Quantifying the Effect of Open-Mindedness on Opinion Dynamics and Advertising Optimization

by

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ABSTRACT

Group opinion dynamics shape our world in innumerable ways. Societal aspects ranging from the political parties we support to the economic decisions we make in our daily lives are all directly affected in some way by group opinion dynamics. This makes understanding and potentially being able to predict the complex inter-relationships between individuals' opinions and group opinion dynamics invaluable both scientifically and economically. We propose an aggregation model incorporating ingroup-outgroup dynamics, as well as media influence, to establish potential causal relationships between various types of social interaction and social phenomena such as the occurrence of group consensus and the hostile media effect. We further apply our model to simplified commercial applications relating to advertisement optimization to determine the optimal proportion of a population to target with advertising in order to maximize opinion shift while fixing cost.
Keywords: Opinion dynamics, aggregation, differential equations, media, ingroup-outgroup dynamics, open-mindedness

DEDICATION

This thesis is dedicated to the reader,

...for being so good looking

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I would like to thank:

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Chapter 1

Introduction

1.1 Opinion dynamics

There are two main forces which shape opinion dynamics: Social forces such as the beliefs of one's peers and exogenous forces such as media influence [8, 28, 4, 6]. Self-thinking behavior also plays an important role in opinion dynamics though for simplicity we will neglect it in our model. Here self-thinking denotes individual's spontaneous thoughts and emotions which are not directly attributable to either the actions of other individuals, or other exogenous inputs. This is perhaps not the most unrealistic decision, since in many situations peers and media dynamics play a larger role than self-thinking [21, 30, 3, 50]. Hence, our modeling effort will focus exclusively on the effects of peer and media influence. In order to model these social factors, we must first understand their nature.

Consensus seeking

One component of peer social influence is the general tendency of many individuals to change their own opinion to align with those of their peers [4]. This phenomenon was famously demonstrated by Asch using his landmark line experiment, in which individuals were asked to compare the relative lengths of lines while surrounded by a group of paid confederates who would give either accurate or false statements about the relative lengths of said lines [4]. It was found that when an individual was in the presence of two or more confederates, 75% of individuals tended to adopt the confederates' opinion and give false statements about the relative lengths of the lines. This trend persisted even when the difference in line lengths was obvious [4]. This consensus-seeking behavior is a feature of virtually all theoretical work on opinion dynamics to date [20, 38, 7, 8, 46, 51, 39, 35].

The tendency to seek consensus with one's peers does not apply evenly to all interactions. Individuals have a greater tendency to change their opinions to align with those more similar to themselves [6, 25]. One of the earlier demonstrations of this phenomenon was made by Berscheid [6]. She designed a setup in which subjects were tricked into observing a confederate argue in favor of a particular opinion slightly different than that of the subject. Prior to this, the subject had either been told or had perceived the confederate holding a similar belief to their own, on either the subject matter being debated or a different subject matter. It was found that individuals were more persuaded by the confederate's arguments when the confederate appeared similar to them, particularly if that similarity was in the subject matter which was being discussed.

This trend of individuals being more susceptible to peer influence from those similar to them has been documented in numerous areas [30, 3, 25, 27, 6]. In the area of American politics it has been found that individuals are in most cases either unresponsive or hostilely responsive to media which has a bias different than their own, but shift in the partisan direction of media which shares their political beliefs [30, 3, 27]. This trend of individuals being affected by opinions similar to their own even extends to indirect messages such as those of hostile sexism found in certain types of pornography [25]. It was found by Hald et al. that experimental exposure to pornography fosters hostilely sexist beliefs, but only amongst subjects with low levels of pretest agreeableness [25].

The above implies that any modeling endeavor should have have a feature incorporated within it, in which individuals are most responsive to other individuals within a certain window of opinion space. This is the case for many models, such as that proposed in [38].

Mathematical consensus-seeking models

In [38] Motsch and Tadmor investigate the requirements for consensus of a broad class of opinion alignment models with properties similar to flocking models used in animal aggregation such as [12]. Opinions of individuals were defined as n-tuples, $(n \geq 1)$ on a scale from zero to ten. Using this framework, Motsch and Tadmor use analytical techniques borrowed from graph theory to establish a requirement for consensus. Additionally they numerically demonstrate that when individuals interact in a heterophilious manner the occurrence of consensus is enhanced. Here 'heterophily' is defined as the tendency of individuals to be more responsive to peer influence of those with different opinions relative to their own. We will discuss this model in greater detail at the beginning of Chapter 2, since it is of special importance to our modeling endeavor.

Other models, such as the class outlined in Section 1 of [20], use the idea of consensus to make direct predictions about results such as elections. By starting from a generally category-based classification of individual's opinions, Galam constructs a bottom up hierarchy from the population based on local majority rules. From this, tree-like networks are built which make deterministic predictions about the outcome at the top based on a random selection of agents at the bottom layer [21, 20].

Another discrete opinion-based model proposed by Galam describes opinion dynamics as an Ising ferromagnetic system, where the alignment of agents between two particular groups is defined analogously to the state classification of atoms in a substance [20]. Here social influence and individual differences are represented in terms of external and internal fields which impact the magnitude and sign of the Hamiltonian function H as shown below:

$$H = -\sum_{i,\,j} J_{i,j} k_i k_j - g \sum_i k_i - \sum_i f_i S_i \;, \tag{1.1}$$

and

$$P(K) = e^{\beta H(K)}/Z \qquad \text{where} \qquad Z = \sum_K e^{\beta H(K)}. \tag{1.2}$$

This function H determines the relative probability P of a certain state of alignments K. Here k_i represents the alignment of the ith individual, the terms f_i, β and g are weighting constants and $S_i = \pm 1$. Galam uses this model to offer a potential explanation of the social phenomenon of *group extremism* found in [15, 50]. In the aforementioned experiments, groups which were forced to reach a consensus opinion tended towards choosing an extreme opinion, relative to the mean opinion of the individuals in the group prior to group interaction. Galam demonstrates that when individuals are forced to choose between distinct sides while additionally trying to align with their peers, extremism tends to result [22, 20]. In this thesis we will demonstrate that by incorporating this dynamic of group allegiance coupled with peer alignment through population heterogeneity, the same result can be found when individual's opinions are represented continuously, as will be shown in Chapters 2 and 3.

Ingroup-outgroup dynamics

The potential explanation for group extremism we shall propose is not without basis as the relationship describing an individual's response to peer opinions is not homogeneous within a population. It has been noted in [30, 27] that when the majority of individuals experience *cross-cutting media* (media differing in partisan bias from their own belief), individuals tend to either not change their opinion, or become more extreme in their beliefs upon exposure. There is however a minority of individuals who moderate when exposed to media which expresses a different partisan perspective than their own [27, 30]. This implies that treating all individuals as equivalent is inadequate to fully describe opinion dynamics.

In order to model this heterogeneous response, we must first consider the psychological mechanism behind its occurrence. Specifically, this means that we must consider the sociological phenomenon of ingroup-outgroup dynamics. *Ingroup-outgroup dynamics* is the process in which individuals identify themselves and others based on their inclusion or lack thereof in a particular group. The process of ingroup-outgroup dynamics is governed by an area in the social sciences known as self-categorization theory [26]. Self-categorization theory holds that under certain circumstances, individuals will categorize themselves as members of an ingroup, with a particular set of social norms and beliefs [26]. This tendency increases under certain circumstances, such as when individuals feel threatened [37]. Individuals who do not share the group's beliefs, are viewed as being part of the outgroup.

Individuals who view themselves as part of an ingroup tend to behave closed-mindedly with respect to those with opposing views associated with the outgroup [26, 50, 15, 30]. When exposed to views belonging to their ingroup, closed-minded individuals feel more certain in their beliefs and become more extreme. This opinion shift still occurs even if the like-minded opinion which they are exposed to is more moderate than their own. Additionally, exposing closed-minded individuals to outgroup opinions heightens their sense of group identity, causing them to counter-argue against the outgroup opinion and feel more certain of their previous belief [50, 26, 37]. This phenomenon explains why clearly identified groups become more extreme when exposed to both like-minded and cross-cutting opinions [26, 50].

One case of this phenomenon documented by Doise was amongst students at an alternative architectural school [15]. In this study by Doise, individuals were exposed to one of two stimuli. The first was a group discussion amongst their fellow students about the merits of their school as compared to a more prestigious school. The second stimulus was a video of members of said more prestigious school discussing the subject's school. In both cases individuals became more polarized in their beliefs about the areas where their school was superior and the areas where the other school was superior. Since this pioneering study, similar results of dual polarization have been noted amongst issues as diverse as: friends' marriage decisions, risk management in business, feminism and verdict determination amongst jurors. In all of these cases the collective belief of a group of individuals after discussion was more extreme than the mean individual opinion pre-discussion [50]. It is additionally worth noting that individual extremism is under some circumstances a prerequisite for closed-minded behavior [52].

Mathematical modeling of non-open-minded individuals

A variation of this type of interaction in the absence of media has been modeled by Boudin et al. and Galam [19, 7]. Galam considers closed-minded individuals in the context of the bottom-up voter model previously discussed [19]. He demonstrates that a minority of closed-minded individuals have the potential to reduce the emergence of a consensus, and postulates that this type of phenomenon could be a potential explanation for the Bush-Gore stalemate in the 2000 American presidential election. This model has minimal relevance to our own other than the psychological dynamics which it incorporates, since it is more of an algorithm for determining a vote as opposed to a model for directly simulating group opinion dynamics over time.

Boudin et al. consider this dynamic using an Eulerian modeling approach [7]. Specifically, this means that rather than considering individuals with specific opinions (a Lagrangian modeling approach), Boudin et al. represent evolving opinion dynamics as a distribution of a population density function in opinion space, whose evolution in time is based on certain assumed binary interactions between individuals. Their model is more similar to ours than Galam's though it differs in that the secondary class of individuals exhibits contradictory behavior rather than closed-minded behavior. Specifically this means that individuals tend to move towards the extreme furthest from their interacting partner. This differs from our closed-minded behavior (which will be explained in detail in Chapter 2) in that in our model, closed-minded individuals move towards the same extreme for all interactions. It should be noted that when we refer to individual interactions in the model of Boudin et al., we are referring to the binary interactions implicit in their model.

Using these contradictory dynamics, Boudin et al. find that consensus does not result in general [7]. This is an interesting result in that it offers a potential explanation for hung elections, though it also raises the potentially more interesting question, namely: How can consensus result when some individuals exhibit these types of consensus-opposing behaviors? This is an important question since, in order for conflicts to be resolved, some sort of consensus must be reached between opposing parties, and in almost all cases some individuals belonging to either party will behave in a closed-minded or contradictory fashion. This will be one of the key questions which we will investigate in this thesis.

Social norms of open-mindedness

One area of opinion interaction where these types of consensus-opposing behaviors are quite common is American politics [3, 30]. Survey data in the social sciences demonstrates that political questions containing the word 'Obama' elicit vastly polarized responses relative to similar questions which lack obvious partisan markers [3]. Additionally, many past and present real world political-military conflicts involved closed-minded individuals on either side [11]. Since several of these conflicts have reached a peaceful consensus of sorts, it is clear that consensus can, under certain circumstances, result even in the presence of closed-minded individuals.

Understanding what these circumstances are is critical to resolving other real world conflicts, and hence will be the primary focus of this thesis. This question has yet to be explored mathematically to the best of our knowledge, but some potential explanations have been given in the social sciences [56, 57, 36]. For example [56] found, through interviewing two forest company managers about grievance resolution, that cooperative norms and open-minded interaction lead to consideration of opposing views and general resolution of conflicts [36, 56]. However this was not the case when competitive, closed-minded norms were present [56]. This implies that open-minded individuals may have the potential to encourage closed-minded individuals to behave open-mindedly through social pressure caused by *open-mindedness social norms*. We will investigate whether this mechanism has the potential to cause consensus in presence of various degrees of ingroup-outgroup dynamics.

1.2 Media dynamics

Two additional questions which the above suggests, are: (1) what role do exogenous social forces such as media play in shaping opinion dynamics in the presence of ingroup-outgroup dynamics, and (2) can social norms of open-mindedness cause consensus in the presence of media? These questions are particularly intriguing since the perception of news media is often influenced by a social phenomenon known as the hostile media effect [60].

The hostile media effect

The hostile media effect is a phenomenon in which highly partisan individuals on either side of a conflict will perceive the media to be hostilely biased against them [9, 13, 10, 33, 60]. First observed by [60] in the Arab-Israeli conflict, it was found that when pro-Arab and pro-Israeli sympathizers were shown the identical news coverage of the 1982 Beirut massacre, each side perceived the coverage to be biased in favor of the other side's beliefs. Since then this phenomenon has been documented to play an important role in conflicts as diverse as: American elections, the Bosnian conflict and the US parcel workers strike of 1997 [10, 9, 13]. Due to this impact on a diverse set of conflicts, it is critical to understand how conflicts in general can be resolved in the presence of the hostile media effect. This resolution process can be represented mathematically as the formation of consensus.

Mathematical modeling of media

Mathematically there has not been much research to date in modeling the effect of closed-minded or consensus-resisting behavior in the presence of media. Most opinion dynamic modeling endeavors incorporating the effect of media primarily focus on the interplay between consensus-seeking behavior and media [8, 35].

One such modeling effort by Boudin et al. represents media opinion interaction using a kinetic model in which individuals seek a consensus [8]. Here media is represented as a background noise which potentially varies in time. Using this framework they demonstrate numerically that an extremist media source can often defeat its own purpose and fail to attract individuals. In addition, they investigate the effect of media in a three party political system.

Another Eulerian modeling effort was carried out by Bullo et al. [35]. They investigated analytically the conditions required for consensus in the presence of media, and find a somewhat analogous sufficiency condition (individuals must be sufficiently close in opinion to interact with each other) to that found in [38] in the absence of media.

In addition to Eulerian techniques for modeling media influence there have also been lattice model-based techniques [46], as well as even SIR modeling endeavors wherein the spread of ideas is represented analogously to the spread of a disease [58]. One such SIR media influence model proposed by Tweedle and Smith? investigates the media's influence on 'Bieber Fever' (the spread of the popularity of the teen idol Justin Bieber). They find that a sustained negative media influence (referred to as the 'Lindsay Lohan effect') would be required to reduce the popularity of Justin Bieber.

1.3 Advertising

Another question which we will consider, is what role do ingroup-outgroup dynamics play in the commercial success of various advertising strategies? This may at first seem like an odd question, since one does not typically imagine consumer preference to be shaped by group identity. That this could occur should not be surprising, though, since numerous studies link product preference to various group-identifying features such as sex, age, personality and political affiliation [40, 59, 41, 32, 61]. Hence, since product preference is correlated with group identity, one can expect that under certain circumstances, individuals would embrace products closed-mindedly for reasons similar to those discussed previously for which they embrace group-identifying opinions in a closed-minded fashion [26]. This idea is further supported anecdotally by the cultural tendency to equate certain beverages with group-identifying traits, such as the perceived manliness of beer [41]. For these reasons it is critical to understand the role which ingroup-outgroup dynamics plays in shaping the commercial success of advertisements.

One specific sub-area of advertising where ingroup-outgroup dynamics is of critical importance is that of guerrilla marketing. *Guerrilla marketing* is an advertising technique in which the advertiser attempts to have maximal impact with minimal investment by creating a novel ad which as a result of its novelty is spread along consumer social networks [5]. These techniques are of increasing importance in recent times due to the rise of social media leading to increased consumer-consumer interactions [47]. Due to this growing importance we will attempt to characterize some of the group psychological regimes under which such techniques are likely to be successful.

Mathematical modeling

Many previous advertising models model the relationship between advertising cost and sales [31]. We will instead use a similar method to that described in [46, 49] and assume that creating consumer preference for one's product is equivalent to generating sales.

As mentioned above, one example of this type of modeling effort was proposed in [46]. Schulze proposes an advertising model using a variation of the Sznajd opinion model originally proposed in [51]. Here opinions are represented as integers on a lattice and individuals are chosen at random for updating. Individuals have their opinions updated to those of their neighbors if and only if all of their neighbors have an identical opinion. Schulze modifies this model to incorporate multiple lattice layers representing aging in the population. Advertising influence is modeled as a global convincing force which probabilistically changes individuals' opinions to that of the advertised opinion. He finds that in this case that under some circumstances advertising can induce a consensus on the advertised opinion.

We will take this idea further in the case of our model and investigate the role of ingroup-outgroup dynamics on the optimal advertising strategy.

1.4 Outline of thesis

In general this thesis will propose an ODE model for opinion dynamics of N individuals in the presence of media, and use this model as a general tool for conducting numerical experiments which lend insight into the general effect of open-mindedness and other social phenomena on opinion dynamics and advertising optimization. These numerical experiments will be supplemented with relevant mathematical and statistical analyses.

Specifically, Chapter 2 will propose our model in the absence of media and demonstrate that under the assumptions of our model, social norms of open-mindedness have the potential to cause consensus in the presence of ingroup-outgroup dynamics.

Chapter 3 will introduce our model in the presence of media and show that, given certain assumptions, consensus can still be caused by social norms of open-mindedness when individuals are also exposed to a single media source and the influence of the hostile media effect.

Chapter 4 will modify the model proposed in Chapter 3 to be relevant for simulating the effect of advertising. Using this modeling framework we will investigate the effect of ingroup-outgroup dynamics on the optimal advertising strategy, and show a variety of associated results.

Our final results chapter, Chapter 5, will analyze the data set found in [30] to estimate realistic parameter values and gain some insight into the causes of the hostile media effect. We will additionally demonstrate the relative validity of our model assumptions by replicating a portion of the Levendusky data through fitted simulation.

Chapter 2

How open-mindedness norms cause consensus in the absence of media

The purpose of this section is to investigate whether open-mindedness social norms have the potential to cause consensus in the presence of ingroup-outgroup dynamics. As mentioned in the Introduction, consensus can under some circumstances result in the presence of ingroup-outgroup dynamics [50]. This is unintuitive since ingroup-outgroup dynamics typically cause individuals to blindly approach the extreme opinion of their respective groups [50]. In this manuscript we propose a mechanism for consensus in which a minority of open-minded individuals facilitate open-mindedness amongst closed-minded individuals in the short term through the creation of an open-mindedness social norm. As mentioned previously the existence of said social norm is supported by [56, 57, 36]. Here we will demonstrate that this can under some circumstances directly cause consensus when the proportion of open-minded individuals is large enough.

For reference, all constants and function mentioned below and in subsequent chapters are stated in Tables A.1 to A.3 in Appendix A.

2.1 Previous research and simulations

The modeling of group opinion dynamics using ODE and PDE aggregation models has grown in popularity over the last few years [38, 16, 8, 35, 39, 7, 1]. Our model is based on that recently proposed by Motsch and Tadmor [38]. Here, opinion dynamics amongst N individuals are modeled as a system of N ODEs of the following form:

$$\frac{dx_i}{dt} = \alpha \sum_j a_{ij}(x_j - x_i), \qquad i = 1, ..., N \tag{2.1}$$

where

$$a_{ij} = \phi(|x_j - x_i|)/\phi_i \qquad \text{, with} \qquad \phi_i = \sum_j \phi(|x_j - x_i|). \tag{2.2}$$

Here x_i denotes the opinion of the ith individual. As was mentioned in the introduction Motsch and Tadmor define opinions of individuals as n-tuples, $(n \geq 1)$ on a scale from zero to ten. In our own model, which is closely related to Motsch and Tadmor's, this scale is compressed and shifted to the interval [-1,1] in order to classify individual's group allegiance based on sign.

Motsch and Tadmor demonstrate that when the interaction function ϕ in (2.2) is heterophilious, (this occurs when ϕ is positive and increasing over some domain), the occurrence of consensus increases (Fig. 2.1 and 2.2). Here consensus is interpreted mathematically as a long-time equilibrium with a single value. This foreshadows the general principle that consensus is more likely when individuals are more open-minded. The aforementioned is evident by noting that heterophilious interactions like those modeled by Motsch and Tadmor imply that individuals are more persuaded to change their opinion when in contact with those somewhat different than themselves. This is equivalent to individuals being more open-minded. Motsch and Tadmor modeled this numerically by taking ϕ in (2.2) to be a step function, or a linear combination of two step functions with varying support. The size of this support determines how willing an individual is to considering opinions different than their own. This can be thought of as a measure of open-mindedness. An example of each of these step functions is shown in Fig. 2.3. We will assume a similar form for the interaction function in our model.

We will extend the results in [38] to our own more complex model and demonstrate that through the creation of open-mindedness social norms, open-minded individuals can cause consensus even in the presence of individuals who possess the propensity to be closed-minded.

It is worth noting explicitly here that a_{ij} (the interaction coefficient) is equal to ϕ evaluated at $|x_j - x_i|$ normalized by the sum over all k of ϕ evaluated at $|x_k - x_i|$. This normalization term is denoted by ϕ_i in the above equation. Due to this normalization the strength of the general interactions between individuals can be represented by the magnitude of α. This will be taken to be 1 in all simulations unless explicitly specified otherwise.

It should be noted explicitly here that Fig. 2.1 depicts the time evolution of each individual's opinion as a separate curve which merges with other individuals' associated curves as the individuals reach an agreement. Observing Fig. 2.1, we see that the heterophilious interaction function results in consensus at equilibrium whereas the simple step function interaction function leads to two clusters of individuals at equilibrium.

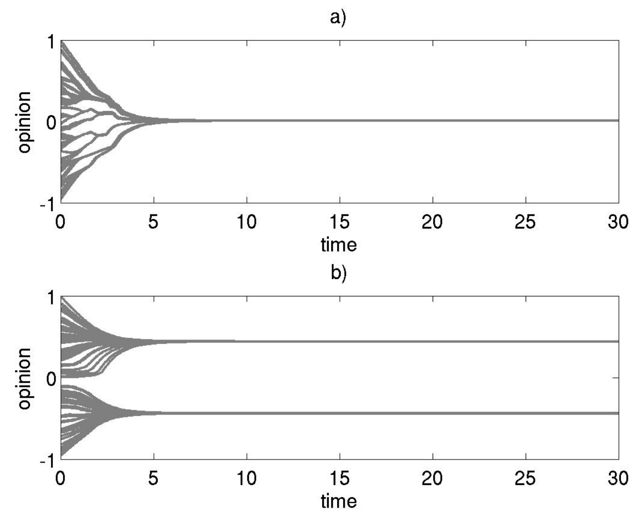

Figure 2.1: The effect of a heterophilious interaction function on consensus: Time series evolution of (2.1) for two different interaction functions a) $\phi = 0.1\chi_{[0,a_s/\sqrt{2}]} + \chi_{[a_s/\sqrt{2},a_s]}$, b)$\phi = \chi_{[0,a_s]}$. Here χ is the characteristic function, $a_s = 0.4$ and $N = 80$. Uniform random initial conditions were used. The heterophilious interaction function in (a) results in a consensus whereas the simple step function interaction function in (b) does not.

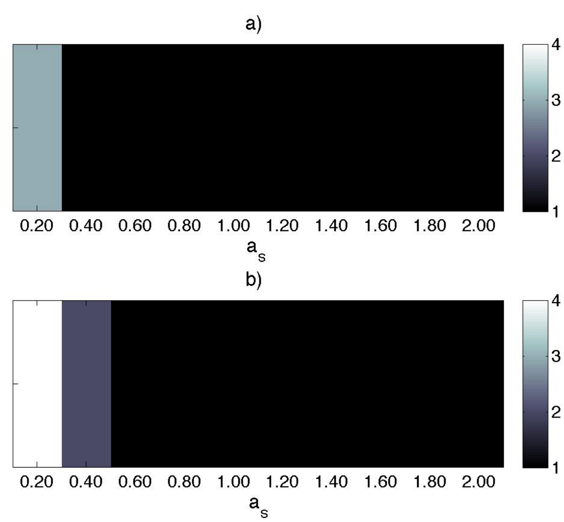

Figure 2.2: Parameter space plot: Average number of clusters when individual-individual interactions are governed by a) heterophilious interaction function $\phi = 0.1\chi_{[0,a_s/\sqrt{2}]} + \chi_{[a_s/\sqrt{2},a_s]}$ and b) a standard step function interaction function $\phi = \chi_{[0,a_s]}$. Averaged over 5 sets of uniform random initial conditions. $N = 80$. The heterophilious interaction function in (a) leads to consensus for a greater range of parameter values.

It should noted explicitly that Fig. 2.2 represents a parameter sweep where the support of ϕ is varied. Specifically, we see that as a_s is decreased more clusters form. Further the heterophilous interaction function used in Fig. 2.2 (a) results in fewer clusters than the step function interaction function (shown in Fig. 2.2 (b)). This implies that heterophilious interaction can lead to increased consensus.

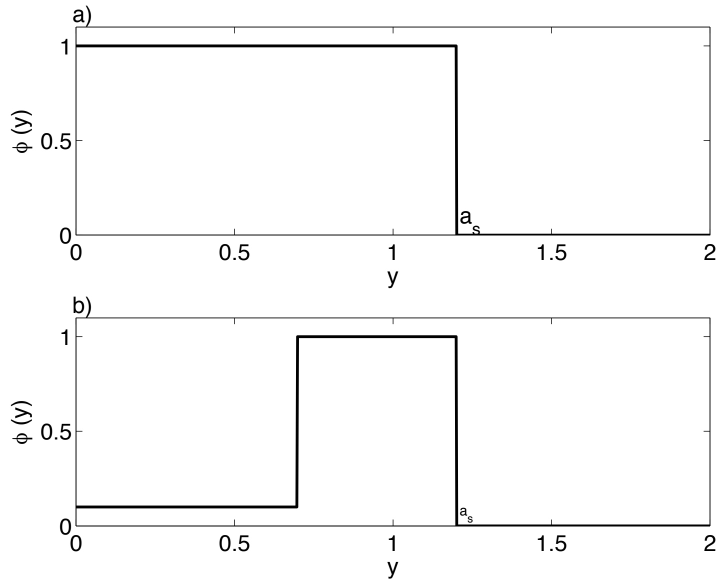

Figure 2.3: Sample interaction function (ϕ): a) Sample step function, this is used in all of the simulations in this manuscript unless explicitly specified otherwise. b) Sample heterophilious interaction function, used in many simulations in [38]. Note that the heterophilious interaction function leads to greater interaction amongst individuals with differing opinions as compared to the step function.

In Fig. 2.3 we see the two general types of interaction functions used in [38]. In addition it should be noted, that the larger the support of ϕ the more responsive open-minded individuals are to the opinions of others.

Numerically the above figures and all other time series in this thesis are simulated using a 4th order Runge-Kutta method for the first three times steps followed by a 4th order Adams-Bashforth method for the remaining time steps. This is done as to limit our evaluation of the right-hand side of the ODE.

2.2 Modeling open-mindedness and closed-mindedness

We will begin our analysis by first defining our model. Using (2.1) to capture the consensus-seeking behavior of open-minded individuals we will incorporate an additional class of individuals who rather than seeking the mean of the group, instead seek the extreme opinion. To account for the fact that this type of closed-minded behavior is usually only possible amongst extremist individuals [52] we will introduce a critical threshold X_c, below which all individuals behave in an open-minded fashion. Numerous studies in the social sciences show that for many issues (including politics), these extreme opinion seeking, closed-minded individuals comprise the majority of the population [30, 50, 26, 15]. We will denote the total number of open-minded individuals by m and the total number of individuals with the propensity to be closed-minded by $N - m$, where N represents the total number of individuals.

Model with open-mindedness social norm

Incorporating the aforementioned features we arrive at the following model stated below:

$$\frac{dx_i}{dt} = f_i, \tag{2.3}$$

where, f_i represents the social force from peer pressure and is defined as follows:

$$f_i = \begin{cases} \alpha_2(N-m-1)\hat{a}_i(1-x_i) + \alpha_3 \sum\limits_{j\in O} \hat{a}_{ij}(x_j - x_i) & \text{if } x_i > X_c \text{ and } i \in \hat{C} \\ \alpha_2(N-m-1)\hat{a}_i(-1-x_i) + \alpha_3 \sum\limits_{j\in O} \hat{a}_{ij}(x_j - x_i) & \text{if } x_i < -X_c \text{ and } i \in \hat{C} \\ \alpha_1 \sum_j a_{ij}(x_j - x_i) & \text{otherwise} \end{cases} \tag{2.4}$$

The fundamental idea of this model is that we distinguish between the set of $m \leq N$ open-minded individuals whose associated indices i belong to the set O, and the $N - m$ individuals who have the propensity to behave closed-mindedly whose associated indices i belong to the set $\hat{C}$. For simplicity we will use the statements of 'an individual belonging to the set of non-open-minded individuals or set of open-minded individuals' and '$i \in \hat{C}$ or $i \in O$' interchangeably. Our convention is to use superscript 'hat' for closed-minded individuals whose indices are in $\hat{C}$. The individuals whose indices are in $\hat{C}$ in fact act closed-mindedly when their opinions are sufficiently extreme $|x_i| > X_c$. It is also important to note that the term *non-closed-minded* refers to individuals in O and individuals in $\hat{C}$ whose opinions are not sufficiently extreme as to be closed-minded.

The normalized interaction function for the open-minded interaction between the ith and jth individual is represented by a_{ij} as in (2.1). Additionally we will explicitly note that the definition of a_{ij} given in (2.4) is defined by (2.2), where the functional form of ϕ is assumed to be a step function with support a_s as is depicted in Fig. 2.3 (a).

We will define $\hat{a}_i$ in (2.4) similarly, except that we must account for the fact that the psychological forces shaping $\hat{a}_i$ are not equivalent to those shaping a_{ij}. This is due to the fact that when an individual's response to peer influence is driven by ingroup-outgroup dynamics, they are driven towards their group's associated extreme opinion, irrespective of whether the said influencing peer is perceived to be a member of their perceived ingroup or a member of their perceived outgroup (this is in the absence of social norms of open-mindedness) [15, 26, 50]. Further, the rate at which they approach this extreme for each interaction will only be function of their distance from said extreme, and possibly of the sign of x_j [52], where the sign of the jth individual's opinion represents the perceived group categorization of the jth individual. For simplicity, though, we will assume that the magnitude of $\hat{a}_i$ is independent of x_j. This is equivalent to assuming that the two psychological responses discussed in Chapter 1 which lead to closed-minded individuals seeking their group's extreme opinion are equivalent in the magnitude of their influence.

Taking the above into account we arrive at the following definition for $\hat{a}_i$ in (2.4):

$$\hat{a}_i = \hat{\phi}(|sgn(x_i) - x_i|)/\hat{\phi}_i \quad \text{, with} \quad \hat{\phi}_i = (N-m-1)\hat{\phi}(|sgn(x_i)-x_i|) + \sum_{j \in O} \phi(|x_j - x_i|). \tag{2.5}$$

For simplicity we will take $\hat{\phi}(|sgn(x_i) - x_i|) = 1$ in all of our simulations unless explicitly stated otherwise. For theoretical purposes though this need not be the case. For example, $\hat{\phi}$ could be defined as the step function depicted in Fig. 2.3 (a) with an associated support of $\hat{a}_s$; in fact, we will use this as our assumed form in several of our stability results stated below.

It should also be noted that $\hat{a}_{ij}$ is almost identical in form to a_{ij} except that it has a different normalization term as shown in (2.6). This is done so as to weight all individual influences strictly in terms of the magnitudes of α_3, α_2 and α_1 .

$$\hat{a}_{ij} = \phi(|x_j - x_i|)/\hat{\phi}_i \qquad \text{where } \hat{\phi}_i \text{ is as defined in (2.5)} \tag{2.6}$$

Additionally it should also be noted that when $m = 0$ (that is O is empty: all individuals belong to the set of non-open-minded individuals and possess the predisposition to closed-mindedness) $\hat{\phi}_i$ is independent of x_j for $j \neq i$.

Continuing to consider (2.4) we note that the critical threshold (X_c) is included in our model as a certain degree of extremism is a prerequisite for closed-minded behavior [52]. This is due to the fact that extremist individuals are better able to counter-argue against outgroup opinions [52]. This transition between open-minded and close-minded behavior in reality is a continuous process in many circumstances (as we will see is supported by Fig. 5.1). For simplicity, though, we will approximate this behavioral transition with a discrete threshold.

The most significant novel feature of (2.4) in terms of our research is the inclusion of the effect of an open-mindedness social norm. We model this effect by having extremist closed-minded individuals behave open-mindedly when and only when they interact with open-minded individuals. As was mentioned previously in Chapter 1, it has been demonstrated empirically in the social sciences that through acting open-mindedly, individuals can prompt open-minded behavior in others through social pressure caused by social norms of open-mindedness [36, 57, 56]. In this chapter we will demonstrate that under the assumptions of our model this mechanism has the potential to cause consensus for a wide range of parameters.

Model without open-mindedness social norm

In the absence of an open-mindedness social norm our model described by (2.4) becomes the model depicted in (2.7). It is easy to see that this differs from (2.4) in that individuals who are acting closed-mindedly move to the extreme, regardless of the open-mindedness of their interacting peers. This difference is implemented mathematically through the removal of the sum term in the cases when $|x_i| > X_c$ and the ith individual belongs to the set of non-open-minded individuals. That is f_i in (2.3) is defined by:

$$f_i = \begin{cases} \alpha_2(N-1)\hat{a}_i(1-x_i) & \text{if } x_i > X_c \text{ and } i\in \hat{C} \ , \\ \alpha_2(N-1)\hat{a}_i(-1-x_i) & \text{if } x_i < -X_c \text{ and } i \in \hat{C} \ , \\ \alpha_1 \sum_j a_{ij}(x_j - x_i) & \text{otherwise} \end{cases} \tag{2.7}$$

Again the interaction coefficient a_{ij} represents the function ϕ described in (2.2) evaluated at $|x_j - x_i|$ and normalized. The function $\hat{a}_i$ will similarly be defined analogously to that in (2.5) except that its normalization will vary slightly. Specifically $\hat{a}_i$ will be defined as:

$$\hat{a}_i = \hat{\phi}(|sgn(x_i) - x_i|)/\hat{\phi}_i, \qquad \text{with} \qquad \hat{\phi}_i = (N-1)\hat{\phi}(|sgn(x_i) - x_i|) \tag{2.8}$$

Considering (2.8) we see that $\hat{a}_i$ is independent of x_j when $j \neq i$. Further $\hat{a}_i$ is simply $1/(N-1)$.

Again it is also worth noting here that all of our numerical simulations in this manuscript use the step functions shown in Fig. 2.3 (a) as the interaction function ϕ unless specified otherwise. Additionally it is worth noting that unless specified otherwise we will assume that $\alpha_3 = \alpha_1 = \alpha_2 = 1$ in all of our numerical simulations. This is fairly reasonable in terms of choosing parameters relevant to politics, although our analysis in later chapters shows that perhaps $\alpha_1 \approx 2\alpha_2$ in order to be relevant to American politics (see Chapter 5).

2.3 Equilibria and stability analysis

Before we consider the stability of equilibria analytically it should first be noted that all equilibrium solutions of Motsch and Tadmor's model [38] are neutrally stable in that perturbations result in the system relaxing to a slightly different equilibrium. In our own model we find numerically, that there seem to exist no unstable equilibria in our simulations in this chapter, or in later chapters (or at least none with a finite basin of attraction). We will demonstrate below that unlike the equilibria in the model proposed in [38], our model does under some circumstances allow for asymptotically stable equilibria. We will not however prove non-existence of other long-term oscillatory states. Some typical equilibria are shown in Fig. 2.4.

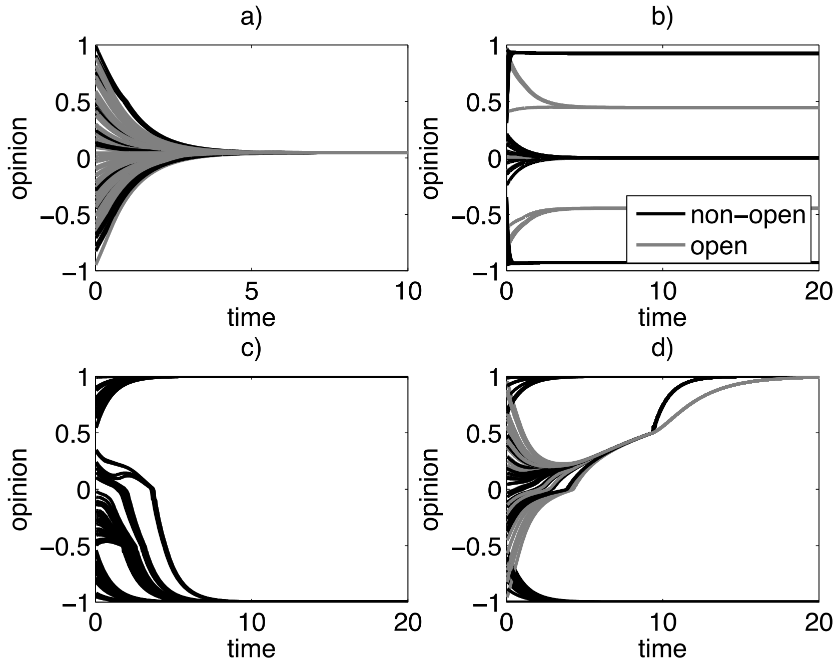

Figure 2.4: Time series showing the approach to various typical equilibrium solutions, with $N = 80$ a) Central consensus equilibrium: $m = 60$, $X_c = 0.5$, $a_s = 1.2$. b) Multi-cluster equilibrium: $a_s = 1.2$, $m = 10$, $X_c = 0.25$. c) Strong polarization in the absence of a social norm of open-mindedness: $X_c = 0.5$, a_s=1, $m = 0$. d) Polarization persists in the absence of a social norm of open-mindedness even when m increases: $X_c = 0.5$, a_s=1, $m = 40$.

These equilibria solutions of (2.3) can be classified macroscopically into three broad classes of equilibrium.

1. The first class (*Type 1*) consists of *consensus solutions* where $x_i^* = \bar{x}$ for all i. Here x_i^* is the equilibrium associated with the ith individual and $\bar{x}$ represents the mean over all x_i^*. These solutions tend to occur when the majority of the population is open-minded or when X_c is large (corresponding to low levels of ingroup-outgroup dynamics) and when a_s is large (see Fig. 2.8 for example of such parameters). These consensus equilibria can occur at moderate values such as in Fig. 2.4 (a) or extreme values such as in Fig. 2.5 (b). The latter case is more common when a relatively larger proportion of the population is closed-minded and when ingroup-outgroup dynamics play a larger role in interactions (smaller X_c). This is in agreement with the psychological literature where extremist consensuses tend to result in group scenarios where individuals have some sort of group-identifying feature associated with their belief [50, 15]. This phenomenon has previously been explored theoretically in [22] using a discrete model. As Fig. 2.5 (b) demonstrates, we are able to replicate this extremist consensus

dynamic using our continuous model when certain constraints are met. Regrettably we do not directly categorize these constraints.

2. The second general class of equilibria (*Type 2*) consists of *polarization equilibria* in which $x_i^* = +1$ for some i and $x_i^* = -1$ for the remaining i. These equilibria tend to occur in the relative absence of open-minded individuals and at very high levels of ingroup-outgroup dynamics; see Fig. 2.4 (c) and 2.5 (a) for examples. Psychologically this type of equilibria would represent cases where there exist two hostilely opposed groups who are unwilling to listen to each other. These equilibria tend to have relatively large basins of attraction as we will investigate below.

3. The third class of equilibria (*Type 3*) consists of *multiple cluster solutions* in which there are several (more than two) different groups of individuals at equilibrium each with a distinct belief on the particular issue being considered. Mathematically this class of equilibria can be characterized as the case when x_i^* takes on at least two distinct values at equilibrium where at least one of these values is not ± 1. These equilibria tend to occur when a_s is small. Physically this corresponds to when open-minded individuals are relatively un-open-minded (as the support of the interaction function is small). For examples of such equilibria see Fig. 2.4 (b).

Before delving into the analysis we must introduce several definitions. The first concerns a graph-theoretic analogue to our model originally made in reference to the models discussed in [38].

Two individuals are said to be *connected* if there exist a path linking them, where path in this instance, is defined as a chain of non-zero interaction coefficients. For example, if a_{12}, a_{24} and a_{47} were all nonzero, one would say that individuals 1, 2, 4 and 7 were all connected. If this relationship holds for all individuals in a population, the population is said to be *connected*. We will denote the path between the ith and jth individual with the strongest minimum link as Γ_{ij}, where 'strongest minimum' refers to the magnitude of the minimum a_{ij} in a path. If no such path exists between two individuals then they are not connected. Using this definition we can more formally state the definition of connected as:

Definition 1. *(Connected) A group of individuals governed by an alignment dynamics such as in (2.1) is said to be connected if there exists a z(t) such that for all paths* $\Gamma_{ij} : \min_{\text{over } a_{pk} \in \Gamma_{ij}}(a_{pk}) \geq z(t) > 0$ *for all i, j.*Further it is said to be uniformly connected if $z(t) > z > 0$, where z is a constant [38].

Continuing to consider graph-theoretic analogues to our model, it is important to note that we can associate an adjacency matrix with a population of individuals governed by our model. Specifically we will let this adjacency matrix be

$$D = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & a_{22} & \dots & \dots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.9}$$

where the entry a_{ij} in D is the interaction coefficient between the ith individual and the jth individual. If the ith and jth individual do not interact (or if the interaction between the ith and jth individual is ignored by the ith individual due to closed-mindedness) then the ijth entry in the matrix D is zero.

The next definition we will consider will be irreducibility where stated more formally:

Definition 2. *(Irreducible) A matrix D is said to be irreducible if its associated directed graph is strongly connected.*

It should be noted here that due to the symmetry of our interaction function ϕ, when a population of individuals is connected as is defined in Definition 2.3, they're also strongly connected when f_i is defined by (2.4). This implies that when f_i is defined by (2.4) and a population is also connected the associated adjacency matrix shown in (2.9) will be irreducible. This is not the case when f_i is defined by (2.7) as all rows associated with closed-minded individuals will be zero.

Theorem 1. *If a matrix A is weakly diagonally dominant, with strict dominance holding for at least one row then it is nonsingular if it is irreducible [53, Theorem 2].*

Definition 3. *(Problem 1) We will define problem 1 to be (2.3), where f_i is defined by (2.4), with a_{ij} defined as in (2.2) and $\hat{a}_i$ as in (2.5). Here ϕ and $\hat{\phi}$ are step functions of the general form depicted in Fig. 2.3 with arbitrary supports a_s and $\hat{a}_s$ respectively.*

Definition 4. *(Class 1) Let $\{x_i^*\}$ be an equilibrium solution to Problem 1. Then $\{x_i^*\}$ is said to be Class 1 if $\phi(x_j^* - x_i^*)$, $\hat{\phi}(x_i^*)$, $\frac{\partial a_{ij}^*}{\partial x_i}$, $\frac{\partial \hat{a}_i^*}{\partial x_i}$ and $\frac{\partial \hat{a}_{ij}^*}{\partial x_i}$ are well-defined for all i and j and $x_i \neq X_c$ for $i \in \hat{C}$. Here a_{ij}^* is defined to be the interaction coefficient evaluated at the equilibrium, with $\hat{a}_{ij}^*$ and $\hat{a}_i^*$ defined similarly.*

Having established the above definitions we can now propose and prove our first theorem. The general idea will be that when open-minded individuals interact with closed-minded individuals this causes neutrally stable equilibria to become attracting. This is proved using an argument which considers the Jacobian associated with Problem 1 as defined in Definition 3 and argues using the Gershgorin circle theorem [24, pg 320] and Theorem 1 that since the Jacobian is irreducible (or rather one of its principal minors is irreducible) and weakly diagonally dominant with negative diagonal elements and strict diagonal dominance holding in at least one row, all of it's eigen values must have negative real parts. Stated more formally we have the following:

Theorem 2. *Assume that the equilibrium solution to Problem 1 is Class 1 as is defined in Definitions 3 and 4. Then said equilibrium solution of (2.3) is linearly stable if all non-closed-minded individuals are connected at equilibrium as defined above, at least one open-minded individual is connected to a closed-minded individual at equilibrium and* $\alpha_3 > 0$*.*

Proof. Before we begin we will first clarify some of our terminology. In the above stated theorem (and in this thesis in general) closed-minded individuals refers to individuals who exist in $\hat{C}$ and have opinion values $|x_i| > X_c$. All individuals not satisfying this criteria are referred to as non-closed-minded. This group of individuals includes open-minded-individuals (who lack the predisposition to closed-mindedness) and moderate opinioned individuals who do possess the tendency to be closed-minded.

Now that we have clarified our definitions we will begin by computing the associated Jacobian for (2.3) when f_i is defined based on (2.4).

It is worth noting explicitly that:

$$\frac{\partial a_{ij}}{\partial x_k} = \frac{\frac{\partial \phi}{\partial x_k}\phi_i - \phi\frac{\partial \phi_i}{\partial x_k}}{\phi_i^2} \tag{2.10}$$

Considering the above, it follows from our assumption that x_i^* is Class 1 for all i, that (2.10) must be well-defined at equilibrium (this is a direct requirement for an equilibrium to be Class 1 so it follows trivially from the assumptions). A less trivial implication of our equilibrium being Class 1 is that $\frac{\partial a_{ij}}{\partial x_k}$ will in fact be zero at equilibrium for all k. This follows by noting that ϕ is assumed to be a step function and equilibrium is assumed to not occur at its jump discontinuity. Therefore, $\frac{\partial \phi(x_j - x_i)}{\partial x_i}$ will be zero since $\phi(x_j - x_i)$ will be constant at equilibrium. A nearly identical argument follows when we consider $\frac{\partial \hat{a}_{ij}}{\partial x_k}$ and $\frac{\partial \hat{a}_{ij}}{\partial x_k}$. Hence, we see that at equilibrium all interaction coefficient partial derivative terms in our Jacobian are zero. Hence, the Jacobian simplifies to:

$$J = \begin{pmatrix} Diag_1 & Non\ diag_{12} & Non\ diag_{13} & ... \\ Non\ diag_{21} & Diag_2 & Non\ diag_{23} & ... \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where,

$$Diag_i = \begin{cases} -\alpha_2(N-m-1)\hat{a}_i^* - \alpha_3 \sum_{j \in O} \hat{a}_{ij}^* & \text{if } |x_i^*| > X_c \text{ and } i \in \hat{C} \\ -\alpha_1 \sum_j a_{ij}^* & \text{otherwise} \end{cases}$$

and

$$Non\ diag_{ij} = \begin{cases} 0 & \text{if } |x_i^*| > |X_c|,\ i \in \hat{C} \text{ and } j \in \hat{C} \\ \alpha_3 a_{\hat{i}j}^* & \text{if } |x_i^*| > |X_c|,\ i \in \hat{C} \text{ and } j \notin \hat{C} \\ \alpha_1 a_{ij}^* & \text{otherwise} \end{cases} \tag{2.11}$$

where as mentioned above a_{ij}^*, $\hat{a}_i^*$ and $\hat{a}_{ij}^*$ represent the interaction coefficients between the ith and jth individual evaluated at the equilibrium state.

Next without loss of generality we will arbitrarily define the isolated closed-minded individuals in the system as individuals 1 through f, here isolated implies that their associated row and column in J is zero except for the diagonal entry and f is a constant describing how many closed-minded individuals meet this criterion. By using this numbering of individuals and by additionally noting that $\hat{a}_{ij}$ and $\hat{a}_i$ are normalized as to sum to 1 (see (2.5) and (2.6) for specifics) , the above Jacobian can be written as

$$J = \begin{pmatrix} -\alpha_2 & 0 & ... & ... \\ 0 & -\alpha_2 & 0 & ... \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & ... & A \end{pmatrix}, \tag{2.12}$$

where A is defined to be the nonzero portion of the Jacobian corresponding to derivatives with respect to the x_i's which are associated with non-closed-minded individuals, and closed-minded individuals with whom open-minded individuals interact (note: if all closed-minded individuals interact with at least one open-minded individual, $A = J$). Further, since J has negative diagonal entries and is weakly diagonally dominant, it's eigenvalues $e_i \leq 0$ for all i [24, pg 320]. The weak diagonal dominance follows by noting that if the ith individual is open-minded

$$-\mathit{Diag}_i = \sum_j \mathit{Non\ diag}_{ij} \tag{2.13}$$

(see (2.11) for details) and if the ith individual behaves closed-mindedly

$$-\mathit{Diag}_i = \sum_j \mathit{Non\ diag}_{ij} + \alpha_2(N - m - 1)\hat{a}_i^*. \tag{2.14}$$

Due to this diagonal dominance, (2.11) will have non-negative eigenvalues if and only if its determinant is zero. This follows by applying the Gershgorin circle theorem [24, pg 320] and noting that the diagonal entries of J are negative. Next we can see by performing a cofactor expansion on (2.12) that $\det(J) = (-\alpha_2)^f \det(A)$. Therefore, $\det(J) = 0$ if and only if $\det(A) = 0$.

At this point for clarity we will note explicitly what we will establish in the below argument. Specifically, we wish to apply Theorem 1 to A to argue that since A is irreducible and weakly diagonally dominant, with strict diagonal dominance holding for at least one row it necessarily must be non-singular. This with the result stated above will give us that all eigen values of J have negative real parts.

For simplicity we will address the diagonal dominance of A first.

Diagonal dominance of A

Considering A more closely we note that (2.13) and (2.14) still apply to the rows of A. Hence, A will be weakly diagonally dominant with at least one strictly diagonally dominant row (this follows from the assumption that at least one open-minded individual interacts with at least one closed-minded individual). Therefore, the weak diagonal dominance of A with strict diagonal dominance holding for at least one row follows trivially from (2.13) and (2.14). Hence, to establish stability we merely must show that A is irreducible.

Irreducibility of A

At this point we will without loss of generality label the closed-minded individuals in A as the first $q-f$ individuals, where q in this instance represents the number of closed-minded individuals ($q \leq N-m$). It should be noted here that $q > f$ by assumption. This is implicit in the assumption that at least one open-minded individual is connected to a closed-minded individual.

To establish that A is irreducible we will apply the assumption that all open-minded individuals are connected at equilibrium where connected is defined as in Definition 2.3. Specifically we will note that since all open-minded individuals are connected the adjacency matrix defined in 2.9 must be irreducible, (see definition 2 and related discussion for clarification).

Contrasting this adjacency matrix with the matrix A we see that:

$$A = \begin{pmatrix} -\alpha_2 & 0 & \vdots & \vdots & \vdots & ... \\ 0 & \ddots & 0 & ... & \underset{\substack{[f+1 \le i \le q] \\ q+1 \le j \le N}}{\alpha_3 \hat{a}^*_{ij}} & ... \\ ... & 0 & -\alpha_2 & & ... & ... \\ ... & \underset{\substack{[q+1 \le i \le N] \\ f+1 \le j < i}}{\alpha_1 a^*_{ij}} & ... & -\alpha_1 & \underset{\substack{[(q+1 \le i \le N] \\ i < j \le N}}{\alpha_1 a^*_{ij}} & ... \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & ... & ... & ... & & -\alpha_1 \end{pmatrix}, \tag{2.15}$$

where rows $q+1$ to N contain identical entries to rows $q+1$ to N of the adjacency matrix defined in (2.9) aside from the diagonal entries and the multiplicative constant. Further we know by assumption that each individual in rows $f+1$ to q is connected to at least one individual in rows $q+1$ to N. Due to the symmetry of our assumed interaction function ϕ, we see that A must be strongly connected. Hence, A is irreducible.

Hence, by Theorem 1 A is non-singular. This paired with the aforementioned weak diagonal dominance implies that all eigenvalues are negative. Therefore when f_i is defined by (2.4) our assumed equilibria are asymptotically stable.

□

At this point we will now extend Theorem 2 to the case when f_i is defined by (2.7). Before we do this though we must define the concepts of block diagonal dominance and block irreducibility. Specifically let A be a $w \times w$ block diagonal matrix defined as follows:

$$A = \begin{pmatrix} D_{11} & D_{12} & D_{13} & ... \\ D_{21} & D_{22} & ... & ... \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.16}$$

where, D_{ii} is a $n_i \times n_i$ square matrix and D_{ij} is a $s_{ij} \times m_{ij}$ rectangular matrix (where $s = m$ if $i = j$), where w, n_i, s_{ij} and m_{ij} are all positive integers.

Definition 5. *(Block diagonal dominance) Let A be a block matrix of the form described in (2.16). Then A is said to be block diagonally dominant if:*

$$(\|D_{ii}^{-1}\|)^{-1} \geq \sum_{j \neq i} \|D_{ij}\|$$

for all i. It is strictly block diagonally dominant if the above inequality is strict. Further we will note that $||s||$ represents an arbitrary matrix norm associated with the subspace to which the matrix s belongs. For our applications we will exclusively use the infinity norm. See [17] for more details.

It is useful to note that in the event that D_{ii} is a single entry, the above definition reduces the normal definition of diagonal dominance under the infinity norm.

Definition 6. *(Block irreducible) Let B be a block matrix of the form described in (2.16). Then B is said to be block irreducible if*

$$B = \begin{pmatrix} \|D_{11}\| & \|D_{12}\| & \|D_{13}\| & ... \\ \|D_{21}\| & \|D_{22}\| & ... & ... \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.17}$$

is irreducible where, $\|D_{ij}\|$ represents an arbitrary matrix norm associated with the subspace containing the matrix D_{ij}. See [17] for more details.

It should be noted that in the case of any D_{ij} being a matrix consisting of a single entry the above is still well defined.

Theorem 3. *If a matrix A is weakly block diagonally dominant, with strict block diagonal dominance holding for at least one row then it is nonsingular if it is block irreducible [17, Theorem 1].*

Definition 7. *(Problem 2) We will define problem 2 to be (2.3), where f_i is defined by (2.7), with a_{ij} defined as in (2.2) and $\hat{a}_i$ as in (2.8). Here ϕ and $\hat{\phi}$ are step functions of the general form depicted in Fig. 2.3 with arbitrary supports a_s and $\hat{a}_s$ respectively.*

Definition 8. *(Class 2) Let $\{x_i^*\}$ be an equilibrium solution to Problem 2. Then $\{x_i^*\}$ is said to be Class 2 if $\phi(x_j^* - x_i^*)$, $\hat{\phi}(x_i^*)$, $\frac{\partial a_{ij}^*}{\partial x_i}$ and $\frac{\partial \hat{a}_i^*}{\partial x_i}$ are well-defined for all i and j and $x_i^* \neq X_c$ for $i \in \hat{C}$. Here a_{ij}^* is defined to be the interaction coefficient evaluated at the equilibrium, with $\hat{a}_i^*$ defined similarly.*

It should be noted that the definition of Class 2 is identical to Class 1 except that Class 2 represents equilibrium solutions where f_i is defined by (2.7) instead of (2.4).

Now that we have laid the framework we can now extend Theorem 2 to the case when f_i is defined by (2.7). The argument will follow much the same as before but in this case the analogous principal minor to (2.15) will no longer be irreducible under our assumptions. Hence instead we must demonstrate that it is block irreducible and invoke an analogous result to Theorem 1 for block irreducibility and block diagonal dominance (specifically we will use Theorem 3). Put more formally we have:

Theorem 4. *Assume that the equilibrium solution to Problem 2 is Class 2 as is defined in Definitions 7 and 8. Then said equilibrium solution of (2.3) is linearly stable if all non-closed-minded individuals are connected at equilibrium as defined above, at least one non-closed-minded individual is connected to a closed-minded individual at equilibrium and* $\alpha_2 > \alpha_1(1-C)$ *where* $C = \sum_{j\in\hat{C},|x_j|>X_c} ||a_{ij}||$ *where* i *is the index associated with the non-closed-minded individual with the largest associated* C.

Proof. Our proof for Theorem 4 will begin much the same as Theorem 2 in that again we will consider the Jacobian associated with the related problem. Specifically we have:

$$J = \begin{pmatrix} Diag_1 & Non\ diag_{12} & Non\ diag_{13} & ... \\ Non\ diag_{21} & Diag_2 & Non\ diag_{23} & ... \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where,

$$Diag_i = \begin{cases} -\alpha_2(N-1)\hat{a}_i^* & \text{if } |x_i^*| > X_c \text{ and } i \in \hat{C} \\ -\alpha_1 \sum_j a_{ij}^* & \text{otherwise} \end{cases}$$

and

$$Non\ diag_{ij} = \begin{cases} 0 & \text{if } |x_i^*| > |X_c| \text{ and the } i \in \hat{C} \text{ and } j \in \hat{C} \\ \alpha_1 a_{ij}^* & \text{otherwise} \end{cases} \tag{2.18}$$

Again as in the case when we considered the Jacobian associated with Problem 1 we see that all partial derivatives terms associated with the interaction coefficients are zero due to the assumed form of ϕ and $\hat{\phi}$.

Proceeding similarly to before we will without loss of generality arbitrarily define the isolated closed-minded individuals in the system as individuals 1 through f. As before, this will allow us to rewrite the above Jacobian as follows:

$$J = \begin{pmatrix} -\alpha_2 & 0 & ... & ... \\ 0 & -\alpha_2 & 0 & ... \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & ... & A \end{pmatrix}, \tag{2.19}$$

where again A is defined to be the nonzero portion of the Jacobian corresponding to derivatives with respect to x_i where the ith individual is either non-closed-minded or is a closed-minded individual with whom a non-closed-minded individual interact. Additionally, as before since J has negative diagonal entries and is weakly diagonally dominant, it's eigenvalues $e_i \leq 0$ for all i [24, pg 320]. The weak diagonal dominance follows by noting that if the ith individual is open-minded

$$\textit{-Diag}_i = \sum_j \textit{Non diag}_{ij} \tag{2.20}$$

(see (2.18) for details) and if the ith individual behaves closed-mindedly

$$\textit{-Diag}_i = \sum_j \textit{Non diag}_{ij} + \alpha_2(N-1)\hat{a}_i^*. \tag{2.21}$$

Again due to this weak diagonal dominance, (2.18) will have non-negative eigenvalues if and only if its determinant is zero. This follows by applying the Gershgorin circle theorem [24, pg 320] and noting that the diagonal entries of J are negative. Next as in Theorem 2 we can see by performing a cofactor expansion on (2.19) that $\det(J) = (-\alpha_2)^f \det(A)$. Therefore, $\det(J) = 0$ if and only if $\det(A) = 0$.

At this point this proof diverges significantly from the proof of Theorem 2 and again for clarity we will note explicitly what we will establish below. Specifically we wish to apply Theorem 3 to argue that since A is block irreducible and block diagonally dominant it must be non-singular. Pairing this with the previously discussed weak diagonal dominance guarantees that all eigen values of J have negative real part. For simplicity we will address the issue of block irreducibility first.

Block irreducibility of A

We will begin by considering the general form of A,

$$
A = \begin{pmatrix}
-\alpha_2 & 0 & \vdots & \vdots & \vdots & \dots \\
0 & \ddots & 0 & \dots & 0 & \dots \\
\dots & 0 & -\alpha_2 & & \dots & \dots \\
\dots & \underset{\substack{[q<i\leq N] \\ f+1\leq j<i}}{\alpha_1 a_{ij}} & \dots & -\alpha_1 & \underset{\substack{[q<i\leq N] \\ i<j\leq N}}{\alpha_1 a_{ij}} & \dots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
 & \dots & \dots & \dots & & -\alpha_1 \\
 & & & & &
\end{pmatrix}. \tag{2.22}
$$

Considering (2.22) we see that it differs from(2.15) in that all non-diagonal terms in the rows associated with closed-minded individuals are zero. As a result the above is not irreducible as connectivity no longer implies strong connectivity. It is however block irreducible as we can see by carefully observing the block matrix representation of (2.22) shown below in (2.23).

It is useful in interpreting the block representation of A to note that it was specifically constructed as to be block irreducible with a strictly block diagonally dominant top row. Specifically,

$$
A = \begin{pmatrix}
A_C & A_{OC_{12}} & A_{OC_{13}} & \dots \\
A_{NC_{21}} & o_{22} & o_{23} & \dots \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix}, \tag{2.23}
$$

where, the $N-q+1 \times N-q+1$ entries in (2.23) are block matrices. It is important to note that all of the block matrices in (2.23) with the exception of A_C, $A_{OC_{1j}}$ and $A_{NC_{i1}}$ simply consist of a single entry. This division is noted in the nomenclature of the block matrices by denoting single entry matrices with a lower case letter. We will now consider each of the capitalized block matrices in greater detail.

Beginning with the most important, we will first consider A_C. The matrix A_C is a $(q-f+1) \times (q-f+1)$ square matrix that contains the diagonal entries in A corresponding to all close-minded individuals contained in A and the diagonal entry corresponding to the non-closed-minded individual who is connected with the most closed-minded individuals (by assumption this non-closed-minded individual must be connected with at least one close-minded individual). Specifically,

$$A_C = \begin{pmatrix} -\alpha_2 & 0 & 0 & ... \\ 0 & -\alpha_2 & 0 & ... \\ \ddots & \ddots & \ddots & \ddots \\ \alpha_1 a_{q+1,f+1} & \alpha_1 a_{q+1,f+2} & ... & -\alpha_1 \end{pmatrix}. \tag{2.24}$$

Next consider the block matrices defined by $A_{OC_{1j}}$. We see that $A_{OC_{1j}}$ is a column matrix of the following form:

$$A_{OC_{1j}} = (0, ..., \alpha_1 a_{q+1,j})^T, \tag{2.25}$$

The formulation of the above follows by noting that the non-diagonal elements of the top $q - f$ rows of A are all zero. Hence, (2.25) contains all of the zero elements in the jth column ($j > q + 1$) and the interaction coefficient between the $(q - f + 1)$th individual (who is guaranteed to be non-closed-minded by construction and hence connected to at least one other non-closed-minded individual based on the connectivity assumption). This guarantees that at least one of the $A_{OC_{1j}}$ terms contains a nonzero entry.

Next considering the block matrix $A_{NC_{i1}}$ in (2.23) we see that it is a row matrix of the following form:

$$A_{NC_{i1}} = (\alpha_1 a_{i,f+1}, ..., \alpha_1 a_{i,q+1}). \tag{2.26}$$

The row matrix described in (2.26) simply contains the interaction coefficients between the ith individual and the individuals whose column indices $f < j \leq q+1$, where all constants are as previous defined. This formulation is chosen largely to maintain the square dimensions of (2.23).

Now that we have defined all of the entries in (2.24) we can now consider the the block irreducibility of A. Specifically by considering:

$$A_{||*||} = \begin{pmatrix} \|A_C\| & \|A_{OC_{12}}\| & \|A_{OC_{13}}\| & ... \\ \|A_{NC_{21}}\| & \|o_{22}\| & \|o_{23}\| & ... \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.27}$$

where $A_{||*||}$ represents the matrix whose ijth entry consists of the infinity norm applied the ijth submatrix of (2.23). First as mentioned previously since all non-closed-minded individuals are connected at equilibrium by assumption the $q+1$th individual (who by formulation is the non-closed-minded individual whose interaction coefficients are contained in the $A_{OC_{1j}}$ blocks) must interact with at least one open-minded individual other than themselves. Hence, at least one non-diagonal entry in the first row of (2.27) must be nonzero. This shows that the associated directed graph to (2.27) must contain an arc directed between the node denoting the first row and the node denoted at least one other row. Further, since our interaction function ϕ is symmetric there must exist an arc from this other row to the first row. All other rows are connected to each other by the rationale given when f_i is defined based on (2.4). Therefore, A is block irreducible.

Block Diagonal dominance of A

Next we will consider the relative block diagonal dominance of A. First we will note that all rows aside from the first are described by (2.20) and are weakly block diagonally dominant. This follows by noting that under the matrix supremum norm $||A_{NC_{i1}}||$ is simply the sum of the interaction coefficients associated with the ith individual interacting with the jth individual where $f < j \leq q+1$. Therefore, since $||A_{NC_{i1}}||_\infty$ is equal to the sum of its entries the row sum associated with the ith row remains unchanged when $i \neq 1$.

We now consider the first row in (2.23) and under what conditions it is strictly block diagonally dominant. The off diagonal row sum is relatively straight forward to calculate as is seen below:

$$\sum_{k \neq j} ||A_{OC_{1k}}|| = \alpha_1(1 - C), \tag{2.28}$$

where, $C = \sum_{j \in \hat{C}, |x_j| > X_c} a_{q+1,j}$. Next considering $(||A_C^{-1}||)^{-1}$ we note that we must first calculate the inverse of A_C. Specifically

$$A_C^{-1} = \begin{pmatrix} -\alpha_2^{-1} & 0 & 0 & \dots \\ 0 & -\alpha_2^{-1} & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots \\ \frac{-a_{q+1,f+1}}{\alpha_2} & \frac{-a_{q+1,f+2}}{\alpha_2} & \dots & -\alpha_1^{-1} \end{pmatrix}. \tag{2.29}$$

, where by considering the supremum norm of A_C^{-1} we have that:

$$||A_C^{-1}|| = \max(\frac{1}{\alpha_2}, \frac{1}{\alpha_1} + \frac{C}{\alpha_2}),$$

therefore,

$$||A_C^{-1}||^{-1} = \min(\alpha_2, \alpha_1(1 + \frac{\alpha_1 C}{\alpha_2})^{-1}), \tag{2.30}$$

where C is as previously defined. Next we consider the requirement for α_2 to be minimal in (2.30).

Specifically this means that:

$$\alpha_2 \leq \alpha_1 (1 + \frac{\alpha_1 C}{\alpha_2})^{-1}$$

Rearranging the above gives:

$$\alpha_2 \leq \alpha_1 (1 - C) \tag{2.31}$$

Hence, since (2.31) is the opposite of the requirement for strict block diagonal dominance we see that strict block diagonal dominance cannot result if α_2 is minimal. Next, we will check the requirement for diagonal dominance when α_2 is larger than $\alpha_1(1+\frac{\alpha_1 C}{\alpha_2})^{-1}$. This gives the requirement that:

$$\alpha_1(1 - C) < \frac{\alpha_1}{(1 + \frac{\alpha_1}{\alpha_2} C)}. \tag{2.32}$$

Rearranging and canceling terms in (2.32) we arrive at the following inequality:

$$\frac{\alpha_1}{\alpha_2}(1 - C) - 1 < 0 \tag{2.33}$$

This is satisfied given our assumption that $\alpha_1(1 - C) < \alpha_2$.

Hence, we see that A has one row which is strictly block diagonally dominant. Therefore, by Theorem 3 A is nonsingular. Hence, all eigen values are negative and $\{x_i^*\}$ is linearly stable.

□

Interpreting the condition for Theorem 2 and 4 qualitatively we see that it encompasses consensus solutions at one extreme, polarization solutions, and cluster solutions in which individuals interact with individuals outside of their cluster at equilibrium. Additionally it is worth noting that the assumption for an equilibrium being Class 1 is not as restrictive as it first appears, as while theoretically there could be N distinct clusters at equilibrium, in general our simulations indicate that there are fewer than six. Hence, it is unlikely that anyone of these clusters would precisely reside at $\pm X_c$.

Further the results of Theorem 2 and 4 are interesting as they imply that closed-minded interaction potentially has the ability to increase stability to a certain extent. Again we will note that in the absence of closed-minded individuals all equilibria are neutrally stable. This is interesting in terms of its implications to the social sciences, as it implies that the extremist consensus equilibria solutions are linearly stable whereas moderate consensus equilibria are not necessarily attracting. This follows from the fact that non-extremist consensus solutions are not Class 1 or Class 2 as they contain no closed-minded individuals. Hence, the above result provides a potential stability-based explanation for why groups such as those described by Sunstein in [50] tend to reach extremist consensuses.

We will now prove some results related to the basin of attraction of one particular case; specifically the case of a polarization solution at which all individual are closed-minded and reach equilibrium at one of the extremes where $\hat{a}_i$ is a normalized step function.

Theorem 5. *Assume that $x_i^* = b_i$ for all i, where $b_i = \pm 1$. Then the equilibrium solution of (2.3) where f_i is defined by (2.7) or (2.4) and $\hat{a}_i$ is $1/N$ is attracting when after some finite time $|x_i| > X_c$ for all i and all individuals are closed-minded.*

Proof. We will begin by defining the equilibrium point associated with x_i as b_i, where b_i is equal to ± 1.

Since $x_i^* = b_i$ for all i, the function

$$E = \sum_i (b_i - x_i)^2 \tag{2.34}$$

is zero at the equilibrium point and positive definite for all other $x_i \in [-1, 1]$. Differentiating (2.34) and substituting in the definition for $\frac{dx_i}{dt}$ given by (2.4) we have:

$$\dot{E} = -2 \sum_i (b_i - x_i) \frac{dx_i}{dt}$$

where

$$\frac{dx_i}{dt} = \begin{cases} \alpha_2 N \hat{a}_i (1 - x_i) & \text{if } x_i > X_c \text{ and } i \in \hat{C}, \\ \alpha_2 N \hat{a}_i (-1 - x_i) & \text{if } x_i < -X_c \text{ and } i \in \hat{C}. \end{cases}$$

Considering the above, we see that when $x_i > X_c$ the top case comes into effect and when $x_i < -X_c$ the bottom case comes into effect. Hence all individuals approaching the $b_i = 1$ equilibrium will be governed by the top equation and all individuals approaching the $b_i = -1$ equilibrium will be governed by the bottom equation. Therefore

$$\frac{dx_i}{dt} = \alpha_2 (b_i - x_i).$$

Hence,

$$\dot{E} = -2\alpha_2 \sum_{i=1}^{N} (b_i - x_i)(\frac{N}{N} b_i - \frac{N x_i}{N}).$$

The above follows by noting that $\hat{a}_i$ is $1/N$. Simplifying the above gives

$$\dot{E} = -2\alpha_2 \sum_{i=1}^{N} (b_i - x_i)^2 \leq 0. \tag{2.35}$$

Observing (2.35) we see that it is negative definite and zero at equilibrium. Hence by Lyapunov's theorem we conclude that the equilibrium $x_i^* = b_i$ is asymptotically stable given our assumptions.

□

Observing Theorem 5 we see that when all x_i are in $(X_c, 1]$ or $[-1, -X_c)$ and individuals are closed-minded that the resulting equilibrium state is attracting. We further conjecture that the result in Theorem 5 holds even when open-minded individuals are present in said equilibrium cluster.

This assertion is supported by Theorem 2 which shows that a polarization equilibrium with both open-minded and closed-minded individuals is in fact linearly stable.

Next we will consider the circumstances under which our model is mean preserving.

Proposition 1. *Assume a system is governed by an alignment dynamic as in (2.4) or (2.7). Then the mean opinion of the population is preserved in time if it is governed by (2.4) and all closed-minded individuals are symmetrically distributed at t=0 and* $\frac{\alpha_3}{\hat{\phi}_i} = \frac{\alpha_1}{\phi_j}$ *for all i, j for which* $a_{ij} \neq 0$*; or if it is governed by (2.7) and all individuals are symmetrically distributed at t=0.*

Proof. The result stated above follows by noting that symmetric pair-wise interactions preserve the mean. Considering the case when all individuals are open-minded we have:

$$\frac{d\text{ (mean)}}{dt} = \frac{1}{N}\sum_i \frac{dx_i}{dt} = \frac{\alpha_1}{N}\sum_i\sum_j a_{ij}(x_i - x_j). \tag{2.36}$$

Next interchanging labels and switching the order of summation we have

$$\frac{d\text{ (mean)}}{dt} = \frac{\alpha_1}{N}\sum_j\sum_i a_{ji}(x_i - x_j) = \frac{\alpha_1}{N}\sum_i\sum_j a_{ji}(x_i - x_j). \tag{2.37}$$

Lastly noting that by definition $a_{ij} = a_{ji}$ we see that the above can be rewritten as

$$\frac{d\text{ (mean)}}{dt} = \frac{\alpha_1}{N}\sum_i\sum_j a_{ij}(x_j - x_i) = -\frac{d\text{ (mean)}}{dt}. \tag{2.38}$$

Since $-\frac{d\text{ (mean)}}{dt} = \frac{d\text{ (mean)}}{dt}$ we see that $\frac{d\text{ (mean)}}{dt}$ must equal zero.

In considering a similar equation to the above in the case of the interactions of closed-minded individuals with other closed-minded individuals and interactions between closed-minded individuals and open-minded individuals we see that the interactions themselves are not symmetric but can be grouped into symmetric pairs by the symmetry assumption in the statement of this proposition.

Specifically for every closed-minded-closed-minded interaction which moves towards an opinion value of 1 at a particular rate there must exist based on the assumed symmetry a similar individual moving towards -1 at an identical rate. The same rationale follows when considering closed-minded open-minded interactions in the case when the alignment dynamic is governed by (2.7). In considering open-minded interaction with closed-minded individuals in the case when the alignment dynamic is governed by (2.4), the assumption that $\frac{\alpha_3}{\hat{\phi}_i} = \frac{\alpha_1}{\phi_j}$ allows for an identical argument to be made as in (2.36). Specifically the aforementioned assumption gives us that $\alpha_3\hat{a}_{ij} = \alpha_1 a_{ji}$. Therefore,

$$\alpha_3\hat{a}_{ij}(x_j - x_i) + \alpha_1 a_{ij}(x_i - x_j) = 0 \qquad \text{for all i and j}$$

which leads to the necessary cancellations. □

2.4 Numerically examining the effect of open-mindedness social norms

At this point we will return to our original question of whether social-norms of open-mindedness can cause consensus in the presence of ingroup-outgroup dynamics. By observing Fig. 2.5 it can be seen for specific parameters that ingroup-outgroup dynamics in the absence of open-minded individuals precludes consensus. This trend in fact extends to all relevant parameters, as is supported by Fig. 2.7 and 2.8 (a), which are discussed below.

It is also worth noting that the equilibrium consensus in Fig. 2.5 is at the extreme opinion. This agrees with the observed psychological phenomenon in which, when groups of individuals reach a consensus, this consensus tend to be more extreme than the initial mean opinion of the group [50, 15]. The result in Fig. 2.5 implies that this trend could be potentially caused by a minority of closed-minded individuals dragging the group opinion towards the extreme. This is not unlike the theoretical explanation given for this phenomenon using a discrete model for opinions by Galam and Mosovici in [22]. This trend cannot be captured using the model (2.1)-(2.2).

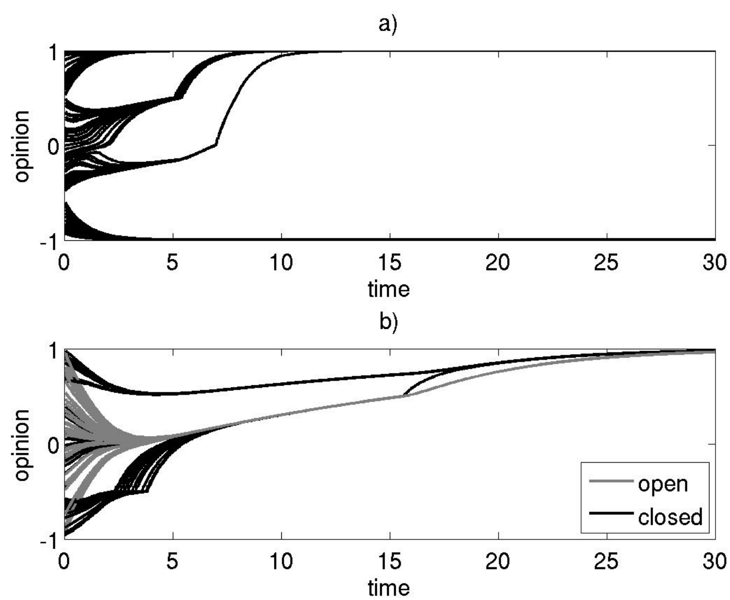

Figure 2.5: The effect of increasing the proportion of open-minded individuals when a social norm of open-mindedness is present. As the proportion of open-minded individuals is increased a consensus results, though it is often more extreme than the initial mean opinion. Here, $X_c = 0.5$, $a_s = 1$, $N = 80$. a) $m = 0$, b) $m = 40$.

At this point we will turn our attention to the relative difference in the occurrence of consensus solutions to (2.3) depending on whether f_i is defined by (2.4) or by (2.7). The former represents the case when there exists a social norm of open-mindedness in interactions between open and closed-minded individuals, whereas the latter represents the case when there is no such norm. We will investigate this variation numerically using a parameter sweep. First though, we must determine what is our parameter space of interest.

In order for our parameter sweep to collectively encompass the largest range of potential psychological traits we will vary X_c, a_s and m; our results are shown in the parameter space plots shown in Figures 2.6 to 2.8. As mentioned, before X_c dictates the critical threshold at which individuals with closed-minded potential, begin to behave in a closed-minded fashion; while a_s represents the support of the interaction function ϕ associated with a_{ij}, which, as was previously mentioned, is the characteristic function $\chi_{[0,a_s]}$. The value of a_s represents how willing non-closed-minded individuals are to consider the opinions of individuals whose opinion is different than their own. This can be thought of as a measure of how open-minded non-closed-minded individuals are. The final parameter varied is m, which represents the number of individuals in the population belong to the set O. This is the set of individuals who lack the propensity to be closed-minded. We will additionally note that in all of our simulations $\hat{a}_i = 1/N$. This represents the fact that for the reasons discussed in Chapter 1 it is reasonable to expect that closed-minded individuals would respond similarly to interactions with all individuals. Furthermore, a_{ij} as previously mentioned is defined as $\phi(|x_j - x_i|)/\phi_i$ where ϕ is defined as the step function depicted in Fig. 2.3 (a).

The collective range of parameters in Figures 2.6 to 2.8 was chosen so as to cover as many parameters as possible while highlighting the region in parameter space where consensus results in the absence of closed-minded individuals.

Fig. 2.6 is identical to Fig. 2.8 (a). It will serve as a reference plot for the specific parameters involved in our parameter sweep (in that the labels represent the actual parameters we used). We will continue to use these parameters in all future parameter sweep plots in this manuscript except in Chapter 4 where 100 sets of parameters is computationally prohibitive. We will also deviate from these parameters when considering the proportion of open-minded individuals required for consensus Figures 2.9, 3.4 and 3.8. This was done so as to focus exclusively on the range of parameters where consensus results. It is also worth noting that the label above the figure in Fig. 2.6 ('0 of 80') denotes the relationship between the value of m (0) and the value of N (80). This be will our labeling convention in all our parameter sweeps.

Considering the range of relevant parameters, we see in Figures 2.7 and 2.8 that when a social norm of open-mindedness exists, consensus can result for all expected parameters if the proportion of open-minded individuals is large enough. This is not the case in the absence of a social norm of open-mindedness (see Fig. 2.7), where consensus is not possible except in the case when $X_c = 1$, which corresponds to no individuals behaving closed-mindedly. This is the case examined in [38] where no ingroup-outgroup dynamics are present. This lack of consensus in the absence of an open-mindedness social norm is caused by the fact that even when the majority of the individuals are open-minded, the minority of closed-minded individuals still approach the extreme and are unswayed

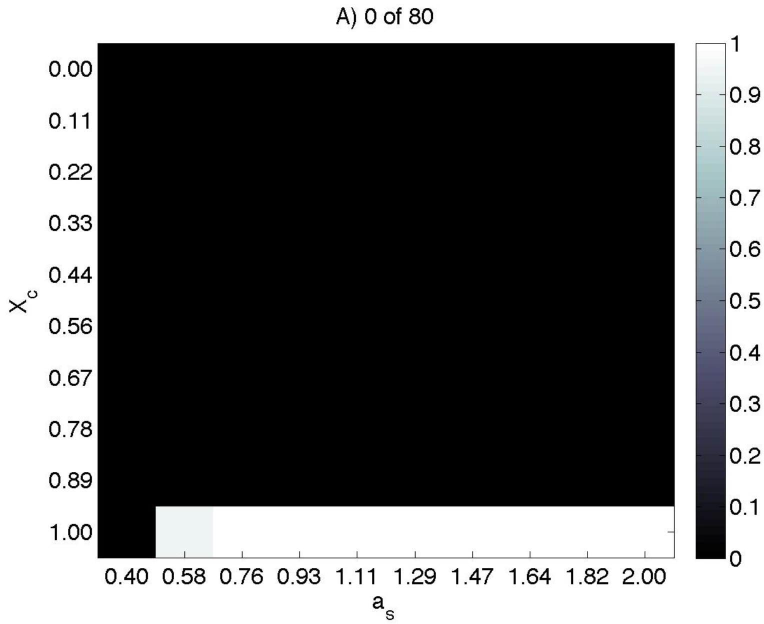

Figure 2.6: Sample parameter space plot depicting the average fraction of uniform random initial conditions resulting in consensus, where f_i is defined by (2.7). Here, $N = 80$ and $m = 0$. Averaged over 20 sets of uniform random initial conditions. The title denotes the number of open-minded individuals used in simulation. x and y axis notes the specific parameter values used in each parameter sweep unless specified otherwise

by the open-minded individuals, as is evident in Fig. 2.4 (c) and (d).

We will also note here that the occurrence of consensus for a given set of parameters can be thought of as a binomial probability where the probability of consensus corresponds to the probability of a success. Hence in estimating the error in the above plot we can use standard error estimates for estimating the error of a binomial probability. Specifically, this means that the above plots also implicitly depict the standard deviation.

Observing Fig. 2.8, we see that when there exists a social norm of open-mindedness, increasing the proportion of open-minded individuals increases the number of parameter values for which consensus occurs. This demonstrates that open-minded individuals have the potential to cause consensus under the assumptions of our model when they create a social norm of open-mindedness when interacting with closed-minded individuals. This is further supported by Fig. 2.9, which shows that when $a_s \geq 0.8$, consensus is possible if the proportion of open-minded individuals is large enough and they create an open-mindedness social norm when interacting with closed-minded individuals.

It is additionally worth noting that the proportion of open-minded individuals required for consensus is independent of a_s. This is equivalent to saying that the proportion of open-minded individuals is independent of their specific open-mindedness as long as their open-mindedness exceeds the critical

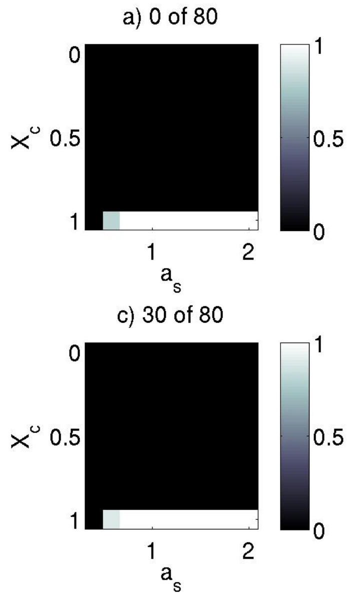

Figure 2.7: Average proportion of uniform random initial conditions which result in consensus when no social norm of open-mindedness is present that is, f_i defined by (2.7); averaged over 20 sets of uniform random initial conditions with, $N = 80$ and a) $m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$. . Increasing the proportion of open-minded individuals has no effect on the set of parameters for which consensus results.

threshold of $a_s > 1$ (Fig. 2.9). This is interesting since it explains why consensus can be reached by two groups so quickly under certain circumstances when previously it seem unreachable. This was the case for instance in the conflicts between Serbia and Montenegro, Georgia and Azkhazia, and Moldova and Transnistria [11]. In these disputes, European influence lead to agreements between hostilely aligned political parties which had exhibited closed-minded behavior in the past [11]. One explanation for this is that these conflicting parties caved to pressure from higher powers out of fear of repercussions [44]. Another explanation though, is that European political authorities who were not personally involved in these conflicts provided an open-minded perspective and encouraged consensus through the creation of open-mindedness norms. It is also worth noting here that these examples still fit with our conclusions even though many of these conflicting parties have since reached separatist agreements.

It should also be noted here that the parameter space plot in Fig. 2.9 is compressed relative to Figures 2.7 and 2.8. This is because it only focuses on parameters where consensus results.

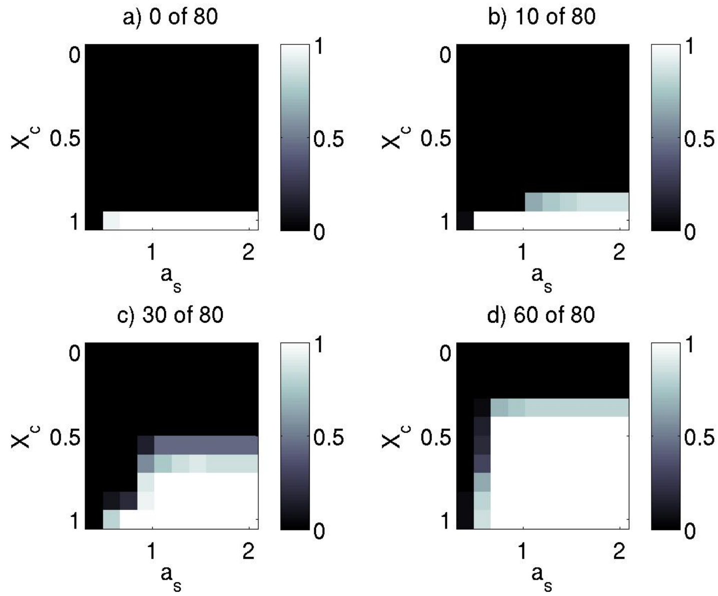

Figure 2.8: Average proportion of uniform random initial conditions which result in consensus in the presence of a social norm of open-mindedness. (i.e. f_i defined by (2.4)), averaged over 20 sets of uniform random initial conditions, with $N = 80$, a) $m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$. $\phi = \chi_{[0,a_s]}$. Increasing the proportion of open-minded individuals increases the number of parameters for which consensus results.

2.5 Summary

The most important result in the above chapter is that under the assumptions of our model, social norms of open-mindedness have the potential to cause consensus in the presence of ingroup-outgroup dynamics if the proportion of individuals in O is large enough. This is an important result since it provides a potential explanation for the empirical observation of groups containing closed-minded individuals reaching consensuses. It is also interesting to note that this consensus solution often tends to be more extreme in nature. This is in agreement with empirical results which show that when individuals discuss a particular issue, the consensus agreement tends to be more extreme than the mean pre-discussion opinion [50, 22]. It also provides support for the theory that group extremism results because closed-minded individuals skew group opinion towards the extreme.

Analytically we see that extremist consensus solutions may be more stable than non-extremist consensus solutions. This may provide an explanation why extremist consensuses occur so frequently [50].

It is also interesting to note that under the assumptions of our model, this proportion of open-minded individuals required for consensus appears to be independent of a_s as long as a_s exceeds a certain critical threshold. This is intriguing as it implies that the specific degree of open-mindedness

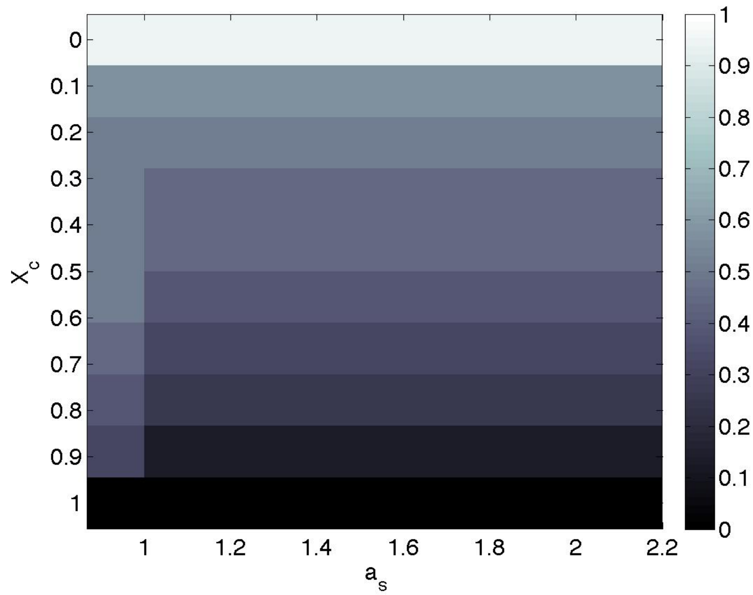

Figure 2.9: Average proportion of open-minded individuals required for consensus. This proportion appears to be independent of a_s once a_s exceeds a certain threshold. Here f_i is defined by (2.4), $N = 80$ and $\phi = \chi_{[0,a_s]}$. The proportion of individuals required to be open-minded for consensus is negatively correlated with X_c. The above plot is averaged over 20 sets of uniform random initial conditions.

amongst open-minded individuals does not matter as long as it exceeds a certain threshold. This is interesting as it potentially explains why under certain circumstances consensuses can be reached or dissolve so quickly [11].

Lastly it is worth noting that our linear stability analysis gives evidence in favor of closed-mindedness increasing stability.

Chapter 3

How open-mindedness norms cause consensus in the presence of media

The hostile media effect is a complex phenomenon in which highly partisan individuals perceive the media to be biased against them, irrespective of its actual bias [60, 10, 9, 13, 42]. This phenomenon has a profound impact in shaping perceptions and as a result in shaping individual actions during conflicts as diverse as: American elections, the Bosnian conflict, the US parcel workers strike of 1997 and even the Arab-Israeli conflict [60, 10, 9, 13]. Due to this impact on a diverse set of conflicts, it is critical to understand how conflicts in general can be resolved in the presence of the hostile media effect. This resolution process can be represented mathematically as the formation of consensus.

In this section we will demonstrate that like in our non-media influence simulations, open-minded individuals have the potential, under the circumstances assumed in our model, to facilitate consensus amongst closed-minded individuals through the creation of an open-mindedness social norms. This implies that to resolve conflicts in the presence of the hostile media effect, it is more effective to try and stimulate the formation of open-mindedness social norms, rather than endeavoring to directly overcome the false perception of the media.

3.1 Summary of previous research

As mentioned previously, the hostile media effect has received a lot of attention in the social sciences [60, 10, 9, 13], though to the best of our knowledge no theoretical works have yet investigated its cause, or ways in which conflicts can be resolved in its presence. This will be the principal purpose of this chapter.

Prior to delving into this modeling endeavor, we will first propose a simpler coupled individual opinion-media model and contrast our results with previous media-opinion modelling efforts such as those described in [35, 39, 55, 8, 1].

One of the first media-opinion models was proposed by Boudin et al. in [8]. Here they construct a consensus-seeking model and investigate the effect of multiple media sources of varying strengths. One conclusion is that an extremist media source does not benefit its cause. This is supported by our model in the case when a_s is small.

Additionally, modeling efforts by Mirtabatabaei et al. [35] have examined what conditions are sufficient for consensus in the presence of media. They demonstrate for their model that when the initial opinion distribution is symmetric and the individual-individual interaction has full support, consensus results.

3.2 Introduction of coupled media opinion model

In this section we see introduce our general for media-opinion dynamics as seen in (3.1). It is of similar form to the animal aggregation model with shepherd term proposed by Albi and Pareschi [2]. Here media influence on individuals is represented as a 'shepherd' term (F^P) where P is an integer denoting the total number of media sources μ_p, where as with x_i μ_p is a real number defined on the interval $[-1, 1]$ denoting the pth media source's bias. The function λ denotes the rate at which the media location changes in time. For simplicity we will assume in all of our simulations, apart from those in Fig. 3.10, that $\lambda = 0$. This implies that the media is fixed in time. This need not be the case, though, as it has been documented that media sources adjust their perspectives in response to pressure from the public [47], and also in response to competition for viewers with other media sources [23]. The first type of influence is particularly intriguing, since it has been theorized that the rise of social media will lead to greater public influence on the media [47]. This means that potentially media could have less of a polarizing effect on individuals' opinions, as is supported by Fig. 3.10.

A general model for opinion dynamics in the presence of media is:

$$\frac{dx_i}{dt} = f_i + F^P(x_i, \mu_p) \tag{3.1}$$

where

$$\frac{d\mu_p}{dt} = \lambda(t, \vec{\mu}, \vec{x}) \tag{3.2}$$

Considering (3.1) we will note that t represents time, $\vec{x}$ represents the vector containing all x_is values, and $\vec{\mu}$ represents the vector containing all μ_p values. Additionally in formulating (3.1) we will assume that opinion interactions are the same as those described previously in (2.4) or (2.7) depending on the particular simulation. Media-individual interactions are also assumed to differ from individual-individual interactions by only a scalar multiple. Hence, in all our simulations the media interaction coefficients (b_{ip} and $\hat{b}_i$ defined in (3.5)and (3.4)) are assumed to have identical forms to those of a_{ij} and $\hat{a}_i$, respectively, unless explicitly stated otherwise. This is supported by the agreement between empirical studies involving individual interaction with media and studies involving individual interaction with groups of other individuals [30, 3, 50, 15, 26, 3]. The definitions of $\hat{b}_i$ and b_{ip} do differ from $\hat{a}_i$ and a_{ij} in that their normalizations are with respect to the total number of media sources, as given in (3.4) and (3.5). Further, it is worth stating that when we assume that individuals respond to media similarly to how they respond to other individuals we are assuming that the closed-minded individuals in our simulations approach the extreme upon exposure to both like-minded and cross-cutting media. Taking this into account we see that the media interaction term becomes

$$F^P = \begin{cases} \sum_p \alpha_{4,p} \hat{b}_i (1 - x_i) & \text{if } x_i > X_c \text{ and } i \in \hat{C} \\ \sum_p \alpha_{4,p} \hat{b}_i (-1 - x_i) & \text{if } x_i < -X_c \text{ and } i \in \hat{C} \\ \sum_p \alpha_{5,p} b_{ip} (\mu_p - x_i) & \text{otherwise} \end{cases}, \tag{3.3}$$

where the constants $\alpha_{4,p}$ and $\alpha_{5,p}$ are defined to be scalars denoting the strength of the pth media source. It is worth noting explicitly that $\alpha_{4,p}$ and $\alpha_{5,p}$ are in general both dependent on p though since in general we will only use a single media source we will use α_4 and α_5 interchangeably with $\alpha_{4,p}$ and $\alpha_{5,p}$. This p dependence is to allow for different media sources to have different degrees of influence independent of their respective partisanships.

The above mentioned constants b_{ip} and $\hat{b}_i$ capture the partisan dependence of the media influence which represents how responsive an individual is to a media source of a particular bias. Specifically

$$b_{ip} = \begin{cases} 0 & \text{if } \sum_p \phi_b(\mu_p - x_i) = 0, \\ \frac{\phi_b(\mu_p - x_i)}{\sum_p \phi_b(\mu_p - x_i)} & \text{otherwise,} \end{cases} \tag{3.4}$$

and

$$\hat{b}_i = \begin{cases} 0 & \text{if } \hat{\phi}_b(\pm 1 - x_i) = 0, \\ \frac{\hat{\phi}_b(\pm 1 - x_i)}{P \hat{\phi}_b(\pm 1 - x_i)} & \text{otherwise.} \end{cases}, \tag{3.5}$$

where in considering (3.4) and (3.5) we note that in all simulations conducted below we will assume that both ϕ_b and $\hat{\phi}_b$ are step functions with identical support to ϕ and $\hat{\phi}$ respectively. Specifically we will refer to the support of ϕ_b as b_s. Further, as in the case of $\hat{a}_i$ we will take $\hat{\phi}_b$ to be constant in all of our simulations. This necessarily implies that $\hat{b}_i = 1/P$.

Examining the effect of open-mindedness social norms in the presence of a media source

At this point it is worth noting that unless explicitly stated otherwise our simulation only involve a single media source. Additionally we will take α_4 and α_5 to be equal to 1 unless explicitly stated otherwise.

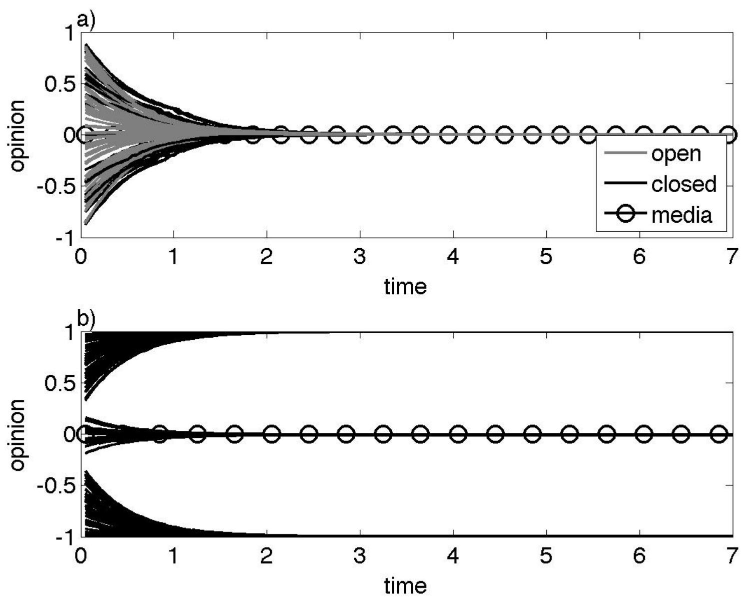

Figure 3.1: The effect of decreasing the proportion of open-minded individuals in the presence of a social norm of open-mindedness and a media source. F^P is governed by (3.3). Parameters: $N = 80$, $\phi = \chi_{[0,a_s]}$, $a_s = 1.2$, $b_s = 1.2$, $X_c = 0.25$, f_i defined by (2.4), $\mu_1 = 0$ a) $m = 60$, b) $m = 20$. In the presence of a social norm of open-mindedness increasing the proportion of open-minded individuals leads to consensus.

As mentioned in the introduction of this section we will extend the result established in the absence of media in Chapter 2. Specifically, we will show that social norms of open-mindedness have the potential under the assumptions of our model to cause consensus in the presence of a media source. Again we will approach this endeavor numerically by contrasting parameter space plots of long-term solutions to (3.1) where f_i is defined by (2.4) or (2.7). The choice of f_i determines whether or not social norms of open-mindedness are present in interactions between members of the set of open-minded individuals and members of the set of non-open-minded individuals. As before we will conduct our parameter sweep over the region in parameter space where consensus results in the absence of ingroup-outgroup dynamics, (see Figure 2.6 for details on the selected parameters).

Observing Fig. 3.1 we see anecdotal support, that like in the no-media-influence case, increasing the proportion of individuals whose indices belong to O results in consensus when open-minded individuals create a social norm of open-mindedness. This is true in general, as is supported by Fig. 3.2. This is interesting since it implies that even in the presence of media, open-mindedness social norms can cause consensus under the assumptions of our model even in the presence of ingroup-outgroup dynamics.

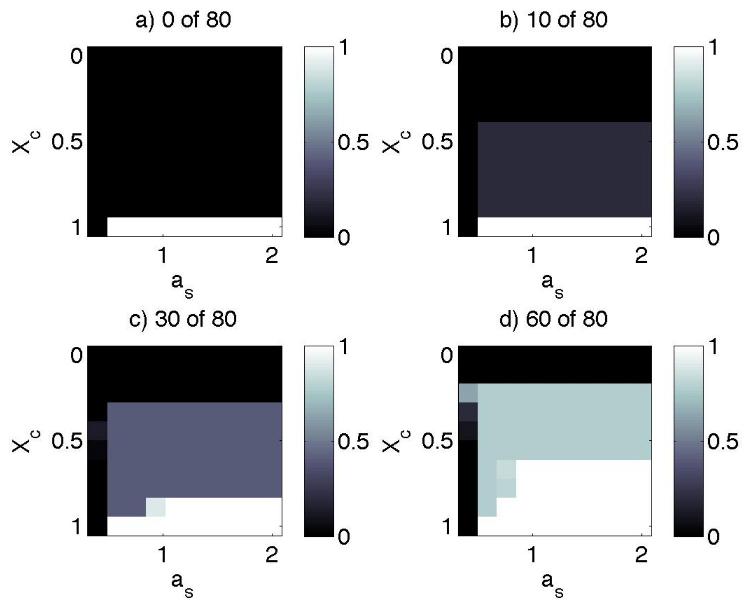

Figure 3.2: Average proportion of uniform random initial conditions which result in consensus in the presence of a media source and a social-norm of open-mindedness. F^P is governed by (3.3). Parameters: $N = 80$ $\phi = \chi_{[0,a_s]}$, f_i defined by (2.4), $\mu_1 = 0$, a)$m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$. Averaged over 30 sets of initial conditions.

Observing Figure 3.2 we see that increasing the proportion of open-minded individuals leads to consensus for a greater proportion of uniform random initial conditions when a social norm of open-mindedness is present with a single media source. It should be additionally noted, as is shown in Fig. 3.2, that consensus is more common for parameter values associated with a lower degree of ingroup-outgroup dynamics (larger X_c). This is not surprising since, as we saw in Chapter 2, ingroup-outgroup dynamics tend to cause polarization, as was found in the absence of media by Boudin et al. in [7] using a related model. Using the mathematical framework developed in [38], this result can be interpreted as that of an open-mindedness social norm increasing connectivity within the population.

Considering Fig. 3.3, we see that like in the no-media-influence case, the absence of a social norm of open-mindedness precludes the possibility of consensus in the presence of ingroup-outgroup dynamics, even when the proportion of open-minded individuals is increased. This is not surprising since [7] found that similar dynamics to our ingroup-outgroup model preclude consensus in the

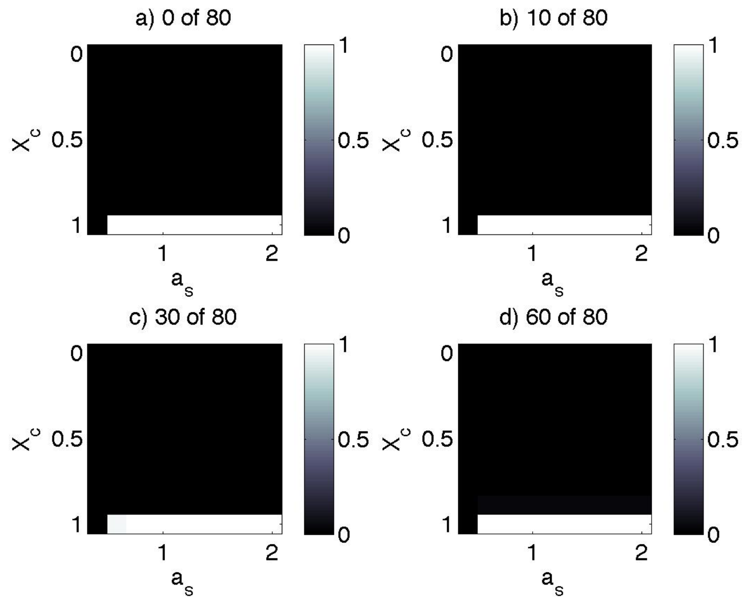

Figure 3.3: Average proportion of uniform random initial conditions which result in consensus in the presence of a media source, and with no social norm of open-mindedness present. The media influence (F^P) is governed by (3.3). Parameters: $N = 80$, $\phi = \chi_{[0,a_s]}$, f_i defined by (2.4), $\mu_1 = 0$, a) $m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$. Averaged over 30 sets of initial conditions.

absence of media.

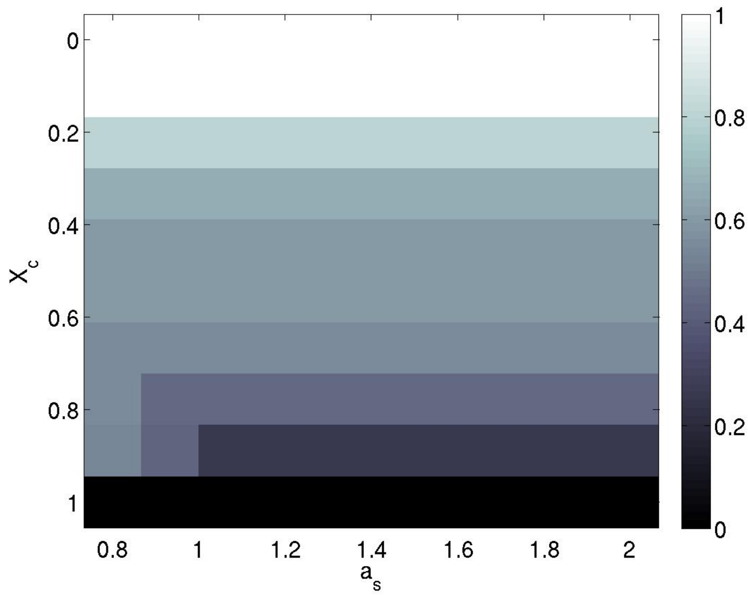

Figure 3.4: Average proportion of open-minded individuals required for consensus. Uniform random initial conditions used. Small X_c requires more open-minded individuals for consensus. The media interaction (F^P) is governed by (3.3). Parameters: $N = 80$ $\phi = \chi_{[0,a_s]}$, α_i=1 $\forall$ i, f_i is defined by (2.4), $\mu_1 = 0$. Averaged over 20 sets of uniform random initial conditions

Observing Fig. 3.4, we see that the proportion of open-minded individuals required for consensus is, as in the no-media-influence case, relatively independent of the specific degree of open-mindedness, as long as open-mindedness exceeds the critical threshold of 1. This, when taken with other figures, further supports our conclusion that social open-mindedness norms have the potential to cause consensus under the assumptions of our model in the presence of a uniform media source as well as ingroup-outgroup dynamics.

It should also be noted here that the parameter space plot in Fig. 3.4 is compressed relative to Figures 3.2 and 3.3. This is as before, to focus on parameters where consensus results.

3.3 Modelling the hostile media effect

In this section we will demonstrate that under the assumptions of our model, open-minded individuals can help polarized populations reach consensus in the presence of the hostile media effect through the creation of open-mindedness social norms. As was mentioned previously, the hostile media effect is the tendency for extremist individuals to perceive media as being hostilely biased regardless of its true bias [33, 10, 60, 13, 9]. This plays a critical role in real life social and military conflicts such the Arab-Israeli conflict [60] and polarization in American politics [10, 13]. In order for any of these conflicts to be resolved a consensus of some sort must be reached between the opposing groups. Here we investigate the effect of social norms of open-mindedness on the occurrence of consensus in the presence of the hostile media effect.

To begin this investigation, we will first define our model for the hostile media effect. As was mentioned above the hostile media occurs in extremist individuals and causes them to perceive the media as being biased against them. We will model this phenomenon by quite literally having extremist individuals perceive the media to be shifted by an amount H away from them in opinion space, relative to the location at which moderate individuals perceive the media to be located. Extremist closed-minded individuals do not base their responses on the actual perceived location of the media for the reason described above and supported by [26, 15]. Incorporating these dynamics we represent the hostile media effect's influence of media as follows:

$$F^P = \begin{cases} \sum_p \alpha_{4,p}\hat{b}_i(1-x_i) & \text{if } x_i > X_c \text{ and } i \in \hat{C} \\ \sum_p \alpha_{4,p}\hat{b}_i(-1-x_i) & \text{if } x_i < -X_c \text{ and } i \in \hat{C} \\ \sum_p \alpha_{5,p}b_{ip}(\mu_p - x_i - H) & \text{if } x_i > X_c \text{ and } i \in O \\ \sum_p \alpha_{5,p}b_{ip}(\mu_p - x_i + H) & \text{if } x_i < -X_c \text{ and } i \in O \\ \sum_p \alpha_{5,p}b_{ip}(\mu_p - x_i) & \text{otherwise} \end{cases}. \tag{3.6}$$

In considering (3.6) we will note that the other aspects of our model will be the same as those given in (3.1); in particular f_i is defined in accordance with (2.4) or (2.7) depending on the presence, or lack thereof, of a social norm of open-mindedness in the interactions between individuals in O and individuals in $\hat{C}$. All interaction coefficients (a_{ij}, $\hat{a}_i$, b_{ip} and $\hat{b}_i$) remain unchanged from the previous section.

Further, as in the previous section λ and μ_1 will be assumed to be zero and all α values will be taken to be 1 for simplicity in all of the below simulations.

3.4 Linear stability

At this point, before numerically considering whether open-mindedness social norms can cause consensus in the presence of the hostile media effect, we will first consider how the hostile media effect affects the local stability of equilibria. Even in the absence of the hostile media effect, the introduction of media greatly increases the diversity of potential equilibrium states. A consensus solution is still possible in the presence of a single media source, and is in fact more common than in the non-media case (Fig. 3.2). A polarized solution is also still possible, though in general there is a greater tendency towards increased clustering. Some sample equilibrium solutions are shown in Fig. 3.9.

Definition 9. *(Problem 3) We will define problem 3 to be (3.1) with $\lambda = 0$. Here f_i is defined by (2.4) or (2.7), where a_{ij} is defined as in (2.2) and $\hat{a}_i$ as in (2.5) or (2.8). Here ϕ and $\hat{\phi}$ are step functions of the general form depicted in Fig. 2.3 with arbitrary supports a_s and $\hat{a}_s$ respectively. Additionally, F^P is defined as in (3.6) with b_{ip} and $\hat{b}_i$ defined in accordance with (3.4) and (3.5) respectively.*

Definition 10. *(Class 3) Let $\{x_i^*\}$ be an equilibrium solution to problem 3. Then $\{x_i^*\}$ is said to be Class 3 if $\phi(x_j^* - x_i^*)$, $\hat{\phi}(x_i^*)$, $\frac{\partial a_{ij}^*}{\partial x_i}$, $\frac{\partial \hat{a}_i^*}{\partial x_i}$, $\frac{\partial a_{\hat{i}j}^*}{\partial x_i}$, $\frac{\partial \hat{b}_i^*}{\partial x_i}$ and $\frac{\partial b_{\hat{i}p}^*}{\partial x_i}$ are well defined for all i and $x_i \neq X_c$ for $i \in \hat{C}$.*

At this point we will extend the results found in Theorem 2 and 4 to the case when one or more fixed media sources are present. Again the approach we will take in proving this result will revolve around establishing that Problem 3's associated Jacobian (or rather one of it's principal minors) is irreducible or block irreducible with diagonally dominant or block diagonally dominant rows. As before we will use these facts to apply either Theorem 1 or Theorem 3 to establish that the Jacobian is non-singular. This paired with the Gershgorin circle theorem and the weak diagonal dominance of the Jacobian will give us that all eigen values have negative real part. More formally this can be stated as follows:

Theorem 6. *Assume that the equilibrium solution to Problem 3 is Class 3 where each of the aforementioned is defined as in definitions 9 and 10 respectively. Then said equilibrium solution of (3.1) is linearly stable if all non-closed-minded individuals are connected at equilibrium as defined previously and one of the following is satisfied: condition 1) f_i is defined by (2.7), at least one non-closed-minded individual is connected to a closed-minded individual at equilibrium and $\alpha_2 > \alpha_1(1-C)$ where C is defined as in Theorem (4),; orcondition 2) $opin$ is defined by (2.4), at least one open-minded individual is connected to a closed-minded individual at equilibrium and $\alpha_3 > 0$, condition 3) f_i is defined by (2.7) or (2.4), at least one non-closed-minded individual has associated nonzero b_{ip}^* and $\alpha_5 > 0$.*

Proof. As before we will begin by calculating the associated Jacobian for (3.1) when F^P is defined by (3.6). As in Theorem 2 and 4 we see that all interaction coefficient partial derivative terms will be zero based on our assumptions. Hence, the Jacobian will simplify to:

$$J = \begin{pmatrix} Diag_1 & Non\ diag_{12} & Non\ diag_{13} & ... \\ Non\ diag_{21} & Diag_2 & Non\ diag_{23} & ... \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$where,$$

$$Diag_i = \begin{cases} -\alpha_2(N-1)\hat{a}_i^* - \sum_p \alpha_{4,p}\hat{b}_i^* & \text{if } |x_i^*| > X_c \text{ and } i \in \hat{C} \\ -\alpha_1 \sum_j a_{ij}^* - \sum_p \alpha_{5,p} b_{ip}^* & \text{otherwise} \end{cases}$$

$$and,$$

$$Non\ diag_{ij} = \begin{cases} 0 & \text{if } |x_i^*| > |X_c|, \text{ the } i \in \hat{C} \text{ and } j \in \hat{C} \\ \alpha_1 a_{ij}^* & \text{otherwise} \end{cases} \tag{3.7}$$

At this point we will note that if condition 1) is satisfied, the argument will follow nearly identically to that in the proof of Theorem 4. If condition 2) is satisfied the argument will follow nearly identically to that in the proof of Theorem 2. Hence for brevity we will only explicitly consider the case when condition 3) is satisfied.

Even in this case the argument will proceed quite similarly to Theorem 2. For simplicity, we consider the case where f_i is defined by (2.7). Specifically as before we will assign our numbering of individuals 1 through N such that J can be written as

$$J = \begin{pmatrix} -\alpha_2 - \sum_p \alpha_{4,p}\hat{b}_i^* & 0 & ... & ... \\ 0 & -\alpha_2 - \sum_p \alpha_{4,p}\hat{b}_i^* & 0 & ... \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & ... & A \end{pmatrix}, \tag{3.8}$$

where A is defined to be the nonzero portion of the Jacobian corresponding to derivatives with respect to x_i's which are associated with non-closed-minded individuals. It should be noted here that previously A contained a certain number of closed-minded individuals this case is contained in argument associated with condition 1 and 2 which are omitted for brevity. Additionally, as before since J has negative diagonal entries and is weakly diagonally dominant, it's eigenvalues $e_i \leq 0$ for all i [24, pg 320]. The weak diagonal dominance follows by noting that if the ith individual is non-closed-minded, (as is the case with all individuals in A)

$$\textit{-Diag}_i = \sum_j \textit{Non diag}_{ij} + \sum_p \alpha_{5,p} b^*_{ip} \tag{3.9}$$

or if the ith individual is closed-minded

$$\textit{-Diag}_i = \sum_j \textit{Non diag}_{ij} + \alpha_2(N-1)\hat{a}^*_i + \sum_p \alpha_{4,p} \hat{b}^*_i \tag{3.10}$$

(see (3.7) for details). Again due to this weak diagonal dominance, (3.8) will have non-negative eigenvalues if and only if its determinant is zero. This follows by applying the Gershgorin circle theorem [24, pg 320] and noting that the diagonal entries of J are negative. Next as in Theorem 2 we can see by performing a cofactor expansion on (3.8) that $\det(J) = (-\alpha_2 - \sum_p \alpha_{4,p} \hat{b}^*_i)^q \det(A)$, where again q is the number of individuals behaving closed-mindedly. Therefore, $\det(J) = 0$ if and only if $\det(A) = 0$.

Considering A in greater detail we see that

$$A = \begin{pmatrix} -\alpha_1 - \xi_{q+1} & & \vdots & \vdots & \vdots & \dots \\ & \ddots & & \dots & & \dots \\ \dots & & -\alpha_1 - \xi_p & & \underset{\substack{[(q<i<N] \\ i<j\leq N}}{\alpha_1 a^*_{ij}} & \dots \\ \dots & \underset{\substack{[q<i\leq N] \\ q<j<i}}{\alpha_1 a^*_{ij}} & \dots & -\alpha_1 - \xi_{p+1} & & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ & \dots & \dots & \dots & & -\alpha_1 - \xi_N \end{pmatrix}$$

$$where, \tag{3.11}$$

$$\xi_p = \sum_p \alpha_{5,p} b^*_{ip}. \tag{3.12}$$

Next noting that A only contains open-minded individuals we see that it trivially follows from our assumption of connectivity and the assumed symmetric functional form of ϕ that A is irreducible. Next we will note that (3.9) applies to A and at least one b_{ip} is guaranteed to be nonzero by assumption. Hence, A is diagonally dominant with strict diagonal dominance holding for at least 1 row. Therefore, by [53] we see that A is nonsingular. Hence all eigen values of J have negative real part. therefore, $\{x_i^*\}$ is linearly stable for all i.

When f_i is defined by (2.4) the argument follows much the same with the modification that there exists a third case in *Non diag*$_{ij}$ (3.7), there is a sum term in *Diag*$_{ii}$ (3.7) and $N-1$ is replaced by $N-m-1$ in (3.7).

□

Considering Theorem 6 we see that media has the potential to increase stability. This is unsurprising, though since the attraction force to media in our model is proportional to the distance from media. Hence, if an individual's opinion is perturbed away from the media, our model implies that it would be drawn right back to said media source. This is perhaps realistic since empirical studies show that the distance from a media source in opinion space is positively correlated with the tendency to move towards said media source upon exposure [30, 3].

It is additionally interesting to note that the hostile media effect does not have the potential to influence stability. This though is largely due to the fact that our assumed functional form of ϕ_b and $\hat{\phi}_b$ leads to stable solutions for all parameter values as is apparent by Theorem 6.

We will also comment here that Theorem 6 is really an extension of Theorem 2 and 4. Specifically Theorem 2 and 4 are sub-cases of Theorem 6 in which $\alpha_{4,p}$ and $\alpha_{5,p}$ are equal to zero. Further as is Theorem 2 and 4, the assumptions of Theorem 6 are not as severe as they first appear, since while in theory there could be N distinct clusters at equilibrium, in practice there are usually fewer than six.

3.5 Numerical investigation of the effect of a social norm of open-mindedness on consensus in the presence of the hostile media effect

Observing Fig. 3.5, we see again anecdotal evidence, that when open-minded individuals create an open-mindedness norm, consensus results when the proportion of open-minded individuals is large enough. This is supported for a much wider range of parameters by Fig. 3.6 and 3.8. This is not the case when open-minded individuals do not create an open-mindedness norm (Fig. 3.7). This implies that under the assumptions of our model, in order to reach consensus between highly polarized groups which are influenced by the hostile media effect, it is more effective to try to increase individuals' open-mindedness than to endeavor to directly overcome the falsely perceived biased perception of the media. Even if a direct advocacy against this bias were successful, it would fail to result in consensus as is evident by our non-hostile media simulations (Fig. 3.2 (a)).

Contrasting Figure 3.6 (whose simulations do incorporate the hostile media effect), with Figure 3.2 where there is no hostile media effect present, we see that the hostile media effect appears to enhance consensus for small values of X_c. This implies that extremist individuals perceiving the media as being

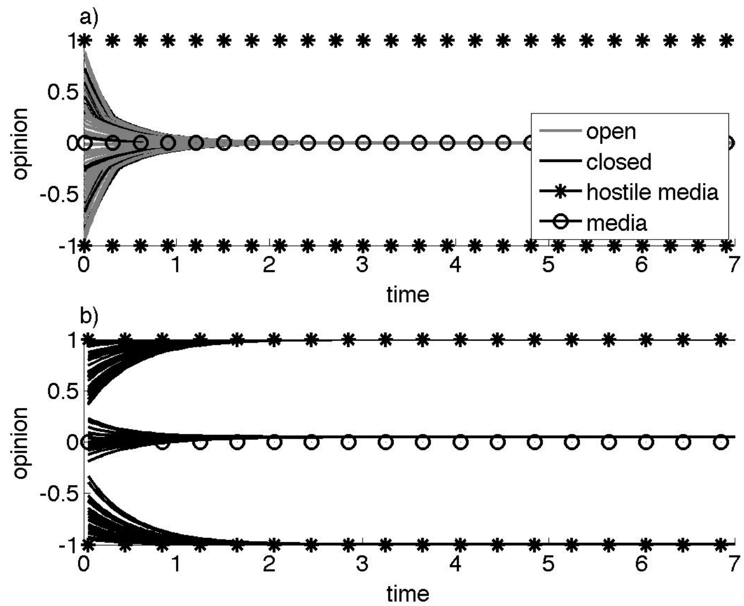

Figure 3.5: The effect of decreasing the proportion of open-minded individuals in the presence of a social norm of open-mindedness and a hostilely perceived media source. Here, $N = 80$, $\phi = \chi_{[0,a_s]}$, α_i=1 $\forall i$, $a_s = 1.2$, $b_s = 1.2$, $X_c = 0.25$, $\mu_1 = 0$, $H = 1$ where, f_i is defined by (2.4). In this simulation the media interaction (F^P) is defined by (3.6) where in a), $m = 60$ and in b) $m = 20$. Increasing the proportion of open-minded individuals leads to increased consensus.

biased against them may under some circumstances actually facilitate conflict resolution. Additionally, it should be noted that consensus is easier to reach at lower levels of ingroup-outgroup dynamics (see Figs. 3.6 and 3.8).

Observing Fig. 3.7 we see that as in the no-media and uniform-media cases, increasing the proportion of open-minded individuals has no effect on consensus in the absence of a social norm of open-mindedness and consensus cannot result in the presence of closed-minded. The bottom rows of Fig. 3.7 corresponds to when only open-minded individuals exist in the population, for the reasons discussed previously in Chapter 2.

It is worth noting that in Fig 3.8 the proportion of open-minded individuals required for consensus is inversely correlated with X_c, though only up to a certain threshold. This implies that the presence of the hostile media effect can under some circumstances actually help facilitate consensus. This is very unintuitive, and considering time-series results, not shown in this thesis, seems largely to be due to open-minded individuals being drawn rapidly towards moderate values by the perceived contrarian media source. This does does not match with the real world circumstance in which the hostile media effect is documented, as the hostile media effect largely occurs in conflicts where individuals are opposed to consensus and ignore media [33, 10, 60, 13, 9]. Hence, we interpret this result and the

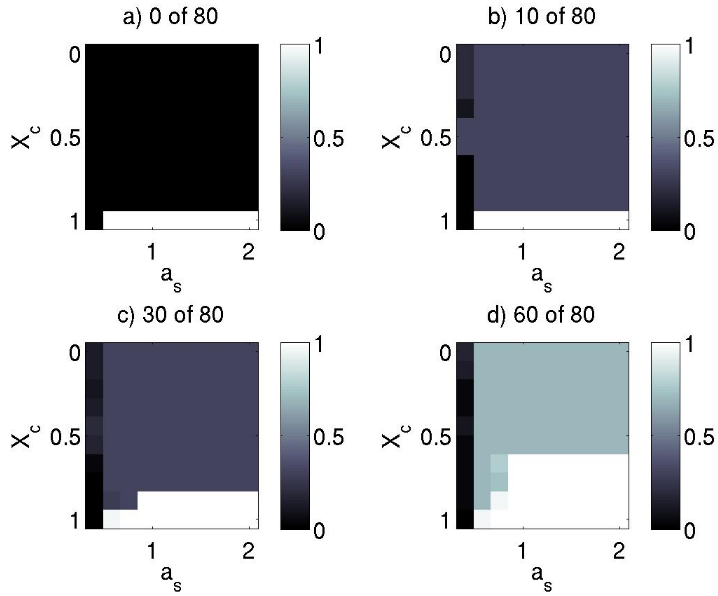

Figure 3.6: The proportion of uniform random initial conditions resulting in consensus in the presence of a social norm of open-mindedness and a hostilely perceived media source. Here, $N = 80$, $\phi = \chi_{[0,a_s]}$, α_i=1 $\forall$ i, $\mu_1 = 0$, $H = 1$ where, f_i is defined by (2.4). In this simulation the media interaction (F^P) is defined by (3.6) where in a) $m = 0$, in b) $m = 10$, in c) $m = 30$ and in d) $m = 60$. The above plot is averaged over 30 sets of initial conditions. Increasing the proportion of open-minded individuals leads to increased consensus.

result of enhanced consensus found in Fig. 3.6 to imply that the hostile media effect does not in fact affect extremist open-minded individuals. This is further supported by our data analysis in Chapter 5 which suggests that the hostile media effect only affects closed-minded individuals (we will discuss this in greater detail in Chapter 5). In this case, when only closed-minded individuals experience the hostile media effect (which they ignore and become more certain in their pre-media exposure beliefs), Fig. 3.8 reduces to Fig. 3.4.

3.6 Summary and future work

The main result of this chapter is that under the assumptions of our model, the presence of social norms of open-mindedness has the potential to cause consensus in the presence of a media source. This is true even in the presence of ingroup-outgroup dynamics and the hostile media effect.

We additionally can conclude that, on some level increasing the strength of media influence increases the stability of equilibrium solutions. This is slightly depressing since it indicates that according to our model, partisan polarization in American politics may continue to persist since the amount of media influence appears to be growing [43, 14, 29].

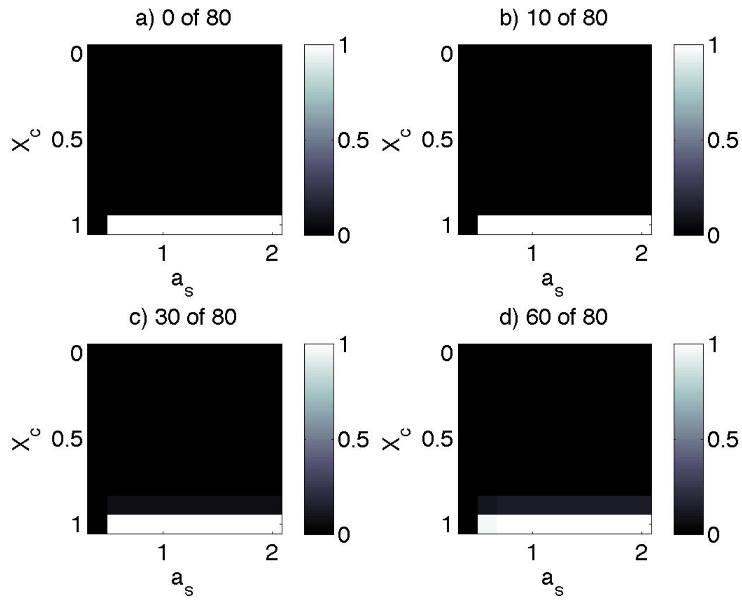

Figure 3.7: The proportion of uniform random initial conditions resulting in consensus in the presence of a hostilely perceived media source in absence of a social norm of open-mindedness. Here, $N = 80$, $\phi = \chi_{[0,a_s]}$, $\mu_1 = 0$, $H = 1$ where, f_i is defined by (2.7) and F^P defined by (3.6). Simulations in the above figure are averaged over 30 sets of uniform random initial conditions and the number of open-minded individuals in each of the above sub-plots is as follows: a) $m = 0$ b) $m = 10$ c) $m = 30$ d) $m = 60$.

Future work

Two areas which are in need of future development are: Simulations with multiple media sources, and simulations incorporating social media through media dependence on individuals' opinions [47]. Sample plots of the first case are shown in Fig. 3.9. Observing said figure, it is clear that a wealth of behavior can result when there are multiple media sources. Fig. 3.10 shows that when individuals can influence the media, the media can under certain circumstances no longer have a significant influence on the system. These results are only case studies and demonstrate that this area of research is in need of further development.

Specifically in representing individual influence on media and media influence on media we assume the following form for λ in (3.1):

$$\lambda(\vec{\mu}, \vec{x}) = \alpha_6 \sum_i c_{ip}(x_i - \mu_p) + \alpha_7 \sum_{q \neq p} \Phi(b_s/2 - |\mu_p - \mu_q|)(b_s/2 - |\mu_p - \mu_q|) sgn((\mu_p - \mu_q)) \quad (3.13)$$

where,

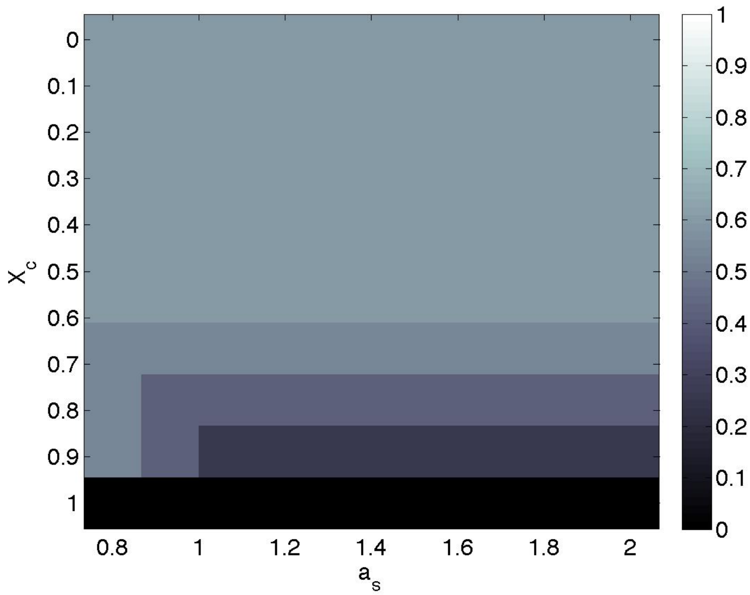

Figure 3.8: The average proportion of open-minded individuals required for consensus in the presence of a hostilely perceived media source. Here, $N = 80$, $\phi = \chi_{[0,a_s]}$, α_i=1, $\forall$ i, $\mu_1 = 0$, $H = 1$ and f_i is defined by (2.4). The media interaction (F^P) is defined by (3.6). All simulations are averaged over 30 sets of uniform random initial conditions.

$$c_{ip} = \phi/\phi_p \qquad \text{where } \phi_p = \sum_j \phi(\mu_p - x_j). \tag{3.14}$$

The first term in the above (c_{ip}) represents individuals influence on the media. We assume that individuals influence the media in the same way as they influence other individuals; Specifically, they draw the media to the mean opinion. We will define c_{ip} similarly to a_{ij} in that it will derive it's form from the step function ϕ with support c_s. The second term represents inter-media competition where, Φ is the heaviside function. The other newly defined constants (α_6 and α_7) are scalars denoting the relative magnitude of each type of influence. All other variables remain the same as previously defined.

The justification for the assumed influence which individuals have on the media is not peer pressure, but rather, profit pressure. To paraphrase, we assume that the media seeks the mean individual opinion in order to maximize its profit. This is a reasonable assumption as empirical studies have determined that profit maximization is the main determinant of a media's bias [23].

The justification for the second term in (3.14) is that when two media sources have the same viewers (as represented by an overlap in the support of the two media's individual-media interaction functions), both media sources experience a repulsion away from each other. As before all interaction

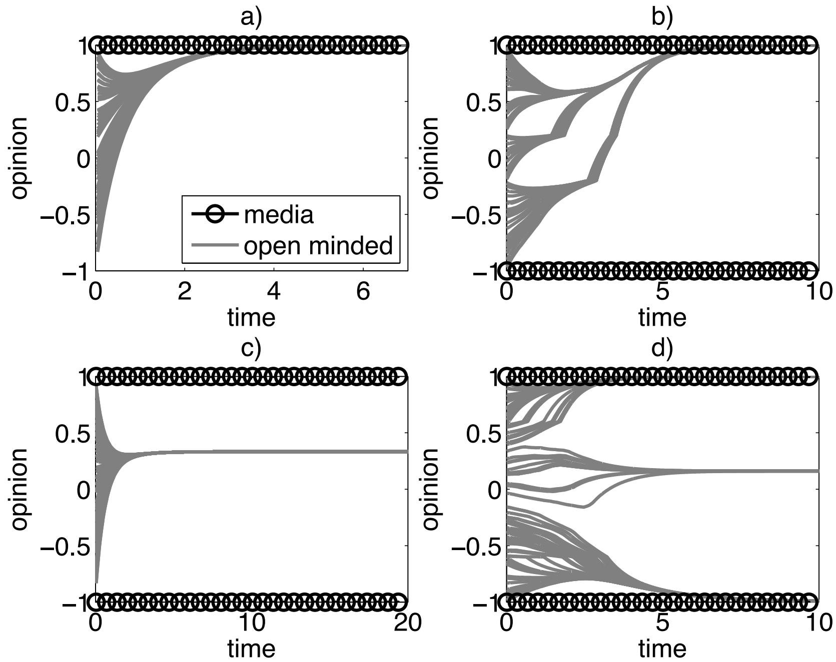

Figure 3.9: Effect of multiple and extremist media sources. Here, $N = 80$. a) Single extremist media source: Leads to consensus equilibrium at the media's bias. We see that under some circumstances an extremist media source has the potential to draw individuals to its bias, where $m = 80$, $a_s = 2$. b) Extremism with two media sources: when one media source is stronger than the other an extremist consensus can result, where $a_s = 1.2$, $m = 80$, $N = 80$, $[\alpha_{5,1}, \alpha_{5,2}] = [1, 0.5]$. c) Moderate consensus with two media: when individuals are more responsive to opinions different from their own, a more moderate consensus can result, where $a_s = 2$, $m = 80$, $N = 80$ and $[\alpha_{5,1}, \alpha_{5,2}] = [1, 0.5]$. d) Multiple clusters with two media sources: when individuals only respond to opinions close to their own multiple clusters can result, where $a_s = 0.4$, $m = 80$, $N = 80$ and $[\alpha_{5,1}, \alpha_{5,2}] = [1, 0.5]$.

functions (ϕ,$\hat{\phi}$,ϕ_b, $\hat{\phi}_b$) are assumed to be step functions.

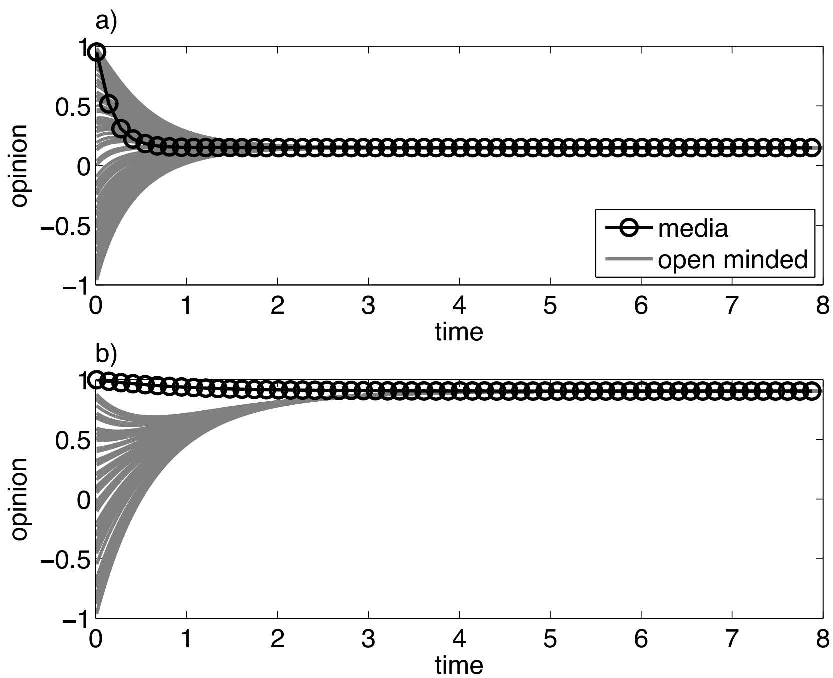

Figure 3.10: Social media influence on media opinion dynamics using (3.1)-(3.3) with (3.13)-(3.14). a) Strong social media influence: We see that when media is responsive to individuals' opinions, under some circumstances it has minimal effect on the population's mean opinion where, $b_s = a_s = c_s$ $a_s = 2$, $\alpha_6 = 5$. b) Weak social media influence: We see that when media is unresponsive to individuals opinions it can under some circumstances have a extremism-inducing effect on the population. Here, $a_s = 2$ and $\alpha_6 = 0.1$ (α_7 is not relevant in this simulation since there is only a single media source).

Chapter 4

Quantifying the effect of open-mindedness in advertising optimization

In this section we will examine the effect of open-mindedness on the commercial application of advertising. For simplicity we will from now on assume the existence of an open-mindedness social norm when open-minded individuals interact with closed-minded individuals. We will make this assumption as it is supported by social science texts [56, 57, 36]. In addition, we conjecture that many of the results below hold in the absence of said social norm.

Numerous mathematical models have been proposed to describe the relationship between advertising and sales [31, 18, 41, 45, 46, 49, 20, 51]. Most these models treat the system as an aggregate [31, 18, 34], though some modeling efforts are agent-based like our own [46]. Additionally, many of the conclusions of these models are inconclusive concerning the specific optimal approach [31, 18, 46]. To the best of our knowledge, though, none of these modelling endeavors explicitly consider the role of ingroup-outgroup dynamics. This is surprising, in that numerous studies have documented that brand loyalty and identity-based emotional sentiment are critical in determining an individual's response to advertising [41, 59].

In this section we will attempt to quantify the effect of open-mindedness on the process of advertising optimization. This will require us to make a simplifying assumption that shifting individuals' opinions in favor of one's product directly translates into increased sales. This is a common assumption of many models [31, 46] and is additionally supported by empirical data [59]. We shall make the simplifying assumption that an advertisement is equivalent to a single media source located at μ_p whose form is given by (3.3).

Further we will note that all of our investigations below will involve parameter sweeps. In order to contrast our results with those earlier in this thesis, we will limit ourselves to the parameter space relevant to consensus used in Chapters 2 and 3 (see Fig. 2.6 for details on the range of parameters investigated). While this potentially excludes some interesting parameters, it still makes the results relevant to a wide range of populations of individuals with varied psychologies.

4.1 Optimizing target proportion given a fixed budget

We will first quantify the effect of open-mindedness on the optimization of advertising with a fixed budget. Here we will assume that an advertisement budget is defined as

$$\text{budget} = \alpha_4 \ \textit{Prop}_c + \alpha_5 \ \textit{Prop}_{nc}, \tag{4.1}$$

where $\textit{Prop}_c$ represents the proportion of closed-minded individuals who are targeted. These are individuals belonging to the set of non-open-minded individuals whose opinion $|x_i| > X_c$. Similarly, $\textit{Prop}_{nc}$ represents the proportion of non-closed-minded individuals targeted. As before these are individuals belonging to the set of non-open-minded individuals whose opinion $|x_i| < X_c$ and individuals belonging to the set of open-minded individuals.

For simplicity we will assume that $\alpha_4 = \alpha_5$. In this case $\textit{Prop}_c$ and $\textit{Prop}_{nc}$ can be combined into a single quantity, which we will refer to as the *target proportion*. This represents the proportion of the total population which experience the advertisement. Hence (4.1) can be rewritten as follows:

$$\text{budget} = \alpha_{4,5} \times \textit{target proportion}. \tag{4.2}$$

For a fixed budget the above presents an obvious trade-off, in which advertisers can either reduce the strength of their advertisement (reduce $\alpha_{4,5}$) and target a larger fraction of the population, or alternatively increase the strength of their advertisement (increase $\alpha_{4,5}$) and target a more limited proportion of the population. This is a realistic design, since most traditional and many untraditional forms of advertisement are priced using roughly this definition [54]. Determining which strategy is optimal for a given population will be the purpose of this section.

Using the above, we model advertising effectiveness using the following model for an advertisement described in (4.3). As mentioned previously, an advertisement is treated as a media source term located at μ_p, where $\mu_p > 0$, (as the advertisement supports its associated product). Other interactions are modeled by (3.1), where f_i is defined by (2.4) or (2.7) and $\lambda = 0$. Specifically we will represent advertising as

$$F^P = \begin{cases} \sum_p \alpha_{4p} \hat{b}_i (1 - x_i) & \text{if } x_i > X_c \text{ and } i \in \hat{C} \cap A, \\ \sum_p \alpha_{4p} \hat{b}_i (-1 - x_i) & \text{if } x_i < -X_c \text{ and } i \in \hat{C} \cap A, \\ \sum_p \alpha_{5p} b_{ij} (\mu_p - x_i) & i \in A \text{ and either } i \notin \hat{C} \text{ or } |x_i| < X_c \text{ ,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Here A represents the set of individuals who are targeted by the advertisement. Additionally we will assume that there is no targeting of niche groups, and that the specific individuals targeted by the advertisement are randomly distributed throughout the opinion space. This is not an unrealistic assumption, as many forms of television and internet media have very diverse sets of viewers [29]. It is worth noting that (4.3) is identical to our previous media model (3.3) defined in Chapter 3, except that only a certain fraction of the population experiences the media. As other definitions remain unchanged from Chapter 3, the stability results also remain largely unchanged from those for (3.3).

Before we numerically determine the optimal *target proportion* we will make the assumption that there is no competition between media sources (under our model's design this implies that our simulations will only involve a single media term). Additionally, we will assume that an advertiser's opinion is located at $\mu_1 = 1$. This is equivalent to assuming that advertisers strongly advocate for their product.

As stated previously we will assume that $\alpha_4 = \alpha_5$, and as before we will take $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in all of our simulations in this section.

Numerical experiments

As was mentioned in the discussion for (4.2) under our assumptions an advertiser with a fixed budget is faced with a trade-off where the proportion of individuals which they target in the population (*target proportion*) is constrained by their selected value of $\alpha_{4,5}$. We will attempt to optimize this trade-off where the measure of optimality is the extent which an advertising strategy is successful in shifting the population's mean opinion to that of the advertiser's (assumed to be 1 in the below simulations). All of the below simulations start with uniform random initial conditions with a population mean opinion of approximately 0. By varying the *target proportion* and $\alpha_{4,5}$ while holding the budget fixed at 0.1 we will attempt to determine what is the optimal *target proportion*. In all of the below simulations as in earlier chapters $a_s = b_s$ and $\hat{\phi}$ is taken to be constant with $N = 80$. Budget will be defined based on (4.2).

Observing Fig. 4.1, we see that it is better under specific parameter regimes (see Fig. 4.1 for details) to target the entire population rather than to concentrate the advertising focus on few randomly selected individuals. This follows by noting that the target proportion of 1 is optimal in Fig. 4.1 (c). We additionally see that this is true on average for all interesting parameters (Fig. 4.2), where the regime of 'interesting parameters' is the same as previously mentioned in Chapters 2 and 3. This is interesting as it implies that under our model assumptions, in the absence of competition, it is on average ideal to make advertising which appeals to everyone equally. We will refer to these cases as cases where a *global targeting strategy* is optimal.

There are notable exceptions to the above rule, as Fig. 4.1 (d) demonstrates. In these cases the advertising media tends to have minimal impact on the mean opinion regardless of the *target*

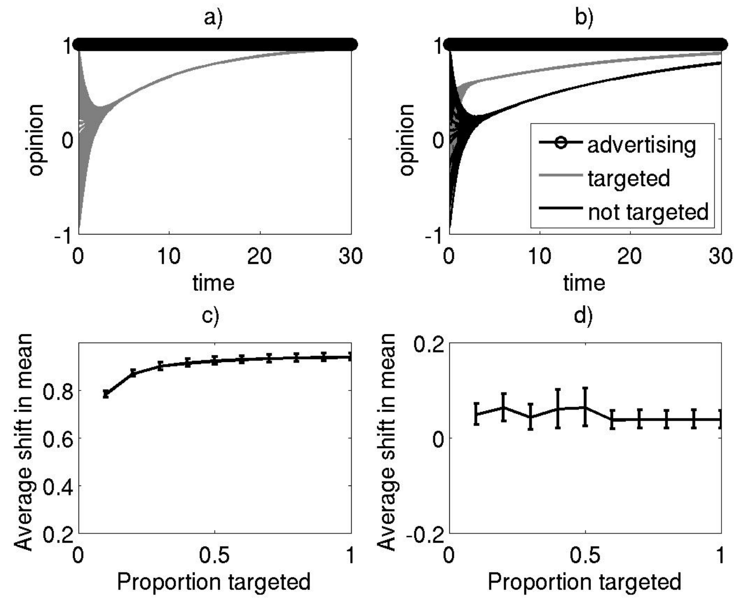

Figure 4.1: The effect of targeting various proportions of the population with a fixed budget (budget=0.1): a) Time series when all individuals are targeted with, $m = 80$ and $a_s = 1.5$. b) Time series when only 10 % of individuals are targeted with, $m = 80$ and $a_s = 1.5$. c) Average shift in mean as a function of the fraction of the population targeted by advertisement with, $m = 80$, $a_s = 1.5$. d) Average shift in mean as a function of the fraction of the population targeted with, $X_c = 1/3$, $a_s = 1.5$, $m = 0$. On average, in c) it is optimal to target the entire population. This is not the case in d).

proportion. These cases in general occur when the interaction function has minimal support (small a_s and b_s) or when X_c is small. This is analogous to individuals being somewhat closed-minded and only being responsive to those close to them in opinion.

Additionally we see in Fig. 4.3 that the global target optimum occurs much less frequently when the number of open-minded individuals in the population is low and when there is a large degree of ingroup-outgroup dynamics present in the population. In these cases, no universal optimal target proportion exists and initial-condition-dependent dynamics dominate. Hence we see that in general under our model's assumptions, open-mindedness results in global targeting strategies being optimal, whereas closed-mindedness leads to more unpredictable advertising dynamics.

Observing Fig. 4.2, we see that given the assumptions of our model, on average it is optimal to target the entire population. Additionally we see that it appears to be slightly more optimal to only target open-minded individuals. It is worth noting though that targeting random individuals and targeting open-minded individual produces nearly equivalent results, particularly when the majority of individuals are closed-minded. This is interesting as in situations like politics, closed-mindedness dominates [30]. Hence our results suggest that under the assumptions of our model, targeting

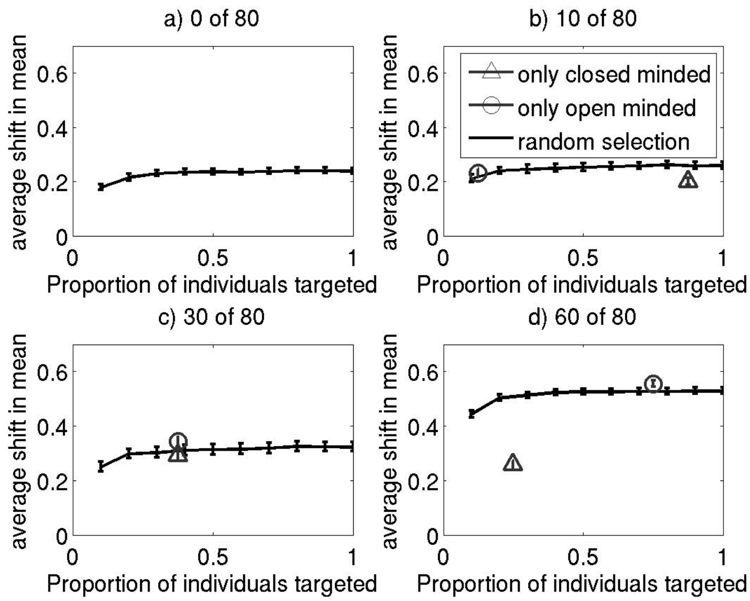

Figure 4.2: Average shift in mean opinion as a function of the proportion of the population which is targeted by the advertisement when the media budget is held constant, averaged over ten sets of initial conditions and the 16 sets of parameters shown in Fig. 4.3. Budget fixed at 0.1. End time taken to be 30. Parameters used: a) $m = 0$ b) $m = 10$ c) $m = 30$ d) $m = 60$.

moderates in the above-mentioned situation is not necessarily of the greatest importance.

Observing Fig. 4.3, we see that while the trend is not uniform, there is a general correlation between open-mindedness (large X_c) and the optimality of a global targeting strategy in the presence of an open-mindedness social norm. This implies that under the assumptions of our model, in cases where there is minimal brand loyalty (modeled by larger X_c) and individuals have weak ties to particular products it is ideal to target all individuals in the population.

The above result depicted in Fig. 4.3 comes with an asterisk, in that the parameter space plot appears to potentially be affected by statistical noise from the relatively few simulations used in its construction (even though in that each parameter space plot contains 1600 simulations). At this time we have no estimate on this noise term, but more extensive computations to improve this statistic are in progress. This statement applies to all parameter space plots in Chapter 4.

4.2 Determining minimum investment for optimal relative returns

In this set of simulations we investigate under which circumstances a specific minimal investment in an advertising budget may be required in order to maximize the associated relative return on investment.

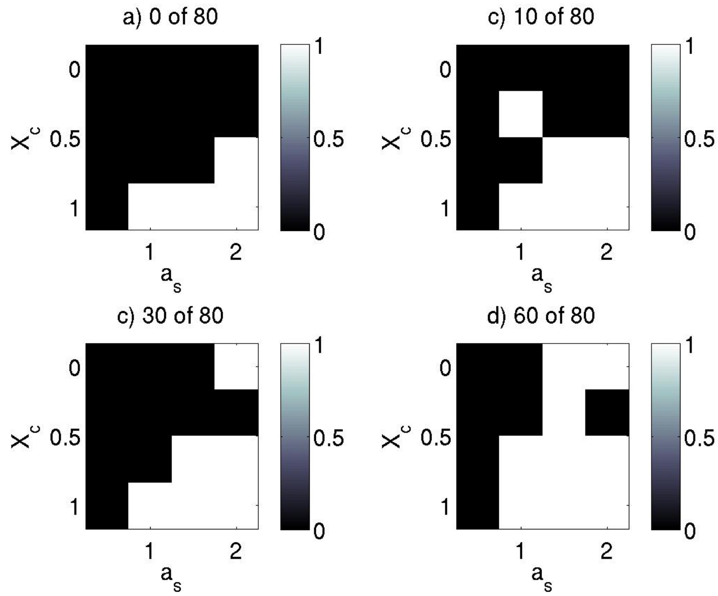

Figure 4.3: Parameters at which it is optimal to target all individuals when the advertising budget is held constant (budget=0.1). Optimality is as is defined earlier. Each of the above tiles represents a line plot such as the one in Fig. 4.1 (c) which has been averaged over ten sets of uniform random initial conditions. The legend label 1 (white) represents cases in which it is clearly optimal to target the entire population, whereas the legend label 0 (black) represents cases when this global target strategy is suboptimal or not statistically different than any other target strategy. We use $N = 80$ for all simulations where, a) $m = 0$ b) $m = 10$ c) $m = 30$ d) $m = 60$.

This is an important trend to investigate, since a smaller company could be operating under a constrained advertising budget yet still desire the greatest relative return for their investment. Using the same model as (4.3) as well as the same simplifying assumptions about the relative magnitudes of α_4 and α_5, we will consider how the relative return on advertising investment varies when various proportions of the population are targeted. We will define the relative return on investment (*RROI*) as:

$$\textit{RROI} = \text{shift in population mean/budget}. \tag{4.4}$$

We will simulate varying the advertisement budget by varying the proportion of individuals targeted. An alternative method would be to vary the value of $\alpha_{4,5}$ in (4.2). This is not as realistic, though, as small companies will at times face cost barriers for targeting a large group of individuals [54]. Hence, in the figures below we will vary the proportion of individuals targeted as opposed to the value of $\alpha_{4,5}$. Further, as before we will assume that there is no competition and that the advertising source's opinion is fixed at $\mu_1 = 1$.

Numerical simulations

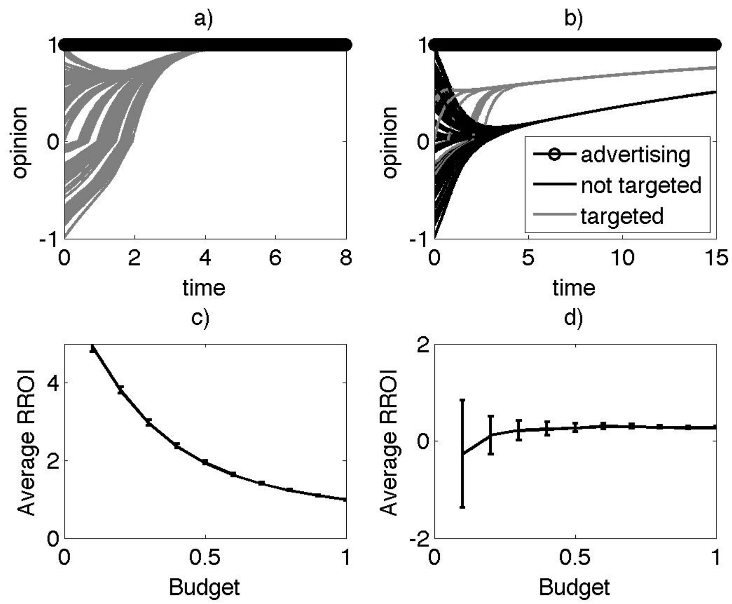

Figure 4.4: Relative return on investment as compared to the advertising budget. Parameters: α_4 and α_5 fixed at 1. a) Sample time series when all individuals are targeted. $X_c = 1$, $a_s = 1$, $m = 60$, $budget = 1$. b) Sample time series when only 10 % of individuals are targeted where, $X_c = 1$, $a_s = 1$, $m = 60$, $budget = 0.1$. c) Average RROI as a function of advertising budget. No cost barrier case. $X_c = 1$, $a_s = 1$ and $m = 60$. d) Average RROI as compared to advertising budget. Possible cost barrier case where, $m = 30$, $X_c = 0$ and $a_s = 2$. In contrasting c) and d) we see the two main types of parameter specific behaviors. Simulations averaged over 10 sets of initial conditions. Error bars represent standard deviation of mean value.

Observing Fig. 4.4 (c), we see that under the assumptions of our model, for the chosen parameter values there is no minimal threshold of investment in order to receive the optimal return on investment. Mathematically we denote this case to be when the the following two conditions are satisfied: 1) the smallest budget value considered (0.1 in our simulations) results in the maximum RROI and 2) this RROI is statistically significantly different from at least one other budget's RROI. This situation, where the minimal investment maximizes the $RROI$ is on average the case, as is shown in Fig. 4.6. There are large exception to this rule, though, as evident by Fig. 4.4 (d). These exceptions in general occur when large levels of ingroup-outgroup dynamics are present in the system, as is evident in Fig. 4.5. This is significant as it implies that under the assumptions of our model, a small company might have difficulty breaking into market with high levels of brand loyalty or when brands are associated with personal identity, as is found to be the case in [59, 41].

Considering Fig. 4.5, we see that increasing m increases the proportion of parameters where advertising faces no cost barrier. Further, from observing Fig. 4.5 we see that when there is large

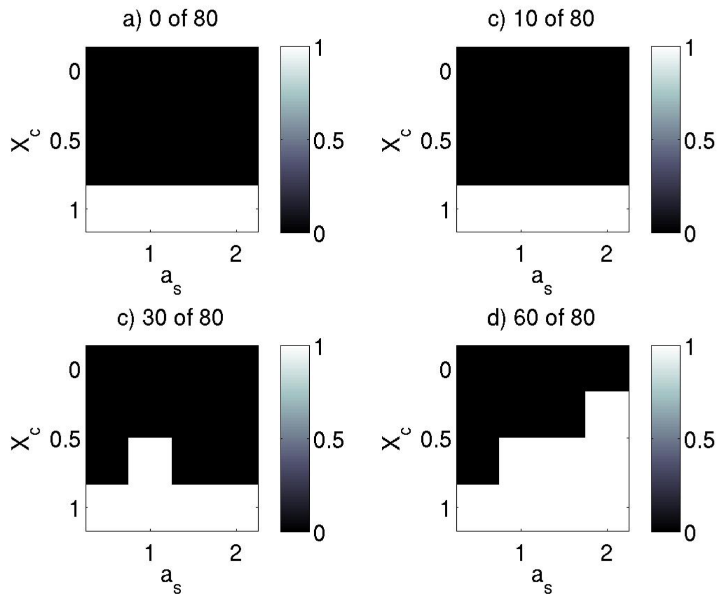

Figure 4.5: Parameters at which, the relative return on investment is maximized by taking the media budget to be the minimum value (0.1 in our simulations). Legend defined such that the value 1, (white) denotes when the optimal relative return on investment occurs when the minimal proportion of the individuals is targeted, whereas the value 0 denotes when this is not the case. Each of the above tiles represents a line plot, such as the one in Fig. 4.4 (c) which has been averaged over ten sets of uniform random initial conditions for each budget. Parameters: $\alpha_4 = \alpha_5 = 1$, $N = 80$, a) $m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$.

degree of ingroup-outgroup dynamics present (small X_c), a minimal investment in advertising is possibly not enough to maximize one's relative rate of return on investment. This implies that in cases such as when brand loyalty is strong and individuals strongly identify with their product choice, that larger investment is required to substantially shift the population's beliefs. This effect can be mitigated, though, by increasing the proportion of open-minded individuals in the population (Fig. 4.5).

This difference in barriers to entry could perhaps explains the relative success and failures of various guerrilla marketing campaigns found in [5]. Here it was found that Lipton's guerrilla marketing efforts were more successful than those of McDonald's, as participants already felt certain in their belief that McDonald's products were unhealthy. This could potentially be explained by the presence of ingroup-outgroup dynamics in the area of fast food advertising, with one group being 'health conscious' people and the other being less 'health conscious' people.

Observing fig 4.6, we see that on average it is optimal, in terms of maximizing the relative rate of return on investment to take the advertising budget to be as small as possible. That is, given the assumptions of our model, averaged over all a_s and X_c the optimal investment to maximize

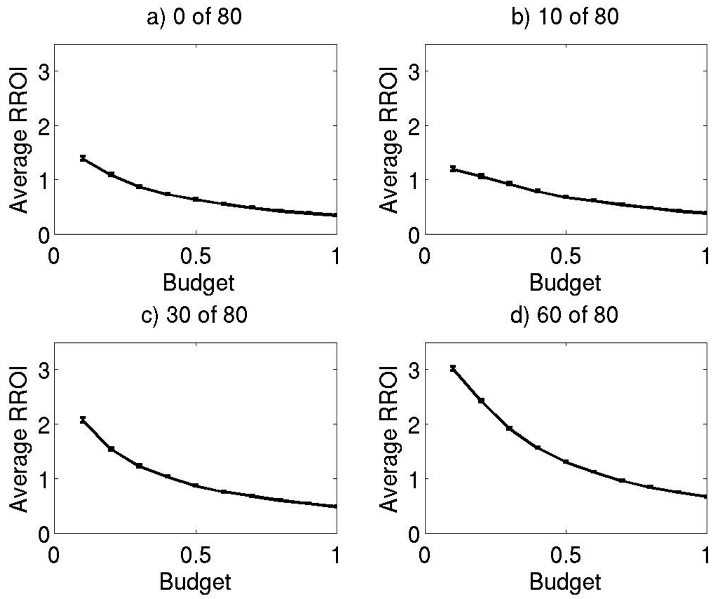

Figure 4.6: Average relative return on investment as a function of the advertising budget, with $\alpha_4 = \alpha_5 = 1$. Simulations averaged over 10 sets of uniform random initial conditions, and over the 16 parameter values shown in Fig. 4.5 where, $N = 80$. Parameters: a) $m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$.

relative returns is the minimal one. The relative returns of this advertisement increase, though, when a greater proportion of the population is open-minded. This implies that on average (but particularly when most individuals are open-minded), there is no economic barrier to entry in terms of advertising.

4.3 Determining the optimal target demographic within an opinion space

In this section we will investigate how open-mindedness affects the optimal target demographic for a given advertisement. Often advertising will address a specific concern customers have with a product [59, 41]. This concern will only be held by individuals with a certain opinion towards said product. Hence these individuals will only be affected by said advertisement when they occupy the interval in opinion space associated with this concern. Optimizing the target demographic can consequently on some level be seen as the process of determining which target interval in opinion space is optimal, where optimality is defined as causing the greatest shift in the population's mean opinion in the direction of the advertised opinion.

We will model the effect of open-mindedness on this phenomenon using a similar method to that which we used to analyze the effect of open-mindedness on budget optimization. We will use the model proposed in (3.3), with the modification that only individuals in a specific target interval experience the media. This modification can be represented by defining the b_{ip} and $\hat{b}_i$ terms in (3.3) as shown in (4.5), where χ, as usual, is the characteristic function:

$$\begin{aligned} b_{ip} &= \chi_{[I]} \\ \hat{b}_i &= \chi_{[I]}. \end{aligned} \tag{4.5}$$

As before f_i is defined by (2.4) and general opinion dynamics are described by (3.1) with $\lambda = 0$. The advertisement target interval I will be defined as follows:

$$I = [I_{min}, I_{max}].$$

where I_{min} and I_{max} represents the lower and upper bounds, respectively, of the target interval. As in Chapters 2 and 3, a_{ij} and $\hat{a}_i$ are defined to be normalized step functions of the forms described in (2.2) and (2.5).

We note explicitly that in all of our simulations below, we assume that there is no competition between advertisers that is, there is a single media source. Additionally we will take all α values to be equal to 1 for simplicity. This is equivalent to assuming that all types of interaction are of equal magnitude. Lastly, as before we will assume in all of our simulations that the advertising source is located at $\mu_1 = 1$. Before we apply these assumptions to our model and numerically estimate the optimal target interval, we will first consider the linear stability of the system.

Linear stability analysis

Our experience from numerical simulations is that in general, when the media interaction functions are defined in accordance with (4.5), similar equilibria result as when the interactions are defined as in Chapter 3. The relative occurrence of equilibria differs qualitatively, though, in that multiple cluster solutions are less common when the target interval I is small.

Definition 11. *(Problem 4) We will define problem 4 to be (3.1) with $\lambda = 0$. Here f_i is defined based on (2.4) or (2.7), where a_{ij} is defined as in (2.2) and $\hat{a}_i$ as in (2.5) or (2.8) and ϕ and $\hat{\phi}$ are step functions of the general form depicted in fig. 2.3 with arbitrary support a_s and $\hat{a}_s$ respectively. Additionally, F^P is defined as in (3.6) with b_{ip} and $\hat{b}_i$ defined in accordance with (4.5).*

Definition 12. *(Class 4) Let $\{x_i^*\}$ be an equilibrium solution to problem 4. Then $\{x_i^*\}$ is said to be Class 4 if $\phi(x_j^* - x_i^*)$, $\hat{\phi}(x_i^*)$, $\frac{\partial a_{ij}^*}{\partial x_i}$, $\frac{\partial \hat{a}_i^*}{\partial x_i}$, $\frac{\partial a_{\hat{i}j}^*}{\partial x_i}$, $\frac{\partial \hat{b}_i^*}{\partial x_i}$ and $\frac{\partial b_{ij}^*}{\partial x_i}$ are well defined for all i, and in addition, $x_i \neq X_c$ for i corresponding to individuals existing in $\hat{C}$ and $x_i^* \neq I_{min}, I_{max}$ for all i.*

Theorem 7. *Assume that the equilibrium solution to Problem 4 is Class 4 where each of the aforementioned is defined as in Definition 11 and 12 respectively. Then said equilibrium solution of*

(3.1) is linearly stable if all non-closed-minded individuals are connected at equilibrium as defined above and that one of the following is satisfied: condition 1) f_i is defined by (2.7), at least one non-closed-minded individual is connected to a closed-minded individual at equilibrium and $\alpha_2 > \alpha_1(1-C)$ where C is defined as in Theorem (4), condition 2) f_i is defined by (2.4), at least one open-minded individual is connected to a closed-minded individual at equilibrium and $\alpha_3 \geq 0$, condition 3) f_i is defined by either (2.4) or (2.7), at least one open-minded individuals has an associated nonzero b^*_{ip} and $\alpha_5 > 0$.

Proof. We will begin this extension of Theorem 6 by deriving the associated Jacobian for (3.1) when f_i is defined based on (3.6) with b_{ip} and $\hat{b}_i$ defined in accordance with (4.5).

As before all partial derivatives associated with interaction coefficient will be zero due to the assumed restrictions on x^*_i. Specifically since the assumed normalized step functions and the characteristic functions are constant apart from at their associated discontinuities (which are excluded as potential equilibria by Definition 12), the partial derivatives of the aforementioned functions at the assumed equilibria must be zero. Hence the Jacobian reduces to:

$$J = \begin{pmatrix} Diag_1 & Non\ diag_{12} & Non\ diag_{13} & ... \\ Non\ diag_{21} & Diag_2 & Non\ diag_{23} & ... \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$where,$$

$$Diag_i = \begin{cases} -\alpha_2(N-1)\hat{a}^*_i - \sum\limits_p \alpha_{4,p}\hat{b}^*_i & \text{if } |x^*_i| > X_c \text{ and } i \in \hat{C} \\ -\alpha_1 \sum\limits_j a^*_{ij} - \sum\limits_p \alpha_{5,p} b^*_{ip} & \text{otherwise} \end{cases}$$

$$and,$$

$$Non\ diag_{ij} = \begin{cases} 0 & \text{if } |x^*_i| > |X_c|\text{, the } i \in \hat{C} \text{ and } j \in \hat{C} \\ \alpha_1 a^*_{ij} & \text{otherwise} \end{cases} \tag{4.6}$$

The rest of proof follows exactly as in Theorem 6.

□

The result in Theorem 7 is closely related to Theorem 6, in that aside from the difference in definition of F^P, Theorem 6 is a special case of Theorem 7 corresponding to the case when I is

$[\mu - b_s, \mu + b_s]$, where b_s is the support of the step function associated with b_{ij} and $\hat{b}_i$ as in Chapter 3.

It is additionally worth noting that if x_i^* resides inside of the interval I, the diagonal dominance of (4.6) is enhanced. Hence, as in Chapter 3 we see that media influence at equilibrium on some level increases the stability of a given equilibrium. The same is true of closed-mindedness.

Numerical simulations

In the below simulations we will attempt to determine the optimal target interval, where optimality is defined as discussed above. For simplicity will make the additional assumption in all of the simulations below that $I_{max} - I_{min} = 0.2$. Hence, while we conjecture that our results generalize to a variety of sizes of target intervals, it should be noted that we will only explicitly consider target intervals of the aforementioned arbitrarily chosen size. Additionally as before all α values will be taken to be 1 for simplicity

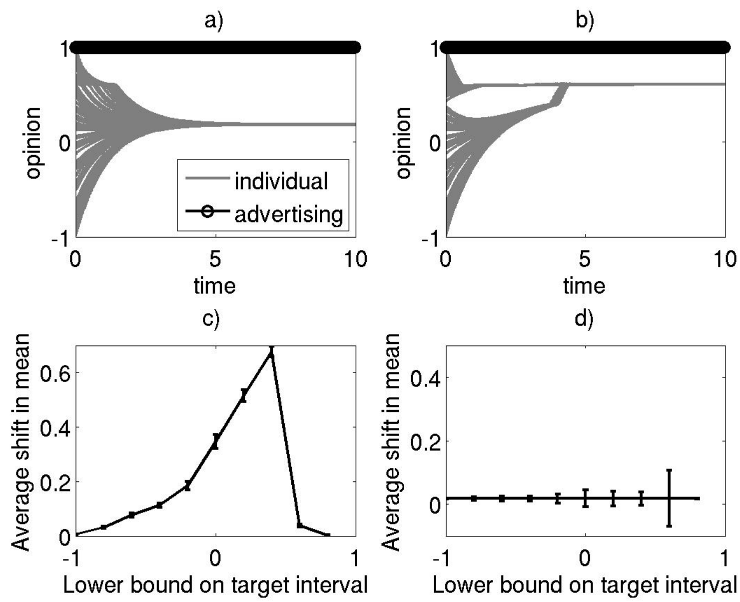

Figure 4.7: Shift in mean opinion as function of the interval targeted I by advertisement. Simulations use uniform random initial conditions. a) Time series in which $I = [0.6, 0.8]$. This target interval fails to shift the mean opinion significantly. Here, $m = 80$ and $a_s = 2$ b) Time series in which $I = [0.4, 0.6]$. This target interval shifts the mean opinion significantly. Here, $m = 80$ and $a_s = 2$ c) Average shift in mean as a function of I_{min}. Averaged over ten sets of initial conditions with $m = 80$, $a_s = 2$. d) Average shift in mean as a function of I_{min}. Averaged over ten sets of initial conditions with $m = 0$, $X_c = 0$ and $a_s = 1$. In many cases there is a clear optimal target interval, though there are exceptions.

Observing Fig. 4.7, we see the two main phenomena which occur as the target interval is varied for a given set of parameters. In the first case depicted in Fig. 4.7 (c) there is a single optimal interval occurring between 0 and the media source, (though for certain rare parameters there are two optimal intervals). This case, that there exists an optimal target interval, is more common as is evident by Fig. 4.8 and 4.9. This is intriguing, since it implies that under the assumptions of our model it is often optimal to target individuals who already have a positive opinion towards one's product. This explains the relative success of targeting previous customers and individuals with a prior preference for a product [59, 41]. The alternative case, as seen in Fig. 4.7 (d), is that all target intervals are essentially indistinguishable, and the final equilibrium state is largely dependent on the initial condition. Unsurprisingly, this second trend tends to occur more often in the presence of large degrees of ingroup-outgroup dynamics, as is evident in Fig. 4.8.

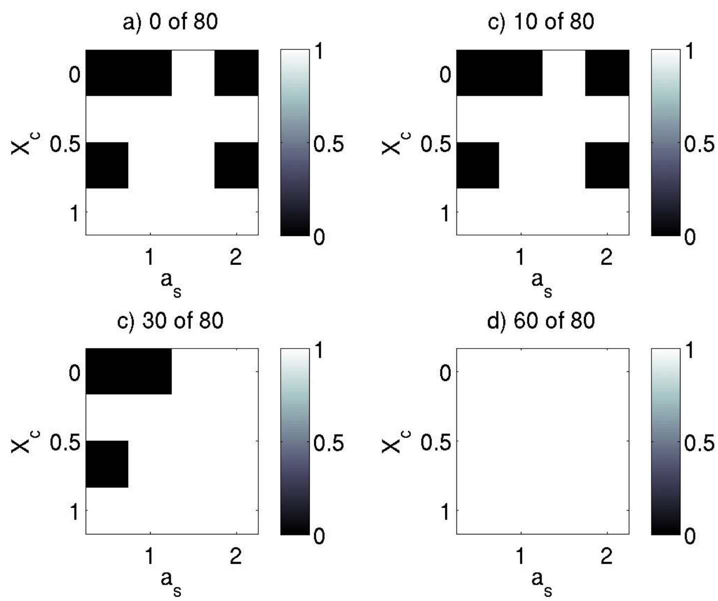

Figure 4.8: Parameters where there exists a clear optimal target interval. White (value 1) denotes parameters where an optimal target demographic exists, while black (value 0) denotes parameters where all demographics are indistinguishable. Each of the above tiles represents a line plot such as the one in Figure 4.7 (c) which has been averaged over ten sets of uniform random initial conditions for each interval. a) $m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$.

Observing Fig. 4.8, we see that, with a few exceptions mostly occurring in the presence of high degrees of ingroup-outgroup dynamics, an optimal target demographic exists. Further when the proportion of open-minded individuals is increased, the proportion of parameter values which have an associated optimal target demographic is enhanced.

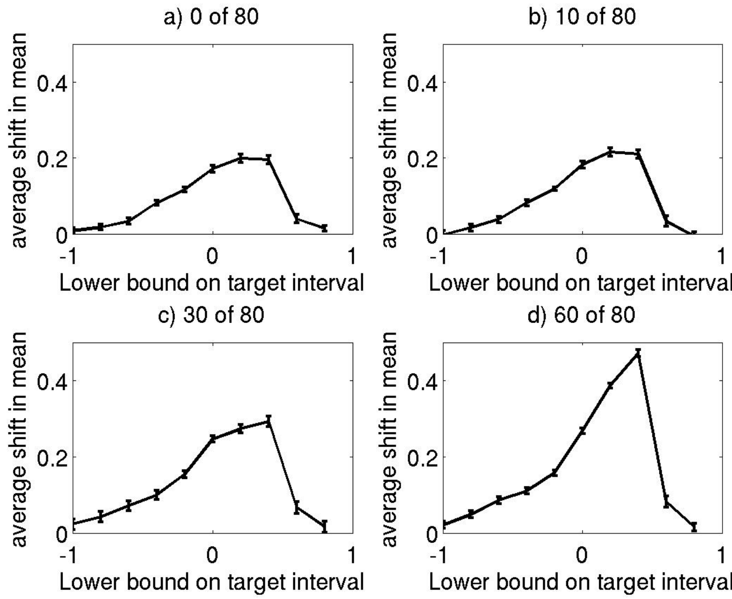

Figure 4.9: Average shift in mean as a function of I_{min}, averaged over ten sets of uniform random initial conditions and over the 16 parameters contained in Fig. 4.8. a) $m = 0$, b) $m = 10$, c) $m = 30$, d) $m = 60$. Increasing the proportion of open-minded individuals increases the average shift in mean, and on average makes the optima associated with the optimal target interval more distinct.

Observing Fig. 4.9 we see that on average the optimal target demographic occurs between 0 and 1 in opinion space. This implies that given our assumptions, it is on average optimal to target individuals who already moderately support one's product. Additionally we see that on average when the proportion of open-minded individuals is increased the distinctness of this optimal target interval is enhanced (Fig. 4.9).

4.4 Asymptotic analysis of reduced case

The result in Fig. 4.9 presents an interesting transition in that targeting individuals close to the advertiser's opinion seems to be optimal but only up to a certain point. We will at this point consider some analysis relating to this transition for the simple case when all individuals are open-minded and a single media source is located at one of the extremes. In this case the system defined by (3.1), (2.4), (3.3) and (4.5) reduces to the equation

$$dx_i/dt = \alpha_1 \sum_j a_{ij}(x_j - x_i) + \alpha_5(\mu_1 - x_i)\chi_{[I]}(x_i). \tag{4.7}$$

Theorem 8. *Let Y be a population of individuals whose opinion dynamics are governed by (4.7).*

Additionally let ϕ be a nonincreasing function with sufficient support as to allow Y to remain uniformly connected for all $t > 0$. Let $\bar{x}_f = \frac{1}{N}\sum_i x_i^*$. Then $x_i^* = \bar{x}_f$ for all i, that is consensus is achieved. Further $\chi_{[I]}(x_i^*)(\mu - x_i^*) = 0$ for all i.

Proof. We will approach this as a proof by contradiction.

Assume that $x_i^* \neq \bar{x}_f$ at equilibrium is true for at least one i. This implies that there must exist at least two distinct equilibrium values. For notational clarity we will denote the above distinct equilibrium values as $w_1, w_2...., w_f$, where $f \leq N$.

Next we let i be such that $x_i^* = w_d$ is closer to one of the extreme values (± 1) than all the other w_q and μ_1. Put more formally $x_i^* = w_d$ where either: $|1 - w_d| < |1 - w_q|$ for all $q \neq d$ and $|1 - w_d| < |1 - \mu_1|$ (case 1), or $|-1 - w_d| < |-1 - w_q|$ for all $q \neq d$ and $|-1 - w_d| < |-1 - \mu_1|$ (case 2). We will denote this w_d as existing in Q.

Since there is only a single media source, we can always find a w_d which exist in Q as long as consensus is not reached at μ_1. This follows by noting that each individual must be closer than the media to one extreme.

Considering $x_i^* \in w_d$ where $w_d \in Q$ we see that if w_d satisfies case 1;

$$\alpha_1 \sum_j a_{ij}(x_j^* - x_i^*) + \alpha_5(\mu_1 - x_i^*)\chi_{[I]}(x_i^*) < 0. \tag{4.8}$$

The above follows by noting that $x_i^* > x_j^*$ for all $j \neq i$ and that $x_i^* > \mu_1$. Additionally by the uniform connectivity assumption at least one of the a_{ij}^* where $i \neq j$ and $x_j^* \neq w_d$ must be nonzero. Hence, $\frac{dx_i}{dt} \neq 0$ which contradicts our assumption that the above is at equilibrium.

The case when w_d satisfies case 2 follows similarly by noting that:

$$\alpha_1 \sum_j a_{ij}(x_j^* - x_i^*) + \alpha_5(\mu_1 - x_i^*)\chi_{[I]}(x_i^*) > 0, \tag{4.9}$$

since $x_i^* < x_j^*$ for all $j \neq i$ and $x_i^* < \mu_1$ (as before at least one a_{ij} where $i \neq j$ must be nonzero by assumption).

Hence, there may only exist one distinct equilibrium value w_d. It trivially follows that:

$$w_d = \bar{x}_f.$$

Next considering the requirement for an equilibrium point, for each i we have

$$\sum_j a_{ij}(x_j^* - x_i^*) = -\alpha_5 \chi_{[I]}(x_i^*)(\mu_1 - x_i^*); \tag{4.10}$$

hence we see that if $x_j^* = x_i^*$ for all j that the right hand side of (4.10) must be zero. This implies that either the consensus occurs at the media's location, or that the media has no influence at equilibrium.

□

The result in Theorem 8 is quite interesting as in some ways it is an extension of [38, Theorem 4.1] to the situation where a media source term is present. This can also be viewed as equivalent to a shepherd term in an animal alignment model such as in [12].

Additionally applying Theorem 8 to (4.7) we see that a consensus equilibrium results when $x_i^* = \bar{x}_f$ for all i. Hence, it is worth noting explicitly that the equilibrium can be defined strictly in terms of $\bar{x}_f$. Further by Proposition 1 we know that with the exception of its media term, (4.7) preserves the population mean opinion. Hence, we can explicitly define the population mean at equilibrium (and as a result the value of x_i^* for all i) in terms of the initial mean and the shift caused by the media, as is shown below:

$$\bar{x}_f = \overline{x}_{initial} + 1/N\sum_{i=1}^{N}\int_0^\infty \alpha_5\chi_I(x_i(t))(\mu_1 - x_i(t))dt, \tag{4.11}$$

Additionally it is important to note that the process of finding the optimal target interval is equivalent to choosing I as to maximize the second term in (4.11).

Theorem 9. *Assume that $a_{ij} = 1/N$ ($a_s = 2$), and that after some finite time t_m we have $x_i(t) \notin I$ for $t > t_m$. Then the equilibrium solution of (4.7) asymptotically converges after time t_m to x^* as defined by (4.11).*

Proof. Taking $z = \frac{1}{N}$ demonstrates that Y is uniformly connected for all $t > t_m$ (this follows since $a_{ij} = 1/N$ for all $t > 0$ and i, j).

Hence the assumptions of Theorem 8 are satisfied. Therefore $x_i^* = \bar{x}_f$, where as before $\bar{x}_f = \frac{1}{N}\sum\limits_i x_i^*$.

Since $\bar{x}_f$ is the consensus equilibrium, the following function E is zero at the equilibrium point and positive definite for all other $x_i \in [-1, 1]$:

$$E = \sum_i (\bar{x}_f - x_i)^2. \tag{4.12}$$

Differentiating (4.12) we have

$$\dot{E} = -2\sum_{i=1}^{N}(\bar{x}_f - x_i)\frac{dx_i}{dt}.$$

Next noting our assumption that after a particular time t_m the media term will cease to contribute, and substituting in the definition for $\frac{dx_i}{dt}$ from (4.7) the above simplifies to:

$$\dot{E} = -2\alpha_1 \sum_{i=1}^{N} \left(\bar{x}_f - x_i\right)\left(\frac{\sum_{j=1}^{N} x_j}{N} - \frac{N}{N} x_i\right).$$

We can simplify the above by noting that in the absence of media and closed-minded influence, the mean opinion is preserved in time by Proposition 1. This gives:

$$\dot{E} = -2\alpha_1 \sum_{i=1}^{N} \left(\bar{x}_f - x_i\right)^2 \leq 0. \tag{4.13}$$

Observing (4.13) we see that it is negative definite . Hence by Lyapunov's theorem we conclude that the equilibrium $x_i^* = \bar{x}_f$ is globally stable under our assumptions.

□

Theorem 9 is an interesting result, since it suggests that if all individuals in the population pass through the advertisement's target interval in opinion space, the consensus equilibrium is globally attracting. This is interesting since it implies that the optimal target demographic found in Fig. 4.7 leads to an asymptotically attracting solution.

It is also worth noting that in the absence of media, Theorem 9 demonstrates that a consensus equilibrium is globally attracting when a_{ij} has full support.

The result of Theorem 9 also extends to the case when the support of a_{ij} is not sufficient to maintain connectivity and clusters form:

Corollary 1. *Assume that all individuals are in the set of open-minded individuals and that a_{ij} is given by the normalized step function discussed in Chapter 2. Further assume that after some time t_m, $x_i \neq I$ for all i. Additionally, assume that x_i^* consists of a series of G equilibrium clusters, where a_{ij}^* is nonzero when $t > t_m$ if and only if $x_i^* = x_j^*$. Then for $t > t_m$ the equilibrium solution of (4.7) asymptotically approaches the equilibria solution $x_i^* = \bar{x}_{fq}$, where $\bar{x}_{fq}$ denotes the mean opinion in the qth cluster, which is the one containing x_i^*.*

Proof. Since $\bar{x}_{fq}$ is the equilibrium solution for individuals reaching consensus in the qth cluster the following function E is zero at the equilibrium point and positive definite for all other $x_i \in [-1, 1]$:

$$E = \sum_{i=1}^{N} (\bar{x}_{fq} - x_i)^2. \tag{4.14}$$

Differentiating (4.14) and substituting in the definition for $\frac{dx_i}{dt}$ when $t > t_m$ given by (4.7), we have

$$\dot{E} = -2\alpha_1 \sum_{i=1}^{N} \left(\bar{x}_{fq} - x_i\right)\left(\sum_{j=1}^{n_q} 1/n_q(x_j - x_i)\right),$$

where n_q is equal to the number of individuals in the qth cluster. Note, $a_{ij} = 1/n_q$ due to the normalization of a_{ij} paired with the assumption that $a_{ij} = 0$ if and only if $x_i^* \neq x_j^*$. Expanding out, the above we have:

$$\dot{E} = -2\alpha_1 \sum_{i=1}^{N} \Big(\bar{x}_{fq} - x_i\Big)\Big(\frac{\sum_{j=1}^{n_q} x_j}{n_q} - \frac{n_q \cdot}{n_q} x_i\Big).$$

Simplifying the above by noting that if individuals only interact with individuals who will reside in their equilibrium cluster, each cluster group can be seen as a separate population. Hence, by Proposition 1 the mean is preserved in time within each cluster group when $t > t_m$. This gives:

$$\dot{E} = -2\alpha_1 \sum_{i=1}^{N} (\bar{x}_{fq} - x_i)^2. \tag{4.15}$$

Observing (4.15) we see that it is negative definite. Hence by Lyapunov's theorem we conclude that the equilibrium $\{x_i^*\}$ is globally attracting where $x_i^* = \bar{x}_{fq}$.

□

Considering Corollary 1 we see that the result of Theorem 9 generalizes to when there are multiple clusters. It is also interesting to note that Corollary 1 implies that when there is no media, cluster solutions are attracting in the absence of closed-minded individuals.

4.5 Summary of results

The above demonstrates three key results. The first is that under the assumptions of our model, when the majority of individuals are open-minded a global targeting strategy is optimal in terms of shifting individuals to an advertised opinion at a fixed cost. The second key result, is that given our assumptions, when individuals are open-minded there is no minimal cost of entry in terms of maximizing the relative return on advertisement investment. This potentially explains why some guerilla marketing campaigns are more successful than others [5]. The final key result, is that in terms of optimal target demographic, on average it is optimal given our model's assumptions to target individuals with a somewhat positive view of one's product. Additionally this trend becomes more pronounced the more open-minded individuals there are in the population. This explains why targeting previous customers can be so successful [41, 59].

One notable fault in the above is that our model does not appear to be numerically tractable to scales in which it would be useful economically (that is, it is $O(N^2)$). This is not inherently the case in that a random sampling algorithm such as one of those used in [2] could allow us to solve for macroscopic behavior of the above systems in $O(N)$ operations. Implementing these types of algorithms will be an area for future work.

Chapter 5

Realistic parameter estimation and experimental replication

The data set which we will analyze was provided by Levendusky in [30]. In his original paper Levendusky applies theories of motivated reasoning to elucidate why partisan media polarizes viewers and why some viewers are much more affected by said programing. Using a series of three experiments, Levendusky uses a least squares framework to test whether exposing particular types of individuals to particular types of media causes a significant shift in their opinions [30] .

The first experiment involved exposing a group of 720 individuals to either a like-minded media source, a cross-cutting media source (a media source opposite to their belief), or a neutral media source. The bias of each media source was determined by the relative partisan skew of its viewership, which as is noted in [29], is highly correlated with the bias of a news program. As an additional check, Levendusky also had experimental subjects identify the skew of the particular program which they were exposed to, and found good agreement with the viewership based approximation for the news source's bias. This subject-based identification will be used in our analysis to quantify the bias of each media source.

Our analysis will focus entirely on the first experiment in [30], as it is the only one which records both pre-and post-exposure opinions of individuals. The other two experiments only record post-exposure opinions, and are used to establish that it is the perceived credibility of a news source that determines its ability to shift opinion, that this opinion shift is due to a perceived strengthening of one's own argument and that these effects persist for days after exposure. The data associated with these experiments are not conducive to fitting to our model, though, so we will not analyze them any further and will instead focus on the data from the first experiment.

This first experiment concludes that like-minded media has a significant effect on an individual's post-exposure opinion, whereas cross-cutting media has little effect unless an individual has a strong opinion. Opinion strength is recorded as a separate variable from the opinion extremism. So theoretically, individuals could feel very strongly about being neutral on a particular issue. Having a strong opinion is defined by ranking the strength of one's opinion to be ≥ 7 on a 1 to 9 scale.

Unfortunately due to the least squares hypothesis test set up to his analysis, Levendusky is unable to determine the magnitude or individual-specific direction of these opinion-shifting forces [30]. This will be one area which we will investigate.

Questions raised by Levendusky results

In addition, the research in [30] raises several questions in regards to the effect of media exposure. The first question is whether or not there exists a critical threshold of strength of opinion which best predicts when individuals exhibit closed-minded behavior and resist or are repelled by the media. This question is of great importance as during political campaigns, campaign resources are finite, and hence the erroneous allocation of either political or financial capital in trying to directly sway the opinions of strongly-opinioned individuals could doom a campaign. Additionally, this issue extends beyond politics to the world of advertising where individuals could for example also have immutably strong opposition to consuming a product [48, 41]. Hence if one could classify certain groups of individuals as being closed-minded using simple survey questions on strength of opinion, as was done in [30], one could greatly reduce wasted advertising cost.

Another important question raised by [30] is whether or not closed-mindedness is actually the most useful predictive measure of an oppositional response to media. Extremism of opinion could be a better predictor, with the closed-mindedness result being merely being due to its correlation with extremism. We will demonstrate that this is not the case.

Finally we will investigate what is the best predictor of the occurrence of the hostile media effect. As was mentioned before, it has long been accepted that the hostile media effect is the tendency for extremist individuals to perceive media as being hostilely biased against them [60, 9, 13, 10, 33]. Below we will demonstrate that in the case of the Levendusky data which pertains to American politics, closed-mindedness is a better predictor than extremism of which individuals will experience the hostile media effect.

We remark that in addition to studying the data set provided by Levendusky [30], we also performed analysis on the data set used by Arceneaux and Johnson [3] (kindly provided by Kevin Arceneaux). Unfortunately this analysis was not successful, as the data was of the form of the Levendusky data in experiment 2 and 3 in that it only recorded post-exposure opinions; and thus no results of this analysis are shown in this thesis.

5.1 Qualitative support for our model found in the data of [30]

In this section we will examine correlations between various variables in the Levendusky data to qualitatively support the formulation of our model. Before we delve into this endeavor though, we will first define these variables. The first variable which we will consider, is the tendency for the ith individual to move away from the media. Specifically, the tendency of the ith individual to move away from the media will be defined as:

$$\text{Tendency to shift away from media}_i = (x_i(0) - x_i(9))(\mu_p - x_i(0)), \tag{5.1}$$

where $x_i(0)$ and $x_i(9)$ represent the initial opinion and the final opinion of the ith individual (the Levendusky experiment involves 9 minutes of media exposure) and as before μ_p represents the media's location. Here the terms initial and final refer to the beginning and end of the media exposure experiment. The formula given in (5.1) implicitly weights individuals closer to the media with less weight than would normally be justified based on the shift alone. This is beneficial since due to measurement noise, such individuals could in reality be on 'the other side' of the media. Hence their direction of movement (towards, or away from the media) is ambiguous. We will use (5.1) as a measure of closed-minded behavior as under our model's assumptions only closed-minded individuals will tend to move away from the media. The media location (μ_p) was determined by the average assigned bias given to the media by the individuals in the study.

The additional variables recorded in the Levendusky data, whose correlations we will consider are as follows:

1. Individuals' pre-media-exposure opinions (recorded on a five point scale from -1 to 1);
2. Individuals' post-media-exposure opinions (recorded on a five point scale from -1 to 1);
3. Individuals' stated degree of left or right partisanship (denoted political id). This denotes party identification as oppose to belief. It was recorded on a seven point scale;
4. The stated strength of individuals beliefs (this was was recorded independently of the actual opinion itself on a nine point scale);
5. The ability of an individual to detect the correct bias in their assigned media program (recorded on a binary, yes-no scale) ;
6. The tendency of an individual to move away from the media, where 'move away' is defined by (5.1);
7. Closed-mindedness. An individual was classified as closed-minded if their stated strength of opinion was greater than 4. The determination of this threshold will be discussed later in this chapter.

Now that we have defined our variables of interest, we will consider the correlations between several of them. Doing so we arrive at the following (table 5.1):

Considering Table 5.1, we note that the p-value represents the probability of the given correlation resulting purely by chance. The Pearson correlational coefficient between data $\{X, Y\}$ is defined as

Pearson's correlation coefficient	value	p-value
a) strength of opinion and tendency to move away from media	0.0411	0.2740
b) **closed-mindedness and tendency to move away from media**	0.0626	0.0958
c) **strength of opinion and extremism of initial opinion**	0.3005	2.9 x 10^{-16}
d) extremism of political id and tendency to move away from media	0.0444	0.2378
e) **extremism of initial opinion and tendency to move away from media**	-0.2172	5.1071 x 10^{-9}
f) **political id and initial opinion**	0.0987	0.0085
g) **closed-mindedness and ability to correctly note bias in media**	-0.0964	0.0102
h) extremism and ability to correctly note bias in media	0.0107	0.7764

Table 5.1: Table of correlation coefficients of various variables in the data of [30]. Statistically significant results are shown in bold

r, where:

$$r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{[n(\sum x^2) - (\sum x)^2][n\sum y^2 - (\sum y)^2]}}$$

and $x, y \in \{X, Y\}$.

Observing Table 5.1 (b), we see that there is a weak but significant correlation between the tendency to move away the media and individuals being closed-minded (as defined by having a strength of opinion greater than 4). This division level is supported by Fig. 5.1 below, and will be discussed in greater detail later in this chapter. This trend, though small, is quite impressive, as this correlation occurs despite closed-mindedness being highly correlated with having an extremist initial opinion (Table 5.1 (c)), since having an extreme initial opinion is highly correlated with the tendency to move towards the media (Table 5.1 (e)), it is quite impressive that the opposite correlation exists when an individual is closed-minded. This supports our model's design which incorporates the dynamic of closed-minded individuals moving away from the media.

It should also be noted that strength of opinion is not significantly correlated with the tendency to move away from the media. This implies that a binary classification of strength of opinion in terms of open-mindedness/closed-mindedness is perhaps under some circumstances more useful than a general classification of opinion strength.

The strange association between extremism and moving towards the media is very unintuitive (Table 5.1 (e)). Rather than being due to some psychological force, though, it is likely due to the ceiling effect in Levendusky's surveys (opinion extremism was ranked on a five-point scale; hence individuals who started at the extreme could only maintain or moderate their opinions). Since none of the media sources used in [30] were completely extremist, this led to the appearance of extremist individuals in general shifting towards the media.

Additionally, observing Table 5.1 we see that closed-mindedness is significantly correlated with the tendency to incorrectly identify the bias of the media source. This trend is not found for extremism of initial belief. This implies that the hostile media effect may in fact be better predicted by closed-mindedness rather than by extremism.

5.2 Parameter estimation

In this section we will try to determine realistic estimates for several parameters. Specifically, we will determine a reasonable approximations for $\hat{b}_i\alpha_4$, $b_{ip}\alpha_5$ and m/N as well as attempt to determine a value for X_c. First though we must define our approximations for $\hat{b}_i\alpha_4$ and $b_{ip}\alpha_5$.

Estimation of $\alpha_4\hat{b}_i$ and $\alpha_5 b_{ip}$

First we will begin by recalling (3.3). In the reduced case occurring in [30] when one individual is interacting with one media source, (3.3) becomes

$$\frac{dx_i}{dt} = F^P(x_i, \mu_p), \tag{5.2}$$

with

$$F^P = \begin{cases} \alpha_4\hat{b}_i(1 - x_i) & \text{if } x_i > X_c \text{ and } i\text{th individual} \in \hat{C}, \\ \alpha_4\hat{b}_i(-1 - x_i) & \text{if } x_i < -X_c \text{ and } i\text{th individual} \in \hat{C}, \\ \alpha_5 b_{ip}(\mu_p - x_i) & \text{otherwise.} \end{cases}$$

By making the assumption that the various media sources used in [30] have a constant effect on individuals independent of any individual's specific opinion we can take $\alpha_4\hat{b}_i$ and $\alpha_5 b_{ip}$ to be constants (note: each individual was only exposed to one media source but several different media sources where used, see [30] for details). As a result of this assumption, in the case that the ith individual is non-closed-minded, we will be able to rearrange (5.2) into a linear function of $|\mu_p - x_i|$. If the ith individual is closed-minded, this linear function will be in terms of $|\pm 1 - x_i|$. This will allow us to use a least squares fit to solve for $\alpha_4\hat{b}_i$ and $\alpha_5 b_{ip}$.

For reasons to be explained later in this section we will introduce two potential rearrangements of (5.2) to perform least squares fits on. The first will be denoted as the *simple linear approximation*, and will make the assumption that the exposure to media in the Levendusky experiment is brief enough that x_i is approximately constant over time (this is of course erroneous). Making this assumption, we see that in the case that the ith individual is non-closed-minded:

$$dx_i = \alpha_5 b_{ip}(\mu_p - x_i)dt \tag{5.3}$$

Here dx_i is equal to the shift in the ith individual's opinion over the time period dt where said individual is being exposed to the media source μ_p. This time period dt is 9 minutes for all individuals. Performing least squares on the data set {YX}, where $dx_i \in Y$ and $(\mu_p - x_i(0))dt \in X$, gives a linear plot whose slope is approximately $\alpha_5 b_{ip}$. Additionally it should be noted that our fit assumes that the intercept is zero. A similar procedure can also be used to determine $\alpha_4 \hat{b}_i$. This method is far from ideal, and will not be the principal method we will use in determining an approximation for our interaction coefficients. It is useful, though, in contrasting the relative strength of the media influence on various groups of individuals, and will be used as a measure of this in several figures.

A superior method can be obtained as follows, where again $x_i(0)$ and $x_i(9)$ denote the opinion of the ith individual at $t = 0$ and $t = 9$ respectively. Time here is measured in minutes. First in the case of non-closed-minded individuals, we have

$$dx_i/dt = k(\mu_p - x_i).$$

Here k is a constant denoting the product $\alpha_5 b_{ip}$. Solving the above ODE gives

$$\mid \mu_p - x_i \mid = Ce^{-kt}.$$

Next, noting that the experiment starts at $t = 0$ and ends at $t = 9$, we have:

$$\mid \mu_p - x_i(0) \mid e^{-9k} = \mid \mu_p - x_i(9) \mid . \tag{5.4}$$

Performing a least squares fit where the x-variable is $|\mu_p - x_i(0)|$ and the y-variable is $|\mu_p - x_i(9)|$ gives a slope from which we can calculate

$$\alpha_5 b_{ip} = k = -\ln(slope)/9. \tag{5.5}$$

Note in the above we are assuming that the x-intercept of our fit is zero. An analogous result for closed-minded individuals follows from performing least squares to find the slope of the following function:

$$\mid (sgn(x_i(0) - X_c) - x_i(0)) \mid e^{-9\hat{k}} = \mid sgn(x_i(9) - X_c) - x_i(9) \mid . \tag{5.6}$$

where $\hat{k} = \alpha_4 \hat{b}_i$.

The above ((5.6) and (5.4)) will be referred to as the *improved linear approximation*. We will note here that the right hand side of (5.6) was treated as $\mid sgn(x_i(0) - X_c) - x_i(9) \mid$ in our actual linear fits. This was done to account for the cases in the data where an individual shifts away from the media to the opposite extreme. This type of massive shift, which is in the direction opposite of that predicted by our model should not be interpreted as supporting large $\hat{k}$. Hence, our assumed least squares set up contains $\mid sgn(x_i(0) - X_c) - x_i(9) \mid$ on the right hand side. Further all of our linear fits assumed that all news programs used in [30] had the same effect on the subjects, aside from variations due to differences in partisan opinion. This is a very reasonable assumption, as the biased news programs used in [30] were chosen specifically such that each program on average was considered equally entertaining and each host was viewed on average as being equally likeable. Additionally we assume that X_c is sufficiently small that all individuals with a closed-minded disposition behave closed-mindedly.

In terms of functioning as a measure of the effectiveness of the media at shifting individuals, (5.5) and (5.6) are unfortunately not ideal, as the error associated with k is $\frac{error_{slope}}{9\times e^{-9k}}$ (found using the normal error propagation method). This increases relative to increasing values of k. This renders the above least accurate when the media has the greatest impact. Since this is the area of greatest interest, we will use the simple linear approximation (5.3) in measuring the strength of the media's influence on a group of individuals (the larger the fitted value of $\alpha_5 b_{ip}$ the more a group of individuals is influenced by the media), and the improved linear approximation ((5.6) and (5.4)) in determining the approximate values for $\alpha_5 b_{ip}$ and $\alpha_4 \hat{b}_i$. An alternative measure of the media's impact on a group of individuals would be the average shift in opinion between the pre-and post-exposure surveys. This is less than ideal, though, since it is more susceptible to the aforementioned ceiling effect in the Levendusky data.

Estimation of m/N

To estimate the proportion of open-minded individuals in the Levendusky data we must develop a classifier by which to label an individual as open-minded or non-open-minded. Since an individuals stated strength of opinion denotes on some level how willing they are to change their opinion we will use this as our variable which determines an individuals classification. Hence, to determine the number of open-minded individuals we simply must determine the critical value of strength of opinion which distinguishes an open-minded individual from a non-open-minded individual.

By making the assumptions that all non-open-minded individuals behave closed-mindedly (X_c is small) and that open-minded individuals are more willing to change their opinions, we can see that the critical value of strength of opinion which distinguishes open-minded individuals from closed-minded individuals will be the value which best distinguishes the degree which individuals tend to shift their opinions upon media exposure. Estimating this degree of shift using the simple linear approximation for $\alpha_5 b_{ip}$ associated with individuals whose strength of opinion resides in a particular interval gives Fig. 5.1. Here the y-variable plotted is the associated simple linear approximation (computed with (5.3)) and the error bars are calculated using the usual method for determining the standard deviation of a linear fit.

Observing Fig. 5.1 we see that the simple linear approximation predictions for individuals' expected shifts in response to media exposure increase substantially when the linear fit is performed on data corresponding to individuals with a stated strength of opinion less than 4. Further, one should note that the plot represents the simple linear approximation to data as a function of the

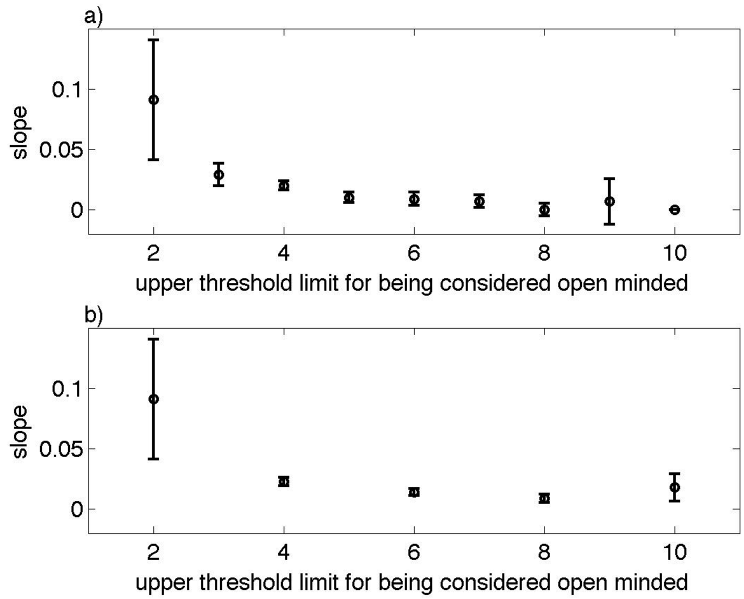

Figure 5.1: Plot of the slope of the simple linear approximation for $\alpha_5 b_{ip}$ (computed using (5.3)), showing an estimate of the degree of shift in response to media exposure as a function of the upper bound on the strength of opinion of individuals used in said linear fit, using data from [30]. Here closed-mindedness is categorized by the strength of belief stated in pre-media exposure survey (as opposed to the actual belief itself). Simple linear approximations were performed on data corresponding to individuals whose stated strength of opinion values resided in the intervals a)$[x-1, x]$ b) $[x-2, x]$, where the x-variable (x) represents the upper bound on strength of opinion used in each fit.

stated strength of opinion of individuals in said data. Each point on the x-axis corresponds to data from a different interval of strength of opinion, where plot (a) corresponds to an interval of strength of opinion of width 1 and plot (b) corresponds to an interval of strength of opinion of width 2. Interpreting Fig. 5.1 in this light, we see that a binary classification of an individual as either open- or closed-minded is somewhat simplistic and would be analogous to fitting Fig. 5.1 to a step function (as with this approximation there would only be two possible slopes). Nevertheless we see that there is a significant difference between individuals with an opinion strength greater than 4 and those with an opinion strength less than 4. Using this as our division point between open-minded and closed-minded individuals, we see that the data set in [30] has far more closed-minded individuals (see Table 5.2).

Once we can divide our data into open-minded/closed-minded individuals we can now apply (5.6) and (5.4) to estimate $\alpha_4 \hat{b}_i$ and $\alpha_5 b_{ip}$. As is shown in Fig. 5.2.

In considering Fig. 5.2, we must note that the above represents the improved linear approximation

proportion of open-minded individuals	proportion of closed-minded individuals
0.35	0.65

Table 5.2: Table of the fraction of open-minded and closed-minded individuals in the data of [30]. Determined using a threshold of strength of opinion of 4.

to $\alpha_5 b_{ip}$ and $\alpha_4 \hat{b}_i$ plotted as to overlap with a weighted scatter plot of the data in [30], where bigger discs represent a larger number of individuals. Since the slope is $e^{-9\alpha_5 b_{ip}}$ in the case of Figure 5.2 (a) and $e^{-9\alpha_4 \hat{b}_i}$ in the case of Figure 5.2 (b) we see that relatively more mild slopes correspond with greater relative αb values. Hence we see that open-minded individuals are more affected by the media as the slope in Figure 5.2 (a) is more mild (as was assumed by our least squares fit).

Next for many applications it could be useful to estimate $\alpha_5 b_{ip}$ and $\alpha_4 \hat{b}_i$ in terms of time as opposed to time viewed. An example of such a situation would be if a politician wished to estimate the impact a speech would have on individuals without knowing the precise amount of time which each individual was exposed to the speech. Specifically, in such a situation the politician would wish to conduct simulations where the constants $\alpha_5 b_{ip}$ and $\alpha_4 \hat{b}_i$ incorporated both the experimentally determined rate at which individuals change their opinions as determined by an experiment like the one in [30] as well as the average rate at which individual are exposed to the media stimuli (since the specific amount for each individual is unknown). To estimate this $\alpha_5 b_{ip}$ and $\alpha_4 \hat{b}_i$ we must take into account the average time which an individual is exposed to news media each day relative to the amount of time subjects in the Levendusky experiment were exposed to news media. Specifically, we will note that according to [29] the average American consumes 70 min of news per day. By comparison, subjects in [30] only watched their respective news clips for 9 min [30]. Using this we can estimate the values of $\alpha_4 \hat{b}_i$ and $\alpha_5 b_{ip}$ for real world applications (denoted below as 'real world αb') as follows:

$$\text{real world } \alpha b = \frac{kT_r}{T_s}, \tag{5.7}$$

where k in (5.7) represents the fitted value for either $\alpha_5 b_{ip}$ or $\alpha_4 \hat{b}_i$ found previously, T_s represents the time which a media source was watched in the Levendusky study and T_r represents the average time per day which individuals spend consuming news. This collectively allows for $\alpha_5 b_{ip}$ and $\alpha_4 \hat{b}_i$ to be described strictly in terms of time as opposed to time spent watching news.

Note the above assumes that viewers only consume approximately one type of partisan news. This is a somewhat reasonable assumption since partisan news programs have largely partisan viewers, (that is, if individuals often sought out contrasting partisan programs the viewership would not be so skewed) [29].

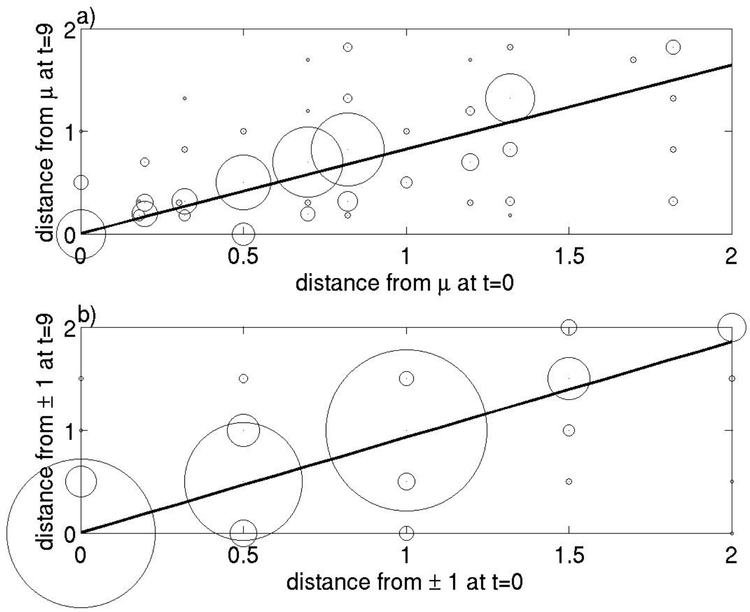

Figure 5.2: Improved linear approximation to $\alpha_5 b_{ip}$ and $\alpha_4 \hat{b}_i$ in (5.2), where both are presumed to be constant in time. The coefficients were extracted by performing a least squares fit to determine the average slopes of the following equations: a) $|\mu_p - x_i(9)| = e^{-9\alpha_5 b_{ip}}|(\mu_p - x_i(0))|$ where it was found that $\alpha_5 b_{ip} = 0.0218 \pm 0.004$; b) $|\pm 1 - x_i(9)| = e^{-9\alpha_4 \hat{b}_i}|\pm 1 - x_i(0)|$, where it was found that $\alpha_4 \hat{b}_i = 0.0085 \pm 0.002$. The relative size of the circles reflects the relative number of data points at each location.

Attempted estimation of X_c

In Fig. 5.3 we construct an estimate for the value of X_c. We note that in our model, individuals with closed-minded dispositions switch from moving towards the media to moving away from said media after passing a certain extremist threshold X_c. Hence we conclude that at X_c individuals should on average maintain a fixed opinion when exposed to media. Therefore, by using a least squares approximation to a closed-minded individual's tendency to shift away from the media, as defined in (5.1) we can take X_c to be the x-intercept of said fit. However, as shown in Fig. 5.3 there is significant scatter in the data and we are not able to obtain a statistically significant value for X_c.

5.3 Replication of a portion of the data in [30]

In this section we will seek to replicate a portion of the Levendusky data using a fitted version of our model. We will attempt to replicate the Levendusky data by running various simulations with fitted parameters on initial data taken from [30], where the maximum allowed strength of an individual's opinion is varied.

constant	Value	1.28 σ
$\alpha_5 b_{ip}$	0.022 (mins viewed)$^{-1}$	0.005 (min viewed)$^{-1}$
$\alpha_4 \hat{b}_i$	0.009 (mins viewed)$^{-1}$	0.002 (min viewed)$^{-1}$

Table 5.3: Realistic parameter values found through improved linear fit on Levendusky data where open-minded and closed-minded individuals are categorized based on Fig. 5.1. Here, σ represents the estimated standard deviation in the αb term and $(-1.28\sigma, 1.28\sigma)$ is the 90 % confidence interval for the αb term.

constant	Value	1.28 σ
$\alpha_5 b_{ip}$	0.1284 (days)$^{-1}$	0.01 (days) $^{-1}$
$\alpha_4 \hat{b}_i$	0.0473 (days) $^{-1}$	0.005 (days)$^{-1}$

Table 5.4: Realistic parameter values found through improved linear fit on Levendusky data where open-minded and closed-minded individuals are categorized based on Fig. 5.1. Fitted coefficients converted for real world applications as to have units strictly in terms of time. Conversion performed using (5.7). Here, σ represents the estimated standard deviation in the αb term and $(-1.28\sigma, 1.28\sigma)$ is the 90 % confidence interval for the αb term.

Since our model is deterministic we will compare the results of these simulations to the Levendusky data in the average sense, by contrasting for a given maximum allowed strength of opinion the simple linear approximation for $\alpha_5 b_{ip}$ as found using our simulated data, to the true simple linear approximation as found using the Levendusky data. We will additionally contrasts these results to the simple linear approximations generated when all individuals are assumed to be open-minded. The results of these simulations are shown in Fig. 5.4 where the maximum allowed strength of opinion of individuals is taken to be the x-variable.

Observing Fig. 5.4 we see that our model can be fitted fairly well to an averaging of the Levendusky data. It performs particularly well when contrasted to simulations using a model which does not incorporate ingroup-outgroup dynamics such as the one in [38] (shown in Figure 5.4 (c)). This result supports our theoretical framework and demonstrates that a simple binary classification of individuals as either closed-minded or open-minded is useful in predicting the average behavior of populations.

5.4 Summary

In analyzing the above discussed data set in the context of our model we were able to verify the effectiveness of our model by replicating some of the Levendusky data through simulation (Fig. 5.4). Additionally we were able to get estimates for the proportion of open-minded individuals in the opinion regime of American politics as well as the magnitude of the influence of media on political belief (Tables 5.2 and 5.3). The first result was particularly intriguing since it potentially explains why polarization is so rampant in American politics as the majority of individuals appear to be closed-

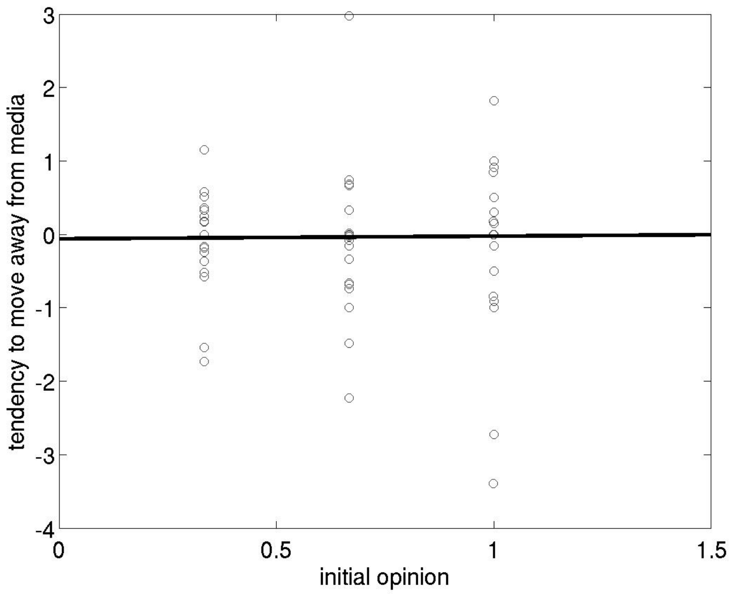

Figure 5.3: Plot of tendency to move away from media as a function of extremism of belief. Here, the tendency to move away from the media is defined by (5.1). X_c is given by the horizontal intercept of the above plot, which is $X_c = 2 \pm 4$. This result is not statistically significant.

minded [30]. Lastly we also found that with respect to the Levendusky data, closed-mindedness is a better predictor of the hostile media effect than extremism (Table 5.1). This is quite surprising since most of the literature attributes the hostile media effect to a perception error on the part of extremists [13, 10, 33, 60]. Our analysis indicates that how strongly an individual feels about their opinion is a much better predictor of their tendency to misidentify a media source's bias than the extremism of their belief on a left-right opinion scale (Table 5.1).

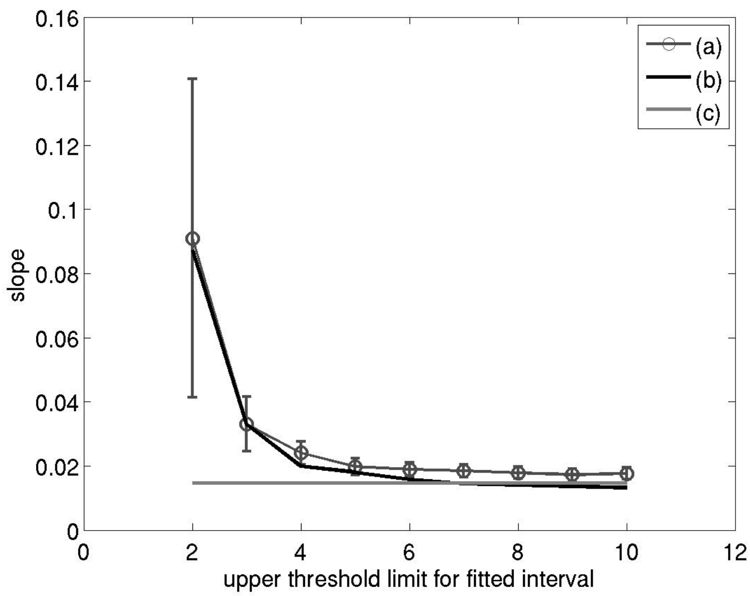

Figure 5.4: Plot of the slope of the simple linear approximation predicting the degree of shift in response to media exposure as a function of the upper bound on the closed-mindedness of individuals used in said linear fit. The data fit was performed on individuals with a strength of opinion within the interval $[0, x]$. a) Simple linear approximation to the Levendusky post-media-exposure data. b) Simple linear approximation to the data produced by simulation with (5.2), using the parameters given in Table 5.3 and initial data given in [30]. We compute $X_c = 0.49$ (found through bisection). The cut-off in strength of opinion for closed-minded designation was taken to be 2. c) Simple linear approximation to the data simulated using (5.2), where all individuals are treated as open-minded both in fitting and in simulation.

Chapter 6

Conclusion

6.1 Methodology

We have proposed an ODE model for opinion dynamics which builds off the earlier work by Motsch and Tadmor [38]. Opinions are represented continuously on the interval $[-1, 1]$. By introducing two classes of individuals (open-minded individuals and individuals with closed-minded potential), we incorporate the feature of ingroup-outgroup dynamics noted in various social science works [26, 15, 30]. Through implementing this feature we are able to capture the dynamics previously modeled discretely in [22] using a continuous representation for opinions.

Using this model as our basic framework for investigation, we have explored the general questions of how consensus can be reached in the presence of closed-minded individuals, as well as the effect of ingroup-outgroup dynamics on the effectiveness of various advertising strategies.

Additionally we have incorporated data into our investigation by replicating through simulation some of Levendusky's data found in [30]. Further, we have used said data to generate realistic parameter estimations for the opinion regime of American politics.

6.2 Results

The tendency for extremism to result in groups is an interesting psychological phenomenon which has garnered a lot of research attention [15, 50, 22]. One theoretical explanation has been given by Galam in [22] using a discrete representation for opinions. We demonstrate that this phenomenon can also occur when opinions are allowed to exist continuously through the interaction between two different classes of individuals (open-minded individuals and individuals with closed-minded potential) see Fig. 2.5. We additionally find analytical stability results lending some support to this theory.

Additionally we investigated the issue of how groups can reach consensuses when some individuals behave closed-mindedly. This is an intriguing question since many conflicts such as several of those occurring on the periphery of Europe have reached peaceful agreements (a consensus of sorts) in the presence of closed-minded individuals [11]. One potential explanation given in the social sciences for this occurrence is that open-minded individuals create a social norm of open-mindedness which stimulates open-minded interaction amongst otherwise closed-minded individuals [36, 56, 57]. We demonstrate theoretically that under the assumptions of our model, this is a plausible explanation for how consensus can be reached in the presence of closed-minded individuals (Chapters 2 and 3). We further show that this is even true in the presence of a media source and the hostile media effect (Chapter 3).

An additional area of investigation was the commercial application of advertising. Specifically, we endeavored to quantify the effect of ingroup-out-group dynamics (in this case as represented by degree of brand loyalty) on the potential success of various advertising strategies. We found three main results, described in Chapter 4. The first result was that under our assumptions, it is on average better to implement a global targeting strategy when costs are fixed and there is no competition. Our second key result is that in the presence of low levels of ingroup-outgroup dynamics there is, under our assumptions, no cost barrier in terms of maximizing relative return on advertising investment. This is not necessarily the case when high levels of ingroup-outgroup dynamics are present. This result potentially explains why certain guerrilla marketing campaigns are more successful than others [5]. Thirdly we considered what is the optimal target demographic for a given advertising campaign, where optimality is represented by the target interval in opinion space where advertising influence has the greatest impact. We found that under the circumstances governed by our model, it is optimal to target individuals with a somewhat favorable view of one's product prior to targeting. This potentially explains why advertisers target previous customers so frequently [59].

Our last area of investigation was an analysis of the data set provided by Levendusky [30]. Through analyzing this data set in the context of our model we were able to verify the effectiveness of our model through replicating some of data through simulation (Chapter 5). Additionally we were able to get statistically significant estimates for the proportion of open-minded individuals in the opinion regime of American politics as well as the magnitude of the influence of media on politically belief. The first result was particularly intriguing since it potentially explains why polarization is so rampant in American politics, as the majority of individuals appear to be closed-minded [30]. Lastly we also found that with respect to the Levendusky data, closed-mindedness is a better predictor of the hostile media effect than extremism. This is quite surprising, since most the literature attributes the hostile media effect to perception error on the part of extremists [13, 10, 33, 60]. Our analysis indicates that how strongly an individual feels about their opinion is a much better predictor of their tendency to misidentify a media source's bias than the extremism of their belief on a left-right opinion scale (Chapter 5).

6.3 Future work

One shortfall of our work is that our simulations only consider relatively few individuals. This is because our simulations are computationally $O(N^2)$, and hence constrained numerically. This hurdle can possibly be overcome through implementing random sampling algorithms such as those discussed in [2] which are $O(N)$. This would be an intriguing future area of research, since it would allow us to extend our results to realistic values of N.

Another potential area for further research would be to incorporate self-thought dynamics into our simulations in the form of random perturbations over time. This would be an interesting area of further research, since currently our model assumes that individuals behave 'sheep-like' and do not think for themselves. While this is not a completely unrealistic assumption, it still prevents the occurrence of some interesting phenomena [16].

Lastly it would be quite interesting to further explore the effect of individual influence on the media. This is quite an intriguing concept, since it has been noted that the rise of social media is ushering in an era where individuals have much greater influence on media [47]. We briefly consider this factor in our analysis (Fig. 3.10) but do not investigate it as much as is merited by its current and future importance.

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Appendix A

List of parameters and functions

A.1 List of constants and functions Chapter 2

Constant	Symbol
The ith individual's opinion	x_i
Number of individuals	N
x critical. The critical threshold for extremism above which individuals whose indices belong to $\hat{C}$ becomes closed-minded	X_c
Open-minded interaction function	ϕ
support of ϕ	a_s
Set of indices of individuals predisposed to closed-mindedness	$\hat{C}$
Set of indices of individuals predisposed to open-mindedness	O
Number of individuals whose indices belong to O	m
Open-minded individual interaction coefficient	a_{ij}
Closed-minded individual interaction coefficient	$\hat{a}_{ij}$
Closed-minded individual with open-minded individual interaction coefficient	$\hat{a}_{ij}$
Closed-minded interaction function	$\hat{\phi}$
Open-minded individual with open-minded individual scalar. Determines relative strength of interaction	α_1
Closed-minded individual with individual with closed-minded disposition scalar	α_2
Closed-minded individual with open-minded individual scalar. Determines relative strength of interaction	α_3

Table A.1: Constants and functions associated with individual opinion dynamics (Chapter 2).

A.2 List of constants and functions Chapter 3

Constant	Symbol
Support of b	b_s
Closed-minded individual media interaction coefficient	$\hat{b}_i$
Open-minded individual media interaction coefficient	b_{ip}
Hostile media shift factor	H
Closed-minded individual media scalar	$\alpha_{4,p}$
Open-minded individual media scalar	$\alpha_{5,p}$
Media location in opinion space	μ_p
Media shift function	λ

Table A.2: Constants and functions associated with media opinion dynamics (Chapter 3).

A.3 List of constants and functions Chapter 4

Constant	Symbol
Proportion of individuals excluding those belonging to $\hat{C}$ where $\|x_i\| > X_c$ who are also targeted by the media	$Prop_{nc}$
Proportion of individuals belong to $\hat{C}$ where $\|x_i\| > X_c$ targeted by media	$Prop_c$
The set of individuals targeted by an advertising source	A
The interval in opinion space over which advertising has influence	I

Table A.3: Constants and functions associated with advertising (Chapter 4).