# Combinatorial Hopf Algebras On Generating Trees And Certain Generating Graphs

by

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# Abstract

Hopf algebras capture how combinatorial objects can be decomposed into their subparts in different ways. Generating trees and generating graphs provide one structured way to understand many combinatorial classes. Furthermore, Hochschild 1-cocycle maps of renormalization Hopf algebras play an important role in quantum field theories but are not well known in combinatorics. In the generalised atmospheric method for sampling self-avoiding polygons, there is a weight function which deals with overcounting and hints at a connection with the 1-cocycle maps. Both of these combinatorial objects can be represented by generating graphs. As a first step towards understanding this connection, we provide two ways to construct Hopf algebras on generating trees through a normalizing map $\tilde{\varphi}$. One is concatenation and deshuffle type and the other is shuffle and deconcatenation type. We also construct an incidence Hopf algebra on certain generating graphs and construct a Hopf algebra on self-avoiding polygons.

*“A mathematician is a machine for turning coffee into theorems”*

— *Alfré Rényi*

*“A comathematician is a device for turning cotheorems into ffee”*

— *Unknown*

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# Contents

# List of Figures

# Chapter 1

# Introduction and Hopf Algebra Preliminaries

## 1.1 Introduction

Hopf algebras emerged in combinatorics because the product and the coproduct can capture the actions of composing and decomposing combinatorial objects respectively. Another reason is that combinatorial objects (permutations, trees, graphs, posets, tableaux and so on) have natural gradings which makes it easier to introduce Hopf algebras on them and many interesting invariants can be expressed as Hopf morphisms ([10]).

Feynman graphs, an important class in quantum field theory, have a Hopf algebra structure [5, 6]. In these renormalization Hopf algebras of combinatorial classes of Feynman graphs (see Section 5.1), there is a family of maps $B_+^\gamma$ which represent insertion into the primitive graph $\gamma$. If this is done naively, there is over counting, and so there are multiplicative factors which deal with the redundancy. As a result, the sum of $B_+^\gamma$ over an appropriate set of $\gamma$ gives a Hochschild 1-cocycle map ([12]). In the GARM method for sampling self-avoiding polygons (see Section 5.2), there is a weight function $W$ that assigns to each sampling sequence a weight. The mean weight of sequences ending at objects with size $n$ gives the total number of objects with size $n$ ([7]). In both cases, a multiplicative factor is used to deal with overcounting as opposed to the more usual additive process like inclusion-exclusion. We think these multiplicative factors may relate to Hochschild 1-cocycle maps. Our original goal for this project was to make this connection precise. Towards that we wanted a common Hopf-algebraic language for both cases. Notice both classes of objects are recursively generated but with redundancy. So generating graphs were a good first place to look. As a simpler case of generating graphs, generating trees are also representations of combinatorial classes that explain local recursive structure. So we started by constructing Hopf algebras on generating trees. We didn't achieve our original goal, but we obtained interesting Hopf algebra structures from generating trees and certain generating graphs. We view this as the first step towards our original goal.

## 1.2 Thesis Organization

In the rest of Chapter 1, we first present the definitions and some propositions for bialgebras and Hopf algebras. Corollary 1.25 is particularly relevant for us because it states that the antipode always exists for a connected and graded bialgebra. Then we show some common examples of combinatorial Hopf algebras which will help in understanding later chapters.

In Chapter 2, we make use of the linearity of the paths from the generating trees of combinatorial classes to simulate word-behavior and construct two Hopf algebras on those classes. We first describe a family of normalizing maps $\tilde{\varphi}_\bullet$ in Section 2.1. The map $\tilde{\varphi}$ tells us how to map paths in the tree up to the root. Since only paths starting at the root in a generating tree correspond to objects in the combinatorial class, $\tilde{\varphi}$ is very important for converting other parts of the tree back into objects. Then we construct a concatenation-deshuffle type Hopf algebra (Theorem 2.20) and a shuffle-deconcatenation Hopf algebra (Theorem 2.37) on the vector space $W$ spanned by the vertices of generating trees. A diagram of this chapter is in Figure 1.1. Throughout the construction, we will use permutations as examples. To give the readers a preview, here we present examples of the concatenation product of the permutations (12) and (21), and the deshuffle coproduct of permutation (231).

**Example 1.1.** Take permutations $u = (12)$ and $v = (21)$, then

$$\psi_c(12, 21) = (1243)$$

as shown in Figure 1.2. This is the result of grafting the path from the root $r$ to (21) to the vertex (12) following analogous steps. Take $w = (231)$, then

$$\begin{aligned}\varphi_{ds}(231) \quad &= \quad r \otimes (231) + (1) \otimes (12) + (1) \otimes (21) + (1) \otimes (21) \\ &\quad +(21) \otimes (1) + (12) \otimes (1) + (21) \otimes (1) + (231) \otimes r\end{aligned}$$

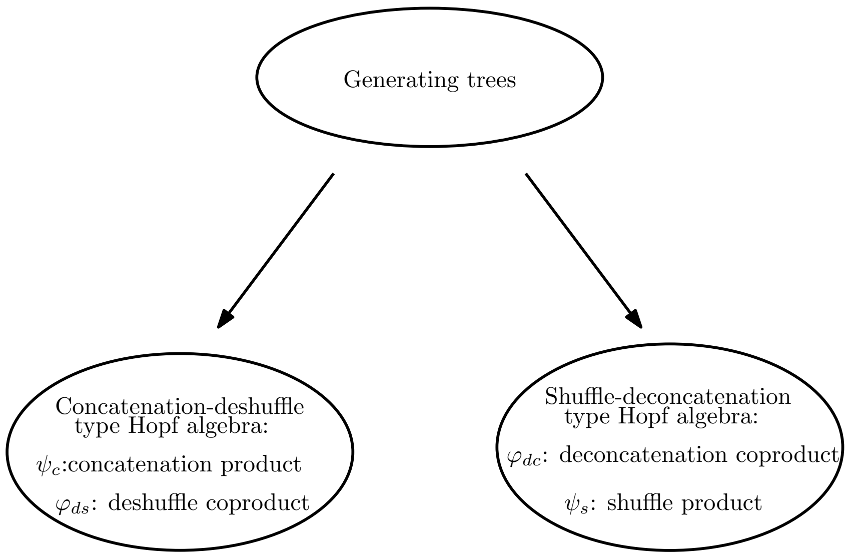
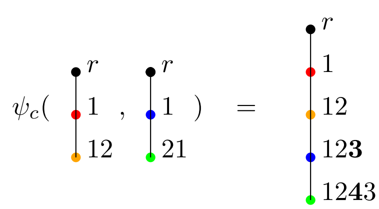



Figure 1.1: A sketch of Chapter 2

as shown in Figure 1.3. This is the sum of breaking up the paths from $r$ to (231) into two parts. The symmetry in the figure is the sum of the first four summands with left terms and right terms flipped.

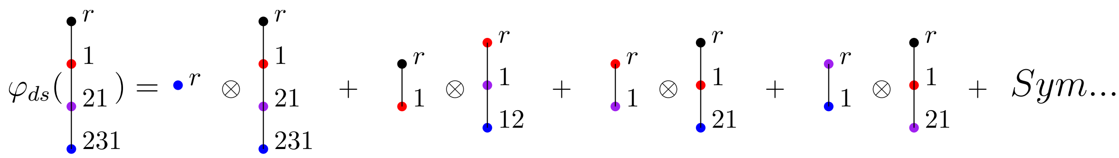



$$\psi_c(\ 12\ ,\ 21\ ) = (1243)$$

Figure 1.2: Path representation of concatenating permutations (12) and (21)

Finally, we show an application to the generating tree of set partitions in Section 2.4 and an application to the generating tree of open partition diagrams in Section 2.5.

In Chapter 3, we define the set of maximum sublattices of a finite lattice $P$ in Definition 3.8 and analyze the structure of a particular type of generating graph $G$ with properties

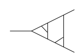


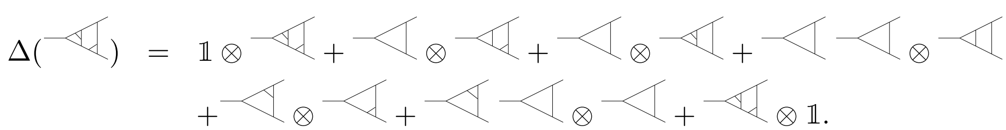

$$\varphi_{ds}(231) = \quad r \otimes (231) \quad + \quad (1) \otimes (12) \quad + \quad (1) \otimes (21) \quad + \quad (1) \otimes (21)$$

$$+ \quad (231) \otimes r \quad + \quad (12) \otimes (1) \quad + \quad (21) \otimes (1) \quad + \quad (21) \otimes (1)$$

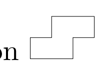
Figure 1.3: Path representation of deshuffling permutation (231)

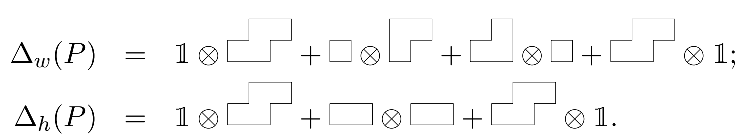
given in Property 3.10. Then in Property 3.16 we describe properties of a map $\phi$ which guarantee we can construct the coproduct $\Delta$. Finally we get to the main result in Theorem 3.22, namely that the polynomials of objects in $G$ with the coproduct $\Delta$, trivial unit and counit results in an incidence Hopf algebra. Throughout the construction, we will use 3-point Feynman graphs with only vertex insertions in the scalar field theory $\phi^3$ as examples. In this case, the Hopf algebra coincides with the usual renormalization Hopf algebra for these graphs. Example 1.2 gives a taste of what the coproduct $\Delta$ looks like.

**Example 1.2.** Apply $\Delta$ to the graph  to get



In Chapter 4, we look at the algebra $U$ of polynomials of self-avoiding polygons. We first present a generating graph of self-avoiding polygons introduced in [7]. Then we construct a width-coproduct $\Delta_w$ and a height-coproduct $\Delta_h$. We prove with $\Delta_w$ and the trivial counit $U$ is a Hopf algebra graded by the widths (Theorem 4.9) and with $\Delta_h$ and the trivial counit $U$ is a Hopf algebra graded by the heights (Theorem 4.11). Example 1.3 gives an example of $\Delta_w$ and $\Delta_h$.

**Example 1.3.** Let $P$ be the self avoiding polygon . Then



Chapter 5 will conclude this thesis with some questions related to our constructions and our original goal.

## 1.3 Bialgebras and Hopf Algebras

Hopf algebras, named after Heinz Hopf, arose in algebraic topology relating to the homology and cohomology of connected Lie groups in the 1940s ([1]). Since the late 1960s, Hopf algebras have been studied from an algebraic point of view starting with the research on the representative rings of Lie groups by Hochschild and Mostow ([2],[3]). By the end of the 1980s, the connection with quantum mechanics (in the form of quantum groups) promoted the research in this area ([4]). Hopf algebras have been studied in their own right as well as for their applications in physics and many fields of mathematics. One important application is in combinatorics, because many combinatorial objects carry natural Hopf algebraic structures and some structures also encode useful combinatorial information.

In this section, we present the definitions of bialgebras and Hopf algebras along with some classic propositions. We follow the text [10] for this presentation. We use the vector space of words as an example when we go through. We assume readers have basic background of linear algebra, groups, rings and tensor products ([8, Chapter XVI]). We assume all algebras in this thesis have a multiplicative identity.

### 1.3.1 Bialgebras

Let $\mathbb{K}$ be a field. We will use $\mathbb{K}$ as the base field for all vector spaces throughout this chapter unless otherwise stated.

Let $\Omega$ be a finite alphabet, then any word $a_1a_2\cdots a_n$ can be viewed as the tensor product $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ where $a_i \in \Omega$ for $i = 1, 2, \ldots, n$. Let $\mathbb{1}$ be the empty word of length 0 and let $W_n$ be the vector space spanned by words of length $n$. Then $W_n$ can be viewed as $\Omega^{\otimes n}$. Thus, $W = \bigoplus_{n=0}^{\infty} \Omega^{\otimes n}$ is the vector space spanned by all words.

**Definition 1.4.** A vector space $A$ over $\mathbb{K}$ is an **algebra** if it is also a ring (with the identity $\mathbb{1}$) with a compatible multiplication, that is, $\lambda(ab) = (\lambda a)b = a(\lambda b)$, for all $\lambda \in \mathbb{K}$ and $a, b \in A$.

In fact, this is equivalent to viewing algebra $A$ (also denoted by $(A, \cdot, u)$) as a $\mathbb{K}$-vector space together with two linear maps. One map is the **unit** $u : \mathbb{K} \to A$ and the other one is the **product** $\cdot : A \otimes A \to A$ such that the following diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[dr, "\cdot"'] & A \otimes A \otimes A \arrow[l, "\cdot\otimes id"'] \arrow[r, "id\otimes\cdot"] & A \otimes A \arrow[dl, "\cdot"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes A \arrow[d, "u\otimes id"'] & A \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "id"] & A \otimes \mathbb{K} \arrow[d, "id\otimes u"] \\ A \otimes A \arrow[r, "\cdot"'] & A & A \otimes A \arrow[l, "\cdot"] \end{tikzcd}$$

Notice the left diagram describes the associativity of the product and the right diagram describes the compatibility between the product and the scalar multiplication.

**Example 1.5.** Let $W$ be the vector space of words over alphabet $\Omega$. Let $u$ be the linear map that maps any $k \in \mathbb{K}$ to $k\mathbb{1} \in W$ and let the product $\cdot$ be concatenation. That is to say, for any words $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ and $b_1 \otimes b_2 \otimes \cdots \otimes b_m$,

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes b_2 \otimes \cdots \otimes b_m) = a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_m.$$

It is easy to check that the above two diagrams commute, and thus $(W, \cdot, u)$ is an algebra.

The commutative diagram formulation suggests considering the dual objects of algebras by reversing the arrows. The resulting objects are called coalgebras.

**Definition 1.6.** Let $\mathbb{K}$ be a field. A **coalgebra** $(C, \Delta, \varepsilon)$ over $\mathbb{K}$ is a vector space $C$ along with two linear maps the **coproduct** $\Delta : C \to C \otimes C$ and the **counit** $\varepsilon : C \to \mathbb{K}$ such that the following diagrams commute:

$$\begin{tikzcd} C \otimes C \arrow[r, "id\otimes\Delta"] & C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta\otimes id"'] \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes C \arrow[r, "\cong"] & C & C \otimes \mathbb{K} \arrow[l, "\cong"'] \\ C \otimes C \arrow[u, "\varepsilon\otimes id"] & C \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "id"] & C \otimes C \arrow[u, "id\otimes\varepsilon"'] \end{tikzcd}$$

If the product describes how to put two elements together into one element, then the coproduct describes how to take an element apart into two elements, possibly in more than one way. The commutativity of the left diagram is also referred to as the **coassociativity** of the coproduct. Now we are going to give a coalgebra structure on words.

**Example 1.7.** Let $W$ be the vector space of words. Let $\varepsilon$ be the linear map that maps $k\mathbb{1}$ to $k$ for any $k \in \mathbb{K}$ and maps $\bigoplus_{n=1}^{\infty} \Omega^{\otimes n}$ to 0. Let $\Delta$ be the deshuffle operation on any single word and extend linearly to $W$. That is for any single word $\omega = a_1 \otimes a_2 \otimes \cdots \otimes a_n$,

$$\Delta(\omega) = \sum_{\substack{\{i_1,i_2,\ldots,i_k\} \text{ subset of} \\ \{1,2,\ldots,n\} \text{ with} \\ i_1<i_2<\cdots<i_k}} (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_k}) \otimes (a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_{n-k}}),$$

where $\{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$ with $j_1 < \cdots < j_{n-k}$. For instance,

$$\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1},$$

and

$$\Delta(acb) = \mathbb{1} \otimes acb + a \otimes cb + c \otimes ab + b \otimes ac + ac \otimes b + ab \otimes c + cb \otimes a + acb \otimes \mathbb{1}.$$

By chasing elements, we could see that both diagrams in Definition 1.6 commute and thus $(W, \Delta, \varepsilon)$ is a coalgebra.

A linear map between two algebras which preserves the algebraic structure is defined to be an algebra homomorphism.

**Definition 1.8.** Let $(A, \cdot_A, u_A)$, $(B, \cdot_B, u_B)$ be $\mathbb{K}$-algebras. A linear map $\phi : A \to B$ is an **algebra homomorphism** if the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\cdot_A"] \arrow[d, "\phi\otimes\phi"'] & A \arrow[d, "\phi"] \\
B \otimes B \arrow[r, "\cdot_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\phi"] & & B \\
& \mathbb{K} \arrow[ul, "u_A"] \arrow[ur, "u_B"'] &
\end{tikzcd}$$

Similarly, by reversing the arrows we can define a linear map between two coalgebras that preserves the algebraic structure as a coalgebra morphism.

**Definition 1.9.** Let $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$ be $\mathbb{K}$-coalgebras. A linear map $\psi : C \to D$ is a **coalgebra morphism** if the following diagrams commute:

$$\begin{tikzcd}
C \otimes C \arrow[d, "\phi\otimes\phi"'] & C \arrow[l, "\Delta_C"'] \arrow[d, "\phi"] \\
D \otimes D & D \arrow[l, "\Delta_D"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "\phi"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\
& \mathbb{K} &
\end{tikzcd}$$

Given two algebras $A$ and $B$, there is a natural algebra structure on the vector space $A \otimes B$ as follows.

**Proposition 1.10.** *Let* $(A, \cdot_A, u_A)$, $(B, \cdot_B, u_B)$ *be* $\mathbb{K}$*-algebras. Then* $A \otimes B$ *is naturally a* $\mathbb{K}$*-algebra with the unit* $u_{A\otimes B} = u_A \otimes u_B$ *and the product* $\cdot_{A\otimes B} = (\cdot_A \otimes \cdot_B) \circ (id \otimes \tau \otimes id)$

*where $\tau$ is the transposition $\tau(a \otimes b) = b \otimes a$. This is equivalent to saying that the following diagram commutes:*

$$\begin{tikzcd}
A \otimes B \otimes A \otimes B \arrow[rr, "id\otimes\tau\otimes id"] \arrow[dr, "\cdot_{A\otimes B}"'] & & A \otimes A \otimes B \otimes B \arrow[dl, "\cdot_A \otimes \cdot_B"] \\
& A \otimes B &
\end{tikzcd}$$

Note that the product $\cdot_{A\otimes B}$ acts coordinatewise, so it is associative by the associativity of both $\cdot_A$ and $\cdot_B$. The commutativity of the second diagram in Definition 1.4 is easy to check by chasing elements.

The commuting diagram says that the product of elements $a \otimes b$ and $a^{'} \otimes b^{'}$ in $A \otimes B$ is the coordinate-wise tensor product $aa^{'} \otimes bb^{'}$. Similarly, given two coalgebras $C$ and $D$, there is a natural coalgebra structure on the vector space $C \otimes D$.

**Proposition 1.11.** *Let $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$ be $\mathbb{K}$-coalgebras. Then $C \otimes D$ is naturally a $\mathbb{K}$-coalgebra with the counit $\varepsilon_{C\otimes D} = \varepsilon_C \otimes \varepsilon_D$ and the coproduct $\Delta_{C\otimes D} = (id \otimes \tau \otimes id) \circ (\Delta_C \otimes \Delta_D)$, i.e., the following diagram commutes:*

$$\begin{tikzcd}
C \otimes D \otimes C \otimes D & & C \otimes C \otimes D \otimes D \arrow[ll, "id\otimes\tau\otimes id"'] \\
& C \otimes D \arrow[ul, "\Delta_{C\otimes D}"] \arrow[ur, "\Delta_C\otimes\Delta_D"'] &
\end{tikzcd}$$

Similarly, note that the coproduct $\Delta_{C\otimes D}$ acts coordinatewise, so it is coassociative by the coassociativity of both $\Delta_C$ and $\Delta_D$. The commutativity of the second diagram in Definition 1.6 is easy to check by chasing elements.

With the definitions and propositions above, we are now ready to define bialgebras. A bialgbra is both an algebra and a coalgebra with compatible operations as follows.

**Definition 1.12.** A $\mathbb{K}$-vector space $A$ is a **bialgebra** if $(A, \cdot, u)$ is an algebra and $(A, \Delta, \varepsilon)$ is a coalgebra such that one of the following equivalent conditions holds:

(1) $\Delta$ and $\varepsilon$ are algebra homomorphisms;

(2) $\cdot$ and $u$ are coalgebra morphisms;

(3) The following four diagrams commute:

$$
\begin{tikzcd}
& A\otimes A \arrow[dl, "\Delta\otimes\Delta"'] \arrow[ddr, "\cdot"] & \\
A\otimes A\otimes A\otimes A \arrow[dd, "id\otimes\tau\otimes id"'] & & \\
& & A \arrow[ddl, "\Delta"] \\
A\otimes A\otimes A\otimes A \arrow[dr, "\cdot\otimes\cdot"'] & & \\
& A\otimes A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\Delta"] & A\otimes A \\
\mathbb{K} \arrow[u, "u"] \arrow[r, "\cong"'] & \mathbb{K}\otimes\mathbb{K} \arrow[u, "u\otimes u"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\otimes A \arrow[r, "\cdot"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & A \arrow[d, "\varepsilon"] \\
\mathbb{K}\otimes\mathbb{K} \arrow[r, "\cong"'] & \mathbb{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "\varepsilon"] & & \mathbb{K} \\
& \mathbb{K} \arrow[ul, "u"] \arrow[ur, "id"'] &
\end{tikzcd}
$$

The $\mathbb{K}$-bialgebra $A$ is also denoted by $(A, \cdot, u, \Delta, \varepsilon)$.

Conditions (1) and (3) are equivalent because the two diagrams in the first row show that $\Delta$ is an algebra homomorphism and the other two diagrams show that $\varepsilon$ is an algebra homomorphism. Similarly condition (2) is equivalent to (3) because the two diagrams in the first column also show that $\cdot$ is a coalgebra morphism and the other two diagrams show that $u$ is a coalgebra morphism.

**Example 1.13.** Let $(W, \cdot, u)$ be the algebra in Example 1.5 and $(W, \Delta, \varepsilon)$ be the coalgebra in Example 1.7. Then $(W, \cdot, u, \Delta, \varepsilon)$ is a bialgebra called the **concatenation-deshuffle bialgebra of words** .

*Proof.* We check that all four diagrams in Definition 1.12 commute. Let $\cdot_t = (\cdot \otimes \cdot) \circ (id \otimes \tau \otimes id)$ be the natural product on $W \otimes W$ as shown in Proposition 1.10. First we claim that for any word $\omega = a_1 a_2 \cdots a_n$ and any single letter $b$,

$$\Delta(\omega b) = \Delta(\omega) \cdot_t \Delta(b).$$

By the definition of $\Delta$ in Example 1.7 and let $a_{n+1} = b$, we have

$$\Delta(\omega b) = \sum_{\substack{\{i_1, i_2, \ldots, i_k\} \text{ subset of} \\ \{1,2,\ldots,n+1\} \text{ with} \\ i_1 < i_2 < \cdots < i_k}} (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_k}) \otimes (a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_{n+1-k}}),$$

where $\{j_1, \ldots, j_{n+1-k}\} = \{1, 2, \ldots, n+1\} \setminus \{i_1, i_2, \ldots, i_k\}$ with $j_1 < \cdots < j_{n+1-k}$. Notice we can separate the sum into two sums according to either $i_k = n+1$ or $j_{n+1-k} = n+1$, thus

$$\begin{aligned} \Delta(\omega b) &= \sum_{\substack{\{i_1, i_2, \ldots, i_{k-1}\} \text{ subset of} \\ \{1,2,\ldots,n\} \text{ with} \\ i_1 < i_2 < \cdots < i_{k-1}}} (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_{k-1}} \otimes b) \otimes (a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_{n-k+1}}) \\ &\quad + \sum_{\substack{\{i_1, i_2, \ldots, i_k\} \text{ subset of} \\ \{1,2,\ldots,n\} \text{ with} \\ i_1 < i_2 < \cdots < i_k}} (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_k}) \otimes (a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_{n-k}} \otimes b). \end{aligned}$$

On the other hand, we have $\Delta(b) = b \otimes \mathbb{1} + \mathbb{1} \otimes b$ and

$$\Delta(\omega) = \sum_{\substack{\{i_1, i_2, \ldots, i_k\} \text{ subset of} \\ \{1,2,\ldots,n\} \text{ with} \\ i_1 < i_2 < \cdots < i_k}} (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_k}) \otimes (a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_{n-k}}).$$

Then we have

$$\begin{aligned} \Delta(\omega) \cdot_t \Delta(b) &= \big((\cdot \otimes \cdot) \circ (id \otimes \tau \otimes id)\big)\big(\Delta(\omega) \otimes \Delta(b)\big) \\ &= \sum_{\substack{\{i_1, i_2, \ldots, i_k\} \text{ subset of} \\ \{1,2,\ldots,n\} \text{ with} \\ i_1 < i_2 < \cdots < i_k}} (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_{k-1}} \otimes b) \otimes (a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_{n-k}}) \\ &\quad + \sum_{\substack{\{i_1, i_2, \ldots, i_k\} \text{ subset of} \\ \{1,2,\ldots,n\} \text{ with} \\ i_1 < i_2 < \cdots < i_k}} (a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes a_{i_k}) \otimes (a_{j_1} \otimes a_{j_2} \otimes \cdots \otimes a_{j_{n-k}} \otimes b). \end{aligned}$$

Thus the claim holds. For any words $\alpha = a_1 a_2 \cdots a_n$ and $\beta = b_1 b_2 \cdots b_m$, following the right hand side of the first diagram we get

$$\begin{aligned} (\Delta \circ \cdot)(\alpha \otimes \beta) &= \Delta(a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m) \\ &= \Delta(a_1 a_2 \cdots a_n b_1 b_2 \cdots b_{m-1}) \cdot_t \Delta(b_m) \\ &= \cdots \\ &= \Delta(a_1) \cdot_t \Delta(a_2) \cdot_t \cdots \cdot_t \Delta(a_n) \cdot_t \Delta(b_1) \cdot_t \Delta(b_2) \cdot_t \cdots \cdot_t \Delta(b_m). \end{aligned}$$

Following the left hand side of the first diagram, we have

$$\begin{aligned}
& \big((\cdot\otimes\cdot)\circ(id\otimes\tau\otimes id)\circ(\Delta\otimes\Delta)\big)(\alpha\otimes\beta) \\
= \ & (\cdot_t)\big(\Delta(\alpha)\otimes\Delta(\beta)\big) \\
= \ & (\cdot_t)\Big(\big(\Delta(a_1)\cdot_t\Delta(a_2)\cdot_t\cdots\cdot_t\Delta(a_n)\big)\otimes\big(\Delta(b_1)\cdot_t\Delta(b_2)\cdot_t\cdots\cdot_t\Delta(b_m)\big)\Big) \\
= \ & \Delta(a_1)\cdot_t\Delta(a_2)\cdot_t\cdots\cdot_t\Delta(a_n)\cdot_t\Delta(b_1)\cdot_t\Delta(b_2)\cdot_t\cdots\cdot_t\Delta(b_m).
\end{aligned}$$

Along with the linearity, the first diagram commutes for $W\otimes W$. The last three diagrams are easy to check in a similar manner. Thus we obtain a bialgebra structure on words. □

### 1.3.2 Hopf Algebras

A Hopf algebra is a bialgebra together with an antipode map $S$ which is the convolution inverse of the identity map.

For $\mathbb{K}$-vector spaces $A, B$ let $Hom(A,B)$ denote the space of all linear maps from $A$ to $B$.

**Proposition 1.14.** *Let $(C,\Delta,\varepsilon)$ be a $\mathbb{K}$-coalgebra and $(A,\cdot,u)$ be a $\mathbb{K}$-algebra. Define a **convolution product** $\star$ on $Hom(C,A)$ as follows,*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[rrr, bend right, "f\star g"'] & C\otimes C \arrow[r, "f\otimes g"] & A\otimes A \arrow[r, "\cdot"] & A .
\end{tikzcd}$$

*Then $Hom(C,A)$ is a $\mathbb{K}$-algebra with the identity $u\circ\varepsilon$.*

*Proof.* It is clear $\star$ is a linear map from the above diagram since $\cdot$ and $\Delta$ are linear maps. Now check $\star$ is associative. Take $f,g,h\in Hom(C,A)$, then

$$\begin{aligned}
(f\star g)\star h &= \cdot\big((f\star g)\otimes h\big)\Delta \\
&= \cdot\Big(\big(\cdot(f\otimes g)\Delta\big)\otimes h\Big)\Delta \\
&= \cdot(\cdot\otimes id)(f\otimes g\otimes h)(\Delta\otimes id)\Delta,
\end{aligned}$$

and

$$\begin{aligned}
f\star(g\star h) &= \cdot\big(f\otimes(g\star h)\big)\Delta \\
&= \cdot\Big(f\otimes\big(\cdot(g\otimes h)\Delta\big)\Big)\Delta \\
&= \cdot(id\otimes\cdot)(f\otimes g\otimes h)(id\otimes\Delta)\Delta.
\end{aligned}$$

Since $\cdot$ is associative and $\Delta$ is coassociative, we have

$$(f \star g) \star h = f \star (g \star h).$$

Check $u \circ \varepsilon$ is the identity as follows,

$$\begin{aligned}(f \star u\varepsilon) &= \cdot(f \otimes u\varepsilon)\Delta \\ &= \cdot(f \otimes u)(id \otimes \varepsilon)\Delta \\ &= \cdot(f \otimes u)(id \otimes 1) \\ &= \cdot(f \otimes \mathbb{1}_A) \\ &= f,\end{aligned}$$

and

$$\begin{aligned}(u\varepsilon \star f) &= \cdot(u\varepsilon \otimes f)\Delta \\ &= \cdot(u \otimes f)(\varepsilon \otimes id)\Delta \\ &= \cdot(u \otimes f)(1 \otimes id) \\ &= \cdot(\mathbb{1}_A \otimes f) \\ &= f.\end{aligned}$$

Finally check $\star$ and scalar multiplication are compatible as follows,

$$\begin{aligned}k(f \star g) &= k\big(\cdot (f \otimes g)\Delta\big) \\ &= \cdot(kf \otimes g)\Delta = (kf) \star g \\ &= \cdot(f \otimes kg)\Delta = f \star (kg).\end{aligned}$$

□

**Corollary 1.15.** *If $A$ is a bialgebra, then $Hom(A, A)$ has a convolution algebra structure.*

**Example 1.16.** Let $(W, \cdot, u, \Delta, \varepsilon)$ be the bialgebra of words as shown in Example 1.13. Then $Hom(W, W)$ is a convolution algebra. Take $f, g \in Hom(W, W)$ and any word $\omega =$

$a_1a_2\cdots a_n$, then

$$\begin{aligned}
(f\star g)(\omega) &= \cdot(f\otimes g)\Delta(\omega)\\
&= \cdot\Bigg(\sum_{\substack{\{i_1,i_2,\ldots,i_k\}\ subset\ of\\ \{1,2,\ldots,n\}\ with\\ i_1<i_2<\cdots<i_k}} f(a_{i_1}\otimes a_{i_2}\otimes\cdots\otimes a_{i_k})\otimes g(a_{j_1}\otimes a_{j_2}\otimes\cdots\otimes a_{j_{n-k}})\Bigg)\\
&= \sum_{\substack{\{i_1,i_2,\ldots,i_k\}\ subset\ of\\ \{1,2,\ldots,n\}\ with\\ i_1<i_2<\cdots<i_k}} f(a_{i_1}a_{i_2}\cdots a_{i_k})g(a_{j_1}a_{j_2}\cdots a_{j_{n-k}}),
\end{aligned}$$

where $\{j_1,\ldots,j_{n-k}\}=\{1,2,\ldots,n\}\setminus\{i_1,i_2,\ldots,i_k\}$ with $j_1<\cdots<j_{n-k}$.

**Definition 1.17.** A **Hopf algebra** is a bialgebra $(A,\cdot,u,\Delta,\varepsilon)$ along with a map $S\in Hom(A,A)$. $S$ is a two-sided inverse of $id_A$ in the convolution algebra, i.e., the following diagram commutes:

$$\begin{tikzcd}
 & A\otimes A \arrow[rr, "S\otimes id_A"] & & A\otimes A \arrow[dr, "\cdot"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "u"] & & A \\
 & A\otimes A \arrow[rr, "id_A\otimes S"'] & & A\otimes A \arrow[ur, "\cdot"'] &
\end{tikzcd}$$

$S$ is called the **antipode**.

**Example 1.18.** Let $(W,\cdot,u,\Delta,\varepsilon)$ be the bialgebra of words as shown in Example 1.13. For any word $\omega=a_1a_2\cdots a_n$, let $S(\omega)=(-1)^n a_n a_{n-1}\cdots a_1$. Then extend $S$ linearly to $W$. Claim $S$ is the antipode. We only need to check the diagram in Definition 1.17 commutes for $\omega$. For $n=0$, we have $S(\mathbb{1})=\mathbb{1}$ which is easy to see satisfies the commuting diagram. For $n>0$, taking $\omega$ through the middle path we get 0 since $\varepsilon(\omega)=0$. Taking $\omega$ through the top path, we have

$$\begin{aligned}
\big(\cdot(S\otimes id_A)\Delta\big)(\omega) &= (S\star id)(\omega)\\
&= \sum_{\substack{\{i_1,i_2,\ldots,i_k\}\ subset\ of\\ \{1,2,\ldots,n\}\ with\\ i_1<i_2<\cdots<i_k}} S(a_{i_1}a_{i_2}\cdots a_{i_k})(a_{j_1}a_{j_2}\cdots a_{j_{n-k}})\\
&= \sum_{\substack{\{i_1,i_2,\ldots,i_k\}\ subset\ of\\ \{1,2,\ldots,n\}\ with\\ i_1<i_2<\cdots<i_k}} (-1)^k(a_{i_k}a_{i_{k-1}}\cdots a_{i_1}a_{j_1}a_{j_2}\cdots a_{j_{n-k}}),
\end{aligned}$$

where $\{j_1, \ldots, j_{n-k}\} = \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\}$ with $j_1 < \cdots < j_{n-k}$. Notice the sum can be separated into two sums according to the parity of $k$ and we can pair them up as follows. For an odd $k$ and a set $\{i_1, \ldots, i_k\}$, if $i_1 < j_1$ we pair $S(a_{i_1} \cdots a_{i_k})a_{j_1}a_{j_2} \cdots a_{j_{n-k}}$ with $S(a_{i_2} \cdots a_{i_k})a_{i_1}a_{j_1} \cdots a_{j_{n-k}}$ otherwise we pair it with $S(a_{j_1}a_{i_1} \cdots a_{i_k})a_{j_2} \cdots a_{j_{n-k}}$. This is a bijection between $\{\{i_1, \ldots, i_k\} : k \text{ } odd\}$ and $\{\{i_1, \ldots, i_k\} : k \text{ } even\}$; thus the sum goes to 0. Similarly, taking $\omega$ through the bottom path also results in 0. So $S$ is the antipode and $W$ is a Hopf algebra.

### 1.3.3 Gradedness and Connectedness

**Definition 1.19.** A vector space $V$ is a **graded $\mathbb{K}$-vector space** if it has a direct sum decomposition

$$V = \bigoplus_{n=0}^{\infty} V_n.$$

Call the elements in $V_n$ homogenous of degree $n$.

If $V$ and $W$ are graded $\mathbb{K}$-vector spaces then $V \otimes W$ is also a graded $\mathbb{K}$-vector space as

$$V \otimes W = \bigoplus_{n=0}^{\infty} (V \otimes W)_n,$$

where

$$(V \otimes W)_n = \bigoplus_{i=0}^{n} V_i \otimes W_{n-i}.$$

The vector space $W$ of words is a graded vector space since $W = \bigoplus_{n=0}^{\infty} \Omega^{\otimes n}$. A homogenous element of degree $n$ is just a linear combination of words with length $n$.

**Definition 1.20.** A linear map $f : V \to W$ between two graded vector spaces is **graded** if $f(V_n) \subset W_n$ for any $n \in \mathbb{N}_{\geq 0}$.

Notice the five maps $\cdot, u, \Delta, \varepsilon, S$ defined on words are all graded maps.

**Definition 1.21.** An algebra, coalgebra, bialgebra or Hopf algebra is **graded** if the underlying vector space is graded and the maps $(\cdot, u)$, $(\Delta, \varepsilon)$, $(\cdot, u, \Delta, \varepsilon)$ or $(\cdot, u, \Delta, \varepsilon, S)$ are graded.

The Hopf algebra $(W, \cdot, u, \Delta, \varepsilon, S)$ shown in Example 1.18 is a graded Hopf algebra.

**Definition 1.22.** A graded $\mathbb{K}$-vector space is **connected** if $V_0 \cong \mathbb{K}$.

For words, it is clear that $W_0 \cong \mathbb{K}$ since the elements in $W_0$ are in the form $k\mathbb{1}$ with $k \in \mathbb{K}$. So $W$ is connected.

In graded and connected bialgebras a lot of things come for free.

**Proposition 1.23.** *Let $(A, \cdot, u, \Delta, \varepsilon)$ be a graded and connected $\mathbb{K}$-bialgebra.*

*(1) $u : \mathbb{K} \to A_0$ is an isomorphism.*

*(2) $\varepsilon \mid_{A_0}: A_0 \to \mathbb{K}$ is the reverse isomorphism.*

*(3) $\mathbb{K}er\ \varepsilon = \bigoplus_{n=1}^{\infty} A_n$.*

*(4) $\forall x \in \mathbb{K}er\ \varepsilon$, $\Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} + \tilde{\Delta}(x)$ where $\tilde{\Delta}(x) \in \mathbb{K}er\ \varepsilon \otimes \mathbb{K}er\ \varepsilon$.*

*Proof.* (1) Because $u$ is a nonzero graded map, $u(\mathbb{K}) \subset A_0$ and by the connectedness $A_0 \cong \mathbb{K}$, we have $1 = dim_{\mathbb{K}} u(\mathbb{K}) = dim_{\mathbb{K}} A_0$. This implies $u(\mathbb{K}) = A_0$. Since $u$ is a linear map, for any $k \in \mathbb{K}$, $u(k) = ku(1)$. Because $u$ is a nonzero map, $u(1) \neq 0$ thus $u$ is injective. So $u$ is an isomorphism.

(2) From (1) we know the following diagram commutes; thus $\varepsilon \mid_{A_0}$ is the reverse isomorphism.

$$\begin{tikzcd} A \arrow[rr, "\varepsilon"] & & \mathbb{K} \\ & \mathbb{K} \arrow[ul, "u"] \arrow[ur, "id"'] & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A_0 \arrow[rr, "\varepsilon|_{A_0}"] & & \mathbb{K} \\ & \mathbb{K} \arrow[ul, "u"] \arrow[ur, "id"'] & \end{tikzcd} .$$

(3) $\varepsilon$ is a graded map and $\mathbb{K} = \mathbb{K}_0$, so

$$\varepsilon(\bigoplus_{n=1}^{\infty} A_n) = 0.$$

Then the result follows from (2).

(4) Notice $\mathbb{1} \in A_0$ since the product is a graded map so $A_0 = \mathbb{K}\mathbb{1}$. Consider the following diagram,

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\cong"] & A & A \otimes \mathbb{K} \arrow[l, "\cong"'] \\ A \otimes A \arrow[u, "\varepsilon \otimes id"] & A \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "id"] & A \otimes A \arrow[u, "id \otimes \varepsilon"'] \end{tikzcd} .$$

If we take $x \in \mathbb{K}er\ \varepsilon$ by the left path, we'll have

$$\Delta(x) = k_1 \mathbb{1} \otimes \frac{1}{k_1} x + y,$$

where $y \in (\mathbb{K}er\ \varepsilon) \otimes A$ and $k_1 \in \mathbb{K}$. Then taking the right path, we get

$$\Delta(x) = \frac{1}{k_2} x \otimes k_2 \mathbb{1} + z,$$

where $z \in A \otimes (\mathbb{K}er\ \varepsilon)$ and $k_2 \in \mathbb{K}$. Combining them together, we have

$$\Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} + \tilde{\Delta}(x),$$

with $\tilde{\Delta}(x) \in \mathbb{K}er\ \varepsilon \otimes \mathbb{K}er\ \varepsilon$. $\square$

From the statement (4) in Proposition 1.23, we know for a graded and connected $\mathbb{K}$-bialgebra $A$ that

$$\tilde{\Delta} = \Delta - id \otimes \mathbb{1} - \mathbb{1} \otimes id$$

is well defined on $\mathbb{K}er\ \varepsilon = \bigoplus_{n=1}^{\infty} A_n$. We expand the definition of $\tilde{\Delta}$ to $A$ by $\tilde{\Delta}(\mathbb{1}) = -\mathbb{1} \otimes \mathbb{1}$ since $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$. Call $\tilde{\Delta}$ the **reduced coproduct** and call the elements in the set

$$\mathbb{K}er\ \tilde{\Delta} = \{p \in A : \Delta(p) = \mathbb{1} \otimes p + p \otimes \mathbb{1}\}$$

**primitives**. Note $\mathbb{1}$ is not a primitive.

**Proposition 1.24.** *The reduced coproduct $\tilde{\Delta}$ is coassociative, i.e.,*

$$(id \otimes \tilde{\Delta})\tilde{\Delta} = (\tilde{\Delta} \otimes id)\tilde{\Delta}.$$

*Proof.* By computation, we have

$$\begin{aligned}
(id \otimes \tilde{\Delta})\tilde{\Delta} &= (id \otimes \tilde{\Delta})(\Delta - id \otimes \mathbb{1} - \mathbb{1} \otimes id) \\
&= (id \otimes \tilde{\Delta})\Delta + id \otimes \mathbb{1} \otimes \mathbb{1} - (\mathbb{1} \otimes \tilde{\Delta}) \\
&= (id \otimes \Delta)\Delta + id \otimes \mathbb{1} \otimes \mathbb{1} - (id \otimes id \otimes \mathbb{1})\Delta - (id \otimes \mathbb{1} \otimes id)\Delta - (\mathbb{1} \otimes \tilde{\Delta}) \\
&= (id \otimes \Delta)\Delta - (\Delta \otimes \mathbb{1}) - (id \otimes \mathbb{1} \otimes id)\Delta - (\mathbb{1} \otimes \Delta) \\
&\quad + id \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes id + \mathbb{1} \otimes id \otimes \mathbb{1},
\end{aligned}$$

and

$$\begin{aligned}
(\tilde{\Delta} \otimes id)\tilde{\Delta} &= (\tilde{\Delta} \otimes id)(\Delta - id \otimes \mathbb{1} - \mathbb{1} \otimes id) \\
&= (\tilde{\Delta} \otimes id)\Delta - (\tilde{\Delta} \otimes \mathbb{1}) + \mathbb{1} \otimes \mathbb{1} \otimes id \\
&= (\Delta \otimes id)\Delta - (\mathbb{1} \otimes id \otimes id)\Delta - (id \otimes \mathbb{1} \otimes id)\Delta - (\tilde{\Delta} \otimes \mathbb{1}) + \mathbb{1} \otimes \mathbb{1} \otimes id \\
&= (\Delta \otimes id)\Delta - (\mathbb{1} \otimes \Delta) - (id \otimes \mathbb{1} \otimes id)\Delta - (\Delta \otimes \mathbb{1}) \\
&\quad + \mathbb{1} \otimes id \otimes \mathbb{1} + id \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes id.
\end{aligned}$$

Thus $(id \otimes \tilde{\Delta})\tilde{\Delta} = (\tilde{\Delta} \otimes id)\tilde{\Delta}$. □

**Corollary 1.25.** *If $(A, \cdot, u, \Delta, \varepsilon)$ is a graded and connected $\mathbb{K}$-bialgebra, then $A$ has a unique antipode $S$, and so $A$ is a graded Hopf algebra.*

*Proof.* Since $S(\mathbb{1}) = \mathbb{1}$ follows from chasing $\mathbb{1}$ through the three paths in Definition 1.17 and

$$A = \bigoplus_{n=0}^{\infty} A_n,$$

we can define $S$ recursively. Suppose we've already defined $S$ on all elements of $A_j$ for $0 \leq j < n$. Now take $x \in A_n$. We know $\Delta(x) = \mathbb{1} \otimes x + x \otimes \mathbb{1} + \tilde{\Delta}(x)$. Write

$$\tilde{\Delta}(x) = \sum_i x_{i,1} \otimes x_{i,2},$$

a finite sum. Since $\Delta$ is a graded map and by (4) from Proposition 1.23, we know $x_{i,2} \in \bigoplus_{l=1}^{n-1} A_l$. From the lower path of the definition of $S$ we have

$$\cdot(id \otimes S)\Delta = u \circ \varepsilon.$$

Applying this to $x$ we can get a recursive formula

$$S = -id - \cdot(id \otimes S)\tilde{\Delta}.$$

Now in order to prove that it is a two-sided inverse of $id$, we have to show that

$$S = -id - \cdot(S \otimes id)\tilde{\Delta}$$

also holds. Since the elements in $A_1$ are all primitives by the fact that $\Delta$ is a graded map, it holds true on $A_1$. Suppose it holds for all elements in $A_k$ for $1 \leq k < n$. Now take an element $x$ in $A_n$. We want to show that $\cdot(S\otimes id)\tilde{\Delta} = \cdot(id\otimes S)\tilde{\Delta}$. The two parts in $\tilde{\Delta}$ both have degree less than $n$, so

$$\begin{aligned}
\cdot(S\otimes id)\tilde{\Delta} &= \cdot\Big(\big(-id - \cdot(id\otimes S)\tilde{\Delta}\big)\otimes id\Big)\tilde{\Delta}\\
&= -\cdot(id\otimes id)\tilde{\Delta} - \cdot(\cdot\otimes id)(id\otimes S\otimes id)(\tilde{\Delta}\otimes id)\tilde{\Delta}\\
&= -\cdot(id\otimes id)\tilde{\Delta} - \cdot(id\otimes \cdot)(id\otimes S\otimes id)(id\otimes \tilde{\Delta})\tilde{\Delta}\\
&= \cdot\Big(id\otimes\big(-id - \cdot(S\otimes id)\tilde{\Delta}\big)\Big)\tilde{\Delta}\\
&= \cdot(id\otimes S)\tilde{\Delta}.
\end{aligned}$$

The third equality is by the associativity of the product and the coassociativity of the reduced coproduct. The last equality holds because of the induction hypothesis of $S = -id - \cdot(S\otimes id)\tilde{\Delta}$ on elements with smaller degrees. We may also notice $S$ is a graded map from the induction process, so $A$ is a graded Hopf algebra. $\square$

In the rest of this thesis, we focus on **combinatorial Hopf algebras** which are graded and connected Hopf algebras [10]. So we present the definitions of combinatorial classes, the trivial unit and trivial counit below.

**Definition 1.26.** A **combinatorial class** $\mathcal{C}$ is a countable set with a size function $|\cdot| : \mathcal{C} \to \mathbb{Z}_{\geq 0}$ with the property that $\mathcal{C}_n = \{c \in \mathcal{C} : |c| = n\}$ is finite for all $n$. Let $c_n$ denote the cardinality of $\mathcal{C}_n$.

Let $V(\mathcal{C})$ be the vector space spanned by the objects in a combinatorial class $\mathcal{C}$. There is a natural grading in terms of the size. The connectedness here means there is only one element with size 0, i.e., $c_0 = 1$ and $\mathcal{C}_0 = \{\mathbb{1}\}$, which is a very natural condition. In this case, if there is a graded bialgebra structure on $V(\mathcal{C})$ (graded by the size), then by Corollary 1.25, the antipode $S$ comes for free and it is a graded Hopf algebra. Usually for combinatorial Hopf algebras, the unit $u$ will be the **trivial unit** with $u(k) = k\mathbb{1}$ and the counit will be the **trivial counit** with $\varepsilon(k\mathbb{1}) = k$ and $\varepsilon = 0$ otherwise, where $k \in \mathbb{K}$.

In the following, we present one more useful proposition which will be used in later chapters. It shows that for an algebra homomorphism, the coassociativity of elements in a base guarantees the coassociativity of all elements.

**Proposition 1.27.** *Let $(A, \cdot, u)$ be a graded and connected $\mathbb{K}$-algebra with $u$, the trivial unit. Assume $A$ is generated by elements in $\mathcal{B}$ as an algebra. Let $\Delta : A \to A\otimes A$ be an algebra* *homomorphism. If for any $a \in \mathcal{B}$, $(id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a)$ then $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$ holds on $A$.*

*Proof.* Because $A$ is generated by $\mathcal{B}$, this is equivalent to proving

$$(id \otimes \Delta) \circ (\Delta) \circ (\cdot) = (\Delta \otimes id) \circ (\Delta) \circ (\cdot).$$

Since $\Delta$ is an algebra homomorphism,

$$\Delta(\cdot) = (\cdot \otimes \cdot)(id \otimes \tau \otimes id)(\Delta \otimes \Delta).$$

Then

$$\begin{aligned}
(id \otimes \Delta)\big(\Delta(\cdot)\big) &= (id \otimes \Delta)(\cdot \otimes \cdot)(id \otimes \tau \otimes id)(\Delta \otimes \Delta) \\
&= \Big( \cdot \otimes \big(\Delta(\cdot)\big)\Big)(id \otimes \tau \otimes id)(\Delta \otimes \Delta) \\
&= \Bigg( \cdot \otimes \Big((\cdot \otimes \cdot)(id \otimes \tau \otimes id)(\Delta \otimes \Delta)\Big)\Bigg)(id \otimes \tau \otimes id)(\Delta \otimes \Delta) \\
&= \Bigg( \cdot \otimes \Big((\cdot \otimes \cdot)(id \otimes \tau \otimes id)\Big)\Bigg)(id \otimes id \otimes \Delta \otimes \Delta)(id \otimes \tau \otimes id)(\Delta \otimes \Delta).
\end{aligned}$$

Notice

$$\begin{aligned}
& (id \otimes id \otimes \Delta \otimes \Delta)(id \otimes \tau \otimes id) \\
= \; & (id \otimes \tau \otimes id \otimes id \otimes id)(id \otimes id \otimes \tau \otimes id \otimes id)(id \otimes \Delta \otimes id \otimes \Delta),
\end{aligned}$$

and

$$\Bigg( \cdot \otimes \Big((\cdot \otimes \cdot)(id \otimes \tau \otimes id)\Big)\Bigg) = (\cdot \otimes \cdot \otimes \cdot)(id \otimes id \otimes id \otimes \tau \otimes id).$$

So we know $(id \otimes \Delta)\big(\Delta(\cdot)\big)$ is just

$$\begin{aligned}
& (\cdot \otimes \cdot \otimes \cdot) \circ (id \otimes id \otimes id \otimes \tau \otimes id) \circ (id \otimes \tau \otimes id \otimes id \otimes id) \\
& \qquad \circ (id \otimes id \otimes \tau \otimes id \otimes id) \circ \Big( \big((id \otimes \Delta)\Delta\big) \otimes \big((id \otimes \Delta)\Delta\big)\Big).
\end{aligned}$$

Similarly, we know $(\Delta \otimes id)\big(\Delta(\cdot)\big)$ is

$$\begin{aligned}
& (\cdot \otimes \cdot \otimes \cdot) \circ (id \otimes \tau \otimes id \otimes id \otimes id) \circ (id \otimes id \otimes id \otimes \tau \otimes id) \\
& \qquad \circ (id \otimes id \otimes \tau \otimes id \otimes id)\Big( \big((\Delta \otimes id)\Delta\big) \otimes \big((\Delta \otimes id)\Delta\big)\Big).
\end{aligned}$$

Since $(id \otimes \tau \otimes id \otimes id \otimes id)$ and $(id \otimes id \otimes id \otimes \tau \otimes id)$ are commutative and along with the condition $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$ on $\mathcal{B}$, applied inductively we get $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$ on $A$. □

### 1.3.4 Examples of Combinatorial Hopf Algebras

In Section 1.3.2, we presented a concatenation-deshuffle Hopf algebra on the vector space $W$ of words in Example 1.18. Now, we are going to present another Hopf algebra structure which is called the shuffle-deconcatenation Hopf algebra on the vector space $W$. After that we will present two Hopf algebras related to renormalization in quantum field theory.

**Example 1.28.** Let $\Omega$ be a finite alphabet, let $V$ be the vector space over $\mathbb{K}$ spanned by words constructed by letters in $\Omega$, as discussed in the beginning of Section 1.3.1. Let $\mathbb{1}$ denote the empty word. Then $V = \bigoplus_{n=0}^{\infty} \Omega^{\otimes n}$ is a graded vector space graded by the length of words. Define the shuffle product $\sqcup\!\sqcup$ recursively; for any words $\alpha = a_1 a_2 \cdots a_n$ and $\beta = b_1 b_2 \cdots b_m$ define

$$\alpha \sqcup\!\sqcup \beta = ((a_1 \cdots a_{n-1}) \sqcup\!\sqcup \beta) a_n + (\alpha \sqcup\!\sqcup (b_1 \cdots b_{m-1})) b_m,$$

with the base cases $\omega \sqcup\!\sqcup \mathbb{1} = \mathbb{1} \sqcup\!\sqcup \omega = \omega$ for any $\omega \in V$. Then extend linearly to $V \otimes V$. For instance,

$$\begin{aligned}
ab \sqcup\!\sqcup cd &= (a \sqcup\!\sqcup cd)b + (ab \sqcup\!\sqcup c)d \\
&= (\mathbb{1} \sqcup\!\sqcup cd)ab + (a \sqcup\!\sqcup c)db + (a \sqcup\!\sqcup c)bd + (ab \sqcup\!\sqcup \mathbb{1})cd \\
&= cdab + acdb + cadb + acbd + cabd + abcd.
\end{aligned}$$

Define the coproduct $\Delta$ to be deconcatenation, that is, for any word $\omega = u_1 u_2 \ldots u_n$ define

$$\Delta(\omega) = \sum_{i=0}^{n} u_1 \ldots u_i \otimes u_{i+1} \ldots u_n.$$

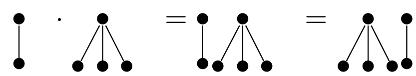
The coassociativity is easy to check. Let $u, \varepsilon$ be the trivial unit and the trivial counit respectively. Notice $V$ is connected and graded as these four maps are all graded maps. By Corollary 1.25, in order to show $(V, \sqcup\!\sqcup, u, \Delta, \varepsilon)$ is a Hopf algebra we only need to show $V$ is a bialgebra. So we need check the four diagrams in (3) of Definition 1.12 commute. The last three diagrams obviously commute and the first diagram can be proved to commute by induction. Here we omit the proof.

**Example 1.29.** (Renormalization Hopf algebra of rooted trees) This Hopf algebra was introduced by Connes and Kreimer in [6]. They used it to keep track of the combinatorial information needed to renormalize integrals in quantum field theory.

A rooted tree is a tree with a special node called the root. Here we draw the root at the top. A forest of rooted trees is a multiset of rooted trees. Let $H_R$ be the vector space over $\mathbb{K}$ spanned by all the forests. Let $\mathbb{1}$ be the empty tree. Then

$$H_R = \bigoplus_{n=0} H_n,$$

where $H_n$ consists of forests with $n$ vertices. Since $H_0 = \mathbb{K}\mathbb{1}$, $H_R$ is a graded and connected vector space. Define the product to be the disjoint union. For example:



With the disjoint union product, we can also view $H_R$ as the polynomial algebra of rooted trees. In other words,

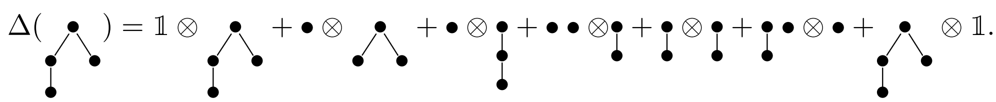
$$H_R = \mathbb{K}[T_1, T_2, \cdots],$$

where $\{T_1, T_2, \ldots\}$ is the set of all rooted trees.

Given a rooted tree $T$, an admissible cut $c$ of $T$ is a set possibly empty, of vertices of $T$ such that no vertex in the set is a descendant of another. Define $P_c(T)$ to be the forest of subtrees rooted at elements of $c$ and $R_c(T)$ be $T$ removing $P_c(T)$. The coproduct on $T$ is defined as

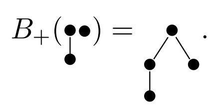
$$\Delta(T) = \sum_{c \text{ admissible cut}} P_c(T) \otimes R_c(T),$$

and extended as an algebra homomorphism to forests.
As an example,



Let $u$ and $\varepsilon$ be the trivial unit and the trivial counit. Readers may check that $\Delta$ is a coproduct and the four diagrams in Definition 1.12 (3) commute. It is easy to see these four maps are all graded maps. So $H_R$ is now a graded and connected bialgebra. Thus by Corollary 1.25, it is a Hopf algebra.

Furthermore, for a forest $F$, let $B_+(F)$ be a linear map that maps $F$ to the new tree of adding a root with children being the roots of every tree in $F$. As an example,



A linear map $L$ is a **Hochschild 1-cocycle** map if it satisfies

$$\Delta \circ L = L \otimes \mathbb{1} + (id \otimes L) \circ \Delta. \tag{1.1}$$

In this case, $B_+$ has been proved to be a Hochschild 1-cocycle map in [6, Theorem 2].

To finish this chapter, we recall the renormalization Hopf algebra in the scalar field theory $\phi^3$. A family of divergent Feynman graphs in this theory will be taken as examples through the construction in Chapter 3.

**Example 1.30.** (Feynman graphs in the scalar field theory $\phi^3$ with the dimension of space-time $D = 6$). A Feynman graph is built from half edges and vertices. Each half edge is adjacent to exactly one vertex and at most one other half edge. Those half edges which are only adjacent to vertices are called external edges while the pairs of adjacent half edges are called the internal edges. In $\phi^3$, the only vertex type is with weight 0 ,the only half edge type is , and the only edge type is with weight 2. An example of a Feynman graph is shown in Figure 1.4.

A Feynman graph $\gamma$ has the **1PI property** if $\gamma$ is a two edge connected graph, in other words, a graph that stays connected after removing any single internal edge. 1PI graphs are important since Feynman integrals are easily reducible to the 1PI case. Notice the graph $\Gamma$ in Figure 1.4 is a 1PI graph.

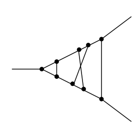


Figure 1.4: A Feynman graph $\Gamma$ in $\phi^3$

The **loop number** of a Feynman graph $\gamma$ is the number of independent cycles and denoted by $l(\gamma)$. The loop number of the Feynman graph $\Gamma$ in Figure 1.4 is 4. The loop number of a disjoint union of graphs is the sum of the loop numbers of each graph.

Another feature of a Feynman graph $\gamma$ is the **superficial degree of divergence** defined as

$$s(\gamma) = Dl - \sum_{\substack{e \text{ internal} \\ \text{edge of } \gamma}} w(e) - \sum_{\substack{v \text{ vertex} \\ \text{of } \gamma}} w(v),$$

where $w$ is the weight function. So $s(\Gamma) = 6 \cdot 4 - 2 \cdot 12 - 0 \cdot 9 = 0$ for $\Gamma$ in Figure 1.4. A Feynman graph $\gamma$ is called **divergent** if $s(\gamma)$ is nonnegative. In $\phi^3$, the divergent graphs are the ones with at most 3 external edges. Furthermore, we can discard those with 0 or 1 external edges since they can't appear as proper subgraphs of a 1PI graph. Physicists are interested in divergent 1PI graphs.

Let $\mathbb{1}$ be the empty graph. Let $H_F$ be the vector space over $\mathbb{K}$ spanned by all the disjoint unions of divergent 1PI Feynman graphs. We can see $H_F$ is a graded vector space graded by the loop number. Thus $H_F = \bigoplus_{n=0}^{\infty} (H_F)_n$, where $(H_F)_n$ consists of graphs with loop number $n$. Let the product be the disjoint union and let $u$ be the trivial unit. Then $H_F$ is a polynomial algebra of 1PI divergent Feynman graphs.

Let $\varepsilon$ be the trivial counit. The coproduct $\Delta$ is defined on divergent 1PI Feynman graph $\gamma$ as

$$\Delta(\gamma) = \sum_{\substack{\gamma' \subseteq \gamma \\ product\ of\ divergent \\ 1PI\ subgraphs}} \gamma' \otimes \gamma/\gamma',$$

where $\gamma/\gamma'$ is the result of taking $\gamma$ and contracting $\gamma'$ which is also in $H_F$ ([15, Proposition 2.21]). Extend $\Delta$ to $H_F$ as an algebra homomorphism. Here we omit the proof of the coassociativity. It is easy to check the four diagrams in Definition 1.12 and thus $H_F$ is a bialgebra. Notice the four maps are graded maps and $H_F$ is obvious connected, so by Corollary 1.25, $H_F$ is a Hopf algebra.

There are a lot of other combinatorial classes, such as permutations and set partitions, which can be equipped with a Hopf algebra structure and some Hopf algebras can capture relevant combinatorial information. For more combinatorial Hopf algebras, readers are referred to the notes by Grinberg and Reiner [10].

# Chapter 2

# Hopf Algebras on Generating Trees

In this chapter, we construct two Hopf algebras on generating trees. As our initial goal is not enumerative, we are less interested in label generating trees [16, 17, 18], but rather the nodes are the objects themselves [23, 20]. This generating tree keeps the information of the objects themselves and gives a sort of linear structure to all these elements coming from the unique path $P_{r,u}$. Thus it is natural to look for structures analogous to those on words. In particular, shuffle and concatenation are two natural operations on words that yield Hopf algebras. However, arbitrary paths in generating trees are not immediately interpretable as objects from the class. Only paths from the root correspond to objects. So to make word operations make sense on objects generated by the generating trees, we need rules to map segments around in the tree. We will construct two different Hopf algebras on the vector space of a generating tree. In Section 2.2, we will give the properties of the maps $\varphi_{ds}, \psi_c$ and construct a "concatenation-deshuffle" type Hopf algebra on the vector space $W$ of the generating tree $T$ (Theorem 2.20). In Section 2.3, we will describe the properties of the maps $\varphi_{dc}, \psi_s$ and construct a "shuffle-deconcatenation" type Hopf algebra on $W$ (Theorem 2.37). We will use the combinatorial class of permutations as an example when we go through. Finally, in Section 2.4, we apply these two methods to the generating tree of set partitions; and in Section 2.5, we apply these two methods to the generating tree of open partition diagrams.

## 2.1 Normalizing Maps

We begin by fixing some notation for convenience. Let $N_{i,j} = \{i, i+1, \ldots, j\}$ and $N_k = \{1, 2, \ldots, k\}$ for $i, j, k \in \mathbb{N}_{\geq 0}$ and $i \leq j$.

**Definition 2.1.** Let $\mathcal{C}$ be a combinatorial class with only one element of size zero denoted by $r$, and let $f$ be a map from $\mathcal{C}_n$ to the set of subsets of $\bigcup\limits_{m>n} \mathcal{C}_m$ such that:

(0) for any $v \in \mathcal{C}_n$ and any $m > n$, $f(v) \cap \mathcal{C}_m$ is a finite set;

(1) for any $a \neq b$ in $\mathcal{C}$, $f(a) \cap f(b) = \emptyset$;

(2) for any $v \in \mathcal{C}$, $\exists\ n \in \mathbb{N}_{\geq 0}$ such that $v \in f^n(r)$.

A **generating tree** of $\mathcal{C}$ is a tree $T = T(V, E)$ with the set of vertices $V = \mathcal{C}$ and edges $(u, v) \in E$ if and only if $v \in f(u)$. Call $r$ the root of $T$. Say $v$ can be generated from $u$ if $v \in f^n(u)$ for some $n \geq 0$, and call $u$ an ancestor of $v$ denoted $u \geq v$.

The map $f$ describes the generating rules.

**Example 2.2.** Let $\mathcal{C}$ be the combinatorial class of all permutations. It has only one size 0 element namely the empty permutation $r = \mathbb{1}$. Let $f((i_1, i_2, \ldots, i_n)) = \{(i_1, i_2, \ldots, i_n, n+1), (i_1, i_2, \ldots, n+1, i_n), \ldots, (n+1, i_1, i_2, \ldots, i_n)\}$ for any permutation $(i_1, i_2, \ldots, i_n) \in \mathcal{C}$. Then we have the generating tree $T = T(V, E)$ shown in Figure 2.1.

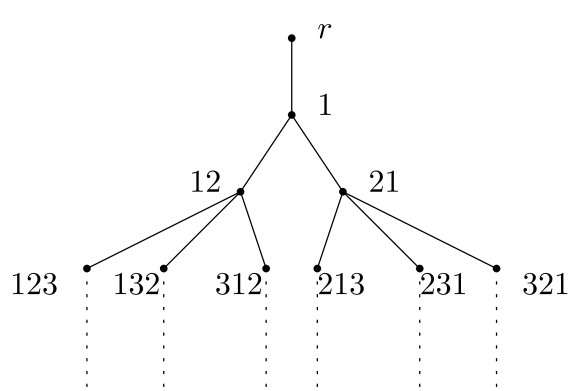



Figure 2.1: A generating tree of permutations

This definition allows us to keep track of how an element $v$ can be generated from the root $r$. Let $P_{r,v}$ denote the unique path from $r$ to $v$. Likewise, we let $P_{u,v}$ be the unique path from $u$ to $v$ if $u$ is an ancestor of $v$. For convenience, sometimes we also regard $P_{u,v}$ as $v_0 v_1 \cdots v_k$ explicitly, where $k = |v| - |u|$, $v_0 = u$, $v_k = v$, and the other $v_i$ are the intermediate steps.

**Definition 2.3.** Let $T = T(V, E)$ be a generating tree of a combinatorial class $\mathcal{C}$ with the root $r$. Let $W$ be the linear space over a field $\mathbb{K}$ spanned by all the vertices $v \in V$. Then $W = \bigoplus_{n=0}^{\infty} W_n$ where $W_n$ is spanned by all the elements with size $n$.

Note $W$ is also $\bigoplus_{n=0}^{\infty} \operatorname{Span} \mathcal{C}_n$, but viewing it in the generating tree, we have the additional information of $P_{r,v}$ associated to each element $v$. Also notice $W_0 = \mathbb{K}r$ which is isomorphic to $\mathbb{K}$. Thus $W$ is a graded and connected $\mathbb{K}$-vector space.

We now introduce a family of maps $\tilde{\varphi}_V$ which will allow us to define $\varphi_{dc}$ and $\varphi_{ds}$ later. Each $\tilde{\varphi}_u$ acts by shifting a path up in the tree to begin at vertex $u$.

**Definition 2.4.** For any $v_0 \in V$, let $\mathcal{A}_{v_0}$ be the subset of $V \times V$ with $(u, v) \in \mathcal{A}_{v_0}$ if $v_0 \geq u \geq v$. Let $\tilde{\varphi}_{v_0} : \mathcal{A}_{v_0} \to V$ be a family of linear maps indexed by $v_0 \in V$ satisfying the following properties.

(0) For any $v_0 \geq u_0 \geq u_k$, $\tilde{\varphi}_{v_0}(u_0, u_k) = v_k$ implies $v_k \leq v_0$, and $k = |P_{u_0,u_k}| = |P_{v_0,v_k}|$. We say $u_i$ and $v_i$ correspond to each other for all $0 \leq i \leq k$, and also say $P_{u_t,u_s}$ and $P_{v_t,v_s}$ correspond to each other for all $0 \leq t < s \leq k$.

(1) For any $u \geq v$, $\tilde{\varphi}_u(u, v) = v$.

(2) For any $u \geq v \geq x \geq y$, $\tilde{\varphi}_u(v, \tilde{\varphi}_v(x, y)) = \tilde{\varphi}_u(x, y)$.

(3) For any $u_{i_1} \geq u_{i_2} \geq u_{i_3} \geq u_{i_4}$ in $P_{u_0,u_k}$, the corresponding $v_{i_1} \geq v_{i_2} \geq v_{i_3} \geq v_{i_4}$ in $P_{v_0,v_k}$ and any $v_{i_0} \geq v_{i_1}$ in $P_{r,v_k}$,

$$\tilde{\varphi}_{v_{i_0}}(u_{i_1}, \tilde{\varphi}_{u_{i_2}}(u_{i_3}, u_{i_4})) = \tilde{\varphi}_{v_{i_0}}(v_{i_1}, \tilde{\varphi}_{v_{i_2}}(v_{i_3}, v_{i_4})).$$

Condition (1) means that attaching any path $P_{u,v}$ to the initial vertex $u$ will result in the same terminal vertex $v$ and hence $\tilde{\varphi}_u$ preserves the path $P_{u,v}$. Condition (2) means that attaching the corresponding path $P_{v,\tilde{\varphi}_v(x,y)}$ of $P_{x,y}$ to any vertex $u$ will lead to the same result as directly attaching $P_{x,y}$ to $u$. Condition (3) shows that attaching corresponding sub-paths twice gives the same result. Here we give a possible family of maps $\tilde{\varphi}_V$ of permutations as an example.

**Example 2.5.** With the same notation as in Example 2.2, for any permutations $v_0 \geq u_0 \geq u_k \in V$, assume $|v_0| = n$, $|u_0| = m$ and $u_k = (l_1, l_2, \ldots, l_{m+k})$ where $n \leq m$. This is equivalent to saying there exist $1 \leq t_1 < t_2 < \cdots < t_m \leq m+k$ such that $(l_{t_1}, l_{t_2}, \ldots, l_{t_m}) = u_0$. Let $\{s_1, s_2, \ldots, s_k\} = N_{m+k} \setminus \{t_1, t_2, \ldots, t_m\}$ where $s_1 < s_2 < \cdots < s_k$. Similarly, there exist $t_{p_1} < t_{p_2} < \cdots < t_{p_n}$ such that $(l_{t_{p_1}}, l_{t_{p_2}}, \ldots, l_{t_{p_n}}) = v_0$. Since $\{t_{p_1}, t_{p_2}, \ldots, t_{p_n}\} \subseteq \{t_1, t_2, \ldots, t_m\}$, $\{t_{p_1}, t_{p_2}, \ldots, t_{p_n}\} \cap \{s_1, s_2, \ldots, s_k\} = \emptyset$. Let $\{q_1, q_2, \ldots, q_{n+k}\} = \{t_{p_1}, \ldots, t_{p_n}, s_1, \ldots, s_k\}$ where $q_1 < q_2 < \cdots < q_{n+k}$. Let

$$l'_{q_i} = \begin{cases} l_{q_i} - m + n, & \text{if } q_i = s_j \text{ for some } j \\ l_{q_i}, & \text{otherwise.} \end{cases}$$

Then we define

$$\tilde{\varphi}_{v_0}(u_0, u_k) = (l'_{q_1}, l'_{q_2}, \ldots, l'_{q_{n+k}}).$$

As an example shown in Figure 2.2, let $v_0 = 132$, $u_0 = 51432$ and $u_2 = 7514362$. Then $t_{p_1} = 3,\ t_{p_2} = 5,\ t_{p_3} = 7$ and $s_1 = 1,\ s_2 = 6$. Thus $q_1 = 1,\ q_2 = 3,\ q_3 = 5,\ q_4 = 6,\ q_5 = 7$ and $\tilde{\varphi}_{132}(51432, 7514362) = 51342$.

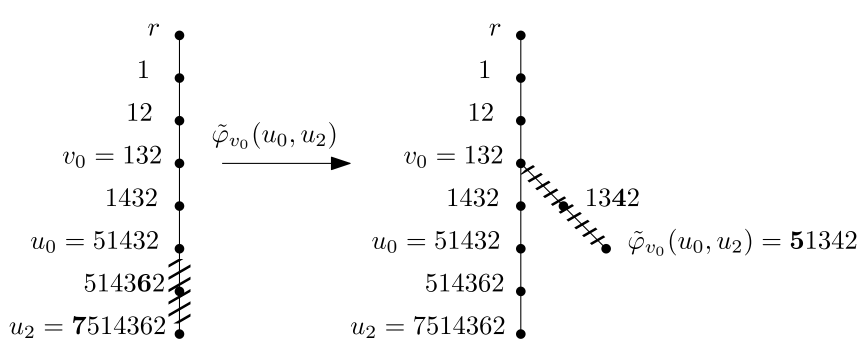



Figure 2.2: Mapping the path between the permutations (51432) and (7514362) to the permutation (132)

This example shows the usual intuition for $\tilde{\varphi}$: paths are mapped up in the tree following analogous steps from the original path but beginning at a different vertex. Now we check that the properties in Definition 2.4 are satisfied. Condition (0) is obvious. Condition (1) is satisfied since if $v_0 = u_0$ then $\{t_{p_1}, t_{p_2}, \ldots, t_{p_n}\} = \{t_1, t_2, \ldots, t_m\}$. Thus $\{q_1, q_2, \ldots, q_{n+k}\} = N_{m+k}$ which implies $\tilde{\varphi}_{v_0}(u_0, u_k) = u_k$. Condition (2) is satisfied because $P_{v,\tilde{\varphi}_v(x,y)}$ and $P_{x,y}$ have the same information of inserting the associated numbers. The same argument works for condition (4) that after two steps the corresponding pieces still contain the same information of inserting the associated numbers into the resulting permutation.

In the following, we will show some properties that $\tilde{\varphi}_V$ have.

**Lemma 2.6.** *With the same notation as in Definition 2.4, assume $\tilde{\varphi}_{v_0}(u_0, u_k) = v_k$. Then for any $v_{i_0} \geq v_{i_3} \geq v_{i_4}$ in $P_{v_0,v_k}$ we have $\tilde{\varphi}_{v_{i_0}}(u_{i_3}, u_{i_4}) = \tilde{\varphi}_{v_{i_0}}(v_{i_3}, v_{i_4})$. Furthermore, if we take $v_{i_0} = v_{i_3}$, then $\tilde{\varphi}_{v_{i_3}}(u_{i_3}, u_{i_4}) = v_{i_4}$.*

*Proof.* Take $u_{i_1} = u_{i_2} = u_{i_3}$, then the corresponding vertices in $P_{v_0,v_k}$ are $v_{i_1} = v_{i_2} = v_{i_3}$. Then for any $v_{i_0} \geq v_{i_1} = v_{i_3}$, applying condition (3) gives us

$$\tilde{\varphi}_{v_{i_0}}(u_{i_3}, \tilde{\varphi}_{u_{i_3}}(u_{i_3}, u_{i_4})) = \tilde{\varphi}_{v_{i_0}}(v_{i_3}, \tilde{\varphi}_{v_{i_3}}(v_{i_3}, v_{i_4})).$$

Applying Condition (2), we have

$$\tilde{\varphi}_{v_{i_0}}(u_{i_3}, u_{i_4}) = \tilde{\varphi}_{v_{i_0}}(v_{i_3}, v_{i_4}).$$

If $v_{i_0} = v_{i_3}$, use Condition (1) to get,

$$\tilde{\varphi}_{v_{i_3}}(u_{i_3}, u_{i_4}) = v_{i_4}.$$

$\square$

This lemma shows that attaching one sub-path of $P_{u_0,u_k}$ to $v_{i_0}$ will result in the same terminal vertex as attaching the corresponding sub-path of $P_{v_0,v_k}$ to $v_{i_0}$. Furthermore, when we attach the sub-path $P_{u_{i_3},u_{i_4}}$ of $P_{u_0,u_k}$ to the corresponding initial vertex $v_{i_3}$, it will result in the same corresponding terminal vertex. This shows the consistency and explains why we call them corresponding vertices.

Before proving the technical results we need, we first prove a similar but simpler result to illustrate the techniques we will use. The following proposition shows that attaching sub-paths of two disconnected paths together to a vertex will be the same as first attaching these two paths together then mapping the corresponding sub-paths to the vertex.

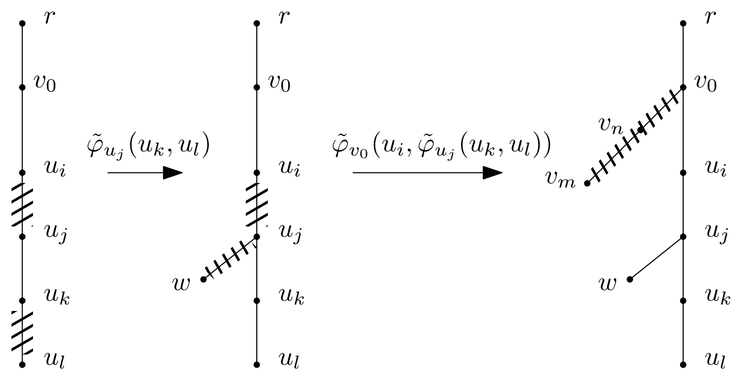



Figure 2.3: The process of attaching $P_{u_i,u_j}$ and $P_{u_k,u_l}$ to $v_0$

**Proposition 2.7.** *With the same notation as shown in Figure 2.3, assume $w = \tilde{\varphi}_{u_j}(u_k, u_l)$ and $\tilde{\varphi}_{v_0}(u_i, w) = v_m$. Suppose $v_n$ corresponds to $u_j$. Take any vertices $v_s \geq v_t \geq v_n \geq v_p \geq v_q$ in $P_{v_0,v_m}$; we have the corresponding vertices $u_s \geq u_t \geq u_j \geq w_p \geq w_q$ in $P_{u_i,w}$. Furthermore, we can find $u_p \geq u_q$ in $P_{u_k,u_l}$ corresponding to $w_p$, $w_q$. Then for any $v_{i_0} \geq v_s$ we have $\tilde{\varphi}_{v_{i_0}}(v_s, \tilde{\varphi}_{v_t}(v_p, v_q)) = \tilde{\varphi}_{v_{i_0}}(u_s, \tilde{\varphi}_{u_t}(u_p, u_q))$.*

*Proof.* Apply condition (3) to $\tilde{\varphi}_{v_0}(u_i, w) = v_m$, we have

$$\tilde{\varphi}_{v_{i_0}}(v_s, \tilde{\varphi}_{v_t}(v_p, v_q)) = \tilde{\varphi}_{v_{i_0}}(u_s, \tilde{\varphi}_{u_t}(w_p, w_q)).$$

Then apply Lemma 2.6 to $\tilde{\varphi}_{u_j}(u_k, u_l) = w$, we get

$$\tilde{\varphi}_{u_t}(w_p, w_q) = \tilde{\varphi}_{u_t}(u_p, u_q).$$

By substituting,

$$\tilde{\varphi}_{v_{i_0}}(v_s, \tilde{\varphi}_{v_t}(v_p, v_q)) = \tilde{\varphi}_{v_{i_0}}(u_s, \tilde{\varphi}_{u_t}(u_p, u_q)).$$

□

The following lemma shows the invariance of attaching several sub-paths of $P_{u_0,u_k}$ to a vertex and attaching the corresponding sub-paths of $P_{v_0,v_k}$ to the same vertex if given $\tilde{\varphi}_{v_0}(u_0, u_k) = v_k$.

**Lemma 2.8.** *With the same notation as in Definition 2.4, assume $\tilde{\varphi}_{v_0}(u_0, u_k) = v_k$ for vertices $v_0 \geq u_0 \geq u_k$, then for any $l \in \mathbb{N}_{\geq 2}$, any $u_{i_1} \geq u_{i_2} \geq \cdots \geq u_{i_{2l-1}} \geq u_{i_{2l}}$ in*

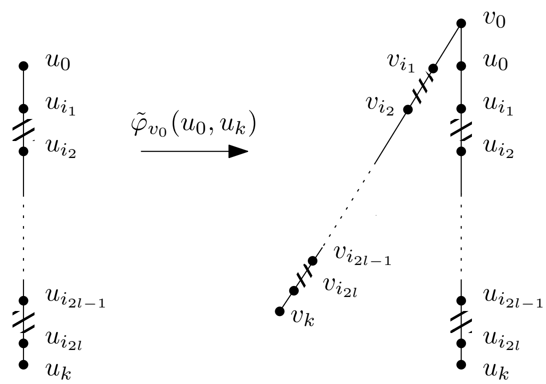



Figure 2.4: Sub-paths in $P_{u_0,u_k}$ and the corresponding sub-paths in $P_{v_0,v_k}$

*$P_{u_0,u_k}$ together with the corresponding $v_{i_1} \geq v_{i_2} \geq \cdots \geq v_{i_{2l-1}} \geq v_{i_{2l}}$ in $P_{v_0,v_k}$ (as shown in Figure 2.4) and any $v_{i_0} \geq v_{i_1}$ in $P_{r,v_k}$, we have*

$$\tilde{\varphi}_{v_{i_0}}(u_{i_1}, \tilde{\varphi}_{u_{i_2}}(\ldots, \tilde{\varphi}_{u_{i_{2l-2}}}(u_{i_{2l-1}}, u_{i_{2l}}))) = \tilde{\varphi}_{v_{i_0}}(v_{i_1}, \tilde{\varphi}_{v_{i_2}}(\ldots, \tilde{\varphi}_{v_{i_{2l-2}}}(v_{i_{2l-1}}, v_{i_{2l}}))).$$

*Proof.* We prove this by induction on $l$. The base case when $l = 2$ is true by condition (3). Suppose for $l < n$ this holds true. Now consider $l = n$. Applying condition (3) for $u_0 \geq u_{i_{2l-2}} \geq u_{i_{2l-1}} \geq u_{i_{2l}}$ together with the corresponding vertices $v_0 \geq v_{i_{2l-2}} \geq v_{i_{2l-1}} \geq v_{i_{2l}}$ and $v_0 \geq v_0$, we get

$$\tilde{\varphi}_{v_0}(u_0, \varphi_{u_{i_{2l-2}}}(u_{i_{2l-1}}, u_{i_{2l}})) = \tilde{\varphi}_{v_0}(v_0, \varphi_{v_{i_{2l-2}}}(v_{i_{2l-1}}, v_{i_{2l}})).$$

Let $u^{'} = \varphi_{u_{i_{2l-2}}}(u_{i_{2l-1}}, u_{i_{2l}})$ and $v^{'} = \varphi_{v_{i_{2l-2}}}(v_{i_{2l-1}}, v_{i_{2l}})$. Applying condition (2), we have

$$\tilde{\varphi}_{v_0}(u_0, u^{'}) = \tilde{\varphi}_{v_0}(v_0, v^{'}) = v^{'}.$$

Consider $u_{i_1} \geq u_{i_2} \geq \cdots \geq u_{i_{2l-3}} \geq u^{'}$ in $P_{u_0,u^{'}}$ and the corresponding vertices $v_{i_1} \geq v_{i_2} \geq \cdots \geq v_{i_{2l-3}} \geq v^{'}$ in $P_{v_0,v^{'}}$. By induction, for any $v_{i_0} \geq v_{i_1}$ we have

$$\tilde{\varphi}_{v_{i_0}}(u_{i_1}, \tilde{\varphi}_{u_{i_2}}(\ldots, \tilde{\varphi}_{u_{i_{2l-4}}}(u_{i_{2l-3}}, u^{'})) = \tilde{\varphi}_{v_{i_0}}(v_{i_1}, \tilde{\varphi}_{v_{i_2}}(\ldots, \tilde{\varphi}_{v_{i_{2l-4}}}(v_{i_{2l-3}}, v^{'})).$$

Substitute $u^{'}$ and $v^{'}$ back we get the result

$$\tilde{\varphi}_{v_{i_0}}(u_{i_1}, \tilde{\varphi}_{u_{i_2}}(\ldots, \tilde{\varphi}_{u_{i_{2l-2}}}(u_{i_{2l-1}}, u_{i_{2l}}))) = \tilde{\varphi}_{v_{i_0}}(v_{i_1}, \tilde{\varphi}_{v_{i_2}}(\ldots, \tilde{\varphi}_{v_{i_{2l-2}}}(v_{i_{2l-1}}, v_{i_{2l}}))).$$

$\square$

**Definition 2.9.** Let $A$ be a subset of edges of $P_{u_0,u_k}$. We can write

$$A = \{P_{u_{i_1},u_{i_2}}, P_{u_{i_3},u_{i_4}}, \dots, P_{u_{i_{2l-1}},u_{i_{2l}}}\},$$

with $P_{u_{i_{2j-1}},u_{i_{2j}}}$, $j = 1, 2 \dots, l$ the connected components of $A$. Let $\kappa(A) = l$ be the number of $A$'s connected components. Define

$$\varphi(A) = \tilde{\varphi}_r(u_{i_1}, \tilde{\varphi}_{u_{i_2}}(u_{i_3}, \dots \tilde{\varphi}_{u_{i_{2l-2}}}(u_{i_{2l-1}}, u_{i_{2l}}))),$$

starting at the bottom.

Notice this $\varphi$ map is well defined on the subset $A$ even if it is not written in the form of connected components. That is to say, for example, if $A = \{(u_0, u_1), (u_1, u_2)\}$, the only connected component of $A$ is $P_{u_0,u_2}$. Then by condition (1),

$$\varphi(A) = \tilde{\varphi}_r(u_0, u_2) = \tilde{\varphi}_r(u_0, \tilde{\varphi}_{u_1}(u_1, u_2)).$$

Note that for any $v \in V$, $\varphi(P_{r,v}) = \tilde{\varphi}_r(r, v) = v$ by condition (1).

Now, we use the $\tilde{\varphi}_V$ defined in Example 2.5 to get the $\varphi$ on permutations.

**Example 2.10.** Take any permutation $u_k = (m_1, m_2, \dots, m_k)$ of size $k$ and any $A = \{P_{u_{i_1},u_{i_2}}, P_{u_{i_3},u_{i_4}}, \dots, P_{u_{i_{2l-1}},u_{i_{2l}}}\}$ subset of edges of $P_{r,u_k}$. Note that for any $s \in N_l$, $P_{u_{i_{2s-1}},u_{i_{2s}}}$ has the information of how $i_{2s-1}+1$ up to $i_{2s}$ are inserted into the permutation $u_k$. By the definition of $\tilde{\varphi}$ in Example 2.5, we know $\varphi(A)$ is the permutation which keeps the order of $i_{2s-1}+1$ up to $i_{2s}$ applied to the elements of $N_p$ with $p = \sum_{s=1}^{l}(i_{2s} - i_{2s-1})$. For instance, let $P_{r,u_7} = 7514362$, then $u_1 = 1,\ u_2 = 12,\ u_3 = 132,\ u_4 = 1432,\ u_5 = 51432,\ u_6 = 514362,\ u_7 = 7514362$. Take $A = \{P_{u_2,u_4}, P_{u_5,u_7}\}$ then $\varphi(A) = \tilde{\varphi}_r(u_2, \tilde{\varphi}_{u_4}(u_5, u_7)) = \tilde{\varphi}_r(12, 614352) = 4213$ as shown in Figure 2.5.

With the techniques in Proposition 2.7 and Lemma 2.8, we can prove an important property of the grafting back-to-root map $\varphi$. This map will contribute to the deshuffle coproduct in Section 2.2. Also this map will be used to bridge the shuffle product and the deconcatenation coproduct in Section 2.3.

**Proposition 2.11.** *Let $A$ be a subset of edges of $P_{u_0,u_k}$ as shown in Definition 2.9. Let $S_0 = 0$ and $S_m = \sum_{n=1}^{m}(i_{2n} - i_{2n-1})$ for $m = 1, 2 \dots, l$. Let $v_{S_l} = \varphi(A)$, as illustrated in Figure 2.6. Let $B' = \{P_{v_{j'_1},v_{j'_2}}, \dots, P_{v_{j'_{2p-1}},v_{j'_{2p}}}\}$ be a subset of edges of $P_{r,v_{S_l}}$ where for each*

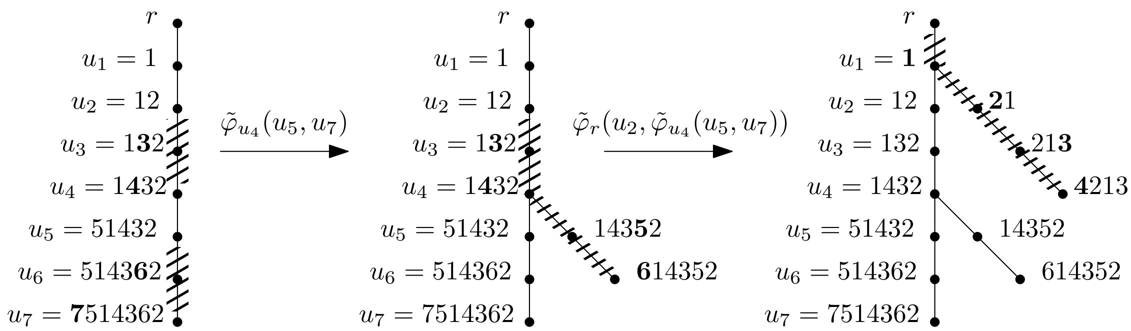



Figure 2.5: Example of $\varphi$ mapping two paths $P_{u_2,u_4}$ and $P_{u_5,u_7}$ to the root $r$

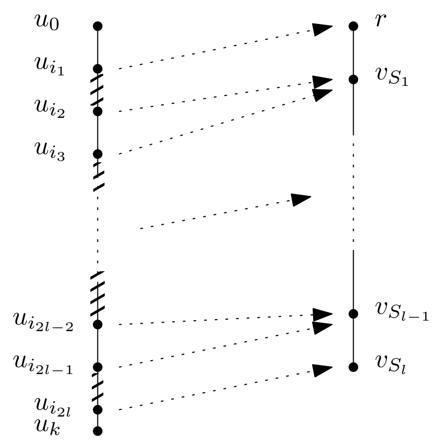



Figure 2.6: $\varphi(A) = v_{S_l}$ in Proposition 2.11

*$m \in N_p$, there exists an $n \in N_l$ such that $S_{n-1} \leq j'_{2m-1} < j'_{2m} \leq S_n$. For $m \in N_p$, let $P_{u_{j_{2m-1}},u_{j_{2m}}} \subseteq A$ correspond to $P_{v_{j'_{2m-1}},v_{j'_{2m}}}$. Let $B = \{P_{u_{j_1},u_{j_2}}, \ldots, P_{u_{j_{2p-1}},u_{j_{2p}}}\}$ then $\varphi(B) = \varphi(B')$.*

*Proof.* We prove this by induction on $\kappa(A)$. The base case $\kappa(A) = 1$ is done by Lemma 2.8. Suppose for $\kappa(A) < l$ the statement is true. Now consider $\kappa(A) = l$, suppose the subset $\{P_{u_{j_{2t-1}},u_{j_{2t}}}, \ldots, P_{u_{j_{2p-1}},u_{j_{2p}}}\} \subseteq B$ consists of those sub-paths in $P_{u_{i_{2l-1}},u_{i_{2l}}}$. Let $u' = \tilde{\varphi}_{u_{i_{2l-2}}}(u_{i_{2l-1}}, u_{i_{2l}})$. For $s = t, t+1, \ldots, p$, let $P_{u'_{j_{2s-1}},u'_{j_{2s}}}$ in $P_{u_{i_{2l-2}},u'}$ be the corresponding sub-path to $P_{u_{j_{2s-1}},u_{j_{2s}}}$ as shown in Figure 2.7.

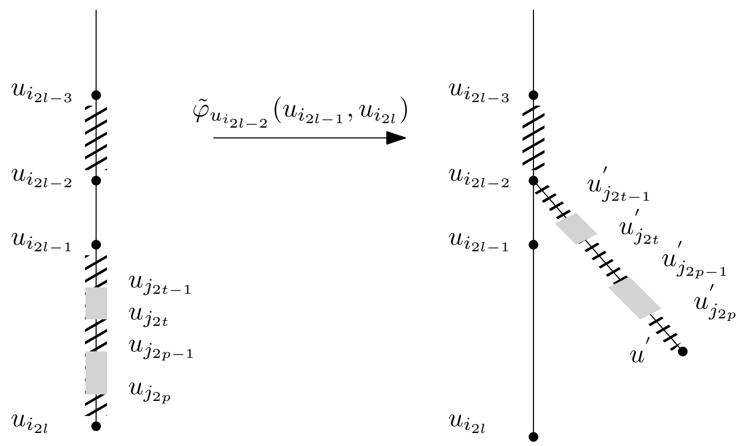



Figure 2.7: $\tilde{\varphi}_{u_{i_{2l-2}}}(u_{i_{2l-1}}, u_{i_{2l}}) = u'$ and its corresponding sub-paths

Then apply Lemma 2.8; for any vertex $w \geq u'_{j_{2t-1}}$, we have

$$\tilde{\varphi}_w(u_{j_{2t-1}}, \tilde{\varphi}_{u_{j_{2t}}}(\ldots, \tilde{\varphi}_{u_{j_{2p-2}}}(u_{j_{2p-1}}, u_{j_{2p}}))) = \tilde{\varphi}_w(u'_{j_{2t-1}}, \tilde{\varphi}_{u'_{j_{2t}}}(\ldots, \tilde{\varphi}_{u'_{j_{2p-2}}}(u'_{j_{2p-1}}, u'_{j_{2p}}))).$$

Now we have

$$v_{S_l} = \varphi(\{P_{u_{i_1}, u_{i_2}}, P_{u_{i_3}, u_{i_4}}, \ldots, P_{u_{i_{2l-3}}, u'}\}),$$

where $\kappa(\{P_{u_{i_1}, u_{i_2}}, P_{u_{i_3}, u_{i_4}}, \ldots, P_{u_{i_{2l-3}}, u'}\}) = l - 1$. Notice $P_{v_{j'_{2s-1}}, v_{j'_{2s}}}$ and $P_{u'_{j_{2s-1}}, u'_{j_{2s}}}$ are corresponding sub-paths, for $s = t, \ldots, p$. By the induction hypothesis and with $w = u_{j_{2t-2}}$, we have

$$\begin{aligned}
& \tilde{\varphi}_r(v_{j'_1}, \tilde{\varphi}_{v_{j'_2}}(\ldots, \tilde{\varphi}_{v_{j'_{2p-2}}}(v_{j'_{2p-1}}, v_{j'_{2p}}))) \\
= \; & \tilde{\varphi}_r(u_{j_1}, \tilde{\varphi}_{u_{j_2}}(\ldots, \tilde{\varphi}_{u_{j_{2t-2}}}(u'_{j_{2t-1}}, \tilde{\varphi}_{u'_{j_{2t}}}(\ldots, \tilde{\varphi}_{u'_{j_{2p-2}}}(u'_{j_{2p-1}}, u'_{j_{2p}}))))) \\
= \; & \tilde{\varphi}_r(u_{j_1}, \tilde{\varphi}_{u_{j_2}}(\ldots, \tilde{\varphi}_{u_{j_{2p-2}}}(u_{j_{2p-1}}, u_{j_{2p}}))),
\end{aligned}$$

which gives us $\varphi(B) = \varphi(B')$. $\square$

## 2.2 Concatenation-Deshuffle Type

In this section, we first define the deshuffle coproduct $\varphi_{ds}$ and introduce the concatenation type product $\psi_c$ which satisfies the properties in Definition 2.16. Then we prove $(W, \psi_c, u_c, \varphi_{ds}, \varepsilon)$ is a Hopf algebra.

### 2.2.1 Deshuffle Coproducts

**Definition 2.12.** For any $v \in V$ define:

$$\varphi_{ds}(v) = \sum_{\substack{A\ subset\\ of\ edges\\ of\ P_{r,v}}} \varphi(A) \otimes \varphi(P_{r,v} \setminus A),$$

where $P_{r,v} \setminus A$ is the subset of edges resulting from the edge set of $P_{r,v}$ with the edges of $A$ removed. Linearly extend $\varphi_{ds}$ to $W$.

Notice $\varphi_{ds}$ is a graded map on $W$ since $\varphi$ preserves the size.

**Example 2.13.** Suppose $P_{r,v} = rv_1v_2\cdots v_k$ and $A = \{P_{r,v_1}, P_{v_{k-1},v_k}\}$, then $P_{r,v} \setminus A = \{P_{v_1,v_{k-1}}\}$. Take $k = 2$. Then the possible subsets of edges of $P_{r,v}$ are $\emptyset$, $\{P_{r,v_1}\}$, $\{P_{v_1,v}\}$, $\{P_{r,v}\}$. Then $\varphi_{ds}(v) = r \otimes v + v_1 \otimes \varphi(P_{v_1,v}) + \varphi(P_{v_1,v}) \otimes v_1 + v \otimes r$.

The following proposition shows that $\varphi_{ds}$ is a coproduct.

**Proposition 2.14.** *The map $\varphi_{ds}$ defined above is coassociative.*

*Proof.* For any $v \in V$,

$$\begin{aligned}
(\varphi_{ds} \otimes id)\varphi_{ds}(v) &= \sum_{\substack{A\ subset\\ of\ edges\\ of\ P_{r,v}}} \varphi_{ds}(\varphi(A)) \otimes \varphi(P_{r,v} \setminus A)\\
&= \sum_{\substack{A\ subset\\ of\ edges\\ of\ P_{r,v}}} \sum_{\substack{B'\ subset\\ of\ edges\\ of\ P_{r,\varphi(A)}}} \varphi(B') \otimes \varphi(P_{r,\varphi(A)} \setminus B') \otimes \varphi(P_{r,v} \setminus A)\\
&= \sum_{\substack{A\ subset\\ of\ edges\\ of\ P_{r,v}}} \sum_{\substack{B \subseteq A\\ (corresponding\\ to\ B')}} \varphi(B) \otimes \varphi(A \setminus B) \otimes \varphi(P_{r,v} \setminus A).
\end{aligned}$$

The last equality is by Proposition 2.11. On the other hand,

$$
\begin{aligned}
(id \otimes \varphi_{ds})\varphi_{ds}(v) &= \sum_{\substack{B\ subset\\ of\ edges\\ of\ P_{r,v}}} \varphi(B) \otimes \varphi_{ds}(\varphi(P_{r,v} \setminus B)) \\
&= \sum_{\substack{B\ subset\\ of\ edges\\ of\ P_{r,v}}} \varphi(B) \otimes \Big( \sum_{C \subseteq P_{r,v} \setminus B} \varphi(C) \otimes \varphi((P_{r,v} \setminus B) \setminus C)\Big) \\
&\overset{\text{Let} A = B \cup C}{=} \sum_{\substack{B\ subset\\ of\ edges\\ of\ P_{r,v}}} \varphi(B) \otimes \Big( \sum_{C \subseteq P_{r,v} \setminus B} \varphi(A \setminus B) \otimes \varphi(P_{r,v} \setminus A)\Big) \\
&= \sum_{\substack{A\ subset\\ of\ edges\\ of\ P_{r,v}}} \sum_{B \subseteq A} \varphi(B) \otimes \varphi(A \setminus B) \otimes \varphi(P_{r,v} \setminus A).
\end{aligned}
$$

Thus, we get $(\varphi_{ds} \otimes id)\varphi_{ds} = (id \otimes \varphi_{ds})\varphi_{ds}$. □

With the $\varphi$ in Example 2.10 we can get the deshuffle coproduct $\varphi_{ds}$ for permutations.

**Example 2.15.** Take $u_3 = 231$ with $u_1 = 1$, $u_2 = 21$. There are 8 possible subsets of edges $\emptyset = \{P_{r,r}\}$, $\{P_{r,u_1}\}$, $\{P_{u_1,u_2}\}$, $\{P_{u_2,u_3}\}$, $\{P_{r,u_2}\}$, $\{P_{u_1,u_3}\}$, $\{P_{r,u_1}, P_{u_2,u_3}\}$, $\{P_{r,u_3}\}$. Then $\varphi_{ds}(231) = r \otimes (231) + (1) \otimes (12) + (1) \otimes (21) + (1) \otimes (21) + (21) \otimes (1) + (12) \otimes (1) + (21) \otimes (1) + (231) \otimes r$.

### 2.2.2 Concatenation Products

The usual concatenation of words $\omega_1$ and $\omega_2$ is $\omega_1\omega_2$ as shown in Example 1.5. Following this concept, we give the definition of the concatenation type product $\psi_c$.

**Definition 2.16.** Let $\psi_c : W \times W \to W$ be a graded bilinear map satisfying the following properties.

(0) (Size preserving): for any $u, v \in V$, $\psi_c(u, v)$ is a vertex in $V$ with size $|u| + |v|$ and for each edge $e$ of $u$ or $v$, we have a unique corresponding edge $e'$ in $\psi_c(u, v)$.

(1) (Identity): for any $u \in V$, $\psi_c(u, r) = u$.

(2) (Associativity): for any $u, v, w \in V$, $\psi_c(\psi_c(u, v), w) = \psi_c(u, \psi_c(v, w))$.

(3) Take $u, v \in V$ and let $x = \psi_c(u, v)$. Let $E_2$ be the corresponding edges of $P_{r,u}$ in $P_{r,x}$ and $E_1$ the corresponding edges of $P_{r,v}$ in $P_{r,x}$. Then $\varphi(E_2) = u$ and $\varphi(E_1) = v$.

(4) Take $u, v \in V$ and let $x = \psi_c(u, v)$. For any $A$ subset of edges of $P_{r,v}$, $B$ subset of edges of $P_{r,u}$ and $A^{'}$, $B^{'}$ the corresponding edges in $P_{r,x}$, we have $\psi_c(\varphi(B^{'}), \varphi(A^{'})) = \varphi(A^{'} \cup B^{'})$.

Condition (3) shows that how the paths $P_{r,u}$ and $P_{r,v}$ concatenate and condition (4) is a technical condition to ensure the compatibility of the map $\phi$ and the map $\psi_c$. We call this concatenation; but actually we don't require $x \leq u$, since we only need the information about which edges of $x$ are coming from $u$, and for all $u, v$ this so-called concatenation process is compatible. Here we give one possible $\psi_c$ for permutations.

**Example 2.17.** With the same notations shown in Example 2.5, take any $u = (i_1, i_2, \ldots, i_m)$ and $v = (j_1, j_2, \ldots, j_n)$ two permutations of size $m, n$ respectively. Define $\psi_c(u, v) = (i_1, i_2, \ldots, i_m, j_1 + m, j_2 + m, \ldots, j_n + m)$. This map is a concatenation of $u$ and $v$. And notice the first $m$ edges of the vertex $\psi_c(u, v)$ correspond to the edges of $u$ and the last $n$ edges correspond to the edges of $v$. For instance, $\psi_c(21, 132) = (21354)$. It is easy to check $\psi_c$ satisfies the above four conditions.

**Proposition 2.18.** *Define $u \cdot v = \psi_c(u, v)$. Then $(W, \psi_c, u_c)$ is an algebra with the identity $r$, where $u_c$ is the trivial product from $\mathbb{K}$ to $W$ with $u_c(k) = kr$.*

*Proof.* Suppose $w = \psi_c(r, v)$. For edges of $P_{r,v}$, the corresponding edges are $P_{r,w}$. So by condition (3), $\varphi(P_{r,w}) = v$ and since $\varphi(P_{r,w}) = w$, we know $w = v$. Combine this with condition (1) to get that $r$ is the identity. From condition (2), we know $\psi_c$ is associative. Since $\psi_c$ is bilinear, we know $\varphi_c$ and $u_c$ are compatible. Thus, $(W, \varphi_c, u_c)$ is an algebra. □

It is easy to see that $\psi_c$ and $u_c$ are both graded maps on $W$. Let $\varepsilon : W \to \mathbb{K}$ be the trivial counit, that is, $\varepsilon(kr) = k$ for $k \in \mathbb{K}$, and $\varepsilon$ is 0 on $\bigoplus_{n=1}^{\infty} W_n$. To prove $(W, \psi_c, u_c, \varphi_{ds}, \varepsilon)$ is a bialgebra, we only need to check $\varphi_{ds}$ is an algebra homomorphism, since the commutativity of the other three diagrams in Definition 1.12.(3) is easy to check.

**Proposition 2.19.** *The coproduct $\varphi_{ds}$ is an algebra homomorphism.*

*Proof.* Notice we only have to prove this on $V \times V$ since $\varphi_{ds}$ is linear and $\psi_c$ is bilinear.

Take $u, v \in V$, let $w = \psi_c(u, v)$ then we have

$$\begin{aligned}
\varphi_{ds}(u) &= \sum_{\substack{B\ subset\\ of\ edges\\ of\ P_{r,u}}} \varphi(B) \otimes \varphi(P_{r,u} \setminus B);\\
\varphi_{ds}(v) &= \sum_{\substack{A\ subset\\ of\ edges\\ of\ P_{r,v}}} \varphi(A) \otimes \varphi(P_{r,v} \setminus A);\\
\varphi_{ds}(u) \cdot \varphi_{ds}(v) &= \sum_{\substack{B\ subset\\ of\ edges\\ of\ P_{r,u}}} \sum_{\substack{A\ subset\\ of\ edges\\ of\ P_{r,v}}} (\varphi(B) \cdot \varphi(A)) \otimes (\varphi(P_{r,u} \setminus B) \cdot \varphi(P_{r,v} \setminus A)).
\end{aligned}$$

Let $E_1$ and $E_2$ be the subsets of $P_{r,u \cdot v}$ corresponding to $P_{r,v}$ and $P_{r,u}$ respectively. Then

$$\varphi_{ds}(u \cdot v) = \sum_{\substack{A' \cup B'\ subset\\ of\ edges\\ of\ P_{r,w}\\ with\ A' \subseteq E_1\\ B' \subseteq E_2}} \varphi(A' \cup B') \otimes \varphi(P_{r,w} \setminus (A' \cup B')),$$

where $A'$ is the subset of edges corresponding to $A$ in $P_{r,v}$ and $B'$ is the subset of edges of corresponding to $B$ in $P_{r,u}$. By Definition 2.16.(3), we know $\varphi(E_2) = u$ and $\varphi(E_1) = v$. By Property 2.11 we get $\varphi(B) = \varphi(B')$ and $\varphi(A) = \varphi(A')$. Using condition (4) in Definition 2.16, we have

$$\varphi(B) \cdot \varphi(A) = \varphi(B') \cdot \varphi(A') = \varphi(A' \cup B').$$

The same argument works for the right hand side of the tensor since $P_{r,u} \setminus B$ and $P_{r,v} \setminus A$ are the complement edges subsets of $A$ in $P_{r,v}$ and $B$ in $P_{r,u}$. So

$$\varphi(P_{r,u} \setminus B) \cdot \varphi(P_{r,v} \setminus A) = \varphi((P_{r,u} \setminus B)') \cdot \varphi((P_{r,v} \setminus A)') = \varphi(P_{r,w} \setminus (A' \cup B')).$$

In both cases the sum runs over all possible subset of edges of $P_{r,u}$ and all possible subset of edges of $P_{r,v}$. Thus we have

$$\varphi_{ds}(u \cdot v) = \varphi_{ds}(u) \cdot \varphi_{ds}(v).$$

□

**Theorem 2.20.** *$(W, \psi_c, u_c, \varphi_{ds}, \varepsilon)$ is a Hopf algebra.*

*Proof.* Combine Propositions 2.14, 2.18 and 2.19 we get that $(W, \psi_c, u_c, \varphi_{ds}, \varepsilon)$ is a bialgebra. Notice the four maps are all graded maps and $W$ is a connected and graded vector space; thus $W$ is a connected and graded bialgebra. Apply Corollary 1.25. Then the result follows. $\square$

**Example 2.21.** Applying Theorem 2.20 to Examples 2.2, 2.5, 2.17, 2.10 and 2.15, we get a concatenation deshuffle type Hopf algebra on permutations.

## 2.3 Shuffle-Deconcatenation Type

In this section, we first give two operations on sets of natural numbers. Then we use these two operations along with $\varphi$ to construct the shuffle type product $\psi_s$ and the deconcatenation type coproduct $\varphi_{dc}$. Then we prove $(W, \psi_s, u_s, \varphi_{dc}, \varepsilon)$ is a Hopf algebra.

### 2.3.1 Reindexing Maps

**Definition 2.22.** Let $m \in \mathbb{N}_{>0}$, $A \subseteq N_m$ and $B \subseteq \mathbb{N}_{>0}$ with $|B| \geq m$. Define $A\langle B\rangle$ to be the subset of $B$ that contains the $a_i^{th}$ smallest element in $B$ for each $a_i \in A$. Likewise, if $A \subseteq B$ define $A\_B$ to be the set of indices of the elements in $A$ relative to $B$, with $B$ ordered from smallest to largest.

For convenience, for $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we say $A \leq k$ if $\forall a \in A,\ a \leq k$.

**Example 2.23.** Let $A = \{1, 4, 7\} \subseteq N_7$ and $B = \{2, 4, 5, 6, 9, 10, 16, 29, 32, 40\}$. Then $A\langle B\rangle = \{2, 6, 16\}$. Let $C = A\langle B\rangle$, then $C\_B = \{1, 4, 7\}$.

By chasing elements, we have the following obvious properties of the operations $\langle\cdot\rangle$ and $\cdot\_\cdot$.

**Proposition 2.24.** *For any $m \in \mathbb{N}_{>0}$, $A \subseteq N_m$ and $B \subseteq \mathbb{N}_{>0}$ with $|B| \geq m$,*

$$(A\langle B\rangle)\_B = A.$$

*For any $C \subseteq D \subseteq \mathbb{N}_{>0}$,*

$$(C\_D)\langle D\rangle = C.$$

Returning to the generating tree $T$, for vertices $u, v \in V$ with sizes $s, t$ respectively there are $s + t$ edges in total in $P_{r,u}$ and $P_{r,v}$. Given a set of $s + t$ edges, there are $\binom{s+t}{s}$ ways to choose which $s$ edges are coming from $P_{r,u}$. Take $A^{(1)}_{s,t} \subset N_{s+t}$ as one choice, denote $A^{(2)}_{s,t} = N_{s+t} \backslash A^{(1)}_{s,t}$. For convenience, we omit the subscripts when it is clear from the context.

**Definition 2.25.** For any $u \in V$ of size $s$, and any $B = \{i_1, i_2, \ldots, i_k\} \subseteq N_s$, define $\overline{B}(u) = \{P_{u_{i_1-1},u_{i_1}}, P_{u_{i_2-1},u_{i_2}}, \ldots, P_{u_{i_k-1},u_{i_k}}\}$ to be the subset of edges of $P_{r,u}$ indexed by $i_j$ with $i_j \in B$.

**Example 2.26.** Let $u = ru_1u_2u_3u_4u_5u_6 \in V$, and take $B = \{2, 3, 6\}$, then

$$\overline{B}(u) = \{P_{u_1,u_2}, P_{u_2,u_3}, P_{u_5,u_6}\} = \{P_{u_1,u_3}, P_{u_5,u_6}\}$$

shown as the shaded area in Figure 2.8.

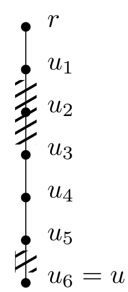



Figure 2.8: Example of $\overline{\{2,3,6\}}(u)$ to illustrate the bar notation

Now we're going to describe a family of maps which lead to the shuffle type product $\psi_s$.

**Definition 2.27.** For $u, v$ two vertices of sizes $s, t$, and $A^{(1)}_{s,t} \subseteq N_{s+t}$, define $\psi_{A^{(1)}_{s,t}}(u, v)$ to be a vertex in $V$ of size $s + t$ such that the following conditions hold.
Let $w = \psi_{A^{(1)}_{s,t}}(u, v)$, then:

(1) $\varphi(\overline{A}^{(1)}_{s,t}(w)) = u$ and $\varphi(\overline{A}^{(2)}_{s,t}(w)) = v$;

(2) For the third vertex $y$ of size $l$ and $B^{(1)}_{s+t,l} \subseteq N_{s+t+l}$,

$$\psi_{B^{(1)}_{s+t,l}}\big(\psi_{A^{(1)}_{s,t}}(u,v), y\big) = \psi_{A^{(1)}\langle B^{(1)}\rangle}\big(u, \psi_{(A^{(2)}\langle B^{(1)}\rangle)_-\left(N\setminus(A^{(1)}\langle B^{(1)}\rangle)\right)}(v, y)\big),$$

where $N = N_{s+t+l}$;

(3) For any $i \leq j \in N_s$ and $p \leq q \in N_t$,

$$\psi_{N_{i,j}\langle A^{(1)}\rangle_-(N_{i,j}\langle A^{(1)}\rangle \cup N_{p,q}\langle A^{(2)}\rangle)}\big(\varphi(P_{u_i,u_j}), \varphi(P_{v_p,v_q})\big) = \varphi\big(\overline{N_{i,j}\left\langle A^{(1)}\right\rangle \cup N_{p,q}\left\langle A^{(2)}\right\rangle}(w)\big).$$

Notice in the RHS of the equation in condition (2), $A^{(2)} \left\langle B^{(1)} \right\rangle$ and $N \setminus (A^{(1)} \left\langle B^{(1)} \right\rangle)$ denote the indices of edges of $v$ and the indices of edges of $v, y$ in combining $u, v, y$ respectively.

**Example 2.28.** Let $u, v, y$ be three vertices with sizes $2, 3, 4$ respectively. Take $A^{(1)} = \{2, 4\}$ and $B^{(1)} = \{1, 4, 6, 7, 8\}$. Then $A^{(1)} \left\langle B^{(1)} \right\rangle = \{4, 7\}$ and $A^{(2)} \left\langle B^{(1)} \right\rangle = \{1, 6, 8\}$. So condition (2) means

$$\psi_{\{1,4,6,7,8\}}(\psi_{\{2,4\}}(u, v), y) = \psi_{\{4,7\}}(u, \psi_{\{1,5,6\}}(v, y)),$$

which is saying after combining $u, v, y$, the edges indexed by $\{4, 7\}$ correspond to the edges of $u$, the edges indexed by $\{1, 6, 8\}$ correspond to the edges of $v$ and the edges indexed by $\{2, 3, 5, 9\}$ correspond to the edges of $y$. Notice $\{1, 6, 8\}_{\_}\{1, 2, 3, 5, 6, 8, 9\} = \{1, 5, 6\}$.

Conditions (1) and (2) in Definition 2.27 will be used to give an associative product. Condition (3) will be used to ensure the compatibility with the coproduct $\varphi_{dc}$ defined later. Here we give an example of a possible family of maps on the permutations.

**Example 2.29.** For permutations $u = (i_1, i_2, \ldots, i_s)$, $v = (j_1, j_2, \ldots, j_t)$ and $A^{(1)}_{s,t} = \{a_1, a_2, \ldots, a_s\} \subseteq N_{s+t}$ with $a_1 < a_2 < \cdots < a_s$. We have $A^{(2)}_{s,t} = \{b_1, b_2, \ldots, b_t\}$ with $b_1 < b_2 < \cdots < b_t$. Define

$$\psi_{A^{(1)}}(u, v) = (a_{i_1}, \ldots, a_{i_s}, b_{j_1}, \ldots, b_{j_t}).$$

For instance, $u = (21)$, $v = (132)$ and $A^{(1)}_{2,3} = \{2, 5\}$ then $\psi_{\{2,5\}}(21, 132) = 52143$.
Condition (1) is satisfied since reducing $(a_{i_1}, \ldots, a_{i_s})$ gives $(i_1, \ldots, i_s)$ and reducing $(b_{j_1}, \ldots, b_{j_t})$ gives $(j_1, \ldots, j_t)$. Condition (3) is satisfied because first truncating a sub-path of $u$ and a sub-path of $v$ then putting them together in the same order will give the same result as first putting them together then reducing the corresponding sub-paths. For instance, let $u = (21)$, $v = (132)$ and $A^{(1)}_{2,3} = \{2, 5\}$ be as above; take $i = j = 1$ and $p = 2, q = 3$. Then $N_{i,j} \left\langle A^{(1)} \right\rangle = \{2\}$ and $N_{p,q} \left\langle A^{(2)} \right\rangle = \{3, 4\}$ which imply $N_{i,j} \left\langle A^{(1)} \right\rangle_{\_}(N_{i,j} \left\langle A^{(1)} \right\rangle \cup N_{p,q} \left\langle A^{(2)} \right\rangle) = \{1\}$. Thus $\psi_{\{1\}}(1, 21) = 132$. On the other side, $\varphi(\overline{\{2, 3, 4\}}(52143)) = 132$. Now check that condition (2) is also satisfied. Take $y = (k_1, k_2, \ldots, k_l)$ and $B^{(1)}_{s+t,l} = \{c_1, c_2, \ldots, c_{s+t}\}$ with $c_1 < c_2 < \cdots < c_{s+t}$. We have $B^{(2)} = \{d_1, d_2, \ldots, d_l\}$ with $d_1 < d_2 < \cdots < d_l$. Let $N = N_{s+t+l}$, then

$$\begin{aligned} \psi_{B^{(1)}}(\psi_{A^{(1)}}(u, v), y) &= \psi_{B^{(1)}}(a_{i_1}, \ldots, a_{i_s}, b_{j_1}, \ldots, b_{j_t}, y) \\ &= (c_{a_{i_1}}, \ldots, c_{a_{i_s}}, c_{b_{j_1}}, \ldots, c_{b_{j_t}}, d_{k_1}, \ldots, d_{k_l}). \end{aligned}$$

Notice $A^{(1)}\left\langle B^{(1)}\right\rangle = \{c_{a_1}, c_{a_2}, \ldots, c_{a_s}\}$ and $A^{(2)}\left\langle B^{(1)}\right\rangle = \{c_{b_1}, c_{b_2}, \ldots, c_{b_t}\}$. Then

$$\begin{aligned}(A^{(2)}\left\langle B^{(1)}\right\rangle)_{-}(N \setminus (A^{(1)}\left\langle B^{(1)}\right\rangle)) &= \{c_{b_1}, \ldots, c_{b_t}\}_{-}\{c_{b_1}, \ldots, c_{b_t}, d_1, \ldots, d_l\} \\ &\overset{\text{Denoted by}}{=} \{e_1, e_2, \ldots, e_t\}.\end{aligned}$$

Similarly, let $\{f_1, f_2, \ldots, f_l\} = \{d_1, d_2, \ldots, d_l\}_{-}\{c_{b_1}, c_{b_2}, \ldots, c_{b_t}, d_1, d_2, \ldots, d_l\}$. This says that the $e_p^{th}$ smallest element in $\{c_{b_1}, c_{b_2}, \ldots, c_{b_t}, d_1, d_2, \ldots, d_l\}$ is $c_{b_p}$ and similarly, $f_q^{th}$ smallest element in $\{c_{b_1}, c_{b_2}, \ldots, c_{b_t}, d_1, d_2, \ldots, d_l\}$ is $d_q$ Thus we have,

$$\psi_{(A^{(2)}\left\langle B^{(1)}\right\rangle)_{-}\left(N\setminus(A^{(1)}\left\langle B^{(1)}\right\rangle)\right)}(v, y) = (e_{j_1}, \ldots, e_{j_t}, f_{k_1}, \ldots, f_{k_l}),$$

and the RHS of condition (2) is

$$\begin{aligned} RHS &= (c_{a_{i_1}}, \ldots, c_{a_{i_s}}, *_{e_{j_1}}, \ldots, *_{e_{j_t}}, *_{f_{k_1}}, \ldots, *_{f_{k_l}}) \\ &= (c_{a_{i_1}}, \ldots, c_{a_{i_s}}, c_{b_{j_1}}, \ldots, c_{b_{j_t}}, d_{k_1}, \ldots, d_{k_l}), \end{aligned}$$

where $*$ is the set $\{c_{b_1}, c_{b_2}, \ldots, c_{b_t}, d_1, d_2, \ldots, d_l\}$.

### 2.3.2 Shuffle Products

Now we can define a shuffle type product $\psi_s$ on $W$.

**Definition 2.30.** For any $u, v \in V$ of sizes $s, t$ respectively, define the product $\psi_s$ to be

$$u \cdot v = \psi_s(u, v) = \sum_{\substack{A_{s,t}^{(1)} \ subset \\ of \ N_{s+t}}} \psi_{A_{s,t}^{(1)}}(u, v),$$

and extend bilinearly to $W \times W$.

**Example 2.31.** Let $u = 12$, $v = 21$, then with the maps in Example 2.29, we have $\psi_s(12, 21) = 1243 + 1342 + 1432 + 2341 + 2431 + 3421$ as shown in Figure 2.9 where the shaded edges correspond to the edges of (12).

From this example, we can see that $\psi_s$ is a shuffle-like operation which shuffles the edges of $u, v$.

**Proposition 2.32.** *$(W, \psi_s, u_c)$ is a graded algebra, where $u_c$ is defined in Proposition 2.18.*

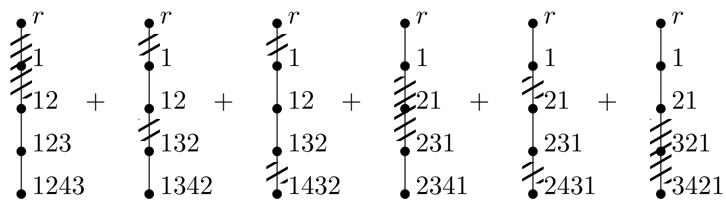



Figure 2.9: Path representation of shuffling permutations (12) and (21)

*Proof.* First we check that the product $\psi_s$ is associative. Let $u, v, w$ be three vertices of sizes $s, t, l$ respectively.

$$\begin{aligned}(u\cdot v)\cdot w &= \Big(\sum_{\substack{A^{(1)}_{s,t}\ subset\\ of\ N_{s+t}}} \psi_{A^{(1)}_{s,t}}(u,v)\Big)\cdot w\\ &= \sum_{\substack{B^{(1)}_{s+t,l}\ subset\\ of\ N_{s+t+l}}}\ \sum_{\substack{A^{(1)}_{s,t}\ subset\\ of\ N_{s+t}}} \psi_{B^{(1)}}\big(\psi_{A^{(1)}}(u,v),w\big),\end{aligned}$$

and

$$\begin{aligned}u\cdot(v\cdot w) &= u\cdot\Big(\sum_{\substack{C^{(1)}_{t,l}\ subset\\ of\ N_{t+l}}} \psi_{C^{(1)}}(v,w)\Big)\\ &= \sum_{\substack{D^{(1)}_{s,t+l}\ subset\\ of\ N_{s+t+l}}}\ \sum_{\substack{C^{(1)}_{t,l}\ subset\\ of\ N_{t+l}}} \psi_{D^{(1)}}\big(u,\psi_{C^{(1)}}(v,w)\big).\end{aligned}$$

Let $\mathcal{AB} = \{(A^{(1)}, B^{(1)}) \in N_{s+t}\times N_{s+t+l} : |A^{(1)}| = s, |B^{(1)}| = s+t\}$ and $\mathcal{CD} = \{(C^{(1)}, D^{(1)}) \in N_{t+l}\times N_{s+t+l} : |C^{(1)}| = t, |D^{(1)}| = s\}$. Then there is a bijection between $\mathcal{AB}$ and $\mathcal{CD}$ given by

$$\begin{aligned}f &: (A^{(1)}, B^{(1)}) \mapsto \Big((A^{(2)}\Big\langle B^{(1)}\Big\rangle)\_\big(N\setminus(A^{(1)}\Big\langle B^{(1)}\Big\rangle)\big),\ A^{(1)}\Big\langle B^{(1)}\Big\rangle\Big),\\ g &: (C^{(1)}, D^{(1)}) \mapsto \Big(D^{(1)}\_\big(D^{(1)}\cup C^{(1)}\Big\langle D^{(2)}\Big\rangle\big),\ D^{(1)}\cup C^{(1)}\Big\langle D^{(2)}\Big\rangle\Big),\end{aligned}$$

where $N = N_{s+t+l}$. Check

$$gf(A^{(1)}, B^{(1)}) = g\Big((A^{(2)}\Big\langle B^{(1)}\Big\rangle)\_\big(N\setminus(A^{(1)}\Big\langle B^{(1)}\Big\rangle)\big),\ A^{(1)}\Big\langle B^{(1)}\Big\rangle\Big).$$

Notice here $A^{(1)}\left\langle B^{(1)}\right\rangle = D^{(1)}$ and $(A^{(2)}\left\langle B^{(1)}\right\rangle)\_\big(N \setminus (A^{(1)}\left\langle B^{(1)}\right\rangle)\big) = C^{(1)}$ which implies $D^{(2)} = N \setminus (A^{(1)}\left\langle B^{(1)}\right\rangle)$ and $C^{(1)} = (A^{(2)}\left\langle B^{(1)}\right\rangle)\_D^{(2)}$. Then

$$\begin{aligned}
gf(A^{(1)}, B^{(1)}) &= \Big(D^{(1)}_(D^{(1)} \cup C^{(1)}\left\langle D^{(2)}\right\rangle),\ D^{(1)} \cup C^{(1)}\left\langle D^{(2)}\right\rangle\Big) \\
&= \Big(D^{(1)}_\big(D^{(1)} \cup (A^{(2)}\left\langle B^{(1)}\right\rangle)\big),\ D^{(1)} \cup (A^{(2)}\left\langle B^{(1)}\right\rangle)\Big) \\
&= (D^{(1)}_B^{(1)},\ B^{(1)}) \\
&= (A^{(1)},\ B^{(1)}),
\end{aligned}$$

and similarly,

$$fg(C^{(1)}, D^{(1)}) = f\Big(D^{(1)}\_(D^{(1)} \cup C^{(1)}\left\langle D^{(2)}\right\rangle),\ D^{(1)} \cup C^{(1)}\left\langle D^{(2)}\right\rangle\Big).$$

Notice here $D^{(1)}\_(D^{(1)} \cup C^{(1)}\left\langle D^{(2)}\right\rangle) = A^{(1)}$ and $D^{(1)} \cup C^{(1)}\left\langle D^{(2)}\right\rangle = B^{(1)}$ which implies $A^{(1)}\left\langle B^{(1)}\right\rangle = D^{(1)}$ and $A^{(2)}\left\langle B^{(1)}\right\rangle = C^{(1)}\left\langle D^{(2)}\right\rangle$. Then

$$\begin{aligned}
fg(C^{(1)}, D^{(1)}) &= \Big((A^{(2)}\left\langle B^{(1)}\right\rangle)_\big(N \setminus (A^{(1)}\left\langle B^{(1)}\right\rangle)\big),\ A^{(1)}\left\langle B^{(1)}\right\rangle\Big) \\
&= \Big((A^{(2)}\left\langle B^{(1)}\right\rangle)_D^{(2)},\ D^{(1)}\Big) \\
&= \Big((C^{(1)}\left\langle D^{(2)}\right\rangle)_D^{(2)},\ D^{(1)}\Big) \\
&= (C^{(1)},\ D^{(1)}).
\end{aligned}$$

Thus, by condition (2) we have the associativity. Now check that $r$ is the identity. $\psi_s(r,u) = \psi_\emptyset(r,u)$ and by condition (1), $\psi_\emptyset(r,u) = \varphi(P_{r,\psi_\emptyset(r,u)}) = u$. Similarly, $\psi_s(u,r) = \psi_{N_s}(u,r) = u$. Since $\psi_s$ is bilinear, $\psi_s$ and $u_c$ are compatible. Furthermore, they are both graded maps, so $(W, \psi_s, u_c)$ is a graded algebra. □

### 2.3.3 Deconcatenation Coproducts

Next we use the $\varphi$ map defined in Definition 2.9 to define the coproduct $\varphi_{dc}$ as follows.

**Definition 2.33.** For any vertex $u \in V$ of size $s$, define

$$\varphi_{dc}(u) = \sum_{j=0}^{s} \varphi(P_{u_j,u}) \otimes u_j,$$

and extend linearly to $W$.

**Proposition 2.34.** *The $\varphi_{dc}$ defined above is coassociative.*

*Proof.*

$$\begin{aligned}
(\varphi_{dc} \otimes id)\varphi_{dc}(u) &= \sum_{j=0}^{s} \varphi_{dc}(\varphi(P_{u_j,u})) \otimes u_j \\
&\overset{\text{Let } w^j = \varphi(P_{u_j,u})}{=} \sum_{j=0}^{s} \sum_{k=0}^{s-j} \varphi(P_{w_k^j,w^j}) \otimes w_k^j \otimes u_j \\
&= \sum_{j=0}^{s} \sum_{k=0}^{s-j} \varphi(P_{u_{k+j},u}) \otimes \varphi(P_{u_j,u_{k+j}}) \otimes u_j.
\end{aligned}$$

The last equality holds by Proposition 2.11 since $P_{w_k^j,w^j}$ corresponds to $P_{u_j,u_{k+j}}$.

$$\begin{aligned}
(id \otimes \varphi_{dc})\varphi_{dc}(u) &= \sum_{k=0}^{s} \varphi(P_{u_k,u}) \otimes \varphi_{dc}(u_k) \\
&= \sum_{k=0}^{s} \sum_{j=0}^{k} \varphi(P_{u_k,u}) \otimes \varphi(P_{u_j,u_k}) \otimes u_j \\
&= \sum_{j=0}^{s} \sum_{k=j}^{s} \varphi(P_{u_k,u}) \otimes \varphi(P_{u_j,u_k}) \otimes u_j.
\end{aligned}$$

Thus $\varphi_{dc}$ is coassociative. □

Now we give the $\varphi_{dc}$ on the permutations.

**Example 2.35.** Take a permutation $u = (a_1, a_2, \ldots, a_s)$ of size $s$. The $\varphi_{dc}(u)$ is the sum over $j \in N_{0,s}$ of the result of reducing the last $(s-j)$ largest numbers of $u$ to $N_{s-j}$ tensor the $j$ smallest numbers of $u$. For instance, take $u = (52413)$,

$$\begin{aligned}
\varphi_{dc}(u) &= \sum_{j=0}^{5} \varphi(P_{u_j,u}) \otimes u_j \\
&= \varphi(r,u) \otimes r + \varphi(1,u) \otimes (1) + \varphi(21,u) \otimes (21) \\
&\quad + \varphi(213,u) \otimes (213) + \varphi(2413,u) \otimes (2413) + r \otimes u \\
&= u \otimes r + (4132) \otimes (1) + (321) \otimes (21) + (21) \otimes (213) + (1) \otimes (2413) + r \otimes u.
\end{aligned}$$

In order to prove that $(W, \psi_s, u_c, \varphi_{dc}, \varepsilon)$ is a bialgebra, we only have to prove that $\varphi_{dc}$ is an algebra homomorphism since the commutativity of the other three diagrams in Definition 1.12 is easy to check.

**Proposition 2.36.** *The coproduct $\varphi_{dc}$ is an algebra homomorphism.*

*Proof.* Let $u, v$ be two vertices of sizes $s, t$ respectively. Then

$$\begin{aligned} u \cdot v &= \sum_{A^{(1)} \subseteq N_{s+t}} \psi_{A^{(1)}}(u, v), \\ \varphi_{dc}(u \cdot v) &= \sum_{A^{(1)} \subseteq N_{s+t}} \sum_{k=0}^{s+t} \varphi(P_{w_k^{A^{(1)}}, w^{A^{(1)}}}) \otimes w_k^{A^{(1)}}, \end{aligned}$$

where $w^{A^{(1)}}$ denotes $\psi_{A^{(1)}}(u, v)$. On the other hand,

$$\begin{aligned} \varphi_{dc}(u) &= \sum_{j=0}^{s} \varphi(P_{u_j,u}) \otimes u_j, \\ \varphi_{dc}(v) &= \sum_{i=0}^{t} \varphi(P_{v_i,v}) \otimes v_i, \end{aligned}$$

and

$$\varphi_{dc}(u) \cdot \varphi_{dc}(v) = \sum_{j=0}^{s} \sum_{i=0}^{t} \sum_{C^{(1)}_{s-j,t-i}, D^{(1)}_{j,i}} \psi_{C^{(1)}}(\varphi(P_{u_j,u}), \varphi(P_{v_i,v})) \otimes \psi_{D^{(1)}}(u_j, v_i).$$

For any $A^{(1)}_{s,t} \subseteq N_{s+t}$ and any $k \in N_{0,s+t}$, we know that there exists a $j$ such that $N_{1,j}\left\langle A^{(1)} \right\rangle \leq k$ and $N_{j+1,s}\left\langle A^{(1)} \right\rangle > k$. Similarly, for $A^{(2)}$ and $i = k - j$, $N_{1,i}\left\langle A^{(2)} \right\rangle \leq k$ and $N_{i+1,t}\left\langle A^{(2)} \right\rangle > k$. Let $C^{(1)} = N_{j+1,s}\left\langle A^{(1)} \right\rangle {}_{-} N_{k+1,s+t}$ and $D^{(1)} = N_{1,j}\left\langle A^{(1)} \right\rangle$. Notice that

$$\begin{aligned} \varphi(\overline{A^{(1)}}(w^{A^{(1)}})) &= u, \\ \varphi(\overline{A^{(2)}}(w^{A^{(1)}})) &= v. \end{aligned}$$

Thus by Proposition 2.11,

$$\begin{aligned} \varphi(P_{r,u_j}) &= \varphi\Big(\overline{N_{1,j}\left\langle A^{(1)} \right\rangle}(w^{A^{(1)}})\Big), \\ \varphi(P_{r,v_i}) &= \varphi\Big(\overline{N_{1,i}\left\langle A^{(2)} \right\rangle}(w^{A^{(1)}})\Big), \\ \varphi(P_{u_j,u}) &= \varphi\Big(\overline{(N_{i+1,s}\left\langle A^{(1)} \right\rangle {}_{-} N_{k+1,s+t})}(w^{A^{(1)}})\Big), \\ \varphi(P_{v_i,v}) &= \varphi\Big(\overline{(N_{j+1,t}\left\langle A^{(2)} \right\rangle {}_{-} N_{k+1,s+t})}(w^{A^{(1)}})\Big). \end{aligned}$$

Then apply condition (3) in Definition 2.27 by noticing

$$N_{j+1,s}\left\langle A^{(1)} \right\rangle {}_{-}(N_{j+1,s}\left\langle A^{(1)} \right\rangle \cup N_{i+1,t}\left\langle A^{(2)} \right\rangle) = C^{(1)},$$

and

$$N_{1,j}\left\langle A^{(1)}\right\rangle \_(N_{1,j}\left\langle A^{(1)}\right\rangle \cup N_{1,i}\left\langle A^{(2)}\right\rangle) = D^{(1)}$$

to get

$$\begin{aligned}
\psi_{C^{(1)}}\big(\varphi(P_{u_j,u}),\varphi(P_{v_i,t})\big) &= \psi_{C^{(1)}}\Big(\varphi\big(\overline{(N_{i+1,s}\left\langle A^{(1)}\right\rangle _N_{k+1,s+t})}(w^{A^{(1)}})\big), \\
&\qquad \varphi\big(\overline{(N_{j+1,t}\left\langle A^{(2)}\right\rangle _N_{k+1,s+t})}(w^{A^{(1)}})\big)\Big) \\
&= \varphi\Big(\overline{(N_{i+1,s}\left\langle A^{(1)}\right\rangle _N_{k+1,s+t}) \cup (N_{j+1,t}\left\langle A^{(2)}\right\rangle _N_{k+1,s+t})}(w^{A^{(1)}})\Big) \\
&= \varphi\Big(\overline{N_{k+1,s+t}}(w^{A^{(1)}})\Big),
\end{aligned}$$

and

$$\begin{aligned}
\psi_{D^{(1)}}(P_{r,u_j},P_{r,v_i}) &= \psi_{D^{(1)}}\Big(\varphi\big(\overline{N_{1,j}\left\langle A^{(1)}\right\rangle}(w^{A^{(1)}})\big),\varphi\big(\overline{N_{1,i}\left\langle A^{(2)}\right\rangle}(w^{A^{(1)}})\big)\Big) \\
&= \varphi\Big(\overline{(N_{1,j}\left\langle A^{(1)}\right\rangle \cup N_{1,i}\left\langle A^{(2)}\right\rangle)}(w^{A^{(1)}})\Big) \\
&= \varphi\Big(\overline{N_{1,k}}(w^{A^{(1)}})\Big).
\end{aligned}$$

Conversely, for any $i, j$ and $C^{(1)}_{i,j}, D^{(1)}_{s-j,t-i}$ we can have $k = i+j$ and $A^{(1)}_{s,t} = D^{(1)} \cup (C^{(1)}+k)$. It is easy to check that these two maps are inverses of each other. Thus we have

$$\varphi_{dc}(u \cdot v) = \varphi_{dc}(u) \cdot \varphi_{dc}(v).$$

□

**Theorem 2.37.** $(W, \psi_s, u_c, \varphi_{dc}, \varepsilon)$ *is a Hopf algebra.*

*Proof.* Note $\varphi_{dc}$ and $\varepsilon$ are graded maps. Combining Propositions 2.18, 2.34 and 2.36, we get that $(W, \psi_s, u_c, \varphi_{dc}, \varepsilon)$ is a connected and graded bialgebra. Then apply Corollary 1.25. The result follows. □

**Example 2.38.** Apply Theorem 2.37 to Examples 2.29, 2.31 and 2.35, we get a shuffle deconcatenation type Hopf algebra on permutations.

Notice that $(W, \psi_c, u_c, \varphi_{ds}, \varepsilon)$ and $(W, \psi_s, u_c, \varphi_{dc}, \varepsilon)$ may not be dual to each other in general. As an example, $\psi_c(12, 21) = (1243)$ but $(12) \otimes (21)$ doesn't show up in $\varphi_{dc}(1243) = (1243) \otimes r + (132) \otimes (1) + (21) \otimes (12) + (1) \otimes (123) + r \otimes (1243)$.

A reason that the other two combinations $(\psi_s,\ \varphi_{ds})$ and $(\psi_c,\ \varphi_{dc})$ don't work is that $\varphi_{ds}$ is not a $\psi_s$-algebra homomorphism, nor is $\varphi_{dc}$ a $\psi_c$-algebra homomorphism. In fact the number of terms on each side of the first diagram in Definition 1.12 isn't equal. As an example, $\psi_c(12, 21) = (1243)$ as above. Then $\varphi_{dc}(1243) = (1243) \otimes r + (132) \otimes (1) + (21) \otimes (12) + (1) \otimes (123) + r \otimes (1243)$ which has 5 terms. Notice $\varphi_{dc}(12) = (12) \otimes r + (1) \otimes (1) + r \otimes (12)$ and $\varphi_{dc}(21) = (21) \otimes r + (1) \otimes (1) + r \otimes (21)$, so $\psi_c(\varphi_{dc}(12), \varphi_{dc}(21))$ has 9 terms.

## 2.4 Application to the Generating Tree of Set Partitions

In this section, we apply both methods in the previous sections to the generating tree of set partitions.

**Definition 2.39.** Let $\mathcal{C}$ be the combinatorial class of set partitions. Let $r$ denote the empty partition. A generating rule is as follows. Given a partition $P = S_1 S_2, \ldots S_k$ of length $m$, where $S_i \cap S_j = \emptyset$ for any $i \neq j$ and $\bigcup_{i=1}^{k} S_i = N_m$,

(1) For each $i \in N_k$, put $m + 1$ in $S_i$;

(2) Add one more part $S_{k+1} = \{m + 1\}$ to P.

Let $T = T(V, E)$ be the generating tree of partitions as shown in Figure 2.10. Let $W$ be the vector space spanned by $V$ over a field $\mathbb{K}$. Notice $W = \bigoplus_{n=0}^{\infty} W_n$ is a graded vector space which is graded by the lengths.

Then we give a possible family of maps $\tilde{\varphi}_\bullet$ as follows.

**Definition 2.40.** Let $v_0 \geq u_0 \geq u_k$ be three vertices in $V$ of length $s, t, t + k$ respectively, then $P_{u_0,u_k}$ has the information of how to add the last $k$ numbers in $u_k$. Define $\tilde{\varphi}_{v_0}(u_0, u_k)$ to be the set partition taking $u_k$ and deleting the $(s + 1)^{st}$ up to $t^{th}$ numbers and relabeling the $i^{th}$ numbers by $i - t + s$ for $i \in N_{t+1,t+k}$.

**Example 2.41.** Take $v_0 = \{1\}$, $u_0 = \{1, 3\}\{2\}$ and $u_3 = \{1, 3, 5\}\{2\}\{4, 6\}$ then

$$\tilde{\varphi}_{\{1\}}(\{1, 3\}\{2\}, \{1, 3, 5\}\{2\}\{4, 6\}) = \{1, 3\}\{2, 4\}.$$

Now we check that these $\tilde{\varphi}$ maps satisfy the three conditions in Definition 2.4. Condition (1) is satisfied because if $v_0 = u_0$, then $s = t$; thus there is no number being deleted and

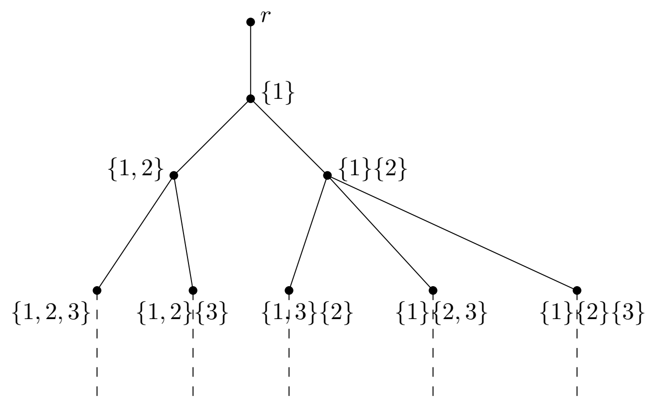



Figure 2.10: A generating tree of partitions

the labels stay the same. If $w_0 \geq v_0$ and has size $l \leq s$, then $\tilde{\varphi}_{w_0}(u_0, u_k)$ is the set partition deleting the $(l+1)^{st}$ up to $t^{th}$ numbers and relabeling the $(t+1)^{st}$ up $(t+k)^{th}$ numbers by $l+1$ up to $l+k$. On the other side, let $Q = \tilde{\varphi}_{v_0}(u_0, u_k)$; $\tilde{\varphi}_{w_0}(v_0, \tilde{\varphi}_{v_0}(u_0, u_k))$ be the set partition deleting the $(l+1)^{st}$ up to $s^{th}$ numbers in $Q$ and relabeling the $(s+1)^{st}$ up to $(s+k)^{th}$ numbers by $l+1$ up to $l+k$. Notice the $(l+1)^{st}$ up to $s^{th}$ numbers in $Q$ are the same numbers in the original set partition. The $(s+1)^{st}$ up to $(s+k)^{th}$ numbers in $Q$ are actually the $(t+1)^{st}$ up to $(t+k)^{th}$ numbers in the original set partition. By computation, they turn out to be relabeled in the same way. Thus condition (2) holds. A similar argument works for checking condition (3).

Then following the Definitions 2.12 and 2.33, we have the two coproducts $\varphi_{ds}$ and $\varphi_{dc}$.

**Example 2.42.** Take the set partition $u = \{1,3\}\{2\}$, all the subsets of edges of $u$ are $\emptyset$, {1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}. Then

$$\begin{aligned}\varphi_{ds}(\{1,3\}\{2\}) &= \{1,3\}\{2\} \otimes r + \{2\}\{1\} \otimes \{1\} + \{1,2\} \otimes \{1\} + \{1\}\{2\} \otimes \{1\} \\ &\quad + \{1\} \otimes \{1\}\{2\} + \{1\} \otimes \{1,2\} + \{1\} \otimes \{2\}\{1\} + r \otimes \{1,3\}\{2\},\end{aligned}$$

and

$$\varphi_{dc}(\{1,3\}\{2\}) = \{1,3\}\{2\} \otimes r + \{2\}\{1\} \otimes \{1\} + \{1\} \otimes \{1\}\{2\} + r \otimes \{1,3\}\{2\}.$$

Now we present a possible way to define the concatenation product $\psi_c$.

**Definition 2.43.** Let $u, v$ be two vertices in $V$ of size $s, t$ respectively. Define $\psi_c(u, v)$ to be the set partition resulting from concatenating $u$ and $v$ with the numbers in $v$ relabeled as $s + 1$ up to $s + t$.

**Example 2.44.**

$$\psi_c(\{1, 3\}\{2\}, \{1, 2\}\{3, 4\}) = \{1, 3\}\{2\}\{4, 5\}\{6, 7\}.$$

We need to check that all four conditions in Definition 2.16 hold. Obviously, conditions (1) and (2) hold. For condition (3), $E_1 = P_{u,\psi_c(u,v)}$ and $E_2 = P_{r,u}$ then $\varphi(E_1) = u$ and $\varphi(E_2)$ is the result of deleting the first $s$ numbers of $\psi_c(u, v)$ and relabelling the last $t$ numbers. Since the first $s$ numbers don't interact with the last $t$ numbers then $\varphi(E_2) = v$. The same reasoning explains condition (4).

The following is a possible way to define the family of maps $\psi_\bullet$ in Definition 2.27.

**Definition 2.45.** Let $u, v \in V$ be set partitions of size $s, t$. Take any $A^{(1)}_{s,t} = \{a_1, \ldots, a_s\} \subseteq N_{s+t}$. Define $\psi_{A^{(1)}}(u, v)$ to be the set partition resulting from putting $u, v$ together with the numbers in $u$ being labeled $a_1$ up to $a_s$ and the numbers in $v$ being labeled $A^{(2)}_{s,t}$ in order.

**Example 2.46.** Take $u = \{1, 3\}\{2\}$, $v = \{1, 2\}\{3, 4\}$ and $A^{(1)}_{3,4} = \{2, 5, 7\}$ then

$$\psi_{A^{(1)}}(u, v) = \{2, 7\}\{5\}\{1, 3\}\{4, 6\}.$$

Then it is clear that condition (1) holds. For condition (2), assume $y$ is another set partition of size $l$, then the LHS of the equation can be explained as first assigning $A^{(1)}$ labels to $u$ and $A^{(2)}$ to $v$. Then for any $B^{(1)}$, assign $A^{(1)} \langle B^{(1)} \rangle$ labels to $u$, $A^{(2)} \langle B^{(1)} \rangle$ labels to $v$ and $B^{(2)}$ labels to $y$. On the right hand side, we can see that $N_{s+t+l} \setminus A^{(1)} \langle B^{(1)} \rangle$ is just the set $B^{(2)} \cup A^{(2)} \langle B^{(1)} \rangle$. Then that means first assigning $(A^{(2)} \langle B^{(1)} \rangle)_{-}(B^{(2)} \cup A^{(2)} \langle B^{(1)} \rangle)$ labels to $v$ when putting $v$ and $y$ together. Thus by applying Proposition 2.24 we have the same result on the RHS. As for condition (3), it holds since first deleting then putting together is the same as first putting together then deleting.

Following Definition 2.30, we get a shuffle type product.

Then by Theorems 2.20 and 2.37, we have two Hopf algebras $(W, \psi_c, u_c, \varphi_{ds}, \varepsilon)$ and $(W, \psi_s, u_c, \varphi_{dc}, \varepsilon)$ on the vector space of all set partitions.

In [19, Page 80, 82], we can see that the product $\cdot$ and the coproduct $\Delta$ in the usual Hopf algebra of set partitions are different from our products and coproducts since their product contains the term of merging two components of a set partition while our products do not. Also, their coproduct does not take apart any component while our coproducts do. For instance,

$$\Delta(\{1,3\}\{2\}) = \{1,3\}\{2\} \otimes r + \{1,2\} \otimes \{1\} + \{1\} \otimes \{1,2\} + r \otimes \{1,3\}\{2\},$$

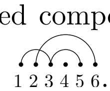
and

$$\begin{aligned}\{1,3\}\{2\} \cdot \{1,2\}\{3,4\} \quad &= \quad \{1,3\}\{2\}\{4,5\}\{6,7\} + \{1,3,4,5\}\{2\}\{6,7\} \\ &\quad +\{1,3\}\{2,4,5\}\{6,7\} + \{1,3,6,7\}\{2\}\{4,5\} \\ &\quad +\{1,3\}\{2,6,7\}\{4,5\} + \{1,3,4,5\}\{2,6,7\} + \{1,3,6,7\}\{2,4,5\}.\end{aligned}$$

## 2.5 Application to the Generating Tree of Open Partition Diagrams

In this section, we apply the two methods in the previous sections to the generating tree of open partition diagrams introduced by Burrill, Elizalde, Mishna and Yen in their paper [14]. We will use the same notation as in the previous sections.

First we present the definition of open partition arc diagrams and their generating rules.

**Definition 2.47.** A partition arc diagram is a graphic representation with labeled vertices ordered along a row and some arcs $(i, j)$ with $i < j$ such that each vertex $i$ is adjacent to at most one larger vertex. Then all of its connected components give a partition. As an example, {{135},{26},{4}} can be represented as 

. An **open partition arc diagram** is a partition arc diagram which allows two additional arc types: semi-openers  and semi-transitories . These open partition diagrams can be viewed as future proper set partitions. Call the number of vertices the size of the diagram. The generating rules are as follows.

Given an open partition arc diagram, one can

(1) Add a fixed point • ;

(2) Add a semi-opener ;

(3) Add a semi-transitory  (provided there is an available semi-arc);

(4) Add a closer (provided there is an available semi-arc).

**Example 2.48.** 1 2 3 4 5 is a valid open partition arc digram and it can generate a proper partition arc diagram 1 2 3 4 5 6 by adding one closer. So 1 2 3 4 5 can be viewed as a future proper set partition.

In order to distinguish vertices of open partition arc diagrams and vertices of a generating tree, we will use endpoints for vertices of arc diagrams.

**Definition 2.49.** Let $\mathcal{C}$ be the combinatorial class of all the open partition diagrams where the size is the number of endpoints. Let $r$ be the empty diagram. Let $T = T(V, E)$ be the generating tree of $\mathcal{C}$ as shown in Figure 2.11. Let $W = \bigoplus_{n=0}^{\infty} W_n$ be the vector space spanned by the vertices in $V$ over a field $\mathbb{K}$. $W$ is graded by the size.

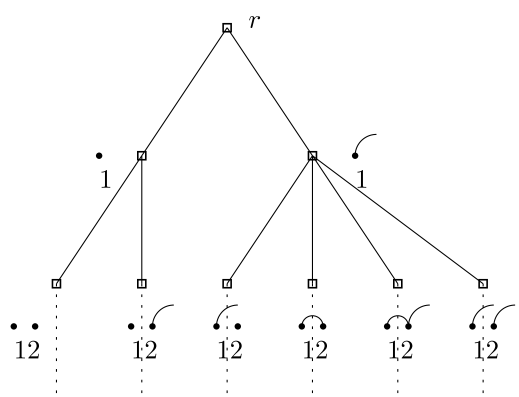



Figure 2.11: The generating tree $T(V, E)$ of the open partition diagrams

We give a possible family of maps $\tilde{\varphi}_V : V \times V \to V$ as follows.

**Definition 2.50.** Let $v_0 \geq u_0 \geq u_k$ be three vertices in $V$ of size $s, t, t+k$ respectively. Then $P_{u_0,u_k}$ has the information of how to add the last $k$ endpoints in $u_k$. Define $\tilde{\varphi}_{v_0}(u_0, u_k)$ to be the diagram taking $u_k$ and deleting the $(s+1)^{st}$ up to $t^{th}$ endpoints together with their outgoing edges (equivalent to reducing the adjacent closers to fixed endpoints or semi-transitories to openers) and relabel the $i^{th}$ endpoint by $i - t + s$ for $i \in N_{t+1,t+k}$.

**Example 2.51.** $\tilde{\varphi}_{1}$( 1 2 3 , 1 2 3 4 5 ) = 1 2 3 .

Now we check that these maps satisfy the three conditions in Definition 2.4. If $v_0 = u_0$, then $s = t$; thus there is no endpoint being deleted and the labels stay the same. So condition (1) is satisfied. If $w_0 \geq v_0$ and has size $l \leq s$, then $\tilde{\varphi}_{w_0}(u_0, u_k)$ is the diagram deleting the $(l+1)^{st}$ up to $t^{th}$ endpoints together with their outgoing edges and relabeling the $(t+1)^{st}$ up $(t+k)^{th}$ endpoints by $l+1$ up to $l+k$. On the other side, let $Q = \tilde{\varphi}_{v_0}(u_0, u_k)$; $\tilde{\varphi}_{w_0}(v_0, \tilde{\varphi}_{v_0}(u_0, u_k))$ be the diagram deleting the $(l+1)^{st}$ up to $s^{th}$ endpoints in $Q$ together with their outgoing edges and relabel the $(s+1)^{st}$ up to $(s+k)^{th}$ endpoints by $l+1$ up to $l+k$. Notice the $(l+1)^{st}$ up to $s^{th}$ endpoints in $Q$ are the same endpoints in the original diagram. The $(s+1)^{st}$ up to $(s+k)^{th}$ endpoints in $Q$ are actually the $(t+1)^{st}$ up to $(t+k)^{th}$ endpoints in the original diagram. By computation, they turn out to be relabeled in the same way. Thus condition (2) holds. A similar argument works for checking condition (3).

With these $\tilde{\varphi}$ maps and following Definitions 2.12 and 2.33, we have the two coproducts $\varphi_{ds}$ and $\varphi_{dc}$. In fact, $\varphi_{ds}$ is the sum over all subsets $A$ of endpoints of the diagram. Given $A$, the left hand side of the tensor is the diagram resulting from deleting the endpoints in $A$ and reducing the rest of the labels. The right hand side of the tensor is the diagram resulting from deleting the endpoints in the complement $A$ and reducing the rest of the labels. $\varphi_{dc}$ is the sum over $i \in N_{0,size}$. Given $i$, the left hand side of the tensor is the diagram resulting from deleting the $1^{st}$ up to $i^{th}$ endpoints and reducing the rest of the labels. The right hand side of the tensor is the diagram resulting from deleting $(i+1)^{st}$ up to the last endpoints and reducing the rest of the labels.

**Example 2.52.** Let $u$ be the open partition arc diagram $\underset{1\,2\,3}{}$, then all the subsets of edges of $u$ are $\emptyset$, {1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}. Then

$$\begin{aligned}\varphi_{ds}(\underset{1\,2\,3}{}) &= \underset{1\,2\,3}{} \otimes r + \underset{1\,2}{} \otimes \underset{1}{} + \underset{1\,2}{} \otimes \underset{1}{} + \underset{1\,2}{} \otimes \underset{1}{} \\ &\quad + \underset{1}{} \otimes \underset{1\,2}{} + \underset{1}{} \otimes \underset{1\,2}{} + \underset{1}{} \otimes \underset{1\,2}{} + r \otimes \underset{1\,2\,3}{},\end{aligned}$$

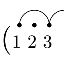
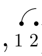
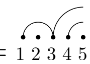
and

$$\varphi_{dc}(\underset{1\,2\,3}{}) = \underset{1\,2\,3}{} \otimes r + \underset{1\,2}{} \otimes \underset{1}{} + \underset{1}{} \otimes \underset{1\,2}{} + r \otimes \underset{1\,2\,3}{}.$$

Here is a possible way to define $\psi_c$.

**Definition 2.53.** Let $u, v$ be two vertices in $V$ of size $s, t$ respectively. Define $\psi_c(u, v)$ to be the diagram resulting from concatenating $u$ and $v$ with the endpoints in $v$ relabeled as $s+1$ up to $s+t$.

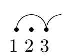
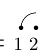
**Example 2.54.**

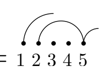
$\psi_c($  ,  $) =$  .

We need to check that all four conditions in Definition 2.16 hold. Obviously, conditions (1) and (2) hold. For condition (3), $E_1 = P_{u,\psi_c(u,v)}$ and $E_2 = P_{r,u}$ then $\varphi(E_1) = u$ and $\varphi(E_2)$ is the result of deleting the first $s$ endpoints of $\psi_c(u, v)$ and relabelling the last $t$ endpoints. Since the first $s$ endpoints don't interact with the last $t$ endpoints then $\varphi(E_2) = v$. The same reasoning explains condition (4).

Here is a possible way to define the family of maps $\psi_\bullet$ in Definition 2.27.

**Definition 2.55.** Let $u, v \in V$ be diagrams of size $s, t$. Take any $A^{(1)}_{s,t} = \{a_1, \dots, a_s\} \subseteq N_{s+t}$. Define $\psi_{A^{(1)}}(u, v)$ to be the diagram resulting from putting $u, v$ together with the endpoints in $u$ being labeled $a_1$ up to $a_s$ and the endpoints in $v$ being labeled $A^{(2)}_{s,t}$ in order.

**Definition 2.56.** Take $u =$  , $v =$  and $A^{(1)}_{3,2} = \{2, 4, 5\}$ then

$\psi_{A^{(1)}}(u, v) =$  .

Then it is clear that condition (1) holds. For condition (2), assume $y$ is another diagram of size $l$, then LHS of the equation can be explained as first assigning $A^{(1)}$ labels to $u$ and $A^{(2)}$ to $v$. Then for any $B^{(1)}$, assign $A^{(1)}\left\langle B^{(1)}\right\rangle$ labels to $u$, $A^{(2)}\left\langle B^{(1)}\right\rangle$ labels to $v$ and $B^{(2)}$ labels to $y$. On the right hand side, we can see that $N_{s+t+l} \setminus A^{(1)}\left\langle B^{(1)}\right\rangle$ is just the set $B^{(2)} \cup A^{(2)}\left\langle B^{(1)}\right\rangle$. Then that means first assigning $(A^{(2)}\left\langle B^{(1)}\right\rangle)_{-}(B^{(2)} \cup A^{(2)}\left\langle B^{(1)}\right\rangle)$ labels to $v$ when putting $v$ and $y$ together. Thus by applying Proposition 2.24 we have the same result on the RHS. As for condition (3), it holds since first deleting then putting together is the same as first putting together then deleting.

Following Definition 2.30, we get a shuffle type product.

Then by Theorems 2.20 and 2.37, we have two Hopf algebras $(W, \psi_c, u_c, \varphi_{ds}, \varepsilon)$ and $(W, \psi_s, u_c, \varphi_{dc}, \varepsilon)$ on the vector space of all open partition diagrams. There could be other Hopf algebras on $W$ coming from other choices of $\tilde{\varphi}$ and $\psi$. In paper [14], Burrill, Elizalde, Mishna and Yen care about $k$-nonnesting partitions. The space $U_k$ spanned by all $k$-nonnesting open partitions arc diagrams (see [14, Definition 2]) are comodules in these two Hopf algebras since these two coproducts won't increase the number of nestings. We hope we can use the method shown in paper [11] to classify all $W$ comodules in these two Hopf algebras we constructed.

# Chapter 3

# Incidence Hopf Algebras On Certain Generating Graphs

While some combinatorial classes possess recursive descriptions in terms of generating trees, more generally, the recursive descriptions can only be represented by generating graphs. This occurs if all well-behaved choices of generating rules which are strong enough to generate all elements also cause some elements to be generated in more than one way. In this chapter, we will give a definition of generating graphs of certain combinatorial classes and describe a family of generating graphs. Then we prove some properties of this family and construct a Hopf algebra on it (Theorem 3.22). We will use the combinatorial class of 3-point Feynman graphs in the scalar field theory $\phi^3$ with only vertex subdivergences as our central example.

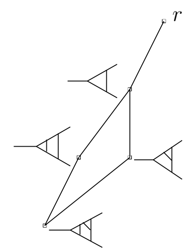
**Definition 3.1.** Let $\mathcal{C}$ be a combinatorial class with only one element of size zero denoted by $r$, and let $f$ be a map from $\mathcal{C}_n$ to the set of subsets of $\bigcup_{m>n} \mathcal{C}_m$ such that:

(1) for any $v \in \mathcal{C}_n$ and any $m > n$, $f(v) \cap \mathcal{C}_m$ is a finite set;

(2) for any $v \in \mathcal{C}, \exists\, n \in \mathbb{N}_{\geq 0}$ such that $v \in f^n(r)$.

A **generating graph** is a graph $G = G(V, E)$ with the set of vertices $V = \mathcal{C}$ and an edge $(u, v) \in E$ if and only if $v \in f(u)$. Call $r$ the root of $G(V, E)$. Say $v$ can be generated from $u$ if $v \in f^n(u)$ for some $n \geq 0$ and call $u$ an ancestor of $v$ denoted by $u \geq v$.

Compared with Definition 2.1, condition (1) in Definition 2.1 is no longer satisfied. As an example, look at the subgraph




of the generating graph shown in Figure 3.1.

**Example 3.2.** Let $T$ be the physical theory $\phi^3$ with the dimension of space-time $D = 6$ as mentioned in Example 1.30. Here we only consider those divergent 1PI graphs with 3 external edges and in which every divergent 1PI subgraph also has 3 external edges. Let $\mathcal{C}$ be the combinatorial class of those Feynman graphs with the number of independent loops as the size. Let $r$ be the empty graph in $\mathcal{C}$. Let insertion of primitive graphs be the generating rule. There are 6 ways to do such an insertion, some of which may give isomorphic results, one for each bijection of the external edges of the inserted graph to the three half edges adjacent to the insertion vertex. Then the generating graph $G(V, E)$ is as shown in Figure 3.1

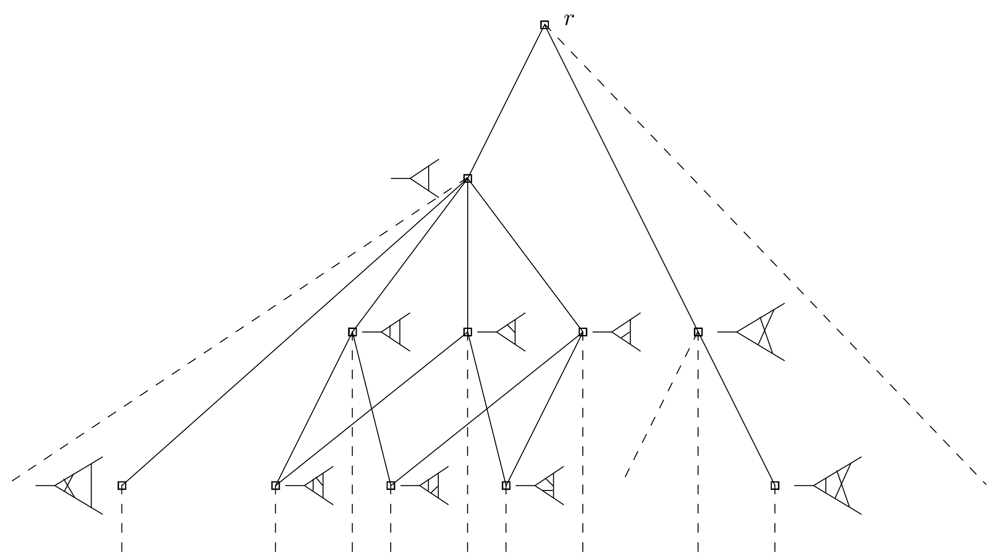



Figure 3.1: A generating graph of $\mathcal{C}$

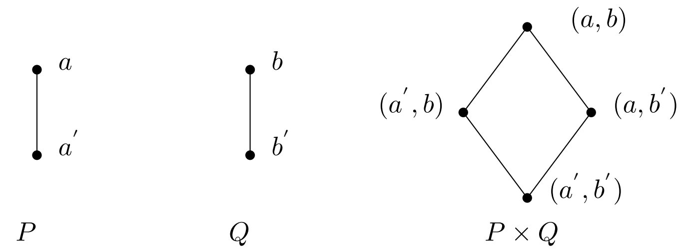



Figure 3.2: Lattices: P, Q and $P \times Q$

In order to describe the family of generating graphs we will work with in this chapter, we need the definition of a lattice and some operations on it ([22]).

**Definition 3.3.** 1) A **lattice** is a partially ordered set in which every two elements have a supremum (or least upper bound denoted by $\vee$) and an infimum (or greatest lower bound denoted by $\wedge$).

2) The **Cartesian product** of two lattices $A$, $B$ is the lattice $A \times B$ with the ordering $(a, b) \leq (a', b')$ if and only if $a \leq a'$ and $b \leq b'$.

**Example 3.4.** Let $P$ be the lattice of $\{a, a'\}$ with $a' < a$ and let $Q$ be the lattice $\{b, b'\}$ with $b' < b$. Then $P \times Q$ is as shown in Figure 3.2.

**Definition 3.5.** Let $G = G(V, E)$ be a generating graph of a combinatorial class $\mathcal{C}$ with root $r$. For $u$, an ancestor of $v$ in $V$, let $G_{u,v}$ be the unique **induced subgraph** from $u$ to $v$ containing all the paths from $u$ to $v$.

Notice that by working in the generating graph, every vertex $v \in V$ contains the extra information of $G_{r,v}$ whereas $v \in \mathcal{C}$ does not. Also every $G_{u,v}$ has a natural poset structure.

**Definition 3.6.** Let $W = W_{\mathbb{K}}(G)$ be the vector space spanned by all the disjoint unions of vertices in $V$ over a field $\mathbb{K}$. For any element $w \in W$, define the size of $w$ to be the sum of the sizes of vertices in $w$. Then $W$ is a graded vector space with $W = \bigoplus_{n=0}^{\infty} W_n$, where $W_n$ is the subspace spanned by the elements of size $n$.

Notice that $W$ can also be viewed as the algebra of polynomials over the vertices in $V$ with the product being disjoint union.

**Example 3.7.** Let $W$ be the vector space of the generating graph $G(V, E)$ of Feynman graphs as defined in Example 3.2.

In the following, we are going to introduce a concept we call maximum independent sublattices of a finite lattice. This construction retrieves the factors of a Cartesian product.

**Definition 3.8.** Let $P$ be a finite lattice with maximum element $a$. Let $P_{u,v}$ be the induced sublattice of $u \geq v$. Let $a_1, a_2 \ldots, a_k$ be all the maximum elements of $P \setminus a$. Define the set of **maximum independent sublattices** of $P$ to be a set of sublattices $\{P_{a,v_1}, P_{a,v_2}, \ldots, P_{a,v_k}\}$ where every $v_i$ satisfies

(1) for any $u_i \in P_{a,v_i} \setminus a$, $u_i \leq a_i$ and $u_i \not\leq a_j$ for any $j \neq i$;

(2) for any $w \in P$ with $w < v_i$, $w$ doesn't satisfy (1).

Call $P$ **irreducible** if it only has one maximum independent sublattice, namely $P$ itself, i.e., its set of maximum independent sublattices is $\{P\}$.

**Example 3.9.** Let $P$ be the lattice as shown in Figure 3.9. Then its set of maximum independent lattices is $\{P_{a,v_1}, P_{a,a_2}\}$. If we delete the vertex $w$, the resulting lattice will be lattice isomorphic to $P_{a,v_1} \times P_{a,a_2}$.

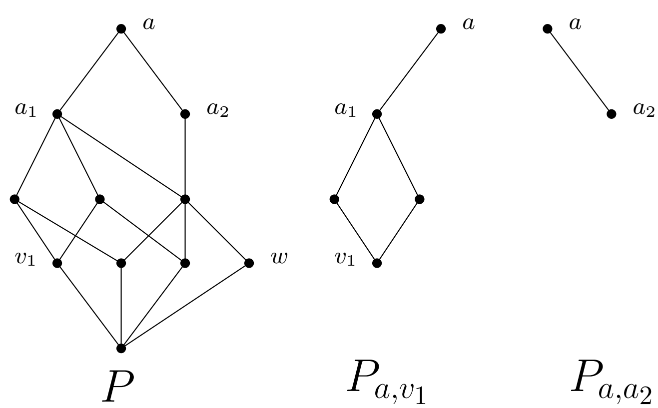



Figure 3.3: Lattice: $P$, $P_{a,v_1}$ and $P_{a,a_2}$

Notice that every sublattice in the set of maximum independent lattices is irreducible since there is only one maximum element $a_i$ in $P_{a,v_i} \setminus a$. Next we will describe the properties that the generating graphs we work with in this chapter have.

**Property 3.10.** *Assume the generating graph $G$ satisfies the following properties:*

*(1) for any $u \geq v$ in $V$, $G_{u,v}$ is a lattice;*

*(2) for any $v \in V$, $G_{r,v}$ is irreducible;*

*(3)* ***independence****: for any $u \geq v$ in $V$, $G_{u,v}$ is lattice isomorphic to the Cartesian product of its maximum independent lattices $G_{u,v_1} \times G_{u,v_2} \times \cdots \times G_{u,v_k}$.*

Given above assumptions, each $G_{u,v}$ has a coordinate system. For any vertex $x \in G_{u,v}$ write its coordinates as $(x_1, x_2, \ldots, x_k)$, where $x_i \in G_{u,v_i}$. Suppose $G_{x,y} \subseteq G_{u,v}$ is irreducible and suppose that, considered in the $G_{u,v}$ coordinate system $G_{x,y}$ has the decomposition $G_{x,y} \cong G_{x_1,y_1} \times G_{x_2,y_2} \times \cdots \times G_{x_k,y_k}$. Since $G_{x,y}$ is irreducible, there exists a unique $i_0$ such that $G_{x,y} \cong G_{x_{i_0},y_{i_0}}$ and $x_j = y_j$ for $j \neq i_0$; otherwise, $x$ would have at least two adjacent descendants in $G_{x,y}$ which contradicts the irreducibility. This means that in the $G_{u,v}$ coordinate system, $G_{x,y}$ varies in the $i_0^{th}$ coordinate and stays constant in the rest of the coordinates.

**Example 3.11.** $G(V, E)$ of Feynman graphs as shown in Example 3.2 satisfies Property 3.10. Condition (1) and (3) are satisfied because for any $u \geq v$, $v$ is obtained by inserting some 3-point 1PI divergent subgraphs at some vertices of $u$ and since inserting into each vertex of $u$ is distinguishable, each one of these subgraphs is associated to one maximum sublattice. Condition (2) is satisfied since every 3-point 1PI divergent graph results in the unique primitive through contracting.

**Definition 3.12.** Say two irreducible $G_{a,b}$ and $G_{c,d}$ have the **parallel relation**, written $\sim$ (or say they are parallel) if there exists a $G_{u,v}$ such that $G_{a,b}, G_{c,d} \subseteq G_{u,v}$ and in $G_{u,v}$'s coordinate system $G_{a,b} \cong G_{a_{i_0},b_{i_0}}$ and $G_{c_{i_0},d_{i_0}} \cong G_{c,d}$ (with notation as above), where $a_{i_0} = c_{i_0}$ and $b_{i_0} = d_{i_0}$ with the same $i_0$.

The idea is that $G_{a,b}$ and $G_{c,d}$ each varies in only one coordinate. For $G_{a,b}$ and $G_{c,d}$ to be parallel, these must be the same coordinates. The $G_{a,b}$ and $G_{c,d}$ only differ from each other in their other coordinates.

**Example 3.13.** Let $G_{a,b}$, $G_{c,d}$ and $G_{u,v}$ be the subgraphs of $G(V, E)$ as shown in Figure 3.4. We can see that $G_{u,v}$ has the maximum independent lattices $\{G_{u,a}, G_{u,c}, G_{u,e}\}$ and $G_{u,v} \cong$

$G_{u,a} \times G_{u,c} \times G_{u,e}$. Then in $G_{u,v}$'s coordinate system, $u, a, b, c, d$ have coordinates

$$\begin{aligned} u &= (u,u,u), \\ a &= (a,u,u), \\ b &= (a,u,e), \\ c &= (u,c,u), \\ d &= (u,c,e). \end{aligned}$$

Thus we can see

$$G_{a,b} = G_{(a,u,u),(a,u,e)} \cong G_{u,e},$$

and

$$G_{c,d} = G_{(u,c,u),(u,c,e)} \cong G_{u,e},$$

which implies $G_{a,b}$ and $G_{c,d}$ are parallel in $G_{u,v}$.

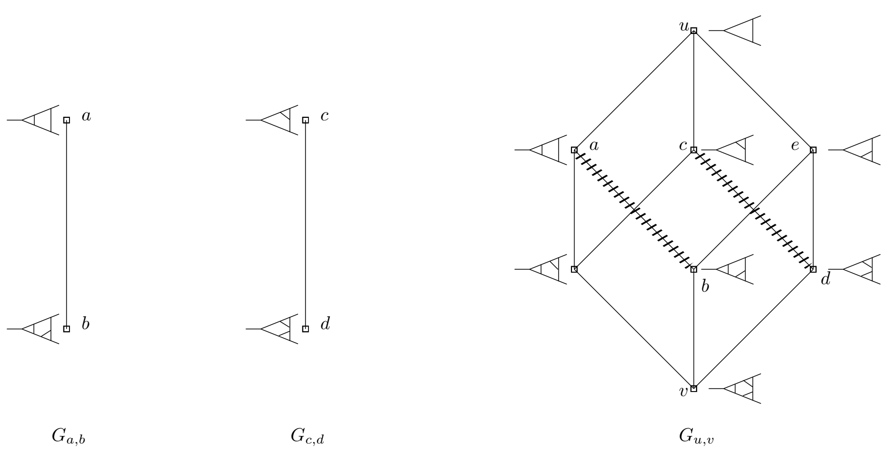



Figure 3.4: $G_{a,b}$, $G_{c,d}$ and $G_{u,v}$

**Lemma 3.14.** *Two irreducible $G_{a,b}, G_{c,d}$ are parallel if and only if $G_{a,b}, G_{c,d}$ are parallel in $G_{a\vee c, b\wedge d}$.*

*Proof.* The necessity is obvious. We only need to prove the sufficiency. Suppose the two irreducible $G_{a,b}, G_{c,d}$ are parallel. Then there exists $G_{u,v}$ and $i_0$ such that $G_{a,b} \cong G_{a_{i_0},b_{i_0}} = G_{c_{i_0},d_{i_0}} \cong G_{c,d}$, where $G_{u,v} \cong G_{u,v_1} \times G_{u,v_2} \cdots \times G_{u,v_k}$ and every vertex $w \in G_{u,v}$ has coordinate $(w_1, w_2, \ldots, w_k)$. Let $x = a \vee c$ and $y = b \wedge d$. Then $x, y \in G_{u,v}$ since $a, c \leq u$ and $b, d \geq v$. So $G_{x,y} \subseteq G_{u,v}$. In the $G_{x,y}$ coordinate system, suppose $G_{x,y} \cong G_{x,y_1'} \times G_{x,y_2'} \times \cdots \times G_{x,y_l'}$ and every vertex $z \in G_{x,y}$ has the coordinate $(z_1', z_2', \ldots, z_l')$. Considered in the $G_{u,v}$ coordinate system, $G_{x,y} \cong G_{x_1,y_1} \times G_{x_2,y_2} \times \cdots \times G_{x_k,y_k}$. For each $G_{x_i,y_i}$ with $x_i \neq y_i$, it can be further decomposed into $G_{x_i,y_1^{(i)}} \times G_{x_i,y_2^{(i)}} \times \cdots \times G_{x_i,y_{n_i}^{(i)}}$ where $\{G_{x_i,y_1^{(i)}}, G_{x_i,y_2^{(i)}}, \ldots, G_{x_i,y_{n_i}^{(i)}}\}$ is the set of maximum independent lattices of $G_{x_i,y_i}$.

Claim: $\{G_{x_s,y_t^{(s)}}\}_{s=1,t=1}^{k,n_s}$ is in one-to-one correspondence with $\{G_{x,y_j'}\}_{j=1}^{l}$.

Proof of claim: For every $G_{x,y_j'}$, since it is irreducible there exists an $s$ such that in the $G_{u,v}$'s coordinate system, $G_{x,y_j'}$ only varies in $s^{th}$ coordinate and is isomorphic to $G_{x_s,(y_j')_s}$. Notice the $s^{th}$ coordinate of $y$ is $y_s$ and $G_{x_s,y_s} \cong G_{x_i,y_1^{(s)}} \times G_{x_i,y_2^{(s)}} \times \cdots \times G_{x_i,y_{n_i}^{(s)}}$. Thus, there exists a $t$ such that $(y_j')_s = y_t^{(s)}$. On the other hand, for every $G_{x_s,y_t^{(s)}}$, there exists a $j$ such that $y_j'$ has the coordinate $(x_1, \ldots, x_{s-1}, y_t^{(s)}, x_{s+1}, \ldots, x_k)$ in the $G_{u,v}$'s coordinate system because of maximality.

Now consider $G_{x_{i_0},y_{i_0}} \cong G_{x_{i_0},y_1^{(i_0)}} \times G_{x_{i_0},y_2^{(i_0)}} \times \cdots \times G_{x_{i_0},y_{n_{i_0}}^{(i_0)}}$ and notice $G_{a_{i_0},b_{i_0}} \subseteq G_{x_{i_0},y_{i_0}}$ is irreducible. Then there exists a $t$ such that $G_{a_{i_0},b_{i_0}} (= G_{c_{i_0},d_{i_0}})$ only varies in $G_{x_{i_0},y_t^{(i_0)}}$. Through the one-to-one correspondence above, there exists a $j$ such that in $G_{x,y}$'s coordinate system $G_{a,b}$ and $G_{c,d}$ only vary in $j^{th}$ coordinate simultaneously. Hence $G_{a,b}, G_{c,d}$ are parallel in $G_{x,y}$. $\square$

From the proof we see that if $G_{x,y} \subseteq G_{u,v}$ then the coordinate system of $G_{x,y}$ in some sense subdivides some coordinates in the coordinate system of $G_{u,v}$. Furthermore, if two irreducible graphs $G_{a,b}$ and $G_{c,d}$ are varying in different coordinates of some $G_{u,v}$ they can't be parallel since they will vary in different coordinates of $G_{a\vee c, b\wedge d}$.

**Proposition 3.15.** *The parallel relation is an equivalence relation.*

*Proof.* The reflexivity and symmetry are clear. We only need to prove the transitivity. By Lemma 3.14, we can suppose now $G_{a,b}, G_{c,d}$ are parallel in $G_{a\vee c, b\wedge d}$ and $G_{c,d}, G_{e,f}$ are parallel in $G_{c\vee e, d\wedge f}$. Let $u = a \vee c,\ v = b \wedge d,\ x = c \vee e,\ y = d \wedge f$, then $G_{u,v} = G_{a\vee c, b\wedge d}$ and $G_{x,y} = G_{c\vee e, d\wedge f}$. We want to prove that $G_{a,b}, G_{e,f}$ are parallel in $G_{a\vee c\vee e, b\wedge d\wedge f}$. Let $p = a \vee c \vee e,\ q = b \wedge d \wedge f$ then $G_{p,q} = G_{a\vee c\vee e, b\wedge d\wedge f}$. Say $G_{u,v} \cong G_{u,v_1} \times G_{u,v_2} \times \cdots \times G_{u,v_k}$

and the coordinate of $u$ is $(u, u, ..., u)$. Since $G_{a,b} \sim G_{c,d}$, there exists an $i$ such that $G_{a,b} \cong G_{a_i,b_i} = G_{c_i,d_i} \cong G_{c,d}$. Since $u = a \vee c$, so in the $i^{th}$ coordinate $a_i \vee c_i = u$ and because $a_i = c_i$ then $a_i = c_i = u$. Similarly, $b_i = d_i = v_i$. We know that $G_{a_i,b_i}$ is one of the maximum independent lattices of $G_{u,v}$. By the discussion above we know that $G_{a,b}$, $G_{c,d}$ and $G_{e,f}$ are varying in the same coordinate, say the $j^{th}$, in the coordinate system of $G_{p,q}$. Since $a \vee c \vee e = p$ so in the $G_{p,q}$ coordinate system with the coordinate of $p$ being $(p, p, \ldots, p)$, one of $a,\ c,\ or\ e$ must have $j^{th}$ coordinate $p$, in which case say it starts varying from $p$. Suppose $G_{a,b}$ starts varying from p which forces the $j^{th}$ coordinate of $u$ to be $p$ as well. Then we know that there is only one coordinate of $G_{u,v}$ merging into the $j^{th}$ coordinate of $G_{p,q}$. Hence $G_{a,b}, G_{c,d}$ are also parallel in $G_{p,q}$ and $G_{c,d}$ also starts varying from $p$. By a similar discussion in $G_{x,y}$ we'll have that $G_{c,d}, G_{e,f}$ are parallel in $G_{p,q}$. Thus, $G_{a,b}, G_{e,f}$ are parallel. □

With the parallel equivalence relation we can describe a linear map that maps all equivalent sublattices up to an object in $\mathcal{C}$ (which is a sublattice starting from the root). For example the two irreducible sublattices $G_{a,b}$ and $G_{c,d}$ shown in Figure 3.4 both describe inserting the object $u$ at the bottom right vertex of $a$ and $c$ respectively. In this case, we believe $G_{a,b}$ and $G_{c,d}$ will be mapped up to $G_{r,u}$.

**Property 3.16.** *Let $\phi$ be a linear map defined on all irreducible $G_{u,v}$ such that $\phi(G_{u,v}) = G_{r,w}$ which satisfies the following conditions:*

*(0) (size preserving): $G_{r,w}$ is lattice isomorphic to $G_{u,v}$;*

*(1) (identity): for any $v \in V$, $\phi(G_{r,v}) = G_{r,v}$;*

*(2) for each $x_1, x_2 \in G_{u,v}$ and the corresponding $y_1, y_2 \in G_{r,w}$, if $G_{x_1,x_2}$ is irreducible then $\phi(G_{y_1,y_2}) = \phi(G_{x_1,x_2})$;*

*(3) if $G_{u,v} \sim G_{x,y}$ then $\phi(G_{u,v}) = \phi(G_{x,y})$.*

Condition (3) shows that mapping an irreducible subgraph twice will give the same result. Condition (4) shows that $\phi$ is well-defined on equivalence classes. By convention, for any $v \in V$, $\phi(G_{v,v}) = r$. By the independence assumption in Property 3.10, we can linearly extend this $\phi$ to all $G_{u,v}$ by defining $\phi(G_{u,v}) = \bigcup_{i=1}^{k} \phi(G_{u,v_i})$ (the disjoint union of the $\phi(G_{u,v_i})$) where $G_{u,v}$ has maximum independent lattices $\{G_{u,v_1}, G_{u,v_2}, \ldots, G_{u,v_k}\}$. Note the identity of disjoint unions is $r$.

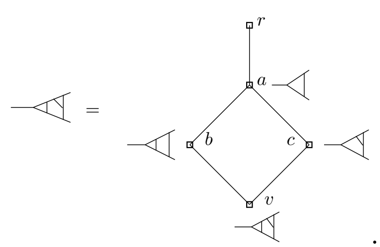
**Example 3.17.** Take $W$ the as shown in Example 3.7. For any irreducible $G_{u,v}$, we know that $v$ is obtained from $u$ by inserting a 1PI divergent 3-point Feynman graph $\gamma$ into a vertex of $u$ and we know that $u$ is the graph resulting from $v$ contracting $\gamma$. Let $\gamma$ inherit the orderings of its external edges from $v$. Then we can define $\phi(G_{u,v}) = \gamma$.

**Lemma 3.18.** *For any $G_{u,v}$, we have $G_{u,v} \cong G_{u,v_1} \times G_{u,v_2} \times \cdots \times G_{u,v_k}$. Take any $y \in G_{u,v}$ with coordinates $(y_1, y_2, \ldots, y_k)$. Then $\phi(G_{u,y}) = \bigcup_{i=1}^{k} \phi(G_{u,y_i})$ and $\phi(G_{y,v}) = \bigcup_{i=1}^{k} \phi(G_{y_i,v_i})$.*

*Proof.* Since $G_{u,y} \cong G_{u,y_1} \times G_{u,y_2} \times \cdots \times G_{u,y_k}$ and every $G_{u,y_i}$ is irreducible, $G_{u,y}$ has the maximum independent lattices $\{G_{u,y_1}, G_{u,y_2}, \cdots, G_{u,y_k}\}$. By the extended definition $\phi(G_{u,y}) = \bigcup_{i=1}^{k} \phi(G_{u,y_i})$.
$G_{y,v} \cong G_{y_1,v_1} \times G_{y_2,v_2} \times \cdots \times G_{y_k,v_k}$ and each $G_{y_i,v_i}$ with $y_i \neq v_i$ has its maximum independent lattices $\{G_{y_i,z_j^{(i)}}\}_{j=1}^{t_i}$. From the proof of Lemma 3.14 we know that the maximum independent lattices of $G_{y,v}$ are in one to one correspondence via the parallel relation to $\{G_{y_i,z_j^{(i)}}\}_{i=1,j=1}^{k,t_i}$. By the Property 3.16 (3), $\phi(G_{y,x_s}) = \phi(G_{y_i,z_j^{(i)}})$. Then by definition $\phi(G_{y,v}) = \bigcup_{i=1}^{k} \phi(G_{y_i,v_i})$. $\square$

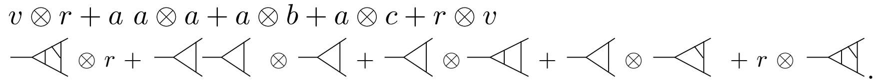
**Definition 3.19.** Let $u, \varepsilon$ be the trivial unit and counit. Let $\cdot$ be the disjoint union and define a linear map $\Delta : W \to W$ as $\Delta(G_{r,v}) = \sum_{u \in G_{r,v}} \phi(G_{u,v}) \otimes G_{r,u}$ for all $v \in V$ and extended to $W$ as an algebra homomorphism.

**Example 3.20.** As an example, let $v$ be




Then

$$
\begin{aligned}
\Delta(v) &= \phi(G_{r,v}) \otimes G_{r,r} + \phi(G_{a,v}) \otimes G_{r,a} + \phi(G_{b,v}) \otimes G_{r,b} + \phi(G_{c,v}) \otimes G_{r,c} + \phi(G_{v,v}) \otimes G_{r,v} \\
&= v \otimes r + \phi(G_{a,b})\phi(G_{a,c}) \otimes a + a \otimes b + a \otimes c + r \otimes v \\
&= v \otimes r + a\ a \otimes a + a \otimes b + a \otimes c + r \otimes v
\end{aligned}
$$

$=$ 

This agrees with the coproduct $\Delta$ in the renormalization Hopf algebra of $\phi^3$.

**Proposition 3.21.** *The vector space $(W, \cdot, u, \Delta, \varepsilon)$ is a bialgebra.*

*Proof.* It is clear that $(W, \cdot, u)$ is an algebra. Now we first need to prove $(W, \Delta, \varepsilon)$ is a coalgebra then we need to prove that the four diagrams in Definition 1.12 commute. In order to show $W$ is a colagebra, we will only check the coassociativity here since it is easy to check the second diagram in Definition 1.6 commutes. Since $\Delta$ is defined as an algebra homomorphism and by Proposition 1.27 we only need to check $\Delta$ is coassociative for every $v \in V$. For any $v \in V$,

$$\Delta(G_{r,v}) = \sum_{z \in G_{r,v}} \phi(G_{z,v}) \otimes G_{r,z}.$$

We know

$$\begin{aligned} (id \otimes \Delta)\Delta(G_{r,v}) &= \sum_{z \in G_{r,v}} \sum_{u \in G_{r,z}} \phi(G_{z,v}) \otimes \phi(G_{u,z}) \otimes G_{r,u} \\ &= \sum_{u \in G_{r,v}} \sum_{z \in G_{u,v}} \phi(G_{z,v}) \otimes \phi(G_{u,z}) \otimes G_{r,u}, \end{aligned}$$

and

$$(\Delta \otimes id)\Delta(G_{r,v}) = \sum_{u \in G_{r,v}} \Delta(\phi(G_{u,v})) \otimes G_{r,u}.$$

Then we have to show that

$$\Delta(\phi(G_{u,v})) = \sum_{z \in G_{u,v}} \phi(G_{z,v}) \otimes \phi(G_{u,z}).$$

In the coordinate system of $G_{u,v}$ we have $G_{u,v} \cong G_{u,v_1} \times G_{u,v_2} \times \cdots \times G_{u,v_l}$, where $\{G_{u,v_1}, G_{u,v_2}, \cdots, G_{u,v_l}\}$ is the set of maximum independent sublattices of $G_{u,v}$. Let $G_{r,\tilde{w}_i} = \phi(G_{u,v_i})$ for $i = 1, 2 \ldots, l$. Then we have $\phi(G_{u,v}) = \bigcup_{i=1}^{l} G_{r,\tilde{w}_i}$. So

$$\Delta(\phi(G_{u,v})) = \sum_{\{\tilde{p}_i \in G_{r,\tilde{w}_i}\}_{i=1}^{l}} \bigcup_i \phi(G_{\tilde{p}_i,\tilde{w}_i}) \otimes \bigcup_i G_{r,\tilde{p}_i}.$$

Let $p_i$ be the vertex in $G_{u,v_i}$ that corresponds to $\tilde{p}_i$ for $i = 1, 2, \ldots, l$. Take $z = (p_1, p_2, \ldots, p_l) \in G_{u,v}$, then by Lemma 3.18

$$\begin{aligned}
\phi(G_{u,z}) &= \bigcup_{i=1}^{l} \phi(G_{u,p_i}) \\
&= \bigcup_{i=1}^{l} \phi(G_{r,\tilde{p}_i}) \\
&= \bigcup_{i=1}^{l} G_{r,\tilde{p}_i},
\end{aligned}$$

and

$$\begin{aligned}
\phi(G_{z,v}) &= \bigcup_{i=1}^{l} \phi(G_{p_i,v_i}) \\
&= \bigcup_{i=1}^{l} \phi(G_{\tilde{p}_i,\tilde{w}_i}).
\end{aligned}$$

Conversely, for any $z \in G_{u,v}$, the coordinate of $z$ is $(p_1, p_2, \ldots, p_l)$ and every $p_i$ corresponds to a $\tilde{p}_i \in G_{r,\tilde{w}_i}$. Thus, $\Delta(\phi(G_{u,v})) = \sum_{z \in G_{u,v}} \phi(G_{z,v}) \otimes \phi(G_{u,z})$. So $\Delta$ is coassociative and hence $(W, \Delta, \varepsilon)$ is a coalgebra. Next we check the four diagrams in Definition 1.12 commute. The first diagram commutes since we defined $\Delta$ to be an algebra homomorphism. The second diagram commutes because that for any $k \in \mathbb{K}$,

$$\Delta \circ u(k) = \Delta(kr) = k\Delta(r) = k(r \otimes r),$$

and

$$(u \otimes u)(k \otimes 1) = (kr) \otimes r = k(r \otimes r).$$

To show the commutativity of the third diagram, we only need to check it commutes for any $k(r \otimes r)$ since $\varepsilon(\bigoplus_{n=1}^{\infty} W_n) = 0$. Take any $k \in \mathbb{K}$,

$$\varepsilon \circ \cdot (kr \otimes r) = \varepsilon(kr) = k,$$

and

$$\varepsilon \otimes \varepsilon(kr \otimes r) = k \otimes 1 \cong k.$$

The commutativity of the last diagram is trivial and thus $(W, \cdot, u, \Delta, \varepsilon)$ is a bialgebra. $\square$

**Theorem 3.22.** $(W, \cdot, u, \Delta, \varepsilon)$ *is a Hopf algebra.*

*Proof.* Notice $W$ is a graded vector space and it is graded by the size of the combinatorial objects in $\mathcal{C}$. Also $W_0 \cong \mathbb{K}$ since there is only one element of size 0, so $W$ is connected. The four maps are easy to check graded so $W$ is a graded and connected bialgebra. By Corollary 1.25, $W$ is a Hopf algebra. □

This Hopf algebra is an incidence Hopf algebra [9] with a mapping back map $\phi$ which induces an order compatible relation.

**Example 3.23.** Combine Examples 3.2, 3.7, 3.11 and 3.17 and apply Theorem 3.22 we know the vector space $W$ of 3-point 1PI divergent graphs in $\phi^3$ with only vertex insertions is a Hopf algebra. Notice in this case, this Hopf algebra coincides with the renormalization Hopf algebra, which is itself also an incidence Hopf algebra.

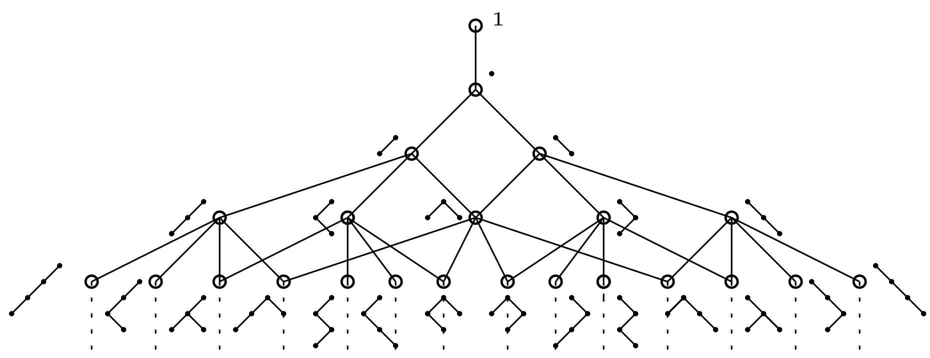
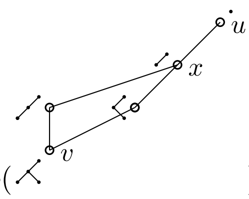
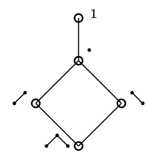


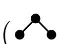
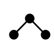
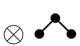

Figure 3.5: A generating graph of plane binary trees

**Example 3.24.** Let $n$ be a positive integer and let $G$ be a generating graph of plane $n-$ary trees where each vertex has $n$ ordered subtrees as children, some of which may be empty. Here we take $n = 2$ as an example, in this case each child is a left or right child. Figure 3.5 is one generating graph of plane binary trees. Let $W$ be the vector space spanned by all finite disjoint unions of plane binary trees over a field $\mathbb{K}$. If $G_{u,v}$ is irreducible, denote by $x$ the only child of $u$ (in $G_{u,v}$) and define $\varphi(G_{u,v})$ to be the binary tree $v$ with $x$ contracted.

For instance,

$\phi($  $) =$  .

Now we check this $\phi$ satisfies the four conditions in Property 3.16. Let $w$ denote the plane binary tree of $v$ with $x$ contracted. Condition (1) is satisfied because $w$ is irreducible and has exactly the same information of how to generate the bottom part from $u$ to $v$. Condition (2) is satisfied because contracting a node is equivalent to doing nothing. So $\phi(G_{r,v}) = G_{r,v}$. Condition (3) is satisfied for the same reason as (1). In this case, the parallel relation describes inserting the identical subtree at the same node but in different levels. So the information is the same; so applying $\phi$ will result in the same tree. Thus, condition (4) is also satisfied. Then following the process described above, we will get a Hopf algebra $(W, \cdot, u, \Delta, \varepsilon)$ for plane binary trees. As an example of this coproduct,

$\Delta($  $) =$  $\otimes r + \bullet \otimes$  $+ \bullet \otimes$  $+ \bullet\bullet \otimes \bullet + r \otimes$  .

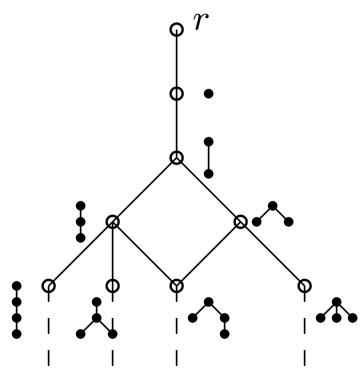
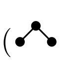
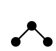
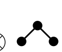


Figure 3.6: A generating graph of rooted trees

In the Connes-Kreimer's Hopf algebra of rooted trees in Example 1.29, as an example

$\Delta($  $) =$  $\otimes r + 2\bullet \otimes$  $+ \bullet\bullet \otimes \bullet + r \otimes$  .

Note this coproduct is the same as the coproduct in Example 3.24 when we forget the left and right.

On the other hand, if we take a naive way to build a generating graph for all rooted trees as shown in Figure 3.6, we could see the generating rule doesn't expose the underlying Hasse diagram of  so we have no way to get the term $\bullet\bullet \otimes \bullet$ or the coefficient 2 for the term $\bullet \otimes$  through our construction process.

So we ask: is there a suitable generating graph representation for rooted trees such that we can use our construction to get the Connes-Kreimer's Hopf algebra? Or how can we extend our construction to make it work even for the naive generating graph in Figure 3.6?

# Chapter 4

# Hopf Algebras On Self-Avoiding Polygons

In this chapter, we will construct two Hopf algebras on the polynomial algebra $U$ of connected self-avoiding polygons. First in Section 4.1, we present a generating graph of self-avoiding polygons introduced by Rechnitzer and Janse van Rensburg in [7]. Then in Section 4.2, we construct two coproducts $\Delta_w$ and $\Delta_h$ based on widths and heights respectively. At last, in Theorem 4.9, we prove with $\Delta_w$ and the trivial counit that $U$ is a Hopf algebra graded by the widths, and in Theorem 4.11, we prove with $\Delta_h$ and the trivial counit that $U$ is a Hopf algebra graded by the heights.

## 4.1 A Generating Graph of Self-Avoiding Polygons

**Definition 4.1.** A self-avoiding walk is a path on a lattice that never revisits the same point. A **self-avoiding polygon** (referred to as a SAP) is a closed self-avoiding walk on a lattice. Given a SAP $P$, define the **length** $l(P)$ as the perimeter of $P$, the **width** $w(P)$ as the horizontal distance between the left most and the right most edges and the **height** $h(P)$ as the vertical distance between the top most and the bottom most edges.

In Rechnitzer and Janse van Rensburg's paper [7, page 3], they give a recursive generating description of self-avoiding polygons as follows. Given a SAP, consider all possible pairs of vertices such that parallel edges of unit length can be inserted to obtain a longer SAP.

Let $\mathcal{C}$ be the combinatorial class of all the SAPs with the length as the size. Let $\mathbb{1}$ be the empty SAP which by convention generates the smallest nontrivial SAP □. Let $G(V, E)$ be the generating graph of all elements in $\mathcal{C}$ where $(P_1, P_2) \in E$ if there exits a pair of vertices of $P_1$ where inserting parallel edges will result in $P_2$.

**Example 4.2.** Take the smallest SAP $P =$ □. There are 8 ways to insert parallel edges into a pair of vertices of $P$ as shown in Figure 4.1.

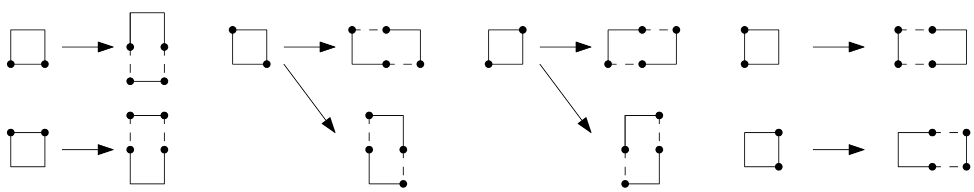


Figure 4.1: Eight ways of inserting parallel edges into P

In fact, there are only two different outcomes as we ignore the duplicated ones.

Then the generating graph $G(V, E)$ of SAPs is as shown in Figure 4.2.

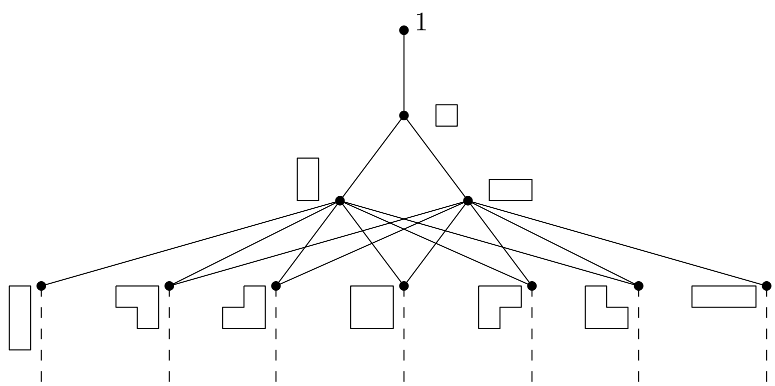



Figure 4.2: Generating graph $G(V, E)$ of self-avoiding polygons

We can see that this generating graph doesn't satisfy the requirements in Chapter 3 because not every $G_{u,v}$ satisfies Property 3.10.

As a counterexample, consider the poset shown in Figure 4.3. $G_{u,v}$ is not a lattice because $v_1$ and $v_2$ do not have an infimum.

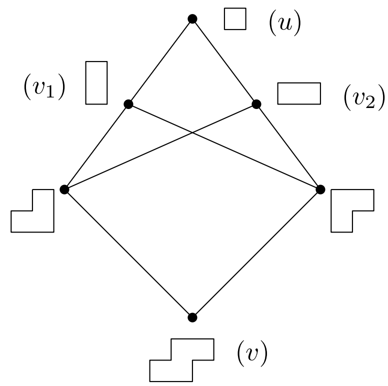



Figure 4.3: A counterexample to Property 3.10

## 4.2 Width Coproduct $\Delta_w$ and Height Coproduct $\Delta_h$

Let $U$ be the $\mathbb{K}$-vector space spanned by all disjoint unions of SAPs. For a disjoint union $s$, let $w(s)$, $h(s)$ be the sums of widths and the sum of heights of the SAPs in $s$ respectively.

There are several ways to grade $U$. From the above generating graph, a natural way is to grade $U$ by the sum of the lengths of every SAP in a disjoint union. We can also grade $U$ by the sum of the widths or the sum of the heights. Denote these two graded vector space by $U = W = \bigoplus_{n=0}^{\infty} W_n$ and $U = H = \bigoplus_{m=0}^{\infty} H_m$ respectively. Unfortunately these latter two grading systems only give diagonal gradings on the generating graph we already have. Notice we do not grade $U$ by the areas because the area grows in various ways from one level to the next level while the length is always increased by 2 and at most one of the width or the height is increased by 1. An example of the area increased by 3 is shown in Figure 4.4.

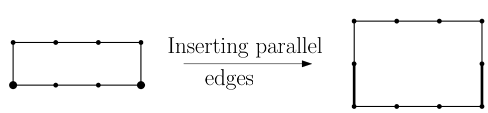



Figure 4.4: Example of the area increased by 3

**Definition 4.3.** Define the **product** $\cdot$ on $U$ to be the disjoint union. Define the **unit** $u$ to be the trivial unit $u : \mathbb{K} \to U$, $k \mapsto k\mathbb{1}$.

Now $U$ is an algebra. Furthermore $\cdot$ and $u$ are graded maps in all three graded vector spaces $U,\ W,\ H$; thus $U,\ W,\ H$ are all graded algebras. Let the **counit** $\varepsilon$ be the trivial counit. We can see that $\varepsilon$ is also graded in $U,\ W,\ H$. In order to have a graded bialgebra, we only need to find a compatible graded coproduct for each graded algebra. In the following, we will construct such coproducts $\Delta_w$ and $\Delta_h$ for $W,\ H$ respectively.

Let $P$ be a self-avoiding polygon. Place $P$ in the first quadrant by aligning its left most boundary at the y-axis and bottom most boundary at the x-axis. An example is shown in Figure 4.5.

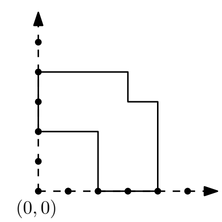



Figure 4.5: A SAP placed in the coordinate system

We can define admissible vertical cuts as follows.

**Definition 4.4.** Let $P$ be a self-avoiding polygon. For $0 < i < w(P)$, if there exists $0 \leq s < t \leq h(P)$ such that $P \cap (i, i+1) \times [0, h(P)] = (i, i+1) \times \{s, t\}$ and $\{i\} \times [s, t] \setminus P = \{i\} \times (m, n)$ for some $s \leq m < n \leq t$ then we say $P$ has an **admissible vertical cut** at $i$ or say $i$ is an admissible vertical cut of $P$.

This definition implies that an admissible vertical cut is actually a place where we can make a vertical cut to get a single valid SAP on each side of this cut. Though it seems asymmetric from left to right, it is not because of the following two cases for the definition of the left side SAP and the right side SAP.

For $i$ an admissible vertical cut of $P$, let $L_i$ be the **left SAP** resulting from contracting the right part of $P$ to $\{i\} \times [m, n]$ and $R_i$ be the **right SAP** resulting from contracting the left part of $P$ to $\{i\} \times [m, n]$. This is formulated as follows.

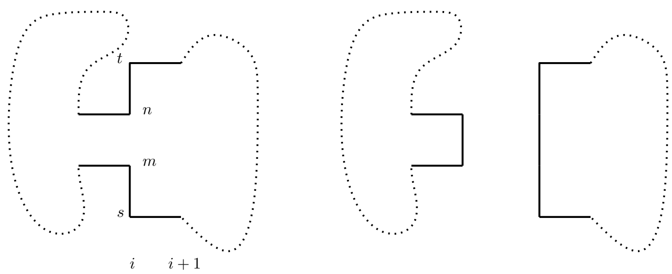
(1) If $(s, t) \neq (m, n)$,

$$\begin{aligned} L_i &= (P \cap [0, i) \times [0, h(P)]) \cup \{i\} \times [m, n], \\ R_i &= (P \cap [i, w(P)] \times [0, h(P)]) \cup \{i\} \times [m, n]. \end{aligned}$$

(2) If $(s,t)=(m,n)$,

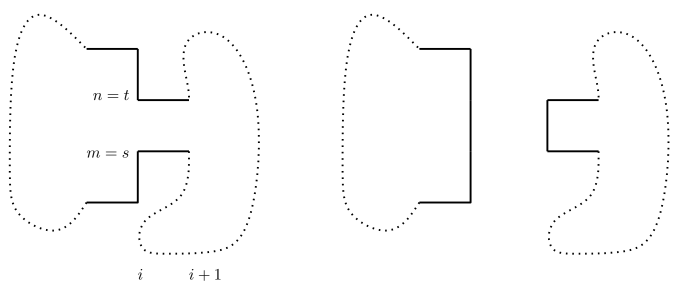
$$\begin{aligned} L_i &= (P\cap[0,i]\times[0,h(P)])\cup\{i\}\times[m,n], \\ R_i &= (P\cap(i,w(P)]\times[0,h(P)])\cup\{i\}\times[m,n]. \end{aligned}$$

**Example 4.5.** Here is an example of an admissible vertical cut at $i$ of case (1),




$P$ $L_i(P)$ $R_i(P)$ ,

and the following is an example of case (2),




$Q$ $L_i(Q)$ $R_i(Q)$ .

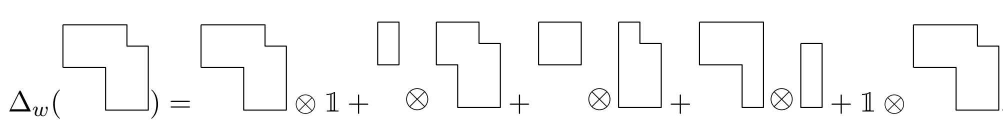
Now we are ready to define the linear map $\Delta_w$ on $W$.

**Definition 4.6.** Define linear map $\Delta_w$ on a single SAP by

$$\Delta_w(P)=\mathbb{1}\otimes P+P\otimes\mathbb{1}+\sum_{\substack{i\ admissible\\ vertical\ cut}} L_i\otimes R_i,$$

and extend as algebra morphism to $W$.

**Example 4.7.** As an example,



Notice that $\Delta_w$ is a graded map on $W$ since it preserves the widths. In order to show that $\Delta_w$ is a coproduct we only need to prove that the first diagram in Definition 1.6 commutes since commutativity of the second diagram is easy to check in this case.

**Lemma 4.8.** *The linear map $\Delta_w$ is coassociative thus is a coproduct.*

*Proof.* Take $P$ a SAP,

$$\begin{aligned}
(id \otimes \Delta_w)\Delta_w(P) &= \mathbb{1} \otimes \Delta_w(P) + P \otimes \mathbb{1} \otimes \mathbb{1} + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} L_i \otimes \Delta_w(R_i) \\
&= \mathbb{1} \otimes \mathbb{1} \otimes P + \mathbb{1} \otimes P \otimes \mathbb{1} + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} \mathbb{1} \otimes L_i \otimes R_i + P \otimes \mathbb{1} \otimes \mathbb{1} \\
&\quad + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} (L_i \otimes R_i \otimes \mathbb{1} + L_i \otimes \mathbb{1} \otimes R_i) \\
&\quad + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} \sum_{\substack{j \text{ admissible} \\ \text{vertical cut} \\ \text{of } R_i}} L_i \otimes L_j(R_i) \otimes R_j(R_i)
\end{aligned}$$

$$\begin{aligned}
(\Delta_w \otimes id)\Delta_w(P) &= \mathbb{1} \otimes \mathbb{1} \otimes P + \Delta_w(P) \otimes \mathbb{1} + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} \Delta_w(L_i) \otimes R_i \\
&= \mathbb{1} \otimes \mathbb{1} \otimes P + \mathbb{1} \otimes P \otimes \mathbb{1} + P \otimes \mathbb{1} \otimes \mathbb{1} \\
&\quad + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} L_i \otimes R_i \otimes \mathbb{1} + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} (L_i \otimes \mathbb{1} \otimes R_i + \mathbb{1} \otimes L_i \otimes R_i) \\
&\quad + \sum_{\substack{i \text{ admissible} \\ \text{vertical cut} \\ \text{of } P}} \sum_{\substack{k \text{ admissible} \\ \text{vertical cut} \\ \text{of } L_i}} L_k(L_i) \otimes R_k(L_i) \otimes R_i
\end{aligned}$$

Notice for any $L_i$, any admissible vertical cut $1 \le k < i$ of $L_i$ is also an admissible vertical cut of $P$ with index less than $i$ and vice versa. Similarly, for $R_i$ and any admissible vertical cut $1 \le j < w(P) - i$ of $R_i$, $i+j$ is also an admissible vertical cut of $P$ and vice versa. Thus the two double sums are summing over two admissible vertical cuts $1 \le l < i < w(P)$ of $P$. Thus the above two equations are equal.

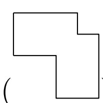
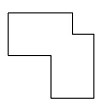
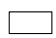
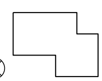
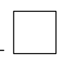
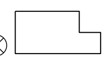
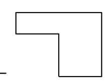
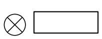
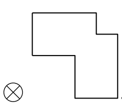
As for a disjoint union of two or more SAPs, $\Delta_w$ is also coassociative by Proposition 1.27 since it is an algebra homomorphism. $\square$

**Theorem 4.9.** *The algebra $(W, \cdot, u, \Delta_w, \varepsilon)$ is a connected and graded bialgebra thus is a Hopf algebra.*

*Proof.* $W$ is clearly graded and connected, and the four linear maps are all graded maps. The commutativity of the last three diagrams in Definition 1.12 is easy to check and the first diagram in Definition 1.12 commutes because we defined $\Delta_w$ as an algebra homomorphism in Definition 4.6. Thus $(W, \cdot, u, \Delta_w, \varepsilon)$ is a connected and graded bialgebra. Applying Corollary 1.25, $W$ is a Hopf algebra. □

Similarly for the width-coproduct $\Delta_w$, we can define **admissible horizontal cuts** in $H$ and get a **height-coproduct** $\Delta_h$. The height-coproduct is the sum of the bottom SAP tensor the top SAP over all admissible horizontal cuts.

**Example 4.10.** As an example,

$\Delta_h($  $) =$  $\otimes \mathbb{1} +$  $\otimes$  $+$  $\otimes$  $+$  $\otimes$  $+ \mathbb{1} \otimes$ .

Analogous proofs lead to the following theorem.

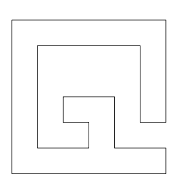
**Theorem 4.11.** *The algebra $(H, \cdot, u, \Delta_h, \varepsilon)$ is a connected and graded bialgebra thus is a Hopf algebra.*

These two Hopf algebras do capture some properties of the generating process since for any admissible vertical cut or any admissible horizontal cut of $P$, $P$ can be generated in the generating grapph $G(V, E)$ by first building the SAP on one side of the cut and then proceeding downwards in the generating graph according to the SAP on the other side of the cut. Also these two Hopf algebras are symmetric with respect to $y = x$. Combining these two Hopf algebras we can define double primitives as follows.

**Definition 4.12.** Define **double primitives** as the elements in the set

$$\mathbb{K}er\ \tilde{\Delta}_w \cap \mathbb{K}er\ \tilde{\Delta}_H = \{u \in U : \Delta_w(u) = \mathbb{1} \otimes u + u \otimes \mathbb{1} \textit{ and } \Delta_H(u) = \mathbb{1} \otimes u + u \otimes \mathbb{1}\}.$$

**Example 4.13.** Here is an example of a double primitive with width 6 and height 6,

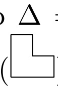
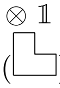
.

Notice that we can not further decompose these double primitives through the intermediate vertices in the generating graph $G(V,E)$. So there are properties that are not captured by either of these two Hopf algebras.

So we consider grading $U$ by both the widths and the heights, that is to say $U$ can be decomposed into $\bigoplus_{w=0}^{\infty}\bigoplus_{h=0}^{\infty} U_{w,h}$. We can see $(U,\cdot,u)$ is still a graded algebra since

$$\cdot(U_{w_1,h_1}\otimes U_{w_2,h_2}) \subseteq U_{w_1+w_2,h_1+h_2}.$$

Then we ask: is there a coproduct $\Delta$ which graded by both widths and heights that will make $(U,\cdot,u,\Delta,\varepsilon)$ into a graded bialgebra?

We tried the linear map $\Delta = \mathbb{1}\otimes id + id\otimes\mathbb{1} + \tilde{\Delta}_w + \tilde{\Delta}_h$, but it is not a coproduct. Readers may see $(id\otimes\Delta)\Delta($$) \neq (\Delta\otimes id)\Delta($$)$.

# Chapter 5

# Conclusion and Questions

In this thesis, we constructed a concatenation-deshuffle Hopf algebra and a shuffle deconcatenation Hopf algebra on the vector space spanned by an augmented generating tree. We also constructed an incidence Hopf algebra on the polynomial algebra of a generating graphs in a certain family. Finally, we constructed a width Hopf algebra and a height Hopf algebra on the polynomial algebra of self-avoiding polygons.

Next we present some questions related to our constructions and then present some questions related to our original goal.

Notice in Section 2.4, we got that our Hopf algebras of set partition appear to be different from the Hopf algebra shown in [19]. But we don't know whether they have a deep connection yet.

**Question 1.** Are the two Hopf algebras of set partition shown in Section 2.4 (or their duals) isomorphic to the Hopf algebra of set partition in [19]?

**Question 2.** What is the classification of finite comodules of the two Hopf algebras of open partition diagrams shown in Section 2.5?

**Question 3.** Is there a way to modify the generating graph of Section 3, perhaps by labeling multiple edges, to allow a similar process to construct Hopf algebras that works for the Connes-Kreimer Hopf algebra of rooted trees and for renormalization Hopf algebras of Feynman graphs with propagator insertions.

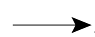
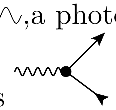
**Question 4.** Is there a coproduct related to both $\Delta_w$ and $\Delta_h$ (Section 4.2) such that $U$ becomes a width and height graded Hopf algebra?

There are some other questions related to our original goal which was understanding the connection of the multiplicative factors between the combinatorial objects described in the following two sections.

## 5.1 Hopf Algebra on Feynman Graphs

Dirk Kreimer and Alain Connes pioneered the Hopf algebraic approach to studying quantum field theory ([5, 6]). They revealed the underlying Hopf algebra structure of Feynman graphs which allows for renormalization. Kreimer showed the relation between a perturbative expansion in quantum field theory and the corresponding Hochschild cohomology ([12]). For more about the Hopf algebra of Feynman graphs, readers are referred to the survey [21].

In physics, Feynman graphs are used to describe the interactions of subatomic particles. Combinatorially, we view Feynman graphs as multigraphs (allowing multiple edges and loops) built from a specified set of half edges. Here we follow the setup in [15, chapter 2] which was already showed in Example 1.30. For a combinatorial physical theory $T$ in the dimension of space-time $D$, there is a set of vertex types, a set of half edge types and a set of edge types. Also the vertices and edges are given power counting weights according to their type. Any internal edge of a Feynman graph in $T$ belongs to one edge type. Physicists are interested in divergent 1PI graphs. Here, we use quantum electrodynamics (QED) as an example to illustrate.

**Example 5.1.** In QED, $D = 4$, the set of half edge type is {, , } which stand for a half photon, a back half fermion and a front half fermion respectively. The permitted edge types are ,a photon, with weight 2 and , an electron, with weight 1. The only vertex type is  with weight 0. One divergent 1PI Feynman graph is shown in Figure 5.1.

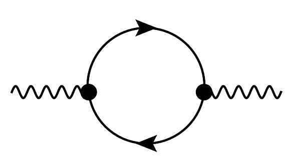


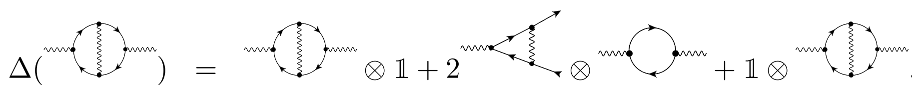
Figure 5.1: Example of divergent 1PI Feynman graph in QED

Let $\mathcal{H}$ be the vector space spanned by all disjoint unions of divergent 1PI graphs over $\mathbb{Q}$ and let $\mathbb{1}$ be the empty graph. The product is the disjoint union and the coproduct $\Delta$ is first defined on connected graphs and then extended as an algebra homomorphism to $\mathcal{H}$. Let $\Gamma$ be a divergent 1PI graph,

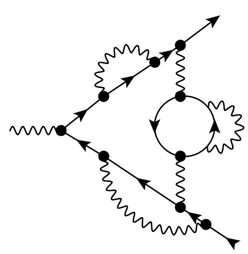
$$\Delta(\Gamma) \quad = \sum_{\substack{\gamma\subseteq\Gamma \\ \textit{product of divergent} \\ \textit{1PI subgraphs}}} \gamma \otimes \Gamma/\gamma,$$

where $\Gamma/\gamma$ is the graph resulting from $\Gamma$ contracting $\gamma$.

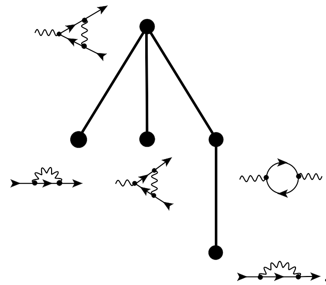
**Example 5.2.** As an example,

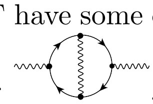
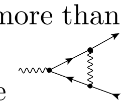
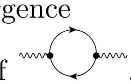


$\mathcal{H}$ can be graded by the loop number. The product and the coproduct are both graded maps. In this Hopf algebra $\mathcal{H}$, for a primitive Feynman graph $\gamma$, $B_+^\gamma$ denotes the map of insertion into $\gamma$. $B_+^\gamma$ is similar to $B^+$ (the Hochschild 1-cocycle map in Connes-Kreimer Hopf algebra of rooted trees as shown in Example 1.29), but not necessarily a 1-cocycle map in $\mathcal{H}$. There are cases where the divergent subgraphs of a divergent 1PI Feynman graph $\Gamma$ are not overlapping; then $\Gamma$ has a unique rooted tree representation such that each vertex of the tree corresponds to a divergent subgraph of $\Gamma$. Call that tree an insertion tree. For example,



has the insertion tree



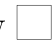
In other cases if the subdivergences of $\Gamma$ have some overlappings, it may have more than one insertion trees. For example, consider . Notice the subdivergence  can be inserted into two places, either the left vertex or the right vertex of . So the graph has two insertion trees. Since $\gamma$ is primitive, the equation (1.1) will have the same tensor of graphs with different coefficients on different sides. Another tricker situation in QCD is that a graph may have overlapping divergent subgraphs with different external structure which allows the graph contracting into two different primitives. In the paper [12], Kreimer used a multiplicative coefficient in the definition of $B_+^\gamma$ to fix the overlapping. Define for a connected Feynman graph $\gamma$,

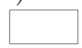
$$B_+^\gamma(X) = \sum_{\Gamma \in \mathcal{H}_{lin}} \frac{\mathbf{bij}(\gamma, X, \Gamma)}{|X|_\vee} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{[\gamma \mid X]} \Gamma,$$

where $\mathbf{bij}(\gamma, X, \Gamma)$ is the number of bijections between the external edges of $X$ and the adjacent edges of the insertion place in $\gamma$ such that $\Gamma$ is obtained, $|X|_\vee$ is the number of distinct graphs obtained by permuting the labels of external edges of $X$, $\mathrm{maxf}(\Gamma)$ is the number of ways of shrinking subgraphs to obtain a primitive cograph, and $[\gamma \mid X]$ is the number of insertion places for $X$ in $\gamma$. Kreimer proved in [12, Theorem 4] that with these coefficients, if we sum over all $B_+^\gamma$ with $\gamma$ a connected primitive graph which has a given external structure, inserting into all insertion places of each $\gamma$ gives the same result as summing over all 1PI graphs with that external structure weighted by their symmetry factors. Kreimer also discussed that with the Ward identities for QED, $\sum B_+^\gamma$ summing over all $\gamma$ with a given external structure and a given loop is a Hochschild 1-cocycle. Later this was proved by van Suijlekom in [13].

## 5.2 Methods for Sampling Self-Avoiding Polygons

In Rechnitzer and Janse van Rensburg's paper [7], they introduced a method called the Generalised Atmospheric Rosenbluth Method (GARM) to sample self-avoiding polygons (SAPs) and some other combinatorial objects. Here we use SAPs on square lattice to illustrate their set up.

Given a SAP $P$, a **positive generalised atmosphere** is defined as inserting parallel edges at a pair of vertices to obtain a longer SAP. Denote by $a_+(P)$ the number of positive generalised atmospheres. For example, let $P$ be the SAW  then $a_+(P) = 8$ as shown in Figure 4.1.

A **negative generalised atmosphere** is defined as contracting a pair of edges so as to obtain a shorter SAW and $a_-(P)$ denotes the number of negative generalised atmospheres. For example, let $P$ be the SAP  then $a_-(s) = 4$ as shown in Figure 5.2.

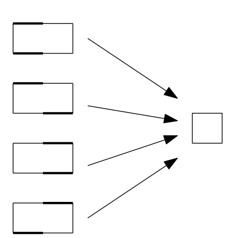
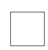


Figure 5.2: Negative generalised atmospheres of $P$

Start with the smallest SAP $\phi_0 =$ , and get a sequence of SAPs $\phi = \phi_0, \phi_1, \ldots, \phi_n$ by inserting a positive atmosphere at each step $0 \leq j \leq n-1$ with the probability $a_+(\phi_j)^{-1}$. Denote by $|\phi| = n$ the length of the sequence $\phi$. Then the probability of obtaining $\phi$ from $\phi_0$ is

$$\Pr(\phi \mid \phi_0) = \prod_{k=1}^{n} a_+(\phi_{k-1})^{-1}.$$

For the sequence $\phi$, define the weight

$$W(\phi) = \prod_{k=1}^{n} \frac{a_+(\phi_{k-1})}{a_-(\phi_k)}.$$

With the weights used as multiplicative factors, it has been proved in [7],

$$\langle W \rangle_n = \sum_{\phi} W(\phi) Pr(\phi \mid \phi_0) = c_{2n+4}.$$

This means the mean weight of all sequences $\phi$ with length $n$ is the number of SAPs of length $2n+4$.

In both the $B_+^\gamma$ in renormalization Hopf algebras and GARM a multiplicative factor is used to deal with overcounting as opposed to the more usual additive process like inclusion-exclusion. We began this project searching for a precise connection to capture this similarity, but were not, so far, able to do so.

**Question 5.** Is there a better common Hopf-algebraic language than the generating graph approach for both cases?

**Question 6.** Do the weights for SAPs relate to Hochschild 1-cocycle maps?

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# Index