

**Objective and subjective probability:
Undergraduate students' descriptions,
examples, and arguments**

by

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Abstract

My thesis addresses several issues of importance to probability education, presented in four separate studies.

The first study attends to definitions and examples of probability offered through resources and produced by undergraduate students. The findings suggest that the everyday notion of probability predates and dominates students' conception of mathematical probability and point out the important role learner-generated examples play in identifying the scope of learners' understanding of probability. The second study examines the distinction between mathematical and everyday aspects of zero-probable and one-probable (extreme) events as featured in a variety of resources and as exemplified by prospective secondary school teachers. Moreover, different types of probability apparent from examples are identified and discussed. The results suggest that the participants use a range of subjective, theoretical, and logical approaches to construct probability examples in everyday and mathematical contexts. The results identified the need for a clear distinction between the notions of 'zero-probable' and 'impossible' in probability instruction and call for pedagogical attention to this issue.

The third study provides an overview of some of the ways in which randomness is defined in mathematics. The study examines and interprets undergraduate students' examples, definitions, and ideas related to randomness by analyzing the participants' written responses, verbal communications, and gestures. The findings were strongly related to those of previous research. The gesture analysis further identified some aspects of randomness that were less apparent in participants' verbal responses.

The fourth study examines undergraduate students' arguments concerning the probability of a fixed and unknown event. The goal of the study was to identify ambiguity caused by the interaction between everyday and mathematical probability in participants' responses. The findings suggest that reflective tasks in which students are asked to examine and reflect on opposing probability arguments may help learners to reconcile some conflicting probability ideas.

Overall, my research provides enhanced understanding of how participants perceived probability related ideas, as evident in their examples, definitions and gestures. Based on the results of my research, I present ideas and tasks for instructional implementation aimed at provoking discussion about different interpretations of probability and strengthening student understanding.

Keywords: Probability; Undergraduate; Randomness; Ambiguity; Definitions; Examples.

To my family.

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Chapter 1.

Introduction

It is strange that the summary of a lifetime of work on the theory of X should begin by declaring that X does not exist, but so begins de Finetti's Theory of Probability.

Robert F. Nau (2001, p. 89)

Bruno de Finetti's theory of probability (1974¹) begins with the thought provoking statement: Probability Does Not Exist. What he means is that probability does not exist in an objective universal sense. Alternatively, he believes, probability exists in a subjective sense, in the form of personal betting odds derived from a given set of information available to a person. His radical work was done in response to the then accepted strict frequentist paradigm that believed in a true objective probability of events in relation to infinite trials of controlled experiments. The dispute between objective and subjective probability was heated after the works of Ramsey and de Finetti. Later on, Savage, Jeffreys, and Lindley joined forces and devised a formal structure to a probability approach, today referred to as Bayesian or subjective probability. Although these works were written between 1930 and 1950, the grounds for the dispute were laid as early as the emergence of probability.

At the dawn of probability in the 17th century, probability was defined based on the notion of equiprobable events underpinned by the principle of indifference. The principle states that if we have no reason to think one alternative is more likely than another, then they should be assigned an equal probability. The question that was raised instantly was in what ways should this indifference be evaluated or decided upon?

¹ de Finetti's original works are in Italian and French, written between 1930 and 1937. According to (Gillies, 2000), were not influential in English speaking countries until they were discovered by L. J. Savage, who edited some of his works for publication in 1951.

Consider a standard two-sided coin. It is possible to toss it for a while and form an opinion about the indifference of one side to the other. It is also possible to obtain technical information about the making of the coin such as the homogeneity of the metal alloy, the preciseness of the mold, or the temperature consistency of the surrounding while cooling down. Another approach is to settle for a uniform state of ignorance and take equiprobability of the two faces as a prior fact about the coin and adjust it only in light of new evidence. Both of these pathways have been pursued and have resulted in different probability perspectives.

The issues surrounding probability is echoed in probability education. Probability and statistics were added to North American curricula only in the twentieth century. According to Scheaffer and Jacobbe (2014), the earliest efforts to include statistics in a junior high school curriculum were made around 1920. A stronger push was made in the 1940s and 50s because of the expanded uses of statistics in sciences and social sciences. Not much progress was made during the 1960s and the 70s with regard to the school curriculum. But the statistics community was laying remarkable groundwork for the future by emphasizing a shift of attention from theory to data analysis: “Building on data analysis as the core focus, the 1980s and 1990s saw great progress in the development of programs and materials in [school] statistics” (Scheaffer & Jacobbe, 2014, p. 2). It was only in the 1980’s that some real progress in statistics education was made, beginning with NCTM’s *Agenda for Action: Recommendations for School Mathematics*. This document included “numerous references to statistical topics including gathering, organizing and interpreting information, drawing and testing inferences from data and communicating results” (Scheaffer & Jacobbe, 2014, p. 6).

I am not trying to review the history of probability education; I am merely pointing out that probability and statistics not only emerged late in mathematics but also in mathematics education. The ambiguous nature of probability, different schools of thought developed around it and the multitude of interpretations of probability may all be factors that could account for the late entry of probability into mathematics education (for example, Hacking (2006), Chernoff (2008), Konold (2002)).

Over two decades after a stable probability curriculum was in place (in the United States), Jones, Langrall and Mooney published a chapter in the *Second Handbook of Research on Mathematics Teaching and Learning* (Jones, Langrall, & Mooney, 2007). Their synthesis of worldwide K-12 curricula suggested that classical (theoretical and frequentist) approaches to probability were the big ideas found in mathematics education and the presence of subjective probability in curricula was abysmal. This sparked my interest and led me to inquire into whether the post-secondary curriculum followed the same suit.

This thesis contributes to the study of the state of probability education in post-secondary education. The scope of this thesis is limited to investigating a few ideas such as: types of probability, the interplay or the tension between everyday probability and mathematical probability and the applicability of probability. I examine undergraduate students' perception of the probabilistic perspectives that are implemented in general-audience-level probability education.

Rather than writing a traditional dissertation, I have chosen the less frequently threaded pathway in writing a thesis: the paper style. I present my work in the form of a set of papers that are connected by common themes and trends.

Instead of single-purpose chapters addressing methodology, data analysis and other components of the research undertaking, my thesis is comprised of four independent studies. Each paper focuses on a moderately important aspect of undergraduate students' perception of probability. However, in the end, as a whole, they enhance and complement the common threads running along all four studies. Each study is written in a paper format and contains the relevant literature review, theoretical considerations, data analysis, and so on. In order to avoid redundancy, I have presented the references at the end.

In this chapter I introduce myself, my interest in the subject, how I developed interest in my research questions, and my journey that resulted in the studies presented in this thesis.

1.1. Educational Background

I hold a BSc and MSc in pure mathematics. I am currently a full-time faculty member at British Columbia Institute of Technology (BCIT), Vancouver, teaching mathematics and statistics courses, including: Technical Mathematics for Architectural and Building Engineering Technology, Statistics for Biomedical Engineering Technology and Calculus for Chemical and Environmental Technology. Back in my home country, Iran, I taught mathematics at middle school, high school, and post-secondary, teaching subjects including Algebra, Calculus, Discrete Mathematics and Geometry. I started my PhD studies in Mathematics Education at Simon Fraser University (SFU) in 2009. Since then, I have taught a number of teacher preparation courses and general undergraduate audience courses for both the Department of Mathematics and the Faculty of Education at SFU.

I wish to share with the reader a brief account of my own probability education since it has shaped my understanding of the field and my interests in probability. In my pre-doctorate academic life I had encountered probability in high school, in the undergraduate program, and briefly in my Master's program.

In Iran, during the time I was a student, we would not learn about any variation of chance or probability before grade 11, a practice that changed only very recently. In Iran, the curriculum is centralized. In other words, the learning outcomes are decided upon by a group of mathematicians and mathematics educators. They write textbooks that cover those outcomes. The textbooks are sent to all schools across the country. They are taught at the same pace and order everywhere.

In a conversation I had with one of the people participating in curriculum development in the Iranian ministry of education I found out that the late appearance of probability in our schooling system is partly due to interpretations of Piaget's stages of cognitive development. In Piaget and Inhelder (1975), the authors linked the development of probabilistic thinking to maturation in proportional reasoning and operational thinking. In Piaget's theory, the acquisition of formal thinking that he deemed essential to developing probability-related strategies is associated only with the third

stage of development. This is the stage where the child can link different concrete operational systems and apply them to different concrete objects.

This was interpreted as a call for keeping probability out of the curriculum until well into the mid-high school years. This is the time by which students have become accustomed to working with abstract structures endowed with certain properties and rules (Boolean algebra, trigonometry and vector spaces are examples of such structures).

Probability is then introduced through an abstract approach: there is the notion of random experiment, a *universe of discourse*, a *sample space* and certain subsets known as *events*. Finally, the probability of events is calculated through enumeration and combinatorial methods. Games of chance, collections of objects and standard randomizers are used to provide contexts for examples and problems.

In recent years, a major change was made to the Iranian probability curriculum. It now starts at lower grades and early concepts of probability are taught in elementary school.

During my undergraduate studies in mathematics, I took two courses in probability and statistics. In the first course, a wide range of topics including discrete and continuous random variables, well-known probability distributions, constructing confidence intervals, hypothesis testing and analysis of variance were covered. But very little or no reflection was done on what all this was based on or, for example, under what assumptions the whole hypothesis testing machinery was operating. The sequel course to the introductory probability was packed with Beta and Gamma functions, the special cases (such as Poisson, or normal distributions), and the limit behaviour of those distributions. We spent so much time on integration and momentum calculations that the course lost touch with probability and, in retrospect, I remember it as another integral calculus course. I finished the program without hearing about different schools of probability or how the discipline emerged in the seventeenth century, dividing mathematicians and philosophers into various schools of thought right from the beginning.

I took a measure theory course during my master's program, where probability theory was revisited and expanded upon. A measure-theoretic approach to probability unifies the discrete and the continuous cases, while providing analogous concepts for the familiar notions of expected value and independence. Different problems are only distinguishable by what measure to use on which sets. The notion of sigma field on a sample space endowed with a unit measure, although very abstract, is a powerful idea. The problem of calculating the probability of an event becomes equivalent to calculating a set measure. This approach helped me resolve two of my unsolved probability dilemmas. The first dilemma arises in situations where the set of desired outcomes is countably infinite and the sample space is infinite but not countable. Picking a random whole number \mathbb{Z} from the set of real numbers (\mathbb{R}) is an example of this situation. Since the early days in my undergraduate program, I knew that the probability of this event is zero. This is because whole numbers are of a smaller cardinality compared to real numbers. The probability in question would be equivalent to dividing \aleph_0 by \aleph_1 , which I approximated to be zero, since I thought of the latter as a considerably greater infinity than the former. The measure theory course once and for all clarified that there is a consistent and objective way of measuring sets and if both sets \mathbb{Z} and \mathbb{R} are measured with the same measure, the former will have a zero measure (exactly zero, and not by means of approximation) while the latter will have a non-zero measure. Another dilemma I had always wondered about was to determine the probability of certain subsets of the set of natural numbers (\mathbb{N}). Consider these examples: the probability of picking:

- A. a multiple of 7 from \mathbb{N}
- B. a prime number
- C. a perfect square
- D. a number with an odd number of digits in its binary expansion.

What all these examples have in common is that the set of desirable outcomes is countable and in one-to-one correspondence with \mathbb{N} . Therefore, the cardinal number of the sample space is equal to the cardinal number of the desirable outcomes, which makes all of the above-mentioned probabilities equal to one (from a weak theoretical perspective). Our intuition of the numbers would suggest otherwise and so does a well-implemented measure-theoretic approach. Intuitively, we know that every 7th number is a multiple of seven, therefore the probability of randomly picking a multiple of seven

should be $1/7$. With regard to examples B and C, we may not have a clear idea of what the probability in question is, but we are confident it is not 1. That is because prime numbers and perfect squares are harder to come by for larger values of n . For example D, an intuitive answer could be $1/2$ since there is no reason for the number of digits to be biased towards even or odd values. A measure-theoretic investigation of these examples, leads to developing interesting ideas such as natural density (Nathanson, 2000), a powerful tool in probabilistic number theory (Tenenbaum, 1995). The natural density of multiples of m is equal to $1/m$, which results in probability $1/7$ for example A. the natural density of prime numbers less than or equal to n is given by $\frac{1}{\ln n}$, a value that tends to zero for large values of n , hence making probability in example B equal to zero. The natural density calculation for the set of perfect squares results in zero, making the event of picking a perfect square number impossible. Finally, it can be shown that as simple as example D seems, it fails the measure-theoretic approach. The upper and lower natural densities don't converge to the same value, therefore the probability of picking a number with an odd number of digits in its binary expansion cannot be calculated. This example points out that not all sets are measurable and thus some probabilities cannot be assigned in an axiomatically consistent way. As seen from the four examples discussed here, the measure approach made some probability calculations possible in situations where the cardinality ideas did not have much to offer.

From the above account of my probability education, one can see that it is severely limited to classical probability and to mathematical contexts. If I were asked to give an example of probability of an event, or a probabilistic situation, it wouldn't cross my mind to give an example of a belief type of probability related to everyday life such as: 'the sun will come up tomorrow' or 'I will die someday'. I was also completely unfamiliar with the notions of subjective probability and the Bayesian school of thought on probability, and with the philosophical issues around probability, and the years of struggle mathematicians and probability theorists went through in order to formally define randomness and to axiomatize probability.

1.2. Research interest in probability education

I developed interest in probability education through three sources: teaching probability, learning about other teachers' (mostly prospective teachers') views of teaching probability, and readings in the history of probability.

When I started my PhD studies in mathematics education, I taught an undergraduate mathematics education course for students in the arts and humanities programs. This course covered basic material from number theory and probability. Through marking the quizzes and assignments of this course, I came to know about the challenges my students faced with probability problems. I noticed that the overall level of their performance in other concepts was considerably higher and they demonstrated more skill and ease in handling those concepts, complicated though they were. This made me notice my own lack of understanding of the body of knowledge around the notion of probability.

For three years I had the chance to work with pre-service teachers taking professional development courses at Simon Fraser University. Each year I asked them to fill out a survey questionnaire and tell me about their mathematical background. One question I asked them was what subjects in mathematics they thought were the most challenging to teach and why. The comments consistently pointed out that most prospective teachers perceived of teaching probability as a difficult task. They found it both a complicated subject and a difficult one to teach. Some prospective teachers mentioned that as students they found probability to be a confusing subject, in which it is easy to be tricked and it is not obvious why certain ways of calculation are correct. It was also mentioned that probability is a difficult concept to teach because students must use abstract and logical reasoning in uncertain and unknown situations.

During my PhD studies, I came across historical and philosophical works on the creation of probability. Gillies (2000), Hacking (2006), Stigler (1990) and Sheynin (1977) are among the sources I found most interesting. As stated earlier, unlike many other mathematical concepts that are more than centuries old, probability is a relatively

new area; it rose late and among controversy. Although existent in earlier works in some form and shape, the official birth of probability is usually ascribed to the 1660s.

Since the very early days, probability has been multi-faceted, “on one side it is statistical, concerning itself with stochastic laws of chance processes, on the other side it is epistemological, dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background” (Hacking, 2006, p. 12). Hacking argues that this duality resulted from a transformation upon two different conceptual structures: knowledge and opinion. One stemmed from evidence of testimony and authority, the other from evidence of things, where one thing points beyond itself. He clarifies the point: “an opinion was probable if it was approved by authority, or at least well testified to. [...] after the Renaissance] a new kind of testimony was accepted: the testimony of nature [...]. A proposition was now probable if there was evidence for it. [...] Probability was communicated by what we should call law-like regularities and frequencies” (p. 44). The evidence of things ultimately formed the basis for what we call classical or theoretical probability. The other type, the evidence of authority and testimony, is the core idea of what we may call subjective probability, according to which “probability is made of judgment” (Jeffrey, 2004, p. XI). The tension between the two probability constructs was never resolved. Although in our time the debate is no longer hot (and perhaps alive), discussions of credible interval versus confidence interval, hypothesis testing versus Bayesian inference, and textbook insistence on very specifically worded ways of interpreting confidence intervals (Foster, 2014) remain reminiscent of the long-fought battle.

1.3. My Thesis Journey

By now the reader knows that I developed an interest in learning more probability and talking to people about probability early in my PhD studies. I wish to continue with giving a brief account of the path I took to finally find the focus of my research and to form the research questions presented in my studies.

My research wanderings can be categorized into three phases:

Phase one consisted of tasks and problems that I found interesting, challenging, and to some extent counter-intuitive, most of which I did not know how to solve at the time. They were typically probability problems involving potentially infinite trials of simple experiments. For example, I collected interview data on a one-dimensional integer random walk where a drunk person is randomly moving to the right or left, each time by one unit on the number line and the probability of reaching a certain number is to be calculated (as a great reference on this problem I suggest Feller (1968)). Another example would be my attempts to study learners' interactions with the infinite monkey theorem: a monkey is punching letters at random on a typewriter; we want to know what the probability of him typing something meaningful is (in some versions of the problem, including Feller (1968), the monkey is to type the entire play of Hamlet).

The list of examples I tried to use for data collection in this period goes on, but what they all have in common is the structure of the task and the general response to it. These tasks address the issue of dealing with *possibly* infinite trials (*possibly* because the drunk man may reach the designated point, and the monkey may punch what we are looking for against all odds, in finite time, thus the process is not necessarily endless, at least not by logical necessity).

The responses I got were very similar. At first, the respondents would assess the probability in question to be very small, next to zero, and upon more reflection the response would change to 1 (probability=1) since, to quote one respondent, "the processes can go on forever, so anything can happen". In this phase of my research, I got a good amount of individual testimonials to Kolmogorov's zero-one law from the respondents, a law that states the probability of tail events can only be zero or one, and that was all I could make of the data.

During the second phase of my wanderings, I focused on tasks that involved infinity. At the time, I worked with prospective secondary school teachers and I presented them with questions regarding choosing numbers from the real number line or the set of all natural numbers. Questions such as: calculate the probability of randomly picking a multiple of five, a perfect square, or a prime number from the set of natural numbers.

The responses I got were more interesting and divided than the first phase and they usually stirred discussion afterwards. I found that questions like these were great teaching tools; they tapped into learners' knowledge of number systems, as well as their understanding of number relations such as factors, multiples and patterns.

Although the tasks and results turned promising with potential for research into respondents' interaction with infinity-related probability problems, my attention was diverted to something that frequently came up in discussions of the tasks. Several prospective teachers would say something like: "if I am asked to pick a number, I always pick 7; so for me the probability of picking a prime number is very high, like 100%. Also I have heard that many people would pick 5 or 7; so these have higher probabilities". Another interesting response: "I think smaller numbers have a higher chance of being picked because people tend to think of smaller numbers".

The notion of everyday context versus mathematical context came up frequently too, as exemplified in this response: "usually when we are asked to think of a number we are told what the number is between, for example, I might ask someone to think of a number between 1 and 10, it is never open-ended". Another person stated that, "when in prize draws and the like we pick numbers, there are only a given number of them, how can one imagine an actual situation where the number of possibilities is infinite?"

I found these points very interesting because they referred to what I now call the distinction between everyday and mathematical probability and the notion of personal versus theoretical probability.

Another observation I made during this phase was that the conventionally correct answer (zero, in the case of picking prime and perfect square problems, see page 6) did not come up very often. Some respondents would state that the probability in question is "very small" (instead of zero); I thought this aversion towards zero may be due to the tension they felt towards assigning zero probability to something that is evidently logically possible (picking a prime number or a perfect square number). Since the respondents did not seem to view the prime number or perfect square number picking events as zero-probable, I thought to ask for their examples of zero-probable events. This task turned out to be an entry point to my third phase of research wanderings. This

phase of my research is reported in the current thesis, although not in actual chronological order. After I examined the responses to learner-generated examples of zero-probability events, I made another discovery. A great number of examples were from everyday contexts: “the Sun will not rise tomorrow” and “a coin lands on edge”, and “rolling a 7 with a standard die” among the frequently repeated ones. They were examples that many probability theorists would not even consider to be appropriate subject matter of probability. For instance, Feller (1968) makes it clear that “there is no room in our probability system for speculations concerning the probability that the sun will rise tomorrow” and “in analyzing a coin-tossing game we are not concerned with the accidental circumstances of an actual experiment” [such as the coin landing on edge as a result of resting on a corner] (p.4). In my view these were not examples of probability situations, since they do not represent a repeated sequence of identical experiments, nor can one formulate them into a desired outcome of a sample space made up of equiprobable events. Even the example of rolling 7 with a standard die did not make perfect sense to me; clearly it is impossible to roll 7, because there is no 7 printed on any of the faces. But how does one actually calculate this probability since $\{7\}$ is not a subset of sample space $\{1,2,3,4,5,6\}$? This sparked my interest in examining students’ notions of the applicability of probability to different contexts including real-life ones and also what they have in mind when referring to random phenomena in general. I conducted classroom discussions on whether or not certain types of problems constitute appropriate subject matter of probability and what types of probability arguments are needed to determine, calculate, or assign probabilities to those occurrences. These discussions informed me of distinct views of probability and randomness apparent in students’ responses and ultimately led me to my research questions and data collection (I collected data from another group of students, since I was looking for original responses).

1.4. Probability: is it as mathematical as the rest of mathematics?

Before I present the structure of the thesis and an overview of the contents, I wish to share with reader a note on the state of probability education. Arguably, under

the research in mathematics education umbrella, probability education is under-represented and it is not as ubiquitous as other mathematical concepts. In the words of the editors of the recent collected work on probability education, Chernoff and Sriraman (2014) “research investigating probabilistic thinking exists, currently, on the fringe of mathematics education” (p. XVII). Moreover, it is mostly populated by task-specific research targeted towards early elementary years and it gets scarcer for the higher age groups. I suggest that this phenomenon to some extent mirrors (or possibly is affected by) the state of probability in the field of mathematics. There has always been a tension between mathematicians and probability theorists about whether or not probability theory is some lesser form of mathematics. In many universities, the department of mathematics engages in studies related to pure and applied mathematics, while the department of statistics is involved with all matters of probability. Also in the Bourbaki project, probability was not included. The project was initiated and developed by mathematicians that “knew what counted as mathematics: they could tell mathematics when they saw it” (Hacking, 2014, p. 56). They may have thought it is implicitly covered in other areas such as integration and measure theory. Probability theorists, including the French ones, never agreed.

I add to the argument above a personal encounter with a prominent mathematics educator, initially trained as a mathematician. The story of this encounter reinforces my point about how the ambiguity (perceived or inherent) in probability may result in reluctance towards research in probability education. This mathematics education researcher was kind enough to engage in a helpful conversation with me about my research interests. After some suggestions on possible directions, he said that he has never been interested in research in probability education since he has always been doubtful of probability theory. He mentioned that in his undergraduate days in a probability theory course, it befuddled him that there are more than one ways to look at and solve a probability problem and not all of them result in the same answer. He referred to a specific classroom experience where the instructor solved a complicated problem and found the answer, and everything seemed correct. Then a student suggested something, upon which the instructor recalculated the probability and came up with a different answer, it seemed correct too, and no one could tell which one was the right answer.

It is very common to have more than one solution to a mathematics problem, but they always (at least in popular view) come to the same final answer, and if not, there has been some mistake somewhere. It seemed to him that finding those “mistakes” is very hard with some probability problems and even with solutions that work, the results are often counterintuitive. I could really relate to what he said since I have had my own share of confusing moments with probability problems; the big offenders through high school being the problems where the order did or did not matter and I had to leave it to chance to make the right decision. It didn't get any better in post-secondary courses either. Here I present, for the amusement of reader, two examples of multi-solution and counter-intuitive problems that I have come across in my probability education:

1. Problem of randomly inscribed triangle: it can be proven² that if three random points are chosen on a given circle, the probability of the three points forming a right triangle is zero. The same applies to equilateral triangle, and a triangle with angles 45° , 65° , 70° . The argument can be extended to show the probability of randomly chosen points forming a triangle with interior angles α , β , γ is zero for any α , β , and γ . A result that is extremely counter-intuitive and confusing, since after all some triangle is being formed.

2. Problem of random chord (Bertrand's Paradox): what is the probability that a random chord of a circle is longer than a side of an equilateral triangle inscribed into a circle?

Several interpretations and solutions to this problem are given, all of which are valid arguments made within the proper probability theory laws. However, one results in $1/2$, another in $1/3$, and another in $1/4$ (see Ghahramani (2005, p. 264), or Wikipedia page on Bertrand's Paradox for more details).

The laboured point I want to make is that probability is a mathematical model that is sensitive to initial information, assumptions, and interpretations of the situation. I do

² Say we are randomly choosing three points on the circle. Given the first point A, there are only two fixed points B, C on the circle that could make an equilateral triangle while there are infinitely many possibilities for the second and third points (only one of which is desirable). This makes the probability of having an equilateral triangle equal to zero.

not see it as a weakness, as much as a call for extra need to be aware of how the mechanism works and what is the scope of applicability of probabilistic methods. Furthermore, probability methods are so far the best tools we have when it comes to dealing with uncertain phenomena. Probability methods are refined, revised, and made more efficient over time. A good teaching practice of a subject like this should ideally make an effort to disambiguate probabilistic concepts and methods by addressing in a dynamic fashion both the epistemological and technical changes taking place in this field.

1.5. The current thesis: what is in it?

I present four stand-alone studies written in paper format. I attend to the meanings and examples of probability in the first study, examples of extreme probability and the type of probability reasoning in the second study, definitions and examples of randomness in the third and, finally, possible conflicts and alignments of different approaches to probability in the last study. The common theme that runs through and brings together my entire work is an attempt to study and understand two seemingly separate notions of probability: the everyday and the mathematical. I try to elaborate on distinctive features of the two, as well as on common features. Also I try to examine learners' senses of the applicability of probability to mathematical and everyday contexts.

In all studies, the participants are students enrolled in post-secondary mathematics education courses at Simon Fraser University. Some are prospective teachers, and some are undergraduate students in liberal arts or social studies programs.

In the first study, I examine definitions and examples of probability from two points of view: through teaching and learning resources in probability education and through the undergraduate students' perspectives. The research questions in this study are formulated based on a model developed by Batanaro and Diaz (2007) where five interrelated components are proposed to analyze various meanings of probability. The model suggests making a distinction among institutional meanings of probability, those

appearing in textbooks and taught in classrooms, and personal meanings of probability, that is, the students' perceptions of probability. Thus, examining students' definitions of probability is suggested as a key component to analyzing probability understanding.

The research questions pertaining to this study are:

- What features of probability are emphasized in students' definition/descriptions of probability?
- What features of probability are emphasized in students' examples of probability?

In the second paper, I investigate through their examples, the learners' notions of extreme events: those with probability zero or one. Research suggests that learning about binary opposites takes place early; therefore, events with probability zero or one are a good starting point for teaching probability. Another reason I find extreme events of research interest is that the terms impossible and certain are frequently used to refer to events with probability zero or one. This creates a tension between the two notions of probability zero, as in never happening, and probability zero, as a matter of definitional necessity (same with certain and probability one). To the best of my knowledge, research in probability education has not addressed this distinction and the conflict resulted from the two notions in situations where probability zero is assigned to events that are not logically impossible and conceivably could happen. Similar to the first paper, I examine definitions and examples of impossible and certain events through some educational resources, followed by studying learner-generated examples of extreme events. The research questions are:

- What aspects of probability (mathematical or everyday) are featured in the participants' examples of extreme events?
- What probability perspectives are involved in the participants' examples of extreme events?

The analysis of data is framed by the two theoretical considerations, Barwell's discursive model, and types of probability. The discursive model outlines the features of a mathematical discourse including specialist use of vocabulary and symbols, the specific syntax such as the way proofs and definitions are formulated, and social conventions, which makes it a favorable model to analyze participants' examples of extreme events.

The third study has been published as a book chapter in Chernoff and Sriraman (2014). It has a slightly different approach and tone since it is co-authored by Rina Zazkis and Nathalie Sinclair. In this work, we first provide an overview of some of the ways in which randomness is defined in mathematics—these aspects of randomness help structure our analysis of learners' uses and descriptions of it. We adopt both discourse analysis and gesture analysis to interpret and probe into understandings of randomness. This study addresses the following research questions.

- How do learners exemplify random phenomena and what features of randomness are present in their examples?
- How do learners communicate ideas of randomness through a broad multimodal discourse?

We classify learner-generated examples of randomness into certain types as a result of which we can address the aspects of randomness that are present in both mathematical randomness and student randomness and those that are missing. In a clinical interview setting, we ask the participants to define and exemplify randomness, both in mathematical and everyday contexts and elaborate on why they think of those as random. Gesture analysis suggests that participants hold separate views of mathematical randomness and everyday randomness.

Prior research has not drawn attention to the ways in which learners are aware of differences among the ways the word 'random' is used in each context or to the distinctions learners attribute to each type of usage. Our study finds that although the notion of randomness is central to the study of probability and statistics, it is rarely explicitly explained. We suggest that teachers (and teaching materials) should attend more explicitly to learners' discourse on randomness and, in particular, to learners' everyday uses of the word.

In the last study, I invited participants to comment on a problem that addresses two of the old issues debated by subjective (Bayesian) and objective (classical) schools of probability: the problem of single events and the problem of past events. I propose that probability is an ambiguous concept since it is developed from two seemingly disjoint ideas, one that believes in objective reality about the uncertainties of the outer

world and the other that tries to obtain a subjective measure of personal tendencies and beliefs about uncertain phenomenon.

Each of these practices is considered valid and self-consistent by some statisticians and practitioners. However, the teaching of probability is to a great extent dominated by the objective paradigm, which creates conflicts and tensions when juxtaposed with learners' personal probabilities. My goal was to identify the instances of ambiguity in participants' responses and to find out if the ambiguity in probability can be exploited to enrich learners' understanding of probability and to create a unified concept of probability that accommodates both subjective and theoretical aspects of their probability conceptions.

1.6. The themes and threads

Although the studies presented in this thesis stand alone and are disjoint, as well as pursuing different goals, these common themes recur in all, sometimes more explicitly than others:

- Distinction between personal, everyday, and mathematical (institutional) meaning of probability-related terms and concepts such as probability, event, impossible, certain, and random.
- The conflicts between probability defined on finite sample spaces and probability on infinite sample spaces.
- The importance of inviting students to create examples of probabilistic concepts both as a research tool and as a learning activity.
- The importance of teaching probability as a modelling tool which has some limitations, a specific scope of applications, and certain underlying assumptions. Also, learners benefit from knowing that there is more than one valid (and useful) way of constructing probabilistic models.

My research has given me an opportunity to learn more about probability, but more importantly, it has earned me awareness about what educators and textbooks often take for granted. I suggest that in any transition from one mode of thinking to the other, it helps the learners to pause and reflect on the nature of the transition, to discuss what properties are changed and what features stay the same. For example, in the transition from the everyday meanings of probability to its mathematical meanings, or

from finite sample spaces to infinite sample spaces, some of the concepts (such as event, random, and probability) acquire new meanings and some objects will assume new properties. It is a worthwhile effort to discuss and draw attention to these new meanings and new properties. What I am hoping to offer in this thesis is enough evidence to support my suggestions along with adequate discussion of transitional moments in learning probability.

Chapter 2.

Undergraduate Students' Definitions and Examples of Probability

Extended attention to probability and statistics in school curriculum has resulted in renewed interest in these topics in mathematics education research. Despite the growing number of studies that explore understanding of probability concepts among students of different ages, the research on undergraduate students' knowledge and beliefs related to probability is still rather limited. My research aims to address this deficiency and looks into undergraduate students' definitions and examples of probability.

2.1. Introduction to the origins of probability

Probability is among the recent branches of mathematics. Unlike geometry or algebra, the laws of probability do not go back to ancient Greece or Indo-Arab mathematical cultures. It was only in the mid-seventeenth century that the systematic study of laws of chance became available and fashionable to the mathematicians and philosophers of the time. Uncertainty, lack of information about certain situations, and ignorance has troubled the people of olden days (as it does today). At all times, efforts have been made to deal with uncertainty and incompleteness of information. Examples of such efforts are: summoning the witnesses to testify to the possible facts, cross examining various evidences and witnesses to decide if they are trustworthy, and consulting the authorities. But it was not until the works of Pascal and Fermat that probability received attention from mathematicians.

Gambling and games of chance played a notable role in attracting mathematicians' attention to the discipline of probability. In the early sixteenth century,

gambling had grown to a full-size entertainment and as early as 1509, many books and pamphlets on the subject of gambling were around (Bellhouse, 1993). There also existed a published literature of cheating (which employs the notions of equal chances and advantageous situations) in the games of chance that was known both to the participants of the gambling practice and to a few of the mathematicians of the time, such as Cardano (Tijms, 2007). Some of these works provide evidence that gamblers in the sixteenth century had a rough concept of probability and also various notions of equality and inequality in play had been around. The important opportunity that gambling and games of chance provided to the mathematicians was to provide models, under which, fairly complicated and challenging combinatorial problems could be set. It is reputed that a French gambler named Chevalier de Mère presented Fermat with the problem of points, a problem of dividing the stakes when the game is untimely interrupted. Fermat is said to have communicated with Pascal about this problem and through their correspondences, a first configuration of an axiomatic probability emerged. According to Garber and Zabell (1979), the influence of Chevalier de Mère on Pascal and Fermat, both prestigious mathematicians, who both worked on the problems from games of chance, generated interest in the subject among mathematicians.

At any rate, the dawn of the seventeenth century marked the beginning of modernity both in society and science and the influence of scientific communities as well as academic correspondence was expanded during this century, which gave birth to the calculus of probability (Sheynin, 1977).

During the eighteenth and nineteenth centuries, the techniques of modern mathematical statistics began to be laid down, all on the foundations of earlier work on probability and chance. Despite these efforts, probability theory became a respectable mathematical field of study only in early 1930. Prior to the notable works of Kolmogorov, von Mises, Chaitin and Weiner, who made efforts to rigorously define probability and randomness in mathematical terms, the study of probability was treated with scepticism by some mathematicians due to the lack of rigorous definitions of notions such as “independent events”, and “randomness”. Mathematicians weren’t the only sceptics; arguably one of the closest fields of science to probability is statistics. We often hear the two terms next to each other and they are taught in K-12 under the umbrella of Data

analysis. It was not the case all the time. Ross (2009) gives an account of how statisticians of early nineteenth century were content to let the data speak for itself, and were not concerned with sampling and sampling-based inference. As a result, probabilistic inference from samples to a population was almost unknown in nineteenth century social statistics.

Even to date, doubts are held as to the value and effectiveness of probabilistic models, perhaps in part because they are treated in most teaching and learning environments as a confusing subject, founded on seemingly shaky grounds.

2.2. The meaning of probability:

In the mathematical context, there are two main interpretations of the notion: probability as a branch of mathematics, often used as short-hand for probability theory, and probability as related to a particular event. Here I focus on the latter, the probability of an event.

The meaning of probability (as in probability of an event) is not uniformly shared even among the people who are involved in probability related practices or those who have an educational background in probability theory. One possible meaning for probability is to consider it as the frequency of occurrence, as a fraction or percentage of the occurrence of a desired outcome over a long run of identical experiments. This interpretation is practical in some situations. For example, when a production line runs smoothly for months with no disruption or change of procedures, the ratio of the number of defective pieces relative to the total number of produced pieces provides a probability estimate for the expected reliability of the production line. We may make statements such as: 'the probability of a defective piece produced by this line is 1%', by this we typically mean that roughly 1% of all productions are defective.

While there are many situations involving uncertainty (most notable among them are experiments dealing with coins, dice, spinners, bag of balls and other such randomizers), in which the frequency interpretation is appropriate, there are other situations in which it is not. Consider, for example, a scholar who asserts that a specific

work of Shakespeare (such as *Hamlet*) is actually written by another person, with probability 80%. Such an assertion conveys information, but not about frequencies at which this event has happened since the event is a one-time event that occurred in the past. This is an expression of the scholar's personal degree of belief arrived at based on the available evidence; a type of probability that is of a subjective nature and is based on personally-held beliefs, and can be used to explain the choices and actions of a rational person. Yet another approach to probability is possible, a probability based on examining the physical features of an object (a six-sided die, for example) and arriving at a conclusion that the shape is perfectly symmetrical and homogenous, the weight is distributed evenly, and no evidence was found that one side will have a greater chance of facing up. This process results in distinguishing N equiprobable outcomes with a probability of $1/N$ assigned to each. This type of probability, which is assigned based on deductive reasoning on what can occur in a probability experiment, is called theoretical probability or *a priori* probability (please note that it is different from *prior* probability in Bayesian methods).

2.2.1. Mathematical probability and natural language probability

Variability in definitions and approaches to probability is not the only important issue with probability. Another important consideration, which pertains to many mathematical terms including probability, is that probability and its nomenclature are used both in common language and in mathematics. A discussion of various lexical ambiguities apparent to some mathematical terms could be found in Durkin and Shire's *Language in Mathematical Education* (Durkin & Shire, 1991). We make a number of probability-related statements every day that do not necessarily map onto their mathematical counterparts. For example when a friend is telling a rather odd story, we might express our surprise (and sympathy) by saying: "no way, it's impossible!" What we actually mean is "the chances of this happening to you are very small". In mathematical probability, the term impossible means that the probability of the event is zero. Based on our everyday experience, we would agree that "the sun will rise tomorrow" has 100%

probability, however, it may not be easy to prove it from a mathematical³ view. Obviously since the sun has risen on Earthlings now for millions of consecutive mornings without a miss, the probability that it won't rise tomorrow is staggeringly low. However, there are a number of factors (not all of them accounted for and even known), such as a sudden explosion of the Sun, or an unforeseen impact from a stray celestial object, that might end the life of Sun any day.

In conversational language we describe our thoughts as random when we are thinking about something unrelated to our main line of thoughts. We may say, "Oh, that was random", when something unexpected (least expected sometimes) happens. In mathematics a random selection means a selection that gives every object an equal chance of being selected and thus any of the outcomes are expected as much as the others. A random sequence of numbers is not one that we did not expect to happen; it is a sequence of numbers that is generated based on a rule that cannot be described in a length shorter than the sequence itself.

This discrepancy between mathematical and conversational use of probability terms both facilitates and hinders students' understanding of probability. Research in probability education suggests that long before their formal introduction to probability, students have dealt with many situations involving uncertainty and they have developed a certain probability vocabulary that is comprehended by other language users in everyday situations. Thus, the classroom experience of probability is nested into this web of meanings; unfortunately some do not fit well, causing serious misunderstandings, confusions and other communication impediments (Konold, 2002).

This overlap between the mathematical terms and everyday language makes it important to define the probability related terms properly.

³ Laplace has argued that, if we observe the Sun rising every morning for $n-1$ days, then we can infer that the probability that it will not rise the next morning is $1/n$, because, out of n days, it has risen on $n-1$, so only 1 day is left for it not to rise. Thus the probability of the Sun rising on day n is $(n-1)/n$. So: the probability of Sun rising on day 2 is: $\frac{1}{2}$ the probability of the Sun rising on day 3 is $\frac{2}{3}$ and so on. Therefore the probability of Sun rising on all days (2,3,...,n) is: $\frac{1}{2} \times \frac{2}{3} \times \dots \times \frac{(n-1)}{n}$ which amounts to $1/n$. Therefore for very large values of n , the probability of the Sun rising on all days is technically zero (Stewart 2009).

2.3. Definitions of probability through teaching resources:

It is an interesting task to look up the definition of probability in various probability and statistics textbooks. The definitions are varied in both the level of mathematical sophistication and in how they incorporate different approaches and views to mathematical modeling of probability. I have examined three kinds of resources, looking for formal or informal definitions of probability:

- teacher education textbooks and resources
- tertiary level probability and statistics textbooks
- online resources.

The goal of examining various resources was to provide illustrative examples of probability definitions, rather than a comprehensive overview.

2.3.1. Teacher preparation textbooks and resources:

The resources examined in this part are:

- A.1. NCTM principles and Standards
- A.2. Integrated resource packages on K-12 British Columbia curriculum
- A.3. Elementary and middle school mathematics (van de Walle, Folk, Karp, & Bay-Williams, 2011)
- A.4. Mathematics for elementary teachers (Beckmann, 2007)
- A.5. Reconceptualizing mathematics for elementary school teachers (Sowder, Sowder, & Nickerson, 2010)
- A.6. Using and understanding mathematics, a quantitative reasoning approach (Bennett & Briggs, 2010)
- A.7. Teaching secondary mathematics (Posamentier, Smith, & Stepelman, 2009) .

A.1. NCTM Principles and Standards

In a chapter on Data Analysis and Probability, the document addresses probability saying: “a subject in its own right, probability is connected to other areas of mathematics, especially number and geometry. Ideas from probability serve as a foundation to the collection, description, and interpretation of data” (p. 47). Then the

authors proceed to address the concepts to be taught at each grade level along with the depth to which the teachers are recommended to cover the concepts. For example, it is recommended that students of pre-K to 2 receive an informal treatment of probability and learn the probabilistic terms such as: unlikely and probably. In grades 3-5 students are to consider chance ideas through experiments involving known theoretical outcomes such as coin flips, or dice rolls. The list of suggested learning outcomes and recommended ideas to be employed by teachers is offered, but a definition of probability is not given.

A.2. BC provincial curriculum document

In British Columbia's mathematics IRPs (integrated resource packages), chance is described as a communication tool which addresses the "predictability of the occurrence of an outcome" (IRP, p. 14) and mathematical probability is defined as a tool that describes "the degree of uncertainty more accurately" (IRP, p. 14). Here both probability theory and probability of an event are referred to. The former mention is more general and potentially addresses all that is involved in making predictions about an outcome and the latter is the specific measure of uncertainty, the probability of an event.

The prescribed learning outcomes section of the same document for grade 6 probability and data analysis suggests that the students should be able to "demonstrate an understanding of probability by identifying all possible outcomes of a probability experiment, differentiating between experimental and theoretical probability, determining the theoretical probability of outcomes in a probability experiment, comparing experimental results with the theoretical probability for an experiment" (p. 42). The other description of probability I could find within the IRP's document comes from grade 9 recommended learning outcomes: the students need to "demonstrate an understanding of the role of probability in society, explain, using examples, how decisions based on probability may be a combination of theoretical probability, experimental probability, and subjective judgment" (p. 86). No definitions of these terms are given in the document.

A.3. Elementary and Middle School Mathematics:

Where the authors lay the big ideas of the whole probability chapter, they say: “The probability of an event occurring is a number between 0 and 1. It is a measure of the chance that the given event will occur [...] The relative frequency of outcomes of an event (from experiments) can be used as an estimate of the exact probability of an event [...] For some events, the exact probability can be determined by an analysis of the event itself. A probability determined in this manner is called a theoretical probability” (van de Walle, Folk, Karp, & Bay-Williams, 2011, p. 473). The probability portrayed here is objective and there are two approaches to finding that true probability: frequentist and theoretical, while an experimental probability is an estimate of the true frequentist probability. Although it is specified that theoretical probability works only for certain types of events, the connection to frequentist probability is not made and it is not clear that if both methods are applicable, which one is preferable and whether they result in the same probability.

A.4. Mathematics for elementary teachers:

In (Beckmann, 2007) we have the definition “the probability of a given outcome is a number quantifying how likely that outcome is. The probability of a given outcome is the fraction or percentage of times that outcome should occur in the ideal world [...] probabilities are always between 0 and 1, or equivalently, when they are given as percentages, between 0% and 100%” (p. 828).

In the pages following the definition, the book elaborates on properties of probability and addresses principles that determine probabilities. Among those principles are:

- If two outcomes of an experiment or situation are equally likely, then their probabilities are equal.
- If an experiment is performed many times, then the fraction of times that a given outcome occurs is likely to be close to the probability of that outcome occurring. The greater the number of times the experiment is performed, the more likely it is that the fraction of times a given outcome occurs is close to the probability of that outcome occurring.

A.5. Reconceptualizing Mathematics for elementary school teachers:

In this book, probability is defined as a feature of an event: “An event is an outcome or a set of outcomes of a designated type. The probability of an event is the fraction of the times the event will occur when some process is repeated a large number of times” (Sowder, Sowder, & Nickerson, 2010, p. 610). This approach is later supplemented by a possibility of skipping the repeated experiments if some theory of the likelihoods arisen by situation is at hand: “A probability that can be arrived at by knowledge based on a theory of what is likely to occur in a situation, such as when a fair coin is tossed, is a theoretical probability” (p. 617).

A.6. Using and understanding mathematics, a quantitative reasoning approach:

In Bennett and Briggs (2010) three types of probability are identified and defined:

A theoretical probability is based on a model in which all outcomes are equally likely. It is determined by dividing the number of ways an event can occur by the total number of possible outcomes.

An empirical probability is based on observations or experiments. It is the relative frequency of the event of the interest.

A subjective probability is an estimate based on experience or intuition (p.435).

A.7. Teaching Secondary Mathematics

A definition of probability or properties of probability is not mentioned throughout the book. The term probability appears in a couple of enrichment tasks such as the birthday problem, the expected number of games of baseball to be played given the odds of national games over a time period, and the net premium of life insurance problems (Posamentier, Smith, & Stepelman, 2009, p. 278). In the latter activity, the author explains that the probability that an 18-year old person will die in next year could be determined by:

$$\frac{\text{Number of 18 – year olds dying during the year}}{\text{Number of alive 18 – year – olds at the start of the year}}$$

The book presents a statement of the law of large numbers, indicating that with a large number of experiments the ratio of success to the number of trials gets very close to the theoretical probability. A definition of probability could not be found.

2.3.2. Tertiary level probability and statistics textbooks

The textbooks chosen for this part of the study are neither a representative sample of all of the textbooks written on the subject of probability and statistics nor indicate any ranking among such books. They are chosen since (to my knowledge) they are commonly used in college and university elementary statistics courses. Also because in my own probability education I have come across a measure-theoretic approach to probability, I have included an advanced probability book that employs a measure-theoretic method. The textbooks used in this section are:

- B.1. Statistical models (Freund, Wilson, & Mohr, 2010).
- B.2. Intro stats (de Veaux, Velleman, & Bock, 2012).
- B.3. Elementary statistics (Triola, 2011).
- B.4. Introductory statistics (Ross, Introductory Statistics, 2010).
- B.5. Introduction to probabilities (Bertekas & Tsitsklis, 2000).
- B.6. A Basic course in probability theory (Bhattacharya & Waymire, 2007).

B.1. Statistical Models

The authors attend to the definition of probability early in the book. It is defined as: “the relative frequency of each category, a number that gives the chance or probability of getting an observation from each category in a random draw” (Freund, Wilson, & Mohr, 2010, p. 13).

B.2. Intro Stats

In De Veaux, Velleman and Bock (2012) probability is described as one of the main contributions of mathematics to statistics. Probability is introduced first through the relative frequency, the number of times event A occurs over the total number of trials. The convergent value of relative frequency is called the empirical probability of event A

given that the experiment involving the event is repeated a large number of times. The book continues on with two more definitions of probability:

“for events that are made up of several equally likely outcomes, we just count all the outcomes that the event contains. The probability of the event is:

$$P(A) = \frac{\text{\# outcomes in A}}{\text{\# of possible outcomes}} \text{ (p. 393).}$$

A third type of probability is introduced next: “We use the language of probability in everyday speech to express a degree of uncertainty about your final grade in a given course basing it on how comfortable you’re feeling in the course or on your midterm grade, and not based on long-run behaviour. We call this third kind of probability a subjective or personal probability” (p. 394).

B.3. Elementary statistics

Triola (2011) gives three definitions of probability all at once:

“1) Relative Frequency approximation of probability: Conduct (or observe) a procedure, and count the number of times that event A actually happens. Based on these actual results, P(A) is approximated as follows: $P(A) = (\text{number of times A occurred}) / (\text{number of times the procedure was repeated})$).

2) Classical approach to probability: assume that a given procedure has n different simple events and that each of these simple events has an equal chance of occurring. If event A can occur in s of these n ways, then $P(A) = s/n$

3) Subjective probabilities P(A), the probability of event A, is estimated by using knowledge of the relevant circumstances” (p. 139 and p. 140).

B.4. Introductory Statistics

In Ross (2010) some big-picture ideas are presented first: “understanding probability is essential to be able to draw conclusions from data. The totality of assumptions necessary to make about the chances of obtaining different data values is referred to as a probability model for the data” (p. 5). Later on, probability is defined as: “a commonly used term that relates to the chance that a particular event will occur when

some experiment is performed” (p. 169). The author returns to probability elsewhere and states that: “It is an empirical fact that if an experiment is continually repeated under the same condition, then, for any event A, the proportion of times that the outcome is contained in A approaches some value as the number of repetitions increase. [...] It is this long-run proportion or relative frequency that we often have in mind when we speak of the probability of an event” (p.176).

B.5. Introduction to Probabilities

This book (an MIT lecture notes series) delivers advanced material on probability to an audience of electrical and computer engineering students at MIT. Probability is defined in this book as a set function satisfying certain properties:

“A probability function assigns to a set A of possible outcomes (also called an event) a nonnegative number $P(A)$ (called the probability of A) that encodes our knowledge or belief about the collective “likelihood” of the elements of A. The probability law must also satisfy certain properties such as countable additivity ($P(\cup_i A_i) = \sum_i P(A_i)$) for a countable number of disjoint events, and normality ($P(\text{sample space})=1$)” (p. 6).

B.6. A Basic course in probability theory

The book presents yet another approach to probability that is comprehensible only by means of advanced mathematical concepts.

A measure-theoretic definition of probability appears on the very first page:

“A probability space is a triple (Ω, \mathcal{F}, P) , where Ω is a nonempty set, \mathcal{F} is a σ -field of subsets of Ω , and P is a finite measure on the measurable space (Ω, \mathcal{F}) with $P(\Omega)=1$. The measure P is referred to as a probability. Ω represents the set of all possible outcomes (intuitively speaking), and \mathcal{F} is referred to as the sample space and the elements in Ω as possible outcomes”.

The authors state that a measure-theoretic framework is essential to understanding the works of Kolmogorov which were done in response to David Hilbert's sixth problem (a pledge to axiomatize those branches of physics in which mathematics is prevalent, a list topped by probability theory) and resulted in a systematic approach to probability theory as a mathematical discipline.

2.3.3. Online resources

The two online resources used to study the definitions of probability are

C.1 Wikipedia (Wikipedia Probability Page, 2014)

C.2 Wolfram Alpha (Wolfram Alpha Probability Page, 2014)

C.1. Wikipedia

Wikipedia is not considered a valid and reliable reference for research purposes since anyone can edit and modify its entries, however, for two reasons I include the probability definition from Wikipedia here:

First, based on my personal experience, the mathematical body of Wikipedia is to a great extent reliable and solid. As a user, I have rarely come across incorrect mathematical definitions, examples, or propositions over my six years of using Wikipedia.

Second, it is one of the resources students refer to when in need of help. When looking up the term probability online, Google search engine often puts the Wikipedia page on top of search results.

Wikipedia defines probability as a measure of the likeness that an event will occur, quoting Webster's definition of probability. Then it classifies probability from the point of view of objectivists and subjectivists.

An objectivist would assign probability based on some objective or physical state of affairs. Frequency-based probability and propensity probability are mentioned as two popular versions of the objective probability. Frequency-based probability is based on the relative frequency in the long run of identical iterations of the experiment. Propensity

probability thinks of probability as a physical propensity, or dispositions, or tendency of a situation to an outcome of a certain kind. Propensity probability is distinguished from frequency probability since the former considers each experiment strongly related to a certain set of generating conditions; when we repeat an experiment, we are performing another experiment with a similar (but not identical) set of generating conditions.

The subjective probability on the other hand is assigned as a degree of belief that could be identified based on individuals' betting behaviour, the price at which one would buy or sell a bet under a given set of conditions.

C.2. Wolfram Alpha

The other online resource I have turned to, looking for definitions of probability is Wolfram Alpha computational knowledge engine. Released in 2009 by a team of computer scientists and experts in science, mathematics, and engineering, the website provides specific and case-based (as opposed to links to other websites or online documents) answers to computational queries. For instance, it responds to the question of probability of 32 coin flips by providing a graph of the probability density function (see Figure 2-1), the probabilities of all heads, all tails, half heads-half tail events, and the expected number of heads.

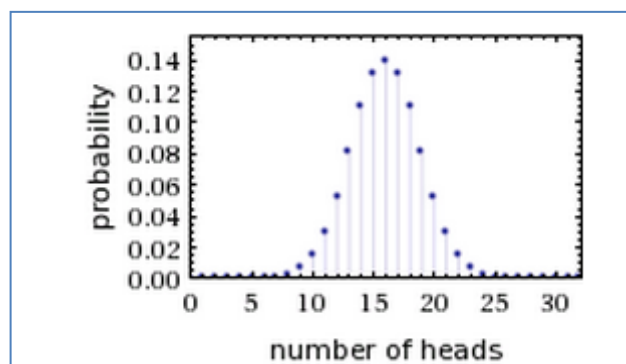


Figure 2-1 The probability distribution of the number of heads in 35 flipped coins.

I have found Wolfram Alpha a great resource to look up theorems and properties related to mathematical concepts. It should be noted that since it is a computational engine (as opposed to an online encyclopedia), the response to a query about a mathematical concept usually doesn't contain a conventional definition of the concept.

Alternatively, the term in question is identified through visuals, examples, and related concepts. I looked into probability search results knowing that a formal definition will not be returned, instead I was interested to know what representations, examples, and probability problems students will see if they search the word probability in Wolfram Alpha.

It turns out that Wolfram Alpha has amazing capabilities in assisting the client with probability related queries at various levels of difficulty. To name a few, it can generate a 1000-step random walk in 2D and 3D, calculate the birthday probability (in a room with n people, what is the probability of getting at least two people with the same birthday), determine probabilities involving coins, dice, and cards, and provides the reader with table of odds from the major lotteries (lotto max and 649). Moreover it produces the probability values (the probability that the random variable X varies between (a, b)) from the known probability distributions including Normal, Poisson, Binomial, exponential, Beta, and Gamma distributions.

2.4. Discussion of probability definitions in resources

In this section I address two issues and discuss each. The first is the lack of definitions of probability of an event within resources. I do not mean to say that the majority of resources do not offer any definitions of probability (as in probability of an event); what I am pointing out is that some resources do not do that. Consider another mathematical notion such as a group, function, or derivative. It is safe to say that the occurrence of definitions of these notions in introductory level resources is quite higher than the occurrence of a definition of probability in similar resources. Next I tend to the diversity of the types of probability as depicted by definitions given in resources.

The different views and approaches to probability through the resources cited previously are summarized in Table 2-1. As seen from the table, definitions of probability through resources are dominated by frequentist and theoretical approaches. Also, very few resources talk about probability as a theory, a model, or a mathematical tool that solves certain types of problems.

Table 2-1 Summary of Different Views

Resource	Theoretical Probability	Frequency Probability	Subjective Probability	Measure-theoretic approach	Probability as a theory	Properties (laws) of probability
A.1. NCTM principles and standards					✓	✓
A.2. BC curriculum documents	✓				✓	
A.3. Van de Walle et all Elementary and Middle School Mathematics	✓	✓				✓
A.4. Beckman Elementary and Middle School Mathematics	✓	✓				✓
A.5. Sowder et all Reconceptualizing Mathematics for elementary school	✓	✓				✓
A.6. Bennett and Briggs Using and understanding mathematics, a quantitative reasoning approach	✓	✓	✓			✓
A.7. Posamentier Teaching Secondary Mathematics						✓
B.1. Freund and Mohr Statistical Models		✓				✓
B.2. De Veaux et all Intro stats	✓	✓	✓			✓

Resource	Theoretical Probability	Frequency Probability	Subjective Probability	Measure-theoretic approach	Probability as a theory	Properties (laws) of probability
B.3. Triola Elementary statistics	✓	✓	✓			✓
B.4. Ross Introductory Statistics		✓				✓
B.5. Bertekas Introduction to Probabilities	✓			✓		✓
B.6. Bhattacharya A Basic course in probability Theory	✓			✓		✓
C.1. Wikipedia	✓	✓	✓	✓	✓	✓
C.2. Wolfram Alpha						✓

2.4.1. Lack of definitions of probability from the resources

In most of the resources I used for this study, the importance of probability as instructional subject matter is emphasized. More often than not examples of the applications of probability in various life encounters are given. However, as indicated in the table, some resources do not give a clear definition of probability (although they go over a wide range of concepts related to probability). I also came across resources that acknowledge how challenging of a task it is to give a concise definition of probability that could be used for practical purposes and, at the same time, holds up to the standards of a mathematical definition and for this reason suggest to treat *probability* just in the same way geometers treat the notion of *point*. For example, Feller (1968) in his introduction to probability theory and its applications suggests that:

We shall no more attempt to explain the true meaning of probability than the modern physicist dwells on the real meaning of mass and energy or the geometer discusses the nature of a point. Instead, we shall prove theorems and show how they are applied. (p. 3)

I suggest three reasons for the lack of definitions of probability in textbooks. The first reason is the everydayness of the term probability. Since it is a term used in normal

language it is treated as a commonly known concept. The second reason is that probability is a difficult concept to precisely define since it has always been subject to philosophical controversy, a situation mathematicians are known to be reluctant to get involved in. Byers (2007) points out: “the business of mathematicians is the doing of mathematics and not reflecting on the subject of what it is that they do” (p. 25). Similar thoughts are offered by mathematician Gowers (2002) . He advocates abstract methods in mathematics and maintains that by applying abstract methods, philosophical dilemmas such as what a number really is, or how can something exist and yet be nothing (referring to zero) will disappear. He suggests that the abstract method can be encapsulated in the slogan: “a mathematical object *is* what it *does*” (p. 16). He contends that asking questions on what a certain mathematical object is, would be similar to asking what a black king in a chess game is. It is a question to be sidestepped and attention must be directed to the rules pertaining to the movements of the black king instead.

A third reason for the absence of a clear definition of probability in the mentioned books is that since the term probability is used both as probability of an event and as the name of a field. Traditionally textbooks do not give definitions of such terms. For example, an Algebra book does not begin with the definition of Algebra and a Geometry book is not expected to define Geometry. True though it is to some extent, we should note that Geometry for instance is the study of properties and invariants of shapes and space whereas probability as a field is about laws of defining and calculating probabilities of events. Perhaps it is a bit unfortunate that the same term is used to describe both the whole study of random phenomena and mathematical approaches to quantify uncertainty (probability as a field) and the numeric measure of uncertainty (probability of an event).

2.4.2. Types of probability as offered by the resources

From the definitions reviewed in Table 2-1, it seems that the theoretical and frequentist view of probability are the predominant definition in resources. Some of the resources however, explicitly define “three types of probability” (A6 and B3 for instance): theoretical probability, frequency probability, and subjective probability.

The formal measure-theoretic and set theoretic definitions of probability given in B5 and B6 do not address the meaning of probability, however if applied to finite sample spaces, those definitions come down to saying: probabilities are numbers between 0 and 1 such that if two events cannot occur simultaneously, the probability of either of them occurring is the sum of the probabilities of each. The measure induced on the sigma field once applied to finite sample spaces would reduce to simple counting measure (the number of desired outcome divided by the number of events in sample space), due to this, the measure-theoretic and set theoretic definitions could be categorized with theoretical probability definitions.

The definition of theoretical probability is one of the two definitions most frequently emphasized through the resources. Enumerating the desired versus possible outcomes seems a straightforward and accessible tool to almost everyone and in finite situations could work just fine. However, the notion of equiprobability of the outcomes has always presented a challenge to this approach. There is supposed to be a finite number of possible outcomes. They are judged “equipossible” and hence “equiprobable”. In Laplace’s terms, “there is no reason to think the occurrence of one of them would be more likely than that of any other” (von Plato, 1994, p. 5). The absolute symmetries of theoretical probability are really hard to come by in the real world. Statistical data show that the real dice and coins are not fair and having a boy is not equiprobable to having a girl. The frequentist approach to probability is also challenged by so many similar considerations. For instance, the premise of a frequentist approach to probability is to have access to infinitely many (or a very large number) of identical experiments and a guarantee that the relative frequency of the desired event will demonstrate a convergent behaviour.

Research in probability education suggests that when teaching probability, all different aspects of it should be considered and interrelated, as students find difficulties in each of these components (Batanaro & Diaz, 2007). In the same way, the different meanings of probability should be progressively taken into account. I end this section by a quote that I think best summarizes this discussion:

The ways in which probabilities are used, in statistical inference and elsewhere, are varied, and they are always open to criticism. We should

guard, however, against the idea that a correct understanding of probability can tell us which of these applications are correct and which are misguided. It is easy to become a strict frequentist—or a strict Bayesian—and to denounce the stumbling practical efforts of statisticians of a different persuasion. But our students deserve a fair look at all the applications of probability. (Shafer, 1992, p. 18)

Given the state of probability definition through teaching and learning resources, it is interesting to examine the meanings the students assign to probability and to see to what extent the institutional and personal meanings of probability are aligned or contrasted. The terms institutional and personal are adopted from Batanero and Diaz (2007), elaborated further upon later in this paper. The institutional meaning refers to a collective meaning of a concept as described and communicated by widely used resources including (but not limited to) textbooks. Whereas the personal meanings of a concept are the meanings individuals form and perceive of the concept as a result of their experience with the concept in everyday life and/or classroom settings. In order to examine the students' personal meanings of probability, I have invited a group of undergraduate students to describe and give examples of probability.

In the following sections, the participants of the study are introduced, the theoretical model is discussed, and the task and research questions are presented followed by analysis of data and discussion.

2.5. The participants

The participants of this study are a group of thirty-three liberal art and social sciences students. At the time of the study, they were enrolled in a course *Mathematical Experience: Numbers and Beyond* offered by the Faculty of Education at Simon Fraser University. The mathematical background of this group of students is not generally very strong. Typically the highest level of mathematics the students have taken before is MATH 190, *Principles of Mathematics for Elementary Teachers*, or grade12 mathematics. In the personal information survey participants were asked to fill at the start of the term, many of them described themselves as “not very confident with

mathematics” and mathematics as being “not their best subject”. The students usually take this course for two reasons: some take it as a “Q-course”⁴ to fulfill the quantitative requirements of their program; some others plan to become teachers and take this course to fulfill the entry into the teacher education program requirement of completing “at least 2 education courses”.

The goal of the course is to explore a variety of mathematical topics related to numbers in order to increase the mathematical literacy of those students, and to increase their capabilities for quantitative reasoning and understanding of numbers in particular. During this course students spend six weeks on concepts from basic number theory, including modular arithmetic, pigeonhole principle, Pascal’s triangle, and Fibonacci numbers. They also spend two to three weeks on rational numbers, irrational numbers, ideas about counting infinite sets, and one-to-one correspondence between sets; subsequently they are introduced to countable infinite sets versus uncountable sets. The last three weeks of this course are spent on probability. The students review the theoretical definition of probability of an event (number of desired outcomes/total number of possible outcomes), and are expected to solve the usual dice and coin probability problems. Through hands-on activities using physical and virtual manipulatives, they explore the experimental probability and apply it to problems, such as the Monty Hall problem and the birthday problem. The focus of instruction in the probability part of the course has been to bring together the experimental probability and theoretical probability. The probability problems mentioned above are approached both experimentally and theoretically.

2.6. Theoretical Model

There is an extensive body of research around probability. Some of the research I am familiar with is problem-specific, addressing conflicts and challenges encountered through famous problems such as Monty-Hall problem. Other researches I have read

⁴ Courses with "Q" designation are designed to assist students in developing quantitative (numerical, geometric) or formal (deductive, probabilistic) reasoning, and to develop skills in practical problem solving, critical evaluation, or analysis.

are concept specific, for example addressing the notion of independent events and conditional probability. There is also a great deal of research done on probabilistic misconceptions, biases, and fallacies: Shaughnessy (1981), Konold, Pollatsek, Well, Lohmeier, and Lipson (1993) , Tversky and Kahneman (1974), and Tversky and Kahneman (1983). Research into learners' explicit definitions and examples of probability is harder to come by. It is also fair to say that the vast majority of research in probability education is conducted around the K-12 age group. Very little work is done with regard to examining undergraduate students' understanding of probability and probabilistic notions. In looking for a theory that has been developed and applied to learners' of higher age groups, I came across a five-component model by Batanero and Diaz (2007). This model is developed as part of a systemic research program for mathematics education at the University of Granada, Spain, and is applied in different works of research in statistics education. Batanero and Diaz (2007) propose five interrelated components of a model to analyze various meanings of probability. Their model suggests that a well-rounded understanding of a concept (probability specifically), should include five elements:

- 1) Knowledge of the field of problems from which the concepts has emerged, in the case of probability, students should learn about games of chance since problems related to chance games were used to develop the first ideas of probability.
- 2) The representations of the concepts: the probability notation, percentages, fractions, frequency tables, and density graphs are among the representations of probability.
- 3) The procedures and algorithms to deal with the problem; in case of basic probability, these procedures are simple counting or using combinatorial tools to calculate the probability of an event.
- 4) The definitions of the concept. These will include the properties and relationships to other concepts, such as different definitions of probability, the numeric properties of it (positivity, additivity, and so on), the relationship between probability and frequency and expectation.
- 5) The arguments and proofs we use to convince others of the validity of our solutions.

In order to refer to these components I have come up with these shorthand phrases for each (not mentioned in the original work): knowledge of the field, representations, algorithms, definitions, and arguments.

This model is used to decide on the task employed for data collection in this study, and also for the initial analysis of the responses. Based on the results of the initial analysis, additional theoretical considerations have been called upon for a more in-depth analysis of the data. I will describe these later in the paper.

2.7. The task

The participants were asked to provide written responses to the following question:

“What is probability? Give an example of the probability of an event.”

Enough time and writing space was given and the responses were collected anonymously.

2.7.1. Some considerations about the task

The task is guided by the five-component model. However, some changes to the original model are made and only parts of it suitable for this study are adopted. The first component of this model (knowledge of the field), informed me as the teacher to engage students in probability problems related to games of chance. The students were introduced to problems that famously played an important part in the development of probability. With regard to the representation, algorithm, and argument components of the model, due to the basic level of the course and the short period of the time allotted to the probability, studying algorithms and arguments was not feasible. The definition component of the model seemed a more helpful tool to address students' concepts of probability. As discussed in earlier sections of this paper, definitions of probability are not as commonly available through textbooks as, for example, the definition of function or limit. Due to presence of the word probability in everyday language, the meaning of the term is often taken as known. While understanding a concept is not reduced to the students' ability to define the word, it is nevertheless important to find out what meanings students assign to concepts. Batanero and Diaz suggest looking into students'

“definitions” of probability (the fourth component of the model). That requires asking the respondents to “define” the term probability.

In mathematics, definitions are rigorous and precise, and the task of defining a term is managed only as a result of the long-term interaction of the person with the concept at hand and concepts related to it. A definition is a final product of a process of inclusion and exclusion of properties as different aspects of a concept are examined and experienced. Students outside of mathematical fields of study are not usually familiar with the requirements of a mathematical definition and if asked to define a term they are likely to try to give a set of properties and features they associate with the concept. Thus, instead of asking for a definition, I asked the participants of the study to describe what probability is and to give an example of the probability of an event. The question in the task is deliberately vague and open to interpretation. For instance the question doesn't ask respondents to describe mathematical probability, nor does it ask to describe the probability of an event.

2.8. Research questions

The research questions pertaining to this study aim at examining how undergraduate students describe probability, in particular:

- What features of probability appear in students' definitions/descriptions of probability?
- What features of probability appear in students' examples of probability?

2.9. Participants' description of probability

As stated earlier, Batanero and Diaz (2007) distinguish between the personal and institutional meanings of probability to differentiate between the meaning that has been proposed for a given concept (probability in this case) fixed by a specific institution (the meaning proposed by the textbook, or the teacher), and the meaning given to the concept by each particular person in the institution.

I considered making such distinction when analyzing participants' responses to the task.

2.9.1. Personal versus institutional probability

My initial analysis of responses attempted to distinguish between the institutional versus personal meanings of probability. It is not easy or possible to draw a clear line between personal meaning(s) the respondents assign to probability and the ones they write down due to their formal probability education. One difficulty is the diversity in the background knowledge of the participants of the study. They have been educated in various educational systems, with difference in curriculum. For instance, not all of them have been educated in British Columbia, not even in Canada. They might have institutional meanings in mind that the researcher could simply take as personal because they are not shared with the researchers' institutional meaning of probability.

I dealt with this problem by limiting the institutional meaning of probability to that mentioned in the textbook used for the course in which the participants are enrolled. *The Heart of Mathematics* (Burger & Starbird, 2010) is used as the textbook. It defines probability as a two-fold concept and offers two definitions of probability:

1) Theoretical probability of event E occurring

$$= \frac{\text{number of different outcomes in E}}{\text{total number of equally likely outcomes}}$$

2) Relative frequency of an outcome

$$= \frac{\text{number of times that outcome occurred}}{\text{total number of times the experiment was repeated}}$$

(p. 586).

I found seven responses that to some extent had the same definition structure as the theoretical probability definition mentioned in the textbook. One response that roughly resembles that of the textbook states that⁵:

$\frac{a}{b}$ ← chance of one of them happening
b ← certain possible outcomes

This definition to some extent adheres to a theoretical view of probability and expresses it as a fraction. The other six responses describe probability as a fraction, which in the outset aligns the respondents' concept of probability with that of the textbook. Responses with no reference to the number of outcomes (desired or possible) or number of experiments performed, had to be deemed personal since they were worded different than the textbook and also did not refer to any of the two different possible ways to define probability (as done in the textbook). The data suggest that most students tend to offer their own description of probability rather than the textbook definition. In search for aspects of probability that fall within the students' definitions, I looked at the data for a second time. In this round, by using two components of the model I looked for representations, manifestations, metaphors, and mathematical primitives. I use the term mathematical primitives, or in short primitives, to refer to concepts that are not defined in terms of previously defined concepts, they are usually understood and motivated by an appeal to everyday experience and intuition.

From the data it seems that the respondents used terms such as chance, likelihood, ratio, fraction, percentage, possibility, and odds to communicate their notions of probability. Table 2-2 summarizes the terms that the respondents used to define or describe probability:

⁵ The excerpt is visually enhanced by writing over it.

Table 2-2 The frequency of the key words in definitions

	Definitions
Fraction or ratio	7
Percentage	6
Likelihood	7
Chance	17
Mathematics	3
Possibility	1
Odds	2

In the first round of data analysis only seven responses were counted as mathematical since I was looking for a fraction style definition that was similar to the definition given in the textbook. The second round of analysis brought to my attention the mathematical terms that were used in seemingly personal definitions. Therefore I categorized more cases as mathematical. The results are presented in Table 2-3.

Table 2-3 Types of probability from data

Mathematical (institutional) probability n=11		Everyday (personal) probability (n=22)
Probability as a fractions (n=7)	Probability as a percent (n=6)	Probability as chance (n=17) or likelihood (n=7)

As a field of study, probability is a mathematical approach to impart structure and order to our vague or incomplete knowledge of random phenomena in a way that enables us to quantify certain aspects of the variation characteristic to the probabilistic matter at hand in order to make decisions based on our knowledge of the situation and the modelling capabilities of the underlying theory.

As a more specific mathematical object however, the probability of an event has certain mathematical features. For instance, it is a non-negative number, and it is less than or equal to one; it is additive over disjoint events and multiplicative over independent events. I tried to extract from the data the features of probability, either proposed as a defining feature, or just as a property.

In some of the probability definitions obtained from such data, the relation between the numeric probability and the likelihood of the event was described as a defining feature of probability:

“Probability is the chances that an event will happen or not happen expressed as a fraction or percentage. The closer this is to 100% (1), the more likely it is. The closer this is to zero, the less likely it is to happen”.

In this definition, probability is viewed as a tool to quantify an intuitively understood notion of likelihood, in the same way the red liquid in thermometer quantifies and determines the temperature. However, it is not clear how this number-likelihood correspondence works how the probability is calculated.

Probability as percentage

Theoretical probability is often defined and calculated as a fraction, it is possible to convert any fraction to percentage but except for fractions such as $\frac{1}{2}$, $\frac{3}{4}$, that have well known percentage equivalents, the theoretical probability is expressed in fraction form. Similarly, the probability of an event calculated via an experimental approach can be expressed both as a fraction and percentage. When virtual manipulatives are used, the experimental probability is typically obtained and expressed as a decimal number that is usually reported and referred to as a percentage. That is because most soft wares available for such purposes (for example, the manipulatives available at the National Library of Virtual Manipulatives and the software Tinkerplots) calculate the probability as a percentage.

Moreover, students learn about percentages early in grade 5 or 6. They use them frequently in the context of discount, tax, tip, and expressing their degree of belief in some event. Students learn to use percentages to refer to chance events before they are introduced to mathematical probability and the colloquial use of probability is often in the context of a percentage.

The participants of this study spent some time with the Excel randomizer function and calculated the experimental probability involved in a couple of simple tasks. As

Excel could be used as a calculator, as well as a sheet to record the results, students calculated and recorded the ratio of desirable outcomes to the total number of iterations of the experiment in the same document. The result of such calculation is returned in a decimal form or a percentage (depending on cell formatting).

Either due to their early, pre-instruction ideas of probability or due to their recent experience with probabilities that are calculated as percentages, the respondents of this study, in six cases, defined probability as being the percentage of an event. Here are two examples of describing probability as a percentage from the data:

“Probability is a percentage that either something happens or not”

“Probability is the percentage that you have to get a certain outcome”.

Although cast in terms of a percentage, the second definition uses the idea of an outcome, which seems more like the mathematical probability. The first seems more associated with everyday events. Drawing a line between mathematical and everyday probability is difficult and perhaps cannot be done objectively. Identifying mathematical terms in responses and using them as inclusion criteria helps the matter to some extent but as seen from the two responses above, it is not a clear cut.

Probability as fraction

As stated before, a fraction is one of the common forms of expressing probability in both theoretical and experimental approaches. The review of resources indicates that where theoretical or frequentist probability is defined, it is in a form of a fraction (except for the measure-theoretic approach).

Seven participants identified probability as a fraction either by using the word fraction or by writing a fraction:

*the chance that an event will happen (or not happen),
expressed as a fraction or %*

(In my count of types of probabilities, this response is counted both as a fraction and percentage). Another example (seen earlier) in which the probability is clearly defined as a fraction is:

$\frac{q}{b}$ ← chance of one of them happening
b ← certain possible outcomes

Again, though both involving fractions, the second seems more mathematical in that it talks about outcomes.

2.9.2. Everyday meanings of probability as described by participants

Words related to probability are ubiquitously used in everyday language among English speakers. Young kids learn to use words such as chance, random, probably, sure, unlikely, impossible, and probably before they come to school. The high occurrence of the word “chance” in participants’ description of probability is noticeable and is attended to here.

Probability as chance

Lakoff and Johnson, in *Metaphors We Live By* (Lakoff & Johnson, 2003), suggest that our conceptual system is largely metaphorical, and the way we think, and what we experience, is pretty much a matter of metaphor. They suggest that since communication is based on the same conceptual system that we use in thinking and acting, language is an important source of evidence for what the system is like.

Lakoff and Johnson (2003) use chance and games of chance as basic metaphors through which other concepts are described and understood. For example, they give instances of how life is metaphorically referred to as a game of chance or a gambling situation, used in expressions such as “I’ll take my chances”, “The odds are against me,” “He’s holding all of the aces” “Play your cards right and you can do it”. These chance metaphors are so commonly and frequently used that they have become

fixed-form expressions, and although these phrases are instances of life viewed as a game of chance, they are used to speak of life in a normal, everyday manner.

The word chance in everyday language is used to describe or to refer to a vast number of situations: chance could be used to refer to a possibility that something will happen, an opportunity that occurs without prior expectation, any occurrence of events in a manner that is not entirely deterministic, or a sequence of events that are not related in a causal way or caused by the same common reason in an obvious way. The word chance is also used to indicate lack of information about a situation or lack of a logical design.

The participants of this study made frequent use of the word chance to describe and define probability. In twenty two of the total thirty three responses, probability is defined as the chance or likelihood of an event (or “something” as seen below).

probability is the chance of something.

In this response, the need for defining “chance” is not felt and it is referred to as a commonly understood notion. However, since probability is directly equated to chance with no additional conditions or specifics it is not clear if the respondent considers probability just as a fancy word for chance or if probability is in some ways different than of chance.

probability is the number of chances for or against something

In this response, chance and probability are not used in an interchangeable way. It is stated that probability is the number of chances, hence once the notion chance is quantified or enumerated, it is called probability.

Unlike in Lakoff and Johnson’s “Life is chance”, in which life is likened to chance games, I do not think the participants have used chance as a metaphor for probability.

According to Low (1988, p. 126), a metaphor is a reclassification that involves “treating X as if it were, in some ways, Y”. The participants of this study define the probability as the chance and not resembling chance in a metaphoric way. They use the everyday notion of chance, which they are familiar with to describe the more technical and field specific term probability. That being said, the repeated use of the phrase “*probability is the chance*” in respondents’ definitions suggests that chance is a commonly understood conceptual notion, and perhaps, a primitive idea that is already available to the learners without need for further clarification, and they conceptualize probabilistic notions by lining them up with their prior experience with chance.

2.10. Participants’ examples of probability

Including a request for examples in the task was almost arbitrary and coincidental rather than an informed theory-based decision. In Batanero and Diaz’s model there is no specific attention to learners’ examples of a concept. The model emphasizes the importance of looking into students’ definitions of probability, the thought processes available to the students when working on probability tasks, and how probability related notions are connected to other mathematical concepts. But examples of probability and how important it is to look into the students’ examples of probability is left out. After the first round of analysis of examples it was evident that there is a need for a more in-depth analysis since the examples not only complemented and enriched the definitions but also added a new perspective on participants’ notions of probability. Therefore I present some additional theoretical considerations to elaborate on my analysis of the data.

2.10.1. Additional Theoretical Considerations

The framework used to analyze the data is a tool for analyzing personal or collective example spaces adopted from Zazkis and Leikin’s (2008) work on exemplifying definitions. Zazkis and Leikin analyzed learner generated examples of definitions of a square. They invited the participants to generate as many examples as possible of definitions of a square; they analyzed the responses using three lenses: accessibility and correctness, richness, and generality. In accessibility and correctness

category, they considered whether examples satisfy the conditions of the task and whether they are generated with ease or struggle. They also considered the mathematical correctness and attended to the logical structure of the definition and minimality. In the richness category, they considered whether the examples vary in type and structure, and whether they are situated in a particular context or drawn from a variety of contexts. For example, a definition that attends to properties of a square other than sides and angles was considered rich. In the generality category, Zazkis and Leikin considered whether the examples are general or specific.

Those categories are slightly modified for the purpose of this study. Zazkis and Leikin had 155 learner-generated examples to work with. Each of their respondents had generated several examples, which made it possible to address the diversity and variety of the example context. In my study, each person generated a maximum of one example, which made the investigation of richness category impractical. Also, since the examples I received were preceded by definitions of probability, it seemed appropriate to look into how individual definitions and examples of probability are linked. Hence, I have analyzed the examples from three perspectives: correctness, generality, and relation of examples to the provided definitions/description.

2.10.2. Data

Analysis of the examples suggests that some of the respondents had a hard time coming up with an example of the probability of an event. The research suggests that generating examples of a concept is far from a trivial task. Zazkis and Leikin point out that the task of generating examples of a concept can be quite a complicated task in which the student experiences dimensions of possible variations associated with the defining features of the concept in question. From the thirty-three responses collected, only twenty-four examples were obtained. Nine students did not provide any examples.

2.10.3. Analysis of data

The data was collected anonymously therefore I could not follow up with the students who did not write an example. The analysis of the examples is conducted based on the twenty-four examples I received.

Correctness of examples

Correctness of the examples are determined based on these criteria: the probabilities are given as numbers (as opposed to described as higher or lower) and correctly assigned (based on laws of probability). Here is an example of a correct example from the data:

Handwritten student response: "ex. what are the chances of me getting 'heads' when I flip a coin? $\rightarrow \frac{1}{2}$
Each time I try my prob. of getting 'heads' stays the same. 50%"

This response highlights some important properties of probability such as its reliance on the conditions under which the experiment is performed, the number of repetitions, and the number of outcomes in each repetition: "each time I try my prob. of getting 'heads' stays the same" relates to both the frequency and theoretical definitions of probability at the same time since the probability presented in this example is both obtained over repeated trials and has the aspect of constancy to it.

Another example of correct probability examples from data is:

Handwritten student response: "If I roll 3 dice, there is a 100% chance that the sum of all my rolls will be more than 2 and less than 19."

A total of 10 examples were considered as mathematically correct, by which I mean the event in question was clearly described and the probability assigned to the event was conventionally correct. The correctness of the other 14 examples could not be verified mostly due to one of the two reasons: either the probability in question was not calculated or the event in question was too general and henceforth the probability could

not be assigned properly. In examples where the probability was not calculated the probability was sometimes described as “high”, or “small” as in the following:

winning is very small but there's still a chance. the probability of

Here is another example⁶:

probability of it raining tomorrow is high.

Some other examples in this category (perhaps due to misreading the task) provide an example of an event and a probability related question with regard to that event but make no comments on the probability in question.

e.g: what is the probability if you pick a heart out of a normal set of cards (52 cards)

Some of the examples in this group only contain a probabilistic situation and no specific event or probabilities:

ex. rolling a die

Generality of examples

Generality is often a valuable feature to encounter in student solutions, definitions, examples, and explanations of why certain things work and why they do not. However it should be treated with care since if not accompanied with the required specifics of a notion (definitions or examples) it may indicate lack of adequate knowledge about the concept. For instance, as stated in Zazkis and Leikin (2008), if a

⁶ The excerpt is visually enhanced by writing over it.

student defines square as a shape with four sides, the definition, although a general one, does not include the essential features of a square and therefore is not completely correct. In probability examples students tend to assume a lot and do not verbalize the specific setting to their intended probabilistic situation and event. Such examples are too general since the specifics such as the conditions or assumptions under which the probability is assigned are missing. For example a response from data states that:

“The probability that it rains today is 80%”

The specifics of assigning an 80% probability to the event of rain are not mentioned. We don't know why it is so, nor do we know what is exactly meant by rain. Does any precipitation that lasts more than one minute count or should it be for a minimum of 2 millimeters? Such examples could be converted to specific and mathematically correct examples of probability if they were accompanied with the conditions surrounding the experiment. For example, if the example above is worded as: the chances of rain on a specific day in a specific city with a certain weather history while having knowledge of weather for the past couple of days is estimated to be 80%, the validity of the probability could be verified since we are presented with the required information to build a probability model around the random experiment. The point here is that examples like this cannot be deemed incorrect since they are subject to personal inferences upon the person's historical memory of the weather. However, without the background information this type of examples is categorized as too general and their correctness is not verifiable. Another example states that

“There is a very low probability of 1/100000 to win Bingo”

Once again it is not specified how this number (1/100000) is reached at, since the idea of winning is not defined by the respondent. For instance, one may ask if getting three numbers right is winning.

In categorizing the examples into general versus specific, I noticed that the context of the examples relates to the specificity of the examples. The examples obtained from the data represented both situations from everyday life and from mathematics.

Five examples were about weather forecast and lifestyle choices (coffee or tea), fifteen examples involved coin flipping, dice rolling, or card game playing, and four examples involved picking random numbers or objects. The examples from everyday life tended to be more general in the sense that the specific underlying assumptions or conditions are not made known as shown before. The examples involving picking numbers and gaming situations (coins, dice, etc.) proved to be more specific:

ex] ~~There is a 1/25 chance that~~ ~~between the whole~~
#s 1-25, there is a $\frac{1}{25}$ chance I will pick the number 3

Also it is noticeable that in examples involving games of chance, more accurate language is used. Here is an example from the data:

"Probability of rolling a 4 on a standard die is $\frac{1}{6}$ "

It is specific about the type of die (standard) and the outcome. Another example states that:

"The chances of me getting heads when I flip a coin is $\frac{1}{2}$, and each time I try, the probability of getting heads stays the same"

Some examples only refer to the outcome without referring to the probabilistic situation or the process involved.

Relation of examples to definitions

When definitions of probability together with examples of probability given by each student are tabulated, it is possible to consider the match or mismatch between definitions and examples, to see if the example given actually satisfies the definition, if they follow the same logic, and if they both have the same mathematical structure to them. From Table 2-4 it is apparent that examples are more mathematical since more fractions and percentages are used to generate examples of probability compared to definitions of probability.

Table 2-4 Keywords used in definitions and examples of probability

	Definitions	Examples
Fraction or ratio	7	11
Percentage	6	11
Likelihood	7	0
Chance	17	5
Mathematics	3	0
Possibility	1	0
Odds	1	0

Compared to definitions, it is noticeable that examples are better worded, more specific, and more mathematically structured. There are instances of good definitions followed by equally good examples obtained from the data—though not very many of them. Here is an example (repeated in part from earlier):

Probability is the chances you have to get a desired response. This will depend on how many possible outcomes you have, how many times you get to try to get the effect, and whether or not your chances stay the same each time.
ex. what are the chances of me getting 'heads' when I flip a coin? $\Rightarrow \frac{1}{2}$
each time I try my prob. of getting 'heads' stays the same. 50%

And another one:

probability is the chance of obtaining a certain order/value/number/choice/object within a set amount of object (2) → 2 outcomes (heads and tails)

(1) → possible (heads/tail)

ex. what is probability of ~~having~~ flipping heads in a coin toss = $\frac{1}{2}$ chance/50% chance

A larger number of students that had difficulty in producing a definition gave examples that showed a better understanding of probability. The probability definitions offered by this group of students typically are general and vague. One might think that based on a definition like this, the student's knowledge of probability must be very limited. The examples, however, show that the student – although not capable of elaborating on the properties and features of probability – is familiar with the concept and its defining features. Here is an example:

It is the mathematical reasoning for chance.
 For example: I have a pen, eraser & keys in my backpack. the probability of me pulling out keys is $\frac{1}{3}$.

“A mathematical reasoning for chance”, however correct, is not laying out any of the characteristics of probability. This description of probability pertains to probability theory (the discipline) and does not specify what types of “reasoning” probability theory will employ to make sense of chance. The example given after the definition, on the other hand, involves an experiment with three presumably equally likely outcomes. Note that the response above puts three small objects inside a rather large space to create an equally likely situation, and assigns a numeric probability to the event of one of the objects pulled out.

Also the mathematical notion used (if any) in example construction tends to match that of the definition. For instance all of the students that have introduced probability as a percentage also gave example of probability in percentage form:

“Probability is a percentage that either something happens or not. Example: 98% probability of rain forecast for tonight”.

Another such example from data:

“Probability is a percentage representation of possible outcomes. If I roll three dice there is 100% chance that I’ll get a total between 2 and 19”.

An example that makes use of fractions both in definition and the example part of the task:

“Probability can often be represented as a fraction, for example there is a one in three chance of winning a game”.

2.11. Discussion and Conclusions

This study primarily aimed at looking into features of probability appearing in students’ definition of probability and features of probability appearing in examples of probability. To answer the first research question I should first mention that mathematical probability (as in probability of an event) has three essential features, regardless of the type of approach, and some additional properties. The defining features are:

- 1) It is a number and it is between zero and one inclusive
- 2) The probabilities of empty set and the set of all events (sample space) are zero and one respectively.
- 3) The probability of a countable union of disjoint sets is equal to the sum of individual probabilities. The additional properties include the inclusion-exclusion law, product rule for independent events, complementary events, and Bayes’ formula for conditional probabilities.

The analysis of data suggests that the first defining feature is a well understood property of probability and it serves as a shared basis for participants’ perception of mathematical probability.

Of note, the second and third features, although essential in mathematical definition of probability, were not mentioned in participants' definitions. These features were not directly sought by the task; therefore the fact that they did not make it to the participants' definitions of probability indicates that they are not perceived as essential and indispensable to the definition.

Also, in the data, there is neither a mention of different definitions of probability (frequentist, theoretical, and subjective probability), nor a deliberately made distinction on everyday versus mathematical probability. To participants of this study, the two terms probability and chance are closely linked and they are both used to make sense of non-deterministic states of events. This may indicate that the respondents consider mathematical probability well aligned with the everyday probability and consider both as an unambiguous well-defined concept.

Answering the second question of this research, the features appearing in examples of probability are found to be similar to those in definitions. The examples are more elaborated and more specific; therefore the numeric aspects of probability are highlighted in examples.

Also, analysis of the examples suggests that the everyday notion of probability predates and predominates students' conception of mathematical probability. As evidenced by the examples from the data, the respondents apply probability to real life occurrences as well as games of chance and mathematical situations. They tend to align probability with chance even though probability is a more technical and a less colloquially used term and that probability is a mathematical construct to quantify certain aspects of chance and random phenomena.

Analysis of student-generated examples of probability suggests that generating examples is a complicated task that makes available to the researcher some new aspects and features of students' awareness of the concept. If students' understanding of probability could metaphorically be likened to an ever changing picture that is partly visible to the teacher, the participants' examples of probability is what clears up the picture and reveals larger parts of the image viewed previously in light of their definitions of probability. The juxtaposition of examples with definitions of probability can inform

both the learner and the instructor of the aspects of the concept that may not be inferred upon solely by means of definitions. For instance some participants that seemed to have no difficulty giving a definition of probability were not able to give an example of the probability of an event. The difficulty the student encounters with the example generation part of the task, informs her of the limitations of her understanding of the concept. On the other hand, a student who struggles with putting a concise and comprehensive definition together and thus is portraying a poor understanding of probability may find it easier to exemplify the defining features of the concept in an example and hence score higher on probabilistic understanding scale.

From the discussion above I suggest that the model developed by Batanero and Diaz could be extended by adding an inquiry into students' examples of probability. As per the discussion, when asked to provide examples of a concept, students will have to extend their reach for the properties of the concept and its association with other concepts at a greater depth that adds dimension to their understanding of the concept. Also generating examples of a concept facilitates the move from general and vague to specific and precise and thus can complement students' definitions and descriptions of a concept.

2.12. Directions for future research

The examples obtained in this study represent finite situations only. That is to say the number of possible outcomes and the number of repetitions of the experiment are both finite. Infinity appears neither in the construction of the sample space nor in the number of samples required to be taken. Considering that the respondents have had little experience with infinity, it is well expected, nevertheless, it seems an interesting possible research direction to take: to see if a more mathematically advanced group of respondents would make use of infinity involved situations with their examples of probability. Also within several examples of probability provided in response to the task, 50% probability made the most appearance and very few examples were close to the endpoints of a probability continuum; that is to say, there were very few examples with 100% probability and no examples of zero probability. Thus another interesting

investigation could be to look into examples of specific extreme probabilities: zero and one.

Chapter 3.

How impossible is the impossible?

3.1. Introduction

In this paper I examine prospective secondary school teachers' understanding of extreme probabilities (zero or one probabilities) through their examples. I discuss the ambiguity arising from everyday notions of probability and mathematical probability and use extreme events (events with probability zero or one) to elaborate on this matter. The examples are used to draw conclusions about how an everyday notion of probability is aligned or is in conflict with mathematical probability. In this study, prospective teachers from a diverse educational background were asked to provide examples of probabilistic situations. The participants were asked to provide examples of 100% probability events and examples of zero probability events. Within the data, I looked for what makes an event impossible, and why an event is perceived as certain.

3.2. About extreme events

I present some textbook terminologies and examples related to extreme probabilities, then discuss why extreme probabilities are of special research interest to me.

Extreme events are those with probability zero or one. The word impossible may seem an apt choice for events with zero probability of happening since this is the word used in everyday English language to refer to the lowest degree of plausibility. However, for reasons discussed below, in the context of mathematical probability it may mean something different. To avoid the confusion, I refer to an event E with probability $P(E)=0$ as a zero-probable event. Likewise, I refer to an event E with $P(E)=1$ as a one-probable

event. The two notions of zero-probable and impossible do not completely coincide and various words such as almost impossible, improbable, implausible, highly unlikely, and unusual are used to address different aspects of zero-probable. In this study, I investigate both ends of extreme probabilities; however, at times the focus is on the zero-probable end more than the other. That is because zero-probable and one-probable events are complementary events, the opposite ends of probability continuum, and thus are very similar in properties. Due to this duality, once the zero-probable is discussed, the reader is able to draw similar conclusions about one-probable events.

As a first step, I need to make a distinction between several everyday concepts and their mathematical counterparts. To this end, I present both everyday and mathematical definitions of impossible and certain.

3.2.1. Everyday impossible/certain

The first three meanings given for the word *impossible* from the Merriam Webster online dictionary are: “unable to be done or to happen, not possible, very difficult”. Collins’ online dictionary defines impossible as: “incapable of being done, undertaken, or experienced, incapable of occurring or happening, absurd or inconceivable; unreasonable”. The word carries various levels of meaning from not being able to happen, something that defies reason and sense, to something that is very difficult. Using the word impossible to express the degree of excitement about an occurrence (which has occurred and thus is not technically impossible) is not uncommon.

The dictionary meaning for the two terms *sure* and *certain* are:

Sure: confident in what one thinks or knows; having no doubt that one is right.
Certain, convinced, definite.

Certain: known for sure; established beyond doubt, unquestionable, indubitable, and undeniable.

3.2.2. Mathematical impossible/certain:

In probability texts, an impossible event is usually defined as an event that can never happen and, subsequently, as an event that will have zero probability of happening. Likewise, an event is sure or certain (I keep using both words sure and certain since there is no consensus within the textbooks I have come across about which word to use) when we have reason to believe that it will certainly happen and thus $P(A)=1$. It is also one of the axioms of probability that the probability of the sample space is equal to one, therefore the sample space is sure to happen.

I wish to emphasize that in mathematics an impossible event is not defined as an event with zero probability (same with sure events and probability one); the fact that impossible events can never occur is an a priori assessment of their likelihood, which results in assigning probability zero to them.

Thus, although there is considerable overlap between the notions of impossible and zero-probable, the two are not congruent. For example, it is impossible to roll a seven with a standard die, thus the probability of rolling a seven is zero. But the probability of picking a rational number from the set of all real numbers is zero without the event being conceptually impossible. As with the two examples, an event has to be zero-probable if it is impossible, but an event can be zero-probable without being impossible. This distinction is not always emphasized or even made in mathematical textbooks, which may result in an interchangeable use of the two terms. Personally, as an ESL learner, I came across the word impossible in the context of mathematical probability before encountering the word in everyday context. Consequently, I used to think the term impossible meant zero-probable.

I present here a couple of textbook definitions of impossible events and the way the books have handled the subtle difference between impossible and zero-probable. I only mention the textbooks that do attend to the task of defining the terms impossible and/or certain events.

3.2.3. Definitions of impossible and certain events in probability textbooks

I present examples from two groups of probability textbooks: (1) books that have carefully made a distinction between impossible events and zero-probable events, and (2) books that have mixed the two up.

The first group. One of the texts in the first group is *Elementary Statistics* (Triola, 2011). Although the book does not explicitly discuss the difference between zero-probable and impossible, it introduces both concepts. The first instances of *impossible* and *certain* are presented in this example: “It is an impossible event for Thanksgiving to be on a Wednesday. And it is certain for Thanksgiving to be on Thursday. That is because Thanksgiving falls [by definition] on the fourth Thursday in November” (p.144). The book adds that, “when an event is impossible, we say that its probability is zero. When an event is certain to occur, we say its probability is 1” (p. 148). Two more examples of such events are mentioned in the same chapter: “It is impossible to get five aces when selecting cards from a shuffled deck. When randomly selecting a day of the week, you are certain to select a day containing the letter y” (p. 151). Triola continues: “When an event is impossible, the probability is zero”. This approach suggests that before we can assign zero probability or probability one to an event we should come to consensus of some sort on the impossibility or certainty of the event in a subjective way.

A textbook that has made a clear distinction between zero-probable and impossible is *Statistical Mathematics* (Aitken, 1947). The book tries to overcome the ambiguity by limiting definitions of impossible and certain to finite sample spaces in order to resolve the issue of zero-probable-but-conceptually-possible events. It offers: “If the number of events in sample space is finite and if event E must inevitably happen in all of the n ways, then $p=1$ [probability of E] and E is certain, while $q=0$ [probability of the complementary event] and E is impossible” (p. 13). Aitken adds that if the sample space is infinite such as “when the system results in events expressible by a continuous variable, we must not suppose that $p=1$ implies certainty, or $p=0$ impossibility. For example, if a point is taken on a line segment, the chance of a particular point P being taken is 0; but some point is taken, and so the point P cannot be regarded as

impossible” (p. 14). The book suggests using the term “almost impossible” for events with probability zero that are not conceivably impossible.

DeGroot and Schervish, in *Probability and Statistics* (2012), introduce impossible as an empty set and prove as a result of the three axioms of probability that the probability of an empty set (impossible event) is zero. They emphasise that although the probability of an impossible event is zero, but “probability zero does not mean impossible” (p. 20).

Some textbooks not only acknowledge this ambiguity but also try to address and clarify the issue by proposing terminologies for various types of zero-probable events. Here are two examples:

- In the book *Measure Theory and Probability* (Adams & Guillemin, 1996) the terms impossible, implausible, and improbable are distinguished as:
- In a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω denotes the sample space, \mathcal{F} the σ -algebra containing events closed under countably many set operations, and μ the probability measure $\mu: \mathcal{F} \rightarrow [0,1]$
 - An event $A \subseteq \Omega$ is impossible if $A = \emptyset$
 - An event $A \subseteq \Omega$ is implausible if $A \notin \mathcal{F}$
 - An event $A \subseteq \Omega$ is improbable if $\mu^*(A) = 0$, where μ^* is the outer measure

In basic terms, if the event contains no outcomes or no intersection with the sample space it is called impossible; if the event is not part of the sigma-field (i.e. cannot be expressed as finite union and/or intersection of events in sample space), it is called implausible and if it is zero-measurable it is called improbable.

A similar approach is commonly adopted in measure theory textbooks; they use the terms almost certain (sure) and almost impossible for events that are zero- or one-probable events, but are not impossible in the sense they could never happen. In measure theory the term almost everywhere (a.e) is a commonly used term to indicate a property that holds true everywhere except for a zero measure set (Halmos, 1974). In some probability theory and model theory textbooks, the same notion has been applied to make a distinction between probability zero and something that never happens (see Stroock (2011), for example). Similarly, it could be used to differentiate between an

event that has a probability one and the event that always happens. For example, imagine that a needle is dropped to the floor in a room with a square floor. Due to gravity there is a guarantee that the tip of the needle will land on some point on the floor. No other alternative is imaginable. If a line (of infinitesimally small thickness) is drawn on the floor, the needle will almost never land on the line. Probability calculation will assign zero to the likelihood of such an event since the measure of the points on the line compared to the measure of the points on the floor is zero, however this event is not logically impossible.

The second group. Here I present examples of resources that use the word impossible to refer to all kinds of events with probability zero and the words certain or sure to refer to events with probability one.

Ask Doctor Math, a popular mathematics forum supported by University of Drexel, responded to a question about statistical impossibility by stating that: “An event is impossible when its probability is zero” (Math Forum- Ask Dr Math, 2007) .

Sowder, Sowder, and Nickerson (2010), in *Reconceptualizing mathematics*, define impossible events as: “the probability of an impossible event is zero” which is a fair statement, but elsewhere they ask the reader to verify if a given event is impossible. To this end the question suggests to determine if the probability is zero:

“Is it impossible? That is to say is probability zero?” [chapter 27, p. 22]

The authors of *Elementary and middle school mathematics* advise teachers to introduce probability via the probability continuum, a line segment drawn on the board with impossible and certain on the two ends, which later maps into zero to 1 probabilities. The overlap between impossible-certain and zero-one probabilities is implied (van de Walle, Folk, Karp, & Bay-Williams, 2011).

From the examples above, it may seem that this is a problem found only in elementary-level books or those written for teacher education purposes. However, it is not uncommon to encounter the same issue in more advanced probability books. For example, in *Probability and Statistics, the Science of Uncertainty* (Evans & Rosenthal, 2010), an undergraduate level textbook, a probability measure is introduced as satisfying

the three axioms of non-negativity, unit property, and countable additivity. Then it is stated that: “The first of these probabilities says that we shall measure all probabilities on a scale from 0 to 1, where 0 means impossible and 1 (or 100%) means certain” (p. 5).

3.2.4. Extreme events from different probability perspectives

Here I present a more in-depth discussion of extreme events and how different approaches to probability result in different understandings of zero probable events. I look into the types of events that are dubbed extreme in each of the three well-known approaches to probability: subjective, theoretical, and frequentist. Chernoff and Sriraman’s (2014) edited book on probability education research points out frequently that these three conceptions of probability dominate teaching and learning of random and stochastic phenomena. Most probability researchers and practitioners agree on the theoretical and frequentist notions, but the subjective probability has a more diverse range of meaning. It may refer to informal, naïve, and sometimes incorrect notions of probability (Jones, Langrall, & Mooney, 2007) or it may refer to a sophisticated axiomatic approach to probability developed in the early twentieth century that quantifies an individual’s degree of belief in the truth of propositions (Jeffrey, 2004) conjoined with alternative tools and approaches commonly known as Bayesian models (Lindley, 2014). Chernoff (2008) suggests distinguishing between the two meanings by referring to the former as personal probability and the latter as subjective probability. In this study, I use the term subjective probability for all of the types of probabilistic inferences derived based on personal tendencies and degrees of beliefs at all levels since the same idea is behind both, rising from basic, crude, and qualitative to advanced, specific, and quantified. In making this choice, I draw an analogy with the use of the same name, geometry, to refer to all levels of it from the kindergarten-level study of shapes, to the research level study of the invariants of shapes in complicated spaces.

Extreme events in subjective probability

In subjective probability an event A is zero-probable if a rational person (based on sound deductions drawn from valid premises) holds firm beliefs that event A will never happen. For instance, if a person believes that they never win the lottery, the

probability of winning the lottery is subjectively zero, and hence the event of winning the prize is considered impossible. A subjectively certain event is an event that a person has maximum degree of belief in its occurrence. Not winning the lottery would be an example of a certain event for the person above.

In *Understanding Uncertainty* (Lindley, 2014), the author states that: “almost all thinking people agree that you should not have probability 0 or 1 for any event, other than one demonstrable by logic, like $2 \times 2 = 4$ ”, (p. 91). The rule that denies probabilities of 1 or 0 is called Cromwell’s rule, named after Oliver Cromwell’s appeal to the Church of Scotland to “think it possible you may be mistaken [for assuming that their actions are driven by the Word of God]” (ibid, p.91). The Rule demands that you should not assign probabilities of 1 to any event that is not demonstrable by logic to be necessarily true, and never to assign probability 0 to any event, unless it can be logically shown to be false. From a Bayesian perspective (the mainstream school of subjective probability), if the a priori probability of an event ($P(A)$) is perceived to be zero, it cancels the effect of all of the evidences, meaning that the posterior probability of A, conditioned to any evidence B, $P(A|B)$ will be zero based on Bayes’ formula: $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$. It means that if someone holds firm beliefs on the impossibility of an event, no amount of evidence or discussion will make a change in their prior. For example, let A be the event that John has committed the crime and B be the event that the fingerprint found on the crime scene matches his. If the prior probability of A is believed to be zero, that is, if John is firmly believed to be innocent of the crime, even a 100% fingerprint match (the evidence) will not change the posterior probability ($P(A|B)$) of John being the guilty party since:

$$P(A|B) = \frac{P(B|A) \times 0}{P(B)} = 0$$

Extreme events in frequentist probability

The frequentist and theoretical view offer very little to supplement our understanding of extreme events. In the frequentist view, probabilities are the stabilized asymptotic frequency of events as the number of independent trials tends to infinity. In this view the probability is defined as $\frac{\text{number of desirable occurrences}}{\text{number of trials}}$ when the number of

trials is very large. This adds to the ambiguity of extreme events even more so: consider event E with $P(E)=0$. Exact zero is only obtained when the numerator is zero; while approximate zero is obtained when an event happens so infrequently that the numerator is dominated by the denominator. Thus, nearly impossible events could be mixed up with zero-probable events. In theory, since the denominator is infinity, the numerator can be any fixed real number, which leads to getting probability zero even for events that are conceivably far from being implausible. In practice, an infinite number of trials is not feasible and we bring to our probability calculation the notion of significant figures and whether the probability is *close enough* to zero or not. For example, the event of *getting all heads* in the experiment of *flipping a fair coin 500 times*, has a theoretical probability of $(\frac{1}{2})^{500}$, which is equal to 3.0549E-151, a number that is expressed as “zero” for any degree of precision below 150 decimal digits, a good enough approximation of zero for all practical purposes. So in a frequentist approach, we may get events with probability zero that are not logically impossible.

In frequency-based probability, an event A is sure to happen if the relative frequency of its occurrence is either equal to one (something that consistently happens all the time), or very close to one. Another way to obtain sure events in frequency method is to ensure the ratio of success to total is approximately 1, within reasonable significant figures. For example, the event of “not getting 500 heads” in 500 flips of a fair coin has a probability of one, accurate to 151 significant figures.

Extreme events in theoretical probability

In theoretical probability, which is based on dividing the sample space into equally likely events and counting the number of elements in event A, an event A is zero-probable if it contains no elements; in other words, its intersection with the sample space is the null set. For example, if a fair standard die is rolled, the event of rolling a 7 is impossible since no subset of the sample space contains 7. In experiments with discrete finite sample spaces, such as the die rolling example, a zero-probable event is equivalent to an event that is logically impossible. In other words, the set-up of the experiment defies the event. For instance, a fair standard die conventionally bears the numbers between one and six, therefore rolling a 7 is impossible. Contrariwise, in

situations involving infinite sample spaces, it is possible to construct examples of an event that has a theoretical probability of zero, but the event is far from being impossible in its everyday sense. For example, in the experiment of picking random numbers from the set of all natural numbers, the event of picking a prime number is zero-probable, since if one starts counting the prime numbers starting from 1 and up, and calculates the prevalence ratio of the prime numbers, the probability converges to zero since the gap between two consecutive prime numbers grows arbitrarily⁷. Arguably, picking a prime number is not logically impossible and most people would consider it a low-probable event.

In theoretical probability, $P(A)=1$ only if the event has the same measure as the sample space, be it a simple counting measure or a Borel measure on subsets of \mathcal{R}^n . That is, in discrete finite sample spaces, a sure event is reduced to the sample space itself. For instance, in an experiment of rolling a standard fair die, the only event with 100% chance of happening is: to role any number between 1 and 6, inclusive; in other words, the only sure event is the sample space itself. If the sample space includes an infinite number of events, it is possible to identify events with 100% probability of occurring without having to enlist all of the possible outcomes. For instance, in the experiment of picking a random number from the set of all real numbers, the event of picking an irrational number has a 100% probability.

3.2.5. On the importance of extreme events

I wish to state briefly why I have chosen extreme events as the subject of my study. I present two reasons: the mathematical complexity of extreme events and the educational significance of extreme events.

Mathematical importance of extreme probabilities

Extreme probabilities are mathematically appealing and elusive at the same time. They create additional complexity to the probability estimation methods and techniques.

⁷ For every $n > 1$, $n! + 2, n! + 3, \dots, n! + n$ are all composite numbers.

Every computer simulation method has limitations and problems when the probability sought after is around the extremes. For example, the central limit theorem allows for a binomial distribution to be approximated by the normal distribution when np and $n(1-p)$ are both greater than 5, even if the sample size is rather small. For very extreme probabilities, though, a sample size of 30 or more may still be inadequate and the approximation works at its worst when the sample proportion is exactly zero or exactly one. Most probabilistic models are reasonably valid when extreme probabilities are carefully avoided (consider naïve Bayes classifiers for instance); fixes and patches are needed to modify the model in order to work with very low or very high probabilities (Lagrange correction for naïve Bayes classifiers). But extreme probabilities do not just cause trouble to probabilistic models; they also have very interesting properties. One of the brilliant (and counter-intuitive) properties of extreme probabilities is captured in Kolmogorov's zero-one law stating that specific events known as tail events will either almost surely happen or almost surely⁸ not happen. One consequence of this law is that in infinite sequences of independent trials (such as coin flipping) any pattern or specific outcome (alternating heads for example) cannot have probabilities other than zero or one (Stroock, 2011).

Educational importance of extreme probabilities

My second reason for investigating extreme probabilities is that from an educational perspective, distinguishing between the binary opposites of certain-uncertain and possible-impossible is often located at the very introductory phases of probability education. Egan (1997) states that the use of binary oppositions provides an initial grasp of phenomena since they are abstract, affective, and they can expand understanding to anything that can be organized in terms of their basic affective concepts. More specific to probability education, van de Walle, Folk, Karp, and Bay-Williams (2011) suggest that young children come to class with all sorts of bewildering ideas of probability, and “to change these early misconceptions, a good place to begin is with a focus on possible and not possible and later impossible, possible, certain” (p. 474). Through observations over a two-year time period of children's thinking in probability contexts, the research

⁸ The word *almost* here is used in measure-theoretic sense, see page 64.

reported by Jones, Langrall, Thornton, and Mogill (1997), and Jones, Thornton, Langrall, Mooney, Perry, and Putt (2000) describes a framework that identifies four important probabilistic constructs and describes how young children think in each of those probabilistic situations. The four levels are associated with subjective thinking, transitional between subjective and naïve quantitative thinking, use of informal quantitative thinking and finally numerical thinking. One of the findings of these studies is that all of the participants of the study were able to recognize every instance of certain and impossible events, even when they consistently used subjective judgments to decide on these. It appears that recognition of certain and impossible events are the starting point for probabilistic thinking. Thus extreme probabilities correspond to the type of events that the learners are familiar with since the very early grades and they potentially serve as a basis or a platform for the quantitative probability in later grades.

Due to the reasons discussed above, I consider extreme events to be significant for research. Based on my experience with teaching probability, the zero-probable or one-probable events are a source of confusion and conflict between the everyday and mathematical probability. They have considerable overlap with what we perceive of as impossible or certain events and when students encounter zero-probable examples that are not logically impossible it creates discomfort and confusion.

3.3. Participants of the study

Participants were 30 pre-service secondary school teachers enrolled in a mathematics education course at Simon Fraser University, a methods course on teaching secondary mathematics. This course is offered (recommended but not required) for prospective teachers that are considering teaching secondary mathematics. Typically students that take this course are confident with their mathematical skills and have taken at least one post-secondary mathematics course. Sixteen of the participants had backgrounds in mathematics, physics, chemistry and engineering, ten came from biology and health studies background, and four had degrees in history and English language. Most of them had taken an introductory probability and statistics course before participating in this study. I chose this group of students as my participants because of their relatively strong mathematical background, as well as their availability.

3.4. Methodology

In this study, I have used learner-generated examples of extreme events to draw on the *mathematicalness* or *everydayness* of the probability-related concepts present in their examples.

Watson and Mason (2005) view examples as “illustrations of concepts and principles” (p. 3). They consider learner-generated examples (LGEs) – an approach in which learners are asked to provide examples of mathematical objects under given constraints – as a powerful pedagogical tool, through which learners enhance their understanding of the concepts involved. They also introduce the construct of *example space* as collections of examples that fulfil a specific function, and distinguish among several kinds of example spaces. Of particular interest in this study are *personal example spaces*, triggered by a task as well as by recent or past experience. When invited to construct their own examples, learners both extend and enrich their personal example spaces, but also reveal something of the sophistication of their awareness of the concept or technique (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006) . In accord with this observation, Zazkis and Leikin (2008) suggest that LGEs provide a valuable research tool as they expose learner’s ideas related to the objects under construction and examples generated by students mirror their understanding of particular mathematical concepts.

Mason (2011), in an analysis of the phenomenology of example construction, suggests that some important aspects of the process of mathematical example construction are: feeling a strong tendency to combine the simplest possible with maximum generality, constructing lots of examples and tinkering with examples to modify them so that they meet some particular constraint, experiencing dimension of possible variation and range of permissible change associated with the examples constructed. Through this process, one explores deeper aspects of the notion, and attention is drawn to the playful aspects of example construction and the ways of tinkering with a basic construction that might be of help for future use. Vinner (2011) finds the role of examples in everyday and mathematical thinking to be very crucial. Unlike in mathematics in which the concept formation is aided by definitions, examples

and proofs, in everyday thinking, examples are the only tool by which we can form and verify concepts and conjectures. Even in mathematics there are important notions such as “proof” that have no agreed upon definitions and the students are supposed to acquire the concept of proof by the many examples they are exposed to.

Zazkis and Leikin (2008) suggest that the task of constructing examples of mathematical concepts can be quite a complex task for students and teachers. However research results find it a well worth effort since:

- The example generating task provides a window into learner’s mind through which significant aspects of conceptualization could be observed (Mason, 2011);
- The example generating task raises the students’ awareness of features of examples that can change and of the range where they can vary (Mason, 2011);
- The processes involved in constructing examples are rich and complex (Antonini, 2011).

3.5. Research Questions

This study addresses the following research questions:

- What aspects of probability (mathematical or everyday) are featured in the participants’ examples of extreme events?
- What probability perspectives are involved in the participants’ examples of extreme events?

3.6. The task

The participants of the study were asked to respond in writing to the task presented in the following table. Ample writing space and time was given.

Table 3-1 The task

Please give an example of an event for the following probability categories. Briefly note your reasoning.	
Probability is exactly one (100%).	Probability is very close to 100% but not 100%
Probability is exactly zero.	Probability is very close to zero but not zero.

3.7. Theoretical considerations, the language of mathematics

Here I attend to theoretical considerations related to the challenges that arose from everyday language versus mathematical language. A discussion of mathematical language is presented first, and then the problem of how to make a distinction between mathematical versus everyday language is addressed. The theoretical framework most relevant to this study is presented next.

3.7.1. Mathematical language

Is mathematics a language? Most linguists would not consider it to be so. Linguist Halliday (1978) prefers to speak of the mathematics 'register', which refers to ways in which the main language (English for example) is used in a specific mathematical context. Although the term 'register' has been taken up in the research literature (for example, Pimm (1987) writes extensively about aspects of mathematical register), the term 'mathematical language' is in use too. After all, *language* is a commonly used metaphor to describe and refer to mathematics: it is often said that mathematics *is* a language and a universal one too. However, as with any other metaphor, it is not to be taken too literally. For example, Keith Devlin's *The Language of Mathematics* (Devlin, 2000) is not about a collection of words and syntax rules. The book discusses how people from different cultures and different times can understand each other's mathematics due to the shared experience of patterns. People *understand* each other and this understanding is an aspect of language. In this study, I use the term mathematical language to refer to the meanings intended or triggered by using mainstream English words in a mathematical and academic (within classroom) context.

3.7.2. Everyday language versus mathematical language

The interplay between everyday language and the mathematical language in a classroom has been of research interest in mathematics education (Pimm, 1987), (Moschkovich, 2002), (Moschkovich, 2003), (Morgan, 1998), (Barwell, 2013).

In the course of developing vocabulary, children experience many aspects and usages of a given word. They are called upon to use language for a variety of new purposes and to use familiar terms in less familiar and quite specialized contexts. This is the source of some learning difficulties, especially when the old meaning and the new meaning are related in some ways, but not quite the same. Research has shown that when familiar terms are exploited by a subject (such as mathematics) to mean a different notion or meaning, this creates difficulties for students. For example Durkin and Shire (1991) indicate that the language that children encounter in both mathematical and musical education is associated with learning problems in those areas of curriculum. Pimm (1987) suggests that the appropriation of everyday language within the mathematical register may be a source of confusion for students. A word like “difference” (as the quality of not being the same), for example, has a wider everyday meaning than the more specific word “difference” in mathematics (as the result of subtraction). Kim, Sfard, and Ferrini-Mundy (2005) investigate how Korean students, who do not have an everyday meaning for infinity, differ from English speaking students. The latter group uses the “infinity” and “infinite” in both colloquial and mathematical context while the former only encounter the concept infinity in mathematical context. Their findings suggest that colloquial discourse has an impact on mathematical discourse. There is a degree of overlap between colloquial everyday language and mathematical language. Probability and related terms such as event, impossible, certain and sample space are examples of concepts that have meaning in both everyday and mathematical registers. These terms refer to meanings different from their everyday meanings, when used in a mathematical context.

3.7.3. Barwell’s discursive model

The model used in this study to make the everyday versus mathematical distinction is adopted from Barwell’s works (Barwell, 2005), (Barwell, 2003), (Barwell, 2013). He offers two models of the nature of mathematics and mathematical language: the formal model and the discursive model. The former sees meaning as fixed and relating to everyday language with no problem, the latter sees meaning as situated in and by interaction. In a discussion of the nature of mathematical discourse, he suggests a number of distinctive features, some of which are particular to mathematics

classrooms. Those features are briefly reviewed here. A mathematical discourse is recognized by the following features:

- It has a specialist mathematical vocabulary which includes:
 - Technical terms specific to mathematics, such as: quotient, equilateral, hypotenuse
 - Specialist use of general terms, such as: line, continuous
 - Mathematical use of everyday terms for unrelated ideas, such as: expression, difference, function
- It involves using specialist syntax, the use of words such as and, or, if and only if to define mathematical relationship
- It involves the use of mathematical symbols such as numerals, variables, integral sign
- It includes specialized ways of talking such as proofs or definitions
- It includes a social dimension, for example the particular ways in which the word problems are to be interpreted.

Barwell has used the features listed above to analyze a radio talk in which two mathematicians give an account of Poincaré's conjecture to non-mathematicians. Since the program is broadcast for a general audience, the mathematicians employed everyday terms and used them to convey mathematical ideas. In my study, the features of mathematical discourse are used to develop a tool for distinguishing between everyday probability and mathematical probability. I found Barwell's discursive model particularly helpful for my study since it outlined some distinctive features of mathematical discourse. However, since Barwell's model is not specific to probability, interpretations and adaptations were needed for the purpose of this study, the specifics of which are discussed below.

Interpreting Barwell's mathematical discursive features

Sorting out the examples into every day and mathematical is not an easy task since mathematical probability involves specialized use of everyday terms. In order to do this to some extent, I have used features of Barwell's discursive model. My interpretations of Barwell's discursive features are presented in table below.

Table 3-2 Interpreting Barwell’s framework in the context of probability

Features of mathematical discourse	Barwell’s examples (general)	My interpretation (specific to probability)
Specialist mathematical vocabulary	quotient, equilateral, hypotenuse, line, continuous, expression	sample space, event, frequency, occurrence, outcome, tends to infinity, limit
Specialist syntax to define mathematical relationship	and, or, therefore	Not applicable to my analysis
Use of mathematical symbols	numerals, variables, integral sign	variables, numbers, percent sign, fractions, $P(A)$, ∞
Use of mathematics classroom conventions	the particular ways in which the word problems are to be interpreted	using examples common to classroom examples of probability

I needed to break down the above categorical features into more specific rules to apply to each example. The first three features I found easy to interpret and identify in the data. So, I focus on aspects of probability that has to do with mathematics classroom conventions.

Mathematics classroom conventions

The most essential feature from the discursive model to my analysis of data proved to be the social dimensions of mathematics classroom, by which Barwell means the specific ways the word problems are to be interpreted. It can also include the assumptions associated with these interpretations, and an expected understanding of properties of an object in mathematical context. For instance, the students are assumed to know what the teacher really means when she draws an imperfect line on the board and refers to it as a straight line. It is to be interpreted as a perfectly straight, infinitely thin, and infinitely long line. Other examples of social dimensions of mathematics are the possibilities offered by the abstraction inherent to some mathematical problems. Mathematical abstraction enables us to focus on one feature without having to be concerned about how and if it can be achieved. For instance, in a mathematics problem it is acceptable to talk about “a pile of 1000 blue shirts” (example from data), focusing on the number of identical shirts without worrying about where to put these shirts and how

we got a thousand of them on our hands and why they are all blue. Same with “rolling a die 10000 times”; it is a mathematically appropriate thought experiment which doesn't have the limitations that everyday life has.

In the course of data analysis, the social conventions of mathematics classroom posed both opportunities and problems to my analysis of examples. One challenging aspect is that teachers and resources often use examples that have both every day and mathematical elements, since they are meant to make sense to the audience. With examples like these, it is difficult to distinguish everyday versus mathematical aspects. Consider a coin flipping situation: it can be both an everyday and a mathematical notion. It is mathematical because games of chance serve as birth grounds for mathematical probability; moreover, making use of coin flip problems is common practice in mathematics classroom. Thus, the grouping tool used in this study places the coin-flipping example closer towards the mathematical probability. However, it is important to note that when referred to in a classroom context, extra assumptions are often made about randomizers such as coin or dice. Unlike in everyday situations, a mathematical die is assumed to be rolled on an infinite plane with no edges, so falling off the table is not a possible outcome of rolling a mathematical die. A mathematical coin not only has two equiprobable sides, but also has no edge since it is infinitely thin. Therefore landing on an edge is not a possible outcome with a mathematical coin. “A drawer full of socks” is a reality, “a drawer with one million socks” in it, not so much (examples from the data). “Picking a blue marble from a bag of five blue and one red marbles”, although a frequently used instructional example, could be envisaged in an everyday context, “a bag of $n-1$ blue and one red marbles” presents a more mathematical context.

Based on the discussion above along with Barwell's discursive model, I arrived at four grouping rules (see section 3.8.1)

3.8. Results of the first round of data analysis

A total of thirty responses were collected. Each respondent was asked to provide four examples (one for each category). Some of the respondents didn't provide all four, which brought the total number of examples down to 114. Two rounds of data analysis

were conducted. The first round sorted the examples based on the everydayness or mathematicalness of the example. They were arranged into three groups: M only, E only, and $M \cap E$, where M is the set of mathematical probability examples, E is the set of everyday probability examples, and $M \cap E$ is the set of examples that have both mathematical and everyday features. In the second round, the examples were examined based on the type of probability featured in them. They were sorted into three groups: Logical impossibility/certainty, Theoretical probability, and Subjective probability.

3.8.1. Grouping tool

In order to sort the examples into M, E or $M \cap E$, I have filled out the chart below for each example from the data.

Table 3-3 Grouping Tool

Example ID	Specialist mathematical vocabulary	Specialist syntax	Use of mathematical symbols	Use of social conventions of mathematics classroom	Grouping

This chart allows for a closer look into the mathematical features (if any) presented in each example and helps with the sorting task to some extent. However, after filling the chart for each data instance I realized that not all features are as significant and informative as others. More specifically, use of numbers is too common to all examples to be indicative of either every day or mathematical probability. The specialist syntax didn't show up frequently enough in the data to be a meaningful indicator either. Based on the discussion presented in section 3.7.3, I came up with these grouping rules:

- Examples featuring randomizers common to probability teaching such as bags of objects, deck of cards, coins, dice are grouped as $E \cap M$.
- If affordances of mathematical abstraction such as possibility of handling large number of objects, large number of repetitions are tapped into, then the example is grouped as mathematical (M).
- If the number of objects or repetitions is denoted by a variable (n) as opposed to a fixed number, the example is deemed more mathematical than every day.

- Any example that cannot be grouped as $E \cap M$ or M is deemed every day.

I present a quantitative summary and examples of each type from the data in what follows.

3.8.2. Quantitative Summary

As mentioned before, a total of thirty responses to the Task were collected. The Task invited each respondent to provide four examples, which could have resulted in 120 examples. However six examples were missing, which brings down the total to 114 examples. Of these, 48 were both everyday and mathematical, 47 had more everyday features than mathematical ones and nineteen had enough mathematical features to make the mathematical list. The results are summarized in a Venn diagram depicted in Figure 3-1.

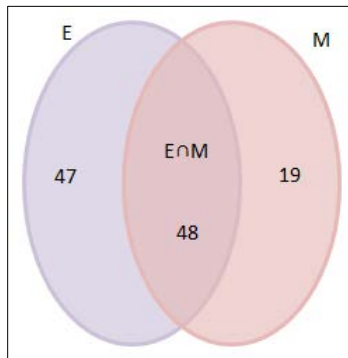


Figure 3-1 Every day and Mathematical Probability

The ellipse on the left of the Venn diagram represents the everyday examples (E). The ellipse on the right (M) is to represent the set of mathematical examples. The overlap region ($E \cap M$) contains examples that are categorized as both every day and mathematical. To refer to examples that have more mathematical features than every day, in other words those that are only in M and not in the overlap, I use the set notation $M - E$; similarly, $E - M$ is the set of examples with more everyday features than mathematical.

3.8.3. Examples from the data

Here I present illustrative examples from each category.

Purely everyday (E-M) examples include situations involving weather, the state of the universe, life and death, personal preferences, facts and conventions.

"In a girls' soccer team, the probability that there are boys on the team is exactly zero" (24-3)⁹.

The probability that *"we'll get hungry at some point in the day"* is exactly one (28-1).

The probability that *"it will rain tomorrow"* is nearly one (29-2).

The probability of *"snow in August"* is nearly zero because it is *"highly unlikely but could happen"* (27-4).

Purely mathematical (M-E) examples include somewhat structured and well-defined situations involving picking a number or object or certain outcomes of randomizers.

"Choosing an even number when selecting from numbers that are divisible by 2" is an exactly-one-probable event (4-1).

The probability of *"choosing an odd number when selecting from numbers that are divisible by 2"* is exactly zero (4-3).

The probability of *"picking 3 red balls from a giant bin filled with millions of red balls and only one blue ball"* is nearly one (17-2).

The probability of *"tossing a coin infinite times and getting all heads"* is nearly zero (6-4).

The examples that are listed under both mathematical and every day (E∩M) commonly stemmed from games of chance situation without the extra mathematical features of infinite repetitions or a very large number of objects.

"Picking 3 red balls from a bin that has 2 red balls and 2 blue balls" is an exactly-zero-probable event (17-3).

⁹ (24-3) refers to the third example given by respondent number 24.

"In a bag of one blue marble P(grabbing a blue marble)" is exactly one (1-1).

"Flipping at least one tail after 50 attempts with a standard coin" is nearly-one-probable (16-2).

The probability of "winning the lottery" is nearly zero, because "chance of winning 649 is so small yet possible" (16-4).

3.9. Discussion of findings in the first round of analysis

The analysis of data suggests that the everyday meaning of probability is more prevalent in prospective teachers' examples of probability. Also their examples are greatly influenced by the instructional examples (those commonly used to teach probability) and textbook examples such as coins, dice and random draws.

Numbers, percentages and fractions were present in both every day and mathematical probability examples across all four groups of exactly-one, nearly-one, exactly-zero and nearly-zero probable events. Situations involving infinity or very large number of objects or trials, on the other hand, made an appearance only in examples of nearly-zero and nearly-one probability.

Barwell's model proved to be useful in distinguishing discursive aspects of probability in everyday and mathematical contexts, the most informative feature being the social aspects of mathematics. However, there is a considerable 40% overlap that could not be further elaborated on. In the process of data analysis, I encountered examples that are either presented in incorrect probability category, or presented in multiple probability categories; in lack of better word, I call these examples *inconsistent examples*. This issue is observed in all categories (mathematical, every day and overlap between mathematical and every day). "The Sun rises tomorrow" (13-2, 29-1), and "we will die at some point" (16-1, 25-1, 27-2) are everyday examples set forth by some participants as exactly-one-probable and by others as nearly-one-probable events. Similarly, "Flipping a coin and getting heads or tails" (9-1, 11-2, 21-1, 23-2) example was presented by different people as either exactly-one-probable or as nearly-one-probable.

Within purely mathematical group (M-E), although I could not find an example presented in more than one probability categories, there were plenty of examples in which the assigned probability was conventionally incorrect. For instance, the probability of “flipping a die infinitely many times and getting straight 6’s” (24-4) was assessed to be nearly zero and not exactly zero. However, a conventional probability calculation results in probability exactly zero for this event. Instances like this suggested that I look for ways to further analyze the probability framework that the respondents had in mind when generating examples of extreme events. This, I attended to via a second round of data analysis.

The second research question of this study calls for examining the types of probability involved in example construction. This type of analysis helps distinguish between non-discursive features of examples that are currently grouped together (every day or mathematical), and enables me to elaborate on the 40% overlap and on the issue of inconsistent examples. Thus I present further theoretical considerations and a second round of analysis to examine the examples based on the probability approach, and/or the type of probability reasoning presented as justification.

3.10. Further theoretical considerations: Types of probability

As mentioned in earlier sections, there are multiple approaches to probability. Probability has been regarded by early and modern developers and practitioners as ranging from being a way of expressing one’s personal degree of belief and tendency in uncertain situations, to being a rigorous, objective mathematical model to quantify and make predictions about probabilistic situations. There is no commonly agreed-upon way of categorizing different types of probabilities.

A review and discussion of dominant philosophical interpretations of probability in mathematics education could be found in Chernoff (2008) and Batanero and Diaz (2007).

I examined the data to identify what type of probability approach may have influenced the responses. Initially I focused on the three major types of probability identified in the literature:

1. Theoretical approach where probability is expressed as the ratio of the number of favourable outcomes over the total number of possible outcomes
2. Frequentist approach where probability is the limit of the relative frequency with which an event occurs in independent repeated trials
3. Subjective approach where probability is assigned based on personal degree of belief in a certain occurrence given the evidence

For reasons explained below, I decided upon an alternate way of categorizing the types of probabilities and arrived at the following categories:

1. Logical impossibility/certainty, stemmed from:
 - Definitions
 - Known facts
 - Sample space
2. Theoretical probability
3. Subjective probability

3.11. Second round of data analysis

In this round of data analysis I have investigated the reason behind probabilities assigned to examples. I have tried to consider the perceptual underpinning of zero-or-one probability as depicted from the data.

3.11.1. Quantitative Summary

In Table 3-4 the number of examples for each type of probability reasoning is given.

Table 3-4 Quantitative summary of types of probability

		Exactly one	Nearly one	Exactly zero	Nearly zero	Total
Logical impossibility/certainty Stemmed from	Definitions	4	0	3	0	7
	Known facts	2	0	4	0	6
	Sample space	16	0	18	0	34
Theoretical probability		0	15	0	16	31
Subjective probability		8	12	5	11	36

From Table 3-4 it is clear that logical impossibility/certainty is where the most of exactly-zero-probable and exactly-one-probable examples stem from, while theoretical probability harbours the most potential for generating nearly-zero-probable and nearly-one-probable examples. Subjective probability type of reasoning seems to be almost equally distributed within the four groups of examples.

3.12. Analysis tool, the inclusion criteria

I wish to share with the reader the criteria for assigning examples to the chosen categories. In particular, I describe what exactly is meant by each of the categories listed above and what types of examples from the data are included. In addition, I address the question that most likely arises when attending to the probability approaches (theoretical and subjective) mentioned above: Why is there no mention of frequentist probability?

3.12.1. Logical impossibility/certainty

In the first category, the logical impossibility/certainty, examples are included that are:

1. Either not accompanied by any explanation, or the justification offered by the respondent denotes the logical impossibility of the event or the impossibility of anything else happening

2. Commonly considered a logical consequence of definitions, the way certain things are made, social conventions, laws of physics, facts that are known to most people, and things that are set and no longer subject to change within reasonable accuracy.

For instance, in “If my friend is a bachelor, the probability is 100% that he is unmarried. This is because logically a bachelor cannot be married” (2-1), the definition of the word bachelor is called upon to create a deliberate logical certainty: if bachelor, then unmarried. This is clearly a logical-definition type.

The probability “that I will start floating off the ground while standing on the surface of the Earth” is exactly zero (13-3), due to the gravity of Earth. Laws of physics have become part of the shared human understanding and knowledge base of how things work, they are relied on as sources of certainty since most people believe them to be proven facts, examined and scrutinized by the scholars, and not merely opinions.

Another example from the data states that “If today is Tuesday, $P(\text{tomorrow being Tuesday})=0$ ” (2-3). It is a social convention for the day following Tuesday to be called Wednesday (and not Tuesday again), and thus Tuesday followed by Tuesday is a logical impossibility.

The above-mentioned examples clearly belong to the logical impossibility/certainty group. Now I attend to the more problematic examples in this category, the ones categorized as sample space group. In this category I have included examples of events that do not belong to the sample space of the proposed experiment (probability=0), or examples of events that include the entire sample space (probability=1). For instance, “rolling a 7 with a standard die”, “obtaining a black diamond in a standard deck of cards”, “getting a sum of at least two after rolling two fair dice” are examples of this type, to name a few.

For the purpose of the discussion of the types of probability brought about in these examples, I focus on one example from this group; the argument can be extended to the other ones as well. Consider the “rolling a 7 with a standard die” example (3-3, 6-3, 12-3, 14-3 and 23-3). It is hard to identify the type of probability that allows for such probability assignment. It is certainly zero-probable because it is against common logic to expect something that wasn't there before (number 7 on a die) to show up out of

nowhere and to expect that the die conjures up surprise numbers each time it is rolled. This is the clear part. What is unclear is whether there is a theoretical way to assign probability zero to this event. In theoretical probability, the sample space S is comprised of equally likely *events* that will cover all of the possible outcomes of a probabilistic situation. The probability of an event A is then assigned by $P(A) = \frac{\text{measure of set } A}{\text{measure of set } S}$ in which the measure could be simply the counting measure (the number of elements in each set) or any other measure. But what is an *event*? When rolling a standard die, is rolling a 7 an *event*? An *event* is usually defined as a collection of outcomes or results of an experiment, and it is regarded to be a subset of sample space since the sample space contains all outcomes and results of an experiment (see Triola (2011), p.139, Burger and Starbird (2010), p. 585, and Freund, Wilson, and Mohr (2010), p. 72 for definition of event). Thus, the empty set and the sample space are *events*. In our example, $S=\{1,2,3,4,5,6\}$ and $A=\{7\}$, which makes A not a subset of S , and thus not an *event* (at least technically). This is not just the case with rolling a number that is not printed on the die; consider for example rolling a die and the die turning into a bird. Is this an event? It sure is not a subset of S . In both of these examples the described situations are not subsets of the sample space; therefore they are not *events*, strictly speaking. In probabilistic situations, where the sample space is a finite set, no non-empty subset of it can be zero-probable (otherwise, the probability of the sample space will have to be zero). Therefore the sample space cannot contain any zero-probable events (except for the empty set).

It is not the goal of this study to explore the problematic aspects of sample space; the interested reader may refer to Chernoff and Zazkis (2010). My aim is to address the types of probabilities accessible to the participants of study when generating examples of extreme events. The laboured point I am trying to make is that the conventional theoretical probability doesn't provide a well-defined, conflict-free platform for generating zero-probable events, and thus the prevalent mindset from which this type of examples are stemming is more likely to be the logical impossibility/certainty. Also supporting my argument is that although the participants were asked to show their reasoning, nowhere in the data did I see an explanation like: the probability of rolling a 7 with a die is zero because it is 0/6. Due to these two reasons I have listed this example along with logical impossibility/certainty as opposed to theoretical probability.

3.12.2. Theoretical probability

In the second group I have included examples that are accompanied by a probability calculation relevant to a theoretical approach or the examples that I thought of as common to teaching and learning theoretical probability. Such examples involve picking objects or numbers, dice rolling, coin flipping, and lottery:

"Out of n balls $n-1$ are green, 1 is red. As n tends to infinity probability of green is almost 100%" (7-2)

Probability of "Randomly picking the number 1 from number line" is nearly zero because it is "one out of infinite" (21-4)

In these examples the set of favourable outcomes is clearly distinguished from the set of all possible outcomes and the number of each set is often mentioned.

3.12.3. Subjective probability

In the third group I have listed examples of probability that could not be classified as logical impossibility/certainty or theoretical probability. The distinctive feature of this group of examples is that the probability assigned to them can't be explained in terms of a logical relation, be it a definition or proven facts or the set of all possible outcomes of an event. Also it is reasonable to say that not everyone will feel the same about the probability of those events. Also they are either too general or too personal. Here is an example: the probability of "getting stuck in traffic on the Port Mann Bridge" is 100% (8-1). True though it may be for the particular person that happens to cross the bridge on weekdays at a specific time, there is no generally agreed-upon reason to believe so. The probability "that in five minutes it will continue to rain" is almost 100% (18-2), the validity of this claim can't be verified since it is neither a matter of definition, nor related to theoretical probability. However, it makes a perfect fit for the subjective category since it is rooted in the person's experience about rain in Vancouver. This person has reason to believe that once rain is observed in Vancouver, it is almost certain to last for the next five minutes.

3.12.4. Frequentist probability, why is it not on the list?

Although there were examples that involved long sequences of coin flipping and die rolling, I did not categorize them under a frequentist approach. Consider the example “tossing a coin infinite times and getting all heads” (6-4), which is presented as a nearly-zero-probable event. The notion of infinite rolls resonates with the way frequency-based probabilities are assigned, but there is a difference. The event in question is not *getting a 6*, it is getting *infinite straight 6's*. The asymptotic ratio of 6's showing up in a hypothetical sequence of infinite rolls of a fair die once observed and documented by a patient probability enthusiast can be used as a basis to assign a frequentist probability to the event of rolling a 6 on a fair die. The probability of rolling infinitely many straight 6's is a different story: in a frequentist approach, one needs to get the probability of rolling a 6 from an infinite die rolling process and to apply laws of probability (which is the common ground for both theoretical and frequentist approaches) in order to calculate the probability of rolling infinite straight 6's. The justification given by the respondent did not bring these up; therefore I tend to think of the above example in the context of theoretical probability. Of note, there are two ways to calculate the theoretical probability of the event in question. One way is to calculate the probability of rolling a 6 through a sample space approach and to apply laws of probability which results in $P = \left(\frac{1}{6}\right)^\infty$. Another way is to consider the infinite sequences that may be obtained from infinite die rolling and count the number of desirable outcomes, count the total number of possible outcomes, and divide. In this case the number of desired outcome is 1 (straight 6's) and the number of all possible outcomes is infinite (6^∞) which results in $P = \frac{1}{6^\infty}$.

3.13. Results of second round of data analysis

The motive for the second round of data analysis was the difficulty I encountered with particular examples from the data as to their everydayness or mathematicalness. The second round of analysis suggests above all that drawing a line between mathematical versus everyday probability is not always possible since they share common logical grounds.

3.13.1. Combined Quantitative Summary

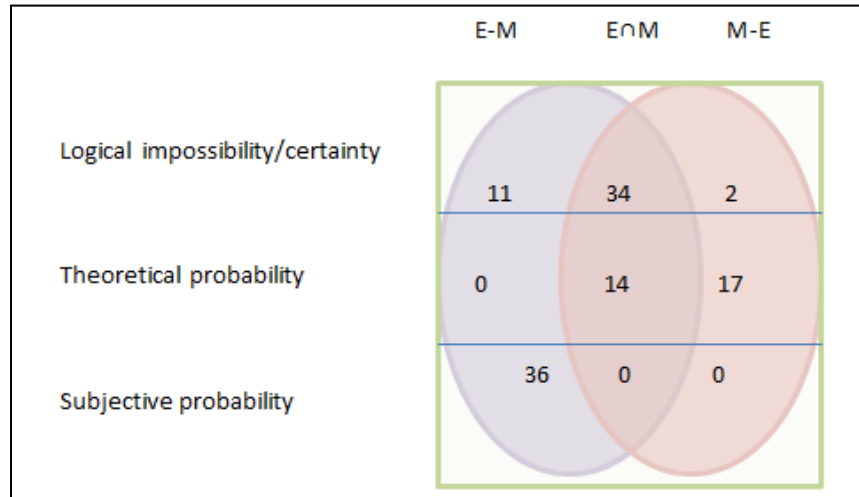


Figure 3-2 Quantitative summary of types of probability cross everyday-mathematical

As shown in Figure 3-2, forty eight (34 + 14) examples have both mathematical and everyday features. Of these, 70% are due to logical underpinnings of both probabilities.

Before further discussion of the findings I share with the reader examples of each category. The examples presented earlier, are revisited here along with the type of probability identified in each.

Purely everyday examples (E-M)

Logical Impossibility:

“In a girls’ soccer team, the probability that there are boys on the team is exactly zero” (24-3).

Subjective:

The probability that “we’ll get hungry at some point in the day” is exactly one (28-1).

The probability that “it will rain tomorrow” is nearly one (29-2).

The probability of “snow in August” is nearly zero because it is “highly unlikely but could happen” (27-4).

Purely mathematical example (M-E)

Logical impossibility/certainty:

“Choosing an even number when selecting from numbers that are divisible by 2” is an exactly-one-probable event (4-1).

The probability of “choosing an odd number when selecting from numbers that are divisible by 2” is exactly zero (4-3).

Theoretical:

The probability of “picking 3 red balls from a giant bin filled with millions of red balls and only one blue ball” is nearly one (17-2).

The probability of “tossing a coin infinite times and getting all heads” is nearly zero (6-4).

Overlap between mathematical and everyday

Logical impossibility/certainty:

“Picking 3 red balls from a bin that has 2 red balls and 2 blue balls” is an exactly-zero-probable event (17-3).

“In a bag of one blue marble P(grabbing a blue marble)” is exactly one (1-1).

Theoretical:

“Flipping at least one tail after 50 attempts with a standard coin” is nearly-one-probable (16-2).

Subjective:

The probability of “winning the lottery” is nearly zero, because “chance of winning 649 is so small yet possible” (16-4).

3.13.2. On the placement of the examples

I address in this section the issue of inconsistent examples, the ones that are either placed inappropriately or placed in different probability categories by different participants. For example, a coin landing on its edge was considered as an exactly-zero probable event by some participants and a nearly-zero-probable event by others. In what

follows I attend separately to everyday, mathematical, and overlap ($E \cap M$) groups among these examples.

Inconsistent examples in everyday category

Examples such as “the Sun will rise tomorrow” and “We will die at some point” are presented in data both as exactly-one-probable and nearly-one-probable. The analysis of types of probability shows that these arguments are made on a subjective basis and thus rooted in personal tendencies and beliefs. On one hand, these two examples seem to be the common denominator of all human experience so far; if we have learned anything as a species it should be that the sun will rise tomorrow and we will die at some point. On the other hand, the more we learn about how things (the solar system and our body) work and the more sci-fi movies we watch, the more open we are to considering alternatives. On this note, Lindley (2014) states that there is no such thing as *uncertainty*, it is only personal uncertainties. That is to say, each person, based on their past experience, knowledge and beliefs, holds various degrees of uncertainty towards an event. New information, even fictitious information, shapes and changes the way we think about uncertain situation. The sunrise and death examples show that even at extremes, where one expects absolute certainty or impossibility, personal beliefs and tendencies divide people, which brings us to the issue of examples that are worded and presented mathematically but different probabilities are assigned to them.

Inconsistent examples in the $E \cap M$ category

An example that was frequently mentioned in the data as nearly-zero probable, exactly-one probable, and nearly-one-probable is the example of “a coin landing [or not landing] on its edge” (3-1, 9-1, 21-1, 4-4, 6-2, 11-2, 23-2, 23-4). This example and its variants showed up in the data eight times. A mathematical coin is Platonic (Chernoff, 2011), it has zero edge thickness, and perfect homogeneity. The conventionally correct probability of a coin landing on its edge is exactly zero, and the probability of otherwise happening (the coin landing on either side) is exactly one. Three people (3-1, 9-1, 21-1) assigned probability zero to this example, although their reasoning is unknown. The correct answer may have been obtained due to considering the Platonic nature of the coin or it may be the result of their everyday experience with an actual coin (called

contextual coin (Chernoff, 2011)) since a coin landing on its edge may not happen in one's life time. Unlike the correct responses, the incorrect ones do reveal something of the respondents' view of probability. In all incorrect responses a possibility, small though it may be, is left for the event of coin landing on its edge. When teachers refer to coin tossing situations in probability classrooms, even though physical manipulatives are made use of, the assumption is to consider an infinitely thin, uniformly designed equiprobable coin, which is repeatedly tossed with identical initial conditions (throws) on an infinite plane. My point is that of the two Platonic and contextual coins, only the latter may conceivably land on its edge. This is what is portrayed in the above-mentioned examples and the notion of a Platonic (mathematical) coin is overshadowed by a contextual (everyday) one.

Inconsistent examples in the mathematical category

There were seven such examples, all categorized as mathematical and theoretical:

- a. *"Out of n balls $n-1$ are green, 1 is red. As n tends to infinity probability of red is almost [and not exactly] 0%"* (7-4)
- b. *"Picking the number 5 randomly from the numbers ranged from $[1, \infty)$ "* is a nearly-zero-probable event (14-4)
- c. *"Randomly picking the number 1 from number line"* is a nearly-zero-probable event (21-4)
- d. *"Getting heads $\infty-1$ times when tossing coin ∞ times"* is a nearly-one-probable event (10-2)
- e. *"Getting heads only once when tossing coin ∞ times"* is a nearly-zero-probable event (10-4)
- f. The probability of *"doing something infinite times such as rolling a dice and getting all 6's"* is nearly zero (24-2).
- g. Probability of *"tossing a coin infinite times and getting all heads"* is almost zero, because "it is still possible" (6-4)

These examples have two things in common; first, they all involve infinity, either as the number of possible outcomes (a, b, c) or as the number of repetitions of an

experiment (d, e, f, g). Second, the probabilities are incorrectly assigned. Conventional probability calculation will result in exactly one for example (d) and exactly zero for the rest. Although all of these examples interestingly display problematic aspects of the calculus of infinity, of particular interest to this study is the last example. With the other examples, I can only speculate why the respondents think the way they think, with the last example though, we have an explanation. The “it is possible” part doesn’t explain why the probability is nearly zero as much as it explains why the probability is not exactly zero: because it is possible. Here I present two arguments pertaining to these examples.

1) The hesitation to assign exactly zero to the events above may be ensued from the strong association between impossible and zero-probable. If the distinction is not clearly made (as discussed in earlier sections), it is easy to take zero-probable for impossible. One may think that if the probability of picking number 5 from the set of real numbers is exactly zero that would mean the event is impossible. But since picking number 5 is not a logical impossibility, and we are open to accept its possibility of occurrence, therefore the probability cannot be exactly zero.

2) In the last example given above, it is deemed possible to get all heads in an infinite sequence of coin flips. I for one find it hard to think about *possibility* divorced from mathematical probability in an infinity-involved situation. Should I be asked, my only tool to speak for possibility or any other degree of plausibility of infinity-involved events is the cold rigorous mathematical probability. That is because I have no prior experience with infinite coin toss and what seems or does not seem logical to happen. From this argument it seems that the respondents’ notions of coin tossing and die rolling even when employed in mathematical sense may be to a great extent framed by the everyday corresponding notions.

3.14. Summary, Pedagogical Implications and Final remarks

With respect to everydayness versus mathematicalness of respondents’ probabilistic notions of extreme events, examples of exactly-zero and exactly-one-probable events display the large extent to which prospective teachers’ notion of probability is framed by the everyday notions. When generating nearly-zero and nearly-

one-probable examples, the respondents made more use of mathematical features, which resulted in more mathematically worded examples. However, further analysis suggested that some of those mathematically worded examples originate from and are driven by an everyday frame of mind. In regard to the types of probabilities presented in examples from the data, a very large number of examples produced by the participants were stemmed from subjective considerations related to everyday life. This stands in discord with the fact that the teaching of probability is dominated by theoretical probability approach.

The results of this study point to the need to reconsider several aspects of probability education. For instance a discussion of when, to what types of problems, and under what assumptions the mathematical probability applies and how it works compared to our everyday notion of probability should be an essential part of probability “chapter” in mathematics classroom.

Also a clear distinction has to be made between zero-probable versus colloquially impossible events, recognizing that the two notions coincide only for finite sample spaces.

A broader understanding of probability for teachers is also called for. For instance, the common definitions of sample space and event, even when mentioned and discussed, may lead to inconsistencies with regard to impossible events as described in this study (recall the confusion about whether rolling a seven with a die, or a coin landing on an edge is an “event”). An extended view of probability in which the theoretical probability is situated within the subjective probability offers a resolution. When a random experience takes place the set of all conceivable outcomes should be distinguished from the set of all possible outcomes, we may call the former *Universe of discourse*, and the latter *sample set*. All of the logically impossible occurrences get probability zero assigned to them on an a-priori basis, and not via theoretical probability. The sample set is comprised of all logically possible outcomes, part of which can be decomposed into equally probable simple events. This part is what we refer to as the *sample space* and the probability of any event is calculated in conjunction with sample space.

This study has also shed some light on the state of ambiguity of probability and its applicability displayed in prospective teachers' examples of extreme events. As a learner of probability I find some notions of probability in need of further clarification. Above all, time is to be spent on probability as a mathematical model, the type of problems that could be handled, and the scope of results that can be expected. As mathematician (Gowers, 2002) suggests, in order to obtain a working mathematical model, we need to decide on specific problems we want to solve, make simplifying assumptions that brings the focus on to the important aspects of the problem we want to solve, decide what level of accuracy we need, and try to achieve it as simply as possible. In the end there are always limitations to what the model can or cannot do. This applies more so to mathematical probability since it has so many common grounds with the everyday experience of learners. A learner of probability could benefit from explicitly knowing what type of events or uncertain situations are subject matter of probability and how the products of probability are to be interpreted and used.

Chapter 4.

“It is very, very random because it doesn’t happen very often”: Examining learners’ discourse on randomness¹⁰

4.1. Introduction

The notion of randomness is central to the study of probability and statistics and it presents a challenge to students of all ages. However, it is usually not defined in textbooks and curriculum documents, as if the meaning of randomness should be captured intuitively. In fact, the word “random” does get used in everyday language, but not always in the same way that it is used in mathematics. Even in mathematics, the notion of randomness has been a challenging one—not only did it emerge relatively recently in the history of mathematics, it has also undergone various attempts to be adequately defined. Given the importance of randomness in the study of probability and its complexity as a concept, our goal in this chapter is to better understand the ways learners use and talk about it.

In this chapter, we first provide an overview of some of the ways in which randomness is defined in mathematics—these aspects of randomness will help structure our analysis of learners’ uses and descriptions of it. We then provide a brief overview of the research in mathematics education and highlight the main resources that learners use to explain randomness. Following this, we present two empirical studies involving prospective teachers and undergraduate students, each aimed at further probing understandings of randomness using different methodological approaches.

¹⁰ This study is a collaboration with Rina Zazkis and Nathalie Sinclair and is published as a book chapter: (Jolfaee, Zazkis, & Sinclair, 2014)

4.2. Randomness in mathematics

Randomness (and probability theory more generally) is a human construct created to deal with unexplained variation. We use probability theory to model and describe phenomena in the world for which there is a lack of deterministic knowledge of the situation, assuming that they had been randomly generated. Thus, what probability is can only be explained by randomness, and what randomness is can only be modeled by means of probability. The notion of probability is heavily based on the concept of a random event and statistical inference is based on the distribution of random samples. Often we assume that the concept of randomness is obvious but, in fact, even today, experts hold distinctly different views of it.

The study of random sequences was revived in the field of mathematics when it became clear that new ideas from set theory and computer programming could be used to characterize the complexity of sequences. Von Mises in 1952 based his study of random sequence on the intuitive idea that a sequence is considered to be random if we are convinced of the impossibility of forecasting the sequence in order to win in a game of chance. This notion of randomness (called by von Mises the *impossibility of a gambling system*) is closely and fundamentally tied to the notion of independence. On the other hand, the inability to gamble successfully encapsulates an intuitively-desirable property of a random sequence: its unpredictability.

Later, in 1966, von Mises suggested that a random sequence does not exhibit any exceptional regularity effectively testable by any possible statistical test. This approach is similar to Kolmogorov and Chaitin's (Li & Vitany, 2008) vision of a random sequence as a highly irregular or complex sequence (also called algorithmically incompressible or irreducible) that the sequence cannot be reproduced from a set of instructions that is shorter than the sequence itself. For example the following sequence of 1's and 0's—10101010101010101010...—is not random because there exists a short description for it: write down 10 infinitely many times. It is noteworthy that this definition relies heavily on language (ordinary or computer-based) and testing methods.

In Durand, Kanovei, Uspenski, and Vereshchagin (2003) it is found rather surprising that algorithms are involved in defining random sequences, since probability

theory does not use the notion of algorithm at all, and proposed more general definitions based only on set theory. One of the nuances their work brings into the definition of randomness is the notion of typicalness according to which a random sequence is a typical representative of the class of all sequences. In order to be a typical representative, it is required to have no specific features distinguishing the sequence from the general population of the sequences.

In contemporary mathematics, the word 'random' is mostly used to describe the output of unpredictable physical processes (Wolfram Alpha Probability Page, 2014). These physical processes are very familiar and include flipping coins and throwing dice. The randomness that is obtained from computer-generation is often called pseudorandom, in part because someone knows the rules of the system that produces the numbers. As such, the very notion of random is intimately tied to the machines that are used to produce individual events (like the result of a flipped coin or the digit between 0 and 9). In addition to unpredictability, the term 'random' usually means random with respect to a uniform distribution. Other distributions are also possible, and this relation of randomness to a particular distribution is what enables the theoretical probabilities that can predict the outcomes of a large number of random events. Without the notion of distribution, one is left simply with unpredictability, which can be found in many everyday situations (like the weather).

In contrast to these more complex definitions, Bennett (1998) proposed a more practical definition of randomness (of a sequence): a sequence is random either by virtue of how many and which statistical tests it satisfies (of which there are many) or by the virtue of the length of the algorithm necessary to describe it (this latter is known as Chaitin-Kolmogorov's definition (Li & Vitany, 2008)). According to Wolfram (2002) this latter characterization (the incompressibility of data) is the most valid definition, showing remarkable consistency. It has also been proven that this definition covers a large number of the known statistical tests in effect. By the early 1990s, it had thus become accepted as the appropriate definition of randomness (of a sequence).

One drawback of these definitions is that they emphasize process (the machines and/or tests that are used to create/test) as opposed to structure. In contrast, in

Structure and Randomness (Tao, 2008) the mathematician Terrence Tao describes sets of objects as random if there is no recursiveness of the information between these set of objects. He compares the notion of pseudorandomness with that of structure to distinguish between sets of objects. Structured objects, on the one hand, are those with a high degree of predictability and algebraic structure, for example the set $A = \{\dots -5, -3, -1, 1, 3, 5\dots\}$ is structured since if some integer n is known to belong to A , one knows a lot about whether $n+1$, $n-1$, etc. will also belong to A . On the other hand, a set of pseudorandom objects is highly unpredictable and lacks any algebraic structure. For example, consider flipping a coin for each integer number and defining a set B to be the set of integers for which the coin flip results in heads; in such a set B , no element conveys information whatsoever about the relation of $n+1$, $n+2$, etc. with respect to B . While this definition of randomness does not explicitly mention an underlying distribution, it does underscore the relation (or lack thereof) elements of a set have with each other.

Coming from a mathematical modeling point of view, Tao (2008) has studied and produced many different examples of randomness (trying to create random sequences is an important part of trying to understand and describe randomness in mathematics). He has proposed three different types of randomness: (1) the first type arises from not knowing the rules of a system (whether the rules are inaccessible or non-existing); (2) the second type arises from a chaotic system, a system which is sensitive to the initial conditions; (3) the third type is known as a stochastic process, where there is some kind of external environmental component whose essentially uncountable agents (external noise) continually affect the system with their actions. These categories of randomness are interesting in that they relate more closely to everyday understandings of randomness, as we demonstrate below.

There is no agreed-upon definition of randomness that can provide necessary and sufficient conditions for all the kinds of randomness in finite or infinite sequences and sets. Even when considering the most famous examples of random sequences of numbers, namely decimal digits of π , we know that its first thirty million digits are very uniformly distributed and its first billion digits pass the “diehard tests” (an old standard for testing random number generators (Marsaglia, 2005)), which means that the sequence is random in the same sense that the outcome of a fair die is random (known range,

unpredictability of the outcomes and uniform distribution). However, the same sequence of numbers doesn't qualify as random according to Kolmogorov-Chaitin's criteria, since there exists a method that describes the sequence in a much shorter length than the sequence itself (Bailey, Borwein, & Plouffe, 1997a), (Bailey, Borwein, & Plouffe, 1997b). Despite these definition issues, the educational research suggests that people's difficulty with randomness lies mainly in their inability to generate and/or recognize examples of randomness. Of course, if they knew the definition, they might not have this difficulty. But, as we show in the next section, people often talk about random behaviours and properties in ways that diverge from the mathematically-accepted discourse.

4.3. Randomness in mathematics education

The psychologists Kahneman and Tversky (1972) discussed perceptions of randomness with respect to sequences of coin flips. They suggested that people's intuitive notions of randomness are characterised by two general properties: irregularity (absence of systematic patterns in the order of outcomes as well as in their distribution) and local representativeness (similarity of a sample to the population). They pointed out that local representativeness is a belief that the law of large numbers applies to small numbers as well and wrote that this belief "underlies erroneous intuitions about randomness which are manifest in a wide variety of contexts" (p. 36).

According to Kahneman and Tversky's own description, "random-appearing sequences are those whose verbal description is longest" (1972, p. 38). By this, the authors refer to the fact that in dictating a long sequence of outcomes one would necessarily use some shortcuts, such as "4 Heads" or "repeat 3 times Tails-Heads." However, short runs and frequent switches—that characterize apparent randomness—minimize the opportunities for shortcuts in verbal descriptions. Therefore, apparent randomness was seen as "a form of complexity of structure" (ibid. p. 38). This is consistent with Kolmogorov and Chaitin's view described above ((Li & Vitany, 2008); (Vitanyi, 1994)).

Researchers in mathematics education continued the tradition, started by psychologists, of using sequences of binary outcomes to make inferences about

people's conceptions of randomness. For example, Falk and Konold (1998) investigated the subjective aspects of randomness through tasks in which people were asked to simulate a series of outcomes of a typical random process such as tossing a coin (known as a generating task) or rating the degree of randomness of several sequences (known as perception tasks). Their findings indicate that perceived randomness has several subjective aspects that are more directly reflected in perception tasks "since people might find it difficult to express in generation what they can recognize in perception" (p. 653).

In continuing the investigation of perception tasks, Batanero and Serrano (1999) studied perceptions of randomness of secondary school students, ages 14 and 17, $n = 277$. They presented students with eight items, four involving random sequences and four involving random 2-dimensional distributions. Specifically, the students were shown four sequences of Heads and Tails of length 40, and asked to guess which were made up and which were generated by actual toss of a coin. Further, the participants were presented with a 2-dimensional 4x4 grid, in which results of the "counters" game were recorded (this game involves placing sixteen counters numbered 1-16 in a bag, choosing a counter, marking its number in a corresponding cell of the grid and returning the counter to the bag, then repeating sixteen times). The participants were asked which grids were the result of an actual game and which, in their opinion, were made up. Their arguments for deciding which results appeared random were recorded and analyzed. Unpredictability and irregularity were the main arguments in support of randomness. This research reveals complexities in the meaning of randomness and it also shows that some students' arguments are in accord with the interpretations attributed to randomness throughout history, such as lack of known cause.

Using computer-based activities, Pratt and Noss (2002) observed 10-11 old children using four separable resources for articulating randomness: unsteerability, irregularity, unpredictability, and fairness. While the last three components were mentioned above, unsteerability, the first component, was described as personal inability to influence the outcomes. Further, children often combined unpredictability with unsteerability, where "unpredictability was usually seen as the outcome of uncontrolled input" (p. 464). Moreover, children used these resources interchangeably where different

settings initially triggered different resources. Pratt and Noss suggested that intuitive notions of randomness had features analogous to di Sessa's (1993) phenomenological primitives (that is, "multitudinous, small pieces of knowledge that are self-evident, not needing justification and weakly connected" (p. 466)).

Researchers appear to agree that the setting and context significantly influences learners' decision making in probabilistic situations (e.g., Chernoff and Zazkis (2010); Fischbein and Schnarch (1997)). However, the notion of randomness was investigated in studies where the context was constructed by researchers. What happens when learners produce their own contexts? In order to answer this question, we designed two empirical studies in which we could analyse learners' perceptions of randomness where the context is not predetermined.

4.4. Learners' discourse on randomness

In order to study learners' discourse on randomness, we conducted two separate empirical studies, each based on different aspects of mathematical knowing. The first draws on Watson and Mason's (2005) suggestions that knowing a particular mathematical concept involves being able to generate a wide variety of examples of the concept, rather than, or in addition to, the ability to provide a definition or description of the concept. Zazkis & Leikin (2008) suggest that learner-generated examples serve not only as a pedagogical tool, as advocated by Watson and Mason (2005), but also as an appropriate research lens to investigate participants' conceptions of mathematical notions. Thus, our first research question attempts to find out how learners exemplify random phenomena and what features of randomness are present in their examples.

Our second study draws on the recent theories of multimodality, which assert that mathematical thinking involves the use of a wide range of modes of expression, including language but also gestures, diagrams, tone of voice, etc. (Arzarello, Paopa, Robutti, & Sabena, 2009). Given the important connections between gestures and abstract thinking we are particularly interested in the kinds of gestures learners use to express their ideas of randomness. Based on Sinclair and Gol Tabaghi's (2010) study of mathematicians' use of gesture, we anticipate that learners' gestures might

communicate some of the temporal and imagistic aspects of randomness that can be difficult to express in words. Thus, our second research seeks to find out how learners communicate ideas of randomness through a broad multimodal discourse.

4.4.1. Study 1: Exemplifying randomness

In this section we consider examples of randomness generated by prospective secondary school mathematics teachers. From 40 participants, we collected 38 different examples. We note with some surprise that only six examples had mathematical context, that is, were related on situation that can be seen as conventional mathematical in the discussion of probability. By “mathematical context” we refer to examples in which the theoretical probability of the exemplified random event can be calculated, such as:

- When playing with a standard pack of 52 playing cards, what is the probability of getting a "King of Hearts" if I choose a card at random?

Though there was no specific request that examples had to be related to a mathematical context we find the paucity of such examples rather surprising given that the data was collected from the population of mathematics teachers in a mathematics-related course. Our further surprise is that most of the mathematical examples (5 out of 6) related to card games or gambling. We speculate that this is because the participants understood or experienced the effects of such randomness on the desired outcome.

In what follows, we examine the examples generated by participants first using Tsonis' (2008) tripartite model and then by using the characteristics of randomness described by Pratt and Noss (2002). We then discuss features evident in our examples that differ from those in the literature.

4.5. Types of randomness according to Tsonis

According to Tsonis (2008), randomness of first type arises from not knowing the rules of a system either in the case of non-existence of such rules, or of the inability to access such rules because of irreversible programs or procedures that have inhibited us from getting access to those rules. The following are illustrative examples of this kind of

randomness from our data:

- Babies being born with autism
- It will rain tomorrow

While there are scientific explanations for the causes of these phenomena (autism, rain), not knowing the explanations resulted in perception of randomness.

Randomness of the second type arises from a chaotic system, a system which remembers its initial conditions and the evolution of the system is dependent on those, in other words the system is very sensitive to the initial conditions only if we had infinite precision and infinite power we will be able to predict such systems accurately. "Rolling snake eyes on a craps table" (an example from our data) is a perfect fit for this category since from a very precise mathematical standpoint it is possible to propose a mechanical model that describe a fair die and build its motion equations and take into account reasonably all of the parameters that matter such as the speed and angle at the tossing point (with a finite accuracy), the viscosity of air, the gravity at the place of experiment, the mass distribution of the coin, the elasticity factors of the table (or the ground at which the coin hits) and eventually show that the outcome of the two dice is uniquely predictable (Strzalko, Grabski, Perlikowski, Stefanowski, & Kapitaniak, 2009). At the same time rolling two dice several times is completely random due to the impossibility of controlling initial conditions when rolled by human, which makes the randomness inherent to dice rolling examples the randomness of second type. On a less pedantic level, this category can include examples in which the unpredictability of the situation is closely related to the idea that small changes in the initial conditions can produce big differences in the outcome. For example:

- A meteor landing on your house
- An earthquake occurring in Vancouver

We find in examples of this kind the perception of a chaotic system, where the outcome is unpredictable.

Randomness of the third type, in which there is some kind of external environmental component whose essentially uncountable agents (external noise)

continually affect the system with their actions. For instance, consider the time a specific person (Sarah) spends in a shopping mall each time: we reasonably understand how the shopping enterprise takes place so it is not randomness of the first type, there are no initial conditions that the system is super sensitive therefore it is not randomness of the second type, the thing is that Sarah's shopping time depends on several environmental (the number is not known) factors such as how many stores are having sales, what are the line ups like in the stores that she shops and whether she runs into a friend and etc. Either of these factors could be considered as simple and rather deterministic systems but the collective behavior of many interacting systems may be very complicated. In terms of our data, this category includes examples in which the main causes and rules are known but the variability or unpredictability arises from the exposure of those rules to external agents. For example:

- someone will miss our next class;
- leaves falling off the tree when it is not autumn.

A note on fitting examples into Tsonis' model. When trying to categorize examples provided by the participants by the types of randomness in Tsonis' model, one important aspect of probabilistic thinking drew our attention: different people with different backgrounds and knowledge find things random at different levels for different reasons. Consider the "babies born with autism" example. At one level this may illustrate randomness of the first type, since the causes of autism (the rules of the system) and why some babies are born with or without it are unknown to a person with no background in biology. At another level, for a person who is aware that autism has a strong genetic basis, and can be explained by rare combinations of common genetic variants, randomness of the second type seems a better fit. Yet again, the same example could be considered as randomness of the third type for someone who knows more about the technicalities of mutation (that mutation could be caused by external agents such as radiation, viruses and some chemicals). Given the ambiguity involved in this categorisation, we considered what could be gained by attending to different attributes of randomness.

4.6. Features of randomness according to Pratt and Noss.

The resource of fairness identified by Pratt and Noss (2002) was the only one not present in our data. However, we provide examples of each of the other three attributes:

Unsteerability.

- It is sunny in Ottawa on Canada Day
- It will rain tomorrow
- A tsunami being triggered by an earthquake

Some examples in this category related to weather conditions and involved the inability to influence the outcome.

Irregularity.

- A particular bus will stall or break down in any given hour of service.
- Someone will miss our next class
- Getting caught talking/texting on your cell phone while driving
- Giving flowers to your wife when it is not valentines, birthday, or an anniversary

The feature of regularity is usually described in terms of sequence outcomes that have no apparent pattern, rather than in terms of a single event. However, in these examples, the sequencing is implied by a routine, such as taking a bus to school daily, attending the class, texting while driving and not getting caught, acknowledging certain dates with flowers. Here irregularity can be seen as a deviation from the routine.

Unpredictability.

- Dying in an airplane accident
- Being struck by lightning
- Getting a royal flush in poker

A large number of examples attended to events that are unpredictable, with a frequent reference to accidents or natural disasters.

Though we attempted to choose examples that best illustrate each of the

features, it is important to note possible overlaps. The event of a particular bus breaking down is unsteerable (by the participants) as well as irregular and unpredictable. The same consideration applies to getting a royal flush in the game of poker.

A notable feature of the generated examples, which does not appear in the definitions of mathematicians or in the mathematics education literature, is the relative rarity of the described random phenomena. This is related both to events with “known” theoretical probability, such as getting a royal flush, and to perceived random events, which are rare based on statistical occurrences, such as earthquakes and being struck by lightning. Theoretically, getting a particular card from a deck of cards has the same features of randomness as getting “heads” when flipping a coin. However, examples like the latter were not present in this pile.

A feature related to relative rarity of the event was that of accident or disaster: the examples mentioned earthquakes, tsunami, being struck by lightning, airplane crash, or dying of a rare disease. Only three examples referred to “lucky” events: a desirable hand in a card game (2) and finding money on the ground. No one mentioned winning a lottery. Further, pushing rarity to the extreme, three examples described events that under normal conditions would be considered impossible:

- being able to win a marathon with no training
- walking on a crack in tiles when your stride length is some constant
- achieving a “perfect” high-five in a massive crowd of people

The connection between randomness and low probability, either calculated or perceived, should be investigated further as it was not featured in prior research.

4.6.1. Study 2: Discussing randomness with interviewer

In the data presented above, there was no opportunity to ask students for further clarifications—for example, we were not able to ask them why their given examples were random. As pointed out above, many of their examples are subject to multiple interpretations; for example, what notion of randomness does this response entail “forest fires caused by lightning”? Is it the time that is considered random or the location of the

next forest that will be on fire by lightning? Would it be non-random if the forest fire was caused by other reasons?

In order to probe in more depth into learners' discourse of randomness, we conducted several clinical interviews with undergraduate students, three of which are analyzed in this section. In the interviews, the participants were initially asked to define randomness and give examples and non-examples of random phenomena. Then, they were asked to explain why they considered those examples random. They were also asked to comment on the differences, if any, between everyday randomness and mathematical randomness. At the end of the interview, they were asked whether they could give examples of situations that are more random (or less random) than their previous examples. The interviews were video recorded so that both speech and gestures were captured and then transcribed.

In the analysis of the interviews, we focus on the participants' different ways of communicating randomness. We highlight features of the randomness that have arisen both in the literature review and in Study 1. We attend closely to the participants' non-verbal ways of communicating about randomness, with particular emphasis on their use of gestures.

Kevin: What's predictable for you may not be predictable for me. Kevin is a prospective teacher with background in biology. When prompted for examples of random events, Kevin responded offering the following two examples: "sneezing for persons without allergies" and "blinking for some people perhaps." In the following exchange, Kevin associates things that have a higher probability of happening as being less random:

I5: Ok, so for people with higher probability of sneezing, would then sneezing be more random for these people or less random?

K6: I would say less random.

I7: So you are saying that the higher the probability, the less random it is?

K8: Yes, I would say so.

Consequently, low probability is associated with the *relative rarity* of the event. Further, he offers the seeing of meteors as another example of random events, this time alluding to his inability to predict such an event:

K10: I guess may be from my perspective, of seeing meteors. If someone that has the science calculates how much stuff is in the air and how much of the sky you can see and probably can figure out the probability, but to me it would be pretty random.

When asked directly whether “seeing a meteor is random because it is rare and it happens only few times in one’s lifetime,” Kevin emphasizes less the predictability of the meteor passing in the sky than the *unpredictability* of the geographic and temporal location of the seer.

K12: No it is the timing, because you can’t really predict when are you seeing it. There is a lot of meteors out there and they do cross the hemisphere just you need to be there at the right place at the right time. You can’t predict when are you going to see it.

The interview next asked Kevin to “give an examples of an ordinary English language use of the word random.” Kevin explicitly associates random with *unpredictability*, but also evokes its *unexpectedness*.

K14: I think the way it is used in common English, the way I use it is unpredictable¹¹ (Figure 4-1 gesture). For example when I see someone that I don't expect to see I would say that was random. Maybe if I had thought about it I would figure out something in terms of probability or something but in everyday usage if you are not thinking about how often you see something and you see it, makes it all random.



Figure 4-1 The open palms gesture.

In his description Kevin also associated the everyday use of the word random with that of *ignorance*, which can be seen both in terms of “not thinking” but also in the gesture used when he says “unpredictable.” When describing the meaning of “random” in the phrase ‘random guy,’ Kevin evokes the *typicalness* sense of random, referring several times to “average person”:

K16: I guess when I use the term random guy it is to describe someone that I don't know. An average person (Figure 4-1 gesture), someone that is not overly tall or really short, and nothing that sticks out. That would be average (Figure 4-1 gesture) and someone that me and the person I'm talking to don't know. So if I'm talking to my friend and I say a random guy I could say an average person (Figure 4-1 gesture) instead.

¹¹ We will underline words that are accompanied by gestures and describe in parentheses the gesture made.

A slightly different meaning emerges when Kevin describes “random thoughts,” which he sees as deviating from the usual:

K20: I think in that situation random would be the thought that wouldn't follow the preceding. If I am thinking about eating, eating, eating and something comes to my mind completely different (Figure 4-2 gesture) probably it would be random in that situation.



Figure 4-2 Left arm moving to the far left of body.

His gesture evokes the idea that the random thought is not in line with the preceding thought, being out of the ordinary. Finally, when asked about whether he thinks that “absolutely everything is random” or that “nothing is random,” Kevin evokes a sense of randomness related to *lack of structure*.

K22: I lean more toward deterministic, if you could crunch all numbers, then you would probably figure out the probability of something happen, I mean given that all of the variables and stuff are known.

Though he never talks specifically about mathematical probability, Kevin's sense of the specific uses of random in everyday language refer to unexpectedness (seeing someone), typicalness (random guy) and relative rarity (random thought). More generally, he speaks of randomness primarily in terms of unpredictability, but also in terms of ignorance and lack of structure. The more things are predictable, the less they are random. Moreover, predictability is a subjective notion (things will be random to him if he can't predict the outcome, even if others can).

Samantha: The more the number of possible outcomes, the more it's random. Samantha is an undergraduate student majoring in Health Sciences. Samantha was initially reluctant to provide an example of random:

S2: Something unexpected, out of ordinary. Right? Random in what context? Like any?

When told that she could describe any context, she asks whether we are talking about the “mathematical context” (gesturing with both hands to the left side of her body) or “randomness in life situations” (gesturing with both hands to the right side of her body). The interviewer asks for both. Samantha responds as follows:

S6: Um, I think, um, in a way they are too similar as to the meaning of random, like what I said random means unexpected (Figure 4-3 gesture) and the outcome you can't predict (moves hands to left like before when talking about the mathematical context) whether it is mathematics or whether it is life situation.



Figure 4-3 Left hand moving from up to down enclosing/ drawing a closed box type space in front of her.

The word “predict” seems to be associated for her with the mathematical context, given the gesture. She describes random in terms of *unexpectedness* and *unpredictability*.

When prompted again for examples in both contexts, Samantha pauses for a long time and then asks for help. The interviewer invites her to “think of a sentence in which you would use the word random.” When again no answer is given, the interviewer proposes she describes what is meant by “random guy” and “random thought.”

S11: Oh I see, I would use the word random like I am walking in a mall and somebody approaches me and says something completely out of ordinary, just he doesn't know me, I don't know him, and he asks me something that I don't know the answer to it and I never expected him to ask me, that would be completely random.

Here Samantha alludes to the *lack of structure* aspect of random, particular in terms of the expectations. What sticks out in this example is that the lack of relevance between different pieces of the set of objects/actions described is what makes it random. She also alludes to the *lack of structure* in her description of random thoughts:

S13: I have those all the times actually, I think about random things (palms parallel, moving right, left, right) when I'm on the skytrain, like how my thoughts just travel (Figure 4-4 gesture) from one thing to another thing but I can't, or I don't plan these thoughts. That happens all the times (Figure 4-5 gesture) when I'm on a train. Because I have one whole hour (Figure 4-5 gesture) to think about anything (Figure 4-5 gesture) and I don't plan for what I'm going to think about (Figure 4-5 gesture) for example I think about trees (Figure 4-5 gesture) and then from trees I randomly jump on my textbook (Figure 4-4 gesture) and from my textbook randomly moving to what am I going to eat for dinner. Does that work?

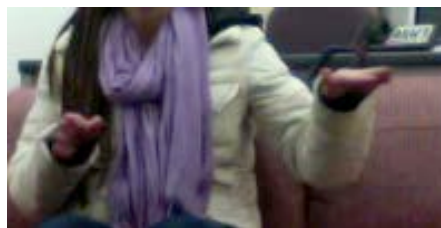


Figure 4-4 Both hands drawing moving arcs

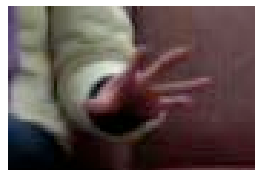


Figure 4-5 Fingers open hand moves away from body

The interviewer prompts Samantha for a mathematical example of random, to which Samantha responds that she took only one mathematics course in university and in that course “we didn't talk about randomness, we talked about probability but we never talked about if it means random.” The interviewer asks whether she could consider the outcome of the roll of a die as random. When she concurs, the interviewer asks what makes it random.

S19: It's unpredictable, you have a set of possibilities (Figure 4-6 gesture) in this case, six different possibilities, but each of those possibilities are unexpected and unpredictable (hands still holding the container making beat gestures at the borders of the container), so each role would be random, I guess. Ok, for example say playing roulette, that would be a game that is random because every time the person spins the ball it is random, the number that the ball will land on is random (Figure 4-6 gesture).

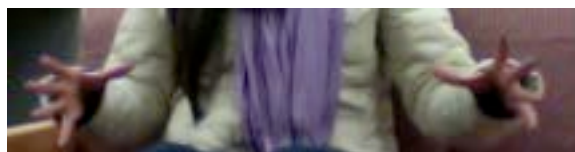


Figure 4-6 Hands enclose a container.

Samantha speaks again of *unpredictability* and *unexpectedness*. But she also emphasizes the way in which randomness is related to having a fixed range of possibilities, which she expresses as well through her container gesture.

The interview then prompts Samantha for “examples of life that are non-random.”

S21: Non-random would probably be like let’s say I’m going to school, all of the events that occur as I’m going to school are not random (Fig. 4.7 gesture). Because they are planned (Figure 4-7 gesture). I would get on the skytrain (Figure 4-7 gesture) and then I would get on the bus (Figure 4-7 gesture) and I would get into my class (Figure 4-7 gesture), like all these events are planned (Figure 4-7 gesture) they are predictable (Figure 4-7 gesture with both hands).



Figure 4-7 Fingers tight, hand moving from up to down, as if putting a stop to a moving thing.

Here Samantha contrasts randomness with things that are planned and predictable. Her repeated gestures evoke also structure and control, which resonates with the *unsteerability* notion of the randomness. The interviewer then asked Samantha to give examples of “events with 100% probability of happening.”

S25: I think it happens with my drawer of socks, because I have a drawer of socks that are all white because I am very organized so I put all of my white socks in one drawer and all of my coloured socks are in another drawer (gesturing a container first to the right and then to the left) so 100% of the time that I go to my white sock drawer, 100% of the time I will pull out a pair of white socks.

I26: Do you see any randomness in this experiment of socks drawer?

S27: There could be, there could be the possibility that my mother put the wrong sock in the wrong drawer (Figure 4-8 gesture) and then I might have a chance of pulling black socks from the white sock drawer (Figure 4-8 gesture).



Figure 4-8 Palm up hand moving slightly away from body.

The interviewer expresses some surprise at the idea that Samantha just puts her hand in and grabs what comes out, to which Samantha responds as follows, using the idea of random as rare, contrasting it with *typicalness*.

S29: (laughs), yes. Well I mean when I look at my white sock drawer it is random to see a black sock, it's not supposed to be there. If there is, that would be random.

At this point, the interview asked Samantha about whether there are things that are more or less random. After a long pause, the interview mentioned the rolling a die situation and asked whether there was “another experiment that is more or less random than this one?”

S32: I think if there was more dice, more than one, um, actually it doesn't work, umm, let's say if there is five dice instead of one, the outcome would be different (Figure 4-8 gesture)? Not sure. It would be more unpredictable (Figure 4.8 gesture)? The outcomes are larger (Figure 4-9 gesture) versus one die when we have six outcomes than if we put more die (Figure 4-9 gesture) it would be 12, 36 and so on, so that would be more random vs. less random, maybe?



Figure 4-9 Both hands enclosing a long container.

In contrast to Kevin, who associated the amount of randomness with the predictability, Samantha associates the range of possible outcomes with randomness. Her gestures indicate the size of the sample space; the bigger the size, the more the situation is random—this kind of thinking is similar to the ‘the more X, the more Y’ intuition, described by Stavy and Tirosh (2000). Moreover, a large sample space is also connected to a *relative rarity* of each particular event, assuming even distribution.

In order to further inspect Samantha’s understanding of random, the interviewer asks her to comment on whether HTHHT is a less probable event compared to other

outcomes since it has a pattern in it and it is not very likely that a coin produces a pattern.

S34: I think that I agree, because when you flip a coin it is 50-50 each time either this or that, it is not like the coin will land straight on the edge, so I don't think that there is too much randomness in flipping a coin.

When asked whether she thinks that a die has more randomness in it since it has six sides, Samantha responds "Yes more randomness if we increase the number of dice." Although Samantha seems unsure about describing and exemplifying randomness, she certainly associates it with being about unpredictability and lack of structure and, in comparing it with non-random, with unsteerability. But she also contrasts it with typicalness. Further, situations are more random the more possible outcomes they have, which is connected to relative rarity. Samantha talks about both random and non-random situations in terms of a range of outcomes; but random events are different because the outcomes can be different.

Tyler: Maybe you can predict it but you can't accurately predict it. Tyler is an undergraduate student majoring in mechanical engineering. He is first asked for examples of random things.

T2: Sure, I suppose everything has a degree of randomness to it. What seat you choose on a bus, what time you head for you know (Figure 4-10 gesture) school in the morning, how much milk I pour (gesturing the pouring of milk) in this coffee, it has elements of randomness to it and of course the obvious ones: playing dice or cards or you know (Figure 4-10 gesture) any sort of video games, RPG's specifically when you are attacking (right hand makes small quick pushes back and forth in front of his face) these little random things (points with whole hand to what looks like an ascending set of stairs with the hand is holding a small object). Even when I'm sipping a coffee the amount of coffee I'm sipping is random because it is not the same every time.



Figure 4-10 As if throwing something to the back.

Tyler's sense of randomness relates here, especially when he talks about the sipping of coffee, to *variability*. When the interviewer asks Tyler to explain why “not being the same every time” is an example of random, Tyler explains that it comes down to a matter of “belief” since there are “two views on how this world works” (the deterministic and the non-deterministic). In the former world, he explains:

T4: [I]f you know all the information about the system you could accurately predict it; meaning it wouldn't be random at all, however (Figure 4-11 gesture) in our world it doesn't really work that way because when you get really really close down like to quantum level to the atoms [...] it is impossible to know the position of atom as well as its direction and velocity and momentum [...] at the same time [...] it is going to have some level of randomness to it.



Figure 4-11 Gesture space moves to his right side of the body as he speaks about real world.

Tyler continues on to explain how the way a drop of water will fall on your hand “is random because depending on how deep it sits on your hand on the molecular level, it might go the other way.” At the quantum level, he describes the behavior of the “is unpredictable,” and asks “and so how can the system as a whole be completely predictable?” Later, he continues on to describe the way in which “when I'm taking a sip

of my coffee there is randomness (hand palm open loosely falls down to his lap) to it and it is almost impossible to say how much exactly.”

Up until now, Tyler’s sense of randomness is related to *unpredictability*, but his talk about the quantum level also involves a level of sense *ignorance* since operations there are unknowable. The interviewer then asked for other examples of randomness and he talks about the exact layout of the table in the room and where a garbage can is placed, saying that “Everything I believe has a degree of randomness to it even if the tiny tiny degree.” He then returns to the idea of *variability*.

Consider M&M’s in a bag. M&M’s are supposed to be made of the same shape (Figure 4-12 gesture) but if you look at each of them you see that one has a little lump to it, other has a distortion to it and however they are manufactured the same way but still there is some degrees of variation (left hand moving from elbow up to down, with wrist rotating as if drawing) between each one.



Figure 4-12 Fingers enclosing a round shape.

The interview prompts Tyler to talk about the use of the word random in common language, mentioning in particular “random thoughts” and “random guy.” For the latter, he describes a random guy as being “out of the normal” and “going against the crowd.” He elaborates that the use of the word ‘random’ in natural speech is:

T10: [...] a little different than its actual meaning, they just mean different, weird and out of ordinary whereas random by definition means unpredictable or means this many (Figure 4-13 gesture) and it is going to be one of them (Figure 4-14 gesture) but we don't know which until you choose. And again about the random thoughts I think the way they use it in speech it doesn't pertain to what they are doing at the moment. If you are playing soccer and you think about your mathematics homework it is random because it doesn't deal with what you are doing at the time.



Figure 4-13 Both hands enclosing a container.



Figure 4-14 Moving right hand toward the left and covering the previously indicated set of things.

In describing unpredictable, Tyler refers, with speech and gesture to the range of possibilities an event can have and to the randomness being linked to the fact that the outcome is not known (until it is chosen). But both examples of non-mathematical uses of the word 'random' relate to *lack of structure*. Tyler also joins Samantha in describing the everyday use of randomness in terms of *not* being typical. This is different from Kevin, who associated randomness with typicality.

The interviewer further prompts Tyler on his definition of probability in terms of unpredictability. Tyler elaborates in more mathematical terms, referring to random events, probability and certainty:

T12: I guess you can predict random events (Figure 4-13 gesture but with palms face down) but you can't say which one for certain it is before it happens. If something is random you don't know exactly what is going to happen maybe you can predict it but you can't accurately predict it, like with 100% certainty. We can say it is going to be probably one of these ten things (repeating above gesture), randomly one of them (Figure 4-14 gesture). So, it could equally be either one (vertical hand movements that partitions the set) or I suppose it doesn't necessarily mean equally but depending on wording I suppose it is implied sometimes?

I13: I don't know, let me ask you this, you said that randomness means not 100% certain, can we come up with examples of randomness with very very high probability, not 100% though?

T14: Sure, like the probability of flipping heads with a coin for 100 times in a row, it is possible; but it is very very random because it doesn't happen very often.

The interviewer asks whether this probability is low and Tyler responds “Yes, what I mean is the probability of this not happening. 99.9% is probability of it not happening.” He continues on to say the event is “very rare but it still might happen, it is random and if it did happen I would say it is very random.” This is the most evident example of connecting randomness to the *relative rarity* of an event.

On gestures that accompany randomness. The sources of randomness identified by the interviewees are similar to, though somewhat more diverse than, those of the exemplification data. For example both emphasize *unpredictability* as a defining feature of randomness. The gestures used by the participants further underscore the way in which the *unpredictability* features strongly in their discourse of randomness. The hands wide open, palm up gesture co-occurs with the word random and seems to communicate a lack of control over the situation, or a lack of knowledge.

Another noticeable aspect of the interviewees' discourse on randomness relates to their strong distinction (expressed through gestures—hands moving to one side, and then to the other; pointing to one side or the other) between everyday randomness and mathematical randomness. Prior research has not drawn attention to the ways in which learners are aware of differences between the way the word 'random' is used in each context or to the distinctions learners attribute to each type of usage. Further, by specifically exploring everyday uses of the word random, we were able to see how the everyday discourse is used in trying to participate in the mathematical discourse so that ideas such as typicality and relative rarity, which are relevant to everyday use, also come to be relevant to mathematical use.

Finally, two of the interviewees made a container-like gesture in several instances when referring to the possible events or outcomes of a phenomenon. Although not explicit in their speech, the gesture suggests that the participants think of the

possible outcomes as belonging to a closed, finite set. This seems particularly relevant to the discourse of mathematical randomness in the sense that one needs information about the distribution of the possible outcomes in order to say something about whether a particular sequence is random. We suggest that this gesture might be helpful to evoke in a didactical situation.

4.7. Concluding remarks

The notion of randomness is central to the study of probability and statistics. However, it is rarely explicitly explained, despite the continued efforts of mathematicians to produce a rigorous definition. Samantha's comment seems to illustrate the issue: "we didn't talk about randomness, we talked about probability but we never talked about if it means random." Researchers in mathematics education have identified some intuitively-constructed features of randomness. However, most studies relied on students considering sequences of outcomes of events, focusing on binary outcomes, such as a coin toss. Our research broadens these approaches, moving away from the context constructed by researchers towards examples and descriptions provided by participants. We focused on (1) how adult learners (prospective secondary school teachers and undergraduate students) exemplify random phenomena and what features of randomness are present in their examples, and (2) how learners communicate ideas of randomness through a broad multimodal discourse.

We first attempted to categorize participants' written examples of randomness using the three types suggested by Tsonis' (2008) model. We found that such classification significantly depends on participants' personal knowledge and experience, rather than on the described event. We then attempted to classify participants' examples using features of randomness identified by Pratt and Noss (2002). We noted that many examples illustrate more than one feature, as there is a significant overlap among unpredictability, unsteerability and irregularity. While these features help characterise participants' intuitions, they do not provide sufficient information on how the notion of randomness is *used* by learners. As such, we extended our study through the use of clinical interviews in which participants' phrasing and gestures could be analyzed.

The following attributes of randomness were featured in participants' responses, extending and refining the list previously compiled by Pratt and Noss (2002): unpredictability, unsteerability, unexpectedness, lack of structure, variability, ignorance, typicalness, and relative rarity. The most common gesture that accompanied the discussion of randomness was that of open hands with palms up, which often communicates in a daily conversations the "I don't know" phrase. It was strongly related to the features of unpredictability and ignorance in participants' talk, which were prominent in this group of participants. Unexpectedness, lack of structure, and variability are related to the feature of irregularity, previously identified by Pratt and Noss (2002).

However, the notion of relative rarity was not mentioned in prior research, but appeared repeatedly both in the interviewers and in the examples provided in writing. That is, events that have low probability of occurring were considered by our participants as random. This was the case both for mathematical randomness and for randomness in everyday usage of the word. Further, the idea of relative rarity, which can be seen as contradicting the idea of typicality, actually appears to co-exist with it. It all depends on the attributes that are associated with randomness.

Based on our findings, we suggest that teachers (and teaching materials) attend more explicitly to learners' discourse on randomness and, in particular, to learners' everyday uses of the word. These can be compared and contrasted with different definitions of randomness available in the mathematics discourse. It seems that the notion of randomness that is related to the passing of certain statistical tests might be the most distinct from learners' discourse on randomness—this very pragmatic view of randomness seems particularly amenable to learning situations involving computer-based technologies.

Chapter 5.

Fixed, but Unknown; Undergraduate Students' Notions of Probability of Single, Past Events

5.1. Introduction

What is the probability of a fixed but unknown past event? This question reflects on two of the old issues debated by subjective (Bayesian) and objective (classical) schools of probability: the problem of single events and the problem of past events. This study will examine undergraduate liberal arts students' answers to the question of probability of a fixed and unknown event. I suggest that reflective tasks where the students are asked to examine and reflect on two opposing probability arguments offer a new space for learners to reconcile some conflicting probability ideas. To situate my study and my task I start by discussing the notion of ambiguity in mathematics. In particular, I suggest that probability is an ambiguous notion for learners; this ambiguity is best witnessed in different subjective and objective interpretations of probabilistic problems. I then turn to the particulars of a small study in which the issues were explored.

5.2. Ambiguity

Mathematics is commonly viewed as a precise discipline, where painstaking efforts are made to define terms in a rigorous and clear manner. The last thing we expect from a mathematical notion is being ambiguous. Any possibility of more than one interpretation for a mathematical expression seems to arise from sloppy use of language rather than any uncertainty in the mathematical ideas, or is it the case really?

Mathematician Bill Byers (2007) believes that this cannot be further from truth. He sustains that mathematics is an ambiguous subject by nature and the ambiguity is where all the new thoughts and creative ideas come from. Defining ambiguity in such a way that it essentially doesn't lose practical usefulness is a tricky task since arguably any word can be ambiguous. Byers' definition of ambiguity is adopted from a definition of creativity offered by Arthur Koestler (notable author and journalist). He swaps the word "creativity" in Koestler's definition with "ambiguity" and defines ambiguity as: "ambiguity involves a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of reference" (p. 28). He suggests that the resolution of the conflict results in creation of ideas that link the two; these new ideas in turn generate a deeper understanding of the concept itself as well as other related mathematical ideas.

Byers' definition makes ambiguity distinct from hazy vagueness resulting from unclear communication. For instance, he discusses the ambiguity in square root of 2: "Square root of two existed for Greeks as a concrete geometric object, but problematic when considered as an arithmetic object" (p. 35). This was due to restricted view of numbers at the time: numbers represented geometric magnitudes, and any two geometric lengths were believed to be commensurable. A conflict resulted when incommensurability of the diagonal of a square with the side length of the square was proven. After this, a more general doctrine of proportionality was developed. This development opened the window for proofs involving incommensurable lengths (Book V of Euclid's *Elements* for example). This, in turn, resulted in a deeper understanding of the notion of numbers as fractional numbers (rational) and non-fractional ones (irrational).

5.2.1. Ambiguity in mathematics education

Gray and Tall (1994) present instances of ambiguities students have to deal with in mathematics classrooms. Those ambiguities involve the mathematical symbol used for both the process and product. For example, $4 + 5$ is used both for the product of the concept of addition (in this case, 9) and for the process of adding (as in counting all). Another example is how the equation of a function $f(x)$ simultaneously tells us how to

calculate the function for a given value x , it also presents the function value for a general value x .

Barwell (2005) proposes that since meaning-making in mathematics is a discursive and social process, ambiguity can be seen as a resource for doing mathematics and for learning the language of mathematics.

Grosholz (2007) suggests that once distinct representations of a concept are compared and contrasted, it often results in generating new knowledge. She calls this a *productive ambiguity*. Foster (2011) offers instances of productive ambiguity in mathematics classroom:

An ambiguity is not the same as an error, a paradox, a contradiction, an absurdity or a fallacy. An ambiguity derives from a significant degree of uncertainty, caused by a lack of specification regarding a particular feature or an unstated assumption, paradigm or frame of reference. This results in an ability to see the same situation in more than one way. (p. 3)

He identifies four types of mathematical ambiguity:

- Symbolic ambiguity such as \times being used both for the product of two real numbers and the product of two vectors
- Multiple-solution ambiguity, such as with inequalities (e.g., $x > 7$, but how much greater?)
- Paradigmatic ambiguity, such as children perceiving thin plastic objects either as idealized two-dimensional shapes or as three-dimensional objects of very slight thickness
- Definitional ambiguity, such as the word radius being used to represent the line segment itself or its length.

In my view, probability involves at least two types of these ambiguities: definitional ambiguity, and paradigmatic ambiguity. The term probability is used to refer both to the name of branch of mathematics (that quantifies uncertain phenomenon) and as the specific numeric measure of uncertainty calculated or assigned to each event. Also probability is perceived both as a manifestation of physical tendencies of an object (as in two-facedness of a coin) and as a person's state of mind and belief about the physical phenomenon (such as our belief in fairness of the coin).

5.2.2. Ambiguity in probability

Ambiguities are to do with assumptions and perspectives, and probability has notoriously been the subject of controversies and interpretations. Following the definition of ambiguity by Byres, I argue that the two major takes on probability are separately self-consistent but in tension and conflict with each other.

The classical approach (a name commonly used to refer to both frequentist and theoretical approach) looks at how procedures perform over all possible random samples; the probabilities relate to the procedure and not to the particular instances or samples. The numeric features of population (called parameters) are fixed but unknown. So, probability statements cannot be made about them. Instead since the parameters are considered to be parameters of the sampling distribution, probability statements can be made about the statistics (estimates of the parameters obtained from a sample). Such statements are converted into confidence statements, which are used in hypothesis testing.

The other approach, known as Bayesian probability (or subjective probability) applies the laws of probability directly to the instance (as opposed to the process); the probabilities are obtained subject to a person's knowledge. Since we are uncertain about the true parameters, they are considered to be random; this allows for direct inferences about the parameters. The probability statements are interpreted as degree of belief. The beliefs are revised after getting new data (see Bolstad (2007) for a detailed comparison between classical and Bayesian methods).

Practitioners of each method claim advantages over the other and at times they deem the other approach as misguided and flawed.

At any rate, both classical and Bayesian approaches have to deal with certain flaws, internal contradictions, or conflicts. Two instances of such ambiguities in probability are the problem of single-case probability and the problem of past event probability. I present a discussion of the two problems and examine them from subjective and objective probability perspectives. Following the discussion, I introduce

the study along with analysis of participants' responses to two arguments on probability of a single past event.

5.3. The problem of single-case probability

The classical frequency-based probability is based on the notion of repeatable experiments and the limit behaviour of the resulting sequence. In many applications of probability we need to quantify our uncertainty about a phenomenon that is not subject to such setting. That is to say it cannot be repeated infinitely many times, or even a large number of times either because they are past events or because of the unrepeatable nature of them. Some examples are: the 2000 US presidential election, Kennedy assassination, and possible death of planet Earth due to collision with a wandering star (examples from Gillies (2000)).

From a classical perspective the probability of a fair coin showing heads is one-half since it is interpreted in the context of infinite coin flip trials, the probability that a specific coin shows heads in a specific flip is not relevant unless *relativized*¹² to the class of infinite coin tosses. Unlike the classical approach, a subjective approach to probability allows probability to be assigned to single instances, because subjective probability is the relation between a person's degree of belief, a relation between knowledge, evidence, and the external phenomenon.

Gillies (2000) addresses this issue at length: the example he uses to illuminate the problem of single-case probability is the question of what is the probability that a 40-year-old English man named Tom lives for another year. This question cannot be answered from a classical (theoretical and frequentist) point of view, unless Tom is considered an instance of the family of all 40-year old English men. As an instance of this category, the frequency of success (living for another year) could be obtained from the historical data and the relevant life expectancy estimates could be calculated. The issue is that Tom can be *relativized* to more than one category: there is the probability-

¹²Gillies uses the term "Relativizing" as relating a single instance to a class consisting of large number of identical independent instances similar to the one singled out.

qua-40-year old male, the probability-qua-heavy smoker, the probability as being from a family of heavy smokers that have lived past 80 years, and the probability of the conjunctions: being a 40-year old heavy smoker English man, a male smoker, English male from a family of long living people and so on. Also we never get to the bottom of the list of the characteristics that *relativizes* Tom, for example he may be the only one in his family that plays football twice a week. Moreover, the frequency data in all these categories may not be available. Gillies sums up this discussion suggesting that with single events, probabilities “will nearly always fail to be fully objective because there will in most cases be a doubt about the way we should classify the event, and this will introduce a subjective element into the singular probability” (p. 120).

5.4. The problem of probability of past events

Is probability only about predicting future outcomes of a probabilistic situation or does it offer help with uncertainties about events in the past? Can we assign probability to events that occurred in the past, and if so, how these probabilities should be interpreted?

For example, a historian may wonder about the chances that the two princes in the tower of London were actually murdered on order by Richard III. A crime enthusiast or an investigator may want to calculate the chances that a certain suspect was the infamous Zodiac killer. Or we may want to know the odds of certain mutations scientists suggest having occurred six million years ago have actually occurred then. Can these probabilities be assigned and interpreted from a theoretical or frequentist point of view? In these examples the outcomes of these events are fixed; they either are one way or the other. The physical world no longer offers changes around these events; tendencies and propensities of the factors do not matter anymore, the dice have fallen. Any statement of probability will be about a relationship between a person and the world about which the statements are being made.

5.5. Undergraduate students' notion of single past events

Today's probability practitioners employ methods and approaches to probabilities that are sometimes a mixture of opposing ideas. The practice of teaching probability both in K-12 and post-secondary education has not responded to this change aptly. While a small number of probability and statistics textbooks try to bring together some of the useful aspects of classical and subjective probability, teaching and learning of probability is dominated by the classical view. None or very little class time is spent on historical highlights of probability development, the difference in subjective and objective probabilities, underlying assumptions, limitations, and what is offered by each probabilistic approach.

In spite of this, my aim is to examine, from undergraduate students' point of view, two sides of an argument where two probabilities are assigned to a single occurrence. Also of interest point to this study is to look into how learners resolve the ambiguity, once they encounter conflicting probability situations.

5.6. Participants

Data were collected from a class of undergraduate students enrolled in a mathematics education course in a Canadian university. This course is one of the core requirements of the liberal arts and social studies programs. The participants' mathematical background is moderately weak; they have not taken post-secondary courses in probability and statistics. This course explores various aspects of numbers such as prime numbers, modular arithmetic, irrational numbers, infinity, and probability. The class spends three to four weeks on probability, covering topics including: theoretical probability, experimental probability, calculating probability of compound events, and conditional probability. At the end of the probability chapter, the students were given the task of this study. The task presents two arguments about the probability of a coin flip event. The participants are asked to take side with the argument they find valid and express their thoughts.

5.7. The task

A group of students are asked to decide on the probability of the following event: A fair coin is flipped and covered so we cannot see the outcome.

What is the probability of having flipped “Heads”?

The responses of two students to this question are presented here:

Thelma:

The probability of the coin showing “Heads” is 50%. It is a fair coin and each of these possibilities has the same chance of happening.

Louise:

The probability of this coin showing “Heads” is either 1 or 0, because it is flipped, and therefore fixed. It is not going to change now. So the probability is unknown, but it is not 50-50 anymore.

Which argument do you agree with? Clearly circle:

Thelma

Louise

Neither

What do you find in favor of your chosen argument(s)? Explain.

What is wrong with the other argument(s)? Explain.

The task of this study underwent several trials and changes before its current version. The first round I started with asking the participants (not the same group of undergraduate students participating in the current study) to calculate the probability that a flipped and covered coin (so that we are unaware of the outcome) has actually landed heads up. However, the participants' responses were brief and one-sided. To most of them the probability in question was 50-50 and the reasons (if given) were in regard to the fairness of the coin and had very little to do with the flipped and covered situation. The responses I got helped me phrase the two arguments of the current task. The second time (reported in this study) I presented both conflicting arguments and asked my next group of participants to take sides with an argument and explain why. The forced decision-making was deliberately included to avoid inconclusive responses, such as agreeing with both arguments.

This task is simple enough for the respondents; it requires no calculation, background knowledge, or formula. Both arguments presented in the task make sense and each has a point. Also the problem is about a single flip and it had already landed, therefore it addresses both single-case and past event issues. As mentioned earlier, the class where I collected data had spent a couple of weeks on probability and covered simple and compound probability calculation, enumeration techniques, and experimental probability. However issues of single events and past events were not discussed in the class, which the participants were taking at the time of data collection.

5.8. Analysis of the task

It is said that the future will resemble the past in some respect. Thus a classical statistician would make use of the recorded historical frequency of a coin in probability assessments about the future behaviour. This argument can be flipped around and said since we have reason to believe in fairness of the coin at the time of the flip the probabilities *have been* 50-50. But the moment the outcome is settled, the space of possibilities has diminished to either {H} or {T}, depending on the actual outcome, the probability of having ruled Heads is therefore zero or one. Hence, a classical response to this question may be either Thelma (concerning a point in time just before the flip) or Louise (right after the coin is settled).

A subjective response however is more likely to be Thelma only, since subjective probabilities change only in light of new information. As long as new information is not obtained, the experiment is not finished. I have come across three different but equally interesting opinions about this problem in the literature: Foster (2014) in his discussion of unnecessary ambiguities in textbook-recommended formulation of the interpretation of confidence intervals¹³ refers to this task and resolves the ambiguity by suggesting:

Whether it [the flipped but covered coin] is heads or tails it is fixed and cannot be changed, yet because we lack knowledge we say that the probability of heads is 0.5, because we know that 50% of the time it will be heads. (p. 31)

He compares this “fixed but unknown” situation to that of Schrodinger’s cat and concludes since the population mean is fixed but unknown, we may express our knowledge of it in a probabilistic way similar to the coin situation. This argument is well aligned with Bayesian methods of probability calculation where “every probability is conditional, at least to the knowledge base” (Lindley, 2014, p. 51); the prior probability of an event is only changed when new information is received on the situation and unless the coin is uncovered, the probability distribution stays the same as before.

Although at the time of reading Foster’s argument I found it reasonable, I questioned the way it was presented as a commonly agreed upon fact: “yet because we *lack knowledge* we say the probability [...]” (p. 31). It is taking for granted that not everyone considers probability to be about what we know or do not know. A great deal of controversies around probability concerns whether it is an objective feature of the phenomenon addressing the physical aspects of the probabilistic situation or is it a temporal feature of our perception of the phenomenon and the knowledge we possess about it at any given time.

¹³ Certain Probability and Statistics textbooks insist on very specific wording in interpreting the meaning of a constructed confidence interval for the population mean. The recommended formulation is: if the procedure involved in the construction of confidence interval is repeated for a large number of times in 95% of the times the constructed confidence interval will contain the population mean. This emphasis is made to avoid interpretations such as: I am 95% confident that the constructed interval will contain the population mean, or in 95% of times population mean will fall within the constructed interval.

Notable author on decision theory Sven Hanson (2010) writes in the very first sentence on his discussion of past events: “the probability that a fair coin tossed yesterday landed heads is either 0 or 1” (p. 207). He suggests to use the term *past probabilities* to refer to such events. He makes a distinction between probabilities and our estimates of them and argues that in the cases similar to the covered coin, the discovery of results will only change our estimate of the chances (of showing Heads) and not the chances.

Both the above mentioned authors see it fit to assign probability to the coin flip event and acknowledge that the probabilities one way or the other are related to our ignorance of the outcome. Another way of responding to this task is to exclude it from what could be considered as the subject matter of probability. A strictly frequentist point of view declares all single-case probabilities irrelevant. Thus both arguments are invalid on the grounds of the event not satisfying the requirements of a mathematical probability setting. For example, Sowder, Sowder, and Nickerson (2010) dismiss problems that lack certain probabilistic features such as repeatability and general ignorance (as opposed to personal ignorance). They suggest that uncertain situations and probabilistic situations are not necessarily the same and are to be treated differently when it comes to probability calculation. For example, they offer, in a room without window it is uncertain that it is raining or not; but this is not a question of probability (because it is actually one or the other). By the same token probabilities of past events could not be determined on a mathematical basis since they are not repeatable and their outcome is already fixed, regardless of our ignorance of those outcomes.

5.9. Analysis of data

I collected twenty-five undergraduate students’ responses to this problem in questionnaire format. As stated in the task, the respondents were given both arguments and they were invited to comment on the validity of each. They could have also chosen the “neither” option and propose a different argument.

The responses obtained from the questionnaire data are summarized in table below.

Table 5-1 Quantitative summary of responses

In favor of Thelma's argument	10
In favor of Louise's argument	12
Neither	3

The responses choosing neither

One of the three respondents that chose 'neither' did not propose a counter-argument. The only explanation he offered is that both arguments are valid and he cannot decide on one. The other two respondents in this group questioned both probabilities suggested by Thelma and Louise and made interesting points. One respondent has appealed to the unknown factors affecting the flip of the coin to reject both probabilities:

"Which side starts may make a difference, so does the force and how many spins".

This response aligns with the problem of single-case events. The respondent does not *relativize* the coin (although mentioned in the task to be a fair coin) with the class of all fair coins flipped under identical circumstances. The other respondent questions the single-case probability problem in a frequentist sense:

"In order to find a probability of a coin being heads or not the coin would have to be flipped say 100 times and deduce the frequency of heads according to the results".

The responses in favour of Louise

Responses in favour of Louise's argument, consistently emphasized that the probability would have been 50-50 if the coin was not yet flipped, but not after the event already happened. One person elaborates on this point: "... because there are no longer *Possible Outcomes*, but an *Actual Outcome*, We cannot be sure if it is zero or one, but it is definitely one of the two". Another respondent highlights the fact that, "The coin is not going to change when you uncover it. It is already determined, just unknown to us". It

seems to me that this group of respondents felt very strongly about the argument they were making. Their responses were unusually loaded with underlined or capitalized words as shown in this reproduced sample:

“The coin is already flipped, so whichever side it landed on has ALREADY HAPPENED. There is no chance of it being one way or the other now because the action is completed- it is simply either Heads (1) or Tails (0)”.

The factors pointed out by this group of respondents can be summarized as:

- Stability of the coin which results in one possible outcome.
- The coin being covered has no impact on the outcome and thus on the probability.
- When there are no possibilities for variation, the probability has to be either zero or one.
- Our ignorance doesn't make a difference in the probability of the coin having landed on Heads.

Based on these points it seems that to this group of participants, probability is an objective feature of the coin, what Hacking (2006) refers to as: aleatory, the probability that is governed by physical phenomenon. Probability is viewed by these participants as a statement about the coin, the emphasis is put on the fact that the coin has landed on one side and therefore any middle-value probability is rejected. As shown from the above-mentioned excerpts, probability one and zero entail the coin showing Heads and Tails respectively. Several responses point out that since the coin is flipped; the outcome is either one or the other and not both. This evokes thought about the state of possibilities before the flip is settled. I suggest that at *any* time (before the flip and after), the coin can assume only one of the landing outcomes and not both. Therefore Louise's argument, which declares the probability to be either zero or one, offers no clearer state of mind about the situation. For instance if someone is to gamble on the outcome of the covered coin and wants to know the odds, it will serve no purpose if he is told that the probability is either zero or one.

The responses favoring Thelma

The responses that favor Thelma's argument can be divided into two groups: in one group respondents write at length to explain the reason for a fair coin producing 50-50 odds, they explain the notion of fairness, equiprobability, and sample space of a coin flip. The part of the task that states the coin is flipped and covered doesn't seem to matter to this group of respondents.

The responses in the other group offer a richer perspective; these respondents make several references to personal aspects of probability and draw links to what "we know", or "don't know":

"I find I am in favour of Thelma's opinion because since we do not know the result of the flip the probability of it being heads is 50%".

Another response explains why what we know matters:

"The probability to guess what the coin has flipped is 50%".

What is notable about this response is that it pictures probability as a numerical tool that helps us make educated guesses about the uncertain events. The notion of guessers is echoed by another response from the data:

"Even though the result of the flip has already been decided, we as guessers can still say that there are 2 different options for the coin to show".

Another response that clarifies and supports the same point and adds human criteria to the notion of random experiment offers:

"Because of the fact that we have not observed the coin thus the outcome has not been recorded. This means the experiment is still incomplete, therefore the chances of heads/tails are still 50-50".

What I notice in this group of responses is that although the reasoning initiates from a classical perspective as they all acknowledge that for a fair coin the probability of

showing Heads is 50%, but the task manages to draw their attention to how the objective chance is embedded in our subjective probability assignments.

5.10. Discussion

Ambiguity is commonly considered as failure in communication. Research in mathematics education suggests that ambiguity is necessary for ideas to move forward because it creates instability in what is known that allows the formation of new knowledge. Gray and Tall (1994) suggest that as much as mathematicians abhor ambiguity and try to stand clear of it, but the ambiguity in interpreting mathematical symbolism is at the root of successful mathematical thinking. Barwell (2003) has pointed out that “once some degree of ambiguity is constructed a space opens up for the students to explore [...]” (p. 4).

In this study, some of the participants develop and acknowledge the coexistence of an objective and a subjective interpretation of probability in an example; on one hand, the probability is a notion that addresses physicality of the coin, on the other hand, the subjective view that is about what “we know as the guesser”. This broadened concept of probability in which physical probabilities are embedded inside a domain of subjective considerations, accommodates for routine probability calculations involving standard randomizers such as coin or dice. Moreover, it allows for probability estimates conditioned on a personal knowledge base when the physical aspects of variability could not be accessed. This study suggests that deviating from the typical probability calculation tasks and engaging students in assessing the validity of conflicting or opposing arguments on probability calculation provides students with new perspectives on probability, where they can enrich and expand their understanding of probability.

Chapter 6.

Summary and Conclusions

This thesis is comprised of four stand-alone studies. The first three studies examine learners' definitions, descriptions, and examples of probability, extreme events, and randomness. In these studies, I have initially attended to the institutional meanings of these notions as reflected in textbooks and teaching resources. Next, I have collected and analyzed undergraduate students' descriptions and examples of these concepts. The last study focuses on one of the ambiguities inherent to probability calculation. In situations where the uncertainty resulting from physical phenomenon and the uncertainty arising from our lack of knowledge about the phenomena do not align, it is often unclear whether a probability can be assigned to a certain outcome or not. In order to address this ambiguity, I examined undergraduate students' responses to two different probabilities assigned to a fixed but unknown event.

In this chapter I present a summary of findings of the four studies. I then address my research contributions, implications, and limitations. In the end I offer some thoughts and considerations for further research.

6.1. Summary of results

The first study, Undergraduate Students' Definitions and Examples of Probability, examines institutional and personal definitions, descriptions, and examples of probability. A review of definitions offered by textbooks and other resources suggests that most post-secondary and teacher education resources fall under one of these two descriptions: they either assume an existing understanding of probability or they offer a limited view of probability. The first group of resources do not give a definition of

probability and, more importantly, do not talk about probability as a mathematical model, nor do they describe how it operates, when it is applicable, when it is used best and what types of problems it cannot solve. The resources that do write about and around probability tend to consider probability to be an objective fact about the uncertainties of the physical world (such as a coin) and thus, prescribe a theoretical and/or frequentist approach to probability. In most resources (those that attend to definitions of probability), subjective probability, either as in an axiomatic approach to probability or as in subjective aspects of theoretical and frequentist probability are not considered or acknowledged.

In the empirical part of this study, a group of undergraduate students enrolled in liberal arts programs were asked to describe and exemplify probability. The research questions pertaining to this study aimed at examining how undergraduate students describe probability, in particular:

- What features of probability appear in students' definition/descriptions of probability?
- What features of probability appear in students' examples of probability?

The analysis of the data suggests that very few mathematical features are apparent in participants' definitions of probability and that a mathematical notion of probability is formed, conjoined and sometimes overshadowed by the everyday use of probability. Also, looking into definitions and descriptions provided by students brings to attention that it can be very difficult for students to formulate a definition of a concept. It seems that an easier task is to provide examples of a concept. In comparison with definitions, there were more mathematical features of probability presented in the participants' examples. This highlights the importance of looking into learner-generated examples of a concept coupled with definitions since the examples reveal aspects of the concept not displayed in definitions. This feature (investigating examples of a concept) is not mentioned or emphasized in the Batanero-Diaz (2007) model of essential components in understanding probability. In light of the results of this study, their model could be extended by adding learner-generated examples as an essential element for investigating understanding of probability.

The second study, “How Impossible Is the Impossible?”, explores secondary school prospective teachers’ notions of probability through their examples of extreme events. The paper reviews the institutional definitions and examples of zero-or-one probable events and the commonly used terminologies to refer to such events, i.e., impossible and certain. The review suggests that extreme events have educational and mathematical importance and that most textbooks and resources offer definitions and examples of zero-or-one probable events and the terms impossible and certain. The more interesting finding of this review is that although zero-probable and impossible are synonymous terms when the sample space is finite (likewise with one-probable and certain), they point to different events and even different notions when the sample space is infinite. This distinction is less frequently acknowledged or highlighted in resources. Learners, too, often miss this distinction and make a one-to-one mapping between impossible and certain, on the one hand, and zero-probable and one-probable, on the other. This mapping may result in conflicts when the calculated probability and the perceived probability of an event do not match. The conflict is likely to happen due to overlap between everyday and mathematical meanings of the terms probability, impossible, and certain. In this study, a group of prospective secondary school teachers were asked to give examples of four events where the probabilities are: exactly one, nearly one, exactly zero, and nearly zero. The study addressed the following research questions:

- What aspects of probability (mathematical or everyday) are featured in the participants’ examples of extreme events?
- What probability perspectives are involved in the participants’ examples of extreme events?

In order to analyze the data, Barwell’s (2003, 2005, 2013) discursive model was adopted. The features of the model relevant to this study were selected and modified, and were used to categorize the responses into everyday and mathematical. To further explain the observations made from the data as well as the mismatch between mathematical probability and perceived probability of an event, a second round of data analysis was conducted. This time the data was examined based on the type of probability represented in the examples.

The combined result of the analyses suggests that the participants use a range of subjective, theoretical, and logical approaches to construct probability examples in everyday and mathematical contexts. The logically impossible-or-certain probability serves as the basis and common ground for mathematical and everyday probability. Also, the purely everyday examples are more subjective and the mathematical examples have more theoretical features. The results highlighted the existing confusion between various approaches to probability among the participants. Furthermore, the results identified the need for a clear distinction between zero-probable and impossible in probability instruction and called for a pedagogical attention to this issue.

The third study, “It is very, very random because it doesn’t happen very often”: Examining learners’ discourse on randomness”, examines prospective secondary school teachers’ descriptions and examples of randomness. Similar to the first and second studies, a review of definitions and examples of the terms random and randomness from a mathematical point of view is presented. As was the case for the first study on definitions and examples related to probability, the majority of the frequently used resources, such as undergraduate probability and statistics textbooks and teacher education textbooks, do not offer explicit definition of the term “random”. However, this time the absence of definition is not simply due to taking things for granted. We¹⁴ reached for a wider range of resources (Terence Tao’s blog, journal articles, and books on the subject of randomness for example). Our review revealed that even within the mathematical domain, there is no single agreed upon definition of randomness. As a result of mathematicians’ efforts to provide a rigorous definition, several approaches, methods, and tests for identifying and creating randomness have been developed—and this work is ongoing.

This study addressed the following research questions:

- How do learners exemplify random phenomena and what features of randomness are present in their examples?
- How do learners communicate ideas of randomness through a broad multimodal discourse?

¹⁴ The third study was co-authored with Rina Zazkis and Nathalie Sinclair

The data presented in this paper were collected in two rounds via questionnaires and interviews. In the first round participants were asked to give examples of random phenomenon. Later, through one-on-one interviews, the participants were asked to elaborate on mathematical randomness versus everyday randomness. Building on the work of Tsonis (2008) and of Pratt and Noss (2002), we classified participants' examples based on the type of randomness featured in each example. Some of our findings were strongly related to the findings of previous research; for instance, as cited in prior research, unpredictability, unsteerability, unexpectedness, lack of structure, variability, ignorance, and typicalness were among common features of examples from our data. We added to this list the notion of relative rarity that appeared repeatedly both in the interviews and in the examples provided in writing. In both mathematical and everyday contexts, participants of the study associated rarity with randomness.

In second part of this study, we interviewed the participants and analyzed their gestures along with their verbal communications. It was a different approach and enabled us to identify some aspects of randomness that either didn't come up or were only implicit in participants' verbal responses. For example, it was revealed that the participants distinguish between mathematical randomness and everyday randomness. Some of the participants kept the gesture space on different sides of their body when talking about each of those randomness types the entire time. The main sources of randomness apparent from participants' gestures include typicalness (evident in the same gesture with every reference to "average"), variability (evident in coffee sipping gesture and quick hand movement from left to right representing a range of values or possibility), and unpredictability (evident in hands away from body gestures when talking about randomness). For some of the participants randomness was contrasted with control (stopping gestures when talking about planned activities), or structure (making container and divider gestures). Compared to examples of randomness, the sources of randomness identified by the interviewees are similar to, though somewhat more diverse than those of the exemplification data. For example, both emphasize unpredictability as a defining feature of randomness. The gestures used by the participants further underscore the way in which the unpredictability features strongly in their discourse of randomness. For example, two of the interviewees made a container-like gesture in several instances when referring to the possible events or outcomes of a phenomenon.

This gesture was followed by a divider gesture, referring to possibilities within the sample space. Although not explicit in their speech, the gesture suggests that the participants think of the possible outcomes as belonging to a closed, finite set and the larger the container is, the more random the outcomes will be (they gestured to a larger box when talking about a more random situation).

Based on our findings, we suggest that teachers (and teaching materials) should attend more explicitly to mathematical and everyday meanings of randomness and, in particular, to learners' everyday uses of the word. Our study suggests that there are similarities between mathematical definitions and everyday notions of randomness (such as typicalness and lack of structure) that could be exploited to enhance a richer understanding of randomness. But we also found that the notion of mathematical randomness defined through passing certain statistical tests might be the most distinct notion of randomness from learners' discourse on randomness, that is to say this never was pointed at by examples or participants' verbal and non-verbal communications. This very pragmatic view of randomness seems particularly amenable to learning situations involving computer-based technologies. With the easy access to computers these days, learners of all ages can be guided through investigative methods of determining if and to what degree a given sequence of numbers (or a sequence of patterns) stands the randomness tests.

In the last study, "Fixed, but Unknown; Undergraduate Students' Notions of Probability of Single, Past Events", undergraduate students were invited to take sides on a subjective versus objective debate by assessing the validity of two arguments presented in the task. Through the problem of flipped but covered coin, I presented to the participants of the study two reasonably valid arguments, one pertaining to the physical world where the coin is flipped, and one to our knowledge. My goal was to examine how learners resolve the ambiguity, once they encounter conflicting probability situations.

Frequentist probability is only defined under well-defined conditions such as repeatable identical experiments conducted a very large number of times, thus it does not apply to single events. Theoretical probability is based on careful examination of how

the set of desirable outcomes weighs against possible outcomes. This method, similar to the frequentist approach, works best when the long run tendencies are investigated. For a single, past event where the wave of possibilities has faded and the only remaining uncertainty stems from our personal knowledge of matters, none of these approaches really apply. In subjective probability (also known as Bayesian probability) every probability is conditional, at least to a knowledge base if not to other conditions. It makes sense to determine the probability of a single or past event since the probability captures the person's knowledge relation with the physical world via taking into account his beliefs, information, observations, new evidence, and so on. A subjective probability calculated based on rational beliefs and logical deductions produces similar results to the other two methods (frequentist and theoretical) in the long run, but certainly has a wider range of applicability and more compatibility with learners' personal notions of probability. My research proposition was that probability is an ambiguous concept and learners' everyday notions of probability can be at conflict with mathematical probability. I also proposed that students can resolve the conflict and accommodate for both notions of probability by nesting mathematical probability into the everyday domain.

The analysis of responses suggested that undergraduate students who responded to the task of this study were strongly divided, manifesting the ambiguity in probability. Also it was revealed that the respondents have a limited view of probability (strictly theoretical or strictly everyday). However, during the course of engagement with the task, some students were able to reflect on both aspects of probability and offer an explanation that accounted for both mathematical and everyday aspects. In light of this finding, this study suggests that deviating from the typical probability calculation tasks and engaging students in assessing the validity of conflicting or opposing arguments on probability calculation (reflective tasks) provides students with new perspectives on probability, where they can enrich and expand their understanding of probability.

Taken together, the four studies contribute to the ongoing research in probability education on several arenas. They provide enhanced understanding of how participants perceive probability-related ideas, as evident in their examples, definitions and gestures. They highlight subtleties and ambiguities in interpreting probability-related concepts and probabilistic events. Furthermore they emphasize the necessity and present ideas for a

different pedagogical implementation, which acknowledges a variety of approaches to probability. Implications for teaching are elaborated upon in section 6.3.

6.2. Limitations and challenges

Once, I heard from a veteran researcher in education that a thesis is the least important work a researcher produces, because real research needs resources and connections and a student has no or very limited amount of them. Although this may be truer for bigger projects, it is still true for my small undertaking presented in this thesis.

To find willing and qualified participants has been consistently a challenge for me. The questionnaire data is often collected from a class that a friendly colleague or I have been teaching at the time, and the subjects for the interviews are found only after many rounds of emails, personal presentations, and several bags of cookies. This has some implications for my data. For example, my interview participants are not randomly chosen; they volunteered either out of pure sympathy, or because they saw value in “donating their time for science” (as Tyler mentioned to me). They were not mathematics-averse and didn’t turn away from talking about mathematics (as many would do!). Therefore, the interview data I collected (reported on in chapter four) relates to a population of mathematically confident undergraduate students. Another challenge related to the interview data was presented by the gesture analysis. We had to make several exchanges of ideas through several rounds of watching the recordings in order to collectively decide about which gestures to address and how to interpret them. Even so, our interpretations are subjective and inevitably framed by what we were looking for.

The questionnaire data were collected from several undergraduate and teacher training classes in the faculty of education at SFU. The participants were approached by the researcher during the class time and were asked to fill the questionnaire if they were willing to participate in research and the responses were collected anonymously. This method of data collection has limited my sample to students enrolled in mathematics education courses. Typically, these students have a weak mathematical background and they are re-learning the K-12 concepts including probability. Another limitation is posed by use of questionnaires. On the bright side, this method results in higher participation

rate and data that includes written responses which are sometimes ideal for analysis purposes. The respondents get a chance to work on the task at their own pace without being pressured by the social conventions of a conversation (as present in interview situations) or being interrupted by the researcher. However, with questionnaire data collected as described above, I wasn't able to follow up with the responses or to have the respondents clarify or elaborate on the interesting or unclear points.

A different type of challenge I faced was lack of appropriate theories (more accurately said, my lack of success with coming across such theories) relevant to examining definitions and examples of probabilistic notions. Research attempts had been made to understand learners' probabilistic thinking (the term statistical thinking is more commonly used in the literature), but has failed to develop a grand model of modes of thinking specific to uncertain situations, (a model similar to modes of thinking in linear algebra (Sierpiska, 2000) for example). I wasn't able to find an all-encompassing model that identifies and characterizes the features of probabilistic thinking, or a model that describes how learners learn probability (similar to van-Hiele's model for learning geometry).

6.3. Implications for teaching

I have been a teacher of mathematics for the past 14 years. A great deal of my teaching time is spent on concepts from Algebra, Calculus, Geometry, and discrete mathematics, leaving a smaller portion for Probability and Statistics. As a high school teacher in Iran, I taught probability from a strictly theoretical point of view, focusing only on combinatorial methods needed to solve the complex numeration situations encountered in game-based probability problems. Being taught in the same manner myself, I considered probability to be an objective, factual feature of the event in question. Even my perception of the Law of large numbers, which I learned about in undergraduate studies, was that it is a thought experiment where infinite trials of a Bernoulli experiment produces a success rate asymptotically close to the predicted rate. Over the course of my PhD studies I learned about research related to data-oriented approach where learners conduct experiments, collect data and study aspects of variability within the data (see Shaughnessy (1992, 2007), Konold (2002), and Konold

(2007)). Subsequently, I started following their advice and changed my teaching focus from calculating probability to investigating variability, predictability, and sense-making with these notions. I have learned to use TinkerPlots¹⁵ and R¹⁶ and I make use of the simulation, data exploration, and data presentation tools to teach the statistical concepts as much as I can fit into my classroom schedule. Another change I incorporated into my teaching, based on the results of my research is to include tasks where students are invited to reflect on arguments about probabilities, random variables, confidence intervals and their interpretations as a supplementary activity to the routine computational tasks.

The findings of my studies suggest that teachers should be aware that learners' conceptions of notions such as probability, event, impossible, and random are not always in sync with those of textbooks. More often than not learners' notions are formed and overshadowed by the everyday counterparts of these concepts. Tasks that involve giving definitions and examples of a concept present opportunities to both teachers and learners to reflect on the scope of their knowledge and perhaps expand the range of variations and possibilities they see for a concept.

Moreover, my findings suggest there is a need for attending to modeling aspects of probability. The modelling aspects of any mathematical construct (consider Euclidean geometry for example) once disclosed and discussed, can enable the learner to see the foundations (axioms) based on which the model is built. This approach (although probably impractical and discouraging for young learners), not only clarifies the blurry regions between everyday use of the notion and mathematical use of that notion, but also enhances learners' understanding of the connections between various mathematical constructs (in the case of Euclidean geometry, if the parallel postulate is replaced with other ones, or the metric requirement is relaxed or modified, other types of geometry are obtained). As a result of attending to modelling aspects of probability in teaching practices, definitions and examples of probabilistic notions will receive the

¹⁵ Tinkerplots is exploratory data analysis and modeling software designed for use by students in grades 4 through university.

¹⁶ R is a strongly functional language and environment for statistical computing and graphics. It is widely used among statisticians and data miners.

proper attention. Furthermore, a model-based teaching portrays one type of probability (for example frequentist) as one possibility for devising models that deal with uncertain phenomenon. Therefore different models can be compared and contrasted and decisions can be made about the efficiency of each model with regard to certain types of problems. The prominent statistician George Box has famously said: “all models are wrong, but some are useful”, he has also said in an introduction to statistical modelling: “*What is it? Why and Where is it Useful?*” This is the type of approach to teaching probabilistic modelling that I suggest teachers to pursue.

6.4. Where to next?

In a recent edited book on probabilistic thinking (Chernoff & Sriraman, 2014), several researchers have put forth a call for research investigating subjective probability that may move the field of mathematics education beyond frequentist and theoretical and into embracing a new interpretation of probability. Given the widespread popularity of Bayesian methods in fields such as machine learning, image processing, and natural language processing, which are integral parts of data analysis in health and ecommerce today, this seems to be the right curricular move. Since the teaching of probability is currently dominated by the frequentist and theoretical approaches¹⁷, some effort should be made by the mathematics education community to stir teaching probability in the new direction. Also with the advances of classroom technology, it is possible to simulate,

¹⁷ I think there are two reasons for the dominance of frequentist approach in teaching probability. After the important works of leading frequentist statistician, Sir Ronald Fisher, in 1920's that contributed to the development of methods such as hypothesis testing, analysis of variance (ANOVA), and deductive statistical inference, these methods were widely used (still are) in applied science and social sciences fields. I suggest that early textbooks on probability and statistics were written under the influence of such practices. Another reason I suggest for dominance of frequentist approaches in teaching practices is due to lack of access of educational institutes to technology. Bayesian methods are best applied when combined with powerful simulation methods. In frequentist approach we have to make assumptions about the distribution of data (in practice, more often than not, students don't even get to see the data, and the so called “data” is usually assumed to follow certain distributions such as normal) and we are trying to make inferences about population parameters (such as mean, mean difference, variance and so on). In a Bayesian probability classroom, students wouldn't be looking for point estimations (or interval estimations) of population parameters, instead, they would try to find the distribution that best describes population parameters. This cannot be efficiently done without actually working with data and using a computer simulation platform.

present (in multitude of ways), and manipulate data in real time in class. As a future direction for research, I am interested in exploring the probability notions formed through data-driven probability teaching.

6.5. The Final Word

Keith Devlin, in his introduction to the latest research in probability education edited book (Chernoff & Sriraman, 2014), writes that what he believes to be the most common probability misconception, the one that has the most significant implications, is that events have a unique probability. The belief in the objectivity of probability and how the true probability is an inherent feature of the event, he suggests, is a result of the way probability is taught through quasi-experimental, pedagogic scenarios (tossing coins, rolling dice, etc.). Probability should be understood “as a measure of our knowledge of the outcomes, [...] as a measure that applies not to the events in the real world, but to our information about that world at any given moment in time” (p. x). In light of the findings of my thesis I can only echo the same opinion as Devlin’s about what probability is, what should it be taken for, and how it should be taught. Although there is some value in controlled experimental approach, we need to make clear to learners that they are motivational activities and the actual applications of probability are in the context of a one-off event. The true power of probability, in Devlin’s words comes from “reflection on the nature of events in the world and what we can know about them based on the information at our disposal” (p. xiii).

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