Simple eigenvalues of graphs and digraphs

by

Krystal Guo

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Name: Krystal Guo

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Examining Committee: Dr. Cedric Chauve, Department of Mathematics
Chair

Dr. Bojan Mohar
Senior Supervisor
Professor

Dr. Jason Bell
Supervisor
Associate Professor, Department of Mathematics, University of Waterloo

---

Dr. Luis Goddyn
Supervisor
Professor

Dr. Karen Yeats
Internal / External Examiner
Associate Professor

Dr. Stephen Kirkland
External Examiner
Professor, Department of Mathematics, University of Manitoba

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ABSTRACT

The spectra of graphs and their relation to graph properties have been well-studied. For digraphs, in contrast, there are relatively few results. The adjacency matrix of a digraph is usually difficult to work with; it is not always diagonalizable and the interlacing theorem does not hold (in general) for adjacency matrices of digraphs. All acyclic digraphs have the same spectrum as the empty graph. This motivates the need to work with a different matrix which captures the adjacency of the digraph. To this end, we introduce the Hermitian adjacency matrix.

Another way to extract more information out of the spectrum is by restricting to specific classes of digraphs. In this thesis, we look at vertex-transitive digraphs with simple eigenvalues. Intuitively, the property of having many simple eigenvalues tends to coincide with having few automorphisms. For example, the only vertex-transitive graph with all eigenvalues simple is K_2. In the case of graphs, we restrict to the cubic vertex-transitive case, where we find combinatorial properties of graphs with multiple simple eigenvalues. We also explore the eigenvectors of vertex-transitive digraphs with all eigenvalues distinct.

Keywords: algebraic graph theory; eigenvalues; linear algebra; directed graphs

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Chapter 1

Introduction

The spectra of graphs and their relation to graph properties have been well-studied. There has been extensive study about the interplay of eigenvalues of a graph and various graph properties, such as the diameter [16, 43] or the chromatic number [17, 33, 32]; see also [45]. The relationship between symmetries of a graph and its eigenvalues has also been investigated extensively, for example in [15, 55, 56]. The eigenvalues of the Laplacian matrix of a graph determine the number of connected components. For connected graphs, the eigenvalues of the adjacency matrix determine whether the graph is bipartite. There are spectral bounds on the independence number and many other properties. As a concrete example, the Delsarte-Hoffman bound for the size of the largest coclique of an undirected graph was first proved by Delsarte for distance regular graphs and generalized to arbitrary regular graphs by Hoffman [28].

1.0.1 Theorem (Delsarte-Hoffman)**.** *Let X be a k-regular graph on n vertices with least eigenvalue of its adjacency matrix equal to τ. Then $\alpha(X)$, the size of the largest co-clique, is bounded by:*

$$\alpha(X) \leq \frac{n}{1 - \frac{k}{\tau}}.$$

There are many methods for finding and bounding the eigenvalues of a graph, with respect to various matrices which depend on the adjacency relation of the graph. The main tools are the Perron-Frobenius theorem and eigenvalue interlacing.

For digraphs, in contrast, there are relatively few results. There is a directed analogue of Wilf's bound on chromatic number [44], however the spectra of digraphs is, comparatively, a less explored area. For a survey of the area, see [12]. The adjacency matrix of a digraph is usually difficult to work with. It is not always diagonalizable. The interlacing theorem does not hold for adjacency matrices of digraphs, in general. All acyclic digraphs have the same spectrum as the empty graph.

This motivates the need to work with a different matrix which captures the adjacency of the digraph. To this end, we study the Hermitian adjacency matrix. For a digraph $X = (V, E)$, the

Hermitian adjacency matrix H is the matrix with entries

$$H_{uv} = \begin{cases} 1 & \text{if } uv \text{ and } vu \in E; \\ i & \text{if } uv \in E \text{ and } vu \notin E; \\ -i & \text{if } uv \notin E \text{ and } vu \in E; \\ 0 & \text{otherwise.} \end{cases}$$

In this definition, i denotes the imaginary unit. The Hermitian adjacency matrix is diagonalizable and the interlacing theorem can be applied. However, the Perron-Frobenius theorem does not hold for this class of matrices, and some strange behaviour occurs, in the sense of being different from the adjacency matrix of graphs.

Another way to extract more information out of the spectrum is by restricting to specific classes of digraphs. In this thesis, we look at vertex-transitive digraphs with simple eigenvalues. Intuitively, the property of having many simple eigenvalues tends to coincide with having few automorphisms. For example, the only vertex-transitive graph with all eigenvalues simple is K_2. In the case of graphs, we restrict to the cubic vertex-transitive case, where we find combinatorial properties of graphs with multiple simple eigenvalues. We also explore the eigenvectors of vertex-transitive digraphs with all eigenvalues distinct.

1.1 Background and known results

The eigenvalues of digraphs have not been extensively studied. However there has been a lot of activity in the study of the eigenvalues of tournaments, with respect to the adjacency matrix and the skew-symmetric adjacency matrix. Some work has also been done for the skew-symmetric adjacency matrix of oriented digraphs.

The skew-symmetric matrix of a tournament T is a $\{0, \pm 1\}$ matrix; for our purposes, it is equivalent to $iH(T)$. There are many results on the eigenvalues of skew symmetric matrices of tournaments. Gregory, Kirkland and Shader used Pick's inequality to bound the maximum absolute eigenvalue of such a matrix in [30]. Skew-symmetric conference matrices are studied in [29] and [22]. More recently, the singular values of the skew-symmetric adjacency matrix of oriented graph has been studied in [1], [14] and [35].

For the skew-symmetric matrix, there is a notion of switching equivalence. If we reverse all arcs across a cut of the digraph, the spectrum will be preserved. This idea of switching has been studied in relation with the switching reconstruction problem [6, 2, 42]. This switching is related to Seidel switching in the Seidel matrix of graphs and a concept that is analogous to two-graphs has been defined as a skew two-graph or oriented two-graph by Cameron in [13] and Moorhouse in [47].

The digraphs studied in this thesis are also called *mixed graphs* in the literature.

1.2 Summary of Results

In this thesis, we study the spectrum of digraphs with respect to various matrices; in particular we investigate the adjacency matrix, the Hermitian adjacency matrix and the A-Laplacian. In addition, we consider simple eigenvalues of these matrices in the case that the digraphs are vertex-transitive, with the intuition that the added algebraic structure and symmetry will help understand the combinatorial structure of the digraphs. In the case of cubic graphs, we find surprising connections with regular maps and Chebyshev polynomials.

1.2.1 Hermitian adjacency matrix of digraphs

The main contribution of this dissertation is the exploration of the Hermitian adjacency matrix of digraphs. As this is a new concept, we give many basic properties, such as the coefficients of the characteristic polynomial. We find that $\mathrm{tr}(H(X)^2)$, for a digraph X, counts the number of edges in the underlying graph and that $\mathrm{tr}(H(X)^3)$ is a linear combination of the numbers of subgraphs isomorphic to four digraphs whose underlying graph is K_3. As we do for the adjacency matrix in Section 5.4, we study digraphs whose H-eigenvalues are symmetric about 0. In the case of the Hermitian adjacency matrix, the picture is less complete than for the adjacency matrix; all oriented graphs and digraphs with bipartite underlying graphs will have H-eigenvalues which are symmetric about 0, but there are also other digraphs with this eigenvalue property which are not of those classes.

It has come to our attention that this matrix was introduced in a recent paper of Liu and Li in [41], where they find several of its basic properties.

We show that the largest eigenvalue of the Hermitian adjacency matrix is upper-bounded by the maximum degree of the underlying graph. Contrary to the case for the adjacency matrix, there does not appear to be a bound on the diameter of the digraph in terms of the number of distinct eigenvalues of the Hermitian adjacency matrix. In fact, we give an infinite families of digraph whose number of distinct eigenvalues is constant, but whose diameter goes to infinity.

In spite of the many unintuitive behaviours that the Hermitian adjacency matrix exhibits, it is still possible to extract combinatorial structure of the digraph from its eigenvalues. In Section 6.10, we find all digraphs whose H-eigenvalues lie in the range $(-\sqrt{3}, \sqrt{3})$. Using interlacing, we find spectral bounds for the maximum independent set and maximum acyclic subgraphs for oriented graphs.

We also study operations on digraphs which preserve the spectrum of the Hermitian adjacency matrix. In particular, given a digraph X with vertex u and a non-adjacent vertex v, the digraphs obtained by adding the arc uv, adding the arc vu and adding both arcs uv and vu are all cospectral under the Hermitian adjacency matrix.

We also find the H-eigenvalues of many families of digraphs, including all oriented cycles and all digraphs whose underlying graph is a star. We give a new proof for finding the eigenvalues of a transitive tournament, with respect to the Hermitian adjacency matrix. We also find all graphs whose H-eigenvalues are the same as those of K_n and the negative of those of K_n.

This matrix was formulated by Bojan Mohar and these results were proved in collaboration with him.

1.2.2 Interlacing bounds on digraphs

The interlacing theorem, stated here in Chapter 4, is a powerful tool when working with the eigenvalues of graphs. We would like to use interlacing to find bounds on combinatorial properties of digraph X in terms of the eigenvalues of some matrix associated with X. We succeed in this endeavour by using the A-Laplacian matrix (defined in Chapter 9 to prove a result which generalizes a result of Haemers [32] for graphs to the class of digraphs with normal A-Laplacian matrices. Since the main theorem is quite technical, we will state Haemer's lemma here, which appears in this dissertation as Corollary 9.3.1.

Theorem ([32]). *Let X be a connected graph on n vertices and let Y and Z be disjoint vertex sets in X with no arcs from Z to Y. Then,*

$$\frac{|Y||Z|}{(n-|Y|)(n-|Z|)} \leq \frac{(\sigma_n - \sigma_2)^2}{(\sigma_n + \sigma_2)^2}$$

where $0 = \sigma_1 < \sigma_2 \leq \cdots \leq \sigma_n$ are the Laplacian eigenvalues of X.

Using the generalization of this theorem to digraphs whose A-Laplacian matrix is normal, we also find an upper bound on the size of the maximum induced acyclic subdigraph.

1.2.3 Simple eigenvalues of vertex-transitive graphs

In the last chapter of this dissertation, we study cubic, vertex-transitive graphs with 1 as a simple eigenvalue and study the combinatorial structure of such graphs, as well as give several families of graphs which such spectral properties. This is joint work with Bojan Mohar.

1.3 Overview

Chapters 2–5 are background chapters containing basic definitions and background materials. The remaining chapters contain original results.

In Chapter 2, we give our main definitions, with regards to digraphs, and notation which will be used through the thesis. We work out preliminary lemmas and theorems concerning eigenspaces and digraph symmetrices for a general class of matrices that capture the adjacency relation of digraphs in Chapter 3. The results in the section are standard for the adjacency matrix, but are proved here for a large class of matrices. We give a thorough introduction to eigenvalue interlacing, which will be an important tool for studying digraph spectra in Chapter 4. Although the main focus of the thesis is on the Hermitian adjacency matrix of digraph, we consider the adjacency matrix of digraphs in Chapter 5; mainly, we study situations that occur for digraphs but not for graphs and also basic properties so that we may compare to the Hermitian adjacency matrix.

The Hermitian adjacency matrix is introduced in Chapter 6. We work out basic concepts for it in Chapter 6 and find operations that preserve the spectrum in Chapter 7. In Chapter 8, we introduce special families of digraphs whose spectra with respect to the Hermitian adjacency matrix merit special attention. In Chapter 9, we use interlacing to find a spectral bound for the maximum acyclic subdigraph. In Chapter 10, we extend the exploration of vertex-transitive digraphs with all simple eigenvalues with respect to a matrix respecting adjacency in Chapter 3 to some results on the simple eigenvalues of cubic vertex-transitive graphs.

Statements of preliminary theorems in linear algebra are given in Appendices A and B. Appendix C contains a table which is referenced by several proofs and which was not included in the main text for reasons of readability.

Chapter 2

Digraphs

2.1 Definitions

In this chapter, we establish the main definitions and notation which will be used for the remainder of this thesis. Concepts which are natural and common for graphs, are defined here for digraphs, for the sake of completeness, but assumed to be understood for graphs. We refer to standard graph theory texts such as [7, 23] for basic definitions which are omitted here. Where it is possible, we have defined concepts for digraphs, such as regularity, so that it generalizes the corresponding concept for graphs.

A *directed graph* or *digraph* is a pair $X = (V, E)$ of finite sets together with two maps

$$h : E \to V$$

and

$$t : E \to V$$

which assign every element $e \in E$ to a *tail* or *initial vertex*, $t(e)$, and a *head* or *terminal vertex*, $h(e)$. The set V is the *vertex set* of X, sometimes denoted $V(X)$ when the context is unclear, and its elements are said to be the *vertices* of X. The set E is the *arc set* of X, sometimes denoted $E(X)$, and its elements are said to be the *arcs* of X. For arc $e \in E$, the head and tail vertices of e are also said to be the *ends* of e. We will work with *strict* digraphs which are digraphs with no loops or parallel arcs, which are arcs with the same head and tail. We note that two oppositely oriented arcs with the same ends are permitted.

We also say that an arc e is *incident* to its ends $t(e) = x$ and $h(e) = y$, and we may write $e = xy$. Thus we may write the arc set E as a subset of the set of all ordered pairs of vertices. If xy is an arc of X, then x is an *in-neighbour* of y and y is an *out-neighbour* of x. The set of all in-neighours of x is the *in-neighbourhood of* x and is denoted $N^-_X(x)$. The set of all out-neighours of x is the *out-neighbourhood of* x and is denoted $N^+_X(x)$.

If $xy \in E(X)$ and $yx \in E(X)$, we say that the unordered pair $\{x, y\}$ is a *digon* of X. Further, if $xy, yx \in E(X)$, we may say that xy and yx are incident to the digon $\{x, y\}$. In this case, since y is both an in-neighbour and an out-neighbour of x, we may say that y is a *neighbour* of x.

We define $G(X)$, the *symmetric subgraph of digraph X,* to be the graph with vertex set $V(X)$ and the edge set being the set consisting of all digons of X. Similarly, we define $D(X)$, the *asymmetric sub-digraph of digraph X,* to be the graph with vertex set $V(X)$ and the arc set being the set of arcs of X which are not incident to any digons of X.

The *underlying graph of a digraph X,* denoted $\Gamma(X)$, is the graph with vertex set $V(X)$ and edge set

$$E = \{\{x, y\} \mid xy \in E(X) \text{ or } yx \in E(X)\}.$$

If a digraph X has every arc incident to a digon, then we say that X is *undirected.* or, more simply, that X is a *graph.* If a digraph X has no digons, we say that X is an *oriented graph.* Given a graph G, the *digraph* of G is the digraph on the same vertex set with every undirected edge replaced by two of arcs with opposite orientation. We will denote the digraph of a graph G as $\vec{D}(G)$.

The *converse* of digraph $X = (V, E)$ is the digraph X^C with vertex set V and arc set E' such that

$$E' = \{xy \mid yx \in E\}.$$

Observe that every digon of X is unchanged under the operation of taking converses. Thus, $G(X^C) = G(X)$ and $D(X^C) = (D(X))^C$.

The *in-degree* of a vertex $x \in V(X)$, denoted $d^-(x)$, is the number of in-neighbours of x. The *out-degree* of a vertex $x \in V(X)$, denoted $d^+(x)$, is the number of out-neighbours of x. The *degree* of a vertex $x \in V(X)$, denoted $d(x)$, is equal to $|N^-_X(x) \cap N^+_X(x)|$. The maximum in-degree (resp. out-degree) of X will be denoted $\Delta^-(X)$ (resp. $\Delta^+(X)$) and the minimum in-degree (resp. out-degree) of X will be denoted $\delta^-(X)$ (resp. $\delta^+(X)$). Note that we may omit the superscripts for graphs, which will then agree with the typical notation for the undirected case.

Suppose X is a digraph such that $G(X)$ is a regular graph and $D(X)$ has in-degree equal to s and out-degree equal to t at every vertex. Then, letting n be the number of vertices of X, we have that $D(X)$ has ns arcs and also nt arcs, both by the handshaking lemma, and so $s = t$. A digraph X is said to be *regular* if $G(X)$ is k-regular and $D(X)$ has in-degree and out-degree equal to t at every vertex. If a digraph X has in-degree equal to out-degree at every vertex, we say that X is *eulerian.*

2.2 Sub-digraphs and other properties

Given a digraph X, any digraph Y whose vertex set is a subset of $V(X)$ and whose arc set is a subset of $E(X)$ is said to be a *sub-digraph of X.*

A *(directed) walk* of X is an alternating sequence of vertices and arcs, $(v_0, a_0, v_1, \ldots, a_{k-1}, v_k)$, such that $h(a_j) = v_j$ and $t(a_j) = v_{j+1}$ for $j = 0, \ldots, k-1$. Since we work exclusively with strict digraphs, we sometimes denote a walk by its sequence of vertices $(v_0, \ldots, v_k)$ or just $v_0 v_1 \ldots v_k$, for convenience. The *length* of a walk $v_0 v_1 \ldots v_k$ is k and its *order* is $k+1$. Given a walk W in a digraph X, we may consider the sub-digraph of X whose vertices and arcs are those appearing in W. Thus, we may speak of the vertex set and arc set of W. If a walk W has all distinct vertices, then it is said to be a *(directed) path.*

For vertices u and v, the *distance* from u to v, denoted $d(u, v)$, is the smallest k such that there exists a walk of length k in X with u as the first vertex and v as the last vertex. We define $d(u, v) = \infty$ when no such walk exist.

A digraph $X = (V, E)$ is *strongly connected* if for every pair of vertices (x, y), there exists a directed walk from x to y in X. A digraph is *weakly connected* if its underlying graph is connected.

2.3 Automorphism group of a digraph

Let $X = (V, E)$ be a digraph. A bijective mapping $\alpha : V \to V$ is an *automorphism* of X if $\alpha(u)\alpha(v) \in E$ if and only if $uv \in E$. The *automorphism group* of X, denoted $\mathrm{Aut}(X)$, is the set of all automorphisms of X, which forms a group acting on V, under composition.

Each automorphism of X acts on V as a permutation, so $\mathrm{Aut}(X) \subseteq \mathrm{Sym}(V)$, the symmetric group acting on V. Let $|V| = n$. We may represent any permutation α of the (totally ordered) n-element set V, as an $n \times n$ permutation matrix P such that $P\mathbf{e}_u = \mathbf{e}_{\alpha(u)}$, where $\mathbf{e}_x$ is the elementary basis vector indexed by the element x of V. More explicitly, in $\mathbb{C}^V$, the vector space of $\mathbb{C}$ of dimension n, with coordinates indexed by the elements of V, the elementary basis vector $\mathbf{e}_x$ is the vector with 1 in the coordinate indexed by $x \in V$ and 0 elsewhere. Thus, we may represent the automorphism group of digraph X by a group of $n \times n$ permutation matrices. We will conflate notation and use $\mathrm{Aut}(X)$ to denote the matrix group of permutation matrices representing the automorphisms of X.

2.4 Chromatic number of a digraph

An *acyclic digraph* is a digraph with no directed cycles. The *chromatic number* $\chi(D)$ *of a digraph* D is the least integer k such that there exists a partition of the vertices of D into k parts such that each part induces an acyclic subdigraph.

Chapter 3

Eigenspaces and automorphisms

In algebraic graph theory, the study of eigenvalues often seeks to exploit the interaction between the automorphism group of a graph and the eigenspaces of a graph. There is a surprising inverse correlation between the number of eigenvalues of a graph and the size of its automorphism group. If the automorphism group of a graph G is arc-transitive, the graph has at most two simple eigenvalues. On the other hand, if a graph on n vertices has at most 2 distinct eigenvalues, the automorphism group is the full symmetric group of n elements. We turn our attention to classes of graphs and digraphs with many automorphisms and many simple eigenvalues, with the intuition that they should not be many in number and with the hope that we may describe them.

In this chapter, we prove that the automorphisms of a digraph fix its eigenspaces under any matrix which "captures" the adjacency of the digraph; this class of matrices will include the adjacency matrix and the Hermitian adjacency matrix of Chapter 6. Then, we specialize to vertex-transitive case for graphs and digraphs.

3.1 Matrices respecting adjacency

Let $X = (V, E)$ be a digraph. We consider the following four relations between ordered pairs (u, v) of distinct vertices of X:

(R0) there are no arcs between u and v, i.e. $uv, vu \notin E$;

(R$+$) $uv \in E$ and $vu \notin E$;

(R$-$) $vu \in E$ and $uv \notin E$; and

(R1) uv and vu are both elements of E.

Observe that every pair of vertices (u, v) must satisfy exactly one of the relations (Ri) for $i \in \{0, +, -, 1\}$. We may summarize this information with a mapping $R_X : V \times V \to \{0, +, -, 1\}$, where $R_X(u, v) = i$ if (u, v) satisfies (Ri).

We are interested in matrices whose entries are related to adjacency in the digraph X. Let M be a matrix with rows and columns, indexed by the vertices of X. We say that M *respects the adjacency of* X if the following conditions hold:

(i) For every pair of vertices (u, v) and (x, y): if $R_X(u, v) = R_X(x, y)$, then $M(u, v) = M(x, y)$.

(ii) $M(u, u) = M(v, v)$ whenever u and v are in the same orbit under the action of $\mathrm{Aut}(X)$.

We may sometimes write that such a matrix M is a *matrix respecting adjacency (of* X*).* The *adjacency matrix* of a digraph X, denoted $A(X)$, is the matrix such that

$$A(u, v) = \begin{cases} 1, & if\, u \neq v \text{ and } R_X(u, v) \in \{+, 1\}; \\ 0, & otherwise. \end{cases}$$

Examples of matrices respecting the adjacency of X where X is a graph include the adjacency matrix $A(X)$, the Laplacian matrix $L(X)$ and the signless Laplacian $Q(X)$. For tournaments, the skew symmetric adjacency matrix is a matrix respecting adjacency. For digraphs in general, the adjacency matrix and the Hermitian adjacency matrix, which we introduce in Chapter 6, also respect adjacency.

3.1.1 Proposition. *If M is a matrix respecting adjacency of digraph X and $P \in \mathrm{Aut}(X)$, then $P^T M P = M$.*

Proof. Observe first that $P^T = P^{-1}$ for a permutation matrix P. Let X be a digraph, M a matrix respecting its adjacency and P be the permutation matrix representing α, an automorphism of X. For vertices u and v of X, we consider the (u, v) entry of the matrix $P^T M P$ and see that:

$$\begin{aligned} (P^T M P)(u, v) &= \mathbf{e}_u^T (P^T M P)\mathbf{e}_v \\ &= (P\mathbf{e}_u)^T M (P\mathbf{e}_v) \\ &= \mathbf{e}_{\alpha(u)}^T M \mathbf{e}_{\alpha(v)} \\ &= M_{\alpha(u),\alpha(v)}. \end{aligned}$$

Since α takes the pair (u, v) to $(\alpha(u), \alpha(v))$, we have $R_X(u, v) = R_X(\alpha(u), \alpha(v))$. Then $M(\alpha(u), \alpha(v)) = M(u, v)$ since M respects the adjacency of X. □

An immediate corollary of Proposition 3.1.1 is that any automorphism P preserves the eigenspaces of M.

3.1.2 Corollary. *If M is a matrix respecting adjacency of digraph X and $P \in \mathrm{Aut}(X)$, then the action of P on $\mathbb{C}^{V(X)}$ fixes the eigenspaces of M.*

Proof. Let M, X, and P be as in the statement of the corollary and let $\mathbf{v}$ be an eigenvector of M with eigenvalue λ. It suffices to show that $P\mathbf{v}$ is also an eigenvector of M with eigenvalue λ.

Proposition 3.1.1 gives that $MP = PM$ and so

$$M(P\mathbf{v}) = (MP)\mathbf{v} = (PM)\mathbf{v} = P(\lambda\mathbf{v}) = \lambda(P\mathbf{v})$$

as claimed. □

3.2 Simple eigenvalues of vertex-transitive digraphs

Let X be a vertex-transitive digraph on vertices $[n] := \{1, \ldots, n\}$. Let M be a matrix respecting adjacency for digraph X. Let λ be a simple eigenvalue of M and let $\mathbf{x} \in \mathbb{C}^n$ be a corresponding eigenvector; for simplicity of notation, we consider $\mathbb{C}^n = \mathbb{C}^{V(X)}$.

Let $P \in \mathrm{Aut}(X)$. By Corollary 3.1.2, the action of P on $\mathbb{C}^n$ fixes the λ-eigenspace of M. Since λ is a simple eigenvalue, we see that $P\mathbf{x} = \alpha\mathbf{x}$ for some $\alpha \in \mathbb{C}$. Since P is a permutation matrix, $\mathbf{x}$ and $P\mathbf{x}$ have the same norm and so $|\alpha| = 1$. Let m be the order of P; that is m is the least positive integer such that $P^m = I_n$. Then $P^m\mathbf{x} = \alpha^m\mathbf{x} = \mathbf{x}$ gives that α is a m-th root of unity. In this way, given a simple eigenvalue λ of M, we may define a mapping $\chi(\lambda, \cdot) : \mathrm{Aut}(X) \to \mathbb{C}$, by $\chi(\lambda, P) = \alpha$ for $P \in \mathrm{Aut}(X)$ where $\alpha \in \mathbb{C}$ is such that $P\mathbf{x} = \alpha\mathbf{x}$ for any $\mathbf{x}$ in the λ-eigenspace of M.

3.2.1 Lemma. *The mapping $\chi(\lambda, \cdot)$ is a character of $\mathrm{Aut}(X)$.*

Proof. We need to show that $\chi(\lambda, \cdot)$ is a homomorphism. Observe that $\chi(\lambda, I) = 1$. Let $P, Q \in \mathrm{Aut}(X)$ and consider $\chi(\lambda, PQ)$. Let $\mathbf{x}$ be an eigenvector of M for λ. Then $P\mathbf{x} = \chi(\lambda, P)\mathbf{x}$ and $Q\mathbf{x} = \chi(\lambda, Q)\mathbf{x}$. Then

$$PQ\mathbf{x} = P(Q\mathbf{x}) = P(\chi(\lambda, Q)\mathbf{x}) = \chi(\lambda, Q)P\mathbf{x} = \chi(\lambda, Q)\chi(\lambda, P)\mathbf{x}$$

and so $\chi(\lambda, PQ) = \chi(\lambda, P)\chi(\lambda, Q)$. □

We may obtain even more information about the entries of $\mathbf{x}$.

3.2.2 Lemma. *All entries of $\mathbf{x}$ have the same norm.*

Proof. For $u, v \in V(X)$, let $P \in \mathrm{Aut}(X)$ be an automorphism taking u to v. We know one exists, since X is vertex transitive. Then $P\mathbf{x} = \chi(\lambda, P)\mathbf{x}$ and so $\mathbf{x}(u) = \chi(\lambda, P)\mathbf{x}(v)$. Since $\chi(\lambda, P)$ has norm 1, we see that $|\mathbf{x}(u)| = |\mathbf{x}(v)|$. □

Then, we may assume that $\mathbf{x}(1) = 1$, by taking the appropriate normalization.The *algebraic conjugates* of an algebraic number α are the roots of the minimal polynomial of α over the rationals.

3.2.3 Lemma. *Under the normalization $\mathbf{x}(1) = 1$, the entries of $\mathbf{x}$ are roots of unity and are closed under algebraic conjugation.*

Proof. Let $u \neq 1$ be a vertex of X. Let $P \in \mathrm{Aut}(X)$ such that P maps 1 to u. Then $\mathbf{x}(u) = \chi(\lambda, P)\mathbf{x}(1) = \chi(\lambda, P)$, which is a root of unity since $\chi(\lambda, \cdot)$ is a character of $\mathrm{Aut}(X)$, a finite group. Then, by taking the kth power of P, for any k, we see that $\mathbf{x}(u)^k$ is an entry of $\mathbf{x}$. $\square$

If M is diagonalizable over $\mathbb{R}$, like the adjacency matrix, then the entries of $\mathbf{x}$ must be real roots of unity by Lemma 3.2.3, which are $\{\pm 1\}$. For $M = A(X)$, we have the following standard theorem, which can be found in [5] or [18].

3.2.4 Theorem (Petersdorf and Sachs [49]). *Let X be a vertex-transitive graph of degree k. If λ is a simple eigenvalue of $A(X)$, then*

$$\lambda = k - 2\alpha$$

for some integer $\alpha \in \{0, \dots, k\}$.

Proof. Let λ be a simple eigenvalue of X and let $\mathbf{v}$ be a corresponding ± 1 eigenvector. Let x be a vertex of X. Without loss of generality, we may assume $\mathbf{v}(x) = 1$. We have that

$$\sum_{y \sim x} \mathbf{v}(y) = \lambda \mathbf{v}(x) = \lambda. \tag{3.1}$$

Let α $(0 \leq \alpha \leq k)$ be the number of neighbours y of x such that $\mathbf{v}(y) = -1$. Then (3.1) implies that $\lambda = k - 2\alpha$. $\square$

3.3 Vertex-transitive digraphs with all eigenvalues simple

Let X be a vertex-transitive digraph on vertices $[n]$. Let M be a matrix respecting adjacency for digraph X. Observe that since $\mathrm{Aut}(X)$ is a subgroup of all $n \times n$ permutation matrices, the action of $\mathrm{Aut}(X)$ on $\mathbb{C}^n$ is faithful.

3.3.1 Lemma. *Suppose M is diagonalizable with all eigenvalues simple. For $P, Q \in \mathrm{Aut}(X)$, if $\chi(\lambda, P) = \chi(\lambda, Q)$ for every eigenvalue λ of M, then $P = Q$.*

Proof. Since M is diagonalizable, it has an orthonormal eigenbasis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbb{C}^n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. For $j \in [n]$, we see from the hypothesis and definition that $\chi(\lambda_j, P) = \chi(\lambda_j, Q)$ implies $P\mathbf{v}_j = Q\mathbf{v}_j$. Since this holds for every j, we see that the action of P and Q agree over a basis of $\mathbb{C}^n$ and, since the action of $\mathrm{Aut}(X)$ on $\mathbb{C}^n$ is faithful, $P = Q$. $\square$

3.3.2 Lemma. *If M is diagonalizable with all eigenvalues simple, then $\mathrm{Aut}(X)$ is an abelian group.*

Proof. Let $P, Q \in \mathrm{Aut}(X)$ and λ an eigenvalue of M. We see that

$$\chi(\lambda, PQ) = \chi(\lambda, P)\chi(\lambda, Q) = \chi(\lambda, Q)\chi(\lambda, P) = \chi(\lambda, QP)$$

since complex numbers commute. Since this holds for every λ, we see by Lemma 3.3.1 that $PQ = QP$. $\square$

A Cayley digraph $X = \text{Cay}(G, A)$ is a *digraphical regular representation (DRR)* of G if $\mathrm{Aut}(X) = G$. This condition is equivalent to requiring that $\mathrm{Aut}(X)$ act regularly on X.

3.3.3 Proposition. *If all eigenvalues of M are simple, then X is a DRR of an abelian group.*

Proof. Recall that we consider $\mathbb{C}^n = \mathbb{C}^{V(X)}$. Let x, y be vertices of X. Since X is vertex transitive, there exists $P \in \mathrm{Aut}(X)$ mapping x to y; that is to say $P\mathbf{e}_x = \mathbf{e}_y$. Suppose $Q \in \mathrm{Aut}(X)$ also maps x to y.

Let λ be an eigenvalue of M with eigenvector $\mathbf{v}$. Since P maps x to y, we must have that

$$\mathbf{v}_x = \chi(\lambda, P)\mathbf{v}_y.$$

But Q also maps x to y, so

$$\mathbf{v}_x = \chi(\lambda, Q)\mathbf{v}_y.$$

Lemma 3.2.2 says that every entry of $\mathbf{v}$ has the same norm, and so every entry of $\mathbf{v}$ is non-zero. Then $\mathbf{v}_y \neq 0$ and we obtain that $\chi(\lambda, P) = \chi(\lambda, Q)$. Since this holds for every choice of λ, Lemma 3.3.1 implies that $P = Q$ and we have shown that the action of $\mathrm{Aut}(X)$ is regular on X.

Since some subgroup of $\mathrm{Aut}(X)$ acts transitively and regularly on X, we have, by Sabidussi's theorem (see [57]), that X is a Cayley digraph of $\mathrm{Aut}(X)$. In addition, since it is $\mathrm{Aut}(X)$ itself that acts regularly on X, we have that X is a DRR of $\mathrm{Aut}(X)$, which is abelian by Lemma 3.3.2. $\square$

Chapter 4

Interlacing for Hermitian matrices

In the theory of eigenvalues of graphs, the method of interlacing is a powerful tool. In this chapter, we state and prove the main theorems about interlacing. The proofs are given here because these theorems, while true for Hermitian matrices, are usually stated only for symmetric matrices. These theorems are found in [27] and [31].

If $(\lambda_i)_{i=1}^n$ and $(\mu_i)_{i=1}^{n-1}$ are sequences of real numbers such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

then $(\mu_i)_{i=1}^{n-1}$ is said to *interlace* $(\lambda_i)_{i=1}^n$. More generally, if $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$ are real numbers with $m < n$, then we say that $(\mu_i)_{i=1}^m$ *interlaces* $(\lambda_i)_{i=1}^n$ if

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$$

for $i = 1, \ldots, m$. If $m = n-1$, we see that this implies that

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}$$

for $i = 1, \ldots, n-1$. For $m < n-1$, this second interlacing is sometimes called *generalized interlacing*. The interlacing is *tight* if there exists $k \in [1, \ldots, m]$ such that

$$\lambda_i = \mu_i \text{ for } i = 1, \ldots, k,$$

and

$$\lambda_{n-m+1} = \mu_i \text{ for } i = k+1, \ldots, m.$$

Note the tight interlacing is a stronger condition than the containment of $(\mu_i)_{i=1}^m$ in $(\lambda_i)_{i=1}^n$, when considered as multisets. We will use these notions mostly where $(\lambda_i)_{i=1}^n$ is the multiset of eigenvalues of an $n \times n$ matrix A and $(\mu_i)_{i=1}^m$ is the multiset of eigenvalues of an $m \times m$ matrix B.

4.1 Interlacing of principal minors

In this chapter, we follow the rational functions approach to interlacing in [27].

4.1.1 Proposition. *[27] If A is an $n \times n$ Hermitian matrix and B an $(n-1) \times (n-1)$ principal submatrix of A, then the eigenvalues of B interlace those of A.*

Proof. Since B is a principal submatrix, B is the matrix obtained from A by deleting, say, the uth row and column. Since A is a Hermitian matrix, we may use spectral decomposition of Section A.4 to decompose A into a sum of idempotents as follows:

$$A = \sum_{\theta} \theta E_\theta$$

where the sum runs over θ distinct eigenvalues of A and E_θ is the idempotent projection matrix onto the θ eigenspace of A in $\mathbb{C}^n$. For t a variable over $\mathbb{C}$, consider the matrix $(tI - A)^{-1}$ where I is the $n \times n$ identity matrix. We have

$$(tI - A)^{-1} = \left(tI - \sum_{\theta} \theta E_\theta\right)^{-1}$$

and since $E_\theta E_{\theta'} = O$ for $\theta \neq \theta'$ and $E_\theta^2 = E_\theta$, we see that

$$\begin{aligned}
\left(tI - \sum_{\theta} \theta E_\theta\right)\left(\sum_{\theta} \frac{1}{t-\theta} E_\theta\right) &= \sum_{\theta} \frac{t}{t-\theta} E_\theta - \left(\sum_{\theta} \theta E_\theta\right)\left(\sum_{\theta} \frac{1}{t-\theta} E_\theta\right) \\
&= \sum_{\theta} \frac{t}{t-\theta} E_\theta - \sum_{\theta} \frac{\theta}{t-\theta} E_\theta \\
&= \sum_{\theta} \frac{t-\theta}{t-\theta} E_\theta \\
&= \sum_{\theta} E_\theta \\
&= I.
\end{aligned}$$

Then

$$(tI - A)^{-1} = \sum_{\theta} \frac{1}{t-\theta} E_\theta.$$

The (u,u)-entry of this matrix is

$$(tI - A)^{-1}_{u,u} = \left(\sum_{\theta} \frac{1}{t-\theta} E_\theta\right)_{u,u}$$

for which we may obtain a different expression using Cramer's rule as follows:

$$(tI - A)^{-1}_{u,u} = \frac{\phi(B,t)}{\phi(A,t)}$$

where $\phi(M,t)$ is the characteristic polynomial of M in variable t. This gives

$$\frac{\phi(B,t)}{\phi(A,t)} = \left(\sum_{\theta} \frac{1}{t-\theta} E_\theta\right)_{u,u}. \tag{4.1}$$

For Hermitian matrices, the idempotent matrices in the spectral decomposition are positive semi-definite. Thus, the diagonal elements of E_θ are non-negative for each θ. If we differentiate both sides of equation (4.1) with respect to t, we obtain

$$\frac{d}{dt}\frac{\phi(B,t)}{\phi(A,t)} = \sum_{\theta} -\frac{1}{(t-\theta)^2}\left(E_\theta\right)_{u,u}.$$

Since each $\left(E_\theta\right)_{u,u}$ is non-negative, we see that

$$\frac{d}{dt}\frac{\phi(B,t)}{\phi(A,t)} \leq 0.$$

Then, between any two poles of $\frac{\phi(B,t)}{\phi(A,t)}$, there is exactly one zero. This shows that the roots of $\phi(B,t)$ interlace those of $\phi(A,t)$, as required. □

The following useful theorem follows from iterated application of Proposition 4.1.1.

4.1.2 Theorem. *If A is a Hermitian matrix and B is a principal submatrix of A, then the eigenvalues of B interlace those of A.*

4.2 Generalized interlacing

In this section we will introduce the generalized interlacing theorems of Haemers [32, 10, 31]. We will provide interlacing properties for matrices following [31] and then, in Section 4.3, we will show the equivalence to the interlacing in Section 4.1.

The following is the main theorem concerning generalized interlacing. Note that we use M^* to denote the conjugate transpose of matrix M.

4.2.1 Theorem. *[32] Let A be a Hermitian $n \times n$ matrix and let S be a complex $n \times m$ matrix such that $S^*S = I$. If $B = S^*AS$, then the eigenvalues of B interlace the eigenvalues of A. Furthermore, if the interlacing is tight, then $SB = AS$.*

To prove Theorem 4.2.1, we need some linear algebra; in particular, we need Rayleigh's Theorem, see [34, Theorem 4.2.2]. We will give the theorem statement but not the proof.

4.2.2 Proposition (Rayleigh)**.** *Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthonormal eigenbasis of a $n \times n$ Hermitian matrix A with corresponding eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$\frac{\mathbf{v}^* A \mathbf{v}}{\mathbf{v}^* \mathbf{v}} \geq \lambda_i \tag{4.2}$$

if $\mathbf{v} \in \langle \mathbf{v}_1, \ldots, \mathbf{v}_i \rangle$ and

$$\frac{\mathbf{v}^* A \mathbf{v}}{\mathbf{v}^* \mathbf{v}} \leq \lambda_i \tag{4.3}$$

if $\mathbf{v} \in \langle \mathbf{v}_1, \ldots, \mathbf{v}_{i-1} \rangle^\perp$. Furthermore, equality holds in 4.2 iff $\mathbf{v}$ is an eigenvector of A for λ_i and equality holds in 4.3 iff $\mathbf{v}$ is an eigenvector of A for λ_i.

We are now ready to prove Theorem 4.2.1.

Proof of Theorem 4.2.1 Recall that A is a Hermitian $n \times n$ matrix, S is any complex $n \times m$ matrix such that $S^*S = I$ and we define B to be $B := S^*AS$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A with corresponding orthonormal eigenbasis $\{\mathbf{u}_1, \ldots \mathbf{u}_n\}$. Let $\mu_1 \geq \cdots \geq \mu_m$ be the eigenvalues of B with corresponding orthonormal eigenbasis $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$. For $i \in \{1, \ldots, m\}$, consider

$$W_i = \langle \mathbf{v}_1, \ldots, \mathbf{v}_i \rangle \cap \langle S^*\mathbf{u}_1, \ldots, S^*\mathbf{u}_{i-1} \rangle^\perp.$$

Since the first term of the intersection has dimension i and the second term has dimension $m-i+1$, then W_i is a non-trivial subspace of $\mathbb{C}^m$. We may choose nonzero $\mathbf{x}_i \in W_i$. Then, since $SS^* = I$, we have that $S\mathbf{x}_i \in \langle \mathbf{u}_1, \ldots, \mathbf{u}_{i-1} \rangle^\perp$. Then, by Proposition 4.2.2, we have that

$$\lambda_i \geq \frac{(S\mathbf{x}_i)^* A (S\mathbf{x}_i)}{(S\mathbf{x}_i)^*(S\mathbf{x}_i)} = \frac{\mathbf{x}_i^* B \mathbf{x}_i}{\mathbf{x}_i^* \mathbf{x}_i} \geq \mu_i. \tag{4.4}$$

If the interlacing is tight, then for some $k \in \{1, \ldots, m\}$, equality holds for all $i = 1, \ldots, k$ in Equation 4.4. By Proposition 4.2.2, equality occur for $i = j$ if and only if $\mathbf{x}_j$ is an eigenvector of B for eigenvalue μ_i and $S\mathbf{x}_i$ is an eigenvector of A with eigenvalues λ_i. Similarly, if we apply the above argument to $-A$ and $-B$, we obtain that $\lambda_{n-m+i} \leq \mu_i$. In the case of tight interlacing, $\lambda_{n-m+i} = \mu_i$ for $i = k+1, \ldots, m$. By Proposition 4.2.2, equality occur for $i = j$ if and only if $\mathbf{x}_j$ is an eigenvector of B for eigenvalue μ_i and $S\mathbf{x}_i$ is an eigenvector of A with eigenvalue λ_i. Thus,

$$SB\mathbf{x}_i = \mu_i S\mathbf{x}_i = \lambda_{n-m+i} S\mathbf{x}_i = AS\mathbf{x}_i$$

where the $\mathbf{x}_i$ are as chosen above. Since AS and SB are $n \times m$ matrices and $\{\mathbf{x}_i\}_{i=1}^m$ is an orthonormal basis of $\mathbb{C}^m$, the action of AS and SB agree over $\mathbb{C}^m$ implies that $AS = SB$. □

In general, we apply Theorem 4.2.1 to a matrix and the quotient matrix of some partition of the matrix into block matrices. Let A be a Hermitian matrix. Let $\mathcal{P}$ be a partition of the rows of

A which induces a partitioning of A into block matrices as follows:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mm} \end{pmatrix}$$

where the A_{ij} are block matrices and the corresponding partition of the rows and columns is $X = \{X_1, \ldots, X_m\}$, as a partition of $[n]$. Note that since we are interested in the eigenvalues of A, which are preserved under permuting the rows and columns simultaneously, we may assume all partitions of the rows of A are of this form. The *quotient matrix of A with respect to partition $\mathcal{P}$* is $\widetilde{B}$ where the ijth entry is

$$\widetilde{B}_{ij} = \frac{1}{|X_i|}\mathbf{1}^T A_{ij}\mathbf{1}$$

the average row sums of the blocks of A, where $\mathbf{1}$ denotes the all ones vector. If A_{ij} has constant row sums for all $i, j \in \{1, \ldots, m\}$, then the partition $\mathcal{P}$ is said to be an *equitable partition* of the rows of A.

4.2.3 Corollary (Haemers)**.** *Let A be a Hermitian matrix and $\widetilde{B}$ be its partition matrix with respect to partition $\mathcal{P}$. The eigenvalues of $\widetilde{B}$ interlace those of A. If the interlacing is tight, then $\mathcal{P}$ is an equitable partition of the rows of A. If $\mathcal{P}$ is an equitable partition of the rows of A, then $\sigma(\widetilde{B}) \subseteq \sigma(A)$.*

Proof. Let $\widetilde{S}$ be the *characteristic matrix of the partition X*; that is, $\widetilde{S}$ has rows indexed by $[n]$ and columns indexed by sets $X_1, \ldots, X_m$ such that

$$\widetilde{S}_{ij} = \begin{cases} 1 & \text{if } i \in X_j\,; \\ 0 & \text{otherwise.} \end{cases}$$

Then, the quotient matrix $\widetilde{B}$ is as follows:

$$\widetilde{B}_{ij} = \frac{1}{|X_i|}\mathbf{1}^T A_{ij}\mathbf{1} = \frac{1}{|X_i|}\left(\widetilde{S}^T A \widetilde{S}\right)_{ij},$$

and

$$\widetilde{B} = D^{-1}\widetilde{S}^* A \widetilde{S}$$

where D is a diagonal matrix defined as follows:

$$D_{ii} = |X_i|.$$

Since D is diagonal, we can let

$$B = D^{\frac{1}{2}} D^{-1} \widetilde{S}^* A \widetilde{S} D^{-\frac{1}{2}} = D^{-\frac{1}{2}} \widetilde{S}^* A \widetilde{S} D^{-\frac{1}{2}}.$$

Let $S = \widetilde{S} D^{-\frac{1}{2}}$. We can see that $SS^* = I_{n \times n}$ and $S^* S = I_{m \times m}$. In addition $B = S^* A S$. Then by Theorem 4.2.1, the eigenvalues of B interlace those of A. Observe that $\widetilde{B}$ is similar and hence cospectral to B. The equality conditions follow. $\square$

Note that if $\sigma(\widetilde{B}) \subseteq \sigma(A)$ it is possible but not necessary for the partition to be equitable.

4.3 Interlacing properties

In this section, we will remark that though Theorems 4.1.2 and 4.2.1 have very different approaches, they are in fact equivalent.

Suppose B is a prinicipal submatrix of Hermitian matrix A. Then we may reorder the rows and columns on A and B so that $B = S^* A S$, where $S = \begin{pmatrix} I & 0 \end{pmatrix}^T$. Hence, the interlacing of prinicipal submatrices follows from Theorem 4.2.1.

If A and B are $n \times n$ and $m \times m$ Hermitian matrices, respectively, where $B = S^* A S$ for some S with orthonormal columns, then we will show that B is a principal submatrix of some matrix which is cospectral with A. Since the columns on S are orthonormal, we may extend S to a $n \times n$ orthogonal matrix; we may choose Q such that

$$\begin{pmatrix} S & Q \end{pmatrix}$$

is an orthogonal matrix. Let

$$\widetilde{A} = \begin{pmatrix} S & Q \end{pmatrix}^* A \begin{pmatrix} S & Q \end{pmatrix} = \begin{pmatrix} S^* A S & S^* A Q \\ Q^* A S & Q^* A Q \end{pmatrix} = \begin{pmatrix} B & S^* A Q \\ Q^* A S & Q^* A Q \end{pmatrix}.$$

We see that B is a principal submatrix of $\widetilde{A}$ and $\widetilde{A}$ is cospectral to A, since A and $\widetilde{A}$ are similar matrices. This shows that Theorem 4.1.2 imples Theorem 4.2.1 and vice versa.

Chapter 5

Adjacency matrix of digraphs

In this chapter, we give background information on the adjacency matrix of digraphs, for the purposes of comparisons with the Hermitian adjacency matrix. For this chapter, eigenvalues of a digraph are the eigenvalues of its adjacency matrix. We refer to standard texts in algebraic graph theory [10, 18] for any definitions omitted here and for further information. We also discuss the known results on the adjacency matrices of tournaments, in particular.

5.1 Incidence matrices

The *in-incidence matrix* of a digraph X is the $|V| \times |E|$ matrix D_h, with rows indexed by vertices and columns indexed by edges such that the (u, e) entry is as follows:

$$D_h(u,e) = \begin{cases} 1 & \text{if } u = h(e); \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, the *out-incidence matrix* of X is the $|V| \times |E|$ matrix D_t, with rows indexed by vertices and columns indexed by edges such that the (u, e) entry is as follows:

$$D_t(u,e) = \begin{cases} 1 & \text{if } u = t(e); \\ 0 & \text{otherwise.} \end{cases}$$

Note that $D_t - D_h$ is the usual *incidence matrix* of X. The *adjacency matrix* of a directed graph X is the matrix A with rows and columns indexed by the vertices of X, such that

$$A(u,v) = \begin{cases} 1 & \text{if there exists an edge } e \text{ such that } u = t(e) \text{ and } v = h(e); \\ 0 & \text{otherwise.} \end{cases}$$

One can easily verify that $A = D_t D_h^T$.

5.2 Characteristic polynomial

Following [10, 18], we find an expression for the characteristic polynomial of a directed graph.

A *basic digraph* is a digraph on n vertices with i isolated vertices and in-degree and out-degree equal to 1 at the other $n-i$ vertices. Let X be a digraph on n vertices. A *basic subdigraph of order* i of X is a spanning subdigraph of X with i isolated vertices and in-degree and out-degree equal to 1 at the other $n-i$ vertices. Note that the set of all basic subdigraphs of the digraph of K_n is in one-to-one correspondence with $\mathrm{Sym}(n)$; to $\sigma \in \mathrm{Sym}(n)$ we associate the basic digraph with arc set $\{1\sigma(1), \ldots, n\sigma(n)\}$, with all loops deleted. Figure 5.1 shows an example of a basic digraph of order 1 in $\vec{D}(K_6)$, which corresponds to the permutation $(123)(4)(56)$.

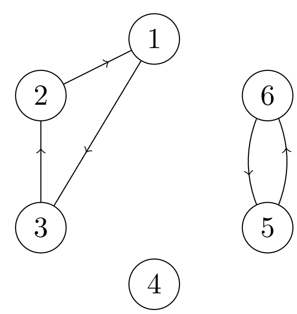


Figure 5.1: Example of a basic subdigraph of $\vec{D}(K_6)$.

5.2.1 Lemma. *Let X be a digraph with n vertices and A be its adjacency matrix. Then the characteristic polynomial of A is*

$$\phi(A,t) = \sum_{i=0}^{n} c_i t^i$$

where $c_i = \sum_C (-1)^{c(C)}$ where the sum runs over all basic subdigraphs of order i of X and $c(C)$ is the number of cycles of C.

Proof. Recall that $\phi(A,t) = \det(tI - A)$. For simplicity, we will let $V(X) = \{1, \ldots, n\}$. For any $n \times n$ matrix M,

$$\det(M) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}.$$

Let $M = tI - A$. Suppose $\sigma \in \mathrm{Sym}(n)$ has i fixed points. The term contributed by

$$\mathrm{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$$

is a constant multiple of t^i. If the basic digraph associated with σ is not a basic subdigraph of X, then the term $\mathrm{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$ is 0. Otherwise, let C be the basic subdigraph of X corresponding to σ.

Consider the cyclic decomposition of σ into disjoint cycles. We may conflate the notation and allow $c(\sigma)$ to denote the number of cycles of length greater than 1 in the cyclic decomposition of σ. A cycle of length m can be written as the product of $m-1$ transpositions. Then $\operatorname{sgn}(\sigma) = (-1)^e$, where e is a the number of even cycles. Observe that the number of odd cycles is equal to $n-i$ mod 2. Then $e = c(\sigma) + n - i \mod 2$ and we have that $\operatorname{sgn}(\sigma) = (-1)^{n-i+c(\sigma)}$.

The number of cycles of length greater than 1 in σ is same as the number of directed cycles in C. Thus, we obtain that

$$\operatorname{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)} = (-1)^{n-i+c(\sigma)}(-1)^{n-i} t^i = (-1)^{c(\sigma)} t^i$$

Therefore the coefficient of t^i in $\det(tI - A)$ is

$$\sum_{\sigma \text{ with exactly } i \text{ fixed points}} \operatorname{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)} = \sum_{\sigma \text{ with exactly } i \text{ fixed points}} (-1)^{c(\sigma)}.$$

Thus we may write the coefficient of t^i in the characteristic polynomial of X as

$$\sum_C (-1)^{c(C)}$$

where the sum runs over all basic subdigraphs of X or order i. □

Note that the above implies that edges which do not occur in any directed cycle of X do not affect the characteristic polynomial and we obtain the next easy corollary.

5.2.2 Corollary. *A directed graph is acyclic if and only if its adjacency matrix is nilpotent.*

Proof. If a graph X is acyclic, then 5.2.1 implies that the adjacency matrix has characteristic polynomial $\phi(A(X), t) = t^{|V(X)|}$ and is hence nilpotent.

Let A be the adjacency matrix of X. If A is nilpotent, then we can simultaneously permute the rows and columns of A so that A is upper-triangular with 0s on the diagonal. We can label the vertices of X as $\{x_1, \dots, x_n\}$ by their order as columns on A. Then, we see that any cycle requires an edge from x_i to x_j where $j < i$, which does not exist. Thus X is acyclic. □

We may observe that a digraph X is acyclic if and only if there is an ordering of the vertices such that $A(X)$ is upper-triangular.

5.3 Bounds on eigenvalues

For a strongly connected digraph X, the matrix $A(X)$ has non-negative entries and so the Perron-Frobenius theorem A.5.1 holds and $A(X)$ has a real, nonnegative eigenvalue ρ with a real nonnegative eigenvector. The following theorem bounds the real parts of the other eigenvalues. The *Perron* *value* of an irreducible matrix M with non-negative entries is the largest real eigenvalue, as in the Perron Frobernius theorem. The eigenvalues of M which are not the Perron value are said to be the *non-principal eigenvalues.* Note that the graph G corresponding to a digraph X in Theorem 5.3.1 differs from $G(X)$ as given in Section 2.1, since it is a multi-graph with parallel edges corresponding to the digons of X. The result in Theorem 5.3.1 is a special case of a theorem of Bendixson in [3].

5.3.1 Theorem. *[3] Let $X = (V, E)$ be a strongly connected, regular digraph on n vertices and let G be the multi-graph on V with an edge e_{uv} for each arc $uv \in E$. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of G. Then all non-principal eigenvalues λ of D satisfy*

$$\frac{\mu_2}{2} \geq \operatorname{Re} \lambda \geq \frac{\mu_n}{2}.$$

Proof. Let A be the adjacency matrix of X. Then $B = A + A^T$ is the adjacency matrix of G. Since X is regular, the all ones vector $\mathbf{1}$ is an eigenvector for both A and B. Let $\mathbf{z}$ be an eigenvector of A with $A\mathbf{z} = \lambda \mathbf{z}$, where $\mathbf{z}$ is orthogonal to $\mathbf{1}$. Then we see that

$$\begin{aligned}
\bar{\mathbf{z}}^T B\mathbf{z} &= \bar{\mathbf{z}}^T (A + A^T)\mathbf{z} \\
&= \bar{\mathbf{z}}^T A\mathbf{z} + \bar{\mathbf{z}}^T A^T \mathbf{z} \\
&= \bar{\mathbf{z}}^T (A\mathbf{z}) + (\overline{A}\bar{\mathbf{z}})^T \mathbf{z} \\
&= \lambda \bar{\mathbf{z}}^T \mathbf{z} + \bar{\lambda} \bar{\mathbf{z}}^T \mathbf{z} \\
&= 2 \operatorname{Re} \lambda \, \bar{\mathbf{z}}^T \mathbf{z}.
\end{aligned}$$

On the other hand, B is symmetric and so has an orthonormal eigenbasis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, where $B\mathbf{v}_j = \mu_j \mathbf{v}_j$. Then $\mathbf{z}$ is orthogonal to $\mathbf{v}_1 = \mathbf{1}$ and there exists $\{\alpha_j\}_{j=2}^n$ such that

$$\mathbf{z} = \sum_{j=2}^{n} \alpha_j \mathbf{v}_j.$$

Then,

$$\begin{aligned}
\bar{\mathbf{z}}^T B\mathbf{z} &= \sum_{i,j \geq 2} \overline{\alpha_i} \alpha_j \overline{\mathbf{v}_i}^T B \mathbf{v}_j \\
&= \sum_{j=2}^{n} |\alpha_j|^2 \mu_j.
\end{aligned}$$

Since $\mu_2 \geq \cdots \geq \mu_n$, we have

$$\sum_{j=2}^{n} |\alpha_j|^2 \mu_2 \geq \sum_{j=2}^{n} |\alpha_j|^2 \mu_j \geq \sum_{j=2}^{n} |\alpha_j|^2 \mu_n$$
$$\mu_2 \bar{\mathbf{z}}^T \mathbf{z} \geq \sum_{j=2}^{n} |\alpha_j|^2 \mu_j \geq \mu_n \bar{\mathbf{z}}^T \mathbf{z}.$$

Then,

$$\mu_2 \bar{\mathbf{z}}^T \mathbf{z} \geq 2 \operatorname{Re} \lambda \, \bar{\mathbf{z}}^T \mathbf{z} \geq \mu_n \bar{\mathbf{z}}^T \mathbf{z}$$
$$\frac{\mu_2}{2} \geq \operatorname{Re} \lambda \geq \frac{\mu_n}{2}.$$

□

5.4 Eigenvalues symmetric about 0

A directed cycle is said to be *even* (resp. *odd*) if it has even (resp. odd) length. Let X be a digraph on n vertices with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. We say that the eigenvalues of X are *symmetric about* 0 if for each $j = 1, \ldots, n$, λ_j and $-\lambda_j$ occur with equal multiplicity in the spectrum of X. In the case of undirected graphs, the graphs with eigenvalues symmetric about 0 are precisely the bipartite graphs. This is also the case for strongly connected digraphs.

5.4.1 Proposition. *A directed graph X has eigenvalues symmetric about 0 if and only if all directed cycles of X are even.*

Proof. A basic subgraph on n vertices of order j with $n - j$ odd has an odd cycle. Observe that there exists a basic subdigraph of X, a digraph on n vertices, of order j with $n - j$ odd if and only if X contains an odd directed cycle. Every directed cycle of X is even if and only if the coefficient of all t^j where $n - j$ is odd is 0 in $\phi(A(X), t)$. Let $m, a \in \mathbb{Z}_{\geq 0}$ be such that $n = 2m + a$, where a is the remainder of n when divided by 2. Then, every directed cycle of X is even if and only if $\phi(A(X), t)$ can be written as

$$\phi(A(X), t) = \sum_{j=0}^{m} a_{n-2j} t^{n-2j} = t^a \sum_{j=0}^{m} a_{2m-2j} t^{2m-2j} = t^a f(t^2)$$

where $f(x) = \sum_{j=0}^{m} a_{2m-2j} x^{m-j}$. If $\lambda_1, \ldots, \lambda_m$ are the roots of f, then the roots of $\phi(H(X), t)$ are $\pm\lambda_j$ for $j = 1, \ldots, m$ and one additional 0 if $a = 1$. Thus the eigenvalues of $A(X)$ are symmetric about 0 if and only if every directed cycle of X is even. □

5.4.2 Proposition. *A strongly connected digraph X has all directed cycles even if and only if the underlying graph is bipartite.*

Proof. Let X be a digraph of order n. It is clear that if $\Gamma(X)$ is bipartite, then every directed cycle of X is even. It remains to show the converse direction.

Suppose, for a contradiction, that X is a strongly connected digraph where all directed cycles have even length and that $\Gamma(X)$ is not bipartite. We will first show that there exists a closed walk of odd length in X.

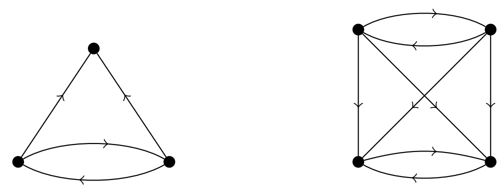
Consider a cycle $C = (v_0, \ldots, v_{m-1}, v_m = v_0)$ in $\Gamma(X)$, whose length, m, is odd. For $j \in \mathbb{Z}_m$, we will find a directed path $P_{j,j+1}$ of odd length from v_j and v_{j+1}. If (v_j, v_{j+1}) is an arc of X, then $P_{j,j+1} = (v_j, v_{j+1})$ and has length equal to 1. Otherwise, $(v_{j+1}, v_j) \in E(X)$ and, since X is strongly connected, there exists a directed path from v_j to v_{j+1} using arcs of X. Let $P_{j,j+1}$ be any such path. Then $P_{j,j+1}$ together with arcs (v_{j+1}, v_j) gives a directed cycle of X. Since all directed cycles have even length, we obtain that $P_{j,j+1}$ has odd length. Now consider the walk W obtained by concatenating paths $P_{0,1}, \ldots, P_{m-1,m}$. Since W starts and ends at $v_0 = v_m$, it is a closed walk consisting of m paths, each of which has odd length. Then W is a closed walk of odd length.

Since there exists an odd closed walk in X, we may choose $W' = (w_0, \ldots, w_k = w_0)$ to be the shortest odd closed walk of X. Since all directed cycles have even length, W is not a directed cycle and hence traverses some vertex v at least twice. Let j be the index such that $w_j = v$ and let v_{j+r} be the next occurrence of v in $(w_0, \ldots, w_{k-1})$. Let $W'' = (v_j, v_{j+1}, \ldots, v_{j+r})$ be the subsequence of path W' from v_j to v_{j+r}. Since W'' is a closed walk of shorter length than W', its length r must be even. We may remove W'' from W' to get

$$W^* = (w_0, \ldots, v_{j-1}, v_j = v_{j+r}, v_{j+r+1}, \ldots w_k = w_0)$$

an odd closed walk with length strictly shorter than that of W', a contradiction. $\square$

We note that the property of being strongly connected in Proposition 5.4.2 is necessary. Every acyclic digraph has all eigenvalues equal to 0 and hence are trivially symmetric about 0, but need not have bipartite underlying graphs. It is also not difficult to find digraphs containing digons with eigenvalues symmetric about 0, where the underlying graph is not bipartite. Figure 5.2 gives two such examples.



D_1 $\phi(A(D_1),t) = t^3 - t$

D_2 $\phi(A(D_2,t) = t^4 - 2t^2 + 1$

Figure 5.2: Two digraphs with eigenvalues symmetric about 0, whose underlying graph is not bipartite.

5.5 Adjacency matrix of tournaments

The tournaments attaining the maximum and minimum Perron values have been well-studied. Brauer and Gentry showed in [8] that, for odd n, the $n \times n$ tournament matrices which attain the maximum Perron value are precisely the regular tournaments. The Brualdi-Li conjecture [11] about the tournament matrices of even order attaining the maximum Perron value has recently been solved by Drury in [24]. The strongly connected tournaments attaining the minimum Perron value have also been studied in [21] and characterized by Kirkland in [39].

Other spectral properties of tournaments have been studied. The number of distinct eigenvalues of a tournament has also been studied. A family of tournaments with all simple eigenvalues is given in [21]. Hadamard tournaments, which coincide with doubly regular tournaments, are studied in [21, 54]. The rank of a tournament matrix has been studied in [20, 9, 21].

Chapter 6

Hermitian adjacency matrix of digraphs

In this chapter, we study the *Hermitian adjacency matrix* of digraphs. We are interested in its spectral properties. In particular, we would like to study properties that two digraphs must share if they are cospectral with respect to the Hermitian adjacency matrix. The Hermitian adjacency matrix recently appeared in [41].

6.1 Definitions

For a digraph $X = (V, E)$, we consider the *Hermitian adjacency matrix* $H := H(X)$, whose entries are given by

$$H_{u,v} = \begin{cases} 1 & \text{if } uv \text{ and } vu \in E; \\ i & \text{if } uv \in E \text{ and } vu \notin E; \\ -i & \text{if } uv \notin E \text{ and } vu \in E; \\ 0 & \text{otherwise.} \end{cases}$$

If every edge of X lies in a digon, then $H(X) = A(X)$, which reflects that X is, essentially, equivalent to an undirected graph.

Since i is a primitive fourth root of unity, we can easily see that

$$H \circ H \circ H \circ H$$

is the adjacency matrix of the underlying graph of X, where $\circ$ denotes the entry-wise matrix product, also called the *Schur product* or *Hadamard product.*

Observe that H is a Hermitian matrix and so is diagonalizable with real eigenvalues. The following lemma contains properties that are true for adjacency matrices which also carry over to the Hermitian case.

6.1.1 Lemma. *For a digraph X on n vertices and $H = H(X)$ its Hermitian adjacency matrix, the* *following are true:*

(i) all eigenvalues of H are real numbers;

(ii) if $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of H corresponding to different eigenvalues.

The eigenvalues of $H(X)$ are the *H-eigenvalues of X* and the spectrum of $H(X)$ is the *H-spectrum of X*. We will denote the H-spectrum by $\sigma_H(X)$ and we will express it as either a multiset of H-eigenvalues or a list of distinct H-eigenvalues with multiplicities in superscripts. We say that digraphs X and Y are *H-cospectral* if $H(X)$ and $H(Y)$ are cospectral matrices. Since $H(X)$ and $H(Y)$ are diagonalizable, X and Y are H-cospectral if and only if $H(X)$ and $H(Y)$ have the same characteristic polynomial; that is

$$\phi(H(X), t) = \phi(H(Y), t).$$

Recall that digraphs X and Y are cospectral (or A-cospectral, if we wish to distinguish between the two matrices) if $A(X)$ and $A(Y)$ have the same characteristic polynomial. To avoid ambiguity, we will refer to eigenvalues and spectrum of X with respect to its adjacency matrix as the *A-eigenvalues* and *A-spectrum*, respectively.

6.2 Characteristic polynomial of H

We first examine one of the coefficients of the characteristic polynomial of $H(X)$ and describe the combinatorial objects that it counts, then we give the formula for the characteristic polynomial of $H(X)$.

6.2.1 Lemma. *For X a digraph and $H = H(X)$ its Hermitian adjacency matrix,*

$$H^2_{u,u} = d(u)$$

where $d(u)$ is the degree of u in the underlying graph of X.

Proof. Since H is Hermitian and has only entries 0, 1 and $\pm i$, we have

$$H_{uv}H_{vu} = H_{uv}\overline{H_{uv}} = 1$$

whenever $H_{uv} \neq 0$. This implies that the (u, u) diagonal entry in H^2 is the degree of u in the underlying graph of X. □

Note that the degree of a vertex x of digraph X in the underlying graph of X is equal to $|N^-_X(x) \cup N^+_X(x)|$. Lemma 6.2.1 immediately gives the following information about the coefficient of t^{n-2} in the characteristic polynomial of a digraph X on n vertices.

6.2.2 Corollary. *The coefficient of t^{n-2} in $\phi(H(X), t)$, is $-e$ where e is the number of edges of the underlying graph of X.*

Proof. Let X be a digraph on n vertices, let Γ be its underlying graph, and let $H := H(X)$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of H. Then the characteristic polynomial of H can be written as

$$\phi(H, t) = \sum_{j=0}^{n} c_j t^{n-j} = (t - \lambda_1) \cdots (t - \lambda_n).$$

Then, the coefficient of t^{n-2} is

$$c_2 = \sum_{1 \le j < k \le n} \lambda_j \lambda_k.$$

Observe that

$$\left(\sum_{j=1}^{n} \lambda_j \right)^2 = \sum_{j=1}^{n} \lambda_j^2 + 2 \sum_{1 \le j < k \le n} \lambda_j \lambda_k = \sum_{j=1}^{n} \lambda_j^2 + 2c_2 = \operatorname{tr}(H^2) + 2c_2$$

The matrix H has all zeroes on the diagonal, and so $\sum_{j=1}^{n} \lambda_j = 0$. Then, we obtain that

$$0 = \operatorname{tr}(H^2) + 2c_2.$$

By Lemma 6.2.1, we have that $\operatorname{tr}(H^2) = \sum_{u \in V(X)} d_\Gamma(u) = 2e$, where e is the number of edges of Γ. Thus $c_2 = -e$. □

Using the definition of matrix determinant, we can write the characteristic polynomial of $H(X)$ in terms of cycles in the underlying graph. Recall the definition of basic digraphs from Section 5.2. Here we will consider basic subdigraph in the digraph of the underlying graph of X. Recall the function R_X defined for a digraph X in Section 3.1. The following result appears in [41] as Theorem 2.8.

6.2.3 Theorem. *[41] Let $X = (V, E)$ be a directed graph with n vertices and $H = H(X)$ be its Hermitian adjacency matrix. Then the characteristic polynomial of H is*

$$\phi(H, t) = \sum_{j=0}^{n} c_j t^j$$

where $c_j = \sum_C i^{2c(C) - n + j + r_-(C) + 2r_+(C)}$, where the sum runs over all basic subdigraphs C of order j in $\vec{D}(\Gamma(X))$ and $r_k(C) = |\{uv \in E(C) \mid R_X(u, v) = k\}|$.

Proof. Let $\Gamma := \Gamma(X)$ be the underlying graph of X. For simplicity, we will let $V(X) = \{1, \dots, n\}$.

Recall that $\phi(H,t) = \det(tI - H)$. Let $M = tI - H$, then

$$\det(M) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}.$$

Suppose $\sigma \in \mathrm{Sym}(n)$ has j fixed points. We let

$$h_\sigma = \mathrm{sgn}(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$$

which is the term contributed by σ in the sum for the determinant. We see that h_σ is a constant multiple of t^j. If the basic digraph associated with σ is not a basic subdigraph of $\vec{D}(\Gamma(X))$, then the term h_σ is 0. Otherwise, let C be the basic subdigraph of $\vec{D}(\Gamma(X))$ corresponding to σ.

Let C be the digraph on vertex set V with arc set $E(C) = \{\ell\sigma(\ell) : \ell \in V\}$. Consider the cyclic decomposition of σ into disjoint cycles. We use $c(C)$ to denote the number of directed cycles in C, which will coincide with the number of cycles of length greater than 1 in the cyclic decomposition of σ. A cycle of length m can be written as the product of $m-1$ transpositions. Then $\mathrm{sgn}(\sigma) = (-1)^e$, where e is a the number of even cycles. Observe that the number of odd cycles is equal to $n - j$ mod 2. Then $e = c(C) + n - j \mod 2$ and we have that $\mathrm{sgn}(\sigma) = (-1)^{n-j+c(C)}$.

For $uv \in E(C)$,

$$M_{u,v} = \begin{cases} -i, & \text{if } R_X(u,v) = + \\ i, & \text{if } R_X(u,v) = -; \text{ and} \\ -1, & \text{if } R_X(u,v) = 1. \end{cases}$$

Then,

$$\begin{aligned} h_\sigma &= (-1)^{n-j+c(C)+r_1(C)}(-i)^{r_+(C)} i^{r_-(C)} \\ &= i^{2(n-j)+2c(C)+2r_1(C)+3r_+(C)+r_-(C)} \\ &= i^{3(n-j)+2c(C)+r_1(C)+2r_+(C)} \\ &= i^{2c(C)-n+j+r_1(C)+2r_+(C)}, \end{aligned}$$

where the third equality is due to $n - j = r_+(C) + r_-(C) + r_1(C)$.

From this, we see that the coefficient of t^j in $\det(M)$ is $\sum_C i^{2c(C)-n+j+r_1(C)+2r_+(C)}$, where the sum runs over all basic subdigraphs of order j in $\vec{D}(\Gamma(X))$. □

We give an example for clarity; Figure 6.1 shows, from left to right, an example of a digraph X, a basic subdigraph C of $\vec{D}\Gamma X$ corresponding to the permutation $\sigma = (123)(4)(56)$, and C with arcs labelled by their contribution to h_σ.

We may verify that the expression for c_{n-2} from Theorem 6.2.3 agrees with the result in Corollary 6.2.2. Using this theorem, we can see the following lemma, which is also given in [41].

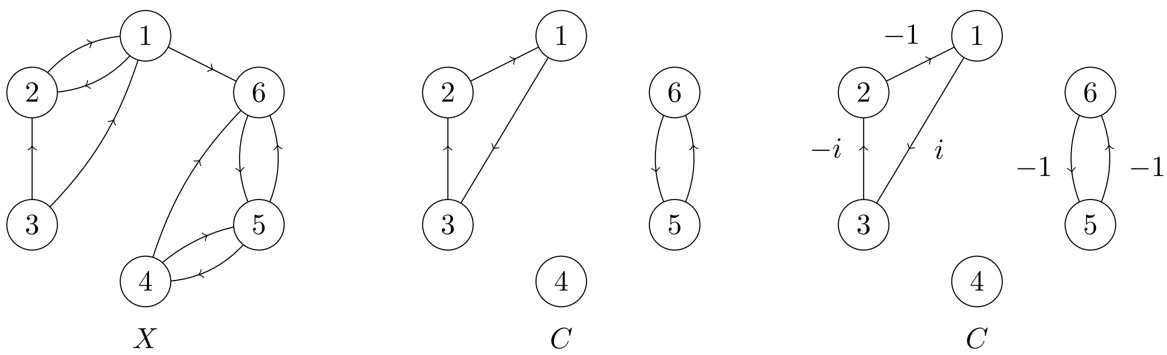


Figure 6.1: An example of a digraph X, a basic subdigraph C of $\vec{D}(\Gamma(X))$, and C with arcs labelled by their contribution to the term in $\det(tI - H(X))$.

6.2.4 Lemma. *[41] Supposed u, v are vertices of a digraph X such that $R_X(u, v) = 1$. If uv is a cut-edge of $\Gamma(X)$, then the spectrum of $H(X)$ is unchanged when $R_X(u, v)$ is changed to $+$ or $-$.*

Proof. Let X_1 be the component of X with edges uv and vu deleted containing vertex u and X_2 be the digraph induced by the vertices $V(X) \setminus V(X_1)$, including v. Let Y_+ be the disjoint union of X_1 and X_2 with the arc uv added. Let Y_- be the disjoint union of X_1 and X_2 with the arc vu added. Let $Y_1 = X$, which is the disjoint union of X_1 and X_2 with both arcs uv and vu added. We will show that the characteristic polynomial of $H(Y_j)$ are equal for $j \in \{+, -, 1\}$, by showing that the contribution of the arcs with both endpoints in $\{u, v\}$ to the coefficients in Theorem 6.2.3 are equal.

Fix $j \in \{+, -, 1\}$ and consider all basic subdigraphs of $\vec{D}(\Gamma(Y_j))$ containing either uv or vu. Since uv is a cut-edge in $\Gamma(X)$, every such basic subdigraph must use both edges. In other words, $\{u, v\}$ is a component consisting of a digon. Then, the contribution of this component to the coefficient in 6.2.3 is either $(-i)i$, $i(-i)$, or $(-1)(-1)$ for $j = +, -, 1$, respectively. In any case, the contribution is equal to 1. □

6.3 Directed walks

Analogous to the results for the adjacency matrix found in standard texts like [5], we may write the (u, v) entry of $H(X)^k$, for a digraph X, as a weighted sum of the walks in $\Gamma(X)$ of length k from u and v. In fact, we will do so in more generality for any matrix M respecting the adjacency of X.

6.3.1 Proposition. *Let X be a digraph and M be a matrix respecting the adjacency of X. For vertices $u, v \in V(X)$ and any positive integer k, the (u, v) entry of the kth power of M is as follows:*

$$\left(M^k\right)_{u,v} = \sum_{W \in \mathcal{W}} \mathrm{wt}(W)$$

where $\mathcal{W}$ is the set of all walks of length k from u to v in $\Gamma(X)$ and for $W = (v_0, \dots, v_k) \in \mathcal{W}$, where $v_0 = u$ and $v_k = v$, the weight is

$$\mathrm{wt}(W) = \prod_{j=0}^{k-1} M_{v_j, v_{j+1}}.$$

Proof. Let $X = (V, E)$. Fix vertices u and v. We proceed by induction. For $k = 1$, the walks of length 1 in $\Gamma(X)$ are precisely the edges of $\Gamma(X)$ and the weight of any such walk is just M_{uv}.

Assume that the statement is true for $k \le m$ and consider when $k = m + 1$. Then,

$$\left(M^{m+1}\right)_{u,v} = (M^m M)_{u,v}$$

and $\left(M^{m+1}\right)_{u,v}$ is equal to the product of the uth row of M^m with the vth column of M. Then

$$\left(M^{m+1}\right)_{u,v} = \sum_{w \in V} (M^m)_{uw} M_{wv}.$$

Let $h_w = (M^m)_{uw} M_{wv}$ for $w \in V$. If $w \nsim v$ in $\Gamma(X)$, then $h_w = 0$. Otherwise, $w \sim v \in \Gamma(X)$ and

$$h_w = \sum_{W \in \mathcal{W}_w} wt(W)$$

where $\mathcal{W}_w$ is the set of all walks from u to v of length $m+1$ in $\Gamma(X)$, where w is the penultimate, or mth, vertex of W. Since every walk from u to v of length $m + 1$ is in $\mathcal{W}_w$ for a unique w, we have that $\cup_{w \in V} \mathcal{W}_w = \mathcal{W}$, where $\mathcal{W}$ is the set of all walks of length m from u to v in $\Gamma(X)$. Then we have that

$$\sum_{w \in V} h_w = \sum_{W \in \mathcal{W}} \mathrm{wt}(W)$$

which concludes the induction. □

We will use Proposition 6.3.1 to find an expression for $\mathrm{tr}(H(X)^3)$ in terms of numbers of sub-digraphs isomorphic to certain types of triangles.

6.3.2 Proposition. *For any digraph X, we have $\mathrm{tr}(H(X)^3) = 6(x_4 + x_5 + x_7 - x_3)$ where x_j is the number of copies of X_j as an induced sub-digraph of X and X_j are digraphs as shown in Figure 6.2.*

Proof. Let $X = (V, E)$ and $H := H(X)$. We apply Proposition 6.3.1 to obtain:

$$\mathrm{tr}(H^3) = \sum_{v \in V} (H^3)_{v,v} = \sum_{v \in V} \sum_{W \in \mathcal{W}_v} \mathrm{wt}(W)$$

where W_v is the set of closed walks at v of length 3 in $\Gamma(X)$ and wt as defined in Proposition 6.3.1. Every closed walk of length 3 has C_3 as its underlying graph. For simplicity, we will refer to any digraph with C_3 as the underlying graph as a *triangle* and we will call a sub-digraph of X isomorphic to a triangle a *triangle of* X. In Figure 6.2, we enumerate all non-isomorphic triangles.

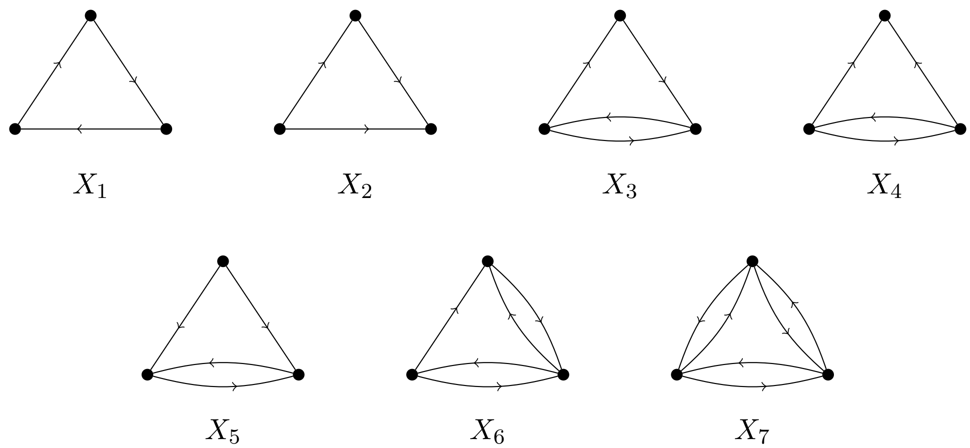


Figure 6.2: All non-isomorphic digraphs on 3 vertices whose underlying graph is a 3-cycle.

Let $X_1, \dots, X_7$ be the digraphs as shown in Figure 6.2. If C is an induced triangle of X, then $C \cong X_j$ for some $j \in 1, \dots, 7$; each such sub-digraph C of X contributes 6 closed walks of length 3, two choices for the direction of traversal and 3 choices for the starting vertex. We now consider closed walks of length 3 on X_j. Observe that such a walk W will have $\mathrm{wt}(W)$ depending only on whether it is a clockwise or counter-clockwise walk with respect to the drawing of X_j in Figure 6.2, and not on the choice of the starting vertex. We summarize the information about the weights of walks in X in Table 6.1

j	$\mathrm{wt}(W)$ W a clockwise walk in X_j	$\mathrm{wt}(W)$ W a counter-clockwise walk in X_j
1	$-i$	i
2	i	$-i$
3	-1	-1
4	1	1
5	1	1
6	i	$-i$
7	1	1

Table 6.1: Weights of closed walks of length 3 on X_j, for $j = 1, \dots, 7$.

Observe that for $j = 1, 2, 6$, the walks on opposite direction will cancel in the sum for $\mathrm{tr}(H^3)$. Thus,

$$\mathrm{tr}(H^3) = 6(x_4 + x_5 + x_7 - x_3)$$

where x_j is the copies of X_j as an induced sub-digraph of X. We may observe that X_j for $j = 4, 5, 3, 7$ are precisely the triangles with an even number of arcs which are not incident to digons. □

6.4 Connectivity

As in the case of the adjacency matrix, it is easy to see that $H(X)$ is an irreducible matrix if and only if $\Gamma(X)$ is connected. The H-spectrum of a digraph X does not determine if X is strongly-connected, weakly-connected or not connected. In Figure 6.3, we give an example of three digraphs; X_1, X_2 and X_3. For $j = 1, 2, 3$, we compute that

$$\phi(H(X_j), t) = t^5 - 5t^3 + 2t^2 + 2t.$$

Observe that X_1 is strongly connected, X_2 is weakly connected but not strongly connected, and X_3 is not weakly connected.

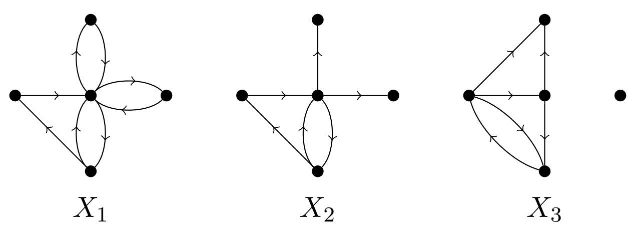


Figure 6.3: H-cospectral digraphs on 5 vertices with different connectivity properties.

6.5 H-Eigenvalues symmetric about 0

We have shown in Section 5.4 that, for a digraph X, its A-eigenvalues are symmetric about 0 if and only if $\Gamma(X)$ is bipartite. Here, we consider digraphs X whose H-eigenvalues are symmetric about 0. Note that the eigenvalues of H are real and can be ordered as $\lambda_1 \geq \cdots \geq \lambda_n$, they are symmetric about 0 if and only if $\lambda_j = -\lambda_{n-j+1}$ for $j = 1, \ldots, n$. The following proposition also appears in [41] as Corollary 2.13.

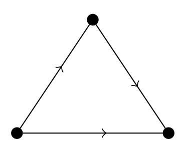
6.5.1 Proposition. *[41] For a digraph X, if $\Gamma(X)$ is bipartite, then the H-eigenvalues are symmetric about 0.*

Proof. First we observe that if a basic digraph on n vertices of order j has $n - j$ odd, then it must contain an odd cycle. Let X be a digraph on n vertices. If $\Gamma(X)$ is bipartite, then $\vec{D}(\Gamma(X))$ contains no odd cycle, and so has no basic subdigraph of order j such that $n - j$ is odd. Thus, in $\phi(H(X), t)$, the coefficient of all t^j where $n - j$ is odd is 0. Let $m, a \in \mathbb{Z}_{\geq 0}$ such that $n = 2m + a$, where a is the remainder of n when divided by 2. Then

$$\phi(H(X),t) = \sum_{j=0}^{m} a_{n-2j} t^{n-2j} = t^a \sum_{j=0}^{m} a_{2m-2j} t^{2m-2j} = t^a f(t^2)$$

where $f(x) = \sum_{j=0}^{m} a_{2m-2j} x^{m-j}$. If $\lambda_1, \ldots, \lambda_m$ are the roots of f, then the roots of $\phi(H(X),t)$ are $\pm\lambda_j$ for $j = 1, \ldots, m$ and one additional 0 if $a = 1$. Then the H-eigenvalues of X are symmetric about 0. □

The converse to Proposition 6.5.1 is not true. For example, the digraph $\widetilde{C_3}$ in Figure 6.4 has eigenvalues $\pm\sqrt{3}, 0$.



$\widetilde{C_3}$

Figure 6.4: A digraph on 3 vertices with H-eigenvalues symmetric about 0 whose underlying graph is not bipartite.

In fact, every oriented graph has H-eigenvalues symmetric about 0. Proposition 6.5.2 below appears in [41] as Corollary 2.13. We give a different proof here.

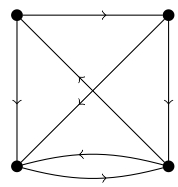
6.5.2 Proposition. *[41] If X is an oriented digraph, then the H-eigenvalues of X are symmetric about 0.*

Proof. Let X be an oriented graph on n vertices and $H := H(X)$. Let the eigenvalues of H be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let $M = iH$. Then, M is a skew-symmetric matrix with purely imaginary eigenvalues $i\lambda_1, \ldots, i\lambda_n$. Since M has rational entries, $\phi(M,t)$ is a polynomial with rational coefficients. Then, by Theorem B.1.1, every eigenvalue μ of M occurs with the same multiplicity as its complex conjugate. Since the eigenvalues of M are purely imaginary, each eigenvalue $\mu \in \{i\lambda_1, \ldots, i\lambda_n\}$ occurs with the same multiplicity as $-\mu$. Then $i\lambda_j = -i\lambda_{n-j+1}$, and H has eigenvalues symmetric about 0. □

There are digraphs with H-eigenvalues symmetric about 0, which are neither oriented nor have bipartite underlying graphs. Computationally, we verified that there are no such graphs on fewer than 4 vertices. Using a Sage computation, on 4 vertices, we find that there are exactly seven H-cospectral classes with H-spectrum symmetric about 0, which contains digraphs which are not oriented, whose underlying graph is not bipartite. One of these classes contains exclusively such digraphs; this class contains 15 non-isomorphic digraphs all of which have underlying graphs isomorphic to K_4, and which contain at least one digon. One graph from this class, D, is shown in Figure 6.5. The characteristic polynomial of D is

$$\phi(H(D), t) = t^4 - 6t^2 + 5$$

which has its roots symmetric about 0.

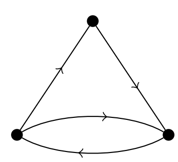


D

Figure 6.5: An example of a digraph on 4 vertices, having H-eigenvalues symmetric about 0, but not oriented and not bipartite.

6.6 Largest H-eigenvalue

For a digraph X, let the eigenvalues of $H := H(X)$ be $\lambda_1 \geq \cdots \geq \lambda_n$. Note that since H is not a matrix with non-negative entries, there is no analogue of the Perron value of the adjacency matrix and the properties of λ_1 are highly unintuitive. Figure 6.6 shows a strongly connected digraph K_3' on 3 vertices with H-eigenvalues $\{1^{(2)}, -2\}$. This shows that, in general, λ_1 is not necessarily simple or largest in magnitude.



K_3'

Figure 6.6: Digraph K_3'.

Instead of considering the largest eigenvalue, we may consider the largest eigenvalue in absolute value (the *spectral radius*), for which we find a bound that is analogous to the adjacency matrix. The spectral radius of matrix M is denoted $\rho(M)$ and

$$\rho(M) = \max\{|\lambda| \mid \lambda \text{ an eigenvalue of } M\}.$$

6.6.1 Proposition. *For a weakly connected digraph X with H-eigenvalues $\lambda_1, \ldots, \lambda_n$, $\rho(H(X)) \leq \Delta(\Gamma(X))$. If equality holds, then $\Gamma(X)$ is a $\Delta(\Gamma(X))$-regular graph and there exists an eigenvector of $H(X)$ corresponding to eigenvalue $\rho(H(X))$ with entries in $\{\pm 1, \pm i\}$.*

Proof. Let $H := H(X)$ and let λ be an eigenvalue of H with eigenvector $\mathbf{x}$. Choose $v \in V(X)$ such that $|\mathbf{x}(v)|$ is maximal. Now we consider the v entry of $H\mathbf{x}$. For simplicity of notation, we will write $N(v) := N_X^-(v) \cap N_X^+(v)$. We obtain

$$(Hx)(v) = \sum_{u \in N(v)} \mathbf{x}(u) + i \sum_{w \in N_X^+(v) \setminus N(v)} \mathbf{x}(w) - i \sum_{x \in N_X^-(v) \setminus N(v)} \mathbf{x}(x).$$

On the other hand, $(H\mathbf{x})(v) = \lambda \mathbf{x}(v)$. Then

$$\begin{aligned} |\lambda \mathbf{x}(v)| = |(H\mathbf{x})(v)| &\leq \sum_{u \in N(v)} |\mathbf{x}(u)| + |i| \sum_{w \in N_X^+(v) \setminus N(v)} |\mathbf{x}(w)| + |-i| \sum_{y \in N_X^-(v) \setminus N(v)} |\mathbf{x}(y)| \\ &\leq \sum_{u \in N(v)} |\mathbf{x}(v)| + \sum_{w \in N_X^+(v) \setminus N(v)} |\mathbf{x}(v)| + \sum_{y \in N_X^-(v) \setminus N(v)} |\mathbf{x}(v)| \\ &= \deg_{\Gamma(X)}(v)|\mathbf{x}(v)| \\ &\leq \Delta(\Gamma(X))|\mathbf{x}(v)|. \end{aligned} \tag{6.1}$$

From this, we obtain that $|\lambda| \leq \Delta(\Gamma(X))$.

If $\rho(H(X)) = \Delta(\Gamma(X))$, then all of the inequalities in (6.1) must hold with equality. From the last inequality of (6.1), we see that every vertex of X has degree $\Delta(\Gamma(X))$ in $\Gamma(X)$. If the second inequality of (6.1) hold with equality, we obtain that all vertices $|\mathbf{x}(z)| = |\mathbf{x}(v)|$ for all $z \in N_X^-(v) \cup N_X^+(v)$. The first inequality of (6.1) follows from the triangle inequality for sums of complex numbers, and so equality holds if and only if every complex number in Z has the same argument as $\lambda \mathbf{x}(v)$, where

$$Z = \{\mathbf{x}(u) \mid u \in N(v)\} \cup \{i\mathbf{x}(w) \mid w \in N_X^+(v) \setminus N(v)\} \cup \{-i\mathbf{x}(y) \mid y \in N_X^-(v) \setminus N(v)\}.$$

Since every element of Z also have the same absolute value as $\mathbf{x}(v)$, we have obtained that

$$\mathbf{x}(v) = \mathbf{x}(u) = i\mathbf{x}(w) = -i\mathbf{x}(y)$$

for every $u \in N(v)$, $w \in N_X^+(v) \setminus N(v)$ and $y \in N_X^-(v) \setminus N(v)$. Suppose v has an out-neighbour w which is not also an in-neighbour. Since $\mathbf{x}(v) = i\mathbf{x}(w)$, we have that $|\mathbf{x}(w)|$ is also maximal. The choice of v was arbitrary amongst all vertices whose entry in $\mathbf{x}$ has maximal absolute value, so we may apply the above argument to any vertex in $N_X^-(v) \cup N_X^+(v)$. Since X is weakly connected, we obtain that $\mathbf{x}(z) \in \{\pm 1, \pm i\}$ for any vertex z of X. □

Note that $\rho(H(X))$, for a digraph X, can be smaller than $\delta(\Gamma(X))$. For example, see digraph $\widetilde{C_3}$ in Figure 6.4 which has eigenvalues $\pm\sqrt{3}, 0$ and $\delta(\Gamma(\widetilde{C_3})) = 2$.

6.7 Diameter

Since the diameter of a graph is reflected in the spectrum of its adjacency matrix, it is natural to consider similar questions for the Hermitian adjacency matrix. If the eigenvalues of the adjacency matrix of a graph are all equal to zero, then the graph has no edges. This is not true of digraphs under the adjacency matrix, since every acyclic digraph is A-cospectral to the empty graph on the same number of vertices. However, an analogous statement is true for the Hermitian adjacency matrix.

6.7.1 Proposition. *For a digraph X, the matrix $H(X)$ has all eigenvalues equal to 0 if and only if $E(X) = \emptyset$.*

Proof. Let n be the number of vertices of X. We may assume $n \geq 2$. Instead of using a Hermitian operator argument, we will derive this as an easy application of Corollary 6.2.2. It is clear that X has no arcs if and only if $\Gamma(X)$ has no edges. By Corollary 6.2.2, this occurs if and only if $c_{n-2} = 0$. If all eigenvalues of $H(X)$ are 0, then $\phi(H(X), t) = t^n$ and $c_{n-2} = 0$. Conversely, if $c_{n-2} = 0$, then $H(X)$ is the all zero matrix and so has all eigenvalues equal to 0. $\square$

Let M be a square Hermitian matrix. Let $\mathcal{M}$ be the matrix algebra generated by $I, M, M^2, M^3, \ldots$ and let $\psi(M, t)$ be the minimal polynomial of M. Then, $\deg(\psi(M, t)) = \dim(\mathcal{M}) + 1$. The degree of $\psi(M, t)$ is the number of distinct eigenvalues of M.

If $M = A(X)$, the adjacency matrix of a digraph, we obtain that the diameter of the graph is at most the dimension of $\mathcal{M}$, whence we obtain a bound on the diameter in terms of the number of distinct eigenvalues.

The case for the Hermitian matrix, where $M = H(X)$ for some digraph X, is very different. It would be convenient if the following were true: if $u, v \in V(X)$ are at distance k, then $(H(X)^k)_{uv} \neq 0$. This would show that the if d is the diameter of X, then $H(X)$ has at least distinct $d + 1$ eigenvalues. However this is untrue. For example, consider the modified directed cycle $\widetilde{C_n}$, obtained from a directed cycle by changing the orientation on one arc to the opposite direction. We may recall that the digraph $\widetilde{C_3}$ was considered in Section 6.5. Here we look at $\widetilde{C_4}$, shown in Figure 6.7.

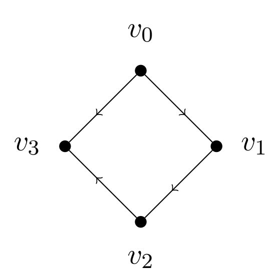


Figure 6.7: Digraph $\widetilde{C_4}$, obtained from C_4 by reversing one arc.

We can compute that

$$\phi(H(\widetilde{C_4}), t) = t^4 - 4t^2 + 4 = (t^2 - 2)^2$$

and we see that $\widetilde{C_4}$ has exactly two distinct H-eigenvalues, but the diameter of $\Gamma(\widetilde{C_4})$ is 2. In fact, we can show that, in general, there does not exist a function of the number of distinct H-eigenvalues which bounds the diameter of the underlying graph.

6.7.2 Proposition. *There exists an infinite family of digraphs $\{X_j\}_{j=1}^{\infty}$ such that the diameter of ΓX_j goes to ∞ as $j \to \infty$ and the number of distinct H-eigenvalues of X_j is a constant C for all $j \geq 0$.*

Proof. We use the necklace digraphs in Section 8.6, by taking X_j to be N_{j-2}. Each digraph N_n has $\Gamma(N_n)$ with diameter n and $H(N_n)$ has exactly 3 distinct eigenvalues. $\square$

6.8 Eulerian digraphs

6.8.1 Proposition. *Let X be a digraph. Then $H(X)$ has the all-ones vector as an eigenvector if and only if $G(X)$ is regular and $D(X)$ is eulerian.*

Proof. Suppose $D(X)$ is eulerian, then the sum along any row of $H(D(X))$ is equal to 0 and so $\mathbf{1}$ is an eigenvector for $H(D(X))$ of eigenvalue 0. If $G(X)$ is regular, then $\mathbf{1}$ is an eigenvector of $H(G(X)) = A(G(X))$ with eigenvalue equal to the valency of $G(X)$, say k. Then

$$H(X)\mathbf{1} = H(D(X))\mathbf{1} + H(G(X))\mathbf{1} = k\mathbf{1}.$$

For the other direction, suppose that $H(X)\mathbf{1} = \gamma\mathbf{1}$ for some $\gamma \in \mathbb{R}$. The row sum along the row indexed by x is equal to $d + (s-t)i$ where $d = d(x)$, $s = d^-(x)$, and $t = d^+(x)$. Then, since γ is a real number, we must have that $s = t$ and $d = \gamma$. $\square$

6.9 Computations on small digraphs

Computation on all isomorphism classes of digraphs of orders 2, 3, 4, 5, and 6 were carried out using Sage open-source mathematical software system [58], for both the adjacency matrix and the Hermitian adjacency matrix. We include some data here to give an idea of how the Hermitian adjacency matrix behaves compared to the adjacency matrix, on small digraphs. We refer to the set of all digraphs that are H-cospectral to a given digraph as an *H-cospectral class.* A digraph is *determined by its H-spectrum* if every digraph that is H-cospectral with it is also isomorphic to it.

Order	2	3	4	5	6
Number of digraphs	3	16	218	9608	1540944
Number of distinct characteristic polynomials	2	6	27	275	10920
Number of H-cospectral classes such that:					
a) characteristic polynomial is irreducible over $\mathbb{Q}$	0	0	0	0	6
b) characteristic polynomial is square-free	1	3	14	214	9980
Maximum size of a H-cospectral class	2	6	21	158	1338
Number of digraphs determined by H-spectrum	1	2	3	5	16
Number of H-cospectral classes containing:					
a) no graphs	0	2	16	242	10769
b) only graphs	1	1	1	1	1
c) at least one graph and a digraph	1	3	10	32	150

Table 6.2: The H-spectra of small digraphs.

Order	2	3	4	5	6
Number of digraphs	3	16	218	9608	1540944
Number of distinct characteristic polynomials	2	7	46	718	35239
Number of A-cospectral classes such that:					
a) characteristic polynomial is irreducible over $\mathbb{Q}$	0	1	12	277	19392
b) characteristic polynomial is square-free	1	5	36	625	33146
Maximum size of a A-cospectral class	2	6	42	592	15842
Number of digraphs determined by spectrum	1	5	23	166	2317
Number of A-cospectral classes containing:					
a) no graphs	0	3	35	685	35088
b) only graphs	1	2	5	15	69
c) at least one graph and a digraph	1	2	6	18	82

Table 6.3: The adjacency matrix spectra small digraphs.

6.10 Digraphs with H-eigenvalues between -1 and 1

In the previous sections, we have seen that the H-eigenvalues of digraphs behave differently and somewhat strangely compared with the A-eigenvalues of graphs. It appears that graph invariants like diameter, minimum degree, and number of connect components cannot be bounded by the H-spectrum. However, since H is a Hermitian, we may use interlacing theorems, which we could not do for the adjacency matrix of digraphs. Using interlacing, we will characterize all digraph with all H-eigenvalues lying between -1 and 1. First we will look at a special case where all H-eigenvalues are equal to 1 or -1, then we will look at the general case. Note that mX where X is a digraph denotes the union of m disjoint copies of X.

6.10.1 Theorem. *A digraph X has the property that $\lambda \in \{-1, 1\}$ for each H-eigenvalue λ of X if and only if $\Gamma(X) \cong mK_2$ for some m.*

Proof. Let X be a digraph on n vertices having the property as in the statement of the theorem. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of $H(X)$. Then $\lambda_i \in \{-1, 1\}$ for $i = 1, \ldots, n$ by assumption. Since the H-eigenvalues of X must sum to $\operatorname{tr}(H) = 0$, the multiplicity of 1 and -1 are equal and X has an even number vertices, say $n = 2m$. Observe that

$$\operatorname{tr}(H(X)^2) = \sum_{i=1}^{n} \lambda_i^2 = n = 2m.$$

Lemma 6.2.1 gives that $\Gamma(X)$ has m edges.

If X has an isolated vertex, then X will have 0 as an H-eigenvalue. Thus, every vertex must have degree at least 1 in $\Gamma(X)$. Then $d_{\Gamma(X)}(v) = 1$ for every vertex v and so $\Gamma(X) \cong mK_2$. □

6.10.2 Theorem. *For a digraph X the following are equivalent:*

(a) $\sigma_H(X) \subseteq (-\sqrt{2}, \sqrt{2})$;

(b) $\sigma_H(X) \subseteq [-1, 1]$; and

(c) every component of X is either a single arc, a digon or an isolated vertex.

Proof. Let X be a digraph on n vertices with H-eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. By assumption $\sqrt{2} > \lambda_1$ and $\lambda_n > -\sqrt{2}$. Let Y be an induced subdigraph of X on 3 vertices and let $\mu_1 \geq \mu_2 \geq \mu_3$ be the H-eigenvalues of Y. Applying the interlacing theorem, we obtain that

$$-\sqrt{2} < \mu_i < \sqrt{2}$$

for $i = 1, 2, 3$. There are exactly 16 digraphs on 3 vertices and so we may compute their H-eigenvalues and determine which digraphs on 3 vertices have all eigenvalues between strictly $-\sqrt{2}$

and $\sqrt{2}$. The digraphs on 3 vertices grouped by H-cospectral classes are found in Section C.1 of the Appendix. Following the naming of the digraphs in Table C.1, we see that Y is isomorphic to one of E_3, Z_5 and Z_6. In other words, Y is either the empty graph on 3 vertices or a graph consisting of an isolated vertex and either one arc or one digon.

Since the choice of Y was arbitrary, the above holds for every induced subdigraph of X on 3 vertices. Then, $\Gamma(X)$ does not vertices of degree 2 or more. Thus $\Delta(\Gamma(X)) \leq 1$ and so $\Gamma(X)$ consists of the union of disjoint copies of K_2 and isolated vertices. The result follows. $\square$

Note that Theorem 6.10.2 implies that X has H-spectrum equal to $\{1^{(m)}, 0^{(k)}, (-1)^{(m)}\}$, where k is the number of isolated vertices and m is the number of components consisting of a single arc or a digon. Since $\vec{P}_2$ and K_2 are H-cospectral, we see that no better characterization of X is possible. We may also see this using an iterated application of Proposition 7.2.7.

6.10.3 Theorem. *If X has the property that $\sigma_H(X) \subseteq (-\sqrt{3}, \sqrt{3})$, then every component of $\Gamma(X)$ is either a path of length at most 3, a 4-cycle or a 5-cycle.*

Proof. Let the H-eigenvalues of X be $\lambda_1 \geq \cdots \geq \lambda_n$ and let $\Gamma := \Gamma(X)$. By assumption $\sqrt{3} > \lambda_1$ and $\lambda_n > -\sqrt{3}$. Again, we consider Y an induced subdigraph of X on 3 vertices and let $\mu_1 \geq \mu_2 \geq \mu_3$ be the H-eigenvalues of Y. Applying the interlacing theorem, we obtain that

$$-\sqrt{3} < \mu_i < \sqrt{3}$$

for $i = 1, 2, 3$. Again, we consult Table C.1 to see that $\Gamma(Y)$ is isomorphic to one of P_3, $K_2 + K + 1$ or E_3. In other words, $\Gamma(Y)$ is acyclic. Since the choice of Y was arbitrary, the above holds for every induced subdigraph of X on 3 vertices. Then, Γ does not contain a triangle.

If Γ contains a vertex of degree at least 3, then Γ contains either a star on 4 vertices or a triangle. We have already shown that Γ does not contain a triangle, so it must contain a star on 4 vertices of X, say x_1, x_2, x_3, x_4. In Theorem 8.5.1, we show that all digraphs on m vertices whose underlying graph is isomorphic to the star have largest H-eigenvalue equal to $\sqrt{m-1}$. Then, the subdigraph of X induced by $\{x_j\}_{j=1}^4$ has maximum H-eigenvalue $\sqrt{3}$, which is impossible by interlacing. Thus every vertex in Γ has degree at most 2.

We have shown that the components of Γ are paths and cycles. Suppose $W \subseteq V(X)$ induced a path of length 4 in Γ. The spectral radius of P_5 is $\sqrt{3}$ and, since all digraphs with P_5 as its underlying graph are H-cospectral by Lemma 6.2.4, W induced a subgraph of X with maximum H-eigenvalue equal to $\sqrt{3}$, which is again impossible. Every cycle of length at least 6 contains an induced path of length 4 and every path of length at least 4 contains an induced path of length 4, so Γ does not have any cycles of length greater than 5 or paths of length greater than 3. $\square$

It is interesting to consider for which values of α the number of weakly connected digraphs whose H-eigenvalues will all lie in the interval $(-\alpha, \alpha)$ or $[-\alpha, \alpha]$ will be finite. We see from Theorem 6.10.3 that there are only finitely many weakly connected digraphs with all H-eigenvalues in the interval $(-\sqrt{3}, \sqrt{3})$. It may be true that the same holds for every α with $0 \le \alpha < 2$. The directed paths show that there are infinitely many weakly connected digraphs whose H-eigenvalues will all lie in the interval $(-2, 2)$, thus $\alpha = 2$ will not give the same conclusion.

6.11 Other applications of eigenvalue interlacing

Interlacing is a powerful tool in algebraic graph theory. Theorem 4.1.2 implies that the eigenvalues of any induced subdigraph interlace those of the digraph itself.

To see a simple example, we find the eigenvalues of the directed cycle D_m in Section 8.1 and we see that D_m has at least one H-eigenvalue equal to 0, and, as an oriented graph, its H-spectrum is symmetric about 0. We let $\eta^+(M)$ denote the number of non-negative eigenvalues of matrix M and $\eta^-(M)$ denote the number of non-positive eigenvalues of matrix M.

6.11.1 Lemma. *If digraph X has D_m as an induced subdigraph, then $\eta^+(H(X)) \ge \lceil \frac{m}{2} \rceil$ and $\eta^-(H(X)) \ge \lfloor \frac{m}{2} \rfloor$.*

Proof. Let $\lambda_1 \ge \cdots \ge \lambda_n$ be the eigenvalues of $H(X)$. By interlacing, we see that $\lambda_{\lceil \frac{m}{2} \rceil} \ge 0 \ge \lambda_{n - \lceil \frac{m}{2} \rceil + 1}$ and the result follows. □

This is not a very strong statement as there are many graphs that satisfy the conclusion, however, it shows, for instance, that any digraph H-cospectral with K_n has no induced directed cycle of length 3 or more, since the H-spectrum of K_n contains only 1 non-negative eigenvalue.

Similarly, the Cveković bound (see [18]) for the largest independent set of a graph extends to the class of digraphs with the Hermitian adjacency matrix.

6.11.2 Lemma. *If X has an independent set of size α, then $\eta^+(H(X)) \ge \alpha$ and $\eta^-(H(X)) \ge \alpha$.*

Proof. Let $\lambda_1 \ge \cdots \ge \lambda_n$ be the eigenvalues of $H(X)$. By interlacing, we see that $\lambda_\alpha \ge 0$ and so $H(X)$ has at least α non-negative eigenvalues. Applying the same argument to $-H(X)$ will show that there are at least α non-positive eigenvalues as well. □

We can also obtain a spectral bound of the maximum induced transitive tournament of a digraph. Gregory, Kirkland and Shader bounded the maximum spectral radius of a skew-symmetric matrix and classified the matrices where equality holds in [30]. We will state a special case of their theorem here, as it applies to the Hermitian adjacency matrix of oriented graphs. Two tournaments are *switching-equivalent* if one can be obtained from the other by reversing all arcs across an edge-cut of the underlying graph. The transitive tournament is defined in Section 8.4.

6.11.3 Theorem. *[30] If X is an oriented graph of order n, then*

$$\lambda_1(H(X)) \le \cot\left(\frac{\pi}{2n}\right).$$

Equality holds if and only if X is switching-equivalent to T_n the transitive tournament of order n.

The following lemma is another immediate consequence of interlacing.

6.11.4 Lemma. *If X is a digraph with an induced subdigraph isomorphic to T_m, then* $\lambda_1(H(X)) \geq \cot\left(\frac{\pi}{2m}\right)$.

In other words, if $m > \frac{\pi}{2\cot^{-1}(\lambda_1(H(X)))}$ then X contains no induced subdigraph switching-equivalent to T_m.

Chapter 7

Cospectrality for the Hermitian adjacency matrix

In this chapter, we study properties of digraphs that are H-cospectral and operations on digraphs which preserve the H-spectrum. In particular, we are motivated to consider digraph operations that preserve the H-spectrum and preserve the underlying graph.

By the computation on small digraphs, as recorded in Tables 6.3 and 6.2, we see that, for small digraphs, the number of H-cospectral classes is smaller than the number of A-cospectral classes on the same number of vertices. Since the support of the Hermitian adjacency matrix of digraph X is the adjacency matrix of the underlying graph of X, we may expect the H-spectrum of X to capture some information about the underlying graph.

By contrast, any two acyclic digraphs on the same number of vertices are A-cospectral; in particular, there are A-cospectral digraphs on 2 vertices with non-isomorphic underlying graphs, where as the smallest pair of H-cospectral digraphs with non-isomorphic underlying graphs have 4 vertices (shown in Figure 7.1).

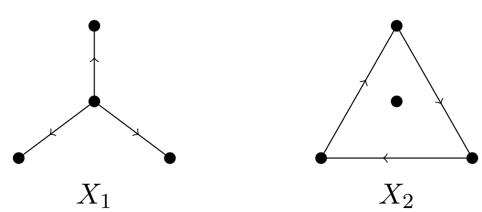


Figure 7.1: The smallest pair of H-cospectral digraphs with non-isomorphic underlying graphs.

There are many cases of H-cospectral digraphs having isomorphic underlying graphs. In Chapter 8, we will see that all orientations of the cycle of odd order n are pairwise H-cospectral and all digraphs with the star of order n as the underlying graph are pairwise H-cospectral. From Table C.1, we see that every pair of H-cospectral digraphs on 3 vertices have the same underlying digraph.

We will try to explain the spectral information about the underlying graph by looking at a few

H-spectrum preserving operations which do not change the underlying graph.

7.1 Converses and local reversal

7.1.1 Lemma. *If X is a digraph and X^C is its converse, then $H(X^C) = H(X)^T = \overline{H(X)}$.*

Proof. It follows from the definition of converse that

$$H(X^C)_{uv} = \begin{cases} 1 & \text{if } uv \text{ and } vu \in E(X); \\ -i & \text{if } uv \in E(X) \text{ and } vu \notin E(X); \\ i & \text{if } uv \notin E(X) \text{ and } vu \in E(X), \text{ and;} \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $H(X^C) = H(X)^T$. The second equality follows from the definition of H. □

7.1.2 Corollary. *A digraph X and its converse are H-cospectral.*

Proof. A standard fact in linear algebra gives that $\det(A) = \det(A^T)$ for any square matrix A. We see the characteristic polynomial of $H(X^C)$ in the variable t is

$$\begin{aligned} \phi(H(X^C), t) &= \det(tI - H(X^C)) \\ &= \det(tI - H(X)^T) \\ &= \det((tI - H(X))^T) \\ &= \det(tI - H(X)) \\ &= \phi(H(X), t) \end{aligned}$$

which gives that $H(X)$ and $H(X^C)$ are cospectral. □

Inspired by this, we now define a local operation on a digraph which will also preserve the spectrum with respect to the Hermitian adjacency matrix. For a digraph X and a vertex v of X, the *local reversal of X at v* is the graph obtained from X by replacing every arc xy incident with v by its converse yx. We can extend this to the *local reversal of X at $S \subset V$* by taking the local reversal at v for each $v \in S$. Observe that the order does not matter. If arc xy is incident to two vertices of S, then it is unchanged in the local reversal at S. We denote by $\delta(S)$ the arcs with exactly one end in S. Note that this operation generalizes the concept of switching-equivalence, defined earlier for tournaments.

7.1.3 Proposition. *If X is a digraph and $S \subset V(X)$ such that $\delta(S)$ contains no digon, then X and the local reversal of X at S are H-cospectral.*

Proof. Let X' be the local reversal of H at S. Let M be a diagonal matrix indexed by the vertices of X given by

$$M_{u,u} = \begin{cases} -1, & \text{if } u \in S; \\ 1, & \text{otherwise.} \end{cases}$$

Consider $MH(X)M$. Applying M on the left of $H(X)$ changes the sign for all columns indexed by vertices of S and applying M on the right changes the sign of all rows index by vertices of S. Then

$$\begin{aligned} (MH(X)M)_{u,x} &= \begin{cases} -H(X)_{u,x}, & \text{if } S \text{ contains exactly one of } u, x; \\ H(X), & \text{otherwise} \end{cases} \\ &= H(X') \end{aligned}$$

since $\delta(S)$ contains no digons. Since $M = M^{-1}$, the matrices $H(X)$ and $H(X')$ are similar and hence cospectral. □

7.2 Operations preserving spectrum

We will use the following theorem about determinants found in [53, Theorem 3.1.3].

7.2.1 Theorem. *Suppose u is a row vector, v is a column vector, a is a matrix entry and A is a square matrix of order at least 2. Then*

$$\begin{vmatrix} A & v \\ u & a \end{vmatrix} = a \det(A) - u(\operatorname{adj} A)v.$$

First, we look at the characteristic polynomial of a directed star with respect to the Hermitian adjacency matrix. In Chapter 8, we find that any pairs of graphs, both of which have a star of the same order as the underlying graph, have the same spectrum with respect to the Hermitian adjacency matrix. Here, we will first use Theorem 7.2.1 to find the H-characteristic polynomial of a directed star, then we will use it to look at when an operation on a graph preserves the H-characteristic polynomial.

Let S_n be the orientation of the star on n vertices where every arc is oriented towards the center vertex. Then $H(S_n)$ the Hermitian adjacency matrix of S_n is

$$H(S_n) = \begin{pmatrix} 0_{n-1 \times n-1} & i\mathbf{1} \\ -i\mathbf{1}^T & 0 \end{pmatrix}.$$

7.2.2 Theorem. *The characteristic polynomial of $H(S_n)$ is $t^{n-2}(t^2 - (n-1))$.*

Proof. We will find the characteristic polynomial of $H(S_n)$ by taking the following determinant:

$$s_n = \begin{vmatrix} t & 0 & \dots & 0 & i \\ 0 & t & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \ddots & i \\ -i & \dots & \dots & -i & t \end{vmatrix} = \begin{vmatrix} tI_{n-1} & i\mathbf{1} \\ -i\mathbf{1}^T & t \end{vmatrix}.$$

From Theorem 7.2.1, we have that

$$s_n = t\det(tI_{n-1}) - (-i)\mathbf{1}^T \operatorname{adj}(tI_{n-1})(i)\mathbf{1} = t^n - \mathbf{1}^T \operatorname{adj}(tI_{n-1})\mathbf{1}.$$

Every minor A_{jk} of the identity matrix is 1 when $j = k$ and 0 when $j \neq k$, since the submatrix contains a row of zeros. Then every minor M_{jk} of $M = tI_{n-1}$ is t^{n-2} when $j = k$ and 0 when $j \neq k$. We obtain

$$s_n = t \cdot t^{n-1} - (n-1)t^{n-2} = t^{n-2}(t^2 - (n-1))$$

as claimed. □

Observe that the same proof would hold when one of the arcs is changed into a digon. With the help of Theorem 7.2.1, we will look into a more general situation where changing an arc into a digon does not change the spectrum.

Let $X = (V, E)$ be a digraph on n vertices with vertices labelled $\{1, \dots, n\}$, with an arc uv such that $vu \notin E(X)$. Without loss of generality, we may assume that $uv = (1, n)$ by relabelling the vertices. Let X' be the digraph obtained from X by adding the arc $(n, 1)$, creating the digon $\{1, n\}$. For a square matrix M, we use M_{kj} to denote the (k, j)*-th minor of* M; that is to say $M_{k,j}$ is equal to the determinant of the matrix with the kth row and the jth column removed. First, we have the following technical lemma.

7.2.3 Lemma. *If X and X' are as constructed above, then $H(X)$ and $H(X')$ are cospectral if and only if*

$$\sum_{\substack{j \in \{2, \dots, n\} \\ R_X(1,j) \in \{+,-,1\}}} \operatorname{Re}(\alpha_j) = 0 \tag{7.1}$$

where $\alpha_j = (-1 - i)A_{1j}\mathbf{v}_j$ where A and $\mathbf{v}$ are such that

$$tI - H(X) = \begin{pmatrix} A & \mathbf{v} \\ \mathbf{u}^T & t \end{pmatrix}.$$

Proof. Let X, X' be as above. We claim that

$$\phi(H(X'),t) - \phi(H(X),t) = \sum_{\substack{j \in \{2,\ldots,n\} \\ R_X(1,j) \in \{+,-,1\}}} \mathrm{Re}(\alpha_j),$$

which will imply the statement of the lemma. By Theorem 7.2.1,

$$\phi(H(X),t) = t\det(A) - \mathbf{u}^T \operatorname{adj}(A)\mathbf{v} = t\det(A) - \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} A_{kj}\mathbf{u}_k\mathbf{v}_j.$$

Similarly, we see that

$$\phi(H(X'),t) = \det(tI - H(X)) = \begin{vmatrix} A & \hat{\mathbf{v}} \\ \hat{\mathbf{u}}^T & t \end{vmatrix}$$

where $\hat{\mathbf{u}}$ is obtained from $\mathbf{u}$ by changing the first entry from $-i$ to 1 and $\hat{\mathbf{v}}$ is obtained from $\mathbf{v}$ by changing the first entry from i to 1. We obtain that

$$\begin{aligned}
\phi(H(X'),t) &= t\det(A) - \hat{\mathbf{u}}^T \operatorname{adj}(A)\hat{\mathbf{v}} \\
&= t\det(A) - \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} A_{kj}\hat{\mathbf{u}}_k\hat{\mathbf{v}}_j \\
&= t\det(A) - \sum_{j,k\in\{2,\ldots,n} A_{kj}\hat{\mathbf{u}}_k\hat{\mathbf{v}}_j - A_{1,1}\hat{\mathbf{u}}_1\hat{\mathbf{v}}_1 - \sum_{k=2}^{n-1} A_{k1}\hat{\mathbf{u}}_k\hat{\mathbf{v}}_1 - \sum_{j=2}^{n-1} A_{1j}\hat{\mathbf{u}}_1\hat{\mathbf{v}}_j \\
&= t\det(A) - \sum_{j,k\in\{2,\ldots,n} A_{kj}\mathbf{u}_k\mathbf{v}_j - A_{1,1}\mathbf{u}_1\mathbf{v}_1 - \sum_{k=2}^{n-1} A_{k1}\mathbf{u}_k - \sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j.
\end{aligned}$$

A similar expansion from $\phi(H(X),t)$ gives:

$$\begin{aligned}
\phi(H(X),t) &= t\det(A) - \sum_{j,k\in\{2,\ldots,n} A_{kj}\mathbf{u}_k\mathbf{v}_j - A_{1,1}\mathbf{u}_1\mathbf{v}_1 - \sum_{k=2}^{n-1} A_{k1}\mathbf{u}_k\mathbf{v}_1 - \sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j\mathbf{u}_1 \\
&= t\det(A) - \sum_{j,k\in\{2,\ldots,n} A_{kj}\mathbf{u}_k\mathbf{v}_j - A_{1,1}\mathbf{u}_1\mathbf{v}_1 - i\sum_{k=2}^{n-1} A_{k1}\mathbf{u}_k + i\sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j.
\end{aligned}$$

Then, we may subtract the two expressions to obtain

$$\phi(H(X'),t) - \phi(H(X),t) = (-1+i)\sum_{k=2}^{n-1} A_{k1}\mathbf{u}_k + (-1-i)\sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j. \tag{7.2}$$

Since A is Hermitian, we have that $A_{kj} = \overline{A_{jk}}$. Since $H(X)$ is Hermitian, we have that $\mathbf{u} = \bar{\mathbf{v}}$.

We apply this to Equation 7.2 to obtain the following:

$$\begin{aligned}
\phi(H(X'),t) - \phi(H(X),t) &= (-1+i)\sum_{j=2}^{n-1} \overline{A_{1j}}\overline{\mathbf{v}_j} + (-1-i)\sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j \\
&= \sum_{j=2}^{n-1} (-1+i)\overline{A_{1j}}\overline{\mathbf{v}_j} + (-1-i)A_{1j}\mathbf{v}_j \\
&= \sum_{j=2}^{n-1} \overline{\alpha_j} + \alpha_j \\
&= \sum_{j=2}^{n-1} 2\,\mathrm{Re}(\alpha_j)
\end{aligned}$$

where $\alpha_j = (-1-i)A_{1j}\mathbf{v}_j$. □

From Lemma 7.2.3, we may trivially obtain the following more tractable statement.

7.2.4 Corollary. *Let X' be the digraph obtained from X by replacing an arc uv with a digon $\{u, v\}$ and let $A = tI - H(X - v)$. If $\mathrm{Re}(A_{u,x}) = 0$ for all $x \neq u$ such that $R_X(u, x) \neq +$, then $H(X)$ and $H(X')$ are cospectral.*

Using the same method, we may consider what happens when an arc uv such that $vu \notin E(X)$ is replaced with arc vu. Let X be a digraph such that $R_X(1, n) = +$ and X^* be the digraph obtained from X by deleting the arc $(1, n)$ and adding the arc $(n, 1)$.

7.2.5 Lemma. *If X and X^* are as constructed above, then $H(X)$ and $H(X^*)$ are cospectral if and only if*

$$\sum_{\substack{j \in \{2,\dots,n\} \\ R_X(1,j) \in \{+,-,1\}}} \mathrm{Re}(\beta_j) = 0$$

where $\beta_j = \alpha_j = (-2i)A_{1j}\mathbf{v}_j$ where A and $\mathbf{v}$ are such that

$$tI - H(X) = \begin{pmatrix} A & \mathbf{v} \\ \mathbf{u}^T & t \end{pmatrix}.$$

Proof. Recall that A is a $n-1 \times n-1$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $(n-1)$-dimensional vectors. By Theorem 7.2.1,

$$\begin{aligned}
\phi(H(X),t) &= t\det(A) - \mathbf{u}^T \operatorname{adj}(A)\mathbf{v} \\
&= t\det(A) - \sum_{j,k\in\{2,\dots,n} A_{kj}\mathbf{u}_k\mathbf{v}_j - A_{1,1}\mathbf{u}_1\mathbf{v}_1 - i\sum_{k=2}^{n-1} A_{k1}\mathbf{u}_k + i\sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j
\end{aligned}$$

Similarly, we see that

$$\phi(H(X^*), t) = tI - H(X) = \begin{pmatrix} A & \tilde{\mathbf{v}} \\ \tilde{\mathbf{u}}^T & t \end{pmatrix}$$

where $\tilde{\mathbf{u}}$ is obtained from $\mathbf{u}$ by changing the 1st entry from $-i$ to i and $\tilde{\mathbf{v}}$ is obtained from $\mathbf{v}$ by changing the 1st entry from i to $-i$. We obtain that

$$\begin{aligned} \phi(H(X^*), t) &= t\det(A) - \tilde{\mathbf{u}}^T \operatorname{adj}(A)\tilde{\mathbf{v}} \\ &= t\det(A) - \sum_{j,k\in\{2,\ldots,n\}} A_{kj}\mathbf{u}_k\mathbf{v}_j - A_{1,1}\mathbf{u}_1\mathbf{v}_1 + i\sum_{k=2}^{n-1} A_{k1}\mathbf{u}_k - i\sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j \end{aligned}$$

Then, we may subtract the two expressions to obtain

$$\phi(H(X^*), t) - \phi(H(X), t) = (2i)\sum_{k=2}^{n-1} A_{k1}\mathbf{u}_k + (-2i)\sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j. \tag{7.3}$$

Since A is Hermitian, we have that $A_{kj} = \overline{A_{jk}}$. Since $H(X)$ is Hermitian, we have that $\mathbf{u} = \bar{\mathbf{v}}$. We apply this to Equation 7.3 to obtain the following:

$$\begin{aligned} \phi(H(X^*), t) - \phi(H(X), t) &= (2i)\sum_{j=2}^{n-1} \overline{A_{1j}}\overline{\mathbf{v}_j} + (-2i)\sum_{j=2}^{n-1} A_{1j}\mathbf{v}_j \\ &= \sum_{j=2}^{n-1} (2i)\overline{A_{1j}}\overline{\mathbf{v}_j} + (-2i)A_{1j}\mathbf{v}_j \\ &= \sum_{j=2}^{n-1} \overline{\beta_j} + \beta_j \\ &= \sum_{j=2}^{n-1} 2\operatorname{Re}(\beta_j) \end{aligned}$$

where $\beta_j = (-2i)A_{1j}\mathbf{v}_j$. □

In this case, we obtain the following corollary.

7.2.6 Corollary. *Let X be a digraph with exactly one pair of vertices u, v such that $R_X(u, v) = +$. Then, the digraph X^* on $(V(X)$ with arc set $E(X) \setminus \{uv\} \cup \{vu\}$ is H-cospectral with X.*

Proof. Let n be the number of vertices of X. We may relabel such that $u = 1$ and $v = n$ and apply Lemma 7.2.5 to obtain that

$$\phi(H(X^*), t) - \phi(H(X), t) = \sum_{j=2}^{n-1} 2\operatorname{Re}(\beta_j)$$

where $\beta_j = (-2i)A_{1j}\mathbf{v}_j$. Since the Hermitian adjacency matrix of any digraph is Hermitian and so has real eigenvalues, we may consider t as a real variable. Since the $(1, n)$ and $(n, 1)$ entries of $H(X)$ (and of $H(X^*)$) are the only entries that are not real numbers, we see that $A_{1j} \in \mathbb{R}$ and $\mathbf{v}_j \in \mathbb{R}$ for each $j = 2, \ldots, n-1$. Then $\mathrm{Re}(\beta_j) = 0$ for all $j = 2, \ldots, n-1$ and so $\phi(H(X^*), t) = \phi(H(X), t)$. □

We demonstrate the use of the lemmas in this section in a specific example. A *pendant vertex* is a vertex of degree 1.

7.2.7 Proposition. *If Y is a digraph with a pendant vertex v such that uv is the only arc of Y incident with v, then the digraphs Y' obtained from Y by adding the arc vu is H-cospectral to Y.*

Proof. Label the vertices of Y and Y' with $1, \ldots, n$ such that vertex v has label n and u has label 1. Then Y and Y' satisfy the hypotheses of Lemma 7.2.3. We see that $\mathbf{v} = \mathbf{e}_1$ in this case and the sum in Equation 7.1 is empty. Thus we obtain that Y and Y' are cospectral. □

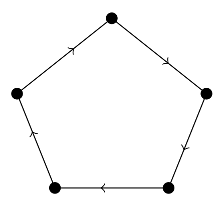
Note that Proposition 7.2.7 also follows from Lemma 6.2.4.

Chapter 8

Hermitian adjacency matrix of special families of digraphs

8.1 Direct cycles

Let D_n denote the directed cycle on n vertices. Figure 8.1 shows D_5.



D_5

Figure 8.1: The directed cycle on 5 vertices.

8.1.1 Lemma. *The eigenvalues of* $H(D_n)$ *are* $2\sin\left(\frac{2\pi k}{n}\right)$ *for* $k = 0, \ldots, n-1$.

Proof. Let A be the adjacency matrix of D_n. It is well-known that the eigenvectors of A are

$$\mathbf{v} = \begin{pmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{pmatrix}$$

for each $\zeta \in \{e^{\frac{2\pi k}{n}} \mid k = 0, \ldots, n-1\}$. Observe that

$$A\mathbf{v} = \zeta\mathbf{v}$$

and

$$A^T\mathbf{v} = \zeta^{n-1}\mathbf{v}.$$

Then, since $H = i(A - A^T)$, the eigenvalues of H are $i(\zeta - \zeta^{n-1})$. Since ζ is a root of unity, we see that $\zeta^{n-1} = \bar{\zeta}$, and so the eigenvalues of H are

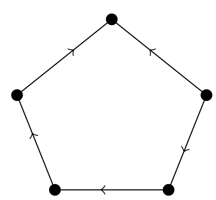
$$i(2i\operatorname{Im}\zeta) = -2\operatorname{Im}\zeta$$

for $\zeta \in \{e^{\frac{2\pi k}{n}} \mid k = 0, \ldots, n-1\}$. Then the eigenvalues of H are $-2\sin\left(\frac{2\pi k}{n}\right)$ for $k = 0, \ldots, n-1$. Since D_n is an oriented graph, its H-eigenvalues are symmetric about 0 and so this is the same as $2\sin\left(\frac{2\pi k}{n}\right)$ for $k = 0, \ldots, n-1$. □

We may observe that the eigenvalues of $H(D_n)$ are symmetric about 0 for any n. Since $\sin(\theta + \frac{\pi}{2}) = \cos\theta$, if $n = 4m$, then $H(D_n)$ has the same eigenvalues as $A(D_n)$.

We will use local reversal to show that all orientations of odd cycles are cospectral and that there are two H-cospectral classes of orientations of even cycles of length $2m$ for any m. First we must consider a specific orientation of a cycle.

The digraph $\widetilde{C}_n$ has vertices $\{x_1, \ldots, x_n\}$ and arc set $E = \{x_j x_{j+1} \mid j = 1, \ldots, n-1\} \cup \{x_1 x_n\}$. We have come across $\widetilde{C}_3$ in Section 6.5 and $\widetilde{C}_4$ in Section 6.7, in Figures 6.4 and 6.7, respectively. Figure 8.2 shows $\widetilde{C}_5$. In order to find the H-eigenvalues of $\widetilde{C}_n$, we will use known theorems about certain types of circulant matrices, which we will use again when finding the H-eigenvalues of the transitive tournament in Section 8.4.



$\widetilde{C}_5$

Figure 8.2: The digraph $\widetilde{C}_5$, obtained from the directed 5-cycle by reversing one arc.

Circulant matrices have been studied extensively, see [34] for more information. A *skew circulant matrix* is a circulant with a change in sign to all entries below the main diagonal. We will follow the notation of [19] and define for $\mathbf{a} = \left(a_0, \ldots, a_{n-1}\right)$ with real entries, the skew circulant matrix of $\mathbf{a}$ as:

$$\mathrm{S}(\mathbf{a}) = \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ -a_{n-1} & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ -a_1 & \ldots & -a_{n-1} & a_0 \end{pmatrix}.$$

The eigenvalues of a skew circulant matrix $\mathrm{S}(\mathbf{a})$ are found in [19, Section 3.2.1].

8.1.2 Theorem ([19]). *The eigenvalues of* $\mathrm{S}(\mathbf{a})$ *are* $\mu_j(\mathbf{a}) = \sum_{k=0}^{n-1} a_k \sigma^{(1+2j)k}$, *where* $\sigma = e^{\frac{\pi i}{n}}$, *for* $j = 0, 1, \ldots, n-1$.

In particular, it will be useful to simplify this statement for skew-symmetric, skew circulant matrices.

8.1.3 Corollary. *Suppose that* $a_j = a_{n-j}$ *for* $j = 1, \ldots, n-1$. *If* n *is odd, then* $\mathrm{S}(\mathbf{a})$ *has eigenvalues*

$$\nu_j(\mathbf{a}) = a_0 + \sum_{k=1}^{\frac{n-1}{2}} 2ia_k \sin\left(\frac{k(1+2j)\pi}{n}\right)$$

for $j = 0, 1, \ldots, n-1$. *If* n *is even, then* $\mathrm{S}(\mathbf{a})$ *has eigenvalues*

$$\nu_j(\mathbf{a}) = a_0 + (-1)^j a_{\frac{n}{2}} i + \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} 2ia_k \sin\left(\frac{k(1+2j)\pi}{n}\right)$$

for $j = 0, 1, \ldots, n-1$.

Proof. Let $\sigma = e^{\frac{\pi i}{n}}$. If $a_j = a_{n-j}$ for $j = 1, \ldots, n-1$, it is clear from the definition that $\mathrm{S}(\mathbf{a})$ is skew-symmetric. Observe that σ is a primitive $2n$-th root of unity and so $\sigma^n = -1$. For any $j \in \{0, \ldots, n-1\}$ and $k \in \{1, \ldots, n-1\}$, consider the contribution of terms with a_k and a_{n-k} in $\mu_j(\mathbf{a})$ of Theorem 8.1.2:

$$\begin{aligned}
a_k\sigma^{(1+2j)k} + a_{n-k}\sigma^{(1+2j)(n-k)} &= a_k\left(\sigma^{(1+2j)k} + \sigma^{(1+2j)(n-k)}\right)\\
&= a_k\left(\sigma^{(1+2j)k} + \sigma^{(1+2j)n}\sigma^{(1+2j)(-k)}\right)\\
&= a_k\left(\sigma^{(1+2j)k} + (-1)^{1+2j}(\sigma^{-k})^{1+2j}\right)\\
&= a_k\left(\sigma^{(1+2j)k} - (\overline{\sigma^{k}})^{1+2j}\right)\\
&= a_k\left(\sigma^{(1+2j)k} - \overline{\sigma^{k(1+2j)}}\right)\\
&= a_k\left(2i\,\mathrm{Im}(\sigma^{(1+2j)k})\right)\\
&= a_k\left(2i\sin\left(\frac{(1+2j)k\pi}{n}\right)\right).
\end{aligned}$$

If n is odd, then we are done. If $n = 2m$, then we have that

$$\mu_j(\mathbf{a}) = a_0 + a_m\sigma^{(1+2j)m} + \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} a_k 2i\sin\left(\frac{k(1+2j)\pi}{n}\right).$$

Observe that $\sigma^m = i$ and we see that

$$\mu_j(\mathbf{a}) = a_0 + (-1)^j a_{\frac{n}{2}} i + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} 2i a_k \sin\left(\frac{k(1+2j)\pi}{n}\right)$$

as claimed. $\square$

It is reassuring that we can see from Corollary 8.1.3 that any skew-symmetric skew circulant matrix with zeros on the diagonal will have purely imaginary eigenvalues. We will use Corollary 8.1.3 to find the eigenvalues of $\widetilde{C}_n$.

8.1.4 Lemma. *The eigenvalues of $\widetilde{C}_n$ are $2\sin\left(\frac{(1+2j)\pi}{n}\right)$ for $j = 0, \ldots, n-1$.*

Proof. Observe that $H(\widetilde{C}_n) = i\,\mathrm{S}(\mathbf{a})$ where $\mathbf{a} = \left(a_0, \ldots, a_{n-1}\right)$ and $a_1 = a_{n-1} = 1$ and $a_j = 0$ for $j \notin \{1, n-1\}$. Then, by Corollary 8.1.3, the eigenvalues of $H(\widetilde{C}_n)$ are $i\nu_j(\mathbf{a})$ for $j = 0, \ldots, n-1$. Let $\sigma = e^{\frac{\pi i}{n}}$ as before. Since $a_{\frac{n}{2}} = 0$, we obtain for $j \in 0, \ldots, n-1$,

$$\nu_j(\mathbf{a}) = 2i \sin\left(\frac{(1+2j)\pi}{n}\right).$$

Then, the eigenvalues of $H(\widetilde{C}_n)$ are $-2\sin\left(\frac{(1+2j)\pi}{n}\right)$ for $j = 0, \ldots, n-1$, which is easily seen to be the same as claimed. $\square$

We consider two digraphs X and Y to be *equivalent under taking local reversals* if there exists a digraph X', obtained from X by taking a series of local reversals, such that $X' \cong Y$.s

8.1.5 Lemma. *All orientations of C_{2m+1} are equivalent under taking local reversals. All orientations of C_{2m} are equivalent to either D_{2m} or $\widetilde{C}_{2m}$ under taking local reversals.*

Proof. Let $n \geq 3$ be an integer. Let X be any orientation of C_n on vertex set $V = \{x_1, \ldots, x_{2m+1}\}$. We may reorder the vertices such that the edges of C_n are $\{x_j x_{j+1}\}$ where the indices are modulo $2n$. We will use geometric intuition and say that the orientation of an arc of X is *clockwise* if it is $x_j x_{j+1}$ for some j and *counter-clockwise* if it is $x_j x_{j-1}$ for some j. We will show that X is equivalent to D_n or to $\widetilde{C}_n$ under local reversals.

Let Y be the graph obtained from X by local reversals such that Y has the most number of arcs with clockwise orientation, say with k arcs oriented clockwise. If $k = n$, then Y is isomorphic to D_n and we are done. We may now assume that $k \leq n-1$. If $k \leq n-2$, there are two arcs a, b both oriented counter-clockwise in Y, then a and b form an edge cut of the graph and we may consider S, the vertices in one of the components of Y with a and b deleted. We may take Y' the local reversal of Y at S. This operation only changes the directions of arcs a and b. Now Y' has $k+2$ arcs oriented clockwise and is equivalent to X under taking local reversals, which contradicts our choice of Y. Then $k = n-1$.

If n is even, then we have obtained that X is equivalent to $\widetilde{C}_n$ by local reversals and we are done.

Now suppose that $n = 2m + 1$. Let a be the unique arc of Y with the counter-clockwise orientation. We may pair up the remaining $2m$ arcs of Y and then take local reversal at the cut sets, as above, to obtain Y' where all $2m$ clockwise arcs of Y have been reversed. Then Y' is again isomorphic to D_{2m+1}. $\square$

The above would imply that every orientation of C_n is H-cospectral with D_n or with $\widetilde{C}_n$. Note that for even n, the spectra of D_n and $\widetilde{C}_n$ are not equal.

8.2 Forests

We see from Lemma 6.2.4 that if uv is a cut-edge in the underlying graph, then changing the adjacency of u and v to $+, -, 1$ does not affect the H-spectrum. We then obtain the following theorem, which also appears in [41] as Corollary 2.21.

8.2.1 Theorem. *[41] If X is a digraph whose underlying graph is a forest, then $H(X)$ is cospectral to $H(\Gamma(X))$.*

Proof. Every edge of $\Gamma(X)$ is a cut-edge and so we may replace every arc uv of X such that $vu \notin E(X)$, while preserving the H-spectrum, by Lemma 6.2.4. Then, X is H-cospectral with $\Gamma(X)$. $\square$

Note that for undirected graphs, the Hermitian adjacency matrix is equal to the adjacency matrix.

8.3 Products of digraphs

Let X and Y be digraphs. The *Cartesian product* of X and Y, denoted by $X \Box Y$, is the graph with vertex set $V(X) \times V(Y)$ such that there is an arc from (x_1, y_1) to (x_2, y_2) when either x_1x_2 is an arc of X and $y_1 = y_2$ or y_1y_2 is an arc of Y and $x_1 = x_2$. The Hermitian adjacency matrix of $X \Box Y$ is

$$H(X \Box Y) = H(X) \otimes I_{|V(Y)|} + I_{|V(X)|} \otimes H(Y)$$

where I_k is the $k \times k$ identity matrix. For definitions of Kronecker products of matrices and vectors, see [18].

8.3.1 Proposition. *If X and Y are digraphs with H-eigenvalues $\{\lambda_j\}_{j=1}^{n}$ and $\{\mu_k\}_{k=1}^{m}$ respectively, then $X \Box Y$ has H-eigenvalues $\lambda_j + \mu_k$ for $j = 1, \ldots, n$ and $k = 1, \ldots, m$.*

Proof. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthonormal eigenbasis of $H(X)$ such that $H(X)\mathbf{v}_j = \lambda_j \mathbf{v}_j$, for $j = 1, \ldots, n$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be an orthonormal eigenbasis of $H(Y)$ such that $H(Y)\mathbf{w}_k = \mu_k \mathbf{w}_k$, for $k = 1, \ldots, m$. Observe that the vectors $\{\mathbf{v}_j \otimes \mathbf{w}_k \mid j \in \{1, \ldots, n\}, k \in \{1, \ldots, m\}\}$ form a basis of $\mathbb{C}^{nm}$. We see that

$$\begin{aligned} H(X \square Y)\mathbf{v}_j \otimes \mathbf{w}_k &= \left(H(X) \otimes I_m\right)\left(\mathbf{v}_j \otimes \mathbf{w}_k\right) + \left(I_n \otimes H(Y)\right)\left(\mathbf{v}_j \otimes \mathbf{w}_k\right) \\ &= H(X)\mathbf{v}_j \otimes I_m \mathbf{w}_k + I_n \mathbf{v}_j \otimes H(Y)\mathbf{w}_k \\ &= \lambda_j \left(\mathbf{v}_j \otimes \mathbf{w}_k\right) + \mu_k \left(\mathbf{v}_j \otimes \mathbf{w}_k\right) \\ &= \left(\lambda_j + \mu_k\right)\left(\mathbf{v}_j \otimes \mathbf{w}_k\right) \end{aligned}$$

for every $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$. □

8.4 Transitive tournaments

The transitive tournaments are an important class of digraphs to study and the spectra of their skew-symmetric adjacency matrices have been studied as skew circulants and Toeplitz matrices in [30, 38]. Here we will find the characteristic polynomials of the Hermitian adjacency matrices of transitive tournaments using a different method, with hopes that this approach may give intuition and tools for further work. We will also consider the matrix as a skew circulant matrix and use existing results to find the actual eigenvalues.

Let T_n denote the transitive tournament on n vertices and $H_n := H(T_n)$ denote its Hermitian adjacency matrix. Let $p_n(t)$ be the characteristic polynomial of H_n in variable t; we will sometimes write p_n for convenience. Figure 8.3 shows T_5.

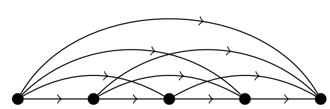

T_5

Figure 8.3: Transitive tournament T_5 on 5 vertices.

8.4.1 Lemma. *The characteristic polynomial $\phi(H_n, t) = p_n(t)$ of the Hermitian adjacency matrix of the transitive tournament of order n satisfies the following recursion:*

$$p_n = 2tp_{n-1} - (t^2 + 1)p_{n-2}. \tag{8.1}$$

Proof. Let $h_n = tI_n - H_n^T$. We will find the following determinant which will give the characteristic polynomial of H_n:

$$p_n = \det(h_n) = \begin{vmatrix} t & i & \dots & i \\ -i & t & \ddots & \\ \vdots & \ddots & \ddots & \\ -i & \dots & -i & t \end{vmatrix}.$$

Cofactor expansion along the first row gives that

$$p_n = t\det(h_{n-1}) + \sum_{k=1}^{n-1}(-1)^k i M_{n-1,k}$$

where $M_{n-1,k}$ is the determinant of the $n-1 \times n-1$ matrix with $-i\mathbf{1}$ as the first column and the next $n-2$ columns are the columns of h_{n-1} with the kth column removed. Note, here the indexing of the columns of h_{n-1} is considered to be $\{1, \dots, n-1\}$. We may simplify the equation to be:

$$p_n = tp_{n-1} + i\sum_{k=1}^{n-1}(-1)^k M_{n-1,k} \tag{8.2}$$

Now we consider the cofactor expansion of the same matrix h_n along the first two rows. Note that there are only 4 different 2x2 sub-matrices in the first two rows:

$$\begin{pmatrix} t & i \\ -i & t \end{pmatrix} = h_2, \begin{pmatrix} t & i \\ -i & i \end{pmatrix}, \begin{pmatrix} i & i \\ t & i \end{pmatrix}, \text{ and } \begin{pmatrix} i & i \\ i & i \end{pmatrix}.$$

The determinants of these 2×2 matrices are $t^2 - 1 = p_2$, $it - 1$, $-it - 1$ and 0, respectively. Note that every term with $\left|\begin{pmatrix} i & i \\ i & i \end{pmatrix}\right|$ as a coefficient contributes 0 to the determinant p_n. Then the cofactor expansion along the first two rows gives:

$$\begin{aligned} p_n &= p_2p_{n-2} + \left|\begin{pmatrix} t & i \\ -i & i \end{pmatrix}\right| \sum_{k=1}^{n-2}(-1)^k M_{n-2,k} + \left|\begin{pmatrix} i & i \\ t & i \end{pmatrix}\right| \sum_{k=1}^{n-2}(-1)^{k+1} M_{n-2,k} \\ &= p_2p_{n-2} + (it-1)\sum_{k=1}^{n-2}(-1)^k M_{n-2,k} + (-it-1)\sum_{k=1}^{n-2}(-1)^{k+1} M_{n-2,k} \\ &= p_2p_{n-2} + \sum_{k=1}^{n-2}(-1)^k\, 2ti\, M_{n-2,k} \\ &= (t^2-1)p_{n-2} + 2ti\sum_{k=1}^{n-2}(-1)^k M_{n-2,k}. \end{aligned}$$

We obtain from Equation (8.2), applied to $n-1$ instead for n, that

$$p_{n-1} = tp_{n-2} + i\sum_{k=1}^{n-2}(-1)^k M_{n-2,k}.$$

We combine the two expressions to obtain that:

$$\begin{aligned}
p_n - 2tp_{n-1} &= (t^2-1)p_{n-2} + 2ti\sum_{k=1}^{n-2}(-1)^k M_{n-2,k} - 2t^2p_{n-2} - 2ti\sum_{k=1}^{n-2}(-1)^k M_{n-2,k} \\
&= (t^2-1)p_{n-2} - 2t^2p_{n-2} \\
&= (-t^2-1)p_{n-2}.
\end{aligned}$$

We may rearrange to obtain:

$$p_n = 2tp_{n-1} - (t^2+1)p_{n-2}$$

as claimed. □

8.4.2 Theorem. *The characteristic polynomial of $H_n = H(T_n)$ is*

$$p_n(t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^j\binom{n}{2j}t^{n-2j} = \frac{1}{2}(t+i)^n + \frac{1}{2}(t-i)^n.$$

Proof. We use the recursion from Lemma 8.4.1 to find the characteristic polynomial. As before, we will write $p_n = p_n(t)$ for simplicity. Let

$$p_0 := 1,\ p_1 := t.$$

We may compute that $p_2 = t^2-1$ and $p_3 = t^3-3t$. Then, p_n for $n = 0, 1, \ldots$ satisfies the recursion in Equation (8.1). Let $G(z)$ be the generating function for the characteristic polynomial p_n;

$$G(z) := \sum_{n=0}^{\infty} p_n z^n.$$

Multiplying both sides of (8.1) by z^n gives:

$$p_n z^n = 2tp_{n-1}z^n - (t^2+1)p_{n-2}z^n.$$

Then, we take the sum for $n = 2, 3, \ldots$ on both sides to obtain:

$$\sum_{n=2}^{\infty} p_n z^n = 2tz\sum_{n=2}^{\infty} p_{n-1}z^{n-1} - (t^2+1)z^2\sum_{n=2}^{\infty} p_{n-2}z^{n-2}$$

which we may simplify as

$$\begin{aligned} G(z) - p_1 z - p_0 &= 2tz(G(z) - p_0) - (t^2+1)z^2 G(z), \\ G(z) - tz - 1 &= 2tzG(z) - 2tz - (t^2+1)z^2 G(z). \end{aligned}$$

Rearranging gives the following:

$$G(z) = \frac{1 - tz}{(t^2+1)z^2 - 2tz + 1}. \tag{8.3}$$

Observe that the denominator in Equation (8.3) factors as

$$(t^2+1)z^2 - 2tz + 1 = (1 - (t+i)z)(1 - (t-i)z).$$

We wish to apply partial fractions to Equation (8.3) by solving for $a_0, a_1, b_0,$ and b_1 in the following:

$$\begin{aligned} \frac{1 - tz}{(t^2+1)z^2 - 2tz + 1} &= \frac{a_0 + a_1 z}{1 - (t+i)z} + \frac{b_0 + b_1 z}{1 - (t-i)z} \\ &= \frac{a_0 - a_0(t-i)z + a_1 z - a_1 z^2(t-i) + b_0 - b_0(t+i)z + b_1 z - b_1 z^2(t+i)}{(t^2+1)z^2 - 2tz + 1} \\ &= \frac{(-a_1(t-i) - b_1(t+i))z^2 + (-a_0(t-i) - b_0(t+i) + a_1 + b_1)z + a_0 + b_0}{(t^2+1)z^2 - 2tz + 1}. \end{aligned}$$

This is equivalent to solving the following system of linear equation in $a_0, a_1, b_0,$ and b_1:

$$\begin{cases} -a_1(t-i) - b_1(t+i) = 0, \\ -a_0(t-i) - b_0(t+i) + a_1 + b_1 = -t, \\ a_0 + b_0 = 1. \end{cases}$$

Solving for $a_0, a_1, b_0,$ and b_1, we obtain:

$$a_0 = -r + 1,\ a_1 = \frac{1}{2}(2r-1)t + ir - \frac{i}{2}$$

$$b_0 = r,\ b_1 = -\frac{1}{2}(2r-1)t + ir - \frac{i}{2},$$

for any $r \in \mathbb{C}$. We may choose a particular solution when $r = 0$ to obtain:

$$a_0 = 1,\ a_1 = \frac{-t - i}{2}$$

$$b_0 = 0,\ b_1 = \frac{t - i}{2}.$$

Then we can write Equation (8.3) as

$$G(z) = \frac{1 - \frac{t+i}{2} z}{1 - (t+i)z} + \frac{\frac{t-i}{2} z}{1 - (t-i)z}.$$

We may expand using the following:

$$\frac{1}{1 - (t+i)z} = \sum_{n=0}^{\infty} (t+i)^n z^n$$

and

$$\frac{1}{1 - (t-i)z} = \sum_{n=0}^{\infty} (t-i)^n z^n.$$

Then we have

$$\begin{aligned}
G(z) &= \left(1 - \frac{t+i}{2} z\right) \sum_{n=0}^{\infty} (t+i)^n z^n + \left(\frac{t-i}{2} z\right) \sum_{n=0}^{\infty} (t-i)^n z^n \\
&= \sum_{n=0}^{\infty} (t+i)^n z^n - \sum_{n=0}^{\infty} \frac{1}{2} (t+i)^{n+1} z^{n+1} + \sum_{n=0}^{\infty} \frac{1}{2} (t-i)^{n+1} z^{n+1} \\
&= 1 + \sum_{n=1}^{\infty} (t+i)^n z^n - \sum_{n=1}^{\infty} \frac{1}{2} (t+i)^n z^n + \sum_{n=1}^{\infty} \frac{1}{2} (t-i)^n z^n \\
&= 1 + \sum_{n=1}^{\infty} \left((t+i)^n z^n - \frac{1}{2} (t+i)^n z^n + \frac{1}{2} (t-i)^n z^n \right) \\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2} (t+i)^n + \frac{1}{2} (t-i)^n \right) z^n.
\end{aligned}$$

Since $G(z)$ is defined such that p_n is the coefficient of z^n, we obtain that

$$p_n = \frac{1}{2} (t+i)^n + \frac{1}{2} (t-i)^n.$$

To obtain the coefficients of p_n, we simplify further as follows:

$$\begin{aligned}
p_n &= \frac{1}{2}\left((t+i)^n + (t-i)^n\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}t^{n-k}i^k + \sum_{k=0}^{n}\binom{n}{k}t^{n-k}i^k(-1)^k\right) \\
&= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}t^{n-k}i^k(1+(-1)^k) \\
&= \frac{1}{2}\sum_{k=0,k \text{ even}}^{n} 2\binom{n}{k}t^{n-k}i^k \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2j}(-1)^j t^{n-2j}
\end{aligned}$$

as claimed. □

We may also consider H_n as a skew circulant matrix, as introduced in Section 8.1.

8.4.3 Lemma. *The eigenvalues of H_n are*

$$\sum_{k=1}^{\frac{n-1}{2}} 2\sin\left(\frac{k(1+2j)\pi}{n}\right)$$

for $j = 0, \ldots, n-1$, when n is odd, and

$$(-1)^j + \sum_{k=1}^{\frac{n-2}{2}} 2\sin\left(\frac{k(1+2j)\pi}{n}\right)$$

for $j = 0, \ldots, n-1$, when n is even.

Proof. Observe that $H_n = i\,\mathrm{S}(\mathbf{a})$ where $\mathbf{a} = \Big(a_0, \ldots, a_{n-1}\Big)$ and $a_1 = \ldots = a_{n-1} = 1$ and $a_0 = 0$. We see that $\mathrm{S}(\mathbf{a})$ is a skew-symmetric skew circulant and so we may apply Corollary 8.1.3 to obtain that eigenvalues of $\mathrm{S}(\mathbf{a})$ are

$$\begin{aligned}
\nu_j(\mathbf{a}) &= \begin{cases} a_0 + \sum_{k=1}^{\frac{n-1}{2}} a_k 2i\sin\left(\frac{k(1+2j)\pi}{n}\right) & \text{if } n \text{ is odd;} \\ a_0 + (-1)^j a_{\frac{n}{2}} i + \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} a_k 2i\sin\left(\frac{k(1+2j)\pi}{n}\right) & \text{if } n \text{ is even.} \end{cases} \\
&= \begin{cases} \sum_{k=1}^{\frac{n-1}{2}} 2i\sin\left(\frac{k(1+2j)\pi}{n}\right) & \text{if } n \text{ is odd;} \\ (-1)^j i + \sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} 2i\sin\left(\frac{k(1+2j)\pi}{n}\right) & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

for $j = 0, \ldots, n-1$. Then eigenvalues of H are $i\nu_j$ for $j = 0, \ldots, n-1$. □

Note that for $n = 2m+1$, the eigenvalue of H_n contributed from $j = m$ is 0.

8.5 Directed stars

In Section 7.2, we found the characteristic polynomials of digraphs with a star as the underlying graph. Here, we will find the eigenvalues directly.

8.5.1 Theorem. *Let X be any digraph whose underlying graph is isomorphic to the star graph on n vertices. Then, the eigenvalues of $H(X)$ are $\pm\sqrt{n-1}$ each with multiplicity 1, and 0 with multiplicity $n-2$.*

Proof. Let X be as in the theorem statement. Then, relabelling so that vertex 1 is the vertex with arcs to or from every other vertex, we may write the Hermitian adjacency matrix $H := H(X)$ as follows:

$$H(X) = \begin{pmatrix} 0 & \mathbf{u}^T \\ \mathbf{v} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{0}$ is the $n-1 \times n-1$ zero matrix and $\mathbf{u}^T = \mathbf{v}^*$. We may observe that $\mathbf{u}^T\mathbf{v} = n-1$. Then

$$H\begin{pmatrix} \sqrt{n-1} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{u}^T\mathbf{v} \\ \sqrt{n-1}\mathbf{v} \end{pmatrix} = \begin{pmatrix} n-1 \\ \sqrt{n-1}\mathbf{v} \end{pmatrix} = \sqrt{n-1}\begin{pmatrix} \sqrt{n-1} \\ \mathbf{v} \end{pmatrix}$$

and $\sqrt{n-1}$ is an eigenvalue of H. Similarly,

$$H\begin{pmatrix} -\sqrt{n-1} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{u}^T\mathbf{v} \\ -\sqrt{n-1}\mathbf{v} \end{pmatrix} = \begin{pmatrix} n-1 \\ -\sqrt{n-1}\mathbf{v} \end{pmatrix} = -\sqrt{n-1}\begin{pmatrix} -\sqrt{n-1} \\ \mathbf{v} \end{pmatrix}$$

and $-\sqrt{n-1}$ is an eigenvalue of H. Observe that the bottom $n-1$ rows of H are scalar multiples of each other, as they all have exactly one nonzero entry, in the first position, and zeroes elsewhere. Then $\mathrm{rk}(H) = 2$, and the remaining $n-2$ eigenvalues of H are equal to 0. □

Note that by Theorem 8.2.1, all forests with isomorphic underlying graphs are H-cospectral.

Let X be a digraph whose underlying graph is Y, the star on n vertices.The proof of Theorem 8.5.1 actually shows that $H(X)$ is similar to $H(Y)$ by a diagonal matrix. If $n \geq 3$, let v be the vertex of Y (and X) of degree greater than 1. If $n \leq 2$, the choice of v is arbitrary. The similarity from $H(X)$ to $H(Y)$ is the matrix P, the $n \times n$ diagonal matrix such that $P(v,v) = 1$ and $P(u,u) = H(X)(u,v)$. It is easy to see that $P^*H(X)P = H(Y)$.

8.6 Necklace digraphs

We construct a family of digraphs for the purpose of studying the relationship between the number of distinct H-eigenvalues and the diameter of the underlying graph. See Section 6.7.

For $n \geq 3$, the nth *necklace digraph,* denoted N_n, is an oriented graph on $3n$ vertices with vertices and arcs as follows:

$$V(N_n) = \{v_j \mid j \in \mathbb{Z}_{2n}\} \cup \{w_k \mid k \in \mathbb{Z}_n\}$$

and

$$E(N_n) = \{v_j v_{j+1} \mid j \in \mathbb{Z}_{2n}\} \cup \{v_{2k}w_k, v_{2k+2}w_k \mid k \in \mathbb{Z}_n\}.$$

Let C_j be the cycle $(v_{2j}, v_{2j+1}, v_{2j+2}, w_j, v_{2j})$. Every v_{2j} lies on two of these cycles, C_j and C_{j-1}. Every other vertex lies on a unique C_j. Figure 8.4 shows N_4 with cycle C_0 highlighted.

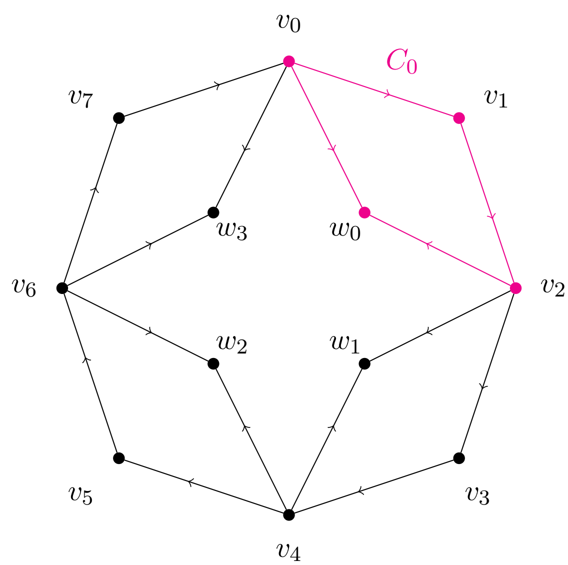


Figure 8.4: N_4 with C_0 in a lighter colour.

We will find the eigenvalues of N_n.

8.6.1 Lemma. *For every $n \geq 3$ and $H := H(n)$, we have $H^3 = 4H$.*

Proof. We will show $H^3(u, v) = 4H(u, v)$. Observe that since the underlying graph $\Gamma := \Gamma(N_n)$ is a bipartite graph of girth 4, we have that $H^3(u, u) = 0$ and $H^3(u, v) = 0$ if $d_\Gamma(u, v)$ is even or $d_\Gamma(u, v) > 3$. We only need to consider the two cases where $d_\Gamma(u, v) \in \{1, 3\}$.
Case 1: u, v are such that $d_\Gamma(u, v) = 1$. Since H and H^3 are Hermitian, we need only check u, v such that $uv \in E(N_n)$. Let $\mathcal{W}$ be the set of all walks of length 3 from u to v. The following are all possible walks of length 3 in Γ, starting at u and ending at v:

(a) $W_1 = (u, v, u, v)$;

(b) $W_2 = (u, v, x, v)$, where $x \in N_\Gamma(v) \setminus \{u\}$;

(c) $W_3 = (u, w, u, v)$, where $w \in N_\Gamma(u) \setminus \{v\}$; and

(d) $W_4 = (u, w, x, v)$, where $w \in N_\Gamma(u) \setminus \{v\}$, $w \in N_\Gamma(v) \setminus \{u\}$, and $wx \in E(\Gamma)$.

These walks are illustrated in Figure 8.5.

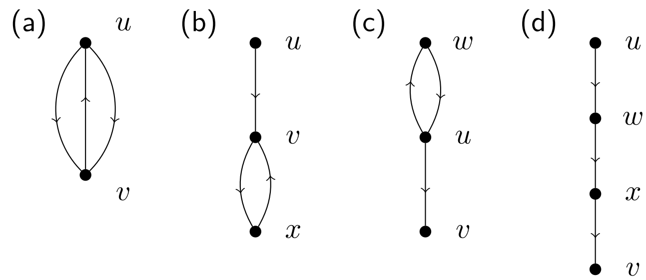


Figure 8.5: All possible walks of length 3 in Γ, starting at u and ending at v.

Every edge of Γ which is traversed once in each direction in any W_j contributes a factor of 1 to $\mathrm{wt}(W_j)$. Thus $\mathrm{wt}(W_1) = \mathrm{wt}(W_2) = \mathrm{wt}(W_3) = i$. To find the weight of W_4, we observe that every arc uv lies on a cycle C_j, for some j, and that W_4 together with arc uv gives a directed 4-cycle in Γ, which must then correspond to C_j. From this we immediately see that, for each such uv, there is exactly one walk from u to v isomorphic to W_4 and that W_4 is a path of length 3 on C_j. We may observe that all such paths either traverse two arcs in the backward direction and one in the forward direction, or all three arcs in the forward direction. In either case, $\mathrm{wt}(W_4) = (\pm i)^2(i) = -i$.

For every arc uv, one of u or v has degree 4 in Γ and the other has degree 2. Then W contains one walk isomorphic to W_1 and either 3 walks isomorphic to W_2 and one isomorphic to W_3, or 3 walks isomorphic W_3 and one isomorphic to W_2. Then, since W_j for $j = 1, 2, 3$ have the same weight, we get

$$
\begin{aligned}
H^3(u,v) &= \sum_{W \in \mathcal{W}} \mathrm{wt}(W) \\
&= \mathrm{wt}(W_1) + \sum_{N_\Gamma(v)\setminus\{u\}} \mathrm{wt}(W_2) + \sum_{N_\Gamma(u)\setminus\{v\}} \mathrm{wt}(W_3) + \mathrm{wt}(W_4) \\
&= i + (3i + i) - i \\
&= 4i \\
&= 4H(u,v)
\end{aligned}
$$

as claimed.

Case 2: u, v are such that $d_\Gamma(u,v) = 3$. In this case, since vertices lying on C_j for any given j can be at distance at most 2, we have that $u \in C_j$ and $v \in C_{j\pm1}$. Also note that either u or v is of degree 4 in $\Gamma(X)$, thus equal to v_{2k} for some $k \in \mathbb{Z}_n$.

Suppose $u = v_{2k}$ for some k. Then $v \in C_{k+1}$ or $v \in C_{k-2}$. See Figure 8.6. In either case, there are two walks from u to v of opposite contributions, and so $H^3(u,v) = 0 = H(u,v)$. □

8.6.2 Corollary. *The spectrum of $H(N_n)$ is $\{0^{(n)}, 2^{(n)}, (-2)^{(n)}\}$.*

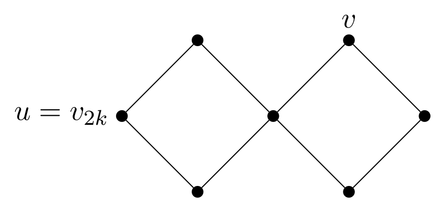


Figure 8.6: An induced subgraph of Γ containing u, v such that $d_\Gamma(u, v) = 3$, as in Case 2.

Proof. Let $H := H(N_n)$. From Lemma 8.6.1, we see that the minimal polynomial of H is $t^3 - 4t = 0$. Since every eigenvalue of a matrix is a root of its minimal polynomial, the distinct eigenvalues of H are 0, 2 and -2. Let q, r, and s be the multiplicities of $0, 2$, and -2, respectively. Since $\mathrm{tr}(H) = 0$, we see that $r = s$. By Lemma 6.2.1, $\mathrm{tr}(H^2) = 2|E(\Gamma(N_n))| = 8n$. Then

$$r(2^2 + (-2)^2) = 8n$$

and so $r = n$. Since $q + 2r = 3n$, we have that $q = n$. □

8.7 Digraphs that are H-cospectral with K_n

In the undirected case, the complete graphs have the property of being determined by spectrum. In the case of directed graphs with the Hermitian adjacency matrix, for each n, we find that there are exactly n digraphs with the same H-spectrum as K_n.

Let $Y_{n,m} = (V, E)$ be the digraph on vertex set

$$V = \{v_j \mid j = 1, \ldots, n\} \cup \{w_k \mid k = 1, \ldots, m\}$$

and arc-set

$$\begin{aligned} E = &\{v_j v_\ell \mid j \neq \ell \in [1, \ldots, n]\} \cup \{w_k w_r \mid k \neq r \in [1, \ldots, m]\} \\ &\cup \{v_j w_k \mid j \in [1, \ldots, n], k \in [1, \ldots, m]\}. \end{aligned}$$

In other words, $Y_{n,m}$ consists of a copy of $\vec{D}(K_n)$ and a copy of $\vec{D}(K_m)$, with all possible arcs from $\vec{D}(K_n)$ to $\vec{D}(K_m)$. Figure 8.7 shows $Y_{2,3}$. We note that $Y_{2,1}$ and $Y_{2,2}$ appeared in Figure 5.2 as examples of digraphs whose A-spectra are symmetric about the origin, but whose underlying graphs are not bipartite.

8.7.1 Proposition. *The digraph $Y_{n,m}$ as defined above is H-cospectral with K_{n+m}.*

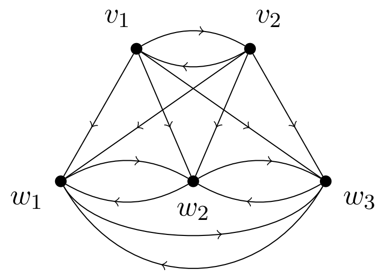


Figure 8.7: Digraph $Y_{2,3}$ of Proposition 8.7.1, which is one of five digraphs H-cospectral to K_5 by Corollary 8.7.2.

Proof. Let $H := H(Y_{n,m})$. Then, we see that

$$H = \begin{pmatrix} J_{n,n} - I_n & iJ_{n,m} \\ -iJ_{m,n} & J_{m,m} - I_m \end{pmatrix} = \begin{pmatrix} J_{n,n} & iJ_{n,m} \\ -iJ_{m,n} & J_{m,m} \end{pmatrix} - I_{n+m}$$

where $J_{r,c}$ denotes the $r \times c$ all-ones matrix. Let $M = \begin{pmatrix} J_{n,n} & iJ_{n,m} \\ -iJ_{m,n} & J_{m,m} \end{pmatrix}$. We see that M has rank 1 and so has 0 as an eigenvalue with multiplicity $n + m - 1$. We also observe

$$M \begin{pmatrix} \mathbf{1}_n \\ -i\mathbf{1}_m \end{pmatrix} = \begin{pmatrix} (n + i(-i)m)\mathbf{1}_n \\ (-in - im)\mathbf{1}_m \end{pmatrix} = (n + m) \begin{pmatrix} \mathbf{1}_n \\ -i\mathbf{1}_m \end{pmatrix}.$$

Since $H = M - I_{n+m}$ and M and I_{n+m} commute and hence have a common eigenbasis, we have that the eigenvalues of H are $n + m - 1$ with multiplicity 1 and -1 with multiplicity $n + m - 1$, which is the same as the H-spectrum of K_{n+m}.

In fact, we may note that H and $H(K_{n+m})$ are similar by a diagonal matrix. Let

$$P = \begin{pmatrix} I_n & 0 \\ 0 & iI_m \end{pmatrix}.$$

Then, we see that

$$PHP^* = \begin{pmatrix} I_n & 0 \\ 0 & iI_m \end{pmatrix} \begin{pmatrix} J_{n,n} - I_n & iJ_{n,m} \\ -iJ_{m,n} & J_{m,m} - I_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & -iI_m \end{pmatrix} = \begin{pmatrix} J_{n,n} - I_n & J_{n,m} \\ J_{m,n} & J_{m,m} - I_m \end{pmatrix}$$

which is exactly $H(K_{n+m})$. $\square$

8.7.2 Proposition. *For each n, there are precisely n non-isomorphic digraphs that have the same H-spectrum as K_n. There are the digraphs K_n and $Y_{a,b}$, where $a = 1, \ldots, n-1$ and $b = n - a$.*

Proof. For $a, b \geq 1$, we observe that the out-degree of a vertex of $Y_{a,b}$ is either $a - 1 + b$ or $b - 1$.

Then, for $c, d \geq 1$, the digraphs $Y_{a,b}$ and $Y_{c,d}$ have different sets of degrees unless $a = c$ and $b = d$. Thus, we observe that $Y_{a,b}$ is isomorphic to $Y_{c,d}$ if and only if $a = c$ and $b = d$. From Proposition 8.7.1, we see that the digraphs

$$\{Y_{a,n-a} \mid a \in [1, \dots n-1]\}$$

are each cospectral with K_n. Together with K_n itself, there are at least n non-isomorphic digraphs in the H-cospectral class containing K_n.

Let X be a digraph which is H-cospectral with K_n. We will now show that X is isomorphic to one of K_n and $Y_{a,b}$, where $a = 1, \dots, n-1$ and $b = n - a$.

Since $[t^{n-2}]\phi(H(X), t) = [t^{n-2}]\phi(H(K_n), t)$, we see that $\Gamma(X)$ has the same number of edges as K_n, and so $\Gamma(X) \cong K_n$, by Corollary 6.2.2.

Also, $\operatorname{tr}(H(X)^3) = \operatorname{tr}(H(K_n)^3)$. By Proposition 6.3.2, we see that $\operatorname{tr}(H(K_n)^3) = 6\binom{n}{3}$, since every 3 vertices of K_n induce a triangle isomorphic to the digraph X_7 in Proposition 6.3.2. Then every 3 vertices of X must induce a digraph isomorphic to one of X_4, X_5 or X_7 in Figure 6.2, which are precisely the triangles with weight equal to 1.

Consider $G(X)$, the symmetric subgraph of X. If there is a path of length 2, say (u, v, w) in a component of $G(X)$, then $\{u, v, w\}$ induce a triangle of X with more than one digon and hence must be isomorphic to X_7. Thus, each connected component of $G(X)$ is complete. If $G(X)$ has 3 or more components, then choosing three vertices from different components will give a triangle with no digons and hence not isomorphic to any X_j for $j \in \{4, 5, 7\}$. Thus, $G(X)$ has at most 2 components. If $G(X)$ has one, then $X \cong K_n$. Otherwise, the vertices of X may be partitioned into $V(X) = A \cup B$ such that A and B induced complete subgraphs of X and every arc whose end points are in different parts does not lie on a digon.

Let $x \in A$. If x has both an in-neighbour y_1 and an out-neighbour y_2, then x, y_1, y_2 induce a triangle isomorphic to X_3, a contradiction. Then, x has only in-neighbours or only out-neighbours. The same also holds for any vertex of B. If x has only in-neighbours, then every vertex of B is adjacent to x in $\Gamma(X)$ and so every vertex of B has x as an out-neighbour in X. By the previous observation, every arc e incident to a vertex y in B has $t(e) = y$. We have show that all arcs with one end in A and one end in B are oriented from B to A. Similarly, if x has only out-neighbours, all arcs with one end in A and one end in B are oriented from A to B. Now we have show that X is isomorphic to one of $Y(|A|, |B|)$ and $Y(|B|, |A|)$. □

8.8 Digraphs with spectrum $\{-(n-1), 1^{(n-1)}\}$

In Section 6.6, we encountered a digraph K_3' with H-spectrum $\{-2, 1^{(2)}\}$. In Section 8.9, we will see a digraph K_4' with H-spectrum $\{-3, 1^{(3)}\}$, used in a construction of digraphs with the least eigenvalue larger than the greatest eigenvalue, in magnitude. The two digraphs are shown in Figure

8.8. We may observe that K_3' and K_4' have H-spectra that is the negative of the H-spectra (and A-spectra) of K_3 and K_4, respectively. Naturally, we may ask if there are other digraphs on n vertices with H-spectrum $\{-(n-1), 1^{(n-1)}\}$. Such digraphs would have a large negative eigenvalue and small positive eigenvalues and thus exhibit extreme spectral behaviour, opposite to the behaviour described in the Perron-Frobenius theorem for undirected graphs. We answer this in the negative and show that K_3' and K_4' are the only non-trivial digraphs with this property.

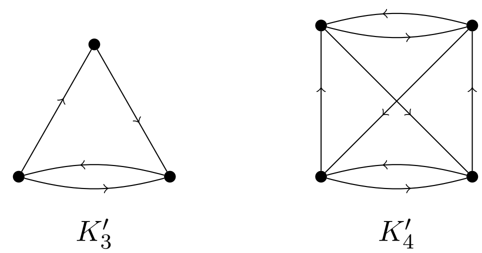


Figure 8.8: K_3' and K_4'.

8.8.1 Theorem. *If X is a digraph such that $\sigma(H(X)) = \{-(n-1), (-1)^{(n-1)}\}$, then $X \cong Y$ where $Y \in \{K_1, K_2, T_2, K_3', K_4'\}$ (where T_2 is the transitive tournament on 2 vertices).*

Proof. Let X be a digraph such that $H := H(X)$ has spectrum $\{-(n-1), (-1)^{(n-1)}\}$. Then X has n vertices, since H is diagonalizable. If $n = 1$, then $X \cong K_1$ and if $n = 2$ then $X \cong K_2$ or T_2. We may assume $n \geq 3$. The characteristic polynomial of H is $\phi(H, t) = (t + (n-1))(t-1)^{n-1}$. Observe that $\phi(H(K_n)) = (t - (n-1))(t+1)^{n-1}$ and so

$$[t^k]\phi(H,t) = \begin{cases} [t^k]\phi(H(K_n), t), & \text{if } n-k \text{ is even;} \\ -[t^k]\phi(H(K_n), t) & \text{if } n-k \text{ is odd.} \end{cases}$$

In particular, $[t^{n-2}]\phi(H,t) = [t^{n-2}]\phi(H(K_n), t)$. Thus, by Corollary 6.2.2, $\Gamma(X)$ has the same number of edges as K_n, and so $\Gamma(X) \cong K_n$.

Also, $\operatorname{tr}(H^3) = -\operatorname{tr}(H(K_n)^3)$. By Proposition 6.3.2, we see that $\operatorname{tr}(H(K_n)^3) = 6\binom{n}{3}$, since every 3 vertices of K_n induce a triangle isomorphic to the digraph X_7 in Proposition 6.3.2. We see that $\operatorname{tr}(H^3) = -6\binom{n}{3}$ and so every 3 vertices of X must induce a digraph isomorphic to digraph $X_3 \cong K_3'$ in Figure 6.2, the only X_j with negative weight.

Consider $G(X)$, the symmetric subgraph of X. If there is a path of length 2, say (u, v, w) in a component of $G(X)$, then $\{u, v, w\}$ induce a triangle of X with more than one digon and hence not isomorphic to K_3', which is a contradiction. Thus, each connected component of $G(X)$ is a copy of either K_1 or K_2. If $G(X)$ has 3 ore more components, then choosing three vertices from different components will give a triangle with no digons and hence not isomorphic to K_3'. Thus, $G(X)$ has at most 2 components. Since $n \geq 3$, we see that $G(X)$ has exactly two components and $n \in \{3, 4\}$. If $n = 3$, then $X \cong K_3'$ since K_3' is an induced sub-digraph.

If $n = 4$, then the deletion of any vertex of X results in a digraph isomorphic to K_3'. Let $x_1, \ldots, x_4$ be the vertices of X such that the deletion of x_4 results in a copy of K_3' with vertex labels as in Figure 8.9.

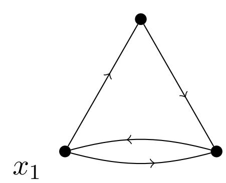


Figure 8.9: Digraph X with vertex x_4 deleted.

We see that x_4 must be in the same component of $G(X)$ as x_3 and so $R_X(x_3, x_4) = 1$. The vertices x_3, x_4 and x_2 induce a triangle of X isomorphic to K_3', so $R_X(x_2, x_4) = +$. The vertices x_3, x_4 and x_1 induce a triangle of X isomorphic to K_3', so $R_X(x_4, x_1) = +$. We have determined the adjacencies of all pairs of vertices of X and we observe now that $X \cong K_4'$. □

8.9 Digraphs with a large negative H-eigenvalue

An unfamiliar aspect of the spectra of the Hermitian adjacency matrix is that there exist digraphs where the least eigenvalue is strictly larger than the largest positive eigenvalue in absolute value. Here, we present an infinite family of digraphs with this property to provide examples and intuition for understanding the Hermitian adjacency matrix. In addition, we use the method of eigenvalue interlacing of Chapter 4, which is not available when working with the adjacency matrix of digraphs.

We define a digraph $X(a, b)$ on $2a + b$ vertices, where $a \geq 1$ and $b \geq 1$. The vertices of $X(a, b)$ consist of $X \cup Y \cup Z$, where $X = \{x_1, \ldots, x_a\}$, $Y = \{y_1, \ldots, y_a\}$ and $Z = \{z_1, \ldots, z_b\}$. The arcs are

$$\{x_j y_j, y_j x_j \mid j = 1, \ldots a\}$$

and

$$\{x_j z_\ell, z_\ell y_j \mid j = 1, \ldots, a \text{ and } \ell = 1, \ldots b\}.$$

We see that K_3' from the previous section is isomorphic to $X(1, 1)$. Figure 8.10 shows $X(1, 3)$ and $X(2, 3)$.

8.9.1 Proposition. *Digraph $X(a, b)$ as constructed above has H-spectrum*

$$\left\{ \frac{-1 + \sqrt{1 + 8ab}}{2}, 1^{(a)}, 0^{(b-1)}, -1^{(a-1)}, \frac{-1 - \sqrt{1 + 8ab}}{2} \right\}.$$

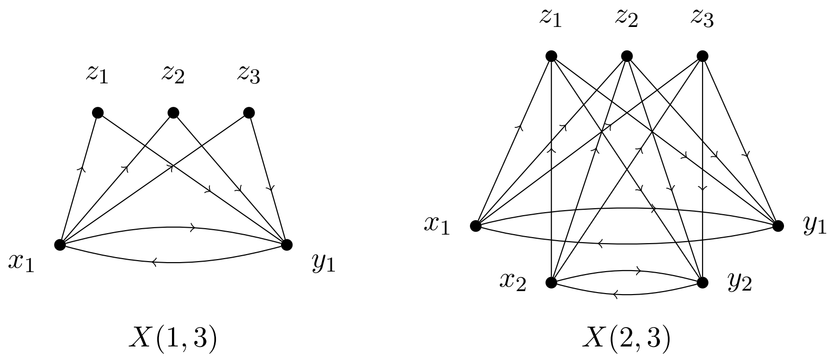


Figure 8.10: Digraphs $X(1,3)$ and $X(2,3)$, constructed as examples of digraphs with a large negative H-eigenvalue.

Proof. Let $H = H(a,b)$. We may write H as follows:

$$H = \begin{pmatrix} \mathbf{0} & I_a & iJ_{a,b} \\ I_a & \mathbf{0} & -iJ_{a,b} \\ -iJ_{b,a} & iJ_{b,a} & \mathbf{0} \end{pmatrix}$$

where we recall that I_n denotes the $n \times n$ identity matrix and $J_{m,n}$ denotes the $m \times n$ all-ones matrix.

Observe that the last b rows are all identical and hence linearly dependent. This implies that $\operatorname{rk}(H) \leq 2a + b - (b-1)$, which implies that H has 0 as an eigenvalue with multiplicity at least $b-1$.

For $j = 1, \ldots, a$, let $\mathbf{v}_j = (\mathbf{e}_j \quad \mathbf{e}_j \quad \mathbf{0})^T$, where $\mathbf{e}_j$ is the a-dimensional j-th elementary vector. We see that $H\mathbf{v}_j = \mathbf{v}_j$ for $j = 1, \ldots, a$ and so 1 is an eigenvalue of H with multiplicity at least a.

Similarly, for $j = 1, \ldots, a-1$, let

$$\mathbf{w}_j = \begin{pmatrix} \mathbf{e}_j - \mathbf{e}_a \\ -(\mathbf{e}_j - \mathbf{e}_a) \\ \mathbf{0} \end{pmatrix}$$

where e_n is defined as above. Then $H\mathbf{w}_j = -\mathbf{w}_j$ and so -1 is an eigenvalue of H with multiplicity at least $a-1$.

We have found $2a + b - 2$ eigenvalues of H. To find the remaining two eigenvalues, we will use the interlacing theorem. Partition the vertices of $X(a,b)$ (and consequently the rows and columns of H) into the sets X, Y and Z. Each block of H, under this partition, has constant row sums and so this is an equitable partition of H. We obtain B the quotient matrix corresponding to this partition as follows:

$$B = \begin{pmatrix} 0 & 1 & ib \\ 1 & 0 & -ib \\ -ia & ia & 0 \end{pmatrix}.$$

We find that the characteristic polynomial of B is

$$\phi(B,t) = t^3 - (2ab+1)t + 2ab = (t-1)(t^2 + t - 2ab).$$

Using the quadratic formula, we see that the roots of $\phi(B,t)$ are 1, τ and σ, where

$$\tau = \frac{-1+\sqrt{1+8ab}}{2}$$

and

$$\sigma = \frac{-1-\sqrt{1+8ab}}{2}.$$

The partition is equitable and so τ and σ are also eigenvalues of H. Since for $a,b \geq 1$, τ and σ are not equal to any of the eigenvalues of H that we have already found. Then H has spectrum $\{\tau^{(1)}, 1^{(a)}, 0^{(b-1)}, -1^{(a-1)}, \sigma^{(1)}\}$. □

Note that $X(a,b)$ has $\rho(H(X(a,b))) - \lambda_1(H(X(a,b))) = 1$, where $\lambda_1(M)$ denotes the largest eigenvalue of matrix M. We now use the Cartesian product to construct digraphs where this difference is much larger. We let $X^{\square n}$ denote the n-fold Cartesian product of X with itself; that is $X^{\square n} = X \square \cdots \square X$, where there are n terms in the product.

8.9.2 Lemma. *The digraph $X_n = K_4'^{\square n}$ has $\rho(H(X_n)) - \lambda_1(H(X_n)) = 2n$.*

Proof. By applying Proposition 8.3.1 n times, the set of H-eigenvalues of X_n is

$$\left\{ \sum_{j=1}^{n} \beta_j \mid \beta_j \in \{-3, 1\} \right\},$$

considered without multiplicity. Then $\rho(H(X_n)) = -3n$ and $\lambda_1(H(X_n)) = n$. □

Chapter 9

Interlacing bounds in eulerian digraphs

9.1 Laplacian matrix of directed graphs

We define the A-Laplacian matrix of directed graphs analoguous to the undirected Laplacian. For a digraph X, let $\Delta^+(X)$ be the diagonal matrix indexed by the vertex set of X with $\Delta^+(u,u)$ equal to the out-degree of vertex u. Let A be the adjacency matrix of X. Then, the *A-Laplacian matrix of digraph X* is

$$L(X) = \Delta^+(X) - A(X).$$

We will write L, Δ^+ and A when there is no ambiguity. We are interested in digraphs where L is normal. Recall that we assume our digraphs have no loops or parallel arcs.

9.1.1 Proposition. *For a weakly connected digraph X, if $L(X)$ is normal, then X is eulerian.*

Proof. Since L is normal, we have that

$$\begin{aligned}
0 &= LL^T - L^TL \\
&= (\Delta^+ - A)(\Delta^+ - A)^T - (\Delta^+ - A)^T(\Delta^+ - A) \\
&= (\Delta^+)^2 - A\Delta^+ - \Delta^+A^T + AA^T - (\Delta^+)^2 + A^T\Delta^+ + \Delta^+A - A^TA \\
&= \Delta^+(A - A^T) + (A^T - A)\Delta^+ + AA^T - A^TA =: M.
\end{aligned}$$

For a square matrix N, let $\operatorname{diag}(N)$ denote the vector consisting of the diagonal entries of N:

$$(\operatorname{diag}(N))_u = N(u,u).$$

Observe that Δ^+ is a diagonal matrix and both A and A^T have zero diagonal. Then

$$\operatorname{diag}(\Delta^+(A - A^T)) = \operatorname{diag}((A^T - A)\Delta^+) = 0.$$

We see that $M = 0$ and so has $\operatorname{diag}(M) = 0$. Then

$$\operatorname{diag}(AA^T) = \operatorname{diag}(A^T A). \tag{9.1}$$

Combinatorially, we have that

$$(A^T A)_{u,u} = |\{w : wu \text{ is an arc of } X\}| = \deg^-(u)$$

and

$$(AA^T)_{u,u} = |\{y : uy \text{ is an arc of } X\}| = \deg^+(u).$$

This and (9.1) implies that $\deg^-(u) = \deg^+(u)$ for every vertex u of X and, since X is weakly connected, so X is eulerian. □

If a weakly connected digraph is Eulerian, then it is also strongly connected. Since there is no confusion, we may say that such graphs are *connected*.

The following lemma can be proved by considering $L + L^T$ as the Laplacian of a multi-graph and appealing to known results about Laplacians of graphs. Here, however, we will give a direct proof to be thorough. Parts (a) and (b) of Lemma 9.1.2 follow from well-known results on M-matrices, see [4].

9.1.2 Lemma. *If X is a connected digraph and $L(X)$ is normal, then*

(a) $L(X)$ has eigenvalue 0 with multiplicity 1;

(b) $\operatorname{Re}(\lambda) > 0$ for $\lambda \neq 0$ an eigenvalue of $L(X)$; and

(c) if $L(X)\mathbf{v} = \lambda\mathbf{v}$, then $L(X)^T\mathbf{v} = \bar{\lambda}\mathbf{v}$.

Proof. First, we will show part (c). Let $\mathbf{v}$ be an eigenvector of $L := L(X)$ with eigenvalue λ. Since L and L^T commute, they may be simultaneously diagonalized, so we many assume $\mathbf{v}$ is also an eigenvector of L^T with eigenvalue θ. Then

$$\mathbf{v}^* L\mathbf{v} = \lambda\mathbf{v}^*\mathbf{v}$$

and

$$\mathbf{v}^* L\mathbf{v} = (L^*\mathbf{v})^*\mathbf{v} = (\theta\mathbf{v})^*\mathbf{v} = \bar{\theta}\mathbf{v}^*\mathbf{v}.$$

Then, $\bar{\theta} = \lambda$.

To show parts (a) and (b), we will first show that

$$\operatorname{Re}(\lambda) \geq 0$$

for all eigenvalues λ of L. Observe that 0 is an eigenvalue of L with the all ones vector as an eigenvector. We will use the property that X is connected to establish that 0 has multiplicity 1 as an eigenvalue and that the other eigenvalues lie in the open right half plane.

To see that all eigenvalues of L lie in the closed right half plane, we must look at some other incidence matrices.

For any digraph X, we may define the two incidence matrices, D_h and D_t of X, with rows indexed by vertices and columns indexed by edges such that the (u, e) entry is as follows:

$$(D_h)_{u,e} = \begin{cases} 1 & \text{if } u = h(e); \\ 0 & \text{otherwise} \end{cases}$$

and

$$(D_t)_{u,e} = \begin{cases} 1 & \text{if } u = t(e); \\ 0 & \text{otherwise.} \end{cases}$$

We see that if X is eulerian, then

$$\Delta^+ = D_t D_t^T = D_h D_h^T$$

and

$$A = D_t D_h^T.$$

Let $N = D_t - D_h$. Then

$$\begin{aligned} NN^T &= (D_t - D_h)(D_t - D_h)^T \\ &= D_t D_t^T - D_t D_h^T - D_h D_t^T + D_h D_h^T \\ &= D_t D_t^T - A + D_h D_h^T - A^T \\ &= L + L^T. \end{aligned}$$

But, NN^T is a square, symmetric matrix with all non-negative eigenvalues since

$$\mathbf{v}^* NN^T \mathbf{v} = ||N^T \mathbf{v}||^2 \geq 0$$

for any vector $\mathbf{v} \in \mathbb{C}^n$, where n is the order of N.

Let $\mathbf{v}$ be an eigenvector of L with eigenvalue λ. Then,

$$(L + L^T)\mathbf{v} = (\lambda + \bar{\lambda})\mathbf{v} = 2\,\mathrm{Re}(\lambda)\mathbf{v}$$

Then $2\,\mathrm{Re}(\lambda)$ is an eigenvalue of NN^T and so $\mathrm{Re}(\lambda) \geq 0$.

Let d be the maximum out-degree of A. Let

$$B = dI - L$$

where I is the $n \times n$ identity matrix. We see that B is a non-negative matrix. Let $\mathbf{v}$ be an eigenvector of L with eigenvalue λ. Then

$$B\mathbf{v} = d\mathbf{v} - L\mathbf{v} = (d - \lambda)\mathbf{v}$$

and $\mathbf{v}$ is an eigenvector of B with eigenvalue $d - \lambda$. Conversely, if $\mathbf{x}$ is an eigenvector of B with eigenvalue θ, then

$$L\mathbf{x} = d\mathbf{x} - B\mathbf{x} = (d - \theta)\mathbf{x}.$$

Then λ is an eigenvalue of L if and only if $d - \lambda$ is an eigenvalue of B with the same multiplicity.

We may regard B as the adjacency matrix of some digraph X' with some number of loops at each vertex. Since X is connected, then so is X'. Then B is irreducible and we may apply the Perron-Frobenius theorem. We obtain that B has a positive real eigenvalue ρ such that $\rho \geq |\theta|$ for all θ eigenvalues of B and ρ has algebraic multiplicity 1. We see that $\rho = d - \lambda$ for some eigenvalue λ of L. Since $\mathrm{Re}(\lambda) \geq 0$ for all eigenvalues of L, we have that $d - \lambda$ is maximized by $\lambda = 0$. Then $\rho = d$ is the Perron value of B and thus has multiplicity 1. Then 0 is an eigenvalue of L with multiplicity 1.

If L has an eigenvalue $\lambda = i\beta$ for some $\beta \neq 0 \in \mathbb{R}$. Then $|d - \lambda|$ is an eigenvalue of B and $|d - \lambda| \geq |d|$, which is a contradiction. Then for $\lambda \neq 0$ an eigenvalue of L, we see that

$$\mathrm{Re}(\lambda) > 0$$

as claimed. □

Observe that if X is not connected, then we may consider the spectrum over the connected components of X to obtain the following corollary.

9.1.3 Corollary. *If X is a digraph and $L(X)$ is normal, then $\mathrm{Re}(\lambda) > 0$ for all eigenvalues λ of $L(X)$ except $\lambda = 0$.*

9.2 Interlacing with the A-Laplacian

In this section, we will consider eulerian digraphs X such that the A-Laplacian matrix of X is normal. Let $\alpha(X)$ denote the size of the largest acyclic subgraph of X. First we will find a spectral bound of disjoint vertex sets Y and Z with no arcs from Z to Y. Then, we will use the bound to find a spectral bound for the maximum acyclic subdigraph of X.

We would like to use interlacing to find bounds of combinatorial properties of digraph X using eigenvalues of some matrix respecting the adjacency of X. In particular, we would like to use the same method as the proof of Lemma 6.1 in [32], which is stated here as Corollary 9.3.1. We achieve this using the A-Laplacian matrix and prove a result which generalizes the original lemma of Haemers to a sub-class of digraphs. However, as this matrix is not symmetric like in the case for graphs, we need to restrict to digraphs X whose the A-Laplacian matrices are normal and we also need a few technical lemmas in order to prove Theorem 9.2.3, the main result of this chapter.

For L, we see that the singular values of $L^T L$ are just $|\lambda|$ for each eigenvalue λ of L.

9.2.1 Lemma. *Let X be an eulerian digraph on n vertices such that the Laplacian L of X is normal. If λ is an eigenvalue of L, then $|\alpha + \lambda|$ is a singular value of*

$$\alpha I + L$$

for each $\alpha \in \mathbb{C}$ and I is the $n \times n$ identity matrix.

Proof. We need to consider the following:

$$\begin{aligned}(\alpha I + L)(\alpha I + L)^* &= (\alpha I + L)(\bar{\alpha} I + L^T)\\ &= |\alpha|^2 I + \alpha L^T + \bar{\alpha} L + L L^T.\end{aligned}$$

Let $\mathbf{v}$ be an eigenvector of L with eigenvalue λ. Then

$$\begin{aligned}(\alpha I + L)(\alpha I + L)^* \mathbf{v} &= |\alpha|^2 \mathbf{v} + \alpha L^T \mathbf{v} + \bar{\alpha} L \mathbf{v} + L L^T \mathbf{v}\\ &= |\alpha|^2 \mathbf{v} + \alpha \bar{\lambda} \mathbf{v} + \bar{\alpha} \lambda \mathbf{v} + |\lambda|^2 \mathbf{v}\\ &= (\alpha + \lambda)(\overline{\alpha + \lambda}) \mathbf{v}\\ &= |\alpha + \lambda|^2 \mathbf{v}.\end{aligned}$$

Then the singular values of $\alpha I + L$ are $|\alpha + \lambda|$ for each eigenvalue λ of L. □

9.2.2 Lemma. *Let X be a digraph such that $L(X)$ is normal. Let*

$$f(\lambda) = \frac{|\lambda|^2}{2\,\mathrm{Re}(\lambda)}$$

and let $\theta \neq 0$ be the eigenvalue of $L(X)$ which maximizes f amongst non-zero eigenvalues of $L(X)$. Let

$$g(\lambda) = \frac{\mathrm{Re}(\lambda)(f(\theta) - f(\lambda))}{\mathrm{Re}(\theta) - \mathrm{Re}(\lambda)}$$

and let μ be the eigenvalue of $L(X)$ which minimizes g such that $g(\mu) \geq 0$, if such an eigenvalue exists in the domain of g, and $\mu = 0$ otherwise. Let $\widetilde{L} = \alpha I + L(X)$, where $\alpha \leq -f(\theta)$. Then $|\alpha|$ is the largest singular value of $\widetilde{L}$. Further,

(a) if X is connected and $\mathrm{Re}(\lambda) \geq \mathrm{Re}(\theta)$ for all $\lambda \notin \{0, \theta\}$, then $|\alpha + \theta|$ is the second largest singular value of $\widetilde{L}$; and

(b) *if X is connected and $-f(\theta) - g(\mu) \leq \alpha$, then $|\alpha + \theta|$ is the second largest singular value of $\widetilde{L}$.*

Proof. Note that f is well-defined for non-zero eigenvalues λ of L, since $\mathrm{Re}(\lambda) > 0$ by Lemma 9.1.2. Observe also that f is positive real-valued and $\alpha \in \mathbb{R}$. The function g is well-defined for λ when $\mathrm{Re}(\lambda) \neq \mathrm{Re}(\theta)$. If there exists an eigenvalue $\lambda \notin \{0, \theta\}$, such that $\mathrm{Re}(\lambda) < \mathrm{Re}(\theta)$, then, we can see that $g(\lambda) \geq 0$, and so μ is non-zero. Also, the range for α in part (b), $[-f(\theta) - g(\mu), -f(\theta)]$, is non-empty.

Since 0 is an eigenvalue of L, Lemma 9.2.1 gives that $|\alpha|$ is a singular value of $\widetilde{L}$. The singular values of $\widetilde{L}$ are of form $|\alpha + \lambda|$ where λ is an eigenvalue of L. Let λ be a non-zero eigenvalue of L. Consider

$$\begin{aligned} |\alpha|^2 - |\alpha + \lambda|^2 &= \alpha^2 - (\alpha + \lambda)(\overline{\alpha + \lambda}) \\ &= \alpha^2 - (\alpha + \lambda)(\alpha + \bar{\lambda}) \\ &= \alpha^2 - (\alpha^2 + \alpha(\lambda + \bar{\lambda}) + |\lambda|^2) \\ &= -\alpha(\lambda + \bar{\lambda}) - |\lambda|^2 \\ &= -2\alpha \,\mathrm{Re}(\lambda) - |\lambda|^2. \end{aligned} \tag{9.2}$$

By definition of α, we have that

$$-\alpha \geq f(\theta) \geq f(\lambda) = \frac{|\lambda|^2}{2\,\mathrm{Re}(\lambda)} \tag{9.3}$$

for all nonzero $\lambda \in \sigma(L)$. From (9.2) and (9.3), we obtain:

$$|\alpha|^2 - |\alpha + \lambda|^2 = -2\alpha \,\mathrm{Re}(\lambda) - |\lambda|^2 \geq 0$$

and we have shown the first part of the statement.

For statements (a) and (b), let δ be as follows:

$$\delta(\lambda) := |\alpha + \theta|^2 - |\alpha + \lambda|^2.$$

Since X is connected, L has only one eigenvalue whose real part is equal to 0 by Lemma 9.1.2. It suffices to show that $\delta(\lambda) \geq 0$ for all nonzero $\lambda \in \sigma(L)$. We expand $\delta(\lambda)$ as follows:

$$\begin{aligned} \delta(\lambda) &= |\alpha + \theta|^2 - |\alpha + \lambda|^2 \\ &= \alpha^2 + \alpha(\theta + \bar{\theta}) + |\theta|^2 - (\alpha^2 + \alpha(\lambda + \bar{\lambda}) + |\lambda|^2) \\ &= |\theta|^2 - |\lambda|^2 + 2\alpha(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda)). \end{aligned}$$

If $\mathrm{Re}(\theta) = \mathrm{Re}(\lambda)$, then $\delta(\lambda) = |\theta|^2 - |\lambda|^2$. In this case,

$$f(\theta) = \frac{|\theta|^2}{2\,\mathrm{Re}(\theta)} = \frac{|\theta|^2}{2\,\mathrm{Re}(\lambda)} \geq f(\lambda) = \frac{|\lambda|^2}{2\,\mathrm{Re}(\lambda)}$$

and, since $\mathrm{Re}(\lambda)$ and $\mathrm{Re}(\theta)$ are positive, $|\theta|^2 \geq |\lambda|^2$ and $\delta(\lambda) \geq 0$.

If $\mathrm{Re}(\theta) < \mathrm{Re}(\lambda)$, then we may consider

$$\delta(\lambda) = |\theta|^2 - |\lambda|^2 - 2\alpha(\mathrm{Re}(\lambda) - \mathrm{Re}(\theta)).$$

Since $(\mathrm{Re}(\lambda) - \mathrm{Re}(\theta)) > 0$ and $-\alpha \geq f(\theta) \geq 0$, we may simplify as follows:

$$\begin{aligned}
\delta(\lambda) &\geq |\theta|^2 - |\lambda|^2 + 2f(\theta)(\mathrm{Re}(\lambda) - \mathrm{Re}(\theta)) \\
&= |\theta|^2 - |\lambda|^2 + \frac{|\theta|^2}{\mathrm{Re}(\theta)}(\mathrm{Re}(\lambda) - \mathrm{Re}(\theta)) \\
&= |\theta|^2 - |\lambda|^2 - |\theta|^2 + \frac{|\theta|^2\,\mathrm{Re}(\lambda)}{\mathrm{Re}(\theta)} \\
&= -|\lambda|^2 + \frac{|\theta|^2\,\mathrm{Re}(\lambda)}{\mathrm{Re}(\theta)} \\
&= \mathrm{Re}(\lambda)\left(-\frac{|\lambda|^2}{\mathrm{Re}(\lambda)} + \frac{|\theta|^2}{\mathrm{Re}(\theta)}\right) \\
&= 2\,\mathrm{Re}(\lambda)(f(\theta) - f(\lambda)) \\
&\geq 0.
\end{aligned}$$

We have shown part (a) and also part (b) when $\mathrm{Re}(\lambda) \geq \mathrm{Re}(\theta)$.

For part (b), we need only consider eigenvalues λ such that $\mathrm{Re}(\theta) > \mathrm{Re}(\lambda)$. In this case, we will use that

$$g(\lambda) \geq g(\mu) \geq 0.$$

Then

$$\begin{aligned}
\delta(\lambda) &= |\theta|^2 - |\lambda|^2 + 2\alpha(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda)) \\
&\geq |\theta|^2 - |\lambda|^2 + (-2f(\theta) - 2g(\mu))(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda)) \\
&= |\theta|^2 - |\lambda|^2 - |\theta|^2 + 2f(\theta)\,\mathrm{Re}(\lambda) - 2g(\mu)(\mathrm{Re}(\theta) - \mathrm{Re}(\lambda))
\end{aligned}$$

Since $\operatorname{Re}(\theta) - \operatorname{Re}(\lambda) > 0$ and $-2g(\mu) \geq -2g(\lambda)$, we obtain

$$\begin{aligned}
\delta(\lambda) &\geq -|\lambda|^2 + 2f(\theta)\operatorname{Re}(\lambda) - 2g(\lambda)(\operatorname{Re}(\theta) - \operatorname{Re}(\lambda)) \\
&= -|\lambda|^2 + 2f(\theta)\operatorname{Re}(\lambda) - 2\operatorname{Re}(\lambda)(f(\theta) - f(\lambda)) \\
&= -|\lambda|^2 + 2\operatorname{Re}(\lambda)f(\lambda) \\
&= 0
\end{aligned}$$

and we obtain that $\delta(\lambda) \geq 0$, as required. □

It is worth observing that if we take $\alpha = -f(\theta)$, then

$$|\alpha + \theta|^2 = \alpha^2 + \alpha 2\operatorname{Re}(\theta) + |\theta|^2 = |\alpha|^2.$$

9.2.3 Theorem. *Let X be a connected digraph on n vertices where $L := L(X)$ is normal. Let f, g, θ and μ be as defined in Lemma 9.2.2. Also, let $\nu \neq 0$ be the eigenvalue of L which minimizes f amongst non-zero eigenvalues of L. Let Y and Z be disjoint vertex sets in X with no arcs from Z to Y. Then,*

$$\frac{|Y||Z|}{(n-|Y|)(n-|Z|)} \leq \frac{|\alpha+\theta|^2}{\alpha^2},$$

where $\alpha = \begin{cases} -f(\theta) - f(\nu), & \text{if } \operatorname{Re}(\lambda) \geq \operatorname{Re}(\theta) \text{ for all } \lambda \notin \{0, \theta\}; \\ -f(\theta) - g(\mu), & \text{otherwise.} \end{cases}$

Proof. Let $\alpha = -f(\theta) - g(\mu)$ and let $\widetilde{L} = \alpha I + L$. In L and $\widetilde{L}$, there is an off-diagonal block of 0s, where the rows are indexed by Z and columns are indexed by Y. This follows directly from hypothesis that there are no arcs from Z to Y. We wish to use interlacing to bound the size of such an off-diagonal block of 0s. Let

$$C = \begin{pmatrix} 0 & \alpha I + L \\ \alpha I + L^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{L} \\ \widetilde{L}^T & 0 \end{pmatrix}.$$

Note that we use 0 in matrices to represent the zero matrix of the appropriate dimensions and hence will omit subscripts. We see that C is symmetric and the eigenvalues of C are

$$\{\pm\sigma \,:\, \sigma \text{ a singular value of } \widetilde{L}\}.$$

Using Lemma 9.2.1, we can write the eigenvalues of C as

$$\{\pm|\alpha+\lambda| \,:\, \lambda \text{ eigenvalue value of } L\}.$$

By Lemma 9.2.2, we see that $|\alpha|$ is the biggest eigenvalue of C and $|\alpha + \theta|$ is the second largest eigenvalue of C.

Since X is eulerian, each row and column of L sums to 0 and so each row and column of $\widetilde{L}$ sum to α. We may partition the rows of $\widetilde{L}$ into rows indexed by $\{Z, V(X) \setminus Z\}$ and the columns of $\widetilde{L}$ into columns indexed by $\{V(X) \setminus Y, Y\}$. Then,

$$\widetilde{L} = \begin{pmatrix} \widetilde{L}_{11} & 0 \\ \widetilde{L}_{21} & \widetilde{L}_{22} \end{pmatrix}.$$

This partition of $\widetilde{L}$ induces a partition of C where all diagonal blocks are square;

$$A = \begin{pmatrix} 0 & 0 & \widetilde{L}_{11} & 0 \\ 0 & 0 & \widetilde{L}_{21} & \widetilde{L}_{22} \\ \widetilde{L}_{11}^T & \widetilde{L}_{21}^T & 0 & 0 \\ 0 & \widetilde{L}_{22}^T & 0 & 0 \end{pmatrix}.$$

We let B be the quotient matrix of C with respect to this partition. Recall from Theorem 4.2.3 that the entries of B are the average row sums of the corresponding blocks of C. We will index the rows and columns of B with $[4]$, for convenience. Since the row and column sums of $\widetilde{L}$ are all equal to α, we see that each row and column sum of $\widetilde{L}_{11}$ and of the matrix $\begin{pmatrix} \widetilde{L}_{21} & \widetilde{L}_{22} \end{pmatrix}$ is equal to α. Then $B(1,3) = B(4,2) = \alpha$ and

$$B(2,3) + B(2,4) = B(3,1) + B(3,2) = \alpha.$$

Observe that $\widetilde{L}_{22}$ is a $n - z \times y$ matrix, where the lower $y \times y$ submatrix is a principal submatrix of $\widetilde{L}$. For $S, T \subseteq V(X)$, let $E(S,T)$ denote the set of edges e such that $t(e) \in S$ and $h(e) \in T$. Let $W = V(X) \setminus (Y \cup Z)$. We will find $B(2,4)$ by taking the sum over all of the entries of $\widetilde{L}_{22}$ as follows:

$$\begin{aligned} (n-2)B(2,4) &= \sum_{j=1}^{n} \sum_{\ell=1}^{n} \widetilde{L}_{22}(j,\ell) \\ &= y\alpha + \sum_{y \ inY} d^+(y) - |E(Y,Y)| - |E(W,Y)|. \end{aligned} \tag{9.4}$$

Since there are not arcs from Z to Y, we have that

$$|E(Y,Y)| + |E(W,Y)| = |E(V(X),Y)| = \sum_{y \in Y} d^-(y). \tag{9.5}$$

Since X is eulerian, we see that $\sum_{y \in Y} d^-(y) = \sum_{y \in Y} d^+(y)$. Then, from (9.4) and (9.5), we obtain that $B(2,4) = \frac{y\alpha}{n-z}$, which implies that $B(2,3) = \alpha - \frac{y\alpha}{n-z}$. By an analoguous argument, we find that $B(3,1) = \frac{z\alpha}{n-y}$ and $B(3,2) = \alpha - \frac{z\alpha}{n-y}$. Thus we have

$$B = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \alpha - \frac{y\alpha}{n-z} & \frac{y\alpha}{n-z} \\ \frac{z\alpha}{n-y} & \alpha - \frac{z\alpha}{n-y} & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}.$$

Since B is the quotient of a symmetric matrix, we see from the proof of Theorem 4.2.3 that B is similar to a symmetric matrix. Thus we may let the real eigenvalues of B be $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n-1} \geq \lambda_{2n}$ be the eigenvalues of C. Observe that C is similar to $-C$ and B is similar to $-B$ by construction of C. Then, we have that

$$\mu_4 = -\mu_1,\ \mu_3 = -\mu_2,\ \lambda_{2n} = -\lambda_1 \text{ and } \lambda_{2n-1} = -\lambda_2.$$

Applying the interlacing theorem gives

$$\lambda_1 \geq \mu_1,\ \lambda_2 \geq \mu_2 \text{ and } \mu_3 \geq \lambda_{2n-1},\ \mu_4 \geq \lambda_{2n}.$$

Recall that $(\lambda_1, \lambda_2) = (|\alpha|, |\alpha + \theta|)$. Then

$$\mu_1\mu_2\mu_3\mu_4 = (-1)^2(\mu_1\mu_2)^2 \leq (\lambda_1\lambda_2)^2 = (|\alpha||\alpha+\theta|)^2. \tag{9.6}$$

On the other hand, we see that

$$\mu_1\mu_2\mu_3\mu_4 = \det(B) = \frac{\alpha^2 y}{n-z}\frac{\alpha^2 z}{n-y}. \tag{9.7}$$

From (9.6) and (9.7), we obtain that

$$\frac{\alpha^2 y}{n-z}\frac{\alpha^2 z}{n-y} \leq (|\alpha||\alpha+\theta|)^2$$

which simplifies to

$$\frac{y}{n-z}\frac{z}{n-y} \leq \frac{|\alpha+\theta|^2}{\alpha^2}$$

as claimed. □

Using Theorem 9.2.3, we can find a bound on the maximum acyclic subdigraph of a digraph whose A-Laplacian is normal.

9.2.4 Theorem. *Let X be a connected digraph on n vertices such $L := L(X)$ is normal. Let α be* *as in Theorem 9.2.3. If X has an induced acyclic subdigraph of order s, then*

$$s \leq \frac{2n}{1+\frac{|\alpha|}{|\alpha+\theta|}} + 1.$$

Proof. Let S be an induced acyclic subdigraph of X of order s. Since S is acyclic, there is an ordering, $(v_1, \ldots, v_s)$, of the vertices of S such that v_j has no neighbours amongst $\{v_i : i \leq j\}$. Then, any partition of the vertices of S into

$$Y = \{v_1, \ldots, v_j\} \text{ and } Z = \{v_{j+1}, \ldots, v_s\}$$

will have the property that there are no arcs from Z to Y. Then, for any $y, z \geq 0$ such that $y + z = s$, we can find Y and Z such that $|Y| = y$, $|Z| = z$ and there are no arcs in X from Z to Y. In particular, we will consider $y = \lfloor \frac{s}{2} \rfloor$ and $z = \lceil \frac{s}{2} \rceil$. Let $\alpha = -f(\theta) - g(\mu)$ where θ, μ, $f(\theta)$ and $f(\mu)$ are as in the statement of the theorem. We may apply Theorem 9.2.3 to obtain that

$$\frac{yz}{(n-y)(n-z)} \leq \left(\frac{|\alpha+\theta|}{|\alpha|}\right)^2.$$

Observe that $y \leq z$ and that $\frac{1}{n-y} \leq \frac{1}{n-z}$. Then

$$\frac{y^2}{(n-y)^2} \leq \left(\frac{|\alpha+\theta|}{|\alpha|}\right)^2$$

and so we see that

$$\frac{y}{(n-y)} \leq \frac{|\alpha+\theta|}{|\alpha|}.$$

We simplify this expression to obtain a bound on y as follows:

$$\begin{aligned}
\frac{y}{(n-y)} &\leq \frac{|\alpha+\theta|}{|\alpha|} \\
y &\leq (n-y)\frac{|\alpha+\theta|}{|\alpha|} \\
y + y\left(\frac{|\alpha+\theta|}{|\alpha|}\right) &\leq n\frac{|\alpha+\theta|}{|\alpha|} \\
y\left(\frac{|\alpha+\theta|+|\alpha|}{|\alpha|}\right) &\leq n\frac{|\alpha+\theta|}{|\alpha|} \\
y &\leq \frac{n}{1+\frac{|\alpha|}{|\alpha+\theta|}}.
\end{aligned}$$

Recall that $y = \lfloor \frac{s}{2} \rfloor \geq \frac{s-1}{2}$ and

$$\frac{s-1}{2} \leq y \leq \frac{n}{1+\frac{|\alpha|}{|\alpha+\theta|}}$$

which gives that

$$s \leq \frac{2n}{1 + \frac{|\alpha|}{|\alpha+\theta|}} + 1.$$

Note that if $|S|$ is even, then the 1 on the right hand side can be omitted. $\square$

9.3 Corollaries

If X is a graph, then the A-Laplacian is the usual Laplacian matrix of a graph. In this case, the A-Laplacian is symmetric and hence normal and so all eigenvalues are real and non-negative. Thus, for λ an eigenvalue of $L(X)$, we see that

$$f(\lambda) = g(\lambda) = \lambda/2.$$

Then, we can recover the original theorem of Haemers [32, Lemma 6.1] for graphs as a corollary of Theorem 9.2.3, which is the following.

9.3.1 Corollary ([32])**.** *Let X be a connected graph on n vertices and let Y and Z be disjoint vertex sets in X with no arcs from Z to Y. Then,*

$$\frac{|Y||Z|}{(n-|Y|)(n-|Z|)} \leq \frac{(\sigma_n - \sigma_2)^2}{(\sigma_n + \sigma_2)^2}$$

where $0 = \sigma_1 < \sigma_2 \leq \cdots \leq \sigma_n$ are the Laplacian eigenvalues of X.

A tournament has normal adjacency matrix if and only if it is regular (see [20]). Then, the A-Laplacian matrices of regular tournaments are normal matrices. Let X be a regular tournament on n vertices. In this case, all of the non-zero eigenvalues have real part equal to $\frac{n}{2}$. We see that

$$f(\lambda) = \frac{|\lambda|^2}{2\frac{n}{2}} = \frac{|\lambda|^2}{n}.$$

Then, $\alpha = -\frac{|\theta|^2}{n} - \frac{|\nu|^2}{n}$, where θ and ν are the largest and smallest eigenvalues of $L(X)$ in magnitude. Theorem 9.2.4 gives the following corollary.

9.3.2 Corollary. *Let X be a connected, regular tournament on n vertices with an acyclic subgraph of order m, then*

$$m \leq \frac{2n}{1 + \frac{|\alpha|}{|\alpha+\theta|}} + 1$$

where $\alpha = -\frac{|\theta|^2}{n} - \frac{|\nu|^2}{n}$ where θ and ν are the largest and smallest eigenvalues of $L(X)$ in magnitude.

9.4 Examples of digraphs with normal Laplacian matrices

We observe that, if the digraph is regular, then the Laplacian matrix is normal if and only if the adjacency matrix is normal. We say that a digraph is *normal* if it has a normal Laplacian matrix and a normal adjacency matrix.

9.4.1 Theorem. *Every Cayley digraph on an abelian group is normal.*

Proof. Consider a Cayley digraph $X = \text{Cay}(G, C)$ where G is abelian and let A be the adjacency matrix of X. Since X is regular, we need only check that A is normal. Then

$$(A^T A)_{u,v} = |\{w : wu \text{ and } wv \text{ are arcs of } X\}|$$

and

$$(AA^T)_{u,v} = |\{y : uy \text{ and } vy \text{ are arcs of } X\}|.$$

Let u, v be vertices of X and suppose they have a common in-neighbour w. Then for some $a_1 \neq a_2 \in C$, we have

$$u = a_1 w \text{ and } v = a_2 w.$$

It is clear that $a_2 u$ is an out-neighbour of u and $a_1 v$ is an out-neighbour of v. In addition,

$$a_2 u = a_2 a_1 w = a_1 a_2 w = a_1 v$$

and so $a_2 u$ is a common out-neighbour of u and v. Conversely, following the same argument, given a common out-neighbour of u and v, we may construct a common in-neighbour of u and v. Then

$$(A^T A)_{u,v} = (AA^T)_{u,v}$$

and so A is a normal matrix. □

In general, the adjacency matrix being normal does not have to coincide with the Laplacian matrix being normal. We present some data on all digraph on 4,5 and 6 vertices. Note that if X has a normal adjacency matrix or a normal Laplacian, it must be eulerian.

Table 9.1: Small digraphs with normal Laplacian and adjacency matrices.

	4 vertices	5 vertices	6 vertices
number of digraphs	218	9608	1540944
eulerian	17	107	2269
regular	5	10	52
normal Laplacian	14	43	194
normal adjacency matrix	14	45	212
normal	14	43	190
connected and eulerian	12	90	2162
undirected	10	31	43

Chapter 10

Simple eigenvalues of cubic vertex-transitive graphs

In this chapter, we turn our attention to simple eigenvalues of cubic vertex-transitive graphs. Cubic vertex-transitive graphs have been studied in [52, 51] and a census of all such graphs with order at most 1280 vertices is maintained by Potočnik, Spiga and Verret in [50].

10.1 Cubic vertex-transitive graphs having 1 and -1 as simple eigenvalues

A cubic graph X has largest eigenvalue 3, which is simple if and only if X is connected. It is well-known that if -3 is also an eigenvalue of X and X is connected, then -3 is a simple eigenvalue and X is bipartite; see [5] for example. By Theorem 3.2.4, the only possible simple eigenvalues of a cubic vertex-transitive graph besides ± 3 are ± 1.

We will index rows and columns of the adjacency matrix $A = A(X)$ of X by vertices of X. We will use function notation where $A(x, y)$ denotes the (x, y)-entry of A. For $\mathbf{v}$ a vector indexed by the vertices of X and a vertex x of X, we write $\mathbf{v}(x)$ for the entry of $\mathbf{v}$ corresponding to x.

A bipartite graph with bipartite (B, C) is said to be*semi-regular* if every vertex in B has degree b and every vertex in C has degree c. Recall from Chapter 4 that a partition $\{V_1, \ldots, V_m\}$ of the vertices of a graph X is *equitable* if the subgraph of X induced by each V_i is regular and the bipartite subgraph of X induced by the edges from V_i to V_j is semi-regular, for each pair i, j such that $i \neq j$. If that is the case, then we define the $m \times m$ *quotient matrix* $B = [b_{i,j}]_{i,j=1}^{m}$ whose entries b_{ij} are number of neighbours of any vertex in V_i in V_j.

10.1.1 Theorem. *If a cubic vertex-transitive graph X has both 1 and -1 as simple eigenvalues, then X is bipartite.*

Proof. Let $\mathbf{v}$ and $\mathbf{u}$ be the ± 1 eigenvectors for eigenvalues 1 and -1, respectively. Let

$$\begin{aligned}V^+ &= \{x \in V(X) \mid \mathbf{v}(x) = 1\},\\ V^- &= \{x \in V(X) \mid \mathbf{v}(x) = -1\},\\ U^+ &= \{x \in V(X) \mid \mathbf{u}(x) = 1\}\end{aligned}$$

and

$$U^- = \{x \in V(X) \mid \mathbf{u}(x) = -1\}.$$

For any automorphism P in $\mathrm{Aut}(X)$, we have that P must either fix both V^+ and V^- or interchange them as sets. Similarly, P either fixes both U^+ and U^- or interchanges them. By using (3.1), we see that V^+ and V^- each induce a 2-regular subgraph of X and U^+ and U^- each induce a 1-regular subgraph of X.

Let $W^{++} = V^+ \cap U^+$, $W^{+-} = V^+ \cap U^-$, $W^{-+} = V^- \cap U^+$ and $W^{--} = V^- \cap U^-$. Consider the subgraph Y induced by the vertices in W^{++}. Since $W^{++} \subseteq U^+$, each vertex of W^{++} has degree 0 or 1 in Y. Since X is vertex-transitive, the automorphism group of X must also act transitively on Y. Then Y is either 1-regular (an induced matching) or an independent set of vertices. The same conclusion applies to W^{+-}, W^{-+}, and W^{--}.

If Y is 1-regular, then we find the quotient matrix of the partition of $V(X)$ induced by $W^{++}, W^{+-}, W^{-+}, W^{--}$ to be

$$B = \begin{pmatrix} 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\end{pmatrix}$$

and this partition is equitable and, by the interlacing theorem (see [10, Theorem 2.5.1], for example), the eigenvalues of B are a sub-multiset of the eigenvalues of A. The matrix B has eigenvalue 1 with multiplicity 2 and so $A(X)$ also has eigenvalue 1 with multiplicity at least 2. This contradicts the assumption that 1 is a simple eigenvalue of X.

Therefore, it must be that W^{++} is an independent set. In this case, by vertex transitivity, the same holds for W^{+-}, W^{-+} and W^{--}. This implies that each vertex in W^{++} has two neighbours in W^{+-}, one neighbour in W^{-+}, and no neighbours in $W^{++} \cup W^{--}$. In particular, the partition of $V(X)$ into sets $W^{++} \cup W^{--}$ and $W^{+-} \cup W^{-+}$ is a bipartition of the graph X. $\square$

10.2 Combinatorial structure

We now consider a cubic vertex-transitive graph X that has $\lambda = 1$ as a simple eigenvalue with eigenvector $\mathbf{v}$ whose entries are in $\{1, -1\}$. We define vertex sets V^+ and V^- as in the previous section.

For $W \subseteq V(X)$, we use $X[W]$ to denote the subgraph of X induced by W. Let M denote the set of edges between V^+ and V^-; that is

$$M = \{e \in E(X) \mid e = xy \text{ for } x \in V^+ \text{ and } y \in V^-\}.$$

10.2.1 Lemma. *For (V^+, V^-) as defined above, the following are true:*

(i) $X[V^+]$ *is the disjoint union of cycles of the same length;*

(ii) $X[V^+]$ *is isomorphic to* $X[V^-]$ *and* V^+ *and* V^- *are blocks of imprimitivity of the action of* $\mathrm{Aut}(X)$*;*

(iii) (V^+, V^-) *is the unique partition of* $V(X)$*, such that both parts induce 2-regular graphs;*

(iv) M *is a perfect matching of* X*; and*

(v) $\mathrm{Aut}(X)$ *acts arc-transitively on* M *and fixes* M *set-wise.*

Proof. For every vertex x in V^+, we have that

$$\sum_{y \sim x} \mathbf{v}_y = \mathbf{v}_x = 1.$$

Since $\mathbf{v}_y$ for all y neighbours of x are either 1 or -1, it follows that x is adjacent to two vertices in V^+ and one vertex in V^-. This implies that M is a perfect matching of X and $X[V^+]$ is a 2-regular graph.

Any partition of $V(X)$ into sets (V_1, V_2) such that the induced graphs $X[V_1]$ and $X[V_2]$ are 2-regular gives rise to an eigenvector for X with eigenvalue 1, by taking the vector $\mathbf{u}$ defined as follows:

$$\mathbf{u}(v) = \begin{cases} 1, & \text{if } v \in V_1; \\ -1, & \text{if } v \in V_2. \end{cases}$$

Since 1 is a simple eigenvalue of X, it follows that $\{V^-, V^+\}$ is the only such partition. Then every automorphism of X must fix V^+ and V^- or must swap V^+ and V^- set-wise. This shows (v). Observe that there is an automorphism of X taking a vertex of V^+ to a vertex in V^-. Such an automorphism must take every vertex in V^+ to a vertex in V^- and every vertex in V^- to a vertex in V^+ and so is an isomorphism from $X[V^+]$ to $X[V^-]$. This shows that (ii) holds. Since $\mathrm{Aut}(X)$ acts transitively on X, the induced action on V^+ is also transitive, so $X[V^+]$ is a vertex-transitive 2-regular graph. Then $X[V^+]$ must be a disjoint union of cycles of the same length. □

We may reorder the vertices of X such that

$$A(X[V^+]) = A(X[V^-]).$$

We may assume that each of $X[V^+]$ and $X[V^-]$ is the disjoint union of m cycles of length k. Let the vertices of V^+ be $v_{i,j}$ where $i = 1, \ldots, m$ and $j = 1, \ldots, k$, such that $\{v_{i,1} \ldots, v_{i,k}\}$ induce a cycle C_i in $X[V^+]$ (and in X). Similarly, we may denote the vertices of V^- as $w_{i,j}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, k$, such that $\{w_{i,1} \ldots, w_{i,k}\}$ induce a cycle D_i in $X[V^-]$.

Let η be the function such that $\eta(i,j) = (r,s)$ when $v_{i,j}$ is adjacent to $w_{r,s}$. Since M is a perfect matching of X, we have that η is well-defined and induces a bijection from V^+ to V^-. The adjacency matrix of X can be partitioned into block matrices, according to V^+ and V^- as follows:

$$A(X) = \begin{pmatrix} B & P \\ P^T & B \end{pmatrix},$$

where $B = A(X[V^+])$ and P is the permutation matrix corresponding to η.

10.3 Relation to regular maps

We recall some preliminary definitions from the area of graph embeddings. Further details may be found in [46]. Let G be a connected multigraph. For each $v \in V(G)$, let π_v be a cyclic permutation of the edges incident to v. Then $\Pi = \{\pi_v \mid v \in V(G)\}$ is said to be an *embedding* of G (on an orientable surface) and it defines a collection of closed walks, called *facial walks* or *faces*, such that each edge is traversed once in each direction by these walks. A triple (v, e, f) where v is a vertex of G incident to edge e incident to face f is said to be a *flag* of (G, Π). An embedding is said to be *regular* if $\mathrm{Aut}(G)$ acts transitively on the flags of (G, Π). If G is arc-transitive and there are two orbits of flags of (G, Π) under the action of $\mathrm{Aut}(G)$, we say that (G, Π) is a *half-regular* map.

Let X be a cubic, vertex-transitive graph with 1 as a simple eigenvalue, as considered in Section 10.2. Let C_i and D_i $(i = 1, \ldots, m)$ be the cycles forming $X[V^+]$ and $X[V^-]$, respectively, as observed at the end of Section 10.2. Let G be the multigraph obtained from X by contracting each cycle C_i, D_i to a single vertex. More precisely, G has a vertex c_i for each cycle C_i of X and a vertex d_i for each cycle D_i of X. There is an edge joining c_i and d_r in G for each edge of X joining a vertex in C_i to a vertex in D_r. We say that G is the *contracted multigraph* of X. Observe that G is connected since X is.

10.3.1 Lemma. *If G is the contracted multigraph of X, then $\mathrm{Aut}(X) \leq \mathrm{Aut}(G)$ and any vertex-transitive subgroup of $\mathrm{Aut}(X)$ acts transitively on the arcs of G. In particular, G is arc-transitive and bipartite.*

Proof. Consider $\alpha \in \mathrm{Aut}(X)$. If we take the contracted multigraph of $\alpha(X)$, we again obtain G. Since the cycles C_i, D_j form blocks of imprimitivity under the action of $\mathrm{Aut}(X)$. Thus α acts on G as an automorphism. so $\mathrm{Aut}(X) \leq \mathrm{Aut}(G)$. Let $\Gamma \leq \mathrm{Aut}(X)$ act transitively on the vertices of X. Then Γ acts arc-transitively on the edges of M, which are in one-to-one correspondence with the edges of G, and thus acts arc-transitively on G. Clearly every edge in G connects some c_i to some d_j, so G is bipartite. □

For a vertex c_i, let the edges incident to c_i be $e_{i,1}, \ldots, e_{i,k}$, where $e_{i,j}$ is the unique edge of G corresponding to the edge of M incident with $v_{i,j}$. Similarly, for a vertex d_i, let the edges incident to d_i be $f_{i,1}, \ldots, f_{i,k}$, where $f_{i,j}$ is the unique edge of G corresponding to the edge of M incident with $w_{i,j}$. Let $\Pi = \{\pi_v \mid v \in V(G)\}$ be a rotation system satisfying the following: for each i, the permutation π_{c_i} satisfies $\pi_{c_i}(e_{i,j}) \in \{e_{i,j-1}, e_{i,j+1}\}$ and π_{d_i} satisfies $\pi_{d_i}(e_{i,j}) \in \{f_{i,j-1}, f_{i,j+1}\}$ where the indices are taken modulo k. Then we say that (G, Π) is a *compatible* embedding of G with respect to X.

Note that since there are two choices for each of the permutations π_v, there are 2^{2m} (possibly non-isomorphic) embeddings of G that are compatible with the graph X.

Conversely, given an embedding (G, Π) where G is a connected multigraph with no loops and Π is a rotation system, we may define a cubic graph $\rho(G, \Pi)$, called the *truncation* of (G, Π), in the following way. The graph $\rho(G, \Pi)$ has vertices (v, e) for every $v \in V(G)$ and every edge e incident to v. In $\rho(G, \Pi)$, every vertex (v, e) is adjacent to $(v, \pi_v(e))$, to $(v, \pi_v^{-1}(e))$, and to (w, e) where w is the other endpoint of e. Every vertex has two neighbours with the same first coordinate and one with the same second coordinate and $\rho(G, \Pi)$ is cubic.

10.3.2 Lemma. *Let X be a vertex-transitive, cubic, connected graph with 1 as a simple eigenvalue and let G be the contracted multigraph of X. An embedding Π of F is compatible with X if and only if $\rho(G, \Pi) = X$.*

We will now consider the action of $\mathrm{Aut}(X)$ on a compatible embedding (G, Π). For a given set Π of local rotations, it is possible that not every element of $\mathrm{Aut}(X)$ preserves the faces of (G, Π). We will restrict our attention to automorphisms of X which preserve the faces of (G, Π). Let $\Gamma \leq \mathrm{Aut}(X)$. Note that Γ also acts on $E(G) = M$. We say that (G, Π) is *compatible* with Γ if, for every $v \in V(G)$ and e an edge incident to v, we have

$$\alpha(\pi_v(e)) = \pi_{\alpha(v)}(\alpha(e))$$

for every $\alpha \in \Gamma$.

10.3.3 Lemma. *Let $\Gamma \leq \mathrm{Aut}(X)$ be a subgroup acting transitively on the vertices of X. If (G, Π) is compatible with Γ, then the flags of (G, Π) lie in at most two orbits under the action of Γ.*

Proof. Consider a flag (v, e, f) of (G, Π) and $\alpha \in \Gamma$ which takes (v, e) to (v', e'). Then, since α preserves the faces of the embedding, $\alpha(f)$ is one of the two faces incident to v' and e'. Since Γ acts transitively on the arcs of G by Lemma 10.3.1, for any (v', e') where v' is incident to e', there exists $\alpha \in \Gamma$ which maps (v, e) to (v', e'). Then, the orbit of (v, e, f), say $\mathcal{O}$, under the action of Γ contains at least half of the flags of (G, Π). Since every orbit of flags has the same order, we have that there are at most 2 orbits of the flags of (G, Π) under the action of Γ. □

We see that if there exists $\Gamma \leq \mathrm{Aut}(X)$ and Π such that Γ acts vertex-transitively on X and (G, Π) is compatible with (X, Γ), then (G, Π) is a regular map or a half-regular map. For example, the cube, considered as a regular map on the sphere, is an embedding compatible with a vertex-transitive, cubic graph that has 1 as a simple eigenvalue, by the result in Section 10.5.1

10.4 Adjacency matrix

For this section, let X be a cubic vertex-transitive graph with 1 as a simple eigenvalue. Let $A := A(X)$ and let P and B be defined as in Section 10.2. In general, we have Lemma A.6.1, but in this specific case, we may do slightly better.

10.4.1 Lemma. *If $B^kP = PB^k$ for some $1 \leq k \leq m$, then*

(i) $BP = PB$ if k is odd, and

(ii) $B^2P = PB^2$ if k is even.

Proof. We may write $B = C + C^{-1}$ where C is a permutation matrix consisting of disjoint cycles of the same order, say q. Let e_i denote the ith elementary basis vector of $\mathbb{R}^X$, which has a 1 in the position indexed by vertex i and zeros elsewhere. We have that

$$Ce_i = e_{i+1}$$

where the indices are modulo q. Then

$$\begin{aligned} B^k &= (C + C^{-1})^k \\ &= \sum_{j=0}^{k} \binom{k}{j} C^{k-j}(C^{-1})^j \\ &= \sum_{j=0}^{k} \binom{k}{j} C^{k-2j} \end{aligned}$$

and

$$B^k e_i = \sum_{j=0}^{k} \binom{k}{j} C^{k-2j} e_i = \sum_{j=0}^{k} \binom{k}{j} e_{i+k-2j}$$

where the indices are modulo q.

For $v \in V(X)$, consider the coefficient of e_v in $B^k e_i$. If v is not in the same component as i in X with the matching deleted, then the coefficient of e_v is 0. If v and i lie in the same component of V^+ (or V^-), the coefficient of e_v is the sum of all $\binom{k}{j}$ for which $k + i - 2j = v \mod q$. If there are no such j, the coefficient of e_v is 0.

Suppose e_v and e_u have the same nonzero coefficient in $B^k e_i$ and $u \neq v$. Then

$$v = i + k - 2j$$

and

$$u = i + k - 2j'$$

for $j \neq j'$ such that

$$\binom{k}{j} = \binom{k}{j'}.$$

Then $j' = k - j$. Then distinct vertices e_v and e_u have the same nonzero coefficient in $B^k e_i$ if and only if $v = i + k - 2j$ and $u = i - k + 2j$ for some j. Let ϕ denote the permutation represented by permutation matrix P.

To prove the statement, suppose first that k is odd and let $k = 2m + 1$. Then $e_{i+1} = e_{i+k-2m}$ has coefficient $\binom{k}{m} \neq 0$ and e_{i-1} has the same coefficient in $B^k e_i$ by above argument. Observe that

$$PB^k e_i = \sum_{j=0}^{k} \binom{k}{j} Pe_{i+k-2j} = \sum_{j=0}^{k} \binom{k}{j} e_{\phi(i+k-2j)}$$

and

$$B^k Pe_i = \sum_{j=0}^{k} \binom{k}{j} e_{\phi(i)+k-2j}.$$

Since $PB^k e_i = B^k Pe_i$ and the e_i's are pairwise orthogonal, we must have that the coefficient of $e_{\phi(i\pm1)}$ in $PB^k e_i$ is equal to the coefficient of $e_{\phi(i)\pm1}$ in $B^k Pe_i$, and distinct from the coefficient of any other e_j. Then, we have that

$$\{\phi(i-1), \phi(i+1)\} = \{\phi(i) - 1, \phi(i) + 1\}.$$

This is true for any i, if we consider ϕ as a map from V^+ to V^-, we see that ϕ sends every edge of V^+ to an edge of V^+ and ϕ is bijective since there is a perfect matching between V^+ and V^-. Then P represents an automorphism of the graph with adjacency matrix B and so $PB = BP$.

Suppose $k = 2m$ is even. We see that $e_{i\pm2} = e_{i\pm k\mp2(m-1)}$ has coefficient $\binom{k}{m-1} \neq 0$ in $B^k e_i$. By a similar argument as above, we obtain that

$$\{\phi(i) + 2, \phi(i) - 2\} = \{\phi(i+2), \phi(i-2)\}.$$

Now

$$B^2 = C^2 + 2I + C^{-2}.$$

Then,

$$\begin{aligned}PB^2e_i &= P(e_{i+2} + 2e_i + e_{i-2})\\ &= e_{\phi(i+2)} + 2e_{\phi(i)} + e_{\phi(i-2)}\\ &= 2e_{\phi(i)} + e_{\phi(i)+2} + e_{\phi(i)-2}\\ &= B^2e_{\phi(i)}\\ &= B^2Pe_i\end{aligned}$$

for all i and we have that $B^2P = PB^2$, which concludes the proof. $\square$

10.5 Families of graphs

10.5.1 Truncations of cubic arc-transitive graphs

The *truncation* of a graph G is a graph $T(G)$ where every vertex v of G corresponds to a clique of order $\deg(v)$ and every edge uv of G gives an edge of $T(G)$ between the cliques corresponding to u and to v, such that each vertex of $T(G)$ is adjacent to exactly one vertex in a different clique. If G is a cubic graph, the truncation of G is also cubic. The following theorem appears as Theorem 2.1 in [59].

10.5.1 Theorem. *[59] If the eigenvalues of a cubic graph X are $\mu_1, \ldots, \mu_n$, the eigenvalues of the truncation of X are*

$$\lambda_i = \frac{1 \pm \sqrt{4\mu_i + 13}}{2}$$

for $i = 1, \ldots, n$ and -2 and 0, each with multiplicity $\frac{n}{2}$.

10.5.2 Theorem. *The truncation of a cubic graph X has 1 as an eigenvalue if and only if X is bipartite. Furthermore, if X is bipartite, then 1 is a simple eigenvalue of the truncation of X if and only if X is connected.*

Proof. We see from Theorem 10.5.1 that the eigenvalues of X which are equal to -3 are precisely those which map to eigenvalues equal to 1 in the truncation of X. A cubic graph X has -3 as an eigenvalue if and only if it is bipartite. The multiplicity of -3 as an eigenvalue of X (and hence of 1 as an eigenvalue of the truncation of X) is 1 if and only if X has exactly one component. $\square$

10.5.3 Corollary. *If X is a connected vertex-transitive cubic graph containing a cycle of length 3, then X has 1 as a simple eigenvalue if and only if $X = T(G)$, where G is a connected, bipartite, arc-transitive, cubic graph.*

Proof. First we show that if X is vertex-transitive, contains a triangle and has 1 as a simple eigenvalue, then X must be the truncation of a graph G. Since X has 1 as a simple eigenvalue, we may partition the vertices of X into cycles V^+ and V^-, as in Lemma 10.2.1. Let v be vertex in V^+. We have that v is incident to a cycle of length 3, say T, in X. Since there is a matching between V^+ and V^-, the triangle T does not use any edge of the matching. Thus v must be incident to a cycle of length 3 in the subgraph of X induced by V^+. Since V^+ induces a vertex-transitive 2-regular subgraph of X by Lemma 10.2.1, $X[V^+]$ must be a disjoint union of cycles of length 3. The same holds for $X[V^-]$. Let G be obtained from X by contracting all the 3-cycles in $X[V^+]$ and $X[V^-]$. We see that G is cubic, since each vertex of X is incident to exactly 1 edge which is not contracted to obtain G. By part (v) of Lemma 10.2.1, G is an arc-transitive bipartite graph, as claimed.

By Theorem 10.5.2, the converse implication is clear. □

10.5.2 Prisms

The *prism of order* $2n$ is the Cartesian product of C_n with K_2. A prism is a cubic vertex-transitive graph, which always has 1 as an eigenvalue, but not always as a simple eigenvalue. We can find exactly which prisms have 1 as a simple eigenvalue.

10.5.4 Theorem. *Let X, P and B defined as in Section 10.2, then $BP = PB$ if and only if X is the prism of order $2n$, where $n \neq 0 \mod 4$.*

Proof. Suppose $PB = BP$. Let H^+ and H^- denote the subgraphs of X induced by V^+ and V^-, respectively. Note that $H^+ \cong H^-$ and $A(H^+) = A(H^-) = B$. Then P represents a permutation ϕ of the vertices H^+ which is a graph automorphism of H^+. We will, however consider ϕ as an isomorphism from H^+ to H^-.

First we will show that H^+ is a cycle of length n. Suppose for a contradiction that it is not. Observe that H^+ and H^- are 2-regular graphs and every automorphism of X must fix each of V^+ and V^- or swaps them. Then, H^+ is vertex transitive since X is. If H^+ is not a single cycle, then H^+ is the disjoint union of cycles of the same length. Let C^+ be one of the cycles and x be a vertex of C^+. Consider $\phi(x)$. Let C^- be the cycle of H^- which contains $\phi(x)$. Since ϕ is an isomorphism, ϕ maps connected components to connected components. Specifically, ϕ map C^+ to C^-. Then, in X, all edges of the matching with one end in C^+ has the other end in C^-. Since X is cubic, the vertices of C^+ and C^- induce a connected component of X. Since X is connected, it follows that $C^+ = H^+$, thus showing that H^+ is a cycle of length n.

Let $Y = C_n \square K_2$. We will show that X is isomorphic to Y by giving an explicit isomorphism. Let the vertices of C_n be $\{c_1, \dots, c_n\}$ and let the vertices of K_2 be $\{1, 2\}$. We will write the vertices of X as

$$V^+ \cup \{\phi(v) \mid v \in V^+\}$$

where $V^+ = \{v_1, \ldots, v_n\}$. Define a map ψ from vertices of X to vertices of Y as follows:

$$\psi(v_i) = (c_i, 1) \text{ and } \psi(\phi(v_i)) = (c_i, 2)$$

for each $v_i \in V^+$. Clearly ψ is bijective. For an edge $v_i v_{i+1}$ of X, we see that $\psi(v_i) = (c_i, 1)$ and $\psi(v_{i+1}) = (c_{i+1}, 1)$. Then $\psi(v_i)\psi(v_{i+1})$ is an edge of Y. Similarly, an edge $\phi(v_i)\phi(v_{i+1})$ of X, we see that $\psi(\phi(v_i)) = (c_i, 2)$ and $\psi(\phi(v_{i+1})) = (c_{i+1}, 2)$. Then $\psi(\phi(v_i))\psi(\phi(v_{i+1}))$ is an edge of Y. For the edge $v_i\phi(v_i$, we see $\psi(v_i) = (c_i, 1)$ and $\psi(\phi(v_i)) = (c_i, 2)$. Then $\psi(v_i)\psi(\phi(v_i))$ is an edge of Y. Then, ψ is an isomorphism as required.

For the other implication, we must check that if $n \neq 0 \mod 4$, then Y has 1 as a simple eigenvalue and we can write the adjacency matrix of Y as

$$A = \begin{pmatrix} B & P \\ P^T & B \end{pmatrix}$$

where $PB = BP$ and P is a permutation matrix.

Consider the spectrum of $Y = C_n \square K_2$ (the prism of order $2n$) for any even n. From [10] or any other standard text in spectral graph theory, we obtain that the eigenvalues of Y are $\theta \pm 1$, for each θ eigenvalue of $C_{\frac{n}{2}}$. Then the eigenvalues of Y are

$$2\cos\left(\frac{2\pi j}{\ell}\right) \pm 1$$

for $j = 0, \ldots, \ell - 1$ and $\ell = \frac{n}{2}$. Note that

$$2\cos\left(\frac{2\pi j}{\ell}\right) - 1 = 2 - 1 = 1$$

when $j = 0$, so Y has 1 as an eigenvalues with multiplicity at least 1. The multiplicity of 1 as an eigenvalue is simple if and only if 0 + 1 is not also an eigenvalue, which is to say, if and only if 0 is not an eigenvalue of C_n. We may see that C_n has 0 as an eigenvalue if and only if n is divisible by 4. Then Y has 1 as a simple eigenvalue if and only if n is not divisible by 4.

Suppose $Y = C_{\frac{n}{2}} \square K_2$ where n is even but not divisible by 8. Then Y has 1 as a simple eigenvalue and we see that the function f such that

$$f((c_i, 1)) = 1 \text{ and } f((c_i, 2)) = -1$$

is an eigenvector with eigenvalue 1. Partition Y into $V^+ = \{(c_i, 1)\}$ and $V^- = \{(c_i, 2)\}$ with respect to f. We may write the adjacency matrix as

$$A = \begin{pmatrix} B & P \\ P^T & B \end{pmatrix}$$

where P is the permutation taking $(c_i, 1)$ to $(c_i, 2)$. Then P commutes with B as required. □

10.5.5 Corollary. *A prism of order $2n$ has 1 as simple eigenvalue if and only if $n \neq 0 \mod 4$.*

10.5.3 Generalized Petersen graphs

We would like to classify which generalized Petersen graphs have 1 as a simple eigenvalue. This turns out to be a difficult task. We record a partial progress here.

The *generalized Petersen graph*, denoted $P(n, k)$ is the graph with vertex set $U \cup V$ where

$$U = \{u_j\}_{j=0}^{n-1} \text{ and } V = \{v_j\}_{j=0}^{n-1}$$

and edges

$$u_j v_j, u_j u_{j+1}, v_j v_{j+k}$$

for every j, where the indices are modulo n. The well-known Petersen graph is $P(5, 2)$. We have the following theorem concerning vertex-transitivity.

10.5.6 Theorem ([25]). *The generalized Petersen graph $P(n, k)$ is vertex-transitive if and only if $(n, k) = (10, 2)$ or $k^2 \equiv \pm 1 \mod n$.*

The following theorem gives the eigenvalues of $P(n, k)$.

10.5.7 Theorem ([26]). *The graph $P(n, k)$ has eigenvalues δ for every root δ of*

$$x^2 - (\alpha_j + \beta_j)x + \alpha_j \beta_j - 1 = 0 \tag{10.1}$$

for $j = 0, \dots, n-1$, where

$$\alpha_j = 2\cos(\tfrac{2\pi j}{n}) \text{ and } \beta_j = 2\cos(\tfrac{2\pi jk}{n}).$$

The eigenvalues of $P(n, k)$ which are equal to 1 are solutions for Equation (10.1) where $x = 1$, which we may simplify as:

$$\alpha_j \beta_j = \alpha_j + \beta_j.$$

We may let $\theta = \frac{2\pi j}{n}$ and rewrite as:

$$2\cos\theta \cos k\theta = \cos\theta + \cos k\theta. \tag{10.2}$$

Observe that $j = 0$ gives a solution to this equation for any k. Hence, every generalized Petersen graph has eigenvalue 1 with multiplicity at least one. From now on, we will only consider the case where $j \neq 0$ and $\cos\theta \neq 1$. We know that $\cos(k\theta) = T_k(\cos\theta)$ where T_k is the kth Chebyshev polynomial of the first kind. Then

$$2\cos\theta\, T_k(\cos\theta) = \cos\theta + T_k(\cos\theta).$$

From this we see that $\cos\theta$ is an algebraic number and, letting $\psi(x)$ denote the minimal polynomial of $\cos\theta$ over $\mathbb{Q}$, we see that $\psi(x)$ divides $2xT_k(x) - x - T_k(x)$. Let

$$f(x) = \frac{2xT_k(x) - x - T_k(x)}{x-1}.$$

Since $T_k(1) = 1$ for any k, we see that $x - 1$ is an irreducible factor of $2xT_k(x) - x - T_k(x)$ and so $f(x)$ is a polynomial of degree k with coefficients over $\mathbb{Q}$. Since $\psi(x)$ is irreducible, $\psi(x)$ divides $x - 1$ or $\psi(x)$ divides $f(x)$. Since we are considering $\cos\theta \neq 1$, we may assume that $\psi(x)$ divides $f(x)$.

10.5.8 Lemma. *If $\cos\theta$ is rational, then $P(n,k)$ has 1 as an eigenvalue with multiplicity greater than 1 if and only if $n = 4m$ for some integer m and $km \in \{m, 3m\}$ or $3km \in \{m, 3m\}$, where the arithmetic takes place over $\mathbb{Z}_n$.*

Proof. We require that $\cos\theta = \cos k\theta = 0$ for some $\theta \neq 0$. Suppose $\cos\theta$ is rational. There is an elementary result (see, for example, [36]) which gives that $\cos\theta \in \mathbb{Q}$ where θ is a rational multiple of π if and only if $\cos\theta \in \{\pm 1, \pm\frac{1}{2}, 0\}$. We will now check these cases explicitly.

We have already considered the case when $\cos\theta = 1$. If $\cos\theta = -1$, Equation (10.2) gives that $\cos k\theta = \frac{1}{3} \in \mathbb{Q}$. Then $k\theta$ must be an irrational multiple of π, which contradicts that k is an integer. Similarly, there are no solutions when $\cos\theta = \pm\frac{1}{2}$. If $\cos\theta = 0$, then Equation (10.2) gives that $\cos k\theta = 0$. This can happen if and only if $n = 4m$ for some m and $\{km, 3km\} \cap \{m, 3m\} \neq \emptyset$, over $\mathbb{Z}_n$. □

Now we consider the case when $\cos\theta$ is not rational. In this case, we find the degree of the extension of the rationals by $\mathbb{Q}(\cos\theta)$. This can be found in an elementary text for algebraic number theory, such as [40], but we include it here for completeness.

10.5.9 Lemma. *The algebraic degree of $\cos(\frac{2\pi j}{n})$ over the rationals is $\frac{\phi(n')}{2}$ where ϕ denotes the Euler totient function and n' is such that $\frac{j}{n} = \frac{j'}{n'}$ and $\gcd(j', n') = 1$.*

Proof. Let $\theta = \frac{2\pi j}{n}$, as before, and let j', n' be such that $\frac{j}{n} = \frac{j'}{n'}$ and $\gcd(j', n') = 1$. Let $\zeta = e^{i\theta}$, which is a primitive n'-th root of unity. Since $\mathbb{Q}(\zeta)$ is the splitting field of a cyclotomic polynomial we see that

$$[\mathbb{Q}(\zeta) : \mathbb{Q}] = \phi(n').$$

We observe that

$$\cos\theta = \frac{\zeta + \zeta^{-1}}{2} \in \mathbb{Q}(\zeta).$$

Since $\mathbb{Q}(\cos\theta)$ is a real extension of the rationals while $\mathbb{Q}(\zeta)$ is not, we see that $[\mathbb{Q}(\zeta):\mathbb{Q}(\cos\theta)] \geq 2$.

Further, we see that

$$(x-\zeta)(x-\zeta^{-1}) = x^2 - (\zeta+\zeta^{-1})x + 1 = x^2 - 2\cos\theta x + 1$$

is a polynomial of degree 2 over $\mathbb{Q}(\cos\theta)$, which has ζ as a root. Hence $[\mathbb{Q}(\zeta):\mathbb{Q}(\cos\theta)] \leq 2$.

We have obtained that $[\mathbb{Q}(\zeta):\mathbb{Q}(\cos\theta)] = 2$ and

$$[\mathbb{Q}(\zeta):\mathbb{Q}(\cos\theta)][\mathbb{Q}(\cos\theta):\mathbb{Q}] = [\mathbb{Q}(\zeta):\mathbb{Q}] = \phi(n').$$

From this we see that

$$[\mathbb{Q}(\cos\theta):\mathbb{Q}] = \frac{\phi(n')}{2},$$

as claimed □

If Equation (10.2) has a solution where $\theta \neq 0$, then, for some $q > 1$ dividing n, we have that

$$\frac{\phi(q)}{2} < \deg(f(x)) = k.$$

We have proven the following theorem.

10.5.10 Theorem. *If the generalized Petersen graph $P(n,k)$ has 1 as an eigenvalue of multiplicity greater than one, then there exists q, a divisor of n, such that $q > 1$ and $\phi(q) \leq 2k$, where ϕ denotes the Euler totient function.*

Proof. Suppose the generalized Petersen graph $P(n,k)$ has 1 as an eigenvalue of multiplicity greater than one, then there exists $j \neq 0$ such that $\theta = \frac{2\pi j}{n}$ satisfies Equation (10.2). If $\cos\theta$ is rational, then we have from Lemma 10.5.8 that $n = 4m$ for some integer m and $km \in \{m, 3m\}$ or $3km \in \{m, 3m\}$ over $\mathbb{Z}_n$. Then $2|n$ and $\phi(2) = 1$. SInce $k \geq 1$, we have that $\phi(2) < 2k$.

If $\cos\theta$ is not rational, then we see that its algebraic order over the rationals is equal to $\phi(n')/2$ where $\frac{j}{n} = \frac{j'}{n'}$ and $\gcd(j', n') = 1$. Then $n'|n$ and $n' > 1$ and

$$\frac{\phi(n')}{2} \leq \deg(f(x)) = k$$

as claimed. □

This result gives a complete characterization when n is prime.

10.5.11 Corollary. *If the generalized Petersen graph $P(p,k)$, where p is an odd prime, is vertex-transitive, then $P(n,k)$ has 1 as a simple eigenvalue unless $P(n,k)$ is isomorphic to the Petersen graph.*

Proof. Since the graphs $P(n,k)$ and $P(n,n-k)$ are isomorphic, we may assume that $k \leq \frac{p}{2}$. Since p is an odd prime, we may assume that $k \leq \frac{p-1}{2}$. Theorem 10.5.10 gives the necessary condition for $P(n,k)$ to have one as a non-simple eigenvalue. We consider when the condition in Theorem 10.5.10 may be satisfied. In this case, the only divisor of p not equal to 1 is p and $\phi(p) = p-1$ gives that $\frac{p-1}{2} \leq k$. Then it must be that $k = \frac{p-1}{2}$. Since $P(p,k)$ is vertex-transitive and $p \neq 10$, we must have $k^2 \equiv \pm 1 \mod p$ by Theorem 10.5.6. Suppose $k^2 \equiv 1 \mod p$. We obtain

$$\begin{aligned} \left(\frac{p-1}{2}\right)^2 &\equiv 1 \mod p \\ p^2 - 2p + 1 &\equiv 4 \mod p \\ 0 &\equiv 3 \mod p \end{aligned}$$

and $p = 3$. In this case, we may check that $P(3,1)$ does in fact have 1 as a simple eigenvalue and so is not an exception. We see that if $P(p,k)$ has 1 as a non-simple eigenvalue then $k^2 \equiv -1 \mod p$. Then

$$\begin{aligned} \left(\frac{p-1}{2}\right)^2 &\equiv -1 \mod p \\ p^2 - 2p + 1 &\equiv -4 \mod p \\ 0 &\equiv 5 \mod p \end{aligned}$$

and $p = 5$. In this case, we obtain that $p = 5$ and $k = \frac{p-1}{2} = 2$ and so $P(p,k) = P(5,2)$ is isomorphic to the Petersen graph, whose multiplicity of 1 as an eigenvalue is greater than 1. □

10.5.4 Regular embeddings of $K_{m,m}$

Let T_k be the graph defined in the following way. The vertices of T_m are $\{v_{i,j}, w_{i,j} \mid i,j \in \mathbb{Z}_m\}$. The edges are

$$\{v_{i,j}, v_{i,j+1}\},\ \{w_{i,j}, w_{i,j+1}\},\ \{v_{i,j}, w_{j,i}\}$$

for all $i,j \in \mathbb{Z}_m$. It is easy to see that T_m is a cubic, vertex-transitive graph with 1 as an eigenvalue, not necessarily simple, by considering the eigenvector that is $+1$ on every vertex $v_{i,j}$ and -1 on every vertex $w_{i,j}$.

Alternatively, we can construct T_k from the regular embedding of $K_{k.k}$, given by Nedela and Skoviera in [48]. We consider $K_{k,k}$ as a Cayley graph on $\mathbb{Z}_{2k}$ with the connection set $\{1,3,5,\ldots,2k-1\}$ as the generating set. The rotation system Π has vertex rotations at each vertex is given by the cyclic permutation $(1,3,5,\ldots,2k-1)$ of the generators. The graph T_k is isomorphic to $\rho(K_{k,k},\Pi)$.

Let B be the $m^2 \times m^2$ matrix such that $B = I_m \otimes C_m$, where C_m is the adjacency matrix of the cycle of order m and I_m is the $m \times m$ identity matrix. Let P be the permutation matrix indexed by $\mathbb{Z}_m \times \mathbb{Z}_m$ such that P takes (i,j) to (j,i). The adjacency matrix of T_m can be written as

$$A := A(T_m) = \begin{pmatrix} B & I \\ I & P^T B P \end{pmatrix}.$$

Observe that $P^2 = I$ and $P = P^T$. By definition, we see that

$$P(e_i \otimes e_j) = e_j \otimes e_i$$

where e_k denotes the kth elementary basis vector. Then for any $m \times 1$ vectors $\mathbf{v}$ and $\mathbf{w}$, we see that

$$P(\mathbf{v} \otimes \mathbf{w}) = \mathbf{w} \otimes \mathbf{v}.$$

10.5.12 Theorem. *The eigenvalues of T_m are*

$$\cos\frac{2\pi j}{m} + \cos\frac{2\pi \ell}{m} \pm \sqrt{\left(\cos\frac{2\pi j}{m} + \cos\frac{2\pi \ell}{m}\right)^2 + 1}$$

for all $(j,\ell) \in \mathbb{Z}_m \times \mathbb{Z}_m$.

Proof. We use the proof methods used in [26] to find the eigenvalues of generalized Petersen graphs, to find these eigenvalues. Let $\mathbf{v}, \mathbf{w}$ be eigenvectors of C_m with eigenvalues λ and θ, respectively. Then

$$B(\mathbf{v} \otimes \mathbf{w}) = (I_m \otimes C_m)(\mathbf{v} \otimes \mathbf{w}) = \mathbf{v} \otimes \theta\mathbf{w} = \theta(\mathbf{v} \otimes \mathbf{w})$$

and

$$P^T B P(\mathbf{v} \otimes \mathbf{w}) = P^T(I_m \otimes C_m)(\mathbf{w} \otimes \mathbf{v}) = P(\lambda\mathbf{v} \otimes \theta\mathbf{w}) = \lambda(\mathbf{v} \otimes \mathbf{w}).$$

Let V be an eigenbasis for C_m in $\mathbb{R}^m$. Then the basis

$$W = \{\mathbf{v} \otimes \mathbf{w} \mid \mathbf{v}, \mathbf{w} \in V\}$$

of $\mathbb{R}^{m^2}$ simultaneously diagonalizes B and $P^T B P$. We construct an eigenbasis U of A over $\mathbb{R}^{2m^2}$ such that the elements of U are

$$\begin{pmatrix} \alpha\mathbf{v} \otimes \mathbf{w} \\ \mathbf{v} \otimes \mathbf{w} \end{pmatrix}$$

where $\mathbf{v}, \mathbf{w} \in V$ with eigenvalues λ and θ, respectively and $\alpha = \delta - \theta$ for each δ a solution to

$$\delta^2 - (\theta + \lambda)\delta + \theta\lambda - 1 = 0. \tag{10.3}$$

Observe that since

$$\left(\frac{\theta + \lambda}{2}\right)^2 = \frac{\theta^2 + \lambda^2}{2} + \theta\lambda > \theta\lambda - 1$$

for any $\lambda, \theta \in \mathbb{R}$, Equation (10.3) always has two distinct solutions for δ. The U consists of $2m^2$ linearly independent vectors in $\mathbb{R}^{2m^2}$. We now verify that each element of U is an eigenvector of A by observing

$$\begin{aligned} A\begin{pmatrix}\alpha\mathbf{v}\otimes\mathbf{w}\\ \mathbf{v}\otimes\mathbf{w}\end{pmatrix} &= \begin{pmatrix}B & I\\ I & P^TBP\end{pmatrix}\begin{pmatrix}\alpha\mathbf{v}\otimes\mathbf{w}\\ \mathbf{v}\otimes\mathbf{w}\end{pmatrix}\\ &= \begin{pmatrix}\alpha\theta\mathbf{v}\otimes\mathbf{w}+\mathbf{v}\otimes\mathbf{w}\\ \alpha\mathbf{v}\otimes\mathbf{w}+\lambda\mathbf{v}\otimes\mathbf{w}\end{pmatrix}\\ &= \begin{pmatrix}(\alpha\theta+1)\mathbf{v}\otimes\mathbf{w}\\ (\alpha+\lambda)\mathbf{v}\otimes\mathbf{w}\end{pmatrix}. \end{aligned}$$

But α has been carefully chosen such that $\alpha+\theta=\delta$ and $\alpha\lambda+1=\delta\alpha$, so

$$A\begin{pmatrix}\alpha\mathbf{v}\otimes\mathbf{w}\\ \mathbf{v}\otimes\mathbf{w}\end{pmatrix} = \delta\begin{pmatrix}\alpha\mathbf{v}\otimes\mathbf{w}\\ \mathbf{v}\otimes\mathbf{w}\end{pmatrix}.$$

Then U is an eigenbasis for A, as claimed, and the eigenvalues of A are the solution for δ in Equation (10.3) where θ and λ range over the eigenvalues of C_k. We use the quadratic formula to see that

$$\begin{aligned} \delta &= \frac{\theta+\lambda\pm\sqrt{(\theta+\lambda)^2-4(\theta\lambda-1)}}{2}\\ &= \frac{\theta+\lambda\pm\sqrt{(\theta-\lambda)^2+4}}{2}. \end{aligned}$$

The eigenvalues of C_m can be found in a standard reference in algebraic graph theory such as [10] and are

$$2\cos\frac{2\pi j}{m}$$

for $j\in\mathbb{Z}_m$. Then, for $j,\ell\in\mathbb{Z}_m$,

$$\begin{aligned} \delta &= \frac{2\cos\frac{2\pi j}{m}+2\cos\frac{2\pi\ell}{m}\pm\sqrt{\left(2\cos\frac{2\pi\ell}{m}-2\cos\frac{2\pi j}{m}\right)^2+4}}{2}.\\ &= \cos\frac{2\pi j}{m}+\cos\frac{2\pi\ell}{m}\pm\sqrt{\left(\cos\frac{2\pi\ell}{m}-\cos\frac{2\pi j}{m}\right)^2+1} \end{aligned}$$

which concludes the proof. □

It is difficult to determine how many eigenvalues of T_m are equal to 1, but we may find two cases where 1 is not a simple eigenvalue of T_m.

10.5.13 Proposition. *The multiplicity of 1 as an eigenvalue of T_{4k} is at least 5 for any $k\geq 1$. The multiplicity of 1 as an eigenvalue of T_{5k} is at least 9 for any $k\geq 1$.*

Proof. We see that 1 appears as an eigenvalue of T_m for every solution (j, ell) to Equation (10.3) where $\delta = 1$, $\theta = 2\cos\frac{2\pi j}{m}$ and $\lambda = 2\cos\frac{2\pi\ell}{m}$, where $j, \ell \in \{0, \ldots m-1\}$. We can simplify this as:

$$\cos\frac{2\pi j}{m} + \cos\frac{2\pi\ell}{m} = 2\cos\frac{2\pi j}{m}\cos\frac{2\pi\ell}{m}. \tag{10.4}$$

Observe that $(j, \ell) = (0, 0)$ is always a solution to (10.4). We find additional solutions to 10.4 when m is divisible by 4 and when m is divisible by 5.

Note that

$$\cos\left(2\pi\frac{1}{4}\right) = \cos\left(2\pi\frac{3}{4}\right) = 0.$$

Then, if $m = 4k$ for some k, then

$$(j, \ell) \in \{(k, k), (k, 3k), (3k, k), (3k, 3k)\}$$

are all solutions to Equation10.4. Then, together with $(j, \ell) = (0, 0)$, there are at least 5 distinct solutions to Equation 10.4, so the multiplicity of 1 as an eigenvalue of T_{4k} is at least 5.

Similarly, if $m = 5k$ for some k, we note that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4} \text{ and } \cos\left(\frac{4\pi}{5}\right) = \frac{-1-\sqrt{5}}{4}$$

and we see that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = \frac{-1+\sqrt{5}}{4} + \frac{-1-\sqrt{5}}{4} = -\frac{1}{2}$$

and

$$\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = \frac{\left(-1+\sqrt{5}\right)\left(-1-\sqrt{5}\right)}{4} = \frac{(-1)^2-5}{4} = -\frac{1}{2}.$$

Then, let $A = \{k, 4k\}$ and $B = \{2k, 3k\}$. For every choice of $j \in A$ and $\ell \in B$, we obtain distinct solution (j, ℓ) to Equation (10.4). Then the multipliticity of 1 is at least $8 + 1 = 9$. $\square$

Chapter 11

Open problems and future work

There are many open problems concerning the spectra of digraphs. One of the motivating questions for this thesis, was the search for a spectral bound for digraph chromatic number. A proper k-colouring of a digraph X is a partition of its vertices into k sets, each of which induced an acyclic subdigraph of X. The chromatic number of digraph X is the least k for which X admits a proper k-colouring. A bound analogous to Hoffman's bound in the undirected case would be a very interesting new result.

One of the difficulties of working with the adjacency matrix of digraphs is that there is no analogue of the eigenvalue interlacing theorem. For a differentiable complex-valued function f, the Gauss-Lucas theorem gives that the roots of f' lie in the convex hull of the roots of f. For a vertex-transitive digraph X, we have that $\phi(A(X-v),t)$ is a scalar multiple of $\phi(A(X),t)$, for any choice of vertex v. The Gauss-Lucas theorem would give that the A-eigenvalues of $X-v$ lie in the convex hull of the A-eigenvalues of X. This is, however, not a very good bound and also cannot, in general, be applied iteratively, as $X-v$ is not vertex-transitive unless X is complete. Finding a true analogue to the interlacing theorem would give a great tool for working with the adjacency matrix of digraphs.

In this dissertation, we study a Hermitian adjacency matrix with entries $\{0,1,\pm i\}$. It is also reasonable to ask about other matrices respecting adjacency which are Hermitian, but with different entries. In particular, for a digraph X, the matrices $H'(X)$ and $H''(X)$ are also matrices respecting the adjacency of X, where

$$H'(X)_{uv} = \begin{cases} 1 & \text{if } uv \text{ and } vu \in E; \\ \frac{1+i}{\sqrt{2}} & \text{if } uv \in E \text{ and } vu \notin E; \\ \frac{1-i}{\sqrt{2}} & \text{if } uv \notin E \text{ and } vu \in E; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H''(X)_{uv} = \begin{cases} 1 & \text{if } uv \text{ and } vu \in E; \\ e^{\frac{i\pi}{3}} & \text{if } uv \in E \text{ and } vu \notin E; \\ e^{\frac{-i\pi}{3}} & \text{if } uv \notin E \text{ and } vu \in E; \\ 0 & \text{otherwise.} \end{cases}$$

In Section 6.10, we find all digraphs with all H-eigenvalues strictly between $-\sqrt{3}$ and $\sqrt{3}$. It is plausible that a classification can also be found for all digraphs with all H-eigenvalues between -2 and 2. In general, which digraphs have all H-eigenvalues in the range $(-\alpha, \alpha)$? In Section 6.10, we also ask for which values of α there are only finitely many weakly connected digraphs whose H-spectrum is contained in $(-\alpha, \alpha)$.

In Section 8.9, we look at digraphs where the spectral radius is significantly larger that the largest eigenvalue. We give a family of digraphs $\{X_n\}_{n=1}^{\infty}$ such that $\rho(H(X_n)) - \lambda_1(H(X_n)) \to \infty$ as $n \to \infty$. For this family, we also have that $\rho(H(X_n))$ and $\lambda_1(H(X_n))$ both approach ∞ as $n \to \infty$. We can ask if there exists a family of digraphs where ρ grows much faster that λ_1; for example, if it is possible to construct a family of digraphs such that $\lambda_1(H)$ is constant and $\rho(H) \to \infty$ and $n \to \infty$.

Intuitively, a digraph X having a large number of simple eigenvalues, with respect to any matrix respecting adjacency, tends to have few symmetries. We could ask if there is a characterization of the digraphs with all eigenvalues simple (with respect to some diagonalizable matrix respecting adjacency) which are also arc-transitive. For the adjacency matrix, the even directed cycles provide examples of such digraphs. For the Hermitian adjacency matrix, the odd directed cycles provide examples of such digraphs.

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Appendix A

Linear algebra

A.1 Definitions

In this section, we give basic linear algebra definitions and concepts. See [34] for a standard reference text.

Let A be a $n \times n$ matrix. If $\mathbf{v}$ is a non-zero vector such that

$$(A - \lambda I)^k \mathbf{v} = 0$$

for some k and some scalar λ, then $\mathbf{v}$ is said to be a *root vector* of A with *eigenvalue* λ. The *index* of $\mathbf{v}$ as a root vector of A is the least integer k such that $(A - \lambda I)^k \mathbf{v} = 0$. If $\mathbf{v}$ is a root vector with index 1, then $\mathbf{v}$ is said to be an *eigenvector* of A. Equivalently, $\mathbf{v}$ is an eigenvector of A with eigenvalue λ if $A\mathbf{v} = \lambda \mathbf{v}$.

Fix λ. We wish to consider eigenvectors of A with eigenvalue λ; this is equivalent to asking for solutions to the equation

$$(A - \lambda I)\mathbf{v} = 0.$$

Observe that the equation has a solution if and only if $|A - \lambda I| = 0$. Then, the eigenvalues of A are roots of $p(\lambda) = |\lambda I - A|$, which is said to be the *characteristic polynomial* of A. The converse is also true; if λ is a root of the characteristic polynomial of A, then there is an eigenvector of A with eigenvalue λ.

A.1.1 Theorem. *Eigenvectors corresponding to different eigenvalues are linearly independent.*

A.1.2 Theorem. *Let A and B be $n \times n$ matrices. The characteristic polynomials of AB and BA coincide.*

A.1.3 Corollary. *Let A and B be $m \times n$ matrices. The characteristic polynomials of AB^T and $B^T A$ differ by a factor of λ^{n-m}.*

We use $\phi(A, t)$ to denote the characteristic polynomial of A in variable t and $\sigma(A)$ to denote the multiset of roots $\phi(A, t)$. The multiset $\sigma(A)$ are said to be the *eigenvalues* of A or the *spectrum* of A.

A.1.4 Theorem. *For a matrix A with spectrum $\sigma(A)$,*

$$\sum_{\lambda \in \sigma(A)} \lambda^k = \operatorname{tr}(A^k).$$

A.2 Polynomials of matrices

Let V be a vector space of dimension n and let A be a linear operator acting on V. Since the space of endomorphism of V has dimension at most n^2, we have that

$$1, A, A^2, \ldots, A^{n^2}$$

are linearly dependent. Then, there exists a polynomial $\psi(A)$ such that

$$\psi(A) = 0.$$

Any such polynomial is said to be an *annihilating polynomial* of A.

A.2.1 Lemma. *There exists a monic annihilating polynomial ψ of least degree.*

The monic polynomial of least degree such that $\psi(A) = 0$ is said to be the *minimal polynomial* of A.

A.2.2 Theorem. *Let f be a polynomial such that $f(A) = 0$ and let ψ be the minimal polynomial of A. Then ψ divides f.*

We will follow the notation of [27] and define the *characteristic polynomial of matrix A in the variable t* to be

$$\phi(A, t) := \det(tI - A).$$

The following is a famous theorem of Cayley and Hamilton.

A.2.3 Theorem. *For any square matrix A over a commutative ring, the characteristic polynomial $\phi(A, t)$ of A satisfies $\phi(A, t) = 0$.*

It is evident that the minimal polynomial of A divides the characteristic polynomial of A. Every eigenvalue of A is a root of the minimal polynomial.

A.3 Similar matrices

If matrices A and B satisfy

$$A = P^{-1}BP$$

for some invertible matrix P, then A and B are said to be *similar* matrices. If A and B are similar, then they represent the same linear operator under different bases.

A.3.1 Theorem. *If A and B are similar, then the following are true:*

(i) $rk(A) = rk(B)$,

(ii) $|A| = |B|$,

(iii) $tr(A) = tr(B)$,

(iv) $p(A) = p(B)$, *where $p(M)$ denotes the characteristic polynomial of matrix M,*

(v) the minimal polynomials of A and B are equal, and

(vi) the eigenvalues of A and B coincide.

A.3.2 Theorem. *Let A and B be real matrices such that $A = P^{-1}BP$ where P is a complex matrix. Then $A = Q^{-1}BQ$ for some real matrix Q.*

A square matrix A is said to be be *diagonalizable* if A is similar to a diagonal matrix. A square matrix N is *normal* if $N^*N = NN^*$, where the $*$ denotes conjugate transpose. A square matrix H with entries in $\mathbb{C}$ is said to be *Hermitian* if $H^* = H$. We may observe that Hermitian matrices are normal.

A.3.3 Theorem. *If A is a normal matrix, then A is diagonalizable.*

A.3.4 Theorem. *If H is a Hermitian matrix, then all eigenvalues of H are real numbers.*

A.4 Spectral decomposition

We will state the spectral decomposition theorem for Hermitian matrices.

A.4.1 Theorem. *If A is a $n \times n$ Hermitian matrix with distinct eigenvalues $\theta_1, \ldots, \theta_m$ and corresponding eigenspaces $V_1, \ldots, V_m$, then*

$$A = \sum_{i=1}^{m} \theta_i E_i$$

where E_i is the projection onto V_i and

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_m.$$

Proof. Since A is diagonalizable, consider an othonormal eigenbasis $\mathcal{B}$ for A. For V_i define E_i to be

$$E_i := \sum_{\mathbf{v} \in \mathcal{B} \cap V_i} \mathbf{v}\mathbf{v}^T.$$

Consider any vector $\mathbf{w} \in \mathbb{R}^n$. Then

$$\mathbf{w} = \sum_{\mathbf{v} \in \mathcal{B}} \alpha_{\mathbf{v}} \mathbf{v}$$

for some $\alpha_{\mathbf{v}}$s. We have

$$\begin{aligned} E_i\mathbf{w} &= \left(\sum_{\mathbf{v} \in \mathcal{B} \cap V_i} \mathbf{v}\mathbf{v}^T \right) \left(\sum_{\mathbf{v} \in \mathcal{B}} \alpha_{\mathbf{v}} \mathbf{v} \right) \\ &= \sum_{\mathbf{v} \in \mathcal{B} \cap V_i} \alpha_{\mathbf{v}} \mathbf{v} \end{aligned}$$

since $\mathcal{B} \cap V_i$ is an orthonormal basis of V_i. If $\mathbf{w} \in V_i$, then $\mathbf{w}$ is fixed by E_i. Otherwise, we have that $E_i\mathbf{w} \in V_i$ is the projection of $\mathbf{w}$ onto V_i, as required. □

For any $n \times m$ matrix A with entries in $\mathbb{C}$, the matrix $N = AA^*$ is Hermitian and hence has real eigenvalues. For such a matrix A, the eigenvalues of AA^* (or, equivalently, A^*A) are said to be the *singular values* of A.

A.5 Perron-Frobenius

The Perron-Frobenius Theorem is an important linear algebra result for algebraic graph theory and can be found in graph theory and linear algebra textbook, including [10, 34].

A square matrix A is *reducible* if there exists a permutation matrix P such that

$$P^T AP = \begin{pmatrix} A_{11} & A_{12} \\ 0_{n-r,r} & A_{22} \end{pmatrix}$$

where A_{11}, A_{12} and A_{22} are block matrices and $O_{n-r,r}$ denotes a $(n-r) \times r$ block of zero entries, for $1 \leq r \leq n-1$. A square matrix A is *irreducible* if it is not reducible.

Let A be an irreducible real $n \times n$ matrix. The *period* d of A is the greatest common divisor of all integers m such that $A^m(j,j) > 0$ for some $j \in [n]$. The period does not depend on the choice of j.

A.5.1 Theorem (Perron-Frobenius)**.** *Let A be an irreducible, real matrix with non-negative entries. Then there exists a unique positive $\lambda \in \mathbb{R}$ with the following properties:*

(i) there exists a real vector $\mathbf{v}$ with positive entries such that $A\mathbf{v} = \lambda\mathbf{v}$;

(ii) λ has geometric and algebraic multiplicity equal to 1 as an eigenvalue of A;

(iii) for each eigenvalue θ of A, we have $|\theta| \leq \lambda$; and

(iv) if A has period d, then A has precisely d eigenvalues θ with $|\theta| = \lambda$, which are

$$\left\{\lambda e^{\frac{2\pi ij}{d}} \mid j = 0, \ldots, d-1\right\}.$$

A.6 Commuting matrices

A.6.1 Lemma. *For real, symmetric, square matrices A and B, if $A^m B = BA^m$ for $m \geq 1$, then $A^{2^h} B = BA^{2^h}$ where 2^h is the largest power of 2 dividing m.*

Proof. It suffices to show for odd m that $A^m B = BA^m$ implies that $AB = BA$. Then, if $A^{2^h m} B = BA^{2^h m}$ for some h, then we may apply the statement to $A' = A^{2^h}$.

The function $f(x) = x^m$ is a bijective function when m is odd. Thus, the eigenvalues of A and A^m are in one-to-one correspondence and A and A^m have the same eigenspaces. Suppose that A and B and both $n \times n$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an eigenbasis for A^m and B, which exists since A^m and B commute. Observe that this is also an eigenbasis for A. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A such that $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$ for $j \in [n]$. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of B such that $B\mathbf{v}_j = \mu_j \mathbf{v}_j$ for $j \in [n]$. Then

$$(AB)\mathbf{v}_j = \lambda_j \mu j \mathbf{v}_j = \mu_j \lambda_j \mathbf{v}_j = (BA)\mathbf{v}_j.$$

for any $j \in [n]$. Then AB and BA agree over a basis of $\mathbb{C}^n$ and so $AB = BA$. □

Appendix B

Polynomials

B.1 Roots of Polynomials

The following standard results about the roots of polynomials with real or rational coefficients are found in [37].

B.1.1 Theorem. *Let p be a polynomial with coefficients in $\mathbb{R}$. If $a + bi$ is a root of p, where $a, b \in re$, then $a - bi$ is also a root of p, with the same multiplicity.*

B.1.2 Theorem. *Let p be a polynomial with coefficients in $\mathbb{Q}$. If α is a root of p, then the algebraic conjugates of α over $\mathbb{Q}$ are also roots of p.*

B.1.3 Corollary. *Let p be a polynomial with coefficients in $\mathbb{Q}$. If $a + \sqrt{b}$ is a root of p, where $a, b \in \mathbb{Q}$ and b not a square, then $a - \sqrt{b}$ is also a root of p, with the same multiplicity.*

Appendix C

Data

C.1 Digraphs on 3 vertices

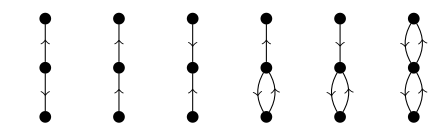
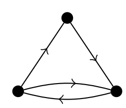
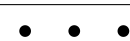
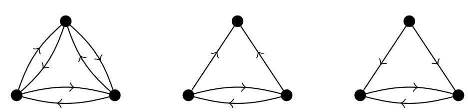
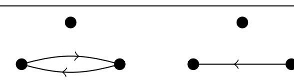
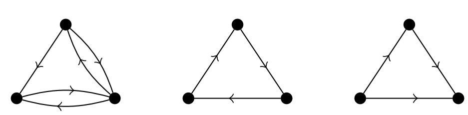
Polynomial $p(t)$	Roots of $p(t)$	All digraphs X such that $\phi(H(X), t) = p(t)$
$t^3 - 2t$	$\sqrt{2}, 0, -\sqrt{2}$	 Z_1 $\vec{P}_3$ Z_2 Z_3 Z_4 P_3
$t^3 - 3t + 2$	$1, 1, -2$	 K_3'
t^3	$0, 0, 0$	 E_3
$t^3 - 3t - 2$	$2, -1, -1$	 K_3 $Y_{2,1}$ $Y_{1,2}$
$t^3 - t$	$1, 0, -1$	 Z_5 Z_6
$t^3 - 3t$	$\sqrt{3}, 0, -\sqrt{3}$	 Z_7 D_3 $\widetilde{C}_3$

Table C.1: H-cospectral classes of all digraphs on 3 vertices.

Nomenclature

$[n]$ $\{1, 2, \ldots, n\}$

$\alpha(G)$ the size of the largest co-clique in a graph G

$\vec{D}(G)$ digraph of graph G

$\Gamma(X)$ underlying graph of digraph X

$\lambda_1(M)$ the largest eigenvalue of matrix M

$\mathbf{1}$ the all ones vector

$\phi(M, t)$ characteristic polynomial of matrix M in variable t

$\rho(M)$ the spectral radius of matrix M

$\sigma(M)$ the multiset of eigenvalues of matrix M

$\mathrm{Sym}(n)$, $\mathrm{Sym}(V)$ the symmetric group acting on $[n]$, or on ground set V

$D(X)$ asymmetric sub-digraph of digraph X

$G(X)$ symmetric subgraph of digraph X

J the all ones matrix

M^* the conjugate transpose of matrix M

T_n transitive tournament on n vertices

X a digraph, unless stated otherwise

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