moi moins délicate et moins pénible.[55]

Gergonne categorized this behaviour as bordering on censorship, suggesting its deleterious effects on scientific publications.

> Certes, il paraît que si la censure s'étendait jusqu'aux recueils scientifiques, et qu'elle fût dévolue à M. Poncelet, les ciseaux n'en demeureraient pas oisifs dans ses mains. (140)[56]

Poncelet's personal accusations, which Gergonne had initially withheld from publication, became exaggerated with footnote commentary. Gergonne declared them "a charge of the most egregious and shameful plagiarism" and as such "completely unrealistic." He claimed to have been protecting Plücker as a "completely unknown man" from such accusations, by choosing originally not to publish this postscript. In these concluding remarks, Gergonne had painted Poncelet as not only ill-mannered, but ultimately valuing his own success over scientific progress.

The mathematical polemic continued indirectly within the context of research publications. In the polemical exchange, Gergonne had dismissed Poncelet's specific corrections blithely: "Si M. Poncelet n'avait pas autant dédaigné l'étude de la dualité de situation, il pourrait prendre ici un ton plus décisif" (142).[57] But Gergonne was more serious than he affected. Shortly afterward, he published "Géométrie de situation. Rectification de quelques théorèmes énoncés dans les *Annales*," in which he created the proper vocabulary with which to express the correlation between dual curves (Gergonne (1827c)). In dual columns, Gergonne defined the term *degree* on the left-hand side,

> A planar curve is of mth degree, when it has m real or ideal intersections with a line.

On the corresponding right-hand side he defined the curve's *class*,

on geometry.

[55]"I have several other claims to address to you on my own account, but these will naturally find a place elsewhere, and the task will become less delicate and painful for me."

[56]"Certainly, it appears that if censorship extended as far as scientific publications, and it was vested in M. Poncelet, the scissors would not remain idle in his hands."

[57]"If M. Poncelet did not disdain studying the duality of position, he could take a more decisive tone here."

> A planar curve is of mth class, when one can draw m real or ideal tangents to it from a point on its plane.[58]

The definitions similarly held for curved surfaces. Now, Gergonne could simply substitute *degree* in the left hand column and *class* in the right hand column for the term *order* to correct his earlier theorems, a process that Gergonne described as "neither very great, nor very difficult." Most importantly, Gergonne preserved the generality of his form of duality.[59] However, not all geometers found Gergonne's rectifications satisfactory since they eluded the uncomfortable result that the order of a curve was not always preserved by its dual. Plücker would address this issue in his *Analytisch-geometrische Entwicklungen* as well as later articles published in Crelle's *Journal* (Plücker (1828a), Plücker (1829b), Plücker (1834)).

The *Bulletin* printed a synopsis of the *mathematical polemic* that described Poncelet's complaints as "baseless" and expressed disbelief that Poncelet would wish to block all research analogous to his (Anonymous (1828b)). Gergonne was described as having been a "zealous propagator" of Poncelet's doctrine for the past ten years. Poncelet's criticisms were dismissed as "vague" and Gergonne's inexactitudes excused based on the "novelty of the research" and "the imperfection of language." Overall, the review was more a summary of Gergonne's footnotes than of Poncelet's text and so it appeared that Gergonne held the upper hand.

Poncelet would not let this be the last word, and responded again in the *Bulletin* (Poncelet (1828c)). He began by limiting the scope of the controversy to a priority dispute and establishing a timeline of his research from 1818 onward with due credit to the earlier contributions made by Delahire, Monge, and Brianchon. By contrast, Poncelet questioned why the "indefatigable activity of Gergonne's pen" [*l'activité infatigable de la plume de M. Gergonne*] had not published until 1826, if he had "completely fixed" [*complétement fixées*] ideas on duality before then. Poncelet suggested that Gergonne was disguising Poncelet's own polar reciprocity behind the more "seductive" [*séduisant*] name of duality. This general principle of duality was "empty of sense in mathematical philosophy and absurd" [*vides de sens en philosophie mathématique, et absurdes*] when divorced from the justification of polar reciprocity. According to Poncelet, the process of polar reciprocity was necessary, useful, and had been covered at length in his *Traité*.

> [...] ni qu'on pût songer sérieusement à en faire une doctrine toute nouvelle, indépendante de la théorie des pôles et polaires, comme l'a tenté, sans succès,

[58] Gray has analyzed Poncelet's correction and Gergonne's rectification as well as Plücker's ultimate resolution in Gray (2010b).

[59] *Ideal* was an adjective first used by Poncelet to describe geometric objects containing real and imaginary points (Poncelet (1822)). Notably, Gergonne used Poncelet's term *ideal* tangents in his definitions without remark–apparently they had by now been more or less accepted in the geometer's lexicon.

M. Gergonne [...] (Poncelet (1828c), 298)[60]

As far as Gergonne's efforts to correct his mathematical errors, Poncelet disapproved of the newly minted definitions of *degree* and *class*, which "tortured" the sense of the words solely in order to support Gergonne's theorem.[61] Gergonne's defence of Plücker likewise failed to convince Poncelet. Poncelet contended that his *Traité* had received insufficient citation precisely in order to preserve the "scaffolding" of double columns, as a matter of "proselytizing" for Gergonne's duality.

Poncelet's argument was little changed from his earlier *Bulletin* notes. The article served more as clarification and condensation: Poncelet was the first to publicize polar reciprocity, later forms of duality were derivative or senseless, and, if his work was left unacknowledged, plagiarized. The most significant departure from Poncelet's two articles was in his description of Gergonne. Whereas Gergonne had originally appeared as a knowledgable editor, who supported and encouraged Poncelet's geometrical methods, he was now portrayed as suspiciously seeking to undermine Poncelet's credit and credibility. Consequently, "Plucker" was merely a consenting pawn. The priority dispute had become very personal.

Later that year Poncelet published his original memoir on polar reciprocity in Crelle's *Journal* (Poncelet (1828a)). The *Académie* had only just approved it in February, nearly four years after Poncelet's submission. Aware of the current tensions, Crelle introduced the article with a very diplomatic footnote referring to the "discussion" between Gergonne and Poncelet and refraining from further comment. The article included a newly written postscript by Poncelet. Poncelet observed that the material treated in his memoir had recently been the research subject of several geometers, and wished to clarify that the version printed here conformed completely to the 1824 manuscript. He explained that he had made progress with his research since that date and included a summary of the several practical applications of his polar reciprocity theory. The civil tone of this research article contrasted sharply with Poncelet's contemporary polemical writings. Poncelet may have felt that Crelle's *Journal*, published outside of France, was not the place to carry on such charged discourse.

By 1828, Gergonne could claim that the principle of duality was now established (Gergonne (1828b)). When he had first begun its study in 1821 "the ideas of duality were not yet as widely known,"

> Aujourd'hui, au contraire, qu'il doit être bien connu que tous les théorèmes de situation marchent par couples, il nous suffira d'avoir démontré l'un d'eux, à

[60] "[...] we cannot seriously dream to make a totally new doctrine of duality, independent of the theory of poles and polars, as was attempted, without success, by M. Gergonne [...]"

[61] Gergonne responded to Poncelet (1828c) in prefacing a series of corrections to a recent article by Chasles (Gergonne (1828c)). He quipped, "M. Poncelet has so frightened us that we do not have the courage to reread this memoir (Gergonne (1827c)), in fear of finding too much to correct."

l'endroit cité, pour que l'autre soit admis sans contestation. (115–116)[62]

In this technical paper on three dimensional geometry, Gergonne took the opportunity to rewrite a theorem from Coriolis in a new style expressing duality. Here, instead of dual columns, Gergonne employed curly brackets, where each dual pair was presented one on top of the other in the text, as shown in Figure 3.2.

THÉORÈME. **Soient, dans l'espace,** $n\left\{\begin{matrix}\text{points}\\ \text{plans}\end{matrix}\right\}$ **quelconques numérotés arbitrairement ainsi qu'il suit**

(1), (2), (3), ... (*n*) . **(1.re *Série*).**

Chacun de ces $\left\{\begin{matrix}\text{points}\\ \text{plans}\end{matrix}\right\}$**, avec celui qui portera le numéro immédiatement supérieur, déterminera une droite; de telle sorte qu'on aura ainsi *n*—1 droites, que l'on pourra désigner respectivement par l'ensemble des indices des deux** $\left\{\begin{matrix}\text{points}\\ \text{plans}\end{matrix}\right\}$ **qui déterminent chacune d'elles, en cette manière**

Figure 3.2: Gergonne's alternative duality format (Gergonne (1828b))

This new format saved space and presented an even more striking picture of the similarity between dual forms. In his introduction and footnotes, Gergonne described Poncelet as a severe critic who had made some reasoned observations on particular theorems, and would never be completely satisfied. So as far as Gergonne was concerned, the general principle of duality could progress without any deep corrections.

However, other geometers could not ignore Poncelet's criticisms so flippantly. Plücker's defence against Poncelet's accusations appeared in the *Bulletin* that year (Plücker (1828c)). He had learned of the "unjust attacks in a letter from Berlin three or four months prior." However, he had not responded because he did not know the exact details. With Poncelet's "tireless" [*infatigable*] reiteration of the same accusations, Plücker had found a new opportunity to defend himself before the *Bulletin* and its readers.

In short, Plücker was not responsible for the material Poncelet had found objectionable. Plücker had willingly given Gergonne "the liberty to change the writing of his memoir" [*la liberté de changer la rédaction de mon mémoire*]. Though Plücker "was absolutely ignorant" [*j'ignorais absolument*] of duality before reading the *Bulletin* in 1827, Gergonne had found the contents favourable to a presentation in double columns. Likewise, Gergonne had chosen to cite Poncelet's *Traité*, which Plücker had only heard of through the Bachelier

[62]"Today, to the contrary, as it must be very well known that all the theorems of position march two by two, it suffices for us to have demonstrated one of them [...] for the other to be admitted without contestation."

catalogue.[63] Poncelet's 1817 article had been his sole inspiration. For the same reason, Plücker explained he had been unable to give thorough citations to Poncelet in the first volume of his text, *Analytisch-geometrische Entwicklung*.[64] Though Plücker seemed sympathetic to the misunderstanding and admired Poncelet's research in general, he dismissed the significance of Poncelet's specific contested results, claiming they were "*all corollaries of a known theorem*," [*tous corollaires d'un théorème connu*] and consequently of little importance (Plücker (1828c), 331).

Plücker recounted that he had only learned of duality through his public involvement in the priority polemic. Since then, he had developed his own theory of duality, which he described as "very different" from that of Gergonne or Poncelet. Plücker advertised that his purely analytic method had "uncovered the secret of duality," and concluded by hoping that Gergonne would publish it, thus demonstrating his continued faith in the editor.[65] Though Plücker would never explicitly criticize either of his predecessors, his decision to reform duality suggests a dissatisfaction with earlier versions.

Plücker followed Gergonne in assuming duality as an open area of research, but Poncelet did not welcome this additional interpretation of duality and quickly responded in the *Bulletin* (Poncelet (1829)). He reasoned that if Plücker had not been responsible for his article, then the polemic was entirely between Poncelet and Gergonne, the author of the double column memoir. As for Plücker, Poncelet "did not have the advantage of knowing him." Though Plücker seems to have accepted Gergonne's editorial choices, Poncelet found in this further means of attack. He suggested that Gergonne had originally withheld publishing Poncelet's complaints, in order to conceal his own interference. Until reading Plücker's explanation, Poncelet would never have imagined that a man of Gergonne's character would "mutilate" the work of a "foreign scholar." Poncelet considered the Pluker article as part of Gergonne's efforts to convince readers that earlier contributions to duality (including Poncelet's *Traité*) were insignificant.

On a less personal level, Poncelet contrasted his practical and original research with

[63] In the midst of Poncelet's counterattacks of 1828, he received a letter from Chasles. Apparently, Poncelet had written to Chasles about his lack of proper citation, to which Chasles begged off with an identical excuse to that of Plücker–he hadn't had access to the original material while in Nice. Chasles claimed to have not seen the *Annales* except those before 1813 and a few after 1822, which he confessed to not have fully read, even though he had contributed to the volume in 1827. This civil private correspondence, under very similar circumstances to the Plücker exchange, illustrates an alternative medium to resolving questions of proper citation.

[64] In the introduction to his *Analytisch-geometrische Entwicklung*, Plücker recognized that despite their bases in "entirely different ideas" [*ganz wesentlich verschiedenen Ideen*] his general analytic method and the method of Poncelet's *Traité* yielded strikingly matching results. Plücker admitted that without an understanding of the underlying groundwork, one might judge "the first method as paraphrasing or a plagiarizing of the second" [*die erste Methode als eine Periphrase, als ein Plagiat der zweiten* (Plücker (1828a)).

[65] Following Plücker's statements, the *Bulletin* editor noted that the publication of Plücker's response had been delayed through editorial forgetfulness, and "some epithets unnecessary to the success of the discussion" [*quelques épithètes inutiles au succès de la discussion*] had been removed. Thus also in the *Bulletin*, Plücker's work remained vulnerable to editorial modifications.

that of Gergonne and Plücker. He explained that his theories had "applications utiles aux arts graphiques, et qui ne doivent pas, en elles-mêmes, être dénuées de toute espèce de mérite, puisqu'elles sont devenues entre les mains de MM. Gergonne et Plucker, le sujet de développements fort étendus."[66] In this attention, Poncelet appealed to a specific and possibly new audience of applied mathematicians, thus turning his work as an engineer into a complementary asset, rather than a competing distraction, to his mathematics.

Gergonne and Plücker had emphasized the importance of the geometric form, exemplified by the use of dual columns, over content. Poncelet contended the opposite, and denigrated the importance of form as an afterthought.

> La facilité et la simplicité même avec lesquelles ces questions dérivent [...] n'en font qu'accoître [sic] le mérite à nos yeux, comme ils l'accroîtront sans doute, aux yeux de tous les vrais amateurs des sciences, qui tiennent encore plus au fond qu'à la forme, et qui n'apprécient pas uniquement les résultats des recherches mathématiques d'après la facilité plus ou moins grande de les démontrer *à posteriori*. (332–333)[67]

Thus, Poncelet's work was more useful and important, and his scientific understanding broader than that of his opponents. Poncelet's response marked the end of direct exchanges on the controversy, but research and publications continued.

3.6 Aftermath: resolution and irresolution

Plücker drew upon the principle of duality in two subsequent articles, and presented theorems in dual column form, but at first without citing Gergonne or Poncelet (Plücker (1828b), Plücker (1829a)). His "secret" of duality appeared in an 1830 article in Crelle's *Journal* (Plücker (1830)). "Über ein neues Coordinatensystem" principally concerned the introduction of homogeneous coordinates, which afforded a new way of addressing "the theory of reciprocity."[68] Plücker advertised his "purely analytic theory" as superior to all previous ones because it was independent of a conic section. In expressing duality in coordinate equations, Plücker maintained the mechanism of finding a dual that had been provided by Poncelet's form of polar reciprocity. At the same time, he achieved a method that could

[66] "[...] useful applications to graphic arts, and must not be stripped of all types of merit, because they have become, in the hands of MM. Gergonne and Plucker, the subject of very extended developments."

[67] "The ease and simplicity with which these questions are derived, [...] only increases their merit in our eyes, as they increase it, undoubtably, in the eyes of all true lovers of science, who value content still more than form, and who do not uniquely appreciate the results of mathematical research according to the ease, more or less great, of proving them *à posteriori*."

[68] For the history of coordinate systems in geometry, including the three variable homogeneous coordinates of Plücker, see Boyer (1956), Coolidge (1940), or Gray (2010b). Sections of of Plücker's article as they pertain to Poncelet's research are summarized in Appendix E.

be applied to curves and surfaces of any degree, thus satisfying the advantage of generality inherent in Gergonne's form of duality. This division of attributes was further reflected in Plücker's citation, where both geometers were equally credited. Plücker continued to pursue questions raised by duality through the 1830s, and through this work he has been credited with "resolving" the theory of duality.[69] One important aspect lay in Plücker's solution to the problem of degree change between a curve and its dual. Plücker specifically addressed this problem in an article published in French in Crelle's *Journal* in 1834, where he claimed that the problem remained unresolved.

> La découverte du *principe de réciprocité* (théorie des polaires réciproques) ou ce qui est identiquement la même chose, celui de *dualité* a fait naître une foule de questions nouvelles, dont l'une que je regarde avec M. Poncelet comme fondamentale n'a pas été résolue jusqu'à ce jour malgré les efforts des plus habiles géomètres. (Plücker (1834), 105)[70]

Plücker continued by explaining Gergonne's initial mistake that the degree of a curve was equal to that of its polar curve, as well as his correction by introducing the word "class". Plücker accepted the value of this new classification, "la méprise d'un homme d'esprit porte ses fruits," by which Poncelet seemed to mean the results from Gergonne's error. However, Plücker remained interested in the explanation behind the fact that the number of points of intersection of an algebraic curve of any degree was *not* always equal to the number of tangents to the same curve passing through a given point. Plücker set aside the analytic interpretation of this problem for his books, the recent *Analytisch-geometrische Entwicklungen* (1828, 1831) and the forthcoming *System der analytischen Geometrie* (1835) (Plücker (1828a), Plücker (1831), Plücker (1835)). Here he presented an informal explanation that one could either consider curves as described by a moving point, in which case they would generally not have cusps but have inflection points; or one could consider curves as enveloped by a moving straight line, in with case they would generally not have inflection points but instead have cusps. Following the principle of duality, one representation of a curve would be the dual of the other. From these two means of generation, which Plücker promised he could prove within a system of analytic geometry, Plücker counted exactly how the degree of a curve would change depending on its number of inflection points and cusps. Plücker thus reunited the duality controversy with the initial methodological controversy from 1817, demonstrating another practical application of analytic geometry.

[69] Plücker's contributions to duality have been examined in detail and praised in texts from Clebsch to Gray (Clebsch (1872), Gray (2010b)).

[70] "The discovery of the *principle of reciprocity* (theory of reciprocal polars), or what is exactly the same thing, that of *duality* has given birth to a series of new questions, of which one that I regard with M. Poncelet as fundamental has not been resolved up to this day despite the efforts of the most able geometers."

The *Annales* ceased publication in 1832. In 1847, at the age of 78, Gergonne gave a very brief paper at the *Académie des Sciences de Montpellier* on the principle of duality in geometry (Gergonne (1847)). Spurred by a contemporary publication on poles and polars, Gergonne recalled that he had deduced a new principle of geometry, called the principle of duality, more than 30 years ago. He took this opportunity to repeat his two philosophical propositions: that duality applied to all non-metric geometry and that the duals of individual results followed directly from substitution. Gergonne asserted that by 1813 the principle of duality was "no longer a mystery" to him. As for dual columns, Gergonne credited their 1824 usage with the proliferation and publicity of the principle.

> A dater de 1824 pour faire mieux saisir cette dualité, j'ai écrit, dans deux colonnes en regard les uns des autres les doubles théorèmes que me fournissait mon principe (1). Malgré cette précaution ce principe a passé presque inaperçu. (Gergonne (1847), 63)[71]

This was Gergonne's only published admission that dual columns may not have been an effective means toward popularizing duality. After his publications, Gergonne noted that research applying the principle of duality in geometry had been pursued by Steiner, Pluker (whose name had reverted to the 1826 misspelling), and Chasles. Poncelet was not in this group, but rather appeared as the "sole antagonist" to Gergonne's duality principle.

According to Gergonne's reconstruction, Poncelet had read no work but his own and consequently attacked dual columns as "awkward," "superfluous," and even sometimes "faulty." Poncelet, Gergonne continued, was an engineer with an admirable reputation. However, Gergonne saw more to science than practical applications.

> Dans le siècle et dans le pays où nous vivons tout ce qui peut contribuer à développer à étendre et à fortifier les facultés de l'intelligence ne saurait être regardé avec indifférence et il est bien connu, d'ailleurs, qu'une nation qui ne cultiverait les sciences que sous l'unique point de vue de leurs applications pratiques et immédiates de leurs résultats matériels, ne saurait se flatter de les voir long-temps fleurir au milieu d'elle. (64)[72]

In Gergonne's account, as an active engineer, Poncelet appeared to have compromised his commitment to mathematical research by emphasizing applications. Gergonne's foreboding conclusion may have also been a comment on his perception of deeper national trends.

[71] "In 1824 to make this duality easier to grasp, I wrote, in two columns facing each other, the double theorems that my principle provided (1). Despite this precaution this principle passed almost unperceived."

[72] "In the century and in the country where we live, everything that can contribute to develop, to extend and to fortify the faculties of intelligence would be impossible to look upon with indifference, and it is well known, moreover, that a nation which cultivated the sciences only from the unique point of view of their practical applications and immediate material results, could not vaunt itself to see them flourish in her midst for long."

Nearly twenty years later, Poncelet republished his *Traité* with supplements including many of his older articles as well as "historical" commentary. Poncelet used this venue to aggressively refuse Plücker's proposed resolution to the duality controversy.

> [...] en 1828, à une communication tendant à prouver que lui aussi (Plucker) était arrivé, par une voie purement analytique, au *principe de réciprocité*, objet de la dispute, il prétend l'avoir présenté sous un point de vue *beaucoup plus général* (préface, p. 6, texte, p. 259). Mais je crains fort qu'il y ait là quelque méprise ou illusion; car, malgré l'honneur qu'il me fait de citer ici mes travaux antérieurs à ceux de MM. Steiner, Gergonne, et Bobillier, imprimés dans les *Annales* de 1827 à 1829, je n'aperçois même dans le *Mémoire* de M. Plucker *sur un nouveau système de coordonnées* et un *nouveau principe de Géométrie* (Crelle, t. V, 1830, p. 1 et 268), que des essais algébriques plus ou moins fructueux, et qui rappellent ceux de ces savants, pour établir une tardive et incomplète prééminence des méthodes soi-disant purement analytiques sur les miennes. (Poncelet (1866), 409)[73]

Poncelet interpreted Plücker's resolution as continuing the 1817 methodological debate, but not contributing to generalizing the principle of duality. He proceeded to list geometers who had given improper attribution to any of his results and theories, among whom Gergonne certainly counted. He died in 1867 shortly after this final attempt to leave a legacy of victory in this academic struggle, alongside his professional success and international acclaim in engineering.

3.7 Conclusion

In contrast to the philosophical exchange between Poncelet and Gergonne over methods in geometry, the two geometers could not agree on the origins, scope, or applications of the principle of duality. Based on their articles from 1817, both geometers theoretically agreed that the choice of problem should determine the choice of method. While we observed in Chapter II (and will see further in Chapter IV), the dependence of a method on the problem at hand was often reversed in practice, both geometers appeared satisfied with the proposed resolution. The different arguments, participants, and conclusions to these three sets of exchanges indicate that these constituted multiple, but interdependent controversies.

[73] "[...] in 1828, he intended to prove that he also (Plucker) had arrived, by a purely analytic view, at the *principle of reciprocity*, the object of the dispute, he claimed to have presented a *much more general* point of view [...]. But I greatly fear that there was some error or illusion in that; because, despite the honour that he does me to cite my works as preceding those of MM. Steiner, Gergonne, and Bobilier, printed in the *Annales* from 1827 to 1829, I only perceive, even in the *Mémoire* of M. Plucker *on a new system of coordinates* and a *new principle of Geometry* [...], more or less fruitful algebraic efforts that recall those of scholars, to establish a late and incomplete preeminence of so-called purely analytic methods over mine."

Arguments from all sides of the duality controversy drew from the earlier publication history. In light of the use of algebraic analysis in geometry and the principle of continuity, we can better comprehend Poncelet's position with respect to introducing new mathematics in a restrained synthetic geometry setting, Gergonne's drive toward mathematical progress with or without complete proof, and Plücker's decision to explore the principle of duality through coordinate equations. Moreover, by direct comparison, we find that repetition and presumed lack of recognition proved a far greater impetus toward professional and public opposition than methodological differences. The oppositional nature of early nineteenth century geometry posited in Chapter I, resulted in part from these concerns of priority and credit.

From its beginning, Poncelet and Gergonne disagreed about the nature of duality and the role of the surrounding controversy. Poncelet classified the duality controversy as a priority dispute and pitted his chronology against Gergonne's (and Plücker's) list of publications. Gergonne labelled the exchange as a mathematical polemic and emphasized the independence of duality from Poncelet's polar reciprocity. Each employed publicity to argue for their respective interpretations and the necessary corollaries.

While Poncelet reached out to five different publications (the *Académie*, his *Traité*, the *Annales*, the *Bulletin*, and the *Journal*) and hence multiple, though overlapping, publics, his publicity strategy remained consistent: duality was no more than the polar reciprocity Poncelet had developed and first published in his *Traité*. He encouraged developing practical applications of the theory of polar reciprocity with due credit, but strongly cautioned against the unproven generality of Gergonne's duality. For Poncelet, the principle of polar reciprocity was established and foundational discussions were closed (Poncelet (1829)).

Gergonne had only one direct publication on the matter, but enjoyed full editorial autonomy.[74] He presented duality as an exciting, new, open geometrical endeavour. Gergonne desired maximal generality, and for this he urged the participation of his readers. To visually communicate a new style of mathematics, he developed dual columns and other formatting devices. He suggested how Poncelet could change the vocabulary and presentation of his polar reciprocity to gain more popular support. In the controversy dispute, Gergonne reserved his harshest criticisms to expose and denounce Poncelet's exclusivity.

The dichotomy between these closed and open representations of duality, is further reflected in Gergonne's decision to bring Plücker into the controversy, while Poncelet simultaneously attempted to dismiss him.[75] In this aspect, Gergonne's publicity strategy

[74]With reference to Gergonne's contributions of anonymous articles to his own journal, Stephen Stigler suggests that Gergonne "contrived controversy" in his journal to appeal to his readers (Stigler (1976), 73).

[75]In a more particular scope, Poncelet's publicity strategies for polar reciprocity ally strikingly with Habermas' assessment of publicity work "aimed at strengthening the prestige of one's own position without making the matter in which a compromise is to be achieved itself a topic of *public discussion*" (Habermas (1991), 200).

ultimately proved more effective. While Poncelet may have obtained a more notable position in the academic hierarchy as an esteemed engineer and regular contributor to the *Académie*, he had no authority over what Gergonne could print in his *Annales*.

At the individual level the controversy remained at a standstill from the mid-1820s until Poncelet's death forty years later. Poncelet and Gergonne often referred to specific (albeit inconsistent and receding) dates marking the origins of their unpublished ideas and continuing through past publications showing evidence of duality. Poncelet and Gergonne continued to trace their respective ideas back, ceding less and less potential influence from the other. Their last bitter attempts to have the final word suggest that the priority dispute would never be resolved satisfactorily. Gergonne continued to believe that duality had not caught on and Poncelet continued to believe that his theory had been obscured. Poncelet and Gergonne remained irresolute. Further, their ultimately personal aggression jeopardized the controversy's potential for resolution. For members of the mathematical community to endorse either position could have dangerous ramifications, as the case of Plücker showed. Gergonne's reaction to Poncelet's accusations against Plücker suggest the severity of plagiarism charges, especially for one without an established career.

Yet, the duality controversy achieved resolution. Partially obscured by the personal vitriol, Poncelet and Gergonne's mathematical criticisms signalled open questions. Each geometer wished to show the superiority of his own form of duality, and their critical exchange promoted duality as a subject of research and reform. In attempting to control their publics, neither Poncelet nor Gergonne was omnipotent. Their attention to publicity (even at the expense of research) further proves the inherent collective dimension of controversies. Poncelet and Gergonne both strove to have the last published word, each addressing duality within their final publications. With this attention to publicity the duality controversy became less a dispute between Poncelet and Gergonne, as an attempt by each to craft a favourable and properly informed public. Moreover dialog on the duality controversy brought to light how Gergonne and Poncelet believed new mathematics should be introduced. Poncelet advocated the use of synthetic geometry and Gergonne similarly chose Euclidean geometry in order to show how a new general principle could be used without risking unproven or controversial material. In line with this, they used familiar problem sets (such as described in Chapter II), to show the efficacy of the new principle. Both geometers also discussed the importance of language in new geometry. For Gergonne, duality would best succeed with a new set of vocabulary, but he admitted that the latter feature would not likely be adopted. Poncelet criticized Gergonne's definitions of degree and class, for redefining old words with new meanings solely in order to show the veracity of a result. These aspects point to the presumed difficulty of introducing new geometry and the necessary efforts involved in securing a public.

In this dimension, their efforts could even be complementary. Although neither Poncelet nor Gergonne expressed satisfaction, the historical literature suggests that both respectively succeeded: Poncelet's *Traité* and the general principle of duality gained appreciative and dedicated audiences. To take one early historical example, Chasles praised the advances of Gergonne and Poncelet in his *Rapport sur les progrès de géométrie* written shortly after Poncelet's death in 1870.

> C'est Gergonne qui a émis l'idée du *principe du dualité* en Géométrie, idée inspirée, on peut le croire, par les beaux résultats de la théorie des *polaires réciproques* de Poncelet, seule méthode que l'on connût alors pour les transformations de cette nature. (Chasles (1870), 59–60)[76]

This distribution of credit between both French geometers can be found in historical summaries up to the present day.[77]

While Gergonne and Poncelet were still living, geometers tried to stay clear of picking sides in the priority dispute, yet were eager to employ the techniques of word and symbol substitution to gain new theorems. The establishment of Crelle's *Journal*, and the consequent widening of the controversy's public to an international arena, further encouraged the resolution of the controversy as much as the reviews in the *Bulletin* had aggravated its beginning. Beginning in 1826, geometers including Charles Sturm, Étienne Bobillier, Michel Chasles, Jakob Steiner, and C. G. J. Jacobi employed duality or reciprocity in their research publications (for instance, see Sturm (1826a), Bobillier (1827), Chasles (1828a), Steiner (1828b), Jacobi (1828)). Attempting to remain uncontroversial, they often cited both Poncelet and Gergonne. Moreover, the duality controversy revealed that *Annales*, the *Bulletin*, and the *Journal* were all available venues (at least until 1832) for publishing this line of research. Almost as soon as it began, the duality controversy transcended its initial antagonists.

The controversy encompassed academic practices condemned by present-day standards as deceitful, vindictive, and unprofessional. However, like Gergonne's dual columns or Poncelet's presentations to the *Académie*, the duality controversy served as a tool in the distribution of knowledge. Though none of these factors should be considered indispensable, each introduced a new, potentially overlapping, public to duality. The controversy revealed a unique time and space where the geometrical community was secure enough to allow such unrestrained public discourse and keen enough to find an invitation within a polemic.

In the duality controversy, we saw how French geometry began to spread to German audiences through the publications and involvements of Julius Plücker. In this particular

[76]"It was Gergonne who expressed the *principle of duality* in Geometry, an idea inspired, we believe, by the beautiful results from the theory of *polar reciprocity* of Poncelet, the only method that was then known for transformations of this nature."

[77]Chasles' accolades in this text are especially noteworthy in contrast to his less generous portrayal of both Gergonne and Poncelet in his *Aperçu historique sur l'origine et le développement des méthodes en géométrie* of 1837, which Poncelet sharply criticized in 1865 (Chasles (1837), 377; Poncelet (1866), 417).

circumstance, we found that Plücker's national and linguistic differences contributed to his unusual reception among French geometers. Similarly, we saw in Chapter I how histories have presented a common narrative of migration from France to Germany, specifically from Poncelet to Plücker and Steiner. The duality controversy suggests one path in this progression, although Plücker's commitment to methods of analytic geometry appears to have preceded and even motivated his involvement with French mathematicians. In Chapter IV, we will follow the chronological development proposed in the historiography, by further examining how Poncelet's geometry was further taken up by the so-called "German geometers": Plücker and Steiner. Drawing on questions raised in Chapters I and II, we will consider how their allegedly analytic and synthetic geometries shaped the terms of the methodological debate, and in particular their attention to the visual and application of the figure.

Chapter 4

Ce principe paraît ne pas être inconnu aux géomètres allemands: Jakob Steiner, Julius Plücker and French geometry

4.1 Introduction

As we saw in the previous two chapters, Poncelet and Gergonne debated the appropriate choice of method in geometry over a series of articles published in the *Annales* in 1817 (Gergonne (1817b), Poncelet (1817c), Gergonne (1817e)). Both geometers resolved that the question should be decided by the problem or theorem at hand, and not a priori. The simplest and most elegant geometric solution depended on the nature of the given objects and desired construction or property. While seeming to resolve the debate, the agreement simply shifted the issues to a more particular domain: for how was one to decide which method was best in any given context? What were the criteria for elegance and simplicity? The compromise strategy only succeeded when a single solution or proof could be found. In instances where Poncelet and Gergonne solved the same problem or proved the same theorem using different methods, neither conceded any advantages to the other (for example, see Brianchon and Poncelet (1820), Gergonne (1821), Poncelet (1821a)). Such repetition, with or without precise acknowledgement, was common among early nineteenth century geometers as evidenced, for instance, by the numerous printed solutions to challenge problems posed in journals.[1]

[1] Historical evidence of this repetition is well-documented in Simon (1906) and Coolidge (1940) Chapter V.2. Norbert Verdier has compared European mathematical journals during the first half of the nineteenth century, including Gergonne's *Annales* and Crelle's *Journal* (Verdier (2009b)). Deborah Kent has investi-

Ten years later, similar themes emerged in reviews of two young geometers: Jakob Steiner and Julius Plücker, who independently arrived at the same results using distinctly different methods and often within a year of each other. Steiner was a young Swiss geometer, who had studied mathematics under the famous pedagog Johann Heinrich Pestalozzi.[2] In 1821 he moved to Berlin, where he worked as a private tutor for Wilhelm von Humboldt and struggled to maintain a position as a mathematics instructor, first at the Friedrich-Werderschen Gymnasium (1821–1822) and then at the Gewerbeschule where he was eventually promoted to a senior position (1825–1835).[3] Meanwhile, Plücker continued to worked at the University of Bonn beginning in 1825, and was appointed as a professor there in 1828 where he would remain until 1833.[4]

A review in the 1828 *Bulletin de Férussac* by Auguste Cournot (1801–1877) of Steiner's 1827 article "Verwandlung und Theilung sphärischer Figuren durch Construction"[5] declared a reemergence of pure geometry, as represented best by Gergonne, Steiner and Poncelet. According to Cournot, these geometers had succeeded through "pure contemplation" of geometric properties in creating a form of "rational analysis" [*l'analyse rationnelle*] more general than algebraic analysis when applied to geometry and extending beyond the "restrained" methods of Greek geometry (Cournot (1827), 299). Cournot particularly admired Steiner for solving long-standing geometric problems and stating (often without proof) a wide variety of geometric theorems. Steiner's frequent choice of coordinate-free, figure based geometric constructions, situated him as an advocate of what was considered pure geometry.[6]

On the other hand, *Bulletin* reviewers occasionally found Steiner's enthusiasm for announcing new problems and theorems excessive. Many of Steiner's articles, with titles like "Einige geometrische Sätze," "Einige geometrische Betrachtungen," or "Démonstrations de quelques théorèmes," were essentially lists of results only sometimes accompanied by proofs

gated the role of posed problems in attempting to create an audience for early nineteenth century American journals (Kent (2008)). Finally, the continued interest in these same areas of geometry research can be seen in contributions by British mathematicians to international scientific journals, primarily in geometry and algebra, later in the nineteenth century documented by Sloan Despeaux in (Despeaux (2008)).

[2]Maarten Bullynck examines Pestalozzi's role in reforming mathematics education during the turn of the nineteenth century in Bullynck (2008).

[3]The many details of Steiner's early life and career have been a popular focus in late nineteenth century biographies including Geiser (1874), Bützberger (1896), Graf (1897), and Lange (1899).

[4]In Chapter I, we noted the role of methodological opposition in several of Plücker's biographies, which also contain expositions of Plücker's life (Bertrand (1867) Clebsch (1872), Dronke (1871), Ernst (1933)).

[5]Translated in the French review as "Transformation et division des figures sphériques, au moyen de constructions graphiques." In the early 1820's, Cournot studied mathematics under Lacroix alongside Lejeune Dirichlet (1805–1859), soon to become a professor at the University of Berlin and whose work would appear in the third volume of Crelle's *Journal.* Although his later work emphasized probability theory and philosophy of science, Cournot achieved his doctorate with a more geometrically oriented thesis on "Le mouvement d'un corps rigide soutenu par un plan fixe" in 1829. A brief biography appears in Granger (2008).

[6]Cournot proposed that certain minds pursued geometry, the theory of extension, for purely ideal enjoyment [*jouissances purement idéales*], rather than applications to common needs [*boisons vulgaires*] or natural philosophy [*philosophie naturelle*].

(Steiner (1826b), Steiner (1826a), Steiner (1828b)). One such article, "Développement d'une série de théorèmes relatifs aux sections coniques," was described in the *Bulletin* by an anonymous reviewer (who may have been Cournot again) as containing too many theorems even to list within the bounds of the review. The anonymous reviewer continued that while Steiner's fruitfulness might be impressive, his complicated presentation required extraordinary patience and fortitude even for a well-disciplined reader.

> Ce mémoire décèle, chez M. Steiner, une grand force de tête, et beaucoup d'habitude des ressources de la géométrie; mais la manière de procéder de l'auteur, et les figures et constructions, souvent assez compliquées, auxquelles il a recours, en rendent la marche lente et la lecture pénible. (Anonymous (1828a), 245)[7]

In conclusion, the reviewer recommended the use of analytic geometry, or some other of "the broad methods [les larges méthodes] from the school of Monge," to render Steiner's results with less difficulty and greater brevity. The reviewer suggested that the *Annales* offered several beautiful examples of the applications of any of these latter methods.

Perhaps by no coincidence, the next issue of the *Annales* contained an article of analytic geometry by Julius Plücker, in which one could find alternative versions of several of Steiner's recent theorems (Plücker (1828b)). In particular, though we will find that they gave quite different statements of the theorem, both Steiner and Plücker credited one result to Gabriel Lamé (1795–1870), and this common source enabled the translatability from one version to the other.[8] Plücker's article ran one-third the length of Steiner's (with far fewer theorems), and its review in the *Bulletin* was correspondingly more succinct (the two reviews fell within two pages of each other). With respect to method, Plücker was credited with demonstrating, without any sort of calculation, a multitude of properties of second order lines (Anonymous (1828a)).

This was not the first common content between Steiner and Plücker. The year before Plücker had proved results in the *Annales* announced by Steiner in Crelle's *Journal*. The *Bulletin* review provided a more in-depth assessment of Plücker's analytic geometry style,

> M. Plucker est fort sobre de calculs, et tous les siens peuvent, en quelque sorte, être suivis de l'oeil ; mais il les choisit avec beaucoup d'art et de goût [;] aussi arrive-t-il très-brièvement, non seulement à la construction du cercle qui en touche 3 autres donnés, mais encore à la construction du cercle qui coupe ceux-là sous des angles donnés, ainsi qu'à la construction du cercle qui en coupe 4

[7] "This memoir indicates, in M. Steiner, a great force of mind, and much habituation to the resources of geometry; but the manner of the author's proceeding, and the often rather complicated figures and constructions, to which he has recourse, make the going slow and the reading painful."

[8] For purposes of abbreviation and clarity, we will refer to the theorem proved by Steiner and Plücker and attributed to Lamé as Lamé's Theorem.

autres sous des angles égaux, ou plus généralement, sous des angles dont les rapports des cosinus soient donnés. (Anonymous (1827a), 173)[9]

The reviewer noted that Plücker's brevity of presentation was all the more impressive from a German. Later in the *Bulletin*, Cournot assessed Plücker's monograph *Analytisch-geometrische Entwicklungen* (1828) as a simple work, in contrast to what he claimed to be "le reproche souvent mérité, que l'on fait aux savants de sons pays, d'affectionner la complication." (Cournot (1828), 179).[10]

However, Cournot also noted the drawbacks to Plücker's style of analytic geometry. While Plücker's minimalist analysis was very useful for demonstrating already known theorems, it was not the best means for mathematical discovery. Indeed, in this text and other articles, Plücker had emphasized his originality of method, rather than of results. His geometric publications contained proofs of theorems originally attributed to other mathematicians, very often Poncelet or Steiner. Because Plücker avoided all calculations that did not concern the final result, and so revealed this result, Cournot suggested that Plücker's analysis strongly resembled "la synthèse" (ibid, 178). Cournot employed the term synthesis to emphasize how Plücker's presentation worked toward a known result, and thus functioned well for proving known theorems.[11]

Overall, the *Bulletin* reviews suggested a dichotomy of advantages between the methods of Steiner and Plücker. Steiner's method yielded almost too many new theorems, and was subject to complications too unwieldy for readers to follow. Plücker's method was concise and simple, but barren of novel results. Steiner and Plücker self-advertised with these descriptions. Steiner described his early articles as full of new and fruitful problems and theorems. Plücker drew attention to his novelty of method. Even their choice of titles reflected these characteristics. While the *Bulletin*'s assessment of Steiner and Plücker was not universally shared, we will see that the overall tone was corroborated in the French context by similar contemporary comments from Gergonne and Poncelet as well as among German scientists who wrote letters of recommendation for Steiner and Plücker.

As we saw in Chapter I, by the end of the nineteenth century, historical accounts would

[9]"M. Plucker is very restrained in calculations, and all of his can, somehow, be followed by the eye; but he chooses them with great art and taste as he very quickly reaches, not only the construction of the circle tangent to three other given circles, but even the construction of the circle that intersects those three in a given angle, as well as the construction of the circle which intersects four others in equal angles, or more generally, in angles whose cosine ratios are given."

[10]"[...] the often merited reproach to scholars of Plücker's country, to love complication."

[11]In the context of this review, Cournot's juxtaposition of analysis and synthesis was somewhat ambiguous due to the numerous possible meanings of the terms. Cournot first described Plücker's "discussion analytique des équations," thus suggesting an approach where algebra was applied to geometry through coordinate equations (similarly to his use of "analytique" in Cournot (1827)). However, by "l'analyse" as a noun Cournot seemed to intend a mode of research or presentation. Finally, "la synthèse" stood for an exposition in which the desired result was known in advance. For Cournot, then, the adjective "analytique" and the noun "l'analyse" carried very different connotations.

point to Steiner and Plücker as both representative and symptomatic of the opposition between geometric methods. However, we have also seen how the qualities ascribed to the synthetic and analytic methods did not always coincide with the individual descriptions of the practising geometers. Yet, as suggested by the above *Bulletin* reviews and will be further demonstrated, the early research of Steiner and Plücker resisted simple categorization. How then did their respective styles become emblematic of the synthetic and analytic methods? Our findings from Chapters II and III suggest that difference in form was emphasized to distinguish the similar findings of Gergonne and Plücker to those of Poncelet. The historical literature has further portrayed the frequent repetition of results in early nineteenth century geometry, as the above examples show with respect to the early career of Steiner and Plücker. Thus, in examining similar results from Steiner and Plücker, we will be able to consider their strategies for differentiation, in particular, with respect to methodological positions. Further, through reviews of their texts we compare how the "German geometers" were viewed in the eyes of their contemporaries.

We will find that Plücker and Steiner were quick to adopt the objects and techniques introduced in Poncelet's "modern pure geometry," and other contemporary developments in French mathematics. The creation of the École polytechnique and its journals in the 1790's, followed by the advent of Gergonne's *Annales* in 1810, provided multiple venues for geometers to share new research. The interconnected contents and citations of these geometers point to a substantial audience interested in new solutions, new proofs, new theories, and new pedagogical strategies. French and German speaking geometers drew upon a common research tradition emerging from "the school of Monge," published in the same mathematical journals, and referenced the same set of contemporary books and authors. These international factors complicate the common historical narrative of the progression of geometry and methodological opposition from Poncelet in France to Plücker and Steiner that we discovered in Chapter I.

Instead, by considering the early publications of Plücker and Steiner we aim to discern what elements of their methodological positions preceded their French interactions. Then, following Poncelet's commitment to the figure, as described in Chapter II, we will examine how the latter two geometers employed geometric figures and whether the figure could serve to categorize pure and analytic geometry in their published research. Likewise, Gergonne dramatically moulded the initial contributions of both Plücker and Steiner, and his vocabulary, use of duality, and commitment to planar geometry problems shaped their later work.

We will focus on two case studies: the solution to a common problem and the proof of a common theorem found in the writings of Steiner and Plücker (and incidentally, also in Poncelet and Gergonne!). Both the problem and theorem concerned relationships between conic sections in the plane, an active area of research for both pure and analytic geometers.

The proximity (in time and space) of the two versions, the common sources, and the comprehensibility of the results recommend these cases as a fruitful site on which to examine and contrast the geometric approaches of Steiner and Plücker between 1826 and 1828.

Our case studies are drawn from constructive planar geometry, whose reemergence as an open area of mathematics, practitioners attributed to the pioneering texts of Monge, and, to a lesser extent, Carnot.[12] Monge's contributions to analytic geometry and his codification of descriptive geometry were celebrated in references from Gergonne, Poncelet, and Plücker, who defined his own "pure analytic geometry" as that which was practiced by Monge (Plücker (1828a)). Likewise, we have seen in prior chapters how Carnot's results on transversal lines, harmonic points, and conic sections fuelled the findings of Gergonne, and Poncelet. Steiner also incorporated Carnot's geometric theorems and responded to Monge's results from descriptive geometry. On a methodological level, Carnot had originally proposed that pure geometry could achieve the same generality as analytic geometry, and his work toward demonstrating this was continued by Poncelet and then Steiner (Nabonnand (2011b)).

Another claimed source for early nineteenth century geometry was ancient Greece, the alleged origin of Apollonius problem. The so-called Apollonius problem was to describe a circle tangent to three given coplanar circles. Based on the number of published solutions, this may have been the most popular problem of the nineteenth century. For example, M. Simon provided around 100 references to solutions for the Apollonius problem, also known as one of the *Taktionsprobleme* in his 1906 history of geometry *Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert*. As we will see, geometers could invoke the Apollonius problem by name and expect to be understood. Steiner and Plücker addressed this elementary geometry problem early in their respective careers. Because of the problem's accessibility and unique reputation, Steiner and Plücker's solutions were reviewed in critiques situating them with respect to each other and their other contemporaries. We will see how this problem illustrated still developing styles of geometry. Steiner's text was figure based, with a plethora of newly minted definitions, but truly elementary in the modern sense of having minimal prerequisites. Plücker's text began with coordinate equations and assumed a common vocabulary and technology beyond elementary geometry. Both articles received positive receptions among French geometers, although we will see how Plücker's text remained eclipsed by Steiner's slightly earlier publication.

In 1828 Steiner and Plücker published proofs of a theorem attributed to Gabriel Lamé, for which reviewers negatively contrasted Steiner's prolix constructions to Plücker's simple equations (Anonymous (1828a), 282–284). Here Plücker exhibited what he would come to describe as an "aesthetic" interest in analytic geometry (Plücker (1839), viii). His empha-

[12]On the influence of Monge in nineteenth century geometry, see Sakarovitch (2005), Taton (1951), Laurentin (2007), and Dupin (1813).

sis on the visual properties of equations would become even more prevalent in his slightly later work on homogeneous coordinates and algebraic curves.[13] Meanwhile, Steiner had seized upon "modern" geometric practices of projection and perspective that would feature prominently in his 1831 monograph, *Systematische Entwicklung*. By contrast, this case study illuminates the particular methods of geometry that would come to be associated with Steiner and Plücker. We benefit particularly from considering Lamé's original proof as a marked point of contrast to both later versions. Like the Apollonius problem, Lamé's theorem has the benefit of being easily communicable without too much geometric prerequisite knowledge. Nevertheless, the proofs contained "modern" techniques that would have required familiarity with recent publications and trends.

In this investigation, we will see how Steiner and Plücker chose different representations, applications and contexts, while remaining within the bounds of geometry in their choice of theorems and problems, their language of objects and relations, and their citations (and lack of citations) to the literature. The conservatism of this geometric framework, and in particular the recycling of well known theorems and problems, enabled a common standard of comparison and retained enough features of older work to remain familiar. In Steiner and Plücker we find this juxtaposition of novelty and tradition, of idiosyncratic methods and common geometric results, and of a desire to push the bounds of imagination and stay true to what they saw as the natural, simple, immediate, and intuitive qualities of geometry. Further, we will see how these multiple geometries gradually incorporated recent innovations, moving toward what was known as "modern" geometry.

For ease of comprehension, we will include figures illustrating several of the descriptions from texts in Gergonne's *Annales*. The texts of Steiner and Poncelet as published outside of the *Annales* suggest that the use or non-use of figures in these particular articles may have been motivated more by choice of venue than by the content itself. Where original figures exist, they will be included (either in their original or in more easily readable reproductions)

[13] Introducing his *Theorie der algebraischen Curven gegründet auf eine neue Behandlungsweise der analytischen Geometrie*, Plücker motivated his approach and argued for an aesthetic criteria in assessing results.

> Das Kriterium für den Wert oder Unwert eines neuen Resultates, wie einer neuen Methode, liegt keineswegs in ihrer möglichen Nutzanwendung, sondern unmittelbar in ihnen selbst: sie müssen, ich glaube mich nicht bezeichnender ausdrücken zu können, ein rein ästhetisches Interesse für sich in Anspruch nehmen. Keine der verschiedenen mathematischen Disziplinen ist einer solchen Eleganz mehr fähig, als die analytische Geometrie, der Einfluss, den, in dieser Beziehung, namentlich Monge's Arbeiten auf mathematische Darstellung überhaupt gehabt haben, ist allgemein anerkannt. (Plücker (1839), viii)
>
> "The criterion for the value or lack of value of new results, such as a new method, lies not in their possible practical applications, but directly in themselves: they must, I do not think it can be expressed more characteristically, possess a purely aesthetic interest. None of the various mathematical disciplines is capable of such elegance as analytic geometry, whose influence that, in this respect, Monge's works have ever had on mathematical representation, is widely recognized."

to coincide with their use in the text. Finally, we will retain the use of emphasis or quotes following the original texts.

4.2 Tangent circles in the plane (1826–1827)

Among Steiner's six articles that appeared in the 1826 inaugural volume of Crelle's *Journal*, his paper "Einige geometrische Betrachtungen" and its sequel, "Fortsetzung der geometrischen Betrachtungen," were the two that generated attention among the French mathematical community (Steiner (1826a)). As we will see, this was not so much due to the novelty of Steiner's results, as much as his development of geometric concepts that seemed valuable in further research and pedagogy.

The first half of Steiner's paper was extracted, translated, edited, and republished the following year in Gergonne's *Annales* under the title "Théorie générale des contacts et des intersections des cercles" (Steiner and Gergonne (1827)). This was an unusual practice for Gergonne and the translation would have involved a substantial effort, thus suggesting the perceived value in Steiner's original contribution. Gergonne lauded Steiner for his solution to the Malfatti problem, the problem of inscribing three tangent circles to a triangle (Figure 4.1), which was generally considered to have not yet received a satisfactory solution.

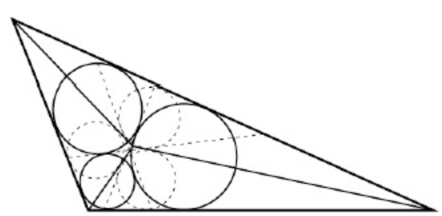

Figure 4.1: Illustration of the Malfatti problem as solved by Steiner

However, Steiner's solution to this problem was not included in Gergonne's translation, another instance of the latter's editorial intervention. Instead, Gergonne developed Steiner's exposition toward a solution of the Apollonius problem. Steiner had advertised the Apollonius problem in his introduction, and researched several aspects of the relationship between three coplanar circles, but not far enough to provide his own solution. By 1826 the Apollonius problem had a long pedigree of previous solutions, including three publications by Gergonne, and Poncelet's solution to the Apollonius problem, which had been published in the *Annales* upon Gergonne's request in 1821 (Poncelet (1821b)). Even so, Gergonne's version of Steiner was received with praise for the latter's definitions and procedure in a *Bulletin* review shortly after its publication.

Later that year, Plücker published an article on tangents and intersections among planar circles and analogous considerations for spheres in space (Plücker (1827)). He acknowledged Steiner's text, but argued for the relevance of his own work as a preferable alternative. Plücker also presented three solutions to the Apollonius problem. While his analytic proofs were new, he attributed two of his three constructive solutions to Gergonne and Poncelet respectively.

The repeated appearance of the Apollonius problem displays the conservatism of geometric content amidst evolving methodological approaches, also documented in Chapters I and II. The problem functioned as a multifaceted prop for achieving diverse dramatic effects. Steiner applied the problem to emphasize the fruitfulness of his definitions and theories and their potential for generalization. Gergonne first applied the problem to show the potential for coordinate equations to solve planar geometric problems, and then in his translation of Steiner the same solution was repurposed to display Gergonne's agility in pure geometry. Poncelet applied the problem to advertise his new principle of continuity and points at infinity. Plücker applied the problem to repeatedly demonstrate ease, elegance and brevity of his analytic approach. The Apollonius problem provided a familiar signpost for prospective readers and a theatre of competition for active geometers. In turn, we can use this problem as a point of comparison for methodological differences.

While our focus will remain on Steiner and Plücker, the historical significance of both of their works will be enhanced by an understanding of their contemporary environment as illuminated by the texts of Gergonne and Poncelet. We will consider these latter texts slightly out of chronological sequence, because Steiner declared his first publication had been derived with complete independence from contemporary results. In Section 4.2.1, we will see how Steiner developed his own set of definitions to describe relationships between coplanar points, lines, and circles. When introducing these purportedly new objects, Steiner relied upon figurative illustrations. Then to better observe the background to Gergonne's adaptation of Steiner, we first review his original proof, and the work of Poncelet. Gergonne's adaptation appeared in 1827 followed by Plücker's proof of the Apollonius problem, in which the latter referenced all three prior geometers.

The progression of results surrounding the Apollonius problem appears relatively consistent in broad outline. In each of the texts considered here, the same geometric objects were employed, though under different forms of representation–from illustrated figures, to virtual figures, to coordinate equations, to mere names devoid of additional description. For all these geometers novelty of results seemed a secondary concern, new theorems appear alongside ancient ones. Nevertheless, each geometer argued for the relevance of his own approach and we will use this comparison to consider when a result was described as elegant, simple, fruitful, general, or worthy of further commentary and favourable review.

Before we turn to the texts, we will briefly clarify two of the geometric terms employed in many discussions of the Apollonius problem, including those of Steiner, Gergonne, and Plücker. Since Viète's *Apollonius Gallus*, geometers were aware that in general the problem of finding a circle tangent to three given coplanar circles had eight solutions. The solutions could be divided into pairs dependent on the type of tangency, exterior or interior.

We will describe a situation in which two circles have no common interior point by calling the circles "exterior circles." Three or more circles will be also called exterior circles if any pair of them are exterior. We will say that two circles are "interior circles" if one lies completely inside the other. Exterior or interior circles may be tangent or not (Figure 4.2). In the texts below, Steiner differentiated these circles as "ausser oder ineinander liegende Kreise," (Steiner (1826a), 168) while Gergonne and Plücker focused on tangency, writing "cercles se touchent intérieurement ou extérieurement" (Steiner and Gergonne (1827), 291). When employing exterior and interior circles, none of the above geometers specified whether the points of tangency might be real or imaginary. However, the application of the Apollonius problem implied that the points would be constructible.

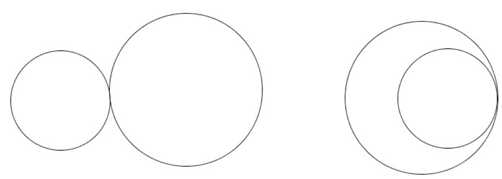

Figure 4.2: Exterior and interior tangent circles

4.2.1 Potenzkreise: Steiner's theory toward a solution to the Apollonius problem

Through an in-depth study of one of Steiner's earliest publications, we will focus on features of his approach to geometry as they relate to both his later method and our understanding of pure and synthetic geometry as informed by the arguments of Poncelet and later commentaries from previous chapters. Immediately, we will see how Steiner often divided his investigations into cases based on specific figures, thus succumbing to the particularity that Poncelet had criticized in "ancient pure geometry" (Poncelet (1817e)). However, Steiner supplemented these particular configurations by claiming his results based on circles extended to general conic sections, as well as by arguing in reverse, by assuming that properties were both necessary and sufficient for a desired condition. Steiner further modified Poncelet's delimitation between pure and analytic geometry, by including calculation in his article that had been labelled as "pure geometry" by Gergonne. Finally, we will find that while Steiner worked toward solving well-known problems, he did so by introducing new definitions. In later reviews of Steiner's work, these new definitions would be considered as innovative and valuable contributions.

Steiner began his “Einige geometrische Betrachtungen” by clarifying its vague title.

> Die in den nachstehenden Paragraphen angefangenen Betrachtungen enthalten die Grundlage der geometrischen Untersuchung über das Schneiden der Kreise. Es lassen sich daraus die Auflösungen fast aller Aufgaben über das Schneiden und Berühren der Kreise entwickeln, und zwar in den meisten Fällen sehr einfach, auch wird durch sie oft zwischen mehreren Aufgaben, welche auf den ersten Anblick keine Gemeinschaft zu haben scheinen, ein gewisser Zusammenhang sichtbar. (Steiner (1826a), 161)[14]

Steiner’s research on this subject over the past three years had been motivated by three sets of problems: finding a circle tangent to three given planar circles (the Apollonius problem), the Malfatti problem, and the fifteenth theorem of the fourth book of Pappus’s *Collectiones mathematicae* (concerning ratios between tangent circles inscribed in a semi-circle). Although acknowledging that each of these problems possessed well documented recent solutions, Steiner avowed that his work was entirely independent, relying only upon theory developed by Viète and Pappus. In fact, Steiner admitted that he had but recently become aware of contemporary French publications in geometry, in particular Poncelet’s *Traité des propriétés projectives des figures*, where many of his own results had already appeared. However, Steiner also promised new findings that would further demonstrate the independence of his research.

> Für die Versicherung, dass der Verfasser Dasjenige, was die Franzosen in dieser Hinsicht getan, vorher nicht gekannt habe, hofft er, werden nicht allein diejenigen seiner Bekannten, welche, bei täglichem Umgange mit ihm, die Entstehung und Entwicklung seiner Arbeiten beobachteten, sondern dem Sachkenner wird auch schon die umfassendere, allgemeinere Entwicklungsweise in den Untersuchungen, aus welcher nicht nur alle jene Betrachtungen, sondern auch eine grosse Menge neuer Resultate von selbst hervorgehen, ein Zeugnis ablegen. (162)[15]

Specifically, Steiner promised his work contained new generalizations to a greater number

[14] “The following paragraphs of introductory remarks contain the groundwork of geometric research concerning the intersections of circles. These permit us to develop the solutions of almost all problems about intersection and contact of circles, and in fact in most cases very simply. As well a certain connection often becomes visible through them between several problems that at first sight seem to have no commonality.”

[15] “For insurance, that the author of this work did not know previously what the French had done in this regard, he hopes to rely on the witness not only of those of his acquaintances who, by daily interaction with him, observed the origin and development of his work, but also, for those who know the subject, the more comprehensive, more general mode of development in the research, from which not only all those matters but also a great quantity of new results issue forth of themselves.”

of given circles, to circles intersecting at given angles rather than in tangent points, and to analogous results for second degree curves and three dimensional surfaces.

Despite his promise to address three sets of problems, at the end of this text Steiner only included his solution of the Malfatti problem, a solution that he left unproven. Though Steiner claimed that his theorems could be applied to many more problems concerning coplanar circles, the details of his solution to the Apollonius problem would only appear in Steiner's anticipated book on circles, spheres, and spherical circles.[16] We will see how the Apollonius problem served as an invisible motivating force for the definitions and results throughout the body of the text, which would become visible in Gergonne's later translation.

Steiner organized his mathematical content into four parts, each devoted to a particular geometric relationship, and a total of nineteen numbered sections, each beginning with a particular configuration of objects and leading to a property or theorem to be referenced by citation to that section, for example "nach (1.):" As Gergonne would observe, Steiner's exposition often delved into the very elementary, too elementary for the average *Annales* reader. In this feature, the entire text served to reinforce Steiner's introductory remarks about the independence of his results. Until Steiner arrived at the final part, in which he posed and solved a series of problems, very few results were assumed as known and there was only one external citation.

The text developed as an exploration, from two points to a point and a circle, to two circles, to three circles, to more complex planar relationships. Steiner modelled his presentation as a discovery beginning from the ground up and resulting in surprising, even when already known, results. Figures served as clarification, and were never invoked as a tool of proof. In our summary, like Gergonne, we aim toward the Apollonius problem and will skip and summarize certain results to this end. Unlike Gergonne, we will still attempt to preserve Steiner's idiosyncrasies and personal style.

Although Steiner presented his research without a clear map of his intended progression, a brief summary here will motivate our understanding of his development. All new definitions will be described and illustrated in detail below, where our summaries of sections will serve as guideposts.

The first part concerned defining the power relationship between coplanar circles and contained five sections each examining a different set of geometric objects: on equations relating to perpendicular lines (1), on the power of a point with respect to a circle (2), on

[16]This book was published in 1931 as *Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln worunter eine grosse Anzahl neuer Untersuchungen und Sätze vorkommen en einem systematischen Entwicklungsgange dargestellt* edited by Rudolf Fueter and Ferdinand Gonseth (Steiner (1931)). In the late nineteenth century, the mathematician Fritz Bützberger had found the unpublished manuscript dating between 1823 and 1826 in a box at the Library of the Naturalist Society of Bern. The manuscript was then rediscovered by the editors, Fueter and Gonseth, amongst Bützberger's papers. The final text ran 360 pages and expanded upon the content from the published article discussed here. We will observe the relevant distinctions in footnotes.

the line of equal power between two circles (3), on the point of equal power between three circles (4), and on power relationships between multiple and orthogonal circles (5). The concept of the *power* of coplanar circles and points derived from a well-known Euclidean relationship: when two chords of a circle intersect, then the rectangles formed by their respective segments are equal (Proposition III.35) and when a tangent and a secant line are drawn from a point to a circle, then the square on the tangent is equal to the rectangle formed by the intersected segments of the secant (Proposition III.36).

In his second part, containing sections 6–8, Steiner continued his examination of circles, now with respect to similitude points and similitude lines. These points and lines were determined with respect to a constant ratio determined by the positions of the circles' centres. In the three sections Steiner progressed from examining ratios between three coplanar points (6), to defining the similitude point and line between two circles (7), and then the similitude line between three circles (8).

Steiner combined the concepts of equal power and similitude to define the common power of coplanar circles in the third part of his paper. He began by examining the relationship between lines of equal power and similitude points (9), which he applied to define the concept of common power between two circles (10), power circles (11), and the power position of a point or circle with respect to a similitude point (12). Finally, Steiner applied these definitions to a series of theorems concerning tangency and power circles (13). The fourth part of his paper concerned solutions and generalizations of the Malfatti problem.

With a sense of Steiner in outline, we now turn to the details of his exposition.[17] Steiner's figures are reproduced in full in Figure 4.27 in Section 4.2.7, the remaining figures are based on Steiner's illustrations, but have been redone for ease of readability. We reference his original numeration (with our present numeration in parentheses).

Equations relating to perpendicular lines (1)

Steiner began with two points M, m situated as pictured in Figure 8 (our Figure 4.3). If the lines Mm and PG remained perpendicular, Steiner concluded that all points P constructed on a perpendicular to Mm with foot G conserved the fixed equality $MP^2 - mP^2 = MG^2 - mG^2$. Conversely, he considered the locus of points P whose distance from each given point M and m when squared was equal to a given quantity. The same fixed equation determined that this locus would be a perpendicular line PG to Mm.

[17] A contemporary presentation of several of Steiner's results on circles can be found in the recent textbook *Geometry by Its History* (Ostermann and Wanner (2012), 98–104).

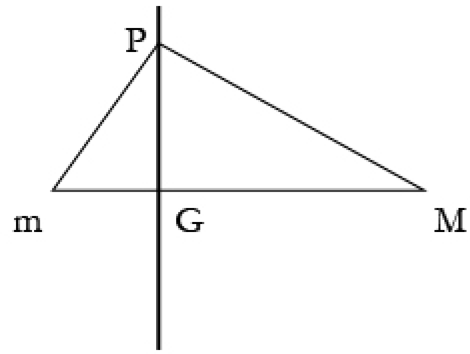

Figure 4.3: Steiner's Figure 8 (reproduced following (Steiner (1826a))

Power of a circle with respect to a point, power of a point with respect to a circle (2)

"In the school books on geometry," Steiner vaguely remarked by way of introduction, "one finds this relationship."[18] Consider, as in Steiner's Figure 9 (our Figure 4.4), a point P exterior to a circle centred at M. Two secants to circle M passing through P would cut the circle at points A, B and C, D respectively, such that the product $PA \times PB$ was equal to $PC \times PD$. This product Steiner called "the power of a circle with respect to a point" or *reciprocally* "the power of a point with respect to a circle." In an accompanying footnote, he pointed to the "ancient" use of power of a hyperbola as a precedent in this choice of vocabulary.[19]

Steiner elaborated three possible cases. If the point P lay outside of the circle, as in Steiner's Figure 9, the power of P would be equivalent to the square of the tangent segment to circle M from P. This distance could also be calculated as PM^2 minus the square of the radius, MA. Thus $PT^2 = PM^2 - R^2$.

If point Q lay inside the circle, as in Figure 10 (our Figure 4.5), then the power of Q with respect to M would be what Steiner described as "the square of half the shortest chord one can draw through Q to the circumference of M," that is, the square of the segment QC. Equivalently, the length of QC was the square of the circle's radius minus the distance MQ^2. So, $QC^2 = R^2 - MQ^2$. The symmetry of the two equations showed how the tangent segment functioned like the circle's semi-chord.

Finally, the power of a point would be zero when the point lay on the respective circle's

[18]Steiner's manuscript more precisely attributes this result to Euclid III.36 (Steiner (1931), 28).

[19]The power of a hyperbola is the area of the rhombus described on the major and minor axes. An explicit definition of the power of a hyperbola as "la moitié du quarré du demi-axe" can be found in the twenty-seventh volume of the *Encyclopédie ou Dictionnaire raisonnée des sciences, des arts et des métiers* (Anonymous (1778), 793).

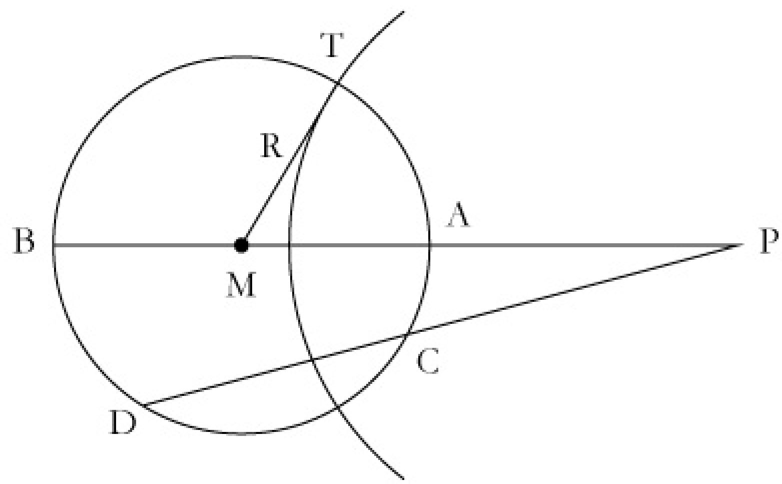

Figure 4.4: Steiner's Figure 9 (Steiner (1826a))

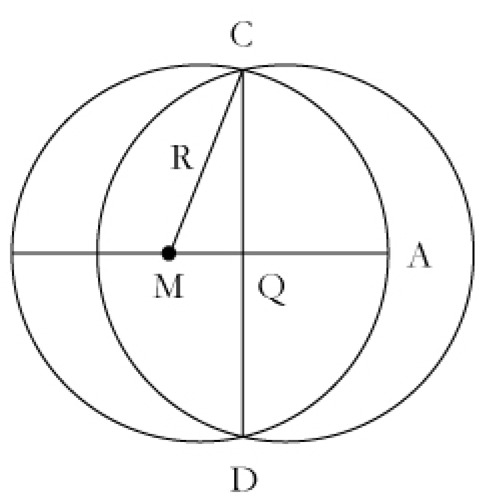

Figure 4.5: Steiner's Figure 10 (Steiner (1826a))

circumference. Steiner gave no pictorial correspondence to this final case, but it followed from either equation.

Line of equal power (3)

Steiner progressively extended his research to more complicated configurations. In Section 3, Steiner combined his results from the two prior sections, considering now two circles of given size and position, which were centred at and denoted by points M and m. He calculated that the locus of points of equal power to both M and m would form what he called the "line of equal power," a line perpendicular to the line containing their centres, Mm.

The case of intersecting circles is illustrated in Figure 10 (our Figure 4.5), although circle m is unlabelled, where the line of equal power contained the common chord of M and m. In the case when the two circles were tangent, their shared tangent line would also be their line of equal power. Generalizing from this configuration and the common power equation (here written with respect to radii as $MP^2 - mP^2 = R^2 - r^2$), Steiner concluded that tangent segments drawn from any point on the line of equal power, PG, to circles M and m respectively would be equal. Thus, reciprocally, circles orthogonal to both M and m would be centred on the line PG.

Point of equal power with respect to three circles (4)

In Section 4, Steiner considered three given circles M_1, M_2, M_3, which pairwise determined three lines of equal power, denoted $l(12), l(13), l(23)$, as represented in Figure 11 (our Figure 4.6).

Figure 11 included only the centres of the three circles without their circumferences, thus illustrating a seemingly general case of ambiguous circle positions. Depending on their size, the three circles might all intersect, all be tangent, not intersect at all, or some combination of those possibilities. Steiner considered each line of equal power in turn. The lines $l(12)$ and $l(13)$ met at a point of equal power for both circles M_1, M_2 and circles M_1, M_3. Therefore the intersection was also a point of equal power for M_2, M_3. So the intersection of any two of the three lines of equal power would be the point, denoted $p(123)$, where all three lines concurred, "the point of equal power for three circles." When the three circles M_1, M_2, M_3 all intersected, their three common chords would then intersect at the point of equal power, as shown in Figure 12 (our Figure 4.7).

When instead the three circles were pairwise tangent, their three common tangent lines would then meet at the point of equal power. This configuration was not illustrated by Steiner.

Steiner considered one other case, Figure 13 (our Figure 4.8), when circles M_1 and M_2

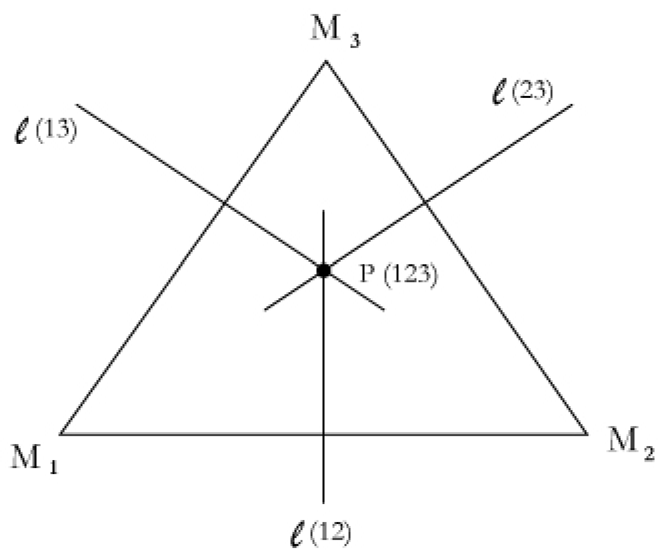

Figure 4.6: Steiner's Figure 11 (Steiner (1826a))

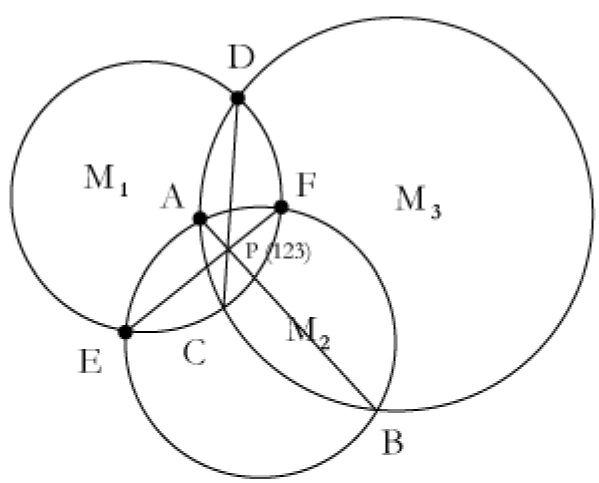

Figure 4.7: Steiner's Figure 12 (Steiner (1826a))

did not meet, but were each intersected by the third circle M_3. Then the common chords between M_1 and M_3 and between M_2 and M_3 would intersect each other on the line of equal power of M_1 and M_2. With this property, Steiner declared "one easily sees" how to find the line of equal power of any two given circles with an arbitrary third circle that intersected the two given. In this situation, the line of equal power would be perpendicular to the line containing the centres and would pass through the point of equal power.

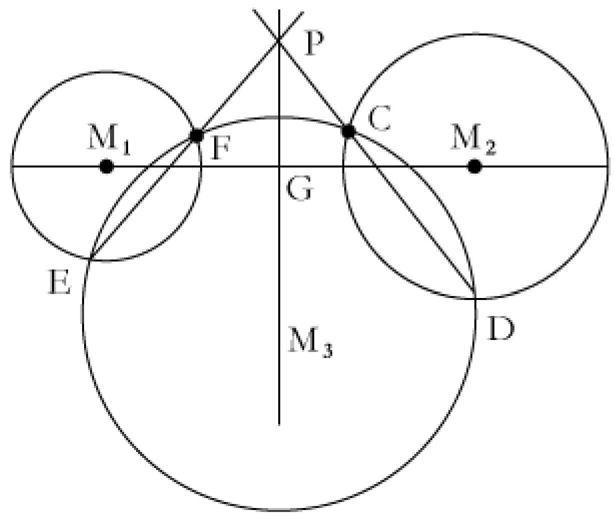

Figure 4.8: Steiner's Figure 13 (Steiner (1826a))

Shifting his focus to the determined line of equal power, Steiner considered some point P on the line of equal power for two given circles M_1, M_2. From this point Steiner drew two tangent lines to M_1 (lines PA and PD) and two tangent lines to M_2 (line PB and PC) Considered pairwise, these four tangent points were also the tangent points of circles tangent to both given circles, such as the circles through A, B and through C, D shown in Figure 14 (our Figure 4.9). Because each tangent segment drawn from the line of equal power was the same length, these same four points A, B, C, D would all lie on a circle centred at P that cut given circles M_1 and M_2 orthogonally. Then by the same reasoning, for a circle drawn with centre P orthogonal to circles M_1 and M_2 at points A, B, C, D, one could draw four circles tangent to both given circles through these points taken pairwise. In this latter property, Steiner's result showed how reversing his argument could lead to a result that could be stated independently of his new definitions.

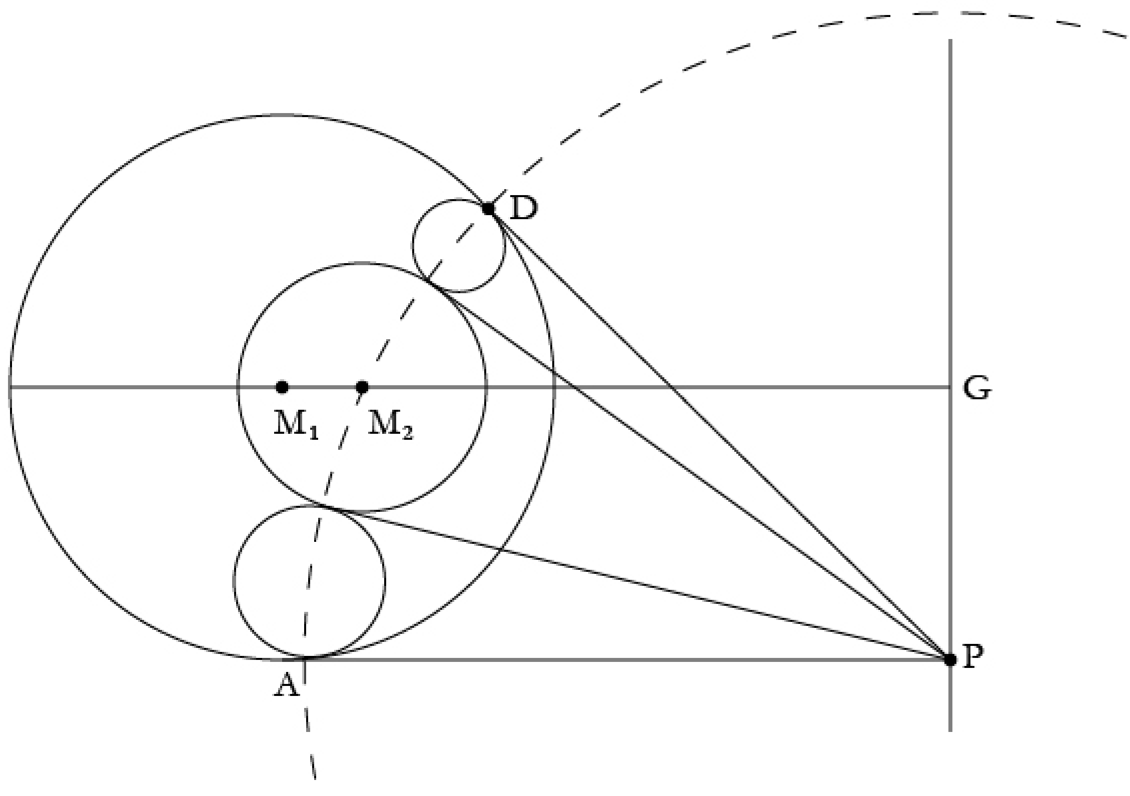

Figure 4.9: Steiner's Figure 14 (Steiner (1826a))

Bundles of circles and orthogonal circles (5)

From the case of two orthogonal circles, Steiner generalized his investigations to the relationship of two bundles [*Schaar*] of orthogonal circles.[20] With reference to Figure 15 (our Figure 4.10), Steiner considered all possible circles $P_1, P_2, P_3, ...$ orthogonal to circles M_1 and M_2.

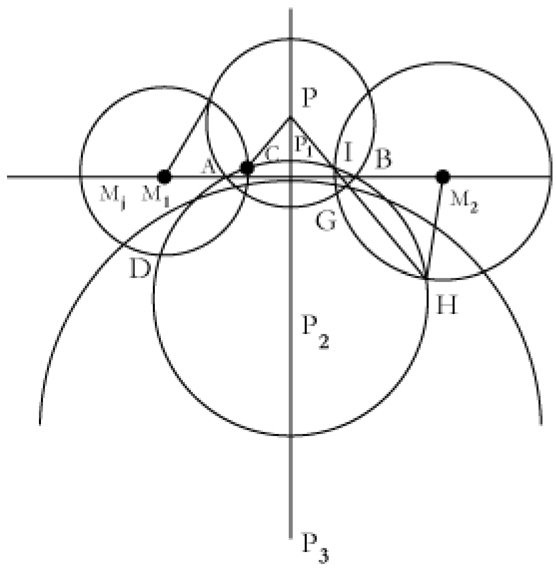

Figure 4.10: Steiner's Figure 15 (Steiner (1826a))

In this situation, the line of equal power for $P_1, P_2, P_3, ...$ would be the axis containing the centres of the circles M_1 and M_2, which Steiner designated as M_1M_2. Reciprocally, the locus of centres of all circles $M_1, M_2, M_3, ...$ orthogonal to the circles previously defined as $P_1, P_2, P_3, ...$ was also the axis M_1M_2. So when any two of the $P_1, P_2, P_3, ...$ circles intersected each other at points A and B, their common chord AB lay on M_1M_2. Further, all chords common to M_1 and any $P_1, P_2, P_3, ...$ would respectively cut M_1M_2 at a unique point that Steiner labelled as M. Steiner declared that "on equal grounds" all chords common to the circle P_1 and $M_1, M_2, M_3, ...$ respectively would meet the line P_1P_2 at a determined point called P. By construction, the circles $M_1, M_2, M_3, ...$ were orthogonal to P_1 and likewise $P_1, P_2, P_3, ...$ were orthogonal to M_1. So, from the definition of orthogonality, the radii drawn from centres $M_1, M_2, M_3, ...$ to their respective intersection points would be tangent to the circle P_1, and the radii from $P_1, P_2, P3, ...$ would be tangent to the circle M_1.

Steiner applied these findings to prove a construction of the common intersection point

[20] Following Louis Gaultier de Tours (1776–1848), Gergonne refers to "une suite de ces cercles" rather than a set (Steiner and Gergonne (1827), 299).

P for a given line M_1M_2 and a circle P_1. Pairs of tangent lines drawn from any points $M_1, M_2, M_3, ...$ on M_1M_2 to circle P_1 would define contact points on the circumference of P_1. The chords drawn by connecting these pairs of contact points would all pass through the same point P.

Steiner thus demonstrated constructively that for a given coplanar circle and line there existed one unique corresponding point, lying on the perpendicular from the given line passing through the circle's centre. His constructions are shown in Figure 16 (our Figure 4.11) for the case where the line M_1M_2 intersected the circle P_1 and by constructing the respective tangents M_1C, M_1D and M_2H, M_2I one determined the secants CD and HI which intersected at P. In this case P fell outside of circle P_1.

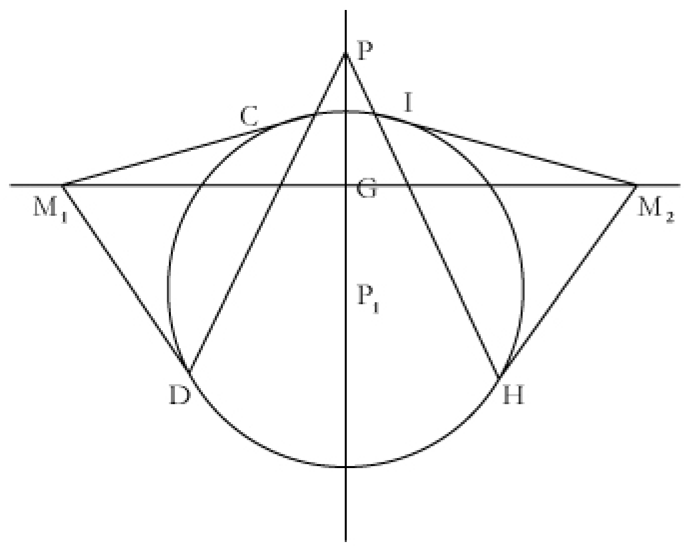

Figure 4.11: Steiner's Figure 16 (Steiner (1826a))

Steiner then described the case where the line P_1P_2 did not meet the circle M_1. Then respective tangents P_1E, P_1F and P_2C, P_2D determined chords EF and CD meeting at M. The constructed point M thus lay inside circle M_1, as shown in Figure 17 (our Figure 4.12).

Curiously, Steiner did not assign a name to this relationship, which was already well-known as polar reciprocity among contemporary French mathematicians, and would be interpreted as such by Gergonne.[21] Steiner concluded his analysis of lines of equal power by

[21] In Steiner's posthumous book on circles, he did include a discussion of poles and polars. The suggested dates for this text would imply that he knew of their use in France in 1826, but chose not to use their terminology in his published article. Although there is no mention of editorial updating in the introduction, we cannot wholly rule out this possibility. As written in the published manuscript,

> Die Franzosen nennen zwei solche zusammengehörige Punkte (M, q oder M_1, q_1) Pôles conjugés;

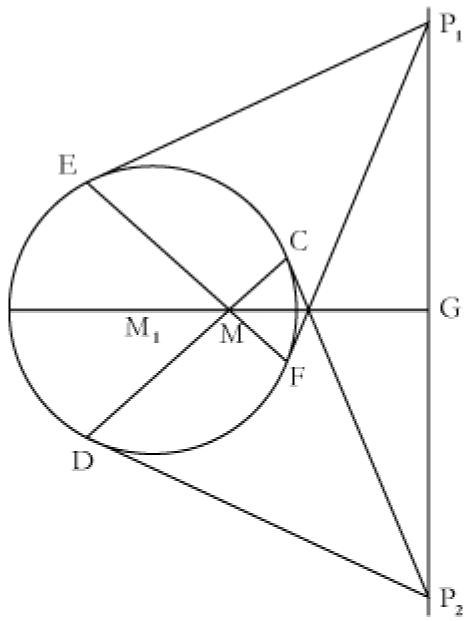

Figure 4.12: Steiner's Figure 17 (Steiner (1826a))

suggesting that the same properties held similarly for second degree curves and analogously on second degree surfaces. He gave no hint about how these results might so be extended.

Already we observe how Steiner exhibited a distinctive presentation style. The content emphasized positional relationships, with suggestive rather than rigorous reasoning. For instance, proofs of concurrence or collinearity were left to the reader and Steiner often argued in reverse without justification. Despite his introduction of new vocabulary, many of his results could be stated without reference to the "power" relationship and thus may have appeared attractive to his readers.

While Steiner's choice of method in this article would be described by Gergonne and Cournot as elementary or pure geometry, this did not obviate the use of computation. Steiner defined "power of a point" with respect to a proportion, rather than a constructive operation, and used a "constant product" to describe relationships between circles and their points or lines of similarity. Nevertheless, the only calculation Steiner employed with these equations was simple addition, and they served more as abbreviations of constructions or

auch nennen sie den Punkt q in Bezug auf die Linie M_1M_2 oder den Punkt q_1 in Bezug auf die Linie Ppp_1 Pôle, und umgekehrt die Linie in Bezug auf den Punkt Polaire. Wir wollen uns bei Lösung der folgenden Aufgabe kürzehalber dieser französischen Benennung bedienen. (Steiner (1931), 54)

"The French call two such associated points conjugate poles; and call the point q pole with respect to the line M_1M_2 or the point q_1 the pole with respect to line Ppp_1, and reciprocally the line polar with respect to the point. We will use the following abbreviation of this French designation in our solution of the following problem."

lengthy verbal relationships than as tools for deriving new geometric information.[22]

By attending to the static figure, Steiner had to consider multiple configurations, which resulted in several cases for each concept. He employed actual figures to represent some, but not all of these discussed configurations. In particular, configurations of interior circles were declared analogous and not displayed pictorially. Much of the textual detail was omitted in the figures themselves. There were no circles in Steiner's Figure 11 and his Figures 9 and 14 only displayed arcs of certain given circles. This enabled a more general representation, as in the case of Figure 11 representing multiple circle relationships at once.

Ratios between three points (6)

In Part II, *On similitude points and similitude lines of coplanar circles*, Steiner set aside the concept of power to investigate a seemingly new relation between two given points, m and M, depicted in the two possible configurations of Figures 18 and 19 (our Figures 4.13 and 4.14).

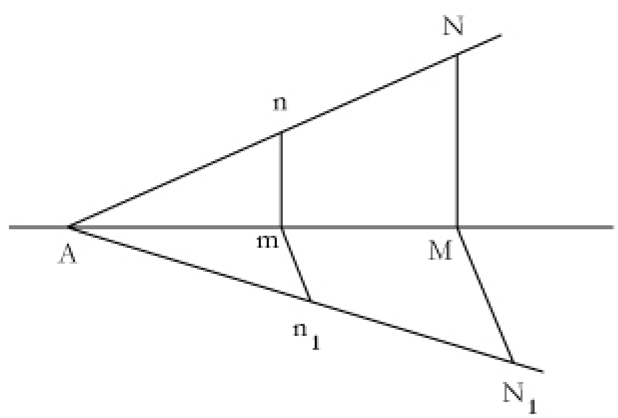

Figure 4.13: Steiner's Figure 18 (Steiner (1826a))

By construction of parallel lines, point I could be determined by the ratio $MN : mn = MI : mI$ and point A by the ratio $MN : mn = MA : mA$. Steiner called these points the "inner and outer similitude points with respect to M and m."

[22]This moderate use of algebra in representing proportional relationships was normal in contemporary elementary geometry textbooks, see for example (Lacroix (1799), Legendre (1800)).

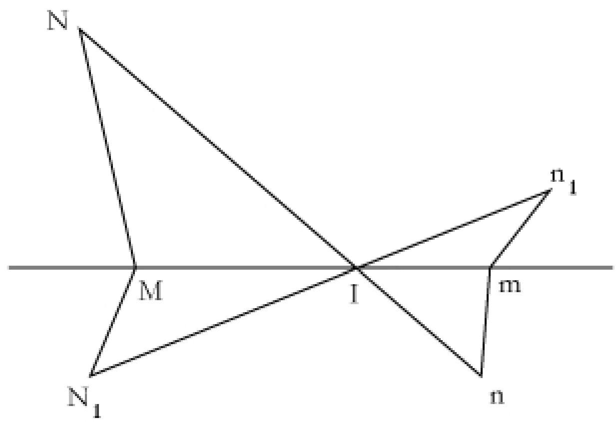

Figure 4.14: Steiner's Figure 19 (Steiner (1826a))

Similitude points and lines of two circles (7)

Steiner extended his definition, by considering the points as circles centred at M and m, still with reference to Figures 18 and 19. Here the pairs of lines MN, mn and MN_1, mn_1 represented pairs of parallel radii, either lying on the same side (Figure 18) or opposite sides (Figure 19) of the axis containing the circles' centres Mm. The lines Nn, N_1n_1 containing the endpoints of these radii would intersect each other at a point A when the points lay on the same side of Mm and a point I when the points lay on the opposite side of Mm. Both points A and I lay on the line Mm.

With reference to Figure 20 (our Figure 4.15), Steiner argued that for any such line An_1N_1 or N_1In_1 passing through A or I as defined with respect to points M and m, such that the pairs of parallel line segments MN_1 and mn_1 were in the same ratio as the radii of circles centred at M and m, then $MN_1 : mn_1 = MN : mn$. So when the radii of circles M and m were of respective length R and r,

$$R : r = MA : mA = MI : mI. \tag{4.1}$$

The points A and I he called, the "similitude points of both circles M, m." Further, any line passing through a similitude point, such as the lines An_1N_1 in Figure 18 or n_1IN_1 in Figure 19, were designated as "similitude lines of circles M, m."

For two exterior tangent circles, their common tangent point would be the outer simil-

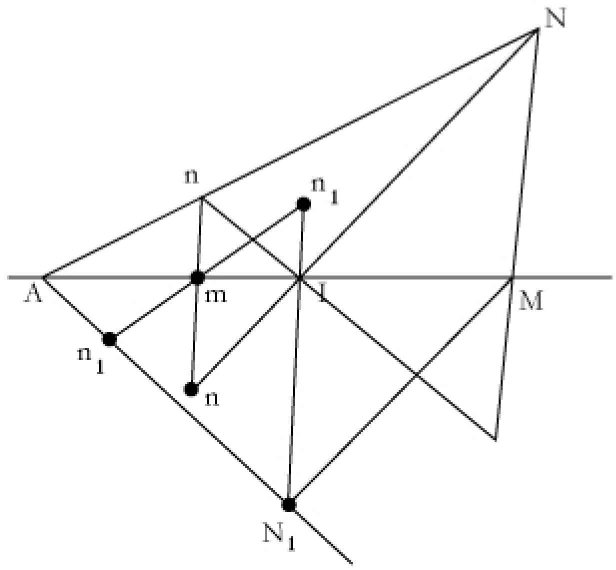

Figure 4.15: Steiner's Figure 20 (Steiner (1826a))

itude point A since the respective radii would be MA, mA. Likewise the tangent point of interior tangent circles would be the inner similitude point I. Focusing on the case of two exterior circles, Steiner argued that any two tangent lines common to both circles would intersect at one of their two similitude points as shown in Figure 21 (our Figure 4.16). Reciprocally, one could "easily" use these similitude points to find a common tangent to the two given circles.

Similitude lines for three given circles (8)

As in Part I of this text, in the second part Steiner followed a pattern in progressive sections from two points, to two circles centred at those points, to three circles. For three given circles M_1, M_2, M_3 considered pairwise, there would be three outer and three inner similitude points. The similitude points of three circles were illustrated in Figure 22 (our Figure 4.17), where the circles were represented by their centres alone as points M_1, M_2, M_3. As in Figure 11, here the absence of drawn circles extended the result to any set of three circles whose centres were not collinear because the possible intersections of the three circles remained unspecified. The position of I's and A's suggested the size of the circles, but by leaving the circles unconstructed the important collinear properties could be seen without distraction and the procedure could be more easily translated to any given set of three circles. Steiner designated the outer and inner points of circle M_1 and M_2 as A_3 and I_3 and likewise for points A_2, I_2, A_1, I_1.

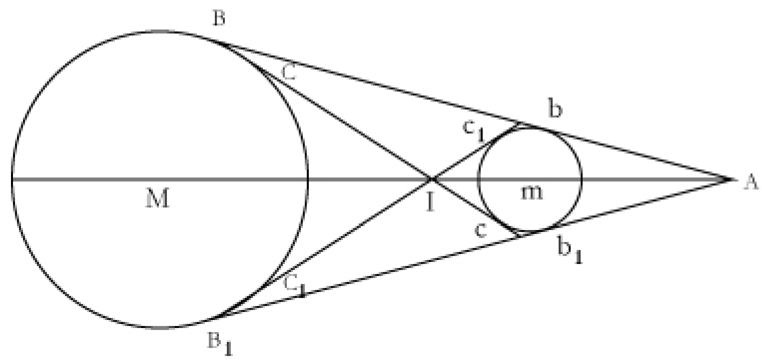

Figure 4.16: Steiner's Figure 21 (Steiner (1826a))

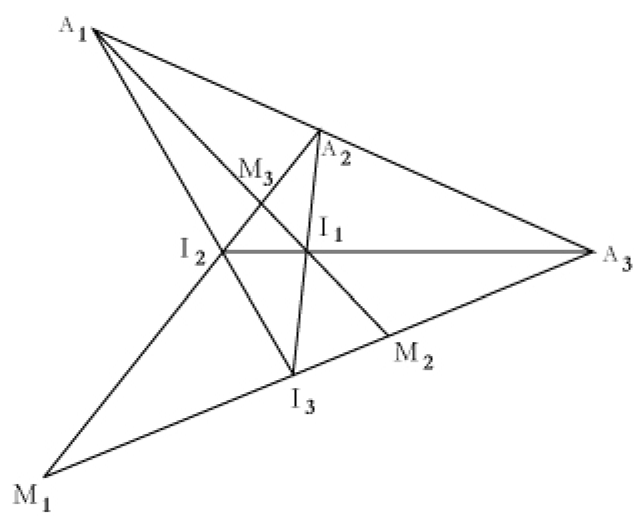

Figure 4.17: Steiner's Figure 22 (Steiner (1826a))

Steiner first considered the line containing points A_3 and A_2. Since A_3 was a similitude point for circles M_1, M_2, the line A_3A_2 would be a similitude line for circles M_1, M_2, and likewise by considering the point A_2, the line A_3A_2 was a similitude line for M_1, M_3. Thus A_3A_2 would be a similitude line for all three circles, and called the outer similitude line since it contained the points A_1, A_2, A_3. Thus all outer similitude points were collinear.

Steiner stated that in "wholly similar ways," one could show that $A_1I_2I_3$, $A_2I_1I_3$, and $A_3I_1I_2$ were collinear, the "inner similitude lines" with respect to the three given circles. When M_1, M_2, M_3 were exterior circles, Steiner showed that all pairs of outer or inner tangent lines to any pair of circles would intersect at a point on one of the similitude lines as shown in Figure 23 (our Figure 4.18).

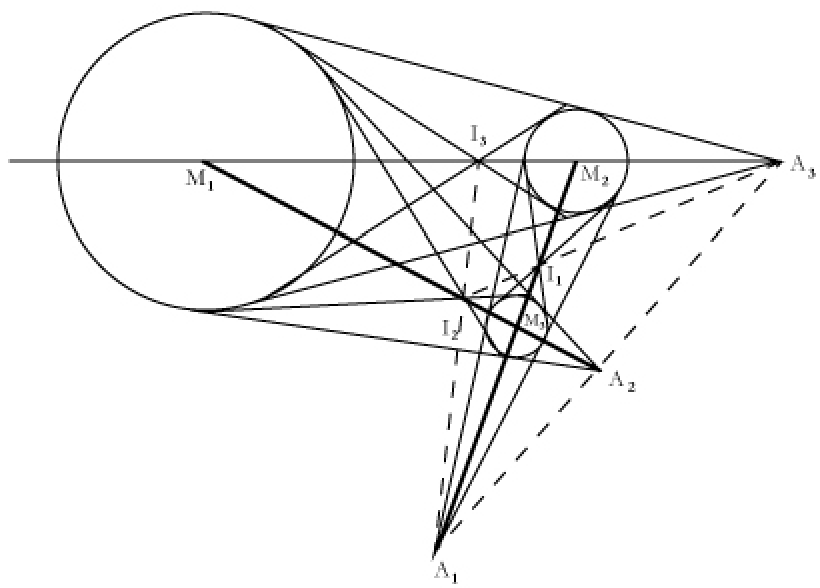

Figure 4.18: Steiner's Figure 23 (Steiner (1826a))

Here in the text, Steiner made his first contemporary and only precise citation to a geometer other than himself. He credited a special case of his result to M. Hirsch, who had previously considered the concurrence of the outer tangent lines in *Sammlung geometrischer Aufgaben* II on page 368.[23]

[23]Meier Hirsch (1770–1851) was a prolific textbook writer in elementary mathematics on algebra, arithmetic and geometry, who also worked as a private instructor in Berlin. His publications of *Sammlung geometrischer Aufgaben* was continued by Ludwig Magnus in the 1830s (Vogel (1972)). Among a series of so-called "mixed" geometry problems and theorems, Hirsch proved this theorem on circles using proportional relationships. The list-like structure of Hirsch's text is paralleled in many of Steiner's early articles, and this

Lines of equal power and similitude points (9)

In Part III, *On circles of common power in a plane*, Steiner synthesized his findings on two coplanar circles. With reference to Figure 24 (our Figure 4.19), if P was any circle orthogonal to circles M_1 and M_2 at respective points A, D and C, B then Steiner could define four new circles by their points of tangency to circles M_1 and M_2 at the pairs of points A, B; D, C; A, C and D, B. Steiner had shown that both tangent points of a circle tangent in the same way to circles M_1 and M_2 would be collinear with A_3 (the outer similitude point of M_1 and M_2) and both tangent points of a circle tangent in different ways to M_1 and M_2 would be collinear with I_3 (their inner similitude point). Thus the circle P determined four lines, each containing three of the above points: $A_3AB, A_3DC, AI_3C, DI_3B$.

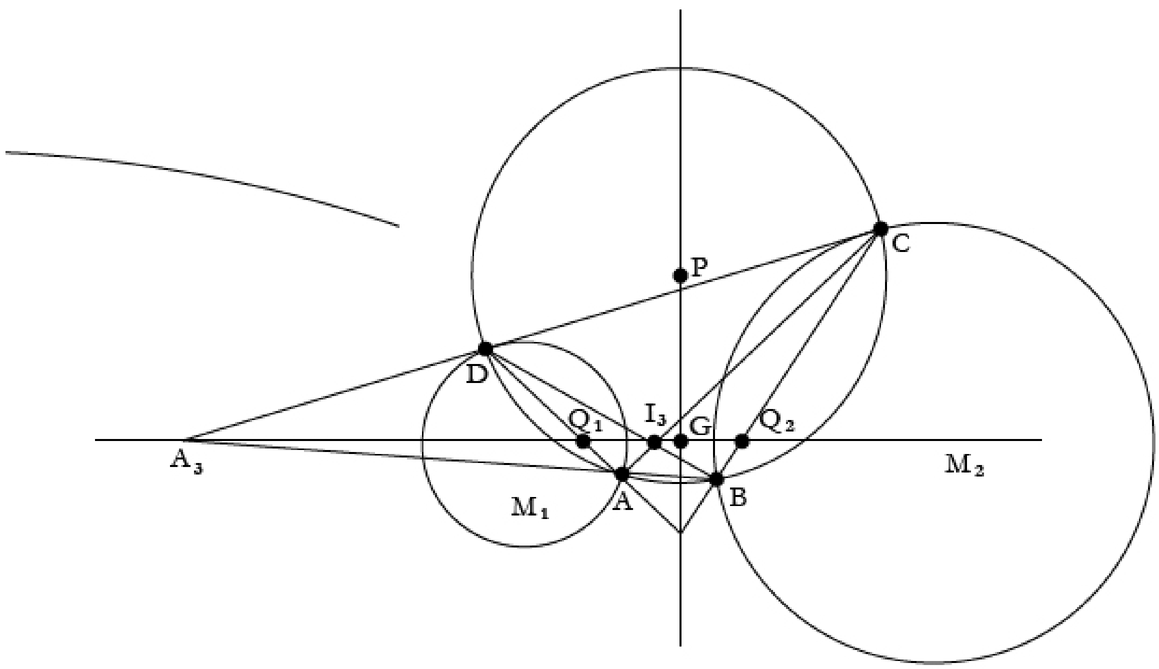

Figure 4.19: Steiner's Figure 24 (Steiner (1826a))

As an "equivalent" result, Steiner presented the following theorem: if from any point P on the line of equal power PG of circles M_1, M_2 one draws four tangent lines PA, PD, PC, PB to the circles, and joins the four tangent points pairwise to form six lines, then the lines BA and CD will intersect at A_3 and the lines AC and BD will intersect at I_3. Further the

reference and resemblance may suggest Steiner's similar pedagogical aims (Hirsch (1807)). However, as we will see in Steiner's later methodological commentary, he negatively assessed most contemporary geometry textbooks as mere compilations of disjointed results–a criticism risked in his own work.

locus of intersection points P of the lines DA and CB will be the line PG itself. Finally, DA and CB will always pass through M_1M_2 at the respective points Q_1 and Q_2 (though Steiner did not name these points, Q_1 and Q_2 were what would be called the poles of the line of equal power with respect to circle M_1 and circle M_2.).

Steiner's exposition here was greatly assisted by the illustrative clarity of Figure 24 to differentiate which points denoted which type of tangency. We will see that this relationship would require very precise language when independent of a visual aid as later presented by Gergonne.

Common power of two circles (10)

Steiner explored the consequences of this result by considering all possible circles P orthogonal to both circles M_1, M_2. These P circles would share the line of equal power $A_3M_1I_3M_2$. So when a straight line A_3AB or AI_3C intersected circle M_1 at A and circle M_2 at B or C, then the product $A_3A \times A_3B$ (the power of A_3 with respect to circle P) or $AI_3 \times CI_3$ (the power of I_3 with respect to circle P) would remain constant as long as the radii M_1A and M_2B or M_1A and M_2C were not parallel to each other. Steiner named this constant product the "common power of circles M_1, M_2 with respect to their similitude point A_3 or I_3."

Power circle of two circles (11)

Again extending the definition from points to circles, Steiner called a circle centred at A_3, whose radius equaled the square root of the respective common powers determined above, the "outer power circle of M_1, M_2." The "inner power circle of M_1, M_2" was centred at I_3. So the power circle [*Potenzkreise*] was uniquely defined with respect to two given circles and one similitude point. This concept would prove crucial in later proofs of the Apollonius problem given by Gergonne and Plücker.

Power position with respect to a similitude point (12)

If three collinear points X, I_3, Y were situated on either side of I_3 such that $I_3X \times I_3Y$ equaled the inner common power with respect to the given circles, then the points X and Y were said to be in "power position" [*potenzhaltend*] with respect to the point I_3. The same relationship held for points X and Y on the same side of the outer similitude point, A_3. Circles could also be in power position with respect to similitude points, when the power of I_3 or A_3 with respect to the new circle equaled the inner or outer common power of with respect to M_1, M_2. Steiner concluded that "clearly" each circle passing through any two points in power position with respect to given circles M_1, M_2 would thus be a circle in power position with respect to the given circles. Further, when the similitude point A_3

or I_3 was in power position to a circle K, then the power circle centred at A_3 or I_3 would intersect K orthogonally.

Relationship between tangency and power circles (13)

From these definitions Steiner constructively showed that each circle tangent in the same way (that is, either all exterior or all interior tangent points) to both exterior circles M_1 and M_2 would be in a power position to the outer similitude points of M_1 and M_2 and orthogonal to the outer power circle of M_1 and M_2. Any circle tangent in different ways (for example, with an interior tangency to M_1 and an exterior tangency to M_2) would be in power position to the inner similitude points and orthogonal to the inner power circle. Steiner suggested that "similar" results followed when M_1 and M_2 were interior circles or intersected one another.

Steiner extended his investigation to all possible circles tangent to the given circles M_1, M_2 (here Steiner did not differentiate the respective position of M_1, M_2, nor did he provide any figures). If all circles in the set $N_1, N_2, N_3, ...$ were tangent in the same way to the given circles, then the outer similitude point A_3 of M_1, M_2 would be the common point of equal power for the entire set of circles. If all such circles were tangent to the given circles in different ways, then I_3 would be the common point of equal power for them. Moreover, the line of equal power, $l(_{12})$, of M_1, M_2 would be the common similitude line for all $N_1, N_2, N_3, ...$ Steiner interrupted his exposition here, promising further developments of these theorems, but for want of time and space proceeded to the problem solving promised in his introduction. The remainder of the paper contained a solution to the Malfatti problem and further generalizations with their solutions.

We note that Steiner solved the original Malfatti problem without reference to any of his newly coined definitions. However, he then suggested a more general case of the Malfatti problem, namely: given any three coplanar circles to describe three new circles each tangent to the others and to two of the given circles (the sides of the triangle had been replaced with circles). To solve this version of the problem, Steiner used both similarity points and power circles. Figure 26 (our Figure 4.20), depicting his construction, shows the three outer similarity points A_1, A_2, A_3 along a line on the right hand side of the figure as well as faintly traced arcs denoting the power circles centred at these points. However, we will not go into further detail about these solutions here since they were not reiterated by Gergonne nor Plücker.

Steiner's power circles united the properties of similitude points and points of equal power. According to Gergonne's assessment, as we will see, Steiner's power circles were apparently new objects and it was this synthesis of planar relationships that enabled Steiner to reach a solution and generalization of the Malfatti problem and promised to lead toward

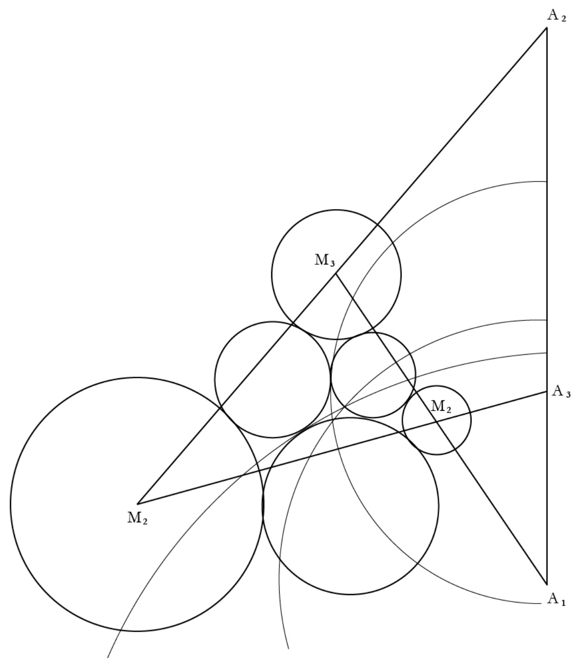

Figure 4.20: A simplified version of Steiner's Figure 26 (Steiner (1826a))

the Apollonius problem.

While breaking chronological order, in order to appreciate Gergonne's subsequent translation of Steiner, it is useful to remind the reader of Gergonne and Poncelet's earlier publications on the subject of the Apollonius problem. Even if Steiner derived his own method and solutions completely independently, when his work reached an audience of French geometers, it was rewritten and reviewed in light of these already published results.

4.2.2 Gergonne and the Apollonius problem (1810–1817)

Prior to 1826, Gergonne had published three separate articles on the Apollonius problem, with essentially the same solution but different degrees of explanation (Gergonne (1810b), Gergonne (1814b), Gergonne (1817a)). His most extended version appeared in the *Annales* in 1817. Gergonne justified his repetition by explaining he had received complaints about lacking a complete justification in his earlier version.

> J'écrivais pour des savans consommés, et je crus devoir être court il paraît que je le fus un peu trop; plusieurs géomètres, qui eurent connaissance de mon mémoire, me firent le reproche, fondé sans doute, que le fil qui m'avait guidé n'y était pas assez apparent, et que mes calculs semblaient plutôt propres à légitimer une construction trouvée par un heureux hasard, qu'à faire découvrir cette construction. II paraît même que, par suite de mon excessif laconisme,

> beaucoup de géomètres n'ont pu suivre mes méthodes et en saisir l'esprit; car on est revenu encore postérieurement sur ces deux problèmes, sur lesquels pourtant j'avais cru ne plus rien laisser à dire. (Gergonne (1817a), 289–290)[24]

Against these reproaches of merely having "legitimatized a graphic construction, discovered in advance," Gergonne would show that an analytic treatment led "naturally" and "absolutely inevitably" to his conclusions. The solution was "elegant," "simple" and "direct"–not "fashioned after the fact." Finally, the exposition would be entirely elementary, even though involving coordinate representation.

Gergonne began by laying out his strategy in detail, which we summarize here. He would successively reduce the problem to determining simpler and simpler geometric objects. The desired circle could be determined by its three tangent points. Finding three tangent points reduced to finding a single tangent point. Finding that point reduced to finding a second line containing that tangent point. Finding that second line reduced to finding any two points on that line, and so on.

Gergonne then chose a set of coordinate axes with the origin at the centre of one of the given circles c and axes through the centres of the remaining two circles c' and c''. With these coordinates, each given circle could be represented by an equation, and there were as many equations as variables. Gergonne introduced the equation of the given circle:

$$x^2 + y^2 = r''^2$$

He then calculated with the remaining equations, to find an additional equation, that of a second line containing the tangent point:

$$\frac{ax + by - r''(r'' - r)}{(a^2 + b^2 - (r'' - r)^2} = \frac{a'x + b'y - r''(r'' - r')}{a'^2 + b'^2 - (r'' - r')^2}$$

These two equations could be combined to solve for one of the tangent points at (x, y). Rather than solving this equation (which Gergonne admitted would be complicated by radicals), he worked backward following the steps of reduction outlined above, showing that analytically simpler lines and points could instead be constructed and still lead to a uniquely determined point. This analytic proof concluded with a planar construction.

In the construction, which we illustrate from Gergonne's "virtual figure," Gergonne differentiated two kinds of tangent lines shared by a pair of circles. The two interior tangents

[24] "I wrote for consummate scholars, and if I thought it right to be brief it appears that I made it a bit too much; several geometers, who knew of my memoir, reproached me without doubt because the thread which had guided me was not very apparent and that my calculations seemed rather to legitimize a construction found by happy accident, than to lead to discovering this construction. It appeared even that, due to my excessive brevity, many geometers had not been able to follow my methods and capture their spirit; because they return again to these two problems, about which I believed that nothing was left to say."

intersected between the two given circles, while the two exterior tangents intersected on one side of both given circles. We will see that Gergonne would later change his use of adjectives in describing these pairs of lines.

Given three coplanar circles c, c', c'', begin by drawing the common exterior tangents to the three circles considered pairwise and the chords of contact defined by the tangent points for each circle. The two chords of contact of circle c will meet at a point M, and their parallel chords on c' and c'' will meet at N. Similarly, the two chords of contact on c' will meet at M' with their parallel chords meeting at N', and the two chords of c'' meeting at M'' with parallels at N'' (Figure 4.21).

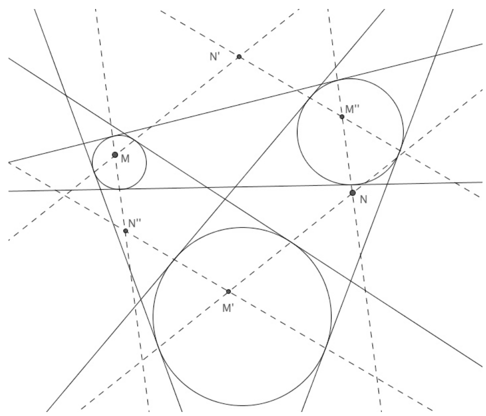

Figure 4.21: Construction of tangent lines and chords of contact

Then draw $MN, M'N', M''N''$. Line MN will meet circle c at points t and θ, $M'N'$ will meet c' at t' and θ', $M''N''$ will meet c'' at t'', θ''. Finally the two circles drawn through t, t', t'' and $\theta, \theta', \theta''$ would be the circles sought (Figure 4.22).

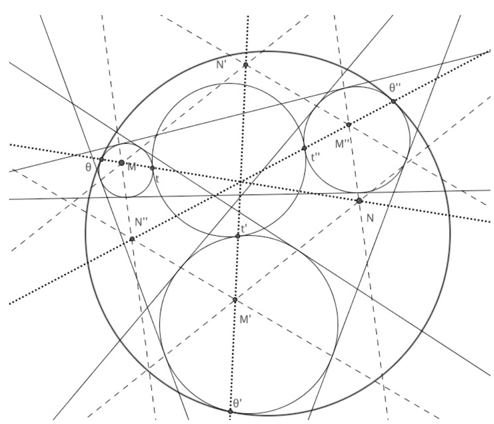

Figure 4.22: Construction of intersection points and tangent circles

The remaining six solutions could be found by substituting combinations of exterior and interior tangents. Gergonne concluded in suggesting that the equations could be modified to include equations of the radical axes of c, c'' and c', c'' with a radical centre O, and showing that the three constructed lines $MN, M'N', M''N''$ concurred at O. We will see shortly that Gergonne's radical axes were exactly the lines of equal power used by Steiner.

Gergonne described his solution as the simplest yet, presenting "une face entièrement nouvelle" for the application of analytic geometry in solving planar problems. Further, the construction could be applied "exactement de la même manière" to the case where the circles lay on the surface of a sphere.

4.2.3 Un rapprochement curieux: Poncelet solves the Apollonius problem (1811, 1821)

Poncelet's first solution from 1811 attests to the ubiquity of the Apollonius problem. The problem had been posed by Jean-Nicholas-Pierre Hachette in the *Correspondance sur l'École polytechnique* the previous year as the sixth of eight problems on circles and spheres for students to solve. Hachette's introduction addressed contemporary use of the figure, as well as anticipated approaches to solving the problem through descriptive geometry.

> Supposant le lecteur habitué à lire dans l'espace, j'ai cru pouvoir supprimer les figures; et suppléer par des caractères aux dessins géométraux: ainsi pour désigner des plans, des sphères ou des points remarquables, je me suis servi de lettres disposées de manière à indiquer leurs positions respectives; cette notation a l'avantage de rendre les explications plus concises et de présenter les mêmes objets plus souvent, en évitant les répétitions de mots. (Hachette (1804), 17)[25]

Hachette provided a solution to the problem by considering the given circles as great circles of three spheres and posing the three dimensional question of finding the sphere tangent to three given spheres whose centre is coplanar with the three given centres. He also cited analytic solutions from Newton in the *Arithmétique universelle* and from Euler and Fuss in the *Mémoires de l'Académie de Pétersbourg*.[26]

Poncelet's solution also appeared in the *Correspondance* and marked his first publication in geometry (Poncelet (1811)). He relied upon results proved in the Hachette article, but his constructions are limited to the plane, and illustrated by two labelled figures that appeared printed at the end of the volume.

[25] "Supposing the reader habituated to reading in space, I felt I could suppress the figures; and supply geometric designs by characters: thus to designate planes, spheres, or remarkable points, I used letters arranged in a manner to indicate their respective positions; this notation has the advantage of rendering the explanations more concise and of presenting the same objects more often by avoiding the repetition of words."

[26] Helena Pycior details the publication of Newton's *Universal Arithmetic*, where the Apollonius problem featured as one of sixty-one geometric examples, as well as Newton's inconsistent stance toward the application of algebra to geometry in Pycior (1997) (167–208).

Poncelet would refer to this solution fifty years later in his *Applications d'analyse et de géométrie*, where he presented several "synthetic" solutions to the Apollonius problem accompanied by in-text figures. The last solution he described in a footnote as analogous to that from 1811 (Poncelet (1864), 38). Even in 1862, Poncelet refrained from the modern terminology of circle relations that he employed in 1821. We recall from the introduction to Chapter II that for Poncelet, "synthetic" denoted a more "restrained" method of geometry than his "modern pure geometry" (Poncelet (1822), xxi).

By contrast, Poncelet's 1821 article advertised an application of Poncelet's new and "controversial" principle of continuity.[27] The article appeared by request of Gergonne. The year before, Poncelet had submitted a memoir on the projective properties of conic sections to the *Académie royale des sciences.* As we observed in Chapter III, this article was reviewed by Arago, Poisson, and Cauchy, the last of whom wrote up a report subsequently published in Gergonne's *Annales.* While criticizing some of Poncelet's methods, Cauchy had complimented Poncelet's "very elegant" solution of the problem to draw a circle tangent to three others (Poncelet and Cauchy (1820), 82). Intrigued, Gergonne requested Poncelet's construction, and accordingly introduced Poncelet's response in a footnote.

> Les constructions dont il va être question sont celles qui ont été annoncées à la page 82 de ce volume. Nous avons pensé qu'elles pourraient offrir un rapprochement curieux avec celles de M. Durrande, insérées également dans le présent volume; et, à notre prière, l'auteur a bien voulu nous les communiquer. (Poncelet (1821b), 317)[28]

The constructions presented could be understood independently, but their proofs rested upon propositions and principles only fully explained in Poncelet's *Traité*, which would appear the following year.[29]

Poncelet began by defining *direct* or *inverse* homologous points as two points, one on each circumference of two coplanar circles, which are collinear with one of the circles' similitude centres and belong to two arcs whose curvature is directed in the same sense, *direct* or the contrary sense *inverse* with respect to this similitude centre. Consequently, two axes, chords, tangent lines, etc. of a pair of circles could be directly or inversely homologous according to the position of their endpoints or points of tangency. Poncelet noted that directly homologous chords and tangents would concur on the common chord

[27] As we saw in Chapter II, Poncelet used similitude centres extensively in his *Traité.* In this article, Poncelet employed, but did not define, similitude and radical objects, for which we can now rely upon Steiner's and Gergonne's expositions or that of Gaultier in Appendix F.

[28] "The constructions which are going to be in question are those that have been announced on page 82 of this volume. We thought that they could offer a curious link to those of M. Durrande, also inserted in the present volume; and, upon our request, the author has kindly communicated them to us."

[29] This delay between construction and proof explains Plücker's comments from 1827, in which it appears that he had read the *Annales* article, but not the *Traité.*

at infinity (a concept briefly explained in Cauchy's review), while inversely homologous lines would concur on the finite common chord between two circles, their *radical axis.* This relationship presented "a very simple means to construct this axis." Poncelet asserted that these properties could be "easily extended" to spherical circles, right cones sharing the same vertex, cylinders with parallel axes, spheres, coplanar curves, or curves of double curvature "soumises ou non à la loi de continuité" provided the set of objects shared a similitude centre.

By means of the radical axis and its relationship to inversely homologous points, Poncelet presented several different construction variations towards finding a circle tangent to three given circles. He began by designating the circles C, C', C'', with a footnote explaining that for practical exactitude it would be convenient to let C be the biggest of the three circles. Poncelet brushed aside the constructive details over which Steiner would labor five years later. The four similitude axes were assumed, and each one corresponded to a type of circle tangency. Whatever axis was chosen in advance would contain whichever three similitude centres were concerned.

Poncelet's first construction began with an arbitrary point M on the circle C. With the chosen similitude axis, Poncelet could determine the points M' inversely homologous to M on circle C', M' inversely homologous to M'' on circle C'' and N inversely homologous to M'' on C. In the case where M and N concurred at the same point, the three points of tangency for the desired circle would be M, M', M''. Otherwise, the line MN would meet the similitude axis at a point P. The polar of P with respect to the circle C would intersect C at its points of contact with each of the two solution circles. One could then either perform a similar construction for C' and C'', or find the respective inversely homologous chords to the one found on C, which would play the same role in finding the two pairs of tangency points. The same operation repeated for each of the four similitude axes would lead to all eight circle solutions. Alternatively, Poncelet summarized what he considered a simpler construction employing polars of the radical centre, with respect to each of the three circles. Poncelet advertised the advantages of all of these constructions in only employing a simple ruler, thus being independent of the compass, once one knew the similitude centres of the given circles.

In order to simultaneously and symmetrically find the three chords containing the tangent points, Poncelet continued finding inversely homologous points to N on C', and so on as above, until one obtained six points. In this construction, the seventh point would always coincide with M, and thus one could create a closed hexagon whose invariant properties would determine the three desired chords. Poncelet claimed this construction was very simple because one only traced straight lines, and did not need to construct the common chords or the radical axis. However, one could even avoid directly employing the similitude axis, instead choosing any three collinear similitude centres. Then by constructing an arbitrary chord of C, finding its homologous inverse on C', and so on, the sixth operation would return to the first chord. The intersections of these chords ultimately determined the chords of contact, as in the prior constructions.

Poncelet suggested that all the preceding constructions and associated propositions continued to hold "in an analogous manner" for three and four spheres, three cones sharing a common summit and three spherical circles. He expanded upon the spherical situation, explaining that the planar circles then followed as a special case.

Referring to his *Annales* article on inscribing and circumscribing polygons from 1817, Poncelet claimed that these propositions were analogous to those found in that paper and it would be "very easy to pass from one to the other by invoking the principle of continuity." Poncelet concluded by advertising his original memoir, where these relationships were more fully explained.

> C'est un rapprochement que je n'ai pas manqué de faire, dans le mémoire dont M. Cauchy a rendu compte à l'Institut. (Poncelet (1821b), 322)[30]

Indeed, Poncelet's *Traité* published in 1822 contained these solutions, which Poncelet favourably compared to those of M. Gaultier because they were "more general and only required the use of a ruler" (Poncelet (1822)).[31] Moreover, in his book Poncelet provided a figure for the homologous point construction as well as the representation of similitude centres and axes for three exterior coplanar circles (our Figures 4.23 and 4.24).

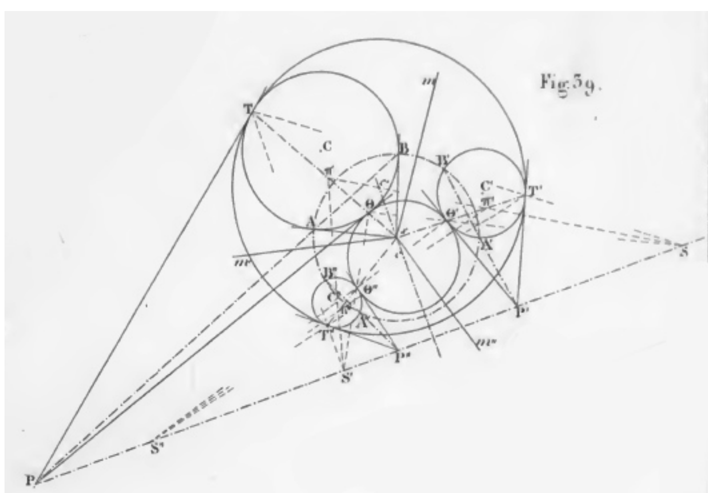

Figure 4.23: Poncelet's Apollonius construction Figure 39 (Poncelet (1822))

[30]"This is a relationship that I did not fail to make in the memoir that M. Cauchy has reported to the Institute."
[31]Gaultier used a ruler and a compass, as we see in Appendix F.

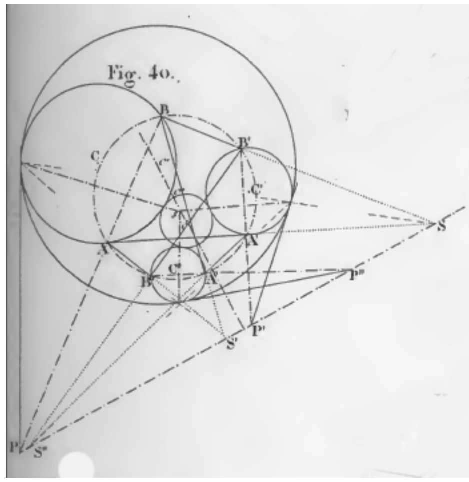

Figure 4.24: Poncelet's Apollonius construction Figure 40 (Poncelet (1822))

Along with the above solution, Poncelet showed how to reach a "very elegant solution" with the same basis as that of Gergonne's from 1814. Poncelet assessed Gergonne's presentation favourably,

> [...] la marche purement algébrique qu'a suivie ce géomètre est entièrement neuve, et parait susceptible de s'appliquer à un grand nombre de questions réputées difficiles dans l'état actuel de l'Analyse. (Poncelet (1822), 138)[32]

Poncelet's positive comments suggest that for him, providing another proof for Gergonne's solution was not intended as a critique of the original analytic approach. The Apollonius problem invited multiple solutions, and the elegance of one did not preclude the advantageous qualities of another. Poncelet's brief references to his law of continuity and lines at infinity signaled new geometric technology, aspects of which we will see featured in our next case study.

4.2.4 Gergonne translates Steiner (1827)

Gergonne's long-term interest in the Apollonius problem, illustrated by his own publications and interest in publishing Poncelet's solutions, may explain why Steiner's article was quickly translated and appeared in the *Annales* under the title "Géométrie Pure. Théorie générale des contacts et des intersections des cercles" in 1827 (Steiner and Gergonne (1827)). Gergonne prefaced his exposition of "Théorie générale des contacts et des intersections des cercles" by explaining the advantages and flaws of Steiner's approach. Steiner had presented the elementary theories of "similitude centres, axes and planes, radical axes, planes

[32] "[...] the purely algebraic path that this geometer has followed is entirely new, and appears susceptible to apply to a great number of questions reported difficult in the current state of Analysis."

and centres, and finally poles, polars, polar planes and conjugate polars" in order to resolve "difficult" and "general" problems with "very elegant" solutions. While the above theories had appeared often in the *Annales*, " l'auteur a tout à fois, étendues et simplifiées d'une manière assez notable" (Steiner and Gergonne (1827), 286).[33] Further, school teachers [*MM. les Professeurs de nos écoles publiques*] would find "abundant resources" for "profitable" student exercises within the text. Gergonne's version of Steiner's "doctrine" promised to be even better than the original German text.

> Nous pensons donc faire une chose très-utile pour le progrès de la géométrie pure, et conséquemment très-agréable à nos lecteurs, en offrant ici, dans un cadre resserré, les principaux points de la doctrine de M. Steiner ; mais sans toutefois le suivre servilement, et en nous permettant de nous écarter un peu de sa marche, toutes les fois que nous penserons qu'il en peut résulter quelque avantage, sous le rapport de la clarté ou de la brièveté, nous exposerons, en un mot, ces théories comme nous pensons qu'elles pourraient et devraient l'être dans les traités élémentaires, en nous rappelant toutefois que nous n'écrivons pas pour des commençans ; c'est-à-dire, en négligeant, pour abréger, des développemens faciles à suppléer pour tout lecteur intelligent. (Steiner and Gergonne (1827), 287)[34]

Here was Steiner clarified, abbreviated, and overall adapted to a mathematically literate French audience. We will see how Gergonne updated his vocabulary, added contemporary (French) references,[35] and set a standard for what could be properly assumed as well known for *Annales* readers. None of Steiner's eighteen figures were reproduced, and we will pay close attention to how Gergonne conveyed similar content with words alone. Perhaps these figures were also well known enough to the audience to be effectively omitted. Notably, this mode of exposition resulted in an abridged Steiner, which was in fact longer than the original German piece.

Structurally, "Théorie générale des cercles" was divided into eleven untitled parts labelled with roman numerals, each of which introduced a particular geometric object or relation. Like Steiner, Gergonne also employed numbered sections that were then used as reference points for individual results through the text, for example (3). Overall, Gergonne

[33] "[...] the author at once had extended and simplified in a very notable manner."

[34] "We thus suppose it will be a very useful thing for the progress of pure geometry, and consequently very agreeable to our readers, to offer here, in a concise framework, the principal points of M. Steiner's doctrine; however, without slavishly following him and permitting ourselves to deviate a bit from his path, whenever we think that it may result in some advantage with respect to clarity or brevity. We will present, in a word, these theories as we think they could and should be in elementary treatments, remembering, however, that we do not write for beginners; that is to say, in withholding, in order to abbreviate, developments which are easy for any intelligent reader to supply."

[35] Steiner's sole contemporary reference to Hirsch did not survive its French translation.

more than doubled the number of sections, using 74 all together, in part because of a larger volume of total content and in part because of shorter section divisions. We will focus our comparison on the common content to both texts, namely the developments of similitude centres (Steiner's similitude points), radical axes (Steiner's lines of equal power), and power circles. However, we will deviate from Steiner's original text, by also including Gergonne's solution to the problem of Apollonius, now independent from its initial analytic proof.

Through Gergonne's re-interpretation, we will elucidate how Steiner's figure-based, exploratory style and German terminology came to be allied with contemporary French geometry. Yet, unlike in the case of Plücker's Pluker article, Gergonne maintained Steiner's original figure-based method, even though his original proofs of the Apollonius problem relied on the use of coordinate equations. This demonstrates a fluidity of Gergonne's commitment to a single geometric method. Moreover, since Gergonne did not use illustrations of figures and applied results from "modern geometry", we will see the potential variation within one method. Without illustrated figures, we may consider how Gergonne communicated Steiner's results, and what was lost and gained in this visual transfer. Finally, to better understand the phenomenon of repetition in early nineteenth century geometry, we attend to Gergonne's arguments for republishing Steiner and which aspects of Steiner's text remained intact. Just as we saw with Plücker in Chapters II and III, Gergonne was the medium by which Steiner's work first became known to a broader French audience.

Gergonne began by describing two arbitrary unnamed coplanar similar polygons. If each pair of the corresponding polygon sides were parallel, then they would be considered homologous and the polygons were *similarly situated*. In particular, the polygons were *directly similar* if the homologous sides were in the same order and *inversely similar* if the homologous sides were in the inverse order. Gergonne credited Monge for having shown that lines connecting directly or inversely homologous vertices would concur at a point called respectively the *direct or inverse similitude centre*. Gergonne noted that here he found the adjectives *direct* and *inverse* preferable to *external* and *internal*, which had been employed previously. Gergonne argued that external and internal could be misleading since a so-called external point could lie inside both polygons and vice versa. As we saw above, in Gergonne's 1817 proof of the Apollonius problem he had used exterior and interior to describe types of tangent lines. Likewise, in Durrande's 1820 proof he used the designators exterior and interior to describe geometric relations. Here Gergonne modified both his and Steiner's vocabulary from exterior and interior to direct and inverse. This evidence suggests that Gergonne derived his choice of vocabulary from Poncelet's article, discussed above (Poncelet (1821b)). This change in vocabulary reflected a divorce from the illustrated circle relationship, and toward a more abstract representation. However, Gergonne continued to use exterior and interior to describe the position of circles.

Gergonne claimed it would be "easy to demonstrate" that any line containing a side of either similarly situated polygon would pass through the similitude centre and contain the other homologous side. Such a line was called a *similitude axis* of the two given polygons. For three similarly situated polygons, which Gergonne named P, P', P'', there could be up to four similitude axes: one direct and three inverse. The direct similitude axis contained the three direct similitude centres for the polygons considered pairwise, and the inverse similitude axes contained two inverse centres and one direct centre. In explaining the construction of centres and axes, Gergonne employed a virtual figure. Objects were described, labelled and manipulated, but not actually pictured. One achieves a sense of the visual language here in reading the first case of Gergonne's proof.

> 4. Soient trois polygones P, P', P'', directement semblables, tracés sur un même plan, et soient d, d', d'' respectivement les centres de similitude de P' et P'', de P'' et P, de P et P'. Si, par les deux points d' et d'', on conduit une droite D, cette droite sera (3) axe de similitude directe de P'' et P, aussi bien que de P et P'; elle sera donc aussi axe de similitude directe de P' et P'' et passera conséquemment (3) par le point d; de sorte que les trois points d, d', d'', seront sur la droite D. (288)[36]

Gergonne's work invoked a figure that the reader might either conceive or construct, and for ease of understanding we include one below in Figure 4.25.

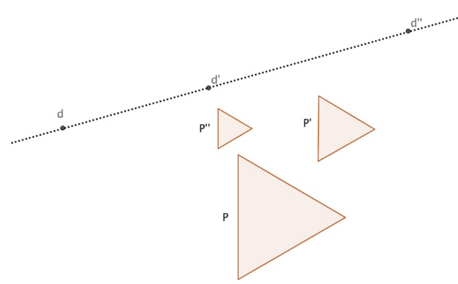

Figure 4.25: The direct similitude axis of three directly similar polygons

Gergonne noted that homologous, regular, evenly many sided polygons had direct and inverse similitude centres and direct and inverse similitude axes because each side of one had to be parallel to both a directly and an inversely homologous side of the other. Gergonne

[36] "Let there be three directly similar coplanar polygons P, P', P'', and let d, d', d'' be the respective similitude centres of P' and P'', of P'' and P, of P and P'. If, by the two points d' and d'', one draws a line D, this line will be (3) the direct similitude axis P'' and P, as well as of P et P'; thus it will also be that of P' et P'' and will consequently pass (3) through the point d; such that the three points d, d', d'', will be on the line D."

then introduced circles as regular, even sided polygons with infinitely many sides and focused on finding centres of similitude. In general, two exterior circles would share two pairs of tangent lines. When the circles were both on the same side of the tangent lines, these lines intersected at the direct similitude centre. When the circles were situated on different sides of the tangent lines, the lines intersected at the inverse similitude centre. Both of these points were collinear with the circles' centres. In the case where two circles were tangent to each other, they would be tangent at their similitude centre, a direct centre for interior circles and an inverse centre for exterior circles.

Next, Gergonne considered three exterior coplanar circles, which he designated as C, C', C''. Pairwise, these circles would have three direct d, d', d'' and three inverse i, i', i'' centres of similitude. All three direct centres of similitude would be collinear, belonging to D the direct similitude axis. Additionally, each direct similitude centre would be collinear with two inverse centres of similitude, belonging to lines I, I', I'', the three inverse axes of similitude. Gergonne's constructive steps again illustrated the use of unambiguous figurative language that the reader might employ to develop their own construction.

While Steiner had limited his study to fixed figures of given magnitude, Gergonne extended his research to circles of variable size and position. He employed visually oriented language in these descriptions. One could "see" that as one of the three given circles varied, the line joining the two variable inverse similitude centres and the line joining the two variable direct similitude centres would rotate on the direct similitude centre defined by the two fixed circles. Similarly, the two lines joining each pair of variable direct and inverse centres would rotate on the fixed inverse similitude centre. Gergonne concluded that this result provided a means of constructing the similitude centre when one could not draw common tangents to two circles. We note that this exception was a relic of figure based geometry, where common tangent lines were undefined for interior circles. The dynamic quality of Gergonne's result further shifted Steiner's material from its original static presentation.

Steiner had generalized from points to circles centred at these points, while Gergonne particularized from similar polygons, to similar regular even-sided polygons, to circles. Their different approaches emphasized different aspects of the concept of similitude. For Steiner similitude depended upon a fixed ratio. For Gergonne similitude followed from homologous sides. Though Steiner's approach seems more abstract and removed from construction, he was able to fully visualize his two primary objects: points and circles. Gergonne's choice to not use figures facilitated defining circles as infinitely sided polygons. As our figure of three equilateral triangles shows, any figure of a polygon would immediately particularize the concept, and many sided polygons could cause unnecessary visual complication for the relationship discussed. Gergonne was further able to take advantage of an independence from the figure by considering variable circles. This form of animation added another dimension to Gergonne's exposition, allowing him to consider multiple cases simultaneously, or extend a result from one configuration to another, not unlike Poncelet's principle of continuity.

The advantages to Gergonne's figure-free presentation were mitigated by the necessary profusion of names and labels corresponding to respective relationships in each possible combination of cases. Steiner's figures enabled an immediate transfer of information, allowing the reader to see concurrence, collinearity, tangent lines, etc. Though Gergonne employed verbs suggestive of visual cognizance, the reader could only see text and whatever figures he or she constructed following Gergonne's instructions. In the situation of a variable figure, the reader would need to imagine the variable objects discussed, perhaps with the aid of multiple figures. Gergonne classified this article as pure geometry, but animated figures exceeded the bounds of circle and straight-edge constructions.[37]

Having fully elaborated the constructions and properties of similitude, Gergonne turned his audience's attention to the relationship between a coplanar circle and point. He spoke confidently with reference to his readership's background knowledge: "On sait que si, par une point situé comme on le voudra sur le plan d'un cercle, on mène à ce cercle une sécante arbitraire ; le produit des distances de ce point aux deux intersections de la sécante avec la circonférence sera une quantité constante, indépendante de la direction de cette sécante" (ibid, 294)[38] Gergonne acknowledged that Steiner had designated this constant product "indistinctly" as the *power of a point with respect to a circle* or the *power of a circle with respect to a point.*

Following Steiner, Gergonne explained how to construct the power of a point that is exterior, on, or interior to a circle. However, as Gergonne continued to describe the relationship between two circles, his technical vocabulary and order of exposition aligned instead with the French geometer Louis Gaultier, who had first coined the adjective *radical* to describe objects in the fixed product ratio that Steiner called *equal power* (Gaultier

[37]Gaultier, whom Gergonne referenced directly, had employed variable figures, but only to the extent of describing when a circle became a line or reduced to a point. In all of these cases, he provided accompanying figures for various stages of the derivation (Gaultier (1813), 153). However, the use of animated figures was common in descriptive geometry, particularly with respect to developable curves, for instance in generating a curve of double curvature by a continuous movement of a line tangent to a given curve (Monge (1798), 40, 106). Carnot employed variable points to transform a primitive system. His first example concerns a triangle ABC, in which one raises a perpendicular from BC to A, called AD, that Carnot initially supposes falls between points B, C. Then to transform the system,

> [...] concevons que le point C se meuve vers le point B, jusqu'à ce qu'il ait passé le point D. La base $\overline{BC}$ est donc variable, ainsi que le segment $\overline{CD}$, tandis que l'autre segment $\overline{BD}$ est constant. (Carnot (1803), 20)
>
> "[...] imagine that the point C moves toward the point B, just until it passes the point D. The base $\overline{BC}$ thus varies along with the segment $\overline{CD}$, such that the other segment $\overline{BD}$ is constant."

We have seen how Carnot's systems were adapted by Poncelet in Chapter II (Section 2.5), and Gergonne may also have been drawing from these more recent publications (Poncelet (1822)).

[38]"One knows that for any secant drawn from a coplanar point through a circle, the product of the distance from the point to the two intersections of the secant with the circumference will be a constant quantity, independent of the secant's direction."

(1813)). With respect to the order of exposition, Gergonne began with circles, again skipping over Steiner's preliminary treatment of points. In particular, a second coplanar circle whose radius squared was the power of its centre with respect to the given circle was called the *radical circle*, and the given circle was declared *primitive*.

In the next section, Gergonne considered the locus of all points such that the difference of the squares of their distances to two fixed points would be constant. He asserted that one could demonstrate "by the elements," referring most likely to Euclid's *Elements*, that such a locus would be a straight line, perpendicular to the line containing the two points. So by definition, all points of equal power with respect to two circles would lie on a straight line perpendicular to the line containing their centres. Gergonne acknowledged Steiner's designation of the *line of equal power*, but decided "we will continue to call it their *radical axis*, after M. Gaultier." Gergonne elaborated four possible cases of two circle relationship–one interior with no common points, exterior with no common points, intersecting, or tangent (interior or exterior) at a point. In the latter two cases, the radical axis would be respectively the common chord or the common tangent–in which case the common power would be 0. For a pair of exterior circles, the radical axis would bisect their common tangent segments. Thus, a third circle perpendicular to both given circles would have its centre on their radical axis. With interior circles, a third circle in which the four smallest half chords to the other circles passed through its centre would have its centre on the radical axis.

With the concept of a radical axis, Gergonne examined many of the same consequences as had been detailed by Steiner with his line of equal power.[39] Likewise, both geometers examined equivalent properties in the concept of a radical centre. In Gergonne's notation, three non-concentric circles C, C', C'' would have three radical axes R, R', R'' when considered pairwise. If r was the point of intersection of axes R' and R'' then r would simultaneously have equal power with respect to circles C'' and C as well as circles C and C'. Thus point r would also lie on line R. So the three radical axes concurred at what Steiner had called the *point of equal power*, and Gergonne would call, again following Gaultier, the *radical centre* of three circles.

Gergonne returned to his earlier technique of a circle of variable size and position and determined that as one of the three circles varied, the radical centre would traverse the radical axis of the two fixed circles. He declared that this result provided a means to construct the radical axis of two non-intersecting circles with the aid of an arbitrary third auxiliary circle intersecting them both. Further, in all cases one could also construct the radical centre of three given circles. By again allowing one circle to move and grow in the plane, Gergonne was thus able to simultaneously consider multiple positional cases and generalize from an initial particular configuration.

[39] A series of circles orthogonal to these circles and centred on their radical axis had also been considered by Gaultier, in the section of his memoir titled "Suites radicales des cercles" Gaultier (1813).

Gergonne posited that three coplanar circles generally would only have one radical centre, except in the case where three or more circles shared two common points. In this exceptional case, all the points through the common chord could be considered as radical centres of all the circles. With a dash of suspense, Gergonne claimed there existed additional situations where circles could share an infinite number of radical centres, "Nous allons même voir que des cercles peuvent avoir une infinité de centres radicaux sans passer par les deux mêmes points" (ibid, 299).[40] As Steiner and Gaultier had observed, all circles $N, N', N'', \ldots$ orthogonal to two given circles M and M' would have radical centres lying on the radical axis of M and M'. Reciprocally, the common radical axis to all such $N, N', N'', \ldots$ would be the line joining the centres of M and M'. Gergonne offered the "familiar example" of the Ptolemaic or Mercator circles, the projection onto a plane of perpendicular spherical lines of meridians and parallels, that is, longitude and latitude. Within these or any sets of perpendicularly intersecting circles, any line of latitude could be considered as the radical axis of any two lines of longitude and reciprocally. Thus, any set of such $N, N', N'', \ldots$ would share an infinite number of collinear radical centres, but without necessarily sharing the same two points, as Gergonne had advertised.

In the following Part VIII, Gergonne veered from Steiner's text to formally discuss poles and polars. Gergonne's choice of content appears motivated toward reproving his construction for the Apollonius problem, now by purely geometric methods. As we saw above, Gergonne's original solution had been framed as evidence in favour of analytic geometry. The same construction, except for a change in point names, remained as simple and elegant in a purely geometric setting, thus in some ways undermining Gergonne's initial intent. Gergonne unabashedly explained his divergence from Steiner's text as an opportunity to showcase his preferred solution.

> Tout amour propre d'auteur à part, cette construction, que nous avons donnée pour la première fois il y a plus de douze ans (*Mémoires de Turin* pour 1814), nous paraît de beaucoup préférable à toutes celles qu'antérieurement et postérieurement on a données du même problème. (ibid, 310)[41]

Having asserted the priority and superiority of his solution, Gergonne returned to Steiner's concept of common power in the tenth part. He carefully elaborated a construction of the circle of common power. Gergonne exhibited a meticulous textual attention to point names and object relations, which Steiner had been able to achieve through the use of a figure.

[40] "We will even see that circles can have an infinity of radical centres without passing through the same two points."

[41] "All self love on the author's part aside, this construction, that we gave for the first time more than a dozen years ago (*Mémoires de Turin* in 1814), appears to us preferable to all those that have been given of the same problem before or afterwards."

Gergonne's virtual figure is illustrated in Figure 4.26. Consider the circles C and C', mutually tangent to a third circle O. The respective points of tangency t and t' would be collinear with the direct similitude point of C and C', d'', if the circles were tangent in the same way; or collinear with their inverse similitude point i'' if the circles were tangent in different ways. The common tangents drawn to circles C and O at t and to circles C' and O at t', would intersect at a point o, lying on the radical axis of C and C'. From this point as a centre, Gergonne instructed the reader to describe a fourth circle ω with radius $ot = ot'$. The constructed circle ω would intersect C and C' orthogonally at t and t'. Then from properties of circles centred on the radical axis, either the tangent segment or the smallest chord drawn to ω through point d'' or i'' would be constant regardless of the situation of the tangent circle O. Moreover, the square of the tangent or the smallest chord will be the constant product $d''t \cdot d''t'$ or $i''t \cdot i''t$.

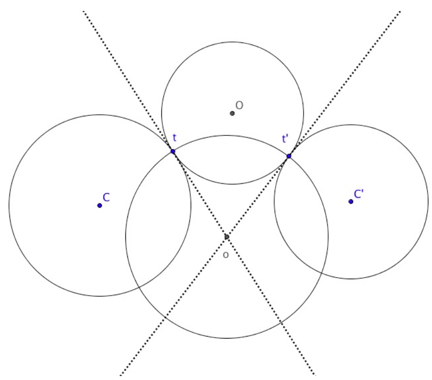

Figure 4.26: Construction of circle ω for C, C' tangent in the same way

Gergonne summarized the results of his construction in a theorem. If one draws an arbitrary common secant through one of the two similitude centres determined by two given circles, then the product of the distances from this centre to the respective intersections of the secant with the circles will be constant regardless of the secant's direction, as long as the corresponding radii are not parallel. This constant is the *common power* of the circles. A further circle centred at one of the centres of similitude of the two given circles, whose radius squared equal the common power of the two circles with respect to this similitude centre was defined as a *circle of common power*. Gergonne noted that Gaultier had "already considered" such circles, but had not given these circles a special designation (Gaultier (1813)). Circles of common power of two given circles C and C' would be orthogonal to all circles tangent to both C and C'.

In the eleventh and final section, Gergonne returned to three given coplanar circles, essentially the hypothesis of the Apollonius problem. Using Steiner's concept of common power, Gergonne derived a theorem, which he attributed to Gaultier. The centres of the eight circles tangent to three given circles are distributed pairwise on the perpendicular lines drawn from their four similitude axes to the radical centre of the three circles. Gergonne thus brought Steiner's research another step toward solving the Apollonius problem, but without elucidating the details of a full proof.

Gergonne's presentation of Steiner's article brought the idiosyncratic German article up to date with contemporary French mathematics. To begin with, the article was now in a language readable by French audiences, or at least those who read the *Annales*. Further, Gergonne had added several citations and modified Steiner's vocabulary to reflect current French usage. While much was added, not all of Steiner's original content survived translation. Most notably, Steiner's figures were not repeated, and with the change of notation his figures as published in Crelle's *Journal* would not have been very useful to a non-German reader. Further, the French article assumed a stronger knowledge base, and skipped over Steiner's foundational exposition. Instead, Gergonne spent more space describing how constructions could be found for all possible cases of circle relationships. Sometimes Gergonne treated the cases individually, and at other times he used a variable circle to derive a general statement. By orienting Steiner's content toward the Apollonius problem, Gergonne obscured its initial appearance as a set of tools applicable to a variety of problems. Most significantly, while Steiner had emphasized that the same properties determined for a circle would likewise hold for any planar conic section, Gergonne only mentioned this generalization in the introduction and hinted at it in the conclusion.

Gergonne's conclusion, designated as a post-script, commented upon another article of Steiner's (Steiner (1826b)) in which he had pointed out several exceptions to a theorem due to Monge. Gergonne defended Monge and dismissed the exceptions as too particular to affect the theorem in its entirety. He further assessed the importance of variable magnitudes in the work of Steiner himself.

> On doit être d'autant plus surpris que M. Steiner ne l'ait pas entendu dans ce sens que précisément un des principaux avantages de ses belles constructions, pour les problèmes de contact, est de se plier sans effort aux cas où tous ou partie des cercles ou des sphères donnés deviennent des points, des droites ou des plans. (314)[42]

Steiner had perhaps failed to recognize the crucial role of this effortless generalization in his own constructions. In the work of Steiner, there was no stated mechanism for generalizing from a sphere to a plane or from a circle to a general conic section. The process was simply assumed as possible. Gergonne continued by comparing ancient and modern geometry, to the advantages of the latter. For example, ancient geometers had not had the "freedom" to consider a straight line as part of the circumference of a circle of infinite radius.

[42]"We are all the more surprised that M. Steiner has not understood it in this sense, as precisely one of the principle advantages of his beautiful constructions, for problems of contact, is to effortlessly conform to cases where all or some of the circles or spheres given become points, lines, or planes."

> [...] mais n'est-ce pas précisément à cette manière plus large d'envisager l'étendue géométrique que les modernes sont en partie redevables de leur supériorité dans la géométrie pure? Supériorité que les mêmes géomètres pourront bien aussi leur contester; mais qui n'en demeurera pas moins un fait patent pour qui ne voudra pas se refuser à l'évidence.[43]

Gergonne took full advantage of the dynamic capabilities of these modern concepts. Thus, Steiner's article, as interpreted by Gergonne, functioned as evidence in favour of the superiority of modern pure geometry, just as Gergonne's earlier Apollonian proofs had served as representatives of the superiority of analytic geometry.

4.2.5 La publicité que ce géomètre a donnée à son travail: Steiner reviewed in the *Bulletin* and attempts at funding in Berlin

Soon after Gergonne's interpretation of Steiner's paper appeared it was reviewed and summarized in the *Bulletin de Férussac*. Here too, the solution to the problem of Malfatti was advertised as finally receiving a satisfactory purely geometric solution, and the content extended beyond Gergonne's translation by actually providing a summary solution. In describing Steiner's exposition, the circle of common power was emphasized as a fruitful innovation in problem solving.

> Les moyens de solution de M. Steiner se tirent uniquement de la théorie connue des centres et axes de similitude, de celle des axes et centres radicaux, de celle des pôles et polaires, et enfin de celle de ce que l'auteur appelle *cercles de commune puissance* de deux cercles donnés, ce qui montre toute la fécondité et toute l'importance de ces diverses théories, et justifie l'attention particulière qui leur a été donnée par plusieurs géomètres français depuis déjà plusieurs années. (Anonymous (1827b), 277)[44]

The *Bulletin* further pointed out how Gergonne had modified Steiner's work to include his solution of the Apollonius problem, which Gergonne "persiste à croire bien préférable à toutes celles qui ont été données antérieurement et postérieurement du même problème."[45] The review continued by explaining Steiner's circle of common power, in order to provide

[43] "[...] but is it not precisely in this broader way of envisaging geometric magnitudes that the moderns are in part owed their superiority in pure geometry? Superiority that the same geometers could also contest; but is no less a patent fact for those who will not deny the obvious."

[44] "Steiner's means of solution rest uniquely on the known theory of similitude axes and centres, of that of radical axes and centres, of that of poles and polars, and finally of that which the author calls *circles of common power* of two given circles, which shows all the fruitfulness and all the importance of these various theories and justifies the particular attention that several French geometers have given them already for several years."

[45] "[...] continues to believe very preferable to all those which have been given before and after of the same problem."

"une idée des procédés de M. Steiner." In this summary Gergonne's vocabulary and notation were employed, and the reviewer showed how the circle of common power could be used to sketch a solution to the Malfatti problem. More so than in the text of Gergonne, a background knowledge in geometric constructions (such as similitude centres, constant ratios, and tangent circles) would be required to follow the review. No intermediate constructions were presented and no figures were employed.

The reviewer concluded by considering some philosophical issues concerning the relationship between analysis and geometry. In particular, he noted the simplification of both methods effected by the development of new concepts and terminology.

> Les analystes s'étant aperçu que certaines fonctions assez compliquées se reproduisaient fréquemment dans leurs calculs, les ont appelées exponentiels, logarithmes, sinus, tangentes, dérivées factorielles, etc.; ils ont créé des signes abréviatifs pour les désigner, et leurs formules en ont acquis beaucoup de clarté et de concision. Puis donc qu'il est certains points, certaines droites et certains cercles dont la considération se représente fréquemment dans les spéculations de géométrie, il est naturel d'en user de même à leur égard, et de les appeler, suivant leurs propriétés, centres de similitude, centres radicaux, polaires, axes de similitude, axes radicaux, cercles de commune puissance, etc. Cette attention doit introduire inévitablement des simplifications analogues dans l'énoncé des théorèmes et dans la solution des problèmes qui appartiennent à la science de l'étendue. (279)[46]

Such a comment appeared to reinforce the importance given to Steiner's development of new vocabulary, such as circles of common power. Although the review had begun by praising Steiner's problem solving, the emphasis ultimately returned to his new tools and their potential for future use. In this respect, the review presented a much more accurate summary of Steiner's original mode of presentation than had Gergonne's "translation."

Meanwhile, Steiner was working as a private teacher in Berlin, and solicited his well-placed acquaintances, Karl Friedrich von Klöden (1786–1856), an educator and the director of the Berlin Gewerbeschule where Steiner then worked, and Friedrich Bessel (1784–1846), a mathematician and astronomer, to write in support of his application for funds from the ministry. Their letters along with Steiner's applications and the ministry's response were

[46]"Analysts perceiving that certain quite complicated functions are reproduced frequently in their calculations, have called them exponentials, logarithms, sines, tangents, factorial derivatives, etc.; they have created abbreviated signs to designate them, and their formulas have acquired greater clarity and conciseness. And thus for certain points, certain lines and certain circles whose consideration is frequently represented in geometric speculations, it is natural to do the same with respect to them, and to call them, following their properties, similitude centres, radical centres, polars, similitude axes, radical axes, circles of common power, etc. This attention must inevitably introduce analogous simplifications in the statement of theorems and in the solution of problems which belong to the science of magnitude."

published in Julius Lange's 1899 biography (Lange (1899), 18–26). Here Steiner emphasized his methodological choices and declared his dedication to synthesis. But while labeled as "synthesis," Steiner's contemporaries recognized his method as personal and unique. Klöden began his review by pointing out the novelty as a positive attribute.

> Dabei ist die Methode durchaus eigentümlich und bisher für diese Aufgaben nicht versucht. (Lange (1899), 18)[47]

Similarly, Bessel described Steiner as "ein erfindungsreicher und origineller Kopf," even in comparison to the work of Poncelet (21).[48] In particular, both Klöden and Bessel characterized recent geometry as almost exclusively confined to analytic approaches. On the one hand, they praised the speed of analytic methods that could address problems "completely inaccessible" [*völlig unzugänglich*] to synthetic methods. On the other hand, synthetic geometry had pedagogical benefits. Steiner's publications revealed the "formative power of geometry" [*bildende Kraft der Geometrie*] and formed more "coherent" and "complete" educational material than the "disjointed problems" of analytical geometry. Like Gergonne, Cournot, and anonymous *Bulletin* reviewers, Klöden and Bessel noted a reemergence of pure or synthetic geometry, now capable of favourable comparison against analysis. Though Steiner was presented as contributing to this reemergence, his method was set apart from past examples of synthetic geometry in the work of Gregory Saint-Vincent, Christian Huyghens, or Newton (21).

Steiner framed his dedication to synthetic geometry as illustrative as his broader philosophical search for systematicity [*Systematiztät*], organic unity [*organischen Einheit*], and intuition [*Anschauung*] based on his early education under Johann Heinrich Pestalozzi (1746–1827). Steiner supported a unified geometric approach, and so dismissed many contemporary results in mathematics, regardless of method, as invented haphazardly.

Steiner provided a narrative in which he studied and rejected combinatorial analysis and differential calculus. Even his first encounters with geometry textbooks had revealed an arbitrary or even empirical approach, as if the individual theorems were the aim of science and the general organic unity remained obscured. Steiner likewise distinguished his form of synthetic geometry from that of the ancients, as more general and complete, but still employing a rigorously genetic path [*streng genetischen Gang*] building up from simple to more complex concepts. In these qualities of generality and completeness, Steiner observed the connection and possible contributions between his method and analytic geometry.

> Dem analytischen Geometer endlich dürfte die Arbeit eine reiche Ausbeute, wohl gar eine Erweiterung seiner Methode gewähren. (ibid, 21)[49]

[47] "The method is quite particular and has not previously been tried for these problems."
[48] "[...] an inventive and original thinker [...]"
[49] The work finally will enable a rich profit for the analytic geometer, as well as an expansion of his method.

While his reviewers and recommenders saw Steiner's work as synthetic geometry in competition with analytic geometry, Steiner viewed synthetic geometry as a much greater and autonomous achievement, whose systematicity, unity, and intuition contrasted against all of mathematics.

> Demzufolge wandelte sich der Begriff der Systematizät, wie er aus den vorhandenen geometrischen Lehrbüchern zu entnehmen ist, ganz und gar um; den Zusammenhang der Anschauungen strebte ich aus der Einheit der Constructionsvermögens selbst an; ohne es mir besonders bewusst geworden zu sein, strebte ich nach der eigentlichen Genesis, die der synthetischen Methode, Anschauungen miteinander zu verknüpfen, zu Grunde liegt, und auf welcher alle geometrische Erfindung beruht. (ibid, 23)[50]

In this respect, Steiner separated his work as more universal and grounded in higher principles, compared to even contemporary synthetic geometry as practised at the École polytechnique.

Steiner presented his work as above all systematic and intuitive, qualities that he seemed to recognize were not immediately apparent in his initial publications. He proposed that his recent articles in Crelle's *Journal* constituted samples [*Proben der Technik*] of his manner of proceeding [*Verfahrensweise*]. His request for funding was motivated in part by wishing to publish a larger text, already written, that fully revealed his geometric approach based on a few theorems of elementary geometry. We saw similar language in Steiner's introduction to "Einige geometrische Betrachtungen," where he suggested a visible connection between seemingly disparate problems.

However, the ministry replied, there was no money available for such enterprises. They suggested Steiner seek employment at a Gymnasium. Steiner would only obtain a position at the University of Berlin in 1832 with the support of Jacobi, and Wilhelm and Alexander von Humboldt.[51] As we will see, Steiner's struggle for full employment was not unique, as the lack of available positions equally affected the analytic geometer, Plücker.

[50] "Accordingly, the concept of systematicity, as it is to be taken from the existing geometry textbooks, is entirely transformed; I strove for the connection of the intuitions from the unity of the means of construction itself; without it being especially known to me, I strove for the actual genesis that lies at the foundation of the synthetic method, seeking to link intuitions with each other. On [this genesis] rests all geometric invention."

[51] On the recruitment of Steiner to the University of Berlin, and his links to the Humboldt brothers, see Kurt Biermann's *Die Mathematik und ihre Dozenten an der Berlin Universität* (Biermann (1973), 38–41). On Alexander von Humboldt's scientific influence through his direct involvement with German mathematicians on the subject of number theory see Pieper (2007).

4.2.6 Un exemple fort remarquable de l'application de cette dernière méthode: Plücker's proof of the Apollonius problem (1827)

Plücker was well aware of the French version of Steiner's article and the associated publicity when he composed his introduction to "Géométrie Analytique. Mémoire sur les contacts et sur les intersections des cercles" (Plücker (1827)). Although Plücker followed the same narrative as Steiner, from similarity to common power, we will see that Plücker's use of coordinate equations, classified by Gergonne as *géométrie analytique*, rendered his objects very different. Further, Plücker had begun publishing and corresponding with contemporary French mathematicians in 1826. Consequently, even just one year later, he was far more in tune with contemporary research. In particular, he appeared comfortable using ideal radical axes–a descriptor only recently introduced by Poncelet in 1820 and just beginning to be acknowledged by other geometers.[52] In fact, Poncelet's influence in this text is evidenced by much of Plücker's choice of vocabulary including ideals, conjugates, and inverse or direct relations as well as one of his constructions. However, in introducing his work Plücker explicitly compared it to Steiner's and strongly suggested that his own presentation was simpler and more rapid.

> [...] mais la publicité que ce géomètre a donnée à son travail ne nous a pas paru un motif suffisant pour renoncer à publier un sommaire du nôtre, qu'on sera peut être bien aise [sic] de lui comparer, et dont on trouvera peut-être même la marche plus rapide et plus simple à quelques égards. (Plücker (1827), 29)[53]

Plücker was also familiar with Gergonne's several solutions of the Apollonius problem in 1827, which he summarized in his *Analytisch-geometrische Entwicklungen* the following year as an analytic determination of the tangency points to the desired circle lying on each of the given circles (Plücker (1828b), 102–105). In his book, Plücker credited Gergonne with simplifying the problem by finding tangency points rather than centres of the desired circles. The solution and proof Plücker provided in his *Entwicklungen* followed Gergonne much more directly than his solution published in the *Annales*. The former version was a very short exposition relying upon well chosen coordinate axes, without any "modern" geometric terminology. In the article we consider here, Plücker would briefly cite Gergonne's solution, but his method differed substantially through the adoption of Poncelet's new geometric objects.[54]

[52]Following Poncelet's introduction, ideal secants, chords, and points of intersection began to appear in the *Annales* in 1826 (including texts by Charles Sturm (Sturm (1826a), Sturm (1826b)), Étienne Bobillier (Bobillier (1827)), and Michel Chasles (Chasles (1828a))).

[53]"[...] but the publicity that this geometer has given to his work does not appear to us as a sufficient motive for renouncing publishing a summary of ours, so that one will be able to better compare them, and then one will perhaps even find our path more rapid and more simple in several respects."

[54]Although Plücker's name remained misspelled as "Pluker" in the title page of this article, we are fairly certain that the article was written by Plücker himself due to numerous footnotes either signed by "JDG"

Plücker organized his article into enumerated sections, each one roughly corresponding to a new construction or proposition that he could then refer to parenthetically. Overall, his pace was more advanced than that of Steiner or Gergonne. His definitions were succinct and clearly motivated toward the problem solving at hand. Plücker began directly with three circles, which he designated as $c = 0, c' = 0, c'' = 0$. Considered pairwise the real or ideal common chord, "that is, *radical axis*," for these circles would have the equations $c' - c'' = 0$, $c'' - c = 0, c - c' = 0$. Since these radical axes each intersected two by two at the same x and y coordinates, they all three would concur in a single point, the *radical centre*.

In a footnote, Plücker attributed this "turn of reasoning" [*tour de raisonnement*], where common points between two coordinate equations were represented through subtracting linear equations, to Gergonne. This use of coordinates would come to be known as "abridged notation".[55] In an extended proof using conventional coordinate equations (occupying three pages with one footnote), Plücker proved that the radical axis to a point on the circle's circumference would be a tangent line through this point.

Plücker asserted that the concurrence of the common chords (Plücker used this designator here, rather than radical axes) could be "easily" applied toward finding a circle passing through two points and tangent to a given circle. He reasoned that the problem would in general have two solutions, and the two desired circles would share a radical centre with the given circle. The radical centre would lie on the common chord of the two solution circles–that is, the line containing the two given points. The common tangents to the given circle and each of the two solution circles would all intersect at the radical centre. Should one of the solution circles be replaced with any other circle passing through the two given points, the radical centre would remain invariant. Thus Plücker constructively determined how to find the unique radical centre of the desired circles by intersecting the line containing the given points with the common chord shared by the given circle and an arbitrary circle containing the two given points. With the use of an arbitrary fixed circle, Plücker achieved the same result as Gergonne had with a variable circle.

Having found the radical centre, tangents from this point to the given circle would determine the tangent points shared with the desired solution circles. Finally, extended radii drawn from these tangent points would intersect the perpendicular raised from the midpoint of the two given points at the centres of the desired circles. Plücker's exposition was wholly descriptive, without any use of letters to name objects.

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or by "Plucker"–a more correct spelling and indicative that Plücker knew of the contents prior to final publication. In the Gergonne-edited Plücker (1826b), the lack of any JDG signed footnotes is conspicuous!

[55]Carl Boyer traced the invention of this "abridged notation" back to Lamé's 1818 text (Boyer (1956), Lamé (1818)). Plücker was among the early adopters, along with Étienne Bobillier (1798–1840), who had employed abridged notation in the article immediately preceding Plücker's. Plücker would frequently employ abridged notation in his *Analytisch-geometrische Entwicklungen* (Plücker (1828a)).

Plücker then considered the situation where the given circle degenerated into a straight line, in which case his initial construction "se trouve en défaut." Returning from abridged notation to coordinate analysis, the second degree equations of any two given circles were

$$(x-a)^2+(y-b)^2=r^2, (x-a')^2+(y-b')^2=r'^2.$$

By taking the difference of these equations and choosing the origin as the radical centre, Plücker derived the equation $r^2-(a^2+b^2)=r'^2-(a'^2+b'^2)$ as that of the radical axis. He concluded that from any point on the radical axis the tangent segments drawn to the two circles would be of equal length. Since the given line would be a common tangent for both solution circles, this line would meet the radical axis midway between the two tangent points. As in the previous construction, the radical axis would not change when any other circle passing through the given points replaced one of the solution circles. Further, the tangent segments drawn from the radical axis to the two circles would still be of equal length. Plücker proceeded to elaborate the geometric construction. Notably, he persisted without labels for objects, thus eliminating the profusion of letters found in Steiner and Gergonne.

> Soit décrit arbitrairement un cercle qui passe par les deux points donnés ; par l'intersection de la droite qui joint ces deux points avec la droite donnée soit menée une tangente à ce cercle ; soit portée la longueur de cette tangente de part et d'autre du même point sur la droite donnée ; on déterminera ainsi ses points de contact avec les cercles cherchés ; alors les perpendiculaires élevées à cette droite par ces deux points couperont la perpendiculaire sur le milieu de la droite qui joint les deux points donnés aux centres de ces mêmes cercles. (Plücker (1827), 33)[56]

Following this introductory construction, Plücker applied these same tools toward finding a circle tangent to three given circles, his solution to the Apollonius problem. In this context, Plücker considered circles of variable size, which he described precisely as represented by coordinate equations with variable coefficients. He began by presenting equations of the three given circles, which he would afterward describe as "primitive."

$$(x-a)^2+(y-b)^2=r^2$$

[56] "Arbitrarily describe a circle through the two given points; from the intersection of the line joining these two points with the given line, draw a tangent to the circle; carry the length of this tangent segment from the same point on the given line; one thus determines the points of contact of the line; then perpendiculars raised to this line through the tangent points will cut the perpendicular on the midpoint of the line joining the two given points at the centres of these circles."

$$(x - a')^2 + (y - b')^2 = r'^2$$

$$(x - a'')^2 + (y - b'')^2 = r''^2.$$

Then, three new circles, respectively concentric to the three given circles, with their radii augmented by a single fixed quantity R would have the following equations:

$$(x-a)^2+(y-b)^2 = (r+R)^2; (x-a')^2+(y-b')^2 = (r'+R)^2; (x-a'')^2+(y-b'')^2 = (r''+R)^2.$$

Plücker chose a coordinate system such that the origin was the radical centre of the three primitive circles. He then calculated with several lines of subtraction, multiplication and addition that when all the radii of three coplanar circles were augmented or diminished by the same quantity R, their radical centre would describe a straight line as R varied. This line passed through the origin and had the linear equation,

$$\{r(a' - a'') + r'(a'' - a) + r''(a - a')\}x + \{r(b - b'') + r'(b'' - b) + r''(b - b')\}y = 0. \quad (4.2)$$

Uncharacteristically Plücker did not enumerate his equations in this article. However, we designate this line as (4.2) as it would reappear frequently in different guises throughout Plücker's exposition.

In his fourth section, Plücker continued to consider the same three given circles. He introduced similitude centres, stating the "known" equations of the direct similitude centres for the three given circles with the same equations as above considered pairwise:

$$\begin{cases} x = \frac{r'a''-r''a'}{r'-r''}, \\ y = \frac{r'b''-r''b'}{r'-r''}, \end{cases} \quad \begin{cases} x = \frac{r''a''-ra'''}{r''-r}, \\ y = \frac{r''b-rb''}{r''-r}, \end{cases} \quad \begin{cases} x = \frac{ra'-r'a}{r-r'}, \\ y = \frac{rb'-r'b}{r-r'}, \end{cases}$$

These points were collinear, lying on the direct *similitude axis* of the three circles of which one could "easily" find the equation:

$$r(b' - b'') + r'(b'' - b) + r''(b - b')x = r(a'' - a) + r'(a'' - a) + r''(a - a')y.$$

Considering this equation with respect to the equation of the line traced by the radical centre found in the previous section (4.2), Plücker concluded the lines were perpendicular, and announced this as a *new* theorem that enabled an alternative construction of the line containing the variable radical centre.

Plücker began his fifth section by constructively describing three new circles derived from his primitive circles. Each pair of the three primitive circles defined a direct similitude centre. Each of the three new circles would be centred on one of these direct similitude centres and share a radical axis with the respective circle pair. Then the three new circles would have the same radical centre as the given circles. Moreover, direct similitude centres were collinear and the circles shared a unique radical axis through the radical centre and perpendicular to the line containing their similitude centres. This new radical axis was "none other than" the line traced by the variable radical centre (4.2). Thus Plücker provided yet another way to find what he now called *that line*. Gergonne interjected with a footnote, these new circles were "what we have called, following M. Steiner, *circles of common power* of three primitive circles, taken two by two." Plücker gave no special designation nor equations for these new circles in this context.

Plücker presented an alternative construction of circles concentric to the three primitive circles. Instead of simultaneously augmenting their radii, the three circles could alternatively augment or diminish, but still by the common variable quantity R. Then their centres of similitude would vary along a line perpendicular to the inverse similitude axis of the three given circles. Plücker observed that he could replace circles with lines or points in this context without altering the results. This comment connected to Gergonne's concluding remarks to his review of Steiner on the advantage of generality in modern geometry (Steiner and Gergonne (1827)).

Plücker declared that applying the properties of these variably concentric circles toward finding the eight solutions of the Apollonius problem was "visible." He limited the problem, "pour fixer les idées," to the situation where the circles sought were tangent to the three given circles in the same way: either all exterior [*les touche tous trois extérieurement*], all enveloping [*les enveloppe tous trois*], or all enveloped [*enveloppé*]. By varying the hypotheses with respect to the "la nature des contacts," Plücker arrived at the general theorem that *the centres of eight circles that are simultaneously tangent to three given circles are distributed, two by two, on the perpendiculars raised to the radical centre of these three circles from their four similitude axes.* He suggested that by "following a path analogous to that of Viète" [*en suivant une marche analogue à celle de Viète*] one could deduce the solution of the Apollonius problem from this theorem. However, his own analysis would be "much more elegant and brief" [*beaucoup plus élégantes et plus brièves*] (Plücker (1827), 37). Indeed, Viète's solution, which was referenced also by Gergonne, contained numerous cases.

Working toward this promised solution, Plücker transitioned to more constructive language and considered the equations of two named circles $(c), (c')$ respectively,

$$(x-a)^2 + y^2 = r^2, (x-a')^2 + y^2 = r'^2.$$ [57]

[57] The second equation is misprinted in the *Annales* as $(x-a')^2 + y'^2 = r'^2$.

Choosing the y-axis as their radical axis, then $a - r^2 = a'^2 - r'^2$. Using this relation, Plücker combined these equations to demonstrate that circles (C), (C') both tangent to circles $(c), (c')$ would share a similitude centre lying on the y-axis, that is, the radical axis of $(c), (c')$. If the two circles were tangent in the same way, then their direct similitude centre would be on the radical axis, otherwise their inverse similitude centre would be. Plücker declared that this relation was "evidently" reciprocal between the circle pairs.

He then introduced a third circle (C''), also tangent to circles (c) and (c'), and deduced from the reciprocal relationship that the radical centre of the three circles would coincide with one of the two centres of similitude of (c) and (c'). The similitude centre was direct or inverse depending on whether the mode of tangency between the circles was the same or different. Plücker concluded with a generalization of these results, "one sees that even if the circles (C), (C'), (C''), ... were very numerous, they would all have their radical centre at the same unique point" (39).

Plücker continued to investigate this unique point in his seventh section, constructing a circle centred around this point with radii equal to a tangent segment drawn to any one of circles (C), (C'), (C''), ... This new circle would intersect (C), (C'), (C''), ... orthogonally, and so Plücker called it their *orthogonal circle*. Gergonne again footnoted this statement, reminding the reader that this circle was Steiner's circle of common power. However, Gergonne's attributions obscure the fact that Plücker designated these circles as orthogonal with respect to circles (C), (C'), (C''), ..., while they are circles of common power, in Steiner's definition of the term, with respect to circles $(c), (c')$. The two geometers emphasized different constructions in defining these circles, which are reflected in their names.

Plücker then examined the relationship between the orthogonal circle and the circles $(c), (c')$. If $(c), (c')$ intersected, one could consider their points of intersection as circles with no radii, to which circles (c) and (c') would both be tangent. Thus the intersection points would be among the circles (C), (C'), (C''), ..., and consequently the orthogonal circle would pass through them, providing an easy construction of the orthogonal circle in this case. In all other cases, the orthogonal circle would share a radical axis with circles $(c), (c')$. This constructive property subtly confirmed that these were the same circles Plücker had first described in Section 5 with respect to $(c), (c')$.

Combining his earlier results, Plücker concluded that any pair of the three circles (C), (C'), (C''), as specified in the previous sections, would have one similitude centre on the radical axis of $(c), (c')$. So three of their six similitude centres will lie on this radical axis. Plücker attributed the general result to Monge, that the four similitude axes of three circles would also be the radical axes of the four pairs of circles at once tangent to these three.[58]

[58]Monge explored properties of curves using cones and their vertical and horizontal projections in his descriptive geometry, however these terms were defined by their three dimensional corresponding objects (see Monge (1798), in particular, pages 96–98). In his descriptions, Monge used neither the term "similitude"

Finally, in Section 9, Plücker began describing "diverses modes" of construction for solving the Apollonius problem. The first solution was presented as original, and derived from his most recent findings, as referenced in parentheses to section numbers. Plücker employed the orthogonal circle (circle of common power) as the primary tool.

> Soient construits le cercle qui coupe orthogonalement les trois cercles donnés, ainsi que leurs quatre axes de similitude. Si ces axes coupent le cercle orthogonal, on achèvera (8) la construction, en décrivant (1) des cercles passant par chaque couple de points d'intersection et touchant en même temps l'un quelconque des cercles donnés. (ibid, 40)[59]

The other two constructions required further theoretical development. Plücker began a new section and constructively described the pole and polar relationship.[60] Let p and p' be the points where circles $(c),(c')$ were respectively tangent to one of the three given circles designated as (C). Then the line pp' would pass through the similitude centre of $(c),(c')$, which by (6) was also the radical centre of the three given circles. Common tangents drawn from points p and p' would concur at the radical centre of circles $(c),(c')$ and (C), and consequently the line pp' contained the pole of the radical axis of (c) and (c') with respect to the circle (C).

These properties led to Plücker's second construction. First, draw the radical centre of three given circles, their four similitude axes and the twelve poles of the four axes with respect to the three circles. Lines emanating from the radical centre to these twelve poles would intersect the given circles at twenty-four points, which considered in sets of three would determine the tangent points with the eight solution circles. Plücker briefly explained how to use poles and polars to derive Gergonne's "elegant construction," which he cited as *Annales*, volume VIII, page 289. In a footnote, Gergonne reminded the reader that this construction had been reproduced in Steiner's article in the *Annales* XVII page 309. However, Gergonne's version, as it appeared in the previously volume, divided the construction into cases. In the single case presented, one only constructed three lines and six intersection points. By contrast, in avoiding cases, Plücker's comprehensive description resulted in all solutions at once.

To derive his third and final construction, Plücker stated that the polar of any point on the line pp' would pass through the pole of pp', which he defined as the intersection of common tangents from points p and p' to circle (C). This point would also be on the polar

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nor "radical", which, as said above, would not be coined for at least another decade.

[59] "Construct the orthogonal circle to the three given circles and construct their four similitude axes. If these axes intersect the orthogonal circle then (8), we will reach the construction by describing (1) circles passing through each pair of intersection points and at once tangent to one of the given circles."

[60] He would first give an description of polar reciprocity using coordinate equations in 1830 as we show in Appendix E (Plücker (1829b)).

of the radical centre of the three circles (C), (C'), (C'') with respect to (C). Following this property, the radical centre would have three polars with respect to the three given circles. Then the four similitude axes would intersect these polars at twelve points. By drawing tangents to the respective circles, one would obtain the twenty-four tangent points belonging to the eight circles sought. As the source of this construction, Plücker cited Poncelet, underscoring Poncelet's lack of analysis.

> Les constructions indiquées, sans analyse, par M. Poncelet (*Annales*, tom. XI. pag. 318) doivent nécessairement rentrer dans les précédentes. (ibid, 41)[61]

Plücker extended these constructions by elaborating modifications necessary when substituting points or lines for the given circles. He emphasized that these offered "no difficulty" and could be demonstrated with "full analytic rigour."

The remainder of Plücker's article concerned more general versions of the same problem, such as finding a circle that intersected three given circles at a fixed given angle or with fixed cosines of the angles, where the angle of two circles was the the angle formed by the intersection of their arcs "whose concavities are oriented in the same way." Plücker followed an analogous path for both cases, beginning again each time with the derivations of section 6 and proceeding to constructions that paralleled his first one. Plücker was thus able to present and prove all of Steiner's promised constructions, and claimed analogous constructions could be found without difficulty for spheres and spherical circles. These "ordinary methods of analytic geometry" [*méthodes ordinaires de la géométrie analytique*] could even be applied to constructions where the orthogonal circle "becomes imaginary, that is, when the square of its radius becomes negative" [*devient imaginaire, c'est-à-dire, lorsque le quarré de son rayon devient négatif*], although this could imply the problem was impossible – for Plücker, an imaginary circle would not suffice as a possible solution (ibid, 44). Plücker concluded in explaining concisely how one could approach the same sets of problems when any second order curve replaced the circles.

> Il suffira pour cela de projeter les données sur un plan tellement situé que les projections soient des cercles; de résoudre le problème plan pour ces cercles et de projeter ensuite les cercles obtenus sur la surface du second ordre. (ibid, 47)[62]

Although this was an entirely geometric description, Plücker's explanation of projection in his final paragraph appears strikingly like his use of coordinate representation throughout his article. Both projection and the use of analysis were treated as tools applied to planar

[61] "The constructions indicated, without analysis, by M. Poncelet (*Annales* vol XI, page 318) must necessarily enter into the preceding."

[62] "It will suffice for this to project the givens onto a plane so situated that the projections will be circles; to resolve the planar problem for these circles and to then project the circles obtained onto the second order surface."

problems. With projection, second order curves were effectively translated onto another surface where they became circles. With analysis, geometric objects were translated into coordinate equations. In both cases, the translated forms could be manipulated accordingly, and then translated back to their original plane. In Plücker's presentation, the use of projection presented a parallel structure to the use of coordinate equations in geometry.

Plücker's use of analytic equations in this 1827 article was minimal and appeared only in the beginning. Within the twelve sections summarized above, his computations were limited to the first four sections and section 6. Further, his use of analysis in geometry remained like that of Gergonne in 1817, as we will see by comparison to his research the following year. While he began by describing the circles with abridged notation, after the first section the circle equations were given in their standard extended form. Further, Plücker only applied analytic geometry to find the radical axes, perpendicular lines and concurrent points. The new geometric objects, such as those mentioned in the *Bulletin* review of Steiner, were primarily described with respect to their positional relationship in purely geometric terms. While one could have derived analytic representations from Plücker's exposition, he did not show the equation of an orthogonal circle, a pole, or a similitude centre, to take three such examples. His brief mention of imaginary circles only allowed imaginary objects as an intermediate step in obtaining a real solution, not as a solution itself, which would have to be represented constructively.

This minimalist computation was viewed as admirable in the subsequent *Bulletin* review.

> Mais d'ordinaire on se trouve beaucoup mieux, dans un grand nombre de recherches, de passer alternativement des ressources que fournit le calcul, à celles qui sont offertes par la géométrie pure, et de ces dernières aux premières, on évite ainsi à la fois et les lenteurs qu'entraînent des calculs compliqués et l'obscurité que fait naître l'accumulation d'un trop grand nombre de lemmes. Dans le 1er. article de la livraison, M. Plucker donne un exemple fort remarquable de l'application de cette dernière méthode [...] (Anonymous (1827a), 173)[63]

The anonymous reviewer suggested that too much calculation made the reader lose sight of the problem, and that Plücker's good taste enabled a comprehensible presentation. We note this review implied that geometric understanding could presumably be achieved without the use of figures.

Plücker's constructions abound with geometric objects, but the majority of these objects were at most enumerated and not named. With the use of coordinate representation and

[63] "But ordinarily it is much better, in a great number of researches, to pass alternatively from resources provided by calculation to those offered by pure geometry, and from the latter to the former, one thus avoids at once both the delays to which complicated calculations lead and the obscurity that the accumulation of great number of lemmas creates. In the first article of the issue, M. Plucker [sic] gives a very remarkable example of the application of this latter method [...]"

the new ideal objects, Plücker was able to avoid delving into various cases, except his initial degenerate straight line case. Although he observed the different situations of interior and exterior circles, for the most part his analysis enabled covering all cases with a single construction or proof. Most notably, while both Steiner and Gergonne had painstakingly listed which similitude points or centres were collinear, Plücker simply stated that three of the six centres would lie on the radical axis. Within Plücker's constructions the only named objects were five circles (C), (C'), (C''), (c), (c') and the two tangent points p, p', which defined the line pp'. Part of this brevity may be attributed to the fact that Plücker's publication came after those of Steiner, Poncelet and Gergonne. Plücker could rely, even without direct citation, on the more elaborate constructions of his predecessors.

As situated in the *Annales* and on the very same content treated by Steiner less than a year earlier, Plücker's article could not escape comparison with Steiner's, a fact that he acknowledged in his introduction. This liaison was also mentioned in the *Bulletin* review, which cited Steiner's treatment of the theory of circle tangency by pure geometry in Crelle's *Journal*. However, without Gergonne's footnotes, any acknowledgement to Steiner in the body of the text was strikingly absent. The last two constructions instead were respectively linked to the earlier work of Gergonne and Poncelet.

With respect to this and earlier Plücker publications, Gergonne concurred that few of the results were original, but this was no fault. Plücker's innovation was in the *form* of his geometry, and the ability to show that analytic geometry could achieve as much as pure geometry.

> Qu'importe, par exemple, que quelques-unes des solutions données récemment par M. le docteur Pluker (pag. 37) soient déjà connues et soient même moins générales et moins complètes que celles que d'autres géomètres ont pu donner des mêmes problèmes, comme la remarque nous en a déjà été faite plusieurs fois ? [...] mais ici le fond est de peu d'importance, et la forme est à peu près tout ; et ce qui recommande principalement le petit mémoire de M. Pluker, c'est qu'il nous montre les deux géométries marchant constamment de front, sans qu'aucune d'elles ait rien à emprunter à l'autre. (Gergonne (1827f), 273–274)[64]

Certainly Gergonne's positive assessment of Plücker was in part aimed at self-promotion, but his choice of points on which to comment coincided with those of the *Bulletin* reviews.

Plücker's article and *Bulletin* review resurfaced the following year in a letter of recommendation written for Plücker by August Leopold Crelle to the Prussian culture minister,

[64] "What does it matter, for example, that some of the solutions given recently by Dr. Pluker (page 37) are already known and are even less general and less complete than those that other geometers were able to give of the same problems, as we have remarked already several times? [. . .] but here the content is of little importance, and the form is nearly everything; and what principally recommends the small memoir of M. Pluker, is that it shows us the two geometries walking constantly abreast, without either of them having to borrow anything from the other."

Karl vom Stein zum Altenstein in July, 1828 (published in Eccarius (1980) as "Document 1: *Gutachten Crelles für den Kultusminister von Altenstein über J. Plückers*"). The letter aimed to promote Plücker's recently published *Analytisch-geometrische Entwicklungen*, which had appeared earlier that year, and provides further evidence of Plücker's perceived intermediary position between analytic and synthetic methods. Moreover, in this letter Crelle provided a thorough description of the differences between the "two different methods of research in studying figures in the plane and in space." His description far exceeds the level of detail observed in either Plücker or Steiner's published records. However, as Crelle was writing a letter of recommendation, some of this detail may be attributed to simply listing Plücker's many merits. First Crelle contrasted the opposing "synthetische, oder graphische, oder anschauliche Methode, mehr oder weniger nach Art der Alten" to the analytic method that uses calculations and is more mechanical. He then outlined the advantages and disadvantages of either approach. The first method benefitted from a clear intuition about the object under investigation and mindful awareness of the operative steps of research. Thus, the synthetic method was most useful, clear, and convincing in simple cases. Whereas, the analytic method seemed too difficult in simple cases, yet required no additional effort to be applied generally to complicated cases. Crelle admitted that the merits of each method had been debated, with Synthesists accusing the analytic method of "bloss mechanische Operationen" and Analysts defending "die grosse Kraft und Fruchtbarkeit der analytischen Operationen" as well as its ease. However, he insisted that both methods had their own "eigenthümlichen Werth." While the analytic method was the method of discovery, the synthetic method achieved the greatest clarity and certainty of insights. For Crelle, the study of mathematics required both qualities, and both qualities had been achieved in the recent work of Plücker.

In advertising Plücker's publication record, Crelle cited Plücker's recent *Bulletin* reviews, in which Plücker had united both methods (Anonymous (1826a), Anonymous (1827a)).[65] Crelle then explained that despite Plücker's claim in his preface that his monograph was purely analytic, it actually combined both methods with success. Crelle also corrected Plücker's accusation from the same preface that Steiner had merely been following in the footsteps of Poncelet, and clarified that Steiner's work had been independently derived and was useful in its own right. Overall, Crelle suggested that the "fremden" prefatory remarks expressed a tendency against synthetic methods and practitioners that was not apparent in the body of Plücker's text.

Crelle concluded by listing the diverse subjects of new geometry that Plücker had included: transversals, ideal chords, polars and poles, similarity points and axes, lines of

[65] It is worth pointing out that this first review (Anonymous (1826a) contained a synopsis of the article by Pluker, which indeed was not analytic geometry, but also was not fully written by Plücker, as we saw in Chapters II and III.

second order, osculation points, imaginary expressions, curves of higher order and other figures in space, etc. In Crelle's description, we find a menagerie of new geometry "betraying" Plücker's capabilities in both synthetic and analytic methods. Thus, although Plücker framed his work as analytic geometry, or even "pure analytic geometry," his contemporaries interpreted different methodological tendencies in his attention to form that would become increasingly apparent in his later works.

4.2.7 The Apollonius problem: conclusions

The texts considered above offer just a sample of early nineteenth century solutions to the Apollonius problem. Even in the early nineteenth century, another solution to the Apollonius problem would hardly qualify as breakthrough research in the absence of methodological novelty. Even so, Steiner's theories on circle relations in the plane offered enough innovation to be seen as worthwhile for a French audience. A strikingly similar publication to Steiner's "Einige geometrische Entwicklungen" had appeared in the *Annales* in 1820 by the young French mathematician J. B. Durrande entitled "Théorie élémentaire des contacts des cercles, des sphères, des cylindres et des cônes" (Durrande (1820)). Gergonne had encouraged Durrande's talents from his first publication in 1816, at the age of 17, until his recent and sudden death. In "Théorie élémentaire des contacts des cercles, des sphères, des cylindres et des cônes" Durrande showed how to derive solutions to the Apollonius problem using what he considered to be "purely elementary considerations" (Durrande (1820)). A comparison between the first few figures of Steiner (Figure 4.27) and Durrande (Figure 4.28) offers a cursory understanding of the common material and approach, compare for example Durrande's Figures 1 and 2 with Steiner's Figures 16 and 17.[66]

Gergonne's interpretation further accentuated the common language and content. In introducing Steiner's work Gergonne cited Durrande's text by volume and page number (though not by name), and moreover followed Durrande in defining and employing poles and polars, though Steiner did not. Despite these connections, the texts of Steiner and Durrande experienced dramatically different receptions. Steiner was propelled to recognition and critical acclaim with two additional posthumous publications of the original German version. Durrande's work remained marginal. Why did Steiner's paper eclipse Durrande's? There are several possible reasons. First, Steiner directed his research to a diverse assortment of problems, including the as yet unsolved Malfatti problem, which had frustrated geometers since 1810. Durrande focused solely on proving known problem solutions. On a deeper level, Steiner emphasized defining new objects of geometry, concepts extended

[66]The uncanny similarity between the two original texts has been discussed in the 1931 foreword to Steiner's unpublished manuscript edited by Rudolf Fueter and Ferdinand Gonseth (Steiner (1931)). The authors mention that Steiner had excerpted large passages from Durrande's 1820 article in January 1824, which were found in his Nachlass.

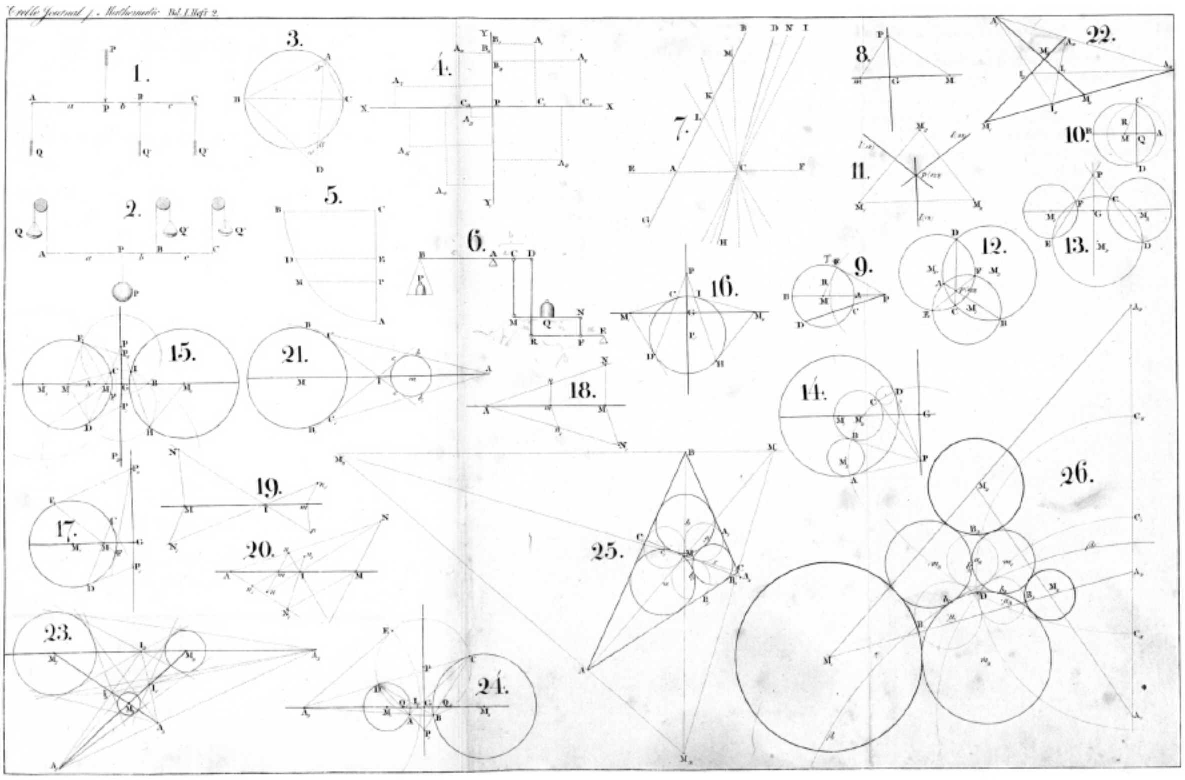

Figure 4.27: Steiner's Figures 8 through 26 from Steiner (1826a)

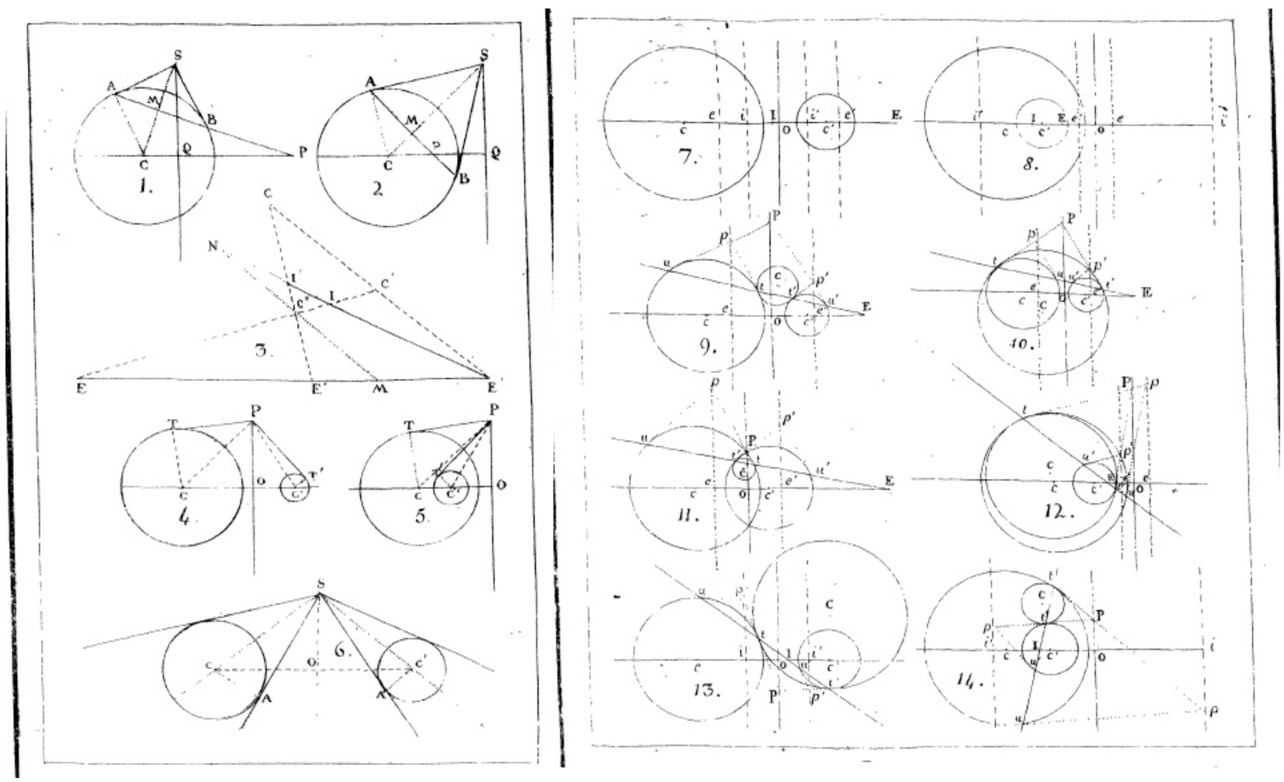

Figure 4.28: Figures from Durrande (1820)

beyond their initial problem solving applications. Durrande reinterpreted modern solutions to known problems in terms of elementary geometry, but did not suggest new tools for practice and research. Further, Durrande presented his results within a highly conservative framework restricted to Euclidean geometry. Although Gergonne criticized Steiner's lack of generality, Steiner's methodological stance appeared receptive to modern geometrical innovations and extended beyond a Euclidean framework. Finally, by 1826 there were more opportunities for publication and circulation of geometry articles. After its initial publication in Crelle's *Journal*, Steiner's findings were reiterated and discussed within the *Annales* and the *Bulletin*. The repetition bolstered his reputation.

Similarly, Plücker achieved recognition less from his particular results, although he did state several new theorems, than from his application of analytic geometry. As in his first publication from 1826, Plücker had drawn inspiration from earlier and apparently unproven Poncelet constructions. Plücker repurposed Poncelet's Apollonius problem solution as well as his use of ideal common chords. For both Plücker and Poncelet, ideal common chords enabled a single construction for the case of intersecting and non-intersecting circles. Significantly, although tentatively, Plücker represented his ideal common chords with abridged notation, which had only been introduced in 1817. Thus Plücker succeeded in combining two of the most innovative contemporary developments in both pure and analytic geometry. These considerations foreshadow Plücker's emerging analytic style, which was already praised for its moderate computation.

Reviews of both geometers focused on the potential for future applications to other areas of geometric research. The Apollonius problem, for all its fame, merely functioned as a showcase to display new geometric objects, techniques, and methods. In this setting, differences of method were emphasized. This extended from the *Annales* subject headings where Steiner's article was labelled as *pure* and Plücker's as *analytic* geometry, to the author's introductions where Steiner drew attention to the "Entstehung und Entwicklung" of his work and Plücker advertised the path toward the solutions, rather than the solutions themselves. The texts themselves reflected this order of priorities. Steiner never even provided a solution to the Apollonius problem, and Plücker's solutions were attributed to Gergonne and Poncelet. All four geometers proclaimed the superiority of their own solutions, but without overtly criticizing the work of their recent predecessors (Newton's and Viète's solutions, on the other hand, were declared overly complicated). Indeed, the Apollonius problem provides a counterexample to the dictate that the choice of method should be based on the problem at hand, as advocated by Poncelet and Gergonne in 1817. Gergonne even provided two different methodological approaches without advocating one over the other.

Distinct methodological approaches arrived at the same constructions and often by employing the same vocabulary. Steiner, Gergonne, Plücker, and Poncelet all invoked similitude points (or centres) and lines (or axes). These objects had become familiar enough in the 1820s that they could be left undefined or described as in the constructions of Plücker and Poncelet. With the widespread adoption of increasingly technical vocabulary, we can simultaneously trace various replacements for the illustrated figure from Gergonne's virtual figures, to Poncelet's advanced summary that eliminated reference to individual constructions, to Plücker's coordinate equations and abridged notation. In this context, Steiner's page of illustrations published in Crelle's *Journal* in 1826 became unnecessary in the French version.

As the *Bulletin* review noted, by assigning names to geometric objects and relations, geometers could avoid repeating their constructions. Plücker scarcely needed virtual figures, not so much because of his use of coordinate equations, but because he could simply write similitude centre or radical axis and at once convey the object's constructive composition and properties. Poncelet was likewise able to quickly convey constructions and minimize the naming of intermediate points and lines. The use of these increasingly complex terms eliminated the need for illustrated figures or painstaking figure descriptions such as those found in Gergonne. Concurrently, the new vocabulary necessitated a greater background knowledge, thus rendering the mathematics less accessible. In this respect, the presence or absence of a figure did not speak to the choice of pure or analytic geometry, just as the absence or presence of arithmetic calculation was independent of method. The case of the Apollonius problem thus forces a modification to Poncelet's claim for the methodological importance of the figure, as described in Chapter II. Here the disappearance of figures did not imply the ability to visualize had lost importance.

Several other factors also advanced the trend away from figurative representation. Both Steiner's 1826 text in Crelle's *Journal* and Poncelet's 1822 *Traité* contained figures, while their respective summaries in the *Annales* did not. Gergonne published the *Annales* as if the entire journal was available to his readership. He casually referenced prior volumes (sometimes, as we have seen, citing an article by volume and page number without naming the author or the text's title), and frequently posed problems based on past content and soliciting future reader responses. Gergonne's decision to leave out Steiner's original figures allies with his unproven statement of the constant power of a point with respect to a circle, this information was assumed as known to the *Annales* readership. This regular readership suggests that much of his circulation may have been by subscription, either individuals or institutions.

Finally, the ability to visualize was independent of illustrations. Steiner delved into three dimensional depictions, without any figures to assist his readers. Gergonne described seeing or conceiving of objects, which the reader must draw or imagine animatedly from their textual descriptions. Recalling Cournot's assessment of Plücker, even analytic geometry could be "followed by the eye." In 1827, geometry remained a graphic discipline, even when the images only existed in the imagination.

Even in Plücker's analytic approach, content in this context remained mostly elementary. Tangents and intersections among coplanar circles could be achieved through Euclidean constructions and despite their new nomenclature, the relationships conveyed by similitude, powers, and radicals were grounded in simple proportions, collinearity, and concurrence. However, Steiner and Plücker both remarked that these same results would continue to hold should circles be replaced by any conic sections, and Plücker described the necessary projective procedure to apply circle specific constructions to any second order curve. This generalization extended the domain of research beyond elementary geometry. Poncelet and Gergonne also applied results from circles to general conic sections, but the concept was novel enough that in 1828 Chasles announced as a new theorem that,

> II suit de là, en particulier, *que tant de cercles qu'on voudra, tracés sur un même plan, peuvent toujours être considères comme les projections stéréographiques d'un pareil nombre de sections planes faites dans une surface du second ordre, et leurs centres comme les projections stéréographiques des sommets des cônes circonscrits à cette surface, suivant ces mêmes sections planes* (*).[67] (Chasles (1828b), 309)

Gergonne modified Chasles' claim with a footnote, signalling both the special position of Steiner and Plücker as *German* geometers and their publications at the forefront of contemporary French geometry.

> Ce principe paraît ne pas être inconnu aux géomètres allemands. M. Plucker [sic] l'invoque formellement, à la pag. 47 du présent volume, et M. Steiner s'en appuye également dans le mémoire dont nous avons donné un extrait à la pag. 285 de notre XVII.e volume, pour transporter ses constructions planes sur des surfaces quelconques du second ordre.[68]

Though Gergonne would later apologize for seeming to have trivialized Chasles' research, his remarks proved prescient in forecasting Steiner and Plücker's early adoption of new principles and practices.

[67]"It follows from there, in particular, *that any circles that one likes, traced on the same plane, can always be considered as the stereographic projections of the same number of planar sections made in a second order surface, and their centres as stereographic projections of the vertices of cones circumscribed to this surface, according to these same planar sections."*

[68]"This principle does not appear to be unknown to German geometers. M. Plucker invokes it formally on page 47 of the present volume, and M. Steiner relies on it equally in the memoir of which we gave an extract on page 285 of our XVIIth volume, to transport planar constructions onto any surfaces of second order."

4.3 Conic sections with four common points (1828)

Our first case study portrayed Steiner and Plücker adapted by and adapting to the standards of French geometry and by 1828, their efforts had succeeded in generating recognition. Since his first publication in 1826, Steiner had contributed six additional articles to Crelle's *Journal* and three to the *Annales*, along with numerous posed problems in both. Plücker had recently published the first volume of his two-part monograph, *Analytisch-geometrische Entwicklungen* and had published a total of three articles in the *Annales* and one in the *Journal*. Both Steiner's and Plücker's articles from this period had been summarized and reviewed in Ferussac's *Bulletin*, and cited in contemporary journals and textbooks (for instance, Anonymous (1826a), Anonymous (1827a), Didiez (1828)).

Our second case study considers different proofs of the same theorem, originally attributed to Lamé, and aims to convey distinctive styles. As opposed to problem solving, geometry theorems were often independent of constructions, more theoretical, and hence better suited toward exhibiting methodological differences. As such our discussion will be deeper, narrower, and more technically involved. We will first examine Lamé's statement and proof of his theorem in order to understand the original context of his theorem as it appeared in the *Annales* in 1817. In the intervening decade between Lamé's first statement of his theorem and Steiner's proof, the subject of Lamé's article, the correspondence between conic sections and their conjugate diameters, remained popular in geometric articles within the *Annales*. The central concept of conjugate diameters has ancient roots and has been traced back to Apollonius. In the Thomas Heath version, Apollonius defined conjugate diameters as those which bisect all chords parallel to the other given diameter (Apollonius (1896), 17). The concept is made clearer through construction. Consider a conic with diameter d. Draw tangent lines to the conic parallel to d, these will intersect the conic at two points P, P' lying on PP', the *conjugate diameter* to d. In order to extend this constructive definition to all conic sections, one would need to include infinite points of intersecting parallel lines and the line at infinity containing them. However, in the texts considered here geometers assumed conjugate diameters as known, and often used projection to extend a simpler construction to a more general result. In particular, the research of Gergonne, Brianchon, and Poncelet employed a variety of geometric methods towards exploring the conditions necessary to determine second order lines and the relationships between conjugate diameters (Brianchon (1817), Brianchon and Poncelet (1820), Gergonne (1821), Poncelet (1822)). For instance, in his 1822 *Traité*, Poncelet included his own proof of Lamé's theorem, which he described as a direct corollary to his more general theorem on polar reciprocity found by passing a planar point to infinity. Though Poncelet's memoir was yet to be read by Steiner or Plücker in 1828, his reference to Lamé gives evidence to the result's popularity.

Thus, in 1828 Steiner's choice of studying the relationship between conic sections in the plane engaged with a set of geometric questions still under lively investigation. Approximately two-thirds of the content in the nineteenth volume of the *Annales* was devoted to articles on the study of geometric curves and surfaces by at least nine different geometers (several articles were signed only as *un abonné*). Steiner's subject matter was well established and represented. His approach, as we will see, was idiosyncratic.

The following issue of the same volume of the *Annales* opened with an article of analytic geometry, "Recherches sur les courbes algébrique de tous les degrés" by " M. le docteur Plucker, professeur à l'Université de Bonn" (Plücker (1828b)). Plücker's ten page article was the first in a two part series, the second of which explored the same questions for algebraic surfaces. In a concise introduction, Plücker explained his intention: "donner quelques exemples d'une méthode à l'aide de laquelle on peut déduire, immédiatement et sans aucune sorte de calcul, un grand nombre de propriétés générales des courbes de tous les degrés, de la simple considération de la constitution algébrique des équations qui les représentent" (97).[69] Plücker's development of coordinate representation without calculation characterized a new type of analytic geometry. As we will see, Plücker's new method involved a combination of abridged notation, where a curve was represented by a single capital letter, and well-chosen coordinate axes that enabled directly studying the coefficients of the equation of the general second-order curve, $Ax^2 + By^2 + 2Cxy + 2Dx + 2Ey + F = 0$.

Unlike the case of the Apollonius problem, both geometers proved Lamé's theorem as an intermediate and not particularly noteworthy result, in markedly different language. While both articles were about relationships between planar curves, the diverse treatments maskedl their parallel content except for their common citation. If the Apollonius problem reinforced the similarities between different geometric methods in their common invocation of illustrated or virtual figures, frequently repeated constructions, and use of the same set of new geometric objects, the case of Lamé's theorem will emphasize their differences.

4.3.1 Lamé's theorem (1817)

Lamé had just graduated from l'École polytechnique in 1817 when his article "Géométrie analitique. Sur les intersections des lignes et des surfaces. Extrait d'un mémoire présenté à l'Académie royale des sciences, en décembre 1816" was published in the *Annales* (Lamé (1817)). As the lengthy title indicates, the article was excerpted from a longer memoir, later revised and published in 1818 as *Examen des différentes méthodes employées pour résoudre les problèmes de géométrie* (Lamé (1818), 30–41).[70] The article's subject matter, the use

[69]"[...] to give several examples of a method, by aid of which one can deduce, immediately and without any sort of calculation, a great number of general properties of curves of any degree, simply from considering the algebraic constitution of their representative equations."

[70]Evelyne Barbin has shown how Lamé demonstrated the potential for progress and discovery in associating analysis and geometry (Barbin (2009)). In particular, Barbin considers the content of Lamé's *Examen des*

of coordinate equations to solve geometry problems, was dear to Gergonne's heart and he may have requested this contribution from Lamé (there is evidence that Gergonne actively inquired after the contents of geometry articles submitted to the *Académie* in the case of Poncelet). This brief article set out to use rectangular coordinate equations in order to find conditions such that:

1. three first or second order curves [*lignes*] on the same plane concur in a point;

2. three first or second order surfaces in space meet on a curve;

3. four first or second order surfaces in space concur in a point.

The result on conjugate diameters was considered by Lamé as an "interesting theoretical and practical consequence."

For his first problem, Lamé used the generality of coordinate equations to represent three curves of second order.

$$\begin{cases} ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \\ a'x^2 + 2b'xy + c'y^2 + 2d'x + 2e'y + f' = 0 \\ a''x^2 + 2b''xy + c''y^2 + 2d''x + 2e''y + f'' = 0 \end{cases}$$

By multiplying the first two equations respectively by the indeterminate constants m and m' and then finding their sum, he derived a single equation. Because of the indeterminacy of m, m' this new equation represented all the lines of second order passing through the intersections of the two first lines.

$$(am+a'm')x^2+2(bm+b'm')xy+(cm+c'm')y^2+2(dm+dm')x+2(em+e'm')y+(fm+f'm')$$

Examining the equation, Lamé remarked that, under various relationships between the coefficients, it could belong to two different parabolas, a circle, or an infinity of ellipses and hyperbolas. The condition for the third line to have the same intersections would be met by setting each coefficient in the new equation equal to its corresponding coefficient in the third equation. That is,

$$am + a'm' = a'', bm + b'm' = b'', cm + c'm' = c'',$$

$$dm + d'm' = d'', em + e'm' = e'', fm + f'm' = f''.$$

Through elimination of indeterminates m, m' one could arrive at four other equations that expressed the concurrence of the three second order lines. Moreover, the three equations

différentes méthodes and his development of abridged notation.

containing a, a', a'', b, b', b'', and d, d', d'' could be simultaneously satisfied, signifying that three straight lines, each defined by one a, b, and d concurred in a point. Each of these equations also belonged to the curve's diameter that bisected all of its chords parallel to the x axis. For this result, Lamé cited an article by Bérard from the *Annales.* Then, since the direction of the x-axis could be chosen with respect to the second order lines, Lamé concluded,

> THÉORÈME. Si plusieurs sections coniques ont quatre points communs ; dans quelque direction qu'on leur mène des diamètres parallèles, les conjugués de ces diamètres concourront en un même point. (Lamé (1817), 233)[71]

At this stage in the text, no figures had been employed, and all geometric objects were represented by their coordinate equations. Having derived his theorem, Lamé then applied it to graphically determine the centre of a conic given five points on its perimeter and the slope of the diameters of a parabola given four points on its perimeter. These two *problems* referenced straight edge figures, copied below. Though the problems themselves featured conic sections, the figures (Figure 4.29) contained only the given points, as was common practice for published geometrical figures at this time.

Each problem had a unique figure, and all pictured lines and points were referenced in the construction. Following these applications, Lamé asserted that one could determine the other elements of the curves, but because these proceedings were "completely foreign to his object" [*tout-à-fait étrangers à notre objet*], he returned to the second problem of his paper.

Within this article, Lamé chose coordinate equations to prove his theorems and then used these theorems to solve associated problems through figures without coordinate representation. Lamé was thus able to employ the generality of algebraic expressions, without sacrificing the practice and application of graphic constructions. These respective advantages of different geometric methods were not the same as those touted by reviewers of Plücker and Steiner ten years later. In fact, Lamé explicitly suggested that the application of algebra to geometry served as the method of discovery in his *Examen des différentes méthodes employées pour résoudre les problèmes de géométrie.* Following a long list of potential applications of algebraic representation of geometric loci beginning with the study of straight lines and extending to second degree curves, Lamé concluded,

> [...] enfin ne rien négliger dans toutes ces applications pour faire remarquer l'accord constant de l'Algèbre avec la Géométrie, accord qui permet de confier au calcul le soin de découvrir de nouveaux théorèmes. (Lamé (1818), 5–6)[72]

[71] Theorem. If three or more conic sections have four common points; then in no matter what direction one draws parallel diameters to these conics, the corresponding conjugate diameters will concur in the same point.

[72] "[...] finally in all these applications do not neglect to notice the constant agreement of Algebra with Geometry, an agreement that permits us to entrust calculation with the task of discovering new theorems."

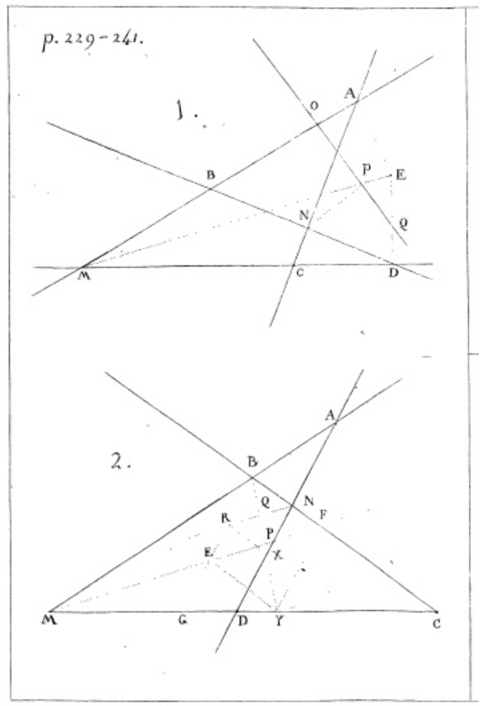

Figure 4.29: Lamé's Figures (Lamé (1817))

In the case of Lamé's theorem, his assessment appears to be correct, as his successors repeatedly credited his algebraic proof as the theorem's source.

4.3.2 Développement d'une série des théorèmes relatifs aux sections coniques: Steiner proves Lamé's theorem (1828)

Steiner's first publications in Crelle's journal were described by Gergonne and in *Bulletin* reviews as employing a method completely unique to the author. As Steiner continued writing for a French audience, his knowledge of the background literature and terminology transformed accordingly. Now writing in French, Steiner cited Gergonne, Carnot and Lamé (and himself).

Unlike the above examples, most articles in the *Annales* and the *Journal* had fairly descriptive titles. When an article was simply labeled as a proof of "several" theorems or solutions to several problems, this was most likely in response to posed problems. Articles of this sort were usually brief and to the point. "Développement d'une série des théorèmes relatifs aux sections coniques" ran thirty pages in twenty-eight sections and included dozens of theorems and problems, which were only designated as such by the use of quotation marks around their statements (Steiner (1828d)). If the reader had any expectations from the title, they could only be based on the content of Steiner's earlier publications. The list-like form of Steiner's article was emphasized by his vague title, lack of preface or introduction, and extensive enumeration of small sections–some as short as a sentence, others extending several paragraphs. The same format featured in at least two of Steiner's recent publications, "Démonstration de quelques théorèmes" and "Einige geometrische Sätze" (Steiner (1828b), Steiner (1826b)), and seemed to go against his earlier systematic intentions or desire to show an organic unity. Yet these shorter publications could also be interpreted as publicizing results that would be systematically developed in his forthcoming book (Steiner (1832)).

The *Bulletin* review later summarized Steiner's article as based on a single proposition: given a triangle and three points on the lines containing the sides of this triangle, then perpendiculars raised through these points would concur in the same point. If one drew a circumference through these points, it would intersect the given lines in three new points from which perpendiculars would also concur. Certainly, this proposition was Steiner's first result, but he only explicitly referenced it in deriving his second result, (whereas, for example, his fifth theorem was referenced three times throughout the text) so its role as the foundation of his research seems to be mostly positional. As well as articles devoted to the proof of new results, Steiner's other primary output at this time were posed problems (without solutions) and theorems (without proof) offered to the readers of the *Annales* and the *Journal* (Steiner (1827c), Steiner (1827d), Steiner (1827a)). These collections served to both secure Steiner's priority, and encourage readers to practice his methods. Structurally, "Développement d'une série des théorèmes" was reminiscent of these catalogs in neither building from general considerations (like in Steiner's *Systematische Entwicklungen*) nor building toward some promised results (like in Steiner's "Leichter Beweis eines stereometrischen Satzes von Euler"). Even so, we will begin at this first derivation, both to introduce Steiner's style with an elementary result and for future comparison with Plücker's proof of the same proposition.[73]

[73]Poncelet, for one, found little merit in Steiner's form of geometry. Although he described an amicable private correspondence with Steiner through the mid-nineteenth century, when retrospectively assessing Steiner's contributions in 1864, Poncelet suggested his colleague confused complication with depth.

> On voit d'ailleurs que ce genre de propositions, dont ce dernier géomètre [Steiner] a souvent fait abus et qu'il a mis à la mode parmi une certaine classe de savants, appartient plutôt à la théorie des nombres et à la Géométrie de situation qu'à la Géométrie proprement dite ou d'intuition; ce qui semble indiquer, à mon sens, une faiblesse, un manque d'initiative des esprits qui, confondant la *complication* avec la *profondeur*, s'éloignent de l'élégante simplicité des anciens géomètres. (Poncelet (1866), 410)
>
> "One sees moreover that this kind of propositions, which the latter geometer [Steiner] has often abused and which has been popular among a certain class of scholars, instead belongs to the theory of numbers and geometry of situation rather than geometry properly named or geometry of intuition, which suggests, in my view, a weakness, a lack of initiative among minds who, confusing the *complication* with *depth*, move away from the elegant simplicity of ancient geometers."

Poncelet suggested that Steiner's frequent publication of new results without proofs was out of fear that Poncelet might publish first and receive priority. Mitigating Poncelet's opinion, we note that in a letter from Jacobi to Steiner from 1833, Jacobi admitted that he did not cite Poncelet as much as Steiner might wish

Steiner began with Figure 1 (our Figure 4.30), for which we will provide a series of step-by-step illustrations following Steiner's construction instructions.

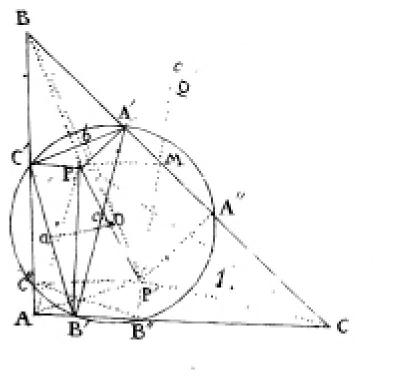

Figure 4.30: Steiner's Figure 1 (Steiner (1828d))

However, we will only go so far as the first theorem, and thus certain features of the finished Figure 1, only described later and not relevant to proving Lamé's theorem, will not be introduced. From a point P in the plane of a triangle ABC drop perpendiculars PA', PB', PC' respectively to BC, CA, AB. Though not specified in the text, we follow Steiner's figure by placing P inside the triangle ABC (Figure 4.31).

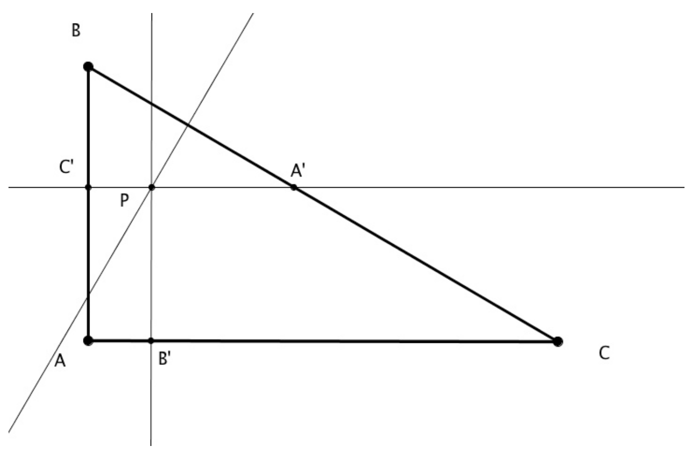

Figure 4.31: Triangle ABC and point P

Then join the vertices A, B, C to P (Figure 4.32).

These new segments determine relationships between the parts of the right triangles.

$$\overline{BA'}^2 - \overline{CA'}^2 = \overline{BP}^2 - \overline{CP}^2,$$

(Jahnke (1903a)). Jacobi's comment suggests that Poncelet had ample support with respect to Steiner's allocation of credit.

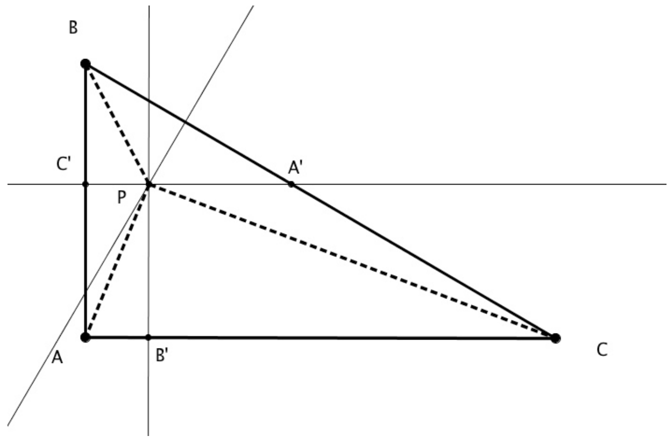

Figure 4.32: Segments AP, BP, CP

$$\overline{CB'}^2 - \overline{AB'}^2 = \overline{CP}^2 - \overline{AP}^2,$$

$$\overline{AC'}^2 - \overline{BC'}^2 = \overline{AP}^2 - \overline{BP}^2.$$

By "adding, reducing and transposing" [*ajoutant, réduisant et transposant*] the above equations Steiner derived,

$$\overline{AB'}^2 + \overline{BC'}^2 + \overline{CA'}^2 = \overline{BA'}^2 + \overline{CB'}^2 + \overline{AC'}^2.$$

This proportional relationship was the necessary and sufficient condition that perpendiculars raised from the points A', B', C' on the three respective sides BC, CA, AB of a triangle ABC all concur in the same point P. Steiner concluded *immediately* that from this resulted (1) that the perpendicular bisectors of the sides of a triangle concurred in a point and (2) that the perpendiculars from each side to its opposite vertex concurred in a point. In Section 2, Steiner introduced a circumscribed circle, and gradually derived further theorems on the ratios of triangles inscribed to conics.

The progression of Section 1 revealed how tenuous the line between pure and analytic geometry might be. Here Steiner began with a figure illustrating the relationship he intended to derive: the concurrence of three lines. From an algebraically computed equation, Steiner determined a criterion for the desired relationship, followed by two exemplary cases. Though Steiner's criterion was constructible, the form of presentation, as well as the use of "adding, reducing, and transposing," suggests an algebraic affinity not present in the figure. Further, the sufficiency of the condition only followed because each of the steps were reversible. Figure 1 served as a representation of the hypothesis and the conclusion, but concealed the intermediary non-constructive steps.

We now jump forward to Lamé's theorem, stated toward the end of the text. The preceding section had concluded with a result on parabolas inscribed to triangles, and with

Section 22 Steiner initiated a new line of research, explicitly referencing only Section 6 and Figure 7. In order to understand his description of the figure manipulated in Section 22, Steiner thus required his reader's knowledge and memory of points only defined in Section 18, where Figure 7 had been introduced. In order to explain Steiner's derivation of Lamé's Theorem, we must determine the necessary results from Section 6, the construction of Figure 7 in Section 18 and its special case in Section 19, before turning to Steiner's proof in Section 22.

Section 6 began by describing the relationships between points pictured in Figure 2 (our Figure 4.33), a circle with centre P circumscribed to a triangle ABC.

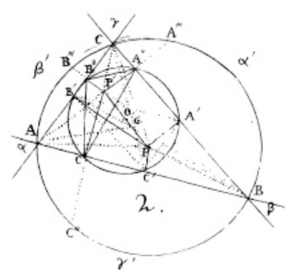

Figure 4.33: Steiner's Figure 2 (Steiner (1828d))

Steiner had shown in Section 2 that perpendiculars raised from AB, AC, BC to P bisect their respective sides at C', B', A'. In Section 6 he employed this result, though without any direct citation.

Drawing $B'C', C'A', A'B'$, he concluded that these lines were respectively parallel to the sides BC, CA, AB of the original triangle. Then, for example, if the line AP' was perpendicular to $B'C'$ it would be perpendicular to BC. So, from a result proved but not referenced from Section 1, P' would be the point where perpendiculars dropped from vertices to opposite sides coincided. Steiner designated the feet of these perpendiculars as A'', B'', C''. Then points $A', B', C', A'', B'', C''$ would all lie on the circumference of a circle centred at O, the midpoint of PP'. These constructive steps are shown in our Figure 4.34.

Steiner referenced Carnot (without a date or a title) in finding a fourth point G on PP' such that $GO : GP :: P'O : P'P$.[74] From this ratio P' and G were the centres of similitude of the two circles centred at O and P.[75] Thus the circle centred at O also passed through

[74] This proportion reappears in many early nineteenth century pure geometry texts. Poncelet attributed the result to Pappus, and also noted Brianchon's treatment (Poncelet (1822),12). Brianchon in turn references "l'illustre auteur de la Géométrie de position," that is, Carnot (Brianchon (1817), 6). Poncelet, Brianchon, and Steiner refer to the division as *harmonic* or *harmonic proportion.* Carnot proved the existence of the fourth proportional point for any three given points in space in *De la corrélation des figures de géométrie* in 1801 (Carnot (1801), 103–125).

[75] The centres of similitude between two circles were defined by precisely this ratio relationship between circles with radii GO and $P'P$ in Steiner (1826a).

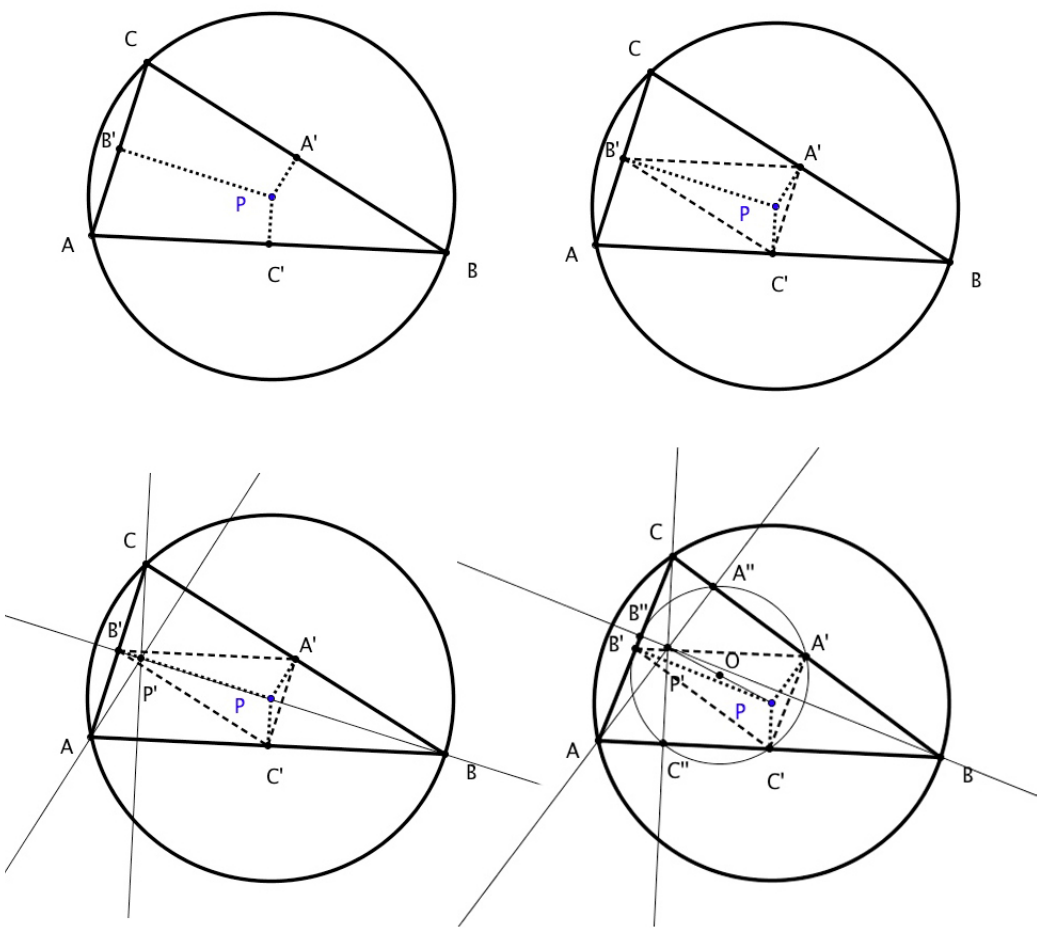

Figure 4.34: Constructing Steiner's Figure 2

the midpoints of the segments $P'A$, $P'B$, and $P'C$. This was the result that Steiner would employ in Section 22.

As we have seen with Figure 1 and Figure 2, each of Steiner's figures carried its own, usually new, definition of points. Often these new points would have overlapping names with the points from prior figures. Since Section 22 cited Figure 7, we must now set aside the point names from Figure 2, except in translating the application of results from Figure 2 as noted. This will hopefully help avoid too much confusion over the changing roles of points. Our exposition, then, is more careful and drawn out than Steiner's, but hopefully to the reader's advantage!

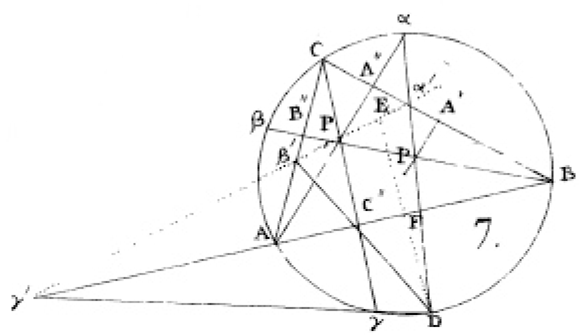

Figure 4.35: Steiner's Figure 7 (Steiner (1828d))

Figure 7 (our Figure 4.35) pictured a circle, but was described in Section 18 as any conic circumscribed to a triangle ABC. As in the previous figures, we will construct Figure 7 step by step with intermediary illustrations along the way (Figure 4.36). However, we note that Steiner's Figure 7 pictured the construction in its entirety and incorporated other elements not featured in Section 18 or 22, which we will leave out of our progressive illustrations.

Consider any conic circumscribed to a given triangle ABC, as in Steiner's Figure 7 we begin with a circle.

Through the vertices of this triangle and through any planar point P', draw lines $AP'A''\alpha$, $BP'B''\beta$, $CP'C''\gamma$. These respectively cut the extended triangle sides opposite the three angles in A'', B'', C'' and the conic at α, β, γ. If through any point D on the circumference, one drew lines $D\alpha$, $D\beta$, $D\gamma$ cutting respective sides BC, AC, AB in α', β', γ' then these three points would always be on a line $\alpha'\beta'\gamma'$ containing P'. This collinearity result followed when one considered $D\beta BCA\alpha D$, or any six such points, as an inscribed hexagon to the conic. Then Pascal's theorem proved the intersections of opposite sides $D\beta$ and CA, βB and $A\alpha$, BC and αD, respectively at β', P, α', were collinear. Choosing a different hexagon and applying the same procedure would ensure the collinearity of all four points. Finally, as D moved along the circumference, the line $\alpha'\beta'\gamma'$ rotated on the point P', and (as Steiner put it) *vice versa*.

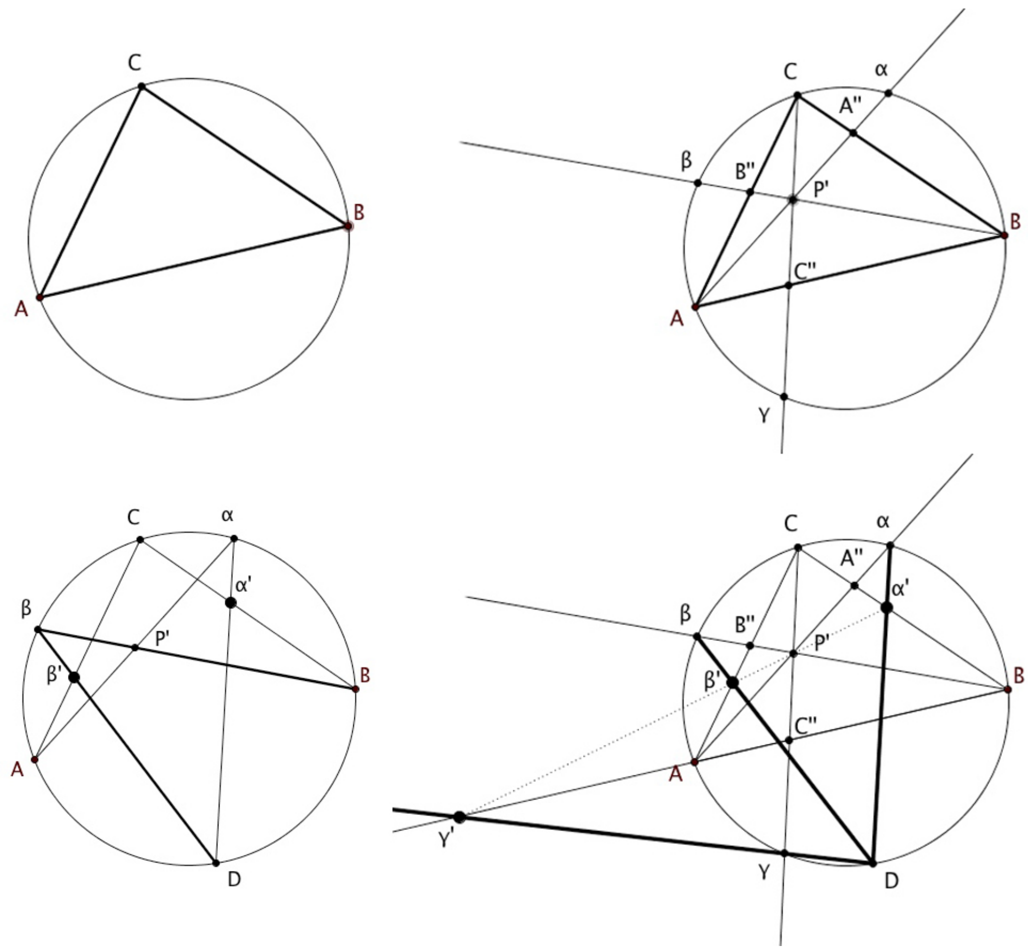

Figure 4.36: Constructive steps toward Steiner's Figure 7

In Section 19, Steiner considered the special case where the conic was a circle, then $P'A'' = A''\alpha$, $P'B'' = B''\beta$, and $P'C'' = C''\gamma$. While Steiner did not state whether or not the conic was a circle in Section 22, he did begin with this particular equality result. This also allied with the application of Section 6, proved only for a given circle not a general conic.

So, in Section 22 points A'', B'', C'' were now the respective midpoints of segments $P'\alpha, P'\beta, P'\gamma$, particular cases of the points constructed in Section 18. If from A'', B'', C'' one drew three lines respectively parallel to $D\alpha, D\beta, D\gamma$, where D was some point on the circumference, they would then pass through each of the midpoints of $P'\alpha', P'\beta', P'\gamma'$ and concur at a point D'. Then, from Section 6, a conic would pass through A', B', C' midpoints of $\beta\gamma$, $\gamma\alpha, \alpha\beta$ (the former BC, CA, AB of Section 6) and A'', B'', C'' the midpoints of $P'\alpha$, $P'\beta$, $P'\gamma$ (the former $P'A, P'B, P'C$ of Section 6). From the construction, the point of concurrence D' would also lie on this circumference and by Pascal's Theorem D, D', P' would be collinear. Steiner declared that from this resulted a theorem due to Lamé (with no date or text cited).

> Quatre points A, B, C, P' donnés sur un même plan déterminent trois systèmes de deux droites AP' et BP', BP' et AC, CP' et AB, qui se coupent respectivement en A'', B'', C''. Si l'on coupe ces systèmes par une droite quelconque $\alpha'\beta'\gamma'P'$ conduite par P', et si, par les points A'', B'', C'', et par les milieux des segmens de cette droite, on mène des droites $A''D', B''D', C''D'$, ces droites concourront en un même point D', et le lieu de ce point sera une conique passant par les points A'', B'', C'' et par les milieux des droites $BC, CA, AB, AP', BP', CP'$, etc. (ibid, 61)[76]

The overall effect of the theorem juxtaposed to the construction is disorienting. Perhaps most jarringly, Steiner had reassigned the point names in his theorem's statement, α, β, γ were suddenly A, B, C (thus better corresponding to Section 6, but in complete disregard of Figure 7). Like in Section 1, where Steiner began by assuming the desired result, here many constructive steps were inverted. The line containing points α', β', γ', and P', found in Section 18, was in the theorem described as any straight line. The midpoints of $P'\alpha', P'\beta', P'\gamma'$ here defined the three lines respectively through A'', B'', C'', on which they had been proved to lie in the proof. The point D remained unmentioned as did the parallel relation and the given circle (or conic).

[76] "Four coplanar points A, B, C, P' determine three systems of line pairs AP' and BP' [sic, Steiner appears to mean BC], BP' and AC, CP' and AB, which intersect respectively in A'', B'', C''. If one intersects these systems by any line $\alpha'\beta'\gamma'P'$ passing through P', and if, by the points A'', B'', C'' and by the midpoints of the segments [$\alpha'\beta'$, $\beta'\gamma'$, $\gamma'P$] of this line, one draws the lines $A''D', B''D', C''D'$, then these lines will concur in the same point D'. And the locus of this point will be a conic passing through the points A'', B'', C'' and through the midpoints of the lines $BC, CA, AB, AP', BP', CP'$, etc."

These latter omissions were precisely what Lamé's Theorem originally featured in 1817. It was only in the constructive proof and Figure 7 that one could recognize that the four common points were α, β, γ, D (or A, B, C, D in the theorem's statement–although D is missing), the diameters were $D\alpha$, $D\beta$, $D\gamma$ with parallel conjugates $A''D', B''D', C''D'$ concurring at D'.

Although in this proof Steiner only examined the case of the circle, Steiner used in numerous other examples parallel projection and central projection in order to extend circle properties to any conic section. However, Steiner's final statement of Lame's theorem referenced a general conic section, without providing an argument of how one might apply projection to extend the proof from the case of a circle. In Section 23, Steiner began a new line of inquiry, and did not further reference Lamé's Theorem, nor use its conclusion, in the remainder of his article.

Throughout Steiner employed a common theme of progressing from a simple case to ever more general elaborations. Each new line of inquiry began with simple particular figures–most commonly a circle and straight lines–that Steiner generalized into any conic sections. He often repeated a three-step argument pattern, first a specific result, then via projection to a broader result for a set of conic sections, and finally via perspective to an even more general result applied to all conic sections.[77] Finding the reciprocal result was another common technique for Steiner. Sometimes he simply stated "Réciproquement" or "vice versa," sometimes he wrote out the reciprocal in full in the same section, and sometimes he devoted an entirely new section to the exposition of the reciprocal. These choices do not appear motivated by the subject matter or the length of the process.[78] The symmetry of Steiner's argument patterns unified his article more than the particular contents, which ranged across objects in the geometric plane, leading to numerous results and then backtracking to different hypotheses. Further, the repetition of generalization could serve as a guide to the reader in following and applying Steiner's system.[79] Though it did not make for easy reading, this linear form of argument perhaps more closely resembled an imitable method of discovery than an argument directed toward a specific result. Here, no one result was placed above another. Some results were employed more frequently, but because Steiner proceeded from particular findings to generalizations, the more specific results appeared more often despite their limited applicability.

[77] For instance, in Section 10 Steiner proved a result for an equilateral triangle circumscribed to a circle. Through parallel projection he extended the result to any triangle and the smallest possible circumscribed ellipse in Section 11. Then in Section 12 he applied central projection or perspective to generalize the result to any triangle and any circumscribed conic section.

[78] Although Steiner had used dual columns in previous articles (such as Steiner (1828b)), and his results from this article were summarized with dual columns in the *Bulletin* review, he did not use them here.

[79] As an extreme example of this pedagogical zeal, Steiner concluded his first monograph, *Systematische Entwicklung* with a list of 85 practice problems that could be mailed to Crelle's *Journal* for review and potential publication.

As in the case of Section 2 and Section 18, each new line of inquiry included a new figure, where Steiner took the opportunity to wipe the slate clean and redefine previously designated points. As we have seen, this practice was not without confusion, perhaps due to some errors in production. Although Steiner included many theorems, not every proposition was illustrated. Steiner referred to figures economically, often using one figure to illustrate several constructions. All of his figures fit onto a single sheet (Figure 4.37).

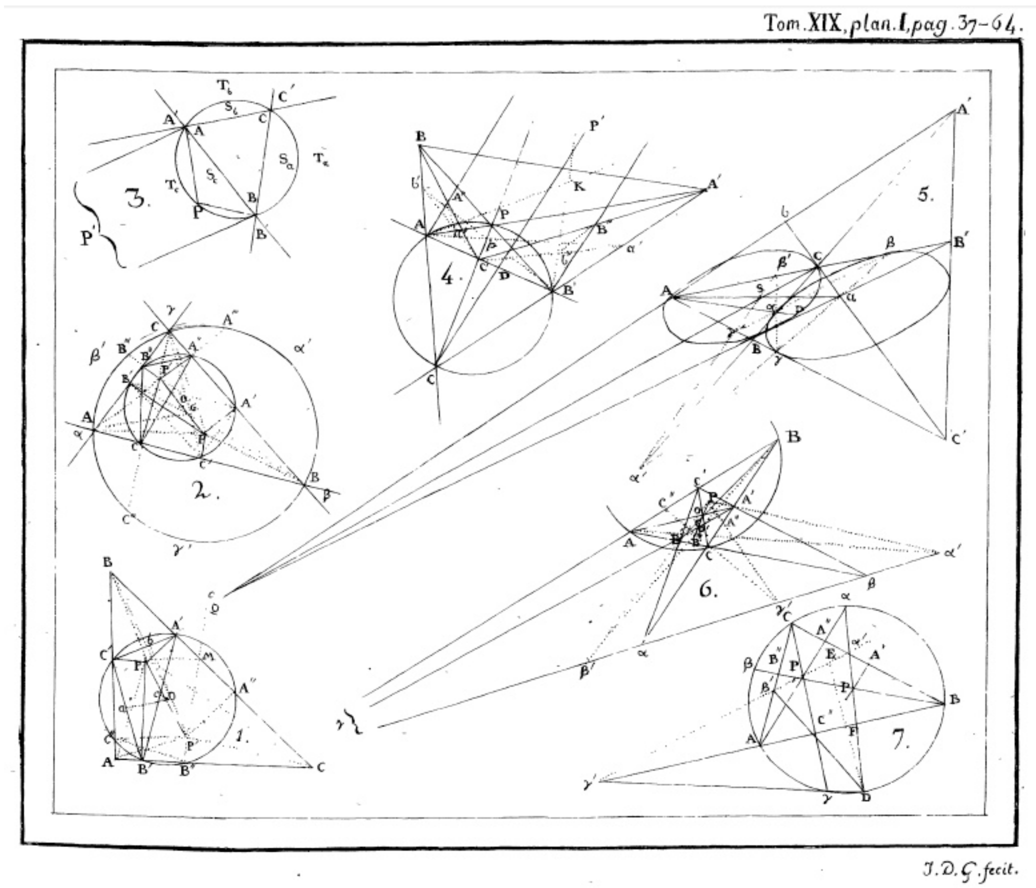

Figure 4.37: Steiner's Figures (Steiner (1828d))

Gergonne designed the prints for all figures in his journal, and clearly took care in his layout. In order to fit all of Steiner's figures on a single sheet, the figures were not arranged linearly, but almost in a clockwise fashion beginning with figure one in the lower left corner. The straight lines in the figures were either solid or dotted, apparently motivated more by visibility concerns than denoting a different step in the construction process. All seven figures included full or partial conic sections. In Figure 5 there were two ellipses, otherwise the conics were circles, or as in Figure 6, partial circles–due to page space constraints. Unsurprisingly, then, when we consider when figures are invoked, they remain linked to preliminary and more particular theorems rather than the general results derived through projection and reciprocity.

Despite the elementary nature of the figures, they contained recent geometric innovations. Specifically, both Figure 3 and Figure 4 indicated an infinite point P' as the intersection of parallel lines. As in the figures of Poncelet, the point at infinity was suggested as being off the page. In generalizing his theorems, Steiner referred to cases where the points of intersection passed to infinity or were situated at infinity. These procedures of projection, perspective, and reciprocity maintained an analogy to constructive results yet extended beyond their constructive limitations.

4.3.3 Recherches sur les courbes algébrique de tous les degrés: Plücker proves Lamé's theorem (1828)

Plücker's research on algebraic curves appeared only 30 pages later in the same *Annales* volume (Plücker (1828b)). Steiner's article had been classified as *pure geometry* and Plücker's was labelled as *analytic geometry*. Though either article might have equally well fit under the heading *geometry of curves*, their disparate subjects contrasted with their spatial proximity. Further, Plücker's research was succinct and driven toward a clearly enunciated goal. Plücker promised to deduce a great number of properties of curves, while refraining from calculation. He would continue these investigations with respect to algebraic surfaces in another essay.

The article comprised three sections. The first motivated the research with a review of Cramer's paradox and a discussion of well-known curve properties. Plücker cited Cramer in his *Introduction à l'analyse des courbes algébriques* as the first geometer to have noted that a planar curve of degree m was completely determined by $\frac{m+1}{1} \cdot \frac{m+2}{2} - 1$ points, but two planar curves of this degree could intersect in up to m^2 points. When $m > 2$, there was an apparent paradox, since different curves could share as many or more points than the number required to completely determine one of them. For instance, when $m = 3$ two planar curves could intersect in 9 points, which should determine a single curve of degree 3. Following Cramer, Plücker explained with professed ease this seemingly "surprising" result by remarking that when completely determining a planar curve one always assumed the given points were chosen "by chance and contain no particular common relation." Turning to his own research in the theory of osculating curves and its geometric interpretation, Plücker had met several similar theorems that initially appear rather "singular," but were "very fruitful with beautiful corollaries." Though these theorems had appeared elsewhere (Plücker did not say where precisely), he would reproduce them here in a more developed manner followed by some of their applications.

The second section contained Plücker's two main theorems and their proofs, these were the only theorems labelled and enumerated in the text. Plücker began with an analytic representation of the properties discussed in Section I, showing that if one represented two curves of any degree $m > 2$ as

$$M = 0, M' = 0,$$

then the equation of the same degree

$$\mu M + M' = 0,$$

in which μ is a constant indeterminate coefficient, would express an infinity of other curves of degree m passing through the m^2 points of intersection of M, M'. But if one found an arbitrary new point on one of the curves, one could find μ from the resulting linear equation, and so the curve containing the new point would be completely determined. Continuing with this line of reasoning in the case where there were $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ given points (one fewer than to completely determine a conic), and invoking the *principle of duality*, Plücker demonstrated his Theorem I in parallel columns.[80]

THÉORÈME I. Toutes les courbes du $m.^{ieme}$ degré qui passent par les $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ mêmes points fixes, se coupent en outre aux $m^2 - (\frac{m+1}{1} \cdot \frac{m+2}{2} - 2)$, autres mêmes points fixes.[81]	THÉORÈME I. Toutes les courbes du $m.^{ieme}$ classe qui passent par les $\frac{m+1}{1} \cdot \frac{m+2}{2} - 2$ mêmes droites fixes, se coupent en outre aux $m^2 - (\frac{m+1}{1} \cdot \frac{m+2}{2} - 2)$, autres mêmes droites fixes.[82]

As an example, and still in parallel columns, Plücker detailed the case of third degree curves passing through the same eight points and thus meeting in a ninth fixed point. Or dually, tangent to the same eight lines and thus tangent to a ninth fixed line. Similarly, fourth degree curves passing through thirteen points will meet in another three fixed points, and so on. Returning to a single column, Plücker noted that some of the points could be of higher order, and so not necessarily unique.

Plücker then continued to consider the situation when the coefficients of the representative equations were not indeterminate, but instead subject to certain conditions, this would serve to lower the number of necessary fixed points for the theorem's hypothesis. Plücker summarized these results in Theorem II–presented in only one column.

> Etant donnés n coefficiens de l'équation générale du $m.^{ieme}$ degré à deux indéterminées, ou encore étant données n équations linéaires entre tous ou partie de ces coefficiens ; toutes les courbes représentées par l'équation générale, ainsi modifièe et passant par les $\frac{m+1}{1} \cdot \frac{m+2}{2} - (n+2)$ mêmes points fixes donnes, se couperont en outre aux $m^2 - \frac{m+1}{1} \cdot \frac{m+2}{2} + (n+2)$ autres mêmes points fixes.

[80]In all theorems in this article, Plücker consistently described both given and deduced points and lines as "fixed," perhaps in order to fix the general equation of the second order curve.

(ibid, 100)[83]

Plücker concluded his second section with a reminder that the theorem required the inequality, $n > \frac{m+1}{1} \cdot \frac{m+2}{2} - 1$.

In the third and final section, Plücker demonstrated the fruitfulness of Section II, by showing assorted examples of easily deducible curve properties, including Lamé's Theorem. In particular, Plücker focused on consequences of Theorem II, which could be applied to second degree curves. In this case, given n coefficients or n linear equations containing the coefficients, then all the curves passing through $4 - n$ fixed points would have n additional fixed points in common.

Plücker began with the a general equation of a second degree curve in two indeterminates with six coefficients.

$$Ax^2 + By^2 + Cxy + 2Dx + 2Ey + F = 0. \tag{4.3}$$

Then if F was supposed known, any combination of four coefficients, equations containing coefficients, and points on the curve (e.g. one coefficient and three points) would give an infinite number of curves passing through the same four points. This relationship formed the basis for all Plücker's subsequent investigations in this article.

Plücker began with the special case of the equilateral hyperbola. In rectangular coordinates, the equation representing equilateral hyperbola would have coefficients such that $A + B = 0$. Such an equation reduced the number of necessary givens in equation (4.3). Drawing from this relationship, Plücker determined that all equilateral hyperbolas sharing three given points would meet in a fourth point. Since systems of two perpendicular lines could be considered as hyperbolas, Plücker extended the property to show that the three heights of a triangle concur in the same point. Though not relevant to the remainder of his argument, Plücker's unexpected triangle result provides a striking contrast to Steiner's proof of the same result merely 70 pages earlier in the same journal and indicates another points of intersection between their respective routes.

Plücker returned to the general conic. From (4.3), Plücker concluded that knowing either ratio $\frac{C}{A}$ or $\frac{C}{B}$ would give two conjugate diameters of the curve, where one would be parallel to one of the coordinate axes. Thus all conics with two conjugate diameters parallel to two fixed lines and passing through three fixed points would meet in a fourth fixed points, and conversely.

In particular, the equation of a diameter whose conjugate was parallel to the x-axis would be $By + Cx + E$. So if the ratio $\frac{E}{B}$ were given then one would know the intersection

[83]"Given n coefficients of the general equation of m^{th} degree in two variables, or given n linear equations among all or part of these coefficients; then all the curves represented by the general equation, so modified and passing by the same $\frac{m+1}{1} \cdot \frac{m+2}{2} - (n+2)$ fixed points, will meet in the same $m^2 - \frac{m+1}{1} \cdot \frac{m+2}{2} + (n+2)$ other fixed points."

point of the given diameter with the y-axis. That is, if the y-axis met the curve, then the intersection point would be the midpoint of the chord intercepted. Further, given any point (a, b) on the diameter's extension, one could conclude that $Bb + Ca + E = 0$. Similarly, if one were given another point (a', b'), now on the extension of the diameter whose conjugate is parallel to the y-axis, $Aa' + Cb' + D = 0$.

Continuing this investigation with a given line of the form $\alpha x + \beta y + \gamma = 0$, the equation of the diameter whose conjugate is parallel to this given line would be $\alpha(By + Cx + E) = \beta(Ax + Cy + D)$, a linear equation relating A, B, C, D, E. One could thus be given one, two, three, or four equations of the same form containing A, B, C, D, E. In the last case, and supposing one of the coefficients as given, the curves would be completely determined except for the last term F. From these considerations, Plücker deduced "sur-le-champ" that all conics passing through three given points and in which the conjugates of the diameters that were parallel to the same fixed line concurred in the same fixed point, would meet at a fourth point. Further, if any number of conics passed through the same four points, the conjugates of their diameters parallel to the same fixed line would concur in a fixed point.

The latter result Plücker attributed to Lamé, giving a precise citation of the *Annales* VII, page 229. He suggested that the theorem could be completed as follows.

> Si la droite, à laquelle les diamètres sont parallèles, tourne sur l'un quelconque des points de sa direction, le point de concours des conjugués de ces diamètres décrira une conique, lieu géométriques des centres de toutes les coniques passant par les quatre points donnés (*).
>
> Si deux coniques sont telles quelles interceptent, sur une même droite donnée, des cordes dont les milieux coïncident ; la même chose aura lieu pour toutes les coniques qui, passant par les quatre points d'intersection de ces deux là, couperont la droite donnée. (ibid, 105)[84]

Finally, Plücker generalized the result for conics passing through $4 - n$ given points, which would then have conjugate parallel diameters meeting in n points. Plücker explained that if $n = 4$, then the conics would be similar and concentric such that the given points of intersection would pass to infinity. Plücker's consideration of this special case shows an attention to Poncelet's innovations in the *Traité des propriétés projectives* (1822) in which he established the common points at infinity among similar, concentric conics.

[84] "If the line to which the diameters are parallel turns on one of its points, the point of concurrence of the diameters describes a conic, the geometric locus of the centres of all conics through the four given points (*)."

"If two conics intercept, on the same given line, chords whose midpoints coincide; then the same thing will occur for all the conics which, passing by the four points of intersection of these two, intercept the given line."

As in Lamé's proof, Plücker used coordinate equations without digressing into computation. Plücker presented as known the necessary equations representing conjugate parallel diameters, from there the reader only needed to count the number of coefficients and the number of givens in order to reach the desired result. The entire article remained within the system of rectangular coordinate equations. The conic section was represented exactly by (4.3), and Plücker seemed to use the two terms of reference (conic and equation) interchangeably in his proofs. Specific conics, such as the equilateral hyperbola, were designated with respect to their coordinate relations in (4.3). Plücker's only mention of constructing the determined conics was to point out that the construction of a fourth point shared by two conics would be very easy and one could then find all the points one desires. Nowhere did he give a constructive procedure, thus there were no problem solutions, only theorems and proofs. That this was geometry, and not algebra, was emphasized in the names of things, equations represented geometric objects. While equations and not figures were the form of representation, the objects remained conic sections, diameters, parallel lines, hyperbolas, and points of intersection. Just as Steiner's figures were nowhere to be seen in his theorem statements, so, too, Plücker made no mention of his coordinate equations and coefficients when drawing conclusions.

Though Lamé and Plücker both used coordinate equations to represent geometric objects, we begin to see how the label "analytic geometry" carried dissonant connotations in the context of the two different publications over ten years apart. Lamé would introduce the use of abridged notation in 1818, but in 1817 he relied upon standard coordinate equations to represent curves and surfaces. In Lamé's exposition, the equations became increasingly complex and numerous. While his paper did not include a great deal of calculation, this was because Lamé left the elimination of variables to the reader and simply showing the end result. In part relying upon the technology later developed by Lamé, Plücker could use symmetry and the best choice of coordinate axes to simplify his equations. Plücker further avoided calculation by emphasizing that the curves could be determined, without showing how one might use a given equation to find the exact coefficients. There was no hidden calculation because it was not necessary to secure the geometric result. Plücker's particular form of analytic geometry without calculation was more apparent than it had been in Plücker (1827) or the as yet unpublished Plücker and Schoenflies (1904).

As pointed out in Cournot's 1828 review, Plücker often progressed toward already known results, rather than making new discoveries (Cournot (1828), 178). However, his method towards these known results incorporated recent developments that had not yet been represented through coordinate equations. Gergonne had introduced dual columns in 1824, but they had been used almost exclusively in non-coordinate geometry since that time. In this article, Plücker's used dual columns to state his theorems and showed how to apply polar reciprocity in his final example, where he provided an equation that represented when one point with coordinates (a, b) lay on the polar of another point (a', b') with respect to the second order curve (4.3).

$$Aaa' + Bbb' + C(ab' + ba') + D(a + a') + B(b + b') + F = 0 \tag{4.4}$$

Plücker deduced from the symmetry of equation (4.4) that the result was reciprocal–the point (a', b') lay on the polar of the point (a, b).[85]

Plücker further incorporated new geometric practices with his use of common points at infinity between any concentric conics. Poncelet had motivated this terminology to explain why "analytic geometry" could in general represent points of intersection for any two second degree equations. In importing this concept to an article considered as analytic geometry, Plücker did not provide a coordinate representation for these points, although the step of passing points or lines to infinity was not strictly constructive either. Nevertheless, it was becoming a common procedure, even in articles that Gergonne designated as "elementary geometry" (Steiner (1828c)). Plücker demonstrated a strong familiarity with contemporary research, and an eagerness to expand the domain of analytic geometry. With the new geometric objects developed by Poncelet and the new abridged notation, the practice of using coordinate equations to solve geometry problems was evolving.

4.3.4 Gergonne's footnote, citation, and text

The asterisk in Plücker's so-called completion of Lamé's theorem referred the reader to a footnote by Gergonne who succinctly stated that "this is precisely what was demonstrated on page 106 of the preceding volume." Gergonne's note was perhaps not *precisely* Plücker's statement, but certainly a special case. There, Gergonne had given a proof that the conjugate of the parallel diameters of all the ellipses circumscribed to the same quadrilateral concur in a point and this point was constantly on the perimeter of the hyperbola which was the locus of centres of the ellipses (Gergonne (1827a)). The first part of the result was attributed to Lamé, although a different text than Plücker's citation. As reference in an accompanying footnote, Gergonne pointed to "une démonstration fort élégante de cette proposition, ainsi que beaucoup d'autres choses intéressantes, dans un petit ouvrage de M. LAMÉ, ayant pour titre : *Examen des différentes méthodes, etc.* ;" (Gergonne (1827a), 106).[86]

Gergonne, although proving the extended version of this theorem, attributed the original statement to an article by Steiner in Crelle's *Journal* from 1827 written in response to a question posed by the geometer Étienne Bobillier in the *Annales* on finding an ellipse

[85] Plücker would develop the implications of this last example when he turned his attention to the use of duality with coordinate representation in 1830 (Plücker (1829b)). See Appendix E.

[86] "[...] a very elegant proof of this proposition, as well as many other interesting things, in a little work of Lamé, *Examen des différentes méthodes, etc.* ;"

closest to a circle given certain limiting conditions (Steiner (1827b)). Despite admiring what he described as Steiner's elegant theorem, Gergonne expressed dissatisfaction with the proof, claiming that it too often relied upon "generally known" results which were not well known at all. Indeed, the criticism from Gergonne's assessment of Steiner in 1826 had reversed. Gergonne did not attempt to reconstruct Steiner's proof, instead taking the material in his own direction. While Steiner's proof had been constructive, Gergonne's proof relied upon calculations with coordinate equations. Accordingly, Gergonne did not consider this a translation or abridgement of Steiner, but claimed it as an independent result. Like Plücker, Gergonne began with the same general form of a conic, (4.3). Unlike Plücker, he then introduced many other variables to serve in algebraic, trigonometric, and differential calculations. These were eventually simplified to reach the desired conclusions. In demonstrating his version of Lamé's theorem, Gergonne derived the equation of a tangent line drawn to the extremities of the diameter ($y - u = m(x - t)$) of the circumscribed ellipse as

$$(A + Cm)x + (Bm + C)y + (D + Em)\sqrt{\frac{-k}{A + Bm^2 + Cm.}} + k = 0,$$

which resulted in the comparatively simple equation of the conjugate diameter, as

$$(A + Cm)(x - t) + (Bm + C)(y - u) = 0.$$

Gergonne then substituted in the fixed line $y = mx$ to find the fixed point given by the equations

$$Ax + mBy + (D + mE) = 0, mx + y = 0.$$

Even in this brief excerpt, we can see that Gergonne delved into algebraic computation. Gergonne's method had neither Plücker's simplicity nor Steiner's use of figures. The proof only applied to ellipses and resulting hyperbola loci, not a general second order curve. Gergonne made no comment on the particularity of his version nor whether one might be able to extend the results to other conic sections.

Thus, readers of the *Annales*, and certainly Steiner among them, might have known that Steiner had already given another version of Lamé's Theorem in an earlier context. In fact, turning to Crelle's *Journal*, Steiner had stated the so-called completion of Lamé's Theorem while determining an optimal ellipse, without proof and with the vaguest of possible references in 1827.

> Alle Kegelschnitte, welche durch vier gegebene Puncte gehen, haben ein System konjugierter Durchmesser, die parallel sind, und ihre Mittelpuncte liegen in der Peripherie eines anderen bestimmten Kegelschnitts K. Dieses ist bekannt.

(Steiner (1827b), 65)[87]

On the one hand, Lamé's theorem and its proof was known. On the other hand, Steiner's extreme brevity (this article runs less than two full pages–Gergonne's analytic proof of the same results goes for ten), certainly demanded a great deal of background knowledge on the part of his reader. However, we suggest that Gergonne's proof in the *Annales* was also motivated by an opportunity to advertise his analytic geometry prowess, or perhaps just utilize the only method readily available to him. Steiner himself made no reference to this earlier result when returning to Lamé's theorem the following year, and as we have seen his statement of the theorem took quite a different form without conics or conjugate diameters in this later version. It is only through Gergonne's reference that we are able to identify the two results as following from the same source. So we find another instance of theorem identification through a common reference, as well as further evidence of the popularity of reproving Lamé's theorem.

4.3.5 Lamé's theorem: Conclusions

That Lamé's theorem was repeatedly re-proven was not a statement against Lamé's original proof. Then why prove Lamé's theorem at all? Certainly, none of these texts aimed at a comprehensive or systematic survey of conic section properties. Steiner's assorted theorems had little to no overarching design. Plücker described his applications as a chance assortment of possible results. Both geometers hinted at selecting examples from a vast wealth of other similar results. With Lamé's theorem, either geometer might have simply cited the well-known result, but chose instead to use the opportunity for proof to show off their particular methods. Yet, neither geometer attempted an entirely self-contained exposition. Steiner pointed the reader vaguely to Carnot for the rationale behind finding a point of determinate ratio. Even less explicitly, Plücker introduced several equations with the preface "one knows."

In deciding to also present a proof of Lamé's theorem, Plücker may have been following Steiner. Plücker's shorter proof, free from construction and calculation, could have contrasted favourably, especially following the *Bulletin* review of Steiner's paper which had recommended the use of analytic geometry. Plücker had also applied his method to show that the perpendicular bisectors of the sides of a triangle concur in a point. This had been Steiner's first theorem, which he proved using proportion equations "adding, reducing, and transposing" in a very computational manner. Plücker proved the same result by considering perpendicular lines as a special case of an equilateral hyperbola. Although Plücker

[87]"All conic sections that go through four given points have a system of conjugate diameters, which are parallel, and their common points lie on the periphery of another determined conic section K. This is known."

may have had Steiner in mind with the inclusion of these two theorems, the overall thesis of his article suggests an independent and long-standing research topic into the relationship between n^{th} degree curves and given points. Lamé's theorem and the theorem on perpendicular bisectors were only evidence of the theory's fruitfulness.

For Steiner, too, proving Lamé's theorem did not motivate his article's content. However, its inclusion by both geometers associated their research with contemporary geometry. If a method was capable of proving well known results, then it could be considered fruitful and worth adopting. In a pedagogical context, the general applicability of a method might carry even more weight. In order to generate support for their methods, they needed to demonstrate the methods' capabilities in familiar geometric territory.

Although Plücker and later Steiner promoted new methods over new results, the resulting theorem statements in their articles obscured the derivation. In both Steiner and Plücker, the theorem's hypothesis stated the minimum required givens to reach the conclusion, without trace of the approach. For Steiner, in the proofs and figures, each point and line was defined with respect to a given conic section, which disappeared entirely in the theorem's statement. Since three non-collinear points uniquely determined a circle, there was no need to explicitly describe it. For Plücker, the representation of geometric objects with coordinate equations was stated as well-known. "All conics" implied the general second degree equation in two variables with six coefficients. The method could not be inferred from the individual results.

Turning to the proof structure, Steiner began with what he wanted to show and worked backward to initial conditions. This strategy worked well in proofs, and could be used to extend a particular result to further cases. As discussed above, Steiner's argument structure was perhaps more imitable in his mode of generalization through projection, perspective, reciprocity, and points at infinity. In this way, Steiner demonstrated an applicable method of discovery.

Plücker began with a general initial theorem pertaining to curves of any degree. He then proceeded to focus on the more particular case of second degree curves. His specific arguments were well-anticipated and his use of coordinate equations straightforward to follow, but did not suggest applicability beyond variations on the same theme of four given conditions to determine a set of conic sections because the number of variables was so crucial to the argument. This was also the case for Lamé's original statement. For Plücker the use of coordinate equations to represent geometric objects opened a path toward simplification for analytic geometry, a contribution that outweighed discovering particular theorems or problems.

4.4 Conclusion

We have seen how Steiner and Plücker produced different representations, applications, and contexts for familiar geometric content. Steiner's geometry was criticized as complicated and laborious, but capable of generating many problem and theorems, both old and new. As constructive geometry, Steiner's geometry could be directly applied to figures in the plane, although, unlike Poncelet's synthetic geometry, we found no descriptions of Steiner's contributions as practical. Plücker's geometry was praised as simple and followable, yet dependent on old results for content. His use of coordinate representation required translation back into geometric figures for solving geometric problems, and this translation was not included in Plücker's theorem and theory driven article, though explicit in his problem solving.

Both of Steiner's articles reviewed here were classified as *pure geometry* by their section heading in the *Annales*, and most likely this designation had been decided by Gergonne. At this point in his career, Steiner seemed open to any number of geometric methods, and even commended Bobillier's use of analytic geometry (Steiner (1828a)). In his articles, he did not describe his own method as pure geometry, and his only published methodological commentary at this time had been with respect to the pedagogy of Pestalozzi. While we have seen that he was much more explicitly methodological in correspondence, his appreciation for synthetic methods was not in opposition to the use of coordinate equations.

In 1832 Steiner published his first book, *Systematische Entwicklungen*, where he developed a public formal statement of his personal geometric method (Steiner (1832)). However, even in this context, Steiner did not exclude or denigrate other geometric methods. To the contrary, in prefacing a list of posed problems and theorems for his readers, he offered the option to employ and practice his method or to follow another method instead (Steiner (1832), 439). In this later text, when Steiner described pure geometry, he intended geometry without the use of sensual mediums (*Versinnlichungsmittel*) such as figures or equations, conducted only by the use of imagination (*Vorstellungskraft*) (ibid, 306). This was certainly not the kind of geometry employed in his solution to the Apollonius problem nor proof of Lamé's Theorem. Steiner's desire for systematicity, unity, and intuition did not entirely manifest in shorter articles.[88]

[88] Steiner's lifelong dedication to these higher principles is attested by Jacobi's 1845 letter of recommendation for Steiner to receive a full professorship, in which Jacobi summarized Steiner's contributions over the past twenty years. Jacobi described *Systematische Entwicklungen* as a holistic and exemplary text for all of mathematics.

> Indem er so den Organismus aufdeckte, durch welchen die verschiedenartigsten Erscheinungen in der Raumwelt miteinander verbunden sind, hat er nicht bloss die geometrische Synthese gefördert, sondern auch für alle anderen Zweige der Mathematik ein Muster einer vollkommen Methode und Durchführung aufgestellt. (Jahnke (1903b), 278)
>
> "Thus he revealed the organism through which diverse manifestations in space are interconnected, he has not only promoted geometric synthesis, but also established a model of a perfect

Plücker's methodological position emerged more powerfully in his *Analytisch-geometrische Entwicklungen* (Plücker (1828a)). In the preface, he described his method as "pure analytic geometry" and criticized Steiner for "following in the footsteps of Poncelet." Yet, as suggested by Crelle's letter of recommendation described above, Plücker's preface was not necessarily indicative of his monograph's contents. Moreover, we suspect that Plücker may have directly belittled Steiner's recent publications, in order to further contrast the independence of his own work that had already suffered by comparison to Poncelet.

As we saw in Chapter I, by the 1870's Steiner and Plücker had become emblematic of the competing methods of analysis and synthesis. Contrary to these stereotypes, neither Steiner nor Plücker exhibited an orthodoxy of method in these early articles. Our two case studies reveal that the line between pure and analytic geometry was but faintly drawn and could be classified neither in terms of calculation nor by the presence of figures. Both Steiner and Plücker used some calculation and both geometers provided constructions based in figure manipulation (in Plücker's monograph and Crelle publications, there are illustrated figures too). The tools of figures and calculation were at times complementary, and certainly served different purposes, one could not replace the other. An analytic proof without translation into figures would not qualify as a geometric solution. Analogously, while the common (ideal) chord between two non-intersecting conics could be easily portrayed in a linear equation, drawing such an object defied common sense (and in part explains why the less figurative term "radical axis" was often preferable). Moreover, these two cases have shown that Steiner and Plücker displayed personal styles that resisted standard methodological classification. Steiner's "synthetic geometry" and Plücker's "analytic geometry" were peculiar to them.

We have commented in several places on the cause of repetition in the repeated solving and proving of the Apollonius problem and Lamé's theorem. With respect to the criteria decided by Gergonne and Poncelet, the theorem or problem alone was not sufficient in determining the best method. Choice would also depend upon the ultimate purpose the theorem or problem intended to serve. Should the geometer (like Steiner or Plücker) intend to use the theorem or problem primarily to promote their own method, then the method chose the content and not vice versa. Repeating well known results functioned as evidence of a method's worthiness and simultaneously advertising familiar material to a potential reader.

At the same time, we must consider the potential effects of repetition on the geometry. From the case studies considered here, two facets of repetition stand out as powerful forces in shaping the image of geometry. First, repeated content enabled the disappearance of the constructive figure from non-elementary texts. New vocabulary continued to replace

method and execution for all other branches of mathematics."

the process of finding geometric relationships, as we saw in the case of similitude and radical objects. These tools allowed abbreviation and appeared to simplify, but to actually draw a radical axis required a step by step constructive definition. A seemingly short construction, like those of Plücker and Poncelet, might in fact be quite labor intensive to execute. Secondly, geometers emphasized the differences between their methods in order to compensate for the similarity of their results. The importance of method only came to the front when geometers described their work, or their work was described by others. In 1839, Crelle wrote a letter of support for Plücker to the ministry, in which various styles of synthetic geometry were lumped together in order to differentiate Plücker's contributions.

> Auch die der analytischen gegenüberstehende, oder vielmehr neben ihr bestehende synthetische Methode, mit welcher Poncelet, Möbius usw. und besonders Steiner so Vieles und Bewunderungswertes geleistet haben, mag zu allen Resultaten ebenfalls gelangen können, indessen kommt es bei der weiteren Entwicklung der Mathematik fast noch mehr als auf die Resultate auf die Vervollkommnung der Methoden an; denn die Methoden sind die Werkzeuge, um immer noch weitere neue Resultate zutage zu fördern; und um der Vervollkommnung der analytischen Methode in der Geometrie hat sich nach meiner Überzeugung Herr Plücker seinerseits schon durch seine früheren Schriften und jetzt wieder durch die vorliegende, ein wesentliches und bedeutendes Verdienst erworben. (Ernst (1933), 31)[89]

From a different perspective, we have seen how French mathematicians associated Steiner and Plücker as German geometers. This same muting of idiosyncrasies between Poncelet and Steiner as synthesists or between Gergonne and Plücker as analysts, each of whom saw the other's work as very different, resulted in an apparent dichotomy. The image of geometry that appeared in the late nineteenth century portrayed their predecessors as separated and ultimately calcified by a spurious methodological divide. In support of this image, late nineteenth century geometers cited the frequent methodological claims made by their French and German predecessors. Yet, the methodological claims did little to describe the qualities of the particular geometer or of the recurring content.

In the midst of this repetition, geometers argued for novelty of form or novelty of method. However, we have not yet examined what constituted the older or even "ancient" geometry, nor what kind of geometry was practiced by contemporaries of Gergonne, Poncelet, Plücker

[89]"The analytical opposed, or rather next to its ruling synthetic method, by which Poncelet, Möbius, etc. and especially Steiner have accomplished so much that is so admirable, can succeed as well in all results, however the further development of mathematics occurs almost more with the improvement of method than the results; because the methods are the tools to promote still further new results; and to the improvement of the analytic method in geometry, in my opinion, Herr Plücker has in turn acquired meaningful and significant worth by his earlier and now the present writings."

and Steiner outside of Gergonne's *Annales.* In order to evaluate these claims for novelty, we look to the context of French books on geometry published in the first third of the nineteenth century.

Chapter 5

Claims for novelty in geometry research in the context of French books (1800–1833)

The historical conception of early nineteenth century geometry, as described in Chapter I, motivated our focus on the the texts of Gergonne, Poncelet, Plücker and Steiner, as epitomizing the progress and innovation within geometry during the first third of the nineteenth century, particularly between 1817 and 1832. As our case studies in Chapters II, III, and IV have demonstrated, Gergonne, Poncelet, Plücker and Steiner knew and responded to each other's work within a close network of researchers, mostly publishing in the *Annales*. While the *Annales* boasted a wide range of French and European contributors over its existence, it does not necessarily provide a model for the majority of French geometry published between 1810 and 1832.[1] By focusing on our particular authors, we have emphasized research articles. Even Poncelet's *Traité* was subsequently reiterated in numerous *Académie* readings and article publications (Poncelet (1828b), Poncelet (1826), Poncelet and Cauchy (1825), Poncelet and Cauchy (1820)).

Alongside the texts considered above, the early nineteenth century also boasted a substantial output of books on geometry. The first third of the nineteenth century includes the time in which Poncelet, Plücker and Steiner received their formal education and wrote their first monographs as well as the entire existence of Gergonne's *Annales*, and we will consider texts published in this interval as contemporary to our above studies. We further limit our corpus to French publications in reflection of the formative training in French geometry and

[1] For a statistical analysis of *Annales* contributors by nationality, subject matter, and professional affiliation, see Otero (1997). Gerini and Verdier document the contents of the *Annales* as compared to the later *Journal de Liouville* in Gérini and Verdier (2007). Similarly, Delcourt compares the contents of the *Annales* and the *Nouvelles Annales* with respect to elementary geometry in Delcourt (2011b).

demonstrated desire for a French audience exhibited by all four of our main actors. We have seen that Poncelet, Gergonne, Plücker, and Steiner presented their work as modern or new, and Chapter III revealed concerns regarding properly introducing new principles of geometry to the mathematical community. However, as the case studies in Chapters II and IV show, geometers often re-appropriated problems, solutions, theorems, and proofs from contemporaries or older sources. By surveying the books published between 1800 and 1833, we will provide a deeper picture of how geometers saw the geometry of their time, in order to better evaluate claims for novelty in geometry research. Further, we will use the context of books to reexamine the questions considered with respect to articles in earlier chapters. Namely, what was the role of the figure? How prevalent was repetition? Were methods differentiated, and if so, how? We will find that these books reveal a different set of interests, authors, audiences, and method definitions than those involved in research articles.

5.1 Defining a corpus: catalog of the Bibliothèque national de France

To obtain an appropriate corpus of contemporary geometry books, we first queried the Bibliothèque nationale de France library catalog for all texts including the keyword "Géométrie" and published between 1800 and 1833 (`www.bnf.fr`). This search returned ninety-two unique titles, about a quarter of which appeared in at least two editions (the elementary geometry texts of Sylvester François Lacroix and Adrien-Marie Legendre were published in fourteen different editions each!).[2]

We refined our corpus by removing about thirty books on practical geometry, including those with the title "géométrie pratique" and geometry applied to industry or design. Practical geometry, as described by F.-J. Servois in *Solutions peu connues de différens problèmes de géométrie-pratiques* concerned the study of executing "diverse geometric operations on the terrain" (Servois (1803), 1). Although texts on practical geometry included preliminary theory, they emphasized geometry in the field, using particular measuring tools and creating working diagrams. Authors of these texts explicitly stated that their audience consisted of surveyors, workers, architects, and designers. Since our case studies so far have focused on geometry written for those who considered themselves as students, teachers, and geometers, we limited our comparative survey of books accordingly. Even with this pre-emptive

[2]Three texts were listed in the BnF catalog, but reported "hors usage," and thus could not be accessed. These were Jules Planche *Observations sur les propositions de géométrie en général, et sur les problèmes en particulier.* Hachette, Paris, 1828; Charles-Félix Fournier, *Élémens de géométrie.* Lefournier et Depériers, Brest, 1829; A. Brocchi *Élémens de géométrie.* Boisgard, Paris, 1833. G. F. Olivier's text *Géométrie usuelle* could only be accessed in its third edition (published in 1835), although appearing after 1833, because the first two editions were published in our time frame (1828, 1832), we included it in our analysis.

exclusion, we will see that nearly every text on geometry included discussions and examples of applications. For this reason, we did not classify texts on geometry *with* applications as "practical geometry," such as Charles Dupin's *Développements de géométrie, avec des Applications à la stabilité des Vaisseaux, aux Déblais et Remblais, au Défilement, à l'Optique, etc.; ouvrage approuvé par l'Institut de France, pour faire suite à la géométrie descriptive et à la géométrie analytique de M. Monge*, which is included in our corpus (Dupin (1813)). We have further seen this fluidity between theory and application in the work of Poncelet, who wrote and worked both as a mathematician and an engineer, and indeed subtitled his *Traité* as also containing "applications to descriptive geometry and to geometric operations on the land" (Poncelet (1822)). However, Poncelet segregated these research interests, as we found with respect to his description of professional obligations that restricted his geometric research in Chapter III (Section 3.5), and can also be seen in the organization of his book. While a potential reader might find practical uses in the articles on geometry in Gergonne's *Annales*, the excluded books on practical geometry were written primarily if not exclusively for the purpose of application to a particular industry. We thus defined practical geometry with respect to a specific intended public that overlapped, but was not representative of, the public who wrote and responded to research articles on geometry.

We next removed any titles dedicated to the theory of parallels or the quadrature of the circle. This included not only texts explicitly designated as such, but also Joseph-Marie Lançon's texts *Découvertes intéressantes dans la géométrie élémentaire* and *Supplément à la géométrie, renfermant des découvertes importantes*, and Laurent Potier Deslaurières' *Nouvelle découverte qui embrasse toute la géométrie, qui donne la solution de ses plus grands problêmes, et qui va reculer les bornes de l'esprit humain* (Lançon (1801), Lançon (1802), Deslaurières (1805)). Both of these authors claimed to offer new discoveries on quadrature of the circle as the primary goal of their texts. In fact, we found Deslaurières' article collected in a 400-page collection of circle quadrature papers written between 1804 and 1876.[3] Although we are removing these texts from our corpus, we note that they bore certain similarities to other geometry publications. Deslaurières presented his text to the *Académie des sciences* (where it apparently received no response) and Lançon had figures engraved by the same artist who engraved Lazare Carnot's well-regarded *Géométrie de position* (Carnot (1803)). Finally, their innocuous titles promising new and interesting discoveries were not unlike the geometry collections entitled *Mélanges de mathématiques* or *Mélanges d'analyse algébrique et de géométrie*, which will be discussed below. These texts on circle quadrature represented a small trend still current in elementary geometry. As Deslaurières commented, the *Institut national* "empêchoit ses Membres de s'occuper de

[3]Marie Jacob analyzes several eighteenth century texts on the quadrature of the circle and the discussion surrounding the subsequent ban by the *Académie royal des sciences* in Jacob (2005) and Jacob (2006).

la découverte de la QUADRATURE DU CERCLE."[4] Nevertheless, this did not prevent schoolteachers and members of other professions from seriously considering the possibility, and as Deslaurières reasonably suggested, "personne n'a pu donner encore la démonstration exacte de son impossibilité: elle restait donc dans l'ordre des choses possibles" (Deslaurières (1805), ii).[5]

Finally, we eliminated from our corpus translations and commentaries of Euclid's *Elements* and a geometry book written as a series of questions and answers for children. Our corpus thus reduced to fifty-two distinct titles, which appear chronologically in Appendix B. When we consulted multiple editions of the same title, these received multiple chronological entries in our table.

We acknowledge that this form of search did not gather all books on geometry published in French between 1800 and 1833. For example, Poncelet's 1822 *Traité des propriétés projectives* was not found in this search because this first edition did not receive any classification within the library catalog and the word "géométrie" is cut-off from the full-title, it reads "Traité des propriétés projectives des figures..."[6] To confirm that our search was representative, we conducted the same search through the Library of Congress online catalog (http://catalog.loc.gov/), which resulted in twenty-one titles. All but one of these was on our list (this was, the 1812 edition of Étienne Bézout's *Cours de mathématiques* originally published in 1752 () (1812 (1752))). The same search through the Catalogue collectif de France (http://ccfr.bnf.fr/portailccfr/jsp/index.jsp), not including the Bibliothèque nationale, returned sixty-five unique texts, eighteen of which are not in our corpus.

For each text we consulted the title page, table of contents, any prefatory remarks, and the sheets of figures (nearly always located at the very end of the volume).[7] When the table of contents indicated methodological commentary or material on imaginary and infinite objects, problems or theorems investigated in our earlier case studies, we reviewed the relevant pages.

After reviewing the contents we separated the titles into five categories for comparative analysis. Based on the titles, we assigned most texts to one or more of three major categories: elementary geometry, analytic geometry, and three-dimensional geometry.

Titles of elementary geometry texts contained the term "géométrie élémentaire," "éléments (élémens) de géométrie," "cours de géométrie," or "géométrie usuelle." We will see

[4] "[...] forbids its Members from occupying themselves with the discovery of the QUADRATURE OF THE CIRCLE."

[5] Despite Deslauriès' insistence, most early nineteenth century mathematicians believed these problems were impossible, as discussed in Lützen (2009).

[6] The 1865 editions were classified as *Géométrie descriptive* and the full title is printed, thus these do show up if there is no date restriction.

[7] Unfortunately, when consulting scanned texts, the figure pages were often poorly copied. While disappointing, this feature was in general not a detriment toward understanding the books' content nor the author's textual use of figures.

that these eighteen texts were all intended for beginning students with no formal knowledge of geometry.

The sixteen texts classified as analytic geometry had titles including the phrase "analytique," "analyse" or "algèbre". We also classified Olry Terquem's *Manuel de géométrie, ou Exposition élémentaire des principes de cette science* as both elementary and analytic geometry following his preface where he described the use of algebra applied to geometry as a feature in his book. We will further discuss Terquem's unusual choice to include algebra in elementary geometry below.

Intriguingly both texts entitled broadly as "Cours de mathématiques," (*Cours de mathématiques* by Charles Bossut (Bossut (1800), first edition 1772) and Nicolas-Louis de LaCaille () (1811 (1741) first edition 1741), dated back to the eighteenth century. Consulting the subtitles and tables of contents revealed that these texts were separated into multiple sections, where elementary geometry preceded "l'application de l'algèbre à la géométrie." As such, we categorized these texts as both elementary geometry and analytic geometry. The fact that no other texts in our corpus intended to cover all of mathematics in a single book suggests that this approach was less prevalent among the nineteenth century authors.

Finally we classified as three-dimensional geometry, all texts entitled "géométrie descriptive", "géométrie perspective", or "géométrie à trios dimensions." As we will see, these twelve texts all concerned techniques for representing solid objects in the plane and interpreting planar drawings as solid objects. Based on their subtitles, Charles Dupin's *Développements de géométrie [...] pour faire suite à la géométrie descriptive et à la géométrie analytique de M. Monge* and Jean-Nicholas-Pierre Hachette's *Éléments de géométrie à trois dimensions* divided into *Partie synthètique* and *Partie algèbrique*, were classified as both analytic and three-dimensional geometry (Dupin (1813), Hachette (1817)). We will find that these two authors clearly separated the two approaches to geometry within their books, and indeed Hachette may have originally published the two parts as distinct volumes.

From consulting the table of contents in the remaining nine texts, we found two distinct additional categories. Five texts concerned what we will call "geometry of the ruler or compass" (the "or" is exclusive in these texts) and four texts contained mixed collections of problems, theorems, and methods. The particular qualities of all five categories will be discussed at greater length below.[8] When multiple editions appeared in our designated time interval, we consulted as many as were available. The exact titles and publication information can be found in our references.[9]

[8] Augustin-Louis Cauchy's *Leçons sur les applications du calcul infinitésimal à la géométrie* is the sole text on calculus returned by our search (Cauchy (1826)). As Cauchy's position as a professor of analysis at the École polytechnique and the course sequence indicates, this area of study was not considered geometry. We include this text in our count because of its title, but beyond the first 35 pages of analytic geometry review the content bears little in common with any other considered here. While an interesting outlier, Cauchy's text did not inform our general survey of early nineteenth century geometry.

[9] We will initially refer to all geometer's of these primary texts by their names as printed in the titles of

Based on this survey, we are able to craft an image of early nineteenth century French geometry as codified in books. This image provides a point of comparison and contrast to contemporary geometry articles. We will first discuss several features common to these texts, beginning with the definition of geometry itself.

5.1.1 The definition of geometry and other common features

Regardless of methodological or subject differences, authors uniformly defined geometry as the study of objects in space. Though not every text included an explicit definition of geometry, we found no significant difference between the definitions in elementary, analytic, or descriptive geometry. While the notion of space was presented as more or less theoretical or physical, these differences did not appear related to the level of abstraction in the accompanying text. As an early example, in his course on mathematics Charles Bossut began by contrasting geometry and analysis, and concluded that geometry was more concrete and grounded in sensual objects that could be seen or touched.

> L'objet de la géométrie est plus déterminé et moins abstrait que celui de l'analyse. Cette dernier branche des mathématiques compare ensemble toutes sortes de quantités, et les symboles qu'elle emploie représentent des rapports généraux et purement intellectuels. La géométrie considère la grandeur en tant seulement qu'elle est étendue et figurée: elle emprunte le secours de la vue ou du toucher pour établir les relations que les lignes, les superficies et les solides ont avec leur unité fondamentale. (Bossut (1800), v)[10]

For Bossut, the solids of geometry had real existence, while one could only imagine the independent existence of a surface or line.

Other definitions more emphatically differentiated the qualities of geometric space and the space where real bodies could be measured. In his text on elementary geometry, Louis Bertrand described how geometrical space was infinite and homogeneous, an abstraction from the physical.

> La Géométrie est relative à l'étendue : l'étendue ou l'espace, abstraction faite des corps qui s'y trouvent, est infini et homogène, ses parties ne sont séparées par aucune limite; mais on peut les concevoir grandes ou petites, figurées de façon ou d'autre; la difficulté est de s'en faire des idées si distinctes, que l'on

their texts in order to maintain consistency. Several of the author's were only known by their surnames or first initials. In all subsequent references we will use only surnames, which are unique to each author here. Authors' full names, dates, and École polytechnique classes (if known) are given in Appendix C.

[10] "The objective of geometry is more determinate and less abstract than that of analysis. This latter branch of mathematics compares all sorts of quantities and the symbols it employs represent general and purely intellectual relationships. Geometry considers size only as it is extended and figured: it makes use of sight and touch to establish the relations that lines, surfaces and solids have with their fundamental unit."

puisse assigner les rapports qu'elles ont, soit entr'elles, soit avec le tout dont elles font partie. (Bertrand (1812), iii)[11]

Similarly, in his analytic geometry, F. H. Francfort described the space of geometry as indefinite and relative.

> L'espace tel que les géomètres le considèrent est une étendue indéfinie dans laquelle on conçoit que tous les corps sont placés. On ne peut donc y déterminer le lieu absolu des corps, mais seulement leurs situations relatives qui sont les seules dont la connaissance nous soit nécessaire, et pour cela on rapporte ces points à des objets fixes dont on suppose que la position en est connue. (Francfort (1831), v)[12]

Although abstract geometry could only be "conceived," in its three dimensional qualities it remained representable as a volume, an area, a length, or a position. In turn, geometry was highly applicable to understanding, representing and manipulating space. Terquem traced its historical dimension with respect to national and imperial divisions.

> La délimitation des divers pays occupés par les nations, la fixation des bornes entre les propriétés territoriales des particuliers, sont des opérations qui ont donné naissance à une science que, d'après son origine, on appelle Géométrie ou Science de la mesure des terres. (Terquem (1829), iii)[13]

Even the etymology of geometry served as a reminder of the connection of geometry to physical measurement. As G. F. Olivier observed, geometry had developed beyond this initial meaning, but remained the science of extension in space.

> La GÉOMÉTRIE tire son nom du principal usage auquel il semble que cette science fut employée dans l'origine, je veux dire, du mesurage des terres, car ce mot signifie art de mesurer la terre. Mais ensuite on l'a appliquée à tout ce qui concerne l'étendue des corps: ainsi la Géométrie est devenue la Science de l'étendue. (Olivier (1835), 1)[14]

[11] "Geometry is relative to extension: extension or space, an abstraction made from bodies that are found there, is infinite and homogeneous, its parts are not separated by any limit; but one can imagine them to be great or small, figured somehow or other; the difficulty is to form distinct enough ideas that one can assign their relationships either among themselves or with the whole of which they form a part."

[12] "The space that geometers consider is an indefinite extension in which we imagine that all bodies are placed. One cannot thus determine the absolute position of bodies in it, but only their relative situation, which is the only necessary knowledge for us, and for that one relates these points to fixed objects whose position is supposed known."

[13] "Demarcating different countries occupied by nations, fixing boundaries between particular territorial properties, are the operations which have given birth to a science that, according to its origin one calls Geometry or the Science of land measure."

[14] "GEOMETRY takes its name from the principal usage for which it seems that this science was originally employed, that is, land measurement, because this word signifies the art of land measuring. But then we have applied it to anything which concerns bodily extension: so geometry has become the science of extension."

Together these definitions suggest a constant rapport between geometry and physical applications. Emphasizing this relationship, authors constantly cited the utility of their approach and results to promote their texts. From its initial role in measuring the earth, geometry remained bounded by sense perception. While not exactly seen and touched, the objects of geometry could be conceived in space, and we will see this theme reiterated in discussions on the relationship between geometry and algebra. Algebra could be *applied to* solving geometry problems or proving geometry theorems, but geometry itself was the study of measurement or position in space. All problems and theorems of geometry concerned either measurement or relative position of geometric objects, and for the majority of practitioners, geometry proper did not include coordinate equations.

5.1.2 Common qualities of French books on geometry: figures, publishers, and the École polytechnique

Most title pages contained not only the bibliographic details, but often also a subtitle indicating the author's intended purpose and a brief description of the author. In these two descriptions, the École polytechnique emerged as a common background institution among about half of authors (Appendix C), as well as the explicit proposed audience for twelve different texts (Lacroix (1803b),) (1809 (1795), Poullet-Delisle (1809), Biot (1810), Dupin (1813), Garnier (1813), Vincent (1826), de Fourcy (1827), Duchesne (1829), Mutel (1831), Reynaud (1833), Olivier (1835)). Among the authors considered, six worked at the École polytechnique, which of course included the figurehead of polytechnique geometry, Gaspard Monge. In addition, twelve of the authors proudly announced having been students at the École polytechnique, a number which can be considered only a minimum for attendees. Eleven of the texts were explicitly intended as preparation for polytechnique exams or to be used in courses. Finally, the École polytechnique publishers and bookstore: Bernard, Madame Bernard, and then Klostermann, published three of the titles.[15]

Most of the books were printed and distributed by the same Imprimerie-Libraire for mathematics in Paris that published Gergonne's *Annales*. The printer and seller Jean Courcier, followed by his widow (born Victoire Félicité Lemaire and referred to as "La veuve Courcier"), and then his son-in-law, Charles Louis Étienne Bachelier, published thirty-five of the books considered here. Throughout the various incarnations of this publishing house, the title page of each geometry text listed the previous owner and current address.[16] The

[15]The creation and influence of the École polytechnique has been a subject of historical interest almost since its inception. Consider Dupin's 1813 introduction in which he expressed admiration for the legacy of former École polytechnique students tinged with pessimism about the institution's trajectory (Dupin (1813), xi). For more contemporary analyses, see Shinn (1980), Dhombres and Dhombres (1989), Langins (1987), Belhoste (2001).

[16]A brief history of Bachelier and his predecessors can be found in Verdier (2011). Verdier also discussed printing, typography, and the general production of Liouville's *Journal* in Verdier (2009a).

remaining books were distributed by a score of different publishers. In contrast to Courcier and Bachelier, often the two operations of printer and publisher were distinct, that is, the book would be printed by an "Imprimerie" and then distributed from a "Libraire" or even several different bookshops. In this case, the "Libraire" served the role of the publisher by distributing and sometimes advertising existent or forthcoming texts. When separate from the publisher, the printer's information appeared before the title page in small font usually toward the bottom of the page. Besides Paris, these books were also distributed from Geneva, Brussels, Reims, Lyon, Metz, Bordeaux, and Nancy. To prevent plagiarism, authors occasionally included a short paragraph declaring that unsigned books were not legal copies, followed by their signatures.[17]

At the end of each text, all but two of the books considered here contained at least a page of figures. Often the figures were signed by the designer, engraver, or lithographer.[18] The most commonly used engraver was "Adam" who provided the figures for various texts by Jean-Baptiste Biot, Charles Julien Dupin, Lacroix, Louis-Etienne Lefébure de Fourcy, and A. Lefévre.[19] Among the thirty-one texts in which some signature was legible, we found twenty-one different artists, most commonly designated as "sculp." As distinct from other scientific illustrations, geometry figures were intended to be reproducible by following the author's textual instructions, usually with a compass and a straight edge. Thus it is possible that some artists might have worked directly from the text in producing their illustrations. We were not able to ascertain how the authors chose the artists or communicated the figures to them. However, the figure designer for Alexandre Vincent's elementary geometry text, Guilliaume Henri Dufour, wrote a text on perspective geometry the following year, indicating a certain degree of cross-pollination between the two groups (Vincent (1826), Dufour (1827)).[20] While almost every geometry text contained figures of polygons and circles, the variety of figures depended on the type of geometry. In M. H. Vernier's *Géométrie élémentaire*, he defined figures as strictly two-dimensional objects:

> Chaque corps occupe une portion de l'espace plus ou moins grande, qui s'appelle *volume*. Le volume est terminé de toutes parts par la *surface*, et la forme de cette surface est la *figure* du corps. (Vernier (1830), 1)[21]

However, these figures could still enclose a volume, as Vernier suggested from his plates containing figures of spheres, cones and cylinders. Within descriptive geometry figures

[17] On the economics of the book trade and the history of French book production up until 1830 see Martin, Chartier and Vivet (1982).

[18] The two books without figures were Joseph Adhémar's 35-page *Cours de géométrie descriptive* and Cauchy's Leçons sur les applications du calcul infinitésimal à la géométrie (Adhémar (1823), Cauchy (1826)).

[19] This may be Jean Adam, one of the engravers who worked on the *Description de l'Égypte* under Napoléon (Beraldi (1885), 13).

[20] Poncelet, for example, designed the figures for his *Traité*.

[21] "Each body occupies a more or less sizable portion of space, which is called *volume*. The volume is bordered on all parts by the *surface*, and the form of this surface is the *figure* of the body."

included two dimensional representations of more complex objects in space, as shown in Figure 5.1.

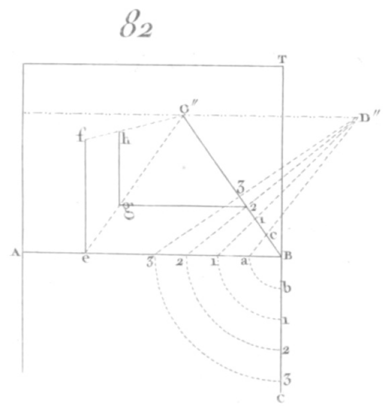

Figure 5.1: Lacroix's three-dimensional perspective, Cloquet Sculp. (Lacroix (1822))

Conic sections were mostly limited to analytic geometry, and although hyperbolas and parabolas were occasionally pictured, as in Figure 5.2, the general conic section was almost always represented by an ellipse. Coordinate axes, on the other hand, were rare and not labelled x and y as they are today.

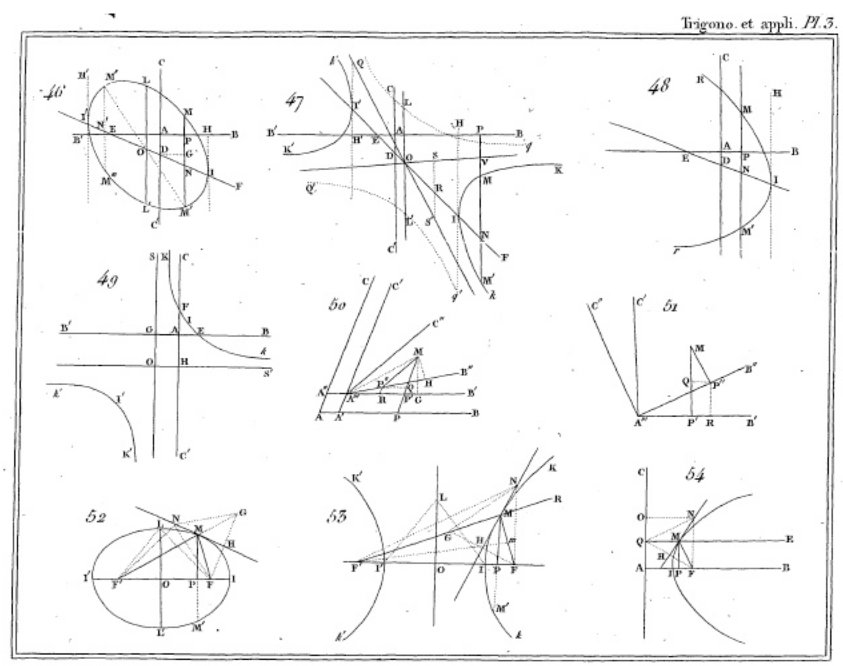

Figure 5.2: Lacroix's lines of second degree, Adam Sculp. Lacroix (1807)

Figures were referenced in text or in the margins where relevant. B. E. Cousinery even included figure citations in a column on the right side of his table of contents shown in Figure 5.3 (Cousinery (1828)).

TABLE DES MATIÈRES.

		Pages.	Planch.	Figures.
	Avant-Propos..	v,		
	CHAPITRE I^{er}.			
	Définitions. — Projection du point.....................	1,		
	Projection d'une droite..............................	2,	1re,	1.
	Projection du plan..................................	3,		
	Notations conventionnelles des épures de perspective......	4,		2.
	Intersection de deux droites..........................	6,		
1er PROBLÈME.	Grandeur d'une droite parallèle au tableau..............	7,		*Ib.*

Figure 5.3: Cousinery's Table of Contents, with figures cited in the right column (Cousinery (1828))

The figures themselves were almost always located at the back of the text, except in cases where the text was divided into two parts and the corresponding figures were placed following each conclusion. The separation of figures and text was likely due to technological restrictions, as plates of figures (as illustrated in Figure 5.2, for example) were printed separately from the text. Regardless of the book's title, classification, publisher, or author's credentials, the presence of these kinds of figures declared a text on geometry.

Having described several common features to most of these books, we now consider the texts categorically, and analyze how geometry was divided and what qualities and constituents comprised each distinct discipline. Drawing from this analysis, we will return to the *Annales*, and other research articles, to offer several conclusions on the audiences, innovations, organizations, and relationships between books and articles in this context.

5.2 What is elementary geometry?

Our survey included seventeen different titles on elementary geometry, about half of which were titled simply "Elements of Geometry."[22] The two most well-known early nineteenth century geometry texts, Legendre's *Éléments de géométrie* and Lacroix's *Élémens de géométrie*, each ran fourteen editions between the years 1794 and 1832.[23] Additionally, the much older texts by Alexis Claude Clairaut (first edition 1741), Bossut (first edition 1772), and LaCaille

[22] As these titles indicate, the alternate spellings "élémens" and "éléments" were used simultaneously by different authors.

[23] We were able to consult Lacroix (1799), Lacroix (1803a), Lacroix (1811), Lacroix (1819), Lacroix (1830) and Legendre (1800), Legendre (1812), Legendre (1832).

(first edition 1741) were all republished in our time period (Clairaut (1830), Bossut (1800), de LaCaille and Labey (1811 (1741)). As further verification of the importance of these authors, in 1906, Max Simon cited the textbooks of Legendre, Bertrand, Bézout, Clairaut and Lacroix as the fundamental texts of early nineteenth century elementary geometry in all "Kultursprachen" (Simon (1906), 27).[24]

To gain an idea of who wrote these texts, we consulted the authors' advertised credentials usually printed after their names on the title page. The authors of elementary geometry texts emphasized their memberships in scientific and scholarly societies of various cities (ten texts), their professorship positions (seven texts), and their association as former students or current faculty at the École polytechnique (five texts). These descriptions are included in Appendix D.

In all but one text, elementary geometry was defined as requiring no prior knowledge except basic arithmetic.[25] To take a typical example, Lacroix's text progressed from planar construction and measurement of lines, circles, and polygons to planes, polyhedra, and the three round bodies in space: spheres, cylinders, and cones. While claiming to adhere to the "ancient method," these geometers represented measurements and proportions through algebraic symbolism. Legendre claimed this approach was more "dextrous" (Legendre (1800), iv). Above all, these authors described their texts as simple, natural, clear, and evident.

The bounds of elementary geometry, as they appear in the table of contents in these books, were well-defined, and variations from this core content were rare. Only Legendre and Vincent included figures on the sphere, and only N. J. Didiez and Terquem included the study of conic sections. In each case, these geometric objects appeared at the end of the text, thus confirming their more advanced status.

Without exception, authors explained in their subtitles and introductions that elementary geometry books were written for beginning students, either for independent study or as formal course textbooks. Although geometry was perceived as less abstract than analysis, the development of mathematical thinking was uniformly portrayed as requiring "courage, perseverance, and dedication" (Bergery (1831), 6). New students had limited prior mathematical experience, and authors spoke of the pervasive problem of student dropout. Criticizing the usual practice of teaching geometry, Clairaut observed, "il arrive communément que les commençants se fatiguent et se rebutent avant que d'avoir aucune idée distincte

[24]The mathematics of Lacroix and Legendre have been studied comparatively by Pierre Lamandé in Lamandé (1993). Lamandé also addresses Lacroix's understanding of numbers in his books on algebra, geometry, and calculus in Lamandé (2004). Liliane Alfonsi's biography of Bézout further describes the influence of his textbooks from the eighteenth century onward (Alfonsi (2011)).

[25]The outlier here is Terquem's *Manuel de géométrie, ou Exposition élémentaire des principes de cette science*, in which he proposed condensing elementary geometry, trigonometry, conic sections, surfaces, projective and descriptive geometry into a one year class with the use of algebra (Terquem (1829)). As further proof of his modern outlook, Terquem wrote a very favourable review of Poncelet's *Traité* for the *Bulletin universal* in 1823.

de ce qu'on voulait leur enseigner" (Clairaut (1830), 1).[26] To secure student interest, the authors employed various teaching methods. They all agreed that geometry needed to be motivated by practical application and utility. Indeed, the importance of learning geometry was couched in terms of national pride and progress, as in Vincent's dedication to "students."

> Cet ouvrage vous appartient à plus d'un titre : c'est pour vous, c'est avec vous que je l'ai composé : recevez-en la dédicace. Puisse-t-il, en vous rappelant les heures de nos entretiens, alimenter en vous cet amour de l'étude qui vous mettra bientôt à même (je l'espère) de payer le tribut que vous devez à l'utilité publique. (Vincent (1826), i)[27]

On a pedagogical level, authors debated whether theorems should appear before or after their proofs, whether problems should be embedded in the text or collected in an appendix (Develey (1812), Legendre (1800), Vincent (1826)), the appropriate use of proof by contradiction (Lacroix (1803a), Schwab (1813), Olivier (1835)), and how much rigour could be obtained without sacrificing the more important quality of simplicity (Lacroix (1799), Vincent (1826), Develey (1812), Clairaut (1830), Mutel (1831), Terquem (1829), etc.). Distinct forms of teaching could be subtle but were still advertised, such as the decision by Louis-Etienne Develey, Auguste Mutel, and Vincent to state propositions without reference to the lettered figure, in order that the wording might more easily be committed to memory, which all three highlighted as important decisions in their introductions. As a further example, LaCaille allowed his text to be more or less advanced through restricting "less useful or less easy" material to small font that the reader could include or ignore depending on preference (de LaCaille and Labey (1811 (1741), iv).

The methodological discourse extended to geometric definitions, in particular, the definition of a straight line. Notably, none of the authors repeated the Euclidean definition, such as that given in François Peyrard's 1804 translation.

> 2. La ligne est une longueur sans largeur.
>
> 3. Les extrémités d'une ligne sont des points.
>
> 4. La ligne droite est celle qui est toute également interposée entre ses points. (Peyrard (1804), 1)[28]

[26] "[...] it often happens that beginners tire and quit before having any distinct idea of what one wanted to teach them [...]"

[27] "This work belongs to you in more than name: it is for you, it is with you that I wrote it: receive it as dedication. If you can, recall the hours of our talks, nourish in yourselves this love of study that you will soon (I hope) contribute the same to the public interest."

[28] "2. The line is a length without breadth. 3. The extremities of a line are points. 4. The straight line is that which is also equally placed between its points."

The straight line was also particularly significant because it was both an elementary concept and, along with the compass, one of the two acceptable tools of construction. In 1800, Legendre claimed that "the definition of the straight line is the most important of the elements" [*La définition de la ligne droite étant la plus importante des éléments*]. In order to avoid the difficulty for students of proving that the straight line was the shortest distance between two points *and* unique, Legendre defined it as such. He claimed this was both a definition and an axiom, on which he would "establish the entire edifice of the elements" [*d'établir l'édifice entier des éléments*] (Legendre (1800), iii).

A similar attention to the importance of the straight line was first devised by Bertrand in 1778. Bertrand began with the concept of homogeneous space, from there defined a plane and then a straight line as the common limit to two halves of a plane divided such that "we can say nothing about the one part which we cannot equally say about the other" [*on ne puisse rien dire de l'une qui ne puisse également se dire de l'autre*] (as quoted by Lacroix (1803a), xiii). Lacroix was impressed enough by this definition that he repeated it in a footnote to his introduction, and even provided in text illustrations suggesting that the characteristic described would not belong to a curved line (Figure 5.4).

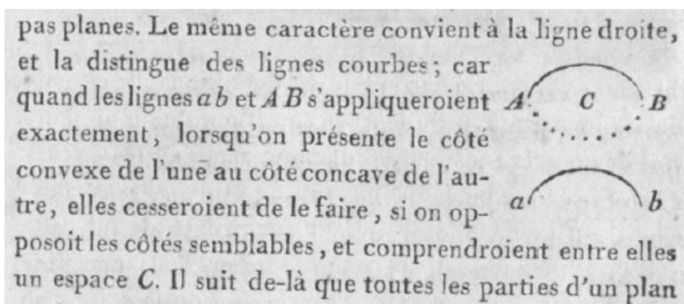
pas planes. Le même caractère convient à la ligne droite, et la distingue des lignes courbes; car quand les lignes *ab* et *AB* s'appliqueroient exactement, lorsqu'on présente le côté convexe de l'unē au côté concave de l'autre, elles cesseroient de le faire, si on opposoit les côtés semblables, et comprendroient entre elles un espace *C*. Il suit de-là que toutes les parties d'un plan

Figure 5.4: Lacroix's illustration of how Bertrand's straight line description excluded curved lines. (Lacroix (1799), ix)

Bertrand claimed in 1812 that his definition "places its object before the eyes" [*met son objet sous les yeux*] and "excluded all superfluity" [*exclut toute superfluité*] (Bertrand (1812), iv) He argued that beginning with points and adding dimensions "reversed the order in which one forms ideas" [*renversent l'ordre dans lequel on forme ces idées*] (v).

A third strategy was avoidance of defining straight lines all together. For instance, Develey instead presented two illustrations for reference.

> La *ligne droite*, qu'il est difficile et peut-être inutile de définir, mais dont chacun a d'ailleurs une idée claire et distincte. La ligne AB, fig. 1 et 2, est une ligne droite. (Develey (1812), 4)[29]

[29]"The straight line, which is difficult and perhaps useless to define, but about which everyone has a clear and distinct idea. The line AB, fig. 1 and 2, is a straight line."

In a footnote, Develey explained that while one could represent lines by "ink, chalk, pencil." These were not geometric objects, only sensible images. He later claimed the properties of uniqueness and the shortest distance between two points as axioms. Similarly, Terquem first defined a line as the limit of a surface, and then claimed that "it is not possible and it is not necessary to define" [*il n'est pas possible et il n'est pas nécessaire de la définir*] a straight line, since it is a "first notion" [*une notion première*] (Terquem (1829), 3).[30] Defining a straight line could exemplify the authors' positions on the dichotomy between the sensible and the abstract, as well as the level of expected rigour and proof.

Within their introductions, authors both acknowledged and criticized the work of their contemporaries in elementary geometry. Develey described the ongoing dialog on the best form of presenting the elements.

> On a beaucoup écrit sur la meilleure forme à donner aux Élémens de Géométrie; je ne voudrais pas répéter ce que d'autres ont dit, et bien mieux que je ne pourrais le faire. Mais avec ces excellentes directions, sommes-nous parvenus à avoir des Élémens parfaits? Je ne le pense pas ; et je suis bien loin de croire que les miens le soient. Quelques auteurs ont fait de grands pas vers cette perfection que nous voyons tous en perspective ; j'ai voulu hasarder aussi quelques efforts ; peut-être un jour quelqu'un plus heureux, mais surtout plus habile que moi, atteindra-t-il le but désiré. (Develey (1812), v)[31]

When evaluating who had succeeded in writing elementary geometry, Legendre was seen as the standard. Develey considered Legendre's *Eléments* as the best so far, though not without fault, but sharply criticized the lack of rigour in Clairaut's "unique" treatment. Lacroix referenced with admiration Bertrand's definition of the straight line, though he chose to adopt a different one (Lacroix (1799), x). Olivier encouraged École polytechnique applicants to supplement his brief text with that of Pierre Louis Marie Bourdon, Lacroix, or Legendre (Olivier (1835), vi). Vincent thanked his former teachers, Lacroix and Francoeur, as well as Legendre, Dupin and Develey, while Legendre himself began each new edition by thanking the various geometers who had recently offered new and relevant material including Lhuilier, Pilatte, Cauchy, and Querret (Vincent (1826), ix; Legendre (1800), Legendre (1812), Legendre (1832)).

Citations to contemporary research articles, correlated with the presence of new mathematical results. Both Terquem in 1829 and Didiez in 1828 demonstrated knowledge of

[30]Terquem also described directions of lines as a first notion that could not be defined (Terquem (1829), 5).

[31]"A lot has been written on the best form to give to the Elements of Geometry; I do not wish to repeat what others have said and very well for I could not do it. But with these excellent directions, do we achieve perfect Elements? I do not think so; and I am far from believing that mine are them. Several authors have taken great steps toward this perfection as we see everything in perspective; I have also attempted some efforts; perhaps one day someone luckier, but above all abler than I, will achieve the desired goal."

Gergonne's *Annales*, and from it adopted the concepts of similitude centre and axis (Didiez (1828), 89, Terquem (1829), 379). While Terquem's references only contained the geometers' surnames without dates or titles, Didiez provided an exact citation to Gergonne's translation of Steiner (*Annales de Mathématiques pures et appliquées*, rédigées par M. Gergonne. t. XVII. p. 288), although Steiner's name remains unmentioned. These texts are unusual. Didiez's text was the first part in a four part series on elementary geometry in the plane, elementary geometry in space, analytic geometry in the plane, and analytic geometry in space. Terquem considered analytic geometry as part of elementary geometry. This attention to algebraic applications may also explain why both included discussions of conic sections.

Intended for novices, all elementary geometry texts included not only the elementary contents of geometry, but also strategies for practising geometry. This tacit knowledge made explicit appeared most prominently in Claude-Lucien Bergery's 1832 text intended for "les instituteurs primaires" in which he explained how to draw lines, what materials were preferable to use and their cost, and gave illustrations of different line types in a construction (Figure 5.5).

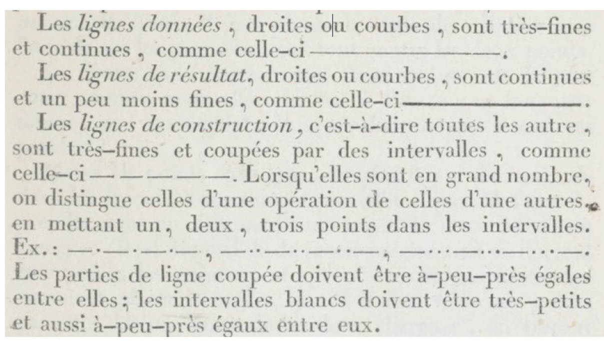
Les *lignes données*, droites ou courbes, sont très-fines et continues, comme celle-ci ————————.
Les *lignes de résultat*, droites ou courbes, sont continues et un peu moins fines, comme celle-ci ————————.
Les *lignes de construction*, c'est-à-dire toutes les autre, sont très-fines et coupées par des intervalles, comme celle-ci — — — — — —. Lorsqu'elles sont en grand nombre, on distingue celles d'une opération de celles d'une autres, en mettant un, deux, trois points dans les intervalles. Ex. : —·—·—·—, —··—··—··—, —···—···—···—. Les parties de ligne coupée doivent être à-peu-près égales entre elles; les intervalles blancs doivent être très-petits et aussi à-peu-près égaux entre eux.

Figure 5.5: How to draw different kinds of lines (Bergery (1831), 9)

With respect to reading the provided figures, Olivier advised students to detach the figure sheets and keep them in a notebook for ease of reference. Although figures were numerous and often cited, Christian Kramp cautioned students to recognize the difference between physical representation on paper and geometric objects.

> Dans l'impossibilité de présenter à nos sens des points, des lignes, des surfaces mathématiques, nous sommes obligés de leur substituer des points, des lignes, des surfaces matérielles, donnés de toutes les trois dimensions, et ayant de plus des qualités physiques, qui ne devroient jamais entrer dans une considération purement mathématique. Nous faisons abstraction de ces dernières autant que nous les pouvons : et quant aux autres, nous tachons de donner à nos lignes aussi peu de largeur, et à nos surfaces aussi peu d'épaisseur que l'imperfection

> inévitable des instruments nous permet de faire. (Kramp (1806), 3)[32]

Each text concluded with several sheets of figures, most often on folding out pages, which gave more room for larger illustrations and could also serve to mark the page in flipping between the image and text. In these elementary geometry books, there was an average of 182 figures and 280 pages of written text. As discussed above, many of these figures were signed by an engraver, draughtsman or lithographer. Figures were enumerated and referenced by number in the text, margins, or table of contents. However, as mentioned above, geometers advertised that their propositions could be read and understood independently of figures. Geometers also needed to exercise caution when using figures in reductio ad absurdum. For this reason, Lacroix attempted to use reductio ad absurdum as little as possible, and when that "form of reasoning" [*forme de raisonnement*] must be used, figures were better avoided.

> [...] mais alors il faut éviter toute construction de figures, ou, s'il en faut absolument, faire du moins en sorte que l'absurdité de la figure ne choque pas trop la vue, parce que cette absurdité empêche l'esprit de suivre le fil du raisonnement, et l'imagination est obligée de faire un effort assez pénible pour redresser la figure de manière à y voir ce que l'on a voulu peindre dans le discours. (Lacroix (1799), xviii)[33]

Likewise aiming to avoid "shocking" figures, Jacques Schwab decided instead to avoid constructions in general and reason with numbers when using proof by contradiction.

> Dans le choix des démonstrations, j'ai préféré celles qui exigent le moins de construction; mais c'est sur-tout dans les propositions où l'on rencontre l'incommensurable, et pour lesquelles on ne peut guère se passer de la *réduction à l'absurde*, que j'ai tâché d'éviter totalement les constructions et d'y substituer un raisonnement sur les nombres; c'est bien l'idée des limites, mais perfectionnée de manière à emporter avec elle la même rigueur que les figures d'Euclide qui choquent la vue. (Schwab (1813), vii)[34]

[32]"Due to the impossibility of presenting mathematical points, lines and surfaces before our senses, we are obliged to substitute material points, lines, surfaces, given all three dimensions and having more physical qualities, which must never enter into a purely mathematical consideration. We abstract from the latter as far as we can: with respect to others, we try to give our lines as little width and our surfaces as little thickness as the inevitable imperfection of our instruments allows."

[33]"[...] but then one must avoid all figure construction, or if one absolutely must do it, at least act as though the absurdity of the figure is not too shocking to see, because this absurdity prevents the mind from following the thread of reasoning and the imagination is obliged to make a rather painful effort to connect the figures in such a way as to see what one wants to illustrate in the discourse."

[34]"In the choice of proofs, I prefer those that require less construction; but especially in propositions where one encounters the incommensurable, and in which one cannot but use *reductio ad absurdum*, where I have tried to totally avoid constructions and instead reason with numbers; this is like the idea of limits, but perfected in order to bring the same rigour as the figures of Euclid that are shocking to see."

Though foundational to elementary geometry, proof by contradiction demonstrates a case where figures could also be problematic.

As in Gergonne's *Annales*, the definition of different methods, such as analytic and synthetic, varied widely according to the author. Develey explicitly rejected employing the terms analysis and synthesis to describe his methods because there was currently "no accord as to their distinct characters" [*n'est point encore d'accord sur le caractère distinctif*] (Develey (1812), v). In a later text on method in mathematics, Develey eventually settled on a definition of analytic and synthetic derived from Greek geometry, synthesis began with the given properties and proceeded "by composition" to prove the truth of the consequence, while analysis began by assuming the consequence as true and then proceeded "by decomposition" to show that the truth rested upon the principal hypothesis (Develey (1831)). However, as if to confirm Develey's initial discomfort, that same year Mutel claimed exactly opposite definitions of the two methods of mathematics. For Mutel, analysis proceeded by deducing consequences from initial conditions and previously proved inquiries to reach the solution. Synthesis, to the contrary, began by showing how to reach a result, and then proving that this result satisfied the given conditions.

> L'analyse consiste à déduire des conditions renfermées dans l'énoncé une suite de conséquences au moyen desquelles on ramène la solution de la question à un petit nombre d'opérations élémentaires, ou bien à une ou plusieurs questions déjà résolues. Par la synthèse au contraire, on prescrit d'abord les opérations à effecteur pour obtenir le résultat que l'on cherche, et on démontre ensuite que le résultat obtenu satisfait à la question. (Mutel (1831), iii)[35]

Even when analysis and synthesis were specified as inverse directions of proof, their meanings varied between individual geometers.

Another pair of definitions, recognized but not adopted by Develey, was described by Bossut and specifically applied to geometry. Synthesis was the "ancient" method in which one directly employed the figure or solid to derive results, while analysis translated the figures into algebraic calculations. Bossut described the advantages of the synthetic method, despite modern preferences for analysis.

> Il y a de précieux avantages attachés à cette méthode. Elle marche toujours le flambeau de l'évidence à la main; et souvent elle fait trouver avec une extrême facilité des théorèmes qui seraient difficiles à découvrir par toute autre voie, ou qui du moins se présenteraient sous une forme moins élégante. Les modernes

[35] "Analysis consists of deducing from conditions contained in the [problem's] statement a series of consequences by means of which one reduces the solution of the question to a small number of elementary operations, or even to one or several already solved questions. Through synthesis, to the contrary, one first specifies the operations to carry out in order to obtain the sought result, and one then demonstrates that the result obtained satisfies the question."

> ont un peu négligé la synthèse, parce-que l'analyse leur a fourni dans la plupart de leurs recherches des secours plus prompts, plus universels, et quelquefois absolument indispensables. Cependant quelques uns d'entre eux l'ont cultivée avec succès, et en ont fait de très belles applications; tels sont principalement Huyghens, Pascal, Newton et Maclaurin. (Bossut (1800), vii)[36]

Other geometers claimed different types of methods or rejected the distinction all together. Clairaut described his geometry as following the "natural method of the inventors," by proceeding consistently from unknown to known (Clairaut (1830), 2). Terquem claimed that rather than focusing on the synthetic-analytic distinction, the best teaching method was one in which "without sacrificing rigour, students can learn the most truths with the least hardship in the least time" (Terquem (1829), iv). While the definitions varied widely, authors were generally consistent in citing at most two possible methods in geometry. A combination of both methods was considered "mixed." Olivier, whose elementary mathematics text incorporated algebra, followed by elementary geometry and finally trigonometry, took a more liberal view of methods and professed to use the shorter and easier method of infinitesimals, while including the more rigorous proof by contradiction and method of limits in a supplementary text.

> J'ai adopté aussi, pour certaines démonstrations, le méthode des *infiniment petits*, que la pensé saisit sans peine, parce qu'elle m'a paru satisfaire l'esprit et surtout parce qu'elle est la moins longue et la plus facile. D'ailleurs on trouvera dans le SUPPLEMENT aux Mathématiques usuelles, les mêmes démonstrations traitées par la *réduction à l'absurde* et par la méthode des *limites*; cette dernière très-rigoureuse est assez courte, mais tient un peu à l'analyse algébrique. (Olivier (1835), ix)[37]

Olivier's hesitation to include algebraic analysis in his beginning geometry text reflected the strict demarcation of what constituted elementary geometry.

[36] "There are precious advantages attached to this method. It always marches with the torch of evidence in hand; and often it finds with extreme ease theorems that would be difficult to discover by another route, or which at least would be presented in a less elegant form. The moderns have neglected synthesis a bit because analysis has furnished them most of their research by more prompt, more universal, and sometimes absolutely indispensable means. However some among them have successfully cultivated synthesis and have made very beautiful applications of it; they are principally Huyghens, Pascal, Newton and Maclaurin."

[37] "I have also adopted, for certain proofs, the method of *infinitesimals*, which is easily grasped because it seems to satisfy the mind and above all it is the shortest and easiest. Moreover, one will find in the SUPPLEMENT to the usual Mathematics, the same proofs treated by *reductio ad absurdum* and by the method of *limits*; the latter is very rigorous and short enough, but depends a little on algebraic analysis."

5.3 Analytic geometry

The second largest category of texts were those on analytic geometry, including sixteen different titles.[38] Several titles received two editions, and the *Essai de géométrie analytique* by Biot and the *Traité élémentaire de trigonométrie rectiligne et sphérique et d'application de l'algèbre à la géométrie* by Lacroix were respectively republished seven and eight times between 1799 and 1827 (we include Biot (1810), Biot (1813), Biot (1826) and Lacroix (1803b), Lacroix (1807), Lacroix (1813), Lacroix (1827)). Authors used the expressions "algebra applied to geometry" and "analytic geometry" interchangeably, and the titles reflect these two alternative descriptions. As an exception that proves the rule, Monge's *Application de l'Analyse à la Géométrie* explained how the calculus of functions could be used to study curved surfaces and surfaces of double curvature. Otherwise, as described in the series of courses by Lacroix, analytic geometry followed the study of arithmetic, elementary geometry and algebra, and preceded differential and integral calculus.

Planar and spherical trigonometry featured in several of these texts, but were often separated from the main analytic geometry content as in Bossut (1800), Bourdon (1825), Lacroix (1803b), or Poullet-Delisle (1809). Otherwise, the content usually began with points, lines, and curves in the plane and then proceeded to points, lines, curves, and surfaces in space. In Antoine Charles Marcelin Poullet-Delisle's comparatively short text, he supposed that readers would find his exclusion of three dimensions "astonishing," but he hoped to treat the subject in a subsequent companion volume (Poullet-Delisle (1809)). By contrast, the texts of Dupin and Jean-Nicholas-Pierre Hachette almost exclusively focused on geometry in three dimensions, considered first "synthetically" and then "analytically" (Dupin (1813), Hachette (1817)). As distinct from elementary geometry, every text in analytic geometry included second degree curves, the conic sections. Because of the use of coordinate equations, texts did not have to be organized by strictly geometric considerations. Reynaud described his contents as beginning with the study of as many equations as unknown variables, then one equation with more than one unknown, and finally more than one equation each with more than one unknown (Reynaud (1819), v–vi). This sounded like algebra, and only those versed in analytic geometry could interpret these descriptions as figures.

Introductions consistently emphasized the difference between, on the one side, elementary, rational, or ordinary geometry and, on the other side, analytic geometry. The authors usually assumed their readers had already learned elementary geometry, and so would be able to compare the advantages of both areas of study. As we saw above, elementary geometry had been positively described as simple, easy to follow, natural, and clear. Analytic geometry was described in these same terms and as elegant, rapid, fruitful, methodical, uni-

[38] Credentials for authors of analytic geometry texts were very similar to those of elementary geometry texts, if slightly more prestigious with polytechnique references (eight texts), academic institution affiliations (nine texts), and professor or adjunct positions (seven texts).

form, general, or universal. Authors took care to offset advantages of analytic uniformity with enough variety to keep students working.

> L'uniformité des méthodes rendant l'esprit paresseux, j'ai dû chercher à les varier et choisir celles qui font naître des idées nouvelles. Les définitions que l'on donne ordinairement des *diamètres*, des *centres*, des *asymptotes* et des *tangentes*, n'étant pas toujours applicables aux courbes des degrés supérieurs, il devenait nécessaire de généraliser ces définitions. (Reynaud (1819), v)[39]

These latter qualities were specifically demonstrated by showing that analytic geometry could solve problems otherwise considered "inaccessible" and had the potential to easily generalize results from two to three dimensions. By contrast, geometry without algebra could be particular and overly complicated.

The profusion of analytic geometry texts in this period was acknowledged by the authors, who often advertised their work as supplementing rather than replacing previous treatments. Poullet-Delisle assured the reader that his publication should not be perceived as a criticism, and only intended to be useful. He professed:

> Je n'ai point ambitionnée d'être neuf: dans un ouvrage de cette espèce, ce serait sans doute une prétention ridicule. (Poullet-Delisle (1809), v)[40]

Authors repeatedly concluded their introductions by requesting constructive feedback from other professors or students. Second editions would then often credit improvements to colleagues' suggestions. In 1813, Jean Guillaume Garnier thanked Gergonne's *Annales Mathématiques*, M. Simon Lhuilier, M. Puissant, and several unspecified articles from *Journaux des Sciences* for helping to make his second edition more methodical, careful, and complete than his first (Garnier (1813)). The ultimate test of a text's success, as Biot observed, was by experiment, "test it on the minds of the students, and verify by this proof the goodness of the chosen methods" (Biot (1810), vi).

Authors differentiated their texts into roughly two overlapping objectives: either providing the necessary tools for immediate application, or preparing students for entrance exams for the École polytechnique, l'École spéciale militaire, l'École de marine, l'École forestière, or for use in l'École centrale des Quatre-Nations (Lacroix's textbook). The pervasiveness of École polytechnique aspirations was epitomized by a book designed to prevent observed difficulties among prospective students written by the entrance examiner, Reynaud. With respect to the first goal, analytic geometry risked being too general or too abstract. Bourdon, who emphasized practical applications to the extent of including instructions on how

[39]"The uniformity of methods makes the mind lazy, I have sought to vary them and choose those which give birth to new ideas. The definitions that one ordinarily gives of *diameters, centres, asymptotes*, and of *tangents*, are not always applicable to curves of higher degrees, it becomes necessary to generalize these definitions."

[40]"I have no ambition to be new: in a work of this kind that would be undoubtably a ridiculous pretension."

to construct different popular styles of archways, began with particular methods. Although these methods "vary with the nature of the problem," Bourdon claimed they were often more simple then general approaches (Bourdon (1825), vi). He also cautioned against abstraction by beginning with "purely geometric" definitions of conic sections as curves formed by the intersections of cones and planes, deriving equations from these properties, and then using transformation of coordinates to show representation by second degree curves. Dupin acknowledged that analytic geometry seemed "incontestably superior" because of its ease and rapidity, but maintained that "rational geometry" was necessary for applications to engineering (Dupin (1813), ix).[41] Accordingly, those in public service should know both methods and choose which one best applied to the problem at hand. Supporting Dupin's view, he received and included in his book a report of approval from the *Institut de France*, written by Carnot, Monge and Poisson who praised his many geometric considerations independent of calculations. Hachette also considered both the "geometry of the ancients" and "modern analysis" as complementary studies (Hachette (1817), vii). The latter yielded more results, though the former was more sensible. Even so, analytic geometry was closely associated with illustrated figure representation, each book contained figures, with an average of about 126 figures and 334 pages of text per volume. Judging by quantity, the figure thus appears as only slightly less important here than in elementary or three-dimensional geometry.

However, most analytic geometers did not advocate equal status to the two methods, nor even consider analytic geometry as more abstract. Lacroix described the synthetic method as necessary, but always "subordinate" to the analytic method (Lacroix (1807), vi). Bossut claimed that, without the analytic method, reasoning could become long, abstract, fatiguing, and liable to lose the attention of beginners (Bossut (1800), 247).

While elementary geometry was described in terms of its contents or the necessary preliminaries, analytic geometry was defined as a process–the application of algebra to solving problems and proving theorems from geometry.[42] As Bourdon succinctly stated, the process had two parts "traduire en Algèbre les questions de Géométrie, et réciproquement, traduire en Géométrie les résultats obtenus par l'Algèbre" (Bourdon (1825), 2). Others, such as Jean-Louis Boucharlat, considered three distinct steps, which included finding the

[41] Dupin proposed a division between rational and analytic geometry. We have seen in Chapters II and III that Poncelet interchangeably employed the terms "pure", "ordinary" and "rational" geometry as opposed to analytic geometry, such as in Poncelet (1822), xi. Poncelet had also cited Dupin's methodological approach with approval in Poncelet (1817c). In Chapter IV, we further noted that Cournot used the phrase "analyse rationnelle" to describe the method employed by Poncelet and Steiner (Cournot (1827)). This thus suggests a well-represented alternative division within geometry

[42] In the paper "On Identities of Algebra in the 19th Century," Caroline Ehrhardt and Frédéric Brechenmacher examine whether algebra can be considered as a mathematical discipline in this time period. With respect to the first third of the century, they conclude that algebra would be best be described as "the practice of solving problems with the help of equations, which leads to concrete values" and not a discipline (Ehrhardt and Brechenmacher (2010)).

value of the unknown through algebra (Boucharlat (1810), vii). Finally, Biot considered the translation into an equation as one step, and the algebraic calculation as the other (Biot (1810), 1). Despite the difference in enumeration, these descriptions were essentially the same, and emphasized the translating procedure as a new process for those who had only separately studied geometry and algebra. Thus, most texts began by explaining this procedure, "mettre le problème en équation." Shorter texts, such as Francfort's 64-page *Essai analytique de géométrie plane* recommended the reader to consult other texts, such as Biot, de Foury, or Bourdon, in order to understand how algebra could be applied to geometry (Francfort (1831)).

Most authors introduced the process of writing geometry as algebra with a small set of emblematic examples. Jean-Nicolas Noël broadly described one such case of finding a triangle.

> On supposera d'abord le problème résolu ; puis on mènera, s'il est nécessaire, des droites propres à donner les triangles qui ont pour côtés les droites connues et inconnues. Représentant ensuite chacune de ces droites par une lettre, et faisant usage des propositions de géométrie qui établissent des relations entre les lignes, on aura les équations du problème proposé. (Noël (1822), 103)[43]

The problem could now be solved algebraically, and then reinterpreted as a geometric figure. Similarly, Bourdon described how to prove a theorem through analytic geometry.

> Si la question proposée est un théorème à démontrer, on traduit algébriquement les relations qui existent entre les différentes parties de la figure, ce qui conduit à des équations auxquelles on fait subir diverses transformations, dont la dernière donne lieu au théorème énoncé. (Bourdon (1825), 2)[44]

In presenting the elements of analytic geometry, Biot considered the form of points, lines, and planes as analogous to the ruler and compass in the drawings of practical geometry (Biot (1810), vi). The potential simplicity of any given problem rested upon the choice of knowns and unknowns. Biot described a "fortunate choice of unknowns" [*choix heureux d'inconnues*] as an art most beautifully exemplified in Newton's *Universal arithmetic.* However, as Lefébure warned, very often complicated questions could demand particular methods and artifices. He thus declared "the first rule to observe" [*la première règle à observer*] was to penetrate the relationships established between the lines, angles, surfaces, solids of

[43] "One will first suppose the problem solved; then one will draw, if necessary, lines belonging to the triangles which have known and unknown lines as sides. Then representing each of these lines by a letter and making use of propositions of geometry which establish relations between lines, one will have equations of the proposed problem."

[44] "If the proposed question is a theorem to prove, one algebraically translates the relations that exist between different parts of the figure, this leads to equations that undergo different transformations, where the last yields the theorem announced."

the problem without distinction of knowns and unknowns (de Fourcy (1827), 3). Then these relations could be expressed by equations and when possible, the values of the unknowns could be deduced.

The discussion on how to choose the best equation underscored how different forms of equations could represent the same geometric object. For this reason, Lacroix explained to his audience of educators, there was a set of imperatives for students, including the need to become familiar with the transformation of coordinates.

> Classer en conséquence les lignes par leurs équations, faire remarquer que ces équations n'ont pas une forme unique, mais qu'elles se compliquent plus ou moins, suivant les relations que les lignes qu'elles représentent peuvent avoir avec celles auxquelles on les rapporte ; en déduire la nécessité de savoir transformer les coordonnées, et employer cette transformation à la classification des lignes, par la simplification de leur équation: (Lacroix (1803b), xiii)[45]

The identification of conic sections and second degree curves was a demonstrative example of the variety of equation forms. Following Lacroix, "les diverses équations des lignes ne sont que les énoncés de leurs diverses propriétés qui se contiennent les unes les autres dès qu'elles sont caractéristiques."[46] These properties could be discovered through a change in coordinates, a calculation explained in every elementary analytic geometry text.

As elementary geometry was considered the "method of the ancients," so analytic geometry was considered "modern." Bossut, whose text originally appeared in 1772, described analytic geometry as producing a "revolution" in "the empire of mathematics" (Bossut (1800), xii). From the opposite perspective, geometry was described by Boucharlat as one of the most beautiful and fruitful uses of analysis (Boucharlat (1810), vii). Lagrange and Monge were only the most recent examples of esteemed modern geometers. Boucharlat dedicated his text to Lagrange and "his sublime conceptions." Dupin dedicated his to Monge, and advertised his text as following the descriptive and analytic geometry of his "illustrious master." Late eighteenth and early nineteenth century geometers credited the origin of modern geometry to Viète and Descartes, admired the work of Newton, and were inspired by both the form and content of Euler's trigonometric and analytic texts. For instance, citing Viète, Descartes, Newton, Euler and Cramer, Lacroix provided a brief history of analytic geometry, which he prefaced in praising the "moderns."

[45] "Consequently to classify these lines by their equations, to note that these equations do not have unique form, but they are more or less complicated according to the possible relationships between the lines which they represent and those with which we compare them; to deduce from this the necessity of knowing how to transform coordinates, and to employ this transformation to the classification of lines, by simplifying their equation :"

[46] "[...] the different equations of lines are only the expression of their diverse properties that one or the other contain as characteristic."

> Vient ensuite l'application de l'algèbre à la géométrie ; cette branche, due entièrement aux modernes, et dont la découverte leur a bientôt donné une immense supériorité sur les anciens, devait nécessairement changer de forme à mesure qu'elle s'étendoit et se perfectionnoit. (Lacroix (1803b), vi)[47]

Biot claimed that Euler had introduced the exhaustive table of contents format in listing all propositions in order to enable ease of reference for students.[48] Hachette recounted how Euler had classified all five types of curved surfaces in 1748: ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic paraboloids, and hyperbolic paraboloids (Hachette (1817), vii).

As citations back to the seventeenth century suggest, claims to modernity in analytic geometry did not necessarily imply recent development nor attention to new research. Analytic geometry exhibited conservative tendencies when compared to the research publications of Gergonne, Poncelet, Plücker and Steiner. Algebraic solutions that indicated imaginary, infinite, and to some extent negative points or curves were dismissed as impossible or absurd.[49] Biot described a parabola as an infinitely elongated ellipse and a hyperbola as an ellipse with an imaginary axis, but when his calculations resulted in a curve defined by imaginary points, he concluded that this was no curve. Solutions that could not be represented on paper were non-existent. Reynaud made an especial point to show that negative and imaginary solutions did not always imply impossibility and that real solutions found algebraically might be impossible geometrically. He proposed that these properties "merit the attention of all students," but his inclusion of only two examples seemed to suggest such cases were rare (Reynaud (1819), 219). The first example concerned division into mean and extreme ratio of a line segment, that is, to find the point X on a given line containing points A and B such that $AB : AX :: AX : BX$. Reynaud set up an algebraic formula with $AB = 2a$, $AX' = x$, and $BX' = 2z - x$ and solved for $x = -a \pm \sqrt{a^2 + (2a)^2}$. He considered several cases, concluding by showing that this proportion excluded the case where the point X was situated to the right of B such that $AB < AX$ and $BX < AX$ because algebraic

[47] "Then came the application of algebra to geometry; this branch, due entirely to the moderns, and whose discovery soon gave them a huge advantage over the ancients, had to change from form to measurement as it was extended and perfected."

[48] This index style was popular in all kinds of geometry textbooks, and J. de Stainville even apologized for not doing so because his text was too long.

> La grosseur de ce volume n'ayant pu permettre n'y joindre une Table détaillée, on s'est vu restreint à présenter seulement dans celle-ci les principaux articles qu'il renferme. (de Stainville (1815), i)
>
> The size of this volume does not permit including a detailed Table, we show restraint in presenting here only the principal articles that it contains.

Nevertheless, his table of contents ran six pages!

[49] As the history of complex numbers in the nineteenth century indicates, imaginary numbers held an ambiguous status within mathematics, and geometry in particular, through the 1820's (Flament (1997), Schubring (2005)).

calculations indicated imaginary values of x. However, Reynaud asserted that the problem had two "geometric" solutions, which he did not elaborate upon. His accompanying figure to the mean and extreme value problem only showed the case of X being situated between A and B (Figure 5.6).

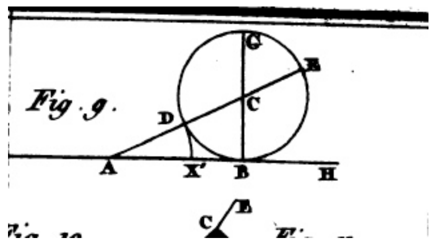

Figure 5.6: Reynaud's Figure 9 for finding mean and extreme values (Reynaud (1819)

Reynaud described the fact that imaginary values did not always indicate the impossibility of geometric problems as an "apparent contradiction" [*contradictions apparentes*] caused by particular geometric conditions, which were "not susceptible to be expressed in the equations of this problem" [*ne sont pas susceptibles d'être exprimées dans les équations de ce problème*] (Reynaud (1819), 20). He thus exhibited a note of caution when interpreting possibility or impossibility based on imaginary solutions to algebraic calculations.

In rare cases, authors slowly began to adopt the mathematical objects and principles being developed and debated within the *Annales*. One example of this gradual incorporation was the case of poles and polars–found in only two of our analytic geometry texts. Biot added a discussion of poles and polars in 1826, his ninth edition, crediting both Monge and the *Annales*. He defined the concepts geometrically with respect to a circle and extended "analogously" to all second order lines (Biot (1810), 197). Biot did not apply these new concepts to proofs or constructions nor reference them again for the remainder of the text. In 1829, Terquem defined poles and polars with respect to a general second order line, and claimed that their properties facilitated "the solution of many problems," but only gave examples within non-analytic geometry (Terquem (1829), 133). The definitions and applications in these books mirrored those found in the *Annales* publications beginning in 1810. By 1813 Gergonne had given an analytic interpretation of poles and polars, which was repeated and refined in the analytic geometry texts of Plücker, Sturm, and Bobillier.

Even so, this was evidence that the nature of analytic geometry was evolving. Lacroix described the development of analytic geometry from the first inventors, who saw it as a means to combine geometry theorems, to its current incarnation as "the general means to deduce the properties of extension from the smallest possible number of principles" (Lacroix (1807), vi). Dupin directly addressed scientific progress, claiming that science was only truly fruitful when elementary texts also progressed. In analytic and descriptive geometry, Dupin attempted to transform "new conceptions, reserved at first to a small number of superior minds" into "general knowledge" (Dupin (1813), vii). The changing content reflected this growing audience.

5.4 Three-dimensional geometry

Descriptive or perspective geometry was defined with respect to objectives. As Monge had first introduced descriptive geometry, writing in 1798,

> La Géométrie descriptive a deux objets: le premier, de donner les méthodes pour représenter sur une feuille de dessin qui n'a que deux dimensions, savoir, longueur et largeur, tous les corps de la nature qui en ont trois, longueur, largeur et profondeur, pourvu néanmoins que ces corps puissent être définis rigoureusement.
>
> Le second objet est de donner la manière de le connaître, d'après une description exacte, les formes des corps, et d'en déduire toutes les vérités qui résultent et de leur forme et de leurs positions respectives. (Monge (1798), 5)[50]

Monge's concise definition illuminated several features in the texts of our corpus on both descriptive and perspective geometry, which we will see reiterated (sometimes word for word) in the writings of his contemporaries and followers. Without exception, in these texts Monge was celebrated as the creator of descriptive geometry.

The separate designations of descriptive or perspective referred to complementary and overlapping practices intended to produce exact representations with a ruler and compass on paper of three-dimensional objects. We will thus refer to them collectively as three-dimensional geometry. The intended audience for these practices were artists, architects, engineers and students destined for the École polytechnique, École militaire, or École de marine. The École polytechnique added descriptive geometry to its entrance exam in 1827, which may explain why half of these texts appeared between 1827 and 1829. Further, of the twelve texts in three-dimensional geometry considered here, eight of them were by authors associated with the École polytechnique as students, professors, or examiners.

Authors argued for the value of their theories based on utility and application. Louis-Léger Vallée characterized the figures executed through the principles of elementary geometry as "arbitrary" representations following "tacit conventions" with little exactitude. He acknowledged that in mathematical proofs, both exact and inexact figures served the same function to "fix in the mind the idea of magnitude" [*à fixer dans l'esprit l'idée des*

[50] "Descriptive geometry has two objectives: the first, to provide methods for representing on a sheet of drawing paper has only two dimensions, namely, length and width, all natural bodies that have three, length, width and depth, provided that these bodies can be rigorously defined.

The second objective is to provide a way to know, following an exact description, the form of bodies and to deduce from it all the truths that result and their respective form and positions."

grandeurs] (Vallée (1819), ix-x). As an example, he pointed to the figure of the cylinder circumscribed to the sphere, engraved on Archimedes' tomb and represented in all elementary geometry texts as a great circle. However, in useful constructions and applications, these three dimensional illustrations required as much rigour as that required in elementary planar geometry. This graphic dimension was also apparent in Monge's quote, and clearly distinguished three dimensional geometry as practicable. These texts contained theory that could be applied in physical space, or that it proved the precision of a technique.

As well as exactitude and precision, authors described their work with terms similar to those used to describe elementary and analytic geometry: as clear and accessible. Unlike the above mentioned geometries, descriptive geometry was also often considered beneficially concise. This feature is confirmed by the average length, only 186 pages of text. Clarity could be achieved by careful consideration of language and figures. E. Duchesne warned that descriptive geometry presented new difficulties to geometry students, but described his *Elements of descriptive geometry* as ideal for beginners because neither too wide in scope nor too detailed (Duchesne (1829), 3). Background knowledge, if any was required, was limited to elementary geometry: relationships between planar polygons and circles and surfaces of revolution. Only Dupin and Hachette addressed analytic geometry, but both segregated their texts into synthetic or descriptive and analytic memoirs that could be consulted independently. Moreover, the extended titles of their two texts, respectively *Développements de géométrie, avec des applications à la stabilité des vaisseaux, aux déblais et remblais, au défilement, à l'optique, etc.; ouvrage approuvé par l'institut de France, pour faire suite à la géométrie descriptive et à la géométrie analytique de M. Monge* and *Éléments de géométrie à trois dimensions. Partie synthétique. Théorie des lignes et des surfaces courbes. Partie algébrique. Traité des surfaces du second degré*,' suggested that descriptive geometry was seen as a separate subject from analytic geometry. As both Dupin and Hachette argued, the former was more sensible and more practical.

With three-dimensional geometry, geometers generalized the particular techniques of artistic practices into a method. Lacroix noted that every two dimensional geometric question had an analog in space, and a two dimensional object, like the circle, was only a particular case of a three dimensional one, like the sphere. Rather than listing numerous examples that the reader could only follow through "mechanical imitation," Lacroix emphasized the importance of a methodical manner of employing perspective techniques (Lacroix (1802), xiv). Hachette was praised by Legendre and Arago for his clarity and method (Hachette (1817)). The distinction lay in the transition from a collection of examples to an interconnected theory.

While based in a mathematically sophisticated method, three-dimensional geometry produced results that could be used by non-mathematicians. Vallée declared one of the principal advantages of descriptive geometry drawings was that they were "often intelligible for people who did not even know Geometry" [*souvent intelligibles pour les personnes mêmes qui ne savent pas de Géométrie*] (Vallée (1819), xvi).[51] Vallée outlined descriptive geometry as producing measurable, recognizable, and manipulable drawings.

> C'est la science qui enseigne les moyens de représenter avec exactitude les grandeurs géométriques, et à faire graphiquement sur ce grandeurs toutes les opérations possibles. (Vallée (1819), xvii)[52]

With similar consideration for those with limited mathematical training, Cousinery explained two forms of projection: *linear projection*, or perspective geometry, which was easy to interpret although the real size of the object was unclear and *orthogonal projection*, or descriptive geometry, where the size was provided but the designs were difficult to conceive as three-dimensional objects. His text emphasized showing objects as they appeared by incorporating linear projection to supplement "the already fruitful theory of projection" (Cousinery (1828), v). With both perspective and descriptive geometry, Cousinery argued that representations could then be understood visually and executed with precision, even by those who did not understand the mathematical reasoning behind them.

Although, the promise of reading three-dimensional geometric representations might not require much mathematics, it did require practice and an understanding of constructive exactitude. Reynaud warned that students had become accustomed to seeing all the necessary lines and points in proofs of planar geometry, and consequently had difficulty attending to points and lines that did not appear in the projective planes. To remedy this he promised to include additional lines on his figures to illustrate how to deduce unknowns from givens, and then show which of these lines were actually necessary for the problem's solution (Reynaud (1833), vi–vii).

Three-dimensional geometry was independent of coordinate representation, and the associated vocabulary of spatial objects usually reflected this independence from equations involving variables. Surfaces were denoted with respect to being curved, oblique, enveloping, ruled, etc. These adjectives emphasized the generation or visual appearance, and not the algebraic properties. In Dupin's text on both analytic and descriptive geometry, he argued against these multiple designations for the same objects. He expressed astonishment that the "denominations of a science where all is harmony and precision are incoherent and often so imprecise," and proposed new geometric nomenclature to describe surfaces and curves modelled in part after the new language of modern chemistry.[53]

[51] Books written exclusively for artists, such as methods of painting in perspective, were excluded from our corpus based on their applied titles as "practical geometry." Nevertheless, we observe an overlap of intended audiences here.

[52] "This is the science that teaches the means to represent geometric magnitudes with exactness, and to graphically perform all possible operations on these magnitudes."

[53] Some reforms to scientific language including chemistry that were proposed and adopted during the late eighteenth and early nineteenth century are described by Pietro Corsi in Corsi (2005).

> Il me semble que pour perfectionner un peu la langue géométrique, imparfaite à tant d'égards, il faudrait consacrer exclusivement la terminaison en *ide* aux surfaces, et la terminaison en *ique* aux lignes courbes. [...] Je ferais plus, je consacrerais le genre féminin aux surfaces, tandis que le genre masculin serait réservé aux longueurs et aux volumes. (Dupin (1813), 291)[54]

He followed this proposal by a list of translations, for example the second degree curve would be *Le Deutérique* and its enclosed area would be *La Deutérique*, and suggested unprejudiced men could judge the advantages and inconveniences of this innovation. Though less comprehensive or considered than Dupin's reorganization, Reynaud also suggested slight abridgements of geometric expressions "consecrated by usage," such as *line* for *straight line* and *arc* for *arc of a circle* (Reynaud (1833), vii). With an eye on potential audiences, Lacroix even proposed using the term "stereography" when teaching three-dimensional geometry to artists so as to "not shock them" by mathematical terms (Lacroix (1802), xx).

Perhaps because "modern geometry" was so closely associated with analytic geometry, the much newer descriptive geometry was rarely classified as such. But descriptive geometry was also not considered the geometry of the ancients. To represent three-dimensional bodies beyond the polyhedra, sphere, cone and cylinder required methods beyond elementary geometry, and the ruler and compass alone did not suffice for all possible geometric configurations in space. Duchesne invoked "a law of continuity" to describe how points form planar curves or curves of double curvature (Duchesne (1829), 69).[55] Surfaces were described and defined by revolution about an object, enveloped by the constant movement of an object, or with respect to the curves formed by intersections with or projections onto planes. These definitions bore little resemblance to analytic discussions of order or degree.

Even without analysis or algebra, authors often assumed some prerequisite geometric study on behalf of their audiences. Hachette explained that when he simply stated propositions, they had either been proved in the Elements (presumably Euclid's) or were evident, thus claiming the evidence of three-dimensional geometry as a sound demonstration (Hachette (1817), 1). In order to help artists understand his text, Vallée only provided "synthetic proofs," and consequently admitted "the method of infinitesimals." Alternative proofs using algebra were relegated to notes at the end of his volume (Vallée (1819), xviii). This led to a more liberal use of infinity and imaginary points than many of his contemporaries. Vallée described "imaginary parts" of spatial objects, defined by points of intersection when

[54]"It seems that to somewhat perfect the geometric language, imperfect in so many ways, one must exclusively consecrate the suffix *ide* to surfaces, and the suffix *ique* to curved lines. [...] I will do more, I will consecrate the feminine gender to surfaces and so the masculine gender will be reserved for lengths and volumes."

[55]As we saw in Chapter III, by the early 1820's the law or principle of continuity began to be a charged concept in many aspects of mathematics, for example in Cauchy's reviews of Poncelet (Poncelet and Cauchy (1820), Poncelet and Cauchy (1825)).

the objects did not share any real points, and explained that one could consider a projection as a perspective where the point of view was "at infinity." Both Vallée and Adhemar also defined asymptotes as tangent lines meeting the curve at a point at infinity (Adhémar (1823), 26). These were not literally constructive operations, and served to expand the potential purview of planar geometry independent of coordinate geometry, just as in the *Annales* the techniques of a dynamic geometry would be employed to describe conic sections and tangent lines. However, these two examples of three-dimensional geometry texts were not intended for professional mathematicians. Vallée's attention to his artistic and mathematical readers shows how courting two audiences could enable exploration of more or less rigorous methods. Similarly, Dufour warned research mathematicians that he spoke the language of artists, "Les savans ne trouveront sans doute rien, dans cet Essai, qui soit digne de leur attention" (Dufour (1827), viii).[56]

The usefulness of three-dimensional geometry for both artists and engineers was proclaimed as a manner of national pride. Monge declared descriptive geometry would advance French machinery, diminish manual labor, and lead to independence from foreign industry (Monge (1798), 2). Over twenty years later, Vallée, whose text was dedicated to the recently deceased Monge, attributed the flowering of French industry to Monge's geometry (Vallée (1819)). Similarly, Hachette desired to "propagate three-dimensional geometry" which would "give a new treasure to the arts, which are the principal source of public prosperity" (Hachette (1817), viii). Gabriel Gascheau described his particular applications to "oblique surfaces because of their frequent usage in the arts where they are employed in the construction of screws, stairs, several types of vaults, etc." (Gascheau (1828), 5). Dupin went beyond representations, explaining the use of descriptive geometry in the stability of boats, the equilibrium of floating bodies, excavation, building embankments, and optics.

Readers could quickly judge the success of three-dimensional geometry by the figures produced, and the volumes contained numerous examples following the text. In these examples, often the plates (*planches*) containing several related figures were numbered and referenced, rather than the individual drawings in the plates. Descriptive geometry figures were also referenced as "épures," which could be translated as a sketch, diagram, or blueprint. Vallée defined épures as a descriptive geometry term for all the figures in which one employs two or more planes of projection. This term revealed the polyvalent nature of these figures, not only to be drawn and viewed, but then utilized to create spatial designs.[57] The importance of figures was made especially apparent by the respective academy reviews of Cousinery and Vallée, each of which accompanied the published text. In Cousinery's review, by Fresnel and Mathieu, his use of projection was praised for generating no more lines

[56] "[...] scholars will not find anything in this essay that is worthy of their attention."

[57] Because of the alternative figure or épure labelling and numbering, it does not make sense to give a comparative average figure count for descriptive geometry texts.

than in orthogonal projection (Cousinery (1828), vi). In Vallée's review by Prony, Fourier, and Arago, the 320 page text was not read in its entirety because it was too long, but the figures were consulted and declared "perfectly designed" with all necessary construction and "no confusion" (Vallée (1819), viii). Lacroix advised that when creating figures the student should draw lines with a pencil and then erase the ones that did not lead to the desired result. He then explained his use of four different kinds of letters in order to designate different parts of his figures. Lacroix spoke to the integrality of figures in geometry, as important as computation in arithmetic. "Des figures chargées de toutes les lignes de construction sont aux planches d'un traité de géométrie ce que des minutes de calcul sont aux exemples d'un traité d'arithmétique" (Lacroix (1802), xi).[58] With projections from points at infinity, not every step of the process could be represented on a plane or even in finite space, but every point from the original three-dimensional body could be faithfully represented and effectively reproduced in the final product.

5.5 Geometry of the ruler or compass

Elementary, analytic, and three-dimensional geometry were each well-defined categories for early nineteenth century textbook authors. As we have seen, many titles or tables of contents referred precisely to one or more of these three geometrical methods. However, these categories were not exhaustive to our corpus. Although no geometer entitled their work with this description, by "geometry of the ruler or compass" we intend titles on geometry that focused on the non-metric positional relationship between geometric objects, such as the articles labelled in Gergonne's *Annales* as "géométrie de la règle" or "géométrie de situation."

The five texts considered here include Carnot's two texts on the correlation of planar figures (Carnot (1801), Carnot (1803)), Charles Julien Brianchon's study of second order lines (Brianchon (1817)), Chasles' research on second degree lines and surfaces (Chasles (1829)), and the 1828 edition of the French translation of Lorenzo Mascheroni's geometry of the compass () (1828 (1797), originally published in 1797). As the title of Carnot's 1803 *Géométrie de Position* suggests, there were existent designations for alternate methods in geometry, but these four authors were not uniform in their descriptions. Carnot also claimed his geometry was elementary, Mascheroni's research was "geometry of the compass" while Brianchon described his work as "geometry of the ruler" and most of these texts could be categorized as "geometry of curves and surfaces," another subject heading found in Gergonne's *Annales*. Indeed, what these texts all emphasized was not so-much results as much as the method employed toward reaching these results.

[58]"Figures filled up with all the construction lines are to the plates of a geometry treatise as the minutiae of calculation are to the examples in an arithmetic treatise."

Originally written in 1798, by its title Mascheroni's text clearly denoted the idea of modifying the rules of elementary geometry. He admired the recent advances in analytic geometry, which rendered ancient achievements child's play.

> Le domaine de la Géométrie continua à s'accroître à l'aide de ces profondes recherches et avec le nouveau secours de l'analyse finie et infinitésimale, au point que les inventions qui d'abord avaient fixé l'admiration des anciens, et mérité les sacrifices de Thalès et de Pythagore, sont devenues l'apanage des enfants de nos jours. (Mascheroni (1828 (1797), 6)[59]

Returning to the "elements," Mascheroni aimed to explore whether the simplest possible constructions of geometry were those involving only the ruler and compass or whether even simpler solutions could be found with the compass alone. Mascheroni defined his approach as determining what propositions from planar Euclidean geometry could be reached through compass constructions. Contrary to his original quest for simplicity, he found that his compass solutions, often reached through trial and error, could be more complicated than those involving both a compass and a straight-edge. Mascheroni explained that his ultimate decision to publish rested instead on the potential for exact applications in astronomical or geographic engineering rather than having achieved greater simplicity.

As his translator, A. M. Carette (a former student of the École polytechnique), observed in his biographical preface to the second French edition, Mascheroni's geometry of the compass met with a popular reception from Bonaparte to Lagrange and Laplace to Delambre's 1808 report on the progress of mathematics (Carette (1828), xii-xv). This recognition, along with the multiple editions and translations, suggests that there was an appreciation for new research related to planar elementary geometry. Though Mascheroni's geometry of the compass did not become an area of active research in the early nineteenth century, its counterpart, geometry of the ruler, was adopted by Brianchon, as well as in *Annales* publications (consider, among the articles we have already consulted, Plücker (1826b), Brianchon (1813), Brianchon and Gergonne (1814), Gergonne and Servois (1810), Gergonne (1813b), Gergonne (1817b)).

The remaining texts in this category were also connected by the work of Carnot, who was cited by the two later authors. Due to Carnot's influence on Poncelet, Gergonne, Plücker and Steiner we must note several important features in his texts. In 1801 Carnot published *De la corrélation des figures de géométrie*, which he expanded into *Géométrie de position* in 1803. Carnot's described his geometry as "elementary," not because it wasn't difficult but because it "did not cede to analytic speculations" [*ne le cède point aux spéculations*

[59] "The domain of Geometry continues to increase by aid of these deep researches and with the new results from finite and infinitesimal analysis, to the point that the inventions which at first had fixed the admiration of the ancients and merited the sacrifices of Thales and Pythagoras have become the property of children in our time."

analytiques] (Carnot (1803), xxx). He continued by first defining elementary geometry as the methodical study of properties belonging to figures composed of straight lines and circles. However, Carnot expressed dissatisfaction with this definition, which appeared like "a collection of propositions that is very incomplete" [*une collection de propositions, qui est très-incomplète*].[60] Instead, Carnot characterized elementary geometry by as a set of solutions to a single general problem:

> *Dans un système quelconque de lignes droites, tracés ou non dans un même plan, quelques-unes d'elles, ou des angles qui résultent de leur assemblage, soit entre elles-mêmes, soit entre les plans qui les contiennent, étant donnés en nombre suffisant pour que toute la figure soit déterminée, trouver tout le reste.* (ibid, xxxiii)[61]

We note that this definition makes no mention of curved lines or circles nor did it concern finding length, area, or volume. Thus, like Mascheroni, Carnot presented new research by limiting the bounds of elementary geometry.

Carnot proposed to solve this general problem by means of an extensive series of tables. The first would show how many linear parts must be known in order for a "primitive" figure to be wholly determined. Then, by "mutations" of the primitive figure, Carnot could find tables for "correlative" figures "qui ne diffère pas essentiellement de la première, mais seulement par quelques modifications ou par la diversité de position des parties correspondantes" (xxxiv).[62] This relationship between the corresponding parts of a proposed "primitive" figure and its associated "correlative" figure, Carnot considered as the central study of "Geometry of position." Primitive figures could be transformed into correlative figures by "insensible degrees" or "mutations," thus possessing many of the same properties of primitive figures, as Carnot would show through proofs and tables through the remainder of his text. Carnot defined various types of correlative relations, and his definition of "direct" and "inverse" relations would be adopted by Poncelet, Gergonne, and Plücker as we saw in Chapter IV.[63]

While never addressing questions of measurement, Carnot's text was full of variables, quantities and calculations. This use of calculation is exemplified by his solution to the Apollonius problem, which we summarize for comparison to the solutions treated in Chapter

[60] Steiner's description of systematicity in synthetic geometry bears strong resemblance to Carnot's aim for completeness in describing elementary geometry here. As we have seen in Chapter IV, Steiner's citations from 1827 indicate that he had read Carnot's *Géométrie de position* by this time.

[61] "*If in any system of straight lines, whether traced or not in the same plane, some of them, or the angles formed by their intersections, either with each other or with the plane that contains them, are given in a quantity sufficient to entirely determine the figure, then find all the remaining [lines or angles].*"

[62] "[...] that do not differ essentially from the first, but only by some modifications or by the difference in position among corresponding parts."

[63] Philippe Nabonnand and Karine Chemla respectively provide valuable analyses of generality and pure geometry in the texts of Carnot, Poncelet, and Chasles in Nabonnand (2011b) and Chemla (1998).

IV (390–391). Referring to an illustrated figure, Carnot assigned variables a, b, c as the radii of the three given circles, x as the radius of the desired circle, and m, n, p, q, r, s as the six line segments defined by the respective point pairs AB, AC, BD, CD, AD, BC. By hypothesis, Carnot knew the segments m, n, s and when one found x, the remaining three segments could be found by addition, $p = b + x$, $q = c + x$, $r = a + x$. The problem thus reduced to finding x. Since $ABDC$ was a quadrilateral, Carnot could use a formula he had derived in an earlier problem for finding the four sides and two diagonals of a given quadrilateral given five of the six line segments:

$$(m^2q^4 + q^2m^4 + n^2p^4 + r^2s^4 + s^2r^4)$$
$$+(m^2n^2s^2 + m^2p^2r^2 + n^2q^2r^2 + p^2q^2s^2)$$
$$-(m^2n^2p^2 + m^2n^2q^2 + m^2p^2q^2 + m^2q^2r^2 + m^2q^2s^2 + m^2r^2s^2 + n^2p^2q^2 + n^2p^2r^2 + n^2p^2s^2 + n^2r^2s^2 + p^2r^2s^2 + q^2r^2s^2) = 0.$$

One could rewrite this equation in terms of the above variables by substituting in the values $p = b + x$, $q = c + x$, $r = a + x$. Carnot did not provide the calculation because the problem had been "résolu d'une manière plus simple par des géomètres de premier ordre, tels que Viete, Newton, Euler, et que la seule synthèse en fournit plusieurs solutions très-élégantes" (ibid, 391).[64] Carnot concluded by stating that his purpose with this solution was only to show one of many applications of the above formula.

Carnot's mention of synthesis in the above quote suggests that he did not consider his own solution as synthetic. Indeed, Carnot designated four different methods in geometry: the synthetic method, the trigonometric method, analytic geometry, and the mixed method. The synthetic or graphic method involved using the known properties of the given figures to find the most appropriate construction, without the use of analysis. Carnot designated descriptive geometry as part of this "graphic method" when it concerned figures on different planes. The trigonometric method utilized relationships between angles and sides of triangles that one could form in the given figure. The analytic method reduced all figures to relationships between abscissa and ordinates, while angular relationships could only be described through sine and cosine relations. Carnot's definition of analytic geometry was more limited than the application of algebra to geometry described above, in that it should not include any geometric properties except those "indispensable to express the conditions of each problem" with coordinate equations [*indispensable pour l'expression des conditions de chaque problème*] (352). Synthetic geometry was advantageous because it concerned real and practical results that could be pictured in the imagination. Analytic geometry was simple and uniform, but could lead to very complicated calculations.

With its use of a figure and trigonometric relations (used to derive the above formula), Carnot considered his solution of the Apollonius problem as exemplifying the mixed method.

[64] "[...] resolved in a much simpler manner by geometers of the first order, such as Viete, Newton, Euler, and as synthesis alone furnishes several very elegant solutions of it."

> La méthode mixte consiste à employer simultanément les ressources de la géométrie graphique, de la trigonométrie et de la géométrie analytique, pour arriver plus facilement au résultat qu'elle veut obtenir. Si cette méthode n'a pas l'avantage d'une certaine uniformité dans ses procédés, elle a plus de moyens pour profiter des diverses propriétés déjà connues des figures, et des simplifications accidentelles, qu'offre dans chaque cas particulier la nature de la question. (353)[65]

Carnot's practice of mixed methods was observed and rejected by Gergonne because of the lack of simplicity and elegance, as noted in Chapter II (Gergonne (1817e)). Moreover, Carnot's frequent use of all four methods stands as a correction to Klein's claim that Carnot initiated the divide between analytic and synthetic methods, as we saw in Chapter I. While Carnot criticized the "very complicated" calculations of analytic methods, he did not divide geometry in half and frequently applied mixed methods in this text.[66] Although Carnot's mixed methods were not adopted as such in the work of Poncelet, Plücker or Steiner, Brianchon would apply them successfully in several articles, including his first published proof of "Brianchon's theorem" in the *Journal de l'École polytechnique* (Brianchon (1810)).

Within his book, *Mémoire sur les lignes du second ordre*, Brianchon cited Carnot surreptitiously as "the illustrious author of the Geometry of position" (Brianchon (1817), 6). Carnot had described his principle of correlation as original, but Brianchon emphasized a progressive lineage by situating his short text with a small bibliography on the historical development of geometry of the ruler and harmonic lines. Brianchon defined geometry of the ruler as having for its object, "les propriétés de situation des systèmes de lignes droites." His list of predecessors began with Pappus, and extended to three present-day geometers: Carnot, Lavit, and Servois. The limitation to ruler constructions in the plane did not preclude Brianchon from invoking points at infinity. He generalized the notion of polygons to include those with vertices at infinity, that is, with parallel adjacent sides. Nevertheless, Brianchon's text was intimately linked to the figure. He attended to this relationship in careful detail, for instance by showing correlation between two figures through the use of backward letters denoting inversely correlated points.

Chasles expressed more caution with points at infinity by associating the concept, and that of imaginaries, with analysis: "Analitiquement parlant, deux sphères ont une seconde courbe d'intersection, toujours imaginaire, située dans un plan à l'infini" (Chasles (1829), 62). Chasles considered his text on second degree lines and surfaces, extracted from *Nou-*

[65] "The mixed method consists in simultaneously employing the resources of graphic geometry, of trigonometry and of analytic geometry to arrive very easily at the result that one wishes to reach. If this method lacks the advantage of a certain uniformity in its procedures, it profits greatly from different known properties of figures, and accidental simplifications that in each particular case suggests the nature of the question."

[66] Carnot's approach to analytic geometry was also connected to his stance on negative numbers, as described by Gert Schubring in Schubring (2005), particularly Chapter V "*Le Retour du Refoulé*: From the Perspective of Mathematical Concepts".

veaux Mémoires de l'Académie Royale de Bruxelles, as pure geometry. As he explained in his title, pure geometry extended beyond elementary geometry to include the polar transformation of conics and second degree cones, general properties of second degree surfaces of revolution, general properties of second degree cones, and the construction of lines of curvature. Chasles was well versed in contemporary research practices and provided over ten precise citations of recent texts and articles, a convention that appears to have been much more common in journals than monographs. He aimed to demonstrate that pure geometry could yield many of the same results as analysis. For instance, he cited an analytic proof from Bobillier, in order to preface his own purely geometric proof that would show the generality of his method (38). Carnot also described his method as uniform, another quality that typically characterized analytic geometry.

These summaries reveal a section of geometry books where new methods were actively considered and developed. With their emphasis on a discipline in flux, these four texts were the least like textbooks and would include the books we have examined in our case studies, such as Poncelet's *Traité* and Steiner's *Systematische Entwicklung.* These authors introduced new principles and alternative techniques. In these texts we find a precedent to claims for novelty. Further, the emphasis lay on new methods and not new results, which reflected that of contemporary research articles.

5.6 Mixed collections

Our final and loosest category includes four books where geometric problems, theorems, or methods were collected. Although we have chosen to classify these texts as mixed collections, they had very little in common with each other. Paul-Marie-Gabriel Treuil's *Essais de mathématiques, contenant quelques détails sur l'arithmétique, l'algèbre, la géométrie et la statique* simply contained an assortment of his recent mathematical researches, including a few problems and theorems on measuring areas and angles and ran only 73 pages (Treuil (1819)). By contrast, Garnier's 1810 *Réciproques de la géométrie, suivies d'un recueil de théorèmes et de problèmes* extended to nearly 400 pages of text and figures, was the second issue since 1807, and received favourable citations from Biot and Francfort (Garnier (1810)). The text could serve as a companion to Garnier's analytic geometry, as the contents included no analytic solutions to theorems and problems involving planar figures, descriptive geometry, and trigonometry. Garnier organized his text as a list of propositions and their proofs, followed by theorems and problems grouped by geometric figures. This composition perhaps reflected Garnier's views on non-analytic geometry as oriented by particular cases more than general method.

With a more modest scope, in *Applications de la géométrie à la mesure des lignes inaccessibles et des surfaces planes* Lefevre addressed a wide assortment of problems, and described his "satisfying" research as "recreational," which he further reiterated in his dedication to his son. His work emphasized physical application, and the specific tools of construction: "les jalons, l'équerre, le graphomètre et avec l'équiangle" (Lefevre (1827), viii). However, Lefevre also included the theoretical side of geometry, referencing both Carnot's work on transversals and Mascheroni's geometry of the compass. Even with an emphasis on recreational problem solving, his focus on inaccessible lines addressed a common class of geometric problems. Poncelet had been interested in similar problems in his *Traité*, where he wrote a section on "Conséquences qui en résultent pour la détermination des droites, ou des points qui appartiennent à un point, ou à une droite, supposés tous deux inaccessibles, invisibles ou placés à l'infini" (Poncelet (1822)). Alluding to Mascheroni, Lefevre presented multiple solutions of the same problem, but based his research exclusively on elementary principles in order to promote its use among those "not habituated to analytic calculation."

Gabriel Lamé's *Examen des méthodes* serves as a fitting culmination to our own examination of a similar topic (Lamé (1818)). Like the texts of Treuil, Garnier, and Lefevre, Lamé motivated his research by solving geometric problems. In his introduction, he explained that the problems led to his methodological reflections, and not vice versa, although the cause and effect might not be apparent. Although both Dupin and Hachette included both elementary and analytic geometry, Lamé's text provides a rare example of these methods employed side-by-side. Consider his list of what he considered the most important aspect of his work, combining analytic and descriptive geometries.

> L'expression analytique de la communauté d'intersection des lieux géométriques ; la détermination complète des courbes et surfaces du second degré par la Géométrie descriptive, lorsqu'on donne un nombre suffisant de leurs points ; la théorie des courbes et surfaces représentées par les équations $x^\alpha : a^\alpha + y^\alpha : b^\alpha = 1$, et $x^\alpha : a^\alpha + y^\alpha : b^\alpha + z^\alpha : c^\alpha = 1$, sont les parties de cet Ouvrage qui me paraissent mériter le plus attention. (Lamé (1818), v)[67]

Lamé's desire to find "general principles for problem solving" resulted in new approaches to analytic geometry. In particular, Lamé was credited with the first use of abridged notation in this text, which greatly helped to alleviate the burden of analytic computation (Boyer (1956), Barbin (2009)). Initially, Lamé claimed there were only two methods in geometry: analytic and synthetic. However, Lamé explained that he did not intend these designations to signify analytic and synthetic geometry, but instead as reverse orders of exposition or demonstration.

[67] "The analytic expression of the cluster of intersections of geometric loci; the complete determination of second degree curves and surfaces by descriptive geometry, when one has a sufficient number of their points; the theory of curves and surfaces represented by the equations [...] and [...], are the parts of this work that seem to merit the most attention."

> Une solution est dite présentée synthétiquement, lorsque énoncée d'abord on en prouve l'exactitude, soit par une méthode inverse de celle qu'à suivie l'Analyse pour la trouver, soit enfin par la démonstration à l'absurde. (ibid, 9)[68]

He later introduced the inverse method, indirect method, and mixed method as well as descriptive geometry, simple geometry, and the theory of transversals. The suggestion of a continuum rather than a dichotomy of methods also appeared in the work of Carnot, Brianchon, Chasles, Poncelet, and in the assorted subject titles of *Annales* articles. For all of these geometers, problem solving motivated exploring the limits of known methods and creating new ones. Collections of problems and theorems could thus serve to summarize past accomplishments, provide material for teachers and students, and promote new approaches to research.

5.7 Conclusions

Through the first third of the nineteenth century there were several possible venues for article length publications by French geometers including *Journal de l'école polytechnique* (1795–1939), *Correspondance sur l'école polytechnique* (1804–1815), *Annales des mathématiques* (1810–1832), *Bulletin des sciences mathématiques* (1823–1831), *Journal für die reine und angewandte Mathematik* (1826–), *Correspondance mathématique et physique* (1825–), as well as publications associated with local and regional scientific academies, most prominently the *Académie des Sciences.* As Gergonne observed, prior to the *Annales* article publication was limited, primarily reserved for those associated with the École polytechnique in the Journal of that school. By 1826 the situation had changed dramatically, and though the *Annales* ceased publication in 1833, it was quickly supplanted by both the *Journal de Mathématiques Pures et Appliquées* (1836) and the *Nouvelles Annales de Mathématiques* (1842–1927).[69] The readers of textbooks and articles may have been identical, but they were addressed as different audiences. There was certainly overlap between authors. If we limit our comparison to the *Annales*, we find Stainville, Brianchon, Chasles, Kramp, Lamé, Noël, and Garnier contributed both *Annales* articles and wrote books considered in our survey.

In 1810, when Gergonne began publishing his *Annales*, he discussed the many functions and advantages of a journal devoted to mathematics.

[68]"A solution is said to be presented synthetically when we first set out to prove its exactitude, either through an inverse method to that which Analysis followed to find it, or through proof by contradiction."

[69]Liouville's *Journal* has been studied by Verdier and Gérini (Verdier (2009a), Gérini (2010b), Gérini and Verdier (2007)). While Delcourt has compared the *Nouvelles Annales* and Gergonne's *Annales* in Delcourt (2011b). Laurent Rollet and Philippe Nabonnand further examined the audience and content of the *Nouvelles Annales* in Rollet and Nabonnand (2013).

> un recueil qui permette aux Géomètres d'établir entre eux un commerce ou, pour mieux dire, une sorte de communauté de vues et d'idées; un recueil qui leur épargne les recherches dans lesquelles ils ne s'engagent que trop souvent en pure perte, faute de savoir que déjà elles ont été entreprises; un recueil qui garantisse à chacun la priorité des résultats nouveaux auxquels il parvient; un recueil enfin qui assure aux travaux de tous une publicité non moins honorable pour eux qu'utile au progrès de la science. (Gergonne (1810a), i–ii)[70]

This public exchange of new ideas aimed toward scientific progress provides a contrast to the slow repetition characteristic of most geometry books. In general, geometry books were written for pedagogical purposes. This was also a motivation for geometry articles, the *Journal de l'école polytechnique* and *Correspondance sur l'école polytechnique* were both directed toward student readers. Similarly, Gergonne introduced the *Annales* as above all consecrated to "recherches qui auront pour objet d'en perfectionner et d'en simplifier l'enseignement" (ii). However, while most articles in the *Annales* and similar journals could certainly be read by students or used by their instructors in creating teaching material, they were not explicitly presented by their authors as such. Instead, the research was framed as an end in itself, or to be used by other participants in the shaping "a community of views and ideas."

Without the community afforded by journal articles, books were intended to be self-sufficient. Book authors noted in their prefaces whether any arithmetic, algebra, or additional geometry might be required in advance, and if so, occasionally cited a few authors who might serve as preliminaries. Articles contained none of these explicit prerequisites, instead adopting in-text references to cite particular concepts or results. Most books contained very few in-text references, which could be explained by frequent introductory comments denying any claim to novelty. There was such little expectation for new content, that, as noted above, Poullet-Deslisle joked that trying to be new would appear as a "ridiculous pretension." In this context, the repetition of results we have seen exemplified in Chapters II, III, and IV appears to have been a common practice. However, when content was repeated in articles, there was also the intent of introducing something else new, such as a method or a principle. Further, despite the prevalent redundancy, we found no evidence of opposition among textbook writers with respect to priority or potential plagiarism. Authors primarily restricted their particular criticisms to pedagogy and order of exposition. When new research did appear, such as in the books of Lamé, Brianchon, or Chasles, the authors also wrote publicity-generating articles with similar content (a few examples include Lamé

[70] "A periodical that allows Geometers to establish a commerce among themselves or, to put it better, a kind of community of views and ideas; a periodical that spares them from vainly engaging in research already undertaken by others; a periodical which guarantees to each the priority of the new results that they come across; a periodical finally, which assures everyone's work publicity, not less honourable for them than useful to the progress of science."

(1817), Brianchon and Poncelet (1820), Chasles (1828b)). This dual publication strategy suggests that books were not perceived as a sufficient medium for introducing new geometry to a wider audience.

Textbooks advertised their specific approaches to an audience of teachers and students as making the content simpler, more accessible, easier to remember. The fruitfulness and elegance ascribed to analytic geometry were described as part of the heritage of Viète, Descartes, Newton, and Euler, more so than contemporary innovations. Chasles' text, originally an article, stands out among the books as citing particular advantages of generality and uniformity in his work over very contemporary proofs and solutions. As a suggestion of the content standards between books and articles, recall that the Apollonius problem appeared frequently in the *Annales* with new solutions by Gergonne by Durrande, and by Plücker (Gergonne (1810b), Gergonne (1814b), Gergonne (1817a), Gergonne (1827d), Durrande (1820), Plücker (1827)). The Apollonius problem also appeared in several textbooks. Only Carnot provided an original solution and proof (Carnot (1801), 109; Carnot (1803), 390). In 1828, Didiez provided a solution without proof, although he had earlier cited an article containing Gergonne's solution, he did not repeat it here (Didiez (1828), 206). Garnier in 1810 and Terquem in 1829 both credited Newton with the "most direct and most central solution" (Terquem (1829), 423). Meanwhile, the solution by Newton had been comparatively rejected by Poncelet, Durrande, and Plücker and roundly criticized by Gergonne as overly complicated. These diverse approaches to the Apollonius problem exemplify the contrasting norms of citations and attention to contemporary research.

The cost of production and limited number of potential purchasers may help to explain why so few books appeared that weren't textbooks, and why only textbooks were reprinted in quick succession. Many of the well-known and widely re-published names in turn of the century geometry–Monge, Lacroix, Legendre–wrote books almost exclusively for a student audience. Textbooks catered to an existent market, while research books were expensive and risked not being sold. As cheaper commodities, journal articles could afford to take more risks. One obvious contrast between articles and books was page length.[71] Likewise, articles included fewer figures, if any, regardless of the geometric method employed, but the author may not have always made the decision to omit textual illustrations. For instance, in the case of Poncelet and Plücker, we have seen how authors would present preliminary results in an article in part to create interest in their books, as Poncelet admitted years later and Plücker noted in his introductions (Poncelet (1866), Plücker (1834)). Comparing the same results by the same authors in articles and in books, we find figures accompanied the latter even when absent in the former. The presence of figure plates did not signify the

[71] The divergent audiences also helps to explain Gergonne's emphasis on the length of Poncelet's *Traité* (Gergonne (1827e)). At 416 pages, the text is only slightly longer than Bossut's or Didiez's texts on elementary geometry and shorter than those by Terquem and Develey. However, if new research typically appeared in article form, 400 pages appears quite long by comparison.

choice of method, although we found that analytic geometry texts contained slightly fewer figures on average than those on elementary or three-dimensional geometry. In general, these textbooks contained geometry problems, which were solved with reference to an illustrated figure.[72]

Textbooks were written for teachers to use with their students, or, less frequently, for immediate student consumption. Discussions of methodology centred around the best method for teaching. The connection to pedagogy may also explain why so few books appeared in categories outside the standard mathematics curriculum of elementary geometry, elementary analytic geometry, and descriptive geometry. There were no courses corresponding to the popular *Annales* subject headings: *Géométrie de la règle*, *Géométrie de situation*, *Géométrie transcendante*, *Géométrie pure*, or *Géométrie des courbes et surfaces.* Textbook authors framed a methodological divide between analytic, elementary and descriptive geometry with respect to prerequisite mathematical knowledge and intended applications. One of the cited advantages of elementary geometry and descriptive geometry was that no algebra was required. Only Terquem proposed introducing geometry alongside algebra. Otherwise, analytic geometry was the next most advanced geometry, to be learned by those who mastered both elementary geometry and algebra, and continued to pursue mathematics.

A minority of our authors referenced synthetic geometry, and when it appeared the definitions were heterogeneous. For several authors, analysis and synthesis only denoted opposite orders of exposition, either from the hypothesis to the conclusion, or by assuming the problem as solved and working backward to the hypothesis (Lacroix (1799), Lamé (1817), Develey (1812), Mutel (1831)). Synthetic could also stand for an older approach to geometry. Bossut, for instance, described the modern neglect of synthesis (Bossut (1800), vii). Biot, similarly, referenced Newton as his only example of beautiful synthetic geometry. Only, Carnot and Hachette followed definitions of "synthetic" at all resembling those suggested in our Chapter I. Carnot provided a distinct definition for the synthetic method of geometry in *Géométrie de position.*

> La méthode synthétique ou graphique consiste à résoudre les questions proposés sans le secours de l'analyse, en cerclant par les propriétés connues des figures qui se présentent, la construction la plus propre à satisfaire aux conditions proposées. (Carnot (1803), 351)[73]

Hachette viewed similar merits and limitations in the synthetic method, which he associated with the geometry of the ancients, while analytic geometry was modern.

[72]The cost of production has been studied by Verdier in his thesis on Liouville's *Journal* (Verdier (2009a)). Jean and Nicole Dhombres addressed these issues from the perspective of books, and particularly textbooks in Dhombres (1985) and Dhombres and Dhombres (1989).

[73]"The synthetic or graphic method consists of resolving the proposed questions without use of analysis, in banding together through the figure's known properties that are presented, the most fitting construction to satisfy the proposed givens"

> Ceux qui ont cultivé la géométrie des anciens et l'analyse moderne appliquée à la géométrie, savent qu'on est encore loin d'obtenir, par la méthode synthétique, les résultats qui se déduisent du calcul. Cependant cette méthode présente le double avantage de rendre la vérité plus sensible, et de conduire, par une suite de raisonnements, des propositions les plus simples aux plus composées. (Hachette (1817), vii)[74]

The primary merit of synthetic geometry, for both Carnot and Hachette, lay in its sensible or visual qualities and its direct connection to the geometric material. These aspects bear resemblance to the role of the figure emphasized in Poncelet's description of pure geometry discussed in our Chapter II.

However, unlike Poncelet, in most books modernity was associated with the application of algebra to geometry. This was not necessarily because geometry was perceived as stagnant. Dupin, who argued that "pure or rational geometry" was more sensible and applicable to engineering, still associated analytic geometry as modern geometry. However, outside the context of descriptive geometry, books equated elementary geometry with ancient geometry and the elements of Euclid. As we discussed above, Carnot presented a different conception of elementary geometry that could lead to new developments and was not subservient to analysis. His approach reappeared in the work of Brianchon, Poncelet, Steiner and Chasles, although not always labelled as elementary. Instead, these latter geometers, in particular Poncelet, would emphasize the modern aspect of their research, equal to that of analytic geometry. Despite his fame among the next generation of geometers, none of Carnot's works were reprinted in the nineteenth century.

If we consider the research publications of Gergonne, Poncelet, Plücker and Steiner within the set of texts they referenced or those cited in the historical accounts we consulted in Chapter I, their work appears to reflect contemporary geometry. As we have seen in Gergonne's *Annales* in Chapters II, III and IV, new content spread quickly through research articles. Polar reciprocity, radical axes, ideal chords, abridged notation, and lines at infinity became common parlance a few years after being introduced. However, when compared to books published during the first third of the nineteenth century, the work of Gergonne, Poncelet, Plücker and Steiner appears different and original. Most of the research articles considered above could not be classified as elementary, analytic, or three-dimensional geometries. When in 1817, Poncelet claimed that he was practicing "modern geometry" without the use of coordinate equations, he challenged the printed standard and the common meaning of modern geometry.

[74] "Those who have cultivated the geometry of the ancients and modern analysis applied to geometry know that we are still far from obtaining, by synthetic methods, the results that are deduced through calculations. However this method presents the double advantage of rendering the truth more sensible, and of leading, by a series of reasonings, to the most simple propositions to the most composed."

Chapter 6

Conclusion

In describing their work, early nineteenth century geometers adopted comparative qualities and methodologies. We began by searching for a single methodological dichotomy, analysis versus synthesis, in the historical literature, and quickly discovered the existence of numerous kinds of divisions. Closer examinations of these divisions in action revealed complications, as the qualities associated with a specific method varied between different geometers, contexts, and intended audiences. Descriptions of one individual's method and descriptions of that same method in general might counterbalance or even contradict each other. Rather than methodological opposition, we found a common dedication to solving planar geometry problems and attention to visualization. Geometers collaborated by developing and modifying new objects, principles, and techniques unhindered and even encouraged by methodological differentiation.

In attempting to understand the methodological division between "pure" and "analytic" geometry the arguments of Poncelet led us to focus on the role of the figure and, more broadly, the visual and aesthetic dimension of geometry. Poncelet denoted the figure as the central object and form of evidence in pure geometry, yet the frequent absence of an actual figure caused us to consider why it was used at all. Early nineteenth century French geometry textbooks almost always contained figures, which were referenced in introducing new definitions, setting up theorems, and displaying solutions to constructive problems. However, when the same authors republished this content in article format, it might only include virtual figures as written descriptions. Comparing results presented with a virtual figure or with an illustrated figure revealed the latter's applicability to summarize at a glance and display an imitable configuration. These qualities aided in representing an initial constructive case of a problem's solution as well as showing possible representations of new definitions. In journal articles, reference to past publications (some of these with initially illustrated figures) could replace the synoptic role of the illustrated figure, as the image became part of the expected prerequisite knowledge. Further, meticulous descriptions with lettered points, lines, and curves could amply provide a usable virtual figure to be constructed or imagined by the reader, and this was even preferred by some geometers when engaged in three-dimensional geometry.

The virtual figure also served geometers in extending an initial simple case of a circle or an ellipse into more complicated and general variations. Further, with the incorporation of ideal, imaginary, infinite, and animated objects the virtual figure maintained the visual character of geometry in difficult or impossible to illustrate configurations. Yet, to substitute the virtual figure for Poncelet's "figure" also diminished apparent differences between methods. The language, constructions, and descriptions of geometry remained visually centred and widespread use of the virtual figure in any kind of geometry depended more on the problem at hand than the chosen method. As shown in both articles and books published during the first third of the nineteenth century, analytic geometry also contained constructive solutions which were always accompanied by virtual or actual figures.

Poncelet had argued that within analytic geometry the figure was lost from view, often as a result of algebraic calculations. However, Plücker gained attention for limiting calculations in analytic geometry, and instead focusing on the representative role of equations and form of the coefficients. Unlike the analytic geometry found in contemporary textbooks, Plücker did not show a translation from a geometric figure to a coordinate equation and back again. In this respect, within Plücker's work the figure was always kept in sight in the form of coordinate equations, and not illustrations or linguistic descriptions.

During their exchange concerning the application of algebra to geometry in 1817, both Gergonne and Poncelet agreed that the qualities of a particular problem should determine the choice of method. Yet, as we saw in direct comparisons of different approaches, neither of them followed this compromise in practice. Instead, when solving the same geometric problem, geometers emphasized their method in order to differentiate similar results. While the choice of method seemed to result in amicable competition between Gergonne and Poncelet or between Poncelet and Plücker, these same geometers engaged in polemical discourse over lack of acknowledgement, priority, and perceived plagiarism. Disputes about citations and recognition also raised larger questions concerning the dissemination of mathematics. In an atmosphere of frequent repetition, authors debated the importance of new forms versus new content. The importance of form became particularly acute during the duality controversy. Poncelet argued that real innovation lay in new results, while Plücker and Gergonne (as well as Crelle, describing Plücker's contributions) assigned greater value to form or method. Dedication to finding new results was further demonstrated by both Poncelet and Steiner in publishing articles with select new theorems and constructions, without initially revealing the underlying proofs. This strategy may have been to secure priority, as their longer monographs included old results, such as the Apollonius problem, to demonstrate the advantages of their new methods.

With numerous geometers drawing upon the same material, each seemed to adopt his own peculiar protocol for citing the work of others. References ranged from vaguely claiming the result as "known" to specifically naming an author, text, year, and page number. Despite self-declared lack of originality, textbooks often contained no specific citations at all. Within articles published in later volumes of Gergonne's *Annales*, geometers assumed a comprehensive background knowledge that allowed assuming results, principles, definitions, and even figures from past publications. The early work of Plücker and Steiner exhibited the potential disadvantage incurred by contributions from less connected locations. When authors, often unintentionally, repeated the recent findings of their contemporaries, offended geometers responded with varied strategies to defend their originality. From the advantage of his editorial position, Gergonne used footnotes and other annotations to remind readers of his priority. Throughout his career Poncelet employed numerous venues to advertise complaints about insufficient citations. Plücker relied upon the novelty of his coordinate based methods to frame his research and offset his appropriation of recent results. Steiner adapted familiar problems into increasingly general variations with frequent particular corollaries. When controversies emerged, they could prove both dangerous and beneficial to participants, as illustrated by Plücker's role in the duality controversy.

As much as geometers repeated, they also developed new approaches. Geometers found inspiration in divisions between pure and analytic methods. As evidenced by Poncelet and Gergonne, by Plücker and Poncelet, and by Steiner and Plücker, inter-methodogical inspiration helped to establish ideal objects, the line at infinity, the principle of duality, and the projectively invariant conic section in geometric practices. When employing a new method geometers could revisit results from recent publications without implied criticism or improper appropriation, such as in the case of Lamé's theorem. The controversy surrounding the principle of duality brought to light Poncelet and Gergonne's notions of how new geometry should be introduced to a broader public. Amidst their disputes, both geometers developed particular vocabulary, used Euclidean or synthetic geometry to verify their theorems, and employed visual forms to build recognition for their respective principles. Their strategies and commentaries implied a perceived conservative audience, either reluctant or opposed to their "modern" geometry. Within research articles their strategies succeeded, as the resultant publicity introduced a wider group of geometers to emergent material. In particular, we saw how the "German" geometers were among the first to apply new principles and objects in their research.

While Steiner and Plücker were labelled as German geometers and compared in this context within the *Annales* and the *Bulletin*, later historiography reflected (now more established) national divisions. German and Italian authors described how geometry had migrated from France to Germany, while French authors emphasized the independent work of Chasles as a counterpoint to the contemporary researches of Plücker, Steiner, Möbius, and von Staudt. However, this linguistic, cultural, or national division was by no means clearcut in the early nineteenth century. The École polytechnique, the school of Monge, and the books of Carnot were claimed as common background by a wide group of geometers, some of whom did not attend the École polytechnique or study directly with Monge including Gergonne, Plücker and Steiner. Further, French mathematics was not necessarily defined by shifting national borders. On the one hand, as shown through Gergonne and Poncelet's relationship with the *Académie des Sciences*, the mathematics and mathematicians of Paris were disconnected from those in the provinces. On the other hand, French textbooks might be written outside of France in regions that would later become part of Belgium, Switzerland, and Luxembourg. Returning to Steiner and Plücker, we observed how Gergonne's editorial interventions introduced them to the vocabulary of French geometry, new French developments, and references to French texts. With attention to this potential audience, through the mid-1830's both Steiner and Plücker would publish frequently in French, even when writing for Crelle's *Journal*. Following the duality controversy, Poncelet also turned to Crelle's *Journal* for publishing several memoirs. Further research in comparing contemporary books written in German besides those of Plücker and Steiner would offer a valuable point of comparison and better understanding of additional publics.

Just as the division between French and German geometers, methodological differentiation was dynamic and subjective. Whether attributed to pure, analytic, synthetic, rational, ancient, modern, or mixed geometries, certain qualities were overwhelmingly admired and others avoided. All four of our main actors described their work as elegant and simple, and attempted to avoid particular cases (exemplified at times by individual figures) as well as excessive calculations. Yet, each also used both individual figures and arithmetical calculations to some extent, often in combination with other more general and direct means. The difference between methods was more a continuum than a bifurcation.

When we direct our attention to the stated reasons geometers gave for choosing one or another geometry, we find variable rationales. Poncelet described synthetic geometry as the particular, restricted, figure-based geometry of ancient Greece whose universal recognition made it the appropriate format for introducing new mathematics. By contrast, Steiner valued synthetic geometry because of its unified, systematic, and organic qualities. With modern pure geometry, Poncelet promoted its direct applicability among engineers and graphic artists. Gergonne, instead, admired the philosophical importance of modern geometrical principles independent of their value in practical situations. Plücker described his method as pure analytic geometry, which he found simple, effortless, and free of computation–in direct opposition of Poncelet's description of the same subject. Poncelet argued that the figure made geometry evident, while Steiner found the figure could obscure an intuitive understanding of geometric objects. Within the rare methodological commentary of these four geometers, the decision to use one or another method was personal.

To add to these mercurial methodological distinctions, we compared the ways in which geometers presented themselves and their work to how they were viewed by contemporary geometers and critics as well as the succeeding generations of students, geometers and biographers. Geometers could differentiate their personal achievements by lumping together the works of others. So, Plücker referred to Poncelet and Steiner as synthetic geometers, and late nineteenth century geometers characterized the early nineteenth century as limited by methodological opposition. At the same time, Poncelet and Steiner actively distinguished their styles of geometry, and our case studies verify their notable differences. Similarly, early nineteenth century geometries displayed a fluidity between methods, with Gergonne, Poncelet, Steiner, and Carnot finding merit in both analytic and pure or synthetic approaches. The above generalizations perhaps reflected those who assigned the categories more than those who fell into them.

Although research geometry remained difficult to classify, as reflected for instance in the many subject titles applied in the *Annales*, our study of books on geometry suggested three primary areas of study: elementary geometry, analytic geometry, and descriptive geometry. The majority of these books did not reference contemporary research and we observed few changes in content over a thirty-year period. In recognition of their new practices, geometers like Poncelet and Gergonne labelled their geometry as modern. As we saw in Chapter I, this claim to modernity extended to later geometers, who often traced the beginning of the new projective geometry to Poncelet's research. When introducing this modern geometry, both Poncelet and Gergonne took care to connect their advances to more familiar traditions. Poncelet claimed that the synthetic method offered a trusted form in which to portray his ideal objects, while he grounded his principle of continuity on the use of continuity in analysis. Similarly, Gergonne explained the best way to introduce the principle of duality was through Euclidean examples. Gergonne also criticized Poncelet for opening his *Traité* with the "controversial" principle of continuity, instead of more readily acceptable polar reciprocity and projection. The application of new methods to well-established geometry problems also served Plücker and Steiner in the choice of the Apollonius problem to test their approaches and generate subsequent publicity. As Gergonne's introduction to Steiner's text showed, finding solutions to old problems could benefit the community of geometry students and teachers. Pedagogy served as an important potential use of new geometry, thus encouraging geometers to demonstrate how their recent work connected to an established curriculum. However, this "modern geometry" also challenged the differentiation between methods as represented in geometry textbooks, which was largely based on expected prerequisite knowledge and intended applications. From Gergonne's use of coordinate representation for elementary planar geometry problems to Poncelet and Steiner's general approach to an unspecified conic section to Plücker's abridged notation without calculation, the domain of one method in geometry could shift dramatically between different texts and audiences. Even the quality "modern" changed meanings in different contexts. While Poncelet and Gergonne associated positive connotations with modern geometry, later German historians complimented early nineteenth century geometers through analogy with Euclid, Apollonius, and Archimedes. A single text could contain both modern and ancient components. Reviews of Poncelet, Gergonne, Plücker and Steiner in the *Bulletin* and Crelle's *Journal* summarized the perceived new advances in methodology, terminology, and applicability, acknowledging that these geometers recycled problems and theorems from ancient and modern sources.

Our investigation of the methodological opposition in early nineteenth century geometry showed less systemic opposition than individual geometers choosing one method over another through personal, varying preferences and perceived qualities. When antagonism arose between persons or methods, it served as motivation for deeper study and wider developments, which in turn integrated disparate objects and practices. The combination of repeated results and evolving forms both challenged and pushed the limits of this visually oriented research geometry.

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Appendix A

Texts in Chapter I corpus, and source texts

A.0.1 Twentieth Century General Histories of Mathematics:

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A.0.2 Our corpus

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CITED BY: Loria (1887), Klein (1926a), Kötter (1901), Ernst (1933), Boyer (1956)

Dronke, A. (1871). *Julius Plücker Professor der Mathematik und Physik an der Rhein.* Friedrich-Wilhelm Universität, Bonn.

CITED BY: Cajori (1893), Ernst (1933)

Ernst, W. (1933). *Julius Plücker; Eine zusammenfassende Darstellung seines Lebens und Wirkens als Mathematiker und Physiker auf Grund unveröffentlichter Briefe und Urkunden.* Rheinischen Friedrich-Wilhelms-Universität, Bonn.

CITED BY: Boyer (1956)

Geiser, C. F. (1874). *Zur Erinnerung an Jakob Steiner.* Verhandlungen der Schweizerischen Naturforschenden Gesellschaft, 56: 215–251.

CITED BY: Loria (1887), Klein (1926a)

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CITED BY: Fano (1907), Cajori (1893), Ernst (1933), Tropfke (1903), Coolidge (1940)

Loria, G. (1887). *Il passato ed il presente delle principali teorie geometriche.* Carlo Clausen, Torino.

CITED BY: Cajori (1893), Kötter (1901), Fano (1907), Simon (1906), Ernst (1933), Boyer (1956)

Kötter, E. (1901). Die Entwickelung der synthetischen Geometrie. *Jahresbericht der Deustchen mathematiker-vereinigung*, 5.2:1–484.

CITED BY: Fano (1907), Schoenflies (1909),Klein (1926a), Ernst (1933), Coolidge (1940), Kline (1972)

Darboux, G. (1904). A survey of the development of geometrical methods. *Bulletin of the American Mathematical Society*, 11(10):517–543.

CITED BY: Klein (1926a), Boyer (1956)

Simon, M. (1906). *Jahresbericht der Deutschen Mathematiker-Vereinigung. Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert.* B. G. Teubner, Leipzig.

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Fano, G. (1907). Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im 19. Jahrhundert. *Encyklopädie der mathematischen Wissenschaften*, 3(4):221–288.

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CITED BY: Kline (1972), Tresse and Schoenflies (1913)

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Tropfke, J. (1903). *Geschichte der Elementar-Mathematik in systematischer Darstellung.* Veit & Comp., Leipzig.

CITED BY: Simon (1906), Coolidge (1940), Boyer (1956)

Appendix B

Texts in Chapter V corpus, organized chronologically

date	author	title	publisher	place	category
1798	Gaspard Monge	Géométrie descriptive	Baudouin	Paris	three-dimensional
1799	Sylvestre François Lacroix	Élémens de géométrie	Duprat	Paris	elementary
1800 (6th edition)	Charles Bossut	Cours de mathématiques	Firmin Didot	Paris	elementary, analytic
1800 (3rd edition)	Adrien-Marie Legendre	Éléments de géométrie	Firmin Didot	Paris	elementary
1801	Lazare Carnot	De la Corrélation des figures de géométrie	Crapelet	Paris	ruler or compass
1802 (2nd edition)	Sylvestre François Lacroix	Essais de géométrie sur les plans et les surfaces courbes: Élémens de Géométrie descriptive	Duprat	Paris	three-dimensional
1803	Lazare Carnot	Géométrie de position	Duprat	Paris	ruler or compass
1803 (3rd edition)	Sylvestre François Lacroix	Élémens de géométrie	Courcier	Paris	elementary
1803 (3rd edition)	Sylvestre François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie.	Courcier	Paris	analytic
1806	Christian Kramp	Élémens de géométrie	Hansen	Cologne	elementary
1807 (4th edition)	Sylvestre François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie.	Courcier	Paris	analytic
1809 (4th edition)	Gaspard Monge	pplication de l'Analyse à la Géométrie à l'usage de l'École Impériale Polytechnique	Vve Bernard	Paris	analytic
1809	Antoine Charles Marcelin Poullet-Deslile	Application de l'agèbre à la géométrie	Courcier	Paris	analytic
1810	Jean Guillaume Garnier	Réciproques de la géométrie, suivies d'un recueil de théorèmes et de problèmes	Courcier	Paris	mixed collections
1810 (2nd edition)	Jean-Louis Boucharlat	Théorie des courbes et des surfaces du second ordre, précédée des principes fondamentaux de la géométrie analytique	Vve Courcier	Paris	analytic
1810 (4th edition)	Jean-Baptiste Biot	Essai de géométrie analytique, appliqué aux courbes et aux surfaces du second ordre	J. Klostermann fils	Paris	analytic
1811 (5th edition)	Nicolas-Louis de LaCaille (Jean-Baptiste Labey)	Leçons élémentaires de mathématiques	Courcier	Paris	elementary
1811 (9th edition)	Sylvestre François Lacroix	Élémens de géométrie	Vve Courcier	Paris	elementary
1812 (second edition)	Louis Bertrand	Élémens de géométrie	J. J. Paschoud	Paris	elementary
1812	Emmanuel Develey	Élémens de géométrie	Vve Courcier	Paris	elementary
1812 (4th edition)	Sylvestre François Lacroix	Essais de géométrie sur les plans et les surfaces courbes: Élémens de Géométrie descriptive	Vve Courcier	Paris	three-dimensional
1812 (9th edition)	Adrien-Marie Legendre	Éléments de géométrie	Firmin Didot	Paris	elementary
1813	Charles Dupin	Développements de géométrie	Vve Courcier	Paris	analytic, three-dimensional
1813 (6th edition)	Sylvestre François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie.	Vve Courcier	Paris	analytic
1813	Jacques Schwab	Élémens de géométrie	Hissette	Nancy	elementary
1813 (5th edition)	Jean-Baptiste Biot	Essai de géométrie analytique, appliqué aux courbes et aux surfaces du second ordre	J. Klostermann fils	Paris	analytic
1813	Jean Guillaume Garnier	Géométrie analytique, ou Application de l'algèbre à la géométrie	Vve Courcier	Paris	analytic
1815	J. de Stainville	Mélanges d'analyse algébrique et de géométrie	Vve Courcier	Paris	analytic
1816 (2nd edition)	Emmanuel Develey	Élémens de géométrie	Vve Courcier	Paris	elementary
1817	Jean-Nicholas-Pierre Hachette	Éléments de géométrie à trois dimensions. Partie synthétique et partie algébrique	Vve Courcier	Paris	analytic, three-dimensional
1817	Charles Michel Potier	Traité de géométrie descriptive	Firmin Didot	Paris	three-dimensional
1817	Charles Julien Brianchon	Mémoire sur les lignes du second ordre	Bachelier	Paris	ruler or compass

date	author	title	publisher	place	category
1818	Gabriel Lamé	Examen des différentes méthodes employées pour résoudre les problèmes de géométrie	Vve Courcier	Paris	mixed collections
1819 (11th edition)	Sylvestre François Lacroix	Élémens de géométrie	Vve Courcier	Paris	elementary
1819	Antoine-André-Louis Reynaud	Traité d'application de l'algèbre à la géométrie, et de trigonométrie	Vve Courcier	Paris	analytic
1819	Paul-Marie-Gabriel Treuil	Essais de mathématiques, contenant quelques détails sur l'arithmétique, l'algèbre, la géométrie et la statique	Vve Courcier	Paris	mixed collections
1819	Louis-Léger Vallée	Traité de la géométrie descriptive	Vve Courcier	Paris	three-dimensional
1822 (5th edition)	Sylvestre François Lacroix	Essais de géométrie sur les plans et les surfaces courbes: Élémens de Géométrie descriptive	Bachelier	Paris	three-dimensional
1822	Jean-Nicolas Noël	Mélanges de mathématiques, ou Application de l'algèbre à la géométrie élémentaire	C. Lamort	Metz	analytic
1823	Joseph Adhémar	Cours de géométrie descriptive	Chaignieau fils ainé	Paris	three-dimensional
1825	Pierre Louis Marie Bourdon	Application de l'agèbre à la géométrie	Bachelier	Paris	analytic
1826 (7th edition)	Jean-Baptiste Biot	Essai de géométrie analytique, appliqué aux courbes et aux surfaces du second ordre	Bachelier	Paris	analytic
1826	Augustin-Louis Cauchy	Leçons sur les applications du calcul infinitésimal à la géométrie	De Bure frères	Paris	infinitesimal calculus
1826	Alexandre Vincent	Cours de géométrie élémentaire	Bachelier	Paris	elementary
1827	Guillaume Henri Dufour	Géométrie perspective	Bachelier	Paris	three-dimensional
1827 (8th edition)	Sylvestre François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie.	Bachelier	Paris	analytic
1827	Louis-Etienne Lefébure de Fourcy	Leçons de géométrie analytique	Bachelier	Paris	analytic
1827	A. Lefevre	Applications de la géométrie à la mesure des lignes inaccessibles et des surfaces planes	Bachelier	Paris	mixed collections
1827 (5th edition)	Gaspard Monge (Barnabé Brisson)	Géométrie descriptive	Bachelier	Paris	three-dimensional
1828	Barthélémy Edouard Cousinery	Géométrie perspective, ou Principes de projection polaire appliqués à la description des corps	Carilian-Goeury	Paris	three-dimensional
1828	N. J. Didiez	Cours complet de géométrie	Bachelier	Paris	elementary
1828	E. Duchesne	Éléments de géométrie descriptive, à l'usage des élèves qui se destinent à l'École polytechnique, à l'École militaire, à l'École de marine	H. Balzac	Paris	three-dimensional
1828	Gabriel Gascheau	Géométrie descriptive	Bachelier	Paris	three-dimensional
1828 (2nd edition)	Lorenzo Mascheroni (trans. A. M. Carette)	Géométrie du compas	Bachelier	Paris	ruler or compass
1829	Michel Chasles	Recherches de géométrie pure sur les lignes et les surfaces du second degré	M. Hayez	Brussels	ruler or compass
1829 (2nd edition)	E. Duchesne	Éléments de géométrie descriptive, à l'usage des élèves qui se destinent à l'École polytechnique, à l'École militaire, à l'École de marine	H. Balzac	Paris	three-dimensional
1829 (3rd edition)	Enrico Giamboni (trans. D. Roux)	Élémens d'algèbre, d'arithmétique et de géométrie, où l'arithmétique et la géométrie se déduisent des premières notions de l'algèbre	Bachelier	Paris	analytic
1829	Olry Terquem	Manuel de géométrie, ou exposition élémentaire des principes de cette science	Roret	Paris	elementary, analytic
1830 (1741)	Alexis Claude Clairaut	Élémens de géométrie	Bachelier	Paris	elementary
1830 (14th edition)	Sylvestre François Lacroix	Élémens de géométrie	Bachelier	Paris	elementary
1830	M. H. Vernier	Géométrie élémentaire à l'usage des classes d'humanités et des écoles primaires	L. Hachette	Paris	elementary
1831	F. H. Francfort	Essai analytique de géométrie plane	Béthune	Paris	analytic
1831	Auguste Mutel	Cours de géométrie et de trigonométrie	Vve Bernard	Paris	elementary
1831	Claude-Lucien Bergery	Géométrie des écoles primaires	P. Wittersheim	Metz	elementary
1832 (14th edition)	Adrien-Marie Legendre	Éléments de géométrie	H. Remy	Brussels	elementary
1832	Alexandre Meissas	Cours de géométrie	A. Pihan Delaforest	Paris	elementary
1833	Antoine-André-Louis Reynaud	Théorèmes et problèmes de géométrie	Bachelier	Paris	mixed collections
1835 (3rd edition)	G. F. Olivier	Géométrie usuelle	Maire-Nyon	Paris	elementary

Appendix C

Authors, dates (if known), and École polytechnique class

Name		Dates (if known)	École polytechnique class
Adhémar	Joseph (Alphonse)	1797-1862	
Bergery	Claude Lucien	1787-1863	X 1806
Bertrand	Louis	1731-1812	
Biot	Jean Baptiste	1774-1862	X 1794
Bossut	Charles	1730-1814	
Boucharlat	Jean Louis	1773-1848	X 1794
Bourdon	Louis Pierre Marie	1779-1854	X 1796
Brianchon	Charles Julien	1783-1865	X 1803
Brisson	Barnabé		
Carette	A. M.		
Carnot	Lazare	1753-1823	
Cauchy	Augustin Louis	1789-1857	X 1805
Chasles	Michel	1793-1880	X 1812
Clairaut	Alexis Claude	1713-1765	
Cousinery	Barthélémy Edouard	1790-1851	X 1808
Lefébure de Fourcy	Louis Etienne	1785-1869	X 1803
de Stainville	Nicolas Dominique Marie Janot	1783-1828	X 1802
Develey	Emmanuel	1764-1839	
Didiez	N. J.		
Duchesne	E.		
Dufour	Guillaume Henri	1787-1875	X 1807
Dupin	Pierre Charles	1784-1873	X 1801
Francfort	F. H.		
Garnier	Jean Guillaume	1766-1840	
Gascheau	Gabriel	1798-1883	X 1816

Name		Dates (if known)	École polytechnique class
Giamboni	Enrico		
Hachette	Jean Nicolas Pierre	1769-1834	
Kramp	Christian	1760-1826	
Labey	Jean-Baptiste	1750-1825	
LaCaille	Nicolas-Louis de	1713-1762	
Lacroix	Sylvestre François	1765-1843	
Lamé	Gabriel	1795-1870	X 1814
Lefevre	A		
Legendre	Adrien Marie	1752-1833	
Mascheroni	Lorenzo	1750-1800	
Meissas	Alexandre André (Nicolas de)	1795-1866	X 1813
Monge	Gaspard	1746-1818	
Mutel	Auguste	1795-1847	X 1813
Noël	Jean Nicolas	1783- 1867	
Olivier	GF		
Potier	Charles Michel	1785-1855	X 1805
Poullet-Delisle	Antoine Charles	1778-1849	X 1796
Reynaud	Antoine André Louis	1777-1844	X 1796
Schwab	Jacques		
Terquem	Olry	1782-1862	X 1801
Treuil	Paul Marie Gabriel	1784-1823	X 1802
Vallée	Louis Léger	1784-1864	X 1800
Vernier	M. H.		
Vincent	Alexandre	1797-1868	

Appendix D

Chapter V author's title page descriptions (if any)

Name	Title page author description
Adhémar	
Bergery	Ancien élève de l'école polytechnique, professeur à l'école d'artillerie de Metz, membre de l'académie royale de la même ville et de plusieurs autres sociétés savantes.
Bertrand	Professeur émérite dans l'Académie de Genève et Membre de celle de Berlin.
Biot	Membre de l'Institut de France, etc. (1810); Membre de l'Institut de France, Adjoint du bureau des Longitudes, Professeur de physique mathématique au Collège de France, et d'astronomie à la faculté des Sciences de Paris, Membre de la Société Philomatique de Paris, des Académies de Lucques, de Turin, de Munich et de Wilna. (1813); Membre de l'Académie des Sciences, Astronome adjoint au Bureau des Longitudes, Professeur de Physique mathématique au Collége de France, et d'Astronomie à la Faculté des Sciences de Paris ; des Sociétés royales de Londres et d'Edimbourg ; de l'Académie impériale de Saint-Pétersbourg ; des Académies Royales de Stockholm, Turin, Munich, [illegible], Berlin, Naples ; Membre honoraire de l'Université de [illegible] de l'Institution royale de Londres, de la Societé philosophique de Cambridge, des Antiquaires d'Écosse, de la Société pour l'avancement des Sciences naturelles de Marbourg, de la Société Helvétique des Sciences naturelles, et de la Société Italienne des Sciences residante à Modène. (1826);
Bossut	Membre de l'Institut National des Sciences et des Arts, etc.
Boucharlat	Licencié ès-Sciences, et Répétiteur à l'École Impériale Polytechnique.
Bourdon	Chevalier de l'Ordre royal de la Légion-d'Honneur, Inspecteur de l'Académie de Paris, Docteur ès-Sciences, etc.
Brianchon	Capitaine d'Artillerie, ancien élève de l'École Polytechnique
Brisson	Ancien Élève de l'École Polytechnique, Inspecteur divisionnaire des Ponts et Chaussées.
Carette	ancien élève de l'école polytechnique, officier supérieur du génie, etc.
Carnot	membre de l'Institut National (1801); De l'Institut national de France, de l'Académie des Sciences, Arts et Belles-Lettres de Dijon, etc. (1803)
Cauchy	Ingénieur en chef des ponts et chaussées, professeur d'analyse à l'École royale polytechnique, professeur adjoint à la faculté des sciences, membre de l'Académie des sciences, chevalier de la Legion d'honneur
Chasles	
Clairaut	Des Académies des Sciences de France, d'Angleterre, de Prusse, de Russie, de Bologne et d'Upsal.
Cousinery	ingénieur des ponts et chaussées, ancien élève de l'école polytechnique
Lefébure de Fourcy	Chevalier de la Légion d'Honneur, Examinateur des Aspirans à l'École royale Polytechnique, à l'École spéciale Militaire, à l'École de Marine et à l'École Forestière, Docteur-ès-Sciences, etc.
de Stainville	Répétiteur-Adjunct à l'École royale Polytechnique.
Develey	Professeur de Mathématiques à Lausanne, membre du Conseil Académique du canton de Vaud, membre correspondant de l'Académie Impériale des Sciences de Saint-Pétersbourg, des Sociétés de Harlem, de Jéna, de Mantauban, de Bordeaux, de Lyon, de Besançon, de la Société économique de Saxe, etc. (1812); Professeur de Mathématiques à Lausanne, Membre correspondant de l'Académie Impériale des Sciences de Saint-Petersbourg, des Académies Royales de Harlem et de Jena, des Sociétés de Montauban, de Bordeaux, de Lyon, de Besançon, de la Société économique de Saxe, etc. (1816)
Didiez	
Duchesne	professeur de mathématiques spéciales au collège de Vendôme
Dufour	Lieutenant-Colonel du Génie, Membre de la Légion-d'Honneur et Secrétaire de la Société des Arts de Genève.
Dupin	Membre de l'Académie Ionienne, Associé étrager de l'Institut royal de Naples, des Académies des Sciences de Turin, Montpellier, etc. ; Correspondant de la première Classe de l'Institut de France, Capitaine du Génie maritime, Membre de la Légion-d'Honneur.
Francfort	
Garnier	Ancien professeur à l'École polytechnique, Docteur de la Faculté des Sciences à l'Université Impériale, et Instituteur à Paris

Name	Title page author description
Giamboni	professeur à Pérouse, ouvrage est traduit de l'italien par D. Roux de Genève
Hachette	Professeur adjoint de la Faculté des Sciences et de l'Ecole Normale, Membre de la Société Philomatique.
Kramp	Professeur de Mathématiques et de Physique à l'Ecole de Cologne ; des Académies d'Erford et de Rovérédo ; de la Société littéraire de Mayence ; et de la Société minéralogique de Jéna.
Labey	professeur de mathématiques pures et examinateur des pour l'ecole imperiale poytechnique
LaCaille	l'Abbé de la Caille
Lacroix	
Lamé	élève ingénieur au corps royal des mines
Lefevre	géomètre en chef du cadastre, membre de plusieurs sociétés royaux des sciences et des arts
Legendre	Membre de l'Institut National (1800); membre de l'institut et de la Légion d'Honneur, de la Société Royale de Londres, etc. (1812); Membre de l'Institut et de la Légion-d'Honneur, de la Société Royale de Londres, etc. (1832)
Mascheroni	
Meissas	ancien élève de l'école polytechnique
Monge	Membre de l'Institut
Mutel	capitaine d'artillerie, ancien élève de l'école polytechnique, auteur de la flore de dauphine et d'un cours d'arithmétique adopté par l'université.
Noël	Professeur des sciences physiques et mathématiques à l'athénée de Luxembourg, correspondant de la société des lettres, sciences et arts, de Metz.
Olivier	Bachelier ès-sciences, professeur de Mathématiques et d'Humanités.
Potier	élève de l'école polytechnique
Poullet-Delisle	Ancien Élève de l'École Polytechnique, Ingénieur des Ponts et Chaussées, Professeur de Mathématiques au Lycée d'Orléans
Reynaud	Chevalier de la Légion-d'Honneur, Examinateur des Candidats de l'École royale Polytechnique et de l'École spéciale militaire, Docteur-ès-Sciences, Membre de plusieurs Académies, etc.
Schwab	
Terquem	Docteur ès sciences, Officier de l'Université, Membre de la Légion-d'Honneur, Professeur aux Écoles royales d'Artillerie, Bibliothécaire du Dépôt central de l'Artillerie, et Membre de l'Académie de Metz
Treuil	Ancien Élève de l'Ecole Polytechnique et de celle des Ponts et Chaussées, Professeur de Mathématiques à l'Ecole spéciale royale militaire, et Professeur de Sciences physiques et de Mathématiques spéciales, au Collège royal de Versailles.
Vallée	Ancien Élève de l'École Polytechnique, Ingénieur au Corps royal des Ponts et Chaussées, Membre de la Société d'Émulation de Cambrai.
Vernier	Professeur de Mathématiques au Collège royal de Louis-le-Grand, ancien élève de l'Ecole normale, Docteur-ès-Sciences
Vincent	Professeur de mathématiques spéciales dans l'Académie de Paris, (Collége royal de Reims), élève de l'ancienne école normale, licencié-es-sciences, membre de plusieurs sociétés savantes

Appendix E

Plücker's interpretation of points and lines at infinity and polar reciprocity with homogeneous coordinates

In Chapter III we saw that Plücker promised to reveal the secret of duality in a later publication through a purely analytic method and without the use of an auxiliary conic section (Plücker (1828c)). Like the case studies in Chapter IV, the 1830 article "Über ein neues Coordinatensystem" provides further evidence of how Plücker adapted Poncelet's findings into his form of analytic geometry. This slightly later text also exemplifies Plücker's particular use of coordinates to represent geometric objects, here with the introduction of homogenous coordinates. Finally, this article shows Plücker's occasional application of illustrated figures to introduce new definitions.

When Plücker finally read Poncelet's *Traité*, he studied it thoroughly and responded with careful accreditation to the original author. At the same time, Poncelet and Plücker continued publishing articles on the same problems and theorems in the same journals, and in 1828 both mathematicians began contributing to Crelle's *Journal für reine und angewandte Mathematik*. Also in 1828, Plücker published the first volume of his *Analytisch-geometrische Entwicklungen*. He advertised the contents with several articles in Crelle's *Journal*, excerpting and referencing his larger monograph. "Über ein neues Coordinatensystem" from 1829 is one such article (Plücker (1829b)). Plücker opened by promoting the "Leichtigkeit" of his new coordinate system in algebraically representing the relationship and position [*Lage*] between points and lines. Since his new coordinate system was independent from magnitude considerations, Plücker classified it as "géométrie de situation" (keeping the French terms in his otherwise German article). As with Plücker's earlier coordinate choices, described in Chapter II, this approach yielded immediate geometric results from the form of the equation.

> Hierhin rechne ich ferner, dass für die Kurven aller Ordnungen sich Gleichungen ergeben, die in Beziehung auf drei Veränderlichen (p, q, r) homogen sind; wonach die geometrische Interpretation des Theorems über die homogenen Functionen unmittelbar die Gleichung der Tangente und osculirender Curven für

> jeden gegebenen Punct der Curve liefert, u. s. w. (Plücker (1829b), 1)[1]

Plücker was explicit about the similarity between his and Poncelet's work. He recognized that despite their bases in "ganz wesentlich verschiedenen Ideen," his general analytic method and the method of Poncelet's *Traité* yielded strikingly matching results. Plücker admitted that without an understanding of the underlying groundwork, one might judge "die erste Methode als eine Periphrase, als ein Plagiat der zweiten." In particular, Plücker pointed out that his paper would derive proofs of significant theorems from Poncelet's writings–that all points at infinity are collinear and that two concentric circles have an imaginary double tangent at infinity [*in unendlicher Entfernung einen imaginären doppelten Contact*].

As in many of his articles from this time period, Plücker presented his main focus as displaying a new method. As compared to the ordinary coordinate analysis, Plücker contrasted the generality, simplicity, flexibility, ease, fruitfulness, and naturalness of a homogeneous coordinate system in three variables. Any new theorems and applications were incidental to the broader methodological goal, for the most part Plücker rederived known results. He would primarily reference work from his own *Entwicklungen* and Poncelet's *Traité*, citing both precisely by section number. Plücker also judiciously credited Poncelet and Gergonne for the development of polar reciprocity, which he described as the most beautiful and most important extension of geometry in contemporary times. From other mathematicians, Plücker included a purportedly new proof of a theorem on the locus of lines about a fixed point from Jakob Steiner, cited vaguely as only an earlier volume of Crelle's *Journal*. He also included a theorem on the polar of a point with respect to a proposed directrix attributed to Étienne Bobillier, "Recherches sur les lois générales qui régissent les courbes algébriques," (Bobillier (1827)). Finally, Plücker often referenced so-called *known* theorems, without further source details.

Plücker claimed that his new method afforded greater generality, and his research on algebraic and transcendental curves of any degree was presented as original. His hints throughout the text suggested the richness of the homogeneous coordinate system. For instance, Plücker concluded his introduction by sketching an analogous homogeneous treatment with a coordinate tetrahedron in space.

The subsequent body of the paper was arranged in 46 numbered sections grouped under six themes. The first theme (sections 1-12) served as an introduction, Plücker introduced his homogeneous points and determined representation of points and straight lines. The following themes were labelled self-explanatorily "On the theory of second order lines" (13-35), "Theory of reciprocity" (36-40) "Coordinate transformation" (41), "General coordinate-determination" (42-45), and "Characteristic property of curves of all degrees" (46).

To establish his new coordinate system, Plücker began with Figure 1. None of Plücker's five articles in Gergonne's *Annales* had contained figures, but they were not uncommon in his *Entwicklung* nor his contributions to Crelle's *Journal*. That said, Plücker's references to figures throughout this article were no more than labels within parentheses, and Plücker did not directly discuss the figure within his description of the coordinate system.

Within an ordinary coordinate system, YAX, Plücker formed a triangle from straight

[1] "Here I further think, that for for curves of all orders the given equations are homogeneous in relation to three variables (p, q, r); for which the geometric interpretation of theorems on homogeneous functions immediately provides the equation of tangent and osculating curves for each point on the curve, etc."

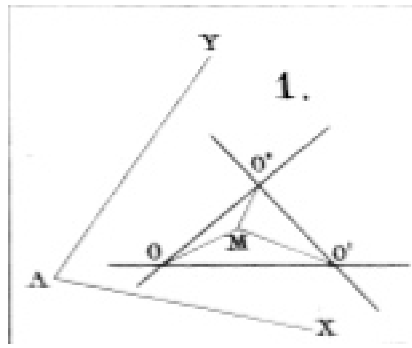

Figure E.1: Plücker's Figure 1, Crelle's *Journal* Bd. 5 Hft. 1 (Plücker (1829b))

lines OO', OO'', $O'O''$ in the plane. Then he defined lengths in the ordinary coordinate system, p, q, r as the distances between any planar point M and the three vertices O, O', O''. With consideration of sign and position, Plücker determined that if the point M was in an acute triangle (as pictured in the figure) $OO'O''$, then one of the distances, say p would be positive, while the other two q and r would be negative. Thus Plücker could find coefficients satisfying the relation,

$$Ap^m + Bp^{m-1}q + Cp^{m-1}r + Dp^{m-2}q^2 + EP^{M-2}QR + \ldots + Yqr^{m-1} + Zr^m = 0 \quad \text{(E.1)}$$

a homogeneous equation in p, q, r. To simplify the three distances p, q, r, into a two coordinate representation, Plücker considered $\frac{p}{q} = \phi$ and $\frac{r}{p} = \psi$. Substituting these into the above equation, he showed

$$A + B\phi + C\psi + D\phi^2 + E\phi\psi + ... + Y\phi\psi^{m-1} + Z\psi^m = 0. \quad \text{(E.2)}$$

Since from the above, ϕ and ψ were both negative, for any chosen point M, the sign of these variables would remain determined and the equation represented all possible mth order lines in the plane. Alternatively, Plücker showed that substituting $\frac{1}{\psi} = \frac{p}{q} = \mu$ and $\frac{1}{\phi} = \frac{p}{r} = \upsilon$ yielded all possible mth order lines represented by an equation of $2m$ degree.

Within either of these above systems of two coordinates, Plücker explained that O,O',O'' would be coordinate vertices, OO', OO'', $O'O''$ would be coordinate axes, and angles $O''OO'$, $OO'O''$, $OO''O'$, or abbreviated as α, α', α'', would be coordinate angles. Plücker noted the lack of exception in this presentation. If two of the coordinate axes were parallel, this simply meant that one of the coordinate vertices would be infinitely distant and one of the coordinate angles would be zero.

Reference to Figure 2 accompanied the following exposition.[2]

A general straight line in Plücker's new three coordinate system would be of the form

$$p + aq + br = 0, \quad \text{(E.3)}$$

[2]When this article was republished for Plücker's complete works, the figures were inserted directly into the text, rather than being located at the end of the issue. Furthermore, in the original Crelle's *Journal* publication, several of the figures were mislabelled with the wrong reference number, this was corrected in the complete works.

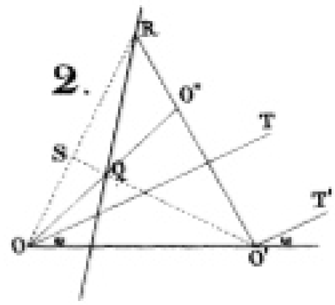

Figure E.2: Plücker's Figure 2, Crelle's *Journal* Bd. 5 Hft. 1 (Plücker (1829b))

or identically,

$$1 + a\phi + b\psi = 0, \mu\upsilon + a\upsilon + b\mu = 0. \tag{E.4}$$

If the straight line respectively met the coordinate lines OO'', $O'O''$ at two points Q and R, these points could have the respective coordinates,

$$Q = (\phi = 0, \psi = -\frac{1}{b}), R = (\phi = -\frac{1}{a}, \psi = 0). \tag{E.5}$$

So then the intersection of lines OR, $O'Q$ at a point S would have the coordinates,

$$S = (\mu = \frac{1}{\psi} = -a, \upsilon = \frac{1}{\phi} = -b). \tag{E.6}$$

Straight lines through through a fixed point on OO', OO'' or $O'O''$ could be represented by the respective linear equations,

$$mp + n(aq + br) = 0, mq + n(p + br) = 0, mr + n(p + aq) = 0, \tag{E.7}$$

where m, n would be any arbitrary indeterminate coefficients.

In the following section, Plücker used these definitions to depict any pair of parallel lines in the plane, which he would then apply to show that all points at infinity in a plane were collinear, lying on a unique line at infinity. Plücker began with OT and $O'T'$ as two parallel straight lines that formed the angle ω with the axis OO' (this is pictured in Figure 2, although not referenced within this section). Then, recalling that angles $\alpha = O'OO''$, $\alpha' = OO'O''$ Plücker obtained the following equations:

$$p\sin(\alpha - \omega) + q\sin\omega = 0, \tag{E.8}$$

for line OT and

$$p\sin(\alpha' - \omega) - r\sin\omega = 0, \tag{E.9}$$

for line $O'T'$. Plücker solved these for the cotangent ratio giving,

$$\cot\omega = \frac{-q + \cos\alpha}{p\sin\alpha} \tag{E.10}$$

$$\cot \omega = \frac{r - \cos \alpha'}{p \sin \alpha'}. \tag{E.11}$$

Using the trigonometric identity in his coordinate system, $(\sin \alpha \cos \alpha' + \sin \alpha' \cos \alpha = \sin(\alpha + \alpha') = \sin \alpha'')$, he could substitute both right hand values from the above two equations to find

$$p \sin \alpha'' - q \sin \alpha' - r \sin \alpha = 0. \tag{E.12}$$

Further, Plücker divided by p to convert these values into the two variable coordinate system of ϕ and ψ,

$$\phi \sin \alpha' + \psi \sin \alpha = \sin \alpha''. \tag{E.13}$$

Since equations (E.8, E.9) represented two parallel lines passing through two arbitrary given points, their intersection on (E.13) contained a point at infinity. Moreover, the form of (E.13) would not change for any two parallel lines through O and O', thus that line represented only infinitely distant points. Plücker concluded that the linear form of the equation showed that all infinitely distant points on a plane were collinear.

> Die lineare Form dieser Gleichung zeigt, dass alle unendlich weit entfernten Puncte einer und derselben Ebene als in gerader Linie liegend zu betrachten sind. (ibid, 8)[3]

In a footnote, Plücker referenced Poncelet's *Traité des propriétés projectives*, where this same theorem had appeared on the "Theory of Projection":

> Tous les points situés à l'infini sur un plan peuvent être considérés idéalement, comme distribués sur une ligne droite unique, située elle même à l'infini sur ce plan.[4]

Plücker found very noteworthy that the theorem resulted directly from his new coordinate system, and with a variable equation representing an "(ideal) line" (the use of parentheses is Plücker's). Plücker observed that while this theorem was the basis of Poncelet's important conclusions, Plücker could now reach the same conclusions through a direct route safe from any objection [*Einwurf*] derived from the linear form of the final equation. Without explicitly criticizing Poncelet, Plücker implied that the theorem had been objectionable before obtaining this secure analytic foundation.

Plücker continued this research to prove, in concordance with Poncelet's *Traité*, that two similar, concentric lines of second order would have a double contact at an infinite distance, which would be real or ideal according to whether both curves were hyperbolas or ellipses. To show this, Plücker derived an equation for the infinitely distant chord of contact

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[3] The linear form of this equation shows that all infinitely distant points in one and the same plane are considered to be lying in a straight line.

[4] "All points situated at infinity on a plane can be ideally considered as distributed on a unique straight line, situated itself at infinity on this plane."

The slight difference in vocabulary is worth noting, though it could be no more than a standard translation. Poncelet's "à l'infini" is equated with Plücker's "unendlich weit entfernten"–that is, the noun becomes an adverb. Plücker did not use the noun *Unendlichkeit* within the text of this paper. However, in "Recherches sur les surfaces algébriques de tous les degrés" (*Annales de Mathématiques* XIX, 1828 pp. 97-106) Plücker included points and poles "à l'infini," thus suggesting this may be attributed to a difference in translation of from French to German mathematical vocabulary.

containing the tangent points. Plücker admitted that in the analytic treatment of second order lines, this theorem was of "no particular importance" since one could immediately obtain the double tangent line from the general relationships, any two general equations of second order could be combined into a linear equation. However, the theorem had particular significance in Poncelet's method where it was used to derive properties of any two doubly tangent curves from those of two concentric circles. Plücker continued by deducing representations of osculating curves as equations in his new coordinate system. This relied upon the use of differentiation to determine tangent equations, followed by an examination of coefficients.

In the following section, Plücker switched subjects to give a very simple analytic treatment of the theory of reciprocity in homogeneous coordinates. He advertised his purely analytic theory as superior to those of Poncelet and Gergonne because it was independent of a directrix conic. Though Plücker had employed the well-known directrix conic version of polar reciprocity earlier in this same article, he now presented a hypothesis that was general a priori. Within Plücker's system, he defined poles and polars entirely through their coordinate equation representation. While he described the pole of a line as easy to construct, he did not explain how the process of construction could be carried out.

First, Plücker considered $b\psi + a\phi = 1$ as any straight line and defined the point ($\psi = b, \phi = a$) as the easy to construct pole of this line. If the point (ψ', ϕ') was on the straight line then the line $\psi'\psi + \phi'\phi = 1$ passed through (b, a) and by the above definition the pole of this line was (ψ', ϕ').

Thus, Plücker concluded that the locus for the poles of all straight lines passing through a given point is a straight line whose pole is this point. This theorem could serve as the basis for the theory of polar reciprocity. Besides this definition, Plücker asserted that everything else in the theory behaved as it had under the ordinary coordinate system. He concluded this paper by suggesting that his theory of reciprocity in the plane with slight modifications could be transferred to space.

The key relationship in Plücker's analytic geometry was that between the coordinate equation and the geometric object. Plücker consistently referred to the equation as representing the point, line, or locus in the plane. Despite the occasional presence of figures in this article, Plücker's references to visualization pointed toward viewing the variables and coefficients. Indeed, for Plücker the figures function much more as supplementary illustrations than instructive diagrams. The secondary nature of the illustrations was further highlighted by Plücker (or the editor's) frequent mislabelling, as well as the presence of unspecified objects in a figure and the absence of figured objects in a description. The figures were static and not manipulated through constructions. That is not to say that Plücker did not include constructions in this paper. He described several constructions undergone by objects in the plane, such as angles formed [bilden] by intersecting lines and lines constructed [construiren] by continuously moving points. However, these processes did not reference any of the accompanying figures, nor did they suggest the reader draw their own–especially since some of the constructions were rhetorical (for instance, constructing all the points of a polar curve) rather than practicable. As we saw in Chapter II, in his 1826 manuscript, Plücker had described coordinate equations as representing geometric objects such as, "un système de deux lignes droites, représentées séparément par les équations" (Plücker and Schoenflies (1904)), though not with the same consistency as here. Overall, Plücker's treatment of the figure, despite the actual physical presence of figures in this paper, appears consistent with that in his earlier writings. On the other hand, this paper contrasted with his earlier work through the emphasis on representing the most general case, sometimes at the expense of requiring substantial calculation.

Whereas Plücker had praised the particularity of analytic geometry in 1826, here he advertised the use of homogeneous coordinates as enabling an even more general geometrical method. Plücker achieved generality in his treatment of straight lines and conics, and further expanded the scope of his geometrical researches to include any general algebraic or transcendental curve. However, by basing his coordinate equations upon an a priori coordinate system, Plücker restricted himself from the clever coordinate manipulation which had characterized his earlier work. Plücker's repeatedly turned to a combination of trigonometric and linear relations to calculate new results. He still described this approach as simple and independent from magnitude considerations, but he no longer advertised the freedom from calculation in his analytic presentation.

Appendix F

Louis Gaultier, radical circles, axes, and centres

Louis Gaultier, often referred to by contemporaries as Gaultier de Tours, is credited with defining the terms "radical axis" and "radical centre," and was often referenced for these contributions in later solutions of the Apollonius problem and other planar geometry applications (Gergonne (1814b), Durrande (1820), Poncelet and Cauchy (1820), Steiner and Gergonne (1827), Plücker (1827), Chasles (1828a)). In addition, Gaultier was the first author that Poncelet cited in prefacing his *Traité* (Poncelet (1822)). Gaultier had attended the École polytechnique and was a professor of descriptive geometry at the *Conservatoire des Arts et Métiers* when he published his article, "Mémoire sur les moyens généraux de construire graphiquement les cercles déterminées par trois conditions, et les sphères déterminés par quatre conditions," in the *Journal de l'École polytechnique* (Gaultier (1813)). Gaultier introduced his text as providing a general solution to the problem of constructing a circle given three conditions and constructing a sphere given four conditions. He showed that there were 107 possible variations of the former problem, which he illustrated with a table at the end of the article (Figure F.1).

The paper was organized into three chapters: Geometric properties of radical circles, spheres and their series; Geometric properties of tangent circles and spheres; Tables of most of the problems to which one can apply the principles developed in the preceding chapter and complete discussion of three problems. While Gaultier solved the Apollonius problem in this article, his work became known instead for his general definitions of circle and sphere relations and not his particular results. In particular, these are referenced in the solutions to the Apollonius problem given by Gergonne, Poncelet, and Plücker in our Chapter IV (Poncelet and Cauchy (1820), Steiner and Gergonne (1827), Plücker (1827)). Thus we will focus on Gaultier's explanatory introduction and the first part of his first chapter, "Notions préliminaires sur les Cercles radicaux et les Sphères radicales," in which he defined radical circles, the radical axis between two circles, and the radical centre between three circles. These definitions will better illuminate the common reference source for the above three authors and provide additional background to the new theories described in the subsequent Bulletin review (Anonymous (1827b)).

Gaultier motivated his attention to the problem of finding a circle given three conditions by comparison with past approaches to the Apollonius problem. He noted that the prob-

Cercles déterminés par trois conditions.

N.os	Combinaisons.	Solutions.	N.os	Combinaisons.	Solutions.
1.	*tp*..........		18.	*pcd*..........	
2.	*td*..........		19.	*lld*..........	2.
3.	*rdd*..........	1.	20.	*cdd*..........	
4.	*ppp*..........		21.	*rpc*..........	
5.	*ppd*..........		22.	*rll*..........	
6.	*pdd*..........		23.	*rcd*..........	
7.	*tr*..........		24.	*plc*..........	4.
8.	*tl*..........		25.	*pcc*..........	
9.	*tc*..........		26.	*lll*..........	
10.	*rpp*..........		27.	*lcd*..........	
11.	*rpl*..........		28.	*ccd*..........	
12.	*rpd*..........	2.	29.	*rlc*..........	
13.	*rld*..........		30.	*rcc*..........	
14.	*ppl*..........		31.	*llc*..........	8.
15.	*ppc*..........		32.	*lcc*..........	
16.	*pll*..........		33.	*ccc*..........	
17.	*pld*..........				

Figure F.1: Gaultier's table showing numerous choices of three given properties to determine a circle using his own abbreviations (Gaultier (1813)) lem had been solved analytically by Newton, Euler and Fusse, while the analogous sphere problem had been solved geometrically by Fermat as well as Monge. However, Gaultier concluded that while the earlier proofs secured the existence of these solutions the graphic constructions provided had not been practicable and thus were unsatisfactory.

> [...] mais les considérations qu'ils ont mises en usage, quoique très-ingénieuses, rendaient les constructions graphiques trop difficiles [...] (Gaultier (1813), 127)[1]

By contrast, Gaultier would provide solutions based on the "most elementary principles" of planar and descriptive geometry that could be constructed by means of the ruler and compass alone. Further, he asserted that these solutions would be so general that they could even lead to imaginary solutions.

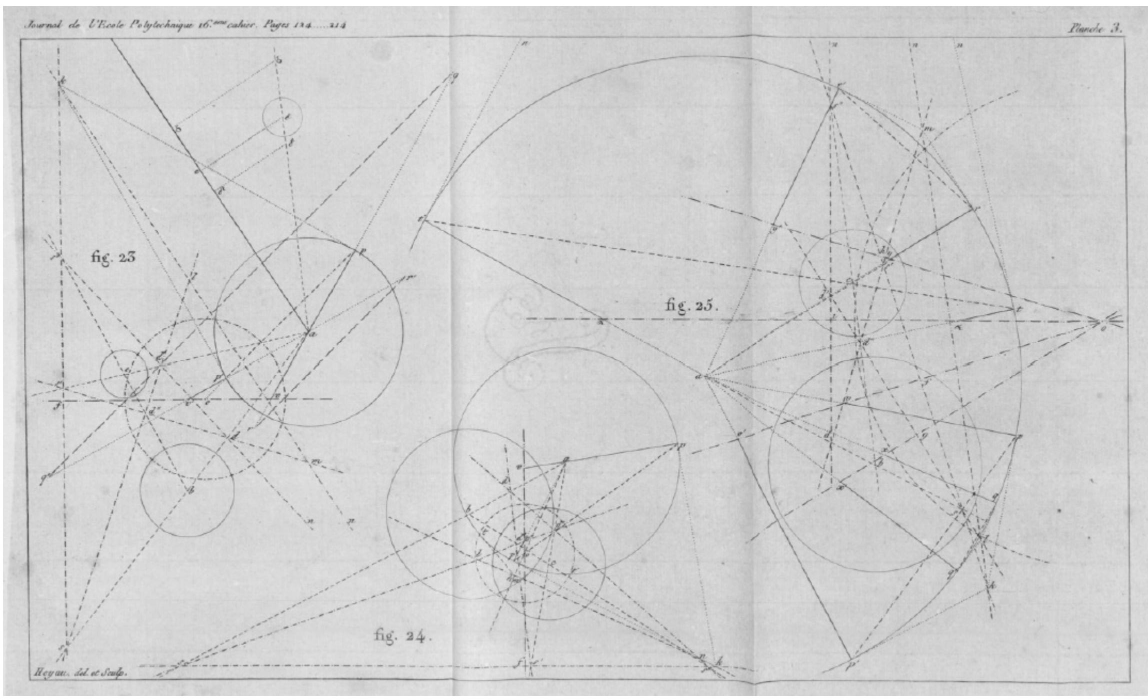

Figure F.2: Gaultier's illustration of solutions to the Apollonius problem (Gaultier (1813))

Gaultier's attention to the graphic applicability of geometry extended to his own use of illustrated figures (Figure F.2). He explained that he would often use one figure to demonstrate several propositions, but would avoid complications by only illustrating the parts of circles that were used in the construction. Therefore certain circles would only appear as arcs or radii. While we have seen that numerous geometers employed this same practice, Gaultier appears extraordinary in making his attention to figure appearance very explicit. He even added a footnote crediting the plates to M. Hoyau, a student at the *Conservatoire*.

> Les Planches de ce mémoire exigeant beaucoup de précision, j'en ai confié l'exécution à M. Hoyau, élève du Conservatoire, qui les avait dessinées pour

[1] "[...] but the considerations that they make use of, although very ingenious, render the graphic constructions too difficult [...]"

le mémoire manuscrit, et qui les a construites immédiatement sur le cuivre. (ibid, 128)[2]

This level of consideration for his audience was perhaps a reflection of Gaultier's teaching position. He further guided his reader by providing textual summaries in the margins of the page.

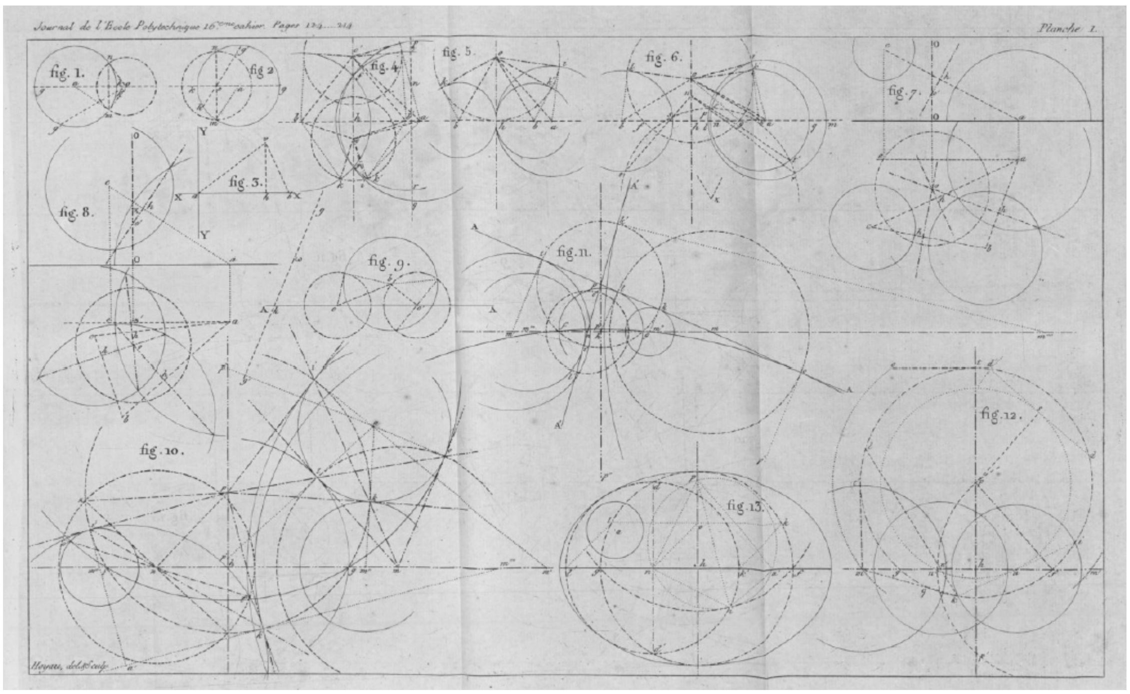

Figure F.3: Gaultier's first sheet of figures, including illustrations of radical circles, axes, and centres (Gaultier (1813))

Gaultier first began with consideration of a circle A, which he called the *primitive circle*. With reference to Figures 1 and 2 (the plate containing these, and all subsequent, figures is our Figure F.3), from a coplanar point o one could draw a secant or chord meeting the circumference of A at points g and k. A *radical circle* was one whose radius om was the length of the square root of $og \cdot ok$, the constant product of the segments of the secant or the chord drawn from its centre to the circumference of the primitive circle. If the point o lay outside of circle A then the radical circle was a *reciprocal radical circle* and if the point o lay within circle A then the radical circle was a *simple radical circle*. He considered the two terms reciprocal and simple as *species* of radical circles.

After examining analogous definitions in the case of spheres, Gaultier considered Figures 4 and 5, when two given circles A and B, centred at a and b respectively, had in common radical circles of the same species. With a few calculations based on the product of the radical circle, he showed that the centres of the latter would all fall on the same line

[2]"The Plates of this memoir requiring a lot of precision, I entrusted their implementation to M. Hoyau, a student at the Conservatory, who had drawn them for the manuscript memoir, and constructed them immediately on copper."

perpendicular to the line *ab*. This line Gaultier designated as the *radical axis* of circles A and B, or for short "axe rad. AB" (139).

He showed three different cases of locating the radical axis of two given circles. When the circles A, B intersected at points f and g (shown in Gaultier's Figure 4), then the line fg would be their radical axis. When the circles A, B were tangent at a point h (shown in Figure 5), then the radical axis would be their common tangent at the point of tangency h. In this case, all the common radical circles would be reciprocal circles. Finally, if the circles A, B had no common points (shown in Figure 6), either exterior circles or one enveloping the other, then their radical axis would be exterior to both circles A and B. To construct the radical axis in this case, one could draw an arbitrary circle Z meeting A and B at points p, r and s, t respectively. By extending the intercepted chords *pr* and *st*, their point of intersection u would lie on the radical axis of AB.

By revolving the circles illustrated in his Figures 4, 5, and 6 about a diameter, Gaultier applied his definitions to spheres in space.

Gaultier's final new definition concerned three circles A, B, C with non-collinear centres, such as in his Figures 7 and 8. The radical axes of A, B and of A, C would intersect in a point o. If one drew a radical circle centred at o to circle A, then it would be a radical circle of B and a radical circle of C. Thus, the circle would be radical to all three circles, and Gaultier designated the point o as the unique radical centre of A, B, C. He noted that when the three given circles were reduced to their centres, then the problem became that of finding a circle through three given points.

Gaultier continued by investigating situations when the given circles had infinite radius and became lines, and analogous results for spheres. He concluded his lengthy preliminary remarks by advising his readers to exercise wisdom when applying these properties to specific problems.

> Enfin il dépend de la sagacité de celui qui veut faire usage des propriétés que nous venons de développer, de ramener les questions aux intersections de lignes droites et de cercles, soit directement, soit en employant les propriétés communes aux deux nouvelles conditions qui complètent la détermination. (ibid, 170)[3]

The following year, Gergonne would use Gaultier's *radical axis* and *radical centre* in his own proof of the Apollonius problem (Gergonne (1814b), 351).

[3]"Finally it depends on the wisdom of those who want to make use of the properties that we have just developed, to reduce questions of intersections of straight lines and circles, either directly, or by employing properties common to two new conditions which completely determine it."