On Laplace–Borel Resummation

of

Dyson–Schwinger Equations

by

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M.Sc., ENS Lyon, 2009

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

in the

Department of Mathematics

Faculty of Science

SIMON FRASER UNIVERSITY

Fall 2014

APPROVAL

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Degree: Doctor of Philosophy

Title of Thesis: On Laplace–Borel Resummation of Dyson–Schwinger Equations

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Date Approved: July 15, 2014

Abstract

In this work we conduct a complex analytic study of Dyson–Schwinger equations, the quantum equations of motion. Focusing on the particular family

$$G(x, L) = 1 - x\, G(x, \partial_{-\rho})(e^{-\rho\, L} - 1)F(\rho)|_{\rho=0}$$

we consider the class of formal solutions

$$G(x, L) = \sum_n \gamma_n(x)\, L^n\,,$$

whose formal and real analytic aspects have already been studied. Taking the point of view of complex analysis we are able to shed some new light on the structure of these solutions and provide useful tools to consider asymptotic questions.

This thesis is built around two functions. The anomalous dimension γ_1 which is closely tied to the energy scaling properties of quantum field theory and the Green function $G(x, L)$, the actual solution to the Dyson–Schwinger equation. We study their dual aspects as formal power series and analytic functions in the variables x and L. Our tool of choice is the Laplace–Borel resummation method which proves suitable to take care of the divergent series occuring naturally in quantum field theory.

Our main results consists of:

(i) conducting a Laplace–Borel analysis of the anomalous dimension γ_1;

(ii) constructing a Laplace–Borel solution to our Dyson–Schwinger equation by using the renomalization group equation.

Contents

Chapter 1

Introduction

The goal of quantum field theory is to describe the propagation and interactions of elementary particles. Those particles are encoded as fields and their behaviours are captured in Green functions or correlation functions. Dyson–Schwinger equations are the analog in quantum field theory of the equations of motion in classical mechanics. To understand their solutions is to understand the corresponding dynamics of elementary particles.

The Dyson–Schwinger equations form a system of integro-differential equations and as a system of non-perturbative equations, they can be used to investigate simultaneously the perturbative and non-perturbative aspects of the theory.

In this thesis we use combinatorial properties of chord diagrams and integral transforms method to gather respectively perturbative and non-perturbative information regarding a special class of Dyson–Schwinger equation.

In this chapter we introduce the Dyson-Schwinger equations and the renormalization group equation in the context of quantum field theory.

1.1 Dyson–Schwinger Equations

For the mathematician, the elementary particles studied in quantum field theory are maps from space-time to some target space chosen to encode the relevant properties of the particles to be captured in our models. These maps are known as fields in the physics literature. Making an analogy with classical mechanics, one can consider the movement of a solid body and to keep track of the direction of its associated angular momentum at any point of space and time. Then we would have a map L from space-time to the unit sphere such that $L(t, x)$ is the direction of the angular momentum of the solid at time t and position x. In this example, the solid body is be the analog of an elementary particle and L is its associated field keeping track of its angular momentum. The same goes for quantum field theory, although some of the features of elementary particles might be more exotic.

As an example let us consider ϕ^3 theory. In the context of quantum field theory, a field is a map $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ where $\mathbb{R}^d$ represent a d dimensional euclidean space-time and the target space does not carry any particular structure.

The properties of the model are fixed by a choice of action functional $S : \mathcal{C}_0(\mathbb{R}^d, \mathbb{R}) \longrightarrow \mathbb{R}$ where $\mathcal{C}_0(\mathbb{R}^d, \mathbb{R})$ is the space of real valued functions with compact support on $\mathbb{R}^d$. For the ϕ^3 theory we take the action to be

$$S(\phi) = \int_{\mathbb{R}^d} \frac{1}{2}\partial\phi^2 - \frac{g}{3!}\,\phi^3\, dvol\,, \tag{1.1}$$

where g is called a coupling constant as it parametrizes the strength of particle interactions.

The study of the classical solutions of this model consists in describing the critical points of the functional S while the study of the quantum properties of the model consists in describing the correlations between the values of the fields ϕ at different points in space-time. For something equivalent, but more common, in practice we can also study the interactions of the particles described by the fields ϕ as a function of their momenta. These are the quantities tied to the collisions experiments of particle physics.

Similar to the action S of the classical model, the partition function captures its quantum properties. The partition function, $Z(g)$, depending on the Planck constant $\hbar$ and the coupling constant g is formally defined by

$$Z(g) = \int_{\mathcal{C}_0(\mathbb{R}^d,\mathbb{R})} D\phi \, \exp\left[-\frac{i}{\hbar}\, S(\phi)\right] , \tag{1.2}$$

where $D\phi$ is formally a density on $\mathcal{C}_0(\mathbb{R}^d, \mathbb{R})$ with the key properties of being invariant by translation and satisfying Stokes theorem. Note that it is possible to take an analytic approach to the partition function giving a precise meaning to the integral by constructing it as a limit porcess using ideas coming from statistical mechanics and renormalization group analysis. There are other ways to approach this question, see the recent [8] and classical [13] for more information.

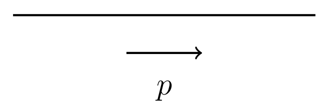
As a formal object Z is the generating function of the correlation functions of the model:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle(g) = \int_{\mathcal{C}_0(\mathbb{R}^d,\mathbb{R})} \frac{D\phi}{Z(g)}\, \phi(x_1)\cdots\phi(x_n)\, \exp\left[-\frac{i}{\hbar}\, S(\phi)\right] . \tag{1.3}$$

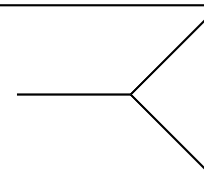
We can manipulate this generating function to obtain functional equations that will be translated as relations among the correlation functions. Those relations are the Dyson–Schwinger equations. The derivation of Dyson–Schwinger equations from the partition function is rather formal and relies on many assumptions on the models so for the sake of clarity we will proceed differently here. We refer again to introductory textbooks such as [4, 21] for more details.

Indeed, there is another more combinatorial way to describe these correlation functions together with their Dyson–Schwinger equations. It corresponds to a diagrammatic interpretation of the physical process encoded in the correlation functions and uses Feynman diagrams.

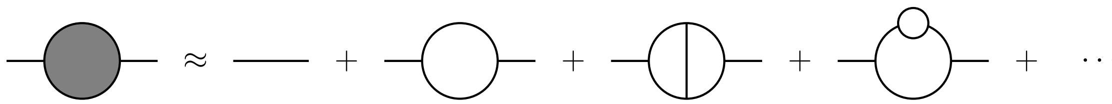
Let us detail the construction for our example. Looking back at the action $S(\phi)$ we distinguish two terms in the integrand. The term $\frac{1}{2}\partial\phi^2$ which is quadratic in ϕ is associated to the propagation of particles. In a Feynman diagram these are represented by edges that carry a momentum:

Monomial		Diagrammatic Representation
$\frac{1}{2}\partial\phi^2$	$\Longleftrightarrow$	

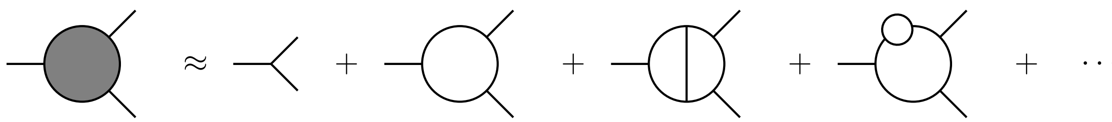
Similarly the cubic term $g\,\phi^3/3!$ signals the presence of interactions involving three particles. Those interactions are represented by corollas with conservation of the momentum:

Monomial		Diagrammatic Representation
$g\,\phi^3/3!$	$\Longleftrightarrow$	

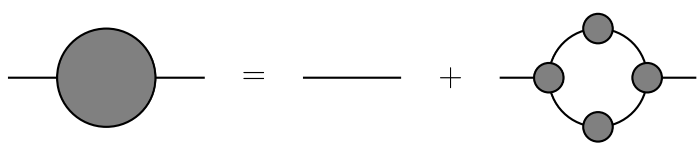
One can then give a diagrammatic representation of the correlation functions by following a simple recipe. Fix a set of external edges corresponding to the particles that will be interacting. Write a formal sum whose terms are all the graphs one can build with the fixed external structure only using the edges and corrolas prescribed by the action S. Instead of having to consider all graphs we can use the familiar $\exp - \log$ scheme to keep only the connected combinatorial objects and a Legendre transform to reduce again the number of graphs. Specifically, taking the logarithm of the partition function eliminates the non-connected graphs and taking its legendre transform eliminates the graphs that are not 2-edge connected. Hence it is always possible to consider only those graphs that are 2-edge connected. As an example, neglecting numerical symmetry factors, the interaction of two particles yields:

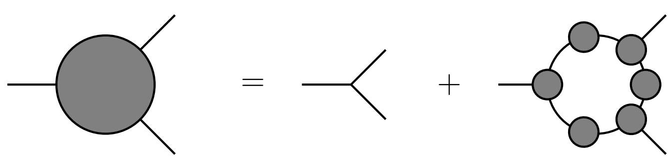


One can do the same for the interaction of three particles:



and so on. Instead of writing the full series of graphs we can replace it by a diagrammatic recursion. If the gray bubbles are place holders for admissible subgraphs with an admissible structure we obtain the combinatorial Dyson–Schwinger equations up to numerical symmetry factors:





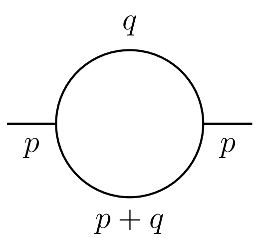
In order to obtain analytic versions of the Dyson–Schwinger equations one introduces the Feynman rules, i.e. a correspondence between the graphs and integral expressions, such that they match with the expression that can be derived from the partition function. Here these rules are

i) an internal edge carrying a momentum p corresponds to a factor $1/||p||^2$;

ii) a vertex with entering momenta p, q, r corresponds to a factor $g\,\delta(p+q+r)$ where δ is the Dirac distribution used here to impose the conservation of momentum at each vertex;

iii) integrate over all the momenta of internal edges.

Applying these rules to a simple graph one has:



$$\longmapsto \qquad g^2 \int_{\mathbb{R}^d} \frac{1}{||q||^2\,||p+q||^2}\,dq$$

Applying these rules to the combinatorial Dyson–Schwinger equations we obtain (after a normalization of the trees to 1) the following analytic version of the Dyson–Schwinger equations, where G_2 corresponds to the two particles interaction and G_3 corresponds to the three particles interaction:

$$G_2(g,p) \;=\; 1 + g^2 \int_{\mathbb{R}^d} \frac{G_2^{-1}(g,q)\,G_2^{-1}(g,p+q)G_3(g,p,q)^2}{||q||^2\,||p+q||^2}\,dq \tag{1.4}$$

and

$$G_3(g,p,q) \;=\; 1 + g^2 \int_{\mathbb{R}^d} \frac{G_2^{-1}(g,r)\,G_2^{-1}(g,p+r)\,G_2^{-1}(g,r-q)\,G_3(g,p,r)\,G_3(g,q,r)^2}{||r||^2\,||p+r||^2\,||r-q||^2}\,dr\,. \tag{1.5}$$

1.2 Renormalization Group Equation

However these Dyson–Schwinger equations are not sufficient to describe a physical theory. Indeed the Feynman rules typically produce divergent integrals which means they need to be renormalized. The renormalization procedure makes the parameters of the correlation functions dependent of the energy of the physical process, but one fundamental requirement for a well defined theory is that the correlation functions are invariant under certain energy reparametrizations. This property is encoded with the condition that they must satisfy a partial differential equation, the renormalization group equation (see chapter 7 in [7]). If μ denotes the energy of the system, then in our case this equation is

$$\mu\,\partial_\mu G_i(g,\mu) \;= \beta(g)\partial_g\, G_i(g,\mu) - \gamma(g)G_i(g,\mu)\,, \tag{1.6}$$

where the function γ is called the anomalous dimension of the fields, β is called the beta function of the system and $i = 2, 3$.

The focus of this thesis is the case of a single Dyson–Schwinger equation and its associated renormalization group equation. While the details of the results are specific to the choice of having to consider only one correlation function there is in principle no obstruction to extend this approach to the more general case of systems of Dyson–Schwinger equations. In the case of a single equation the renormalization group equation takes the form

$$\mu\partial_\mu G = \gamma(g)(sg\partial_g - 1)\,G\,, \tag{1.7}$$

where s is a normalization parameter.

The anomalous dimension γ and the correlation function G satisfy equations that can be solved perturbatively using power series expansions. As a rule these perturbative series are only asymptotic series. Our goal is to use integral transform methods, namely the Laplace–Borel resummation method, to try to reconstruct non-perturbative solutions from these perturbative series.

1.3 Results

In Chapter 2 we introduce some preliminary constructions regarding different represenations of Dyson–Schwinger equations. Expanding on the work of [24] we explain the transition from the integral form of the Dyson–Schwinger equation,

$$G_2(g,p) \;=\; 1 + g^2 \int_{\mathbb{R}^d} \frac{G_2^{-1}(g,q)\,G_2^{-1}(g,p+q)G_3(g,p,q)^2}{||q||^2\,||p+q||^2}\,dq\,, \tag{1.8}$$

to an infinite order differential equation

$$G_2(g,p) \;=\; 1 + g^2\,G_2(g,\partial_{-\rho})^{-1}\,[e^{-L(p)\,\rho} - 1]\,F(\rho)|_{\rho=0}\,. \tag{1.9}$$

We then explain the principles of the Laplace–Borel resummation technique and construct a toolbox of useful properties in chapter 3 before moving on to the main results of this work.

In Chapter 4 we study the resummation of the anomalous dimension γ building on previous results such as [24, 20, 1, 2]. Starting with the combinatorial interpretation of γ as a chord diagram expansion (see [20]) we get some idea of the structural properties of the Borel transform of γ. Then using the P-differential equation (see [24, 1, 2]) we give a resummation formula for a large class of parameter functions P. In any case this formula clearly shows that the singularities of the Borel transform of γ, known in physics as instantons and renormalons, prevent us from having a unique non-perturbative solution corresponding to the perturbative expansion.

Finally in Chapter 5 we use the fact that G must satisfy the renormalization group equation to obtain a resummation formula for the perturbative expansion of G in terms of the parameter $L(p)$.

This thesis fits in a larger area of research in physics that aims at extending WKB and Laplace–Borel resummation methods, already used successfully on the Schroedinger equation, to other problems where a perturbative solution is always available but never convergent. One can find references regarding those efforts in the non exhaustive list of papers:

- Schroedinger equation, A. Voros (1983) and A. Getmanenko (2008) [23, 19];
- WKB analysis related to Dyson–Schwinger equations, D. Kreimer and A. Youssef (2013) [18];

- trans-series solutions in quantum mechanics and quantum field theory, G. Basar, G.V. Dunne and M. Unsal (2013, 2014) [3, 12];
- Laplace–Borel analysis in Chern–Simons theory, S. Garoufalidis (2007) [16].

Chapter 2

Rewriting Dyson–Schwinger Equations

In this chapter we introduce a functional representation of Dyson–Schwinger equations. The goal is to revisit the traditional manipulations of Dyson–Schwinger equations leading to equivalent representations but this time keeping track of the analytic conditions that have to be fulfilled for these equivalence to be more than just formal. This process give rise to Theorem 2.1.1 which can be seen as an analytic version of the correspondence described in [24] Section 3.3.

2.1 From Integral to Differential Representation

In this section we take the following integral equation as a model for our Dyson–Schwinger equation is

$$\phi(\hbar, p) \;=\; \phi_0(p) \;+\; \hbar \int_{\mathbb{R}^d} A(\phi(\hbar, q))\, K(p, q)\, dq \tag{2.1}$$

where dq is the volume form on $\mathbb{R}^d$ and A is some analytic function. Working in the context of semi-classical analysis $\hbar$ is an expansion parameter and does not correspond to the Planck constant. Most of the time it is some power of the coupling constants.

We assume the integral operator is well defined, that is, the corresponding Feynman rules have already been renormalized and the parameters $\hbar$ and p stay away from potential singularities.

This integral form of the Dyson–Schwinger equation is most useful for proving properties of the corresponding solutions by induction using fixed point methods, however in practice the momentum variable $p \in \mathbb{R}^d$ is not necessarily the most relevant and we prefer to use quantities of the form $L(p) = \log(||p||^2/\mu^2)$ which is dimensionless and compatible with scaling properties coming from the action of the renormalization group. This is why we want to rewrite 2.1 introducing an infinite order differential operator to emphasize the role of these new quantities.

So let us formalize all this. Let $L : \mathbb{R}^d \longrightarrow \mathbb{R}$ and associate to this function an ansatz for the solution to the equation 2.1

$$\widetilde{\phi}_L(\hbar, p) \;=\; \sum_{n=0}^{+\infty} \gamma_n(\hbar)\, L(p)^n \,. \tag{2.2}$$

Based on this ansatz we can formally derive another form of the Dyson–Schwinger equation 2.1 as in [24].

Our next theorem gives the conditions under which this correspondence holds at the analytic level. It is convenient to introduce the set of parameters $\hbar$ making the integrand of the Dyson–Schwinger equation singular

$$Sing(DSE) \;=\; \left\{\hbar \in \mathbb{C} \,|\, \exists q \in \mathbb{R}^d \text{ s.t. } A(\phi(\hbar, q)) = \infty\right\} \tag{2.3}$$

and the functions $A\Gamma_k(\hbar)$, coefficients of a series expansion defined by the relation

$$A(\widetilde{\phi}_L(\hbar, p)) \;=\; \sum_{k=0}^{+\infty} A\Gamma_k(\hbar)\, L(p)^k \,. \tag{2.4}$$

Theorem 2.1.1. *Assume that $\widetilde{\phi}_L$ is a solution of 2.1. Then it is also a solution of*

$$\widetilde{\phi}_L(\hbar, p) \;=\; \phi_0(p) + \hbar A(\widetilde{\phi}_L(\hbar, \partial_\rho)) \int_{\mathbb{R}^d} e^{\rho\, L(q)}\, K(p, q)\, dq\Big|_{\rho=0} \tag{2.5}$$

for all $\hbar \in \mathbb{C} \setminus Sing(DSE)$ if the following conditions are satisfied:

(i) for all $k \in \mathbb{N}$, $A\Gamma_k(\hbar) \in \mathcal{O}(\mathbb{C}\backslash Sing(DSE))$ *and* $\int_{\mathbb{R}^d} K(p,q)L(q)^k\,dq$ *is absolutely convergent;*

(ii) for all $q \in \mathbb{R}^d$ *the series* $\sum A\Gamma_k(\hbar)L(q)^k$ *and* $\sum A\Gamma_k(\hbar)\int_{\mathbb{R}^d} K(p,q)L(q)^k$ *are absolutely convergent;*

(iii) $\int_{\mathbb{R}^d} K(p,q)e^{\rho\,L(q)}\,dq$ *is an analytic function of* ρ *in some neighborhood of the origin;*

Proof. We will go through the proof of the formal correspondence between the two equations as in [24] and check at each step that our operations make sense at the analytic level. So let us start with the Dyson–Schwinger equation in its integral form:

$$\phi(\hbar,p) \;=\; \phi_0(p) \,+\, \hbar \int_{\mathbb{R}^d} A(\phi(\hbar,q))\,K(p,q)\,dq\,. \tag{2.6}$$

Using the ansatz inside the integral one has for all appropriate $\hbar$

$$\int_{\mathbb{R}^d} A(\phi(\hbar,q))\,K(p,q)\,dq \;=\; \int_{\mathbb{R}^d} \sum_{k=0}^{\infty} A\Gamma_k(\hbar)\,L(q)^k\,K(p,q)\,dq\,, \tag{2.7}$$

and we need to discuss how to swap the summation and integration symbols. Indeed swapping the order of summation and integration one obtains:

$$\sum_{k=0}^{\infty} A\Gamma_k(\hbar) \int_{\mathbb{R}^d} K(p,q)\,L(q)^k\,dq\,. \tag{2.8}$$

By (i) we know that any term of the sum is finite and using (ii) the whole sum exists, then by Lebesgue dominated convergence theorem we do have the equality:

$$\int_{\mathbb{R}^d} \sum_{k=0}^{\infty} A\Gamma_k(\hbar)\,L(q)^k\,K(p,q)\,dq \;=\; \sum_{k=0}^{\infty} A\Gamma_k(\hbar) \int_{\mathbb{R}^d} K(p,q)\,L(q)^k\,dq\,. \tag{2.9}$$

Next we want to use the trick which consists in writing $L(q)^k$ as $\partial_\rho^k e^{\rho\,L(q)}\big|_{\rho=0}$. Observe that by assuming (iii) we get the equality:

$$\int_{\mathbb{R}^d} K(p,q)\,\partial_\rho^k\,e^{\rho\,L(q)}\,dq = \partial_\rho^k \int_{\mathbb{R}^d} e^{\rho\,L(q)}\,K(p,q)\,dq\,. \tag{2.10}$$

Putting everything together the Dyson–Schwinger equation 2.1 becomes:

$$\phi(\hbar, p) = \phi_0(p) \,+\, \hbar \sum_{k=0}^{\infty} A\Gamma_k(\hbar)\, \partial_\rho^k \int_{\mathbb{R}^d} e^{\rho\, L(q)}\, K(p,q)\, dq\Big|_{\rho=0}\,, \tag{2.11}$$

which we can simply repackage as

$$\widetilde{\phi}_L(\hbar, p) = \phi_0(p) + \hbar A(\widetilde{\phi}_L(\hbar, \partial_\rho)) \int_{\mathbb{R}^d} e^{\rho\, L(q)}\, K(p,q)\, dq\Big|_{\rho=0}\,. \tag{2.12}$$

□

Note that it is customary to introduce a Laurent series $F(\rho)$ defined by the factorization

$$\int_{\mathbb{R}^d} e^{\rho\, L(q)}\, K(p,q)\, dq \;=\; (e^{\rho\, L(p)} - 1)\, F(\rho)\,, \tag{2.13}$$

such that the Dyson–Schwinger equation is

$$\widetilde{\phi}_L(\hbar, p) = \phi_0(p) + \hbar A(\widetilde{\phi}_L(\hbar, \partial_\rho))\, (e^{\rho\, L(p)} - 1)\, F(\rho)\Big|_{\rho=0}\,. \tag{2.14}$$

This is the representation that will be used in chapter 4 for the main reason that in practice working with the parameter L is more natural than working directly with the momentum parameter p.

We also want to point out that 2.5 can be seen as a differential equation with a recursively defined infinite order differential operator $A(\widetilde{\phi}_L(\hbar, \partial_\rho))$. This is a perspective on these equations that we have not seen exploited so far but that could lead to yet another method to study structural properties of their solutions (see [6] for some examples of properties of infinite order differential operators).

2.2 In Practice

The previous theorem requires many conditions but they are all very natural. Let us consider a simplified situation to illustrate them. Fix the dimension to be $d = 1$ and let $\mu \in \mathbb{R}_+$, we take for all $p \in \mathbb{R}$

$$L(p) = \log(|p|^2/\mu^2)\,. \tag{2.15}$$

Our first problem is to be sure that for all $k \in \mathbb{N}$ the following integrals are convergent:

$$\int_{\mathbb{R}} K(p,q)\, \log(q^2/\mu^2)^k \, dq\,. \tag{2.16}$$

Looking at L we see that the integral might fail to be convergent due to divergences when $|q| \to 0, \infty$. To prevent this from happening we must choose $K(p,q)$ so that its behaviour at 0 and ∞ dominates the logarithm. Such examples include kernels of the form

$$K(p,q) = \alpha(p)\, q\, e^{-q^2} \text{ or } K(p,q) = \frac{\alpha(p)}{1+q^2}\,. \tag{2.17}$$

In the first case the decreasing exponential dominates the logarithm at ∞ and the polynomial factor q ensure that the integrand stays finite at 0. For the case where K is rational in q the dominating powers are chosen again in order to behave like $1/q$ at ∞ and q at 0. Next let us look at the other integral condition, that is the analycity in some neighbourhood of the origin of the function of ρ

$$\int_{\mathbb{R}} K(p,q) \left(\frac{q^2}{\mu^2}\right)^{\rho} dq\,. \tag{2.18}$$

The integrand being an entire function of ρ it is sufficient for the integral to be convergent in order to have an analytic function. For the exponential kernel we have

$$\int_{\mathbb{R}} \alpha(p) \left(\frac{q^{2\rho+1}}{\mu^{2\rho}}\right) e^{-q^2}\, dq\,. \tag{2.19}$$

The exponential takes care of the convergence at ∞ whatever the choice of ρ and we are left with examining what happens at the origin. But to avoid any divergence it is sufficient to choose ρ such that $2\Re(\rho) + 1 \geq 0$ so there is definitely a neighbourhood of the origin for which the integral is convergent and the condition is satisfied.
We can do the same kind of analysis if we pick the rational kernel:

$$\int_{\mathbb{R}} \alpha(p) \left(\frac{q^{2\rho+1}}{\mu^{2\rho}(1+q^2)}\right) dq\,. \tag{2.20}$$

Now we need two conditions on the real part of ρ in order to enforce the convergence at 0 and ∞, they are simply $2\Re(\rho) < 1$ and $2\Re(\rho) > 1$. So as long as $|\Re(\rho)| < 1/2$ the integral is convergent that is there is a neighbourhood of the origin on which the integral is analytic.

Of course by restricting ourselves to $d = 1$ we greatly simplified the analysis of the conditions, but given the general forms of the Feynman rules in physical models it is mostly the case that by partial integrations in spherical coordinates we can schematically reduce the problem to something close to these examples.

Checking condition (ii) of the theorem is a much more complicated problem. Indeed making the particular choice $A(y) = 1/y$ and assuming that $\gamma_0 = 1$ in order to simplify the notations, one obtains the following expression for the coefficients $A\Gamma_k(\hbar)$:

$$A\Gamma_k(\hbar) = \sum_{i=1}^{k} (-1)^i \sum_{\substack{p_1+\cdots+p_i=k \\ p_1,\cdots,p_i \geq 1}} \prod_{j=1}^{i} \gamma_{p_j}(\hbar)\,. \tag{2.21}$$

Thus going further requires to get some information about the coefficients $\gamma_i(\hbar)$. These are computed by induction using the formula

$$\gamma_n(\hbar) = \frac{1}{n}\,\gamma_1(\hbar)\,(\hbar\,\partial_\hbar - 1)\,\gamma_{n-1}(\hbar)\,. \tag{2.22}$$

The easiest way do obtain this formula consists in applying the renormalization group equation 1.7 to the ansatz $\widetilde{\phi}_L$:

$$\partial_L \sum_n \gamma_n(\hbar)\,L^n \;=\; \gamma_1\,(\hbar\partial_\hbar - 1)\sum_n \gamma_n(\hbar)\,L^n\,, \tag{2.23}$$

that is for all $n > 1$

$$n\,\gamma_n(\hbar) \;=\; \gamma_1(\hbar)\,(\hbar\partial_\hbar - 1)\gamma_{n-1}(\hbar)\,. \tag{2.24}$$

So we see that giving suitable estimates in order to verify the convergence of the series appearing in condition (ii) can only be done once we have gathered information about the function γ_1. This is the object of the second part of this work.

As an example, if we assume that γ_1 is exponentially bounded on a suitable domain, we would have

$$|\gamma_n(\hbar)| \leq\, C\,\frac{A^n}{n!}\,Pol_n(\hbar)\,e^{n\,\alpha\,\hbar} \tag{2.25}$$

with Pol_n a polynomial of degree $n-1$ and A, C some positive constants. We can use those inequalities to get a crude estimates on $A\Gamma_k(\hbar)$:

$$|A\Gamma_k(\hbar)| \leq \sum_{i=1}^{k} A^i\, e^{i\,\alpha\,\hbar} \sum_{\substack{p_1+\cdots+p_i=k \\ p_1\,,\cdots,\,p_i\geq 1}} \prod_{j=1}^{i} \frac{Pol_{p_j}(\hbar)}{p_j!}\,. \tag{2.26}$$

There must be a polynomial P_k of degree k that bounds the sum of products of Pol_{p_j}, so ignoring the denominators we have

$$|A\Gamma_k(\hbar)| \leq 2^{k-1}\, P_k(\hbar) \sum_{i=1}^{k} A^i\, e^{i\,\alpha\,\hbar} \tag{2.27}$$

$$\leq \frac{A}{2}\, e^{\alpha\,\hbar}\, 2^k\, P_k(\hbar)\, \frac{1-A^k\, e^{k\,\alpha\,\hbar}}{1-A\, e^{\alpha\,\hbar}}\,. \tag{2.28}$$

Thus we see that in this case the coefficients of the series appearing in condition (ii) can be bounded by expressions that depend geometrically on k providing the necessary tool to check for convergence. So we see that the knowledge of γ_1 is the essential prerequisite to the application of our theorem.

To summarize, for the rest of the thesis we will assume that 2.5 is a valid form for our Dyson–Schwinger equation and we will proceed to look for actual solutions.

Chapter 3

Technical Preliminaries

In this chapter we introduce some basic tools of complex analysis necessary to carry out Laplace–Borel resummation.

Even if familiar with complex analysis the reader is encouraged to go through these results in order to understand the fundamental ideas behind Laplace–Borel resummation and how we intend to use it to explore Dyson–Schwinger equations.

The most important result in this section is theorem 3.1.2 which gives a correspondence between an asymptotic series and an analytic object obtained by the processes of Laplace–Borel resummation.

3.1 Laplace–Borel Resummation

In this section we introduce the Laplace–Borel resummation method. This is a vast subject but we focus here on the properties useful in the rest of this text. More details can be found in [9, Chapters 3,5].

An important technical point: from this point and until the end of this thesis analytic functions are considered by default to be multivalued unless stated otherwise.

This choice is due to the fact that the entire theory of Laplace–Borel resummation revolves around considering all the analytic continuations of the Borel transforms which are by construction only germs of analytic functions.

First of all let us introduce the notion of Gevrey-1 asymptotic expansion of an analytic function.

Definition 3.1.1. Let f be an analytic function on a domain $\mathbb{D}$ of $\mathbb{C}$ and $\widetilde{f} = \sum_n \widetilde{f}_n \, z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]]$ be a formal power series. We say that $\widetilde{f}$ is a Gevrey-1 asymptotic expansion of f on $\mathbb{D}$ if there are some $A, B > 0$ such that for all $z \in \mathbb{D}$, $n \in \mathbb{N}^*$

$$\left| f(z) - \sum_{k=0}^{n-1} \widetilde{f}_n \, \frac{1}{z^{n+1}} \right| \leq A \, B^n \, n! \, \frac{1}{|z|^{n+1}} \, . \tag{3.1}$$

When $\widetilde{f}$ is a Gevrey-1 asymptotic expansion of f we write $f(z) \sim_1 \widetilde{f}$.

The Laplace–Borel resummation process consists in constructing an analytic function f starting from some formal power series $\widetilde{f}$ such that $f(z) \sim_1 \widetilde{f}$.

The first step consists in defining the formal Borel transform of a formal power series as

$$\mathcal{B} : \mathbb{C}[[1/z]] \longrightarrow \mathbb{C}\delta \oplus \mathbb{C}[[\zeta]] \tag{3.2}$$

$$C^{te} + \sum_{k=0}^{+\infty} \widetilde{f}_k \frac{1}{z^{k+1}} \mapsto C^{te}\delta + \sum_{k=0}^{+\infty} \widetilde{f}_k \frac{\zeta^k}{k!} \tag{3.3}$$

where δ is a formal version of the Dirac distribution meaning that it will serve as a neutral element for the convolution product of formal power series.

A subclass of formal power series is of particular interest when considering the formal Borel transform.

Definition 3.1.2. A formal power series $\widetilde{f} \in \mathbb{C}[[1/z]]$ is said to be of Gevrey class 1 if there are $A, B > 0$ such that for all n large enough $|\widetilde{f}_n| \leq A \, B^n \, n!$.

Indeed one can immediately see that if $\widetilde{f}$ is a formal power series of Gevrey class 1 then $\mathcal{B}[\widetilde{f}] \in \mathbb{C}\delta + \mathbb{C}\{\zeta\}$, i.e. its formal Borel transform is associated to a germ of analytic function at the origin.

Example 3.1.1. The Euler series $E(z) = \sum (-1)^k \, k! \, z^{-1-k}$ is of Gevrey class 1 while $\sum 2^{\binom{k}{2}} \, z^{-(k+1)}$ is not.

Now we define the Laplace transform of a function along the positive real axis to be

$$\mathcal{L}_0[\widehat{f}](z) = \int_{\mathbb{R}_+} e^{-\zeta\, z} \, \widehat{f}(\zeta) \, d\zeta \tag{3.4}$$

for any function $\widehat{f}$ locally integrable along $\mathbb{R}_+$ and satisfying some exponential bound for $\zeta \in \mathbb{R}_+$ large enough. Let $\mathbb{H}_c = \{z \in \ | \Re(z) > c\}$ be a half plane, we have the following result.

Lemma 3.1.1. *Assume that $\widehat{f}$ is locally integrable over $\mathbb{R}_+$ and there are $A, c_0 > 0$ such that for all $\zeta \in [1, +\infty[$ we have $|\widehat{f}(\zeta)| \leq A \, exp(c_0 \, |z|)$. Then for all $c > c_0$ the Laplace transform $\mathcal{L}_0[f]$ is analytic in $\mathbb{H}_c$.*

Proof. Since the integrand of $\mathcal{L}_0[\widehat{f}](z)$ is an entire function of z, the Laplace transform is analytic as long as the integral is convergent. The Laplace transform can be decomposed as

$$\mathcal{L}_0[\widehat{f}](z) = \int_0^1 e^{-\zeta\, z} \widehat{f}(\zeta) \, d\zeta + \int_1^{+\infty} e^{-\zeta\, z} \widehat{f}(\zeta) \, d\zeta \,, \tag{3.5}$$

where the first term is convergent and the second term is bounded by

$$\int_1^{+\infty} A \, e^{-\zeta(z - c_0)} \, d\zeta \,. \tag{3.6}$$

Thus the condition $\Re(z - c_0) > 0$ ensure that the Laplace transform is convergent. □

So the Borel transform of a Gevrey-1 formal power series provides a germ of an analytic function in a dual variable and under certain growth conditions taking its Laplace transform gives us back an analytic function. This is the process of Laplace–Borel resummation. In particular one can observe that

$$\mathcal{L}_0 \mathcal{B}[\widetilde{f}_n / z^{n+1}] = \frac{\widetilde{f}_n}{n!} \int_{\mathbb{R}_+} e^{-\zeta\, z} \zeta^n \, d\zeta = \frac{\widetilde{f}_n}{z^{n+1}} \tag{3.7}$$

so that if $\widetilde{f} \in \mathbb{C}\{z^{-1}\}$ then $\mathcal{L}_0\mathcal{B}[\widetilde{f}]$ is equal to the Taylor series of $\widetilde{f}$. Thus when there is nothing to resum, the original series is convergent, the Laplace–Borel summation process is equivalent to the identity (up to taking a Taylor series).

With slightly more general hypotheses we get a much stronger result. For $\epsilon > 0$ define the half-strip St_ϵ to be the set of points $\zeta \in \mathbb{C}$ such that the distance from ζ to $\mathbb{R}_+$ is strictly smaller than ϵ. For any $c_0 \geq 0$ we also introduce $\widehat{E}_{c_0}(\mathbb{R}_+)$ the set of germs of analytic functions $\widehat{f} \in \mathbb{C}\{\zeta\}$ such that for some $\epsilon, A > 0$ the germ $\widehat{f}$ has a unique analytic continuation on St_ϵ and for all $\zeta \in St_\epsilon$ this analytic continuation satisfies $|\widehat{f}(\zeta)| \leq A\, e^{c_0\,|\zeta|}$. The previous observation then generalizes to the following theorem.

Theorem 3.1.2. *If $\widehat{f} \in \widehat{E}_{c_0}(\mathbb{R}_+)$ then for all $c > c_0$ and all $z \in \mathbb{H}_c$ we have*

$$\mathcal{L}_0[\widehat{f}](z) \;\sim_1\; \sum_{k=0}^{+\infty} \widehat{f}^{(k)}(0) \frac{1}{z^{k+1}}\,. \tag{3.8}$$

Proof. Since $\widehat{f} \in \widehat{E}_{c_0}(\mathbb{R}_+)$, it must be analytic in some half-strip St_ϵ.

For all $\zeta \in St_\epsilon$ and $n \in \mathbb{N}^*$ consider the expression:

$$R_n(\zeta) = \widehat{f}(\zeta) - \sum_{k=0}^{n} \widehat{f}^{(k)}(0)\,\frac{\zeta^k}{k!}\,. \tag{3.9}$$

Observe that:

- R_n and all its derivatives are vanishing at the origin;
- $R_n^{(n+1)} = \widehat{f}^{(n+1)}(\zeta)$;

Then applying the Laplace transform to $R_n^{(n+1)}(\zeta)$ together with the previous lemma one sees that:

$$\begin{aligned}
\mathcal{L}_0[\widehat{f}^{(n+1)}](z) &= \mathcal{L}_0[R_n^{(n+1)}](z) && (3.10)\\
&= z^{n+1}\,\mathcal{L}_0[R_n](z) && (3.11)\\
&= z^{n+1}\left\{\mathcal{L}_0[\widehat{f}](z) - \sum_{k=0}^{n-1} \widehat{f}^{(k)}(0)\frac{1}{z^{k+1}} - \frac{\widehat{f}^{(n)}(0)}{z^{n+1}}\right\}\,, && (3.12)
\end{aligned}$$

which gives the relation

$$\mathcal{L}_0[\widehat{f}](z) - \sum_{k=0}^{n-1} \widehat{f}^{(k)}(0) \frac{1}{z^{k+1}} = \frac{1}{z^{n+1}} \left\{ \mathcal{L}_0[\widehat{f}^{(n+1)}](z) + \widehat{f}^{(n)}(0) \right\} . \tag{3.13}$$

So as a first step we obtain the bound:

$$\left| \mathcal{L}_0[\widehat{f}](z) - \sum_{k=0}^{n-1} \widehat{f}^{(k)}(0) \frac{1}{z^{k+1}} \right| \leq \frac{1}{|z|^{n+1}} \left\{ \mathcal{L}_0[|\widehat{f}^{(n+1)}|](z) + |\widehat{f}^{(n)}(0)| \right\} . \tag{3.14}$$

Now consider any $\zeta \in \mathbb{R}_+$ and the disk $\mathbb{D}(\zeta, \epsilon) \subset St_\epsilon$. Since $\widehat{f}$ is analytic in $\mathbb{D}(\zeta, \epsilon)$ we can use Cauchy's inequality to bound $\widehat{f}^{(n)}(0)$. Since $\widehat{f} \in \widehat{E}_{c_0}(\mathbb{R}_+)$, we have in $\mathbb{D}(\zeta, \epsilon)$:

$$\left| \widehat{f}^{(n)}(y) \right| \leq \frac{n!}{\epsilon^n} \sup_{y \in \partial \mathbb{D}(\zeta, \epsilon)} |\widehat{f}(y)| \leq \frac{n!}{\epsilon^n} A \, e^{c_0(\zeta + \epsilon)} . \tag{3.15}$$

In particular at $\zeta = 0$ we have $\left| \widehat{f}^{(n)}(0) \right| \leq n! \, A \, e^{c_0 \epsilon} / \epsilon^n$ from which we get that for all $z \in \mathbb{H}_c$

$$\mathcal{L}_0[|\widehat{f}^{(n+1)}|](z) \leq \frac{(n+1)! \, A}{\epsilon^{n+1}} \int_{\mathbb{R}_+} e^{-\zeta(z - c_0) + c_0 \epsilon} \, d\zeta \leq \frac{(n+1)!}{\epsilon^{n+1}} \frac{A e^{c_0 \epsilon}}{c - c_0} . \tag{3.16}$$

But $(n+1)! \leq 2^n \, n!$ so eventually we get for all $n \in \mathbb{N}^*$ and all $z \in \mathbb{H}_c$:

$$\left| \mathcal{L}_0[\widehat{f}](z) - \sum_{k=0}^{n-1} \widehat{f}^{(k)}(0) \frac{1}{z^{k+1}} \right| \leq A' \, B^n \, n! \tag{3.17}$$

that is

$$\mathcal{L}_0[\widehat{f}](z) \sim_1 \sum_{k=0}^{+\infty} \widehat{f}^{(k)}(0) \frac{1}{z^{k+1}} . \tag{3.18}$$

□

This is the general link between a formal power series and its Laplace–Borel sum. Let us reformulate that with a precise definition.

Definition 3.1.3. A formal power series $\widetilde{f} \in \mathbb{C}[[1/z]]$ is said to be fine-summable along $\mathbb{R}_+$if it satisfies the following conditions:

(i) $\mathcal{B}[\widetilde{f}] = \widehat{f} \in \mathbb{C}\delta \oplus \mathbb{C}\{\zeta\}$;

(ii) there is some $c_0 \geq 0$ such that $\widehat{f} \in \widehat{E}_{c_0}(\mathbb{R}_+)$;

When these conditions are satisfied we call $\mathcal{L}_0\mathcal{B}[\widetilde{f}]$ the Laplace–Borel sum of $\widetilde{f}$ along $\mathbb{R}_+$. Moreover $\mathcal{L}_0\mathcal{B}[\widetilde{f}] \in \mathcal{O}(\mathbb{H}_c)$ for all $c > c_0$ and for all $z \in \mathbb{H}_c$ we have

$$\mathcal{L}_0\mathcal{B}[\widetilde{f}](z) \sim_1 \widetilde{f}(z).$$

3.2 Properties of the Resummation

Here we have the notion of fine-summability along $\mathbb{R}_+$ but it can be generalized easily to any direction $e^{i\theta}\mathbb{R}_+$ of the complex plane.

For that we have to consider the Laplace transform along the direction $e^{i\theta}\mathbb{R}_+$:

$$\mathcal{L}_\theta[\widehat{f}](z) = \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta z}\widehat{f}(\zeta)\dot{d}\zeta = \int_{\mathbb{R}_+} e^{-\zeta z e^{i\theta}}\widehat{f}(\zeta e^{i\theta}) e^{i\theta} d\zeta . \tag{3.19}$$

It is then quite easy to adapt all the previous results to this new setting by a change of variables. As an example we got the following existence lemma for the Laplace tranform along $e^{i\theta}\mathbb{R}_+$.

Lemma 3.2.1. *Assume that $\widehat{f}$ is integrable along $e^{i\theta}\mathbb{R}_+$ and that there is $c_0, A > 0$ such that $|\widehat{f}(\zeta)| \leq A\, exp(c_0|\zeta|)$ for all $\zeta \in e^{i\theta}[1, +\infty[$. Then the Laplace tranform $\mathcal{L}_\theta[\widehat{f}]$ exists and is analytic in the tilted half plane $\mathbb{H}_c^\theta = \left\{z \in \mathbb{C} \,|\, \Re(z\, e^{i\theta}) > c\right\}$ for all $c > c_0$.*

Proof. This is the same argument as in 3.1.1. □

In the same fashion all the other results remain the same mutatis mutandis. In particular if $\widetilde{f}$ is fine summable along $e^{i\theta}\mathbb{R}_+$ we get $\mathcal{L}_\theta\mathcal{B}[\widetilde{f}](z) \sim_1 \widetilde{f}(z)$ for appropriate values of z. Thus by choosing different θ we obtain a priori different resummations of the same original series $\widetilde{f}$. As a rule the results for different θ are not independent. In the simplest situation one can go from one resummation to the other by analytic continuation.

Let $[\theta_1, \theta_2]$ be some interval in the real line and $\widehat{E}(\theta_1, \theta_2)$ be the set of germs of analytic functions $\widehat{f} \in \mathbb{C}\{\zeta\}$ such that for each $\theta \in [\theta_1, \theta_2]$ there is $c(\theta) > 0$ such that $\widehat{f} \in \widehat{E}_{c(\theta)}(e^{i\theta}\mathbb{R}_+)$. Thus if $\widehat{f} \in \widehat{E}(\theta_1, \theta_2)$ then for any $\theta \in [\theta_1, \theta_2]$ the Laplace transform of $\widehat{f}$ in the direction $e^{i\theta}\mathbb{R}_+$, $\mathcal{L}_\theta \widehat{f}(z)$, defines an analytic function in a half plane $\mathbb{H}^\theta_{c(\theta)}$. The following theorem shows that it is possible to glue these different functions.

Theorem 3.2.2. *Assume that* $\widehat{f} \in \widehat{E}(\theta_1, \theta_2)$ *with* $0 < \theta_2 - \theta_1 < \pi$. *Then* $\mathcal{L}_{\theta_1} \widehat{f} \in \mathcal{O}(\mathbb{H}^{\theta_1}_{c_1})$, $\mathcal{L}_{\theta_2} \widehat{f} \in \mathcal{O}(\mathbb{H}^{\theta_2}_{c_2})$, $\mathbb{H}^{\theta_1}_{c_1} \bigcap \mathbb{H}^{\theta_2}_{c_2} = \mathbb{H}_{12} \neq \emptyset$ *and we have*

$$\mathcal{L}_{\theta_1} \widehat{f}\,\Big|_{\mathbb{H}_{12}} = \mathcal{L}_{\theta_2} \widehat{f}\,\Big|_{\mathbb{H}_{12}} . \tag{3.20}$$

Proof. First of all the intersection $\mathbb{H}_{12}$ is not empty for geometric reasons. Observe that $\partial \mathbb{H}^{\theta_1}_{c_1} = c_1 + e^{i(\frac{\pi}{2} - \theta_1)}\mathbb{R}$ and $\partial \mathbb{H}^{\theta_2}_{c_2} = c_2 + e^{i(\frac{\pi}{2} - \theta_2)}\mathbb{R}$ and since $\theta_2 - \theta_1 < \pi$ there is some point z^* at the intersection of these two lines. Thus $\mathbb{H}_{12} \neq \emptyset$ and our claim does make sense.

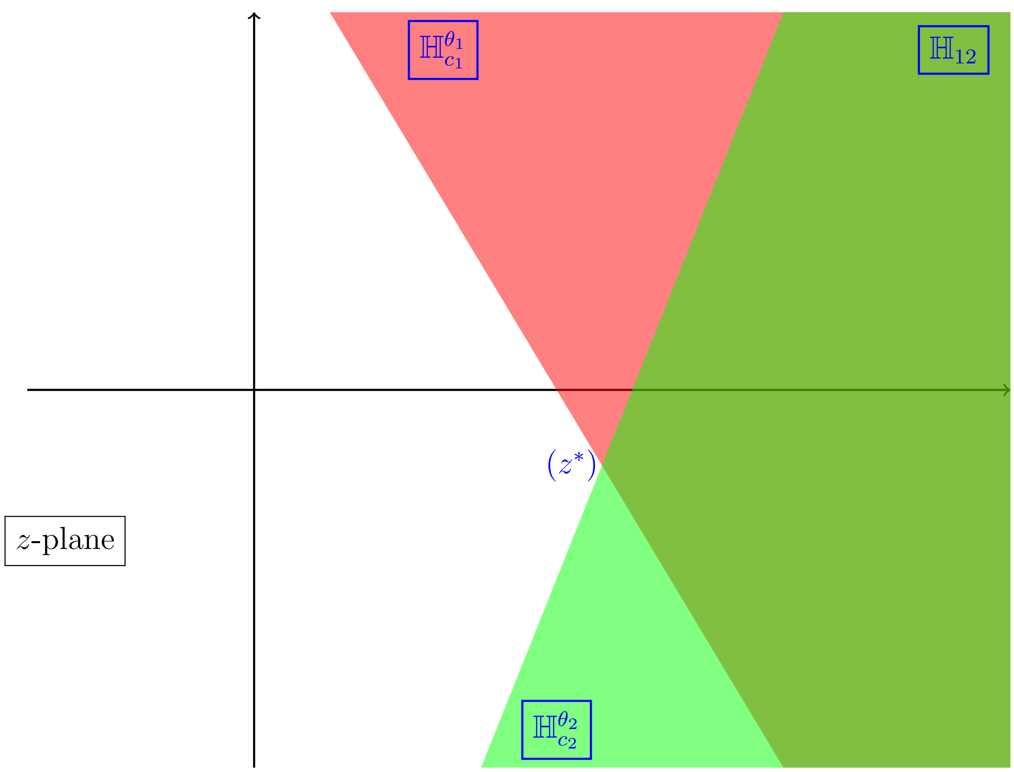


Figure 3.1: Domains of analyticity of the Laplace transforms.

Now let us consider what happens in the ζ-plane. By hypothesis $\widehat{f}$ defines an analytic function in the sector contained between the lines $e^{i\theta_1}\mathbb{R}_+$ and $e^{i\theta_2}\mathbb{R}_+$. Inside this sector we can consider the domain $\mathcal{U}$ enclosed by the curve

$$\{re^{i\theta_1}\,|\,r\in[0,R]\}\cup\{re^{i\theta_2}\,|\,r\in[R,0]\}\cup\{Re^{i\theta}\,|\,\theta\in[\theta_1,\theta_2]\}\,. \tag{3.21}$$

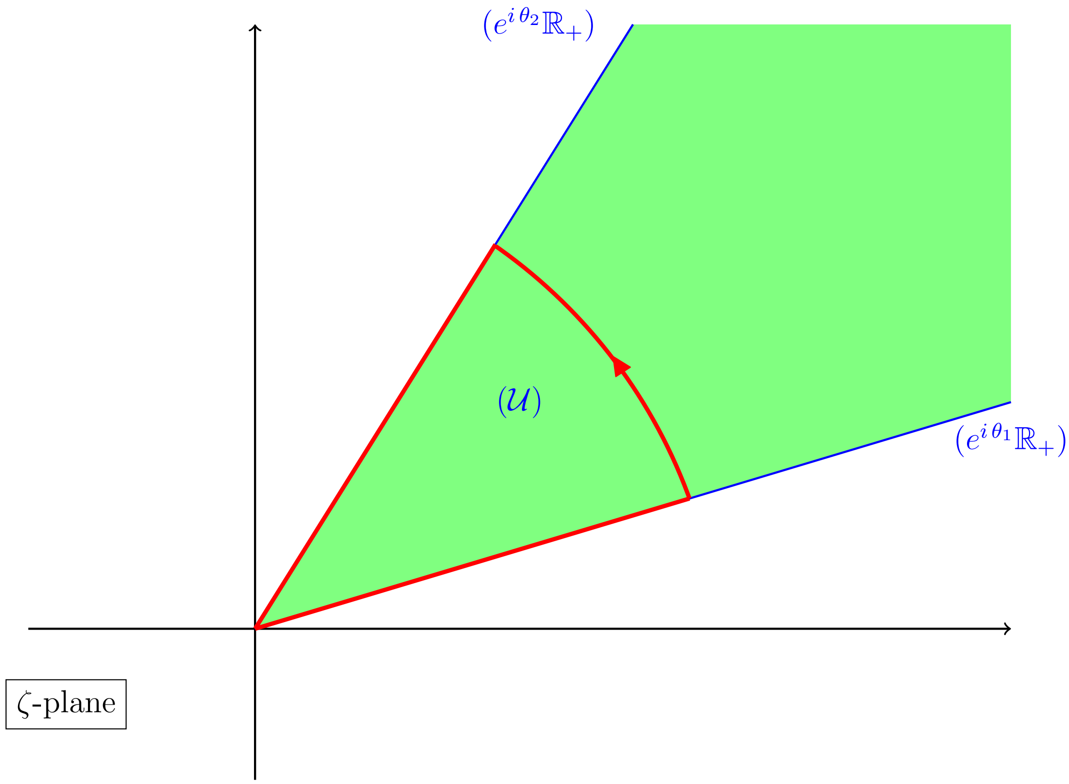


Figure 3.2: Graphic representation of the contour of integration.

Then by Cauchy's theorem we have the formula:

$$\int_{\partial\mathcal{U}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta = \int_0^{Re^{i\theta_1}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta - \int_0^{Re^{i\theta_2}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta + \int_{\substack{|\zeta|=R\\ \theta_1<\arg(\zeta)<\theta_2}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta = 0\,. \tag{3.22}$$

So when we take the limit as $R\longrightarrow\infty$ we obtain

$$\mathcal{L}_{\theta_1}\widehat{f}(z) - \mathcal{L}_{\theta_2}\widehat{f}(z) + \lim_{R\longrightarrow\infty}\int_{\substack{|\zeta|=R\\ \theta_1<\arg(\zeta)<\theta_2}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta = 0\,. \tag{3.23}$$

Then in order to prove our theorem we need to show that for all z in some open subset of $\mathbb{H}_{12}$

$$\lim_{R\longrightarrow\infty} \int_{\substack{|\zeta|=R \\ \theta_1<\arg(\zeta)<\theta_2}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta = 0\,. \tag{3.24}$$

Indeed by the identity theorem of analytic functions, if that is the case then $\mathcal{L}_{\theta_1}\widehat{f}(z) = \mathcal{L}_{\theta_2}\widehat{f}(z)$ for all $z \in \mathbb{H}_{12}$. From the fact that $\widehat{f}$ is exponentially bounded we get the following estimates:

$$\left| \int_{\substack{|\zeta|=R \\ \theta_1<\arg(\zeta)<\theta_2}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta \right| \leq C^{te}\,(\theta_2-\theta_1)\,e^{-R(\Re(z)-c)}\,. \tag{3.25}$$

Then one can pick D, an open subset of $\mathbb{H}_{12}$, such that for all $z \in \mathbb{D}$, $\Re(z) > c$ and from our last estimates we have on this set

$$\lim_{R\longrightarrow\infty} \int_{\substack{|\zeta|=R \\ \theta_1<\arg(\zeta)<\theta_2}} e^{-z\zeta}\,\widehat{f}(\zeta)\,d\zeta = 0\,. \tag{3.26}$$

So we get $\mathcal{L}_{\theta_1}\widehat{f}\,\big|_D = \mathcal{L}_{\theta_2}\widehat{f}\,\big|_D$ as desired. □

One can use the previous theorem to glue different resummations of the original series into analytic functions.

Example 3.2.1. Let us go over the classical example of the resummation of the Euler series. The Euler series $\widetilde{f}(z) \in \mathbb{C}[[1/z]]$ is the formal power series solution to the differential equation

$$\frac{df}{dz} - f = -\frac{1}{z}\,. \tag{3.27}$$

A quick calculation show that the series must be

$$\widetilde{f}(z) = \sum_{n=1}^{+\infty} (-1)^{n+1}\,(n-1)!\,\frac{1}{z^n}\,. \tag{3.28}$$

This series is of Gevrey class 1 and its Borel transform is simply $\widehat{f}(\zeta) = 1/(1+\zeta)$. It has only one singularity at -1 and as a rational function it is exponentially bounded everywhere else. Thus for any $\theta \neq \pi$ we have $\widehat{f} \in \widehat{E}_0(e^{i\theta}\mathbb{R}_+)$ and the Euler series is fine summable along each of these directions:

$$f_\theta(z) = \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta\, z}\frac{d\zeta}{1+\zeta}\,. \tag{3.29}$$

Let us pick the solution f_0 corresponding to an integration along the positive real axis. We know that $f_0 \in \mathcal{O}(\mathbb{H}_0^0)$ and from theorem 3.2.2 a rotation of the line of integration for the Laplace transform corresponds to an analytic continuation of f_0 for rotation angles smaller than π. Let us illustrate this process starting from f_0 and proceeding to do an analytic continuation by a clockwise rotation of the contour of integration.

On the next page we read the figure as follows, on each line one can see:

- on the left the contour of integration in the Borel plane along which we compute the Laplace transform;
- on the right side the half space in which the Laplace–Borel transform along the chosen contour is analytic;

Going from top to bottom we observe the effect of a counterclockwise rotation of the contour of integration.

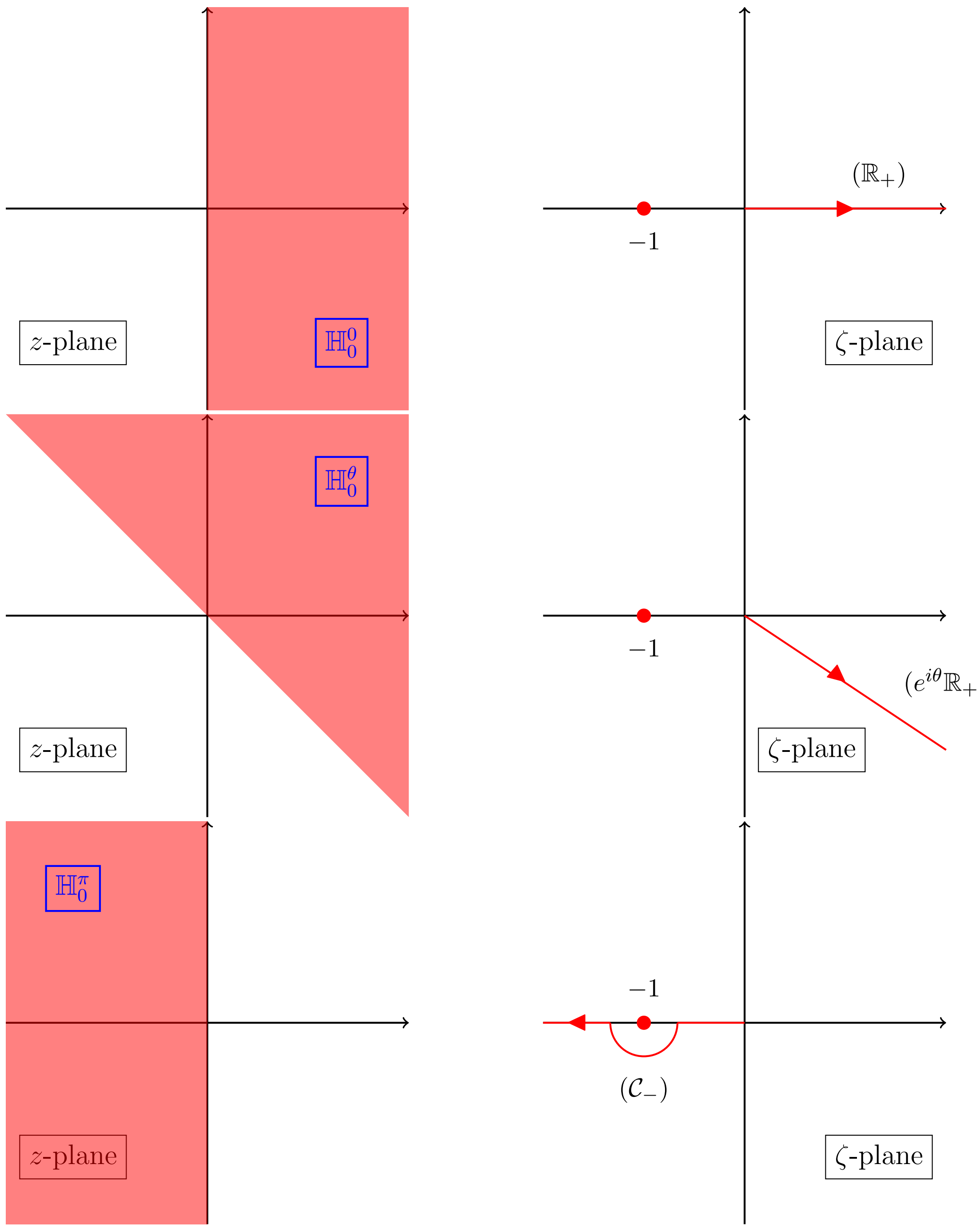


Figure 3.3: Illustration of the principle of analytic continuation by clockwise rotation of the contour of integration.

Thus the clockwise analytic continuation of f_0 gives rise to the function $f_+ \in \mathcal{O}(\mathbb{C} \setminus i\mathbb{R}_-)$ and by the same argument a counterclockwise analytic continuation of f_0 yields another analytic function $f_- \in \mathcal{O}(\mathbb{C} \setminus i\mathbb{R}_+)$. These two functions have in common the domain of definition $\mathbb{C} \setminus i\mathbb{R}$ so it makes sense to compare them. To do so we choose representatives of f_+ and f_- corresponding to the contours of integrations $\mathcal{C}_+$ and $\mathcal{C}_-$ depicted below.

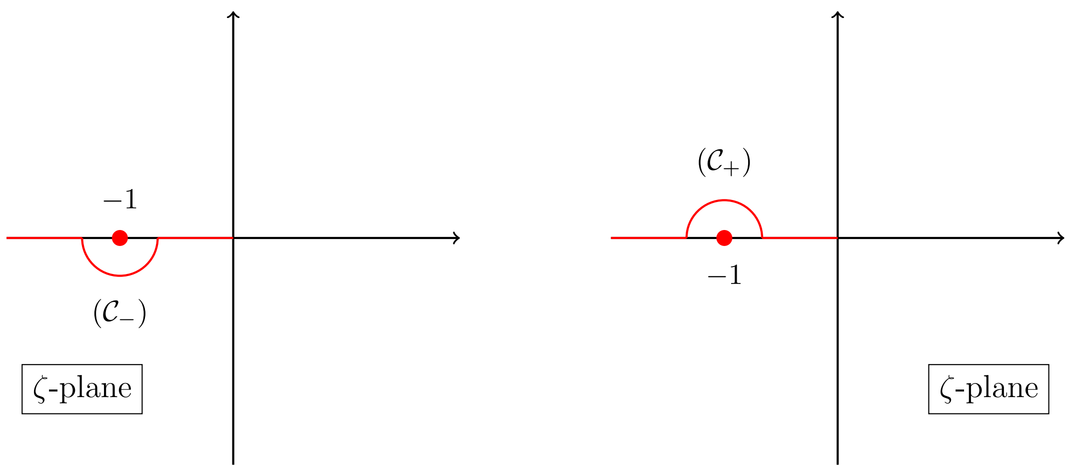


Figure 3.4: The contours of integration avoiding the singularity at -1 from the left and from the right.

Computing their difference reduces to a residue calculation:

$$f_+(z) - f_-(z) = 2i\pi \mathrm{Res}\left(\zeta \mapsto \frac{e^{-\zeta z}}{1+\zeta}; \zeta = -1\right) = 2i\pi e^{-z}\,. \tag{3.30}$$

This relation holds on $\mathbb{C} \setminus i\mathbb{R}$; it is a connection formula whose origin is the presence of a singularity at $\zeta = -1$ for the Borel transform of the Euler series. This is the Stokes phenomenon. Observe that the domain $\mathbb{C} \setminus i\mathbb{R}$ can be divided in $\Re(z) > 0$ and $\Re(z) < 0$. On the right half-plane the above formula shows that at infinity the two functions f_+ and f_- get exponentially close while at infinity on the right half-plane there difference gets arbitrarily large. The line $i\mathbb{R}$ separates these two asymptotic behaviours and is called a Stokes line.

Example 3.2.2. Not all divergent power series are fine-summable, even if their growth is controlled. Indeed any Gevrey-1 series whose Borel transform is lacunary will not be fine-summable along any direction of the complex plane. As an example fix p an integer strictly larger than 1 and consider the formal power series

$$\widetilde{f}(z) \;=\; \sum_{\substack{n=p^k\\ k\in\mathbb{N}}} n!\,\frac{1}{z^{n+1}}\,. \tag{3.31}$$

The series $\widetilde{f}$ is obviously of Gevrey class 1 and its Borel transform is the germ of analytic function

$$\widehat{f}(\zeta) \;=\; \sum_{k=0}^{\infty} \zeta^{p^k}\,. \tag{3.32}$$

However we claim that $\widehat{f}$ has singularities at each p^kroot of unity for all $k \in \mathbb{N}$. First observe that $\widehat{f}$ has a radius of convergence equal to 1 with a singularity at 1. Moreover for any $k \in \mathbb{N}^*$ it satisfies the functional equation

$$\widehat{f}(\zeta^{p^k}) \;=\; \widehat{f}(\zeta) - \sum_{i=0}^{k-1} \zeta^{p^i}\,. \tag{3.33}$$

Thus if 1 is a singular point of $\widehat{f}$ so are all the solutions of the equation $\zeta^{p^k} = 1$ for all $k \in \mathbb{N}$.

So $\widehat{f}$ is singular on a dense subset of $\mathbb{S}^1$ and one cannot find any $\epsilon > 0$ such that $\widehat{f}$ has a unique analytic continuation on a half-strip St^{θ}_{ϵ} around any direction $e^{i\theta}\mathbb{R}_+$. Hence $\widehat{f}$ fails at condition (ii) in the definition of fine-summability along all the direction in $\mathbb{C}$.

We conclude this section with some technical properties of the convolution product, the Laplace and the Borel transforms that are used in many calculations in the following two chapters. Recall that if $\widehat{f}$ and $\widehat{g}$ are analytic functions in a neighbourhood of the origin then for $|\zeta|$ small enough their convolution product is give by the line integral

$$\widehat{f} * \widehat{g}(\zeta) = \int_0^{\zeta} \widehat{f}(t)\,\widehat{g}(\zeta - t)\,dt\,. \tag{3.34}$$

In the following statement we take the point of view of the Laplace transforms. Using the relation between the Laplace and Borel transforms given above (see theorem 3.1.2) one can get equivalent properties from the point of view of the Borel transform.

Property 3.2.3. *Let $\widehat{f}$ and $\widehat{g}$ be germs of analytic functions in a neighbourhood of the origin without constant terms and $\theta \in \mathbb{S}^1$ be such that the Laplace transforms below are well defined. We have the following relations:*

(i) $(\widehat{f} * \widehat{g})' = \widehat{f}' * \widehat{g} = \widehat{f} * \widehat{g}'$;

(ii) $\mathcal{L}_\theta[\widehat{f} * \widehat{g}] = \mathcal{L}_\theta[\widehat{f}]\mathcal{L}_\theta[\widehat{g}]$;

(iii) $\mathcal{L}_\theta[-\zeta \widehat{f}] = \frac{d}{dz}\mathcal{L}_\theta[\widehat{f}]$;

Proof. Without loss of generality we work with $\theta = 0$ in order to simplify the notations.

(i) The first equality follows immediately from the application of the formula of derivation of an integral depending on a parameter. The second equality result from the commutativity of the convolution product.

(ii) By definition the Laplace transform of a convolution product is given by

$$\mathcal{L}_0[\widehat{f} * \widehat{g}](z) = \int_{\mathbb{R}_+} e^{-\zeta z}\, \widehat{f} * \widehat{g}(\zeta)\, d\zeta \tag{3.35}$$

$$= \int_{\mathbb{R}_+} \int_0^{\zeta} e^{-\zeta z} \widehat{f}(t)\widehat{g}(\zeta - t)\, dt\, d\zeta \tag{3.36}$$

$$= \int_{\mathbb{R}_+} \int_0^{\zeta} e^{-t z} \widehat{f}(t)\, e^{-(\zeta - t)z} \widehat{g}(\zeta - t)\, dt\, d\zeta\,. \tag{3.37}$$

So our domain of integration is $\{0 < \zeta < +\infty\,, 0 < t < \zeta\}$, that is the part of the left upper quandrant of the (t, ζ) plane between the lines $\zeta = 0$ and $\zeta = t$. But this domain can also be parametrized as $\{0 < \zeta < +\infty\,, \zeta < t < \infty\}$. We then rewrite our integral as

$$\mathcal{L}_0[\widehat{f} * \widehat{g}](z) = \int_{\mathbb{R}_+} \int_{\zeta}^{+\infty} e^{-t z} \widehat{f}(t)\, e^{-(\zeta - t)z} \widehat{g}(\zeta - t)\, dt\, d\zeta \tag{3.38}$$

$$= \int_{\mathbb{R}_+} e^{-\zeta z}\, \widehat{f}(\zeta)\, d\zeta \int_{\mathbb{R}_+} e^{-\zeta' z}\, \widehat{g}(\zeta')\, d\zeta' \tag{3.39}$$

$$= \mathcal{L}_0[\widehat{f}](z)\, \mathcal{L}_0[\widehat{g}](z)\,, \tag{3.40}$$

with the change of variables $\zeta - t = \zeta'$.
(iii) Let us look at the Laplace transform

$$\mathcal{L}_0[\widehat{f}](z) = \int_{\mathbb{R}_+} e^{-\zeta\, z}\widehat{f}(\zeta)\, d\zeta\,. \tag{3.41}$$

As we have seen before, when this integral is convergent it defines a function analytic in z in some right half-plane. Changing the integrand with a multiplication by $-\zeta$ does not decrease the domain of analyticity of the function. Thus we can exchange the order of integration and differentiation with respect to the variable z to obtain

$$\frac{d}{dz}\,\mathcal{L}_0[\widehat{f}](z) = \int_{\mathbb{R}_+} -\zeta\, e^{-\zeta\, z}\widehat{f}(\zeta)\, d\zeta = \mathcal{L}_0[-\zeta\,\widehat{f}](z)\,. \tag{3.42}$$

$\square$

A formal power series can always be interpreted as the asymptotic expansion of some analytic function. Such a function is not unique but theorem 3.1.2 shows that up to exponential terms it is given by the Laplace–Borel resummation process. We are now ready to apply this method to the case of Dyson–Schwinger equations.

Chapter 4

Summability of the Anomalous Dimension

4.1 Introduction

In the previous part we discussed the possibility of looking for solutions to Dyson–Schwinger equations in the form of power series as

$$\widetilde{G}(x, L) = \sum_{n=0}^{+\infty} \gamma_n(x)\, L^n\,. \tag{4.1}$$

In this section we want to explore some of the analytic properties of the coefficients γ_n. Observe that due to the recursive nature of the Dyson–Schwinger Equation 2.5 the γ_n satisfy the recursion

$$\gamma_n(x) = \frac{1}{n}\,\gamma_1(x)(x\partial_x - 1)\gamma_{n-1}(x) \tag{4.2}$$

for all $n > 1$. So we see that the properties of the coefficients $\gamma_{n>1}$ can be read directly from those of γ_1. Hence our work will be focused on the function γ_1 called the anomalous dimension associated to the Green function $G(x, L)$.

We start by using a combinatorial description of the series expansion of γ_1 as described in [20] to obtain some preliminary results regarding the location of the singularities of its Borel transform $\widehat{\gamma}_1$. It is this example that guides us to a general analysis of the P-differential equation. Our main results in theorems 4.3.2 and 4.3.3 proves the Laplace–Borel summability of γ_1 for a large class of P-differential equation.

In particular this result shows that to the asymptotic expansion of the anomalous dimension corresponds several analytic objects. Thus going from the formal to the analytic solutions to our problem one must introduce additional conditions to be able to select one non-perturbative expression for the anomalous dimension.

4.2 Chord Diagrams

Suppose we are looking for a formal power series solution to the Dyson–Schwinger equation, say $\widetilde{G}(x,L) \in \mathbb{C}[[x,L]]$ and that we would like a combinatorial description of this solution. Let $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ be some family of combinatorial classes, we try the ansatz

$$\widetilde{G}(x,L) = \sum_n \sum_{X\in\mathcal{C}_n} \widetilde{G}_X \, x^{|X|} \, L^n \,. \tag{4.3}$$

From experience we observe that the number of terms in the coefficient of $x^{|X|}$ relative to the combinatorial class $\mathcal{C}_n$ grows like $|X|!!$, where $|X|$ is the number of chords in the diagram, which suggests to look at restricted class of matchings to describe $\mathcal{C}_n$.

Using the recursive form of the Dyson–Schwinger equation **??** we can obtain an arbitrary long series expansion of the $\widetilde{G}(x,L)$. Here is the beginning of this expansion in terms of the coefficients f_j of the Laurent series F:

$$\widetilde{G}(x,L) = 1{+}f_0xL + f_0f_1x^2L - \frac{1}{2}f_0^2x^2L^2 + (f_0f_1^2 + 3f_0^2f_2)x^3L \tag{4.4}$$

$$-\frac{1}{2}(f_0f_1^2 + 3f_0^2f_2)x^3L^2 + \frac{1}{6}3f_0^3x^3L^3 \tag{4.5}$$

$$+ (15f_0^3f_3 + 11f_0^2f_1f_2 + f + f_0f_1^3)x^4L \tag{4.6}$$

$$-\frac{1}{2}(18f_0^3f_2 + 9f_0^2f_1^2)x^4L^2 + \frac{1}{6}(23f_0^3f_1)x^4L^3 + \cdots \tag{4.7}$$

It happens that the correct interpretation makes use of combinatorial quantities that can be read on rooted connected chord diagrams as described in [20]. Let us briefly extract from [20] some elementary constructions necessary to provide a combinatorial meaning to the asymptotic expansion of the anomalous dimension γ_1.

Definition 4.2.1. A **chord diagram** of order n is the data of $2n$ points $(p_1, \cdots, p_{2n})$ arranged on a circle, together with n distinct pairs of distinct points $\{(p_{i_1}, p_{i_{n+1}}), \cdots, (p_{i_n}, p_{i_{2n}})\}$ forming a partition of $(p_1, \cdots, p_{2n})$, where $(p_{i_k}, p_{i_{n+k}})$ is represented by a chord joining the points p_{i_k} and $p_{i_{n+k}}$ on the circle. Additionally we say that:

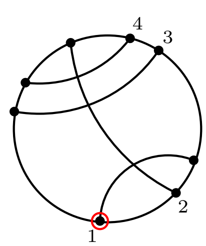
- Two distinct chords $(p_{i_k}, p_{i_{n+k}})$ and $(p_{i_l}, p_{i_{n+l}})$ intersect each other if and only if $p_{i_k} < p_{i_l} < p_{i_{n+k}} < p_{i_{n+l}}$ or $p_{i_l} < p_{i_k} < p_{i_{n+l}} < p_{i_{n+k}}$;
- A chord diagram is rooted when we distinguish one of the points $p_1, \cdots, p_{2n}$ on the circle;
- A chord diagram is disconnected if we can partition the set of chords in two sets so that no chord of the first set intersect any chord of the second. Otherwise we will say that the chord diagram is connected.

In the following $\mathcal{RCCD}$ denotes the set of rooted connected chord diagrams, it corresponds to $\bigcup_{n\in\mathbb{N}^*} \mathcal{C}_n$ in our ansatz. We say that a diagram with n chords has degree n and we denote by $\mathcal{RCCD}_n$ the family of these diagrams. Unless stated otherwise our chord diagrams are oriented counterclockwise, meaning that the points on the outer circle are numbered $p_1, p_2, ..., p_{2n}$ with p_1 corresponding to the root and continuing counterclockwise from this point.

Figure 4.1: (A) is a rooted connected chord diagram of degree 4 while (B) has degree 3 but is not connected. Here the root is the circled vertex, the root chord is numbered 1 and the rest is labelled in the counterclockwise order.

Definition 4.2.2. Let X be a rooted connected chord diagram of degree n. We denote by $\mathcal{I}(X)$ the labeled directed graph whose set of vertices $\{1, 2, ..., i, ..., n\}$ corresponds to the set of chords of X where i stands for the i^{th} chords in the counterclockwise order and there is a directed edge from i to j if the i^{th} chord intersects the j^{th} chord with $i < j$. The graph $\mathcal{I}(X)$ is called the **directed intersection graph** of X.

Here is an example of a rooted connected chord diagram and its directed intersection graph:



$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] \arrow[rr, bend left] & 3 & 4 \end{tikzcd}$$

Definition 4.2.3. Let X be a rooted connected chord diagram and $\mathcal{I}(X)$ its directed intersection graph. A chord i is said to be **terminal** if the vertex i of $\mathcal{I}(X)$ has no outgoing edges.

Hence a terminal chord does not intersect any chord with a larger index. As one can see in the examples presented below, the linear order on the vertices of the directed intersection graphs makes it easy to observe the gaps between terminal chords.

But unfortunately things are not straightforward and at this point we need to relabel the chords of our diagrams in a new order called the intersection order if we want to capture combinatorially the recursive nature of our Dyson–Schwinger equation.

This order is defined recursively directly on the chord diagrams or on their intersection graph. We choose this second option and we express this new order as a permutation of the counterclockwise order.

Definition 4.2.4. Let X be a rooted connected chord diagram of degree n with its sequence of chords $(1, 2, ..., n)$ in the counterclockwise order and $\mathcal{I}(X)$ its intersection graph. Apply the following recursive procedure:

1) consider the graph $\mathcal{I}(X)$ and delete the edges going out of its smallest vertex, the vertex 1;

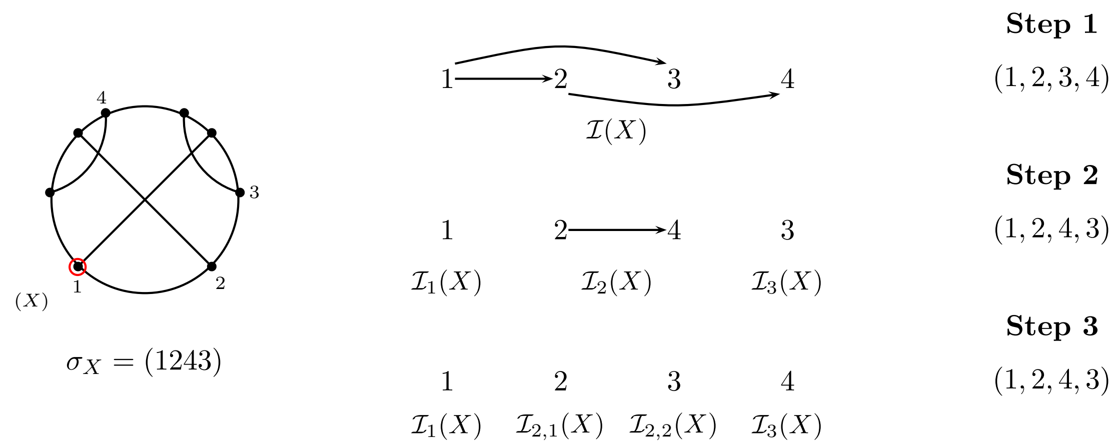
2) obtain k connected components $\mathcal{I}_1(X) = \{1\}, \mathcal{I}_2(X), ..., \mathcal{I}_k(X)$ where the smallest vertex of $\mathcal{I}_p(X)$ is larger than the smallest vertex of $\mathcal{I}_q(X)$ when $q < p$;

3) then each connected component $\mathcal{I}_p(X)$ is associated to its sequence of vertices $(x_{1,p}, x_{2,p}, ...)$ in counterclockwise order. This defines a permutation $(1, 2, ..., n) \mapsto (1, x_{1,1}, x_{2,1}, ..., x_{1,2}, x_{2,2}, ..., x_{1,k}, x_{2,k}, ...)$;

4) apply this procedure recursively to each $\mathcal{I}_p(X), (x_{1,p}, x_{2,p}, ...)$ until we are left with n singletons;

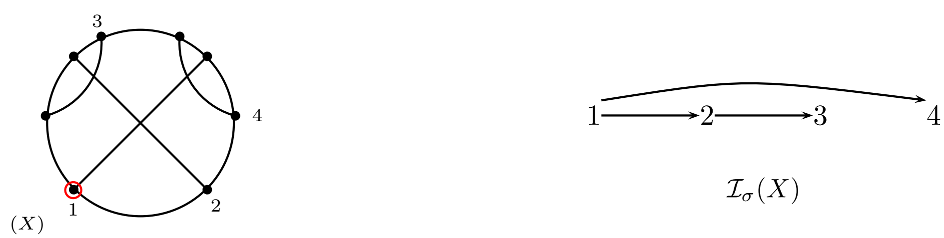
This defines a permutation $\sigma_X : (1, 2, ..., n) \mapsto (\sigma_1, \sigma_2, ..., \sigma_n)$ that we call the **intersection order** of X.

This procedure is easily understood on an example:

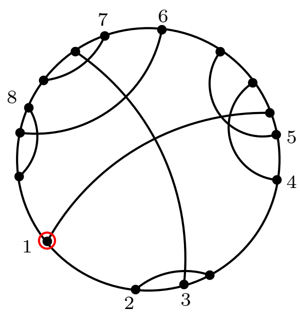


where each step corresponds to rearranging the elements of each block with respect to their smallest label as prescribed in the definition.

So finally we obtain the chord diagram and its corresponding intersection diagram now labeled in the intersection order given by $\sigma_X = (1243)$:

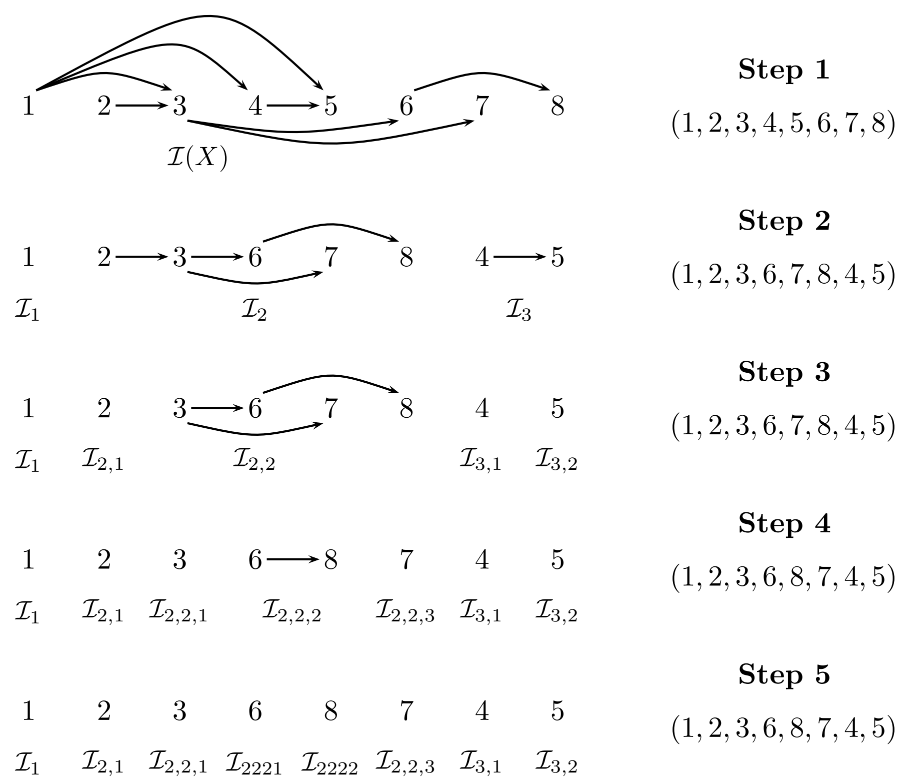


A more general example:

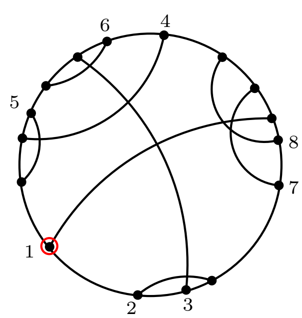


(X)

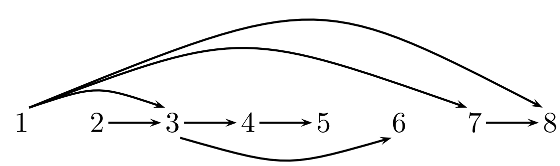
$\sigma_X = (12368745)$



So finally we get the following chord diagram together with its intersection graph in the intersection order given by $\sigma_X = (12368745)$:



(X)



$\mathcal{I}_\sigma(X)$

If $\mathcal{I}(X)$ is a directed intersection graph we denote by $\mathcal{I}_\sigma(X)$ the graph obtained by relabelling the vertices with the permutation σ_X. This operation is an automorphism of the graph $\mathcal{I}(X)$ so if i was a terminal chord of X in the counterclockwise order, $\sigma_i = \sigma_X(i)$ is a terminal chord of X in the intersection order.

For chord diagrams of small degree the intersection order and the counterclockwise order coincide most of the time. It is only at higher degrees that we start to see the differences between these orders.

Definition 4.2.5. Let X be a rooted connected chord diagram with intersection order $\sigma_X = (\sigma_1, ..., \sigma_n)$. The sequence of terminal chords of X in the intersection order is denoted by $Ter_\sigma(X) = (\sigma_{i_1}, ..., \sigma_{i_k})$ with $\sigma_{i_1} < \sigma_{i_2} < ... < \sigma_{i_k}$. We associate to $Ter_\sigma(X)$ the sequence of consecutive gaps between terminal chords in the intersection order:

$$\delta(X) = (\sigma_{i_2} - \sigma_{i_1}, \sigma_{i_3} - \sigma_{i_2}, ..., \sigma_{i_k} - \sigma_{i_{k-1}})\,. \tag{4.8}$$

We denote by $b(X)$ the first element of $Ter_\sigma(X)$ i.e. the smallest terminal chord in the intersection order.

It is easier to handle sequences of gaps $\delta(X)$ with constant lengths over the chord diagrams with constant degree. So if X has degree n and $\delta(X) = (\delta_1, ..., \delta_k)$ we introduce:

$$\bar{\delta}(X) = \; (\underbrace{0, ..., 0,}_{n-k-1 \text{ times}} \; \delta_1, ..., \delta_k)\,, \tag{4.9}$$

so that $\bar{\delta}(X)$ has length $n-1$ if X has degree n. The gaps are then linked to the size of the diagram by a simple relation.

Lemma 4.2.1. *For X a rooted connected chord diagram let $g(X)$ be the sum of its gaps i.e. the sum of the elements of $\bar{\delta}(X)$. We have the relation:*

$$g(X) + b(X) = |X|\,. \tag{4.10}$$

Proof. Using the notation of the previous definition the sum over all the gaps reduces to a telescopic sum:

$$g(X) = \delta_1 + \cdots + \delta_k = \sigma_{i_2} - \sigma_{i_1} + \sigma_{i_3} - \sigma_{i_2} + \cdots + \sigma_{i_k} - \sigma_{i_{k-1}} = \sigma_{i_k} - \sigma_{i_1}\,. \tag{4.11}$$

But by definition $\sigma_{i_1} = b(X)$ and $\sigma_{i_k} = |X|$ since the last chord is always terminal. So we get $g(X) = |X| - b(X)$. $\square$

The main result of [20] consists in solving explicitly in the ansatz 4.3 to get a formal power series expression for the anomalous dimension:

$$\widetilde{\gamma}_1(x) = \sum_{X \in \mathcal{RCCD}} f_X \, f_{b(X)-1} \, x^{|X|} \,, \tag{4.12}$$

where the monomials f_X are products of the coefficients f_j of the series expansion of F defined by

$$f_X = f_0^{|X|-k-1} \, f_{\delta_1} \, f_{\delta_2} \dots f_{\delta_k} \,. \tag{4.13}$$

We sketch the main idea for the proof of the chord diagrams expansion. It all starts by observing that the rooted connected chord diagrams satisfy Stein's recursion [22], that is if c_n is the number of rooted connected chord diagrams on n chords then

$$c_n = (n-1) \sum_{k=1}^{n-1} c_k \, c_{n-k} \,. \tag{4.14}$$

This recursion can be generalized to keep track of the smallest terminal chord b(X) of the chord diagrams. One finds that if γ_k is the generating function of such chord diagrams with smallest terminal chord larger or equal to k then

$$k \, \gamma_k = \gamma_1 (2x\partial_x - 1) \gamma_k \,. \tag{4.15}$$

The rooted connected chord diagrams admit a recursive decomposition, the root-share decomposition, into smaller diagrams as well as a labelled rooted trees representation. We do not describe these transformations but instead refer to [20] for a complete treatment. The crucial point is that every rooted connected chord diagram X has a root-share decomposition in two sub-diagrams X_1 and X_2 such that the following equality holds:

$$f_X f_{b(X)-k} = f_{X_1} f_{b(X_1)-1} f_{X_2} f_{b(X_2)-k+1} \,. \tag{4.16}$$

This factorization is proved using the rooted tree representation of the chord diagrams and is all that we need to show that the chord diagrams expansion does solve the Dyson–Schwinger equation.

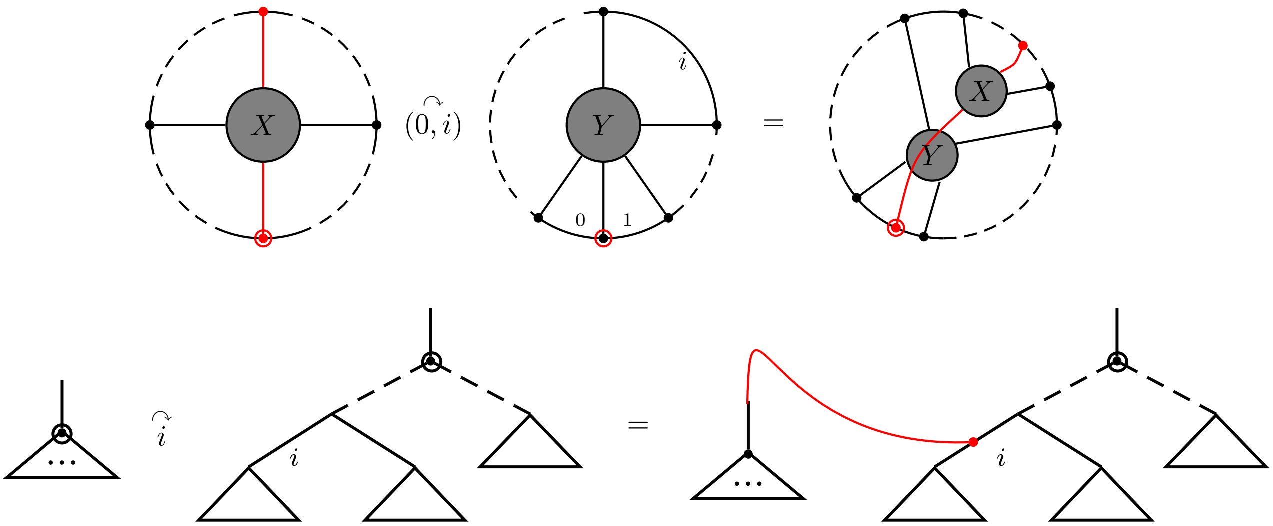


Figure 4.2: Root-share decomposition and tree representation.

In [20] we proved that when the series F is bounded by a geometric series then the anomalous dimension $\widetilde{\gamma}_1$ necessarily has a convergent Borel transform, so let us study an example of the limiting case. For the rest of this section we fix a particular Dyson–Schwinger by the choice of

$$F(\rho) = \frac{1}{\rho} + \frac{a}{1 - a\rho} \tag{4.17}$$

where $a \in \mathbb{C}^*$. With this rational function $\widetilde{\gamma}_1$ takes a very simple form.

Theorem 4.2.2. *With $F(\rho)$ as introduced above and c_n the number of rooted connected chord diagrams of order n the series representation of the anomalous dimension is*

$$\widetilde{\gamma}_1(x) = \frac{1}{a} \sum_{n=1}^{+\infty} c_n \, a^n \, x^n \, . \tag{4.18}$$

Proof. Let us look at the power series expansion of $F(\rho)$:

$$F(\rho) = \frac{f_0}{\rho} + \sum_{n=0}^{+\infty} f_{n+1}\,\rho^n \tag{4.19}$$

$$= \frac{1}{\rho} + a \sum_{n=0}^{+\infty} a^n\,\rho^n\,. \tag{4.20}$$

That is $f_0 = 1$ and for all $n > 0$ we have $f_n = a^n$ which implies that for all $X \in \mathcal{RCCD}$

$$f_X \;=\; 1^{|X|-k-1}\,a^{\delta_1+\cdots+\delta_k} \;=\; a^{|X|-b(X)} \tag{4.21}$$

by application of lemma 4.2.1. Thus the series expansion $\widetilde{\gamma}_1$ is nothing but

$$\widetilde{\gamma}_1(x) = \sum_{X\in\mathcal{RCCD}} a^{|X|-b(X)+b(X)-1}\,x^{|X|} \tag{4.22}$$

$$= \frac{1}{a}\sum_{n=1}^{+\infty} c_n\,(a\,x)^n\,, \tag{4.23}$$

where c_n is the number of rooted connected chord diagrams of order n. □

In other words $\widetilde{\gamma}_1$ is nothing but a rescaled version of the generating function of rooted connected chord diagrams. We now use a combination of analytic techniques together with this combinatorial interpretation to obtain information about the analytic object represented by the formal power series $\widetilde{\gamma}_1$.

Our first observation concerns the growth rate of this formal power series.

Lemma 4.2.3. *The formal power series $\widetilde{\gamma}_1$ is a divergent series of Gevrey class* 1.

Proof. From [22] we know that the number of rooted connected chord diagrams satisfies the recursion

$$c_1 = 1 \text{ and for } n > 1,\; c_n = (n-1)\sum_{k=1}^{n-1} c_k\,c_{n-k}\,. \tag{4.24}$$

From this relation we see that $c_n > (n-1)!$ so the series has a radius of convergence equal to zero. However c_n is bounded by the total number of chord diagrams with n chords that is

$$c_n < (2n-1)!! \le (2n)!! = 2^n\,n!\,. \tag{4.25}$$

So the coefficients of $\widetilde{\gamma}_1$ do not grow faster than $n!$ which proves that the series is of Gevrey class 1. $\square$

This particular result actually follows from a more general observation.

Proposition 4.2.4. *Assume that there is a positive constant C such that for all $n \in \mathbb{N}$ the coefficients of $F(\rho)$ satisfy $|f_n| \leq C^{n+1}$, then for all $k \in \mathbb{N}^*$ the formal power series $\gamma_k(x)$ is of Gevrey class 1.*

Proof. We need to show that for all $k \in \mathbb{N}^*$ and $n \in \mathbb{N}$

$$\left| \sum_{|X|=n\,,\, b(X)\geq k} f_X\, f_{b(X)-k} \right| \leq A_k\, B_k^n\, n!$$

with $A_k, B_k \in \mathbb{R}_+$. To do so we write the monomial $f_X = f_0^{p_0(X)} \cdots f_n^{p_n(X)}$ with $p_i(X)$ the number of times the factor f_i appears in the product f_X.

We have $p_0(X) + \cdots + p_n(X) = n - 1$, the length of $\bar{\delta}(X)$. So we get:

$$\begin{aligned}\left| \sum_{|X|=n\,,\, b(X)\geq k} f_0^{p_0(X)} \cdots f_n^{p_n(X)}\, f_{b(X)-k} \right| &\leq \sum_{|X|=n\,,\, b(X)\geq k} |f_0|^{p_0(X)} \cdots |f_n|^{p_n(X)}\, |f_{b(X)-k}| \\ &\leq \sum_{|X|=n\,,\, b(X)\geq k} C^{p_0(X)+2\,p_1(X)+\cdots+(n+1)\,p_n(X)}\, C^{b(X)-k+1}\,.\end{aligned}$$

We can decompose the exponent of C as

$$\begin{aligned}p_0(X)+&2p_1(X) + \cdots + (n+1)p_n(X) \\ &=[p_0(X) + \cdots + p_n(X)] + [p_1(X) + 2p_2(X) + \cdots + np_n(X)] \\ &=(n-1) + g(X)\end{aligned}$$

where $g(X)$ is the sum over the gaps of X. Using Lemma 4.2.1 in the second inequality we get

$$\left| \sum_{|X|=n\,,\, b(X)\geq k} f_X\, f_{b(X)-k} \right| \leq \sum_{|X|=n\,,\, b(X)\geq k} C^{g(X)+n-1+b(X)-k+1} \leq C^{2n-k}\, c_{n,k}$$

with $c_{n,k}$ the number of rooted connected chord diagrams of degree n with smallest terminal chord larger than k. So for all $k \in \mathbb{N}$ this is bounded by the total number of rooted chord diagrams of degree n i.e.

$$c_{n,k} \leq (2n-1)!! \leq (2n)!! = 2^n \, n! \, .$$

Putting everything together we get the desired bound:

$$\left| \sum_{|X|=n \, , \, b(X) \geq k} f_X \, f_{b(X)-k} \right| \leq C^{-k} \, (2C^2)^n \, n! \, .$$

□

For the γ_k satisfying the conditions of this proposition, applying the Borel transform gives series with a non zero radius of convergence so that they are amenable to a study of their Borel summability properties. In particular, for γ_1, this observation opens the road for a study of the global analytic properties (value distribution, asymptotics at infinity) of the β function of this model.
In order to use the standard results concerning Laplace–Borel resummation we proceed to the change of variables $x = 1/t$. That means we consider the formal power series

$$\widetilde{\gamma}_1(t) = \frac{1}{a} \sum_{n=1}^{+\infty} c_n \, a^n \, \frac{1}{t^n} \, . \tag{4.26}$$

Then by definition of the Borel transform with respect to the variable t we have

$$\widehat{\gamma}_1(\zeta) \;=\; \mathcal{B}[\widetilde{\gamma}_1](\zeta) \;=\; \frac{1}{a} \sum_{n=1}^{+\infty} \frac{c_n \, a^n}{(n-1)!} \, \zeta^{n-1} \, . \tag{4.27}$$

To reconstruct an analytic function from $\widetilde{\gamma}_1(t)$ we need to learn a few things about the singular points of its Borel transform $\widehat{\gamma}_1(\zeta)$. Let us start by locating the singularity closest to the origin.

Lemma 4.2.5. *The dominant singularity of* $\widehat{\gamma}_1(\zeta)$ *is* $\zeta_0 = 1/2a$.

Proof. First of all observe that by choosing $a = 1$ we can consider that $\widehat{\gamma}_1(a = 1; \zeta)$ is a shifted exponential generating function of the rooted connected chord diagrams.

From [15] we know the asymptotic estimates of c_n:

$$\frac{c_n}{(n-1)!} \sim C^t \left(\frac{1}{2}\right)^{-n} n^{-1/2} . \tag{4.28}$$

By applying Pringsheim's theorem this indicates that $\zeta_0 = 1/2$ is the closest singularity of $\widehat{\gamma}_1(a = 1; \zeta)$. From there we obtain the general case simply by observing that $\widehat{\gamma}_1(a = 1; a\,\zeta) = \widehat{\gamma}_1(\zeta)$, that is considering a general $a \in \mathbb{C}^*$ is nothing but a rescaling of the variable. Thus $\zeta_0 = 1/2a$. □

To get more information we need to go beyond the series expansion and look for a functional equation satisfied by the anomalous dimension.

Proposition 4.2.6. *The series $\widetilde{\gamma}_1(t)$ and $\widehat{\gamma}_1(\zeta)$ satisfy respectively the following equations:*

$$-\partial_t \,\widetilde{\gamma}^2 \;=\; \frac{1}{a\,t}\,\widetilde{\gamma} + \frac{1}{t}\,\widetilde{\gamma}^2 \tag{4.29}$$

and

$$\widehat{\gamma} \;=\; a \,+\, a\,\zeta\,\widehat{\gamma} \,*\, \widehat{\gamma}' . \tag{4.30}$$

Proof. We start with $\widetilde{\gamma}_1(t)$. Everything follows from rewriting the recursion satisfied by c_n (see [22]) in terms of a differential equation. Observe that

$$\widetilde{\gamma}_1(t) = \sum_n c_n\, a^{n-1}\, \frac{1}{t^n} , \tag{4.31}$$

$$\widetilde{\gamma}_1^2(t) = \sum_n a^{n-2} \sum_{k=1}^{n-1} c_k\, c_{n-k}\, \frac{1}{t^n} , \tag{4.32}$$

$$-\,t\,\partial_t\,\widetilde{\gamma}_1^2(t) \;=\; \sum_n n\,a^{n-2} \sum_{k=1}^{n-1} c_k\, c_{n-k}\, \frac{1}{t^n} \tag{4.33}$$

$$\frac{1}{a}\,\widetilde{\gamma}_1(t) \;=\; \sum_n c_n\, a^{n-2}\, \frac{1}{t^n} . \tag{4.34}$$

By substitution in the equation

$$\frac{1}{a}\widetilde{\gamma} = -t\,\partial_t\,\widetilde{\gamma}^2 - \widetilde{\gamma}^2 \tag{4.35}$$

we recognize the recursion

$$c_n = (n-1)\sum_{k=1}^{n-1} c_k\, c_{n-k}\,. \tag{4.36}$$

Now to obtain an equation satisfied by the Borel transform $\widehat{\gamma}_1$ we can simply compute the Borel transform of the differential equation

$$-\partial_t\,\widetilde{\gamma}^2 \;=\; \frac{1}{a\,t}\,\widetilde{\gamma} + \frac{1}{t}\,\widetilde{\gamma}^2\,. \tag{4.37}$$

Since $\mathcal{B}[t \mapsto 1/t](\zeta) = 1$ we introduce $\mathbb{1}$ as the constant function equal to 1 everywhere. Using the properties described in property 3.2.3 we obtain

$$\zeta\,\widehat{\gamma} * \widehat{\gamma} \;=\; \frac{1}{a}\,\mathbb{1} * \widehat{\gamma} + \mathbb{1} * \widehat{\gamma} * \widehat{\gamma}\,. \tag{4.38}$$

This expression can be simplified by taking its derivative with respect to ζ:

$$\widehat{\gamma} * \widehat{\gamma} + \zeta\,\widehat{\gamma} * \widehat{\gamma}' \;=\; \frac{1}{a}\,(\widehat{\gamma} - a) + \widehat{\gamma} * \widehat{\gamma}\,, \tag{4.39}$$

that is

$$\widehat{\gamma} \;=\; a \;+\; a\,\zeta\,\widehat{\gamma} * \widehat{\gamma}'\,. \tag{4.40}$$

□

With the functional equation satisfied by $\widehat{\gamma}_1$ we can obtain an interesting property regarding its set of singularities.

Lemma 4.2.7. *Assume that ζ_0 and ζ_1 are non-zero distinct singular points of $\widehat{\gamma}_1$. Then the lattice $(\zeta_0\,\mathbb{N} + \zeta_1\,\mathbb{N}) \setminus \{0\}$ is contained in the set of singular points of $\widehat{\gamma}_1$.*

Proof. We only treat the case concerning singularities of the type $(\zeta - \zeta_i)^{-\alpha_i}$ with $\alpha_i \in \mathbb{R}_+$ as these are the ones appearing in our problem. However this result holds for a larger class of singularity types.

Assume that $\widehat{\gamma}_1$ has a singular point at ζ_i. Then one can use the decomposition

$$\widehat{\gamma}_1(\zeta) = \frac{A_i(\zeta)}{(\zeta-\zeta_i)^{\alpha_i}} + B_i(\zeta) \tag{4.41}$$

where A_i and B_i are regular at ζ_i. Then

$$\widehat{\gamma}_1'(\zeta) = \frac{A_i'(\zeta)}{(\zeta-\zeta_i)^{\alpha_i}} + \frac{-\alpha_i\, A_i(\zeta)}{(\zeta-\zeta_i)^{\alpha_i+1}} + B_i'(\zeta) \tag{4.42}$$

and we are able to express the convolution product $\widehat{\gamma}_1 * \widehat{\gamma}_1'$ as

$$\widehat{\gamma}_1 * \widehat{\gamma}_1'(\zeta) = I_{ij}(\zeta) + II_{ij}(\zeta) + B_i * B_j'(\zeta) \tag{4.43}$$

with

$$I_{ij}(\zeta) = \int_0^\zeta \frac{B_j(\zeta-t)\, A_i'(t)}{(t-\zeta_i)^{\alpha_i}}\, dt + \int_0^\zeta \frac{B_i'(t)\, A_j(\zeta-t)}{(\zeta-t-\zeta_j)^{\alpha_j}}\, dt + \int_0^\zeta \frac{-\alpha_i\, A_i(t)\, B_j(\zeta-t)}{(\zeta-\zeta_i)^{\alpha_i+1}}\, dt\,, \tag{4.44}$$

$$II_{ij}(\zeta) = \int_0^\zeta \frac{A_i'(t)\, A_j(\zeta-t)}{(t-\zeta_i)^{\alpha_i}(\zeta-t-\zeta_j)^{\alpha_j}}\, dt + \int_0^\zeta \frac{-\alpha_i\, A_i(t)\, A_j(\zeta-t)}{(t-\zeta_i)^{\alpha_i-1}(\zeta-t-\zeta_j)^{\alpha_j}}\, dt\,. \tag{4.45}$$

Thus we see that the terms I_{ij} and $B_i * B_j'$ do not introduce any new singular points (although it may change the nature of existing ones) while $II_{ij}(\zeta)$ is introducing a new singularity at $\zeta_i + \zeta_j$. Indeed the integrands appearing in II_{ij} contain terms of the form

$$\frac{1}{(t-\zeta_i)^a\,(\zeta-t-\zeta_j)^b} = \frac{1}{(t-\zeta_i)^a(\zeta-\zeta_i-\zeta_j)^b} + \frac{1}{(\zeta-\zeta_i-\zeta_j)^a(\zeta-t-\zeta_j)^b}\,. \tag{4.46}$$

These integrals do not cancel each other since their integrand do not have the same singularities or more precisely can have the same singularities but placed in different sheets of the Riemann surface of the logarithm. So we see that if $\widehat{\gamma}_1$ has singularities at ζ_i and ζ_j then $\widehat{\gamma}_1 * \widehat{\gamma}_1'$ is singular at ζ_i, ζ_j and $\zeta_i+\zeta_j$. But since $\widehat{\gamma}_1(\zeta) = a + a\,\zeta\,\widehat{\gamma}_1 * \widehat{\gamma}_1'(\zeta)$ then $\widehat{\gamma}_1$ is also singular at $\zeta_i + \zeta_j$.

Thus proceeding by induction $\widehat{\gamma}_1$ has singular points at $a\zeta_0 + b\zeta_1$ for all $a, b \in \mathbb{N}$ such that $(a,b) \neq (0,0)$. $\square$

It is a general property of the convolution product of analytic functions to create a lattice of singular points generated by the first singularity one encouters along every direction of the complex plane.

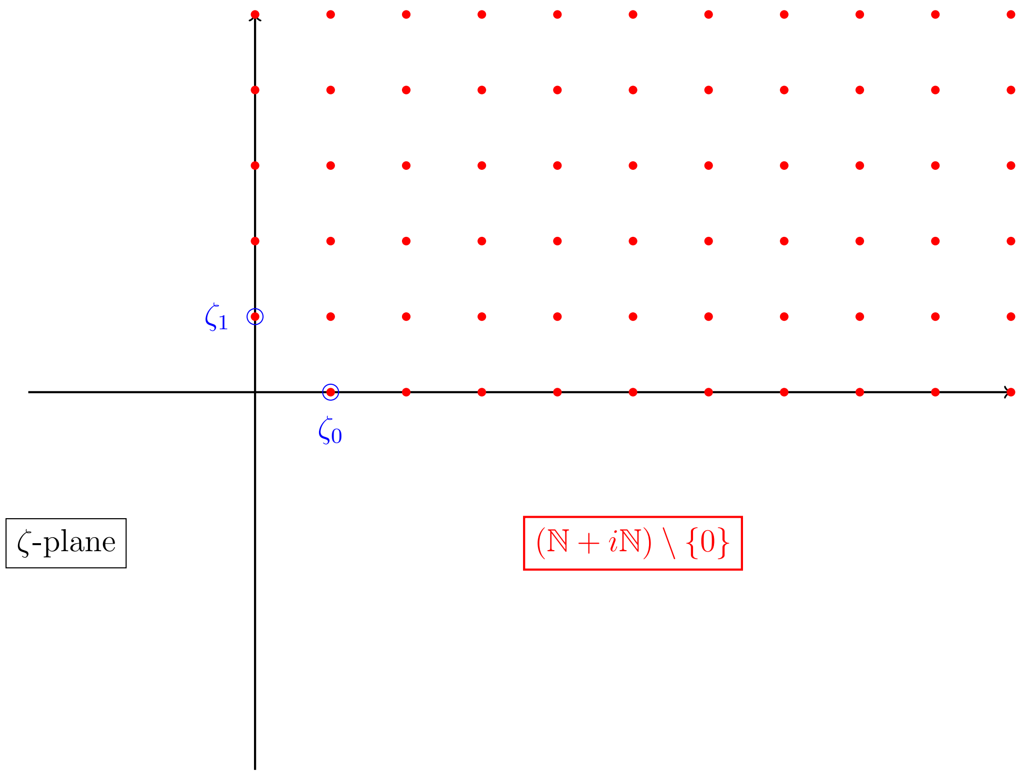


Figure 4.3: A pattern of singular points generated by $\zeta_0 = 1$ and $\zeta_1 = i$.

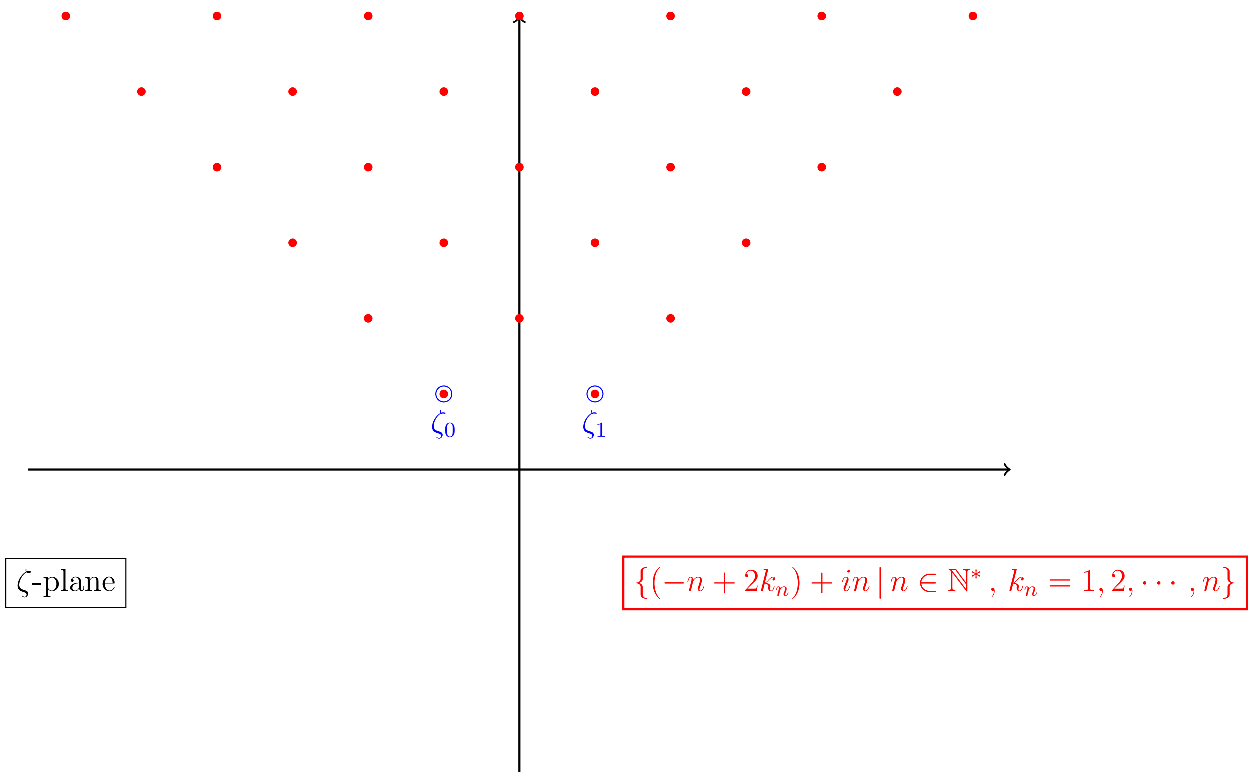


Figure 4.4: A pattern of singular points generated by $\zeta_0 = 1 + i$ and $\zeta_1 = -1 + i$.

Coming back to $\widehat{\gamma}_1$, lemma 4.2.7 combined with our knowledge of the existence of a singular point at $1/2a$ shows that $\widehat{\gamma}_1$ is analytic in a neighborhood of the origin with singular points on $\frac{1}{2a}\mathbb{N}^*$. Unfortunately we do not have a general closed form representation for the analytic continuation of $\widehat{\gamma}_1$ in this particular case so we are not able to prove directly that these are the only singularities of the Borel transform of the anomalous dimension. However what follows in the next section seems to indicate that is the case. To see that let us do a small modification to the differential equation satisfied by $\widetilde{\gamma}_1$. With p some non zero number let

$$-\partial_t \widetilde{\gamma}^2 \;=\; \frac{1}{at}\,\widetilde{\gamma} + \frac{1}{t}\widetilde{\gamma}^2 - \frac{p}{t}\,. \tag{4.47}$$

Looking for a formal power series solution to this differential equation in the form of

$$\widetilde{\gamma}(t) = \sum_{n=1}^{+\infty} c'_n \frac{1}{t^n}\,, \tag{4.48}$$

we see that similarly to the chord diagram recursion it corresponds to $c'_0 = 0$, $c'_1 = p$, $c'_2 = p^2$, $c'_3 = 4p^3$ and in general

$$c'_n = (n-1) \sum_{k=1}^{n-1} c'_k\, c'_{n-k}\,. \tag{4.49}$$

Thus an immediate induction shows that these new coefficients are related to the number of rooted connected chord diagrams by $c'_n = p^n\, c_n$. That means if $\widehat{\gamma}$ is the Borel transform of $\widetilde{\gamma}$, then it has its first singularity at $1/2ap$ which propagates through the convolution product to every $n/2ap$ for $n \in \mathbb{N}^*$.

So there seems to be a general pattern here and motivated by these combinatorial observations we will apply purely analytic techniques to investigate this phenomenon in the next section. For now we can gather the following wisdom from this part:

- if the asymptotic series expansion of the anomalous dimension satisfies a non-linear equation its Borel transform must satisfy a convolution equation;

- the convolution product acts as a propagator of singularities in the Borel space creating a lattice of singular points for the Borel transform;

- the convolution product of germs of analytic functions is an analytic function on the universal cover of $\mathbb{C}$ punctured at its singular points;

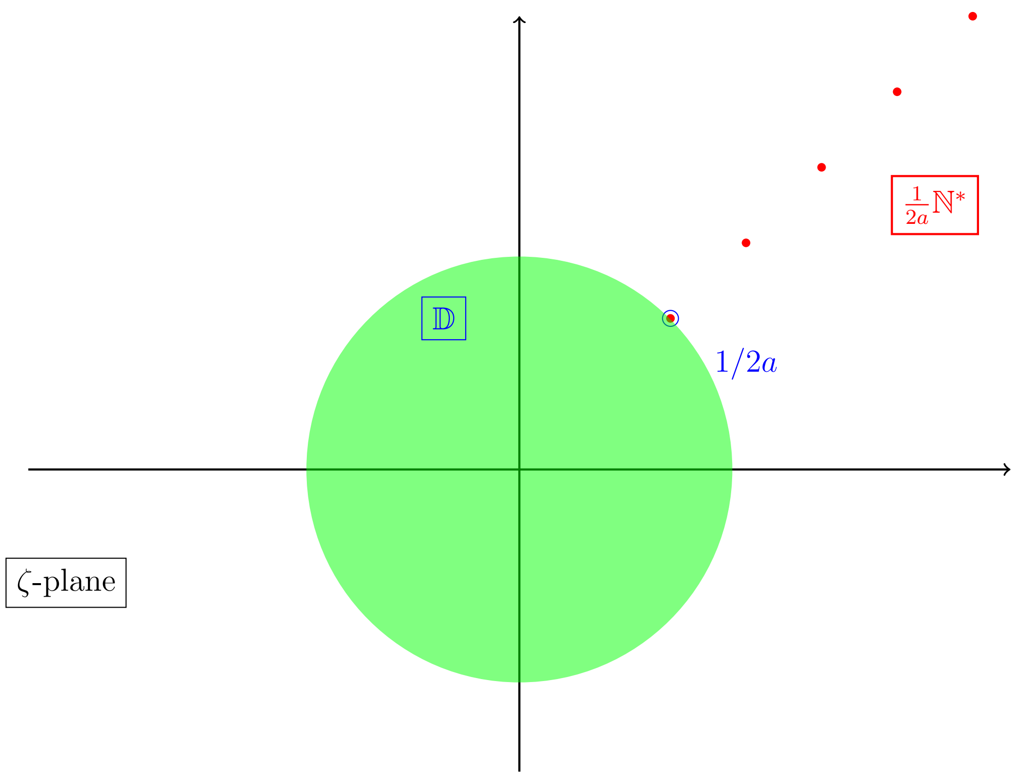


Figure 4.5: The singular points generated by the first singularity at $1/2a$.

4.3 Resummation of the P-differential Equation

More generally, the anomalous dimension satisfies a non-linear differential equation depending on a function P (see [24]), the so-called P-differential equation:

$$2s\,\gamma\,x\,\partial_x\,\gamma = \gamma^2 + \gamma + P(x)\,. \tag{4.50}$$

One must be careful of the fact that P does not have a straightforward interpretation. Indeed we can show (see [20]) that $P(x)$ has a series expansion expressed in terms of chord diagrams:

$$P(x) = \sum_{X \in \mathcal{RCCD}} f_X\,(f_{b(X)-2} - f_{b(X)-1})\,x^{|X|}\,. \tag{4.51}$$

Hence due to the rapid growth of the number of rooted connected chord diagrams this series is most of the time divergent with a convergent Borel transform. For the rest of this section we assume that P has been resummed to an analytic function so that the P-differential equation is well defined.

Another note of caution concerns the algebraic manipulations of the P-differential equation. The nature of Laplace–Borel resummation techniques consists in using at the same time expressions that can be seen as formal power series, germs of analytic functions and their analytic continuations. In order to ensure that one can treat all these objects on an equal footing in the algebraic manipulations we cannot use any simplification coming from the division of functions / series.

That being said, following Ecalle (see [14]), we start by doing a change of variables to place the P-differential equation in a canonical form that greatly simplifies the shift to the Borel space. Note that [14] contains general claims concerning the classification of analytic differential equations, however we do not make use of those results for two reasons: the lack of explicit constructions of the solutions and the absence of proof of many key results of the classification. Fortunately our problem is well suited for a direct approach leading to quantitative results.

So as in the previous part we exchange x for $1/t$,

$$-2s\,\widetilde{\gamma}\,t\,\partial_t\widetilde{\gamma} = \widetilde{\gamma}^2 + \widetilde{\gamma} + P(t)\,, \tag{4.52}$$

and continue with the following transformation.

Lemma 4.3.1. *Let* $P(t) = \frac{p_1}{t} + \mathcal{O}(\frac{1}{t^2})$ *with* $p_1 \neq 0$. *Under the change of variables* $\widetilde{\gamma} = \frac{-p_1}{t}(1+g)$ *we have the following correspondence:*

$$\partial_t\widetilde{\gamma} = \frac{\widetilde{\gamma}^2 + \widetilde{\gamma} + P}{-2\,s\,t\,\widetilde{\gamma}} \tag{4.53}$$

$$\Leftrightarrow$$

$$\left(\partial_t - \frac{1}{2\,s\,p_1}\right) g = G(t,g) \tag{4.54}$$

where $G \in \mathbb{C}\{\frac{1}{t}, g\}$ *and does not contain linear terms in* g.

Proof. Under the change of variables we have:

$$\partial_t \widetilde{\gamma} = \frac{p_1}{t^2}(1+g) - \frac{p_1}{t}\partial_t g \tag{4.55}$$

and

$$\frac{\widetilde{\gamma}^2 + \widetilde{\gamma} + P}{-2\,s\,t\,\widetilde{\gamma}} = \frac{1}{2s}\left[\frac{p_1}{t^2}(1+g)^2 - \frac{1}{t}(1+g) + \frac{1}{t} + \mathcal{O}(\frac{1}{t^2})\right]\left(1 - \frac{g}{1+g}\right). \tag{4.56}$$

We rearrange the terms in order to get an equation of the form:

$$(\partial_t - \alpha)\, g = G(t,g)\,. \tag{4.57}$$

We find that $\alpha = 1/2sp_1$ and

$$G(t,g) = \left(1 - \frac{1}{2s}\right)\frac{1+g}{t} + \frac{1}{2sp_1(1+g)}\left(p_1 g^2 + 1 - \frac{tP}{p_1}\right) \tag{4.58}$$

where G does not contain any linear term in g since $G(t,g) = \mathcal{O}(1/t)$. □

Recasting the P differential equation in this form is particularly appropriate to conduct a Laplace–Borel study of the formal solution. For the rest of the analysis it is convenient to expand $G(t,g)$ as a series in g, that is

$$G(t,g) = \sum_{n=0}^{+\infty} \psi_n(t)\, g^n\,, \tag{4.59}$$

with

$$\psi_0(t) = \left(1 - \frac{1}{2s}\right)\frac{1}{t} + \frac{1}{2sp_1}\left(1 - \frac{tP}{p_1}\right) \tag{4.60}$$

$$\psi_1(t) = \left(1 - \frac{1}{2s}\right)\frac{1}{t} - \frac{1}{2sp_1}\left(1 - \frac{tP}{p_1}\right) \tag{4.61}$$

$$\cdots \tag{4.62}$$

$$\psi_n(t) = (-1)^n\frac{1}{2sp_1}\left(1 - \frac{tP}{p_1}\right) + \frac{(-1)^{n-2}}{2s} \tag{4.63}$$

$$\cdots \tag{4.64}$$

Taking the Borel transform of equation (2.51) with respect to t immediately gives:

$$\widehat{g}(\zeta) = \frac{\widehat{\psi}_0(\zeta)}{\lambda_0 - \zeta} + \frac{1}{\lambda_0 - \zeta}\,\widehat{\psi}_1(\zeta) * \widehat{g}(\zeta) + \frac{1}{\lambda_0 - \zeta}\,\widehat{\psi}_2(\zeta) * \widehat{g} * \widehat{g}(\zeta) + \cdots \tag{4.65}$$

where $\lambda_0 = 1/2sp_1$, $\widehat{g}$ is the Borel transform of g and $\widehat{\psi}_n(\zeta)$ the Borel transform of ψ_n. We are now ready to study the properties of $\widehat{g}$. We start with a strong assumption on the Borel transform of P.

Theorem 4.3.2. *Assume that $p_1 \neq 0$, $\widehat{P}(\lambda_0) \neq 0$ and $\widehat{P}(\zeta)$ is an exponentially bounded entire function. Then $\widehat{g}$ is singular on $\mathbb{N}^*/2sp_1$ and is exponentially bounded away from its singular points along any ray avoiding a singular direction.*

Proof. As we just observed, the Borel transform $\widehat{g}(\zeta)$ satisfies the convolution equation

$$\widehat{g}(\zeta) = \frac{\widehat{\psi}_0(\zeta)}{\lambda_0 - \zeta} + \sum_{n=1}^{+\infty} \frac{1}{\lambda_0 - \zeta}\,\widehat{\psi}_n * \widehat{g}^{*n}(\zeta) \tag{4.66}$$

which can be solved by iteration to obtain an expression of $\widehat{g}$ as an infinite sum of convolution products of the functions $\widehat{\psi}_n$. We choose to rearrange this series as follows:

$$\widehat{g}(\zeta) = \frac{1}{\lambda_0 - \zeta} \sum_{n=1}^{+\infty}\sum_{p=0}^{n} \sum_{\substack{i_1+\cdots+i_n=p \\ i_1\geq\cdots\geq i_n\geq 0}} C(i_1,\cdots,i_n)\widehat{\psi}_{i_1} * \left(\frac{1}{\lambda_0 - t_1}\widehat{\psi}_{i_2}\right) * \cdots * \left(\frac{1}{\lambda_0 - t_{n-1}}\widehat{\psi}_{i_n}\right)(\zeta)\,, \tag{4.67}$$

where the t_k are integration variables and the $C(i_1, \cdots, i_n)$ are combinatorial factors defined inductively. We simplify the notations by introducing

$$\widehat{\psi}_{i_1\cdots i_n}(\zeta) = \widehat{\psi}_{i_1} * \left(\frac{1}{\lambda_0 - t_1}\widehat{\psi}_{i_2}\right) * \cdots * \left(\frac{1}{\lambda_0 - t_{n-1}}\widehat{\psi}_{i_n}\right)(\zeta)\,. \tag{4.68}$$

This way of grouping the terms is the most appropriate to study at the same time the convergence of the series as well as the singularities of the resulting function.

We start by looking for the singularities of $\widehat{g}$. First of all the immediate generalization of lemma 4.2.7 shows that the terms $\widehat{\psi}_{i_1\cdots i_n}$ are singular at $\lambda_0, 2\lambda_0, \cdots, (n-1)\lambda_0$ for all admissible sets of labels $(i_1, \cdots, i_n)$. That means it is sufficient to show that for n large enough the following series of functions is not identically zero:

$$\sum_{p=0}^{n} \sum_{\substack{i_1+\cdots+i_n=p \\ i_1\geq\cdots\geq i_n\geq 0}} C(i_1,\cdots,i_n)\widehat{\psi}_{i_1\cdots i_n}(\zeta)\,. \tag{4.69}$$

Indeed we can focus on these series as when n is growing the set of singular points is strictly increasing and cancellations cannot occur between terms having different sets of singular points. But given the form of the $\widehat{\psi}_n$ functions we do not get any complete cancellations of those series.

So we are left we studying the convergence of the series and at the same time the fact that it stays exponentially bounded away from its singularities. Note that since $\widehat{P}$ is exponentially bounded so are the $\widehat{\psi}_n$ and $\widehat{\psi}_n(\zeta)/(\lambda_0 - \zeta)$ away from λ_0. Now let us consider the convolution product

$$\left(\frac{\widehat{\psi}_n}{\lambda_0 - t_1}\right) * \left(\frac{\widehat{\psi}_m}{\lambda_0 - t_2}\right)(\zeta) \;=\; \int_0^\zeta \frac{\widehat{\psi}_n(t)\widehat{\psi}_m(\zeta - t)}{(\lambda_0 - t)(\lambda_0 - \zeta + t)}\, dt\,. \tag{4.70}$$

This integral is well defined for any path of integration joining 0 to ζ while avoiding the singularity at λ_0. Equivalently one can say that this convolution product defines an analytic function for any ζ in the universal cover of $\mathbb{C} \setminus \{\lambda_0\}$. So one has a first inequality

$$\left|\left(\frac{\widehat{\psi}_n}{\lambda_0 - t_1}\right) * \left(\frac{\widehat{\psi}_m}{\lambda_0 - t_2}\right)(\zeta)\right| \leq \int_0^\zeta C^2\, e^{\tau\,|t|}\, e^{\tau\,|\zeta - t|}\, |dt| \tag{4.71}$$

for some positive constants C and τ. Since the integrand is non singular along the ray of integration from 0 to ζ we have along any ray avoiding a singular direction

$$\left|\left(\frac{\widehat{\psi}_n}{\lambda_0 - t_1}\right) * \left(\frac{\widehat{\psi}_m}{\lambda_0 - t_2}\right)(\zeta)\right| \leq C^2\, \zeta\, e^{\tau\,|\zeta|}\,. \tag{4.72}$$

Now this argument generalizes to the n-fold convolution product $\widehat{\psi}_{i_1 \cdots i_n}$ and we get

$$|\widehat{\psi}_{i_1 \cdots i_n}(\zeta)| \leq\; C^n\, \frac{\zeta^{n-1}}{(n-1)!}\, e^{\tau\,|\zeta|}\,. \tag{4.73}$$

This shows that the generic term of the series representation of $\widehat{g}$ is exponentially bounded for all ζ away from its singular points and geometrically bounded in the parameter n since the sum of the combinatorial coefficients $C(i_1, \cdots, i_n)$ grows at most exponentially (universal class of rooted trees). Thus the series is convergent and $\widehat{g}$ is exponentially bounded as desired.

$\square$

So if $\widehat{P}$ is exponentially bounded and does not have any singular point the previous theorem shows that $\widehat{g}$ is Laplace transformable along any direction except $\arg(1/2sP'(0))$. In particular that means we have established a resummation formula for the anomalous dimension:

$$\gamma_\theta(t) = \frac{p_1}{t}\left(1 + \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta\, t}\,\widehat{g}(\zeta)\, d\zeta\right)\,, \tag{4.74}$$

where $\widehat{g}$ is explicitly given in terms of $\widehat{P}$.

As an example we can consider the case where $P(t) = p_1/t$ which means that

$$\psi_0(t) \;=\; \psi_1(t) \;=\; \frac{\left(1 - \frac{1}{2s}\right)}{t} \tag{4.75}$$

and for all $n \geq 2$ we have $\psi_n(t) = (-1)^{n-2}/2s$. Thus the Borel transform of g only involves convolution products of the

$$\mathbb{1} \,*\, \left(\frac{1}{\lambda_0 - t_1}\right) \,*\cdots*\, \left(\frac{1}{\lambda_0 - t_n}\right)(\zeta) \tag{4.76}$$

with singular points at the integer multiples of λ_0. The resummation of the anomalous dimension in the direction θ is then a generalization of the solution corresponding to the Lambert function in the real case that was studied in [17]. To be more specific, assuming that $p_1 \in \mathbb{R}_+$ then for all $\theta \neq 0$ the Laplace–Borel transform γ_θ is analytic on $\mathbb{H}_0^\theta$ and we can observe the following things:

i) because of the singular points on $\mathbb{R}_+$ in the Borel plane, the Laplace–Borel resummation of $\widetilde{\gamma}$ is not unique. Similarly to the example 3.2.1 we have two distinct analytic countinuations of the solution with $i\mathbb{R}$ as a Stokes line;

ii) our last theorem shows that $\widehat{g}$ can be analytically continued along any path joining 0 to ∞ avoiding λ_0, $2\lambda_0$, $\cdots$, so we have access to more Laplace–Borel resummations of $\widetilde{\gamma}_1$ which can be described by a sequence of deformations of $\mathbb{R}_+$ that specifies how one is to avoid each singular point (see figure 4.3).

This illustrates the fact that we cannot speak of a unique resummation for those divergent series appearing in quantum field theory. Instead we have a family of resummations that may or may not correspond to a non-perturbative physical solution.

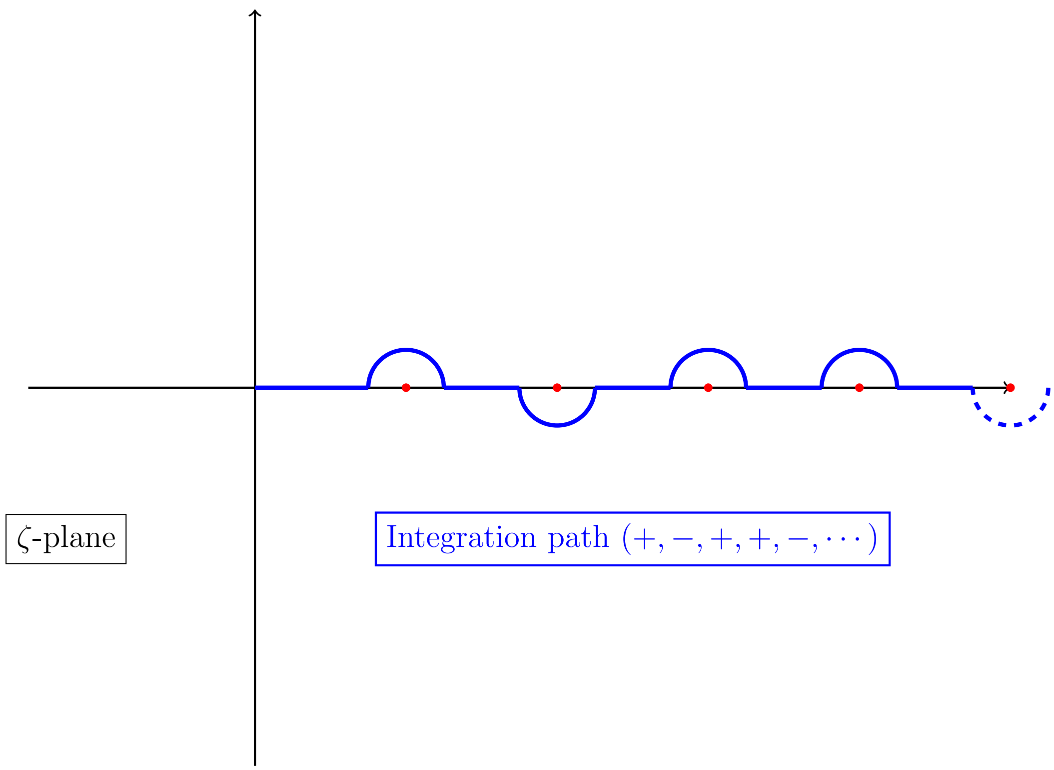


Figure 4.6: A example of admissible path of integration along the singular direction.

Moreover as we pointed out at the beginning of this section P is given to us as an asymptotic series which means that its Borel transform cannot be an entire function. But looking at the proof of 4.3.2 it is immediately possible to accommodate more realistic hypotheses for $\widehat{P}$.

Theorem 4.3.3. *Assume that* $p_1 \neq 0$ $\widehat{P}(\lambda_0) \neq 0$ *and* $\widehat{P}(\zeta)$ *is an exponentially bounded function with singularities on* $(\lambda_1\mathbb{N} + \cdots + \lambda_k\mathbb{N}) \setminus \{0\}$*. Then* $\widehat{g}$ *is singular on* $(\lambda_0\mathbb{N} + \lambda_1\mathbb{N} + \cdots + \lambda_k\mathbb{N}) \setminus \{0\}$ *and is exponentially bounded away from its singular points along any ray avoiding a singular direction.*

Proof. Indeed we can apply the same arguments as in 4.3.2 but this time the monomials of convolution $\widehat{\psi}_{i_1\cdots i_n}$ are mixing the singularities of $\widehat{P}$ with the one generated by λ_0. □

So we see that if we were able to gather some global information concerning the parameter function P of the P-differential equation then we could immediately deduce additional property of the anomalous dimension via its Laplace–Borel resummation formula. We hope that in the future a better understanding of the meaning of P will allow us to carry out such a study.

Thinking about these results we can establish a strategy in three points for future applications:

i) understand the Borel transform of P for realistic models;

ii) use the previous results to obtain a detailed description of the Borel transform of γ_1;

iii) determine additional constraints that non-perturbative physical solutions should satisfy and see if it possible to choose among all our resummations one that fits those conditions.

We also believe that the results of the theorems 4.3.2 and 4.3.3 should hold not only along non-singular rays but more generally on any compact domain of the universal cover of $(\lambda_0\mathbb{N} + \lambda_1\mathbb{N} + \cdots + \lambda_k\mathbb{N}) \setminus \{0\}$. We recommend the recent results of [11] as a starting point in this direction.

Moreover the theorems 4.3.2 and 4.3.3 indicates that the anomalous dimension is a resurgent function (see [14]) that is a highly structured kind of asymptotic series. In particular the qualitative properties of the resummations for different choices of analytic continuations can be studied using the ideas of Ecalle's alien calculus but a lot of technical work still needs to be done in this direction.

Chapter 5

Renormalization Group Equation

5.1 The Renormalization Group Approach

Dyson–Schwinger equations, in particular in their form

$$G(x,L) = 1 - x\,G(x,\partial_{-\rho})(e^{-\rho L}-1)F(\rho)\Big|_{\rho=0}\,, \tag{5.1}$$

produce formal power series solutions $\widetilde{G}(x,L)$ known as perturbative expansions (see chapter 1). On the other hand quantum field theory teaches us that for a renormalizable (the interesting ones) theory $G(x,L)$ must also be a solution of a partial differential equation, the renormalization group equation, that takes the following form in the simplest situation

$$\partial_L\,G(x,L) \;=\; \gamma(x)\,[s\,x\,\partial_x - 1]G(x,L)\,. \tag{5.2}$$

In this chapter we use the renormalization group equation to gain some insight into the analytic information contained in the formal power series solution $\widetilde{G}(x,L)$. In particular

- we show that under reasonable hypotheses the formal solution to the renormalization group equation can be studied by Laplace–Borel methods;
- we give a formula for the resummation of $\widetilde{G}$;

Our objective is to offer an alternative way for the extraction of analytic properties of the Green function given the anomalous dimension and the specific initial conditions. Our main result is theorem 5.2.4 which gives an integral formula for the resummation of $\widetilde{G}$.

5.2 Laplace–Borel Solution

We consider a particular case of the renormalization group equation of the form

$$\begin{cases} \partial_L G = \gamma(x)\,[s\,x\,\partial_x - 1]G\,, \\ G(x,0) = G_0(x)\,, \\ G(0,L) = {}_0G(L), \end{cases} \tag{5.3}$$

and to simplify the notation we will often denote the differential operator $s\,x\,\partial_x - 1$ by ${}_sD_x$. Here $\gamma(x)$ is the anomalous dimension, some analytic function with properties to be detailed below.

We are looking for a formal power series solution of 5.3 in the parameter L. Note that for an appropriate choice of boundary conditions it corresponds to a solution of the Dyson–Schwinger equation (4.1):

$$\widetilde{G}(x,L) \;=\; G_0(x) \,+\, \sum_{n=1}^{+\infty} \gamma_n(x)\,L^n\,. \tag{5.4}$$

Our goal is to study the summability of $\widetilde{G}(x,L)$ in the parameter L as a function of the properties of the anomalous dimension $\gamma(x)$.

First let us give conditions on the pair (γ, G_0) for the series $\widetilde{G}(x,L)$ to be of Gevrey class 1 in L.

Proposition 5.2.1. *Let $\widetilde{S}$ be some open sector of $\mathbb{C}$ whose apex is at the origin. The series $\widetilde{G} \in \mathcal{O}(\widetilde{S})[[L]]$ is of Gevrey class k on $\widetilde{S}$ if the following conditions are statisfied:*

(i) *the function G_0 is analytic on some open disc $\mathbb{D}_0 \subset \mathbb{C}$ centered at the origin and containing $\widetilde{S}$;*

(ii) *the function γ is analytic on $\widetilde{S}$ and has an asymptotic expansion of Gevrey class k at the origin;*

Proof. A simple calculation shows that the generic term of the series $\widetilde{G}$ is

$$\gamma_n(x) \;=\; \frac{1}{n!}\,[\gamma(x)\,{}_sD_x]^n\,G_0(x)\,. \tag{5.5}$$

We want to prove that there are some positive functions A, B on $\widetilde{S}$ such that for n large enough

$$|[\gamma(x)\,{}_sD_x]^n\,G_0(x)| \leq A(x)\,B(x)^n\,(n!)^{k+1}\,. \tag{5.6}$$

The proof relies on the decomposition of the general term $[\gamma(x)\,{}_sD_x]^n\,G_0(x)$ as

$$[\gamma(x)\,{}_sD_x]^n\,G_0(x) \;=\; {}_sD_x[G_0](x)\,\gamma(x)^{n-1}\,(s\,x)^{n-1}\,\partial_x^{n-1}\gamma(x) \;+\; R_n(x)\,, \tag{5.7}$$

where R_n only involves derivatives of γ of order smaller that $n-1$. We know that

- from (i) we can use Cauchy's estimates to get on $\widetilde{S}$ the inequality $|\partial_x^n\,G_0(x)| \leq R_{\widetilde{S}}^{-n}\,M_{\widetilde{S}}\,n!$ where $R_{\widetilde{S}}$ and $M_{\widetilde{S}}$ are some positive constants for all n;
- from (ii) we can use the Gevrey-k asymptotic expansion to get on $\widetilde{S}$ for n large enough $|\partial_x^n\gamma(x)| \leq \widetilde{A}(x)\,\widetilde{B}(x)^n\,(n!)^{1+k}$ for some positive functions $\widetilde{A}$ and $\widetilde{B}$;

Due to Leibniz rule each term of the expansion of $[\gamma(x)\,{}_sD_x]^n\,G_0(x)$ involves at most p derivatives of G_0 and q derivatives of γ where $p+q$ is at most n. So from the above estimates no term in the expansion can involve more than $1+k$ power of $n!$. Since the total number of terms in $R_n(x)$ is geometric in n we get the desired bound of the form

$$|[\gamma(x)\,{}_sD_x]^n\,G_0(x)| \leq A(x)\,B(x)^n\,(n!)^{k+1}\,. \tag{5.8}$$

□

So we see that the anomalous dimension γ controls the growth of the coefficients of the formal solution, from now on we assume it is of Gevrey class 1 in some appropriate domain.

Now we want to learn things about the Borel transform of $\widetilde{G}(x,L)$ with respect to the variable $\Lambda = 1/L$ denoted by $\widehat{G}(x,\zeta)$:

$$\widetilde{G}(x,\Lambda) \;=\; G_0(x) \,+\, \sum_{n=1}^{+\infty} \gamma_n(x)\, \frac{1}{\Lambda^n}\,, \tag{5.9}$$

$$\widehat{G}(x,\zeta) \;=\; \sum_{n=1}^{+\infty} \frac{\gamma_n(x)}{(n-1)!}\, \zeta^{n-1}\,. \tag{5.10}$$

Through a series of change of variables we are going to relate $\widehat{G}$ to a solution to an inhomogeneous heat equation. Our starting point is the renormalization group equation

$$\begin{cases} \partial_L G = \gamma(x)\,[s\,x\,\partial_x - 1]G\,, \\ G(x,0) = G_0(x)\,, \\ G(0,L) = {}_0G(L). \end{cases} \tag{5.11}$$

Our first change of variables consists of taking $L = 1/\Lambda$ and $H(x,\Lambda) = \Lambda^{-1/2}\, G(x,\Lambda^{-1})$. Expressing the initial condition as a limit we get the correspondence:

$$\begin{cases} \partial_\Lambda H \,+\, \frac{1}{2\,\Lambda}\, H(x,\Lambda) \;=\; -\frac{1}{\Lambda^2}\,\gamma(x)\, {}_sD_x H\,, \\ \lim\limits_{\Lambda\to\infty} \Lambda^{1/2}\, H(x,\Lambda) = G_0(x)\,. \end{cases} \tag{5.12}$$

Next we take the Borel transform $\widehat{H}(x,\zeta)$ of $H(x,\Lambda)$, that is we can express the latter as a Laplace tranform of the form

$$H(x,\Lambda) \;=\; \int e^{-\zeta\,\Lambda}\, \widehat{H}(x,\zeta)\, d\zeta\,. \tag{5.13}$$

Then using 5.12 we obtain an equation for $\widehat{H}$:

$$-\zeta\,\widehat{H} \,+\, \frac{1}{2}\,[1] \,*\, \widehat{H} \;=\; -\gamma\,[\zeta] \,*\, {}_sD_x\,\widehat{H}\,, \tag{5.14}$$

where $*$ is the convolution product of analytic functions. Using the fact that $\partial_\zeta(f * g) = f' * g = f * g'$ we can twice take the derivative with respect to ζ to get rid of the convolution product. We arrive at:

$$\zeta\,\partial_\zeta^2\,\widehat{H} \,+\, \frac{3}{2}\,\partial_\zeta\,\widehat{H} \;=\; \gamma(x)\,{}_sD_x\,\widehat{H}\,. \tag{5.15}$$

It is more convenient not to consider the initial condition at this point. We will come back to it later. Things can be further simplified by choosing $z = 2\,\zeta^{1/2}$ and $\widehat{I}(x,z) = \frac{z}{2}\,\widehat{H}(x, z^2/4)$. An easy calculation shows that 5.15 becomes $\partial_z^2\,\widehat{I} \;=\; \gamma(x)\,{}_sD_x\widehat{I}$ and we are left to work out their initial conditions. One has

$$\lim_{\Lambda\to\infty} \Lambda^{1/2}\,H(x,\Lambda) \;=\; \int e^{-y}\,y^{-1/2} \lim_{\Lambda\to\infty} \widehat{I}(x, 2\,y^{1/2}/\Lambda^{1/2})\,dy \;=\; \sqrt{\pi}\,\widehat{I}(x,0)\,, \tag{5.16}$$

so by 5.12 we arrive at $\widehat{I}(x,0) = G_0(x)/\sqrt{\pi}$. But since we have to deal with a second order equation in the varaible z we need one more condition. We know that

$$\partial_z\,\widehat{I}(x,z) \;=\; \frac{1}{2}\,\widehat{H}(x,\zeta) \;+\; \zeta\,\partial_\zeta\,\widehat{H}(x,\zeta) \tag{5.17}$$

and using 5.12 one computes

$$\begin{aligned}
\int e^{-\zeta\,\Lambda}\,\partial_z\,\widehat{I}(x,z)\,d\zeta \;&=\; \int e^{\zeta\,\Lambda}\left(\frac{1}{2}\widehat{H}(x,\zeta) + \zeta\,\partial_\zeta\widehat{H}(x,\zeta)\right)\,d\zeta \\
&= \frac{1}{2}H(x,\Lambda) \;+\; \int e^{-\zeta\,\lambda}\zeta d\widehat{H}(x,\zeta) \\
&= -\frac{1}{2}H(x,\Lambda) \;+\; \Lambda\int e^{-\zeta\,\Lambda}\zeta\widehat{H}(x,\zeta)\,d\zeta \;\;\text{using integration by parts,} \\
&= -\frac{1}{2}H(x,\Lambda) - \Lambda\partial_\Lambda H(x,\Lambda) \\
&= \frac{1}{2\,\Lambda}\,\gamma(x)\,{}_sD_x\,H(x,\Lambda)\,.
\end{aligned}$$

But on the other hand with the change of variables $y = \zeta\,\Lambda$ we have

$$\lim_{\Lambda\to\infty} \Lambda^{1/2}\int e^{-\zeta\,\Lambda}\,\partial_z\widehat{I}(x,z)\,d\zeta \;=\; \int e^{-y}\,y^{\frac{3}{2}-1} \lim_{\Lambda\to\infty} \partial_y\widehat{I}(x, 2\,y^{1/2}/\Lambda^{1/2})\,dy \;=\; \frac{\sqrt{\pi}}{2}\,\partial_y\widehat{I}(x,0)\,. \tag{5.18}$$

Thus since $\lim\limits_{\Lambda\to\infty} \frac{1}{2\,\Lambda^{1/2}}\,\gamma(x)\,{}_sD_xH(x,\Lambda) = 0$ we can conclude that $\partial_z\widehat{I}(x,0) = 0$. So 5.15 is equivalent to:

$$\begin{cases} \partial_z^2\,\widehat{I}(x,z) \;=\; \gamma(x)\,{}_sD_x\,\widehat{I}(x,z)\,, \\ \widehat{I}(x,0) \;=\; \frac{1}{\sqrt{\pi}}\,G_0(x)\,, \\ \partial_z\widehat{I}(x,0) \;=\; 0\,. \end{cases} \tag{5.19}$$

Looking at this last equation we see that it is very close to a heat equation where z plays the role of a spatial variable and x is related to a time variable:

$$s\,\gamma(x)\,{}_s\partial_x\widehat{I} \,-\, \partial_z^2\,\widehat{I} \;=\; \gamma\,\widehat{I}\,. \tag{5.20}$$

Indeed we can introduce the variable t such that $x = f(t)$ with

$$t(x) = \int_{x_0}^{x} \frac{d\log(u)}{s\gamma(u)} \tag{5.21}$$

such that $t(0) = 0$ and with $\widehat{J}(t,z) = \widehat{I}(x,z)$ we obtain the inhomogeneous heat equation:

$$\begin{cases} \partial_t \widehat{J}(t,z) - \partial_z^2 \widehat{J}(t,z) \; = \; \gamma(f(t))\, \widehat{J}(t,z) \;, \\ \widehat{J}(t,0) \; = \; \frac{1}{\sqrt{\pi}}\, G_0(f(t))\,, \partial_z \widehat{J}(t,0) \; = \; 0\,. \end{cases} \tag{5.22}$$

Finally, going through our series of change of variables, the initial condition for this heat equation $\widehat{J}(0,z) \; = \; {}_0\widehat{J}(z)$ is defined by the relation

$$L^{1/2}\, {}_0G(L) \; = \; \int e^{-\zeta/L}\, {}_0\widehat{J}(2\,\zeta^{1/2})\, \frac{d\zeta}{\zeta^{1/2}}\,. \tag{5.23}$$

Now fixing $\phi, \alpha \in \mathbb{S}^1$ there is a semi-group of operators $\{\,{}_\phi K_t\}_{t \in e^{i\alpha}\mathbb{R}_+}$ such that by Duhamel's principle 5.22 is equivalent to the integral equation

$$\widehat{J}(t,z) \; = \; {}_\phi K_t \; \circledast \; {}_0\widehat{J}(z) \; + \; \int_0^t \; \gamma(f(\tau))\, {}_\phi K_{t-\tau} \; \circledast \; {}_\tau\widehat{J}(z)\, d\tau \tag{5.24}$$

where the time integral is along the line segment $[0,t] \subset e^{i\alpha}\mathbb{R}_+$ and

$$ {}_\phi K_t \; \circledast \; {}_\tau\widehat{J}(z) \; = \; \int_{e^{i\phi}\mathbb{R}_+} p_t(x,z)\, \widehat{J}(\tau,y)\, dy \tag{5.25}$$

with the heat kernel $p_t(x,y) \; = \; \frac{1}{(4\,\pi\,t)^{1/2}} \left\{ \exp\left(-\frac{|y-z|^2}{4t}\right) \; - \; \exp\left(-\frac{|y+z|^2}{4t}\right) \right\}$. We focus on a particular class of initial conditions. Let Ω be a domain in $\mathbb{C}$ containing a neighborhood of infinity. We introduce the set of square exponentially bounded analytic functions at infinity

$$E^2_\infty(\Omega) \; = \; \left\{ F \in \mathcal{O}(\Omega) \,\middle|\, \exists A, B > 0 \; \text{ such that } |F(z)| \le A\, e^{B\,|z|^2} \; \text{ when } |z| \to \infty \right\}\,. \tag{5.26}$$

Now for any complex valued function F we denote by $Sing(F)$ its set of singular points and $aSing(F)$ the set of arguments of those singular points. Using the semigroup property of the heat flow we get an explicit solution to the integral equation 5.24.

Lemma 5.2.2. *Assume that* ${}_0\widehat{J} \in E^2_\infty(\Omega)$ *with* $0 \in \Omega$. *For each* $(\phi, \alpha) \in (\mathbb{S}^1 \setminus aSing({}_0\widehat{J})) \times (]-\frac{\pi}{2}, \frac{\pi}{2}[\setminus aSing(\gamma \circ f))$ *the equation 5.24 has*

$$\widehat{J}(t,z) \;=\; {}_\phi K_t \circledast {}_0\widehat{J}(z) \;\; \widehat{\exp} \int_0^t \gamma(f(\tau))\, d\tau \tag{5.27}$$

as a solution for all $(z,t) \in \Omega \times e^{i\alpha}\mathbb{R}_+$ *with*

$$\widehat{\exp} \int_0^t \gamma(f(\tau))\, d\tau \;=\; 1 \,+\, \sum_{k=1}^{+\infty} \int_0^t \gamma(f(\tau_1)) \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} \gamma(f(\tau_k))\, d\tau_k \,\cdots\, d\tau_1\,. \tag{5.28}$$

Proof. We are solving equation 5.24 by iteration, using mainly the fact that for all $t, \tau, \sigma \in e^{i\alpha}\mathbb{R}_+$ such that $|t| \geq |\tau| \geq |\sigma|$ we have ${}_\phi K_{t-\tau} \circledast {}_\phi K_{\tau-\sigma} \;=\; {}_\phi K_{t-\sigma}$. Thus one has

$$\widehat{J}(t,z) \;= {}_\phi K_t \circledast {}_0\widehat{J}(z) \,+\, \int_0^t \gamma(f(t_1))\, {}_\phi K_{t-t_1} \circledast {}_{t_1}\widehat{J}(z)\, dt_1 \tag{5.29}$$

$$= {}_\phi K_t \circledast {}_0\widehat{J}(z) \,+\, \int_0^t \gamma(f(t_1))\, {}_\phi K_{t-t_1} \circledast {}_\phi K_{t_1-0} \circledast {}_0\widehat{J}(z)\, dt_1 \tag{5.30}$$

$$+ \int_0^t \gamma(f(t_1)) \int_0^{t_1} \gamma(f(t_2))\, {}_\phi K_{t-t_1} \circledast {}_\phi K_{t_1-t_2} \circledast {}_{t_2}\widehat{J}(z)\, dt_2\, dt_1 \tag{5.31}$$

$$= {}_\phi K_t \circledast {}_0\widehat{J}(z) \left[1 + \int_0^t \gamma(f(t_1))\, dt_1\right] \tag{5.32}$$

$$+ \int_0^t \gamma(f(t_1)) \int_0^{t_1} \gamma(f(t_2))\, {}_\phi K_{t-t_2} \circledast {}_{t_2}\widehat{J}(z)\, dt_2\, dt_1 \tag{5.33}$$

and so on. This gives the limiting formula:

$$\widehat{J}(t,z) \;=\; {}_\phi K_t \circledast {}_0\widehat{J}(z) \left[1 \,+\, \sum_{k=1}^{+\infty} \int_0^t \gamma(f(t_1)) \int_0^{t_1} \cdots \int_0^{t_{k-1}} \gamma(f(t_k))\, dt_k \,\cdots\, dt_1\right]. \tag{5.34}$$

Now we need to make sure that this expression makes sense. For that we have to pick the parameters (ϕ, α) representing our lines of integration so that they avoid singular points of the integrands. This ensure the local integrability of all the expressions in 5.27.

Next observe that for any finite t we get the following volume bound

$$\left|\int_0^t \gamma(f(t_1)) \int_0^{t_1} \cdots \int_0^{t_{k-1}} \gamma(f(t_k))\, dt_k \,\cdots\, dt_1\right| \;\leq\; \sup_{\tau \in [0,t]} |\gamma(f(\tau))|^k\, \frac{|t|^k}{k!}\,. \tag{5.35}$$

Thus the series denoted by $\widehat{\exp} \int_0^t \gamma(f(\tau))\, d\tau$ is absolutely convergent for any finite t.

Finally we have to make sure that the integral ${}_\phi K_t \circledast {}_0\widehat{J}(z)$ is convergent. Our restrictions on ϕ already take care of local integrability thus we only need to observe the behaviour of the integrand as $|y|$ goes to infinity.

$$ {}_\phi K_t \circledast {}_0\widehat{J}(z) = \frac{1}{(4\,\pi\, t)^{1/2}} \int_{e^{i\phi}\mathbb{R}_+} \left\{ \exp\left(-\frac{|y-z|^2}{4t}\right) - \exp\left(-\frac{|y+z|^2}{4t}\right) \right\} \widehat{J}(0,y)\, dy\,. \quad (5.36) $$

To be sure that the integral is convergent for any square exponentially bounded ${}_0\widehat{J}$ the exponential terms of the heat kernel must be decreasing at infinity which means that we have to restrict t to $\Re(t) > 0$. Or in terms of the angular parametrisation we must have $\alpha \in] - \frac{\pi}{2}, \frac{\pi}{2}[$.

□

We can be more specific concerning the properties of these solutions. In particular we have some smoothing property of the heat flow on the initial condition ${}_0\widehat{J}$.

Property 5.2.3. *Let* ${}_0\widehat{J} \in E^2_\infty(\Omega)$ *with* $0 \in \Omega$ *and* $(\phi, \alpha) \in (\mathbb{S}^1 \setminus aSing({}_0\widehat{J})) \times (] - \frac{\pi}{2}, \frac{\pi}{2}[\setminus aSing(\gamma \circ f))$. *Then for* $t \in e^{i\alpha}\mathbb{R}_+$ *with* $\Re(t^{-1})$ *large enough the solution* $\widehat{J}(t, \_)$ *of 5.24 is an element of* $E^2_\infty(\mathbb{C})$.

Proof. The z dependence of $\widehat{J}(t,z)$ is concentrated in the convolution product ${}_\phi K_t \circledast {}_0\widehat{J}(z)$, thus it is enough to focus on its properties. But by definition

$$ {}_\phi K_t \circledast {}_0\widehat{J}(z) = \int_{e^{i\phi}\mathbb{R}_+} p_t(y,z)\, {}_0\widehat{J}(y)\, dy \quad (5.37) $$

with the heat kernel $p_t(y,z)$ an entire function of the variable z. Thus as long as the integral is convergent it defines an entire function of z.

Now concerning the behaviour of this convolution product for large $|z|$ one has the following estimates:

$$ \left| {}_\phi K_t \circledast {}_0\widehat{J}(z) \right| \le \frac{e^{-\Re(t^{-1})|y|^2/4}}{|\pi\, t|^{1/2}} \int_{e^{i\phi}\mathbb{R}_+} e^{-\Re(t^{-1})|y|^2/4}\, |\, {}_0\widehat{J}(y)|\, |dy|\,. \quad (5.38) $$

So all we need to show is that the integral of the right hand side of the inequality is finite. But ${}_0\widehat{J} \in E^2_\infty(\Omega)$ so there are some positive constants a, b such that

$$e^{-\Re(t^{-1})|y|^2/4}\,|\,{}_0\widehat{J}(y)| \;\leq\; a\,\exp\left(-\frac{\Re(t^{-1})}{4}|y|^2 + b|y|^2\right)\,. \tag{5.39}$$

So by choosing t such that $\Re(t^{-1})$ is large enough we are sure of the convergence of the integral at $|y| \longrightarrow \infty$ which concludes the proof.

□

At this point we know enough about the Borel transform to be able to express the solution of our original problem as a Laplace transform.

Theorem 5.2.4. *For each $\theta \in \mathbb{S}^1$ and $(\phi, \alpha) \in (\mathbb{S}^1 \setminus aSing({}_0\widehat{J})) \times (]-\frac{\pi}{2}, \frac{\pi}{2}[\setminus aSing(\gamma \circ f))$, the following is a solution to 5.3 for $\Re(1/t(x))$ large enough and with appropritate initial conditions*

$$G_{\phi,\alpha,\theta}(x, L) \;=\; \frac{1}{L^{1/2}}\;\overset{\frown}{\exp}\int_0^{t(x)} \gamma(f(\tau))\,d\tau \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta/L}\;{}_\phi K_{t(x)} \;\circledast\; {}_0\widehat{J}(2\zeta^{1/2})\,\frac{d\zeta}{\zeta^{1/2}}\,. \tag{5.40}$$

Proof. One obtains this formula by following the change of variables:

$$G_{\phi,\alpha,\theta}(x, 1/\Lambda) = \Lambda^{1/2}\, H(x, \Lambda) \tag{5.41}$$

$$= \Lambda^{1/2} \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta\,\Lambda}\, \widehat{H}(x, \zeta)\, d\zeta \tag{5.42}$$

$$= \Lambda^{1/2} \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta\,\Lambda}\, \widehat{J}(t(x), 2\,\zeta^{1/2}) \frac{d\zeta}{\zeta^{1/2}}\,, \tag{5.43}$$

or expressed in the L variable

$$G_{\phi,\alpha,\theta}(x, L) = \frac{1}{L^{1/2}} \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta/L}\, \widehat{J}(t(x), 2\,\zeta^{1/2}) \frac{d\zeta}{\zeta^{1/2}} \tag{5.44}$$

$$= \frac{1}{L^{1/2}}\ \widehat{\exp} \int_0^{t(x)} \gamma(f(\tau))\, d\tau \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta/L}\ {}_{\phi}K_{t(x)} \circledast {}_0\widehat{J}(2\zeta^{1/2})\, \frac{d\zeta}{\zeta^{1/2}}\,. \tag{5.45}$$

The Laplace integral is convergent due to the property 5.2.3 so this formula defines a Laplace–Borel resummation of the series $\widetilde{G}(x, L)$. □

We see that our resummation process yields a family of solutions depending on many parameters such as θ, α, ϕ as well as the initial conditions. We are then in the presence of what can be assimilated to a Stokes phenomenon, although its origin is somehow different from the one presented in the canonical example 3.2.1. The next part illustrates this phenomenon.

Moreover the parameter t in the heat equation controls the convergence of the integral solution. But since t is related to x that means our solutions are not valid for all values of the coupling constant. From property 5.2.3, if the initial condition ${}_0\widehat{J}$ is rational we do not have any constraints on the size of the coupling constant but as soon as the initial condition grows exponentially we do have a constraint.

5.3 Connection Formulas

For the sake of clarity of the argument let us work on a particularly simple example. Let $c \in \mathbb{R}$, define the anomalous dimension to be $\gamma(x) = c/x$ and the initial condition for the heat equation in the Borel plane is ${}_0\widehat{J}(z) = 1/(1+z)$. With this particular choice we obtain $t(x) = c\,x/s$, $f(t) = s\,t/c$ and the path order exponential reduces to

$$\overset{\frown}{\exp} \int_0^{t(x)} \frac{d\tau}{s\,\tau} = 1 + \sum_{k=1}^{+\infty} \frac{1}{s^k}\, Li_k(c\,x/s) \tag{5.46}$$

where Li_k is the k^{th} polylogarithm. The initial condition gives the convolution

$${}_\phi K_{t(x)} \circledast {}_0\widehat{J}(z) = \int_{e^{i\phi}\mathbb{R}_+} p_{t(x)}(y,z) \frac{dy}{1+y} \tag{5.47}$$

which is well defined for all ϕ but $\phi = \pi$. Note that since ${}_0\widehat{J}$ is rational we do not have any constraints on the size of the coupling constant x.

Because of the singularity of ${}_0\widehat{J}$ at -1 we really have two analytic continuations of ${}_\phi K_{t(x)} \circledast {}_0\widehat{J}(z)$, one corresponding to a path of integration avoiding -1 from the left, the other from the right as in example 3.2.1. If we denote these two functions respectively by ${}_+K_{t(x)} \circledast {}_0\widehat{J}(z)$ and ${}_-K_{t(x)} \circledast {}_0\widehat{J}(z)$ then by property 5.2.3 they share the same domain of analyticity for all x and their difference is given by:

$${}_+K_{t(x)} \circledast {}_0\widehat{J}(z) - {}_-K_{t(x)} \circledast {}_0\widehat{J}(z) = 2i\pi\, p_{t(x)}(-1,z) \tag{5.48}$$

by application of the residue formula.

If we denote the corresponding Green functions by $G_{+,\alpha,\theta}$ and $G_{-,\alpha,\theta}$ then we obtain a connection formula relating these two Laplace–Borel resummations:

$$G_{+,\alpha,\theta}(x,L) - G_{-,\alpha,\theta}(x,L) = \frac{2i\pi}{L^{1/2}} \left[1 + \sum_{k=1}^{+\infty} \frac{1}{s^k}\, Li_k(c\,x/s)\right] \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta/L}\, p_{t(x)}(-1, 2\,\zeta^{1/2}) \frac{d\zeta}{\zeta^{1/2}}\,. \tag{5.49}$$

This time the Laplace transform plays a lesser role as the parameter θ is only constrained by a choice of determination of $\zeta^{1/2}$ which we are free to choose at our conveniance.

Thus contemplating our resummation formula for the power series expansion $\widetilde{G}$ of the Green function

$$G_{\phi,\alpha,\theta}(x,L) = \frac{1}{L^{1/2}} \ \widehat{\exp} \int_0^{t(x)} \gamma(f(\tau))\, d\tau \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta/L} \int_{e^{i\phi}\mathbb{R}_+} p_{t(x)}(y, 2\zeta^{1/2})\, dy\, \frac{d\zeta}{\zeta^{1/2}} \tag{5.50}$$

we see that once more the aymptotic series alone does not define a unique non-pertubative solution by way of resummation. Instead we get a famlily of solutions parametrized by paths of integrations which, while they share the same domain of analycity, are not necessarily analytic continuations of each other as can be seen with our example of connection formula.

We feel that the non-uniqueness of a non-perturbative solution is raising some interesting questions which would require more input from the side of theoretical physics, especially:

i) what are the additional requirements one should impose on the Green functions to single out one of the resummations as being the physical solution?

ii) are some connection formulas, both the anomalous dimension and the Green function, related to phenomena like S-duality?

iii) can we compare these non-perturbative features to the results of constructive quantum field theory?

In both cases combining integral representations of the non-perturbative Green functions and the anomalous dimensions with a better understanding of the singularities of their Borel transform offers strong tools for a better understanding of non-perturbative results in quantum field theory.

Chapter 6

Conclusion

We saw in Chapter 2 that Dyson–Schwinger equations are fixed point integro-differential equations and that in our special case they can be reduced to

$$G(x, L) = 1 - x\, G(x, \partial_{-\rho})(e^{-\rho\, L} - 1)F(\rho)|_{\rho=0}\,. \tag{6.1}$$

In Chapter 4 and in [20] we understood the formal power series solution to this equation as a series indexed by rooted connected chord diagrams:

$$\widetilde{G}(x, L) = \sum_{n=0}^{+\infty} \gamma_n(x)\, L^n \;=\; 1 - \sum_{n=1}^{+\infty} \sum_{\substack{X \in \mathcal{RCCD} \\ b(X) \geq n}} f_X\, f_{b(X)-n}\, x^{|X|}\, \frac{(-1)^n}{n!}\, L^n\,. \tag{6.2}$$

This representation of the perturbative solutions is a description alternative to the traditional Feynman diagrams expansion. In general it is only an asymptotic series and does not have an immediate analytic interpretation however as it is of Gevrey class 1, it is amenable to being studied in the Borel plane (4.2.4).

Thinking about this result two natural questions come to our mind. What class of Dyson–Schwinger equations admit similar formal solutions expressed as chord diagrams expansions? The chord diagrams expansions being asymptotic series how can we use them for numerical applications? While the first question seems wide open we can briefly comment on the second point.

Asymptotic series such as the chord diagrams expansion exhibit a typical behaviour where the general term of the series decreases monotonically up to a certain rank before blowing up. This rank depends on the value of the parameters of the series and is called the least term. Approximating the value of the function by the partial asymptotic series up to the least term gives a good agreement with the true value of the approximated quantity at this point. Estimating the rank corresponding to the least term is then of a certain importance when it comes to the question of numerical applications of the asymptotic formula. We point out that for the chord diagrams expansions this comes down to the combinatorial problem of understanding the distribution of gaps for the chord diagrams with a fixed number of chords. Indeed since the general term of the numerical series at point x_0 is

$$\sum_{|X|=n} f_X f_{b(X)-1}\, x_0^n\,, \tag{6.3}$$

understanding the typical profile of the gaps is a good way to approximate the sum over the diagrams of size n in order to study the rank of the least term.

Going back to Chapter 4, the first coefficient $\gamma_1(x)$ of the chord diagrams expansion of $G(x, L)$ has a nonperturbative interpretation as the anomalous dimension of the theory which encodes the scaling behaviour of our model. To go beyond its pertubative expansion one can use the fact [24] that this anomalous dimension must satisfy a non-linear differential equation whose solutions can be studied using Laplace–Borel transforms methods:

$$-2s\,\gamma\, x\, \partial_x\, \gamma = \gamma^2 + \gamma + P(x)\,. \tag{6.4}$$

Analyzing this differential equation we can prove that the chord diagram expansion of γ_1 as an asymptotic series of Gevrey class 1 is actually Laplace–Borel summable and possess non-perturbative corrections coming from a lattice of singular points in its Borel plane. This is Theorem 4.3.3:

Theorem. *Assume that $p_1 \neq 0$, $\widehat{P}(\lambda_0) \neq 0$ and $\widehat{P}(\zeta)$ is an exponentially bounded function with singularities on $(\lambda_1\mathbb{N} + \cdots + \lambda_k\mathbb{N}) \setminus \{0\}$. Then $\widehat{g}$ is singular on $(\lambda_0\mathbb{N} + \lambda_1\mathbb{N} + \cdots + \lambda_k\mathbb{N}) \setminus \{0\}$ and is exponentially bounded away from its singular points.*

In the most realistic case the distribution of these singular points is not wild and the non-perturbative corrections are amenable to more quantitative studies.

This result gives a key to the non-perturbative understanding of the rest of the γ_k coefficients thanks to the recursive formula

$$\gamma_n(x) = \frac{1}{n}\,\gamma_1(x)(x\partial_x - 1)\gamma_{n-1}(x)\,. \tag{6.5}$$

Thus we might ask then if similar techniques can be used to say something regarding the resummation of $G(x, L)$ in terms of the L variable. We answer this question in chapter 5 as it can be done taking as a starting point the renormalization group equation satisfied by $G(x, L)$:

$$\partial_L\, G(x, L) \;=\; \gamma(x)\,[s\,x\,\partial_x - 1]G(x, L)\,. \tag{6.6}$$

Working in the Borel plane the problem is reduced to the non-homogeneous heat equation

$$\partial_t \widehat{J}(t, z) - \partial_z^2 \widehat{J}(t, z) \;=\; \gamma(f(t))\, \widehat{J}(t, z)\;. \tag{6.7}$$

Solving this equation by iteration we simply follow back the flow of transformations to get a Laplace–Borel resummation formula in 5.2.4.

Theorem. *For each* $\theta \in \mathbb{S}^1$ *and* $(\phi, \alpha) \in (\mathbb{S}^1 \setminus aSing({}_0\widehat{J})) \times (] - \frac{\pi}{2}, \frac{\pi}{2}[\setminus aSing(\gamma \circ f))$, *the following is a solution to 5.3 for* $\Re(1/t(x))$ *large enough and with appropritate initial conditions*

$$G_{\phi,\alpha,\theta}(x, L) \;=\; \frac{1}{L^{1/2}}\; \widehat{\exp} \int_0^{t(x)} \gamma(f(\tau))\, d\tau \int_{e^{i\theta}\mathbb{R}_+} e^{-\zeta/L}\; {}_\phi K_{t(x)} \;\circledast\; {}_0\widehat{J}(2\zeta^{1/2})\, \frac{d\zeta}{\zeta^{1/2}}\,. \tag{6.8}$$

As was already observed in Broadhurst and Kreimer in terms of the propagator-coupling duality [5], the anomalous dimension is really the important object to capture the singularities.

These results fall into the larger corpus of recent attempts to use integral transforms techniques to extract non-perturbative information from the asymptotic series that are perturbative solutions to problems of fundamental physics. While theorems

4.3.3 and 5.2.4 only apply to one of the simple case of Dyson–Schwinger equations we see that extracting quantitative information is already a difficult task. We believe it is possible to carry similar studies for more general systems of Dyson–Schwinger equations but it would probably require more technical tools such as the ones coming from Ecalle's alien calculus.

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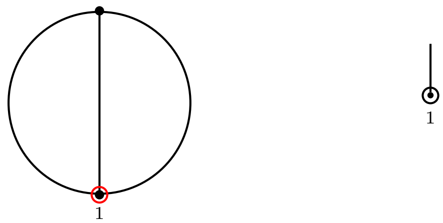
Appendix A

Appendices

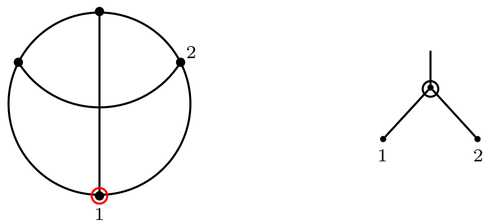
A.1 The Combinatorial Objects

The following table contains all the rooted connected chord diagrams up to 4 chords together with their corresponding rooted planar binary trees, smallest terminal chords $b(C)$ and monomials f_C. The chords of the diagrams and the leaves of the trees are labelled in the intersection order.

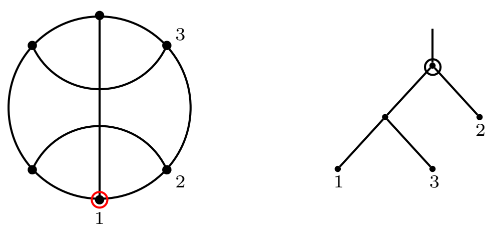
It serves as a reference to chapter 4 and is extracted from [20]. We refer to this article for the construction of labelled rooted planar binary trees.



$b(C) = 1\ ,\ f_C = 1$



$b(C) = 2\ ,\ f_C = f_0$



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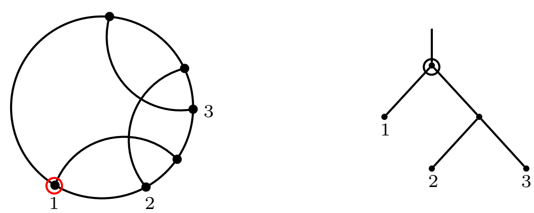
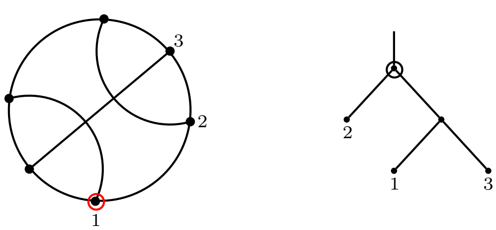


$b(C) = 3\ ,\ f_C = f_0 f_0$



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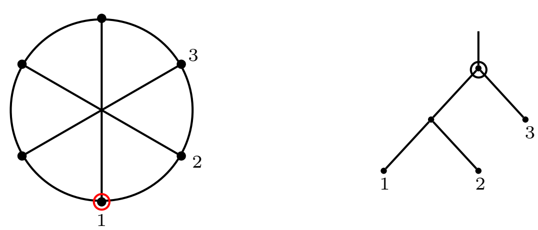

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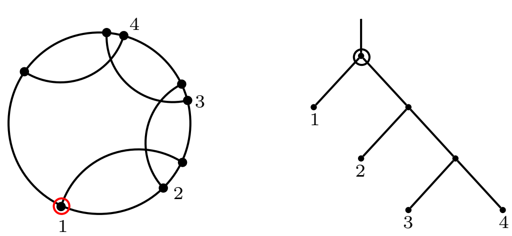

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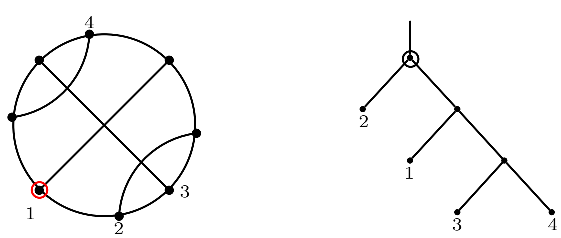

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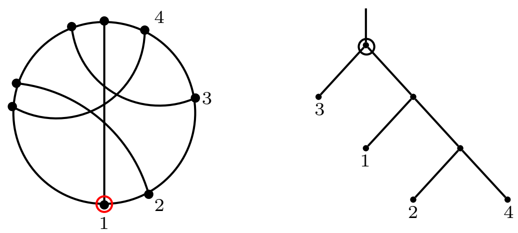

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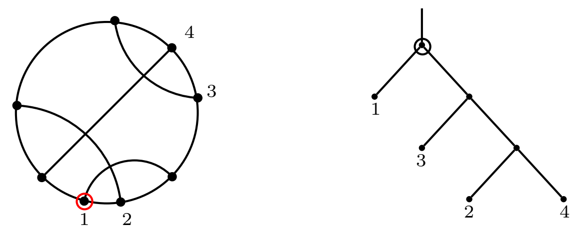

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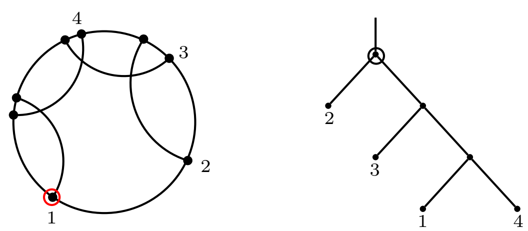

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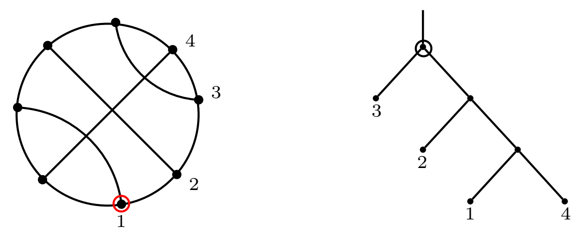

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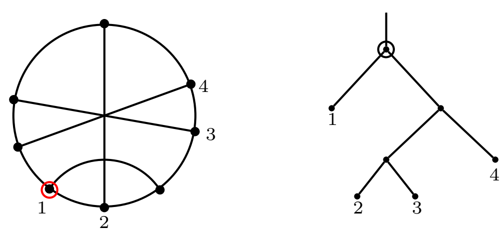

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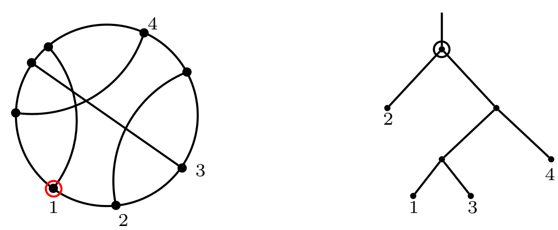

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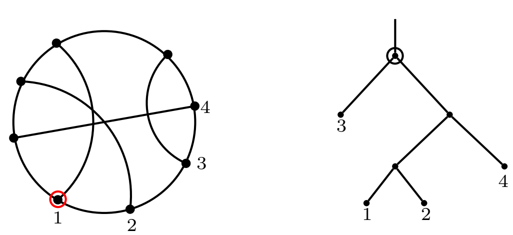

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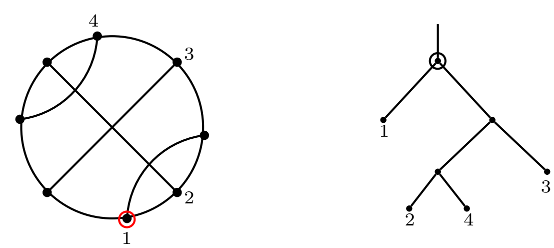

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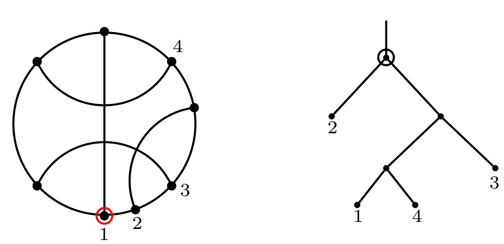

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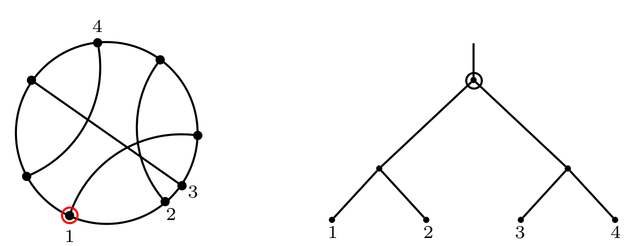

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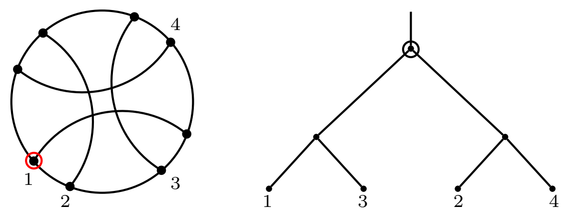

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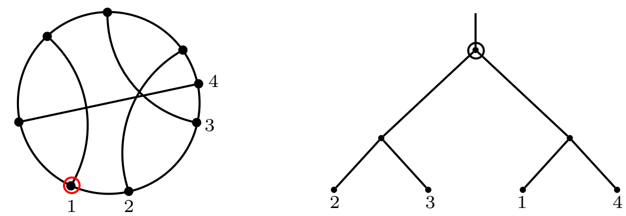

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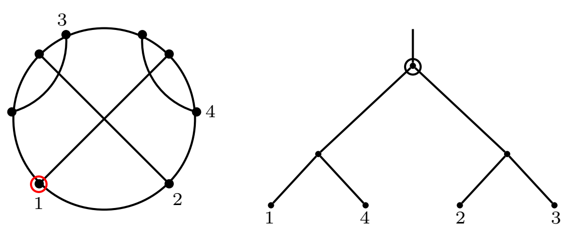

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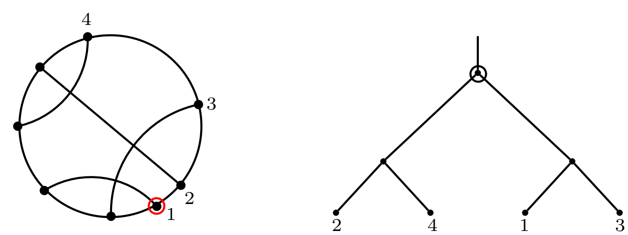

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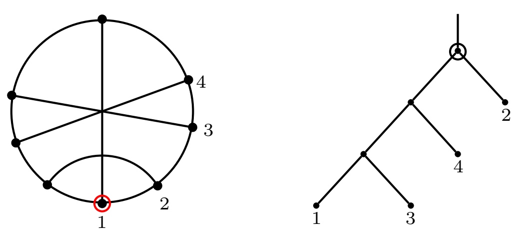

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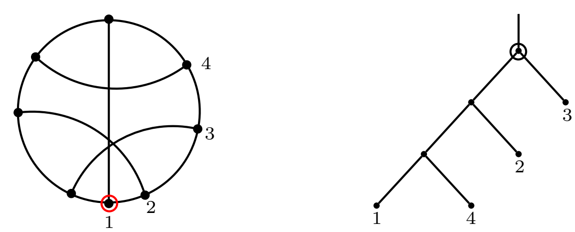

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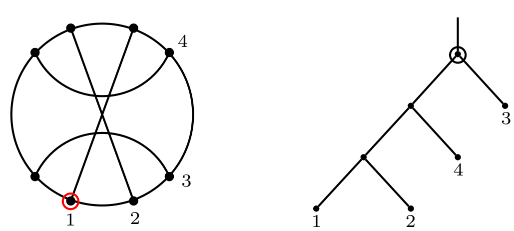

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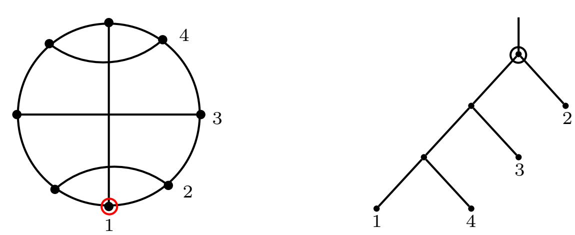

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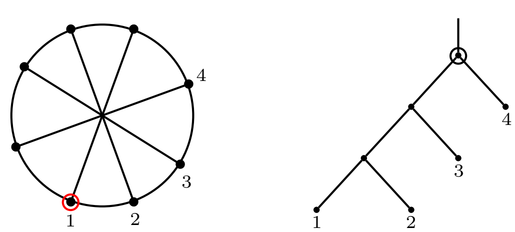

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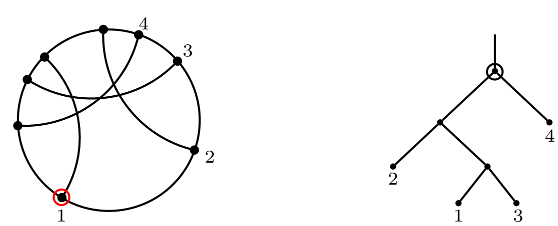

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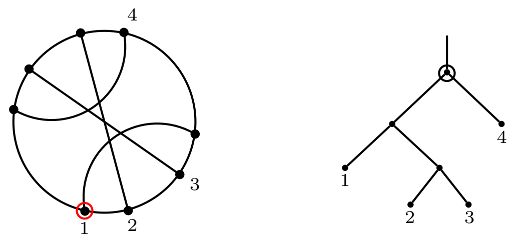

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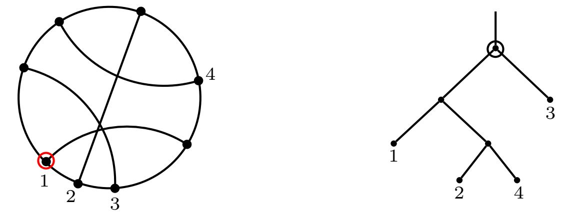

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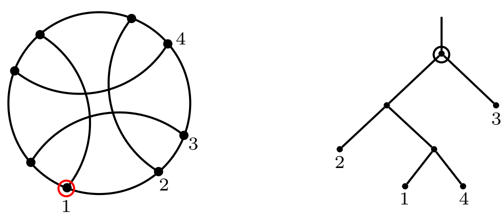

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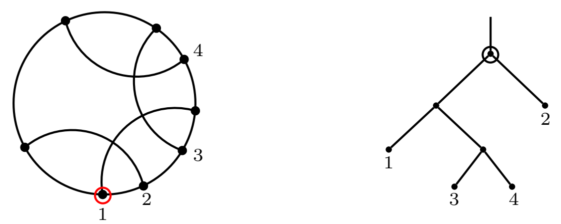

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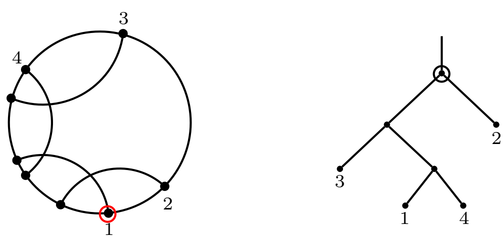

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