

Dealing With Abstraction: Reducing Abstraction in Teaching (RAiT)

by

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Abstract

One of the most important challenges for mathematics teachers involves dealing with mathematical abstraction, specifically; figuring out efficient ways to translate abstract concepts into more easily understandable ideas for their students. Reducing abstraction is one of the theoretical frameworks originally proposed by Hazzan (1991) to examine how learners deal with mathematical abstraction while working with new mathematical tasks or concepts. However, very little is known about how teachers deal with mathematical abstraction while implementing mathematical tasks in the classroom. To complement this body of research, my study seeks to understand the features of teaching practices in real classroom situations with regard to dealing with mathematical abstraction.

In this study, the level of abstraction involved in a situation has been interpreted from three distinct perspectives: 1) as the quality of the relationships between the mathematical concept and the learner; 2) as a reflection of the process-object duality; and 3) as the degree of complexity of a mathematical task or concept. Upon close analysis of the primary (classrooms observation) and secondary (TIMSS 1999 Public Release video lessons) data, various behaviours and strategies used by teachers to reduce abstraction while implementing tasks have been identified in each of the above three categories. As a result, a framework of “Reducing Abstraction In Teaching” (RAiT) has emerged, thus offering a new perspective on and an application of the notion of reducing abstraction.

While reducing abstraction in teaching is often intended to make the mathematical concept or object more accessible to students and, thus, to achieve meaningful learning, this study discovered and exemplified some instances in which RAiT activity may not necessarily be supportive for that purpose. Hence, this study suggests a need for teachers to pay attention to the possible deficiencies of students’ understanding that may arise as a consequence of some of the strategies of reducing abstraction in teaching. Finally, the study concludes with a number of recommendations and suggestions, including avenues for future research.

Keywords: Dealing with abstraction in teaching; Abstraction in mathematics education; reducing abstraction; task implementation and abstraction; mathematical abstraction

Dedication

To the loving memory of my father

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1. Introduction

1.1. Context and Motivation

My educational journey began in the late seventies in a rural primary school in Nepal. At that time, Nepali schools (particularly those in rural areas) suffered from an acute shortage of teachers. As a result, there were thousands of children at school without any teachers, let alone trained teachers. The case was even worse in the secondary schools, particularly in the areas of mathematics and science. I myself experienced firsthand how challenging it was to try to learn without a teacher. I still remember vividly those old days when my classmates and I had to do self-study for most of the content areas in mathematics and science, and prepare for the national level high school exit examination (the certificate required to leave school) on our own. This acute shortage of mathematics teachers in the rural schools (particularly at the secondary level) motivated me to pursue mathematics courses in my tertiary education, which led to my being hired as a mathematics teacher at the high school level.

As a beginning mathematics teacher, I was full of confidence and enthusiasm to share my mathematical knowledge. I tried my best to explain mathematical concepts clearly to my students, modeling my efforts after my teachers in school and university. While the resources available to me were limited (textbook, blackboard, different colours of chalk but no technology), I made the most of them, showing my students all the steps and procedures (those different colours of chalk did come in handy) needed to solve problems. Since classroom interaction between teachers and students was not the part of the classroom culture in Nepal, my job was essentially to deliver lecture for the entire class time and to likewise fill the blackboard (making it as colourful as possible) from one corner to the other. Students were there to listen and to copy the solution from the blackboard step by step. As much as possible, I incorporated useful examples, clever tricks and mnemonics, with a view that these teaching methods would help my students to retain the information given during class. I also assigned a lot of questions designed

to make students practice the different skills learned in class. However, the work I received back from many students was not what I expected. Of course, a few of them obtained the correct answers using all the steps and procedures exactly as I had shown them. A few others tried to follow the steps and procedures, but obtained answers that made no sense. Others did not even know where or how to begin. What surprised and frustrated me about this was that even after explaining and showing the procedures again, the work these students produced was not qualitatively different from their previous work. At this point, I knew the job was not going to be easy.

Since I was not ready to accept the fact that so many of my students did not have the ability to succeed in mathematics, I decided to talk to them and find out what made learning mathematics so difficult. It should be noted that I did this even though I knew this kind of practice was not usually pursued in the classroom at the time. After listening to my students, I came to realise that their difficulties and frustration were closely related to what Ferguson (1986) calls “abstraction anxiety“. They did not see any connection between mathematics and their day-to-day lives. As a result, mathematics for them was a collection of strange symbols and rules that were to be memorized. Solving a mathematical task meant the use of these rules, symbols and procedures. Further, every student had to perform exactly the same way as prescribed by the teacher or textbook. Their comments about their experience of and attitudes towards mathematics constituted for me an important point of reflection, discouragement and, at the same time, a new source of determination and hope.

At this point, I decided that while implementing mathematical tasks, I would try to make these more meaningful so that my students could more easily grasp and make sense of concepts they were learning in the classroom. In this regard, one of the professional development workshops that I attended during that year proved to be very helpful. From that workshop, I learned various methods of making mathematical concepts more easily understood by students. Some of the strategies I used included connecting mathematical activities to students’ real world experiences and using their everyday language alongside more formal mathematical language. For example, while teaching quadratic equations, I used a soccer ball (football) and threw it up in the air to show that the path taken by the ball (due to gravity) would describe a parabolic path. This activity provided a rich opportunity for students to learn many concepts, such as

quadratic equations, graphs, maximum and minimum values and the relationships between those ideas. Most importantly, they saw that the quadratic equations they had been studying had a basis in real life. Through such activities, I witnessed a change in my students' attitude towards mathematics; they began to seem more interested and involved in mathematics and problem solving.

During the time that I taught at that high school, I experienced many facets of teaching and encountered various learning problems in mathematics. Nevertheless, I started to enjoy being with my students; I became motivated to teach. We developed a kind of solidarity, my students and I, as we teamed together to improve their grasp of the material. This firsthand teaching experience helped me learn to connect with my students and understand the landscape of how mathematics are taught in a classroom environment. As much as possible, I tried to design classroom activities in such a way as to make the tasks or concepts easily accessible to the students, and as a result, make the learning experience as meaningful for them as possible. Suffice to say, I found this very challenging, as I myself had never experienced this kind of teaching or learning during my high school or university education.

1.2. Front and Back of Mathematics: Learning and Teaching Challenges

Although I managed to do well in mathematics in my high school and university mathematics courses, I rarely had an opportunity to discuss my understanding of mathematical ideas with teachers or classmates. Nor did I have an opportunity to be shown the connections between the mathematical world and the world that I lived in, which could have led me to form a more personal interpretation of mathematics. Instead, for the bulk of my own education, my mathematical learning experience consisted mostly of the internalization of the objective truth of mathematics. For me, as a student, learning mathematics was narrowly defined as the ability to follow exactly the same steps and procedures as given by the teacher or textbook, page by page and line by line. We were often reminded (by our teachers) to realize the fact that mathematics is formula-based, symbolic and algorithmic, which requires the right attitude, good memory skills and a certain kind of thinking style in order to be successful. Perhaps mathematics, as known

by our teachers (at high school and university), was all about theorems, the game of rules and algorithms. For my generation, success in mathematics meant passing exams full of mathematical problems that required us to memorise algorithms and procedures and to obtain the correct answers, regardless of whether or not we understood why and how the method worked.

In short, my own mathematics learning experience can be likened to what Hersh (1997) calls the exclusive hegemony of the 'frontal aspect' of mathematics. Hersh argues that mathematics had 'front' and 'back', where the frontal aspect of mathematical knowledge refers to the product of mathematical activity in its finished and singular best product (i.e. finished mathematics), thereby separating it from the construction, conjecture, intuitive and developmental aspects of mathematics (i.e. mathematics in the making), which are often depicted as the 'back' aspect of mathematics. Hersh (1991) elaborates:

[...] the "front" of mathematics is mathematics in "finished" form, as it is presented to the public in classroom, textbooks, and journals. The "back" would be mathematics as it appears among working mathematicians, in informal settings, told to one another in an office behind closed doors.

Compared to "backstage" mathematics, "front" mathematics is formal, precise, ordered and abstract. It is separated clearly into definitions, theorems, and remarks. To every question there is an answer, or at least, a conspicuous label: "open question". The goal is stated at the beginning of each chapter, and attained at the end.

Compared to "front" mathematics, mathematics "in back" is fragmentary, informal, intuitive, and tentative. We try this or that, we say "maybe" or "it looks like" (p. 128).

Indeed, the front and back aspects of mathematics exist side by side, and the growth of mathematical knowledge is possible only through their perpetual reflexive and interactive relationships. By overemphasising the frontal aspect and neglecting the back, my mathematics education provided the absolutist or formalist picture of mathematics, in which mathematics is viewed as a body of pure knowledge, thereby masking the fallibilist or constructivist picture embedded in the back aspect of mathematics. As a result, *mathematics as collections of unchangeable formulae and definitions* and *mathematics teaching as reproduction of theorems* were some of the images I

constructed during my studies at the high school and university levels. As such, in my teaching, I continually asked myself the question: “Am I creating the same sort of mathematical experience for my students that I went through?”

My decision to provide a different type of mathematical experience for my students demanded a shift in perspective in my own way of thinking about the subject. This was one of the toughest challenges for me because my own understanding of mathematics consisted of formulae, algorithms, procedures and theorems that provided the notion of mathematics as a body of pure knowledge. Still, I tried my best to provide this different kind of mathematical experience to my students by attending not only the front aspect of mathematics, but also revealing the messy parts—the ones that consisted of informal, intuitive and tentative ideas. I structured each unit so that it started with familiar ideas, moved from exploration to explanation, and focused on changing my role from knowledge transmitter to activity facilitator. One of the most satisfying things for me as a teacher was to witness my students showing interest in, and appreciation for mathematics as they began to make sense of it. This experience had a positive impact on my students’ mathematics achievement, as measured by not only the standardized tests, which focused more on procedure, completeness and correctness, but also as measured by more meaningful assessment that focused on conceptual understanding, such as writing journals and essays around mathematical concepts. During those years of teaching, I realized that learning so much about teaching and about how people learn mathematics, that this new knowledge had completely changed my way of thinking about mathematics.

My bachelor’s and master’s degrees in pure mathematics provided me with an opportunity to accumulate a vast array of subject matter knowledge, even though it were not necessarily organised into a coherent whole. However, I always felt something was lacking in my understanding of teaching and learning mathematics (at that time, in Nepal, neither education nor pedagogy courses were required in order to become a teacher). This needful lack that I sensed in my own abilities and understanding was remedied through furthering my education in the Masters of Education program at Endicott College, USA as well as through my recent doctoral program in mathematics education at Simon Fraser University, Canada. These programs provided me with the opportunity to reflect upon my experiences and integrate them with emerging theoretical

perspectives of teaching and learning mathematics. The mathematical knowledge that I had accumulated earlier, which was mostly abstract, impersonal and fragmented, now emerged as a meaningful and coherent whole. Furthermore, with my exposure to educational and pedagogical theories in mathematics education in recent years, the knowledge that I gained from trial and error during my early teaching career finally began to make more sense.

1.3. Mathematical Abstraction and Teaching Challenges

In recent years, the notion of mathematical abstraction in the learning and teaching of mathematics has received growing interest from the research community among psychologists and mathematics educators. While there is no consensus as to the unique meaning of abstraction (Hazzan & Zazkis, 2005), it is generally said that mathematics is an abstract subject, and that dealing with mathematical abstraction is fundamental in the learning and teaching of mathematics. Whitehead (1925 / 2011) for example, states that “so long as you are dealing with pure mathematics, you are in the realm of complete and absolute abstraction.[...] Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about” (p.27). Gowers (2002) also asserts that “ there are (at least) two senses in which mathematics is an abstract subject: it abstracts the important features from a problem and it deals with objects that are not concrete and tangible” (p. 28). As such, a persistent challenge in teaching mathematics is to find ways to make abstract mathematical ideas accessible to students. If teachers understand more clearly what processes their students go through while learning new concepts, teachers may be able to teach more effectively and students, in turn, may learn better. In this regard, Hazzan’s (1999) research claimed that since students usually do not have enough resources to deal with the same abstraction level of a new mathematical concept as given by the authorities (teacher and textbook), they tend to reduce abstraction of the concept in an attempt to make the concept more accessible.

This idea is in line with constructivist philosophy and genetic epistemology, according to which children start as concrete thinkers with limited cognitive ability to understand abstraction. As they grow, their cognitive level for abstract thinking gradually

develops. In the constructivist tradition, knowledge is viewed as something that is not directly transmitted from the knower to the learner but actively created by the learner based on their previously acquired knowledge, experience and level of thinking as well as their familiar contexts (Piaget, 1970; Driver, Asako, Leach, Mortimer & Scott, 1994; Hershkowitz, Schwarz & Dreyfus, 2001).

From this perspective, if we agree that one of the goals of teaching is to make the mathematical concept or task more accessible to the students in order to promote learning, it then follows that teaching should involve two things: on the one hand, teaching activities should involve some way of reducing the level of abstraction of mathematical concepts (or tasks) so that the concept is accessible to the student. On the other hand, since student understanding is not supposed to remain at this lower level, it is the teacher's job not only to initially reduce the level of abstraction, but, once a basic understanding is gleaned, to then assist students in raising the level of their understanding. Hence, a question worthy of further exploration is raised: in their efforts to make the mathematical concept (or task) accessible to their students, how do teachers deal with mathematical abstraction while implementing mathematical task in the classroom? This is the question that is being explored in this study. I will revisit this question in chapter 3 after providing an overview of how the notion of abstraction is understood in mathematics and the mathematics education community in chapter 2.

1.4. Task, Task Implementation and Teaching Challenges in Reformed Mathematics

Mathematics, as it is widely understood, plays a key role in describing and understanding the world in which we live. NCTM (2000) states:

In this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures. Mathematical competence opens doors to productive futures. A lack of mathematical competence keeps those doors closed. [. . .] All students should have the opportunity and the support necessary to learn significant mathematics with depth and understanding. (p. 50)

And yet, studies show that many students have unpleasant experiences with mathematics in their elementary school, high school and college level classes (1999; Ferguson, 1986). Researchers point out that one of the key factors related to this issue concerns teaching practices. For example, in his study of mathematics classrooms across the United States, Cajori (1890) said, “[There were] no explanations of processes either by master or pupil [...] the problems were solved, the answers obtained, the solutions copied” (p. 10). Despite many reform efforts in the last century, researchers in mathematics classrooms found that school mathematics practices in the late twentieth century remained similar to what Cajori described over a hundred years ago. Mathematics is viewed as a body of knowledge apart from human experience. In order to facilitate learning in this tradition, teachers are expected to deliver carefully sequenced mathematics lessons through lectures, explanation and demonstration. Students are expected to repeatedly practice these skills and procedures and then tested on them to determine what has been learned (Ellis & Berry, 2005). As said earlier, I myself have witnessed these kinds of teaching practices as a student in high school as well as university, and based on my experience, I suspect that this style of mathematics teaching is probably one of the factors to blame for the unpleasant experiences with mathematics reported by many students.

Over the last two decades, there have been fundamental changes (or at least some efforts to change) in both the content of school mathematics and the way it is taught. This is due to the mathematics education community being influenced by various factors, including knowledge gained from research, advances in technology that have impacted curriculum, and most importantly, the constructivist and social-constructivist theories of learning, which have helped educators understand how learners learn mathematics. As a result, the National Council of Teachers of Mathematics (NCTM) released two important documents: “Curriculum and Evaluation Standards for School Mathematics” (1989) and “Principle and Standard for School Mathematics” (2000), which provide guidance and direction for teachers and all stakeholders of school mathematics, at least in the U.S. and Canada. Further, NCTM released the Professional Standards for Teaching Mathematics (1991) to “[spell] out what teachers need to know to teach toward new goals for mathematics education and how teaching should be evaluated for the purpose of improvement” (NCTM, 1991, p. vii). These documents have been significantly

influential in the reform movement of mathematics education, not just in the U.S. and Canada, but throughout the world.

With the release of these documents, many of the core beliefs rooted in the traditional ideology of teaching and learning were challenged. The interpretation of how learning takes place evolved from a cognitive and information processing framework to a constructivist orientation. A call for change in curriculum, teaching practices and assessment was advocated, with an emphasis on problem solving, communication, connections and reasoning. To foster these skills, the NCTM (2000) focused on the nature of mathematical tasks and the role of teachers as follows:

[S]tudents confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. (p. 3)

As a result, mathematics educators began to emphasize not only the 'front' aspect of mathematics (finished mathematics), but also the 'back' aspect. That is, those teaching math began to focus on "seeking solutions, not just memorizing procedures; exploring patterns, not just memorizing formulas; formulating conjectures, not just doing exercises" (National Research Council [NRC], 1989, p. 84). Teachers were advised to assign their students tasks that "stimulate students to make connections and develop a coherent framework for mathematical ideas; call for problem formulation, problem solving and mathematical reasoning; [and] promote communication about mathematics" (NCTM, 1991, p. 25). In this regard, NCTM (2000) further states, "most mathematical concepts or generalizations can be effectively introduced using a problem situation," and that one of the important aspects of problem-solving is to design "worthwhile mathematical tasks" (p. 18-19) through which students can engage in thinking and reasoning about important mathematical concepts. Consequently, much has been written on mathematical tasks and how they influence the way students think and

engage in learning mathematics (Hiebert & Wearne, 1993; Stein & Lane, 1996; Birky, 2007).

Evidence shows that the nature of mathematical task plays an important role in students' thinking and understanding; it does not however necessarily guarantee the enhancement of learning. For example, as the TIMSS (1999) video study report shows, American curricula already included the recommended types of mathematical problems set forth in the NCTM standards, and U.S. teachers assigned those problems at rates not dissimilar from those in other countries (Hiebert, Galimore, Garnier, Givvin, Hollingsworth, Jacobs, Chui, Wearne, Smith, Kersting, Manaster, Tseng, Etterbeek, Manaster, Gonzales, & Stigler, 2003). And yet, the achievement of U.S. students does not reflect this fact. The TIMSS (1999) video study team found that in American eighth grade mathematics classrooms, the problems that were intended to engage students in what Skemp (1976) would call 'relational understanding' (problems rooted in making connections) were almost always implemented in a way that required the use of procedures, recall of facts or answers which resulted in 'instrumental understanding' (not making connections) on the student's part. Skemp (1976) distinguishes relational understanding from instrumental understanding in that the former involves "knowing both what to do and why", whereas the later describes knowing "rules without reasons". In contrast, some of the teachers from other countries, whose students scored higher in the TIMSS 1999 achievement test, implemented the tasks initially requiring procedures and recall of facts in a way that fosters relational understanding or 'making connections' (Hiebert et al., 2003, Birky, 2007).

This suggests that it is not just the nature of the mathematical task, but rather how the task is implemented by teachers in the classroom that impacts student learning. Hiebert et al. (2003) write, "[In order] to better understand, and ultimately improve, students' learning, one must examine what happens in the classroom" (p.2) rather than just the nature of problem. Even though teaching is not the sole cause of students' learning, and the relationships between classroom teaching and learning are neither simple nor straightforward, evidence suggests that teaching style makes a difference in students' learning (Brophy & Good, 1986; National Research Council, 2001).

Much has been written on teaching and, particularly, on teaching mathematics (Richardson, 2001; English, 2010). However, research on what teaching looks like in actual mathematics classrooms, especially on how teachers deal with mathematical abstraction in the course of their teaching, is slim. Hazzan (1999) and, later, Hazzan and Zazkis (2005) examined how students deal with abstraction in learning new mathematical concepts in university and school mathematics settings. However, they did not explore how teachers deal with mathematical abstraction in teaching. Others who have analyzed teaching practices in mathematics either focus on the nature of the task and how it is implemented, particularly from a cognitive viewpoint, such as the cognitively high or low level of the task (Henningsen & Stein, 1997; Hiebert et al., 2003; Stein, Grover & Henningsen, 1996), or on “making connection” or “non-making connections” type problems (Hiebert et al., 2003; Birky, 2007). To complement this body of research, my study seeks to understand the features of teachers’ task implementation behaviour in real classroom situations with regard to dealing with mathematical abstraction in teaching.

1.5. Research Questions

The purpose of this study, therefore, is to explore the meaning of abstraction as used in mathematics education and to address the following research questions:

- How do teachers deal with mathematical abstraction in teaching?
- Do they reduce abstraction level of a concept (or a task) while implementing a mathematical task? If they do, what are the approaches of reducing abstraction in teaching mathematics in the data sample of this study?

1.6. Overview

In the following chapters, I present the details of this study. Chapter 2 describes literature relevant to the notion of abstraction in mathematics and mathematics education. More specifically, in this chapter, I describe different interpretations of abstraction in mathematics education and the challenges and opportunities that the very

nature of mathematical objects provides in teaching and learning the subject. Chapter 3 elaborates upon the theoretical perspectives that have shaped my research question. This chapter begins with a description of the existing theoretical frameworks of reducing abstraction closely related to my study, and proceeds with a discussion pointing to its shortcomings for the approach and perspective of my study. I then propose a modification to and extension of the existing framework. Chapter 4 describes the methodology incorporated in this study, which is qualitative in nature. More specifically, this chapter provides the details of the modified analytic induction methodology and the rationale behind the choice of this methodology. Chapter 5 presents the data analysis of TIMSS's 1999 video lessons along with a discussion of this. In the course of this work, the Reducing Abstraction in Teaching (RAiT) framework has emerged. In chapter 6, this RAiT framework has been used to analyse and interpret primary data obtained through my classroom observation of three instructors in university preparation mathematics courses. Finally, Chapter 7 presents a summary of the study, the implications of its results and avenues for further research, followed by the limitations of the study and final reflection.

2. Abstraction in Mathematics and Mathematics Education

Abstraction has been an object of discussion across several disciplines. Particularly in philosophy and the philosophy of mathematics, it has been the central topic of intense inquiry as far back as the days of Plato and Aristotle. As Hershkowitz, Schwarz and Dreyfus (2001) put it, “not only did Plato and his followers see in abstraction a way to reach 'eternal truths', but modern philosophers such as Russell (1926) characterized abstraction as one of the highest human achievements” (p. 196). In this chapter, rather than providing a detailed review of research on abstraction in philosophy and other disciplines, I will focus on the notion of abstraction as used in mathematics and mathematics education.

2.1. Abstraction in Mathematics: Historical Roots

Abstraction is often seen as the fundamental characteristic of mathematics. For Aristotle, “mathematical objects are the result of abstraction” (Lear, 1982, p.161). Similarly, for Davis and Hersh (1983), abstraction is “the life’s blood of mathematics” (p. 113). But, what constitutes a mathematical object, and how do (or can) we know them? These have been the central questions in the philosophy of mathematics from time immemorial. Plato’s answer for this question is the following: mathematical objects, like circles and triangles, are *forms*, which can be accessed only by intellect. What we experience through our senses is merely the imperfect reflection of these perfect forms. Hence, in Platonic view, “there are two separate realms accessible to human cognition: A transient, changing realm perceptible to the senses, and a timeless, eternal realm that is conceivable to the intellect” (Campbell, 2004, p. 12). In the theory of recollection, Plato maintained that before we are born, our soul has knowledge of the *form*, but somehow we forgot this knowledge during the traumatic experience of our own birth. Therefore, gaining knowledge means remembering the knowledge that we already

possessed in our previous life, which can be encouraged through a dialectical process (Allen,1959).

Plato supported his theory of recollection by illustrating the dialogue between Socrates and a slave boy. Through a sequence of questions, the ignorant slave boy was able to acquire the knowledge of a simple geometrical theorem (of how to construct a square with an area twice that of the given square). Since Socrates did not tell the theorem to the boy, Plato, in *Meno*, draws the conclusion that the boy must have had knowledge of it already (Allen, 1959). Although this theory's reliance on otherworldly *forms* may seem problematic for us today, it does have some advantages. For instance, it does justice to the ontological and epistemological status of the abstract nature of mathematical objects by explaining why we need to apprehend via our intellect rather than merely physical experience in order to justify a mathematical statement. In fact, the public image of mathematics in the present day still seems to be largely influenced by Platonic philosophy of mathematics, in which mathematics is viewed as an objective reality existing in the Platonic realms.

In contrast, Aristotle took a different position, holding that mathematical objects, like circles and triangles, are abstractions from our experience (Lear, 1982). For Aristotle, abstract mathematical concepts depend on some sensible bodily experience for its subsistence. That is, we access the idea of a perfect triangle from our interaction with various roughly triangular objects. For this to happen, however, he introduces the notion of a '*qua-operator*' that allows us to study specific triangles that we encounter, *qua* their triangularity. That is, we deliberately ignore some features of specific triangles to form our concept of the triangle. This act of deliberately ignoring some attributes of the objects to form the concept of those objects is what Lear (1982) calls *legitimate separation* (separation of the correct things) in thought. In the concept of the triangle for example, we focus on a particular triangle existent in our ordinary experience, and attend to some features of the triangle that share commonality with all other triangles, while ignoring other distinguishing features such as size, colour and height. Distinctly of the Aristotelian tradition, Rand (1990) summarizes the idea of legitimate separation in the process of abstraction in this way:

If a child considers a match, a pencil and a stick, he observes that length is the attribute they have in common, but their specific lengths differ [...]. In order to form the concept “length,” the child’s mind retains the attribute and omits its particular measurements”. That is, “length” must exist in some quantity, but may exist in any quantity [...] the term “measurements omitted” does not mean, in this context, that measurements are regarded as non-existent; it means that measurements exist, but are not specified. (pp. 11-12)

This statement is equivalent to the basic principle of algebra, which states that ‘algebraic symbols must be given some numerical value, but may be given any value’. In the equation $(x + y)^2 = x^2 + 2xy + y^2$ for example, any number may be substituted for the symbol ‘x’ and ‘y’ without affecting the truth of the equation (Rand, 1990).

The difference between the approaches taken by Plato and Aristotle in philosophy can be seen as an early example of tension between rationalists and empiricists among early philosophers, the former giving primacy to abstract concepts and ideas and the latter giving primacy to experience. However, a common feature in both traditions is that they recognize mathematics as a subject that deals with abstract objects, even though they each take what amounts to a different psychological position regarding how those mathematical objects are accessible via human cognition. In Platonic tradition, for example, mathematical objects are thought to exist independently in their own right, and knowledge of them is purely an act of intellect. In contrast, Aristotelian tradition views mathematical objects as being immanent within objects of experience whilst knowledge thereof can be gained via a process of abstraction.

2.2. Abstraction in Mathematics Education Research

Abstraction in the mathematics education research has been discussed from various view points. Before going into the details of the various interpretations of abstraction in the mathematics education research, in the following section I briefly describe some of the issues related to mathematics that gives the image of mathematics as an abstract subject.

2.2.1. What Does It Mean to Say that Mathematics is “Abstract”?

Mathematical objects are often thought of as being abstract in nature. Literature shows that there are mainly three issues relating to mathematics that contribute to the image of mathematic as an abstract subject. The first one is concerned with mathematics as self-containment system disconnected from the physical and social world. As Mitchelmore and White (2004) put it, “mathematics has become increasingly independent of experience, therefore more self-contained and hence more abstract” (p. 329).

To emphasise the special meaning of abstraction in mathematics, Mitchelmore and White (2004) distinguishes two modes of abstraction: *abstract-apart* and *abstract general*. According to them, *abstract-apart* refers to the mode of abstraction “in which the mathematical idea is separated and apart from any context” (p. 57). In this sense, the object of abstraction shares much with the Platonic view of mathematical objects, in which a mathematical object is thought to exist in isolation, independent of any other sensible features of an actual object. In other words, its meaning is defined in the world of mathematics and it has no meaning in the physical world. On the other hand, *abstract general* ideas are those mathematical ideas that embody all features and contexts of the objects of thought, although not all features of the objects are specified. They state that the essence of abstraction in mathematics seems to be its self-containment:

An abstract mathematical object takes its meaning only from the system within which it is defined. Certainly abstraction in mathematics at all levels includes ignoring certain features and highlighting others. But it is crucial that the new objects be related to each other in a consistent system which can be operated on without reference to their previous meaning. (p. 330)

As can be seen, mathematical language may consist of symbols and syntax that do not have precise meaning outside of the mathematical world. For example, even though we occasionally use symbols like x, y , and K in the “real” world, objects such as $\sqrt{-1}$ and complex numbers such as $x + iy$ are not known outside of the mathematics world. Even if mathematics uses everyday words, their meaning is defined within the system in relation to other mathematics terms and concepts, which often results in a different meaning than the everyday meaning. For example, in everyday language,

“angle” means “point of view”, but this has a different meaning in mathematics. In other words, objects of mathematics are unique unto themselves.

The second issue concerning abstraction in mathematics is that mathematics is often perceived as “the rules of the game” by students. This is because a large part of school mathematics consists of rules or formulae, symbols and syntax for manipulation of mathematical objects that have little or no connection to their everyday lives (Mitchelmore and White, 2004; Schoenfeld, 1989). Different kinds of problems and situations demand different rules and formulae, and students are expected to learn those rules and formulae and use them appropriately.

The third issue here is the hierarchical nature of mathematics. As Bastik (1993) pointed out, mathematical objects follow a hierarchical system within the school curriculum, such as: “counting comes before addition, which is the foundation for multiplication, which leads on to algebra, which is the basis for [...]” (p. 93). The author maintains that abstraction in mathematics describes the difficulty and complexity non-mathematicians associated with increasing hierarchical levels of mathematics. In essence, the more we move to the higher levels within the hierarchy, the more abstract mathematical entities become.

In a nutshell, the essence of abstraction in mathematics (which gives rise to the public image of mathematics as an abstract subject) is related to the issues of its self-containment, disconnectedness, algorithmic and hierarchical nature. In fact, mathematics is still largely influenced by the *abstract apart* view of abstraction (in schools and universities) in which decontextualization and disconnectedness are seen to be the essential properties of abstraction. Frorer, Hazzan and Manes (1997), for example, investigated how students view abstraction in mathematics. They found that adjectives such as “hidden, complex, requiring deep thought, not concrete, apart from actual substance or experience, not easily understood, a mental construction, a theoretical consideration” (p. 218) were the words student often associated with abstraction. Implicit in their use of these adjectives are the students’ unpleasant mathematics learning experiences. As Ferguson (1986) claims, abstraction (related to such negative learning experiences) is one of the contributing factors for mathematics anxiety among students.

2.2.2. Abstraction and Anxiety

Mathematics anxiety is one of the research fields in mathematics education with vast literature (Suinn, Edie, Nicoletti & Spenelli, 1972; Brush, 1978; Ferguson, 1986; Lyons & Beilock, 2012). Ferguson (1986) distinguishes three factors that consist of mathematics anxiety: 1) *mathematics test anxiety* 2) *numerical anxiety* and 3) *abstraction anxiety*. Mathematical test anxiety is associated with anticipating, taking, and receiving mathematics tests. Numerical anxiety is related with number manipulation. Abstraction anxiety concerns with anxiety related to specific mathematics content.

The Mathematics Anxiety Rating Scale (MARS) is a popular instrument in the investigation of mathematics anxiety (Suinn, Edie, Nicoletti & Spenelli, 1972). Ferguson sought to assess student anxiety about abstraction, for which she modified the Mathematics Anxiety Rating Scale (MARS) by adding ten questions specially designed to assess abstraction anxiety. Hence, Ferguson's new mathematics anxiety inventory (which she calls Phobos Inventory) consisted of thirty items in which 10 of the items related to mathematics test anxiety (adapted from MARS), 10 related to numerical anxieties (adapted from MARS) and 10 items related to abstraction anxiety. Following is the list of ten items Ferguson (1986) designed to assess abstraction anxiety.

1. Having to work a math problem that has x 's and y 's instead of 2's and 3's.
2. Being told that everyone is familiar with the Pythagorean Theorem.
3. Realizing that my psychology professor has just written some algebraic formulas on the chalk board.
4. Being asked to solve the equation $x^2 - 5x + 6 = 0$.
5. Being asked to discuss the proof of a theorem about triangles.
6. Trying to read a sentence full of symbols such as $A = \{x: |x - 2| = 3, x \in I\}$.
7. Listening to a friend explain something they have just learned in calculus.
8. Opening up a math book and not seeing any numbers, only letters, on an entire page.
9. Reading a description from the college catalog of the topics to be covered in a math course.
10. Having someone lend me a calculator to work a problem and not being able to tell which buttons to push to get the answer.

Although Ferguson defines abstraction anxiety as "the anxiety associated with specific mathematics topics" (ibid., p. 149), she does not provide a clear definition of abstraction. However, examining the items she used to measure abstraction anxiety, it appears that abstraction anxiety in her works includes three views of abstraction:

1. Relational view of abstraction (Abstraction anxiety due to students' unfamiliarity of the problem or concepts such as item 2, 5, 7 and 10)
2. Objective view of abstraction (Abstraction anxiety due to the decontextualized or disconnectedness nature of a problem from the real world such as item 1, 3, 4 and 8)
3. Abstraction related to the level of complexity or difficulty (Abstraction anxiety due to the complexity or difficulty of a problem or task such as item 6 and 9)

In her study, Ferguson observed that numerical anxiety and mathematics test anxiety are not the only factors in the mathematics anxiety construct. Abstraction anxiety also plays a significant role in the construct.

2.2.3. *Abstraction as Product and Process*

In the mathematics research community, many researchers and educators agree on the fact that abstraction is one of the most important aspects of mathematics and, as such, abstraction has long been the topic of discussion in the mathematics education (e.g., Piaget, 1980; Ginsburg & Asmussen, 1988; Dienes, 1989). Piaget (1980) observes that abstraction constitutes the skills required for learning elementary through advanced mathematical concepts: "[Abstraction] alone supports and animates the immense edifice of logico-mathematical construction" (p. 92). Ferrari (2003) also notes the role of abstraction in learning mathematics and states "abstraction has been recognized as one of the most important features of mathematics from a cognitive viewpoint as well as one of the main reasons for failure in mathematics learning" (p. 1225). However, difficulty with regard to the precise meaning of abstraction still exists, as noted by Hazzan and Zazkis (2005):

There is no consensus with respect to a unique meaning for abstraction; however, there is an agreement that the notion of abstraction can be examined from different perspectives, that certain types of concepts are

more abstract than others, and that the ability to abstract is an important skill for a meaningful engagement with mathematics. (p.102)

The sources of this definitional conflict can be traced partly to the language, as ‘abstraction’ can be used as a verb, noun, or adjective, each giving different perspectives to the meaning of abstraction. Mitchelmore and White (1995), in their extensive literature review, note that abstraction in mathematics education literature has been discussed mainly from two perspectives:

1. Abstraction as the product or object: From this perspective, a new mental object results from the process of abstraction. Borrowing the words of Davis and Hersh (1981), abstraction in this sense occurs “when the perception of three apples is freed from apples and becomes the integer three” (p.126). Here, the abstract object *the integer three* is created through the process of abstraction from three apples.
2. Abstraction as the process: Abstraction in this sense is the process itself by which one focuses on common features of the objects that are relevant to the concept by eliminating other features that are irrelevant. In other words, as Davydov (1990) puts it, abstraction in this sense is the process of “separating a quality common to a number of objects/situations from other qualities” (p.13).

In the following sections, I use these distinctions to examine different views on abstraction in mathematics education.

Abstraction as the product or object

Skemp (1986) defines a mathematical concept as “the end-product of [...] an activity by which we become aware of similarities [...] among our experiences” (p.12). The emergence of the end-product from an activity is the mathematical object (in our case), which is a “static object-like representation which squeezes the operational information into a compact whole and turns the cognitive schema into a more convenient structure” (Sfard, 1991, p. 26). For example, the algebraic representation $T_n = 4n + 3$ for $n \geq 1$ is the mathematical object which emerged from the activity of finding different terms in the sequence 7, 11, 15, 19, 23... and so on. This algebraic notation is an abstraction or a mathematical object that squeezes the operational information of adding four each time and allows us to find any term in the sequence without having to add four repeatedly. Unlike the everyday meaning of the word ‘objects’, mathematical objects are

quite different in nature and appearance which distinguishes mathematics from other subjects.

On the nature of mathematical objects

In everyday language, the word 'objects' refers to the things that are tangible, material and real, but this is not the case in the philosophy of mathematics. In mathematics, 'objects' can refer to mental constructs or abstract objects, such as properties, classes, propositions or relationships. The public image of mathematics is, therefore, that mathematics is disconnected and decontextualized from real life. Hence, one of the questions worthy of discussion is how such abstract mathematical objects, given their decontextualized and disconnected nature, are so useful and applicable in modelling every aspect of real life situations? In other words, if mathematical entities exist independently, without any reference to the world that we live in, how are they so useful in real life situations, for instance, in business, physics, astronomy, engineering, and so on? This is an ontological question about mathematical objects.

The ontological (and epistemological) complexity of mathematical objects has a historical root, and discussion surrounding this issue maintains considerable momentum in the philosophy of mathematics. Davis and Hersh (1981) nicely articulated the difficulty a mathematician encounters when asked about the nature of the objects he or she deals with:

The typical working mathematician is a Platonist on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality whose properties he is attempting to determine. But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all. (p. 321)

As stated earlier Platonic view with regard to mathematical objects such as circles and triangles holds that these mathematical objects are abstract entities which exist independent of all our rational activities. What we experience through our senses is merely the imperfect reflection of these perfect *forms*. Hence, mathematical knowledge is possible not through our experience but only through our intellect. In contrast, Aristotelian view rejects the out worldly existence of such *forms* and holds that mathematical objects, like circles and triangles, are abstractions from our experience.

For Aristotle, abstract mathematical concepts depend on some sensible bodily experience for its subsistence. For example, the concept of a perfect triangle is possible from our interaction with various roughly triangular objects.

Thus, the distinction between Platonic and Aristotelian views is an early example of tensions between philosophical theories with regard to the nature of mathematical objects. Piaget (1979), for example, observes that “it was never possible to agree upon what in fact mathematical entities are” (Cited in Sierpinska & Kilpatrick, 1998, p. 177). Cañón (1993) also notes that “the ontology of mathematical entities and, even more so, its epistemology is interpreted in an incredibly disparate way (in the literature) and it remains a mystery” (cited in Godino & Batanero, 1998, p. 2).

This debate has recently taken a different turn in terms of their meaning and interpretation, at least in mathematics education. Noss and Hoyles (1996), evoking the work of Lave (1988), introduce the term ‘situated abstraction’, which gives meaning to mathematical object in relation to the situation and context from which the mathematical object or concept has been abstracted. They use the phrase ‘situated abstraction’ to “emphasize connection with situations, not seeking to challenge the utility of formal mathematical abstraction, but maintaining that abstraction can take place *in situ* rather than only within a self-contained system” (p. 122). This idea shares much with the “abstract general” idea (White & Mitchelmore, 2010) as discussed earlier. As opposed to abstract-apart concept (concepts in isolation), abstract-general concepts represent some “common feature of the situations in which they arise. There are strong links between the abstract object and the contexts from which the concept has been generalized, allowing the abstract object to be applied to similar contexts” (p. 207).

In fact, many fundamental mathematical objects, especially elementary concepts like counting, numbers and their operations, model reality. Later developments, such as algebra and number theory, are built on these fundamental ideas, and thus have a basis in reality. Hence all mathematical objects have some link to reality, giving meaning to the ontological status of mathematical objects. This shift in perspectives allows for a view of mathematics as socially created and situated, and retaining its applicability across different settings.

Abstraction as a process

The Oxford English Dictionary defines abstraction (n.) as “the act or process of separating in thought, of considering a thing independently of its associations; or a substance independently of its attributes; or an attribute or quality independently of the substance to which it belongs”. From this definition it is clear that abstraction can be viewed as a process. But for Walkerdine (1988) and Noss and Hoyles (1996), the use of words such as ‘separate’ and ‘independent’ seems to be problematic, as they provide a negative connotation of the notion of abstraction, such as abstraction as a process of decontextualization. This problem was also acknowledged by Thomas Aquinas (1245–1274), who long ago tried to resolve this problem by distinguishing two modes of abstraction. Aquinas writes:

Abstraction may occur in two ways. First [...] we may understand that one thing does not exist in some other, or that it is separate from it. Secondly [...] we understand one thing without considering another. Thus, for the intellect to abstract one from another thing which are not really abstract from one another, does, in the first mode of abstraction, imply falsehood. But, in the second mode of abstraction, for the intellect to abstract things which are not really abstract from one another, does not involve falsehood.

[...] If, therefore, the intellect is said to be false when it understands a thing otherwise than as it is, that is so, if the word otherwise refers to the thing understood.[...] Hence, the intellect would be false if it abstracted the species of a stone from its matter in such a way as to think that the species did not exist in matter, as Plato held. But it is not so, if otherwise be taken as referring to the one who understands. (*Summa Theologiæ* I. 85. 1 ad 1; Aquinas, 1999, p. 157) (cited in Long, 2006, p. 7)

Aquinas thus saw that ‘separation’ in the process of abstraction can be viewed from two perspectives. In the first mode of abstraction, all distinguishing features of the particular object from the concept are excluded, as if these features are non-existent. In the second mode of abstraction, features of the particular objects from the concept are omitted; however, these attributes are considered existent in the concept. These modes of abstraction are often called *precisive* and *non-precisive*, respectively (Long, 2006). Maurer (1968) explains these two distinct modes of abstraction as follows:

Precision is a mode of abstraction by which we cut off or exclude something from a notion. Abstraction is the consideration of something without either including or excluding from its notion characteristics joined

to it in reality. Abstraction without precision does not exclude anything from what it abstracts, but includes the whole thing, though implicitly and indeterminately. (cited in Long, 2006, p. 7)

A similar position is held by Gilson (1937) when he asserts that “to abstract is not primarily to leave something out, but to take something in, and this is the reason why abstractions are knowledge” (pp. 144-45). This view of abstraction is similar to the notion of abstraction as found in the work of Walkerdine (1988). According to Walkerdine, abstraction is the process in which one focuses on features that are relevant to the concept by “suppressing” other details that are irrelevant. For Skemp (1986), on the other hand:

Abstracting is an activity by which we become aware of similarities [...] among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class [...]. To distinguish between abstracting as an activity and abstraction as its end-product, we shall hereafter call the latter a *concept*. (p. 21)

In contrast with the views described above, Hershkowitz, Schwarz and Dreyfus’s (2001) account of abstraction sees the abstraction process from socio-cultural perspectives within mathematics education. They maintain that abstraction is a process of “[...] vertically reorganizing previously constructed mathematics into a new mathematical structure” (p. 2). This vertical reorganization activity “[...] indicates that abstraction is a process with a history; it may capitalize on tools and other artifacts, and it occurs in a particular social setting” (p. 2).

What is implicit in the account of abstraction by Hershkowitz et al. (2001) is that abstraction is a process that occurs within a context. Abstraction in this view takes in to account various things such as students’ prior knowledge, the learning environment, technological tools, mathematics curriculum, classroom norms and other social components. Based on this idea of abstraction, they developed a theoretical-methodological model of abstraction called RBC (recognizing (R), building-with (B) and constructing (C)). According to this model, a new construct that emerges as a result of abstraction can be described and analysed by means of three observable epistemic actions:

Recognizing a familiar mathematical structure occurs when a student realizes that the structure is inherent in a given mathematical situation [...]. *Building-With* consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. *Constructing* is the central step of abstraction. It consists of assembling knowledge artefacts to produce a new knowledge structure to which the participants become acquainted. (Schwarz, Hershkowitz & Dreyfus, 2002, p. 83)

According to this model, the first step in concept formation for learners is to familiarize themselves with the new and unfamiliar concept utilizing their existing knowledge schemata. It is therefore, suggested that for meaningful learning, teachers should emphasize in their teaching the connection between abstract mathematical concepts and their familiar real world (at least at the school level). This way, once students grasp the meaning in mathematical objects and have concretised such objects in their minds, the teacher can then move to another level of abstraction. What is implicit in this perspective is that learning progression moves from concrete to abstract ideas. However, some researchers, such as Kaminski, Sloutsky and Heckler (2008), suggest that if a concept is introduced with the use of generic instantiations rather than concrete examples, students may be better able to apply that concept in various situations. They argue that “grounding mathematics deeply in concrete contexts can potentially limit its applicability” (p. 455). This shows that debate surrounding the effectiveness of the *abstract-concrete* order and the *concrete-abstract* order of teaching and learning mathematics continues in mathematics education.

2.3. Abstract-Concrete Dichotomy and Mathematics Teaching

Understanding what a mathematical concept is and how it is learned and taught has been the focus of much research within the mathematics education research community in recent years. Even though the concrete (to semi-concrete) to abstract order of teaching is widespread, the debate around the idea that whether instruction should support abstract-concrete order or concrete–abstract order as a teaching sequence still exists.

2.3.1. From Abstract to Concrete

Kaminski and Sloutsky's (2008, 2009) focused their attention primarily on the issue of mathematical knowledge transfer in which they observed some problematic aspects of concrete-abstract learning progression. Vygotsky's theoretical perspective however, provides an extensive treatment to the issue of learning progression and particularly, on child's concept formation in which he argues that concept development does not necessarily follow the concrete- abstract path. According to him, scientific concepts develop from abstract to concrete where as spontaneous concepts ascent from concrete to abstract.

Vygotsky (1987) classifies concepts in to two types, spontaneous concepts and scientific concepts, and pointed out that a major difference between spontaneous concepts and scientific concepts lies in the degree of conscious comprehension of the substance of the concepts. According to him, spontaneous concepts are the children's intuitive knowledge or complexes (nonconscious concepts) based on their everyday experiences. Scientific concepts, on the other hand, are conscious concepts (such as mathematical concepts) taught in school and are mediated by other concepts.

Vygotsky's thesis is based on the experiment in which he observed that "the child formulates Archimedes' law better than he formulates his definition of what a brother is" (Vygotsky, 1987, p. 178). Based on this observation, he argued that the spontaneous concept (such as brother) and scientific concepts (such as the concept involved in Archimedes' law and taught in school) are learnt in different ways. The concept of 'brother' was developed through day to day experience and has not been subject to defining it in a scientific way. Archimedes' Law on the other hand, was introduced and developed in a scientific way, with a verbal definition, and with relation to other concepts. Thus, in Vygotskian theoretical approach scientific concepts formation and spontaneous concept formation follow different developmental trajectories; the former from the abstract to the concrete (from the concept to the thing) and the later from the concrete to the abstract:

From the very beginning the child's scientific and his spontaneous concepts for instance "exploitation" and "brother" develop in reverse directions: starting far apart, they move to meet each other. [...] The child becomes conscious of his spontaneous concepts relatively late; the ability

to define them in words, to operate with them at will, appears long after he has acquired the concepts. He has the concept (i.e. knows the object to which the concept refers), but is not conscious of his own act of thought. The development of a scientific concept, on the other hand, usually begins with its verbal definition and its use in non-spontaneous operations with working on the concept itself. It starts its life in the child's mind at the level that his spontaneous concepts reach only later. (Vygotsky, 1986, p. 192)

One important insight from Vygotsky is that in the formation of spontaneous concepts the movement in child's mind is from particular instances to more general concepts whereas in school (for scientific concepts) a set of abstract principles and definitions are introduced first and as the education continues, children are to find out how these abstract concepts are applied to the specific context. Each of the concepts however has its own strength and weakness. Vygotsky (1986) writes:

When the child learns a scientific concept, he quickly begins to master the operations that are the fundamental weakness of the everyday concept. He easily defines the concept, applies it in various logical operations, and identifies its relationships to other concepts. We find the weakness of the scientific concept where we find the strength of the everyday concept, that is, in its spontaneous usage, in its application to various concrete situations, in the relative richness of its empirical content, and in its connections with personal experience. Analysis of the child's spontaneous concept indicates that he has more conscious awareness of the object than of the concept itself. Analysis of his scientific concept indicates that he has more conscious awareness of the concept than of the object that is represented by it. (p. 218)

As Vygotsky pointed out, one of the weaknesses related to spontaneous concepts is that since spontaneous concepts are not based on systematic logic and reason but rather on children's everyday experience, the child is more conscious of the substances of the concept rather than the concept itself. As such, in the context of mathematics education, Vygotsky's theory supports the idea that algebraic concepts (which are scientific) are to be introduced before arithmetic concepts (as those are closely related to child's everyday experience) in school curriculum which is not the traditional order.

Dialectical relationship of abstract and concrete

From what has been said, Vygotskian theory of concept formation appears (at the surface) to suggest the abstract-concrete order of teaching, but it is not the case. Vygotsky (1986) pointed out that development of a true concept (conceptual understanding) is only possible when the everyday concepts (lower form) and the scientific concepts (higher form) come into dialectical relationship:

In working its slow way upward, an everyday concept clears a path for the scientific concept and its downward development. It creates a series of structures necessary for the evolution of a concept's more primitive, elementary aspects, which give it body and vitality. Scientific concepts in turn supply structures for the upward development of the child's spontaneous concepts toward consciousness and deliberate use. (p. 194)

Starting far apart, they move to meet each other. (p. 192)

Taking this idea further, Davydov's (1990) introduced dialectical materialistic account of abstraction which offers a way of resolving the debate about how to teach abstract mathematical concepts. Davydov thought that the concrete is correlated with the abstract, and that learning does not follow a trajectory from concrete to abstract, but rather a dialectical, two-way relationship between the concrete and the abstract. In his view, the abstraction process results in the discovery of the essence, which ultimately needs to ascend back to the concrete. In other words, in Davydov's (and also in Vygotsky's) notion of abstraction, conceptual understanding occurs when the abstract concepts descend overtime to the level of experience and become concrete, and spontaneous concepts ascend to the object of self-consciousness and become conceptualized, each meeting in the middle. Davydov writes, "to know essence means to find the universal as a base, as a single source for a variety of phenomena, and then to show how this universal determines the emergence and interconnection of phenomena—that is, the existence of concreteness" (p. 289).

The Problem

The *abstract-concrete* order of learning seems to rely on two assumptions. The first is a Platonic philosophy in which mathematics is viewed as an objective reality existing in the platonic realm, which is not accessible to our senses but can be revealed

by a good teacher (Ernest, 1991). The second is based on the assumption that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). Based on this theory, The New Mathematics movement (1960-1970), for example, was introduced from the perspective that learning abstract structures would allow for their correct usage later on (Moon, 1986). Hence, the views about teaching as the transfer of abstract, decontextualized mathematical concepts had become one of the characteristics of most school teaching in the sixties (Brown, Collins & Duguid, 1989). However, as Brown et al. (1989) point out, this method of teaching has proven to be ineffective and "much of what is taught turns out to be almost useless in practice" (p. 32). Consequently, the reformed mathematics was advocated for the improvement of mathematics education which supports the idea that since the progression of learning move from concrete to abstract the instruction should follow the same route- that is, from concrete to abstract.

2.3.2. *From Concrete to Abstract*

The position that mathematics instruction should ascend from concrete (to semi-concrete) to abstract as a teaching sequence is based on Piaget's (1970) notion of developmental psychology, according to which children develop abstract thinking slowly, starting as concrete thinkers with little ability to create or understand abstractions. In this tradition, as Ozmantar (2005) puts it, "abstraction is considered as higher-order knowledge ... arising from the recognition of commonalities isolated in a large number of specific instances and associated with an ascending developmental process from the concrete to the abstract" (p. 79). Mitchelmore and White (1995) therefore, recommend that teaching should start with concrete contexts and then ascend to the abstract, at least at the elementary level. They write:

Teaching mathematics through the process of abstraction would look very different from traditional mathematics teaching. It would start with applications instead of leaving them to the end. Students would build up their understanding of each context separately, gradually becoming aware of similarities, and symbols would be used to summarise and clarify the similarities. Finally, ways would be found to manipulate the symbols abstractly in order to deal more efficiently with the original applications and investigate new ones. (p. 66)

Based on this idea, the genetic approach to teaching mathematics is widespread (Safuanov, 2004). The core principle of the genetic approach to teaching advocates that mathematics is best learned when learners are exposed to or engaged with (at least in part) the mathematical process that is most closely related to the path followed by the original inventors or discoverers of that knowledge, akin to what Sawyer (1943) once said: “The best way to learn (mathematics) geometry is to follow the road which the human race originally followed: Do things, make things, notice things, arrange things, and only then reason about things” (p. 17). Sawyer also argued that abstract concepts cannot be taught effectively in an abstract way. In his book *Concrete Approach to Abstract Algebra*, Sawyer (1959) therefore suggested:

Professor[s] may choose familiar topics as a starting point. The students collect material, work problems, observe regularities, frame hypotheses, discover and prove theorems for themselves. The work may not proceed so quickly; all topics may not be covered; the final outline may be jagged. But the student knows what he is doing and where he is going; he is secure in his mastery of the subject, strengthened in confidence of himself. He has had the experience of discovering mathematics. He no longer thinks of mathematics as static dogma learned by rote. He sees mathematics as something growing and developing, mathematical concepts as something continually revised and enriched in the light of new knowledge. (p. 2)

This notion of teaching aligns with the objectives of most of the reformed mathematics curricula with which I’m familiar, including the National Council of Teachers of Mathematics (NCTM) principles, standards, and visions. This modern reformed movement has roots in the constructivist philosophy and genetic epistemology, according to which knowledge is viewed as something that is “not directly transmitted from knower to another, but is actively built up by the learner” (Driver et al., 1994, p. 5). Incorporating the basic notion of these theories of learning and ever changing role of technology in education, the National Council for Accreditation of Teacher Education (NCATE, 2003) recommends that teachers introduce mathematical concepts by using various pedagogical tools as well as learners’ familiar contexts in order to enhance mathematical learning. In fact, as Freiman, Vézina, & Gandaho (2005) state, the “use of technology becomes an important didactical resource for communication in the mathematics classroom” (p.178) in the recent years. In this regard, some of these recommendations for teachers as documented in NCATE (2003) include:

- selects and uses appropriate technological tools, such as but not limited to, spreadsheets, dynamic graphing tools, computer algebra systems, dynamic statistical packages, graphing calculators, data-collection devices, and presentation software.
- selects and uses appropriate concrete materials for learning mathematics

The reasoning behind the use of these types of recommendations or pedagogical tools such as technology and concrete (manipulative) materials is that they mediate between the new (abstract) concept and students' familiar context, object or concept. In this process, abstraction level is reduced and the concept becomes less abstract for students. Consequently, the use of concrete materials (such as manipulatives) in the teaching of mathematics has been widespread in the last few decades.

The Problem

There is, however, considerable research showing that merely using concrete objects to teach mathematics does not guarantee that students will learn mathematics (Ball, 1992; Uttal, Liu, & DeLoache, 1999; Vitale, Black, & Swart, 2014). As stated earlier, Kaminski and Sloutsky (2009) for example, reported that the use of concrete objects or examples does not necessarily promote students' abstract thinking and "ability to recognize novel instantiations and successfully transfer knowledge" (p. 154).

In their research, Kaminski and Sloutsky (2009) investigated the effects on knowledge transfer from one generic instantiation versus concrete instantiations. One group of undergraduate students were given many concrete examples that situated the mathematics in context. In contrast, the other group were given generic examples on the same concept without context. The researchers found that students seemed to learn concepts quickly when they were presented with real objects such as marbles or containers. However, the knowledge gained using this method appeared to be bound to that specific learning domain and could not be easily recognised or transferred in other situations.

Noss and Hoyles (1996) maintain that the problem with all accounts of abstraction discussed above is that the distinction between abstract and concrete is drawn on the basis of traditional views of abstraction, in which decontextualization and disconnectedness are assumed to be the essential properties of abstraction. For them,

the scope of this view of abstraction is too narrow and ontologically problematic. They suggest:

Where can meaning reside in a decontextualized world? If meanings reside only within the world of real objects, then mathematical abstraction involves mapping meaning from one world to another, meaningless, world—certainly no simple task even for those with the capacity to do it. If meaning has to be generated from within mathematical discourse without recourse to real referents, is this not inevitably impossible for most learners? (p. 21)

Likewise, van Oers (2001) observes that the distinction between abstract and concrete as commonly understood is misleading: “The split between the concrete and the abstract actually creates a misleading divorce between the perceptual-material and the mental conceptual world” (p. 287). He therefore, suggests that we take into account the inner relationship between the abstract mental-conceptual world and concrete perceptual-material world in order to allow for meaningful insight into the abstraction of the concrete world.

Rather than speaking in terms of objective view (disconnected and decontextualized) of abstraction, Sfard (1991) refers to a shift that occurs through reification in the move from process to object. What follows is the brief overview of the theory of process-object duality found in Sfard’s work.

2.3.3. Process - Object Duality

According to Sfard, a mathematical entity can be viewed as a process (operational conception) and as an object (structural conception) and that object conception is more abstract than process conception. She writes:

Of the two kinds of mathematical definitions, the structural descriptions seem to be more abstract. Indeed, in order to speak about mathematical *objects*, we must be able to deal with *products* of some processes without bothering about the processes themselves. [...] It seems, therefore, that the structural approach should be regarded as the more advanced stage of concept development. (Sfard, 1991, p.1)

In the process of concept formation, operational conception precede the structural conception and that transition from operational to structural conception involves three steps: interiorization, condensation, and reification. Interiorization refers to the stages in which a learner get acquainted with the process which will give rise to a new concept such as counting leads to natural numbers. Condensation is similar to its everyday meaning which involves squeezing lengthy sequence of operation in to manageable unit. In this stage, a learner can view the former process as a separate entity and use it as a sub process to other processes. In the reification stage, a learner can “see this new entity as an integrated, object-like whole (Sfard, 1991, p. 18). In other words, reification occurs when a learner is able to view the process structurally as an object.

Although operational conception and structural conception seem to be incompatible in surface, Sfard points out that these two approaches are complementary to each other for conceptual understanding of a mathematical concepts and that there is an “intricate interplay between operational and structural conceptions of the same notions” (ibid., p.1). Hence, from this perspective, teaching is an activity that supports the transition from operational conception to structural conception, and at the same time help students to be able to integrate the operational and structural aspect of a mathematical concept.

In her later writings, Sfard refers to a discursive shift in which abstract mathematical concepts or objects are seen to be discursively constituted. In her own words, “mathematics begins where the tangible real-life objects end and where reflection on our own discourse about these objects begins” (Sfard, 2008, p. 129). Sfard defines thinking as “an individualized version of (interpersonal) communicating” (ibid, p. 81) and that “patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual” (ibid., p. 78). Based on these assumptions, she coined the term “commognition” by combining two words *communication* and *cognition* which “stresses the fact that these two processes are different (intrapersonal and interpersonal) manifestations of the same phenomenon” (ibid., p. 296). She argues that “changes in all forms of human doing are a function of changes in commognition, thus in discourses” (ibid., p. 116). Hence, from the commognitive standpoint, learning mathematics is defined as “individualizing mathematical discourse, that is, as the

process of becoming able to have mathematical communication not only with others, but also with oneself” (Sfard, 2006, p. 162).

One of the important ideas in Sfard’s commognitive framework is objectification. Objectification is defined as “a process in which a noun begins to be used as if it signified an extradiscursive, self-sustained entity (object), independent of human agency” (Sfard, 2008, p. 412). The process consists of two steps: reification and alienation. Reification refers to how talk about processes and actions is transformed into talk about objects. Alienation is presenting the objects in an impersonal way as if they existed independently without the participation of human being. When these two processes are taken together what was previously something to do (*processes*) transforms itself into discursive object (*structural*). For example, in the process of objectification, the process such as counting a set of objects and ending on the word four transforms into discursively constructed objects such as number four. She maintains that this act of objectification is central to the development of human thought without which mathematics could not progress as a discipline.

2.4. Towards Relational View of Abstraction and Teaching

Wilensky (1991) takes a different perspective and, instead of locating the assessment of abstraction solely in the object, he redefined abstraction as the relationship between the person and object of thought, thereby promoting a more subjective notion of abstraction. He maintains that even physical objects, tangible things and pictures of such things are not necessarily concrete from the perspective of the learner. For example, snow, which is the frozen ice crystal of rain, is a physical object that is tangible. But if a person is living in a place where snowfall never occurs or has otherwise never seen snow, he may find the concept abstract—because snow for him is just a mental object abstracted by the properties of snow as told by someone else or read about in a book. Wilensky says:

Concreteness is not a property of an object but rather a property of a person's relationship to an object. Concepts that were hopelessly abstract at one time can become concrete for us if we get into the "right relationship" with them. Concreteness, then, is that property which measures the degree of our relatedness to the object, (the richness of our

representations, interactions, connections with the object), how close we are to it, or, if you will, the quality of our relationship with the object. (Wilensky, 1991, p.198)

From this perspective, mathematical concepts are neither more nor less abstract in their own right; rather, this depends on the internal connection with the concept on the part of the learner. An individual who is not familiar with a concept may find that concept abstract, whereas the same concept may be quite concrete for another individual. For example, “to a topologist a four dimensional manifold is as concrete as potato” (Noss and Hoyles, 1996, p.46), whereas the same concept for a layman is very abstract and challenging. He goes on to say, “concepts that were hopelessly abstract at one time can become concrete for us if we get into the ‘right relationship’ with them” (Wilensky, 1991, p.198). Therefore, in the process of learning, when the abstract mathematical objects are being in a right relationship with the learner, a previously abstract concept becomes familiar and meaningful similar to what Mitchelmore and White (1995) argue:

A concept—often formed over some time and with great difficulty eventually becomes an object in its own right [...] At first, children can only think about numbers using specific objects such as counters or fingers. But by secondary school, most children have learnt to compute with numbers without thinking about what they might mean. What was once a difficult, abstract idea has become a familiar, almost concrete object. (p. 53)

Hence, in Wilensky’s (1991) account of abstraction, the debate surrounding the teaching trajectory of abstract-concrete or concrete-abstract order as commonly understood collapses and takes a different turn. This turn is directed towards establishing a ‘right relationship’ between learners and new (unfamiliar) mathematical concepts in order for meaningful learning to take place. How can this be done?

There is no direct answer for this question. However, one way of accomplishing this may be to use learners’ existing knowledge, experience and level of thinking, as well as their familiar contexts while introducing new mathematical concepts. In so doing, the level of abstraction of the concept is reduced and presented to the students so that students can deal with mathematical concepts or objects within their comfort zone, thereby promoting a richer connection between the learner and the concept.

This notion is closely aligned with the constructivist perspective on learning, which views knowledge as something to be actively constructed by the learner based on their existing repertoire of knowledge and experience. In fact, as Cornu (1991) states, “for most mathematical concepts, teaching does not begin on virgin territory” (p.154), but rather, all students come with certain ideas, intuition, and knowledge already formed in their mind on the basis of their previous experience. Cornu argues that “these ideas do not disappear following the teaching, contrary to what may be imagined by most teachers” (p.154). Therefore, he suggests, it is important for teachers to become explicitly aware of this difficulty on the part of their students and to attempt to reconstruct their knowledge structure to accommodate the new concepts. Bain (2004) adds:

Students bring paradigms to the class that shape how they construct meaning. Even if they know nothing about our subjects, they still use an existing mental model of something to build their knowledge of what we tell them, often leading to an understanding that is quite different from what we intend to convey. (p. 27)

In this regard, Hazzan (1999) observed that when encountered a new and unfamiliar mathematical concept or problem, learners exhibit various tendencies of making unfamiliar familiar based on their previous knowledge and experience (with similar concepts or situations) often resulting to an understanding that is quite different from what the teacher intended to convey. She calls this learners’ coping strategy *reducing abstraction*. What follows is a brief overview of reducing abstraction framework, the detailed introduction of which is presented in chapter 3.

2.5. Reducing Abstraction: A Brief Introduction

Reducing abstraction is a theoretical framework propounded by Hazzan (1999) that examines learners’ activity while learning new mathematical concepts. This framework is based on her work in which she observed that while learning a new and unfamiliar mathematical concept, students tended to work in a lower level of abstraction as a way of coping strategy in order to make the concept mentally accessible. She concluded that the reason for students working in a lower level of abstraction is related to their existing constructs. Since the students usually do not have enough resources to

deal with the same abstraction level of the concept as intended by the authorities (e.g., textbook or teacher), they often reduce abstraction level of the concept inappropriately resulting in an understanding that is quite different from what is expected by the authorities.

Drawing from the literature, Hazzan (1999) defines level of abstraction from three perspectives: 1) relational view point (abstraction level is measured based on the relationship between the concept and the thinking person) 2) process-object duality view point (process conception of a mathematical concept is less abstract than object conception). 3) degree of complexity view point (the more complex or compound a mathematical entity is the more abstract it is).

I will attend to these three interpretations of abstraction in more detail in chapter 3. Suffice it to say here that Hazzan found that since students frequently do not have enough resources to cope with the unfamiliar and abstract concepts, they tend to reduce the abstraction level (often inappropriately) in an attempt to make the unfamiliar familiar.

Of note here is that Hazzan amalgamates these three viewpoints under one umbrella in her theoretical framework of reducing abstraction. A close look at this theoretical framework reveals that these three viewpoints of interpretation of abstraction are in conflict with each other. For example, viewpoint (1) is in conflict with view point (3) because the former takes in to account the subjective aspect of the learners according to which the abstraction is interpreted based on the quality of relationship between the learner and the concept at hand. In contrast, viewpoint (3) ascribes complexity to the concept and does not take the subjective aspect of the learner into account. Likewise, viewpoint (2) considers the object conception to be at a higher level of abstraction but it may not always be the case if we look this issue from view point (1). For example, the circle is an object and often easier for people to think about as an object than that as a process—as is the case for many geometrical concepts. Despite these contradictions, it may be helpful in interpreting learning and teaching activities with regard to dealing with abstraction.

2.6. Summary

The literature reveals that there is no consensus on a single definition of abstraction (Hazzan & Zazkis 2005). The source of definitional conflict seems to arise partly due to the language, as 'abstraction' can be used as a verb, noun, or adjective, allowing for different interpretations in the meaning of the term. However, as Mitchelmore and White (1995) noted, abstraction in mathematics education literature has been discussed mainly from two perspectives: 1) abstraction as the product 2) abstraction as the process.

With regard to abstraction as the product, the literature also revealed that the notion of abstraction in mathematics education has been used mainly from two perspectives: 1) Relational view of abstraction in which abstraction is viewed as the degree of relatedness of the person and the object of thought (e.g., Wilensky, 1991). From this perspective, abstraction is not a property on its own right, it represents the quality of the relationship between the concept and the thinking person. 2) Objective view of abstraction in which an abstract object or concept is the one that is disconnected or decontextualized from the real world similar to what Hiebert and Lefevre (1986, pp. 4-5) state: "the term abstract is used here to refer to the degree to which a unit of knowledge [...] is tied to specific contexts. Abstractness increases as knowledge becomes freed from specific contexts" (pp. 4-5).

With regard to the meaning of abstraction as process, it has been discussed mainly from two modes of abstraction: 1) *precisive* and 2) *non-precisive*. In the first mode, abstraction is viewed as the process in which all distinguishing features of the particular object from the concept are separated, as if these features are non-existent. In the second mode of abstraction, features of the particular objects that are irrelevant are omitted from the concept; however, these attributes are considered existent in the concept. Hershkowitz, Schwarz and Dreyfus's (2001) interpret abstraction as a process but they do so from socio-cultural perspectives within the mathematics education. They maintain that abstraction is an activity of vertically reorganizing previously constructed knowledge into a new knowledge structure.

Since mathematical concepts are commonly thought to be abstract, the debate over what is the most effective way of teaching mathematics, e.g., whether proceed from abstract to concrete or the other way around, has been an important topic of discussion in the mathematics education. In this chapter, I have provided description of both views including their advantages and disadvantages in the context of teaching and learning mathematics. The debate about abstract-concrete versus concrete-abstract teaching trajectory collapses with Wilensky's (1991) interpretation of abstraction. In Wilensky's view, abstraction is interpreted not as the property of the object itself, but rather the relationship between the person and concepts he is trying to learn.

One of the important issues discussed about abstraction in the mathematics education is that in order to achieve meaningful learning, learners should be able to appropriately deal with the abstraction of a mathematical concept and bring the concept within their mental reach. As such, effective teaching should aim to bring the unfamiliar or abstract mathematical ideas within the reach of students coping capabilities. One way of doing this is to reduce the abstraction level of the new concept by making the concept easily accessible to the students, similar to Mitchelmore and White's (1995) suggestion. In order for the meaningful learning, they point out the importance of providing smooth transition from students' existing knowledge schema to the new knowledge through teaching. They therefore, suggest that teaching should start with concrete contexts and then ascend to the abstract, at least at the elementary school level. In so doing, the abstract concept becomes less abstract and more accessible to the students. In this regard, Hazzan's study revealed that in order to make new and unfamiliar concept accessible, students tend to use various strategies of reducing abstraction often resulting inappropriate reduction of abstraction.

As stated earlier, Hazzan combines three different perspectives of the interpretation of abstraction under one umbrella in her framework of reducing abstraction. Even though these three perspectives seem very different and incommensurable, I find them helpful in examining teachers' activities with regard to their ways of dealing with abstraction while implementing mathematical task in classroom. Hence, I use the abstraction in the sense of Hazzan in this study.

3. Towards a Theoretical Framework

The literature review in Chapter 2 sets forth the basis for my research: it revealed the different faces of abstraction in mathematics and mathematics education; it also provided an insight into the challenges and opportunities that the very nature of mathematical objects can offer in teaching and learning mathematics. Further, the literature revealed a great deal of informed speculation about how mathematics is taught well or poorly, particularly in elementary through high school mathematics classrooms. Not surprisingly though, very little of that speculation has been validated by empirical and experimental research, confirming that much is left to be done.

The initial motivation for this research project came from my own subjective experience of my move from a student of pure mathematics to a teacher of mathematics (mathematics education). This transition provided me with the opportunity to experience firsthand the problems and challenges related to teaching (and learning) mathematics. However, my increasing awareness about the various theoretical perspectives in mathematics education greatly influenced the perspective that I took to analyse and interpret the data when seeking to answer to my research questions.

In this chapter, I briefly describe the most significant of these theoretical perspectives related to my study. First, I attend to the ‘reducing abstraction’ theoretical framework as propounded by Hazzan (1999). Since the nature of (instructional) mathematical tasks is an important factor in teaching and learning mathematics, I attend to this topic. I also elaborate on the notion of instructional tasks and various theoretical perspectives on task implementation, particularly the mathematical task framework of Stein, Smith, Henningsen and Silver (2000). It goes without saying that various theoretical viewpoints mentioned above have greatly influenced and contributed to my study; none of them, however, served the purpose of this study in their exact original form, which necessitated a modification or extension of the existing framework in order to address the questions raised in this study. Finally, I present a description of the

development of a new theoretical framework with regard to teachers' task implementation behaviour and their approaches to dealing with abstraction in teaching.

3.1. Reducing Abstraction: Examining Learners' Activity

As stated earlier, Hazzan's (1999) study on how students learn abstract algebra is an important piece of work and provides a window to examine learners' activity while learning new mathematical concepts. Her finding is that learners usually do not have the mental constructs or resources '*to hang on to*' or cope with the same abstraction level of the new (unfamiliar) concept as intended by the instructor or textbook, and hence, they tend to reduce the level of abstraction in order to make the concept more mentally accessible. This activity of reducing abstraction usually happens *unconsciously*. Hazzan and Zazkis (2005) suggest:

Reducing abstraction is a theoretical framework that examines learners' behavior in terms of coping with abstraction level. It refers to situations in which learners are unable to manipulate concepts presented in a given problem; they therefore, unconsciously reduce the level of abstraction of the concepts involved to make these concepts mentally accessible. (p.101)

The reducing abstraction framework is based on the following three interpretations of abstraction (also referred to as different categories):

1) Abstraction level as the quality of the relationships between the object of thought and the thinking person

According to this interpretation, the level of abstraction is measured by the relationship between the learners and the concept (mathematical object). It is based on Wilensky's (1991) interpretation that mathematical concepts are neither more nor less abstract in their own right, but rather that this depends on the connection with the concepts on the part of the learner. From this perspective, when learners encounter new and unfamiliar (and therefore abstract) mathematical concepts, they try to make the concepts more accessible by using their existing resources and past experiences with other (familiar) mathematical objects, which often results in reducing abstraction

(Hazzan, 1999). For example, Hazzan and Zazkis (2005) mention that when asked to add numerals such as 12 and 14 in base 5, Sue (one of the students in their study) avoids base 5 additions by converting to base 10, performing the operation in base 10 and then converting the result back to base 5, thus reducing level of abstraction from unfamiliar base 5 addition to familiar base 10 addition.

2) Abstraction level as a reflection of the process-object duality

Reducing abstraction in this category refers to the tendency of students to work with the problem by following a step-by-step procedure (process conception) rather than a meaningful mathematical concept (object conception). In other words, it refers to the behaviour of students when there is an emphasis on the procedures and techniques required in order to get the answer, rather than constructing meaningful mathematical objects. For example, Hazzan and Zazkis (2005) observe that when asked whether $33 \times 52 \times 7$ is divisible by 7, Mia (one of the students in their study) calculated the products ($= 1575$) and then divided it by 7 (*i.e.* $\frac{1575}{7}$) to get the answer, rather than analyzing the object for divisibility. This, from Hazzan's (1999) perspective, is an act of reducing abstraction in line with Sfard's (1991) theory of process-object duality, according to which the process conception is less abstract than an object conception. Rather than speaking in terms of abstraction, Sfard (2008) refers to the discursive shift that occurs through reification in the move from process to object.

3) Abstraction level as the degree of complexity of mathematical concepts

This refers to the idea that “the more compound an entity is, the more abstract it is” (Hazzan, 1999, p. 82). In fact, one of the reasons students view mathematics as being abstract is because it is complex and difficult. As stated earlier, the investigation of Frorer et al. (1997) into how students view abstraction in mathematics revealed that ‘complex’, ‘requiring more steps’, ‘difficult’ and ‘not easily understood’ were some of the adjectives students used to refer to abstraction in mathematics.

In this interpretation of abstraction, abstraction level is measured by the degree of complexity inherent in the mathematical concept. This refers to the tendency of

students working with a particular case rather than a general one. The following example illustrates reducing abstraction in this category:

Instructor: Do you think there is a number between 12358 and 12368 that is divisible by 7?

Nicole: I'll have to try them all, to divide them all, to make sure.
Can I use my calculator? (Hazzan & Zazkis, 2005, p. 112)

Here, Nicole was expected to consider the interval of ten numbers and examine the divisibility by 7, but she preferred to consider each number separately to check the divisibility. That is, she is reducing the level of abstraction by working with a subset (a particular number), rather than working with the larger set (an interval of numbers) itself or properties of numbers.

Hazzan (1999) observes that sometimes reducing the level of abstraction can be used inappropriately by students and become misleading. However, in some cases, it may be an effective strategy when working with new mathematical problems. Further, with regard to the level of abstraction in the framework, Hazzan (1999) notes that these three interpretations of the level of abstraction are not disjointed, but rather they are inter-related, and that reducing abstraction in one category may even emerge from another category. For example, a learner trying to cope with a new (unfamiliar) concept in a less complex manner (category 3) or as a process (category 2) can be interpreted as an attempt by the learner to make the concept more familiar (category 1). Hazzan (1999) mentions that the relationship between the first and the second category may be interpreted as "the more one works with an unfamiliar concept initially being conceived as a *process*, the more familiar one becomes with it and may proceed toward its conception as an *object*" (p. 79). As for the second and third category, "when the set concept is conceived as an *object*, a person becomes capable of thinking about it as a whole [...] when one deals with the elements of a set instead of with the set itself we may interpret this as a *process* conception of the concept" (ibid., p. 83). In other words, based on the perspective one takes, one category of reducing abstraction can be thought of as reducing abstraction in another category. Sometimes there are "cause-effect relationships" between several levels of reducing abstraction (Raychaudhuri, 2001). Hazzan's framework of reducing abstraction can be summarized the following way (see Figure 1).

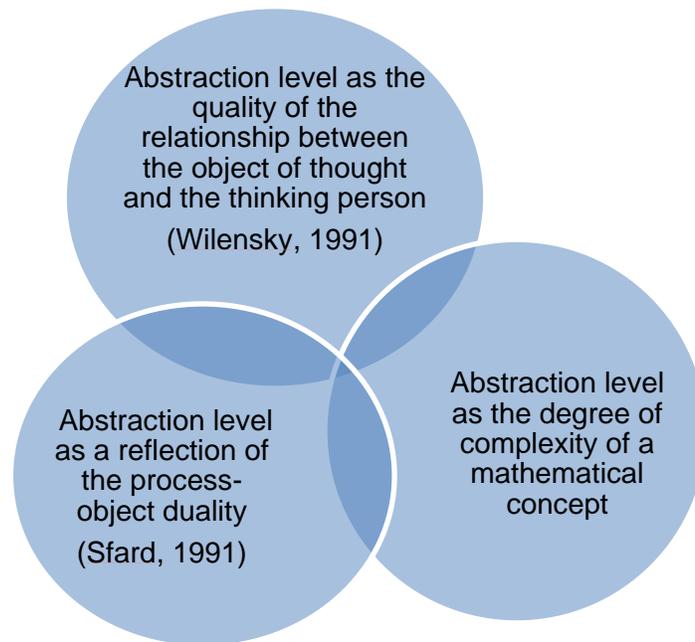


Figure 1. Hazzan's Reducing Abstraction Framework

Later, using Hazzan's framework, Raychaudhuri (2001) sought to examine her students' activity with regard to their ways of dealing with abstraction while learning differential equation. She found that in addition to the three categories of reducing abstraction, her students exhibited another, different tendency while solving differential equations. To incorporate this different tendency of her students, she introduced a new interpretation of abstraction, extending the framework to a fourth category:

4) Abstraction level as situating the object in the cognitive structure

This refers to the tendency of students working with a mathematical task to ignore the situatedness or interconnection of the objects or concepts within other objects or concepts, and instead treating the object or concept as stand-alone. Raychaudhuri (2014) writes:

Abstracting here is interpreted as viewing the concept as situated in a connected structure of cognition. Confronted with new concept, the students might reduce abstraction by attempting to make connections between the new and the old in a piece-meal fashion, or failing to incorporate into one's cognitive structure the connections and thereby treating it as stand-alone. (p. 3)

In other words, students reduce abstraction in this category by not incorporating the connection a concept or an object has with other concepts or objects and treating it as stand-alone, thereby losing the integrated meaning of the concept. This tendency looks similar to the third category above (general vs. particular) on the surface, but Raychaudhuri (2014) clarifies the distinction by saying that in contrast to that third category, where learners choose to work with a particular case rather than a general one, learners in this fourth category choose to work with the concepts in a disconnected fashion. She adds, “although one may be tempted to think of the cases presented here as learners trying to reduce the degree of complexity of a concept, we mention that the distinct nature of these cases is that of a lack of connections between different concepts” (ibid., pp. 16-17).

3.2. Reducing Abstraction in Teaching (RAiT)

As previously mentioned, my literature review revealed that a great number of studies have been done about abstraction in mathematics education and how mathematics can be taught effectively. However, the studies that aimed to examine the how learners deal with abstraction in learning new mathematical concepts are slim (e.g., Hazzan, 1999, 2003; Raychaudhuri, 2001, 2014; Hazzan & Zazkis, 2005). These researchers examined learners’ coping strategies with regard to dealing with mathematical abstraction. They found that since learners often do not have sufficient mental resources to cope with the same level of new (unfamiliar) mathematical concepts as intended by authorities (such as the teacher or textbook), they tend to reduce the abstraction level to make the unfamiliar concepts more accessible.

Not surprisingly though, all the researchers who conducted their research from the reducing abstraction viewpoint explored learners’ tendency of coping with the abstraction level of concepts while learning new mathematical content. There was, however, no study found in my literature review that specifically looked at teachers’ ways of dealing with the abstraction level of concepts while implementing mathematical tasks in the classroom. Hence, a question worth exploring is: how do teachers deal with mathematical abstraction in their teaching practices? Do teachers reduce abstraction in teaching? If they do, what is the nature of reducing abstraction in teaching? In the

following section, I first provide the rationale for reducing abstraction in teaching and then look closely at the teaching activities and task implementation behaviours of teachers with the intent of answering these questions.

3.2.1. *Why reduce abstraction in teaching?*

Much has been written on teaching, particularly on the distinction between traditional and reformed teaching pedagogy (Cicchelli, 1983; Huang & Leung, 2005; Mascolo, 2009). Traditional teaching methods refers to the teacher-centered approach in which teachers serve as the source of knowledge while learners serve as passive receivers. The activity consists mainly of teacher explanations and demonstrations of procedures, followed by student practice of those procedures, with an emphasis on basic facts and skills. In contrast, reformed teaching refers to the student-centered approach which encourages inquiry-based activities and challenges students to make sense of the mathematical ideas through exploration and projects (often in real life context). In reformed classroom, teachers draw on a range of representations and tools (such as graphs, diagrams, models, images, stories, technology, everyday language etc.) to support students' mathematical development (National Research Council, 2001; Silver and Mesa, 2011).

In recent decades, the traditional, teacher-centered approach has been seriously challenged by proponents of the students-centered approach. Reformed teaching approaches have their origin in a constructivist developmental theory of learning (Piaget, 1948/1973; Fosnot, 2013), which maintains that learners actively construct their knowledge and understand through their actions upon and experiences in the world. Piaget's theory of cognitive development is perhaps the most influential theory that align with the constructivist tradition and stands in contradiction to both the rationalist and empiricist approaches to the acquisition of knowledge. In the rationalist tradition, knowledge is viewed as either an innate property or a logical product of the mind (Chomsky, 1980). Empiricists, on the other hand, argue that knowledge is acquired from sensory experience (Hume, 1777/1993; Locke, 1689/1996). In contrast to the rationalist approach, Piaget maintained that new knowledge is constructed by the learner based on his existing knowledge (which is acquired through interaction with the world). In contrast to the empiricist approach, Piaget held that knowledge is acquired not just from

experience, but that it is constructed from the analysis of these experiences and by assimilating them into an existing conceptual structure. Elaborating on the meaning of constructivism, Noddings (1990) states:

Constructivism [... is] an active knowing mechanism that knows through continued construction. This active construction implies both a base structure from where to begin construction (a structure of assimilation) and a process of transformation or creation which is the construction. It implies, also, a process of continual revision of structure (a process of accommodation). (p. 9)

Piaget held that the learner must be in dynamic equilibrium with his/her *cognitive structures* or environment. Learning occurs when this equilibrium is disturbed by new knowledge or experience conflicting with the old. The resulting cognitive conflict (disequilibrium) motivates learners to reconstruct their existing knowledge in order to accommodate the new knowledge. This process resolves the conflict and contradiction engendered by the new knowledge and experience and gives rise to a new and higher mode of knowledge. Piaget termed this process as *equilibration*.

An underlying idea in constructivism is that all new knowledge develops from existing knowledge, as Mascolo (2009) maintains:

It is not possible to learn anything totally anew. [...] Any given act of knowing involves the assimilation of a to-be-understood object into existing knowledge structures as well as the simultaneous attempt to accommodate or adjust one's knowledge structure around the new experience. [...] Without the capacity to assimilate objects with existing knowledge, there would simply be no way to make sense of the world. (pp. 5-6)

The reducing abstraction framework shares much with the constructivist approach in that it also assumes the learners' existing knowledge as the basis for any new knowledge. In the case of mathematics learning, reducing abstraction refers to the idea that when encountering new and unfamiliar (and therefore, abstract) mathematical concepts, learners tend to make these familiar or concrete by utilising their existing knowledge structure and past experience with similar mathematical objects. Since learners often do not have sufficient mental resources to deal with the same level of abstraction of the concepts as set forth by their instructor or textbook, their attempt to

make sense of the new concept often results in the reduction of the abstraction level of the concept.

The constructivist theory has important implications for teaching and learning mathematics. If learners construct knowledge based on their existing knowledge, there is no way to simply teach or transmit knowledge to the learners. Therefore, any attempt to teach a new concept must take into account the student's existing knowledge and experience. It follows then that teaching mathematics should be directed towards establishing a *right relationship* (in the sense of Wilensky) between the learners and the new (unfamiliar) mathematical concept so that students can construct the meaning of the mathematical objects and concepts that their teacher intended to convey.

As has been previously mentioned, Wilensky (1991) argues that abstraction is not a property of the concept or object itself. Rather, this depends on the relationship between the person and the object of thought. He writes, "concepts that were hopelessly abstract at one time can become concrete for us if we get in the *right relationship* with them" (p. 198). From this perspective, "a good teacher is one who is able to engage the student's existing ways of knowing and introduce novelty in such a way as to prompt transformation in the structure and content of a student's knowledge and skills" (Mascolo, 2009, p.6). In other words, teaching new mathematical concepts requires that teachers use learners' previously acquired knowledge, experience and skills. In doing so, a richer connection between the learner and the concept can be established.

Hence, the idea of 'reducing abstraction in teaching (RAiT)' points out that while introducing new mathematical concepts, it is necessary for teachers to use learners' previously acquired knowledge, experience and level of thinking, as well as their familiar contexts. In fact, as Cornu (1991) states, "for most mathematical concepts, teaching does not begin on virgin territory" (p. 154); rather, all students come with certain ideas, intuition, and knowledge already formed in their mind on the basis of their previous experience. Cornu argues, "these ideas do not disappear following the teaching, contrary to what may be imagined by most teachers" (p. 154). Therefore, he says, it is important for teachers to become explicitly aware of this difficulty of their students and to attempt to reconstruct students' knowledge structure to accommodate the new knowledge. Along the same line, Bain (2004) asserts:

Students bring paradigms to the class that shape how they construct meaning. Even if they know nothing about our subjects, they still use an existing mental model of something to build their knowledge of what we tell them, often leading to an understanding that is quite different from what we intend to convey. (p. 27)

These ideas share much with many other educators and psychologists (see Diesterweg, 1835; Piaget, 1970, Safuanov, 2004). As previously mentioned, Piaget's idea of developmental psychology and genetic epistemology, for example, asserts that children tend to think very concretely, specifically in their early stages. Based on this idea, the genetic approach to teaching mathematics is widespread.

The first educator who used the term "genetic teaching" probably was the prominent German educator F.W.A. Diesterweg (1790-1866). In his article "Guide to the Education of German Teachers", Diesterweg (1835) writes:

[T]he formal purpose requires genetic teaching of all subjects that admit such teaching because that is the way they have arisen or have entered the consciousness of the human ... Though a pupil covers in several years a road that took milleniums for the mankind to travel [sic]. However, it is necessary to lead him/her to the target not sightless but sharp-eyed: he/she must perceive truth not as a ready result but discover it. (cited in Safuanov, 2004, p. 153)

Safuanov, who believes in the genetic approach to teaching, suggests that teaching should involve with the process of introducing new abstractions, concretising or semi-concretising them, and then repeating this process at a slightly more advanced level. Safuanov (2004) writes:

Strict and abstract reasoning should be preceded by intuitive or heuristic considerations; construction of theories and concepts of a high level of abstraction can be properly carried out only after accumulation of sufficient (determined by thorough analysis) supply of examples, facts and statements at a lower level of abstraction. (p. 154)

Hence, my literature review led me to the idea of reducing abstraction as a teaching activity. That is, the concepts are concretized and presented to the students at a lower level of abstraction, temporarily mediating through concepts, objects, discourses or situations in a lower level of abstraction. The goal is, however, to move to the higher

level of abstraction by stepping up from the lower level. When the student assimilates the new and abstract concepts or ideas in to his or her existing knowledge structure, the previously abstract concept becomes less (or no longer) abstract for the student. Stepping on the newly concretized abstract concept, a teacher can then introduce another concept, and the cycle continues as illustrated in Figure 2.

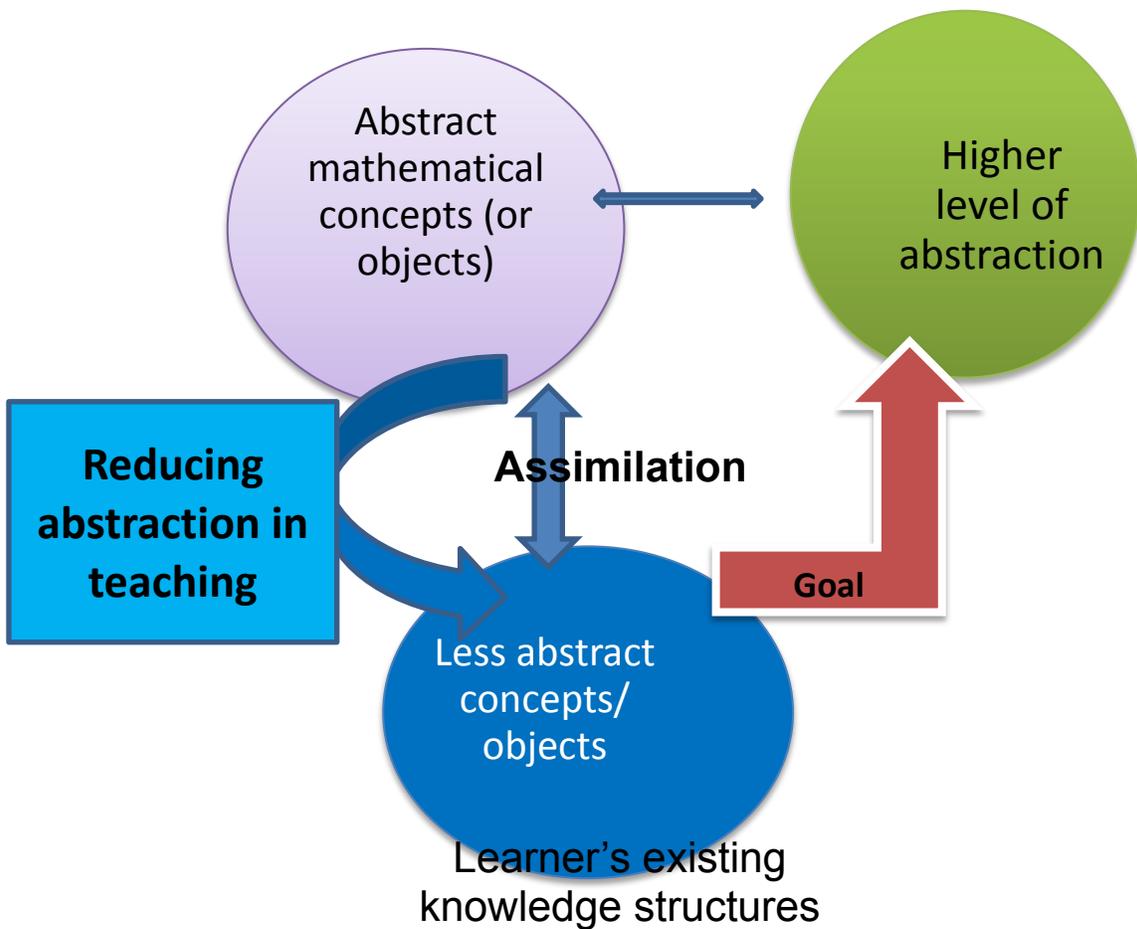


Figure 2. Teaching Model in RAiT

Teaching in this way not only helps students to understand mathematics better, but also helps them to learn how to deal with mathematical abstraction that is important in order to make learning advances in mathematics later on. But how can this be done? What are the ways? This is not an easy question to answer. Teaching is a complex system involving various factors that directly influence the quality of teaching and learning mathematics. Two important factors greatly discussed in the literature are the 'nature of instructional task itself' and 'how it is implemented' in the classroom by

teachers. Looking at this issue from the perspective of this study (dealing with abstraction), mathematical tasks involve the abstraction that the teacher and student have to deal with in the teaching and learning process.

With regard to teaching, it involves a task (or concept) and its implementation, the goal of which is to make that task (or concept) more accessible to students. This activity often results in reducing the level of abstraction of the concept while presenting it to the students. Hence, the nature of the task and how it is implemented in class are closely related issues involved in understanding how teachers deal with mathematical abstraction. What follows is a brief description of the role of mathematical tasks and teachers' task implementation behaviour in mathematics classrooms.

3.3. Mathematics Task, Task Implementation and Learning

3.3.1. *Mathematical task*

The role of the mathematical task has been explored by many researchers under different terms such as “instructional tasks” (Hiebert & Wearne, 1993) and “academic tasks” (Doyle, 1983). It is widely accepted that mathematical tasks play an important role in engaging students in thinking and learning because “tasks convey messages about what mathematics is and what doing mathematics entails” (NCTM, 1991, p. 24). According to NCTM (2000), “most mathematical concepts or generalizations can be effectively introduced using a problem situation” (p. 334). They go on to assert that “in effective teaching, worthwhile mathematical tasks are used to introduce important mathematical ideas and to engage and challenge students’ intellectually” (p. 18). Hence, a worthwhile mathematical task is one that provides a richer opportunity for students to deal with mathematical abstraction, so that, with the help of a teacher, they may be able to establish a right relationship with the concept, thereby enhancing their thinking and reasoning about important mathematical concepts.

Since different tasks have different features and different levels of abstraction, their influence is felt differently in the ways students deal with abstraction and engage in learning mathematics (Henningesen & Stein, 1997). For example, tasks that ask for memorized facts, rules or procedures lead to one type of opportunity for student thinking.

In contrast, tasks that demand students to engage in problem solving and mathematical reasoning lead to still another kind of opportunity for students. Lappan and Briars (1995) therefore, maintain:

There is no decision that teachers make that has a greater impact on students' opportunities to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages students in studying mathematics. (p. 138)

NCTM also recognizes a good mathematical task as a necessary element for effective teaching, as it provides an intellectual environment for the student to learn important mathematical concepts. In "Principles and Standards for School Mathematics (2000)", it is stated:

Well-chosen tasks can pique students' curiosity and draw them into mathematics [...] worthwhile tasks should be intriguing; with a level of challenge that invites speculation and hard work. Such tasks often can be approached in more than one way. [...] When challenged with appropriately chosen tasks, students become confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and willing to persevere. [...] When students work hard to solve a difficult problem or to understand a complex idea, they experience a very special feeling of accomplishment, which in turn leads to a willingness to continue and extend their engagement with mathematics. (pp. 18-19)

A worthwhile mathematical task not only prompts student to learn mathematics, but also provides a rich opportunity for the teacher to deal with mathematical abstraction in his/her teaching practices and, therefore, allow him or her to become a more effective teacher. While implementing a worthwhile task, NCTM (2000) suggests that teachers have to learn:

What aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge. (p. 19)

3.3.2. Task implementation

A worthwhile mathematical task certainly plays an important role for effective teaching and learning; it is however not a sufficient condition. Studies show that even a rich task in and of itself does not guarantee enhancement of learning if it is not implemented properly by the teacher (Hiebert et al., 2003; Birky, 2007). The TIMSS (1999) video study team, for example, found that the reformed mathematics curriculum in the U.S. includes features of mathematical tasks mentioned in the NCTM standard, and that teachers assigned problems similar from those in other countries whose students achieved higher scores on the TIMSS 1999 Achievement Test (Hiebert et al., 2003). However, the achievement of U.S. students reinforces the idea that the introduction of rich tasks alone does not guarantee improved learning.

As previously mentioned, Hiebert et al. (2003) found that in American eighth grade mathematics classrooms, the problems that were intended to engage students in *relational understanding* (problems concerning making connections) were often implemented in a way that emphasized the use of rules, procedures and memorized facts. Teachers from the countries whose students scored higher on the TIMSS 1999 achievement test, on the other hand, often implemented the exercises initially requiring procedures, recall of facts or answer (non-making connections problems) in a way that fostered relational understanding (Skemp, 1976) or “making connections” (Hiebert et al., 2003).

Wood, Shin and Doan (2006), in their analysis of three U.S. classrooms taught by competent teachers in a reformed mathematics curricula, also found that “the presentations are divested not only of reasons, but are also completely devoid of any richness of thought that allows the learner to reason and gain insight into what one is doing mathematically when using the procedure” (p. 83).

In summary, it is generally agreed upon that the nature of the mathematical task shapes the students’ way of thinking and influences students’ learning (Doyle, 1983; Henningsen & Stein, 1997). Cognitively demanding tasks (with a higher level of abstraction), contrary to procedural ones, can provide a richer context and thus encourage higher order thinking on the part of students. Nevertheless, the literature also revealed that the quality of the task by itself does not guarantee the quality of learning

(Hiebert et al., 2003). Rather, teachers' task implementation behaviour and how the task is implemented during teaching significantly impacts student understanding. For example, cognitively demanding tasks (Henningsen & Stein, 1997) or tasks that require making connections (Heibert et al., 2003) intended to foster relational understanding can be implemented in a way that leads to procedural or instrumental understanding. On the contrary, teachers can implement cognitively low-level tasks in a way that fosters higher order thinking and relational understanding. Hence, not only the nature of the mathematical task and its level of abstraction, but also the teacher's approaches toward task implementation and how the abstraction of the task is dealt with during the lesson must be taken in to account in determining the quality of teaching and learning.

3.4. Dealing with Abstraction: Towards a RAiT Framework

As described above, the tasks used in the classroom certainly play a key role in learning mathematics, as these convey messages about what it means to do mathematics (Hiebert & Wearne, 1993; Watson & Mason, 2005; Zaslavsky, 2005); however, as we've seen, the quality of the task does not guarantee the quality of learning. As Christiansen and Walther (1986) argue, "the setting of the tasks together with related actions performed by the teacher constitute the major method by which mathematics is expected to be conveyed to the students" (p. 244). As such, teachers' actions with regard to task implementation have been the topic of discussion among researchers and educators.

My literature review revealed that studies on the role of mathematical tasks in teaching and learning were conducted from the cognitive perspective. Doyle (1988), for example, categorized mathematical tasks and their implementation based on the cognitive process involved in the tasks. According to Doyle, mathematical tasks used in teaching fall under one of the following categories:

- a. Memory tasks (in which students are required to recall facts or information from previously encountered situations),
- b. Procedural or routine tasks (in which students are expected to use procedures or methods they were shown earlier)

- c. Comprehension or understanding tasks (those which engage students in a cognitively high level of thinking, as these problems require students to make decisions regarding the strategies and information they should use);
- d. Opinion tasks (those which require the students to tell their preferences).

He also mentioned that comprehension or understanding tasks are cognitively demanding and call for conceptual structure as opposed to the other categories.

Stein and Lane (1996) and Henningsen and Stein (1997) focus their attention on two aspects of tasks: first, task features and second, the cognitive demand of the task. 'Task features' refers to aspects of tasks, such as "multiple solution strategies, multiple representations, and mathematical communication". Cognitive demand, on the other hand, refers to the kind of "thinking processes entailed in solving the task as announced by the teacher (during the setup phase) and the thinking processes in which students engage (during the implementation phase)" (Henningsen & Stein, 1997, p. 529). According to these researchers, a task passes through three phases: first, as written by curriculum developers; second, as 'set up' by the teacher in the classroom; and third, as implemented in the classroom by the students during the lesson. They defined 'task set up' as the task that is announced by the teacher and 'task implementation' as the way students engaged and actually worked on the task during the lesson, which ultimately influences what and how they learn.

From the observation of task features and how the tasks are implemented, Henningsen and Stein (1997) identified six kinds of cognitive thinking processes that occur in a mathematics classroom. These six processes have been divided into two categories depending on the nature of the cognitive demand of the task—namely whether it is high- or low-level. High-level tasks involve 'doing mathematics' and 'procedures with connections', whereas low-level tasks refer to 'procedures without connections', 'memorization', 'unsystematic and/or non-productive exploration' and no mathematical activity (i.e. a task that requires no mathematical thinking) (Henningsen & Stein, 1997).

Using this framework, Stein and Lane (1996) found that when teachers used cognitively demanding tasks that engage students in "doing mathematics or using

procedures with connection to meaning” (p. 50) and maintained the cognitive level of the task during implementation phases, there was a significant gain in students’ learning outcome. However, when the tasks were implemented in a procedural manner, and in a less cognitively demanding way, students’ level of understanding was relatively low. A larger project, QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning), also supported this result (Silver and Stein, 1996).

QUASAR is a U.S. national project designed to improve mathematics instruction for middle school students in the economically disadvantage communities in six different sites across the country. QUASAR researchers found that middle school children benefited substantially when students were engaged in cognitively challenging tasks. However, as Henningsen and Stein (1997) report, tasks that were intended to engage students in cognitively high-level activity were frequently implemented in ways that reduced the complexity or richness of the task. They noted that this was problematic because if the complexity or richness of the task is reduced, “the cognitive demands of the task are weakened and students’ cognitive processing, in turn, becomes channelled into more predictable and mechanical forms of thinking” (p. 535). Tzur (2008) also found significant deviations between the intended level of the task (as intended by the developers) and the actual implementation of the task by teachers. A related issue, then, is to identify the factors that influence the design and implementation of the task, including activities for promoting student learning.

A growing body of research has focused on this issue and identified various factors that play key roles in the way tasks are implemented in the classroom, including teachers’ experience, knowledge and beliefs (Ben-Peretz, 1990; Corey & Gamoran, 2006; Henningsen & Stein, 1997). Charalambous (2010), for example, investigated the relationship between the task and the Mathematical Knowledge for Teaching (MKT). He found that teachers with high MKT largely maintained the cognitive demand of curriculum tasks at their intended level during task implementation and enactment, whereas teachers with low MKT proceduralized even the intellectually demanding tasks by “rushing to give them rules and formulas to follow, hence eliminating any likelihood that they engage with the content I higher level” (p. 264)

Putnam, Heaton, Prawat and Remillard (1992) argue that teachers' knowledge and beliefs about learners, mathematical content knowledge, and knowledge and belief about mathematics (how they view mathematics themselves) significantly influenced their tendency to change the nature of the tasks and the way the tasks were implemented. It is therefore suggested that to achieve effective learning for students, teachers need to know and understand not only the mathematics of the lesson, but also the mathematics suitable for teaching, for example, how to design good tasks, identify students' difficulties in learning and initiate a productive discussion of mathematics in class.

A closely related idea to promote effective teaching pointed out by Shulman (1987) is pedagogical content knowledge (PCK), which suggests that knowledge needed for teaching a specific subject requires knowledge and understanding of the both content and pedagogy. In addition to knowledge of the subject matter and the discipline, PCK involves knowing the "most useful forms of representation of these ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations-in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

Implicit in these ideas is that for effective teaching to take place, teachers should have sound knowledge of both the subject and how to design and implement mathematical tasks in such a way that their students can cope with the level of abstraction of the task. As such, there is a growing interest in this issue in the mathematics education research community, and as a result, various theoretical frameworks have been developed to study the relationships between the nature of mathematical task, task implementation behaviour and the quality of teaching and learning.

Prior to 2003, the team responsible for the TIMSS curriculum studies and achievement test, for example, developed their own framework, in which they initially categorized the cognitive complexity of tasks in to five categories: knowing, using routine procedures, investigating and problem-solving, mathematical reasoning, and communicating (Schmidt et al., 1996). In 2003, TIMSS modified the framework to more clearly describe specific students' behaviour, skills and abilities relevant to each

cognitive domain. In this new framework, mathematical tasks were classified into four cognitive domains: knowing facts and procedures, using concepts, solving routine problems, and reasoning (Mullis et al., 2003).

The TIMSS (1999) video study team categorised the problems from two perspectives: first, what kind of mathematical behaviour is intended by the problem statement, and second, what kind of mathematical behaviour is actually observed during the implementation? Likewise, the team classified teachers' task implementation behaviour which can be put into three broad categories:

1. Using procedures: The use of routine algorithms and procedures without much explanation; answer oriented.
2. Stating concepts: Instead of developing the concept, it is stated by the teacher without describing mathematical relationships.
3. Making connections: Mathematically rich discussion with a focus on constructing relationships among mathematical ideas, facts, or procedures. (Hiebert et al., 2003)

Since my study aimed to explore how teachers deal with mathematical abstraction during task implementation, none of the frameworks discussed above fit the purpose of my study. There are various reasons for this. For starters, the TIMSS (1999) video team analysed the data from a different theoretical perspective, focusing on the features of problems and the way they were presented in the classroom, thereby limiting their analysis of task implementation behaviour to the three categories mentioned above. Henningsen and Stein's (1997) framework, on the other hand, is cognitively oriented, concerning itself chiefly with the cognitive level of the task and whether that cognitive level is maintained or reduced in the processes of implementation.

Here I would like to direct the reader's attention to an important point of clarification; the cognitive level of a concept or a task should not be confused with the abstraction level of the concept or task. Abstraction level in this study is defined on the basis of the three interpretations of abstraction as previously discussed: 1) Abstraction level as the quality of the relationships between the object of thought and the thinking person; 2) Abstraction level as a reflection of the process-object duality; and 3) Abstraction level as the degree of complexity of mathematical concepts. A detailed

description of each of these (given from the task implementation viewpoint) is given in Chapter 5.

Further, it should be noted that the subjective phenomenon of a concept involved in a task is one of the important aspects in the interpretation of its abstraction level. For example, a concept that is too abstract for one learner may be concrete for another depending on one's existing knowledge, experience and familiarity with the concept. On the contrary, cognitive level as used in Henningsen and Stein's (1997) framework is related solely to an objective aspect of the task; here this refers to the kind and level of thinking required of students in order to successfully engage with and solve the task without taking into account the subjective aspect, such as the students' familiarity with the concept. Moreover, their definition of cognitive level involves both the knowledge and context of students (e.g., what has been done or learned previously in the classroom), which is impossible to implement in my study due to it being primarily based on the observation of the TIMSS (1999) public release video lessons as well as a few classroom observations, where data regarding the knowledge level and context possessed by students was not available.

Further, the interpretation of task implementation in my study is different from those given by Henningsen and Stein, (1997) in that they define task implementation as worked out by the students in the classroom, whereas in my study, it is defined as how the teacher presented the task to the whole classroom, regardless of how it was worked out by the students, a detailed description of which is given in Chapter 5.

The framework closely related to this study is that of reducing abstraction propounded by Hazzan (1999). However, Hazzan's framework falls short of addressing the questions raised in this research project. This is because Hazzan's framework concerns students' behaviour in dealing with mathematical abstraction when learning a new concept. In contrast, the goal of my study is concerned chiefly with teachers' behaviours in dealing with abstraction while implementing mathematical tasks, thereby examining how abstraction is dealt with as a teacher's activity rather than learner's activity.

This shift in perspective necessitated a different interpretation of the notion of reducing abstraction, because a learner's goal is to learn mathematics for himself or herself, whereas a teacher's goal is to help his or her students to learn mathematics. For example, Hazzan (1999) discusses the idea that the mental process of making unfamiliar concepts more familiar by reducing levels of abstraction happens *unconsciously*; this often occurs when learners do not have a '*mental construct to hang on to*' in order to cope with the same level of abstraction as given by the instructor or textbook (see Hazzan, 1999; Raychaudhuri, 2001; Hazzan & Zazkis, 2005). However, in examining the notion of reducing abstraction as a teacher's activity, the choice of words and phrases such as '*unconscious*', '*lacks the mental construct*', and '*to hang on to*' are inapplicable in most cases. The assumption here is that teachers are the experts and thus usually have sufficient mental resources to deal with the abstraction of the mathematical concept at the same or even at a higher level than that given by the textbook. Therefore, the act of reducing abstraction for teachers, in most cases, is a pedagogical choice.

Equipped with knowledge of various theoretical perspectives, particularly that of Hazzan (1999), Wilensky (1991), Sfard (1991) and Henningsen and Stein (1997), I examined the data obtained from the TIMSS (1999) public release video lessons. As stated in Chapter 1, I want to restate the research questions that guided this work: in their efforts to make abstract mathematical concepts (or task) accessible to their students, how do teachers deal with mathematical abstraction while implementing mathematical task in the classroom?

Using the methodology adopted for this study (see Chapter 4), I analysed the teaching practices captured in the TIMSS 1999 public release video lessons, paying particular attention to the instructional tasks and to teachers' task implementation behaviours. In so doing, a new theoretical framework emerged which I call 'Reducing Abstraction in Teaching' (RAiT), a detailed description of which is given in Chapter 5.

4. Collecting and Engaging With the Data

This chapter describes the methodological perspective of this study. The research questions for this study initially emerged, in a rather dormant form, from my own subjective experience as a teacher, a detailed account of which was given in Chapter 1. My search for deeper understanding of the issues raised in this project was greatly substantiated by the data I collected and the methodology used to analyse and interpret the data. In this chapter, I begin with a description of the different methodological perspectives used in educational research. I then describe my research instruments, data sources and my rationale for the choice of these data sources. This is followed by the research methodology (modified analytic induction) employed in this study and, finally, I provide a description for the development of a coding system.

4.1. Qualitative Versus Quantitative Research

Research on education has been contributing to the growing bodies of knowledge in this field. This knowledge growth does not occur on its own; rather, “it is produced through the inquiries of scholars—empiricists, theorists, practitioners—and is therefore a function of the kinds of questions asked, problems posed, and issues framed by those who do research” (Shulman, 1986, p. 3). The work of this diverse community of scholars who do research can generally be divided into two paradigms of study: quantitative and qualitative research.

Quantitative and qualitative research programs are based on different philosophical perspectives. Gall, Gall & Borg (2007) say, quantitative research falls under the ‘positivist’ paradigm, which is based on the belief that “physical and social reality is independent of those who observe it, and that observation of this reality, if unbiased, constitutes scientific knowledge” (p. 16). Therefore, the role of researchers is to find the objective reality that is out there to be discovered. On the other hand,

qualitative research falls under the 'postpositivist' paradigm, according to which "social reality is constructed and it is constructed differently by different individuals" (Gall et al., 1996, p. 19). Further, as Denzin and Lincoln (2005) stated, "qualitative researchers study things in their natural settings, attempting to make sense of, or to interpret, phenomena in terms of the meanings people bring to them" (p. 3). Researchers in this camp believe that social reality is not independent of the participant, but is continuously constructed by the participant in the local situation. In the context of educational research, Creswell (2005) defines quantitative and qualitative research as follows:

Quantitative research is a type of educational research in which the researcher decides what to study, asks specific, narrow questions, collects numeric (numbered) data from participants, analyses these numbers using statistics, and conducts the inquiry in an unbiased, objective manner. Qualitative research is a type of educational research in which the researcher relies on the views of participants, asks broad, general questions, collects data consisting largely of words (or text) from participants, describes and analyses these words for themes and conducts the inquiry in a subjective, biased manner. (p. 39)

In a nutshell, the goal of quantitative research is "collecting facts of human behavior, which when accumulated will provide verification and elaboration on a theory that will allow scientists to state causes and predict human behavior" (Bogdan & Biklen, 1998, p. 38). In contrast, the goal of qualitative research is to "better understand human behavior and experience, [...] grasp the processes by which people construct meaning and to describe what those meaning are" (ibid., p. 38).

In the field of education, both quantitative and qualitative approaches have been used depending on the nature and objectives of the investigation. While selecting a research methodology, I was influenced by Lederman (1992), who says that "we must let the research questions direct the research approaches and data analysis procedures" (p. 1012).

4.1.1. Research Questions

The main research questions for this study (which was also stated in Chapter 1) are:

- How do teachers deal with mathematical abstraction in teaching?
- Do they reduce abstraction level of a concept involved in a task while implementing a mathematical task? If they do, what are the approaches of reducing abstraction in teaching mathematics in the data sample of this study?

While selecting the research methodology appropriate to the aim of my research project, I considered Denzin and Lincoln's (2005) suggestion that if the study involves human experience, perceptions, intentions, life story, interviews, observations, interactions with all its context-bound subtle nuances, a quantitative positivist approach is of little help. Further, my literature review informed me that qualitative research has been used extensively to investigate teachers' practices and beliefs (e.g., Thompson, 1992; Burnaford, Fischer, & Hobson, 2000) as well as their task implementation behaviour (e.g., Birky, 2007; Hiebert et al., 2003). My study also involves human and social phenomena, as it aims to investigate teachers' practices and their task implementation behaviour. In particular, since my study aims "to know not how many or how well, but simply how" (Shulman, 1981, p. 7) teachers deal with mathematical abstraction, I began to investigate qualitative research methodology, which led me to analytical induction.

4.1.2. Analytic induction as a methodology

Analytic induction is a qualitative research methodology first coined by Znaniecki (1934). According to Buhler-Niederberger (1985), analytic induction is a method of "systematic interpretation of events which includes the process of generating hypotheses as well as testing them. Its decisive instrument is to analyze the exception, the case, which is deviant to the working hypothesis" (cited in Flick, 2014, p. 497). Analytic induction is oriented towards examining theory or knowledge by examining cases, and as new or negative cases are found, the hypothesis is modified in such a way that cases are either consistent with the explanation or are considered to be outside of the domain of the study.

Although analytic induction has the same roots in the development of the research methodology as grounded theory, these approaches differ in some aspects. The analytic induction method "begins with an analyst's deduced propositions or theory-derived hypotheses" (Patton, 2002, p. 454) and "is a procedure for verifying theories and

propositions based on qualitative data” (Taylor & Bogdan, 1984, p. 127). In contrast, grounded theory attempts to approach data without any hypothesis or pre-existing theoretical framework, and then to let the theory emerge from the data itself (Strauss & Corbin, 1990).

Originally, analytic induction had a very strict definition and was identified as a research methodology with the goal of producing research that searches for ‘universal’ generalization (Znaniecki, 1934). In recent years, the traditional approach to analytic induction has been criticised, as it may not be practical or possible to search universal generalization in the social sciences. Therefore, a modified approach to analytic induction has been recommended by various commentators (Gilgun, 2001; Bogdan & Biklen, 1998).

The rationale for using modified analytic induction in this study comes from the suggestion that this methodology allows for the possibility of developing “descriptive hypotheses that identify patterns of behaviors, interactions, and perceptions” (Gilgun, 1995, p. 269) rather than ‘universal’ generalizations. Furthermore, modified analytic induction allows the researcher to begin with his or her pre-existing theoretical perspectives (often having emerged from the literature) on the topic and to then test his or her theoretical frameworks with empirical cases and, thus, to amend and improve the theory (Bogdan & Biklen, 1998).

Analytic induction, as the name suggests, is not a truly inductive methodology. As Katz (2001) commented, this methodology is poorly labeled because it is not a purely inductive method of research. Rather, it is a process that requires researchers to work fluidly, moving back and forth between their ideas and their evidence.

Inductive analysis, in general, refers to approaches that primarily use detailed readings of raw data from which then patterns, themes or categories are discovered within the data. In contrast, deductive analysis refers to the method in which data are analyzed through a predetermined lens or framework and then used to test “whether data are consistent with prior assumptions, theories, or hypotheses identified or constructed by an investigator” (Thomas, 2006, p. 238). However, in analytic induction,

the researcher begins deductively and then looks at the data inductively. According to Patton (2002):

[...] with analytic induction, qualitative analysis is first deductive or quasi deductive and then inductive as when, for example, the analyst begins by examining the data in terms of theory-derived sensitizing concepts or applying a theoretical framework developed by someone else. [...] After or alongside this deductive phase of analysis, the researcher strives to look at the data afresh for undiscovered patterns and emergent understanding (inductive analysis). (p. 454)

In studies where no theoretical framework is available a priori, categories must be generated inductively from the data. Since my study is closely related to the theoretical perspectives as described in the literature (Wilensky, 1991; Sfard, 1991; Henningsen & Stein, 1997), and particularly that of Hazzan (1999), “modified analytic induction” as suggested by Bogdan and Biklen (1998) would best apply for the analysis of the data in this study.

4.2. Data Sources

The data collection for this study came from both primary and secondary sources. The primary data collection involved mathematics classroom observation in classes taught by three different teachers. The secondary data came from the TIMSS 1999 public release video (mathematics) lessons. In the following sections, I provide a description of the components of each of the data sources, including the rationale for my choice of the data, beginning with the secondary data.

4.2.1. *TIMSS (1999) public release video lessons*

Videotape methodology in mathematics education

Video analysis has evolved as one of the powerful methodological tools for research in the field of the social sciences (Knoblauch, Schnettler, Raab, & Soeffner, 2006) and teacher education (Stephens, Leavell, Fabris, Buford, & Hill, 1999; Wang & Hartley, 2003), and more importantly in the teaching of science and mathematics (Brückmann et al., 2007; Goldman, Pea, Barron, & Denny, 2014; Stigler and Hiebert,

2009). Particularly when investigating teaching practices in mathematics, video can be one of the most rich and powerful data sources, as it allows us to extract a variety of data such as gesture, movement, interaction, behaviour, attitude, time frame and so on, which are otherwise extremely difficult to collect. Video can be slowed down, stopped and replayed, allowing the researcher to focus on a very short analytical unit. It also allows researchers to view the data multiple times, and with multiple perspectives.

Although videotape methodology offers multiple benefits, it also has its own challenges. As Roschelle (2000) pointed out, the equipment is technical and often expensive. Also, data reduction can become challenging, as video data often results in an enormous amount of information and is extremely time-consuming to watch. Transcription is another time-consuming activity. Further, there is a risk that the researcher may rely too much on the transcript, which can eliminate the advantages of videotape methodology, as transcription cannot carry the richness of the original source as much as a video. Roschelle (2000) also cautioned against the possibility of ethical challenges that may arise whenever recognizable images of research participants are presented. However, as Stigler and Hiebert (1997) note, “the benefits of video are well worth the methodological challenges” (p. 55).

Using Existing Video Data for Research

The use of existing videos as data source is increasingly common for research in education. If existing videos are available and serve the purpose of the study, there are certain advantages in using them. One of these is that “it can allow researchers to access data on a scale that they could not hope to replicate first-hand; and the technical expertise involved in developing good surveys and good datasets can lead to data that is of the highest quality” (Smith, 2008, p. 21).

In addition, the advantages of pre-existing data also relate to the ability to save time and money, as well as the quality of the data. Obtaining secondary data is almost always faster and less expensive. As Hakim (1982) said, the availability of low cost, high quality datasets in secondary analysis of those datasets ensures that “all researchers have the opportunity for empirical research that has tended to be the privilege of the few” (cited in Smith, 2008, p. 40). Further, as Glaser (1963) noted, it is not the cost of the data analysis, but rather the cost of the data collection that is sometimes beyond the

reach of an independent researcher. Thus, this high level of accessibility inherent in secondary data enables researchers—particularly novice researchers—to carry out research successfully and independently. The volume and quality of data used in this study, such as the TIMSS 1999 video lessons, were all but impossible for me to generate as an individual researcher. Hence, it is for this reason I used the TIMSS 1999 public release video lessons as one of my data sources.

TIMSS 1999 video study: Context and Procedure

TIMSS 1995 was the first major international video-based classroom research project, in which 231 eighth-grade mathematics lessons were videotaped from three participating countries: Japan, Germany and the United States (Stigler & Hiebert, 1997). The TIMSS 1999 video study, a follow-up to their 1995 study, was a more ambitious research project conducted by LessonLab, Inc. under contract to the U.S. Department of Education (Hiebert et al., 2003). This is perhaps one of the most well-known and largest video studies in education, with seven countries participating.

For the TIMSS 1999 research project, researchers randomly selected 90 to 140 schools from each of the six participating countries: Australia, Hong Kong, the Czech Republic, the Netherlands, Switzerland, and the United States (Jacobs et al., 2003). Likewise, they randomly selected one eighth-grade mathematics teacher from each participating school, and one lesson was videotaped from each teacher. Because Japan did not participate in the TIMSS 1999 study, TIMSS researchers included 50 Japanese lessons videotaped earlier in the 1995 study for reanalysis. Therefore, the TIMSS 1999 video study consisted of a sample of 638 eighth-grade mathematics lessons: 87 from Australia, 100 from the Czech Republic, 100 from Hong Kong SAR, 50 from Japan, 78 from the Netherlands, 140 from Switzerland, and 83 from the United States (Heibert et al., 2005). This sample is intended to be representative of standard teaching methods in each of the participating countries.

Because I did not have access to all the video lessons, my analysis is based on the Public Release video lessons and their transcripts, which were translated into English by the TIMSS 1999 video study team. The Public Release video lessons consist of a sample of four video lessons from each of the seven participating countries, totalling

28 video lessons, and are available for the public (<http://timssvideo.com/timss-video-study>) free of cost for the purpose of education, research and training.

These videos have been used widely for research and training purposes. They were analysed by the TIMSS videos study team itself and others using different theoretical frameworks (Hiebert et al., 2003; Jacobs et al., 2003; Birky, 2007). Hiebert et al. (2005) noted that the focus of the TIMSS 1999 video study mainly falls into three broad categories: a) structure and organization of daily lessons, b) nature of mathematics presented; and c) the way in which mathematics was worked during lessons. Using this as a general framework, they developed 75 different codes revolving around these three thematic categories of teaching.

Later, many other researchers analysed the data from different theoretical frameworks, focusing on different aspects of teaching and learning mathematics (Hiebert et al., 2003; Birky, 2007). On the one hand, these studies answered some of the questions related to teaching (and learning) mathematics (from the national and international perspectives), but on the other hand, they also raised many questions and issues within the field. Addressing some of the questions that the previous study did not answer, my study aims at generating insights into the teaching of mathematics, particularly concerning the issue of how teachers deal with abstraction in teaching mathematics.

4.2.2. Classroom observation (Primary data)

This part of the data collection process consisted of my observation of mathematics classrooms taught by three different instructors. My motivation for this part of the data is related to two reasons: First, one of the questions (which came as the by-product of the study during the analysis of first part of data) that I find worth exploring is “why did the teachers implement the task the way they did”? That is, what are the factors associated with various task implementation behaviours? Although the secondary data provided great deal of information regarding various ways of dealing with abstraction while implementing task, it fell short to answer these questions because interviews with these teachers were not possible with this section of the data. Second, as stated earlier, all the data in the first part came from eight grade mathematics classrooms (TIMSS 1999

video lessons). But this part of data (primary data) came from university preparatory mathematics course extending the scope and applicability of the RAIT framework to different levels of mathematics.

In this study, I performed the observation as a nonparticipant observer. A nonparticipant observer is one who watches the interactions of the individuals at the research site and takes field notes on the activities being observed. Through classroom observation, I had an opportunity to collect information first-hand about actual behaviour and interaction as they occurred in the natural classroom setting.

The process of gathering the data consisted of nine classroom observation visits (three for each instructor) to college preparatory mathematics courses (each lasted about an hour and half) taught by three different teachers, all of whom were experienced and professionally trained mathematics educators. I attended these classes and took extensive field notes. I also audio-taped all classroom interaction, and as much as possible, I noted all the phrases, statements or sentences the instructors used to explain concepts, including some observable behaviour such as students' gestures and responses that I found relevant for my study. Furthermore, when a situation arose which required further clarification and justification about the action and behaviour of the teachers as related to task implementation, I sought clarification and justification from the participating teachers through informal interviews/ conversation after the classroom observation. In Chapter 6, I present three representative examples with analysis and interpretation using the framework detailed in Chapter 5.

The actual data collection process - Primary Data

Earlier in the study (prior to primary data collection), I did an initial analysis of the secondary data (the TIMSS 1999 public release video lessons), through which various methods of reducing abstraction while implementing tasks used by teachers were identified and put into different thematic categories. Keeping the initial analysis and interpretation of the secondary data in mind, the primary data collection phase began during the Fall 2011 term. In the beginning of Fall 2011 (September), I contacted four instructors (personal contacts) who were teaching university preparatory mathematics courses at local universities and colleges. University preparatory mathematics courses

were usually offered to the students whose mathematical background and skills were deemed to be insufficient to allow them to enroll in university courses such as Calculus I.

Three of the four instructors voluntarily agreed to participate in the study. In the beginning of Fall 2011, I examined the curriculum and invited the instructors to participate in informal conversations in which they were asked to discuss the main difficulties faced by their students. More specifically, I sought through these conversations to learn more about their concerns as related to the difficulty of their students with regard to mathematical abstraction and how they, as instructors, deal with this when teaching. This part of the conversation is similar to the idea of unstructured interview (in qualitative research methodology) in which neither the question nor the answer categories are predetermined. Rather, it relies entirely on the spontaneous generation of questions in the natural flow of an interaction (Patton, 2002). However, to initiate the conversation, I put the following as an opening question in each conversation: I am interested in learning your concerns as related to the difficulty of your students while learning new a (abstract) mathematical concept. Could you tell me, as a teacher, how do you deal with such difficulty (related to mathematical abstraction) in teaching? The conversation then unfolded itself and lasted about 15-20 minutes. These conversations were not recorded but I noted important points of the conversation related to my study.

Also at these meetings, following a discussion about and elaboration on the nature of my research project, we set up the dates and times for classroom observation visits. During the Fall 2011 term, I observed three different classes for each participating instructor. During each classroom observation visit, I took extensive field notes, collected the materials that the instructor had prepared, and noted specific examples and explanations that they used in introducing the material to their students, as well as in addressing student questions. I also audio recorded the classroom interaction. When a situation arose during my observation which required further clarification and justification about the action and behaviour of the teachers as related to task implementation, I sought clarification and justification from the participating teachers through informal conversation/interview after the classroom observation.

4.3. Analysing the Data

Handling qualitative data is not usually a step-by-step process; rather, it is an iterative process which calls for subjective interpretation of the content of the data through the systematic process of coding and identifying thematic units or patterns (Sipe & Ghiso, 2004). Since we bring “our subjectivities, our personalities, our predispositions, [and] our quirks” (ibid., p. 482–3) to the coding process, “coding is not a precise science; it’s primarily an interpretive act” (Saldaña, 2012, p. 4). Hence, my approach to developing the code, analysis of the data and the interpretation of the result is influenced by various factors such as the existing literature and the conceptual framework, including my personal experience and disposition as a mathematics student and, later, a mathematics educator.

4.3.1. *Brief summary of coding system*

Coding is an integral part of the analytic induction approach as in other qualitative research methodologies (e.g., grounded theory), the main purpose of which is to simplify or reduce the data/transcript to a manageable level and achieve conceptual schema. Coding in analytic induction is similar to that in other qualitative methodologies. Gilgun (2001) pointed out that there is no specific coding guidance available for analytic induction, and hence it is possible to use the coding schemes of other qualitative methodologies such as grounded theory, developed by Strauss and Corbin (1990).

For the initial coding of teachers’ behaviours regarding reducing abstraction, I was guided by both inductive and deductive reasoning. I began with the process of identifying important teacher behaviour during task implementation and attaching these with some descriptive words and phrases in an ad hoc manner. I did this by a process of examining the transcripts in light of the literature as described in chapter 3. This coding process not only helped me to break the data apart in analytically relevant ways, but also led me “from the data to the idea, and from the idea to all the data pertaining to that idea” (Richards & Morse, 2012, p. 154).

For this study, I intended to develop a coding system that characterized important teacher behaviours during task implementation or public discussion of a

problem that influenced whether or not the abstraction level of the concept involved in a task was reduced for the students. In pursuit of that goal, having been informed by the literature and having kept its theoretical perspectives in mind, I read the transcripts and watched the videos carefully, with an eye towards identifying teachers' key actions as well as searching for and developing an understanding for the meaning for each of these actions. Specifically, I looked at teachers' task implementation behaviour and the way they dealt with mathematical abstraction in their teaching. However, in order to develop the codes and thematic units related to teachers' behaviours in task implementation, it became necessary to understand more clearly the interpretation of reducing abstraction, as well as what characteristics of teachers' behaviour qualify as reducing abstraction. In this regard, I initially followed Hazzan's (1999) threefold interpretation of abstraction of her framework of reducing abstraction as discussed earlier (Section 3.1). Since Hazzan described reducing abstraction as students' actions in learning, whereas I focus on teachers' action in teaching, Hazzan's framework fell short in addressing the issues raised in this study. This shift in perspectives necessitated some modification and reinterpretation of the categories of Hazzan's reducing abstraction. Hence, the reducing abstraction categories in teaching are my interpretation of teachers' action in light of the literature. Below, I provide a brief description of each of the categories:

Category 1: Abstraction level as the quality of the relationships between the mathematical concept and the learner

Teachers' task implementation behaviour in which teacher makes an attempt to establish a right relationship (in the sense of Wilensky, 1991) between the students and the abstract mathematical concept was considered as reducing abstraction in this category. According to Wilensky, abstraction is not an inherent property of an object; rather, it is based on the relationship between a person and the object of his or her thought. Even a concept that is hopelessly abstract at some point can become concrete for us, if we are able to achieve a right relationship with the concept.

Category 2: Abstraction level as a reflection of process-object duality

Teachers' task implementation behaviour in which the teacher shifts the emphasis to a process or to the correctness of the answer rather than the concept itself is considered as reducing abstraction in this category.

Category 3: Abstraction level as the degree of complexity of a mathematical concept or task

Here abstraction is determined by the degree of complexity. Hence, teachers' task implementation behaviour in which a teacher attempts to reduce the complexity or complication of a problem in various ways is considered as reducing abstraction in this category.

With these three thematic categories borne in mind, I repeatedly watched the videos and read and reread the transcripts. Because of the dynamic nature of video, the video data were very helpful in my attempt to understand what was going on in the classroom, which might otherwise be impossible to determine using only the transcript. While watching the videos, I paid close attention to the classroom norms, and to social and cultural artefacts including some observable behaviour of the teachers and students such as gestures I deemed relevant for this study. These too were noted.

Upon close analysis of the data (secondary data), I identified various approaches of teachers in dealing with abstraction while implementing a task, each of which seemed to fall under one of the three thematic categories I listed above. I note, following Saldana (2008), that the act of coding requires me to wear my analytical lens, and my interpretation of the data depends on the type of filters that covers this lens. In particular, having filtered the data through the theoretical perspective with three interpretation of abstraction (three categories) as discussed earlier, I identified five reducing abstraction behaviours that fall under the first category, two behaviours under the second category and three behaviours under the third category, resulting a new theoretical framework of Reducing Abstraction in Teaching (RAiT).

Of note here is that my intention in this study is to explore how teachers deal with abstraction (in implementing a task) rather than provide a summary of occurrences. Hence, this study does not address the issues related to the frequency of teachers'

reducing abstraction behaviour. Below, I describe each of the subcategories briefly. (A more detailed description is provided in Chapter 5).

Category 1: Abstraction level as the quality of the relationships between the mathematical concept and the learner

1a) Reducing abstraction by connecting unfamiliar mathematical concept to real-life situations

Reducing abstraction here refers to the teachers' behaviours in which an attempt was made to reduce the abstraction level of the task by connecting the abstract (unfamiliar) mathematical concept to students' familiar real-life situations. I follow the TIMMS (1999) video study team's definition for 'real-life situation', which states that "real life situations are those that students might encounter outside of the mathematics classroom. These might be actual situations that students could experience or imagine experiencing in their daily life, or game situations in which students might have participated" (LessonLab, 2003a, p. 59).

1b) Reducing abstraction by experiment and simulation

Reducing abstraction in this subcategory refers to teachers' acts of task implementation in which the abstraction level of the concept is reduced through experimental learning and simulation. Many researchers point out the benefit of experimental learning in understanding abstract mathematical concepts. In experimental learning, students are actively involved in the entire process of learning, from data production, to report writing (Garfield & Ben-Zvi, 2007). As such, experimental learning provides a situation in which students become familiar with every steps of the problem-solving process and, hence, the new and abstract concept becomes less abstract. However, in some situations, this approach is impossible, complicated, or time consuming to carry out. In such cases, educational simulation and modelling may serve as an alternative approach, and are often used in teaching abstract mathematical concepts, as this approach seems to substantially reduce the abstraction level of the concept.

1c) Reducing abstraction by storytelling

Reducing abstraction in this subcategory aligns with Haven's (2000) statement, "telling a story creates more vivid, powerful and memorable images in a listener's mind than does any other means of delivery of the same material" (p. xvii). Therefore, if mathematics is presented in the form of story, it is likely that students will struggle less to cope with the abstract mathematical concepts than they would if only their rational and logical capabilities are drawn on during instruction (Zazkis & Liljedahl, 2009). In implementing an abstract mathematical task or concept, teachers' use of story as a pedagogical tool was coded as reducing abstraction in this subcategory. Only those stories in which substantive mathematics were involved were coded. Stories from teachers or students not related to mathematical tasks or concepts were not coded.

1d) Reducing abstraction by using familiar but informal language rather than formal mathematical language

Reducing abstraction in this subcategory comes from Zazkis's (2000) suggestion that "the non-mathematical code, at least initially for students, has a greater power to communicate their intended meaning" (p. 39). One key idea about teaching, according to this view, is that (initially) building concepts by using informal, everyday language in teaching abstract concepts would make such concepts more mentally accessible to the students. Hence, if a teacher used informal, everyday language in introducing new (abstract) mathematical concepts, this act is coded as reducing abstraction in this subcategory.

1e) Reducing abstraction through the use of pedagogical tools (such as model, manipulative, graphical representation, metaphor, analogy, gesture and so on)

Researchers in mathematics education support the idea that the incorporation of tools such as models, metaphors, metonymies, analogies, gestures, manipulatives and so on is pedagogically effective in teaching mathematics (see Lakoff & Núñez, 2000; Sfard, 1991; Edwards, Radford & Arzarello, 2009). This is because these tools are widely believed to act as a bridge between abstract mathematical tasks or concepts teachers are trying to teach (target domain) and the students' familiar situations and concrete knowledge (source domain). Hence, a teacher's act is coded as reducing abstraction in this subcategory if a teacher utilized these tools while implementing

mathematical tasks. To qualify a teacher's behaviour as reducing abstraction in this subcategory, task implementation must involve both a visible source domain and a target domain.

Category 2: Abstraction level as a reflection of process-object duality

2a) Reducing abstraction by shifting the focus onto procedure

Reducing abstraction in this subcategory refers to a teacher's task implementation behaviour in which the focus is shifted to canonical procedures (i.e. how to do it) even though the original mathematical problem or statement implies or includes a focus on concepts, meaning, or understanding. This idea was adopted from Sfard's (1991) theory of process-object duality, in which she maintains that process conception is less abstract than object conception.

2b) Reducing abstraction by shifting the focus onto the end-product (answer)

Reducing abstraction in this subcategory involved a teacher's task implementation behaviour in which the teacher shifts the focus onto the end-product (answer) and its accuracy, even though the original mathematical problem or statement implies or includes a focus on concepts or meaning.

Category 3: Abstraction level as degree of complexity of the mathematical problem or concept

3a) Reducing abstraction by shifting the focus onto the particular rather than the general

Reducing abstraction in this category refers to a situation where the presented materials and discussions limit the scope by focusing on particular cases rather than on general ones. By focusing on a particular case rather than the general case, task implementation strategy of this kind may reduce the abstraction level by making the task less compound or less complex for the student.

3b) Reducing abstraction by stating the concepts rather than developing it

When mathematical concepts are involved in a lesson, they can either be developed or stated. Reducing abstraction in this subcategory refers to a situation where the complexity of the (introducing) concept or (solving) problem is reduced by stating the

concept without describing mathematical relationships or noting why the concept is appropriate.

3c) Reducing abstraction by giving away the answer in the question or providing more hints than necessary

Reducing abstraction in this subcategory comes from the observation of Brousseau (1997), who observed that when students could not easily find the answer to a task, Topaze, a school teacher, tended to give the answer away within the question itself in a slightly indirect way, thereby lowering the intellectual difficulty of the task. Such teachers' task implementation behaviour can make the task less complex and less difficult, however this detracts from the learning opportunity for the students.

4.4. Moving forward

As previously discussed, the data is comprised of all TIMSS 1999 public release video lessons, totalling of 28 video lessons and their English transcripts translated by the TIMSS video lesson team. The process of coding and analysing the data proved to be extremely useful in my attempt to organize and manage such a large quantity of data. This process helped shed light on the various issues raised in this study, as well as allowing me to organize the data into meaningful codes and themes. As a result, the theoretical framework that I call Reducing Abstraction in Teaching (RAiT) has emerged with three thematic categories and many subcategories as defined above, a detailed discussion of which follows in Chapter 5.

This study employed the qualitative, interpretive approach, in which basic assumptions rest on the idea that realities are multiple and socially constructed (Lincoln and Guba, 1985) by our understanding and interpretation of phenomena and making meaning out of this process. Hence, it is important to note that these three interpretations of abstraction should not be thought of as independent or scientifically precise categories and subcategories, nor do they capture all possible interpretation for abstraction. However, I assigned the teachers' acts of reducing abstraction to the category that was deemed to fit best from the qualitative interpretive perspective.

Although the process of designing the coding scheme shed light on the various issues raised in this research, not surprisingly, it also raised further issues. For example, what were the factors associated with each of those teacher's task implementation behaviour? Why did teachers reduce the abstraction level of the concept involved in a task the way they did? Interviews with the teachers might help bring to light the various factors associated with teachers' actions related to reducing abstraction. As stated earlier, these questions remained unanswered while working with secondary data (e.g., the TIMSS 1999 video lessons/ transcripts) because interviews with these teachers were not possible with this section of the data. It is primarily for this reason that the primary data was collected, analysed and presented in chapter 6 . The method of analysis used is similar to that of the TIMSS video lessons. However, when a situation arose which required further clarification and justification about the action and behaviour of the teachers as related to task implementation, I sought clarification and justification from the participating teachers through informal interviews.

While analysing the primary data, I observed a pattern related to how teachers tailor their actions to deal with abstraction in their lessons very similar to those observed in the TIMSS 1999 video lessons. However, this part of the data sheds light on issues that were not possible to understand from the analysis of secondary data alone. In the following chapters, I first attend to the secondary data (the TIMSS 1999 Public Release video lessons) and then present the results and analysis in Chapter 5, leaving the analysis of primary data to Chapter 6.

5. Revisiting the TIMMS 1999 Video Lessons

The main focus of this study was to explore and describe how teachers in a sample of the TIMSS 1999 Public Release video lessons from seven countries deal with mathematical abstraction while implementing mathematical tasks. Towards that end, this chapter begins with an examination of the nature of teachers' task implementation (presentation) behaviour captured in the TIMSS 1999 Public Release video lessons using the coding system developed and discussed in Chapter 4. In the following, I will briefly outline Chapter 5. First, I provide a description of the issues related to task implementation behaviour and reducing abstraction in teaching. I then briefly discuss the previous work done by the TIMSS 1999 video study team itself as well as others using different theoretical frameworks (Stigler & Hiebert, 2009; Hiebert et al., 2003; Jacobs et al., 2003; Birky, 2007), followed by my motivation and rationale for the choice of the theoretical perspective. Finally, using the framework and the coding system developed in Chapter 4, I analyse teachers' actions with regard to dealing with mathematical abstraction in teaching.

5.1. Task Implementation Behaviour and Reducing Abstraction

In order to analyze teachers' task implementation behaviour, which involves how teachers deal with mathematical abstraction, it is imperative to first understand more clearly what features of teachers' actions and classroom discussions qualify as dealing with abstraction in general and reducing abstraction in particular. In this regard, Hazzan (1999) maintained that, for learners, reducing abstraction can be defined as an attempt on the part of learners to establish a *right relationship* (in the sense of Wilensky, 1991) to new (unfamiliar) mathematical concepts. Since students often do not have sufficient resources to cope with the same level of abstraction of the task or concept as expected by the authorities (i.e. teachers or textbooks), they tend to reduce the level of abstraction

to make it more mentally accessible. Students' reduction of abstraction led Hazzan (1999) to the conclusion that "the mental mechanism of reducing the level of abstraction enables students to base their understanding on their current knowledge, and to proceed towards mental construction of mathematical concepts conceived on a higher level of abstraction" (p. 84). However, as Hazzan observed, the problem inherent in this process is that students often reduce abstraction level inappropriately.

From this (and earlier) discussion, it becomes clear that in order for meaningful learning to occur, the abstraction level of the task or concept has to match with the existing knowledge schema of the learner at least initially so that the mathematical concept or object is more accessible to the learners. Hence, the notion of reducing abstraction in teaching (RAiT) refers to teachers' task implementation behaviour in which teachers attempt to make things easier for their students to learn by making the concept more concrete and less abstract.

This is in line with the widely accepted idea that to introduce new mathematical concepts effectively, it is necessary for teachers to use learners' previously acquired knowledge, experience and level of thinking, as well as their familiar contexts. This suggestion comes in part due to the fact that, in so doing, a richer connection, or borrowing the words from Wilensky (1991), a *right relationship* between the learner and the concept may be established.

Literature on teaching mathematical concepts (particularly in abstract algebra) also shows that there is consensus among mathematics education researchers on the point that when teaching new concepts, such concepts should be adequately mediated through concrete examples and less complex ideas for meaningful learning to occur (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron, Hazzan, & Zazkis, 1995). Hence, as stated previously in Chapter 3, for meaningful learning to be achieved, teaching mathematics should involve with the process of introducing new abstractions, concretising or semi-concretising them, and then repeating this at a slightly more advanced level, raising students' understanding to a higher level. As such, a teacher may use many strategies and methods, including connections to real-life situations and use of pedagogical tools in their teaching in order to make new concepts more accessible to their students.

My objective in this chapter is to explore various forms of teachers' task implementation behaviour, which suggests important features of teachers' pedagogical moves while dealing with mathematical abstraction. By reading and re-reading transcripts, watching and re-watching the Public Release video lessons of TIMSS (1999), and by applying the methodology developed in Chapter 4, I identified various ways that teachers deal with abstraction while implementing mathematical tasks. In the following sections, I describe and illustrate teachers' different task implementation behaviours and their approaches of how to deal with mathematical abstraction in teaching.

5.2. Category 1 (RAiT-1): Abstraction Level As The Quality Of The Relationships Between The Object Of Thought And The Thinking Person

As stated previously, Wilensky (1991) suggested that abstraction is not an inherent property of an object; rather, it is based on the relationship between the person and the object of thought. From this perspective, reducing abstraction refers to the situation where a teacher makes an attempt to establish a *right relationship* (in the sense of Wilensky) between the students and abstract mathematical concepts.

Teaching activity often involves an act of introducing high-level of unfamiliar mathematical concepts from a fairly basic level (by making it familiar to the students) so that the concept will be more accessible to the students. For this purpose, teachers use various ideas and methods in teaching, such as: connecting unfamiliar mathematical concepts to familiar real-life situations, using pedagogical tools that map unfamiliar mathematical concepts (target domain) to the familiar (source) domain, or using informal (everyday) language to describe formal mathematical language, all of which reduce the abstraction level of the concept for students. The goal is, however, to move to the higher level of abstraction by stepping up from the lower level. As stated earlier, when the student assimilates the concepts or ideas and is able to maintain a *right relationship* (in the sense of Wilensky) to the concept, the previously abstract concept becomes concrete (or less abstract) for the student. Stepping on the newly concretized abstract

concept, a teacher can then introduce another concept, and the cycle continues as in Figure 3.

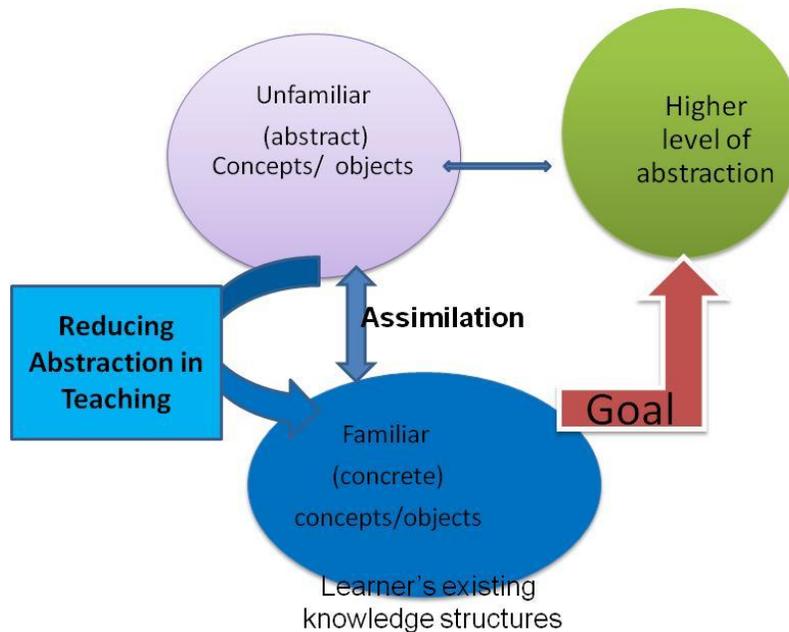


Figure 3. Teaching Model: RAiT-1

Towards that end, teachers use various methods and strategies to deal with abstraction in an effort to make unfamiliar mathematical concepts more familiar for their students. Close observation of the TIMSS (1999) video lessons and their transcripts revealed various approaches of teachers dealing with abstraction while implementing tasks which I briefly introduced in Section 4.4. In this chapter, I provide a more detailed explanation of each category and exemplify each subcategory with an excerpt from the TIMSS video lessons, as well as analyse the excerpt.

5.2.1. RAiT- 1a. Reducing abstraction by connecting unfamiliar mathematical concept to real-life situations

Reducing abstraction in this subcategory refers to the situation where teachers develop a mathematical concept by connecting an abstract (unfamiliar) mathematical concept to a familiar real-life situation.

What is 'real' in real-life situation?

Defining boundaries between what is considered to be real and what is not is a philosophical debate that has raged on since the days of Plato and Aristotle. This debate is beyond the scope of this study. However, in the mathematics education community, as King (2006) argues, the debate about drawing a fine line between what is and what is not real-life may be interesting, but at the same time it is also unproductive. It may be, as Pimm (1997) suggested, more productive to agree that mathematical tasks need not be real but only "offer [the] possibility of productive intellectual and emotional engagement" (p. xii). This does not mean that the connection between mathematics and students' real-life experience should be disregarded. Instead, as Brown (2001) suggested, mathematics educators need to broaden our conception of what counts as real-life:

If we can speak of what is "real" in a more vibrant sense than what "exists" or what we can "touch" and "see" then we not only legitimize more interesting connections between mathematics and the real world connections as a slave against an otherwise "unreal" world of mathematics. (p. 191)

Realistic Mathematics Education (RME) (in the Netherlands) is commonly known as a real world mathematics education that emphasises the importance of connection of mathematical concepts to real life situation in teaching and learning mathematics. As Van den Heuvel-Panhuizen (2000) observe, the word "realistic" in RME is not understood appropriately either within or outside the Netherlands. They say:

The Dutch reform of mathematics education was called 'realistic' is not [sic] just because of its connection with the real world, but is related to the emphasis that RME puts on offering the students problem situations which they can imagine. The Dutch translation of 'to imagine' is 'zich REALISERen.' It is this emphasis on making something real in your mind, that gave RME its name.[...] This means that the context can be one from the real world but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can provide suitable contexts for a problem, as long as they are real in the student's mind. (p. 4)

This is a broad view of what counts as real-life, according to which real-life context refers not only to situations that are likely to be encountered by students in their actual lives but also to some hypothetical contexts which bear some real element not too

far removed from a real world situation, as long as the situation is real in student’s mind. Along the same lines, TIMMS (1999) addressed the debate over defining real-life situations (as stated earlier in Section 4.3.1) in the following way:

Real life situations are those that students might encounter outside of the mathematics classroom. These might be actual situations that students could experience or imagine experiencing in their daily life, or game situations in which students might have participated. (LessonLab, 2003, p. 59)

The TIMSS (1999) report revealed that teachers from all participating countries used real-life connections, but in varying degrees, with the Netherlands at the top of the list with 42% of the problems connected to real-life situations. The following table shows the average percentage of problems per lesson set up with a real-life connection and problems with mathematical language or symbols only.

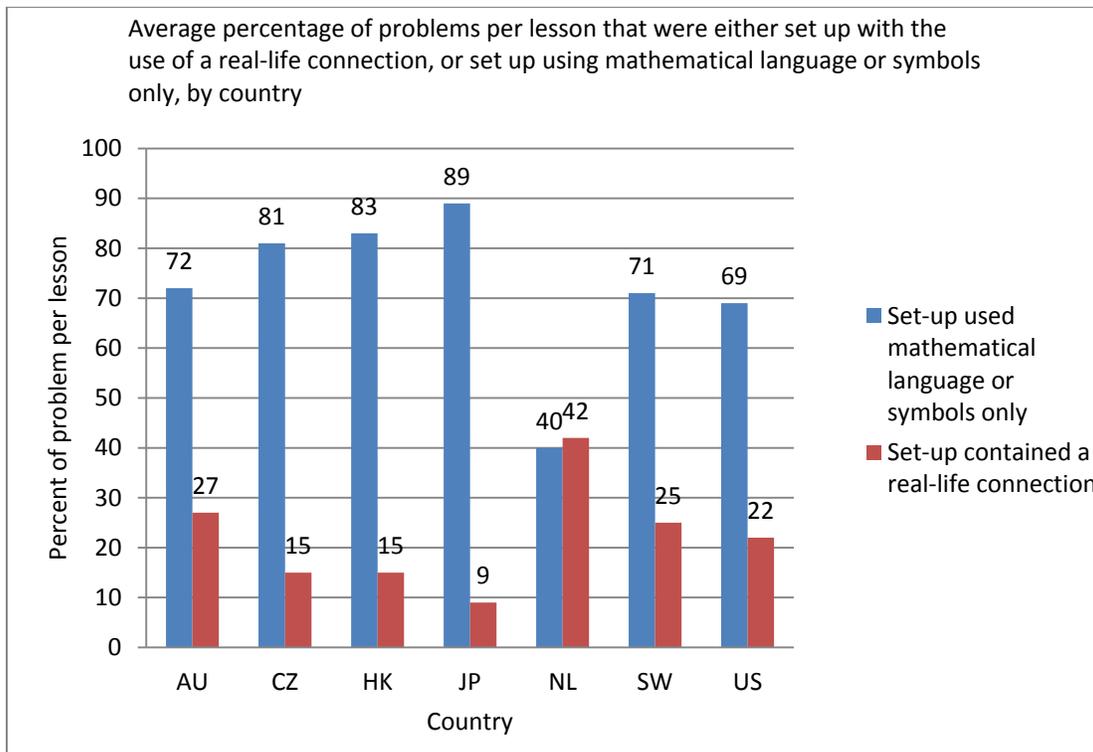


Figure 4. Real life connection problem (in TIMSS 1999 video study)

Note: Adapted from Heibert et al., 2003, p. 85

Real life connection in teaching (and learning) mathematics

The role of an appropriate relationship between mathematics and real world experience in doing and understanding mathematics has long been discussed (Davis & Hersh, 1981). Pedagogical theories in mathematics education seem to support the idea of connecting mathematics with students' real-life situations in order to enhance learning. Boaler (1993), for example, says, "if the students' social and cultural values are encouraged and supported in [the] mathematics classroom through the use of context and through the use of acknowledgement of personal route and direction, then their learning will have more meaning for them" (p. 6). Burkhardt (1981) also observe significant benefits to connecting mathematical problems in the context of real-life situations. They maintain that real-life problems connect better with students' intuitions about mathematics, show the relevance of mathematics and provide a sense of familiarity to the students. Bevil (2003) also finds that intermediate and middle school students who were exposed to the real world mathematical applications curriculum performed significantly better academically than their counterparts who were exposed to the traditional curriculum.

The Realistic Mathematics Education (RME) movement also recognizes the ability to connect mathematics to real-life as one of the important aspects of learning mathematics, which they call horizontal mathematization. The idea of mathematization comes from the work of Treffers (1987) and Freudenthal (1991), who view learning mathematics as two types of mathematization: horizontal and vertical. According to Freudenthal, horizontal mathematization involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols. In other words, horizontal mathematization suggests that experience involving real-life problems and contexts should serve as a starting point for learning mathematics, whereas vertical mathematization involves the process of constructing new mathematical structures within the world of mathematics and by mathematical means. This process usually utilizes previously constructed mathematical constructs to lead to a new construct.

The central idea of horizontal mathematization is that connection between mathematics and real-life situation provides a basis for learners to invent mathematical

tools or construct that can help them in the process of solving problems. These tools or constructs then can be utilized as the foundation for vertical mathematization. By doing so, students can see mathematics belonging to their familiar situation or problem type, which provides them a 'feeling of familiarity' or the state of '*at-homeness*' (Cockcroft, 1982). Hence the distance between the concepts and the learner diminishes, in which case the concept become less abstract in the sense of Wilensky (1991). However, as some psychologists and mathematics educators (e.g., Brownell, 1935; Prawat, 1991) have argued, in some instances, emphasizing the connections between mathematics and real-life situations may not be beneficial as it can distract students from the important ideas and relationships within mathematics. This often occurs if the process of mathematization is solely left to the students. Hence, teachers' pedagogical knowledge comes into play in such situations to reduce the distraction and lead students in the direction of understanding the concept.

I now present some representative examples from the data that illustrate the behaviour of teachers connecting mathematical concepts to real-life situations, which results in the reduction of abstraction for the mathematical task or concept in use.

Example 1:

The following dialogue is excerpted from one of the eighth-grade Swiss mathematics lessons, which focuses entirely on solving equations. It is one of the lessons in the series in which the already introduced topic is practiced in order to provide students with opportunities to become more fluent and efficient with the content and concept of the properties of equality (addition, subtraction, multiplication and division) of real numbers.

Consider the classroom dialogues between students and teacher while solving the equation $4x + 5 = 2x - 13$:

23:09 T: It's better to get the correct answer than to finish quickly with all the results wrong, you're right... So here we get to two X is equal to...

23:18 T: because minus two X minus five.

23:20 S: Minus one.

23:21 T: Minus?

23:22 S: Minus (inaudible).
23:23 S: No, minus 13.
23:25 S: Minus 18.
23:26 S: But how much is it when one does minus 13 minus five?
23:27 S: Minus 18.
23:28 T: Well, minus 13 minus five, you've got 13 objects missing,
 someone asks you for five more, you've got 18 missing.
23:32 S : Ha, yeah.
(TIMSS, 1999, SW4)

The abstract nature of algebraic expressions posed many problems to students in understanding algebra and, as such, many researchers focused their attention to this issue. One of the underlying causes of the students' problems in working with algebraic expression is related to the concept of numbers and operations, particularly with negative numbers. Vlassis (2002), for example, found that many errors made by students while solving equations are due to the presence of negative numbers, and concluded that it is the degree of abstraction of the negative numbers involved in the algebraic equation rather than the presence of variables that created the difficulties for the students. He wrote:

The negatives place the equation ('arithmetical' or 'non-arithmetical') on an abstract level. It is no longer possible to refer back to a concrete model or to arithmetic. The "didactical cut" does not seem to depend upon the structure of the equation (unknown on both sides of the equation), but upon the degree to which the equation has been made abstract by the negatives. Arithmetical equations with negatives therefore also represent an obstacle for those students who are unable to give them a concrete meaning (p. 350).

The *Didactical cut* (Fillooy & Rojano, 1989) refers to the students' inability to spontaneously operate with or on the unknown. This is related to the cognitive obstacle students experience in the transition process from arithmetic to algebra. In the dialogue above, the student's question "but how much is it when one does minus 13 minus 5" clearly indicates that this student could not make sense of what minus 13 minus 5 means (00:23:17). This supports Vlassis's (2002) assertion that "the negatives place the equation [...] on an abstract level" (p. 350). The difficulties of students with negative numbers are due in part to the abstraction and the lack of connection to the real world. In

addition, literature suggests that one of the main reasons for the difficulty students have with negative numbers is related to the meaning of the minus sign (Vlassis's, 2002). The minus sign has two different functions: a process as subtraction, and an object as a negative number or an additive inverse to a number. Hence, understanding minus signs involves a two-step process: first, to distinguish the different meanings of the sign, and then to understand that the meanings are interchangeable depending on the context.

In the above dialogue, “but how much is it when one does minus 13 minus five?” shows that the student lacks the understanding of the different meanings of the sign. Consequently, she could not treat the expression as an addition of assigned numbers, that is, that minus 13 minus 5 is the same as negative 13 plus negative 5. When the teacher directs her attention to a real-life situation by saying, “minus 13 minus five, you've got 13 objects missing, someone asks you for 5 more, you've got 18 missing”, she seemed to be able to make meaning of the negative numbers and their operations, which is indicated by her expression “Ha, yeah” (00:23:32).

This act of bringing up the situation, which connected the concept of negative numbers to an everyday life context helped the student understand the meaning of negative numbers and their operation in two ways: first, the real-life situation to which she was familiar provided her a basis for the interpretation of the mathematical situation involving negative numbers and operation. At that moment in the lesson, minus 13 minus 5 was no longer an abstract and strange problem; rather, it became simply a phenomenon in her real life, such as missing objects (that is, negative can be thought of as missing or given away). Second, the connection of a real-life situation helped the student understand that a formal mathematical algorithm (the different meanings of the minus sign, in this case), such as ‘minus 13 minus 5’ (-13-5) is the same as ‘negative thirteen (13 missing objects) plus negative five (giving away five more objects).

This shows that the teachers’ pedagogical move of connecting an unfamiliar mathematical concept (in this case, negative numbers and operations with them) to a student’s familiar, real-life situations (operation on concrete objects) reduced the level of abstraction of the concept of the negative sign and negative numbers, thereby helping students understand the concept better.

5.2.2. RAI-1b. Reducing abstraction by experiment and simulation

As stated earlier in Section 4.3.1, reducing abstraction in this subcategory refers to the teachers' activities in which an unfamiliar task (or concept) is implemented in the class through experiment and/or simulation.

Experiment and Simulation in mathematics/ Statistics education

The literature shows that the conventional methods of teaching mathematics, which involve telling, looking and listening, have not been successful. According to the National Research Council (1989), "much of the failure in school mathematics is due to a tradition of teaching that is inappropriate to the way most students learn" (p. 6). One of the reasons for such failure is that in traditional method of teaching, the role of students in general is as passive receivers of knowledge. In contrast, experimental learning requires students' active participation in the whole process of problem solving (from data production to the final solution), and hence they become familiar with every step of problem solving. Such experience not only makes the abstract task familiar, but also provides a sense of ownership and a feeling of *at-homeness* for the student with the mathematics they are learning. However, in some situations, this approach is impossible, complicated, or too time-consuming to carry out. In such cases, educational simulation and modelling may serve as an alternative approach, and one which is often used in teaching abstract mathematical concepts, as this approach often reduces the abstraction level of the concept.

Experimental learning is one of the strategies widely used in overcoming difficulties in teaching and learning some of the abstract concepts in mathematics and statistics. Through such contextual and experimental learning, students come to know that mathematics/statistics relates usefully to their real life and that it is not just the game of symbols, rules and formulae (Garfield & Ben-Zvi, 2007). In addition, students come to know that "data do not exist in a vacuum, but that there is a story behind the numbers; students learn that statistical analysis involves developing the narrative of the data" (Libman, 2010, p. 9) "unlocking the stories in the data" (Garfield & Ben-Zvi, 2007, p. 19). This strategy has been reported as effective by many researchers in teaching mathematical and statistical concepts (Gnanadesikan, Scheaffer, Watkins & Witmer,

1997). Statistics in general falls under the school mathematics curriculum and, hence, is taught by mathematics teachers. Basic statistical concepts such as mean, median, mode and quartiles are some of the most important concepts in statistics. They have much application in real-life situations and though they seem simple and straightforward in appearance, as Green (1992) points out, they may be very challenging concepts for the novice:

Statistical concepts provide a fascinating area to explore. What the statistician regards as straightforward and obvious (terms such as average, variability, distribution, correlation . . .) are the distilled wisdom of several generations of the ablest minds. It is too much to expect that there will not be a struggle to pass on this inheritance. (cited in Batanero, Godino, Vallecillos, Green & Holmes, 1994, p. 543)

Researches show that the basic concept of statistics (such as mean, median and mode) raise difficulties in students' understanding and that (even after many years of formal studies in statistics) students often find these concepts abstract and challenging (Garfield & Ben-Zvi, 2007, 1988; Batanero et al., 1994). Batanero et al. (1994), for example, found that many students' and even teachers' understanding of the mean is superficial in that this is understood to be more algorithmic and computational rather than conceptual. More recently, Ekol (2013) observed that students' understanding of measures of variability (such as mean and standard deviation) is more procedures and calculation driven. However, when students were taught using dynamic geometry software (e.g., geometer's sketchpad), they developed deeper understanding of measures of variability. Based on this observation, Ekol suggested that one way of introducing more challenging concepts in statistics effectively to students is mediating through dynamic learning tools.

In fact as Batanero et al. (1994) stated, the algorithmic and calculation driven knowledge is problematic because "knowledge of computational rule not only does not imply any deep understanding of the underlying concept, but may actually inhibit the acquisition of a more complete conceptual knowledge. Learning an algorithm or computational formula is a poor substitution for an understanding of the basic underlying concept" (ibid., p. 533).

In a similar way, research shows that the case with understanding and using box plots is not much different. Leavy (2006) for example, found that “many preservice teachers were not adept at constructing, or in some cases even aware of, stem and leaf plots and box and whisker plots” (p. 106). Although, box plots have been included in school mathematics curriculum in many countries, the age of the students to which they are taught differ considerably. NCTM, for example expects students of grades 6-to-8 to be familiar with box plots (NCTM, 2000). However, in France, this is taught to 16-to-17-year-olds; in Australia, it is introduced to 15-to-16-year-olds, and in China and Israel, it is not included in the school curriculum at all (Bakker, Biehler & Konold 2004). Considering the complexity and difficulties involved in constructing and understanding box plots, some educators suggest not introducing box plots to young students. Bakker, Biehler and Konold (2004), for example, maintain that “it is unwise to teach box plots to students younger than 14 or 15 years old, especially given the fact that many students of this age are still struggling with the meaning and usage of percentages” (p. 171).

As discussed above, the basic concepts of statistics (mathematics), which can seem simple and intuitive, are in fact challenging for many students. One of the reasons for lack of students’ understanding of basic statistical concepts can be attributed to teaching approach. In the traditional teaching approach, for example, a teacher uses a numerical dataset from a textbook or some other source. This dataset is often remote from students’ experience and interest, which reduces students’ involvement in the process of data collection and manipulation. This experience, in turn, can lead to the impression among students of mathematical/statistical concepts as being abstract and a game of symbols, rules and formulas.

As such, several teaching strategies are suggested for overcoming difficulties in teaching and learning statistical/mathematical concepts. One of these strategies is to promote experimental learning in which students are actively involved in the entire process of learning, from data production to report writing. Such contextual and experimental learning provides students a basis for relating mathematical/statistical concepts to their real life.

Many researchers have pointed out the benefit of students gaining real-life statistical (and mathematical) experience from solving real problems with real data

(Garfield & Ben-Zvi 2007; Moore & Cobb, 2000). This beneficial effect can be attributed to the fact that such real-life statistical experience provides concrete instantiation to the otherwise abstract mathematical ideas, thereby making the ideas more accessible to the students. Moore and Cobb (2000) for example, maintain that such activities provide “more experience with the process of searching for patterns at a low level of abstraction before formulating a more abstract statement and then assessing its validity” (p. 622).

Such an approach, however, is not always possible due to the various factors involved in the process, such as time, cost and risk factors including the complication involved in carrying out the experiment. In such cases, educational simulation and modelling can be carried out instead, which may serve the purpose of teaching abstract mathematical/statistical concepts while keeping many of the benefits of experimental learning. As such, this has been widely used in education as a pedagogical tool. In what follows, I provide a representative example of how a teacher uses simulation in teaching some basic statistical concepts such as mean, median, mode, and box and whiskers plot in an attempt to make these concepts easily accessible for the students.

Example 2:

This example comes from an Australian eighth-grade mathematics lesson that focuses on some of the basic statistical concepts related to data collection and representation, particularly on constructing box-and-whisker plots. Since there is no national mathematics curriculum in Australia, each state and territory is responsible for its own curriculum, generally consisting of five content strands: Number, Measurement, Space, Algebra and Chance and Data (statistics) (Hiebert et al., 2003). Although statistics at the tertiary level is considered as a distinct discipline, some basic statistical content is taught in school-level mathematics curriculum. This lesson from the chance and data content strands covers some basic concepts from statistics. In it, students are expected to be able to calculate the mean, median, mode and quartiles of a data set, and to represent the data in box-and-whisker plots. In previous lessons, the frequency distribution table was taught and students were familiar with the sigma notation. This lesson begins with a review of work covered in the previous lesson, particularly going over homework, which is a typical practice in Australian mathematics classrooms (Heibert et al., 2003). The activity begins with the teacher telling the following story:

03:33 T: All right, here's the story.

03:37 T: My wife bakes hot chocolate chip cookies, which I like.

03:50 T: But lately, the number of cookies or the number of chocolate chips in the cookie... has been decreasing.

04:00 T: We're going to simulate an experiment here whereby we have to find out how many chocolate chips I've got to put into a mixture to create six cookies-

04:18 T: Which you have on that sheet that I've given you- so that I can be pretty sure that each cookie is going to end up with at least three, yes?

04:30 T: Now, anyone got any ideas as to how many chocolate chips I would have to put into my mixture so that I would end up with at least three in each cookie?

04:42 S: Eighteen?

04:43 S: Eighteen.

04:44 T: What about if I had 18- now, they're mixed up in this mixture for the six cookies- are you sure that each time you scoop out some of that mixture, you're gonna get three?

04:56 T: So you- you're riding on the bare minimum there, aren't you? You're hoping- you're hoping that you're going to get three each scoop.

05:04 T: What we're going to do is this little simulation exercise and it's just to gather some statistical data so we can carry on with our statistics.

05:13 T: You're gonna work in pairs. One person's going to roll the dice.

05:18 T: The number of the dice indicates the cookie and your cookies are numbered one to six. For example if I rolled a five that means I've got one chocolate chip for cookie number five.

05:30 T: You're going to have to roll the dice sufficient times to end up with a minimum of three chocolate chips in each cookie. Do you understand?

05:40 S: Yeah.

05:48 T: All right. So you know what to do? One person's rolling the die. Each time that number comes up, you put a stroke into your cookie.

05:58 T: Now, it may be that you end up with ten chocolate chips in cookie number five. That doesn't matter.

07:33 T: You must keep rolling until every cookie has got three chocolate chips in it.

08:14 T: Once you have finished, count up the total number of cookies-or sorry, the total number of chocolate chips that you required to get three chocolate chips into each cookie.

(TIMSS, 1999, AU3)

As stated earlier, one of the factors well-documented in the literature is that of difficulties encountered by students in understanding even the basic concepts of statistics and this being related to a teaching approach in which students are not provided with the opportunity to engage in the process of data collection and manipulation (Garfield & Ben-Zvi 2007; Batanero et al.,1994). As a result, the students lack interest and motivation. In this lesson, by encapsulating the mathematics in the form of a story, students' interest and enthusiasm seem to have been engaged and, as a result, a familiar and positive learning atmosphere was created. A detailed discussion about the use of storytelling in teaching mathematics will come later in this chapter. The focus in this section, however, is on the use of simulation.

The basic idea in simulation is to imitate something real in which certain key behaviours or characteristics of physical or abstract systems are retained. It “refers to the artificial replication of sufficient components of a real-world situation to achieve specific goals” (Fanning & Gaba, 2009, p. 459). To carry out the real experiment as described in the “chips and cookies” story is not likely possible in the classroom. Simulation, on the other hand, is more economical, less time-consuming and carries less physical risk than the actual experiment, while still preserving the flavour of the real-life situation.

The main purpose of the simulation activity here was to collect the data for computing the mean, median, mode and (lower and upper) quartiles in order to construct the box and whiskers plots. In this simulation activity, students were to find the minimum of how many chocolate chips were required in order to be certain that each of the six cookies had at least three chocolate chips in it. Each number in the dice corresponds to the cookie's number and cookies were numbered one to six. For example, if the die rolled a three, that means the student got one chocolate chip for cookie number three, and so on. The students were instructed to roll the dice a sufficient number of times until they got three chocolate chips for each cookie. They were then required to enter the resulting data into a frequency distribution table.

After tabulating the data in a frequency distribution table, the class computed the mean, median, mode and (lower and upper) quartiles. Students initially seemed to struggle to make sense of medians and quartiles, but with the help of gestures and an explanation of the concept in context, the teacher helped the students understand them better. (I will attend more to gesture in the next section.) The class then constructed box and whiskers plots. Box and whisker plots (usually called box plots) are a graphical technique in the field of exploratory data analysis, in which numerical data are organized and displayed using five values: the lowest value, lower quartile, median, upper quartile, and the largest value. The conversation then continued as follows:

36:51 T: This is for the data we're going to work on, your chocolate chip...cookie one, right?

37:02 T: And just before we go on there... How many chocolate chips do you think you would need to ensure that you ended up with three?

At this moment students gave different answers but their answers differed significantly. Then the teacher encouraged students to think deeper.

37:26 T: Well let's face it, you know...

37:27 T: If we if we put these chocolate chips into the mixture and mixed them around, we would expect them to get mixed a little bit and not sit in one corner, wouldn't we? Right?

37:37 T: But using- using the information that we gathered from that little thing-

37:42 S: Twenty-nine?

37:43 T: Twenty-nine?

37:44 S: About thirty.

37:45 T: So where are you getting these figures from?

37:47 S: From the mean.

37:48 T: The mean!

37:49 T: The mean? Okay. Let's just have a look at those three numbers again: a mean of thirty point five seven, a median of twenty-eight point five, and a mode of twenty-five.

(TIMSS, 1999, AU3)

Next, the class discussed the meaning of each of the new concepts (mean, median, mode and quartiles), referring back to the "chips and cookies" story. In this

case, the students seemed to overcome some of the difficulties and challenges related to box and whisker plots as described in the literature (see Bakker, Biehler & Konold, 2004).

This simulation activity provided an opportunity for the students to be actively involved in the entire process of learning: from data production to displaying the data in box plots. Such simulated real-life statistical experience from solving real problems with real data (familiar context) substantially reduces the abstraction level of the concept. The previously abstract statistical concepts (i.e. mean, median, mode and quartiles) were no longer abstract or remote from students' everyday lives; rather, these concepts became part of their life stories.

5.2.3. RAiT-1c. Reducing abstraction by storytelling:

Reducing abstraction in this subcategory refers to the teaching activity in which an abstract mathematical task or concept is introduced to the students mediating through story. Here, story serves as a pedagogical tool to make an unfamiliar and abstract concept familiar to the students, thereby reducing the abstraction level of the concept.

Storytelling and mathematics education

"Humans are storytelling organisms who, individually and socially, lead storied lives" (Connelly & Clandinin, 1990, p. 2).

Students often perceive mathematics as an abstract subject remote from their every day experience and consisting of a collection of facts, rules and skills. Consequently, mathematics teachers continue to search for ways to help teach mathematical concepts. One of the ways to bring abstract mathematical concepts to life for students more concretely is to teach mathematics through telling a story. When students are intimidated by abstract concepts, or may doubt their ability to understand such concepts, storytelling may transform the perceived abstract and objective mathematics into a subject imbued with imagination, myth and subjective meaning and feeling. This Indian proverb nicely captures the power of story in the learning process: "Tell me a fact and I'll learn. Tell me the truth and I'll believe. But tell me a story and it

will live in my heart forever". Bruner (1986) characterized this educational approach as one that "emphasizes the learner and how they construct a representation of reality through their interactions with the world and their discussions with others" (p. 34).

Storytelling as a pedagogical tool is hardly a new idea. As Livo and Rietz (1986) put it, "the telling of stories is an old practice, so old, in fact that it seems almost as natural as using oral language" (pp. 7–8). In aboriginal societies, "storytelling is a very important part of the educational process. It is through stories that customs and values are taught and shared" (Little Bear, 2000, p. 81).

Creating interest with a story and engaging students in the story's mathematics is an important pedagogical step in teaching abstract mathematical ideas. In using a mathematical story during instruction, not only are students' rational and logical thinking called upon, but also their imaginative and fantasizing capabilities (Schiro, 2004). When this happens, it is likely that students struggle less to cope with abstract mathematical concepts as opposed to when only their rational and logical capabilities are drawn upon during instruction.

Example 3:

This example comes from an Australian eighth-grade mathematics lesson that focuses on some of the basic statistical concepts of data collection and representation, and particularly on constructing box and whiskers plots. Part of the data from this lesson has already been analysed and presented above (Example 2). Of note here is that in this section, the data has been analysed from a different perspective (storytelling). This confirms (as stated earlier) that behaviours that reduce abstraction are not disjointed from one another, but rather they intersect, and one category or subcategory may overlap with another depending on the perspective one takes.

This lesson begins with a review of work covered in previous lessons and going over the homework. The teacher then tells the "chips and cookies" story to model an activity for the rest of the lesson as given in the previous example. (Please refer to the conversation from 03:33 to 05:04 as given in Example 2.)

The traditional way of introducing basic statistical concepts such as mean, median, mode and quartiles is to consider the data as taken for granted, in which a set of (abstract) data is taken directly from the textbook or another source. Here the teacher tells a story about familiar objects and context, describing a special situation in which the numbers of chocolate chips in the cookies are decreasing for some reason. When introducing the problem this way, students' interest and motivation were engaged and they seemed to be more curious and enthusiastic, as evident from watching the video. In fact, as Zazkis and Liljedahl (2009) tell us, "creating interest with a story is an important initial step. Describing a chain of events may engage students, create excitement, mystery or suspense, and motivate thinking about a particular problem" (p. 4).

The "chips and cookies" story made the mathematics less abstract by providing students a very concrete, physical way of thinking about mathematics in two respects: the first involved the use of a concrete model of mathematical ideas by having certain (concrete) characters (the teacher and his wife) in the story who actively created a mathematical situation. Further, it involved an activity in which the dice (physical materials) were rolled and each time this happened, the cookie represented by the number rolled (a concrete representations of numbers) got a chocolate chip. For example, if the number four was rolled with a die, then cookie number four got one chocolate chip. Second, the story provided an opportunity for the students in which they could imitate the role of the characters in creating a mathematical problem and trying to solve it, similar to what Zazkis and Liljedahl (2009) write: "thinking and acting like their heroes [...] create[s] empathy and make[s] the material more accessible and memorable[...] more relevant and more vivid and [...] may help students relax as they provide something to hold to when moving to general theory or technical detail" (p. 4).

Further, as Schiro (2004) points out, "one of the powerful parts of children's mind is their imaginative and fantasizing capabilities" (p. 77). In storytelling, rational and logical capabilities marry with the imaginative and fantasizing capabilities, and as a result, a powerful learning environment is created in which students struggle less to cope with abstract mathematical concepts. In this example, students could imagine themselves in the role of the 'wife' and mix the chocolate chips into the dough, which provided a familiar context for the students.

After the data was collected and the mean, median, mode and quartiles had been computed, box and whisker plots were constructed. The teacher then returned to the “cookies and chips” story, providing an opportunity for the students to connect the meaning of each of the new concepts (mean, median, mode and quartiles) into the story. As such, the story continues as follows:

- 38:31 T: What about if you were a chocolate chip cookie manufacturer, yeah? And you were looking at ensuring that every cookie had three chocolate chips in it?
- 39:11 T: When you buy those chocolate, uh, chip cookies, they don't all have the same number. There's no fellow sitting there poking in the same number of, uh, chips is there?
- 39:22 T: So all of the ingredients are mixed together.
- 39:24 T: And they would have some quality control and quantity control to ensure that there are a minimum number of chocolate chips.
- 40:14 T: In your home situation, most likely you wouldn't look at the thirty point five seven. Yeah?
- 40:20 T: If you were a bulk manufacturer, uh, of these things and you wish to ensure that you had them, uh...
- 40:30 T: I guess the thirty point five seven, which you'd take is th-uh, thirty or thirty-one, would be the number that you'd pick.

(TIMSS, 1999, AU3)

Here students were required to act and think from three different roles: as a manufacturer, a buyer and simply as a consumer in a home situation who must decide how many chips must be mixed into the dough in order to make six cookies. In doing so, not only were their rational and logical thinking called upon, but also their imaginative and fantasizing capabilities thereby, promoting higher order thinking. Further, the mean, median, mode and quartiles were no longer abstract concepts remote from their everyday lives, but rather something meaningful and less abstract.

5.2.4. *RAiT-1d. Reducing abstraction by using familiar but informal language rather than formal mathematical language*

As stated earlier (in Chapter 4), reducing abstraction in this subcategory refers to the tendency of teachers to use informal language rather than formal mathematical

language to describe definitions, properties and concepts. Because students are usually familiar with everyday language, teachers' use of informal (everyday) language to explain unfamiliar formal mathematical concepts provides a sense of familiarity for the students, thereby reducing the abstraction level of the concepts.

Use of informal language in teaching mathematics

Language in the context of the mathematics classroom, in general, can be divided into two categories: informal everyday language and formal mathematical language. Students often find informal language to be easy, familiar and friendly. In contrast, formal mathematical language is often perceived by students as abstract and difficult. Of course, mathematics has its own system of language. As Winslow (1998) maintains, "mathematics can be singled out, among other forms of human imagination and ingenuity, by the very specific linguistic register in which its ideas are formulated" (p. 19).

Mathematics utilizes formal or mathematical terminology to communicate ideas within the discipline that are foreign to other disciplines. Hence students' proficiency in formal mathematical language is the key to mathematics learning, as the "students' non-mathematical informal code is frequently vague and inadequate for describing mathematical relationships" (Zazkis, 2000, p. 39). Still, students are not usually familiar with formal mathematical language, and requiring students to use rigorously formal mathematical language may restrict their ability to understand and express their mathematical ideas, which in turn may negatively affect their learning. On the other hand, the use of overly informal language may inhibit the development of formal mathematical language.

This tension between formal and informal language in teaching mathematics still exists in the literature (Zazkis, 2000; Rubenstein, 1996; Mitchell, 2001). Some researchers argue that informal language in mathematics education may give the mathematics a different meaning by placing it into a different context, and hence may be detrimental to formal education in mathematics if not used carefully and appropriately (Mitchell, 2001). Others maintain that appropriate use of everyday language in teaching mathematics is pedagogically helpful. Anghileri (2000), for example, believes that "mathematical understanding involves progression from practical experiences to talking

about these experiences, first using informal language and then more formal language” (p. 8). Rubenstein (1996) described a situation where a student finds the concept of *angle bisector* as abstract and difficult, and proposed an easier term, ‘*midray*’, to refer to *angle bisector*. As the students began to understand the meaning of the angle bisector, the teacher later extended the concept to develop *midline* and *midplane*.

Herbel-Eisenmann (2002) analyzed the developing language of students’ in understanding a mathematical concept. She observed that students’ developing language move through three categories: from ‘contextual language’ to ‘bridging language’ and finally to ‘official mathematical language’. She refers contextual language as the language that depends on specific context or situation (similar to ‘everyday language’). Bridging language is the transitional mathematical language and refers to a particular process or representation without the contextual reference. Official mathematical language refers to the language that is “part of the mathematical register and would be recognized by anyone in the mathematical community” (p.102). In her study, Herbel-Eisenmann concluded that allowing students to develop their own words for a mathematical concept before introducing formal mathematical language is more natural and makes mathematical ideas more understandable for students.

According to this view, one way of teaching mathematics is to first build concepts through the use of student-invented mathematical language (or everyday language), and then to attach the formal mathematical vocabulary needed to establish the ideas within the discipline.

This suggestion comes due in part to the fact that “children construct new ideas and communicate these in language [...] with which they are familiar, and that may be idiosyncratic initially, and learn later the conventional language formats for expressing these ideas” (Munro, 1989, p. 121). As students extend their mathematical terminology and build their mathematical proficiency, they become more confident of their ability to learn and apply mathematics. For this reason, it is often suggested to “use informal everyday language in mathematics lessons before or alongside technical mathematical vocabulary” (Department for Education and Employment, 1999, p. 2) in order to arrive at a meaningful understanding of mathematical concepts.

The Department for Education and Employment (DfEE, 1999) document, however, reminds us:

[A]lthough this can help children to grasp the meaning of different words and phrases, a structured approach to the teaching and learning of vocabulary is essential if children are to move on and begin using the correct mathematical terminology as soon as possible (p. 2).

This calls for a transition from informal language to formal mathematical language in a planned and systematic way as an important step in the teaching and learning of mathematics. If done appropriately, the use of informal everyday language in teaching mathematics may reduce the abstraction level of concepts, thereby making those concepts more accessible to students.

Example 4:

This lesson comes from an eighth-grade mathematics classroom in the Netherlands. The lesson is about the Pythagorean theorem. The teacher begins the class by asking whether anybody has ever heard about Pythagoras. Students' responses as "Uh, Uncle Pete" shows that they have heard about Pythagoras before and they like to call him "Uncle Pete". Some of the students immediately associated "Uncle Pete" to the equation $a^2 + b^2 = c^2$. The class then moved on to a brief discussion of different types of triangles, particularly the isosceles and equilateral triangles and their basic properties. After the teacher stated the fact that Pythagoras theorem is applicable only to the right triangle, the focus then shifted to the right triangle. The ensuing conversation went this way:

04:33 T: We know that the edges of a triangle- or any figure- are called "sides".

04:38 T: In a right-angled triangle, this side is attached to a right angle. So what should we call this side? A right-angled side.

04:47 T: Yes? Because this side is attached to a right angle so you call that a right-angled side.

05:00 T: Do we have any other right-angled side in there?

05:02 S: Yes.

05:03 T: Yes, all the way on the other side. That one is attached to the right angle as well, therefore you call that a right-angled side as well.

05:19 T: Then I still have one side left. It isn't so obvious because it is lying flat. But if you see this triangle, what can we call that side?

05:28 S: The long side.

05:29 T: The long side. That is correct. Or in a different way?

05:33 S: The right side?

05:34 T: It is actually at an angle. If you see it in such an... like a diagonal so you call this the sloped side or the hypotenuse, is what you call this one.

05:45 T: These are just names, you know, you may also keep calling this "the long side", no problem.

(TIMSS 1999, NL2)

Mathematical vocabulary (and mathematical language), which may seem simple and familiar for teachers, often poses challenges to students (Rubenstein & Thompson, 2002). Unlike common, everyday language, which students see, hear and use in day-to-day life while speaking, reading, watching television and so on, mathematical language (including mathematical vocabulary) is limited largely to educational and professional settings. For example, the word 'hypotenuse' is commonly used in mathematics classrooms and in the mathematical community, but rarely used in everyday language. Consequently, students encounter difficulties in learning mathematical language and the technical terms and phrases used in the discipline of mathematics.

In this example, instead of giving the mathematical definition of 'hypotenuse', the teacher used words drawn from everyday language such as 'the long side', the side 'like a diagonal' and 'sloped side' to explore the concepts of the technical vocabulary 'hypotenuse'. This activity is in line with Rubenstein & Thompson (2002) assertion that "a major premise of all [vocabulary] strategies is to connect new terms or phrases to ideas children already know. Children should first do activities that build concept, then express their understanding informally, and finally, when ideas solidify; learn the formal language" (p. 108).

In the context of mathematical vocabulary, Peirce and Fontaine (2009) distinguish between technical vocabulary words and sub-technical vocabulary words; the former has a precise mathematical meaning, and the latter has common meaning but also a mathematical denotation. 'Hypotenuse', for example, is a technical term having a

precise meaning and used only within the mathematics register. Phrases such as ‘the long side’, the side ‘like a diagonal’ and ‘sloped side’ are sub-technical vocabulary often used in everyday language; they have mathematical denotation but lack precision in their meaning or are ambiguous. In other words, the precise meaning of the term ‘hypotenuse’ as a side opposite to a right angle of a triangle in a plane (a strictly mathematical sense) is unfamiliar and abstract idea for students where as the terms such as ‘like a diagonal’ or ‘sloped side’ are familiar and often used in everyday situation—the latter being at the lower level of abstraction and the former being at the higher level from the perspective that the abstraction is interpreted in this category.

From a teaching perspective, there is a danger that students may adopt sub-technical terms as if they are mathematical terms if the technical vocabulary is not taught explicitly. For example, a student who interprets a hypotenuse as a side ‘like a diagonal’ or a ‘sloped side’ may not recognize the hypotenuse in the other situations where the hypotenuse is a vertical or horizontal straight line as illustrated in Figure 5.

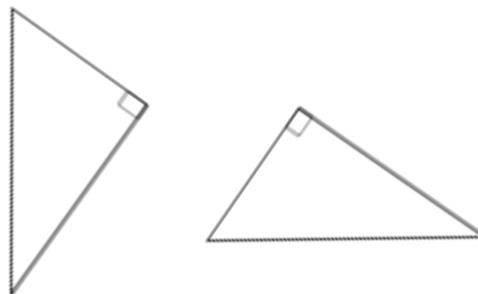


Figure 5. Right Triangles

To overcome such confusion or ambiguity, the teacher used technical (formal) and sub-technical (informal) language interchangeably during the lesson. In doing so, the teacher provided an opportunity for the students to explore and extend the meaning of the mathematical vocabulary from one instance to another, mediating through sub-technical (everyday) language. The goal, however, is to move towards more technical terms and language similar to what Rubenstein and Thompson (2002) suggested:

Simply withhold the formal terminology. Let students use materials to explore ideas, suggest their own terms, and explain their rationales. Of

course formal terms must be introduced eventually, and students must be able to translate between informal and standard words, but by inventing they realize that terms come from people *thinking* about new ideas. This realization aligns with an important goal we have for students: to be thinkers and creators in the world of the future. (p. 109)

These suggestions arise due in part to the fact that if a conceptual foundation is formed, students' informally created language (sub-technical) and the more technical mathematical language will be related, and when students realize this connection they can easily translate the informal language to the new and formal technical language.

The example above illustrates a situation in which the teacher makes a pedagogical move to introduce the unfamiliar concept of a 'hypotenuse' by reducing the level of abstraction, in that the use of informal language such as 'the long side', a side 'like a diagonal' and 'sloped side' are more familiar to the students. However, this approach is not free from dangers as noted earlier.

5.2.5. *RAiT-1e. Reducing abstraction through the use of pedagogical tools (such as model, manipulative, graphical representation, metaphor, analogy, gesture, and so on)*

The benefit of incorporating pedagogical tools such as models, metaphors, metonyms, analogies, gestures and manipulative in teaching and learning mathematical concepts has been well documented in the literature (see Lakoff & Núñez, 2000; Sfard, 1991; Edwards, Radford & Arzarello, 2009). In the context of teaching mathematics, these tools act as a vehicle to lead students from unfamiliar mathematical ideas to the familiar and concrete domain, and then back to the mathematical concepts instructors are attempting to teach.

This kind of pedagogical move in task implementation generally reduces the abstraction level of the task as these tools allow teachers to build a bridge between unfamiliar abstract mathematical concepts (target domain) and a familiar/concrete situation (source domain). Hence, reducing abstraction in this category refers to a teacher's act in which an unfamiliar and abstract concept (target domain) is presented to the students at a fairly basic level with the use of familiar pedagogical tools and artefacts (such as models, manipulative, metaphors, analogies, gestures, etc.). These

pedagogical tools are related to the world of mathematical entities though a set of principles that map elements of the source domain (often familiar and thus concrete) to the target domain (often unfamiliar and thus abstract). Mapping refers to some kind of correspondence between one domain and the other that preserves structure and meaning, on the basis of which inferences between the source and target objects can be drawn. In other words, “[it] is the map [...] [T]hey are packages that relate objects and their transformations to other objects and their transformations” (Cuoco & Curcio, 2001, p. X).

Some of the representative task implementation behaviours of teachers identified from the data as effective ways of reducing abstraction in this subcategory are discussed below.

Use of Graphical Representation

Graphical representation is widely used in teaching mathematical concepts. As the TIMSS 1999 report revealed, use of graphical representation in teaching eighth-grade mathematics lessons was a common task implementation behaviour in all seven of the participating countries, but with varying frequency, as shown in Figure 6:

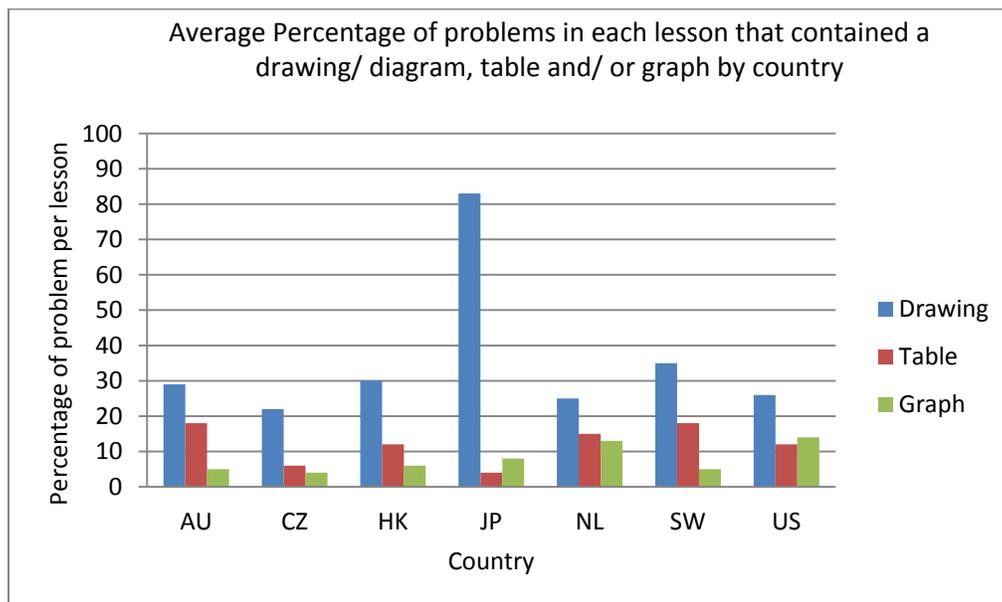


Figure 6: *Average Percentage of problem in each lesson that contained graphical representation.*

Note: Adapted from Hiebert et al., 2003, p. 86

Below, I present an example of how a Swiss teacher uses graphical representation in order to make abstract concept less abstract.

Example 5:

This lesson comes from a Swiss eighth-grade mathematics classroom, the topic of which is on solving one variable linear equation by applying the properties of equality (addition, subtraction, multiplication and division) of real numbers. The teacher stated that the lesson is a review of the previous lesson in order to provide students with opportunities to become more familiar and efficient with the content they had already encountered.

Consider the following conversation between students and teacher while solving the equation $4x = 4x - 1$:

38:57 T: The G part, what do we have that's tricky? We have four X is equal to four X... minus one. So how are we going to see that?

39:04 S: It's not possible!

39:09 T: It's impossible. Valentine?

39:11 S: Because it would need a place, for example four X is equal to three X minus one. Otherwise we can't take away the X's. We can't take away one.

39:21 T: Let's try to take it away, let's try to be as methodical as we can, then let's see what happens. Let's see, where do we find, so-

39:28 S: Times four X.

39:30 T: You're okay that this four X is bothering you.

39:32 S: Yeah.

39:33 S: We do minus four X.

39:34 T: So we do minus four X.

39:35 S: So it does zero equals zero.

39:35 T: We find zero X. Therefore zero is equal to minus one. Is that possible?

39:42 S: No!

39:43 T: No. Therefore when you get to a case like this...

(TIMSS 1999, SW 4)

At this stage, some of the students seemed to be struggling to make sense of what it meant when they get $0 = -1$ while solving the equation. At this point, the teacher referred to the graphical method, which allowed students to see clearly that the lines were parallel, and that, thus, there was no point of intersection. That is, there was no solution. The lesson continues:

- 40:27 T: It starts from zero. Right. We'll put it in one color. That one there. Here, it starts from zero and each time it advances it goes up how much?
- 40:35 S: Four.
- 40:36 T: Up four. Okay... it goes up like this.
- 40:45 T: And then it goes down also each time. Each time I go backwards, I go down four. Are we okay?
- 40:49 S: Yeah.
- 40:49 T: That's okay? That's the function of G, the left element. The right element... what nice little colors here would
- 40:59 T: distinguish this from blue? We don't have much... a little orange, a green. We'll see the difference but I don't have anymore orange... So the
- 41:12 T: function of the right side. I'll write it here... it says- it says from R to R, X goes on four X minus one. Therefore it means I go from where?
- 41:25 S: Minus one.
- 41:25 T: I go from minus one on- there and then, each time I advance one, I go up four.
- 41:31 S: Well it would do a (inaudible).
- 41:33 S: It is parallel.
- 41:33 T: It is, well, parallel. Therefore when do they meet each other, lines?
- 41:37 S: Never.
- 41:37 T: Never. Therefore the- what we're looking for as a solution, it's the points that meet each other. Therefore there's none.

(TIMSS, 1999, SW4)

Here the teacher brought up the graphical representation in order to show why there was no solution for this ($4x = 4x - 1$) equation as shown in Figure 7.

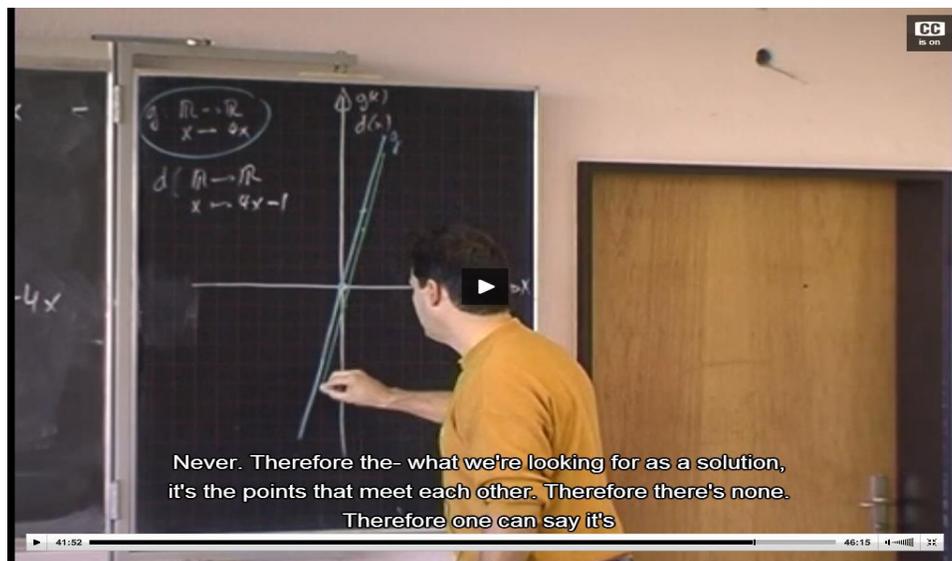


Figure 7. Graphical Representation

Students were then able to see in the graph that the lines were parallel and hence there was no solution. The teacher's act of mapping the unfamiliar target domain (solution of equations as an empty set) to the students' familiar source domain (visual representation of the parallel lines) in order to make the concept more mentally accessible can be interpreted as having reduced abstraction in this category.

Use of Analogy, Metaphor and Mnemonics

Much has been written in the literature about the key role that analogy and metaphor can play in learning abstract concepts in various disciplines (Gentner, Holyoak, & Kokinov, 2001; Richland, Holyoak & Stigler, 2004; Lakoff & Núñez, 2000; Sfard, 1991). Many researchers have argued that analogical reasoning may be beneficial to learning abstract concepts, as it makes the unfamiliar familiar (Duit, 1991; Gentner, Holyoak, & Kokinov, 2001). Taking the constructivist perspective, Duit (1991), in his extensive literature review, asserted that analogies and metaphors are powerful tools in learning; they facilitate understanding of the abstract concept by resorting to the similarities of a concrete situation in the real world, thereby allowing learners to visualize the abstract and promoting students' interest and motivation for learning. Along the same line, Richland, Holyoak & Stigler (2004) maintain that "analogical comparison can result in formation of abstract schema to represent the underlying structure of source

and target objects, thereby enhancing reasoners' capacity to transfer learning across contexts" (p. 38).

An analogy is a process that compares unfamiliar facts and situations with those that are known and familiar. The familiar is the source domain or analogue that is used as a basis for comparison, and the unknown and unfamiliar is the target domain which is to be explained. Thus, analogy refers to the relationship between two concepts or processes that allows inference about the least known things on the basis of known and familiar things.

Analogy involves two levels of relation between the source and target domains: surface similarity and relational similarity. Surface similarity refers to the resemblance of objects or attributes between the source and target, whereas relational similarity refers to the resemblance in the structure or relations between the elements of the source and target (Gentner & Markman, 1997). For example, a student who notices the similarity between the addition of one set of numbers and addition of another set of numbers at first may not notice the similarity between addition of numbers and variables because variables are different than numbers at the surface level. However as the student learn more about variables, he or she notices that the information he or she knew about addition of numbers can be transferred to inform his or her reasoning about addition involving variables because addition of numbers and addition of variables are structurally similar. Many educators have argued that similarity only at the superficial level is insufficient for a successful analogy, and that relational similarity is the crucial feature of analogical thinking (Gentner & Markman, 1997).

Similar to the analogy, metaphor is the term related to establishing the comparison between different domains, usually relating one that is familiar (source domain) to one that is unfamiliar (target domain). Johnson (1987) elaborated the meaning and role of metaphor as follows:

Understanding via metaphorical projection from the concrete to the abstract makes use of physical experience in two ways. First, our bodily movements and interactions in various physical domains of experience are structured (as we saw with image schemata), and that structure can be projected by metaphor onto abstract domains. Second, metaphorical projection is not merely a matter of arbitrary fanciful projection from

anything to anything with no constraints. Concrete bodily experience not only constrains the "input" to the metaphorical projections but also the nature of the projections themselves, that is, the kinds of mappings that can occur across domains. (p. xv)

Although analogy and metaphor appear similar on the surface, Duit (1991) pointed to some differences between analogy and metaphor in that an analogy explicitly compares the structural or functional similarities (e.g., a circle is like an orange) whereas a metaphor compares literally, highlighting features or relational qualities (e.g., calling a teacher 'the captain of the ship'). Particularly in an analogy, words such as '*like this/that*', '*same as*', '*similar to*', etc. are often used to compare the source domain and the target domain, while in metaphor such words are omitted.

Another instructional tool that can be used similarly in teaching mathematics is a mnemonic device. Mnemonics devices are memory aids that consist of acoustical keywords such as acronyms, short poems or memorable phrases or images that can help learners to retrieve information. Mnemonics are often used in teaching unfamiliar abstract concepts, as they help students remember specific information by linking new and abstract concepts to their prior knowledge through the use of acoustic or visual mental cues (Mastropieri & Scruggs, 1989). An example of a classic mnemonic device for teaching the order of operation is the acronym PEMDAS, or Please Excuse My Dear Aunt Sally, which reminds the students about using parenthesis, exponents, multiply, divide, add and subtract as the order of operation to follow while working with problems involving different operations.

Even though some distinctions can be drawn between the meanings of the terms 'analogy', 'metaphor' and 'mnemonic', each helps students to understand abstract ideas in a similar manner, by providing a potential bridge between concepts that are abstract and unfamiliar and experiences that are concrete and familiar. Therefore, analogy, metaphor and mnemonic devices are used as instructional tools in teaching abstract mathematical ideas.

Example 6:

This lesson comes from an Australian eighth-grade mathematics classroom and focuses on the concept of an 'exterior angle of a polygon'. The lesson begins with a

discussion that reviews and links previous work with the current topic. At the start of this lesson, the teacher instructed students to read the materials from the textbook in pairs and to then identify some important vocabulary for the lesson with which they were unfamiliar. Students identified five such terms that they didn't know the meaning of, including: Exterior angle, Polygon, Convex, Quadrilateral and Rays. A discussion ensued about the meaning of each of these terms. After the discussion on the first four terms (Exterior angles, Polygon, Quadrilateral and Rays), it came time to discuss the fifth term, 'Convex'. The dialogue proceeded this way:

- 14:52 T: Now what haven't we explained yet?
- 14:59 S: Convex polygon.
- 15:01 T: Convex. Anyone know what convex is? Anyone know- there's a word that's the opposite of convex.
- 15:09 S: Oh.
- 15:16 S: Doesn't convex mean, its like, um, when you have glasses there's uh, (inaudible) convex lens is the one which is curved?
- 15:26 T: Curved. Yes. Curved which way? That's the key thing.
- 15:30 S: Outside.
- 15:31 S: Inside.
- 15:32 S: Outside.
- 15:33 T: Yeah, curved inside. The- the other word that means the opposite of that also starts with that.
- 15:41S: Concave.
- 15:42 T: Concave. Yeah.
- 15:44 T: Now convex polygon is the sort we just drew (pointing to the figure ). But for example, a concave polygon... looks like that (pointing to ).
- 15:57 T: And the easy way to remember it is, is that part of it is caved in. Okay? It's pointing inwards. Convex polygon, everything is point outwards.

(TIMSS, 1999, AU1)

Here, 'convex polygon' is a new and unfamiliar (and therefore abstract) concept for the students. When students encounter such new and unfamiliar concepts, one way they cope with this type of situation is to call upon their analogical thinking to make

sense of the unfamiliar and abstract concept using the basis of familiar and concrete ideas via the similarity principle. This can be carried out in two ways: through relational similarity and surface similarity. As stated earlier, surface similarity is the one in which objects or attributes between the source and target domains looks similar on the surface, whereas in relational similarity, objects or attributes are similar relationally or structurally.

Here in our example, when asked about convex polygons, students tended to associate this concept with their familiar and concrete object, 'convex lenses'. In fact, the convex lens and the convex polygon are similar in both the surface and structural level, which helped them make sense of the convex polygon. Once students had grasped the concept of a convex polygon, the teacher then extended the idea to the concave polygon, which is just the opposite of a convex polygon.

Further, in the case of concave polygons, the teacher reminded the students to think of this concept as something that is 'caved in'. The keyword 'caved' worked here as a mnemonic device because it provided a link between the unfamiliar abstract idea (target concept and vocabulary) of a concave polygon to familiar 'caved in' objects (such as tea cups).

As discussed earlier, mnemonic devices used in instruction link new information to students' prior knowledge and experience through visual cues, acoustic cues, or both. In our example above, the word 'caved' is a pedagogically helpful mnemonic device because it linked the unfamiliar concept of a concave polygon to students' familiar objects through both acoustic and visual cues. Indeed, the word "caved" is acoustically similar to the target word—concave polygon, which made this an especially useful mnemonic device to use here. Furthermore, a concave polygon can be visualized as an object that is 'caved in', such as a tea cup. This familiar, visualized object then provides a link to the unfamiliar abstract idea of a "concave polygon". When the students need to recall the new information about concave polygons, the visualization can be invoked and the students are able to more easily retrieve the concept of a concave polygon.

The use of analogy as well as mnemonic devices in the above teaching activity reduced the abstraction level of the new concept 'convex polygon' and 'concave

polygon' (i.e. least known things) through the mediation of known and familiar things such as convex lenses and "caved in" objects respectively.

Example 7

While in Example 6, structural similarity was discussed, Example 7 shows a surface similarity analogy that focuses on graphing linear equations from a U.S. eighth-grade mathematics class. Here the teacher showed some examples on the board of how to graph linear equations and asked students to practice the skills they had just learned. While the students worked in groups, the teacher walked around the class and helped the groups. One of the groups drew the teacher's attention to make sure their graphs were correct. The conversation then proceeded as follows (Note that the students were assigned to draw the graph of the equation $y = \frac{2}{3}x + 8$ on which the following conversation ensued):

26:35 T: How come it's going down, guys?

26:36 S: I don't know, it... does it.

26:38 T: Aren't all these slopes positive? So all of these should be going up.

26:42 S: How's it going down?

26:44 T: Look at it. It's going down.

26:46 S: It's going up.

26:47 T: How do you read a book? How do you read a book, sir?

26:48S: Left to right.

26:49 T: You read from left to right. So let's start on the left side of the line.

26:52 S: Yeah.

26:53 T: And as I move to the right I'm going in which direction?

26:56 S: Down.

26:57 T: Right.

(TIMSS 1999 US1)

Here the graph drawn by the students was not what the teacher expected. Even for the positive slope of the linear equation, the graph produced by the students was going down from left to right. This was not surprising because 'going up' or 'going down'

are relative concepts which depend on what perspective you take to look at them. For example, the graph which is 'going up' from 'left to right' (LR) direction can be thought of as 'going down' from 'right to left' (RL) direction. While determining the slope of a line graph (positive or negative), mathematics teachers tend to consider the direction of thinking from left to right as a general mathematical convention, and they often forget that this idea may be new to students, and thus, this may require additional help and explanation. The above example clearly shows the difficulty students encountered while determining whether the slope is positive or negative. The teacher then used the analogy that the direction of the graph (unfamiliar, target domain) should be thought of as the direction of reading a book (e.g., left to right) (familiar, source domain). From the latter part of the lesson, it was evident that this analogy was helpful in allowing the students to retain the information that the line with a positive slope goes up (LR order).

A point to note here is that the teacher's suggestion that the graph should be thought of as the direction of reading a book worked well in this example but may not work in other contexts and cultures. Mathematics teachers (in many countries) often forget that the left to right order in writing and reading languages is a learned skill and by no means universal. For example, English and other contemporary European languages are written and read from left to right. In contrast, languages from the Semitic language family such as Hebrew and Arabic are written from right to left. Chinese and Japanese were historically written as vertically from top to bottom, but in modern times they are also written horizontally in either order (from LR or from RL) (Chan & Bergen, 2005). Therefore, the analogy that the direction of a graph or line should be thought of from left to right, as in the case of reading a book, may work in some contexts and cultures in which languages are written and read horizontally from left to right (LR), but does not work in other contexts or cultures in which the language does not follow the LR order. This example shows that activity intended to reduce abstraction can be context specific and culturally embedded.

Use of Gestures

Many studies have attempted to understand the role of gesture along with spoken language in teaching and learning processes (Arzarello & Edwards, 2005; Cook & Goldin-Meadow, 2006; Sinclair and Tabaghi, 2010). In recent years, gestures and

bodily movement are considered to be a source of information and closely related to thinking and communication. Roth (2001) for example, maintains that “humans make use not just of one communicative medium, language, but also of three mediums concurrently: language, gesture, and the semiotic resources in the perceptual environment” (p. 9). In the context of mathematics, Sinclair and Tabaghi (2010) observed that in describing mathematical concepts, mathematicians use talk, gesture and diagram “in ways that blur the distinction between the mathematical and physical world” (p. 223). Based on their study, they come to the conclusion that “gesture and talk contribute differently and uniquely to mathematical conceptualisation” (ibid., p. 223).

Likewise, Arzarello and Edwards (2005) state, mathematical ideas are abstract and that for a better understanding of abstract ideas, one “has to come to terms with our need for seeing, touching, and manipulating. It requires perceivable signs” (p. 127), and through the mediation of gesture, abstract mathematical ideas come within the reach of the learners. As such, the appropriate use of gesture in teaching and learning abstract mathematical concepts have been widely accepted and even encouraged.

Gesture is defined here as the movement of the hands or other body parts associated with mathematics and used during teaching and learning processes. The following example illustrates one of the teaching activities in which a mathematical concept was made less abstract for students through the use of gesture.

Example 8

This lesson comes from an eighth-grade Australian classroom on the topic of data collection and representation, focusing particularly on measures of central tendency. Part of the data from this lesson has already been analysed and presented in earlier sections (See sections 5.2.2 & 5.2.3). The focus of the discussion here is on the role of gesture in teaching the concepts of points of central tendency. Consider the following conversation:

16:44 T: Okay, would you now find for me the three points of central tendency, please.

(Students were working quietly for about a minute.)

17:50 T: Finished, Josh?

17:51 T: (inaudible)

18:01 T: Is there anyone else having difficulty with the median?

At this point, students were working to find the mean, median and mode. However, some of the students were struggling to find the median. Their understanding of the median was similar to the case reported in the study of Barr (1980), who found that many students of age 17-to-21 do not have a clear idea of the median. Here the students interpreted the median as a central value of something without knowing what exactly this “something” was or otherwise encountering difficulties finding the median. One of the difficulties in this case was caused by the even number of items in the data. When there is an even number of items, the median value is the average of the two middle values and hence does not belong to the data set. The conversation continued as follows:

18:06 T: Okay. Just stop what you're doing, listen in for a moment please. What is the median score?

18:12 T: The middle score. How many pieces of data do we have?

18:14 S: Eleven.

18:15 S: Fourteen.

18:16 T: Fourteen.

18:17 T: Fourteen pieces. Okay, now think of it, 14 pieces of data balanced, how many on each side?

18:23 S: Seven.

18:24 T: Have we got any piece of data for the center?

18:25 S: No.

18:26 T: So we haven't got an actual median which is one of our pieces of data. What are we going to have to do?

18:32 T: Get the seven and eight pieces of data, add them together, and (inaudible).

18:37 T: Find their average, okay. Do you understand that, girls?

(TIMSS 1999, AU3)

At this point, the teacher extended his both hands to represent all fourteen pieces of data and then asked the student how many pieces of data were there on each side.

His gesture (see Figure 4 below) helped students to associate the concept of a median to something concrete that they could easily see and understand. For example, students saw that each of his hands represented seven pieces of data and hence the middle part of him—his head or body—represented the median value, which is in the middle. Further, the students also learned that the median of a data set that involves two middle values (in the case of an even number of items) is the average of the two middle values (the average of the 7th and 8th items in the above example).



Figure 8. Gesture for median

When the median was found, the students were instructed to compute both the lower and upper quartiles.

- 21:29 T: All right. Finding the lower quartile.
- 21:32 T: Well, remember the lower quartile is going to be the midpoint of that lower range, yeah? So have we used one of those pieces of data to obtain this median?
- 21:43 S: No.
- 21:44 T: No, so all pieces of data that are in that lower range are in play. So how many pieces are there?
- 22:02 S: Seven.
- 22:03 T: Seven. Now we used the seventh and eighth to generate that median, yeah?
- 22:08 T: But we haven't used a piece of data, one of our pieces of data as the median, have we? It was the average of those two. So the seventh and eighth piece of data is still in play.
- 22:22 T: One to seven, so what is the midpoint there?

22:26 T: That's it... Anybody else having trouble finding the first and third quartile?

22:31 S: Yes.

22:33 T: Listen in, listen in then.

At this point, students seemed to be confused and could not find the values of the quartiles. The teacher, using a hand gesture, explained the meaning of quartile concretely in that each of his arms represented seven pieces of data and that the lower quartile and upper quartile are simply the middle point in his right and left arms, respectively, as shown in the figure below.



Figure 9. *Gesture 1: Lower quartile*



Figure 10. *Gesture 2: Lower quartile*

It was evident from the data that the use of gesture played an important role in students' understanding of the concepts of medians and quartiles. Similar to Radford's (2005) observation, the use of gesture in our example above helped students "to make their intentions apparent, to notice abstract mathematical relationships and to become aware of conceptual aspects of mathematical objects" (p. 143). Further, the information conveyed by the gesture was also conveyed by speech, thereby making the situation more conducive for learning. This teaching activity illustrated how gesture can be used in reducing the abstraction of the concept of the points of central tendency (particularly the median and quartile), thereby making these concepts less abstract for the students.

In this section, I have provided some representative examples that illustrated various activities or approaches of teachers dealing with abstraction in teaching mathematics. As evidenced from the data, these activities helped students make abstract mathematical concepts more accessible thereby, attempting to establish a qualitative relationship or what Wilensky (1991) would call it a 'right relationship' between the concept and the learner.

But this is not the only perspective for interpreting or determining the level of abstraction in a mathematical activity. Sfard's (1991) theory of process-object duality, for example, maintains that mathematical concepts or abstract notations can be viewed from two different angles: one as a 'process' and the other as an 'object', and that process conception is on the lower end of the abstraction continuum and object conception is on the higher end. It is this perspective of determining the level of abstraction that I turn to next.

5.3. Category 2 (RAiT-2): Abstraction Level As Reflection Of The Process-Object Duality

Drawing from the literature, Hazzan (1999) provided the second interpretation of abstraction level based on process-object duality. Process-object duality refers to the idea of seeing mathematical entities in which mathematical objects or abstract notations can be viewed both as a process and as an object. For example, the mathematical

notation $\frac{2}{5}$ can be viewed both as the process of division and as the concept of a fraction (Gray & Tall, 1991/1994).

Much has been written about the apparent process-object duality of mathematical concepts. Some of the most influential work on the issue of process-object duality in mathematics education comes from that of Gray and Tall (1991), Dubinsky (1991) and Sfard (1991). Gray and Tall (1991/1994) develop the idea of 'procept', which refers to the amalgam of concept and process represented by the same symbol. This notion of process-object duality led Gray and Tall to identify two different ways of thinking: procedural and proceptual. Procedural thinking is that in which the focus is on the process and the algorithm, whereas proceptual thinking refers to the ability to manipulate mathematical symbols flexibly as process or concept. Gray and Tall (1991) drew distinctions between those who progress well and those who struggle in mathematics based on the manner of their thinking or 'proceptual divide'. They maintained that:

The more able are doing qualitatively different mathematics from the less able... in that more able treat such symbolism flexibly, as process or concept, whichever is more appropriate in a given context, the less able tend to conceive of mathematics more as separate procedures to carry out computations. The manipulation of concepts is cognitively easier than the coordination of procedures, and so the more able are doing an easier form of mathematics than the less able. (p. 1)

Dubinsky (1991) also recognizes the process-object duality of mathematical concepts and proposed a path that students follow in the development of mathematical concepts, which became known as APOS theory. APOS is the acronym given to the mental structure of action (A), process (P), object (O) and schema (S). These mental structures are constructed by learners using mental mechanisms such as interiorization, coordination and encapsulation. According to this theory, "mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions, processes, and objects and organizing them in schemas to make sense of the situations" (Dubinsky & McDonald, 2001, p. 2).

To illustrate the concept formation in APOS theory, let us use the following example. Let's say a student sees a function, $f(x) = 2x + 5$ and, guided in a

mechanical way by the external clues of the expression and is then able to say $f(x) = 9$ when $x = 2$ by substituting the given value of x in the function without being able to discuss the function in general; this is considered to be action conception. As these actions are repeated and reflected on, the student no longer relies on the external clues but progresses to the process conception, in which the student is able to think through the function without having a specific example and to carry out the steps without necessarily having to perform each of them. In process conception, students can think of function as input and output, but are not able to see the process as a whole or as an object. Upon reflection, a student may encapsulate his or her process conception into an object conception in which the function now becomes a single unit that can be acted upon.

Along the same line, Sfard (1991) talks about process-object duality from structural-operational perspectives. From this perspective, abstract notation such as a number, function, etc. can be viewed from two fundamentally different ways: operationally (as processes) and structurally (as objects), and operational conception precedes structural conception in the process of concept formation. According to Sfard (1991), the process of moving from operational to structural conception involves three steps:

A constant three step pattern can be identified in the successive transitions from operational to structural conceptions: first there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three steps will be called interiorization, condensation and reification.

Condensation means a rather technical change of approach, which expresses itself in an ability to deal with a given process in terms of input/output without necessarily considering its component steps.

Reification is the next step: in the mind of the learner, it converts the already condensed process into an object-like entity... The fact that a process has been interiorized and condensed into a compact, self-sustained entity, does not mean, by itself, that a person has acquired the ability to think about it in a structural way. Without reification, her or his approach will remain purely operational. (pp. 64–65)

Though the construction of object conception of a mathematical concept may seem simple for teacher, it is in fact a challenging activity for students. In their study about the nature of students' mental construction of two variable functions in a multivariable calculus course, Martínez-Planell and Gaisman (2012), for example, found that only one out of their 13 students was able to encapsulate the process conception and constructed object conception. Sfard (1991) also recognises the "inherent difficulty of reification" (similar to encapsulation) and suggests that:

The ability to see something familiar in a totally new way is never easy to achieve. The difficulties arising when a process is converted into an object are, in a sense, like those experienced during transition from one scientific paradigm to another. (p. 30)

Although there are some subtle differences between those ideas given by Sfard, Gray and Tall, and Dubinsky with regard to the duality of a mathematical concept, these ideas have much in common. In particular, they all seem to agree on two points: first, the ability to think of a mathematical concept structurally (in Sfard, 1991) or proceptually (in Gray & Tall, 1991, 1994) or as an object (after encapsulating the process) all refer to the very similar mental mechanism of concept development. Second, the process conception of a mathematical concept in each of the theories precedes the object conception in the process of concept development. That is, the object conception has its genesis in process conception, and thus, process conception can be interpreted as being on a lower level of abstraction than the object conception.

An issue closely related to the discussion of these kinds of conceptions (process or object) and to understanding mathematics learning was discussed by Skemp (1976) decades ago. He coined the terms 'instrumental understanding' and 'relational understanding' to distinguish the quality of the understanding of a concept. Relational understanding refers to the object conception of a mathematical concept, whereas in the case of instrumental understanding, the focus shifts to process conception. According to Skemp (1976), relational understanding involves "knowing both what to do and why" (p. 2) and "building up a conceptual structure" (p. 14). Instrumental understanding, on the other hand, shifts the focus to the algorithm and the end-product (answer) without any understanding of the underlying mathematical concept. This type of understanding

involves ‘rules without reasons’ and can usually be carried out at a lower level of thinking. According to Skemp (1976):

[...] instrumental mathematics is usually easier to understand; sometimes much easier [...] If what is wanted is a page of right answers, instrumental mathematics can provide this more quickly and easily. So the rewards are more immediate, and more apparent. It is nice to get a page of right answers, and we must not underrate the importance of the feeling of success which pupils get from this. [...] Just because less knowledge is involved, one can often get the right answer more quickly and reliably by instrumental thinking than relational. (p. 23)

This tendency can also be seen widely in mathematics teaching practices. Teachers telling *minus times minus equals plus* and *to divide by a fraction, you invert and multiply* without further explanation are some of the examples of this that can be found in many mathematics classrooms (Skemp, 1976; Sfard, 2007).

Looking at this issue from the perspective of RAI_T, teaching activity that focuses on process conception rather than object conception can be interpreted as a way of reducing abstraction. Thus, reducing abstraction in this category refers to a situation where there is an emphasis on process or the final answer rather than on mathematical concepts or objects. In other words, this category refers to a situation where the teaching activities exhibit a tendency towards instrumental understanding rather than conceptual or relational understanding on the part of students. Two related ways of reducing abstraction were identified from the data, namely 1) reducing abstraction by shifting the focus to procedure, and 2) reducing abstraction by shifting the focus to the end product. Both of these are illustrated below.

5.3.1. RAI_T-2a. Teacher reducing abstraction by shifting the focus to procedure

Reducing abstraction in this category refers to a situation where the teacher shifts the focus towards canonical procedures (how to do it) even though the original mathematical problem or discussion implies or includes a focus on concepts or understanding. The following example illustrates a teacher’s implementation behaviour to reduce abstraction in this subcategory:

Example 9

This example comes from a Czech eighth-grade mathematics lesson that focuses on the concept of exponents and laws related to this concept. The lesson begins with a brief discussion about the laws of exponents, which is a follow-up from the previous lesson. The teacher then assigns the problem from previously covered topics. Students have to solve the assigned problems on the board in front of the class and provide a description out loud about the steps, process and formula they used to solve their particular problem. The problem assigned to the student is $\frac{1.2^{11}}{1.2^9}$. The conversation ensued as follows:

- 19:59S: One point two to the eleventh power divided by one point two to the ninth power.
- 20:09T: Tell him the instructions on what to do with the problem.
- 20:12T: And you're supposed to calculate it.

The student begins solving the problem on the board in front of the class. The teacher then asks the student to choose the rule(s) for this problem from the list. Note that the teacher already wrote the list of properties of exponents on the board and made this visible to all students from the beginning of the class. The student is expected to point out the particular rule used for solving the problem. The conversation continued:

- 20:14T: So, which rule will you follow? Stand aside and tell us which rule you will follow.
- 20:21S: This one (Pointing to the rule from the list).
- 20:24T: We're dealing with power of quotient with the same...
- 20:29S: Same main number.
- 20:29T: Very good. Equals! You're supposed to calculate the problem, you just simplified it.
- 20:40T: Calculating the problem means, you come up with a number. You can calculate it in your head.

Teacher is instructing the students what to do while he is writing on the board.

- 20:48T: First, you calculate 12 squared, which is how much? Just ignore the numbers that are to the nearest tenths for now.
- 21:03T: That's 144, and you separated the two because of the decimal numbers (inaudible). Very good.

(TIMSS, 1999, CZ4)

Solving this problem involves two things: first, simplification of the fraction $\frac{1.2^{11}}{1.2^9}$ and second, number calculation. The simplification process involves the use of the division rule of exponents which yields $\frac{1.2^{11}}{1.2^9} = 1.2^{(11-9)} = 1.2^2$. The calculation of the number involves finding the value of 1.2^2 , which can be carried out by multiplying 1.2 by itself. For the first part, the teacher instructs the student to point out the correct rule (without further discussion or justification) from the formula list that can be used to simplify the fraction. For the second part, the teacher instructs the student to follow the trick or procedure in which 1.2 is considered as 12 and that multiplying 12 by itself gives 144. Then 144 is separated as 1.44 because of the two decimal places needed (i.e., counting the decimal places in multiplicand and multiplier and locating the decimal point in the answer). Although, this rule produces correct answers while multiplying two decimal numbers if the rule is followed correctly (procedural knowledge), it does not, however, help students understand the underlying mathematical idea behind that rule and why the rule works.

As evidenced in the data, there were some mathematically important opportunities to lead the discussion not only towards process conception but also towards a more meaningful understanding of the idea of an exponent and what the mathematical object $\frac{1.2^{11}}{1.2^9}$ means. As stated earlier, a mathematics entity such as $\frac{1.2^{11}}{1.2^9}$ can be viewed both as a process and as an object. But there is no discussion of the object conception of $\frac{1.2^{11}}{1.2^9}$. Instead, the discussion focused on the process conception. That is, the problem was solved algorithmically, with the discussion focusing on procedures and rules rather than the underlying mathematical concepts. As discussed earlier, this particular task implementation behaviour exhibited the process conception of the mathematical concept rather than object conception, thus reducing abstraction in this subcategory.

5.3.2. *RAiT-2b. Reducing abstraction by shifting the focus to the end-product (answer)*

Reducing abstraction in this subcategory involves a situation where the teacher shifts the focus to the end product (answer) and its accuracy rather than the concepts or

understanding. The following is a representative example that illustrates the ways of reducing abstraction in this subcategory.

Example 10:

The following excerpt was taken from an Australian eighth-grade mathematics lesson. This lesson focuses on developing students' understanding of congruent triangles and their properties and begins with a brief review of congruent triangles, for which the teacher draws two congruent triangles on the board and assigns ten quick questions to the students. Consider the following excerpt from the lesson:

- 00:34 T: All right, this one is A B C, and this one is D E and F. All right, they are congruent. So I'm going to ask you- ask you 10 quick questions on that.
- 00:49 T: All right, number one, which angle is equal to angle A?
- 01:04 T: Which angle is equal to angle D?
- 01:15 T: Number three, if angle A is 30... and angle D is 60... uh, what size is angle E?
- 01:35 T: And number four, what size is angle D?
- 01:49 T: Which line is equal to B C?
- 01:56 S: I'm sorry?
- 01:57 T: Which line is equal to B C?
- 02:03 T: Number six, which line is equal to E F?
- 02:14 T: And... okay, if A C, if A B is five, all right, and B C is three, and A C is four centimeters, all right, how long is D F? ...

(TIMSS, 1999, AU2)

The teacher allowed students some time to work on the problems, and then asked them for their answers for each of the ten questions:

- 03:25T: Okay, who wants to answer them? Okay.
- 03:28S: Number one angle F.
- 03:30T: Yep.
- 03:31S: Number two angle C.
- 03:33T: Yes.
- 03:33S: Number three 60 degrees.

03:34T: Yes.

03:35S: Number four 90 degrees

Students continued answering...

04:33T: Who got them all right? That's pretty good. Who got nine right?

04:42 T: Yeah, right. Number 10 was A B.

04:45 S: Nine.

04:46 T: All right, okay, I've got a worksheet for you to do and you've got to work in groups of three or four.

(TIMSS, 1999, AU2)

Here, the focus of the teaching activity seems to move towards what the correct answer is, how many students were able to complete solving the problem, and how many got the answer right rather than how the students arrived at those results. Of note here is that this teaching activity was part of the review of what had been learned in the previous lesson. Asking few questions with a focus on the answer regardless of how the answer was derived by the students is a common review strategy found in all participating countries in the TIMSS 1999 video study (Hiebert et al., 2003).

Although such teaching behaviour is considered as one of the teaching strategies to assess students' understanding of the material already taught, getting the correct answer is not necessarily proof of students' understanding of the material (see Erlwanger, 1973). Regarding the kind and quality of students' understating (about congruent triangles), nothing can be said at this point because the data from the previous lesson was not available, and it was not clear from the above excerpts. However, considering the above conversation between the students and teacher, this teaching activity seemed to focus at achieving instrumental understanding with its emphasis on completeness and end product (answer) rather than conceptual understanding, thus reducing the level of abstraction from the perspective of this category.

5.4. Category 3 (RAiT-3): Abstraction Level As The Degree Of Complexity Of Mathematical Task /Concept

In this category, abstraction is interpreted by the degree of complexity of the task or concept. The working assumption here is that “the more compound an entity is, the more abstract it is” (Hazzan, 1999, p. 82). From this perspective, reducing abstraction refers to the teacher’s behaviour of reducing the complexity of the problem in various ways, including task implementation at a lower cognitive level. Prior research also identified some of the teachers’ tendencies associated with implementation of tasks at a lower cognitive level. Stein, Grover and Henningsen (1996), for example, found that some of the teachers’ behaviours of implementing tasks at a lower cognitive level were associated with the tendency of teachers stating the concept rather than developing it and telling how to solve or solving the problems for students,

I have also identified three task implementation approaches intended to reduce the complexity of the problem for their students. What follows is a discussion of each of them.

5.4.1. *RAiT-3a. Reducing abstraction by focusing on the particular rather than the general*

Reducing abstraction in this category refers to a situation where teaching activities focus on particular cases rather than general ones. This may also occur during concept development when teaching activities focus on some elements or subsets rather than the set as a whole, thus working with a less compound situation or object.

Sometimes this tendency of reducing abstraction in teaching may be pedagogically beneficial, as some concepts are difficult and, thus, more challenging for students if the concept is introduced as a whole. In such cases, developing the concept from a particular case, and with repetition and increasing abstraction, the larger concept can then be introduced more easily. However, in some cases, this strategy may not be pedagogically effective if not implemented appropriately because it may provide a partial picture of the concept rather than one of the complete whole, as Tall (1991) pointed out. Tall (1991) observes that sometimes a teacher attempts to:

[...] [s]implify a complex topic by breaking it into smaller parts which are logical from [a] mathematical point of view. The problem however is that student may see the pieces as they are presented, in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available. (p. 17)

As a result, students' construction of knowledge is fragmented and incomplete, as they usually do not succeed in bringing all the pieces together to form a coherent whole. Below, I present a representative task implementation behaviour that exhibited reducing abstraction in this subcategory.

Example 11:

This U.S. eighth-grade mathematics lesson focuses on the concept of the exponent and the laws of exponents. The lesson begins with an organisational segment, where the teacher handed out various materials, including the chapter assignment and a worksheet to use later in the lesson. The conversation continued as follows:

- 04:00 T: Remember on the...the...the composition of your exponent it always has a base number.
- 04:05T: And then the power or the exponent is the number that it is risen to. Okay.
- 04:09T: The exponent stands for the number of times that the base is going to be multiplied.
- 04:16T: So if we have two cubed, this tells me that the base is gonna be multiplied three times.
- 04:21T: Two, times two, times two.

(TIMSS 1999, US3)

The conversation then shifted to the concept of exponential growth and how to graph this. In particular, she showed what the graph of 2^x looks like, but by considering only positive integers for the values of x and finding corresponding y values.

- 06:02 T: We found that two to the one was two.
- 06:06 S: Two.
- 06:07 T: Two to the two was four.
- 06:08 T: Four.
- 06:09 T: Two to the three was?

06:10 S: Eight.
06:11 T: And two to the fourth was?
06:13 S: Sixteen.
06:15 T: What do you think that graph's gonna look like?
06:18 S: It- a curve.
06:20 T: Yeah. It's gonna be a curve. Think about that.
06:25 T: Two to the first power, we get just two. Right?
06:29 T: Then we go up to four. We go up to eight. We go up to sixteen. We get some significant growth. Okay.
06:38 T: Yes?
06:39 S: Become a parabola if you go to the negative side.
06:42 S: That's a good question. We're gonna explore that in this unit. Okay.
06:46 S: We're gonna start dealing with the negative exponents.

Even though the teacher pointed out that the class will be exploring the graph of an exponential function and that they were going to work on negative exponents, there is no discussion throughout the lesson as to what a negative exponent means. Neither was there any explanation of the meaning of zero or decimal exponent at this point. The focus of the conversation then shifted to the laws of exponents, with all the examples and justifications shown using positive integers. The students were then assigned seat work. The first occurrence of conflict they encountered with their new understanding of exponents surrounded the concept of zero exponents. The conversation continued as follows:

41:03 T: That's A to the zero equals one. A to the zero power equals one.
41:07 S: Zero- A to the zero, but then it would still be zero.
41:09 S: Wouldn't it be zero?
41:12 S: Unless you can-
41:13 S: Because anything times zero is zero. Right?

As this conversation shows, A to the zero exponents is not consistent with the definition of the exponent given by the teacher, so students are struggling to make sense of what it means to say A to the zero exponent:

- 43:39 T: Does A to the zero mean A times zero?
43:41 S: No.
43:42 S: No it doesn't. What does it mean?
43:43 S: A would have to be one then, right?
43:45 T: No. 'Cause I can tell you that a million to the zero power is one.
43:47 S: Oh. So A times- A times itself to-
43:50 T: Ten thousand to the zero power is one.
43:53 S: What?

Even though the teacher stated that any number (such as a million or a thousand or any other number) to the zero power is one and therefore A to the zero power is one, but students did not find this explanation helpful. The teacher then reminded the students of the definition of the exponent and provided the following explanation:

- 45:15 T: No. 'Cause what's two to the one power?
45:18 S: Two.
45:19 S: Two.
45:20 T: Okay. What's two to the second power?
45:22 S: Four.
45:23 T: Okay. And how did you do that?
45:26 S: I multiplied two by itself.
45:28 T: You multiplied it twice. Okay.
45:32 T: So here, this is not saying A times zero, this is say A to the zero power.
45:36 S: Told you.
45:37 T: A to the zero power.
45:39 S: That means you're not timesing it by anything so it stays A. So that would be one, right?
45:43 T: Well, not necessarily. What if A was equal to 1,000?
45:47 S: Then it'd be A to, I don't know.

The majority of the students still did not find this explanation, with its examples that a number to the zero exponent equals one, as logical or meaningful. The conversation continued:

- 46:39 S: Oh, 'cause you're just timesing it by nothing, right? So it would be one, right? Or, you just get it.

- 46:44 S: How do you get zero equals one.
- 46:46 T: So no matter what number you're gonna put in to the zero power, you're gonna get one.
- 46:50 T: But using those laws right there, if you start thinking about it...
- 46:56 T: Work out a problem so that the solution is A to the zero and the light will go off. It will.

(TIMSS 1999, US3)

Some of the issues related to the students' difficulty in understanding zero exponents has to do with the commonly understood definition of an exponent, as well as related teaching activities and strategies, particularly the kind of examples and justification used in such teaching activities. For example, in our case above, all the examples and justifications provided by the teacher for the rule of exponents involved positive integers, so when the students encountered a zero exponent, this did not fit very well with the commonly understood definition of an exponent. Other studies have also reported similar reasons for students' difficulties with understanding exponents. Lochhead (1991), for example, observed that in the case of the negative exponent, "it ignores the original definition in which the superscript was the number of times the base number was to be multiplied by itself, and there is no way to multiply a number a negative number of times" (p. 77). In fact, exponents such as 2^0 , 2^{-3} or $2^{\frac{1}{2}}$ make no sense from the view of the common definition, as it is meaningless to say that 2 is multiplied by itself zero times, or that 2 is multiplied by itself negative three times, or that 2 is multiplied by itself one-half times.

In this example, the concept of exponents was introduced with the common definition, which worked well with all positive integers and seemed to help students in some particular cases (exponent with positive integers). This teaching activity can be interpreted as reducing abstraction in this subcategory as the focus of the activity shifted to a subset or a particular case (exponent with positive integers) rather than a more complex object, such as a set of real (and complex) numbers. Careful observation of the use of the commonly understood definition in teaching, however, shed light on the possible difficulties and kinds of understanding students might develop in the learning process of grasping what exponent really means. In fact, reducing abstraction activity, in

this case, seemed to allow for the development of a partial picture of the concept of an exponent rather than a complete whole.

5.4.2. RAI-3b. Reducing abstraction by stating the concept rather than developing it

When concepts are involved in the lesson, they can either be developed or be stated. Reducing abstraction in this category refers to a situation where the complexity of the concept or problem is reduced by stating the concept rather than developing it. For example, while presenting a lesson and working on the problems in an arithmetic series, the teacher may either develop the idea or derive the formula by induction (or otherwise) that the sum of up to n^{th} term of an arithmetic series can be computed by $S_n = \frac{n}{2}\{2a + (n - 1)d\}$. Alternately, the teacher can simply remind students, without any explanation, that the formula to find the sum of n terms is $S_n = \frac{n}{2}\{2a + (n - 1)d\}$ and that they are required to memorize the formula. For various reasons, teachers sometimes tend to opt for the second type of implementation behaviour. As a result, the complexity involved in solving the task or in developing the concept is reduced or removed by transforming the task into a trivial memorization and 'plug in number' type problem. The following example illustrates this type of reducing abstraction behaviour.

Example 12:

Consider the following conversation from an eighth-grade U.S. mathematics lesson, the focus of which was on exponents and computation with exponents. This is the first in the series of lessons on exponents, and students were not allowed to use calculators.

27:59T: Okay. And what about number nine? Ryan, you have that?
A B-

28:05S: It's A to the fourth and B to the fourth.

28:07T: Okay. Did someone find the rule?

28:09S: We did.

28:10T: Blake?

28:12S: It's A to m times B to m

28:15T: Okay. So when we have our base in here being raised to a power, each individual term is raised to that power. So we get-

28:33 A to the m, B to the m. Okay.

28:39 T: And remember when we have two variables next to each other, they're being multiplied.

28:45 T: Okay. So there's the third rule of exponents for multiplication that you need to know. Okay.

Note that the teacher refers to the rules of exponent for multiplication in this order: Rule 1. $a^m \cdot a^n = a^{m+n}$ Rule 2. $(a^m)^n = a^{m \cdot n}$ Rule 3. $(a \cdot b)^m$

28:52 T: When we have the same base being raised to a power, we just add the powers.

28:58 T: When we have a base raised to a power, raised to a power, we multiply the exponents.

29:04 T: When we have two terms raised to a power within parentheses, we raise each term to that power. Okay.

(TIMSS, 1999, US3)

As discussed above, providing justification and clarification of the concept is, in general, a more complex phenomenon (and time consuming) than simply stating the concept. Here the teacher proceeded by stating the rules, such as if “we have the same base being raised to a power, we just add the powers”, and “if we have a base raised to a power, raised to a power, we multiply the exponents” and so on. Note that in some cases, rules appear a bit complicated to be developed (e.g. quadratic formula) and, therefore, formulas are given. It is not the case here that the rules development is not beyond students' level of mathematics. Therefore, such task implementation behaviour can be interpreted as reducing abstraction in this category, as stating the concept is a less complex phenomenon than developing the concept.

Of note here is that the previous example (Example 11) also can be seen in this category where teacher states that A to the zero power is one (i.e., $A^0 = 1$) without further discussion- an example of overlap of categories.

5.4.3. RAIT-3c. Reducing abstraction by giving away the answer in the question or providing more hints than necessary

Reducing abstraction in this subcategory is similar to what Brousseau (1997) calls the 'Topaze effect'. The name of the 'Topaze effect' comes from a play by Marcel Pagnol, written in Paris in 1928. In the play, Topaze is a school teacher. When a student cannot easily find the answer, the teacher gives away the answer within the question itself in a slightly indirect way, thereby lowering the intellectual difficulty of the tasks (see Brousseau, 1997). This kind of task implementation behaviour can be interpreted as reducing abstraction in teaching in this subcategory, as such teaching activity makes a complex problem less complex for students. The following example illustrates reducing abstraction in this way.

Example 13:

This Hong Kong eighth-grade mathematics lesson focuses on the concept of the polygon and deriving the sum of the interior angles of a polygon. The lesson begins with a review of convex and concave polygons, and then in a later part of the lesson, new material (regarding the sum of the interior angles in polygons) is introduced and practiced. Here is the transcript of the classroom conversation about equilateral and equiangular polygons:

- 07:15 T: Sorry. It is uh, uh, you have learned it in Form One. It is the... equilateral polygon and also... equiangular polygon.
- 07:32 T: What are they? What is meant by equiangular polygon? And what is meant by equilateral polygon?
- 07:32 T: What is meant by equiangular polygon? And what is meant by equilateral polygon?
- 07:50 T: That means- okay, I- I- I- I- I- give you some time to think. What does it mean by equilat- equilateral tri- polygon?
- 07:58 T: Equiangular polygon?
- 08:00 T: Do you still remember it?
- 08:02 S: Yes.
- 08:03 T: Yes. What does it mean? For- for- for the first type. I'll give you some hints. All the sides are...?
- 08:10 S: Are equal.

08:10 T: All the sides are equal. Okay? All the sides of the polygons are equal.

08:16 T: How about equiangular polygon?

08:17 S: (inaudible)

08:18 T: All the sizes of the angle?

08:21 S: Are equal.

08:21 T Equal. Get it?

(TIMSS, 1999, HK3)

Here the teacher seemed to open up a dialogue on the concept of concave and convex polygons, as well as equilateral and equiangular polygons. However, he negotiated the challenging aspects of the problem in two ways: first, by telling them the meaning in a slightly disguised form (8:03) rather than allowing students to discover the meaning by themselves. For example, at 8:03, he asked the question, “what is meant by equilateral polygon?” giving a hint by saying “all the sides are...?” A closer look at this hint reveals that the answer is already there in the hint. The only word suitable to fill the blank in this case is the word “equal”. If the blank is filled by the word unequal (i.e. all the sides are unequal) it does not make sense. It is logically, grammatically and structurally incorrect to say all the sides are unequal. Similar here is the case with equiangular polygons. The teacher asked what it is meant by ‘equiangular polygon’, but he gave away the answer in a slightly disguised form within the hint when he said “all angles are...” (at 8:18) because, as in the previous case, the only suitable word to fill the blank is the word “equal”.

This act of a teacher’s task implementation reduced the complexity of the problem for the students, which can be interpreted as reducing abstraction in this category. This tendency of reducing abstraction, however, may not be pedagogically beneficial, as it denies students the opportunity to progress on their own.

5.5. Summary

In this chapter, I described various approaches teachers use in dealing with abstraction in their teaching practices as captured in the TIMSS 1999 Public Release

video lessons. I also included a brief discussion about how the issue of such teaching activities are dealt within the literature. The chief goal of teaching activities is to promote students' learning, and one way of achieving this goal involves making new (abstract) concept much more accessible to the students. Towards that end, teachers exhibited various approaches of dealing with abstraction in ways that reduced the abstraction of the concepts or tasks while presenting to students. These approaches seemed to belong to three broad categories and their various subcategories as illustrated above. As a result, Reducing Abstraction in Teaching (RAiT) has taken its shape as a theoretical framework (see Figure 11) that examines how teachers deal with mathematical abstraction while implementing tasks.

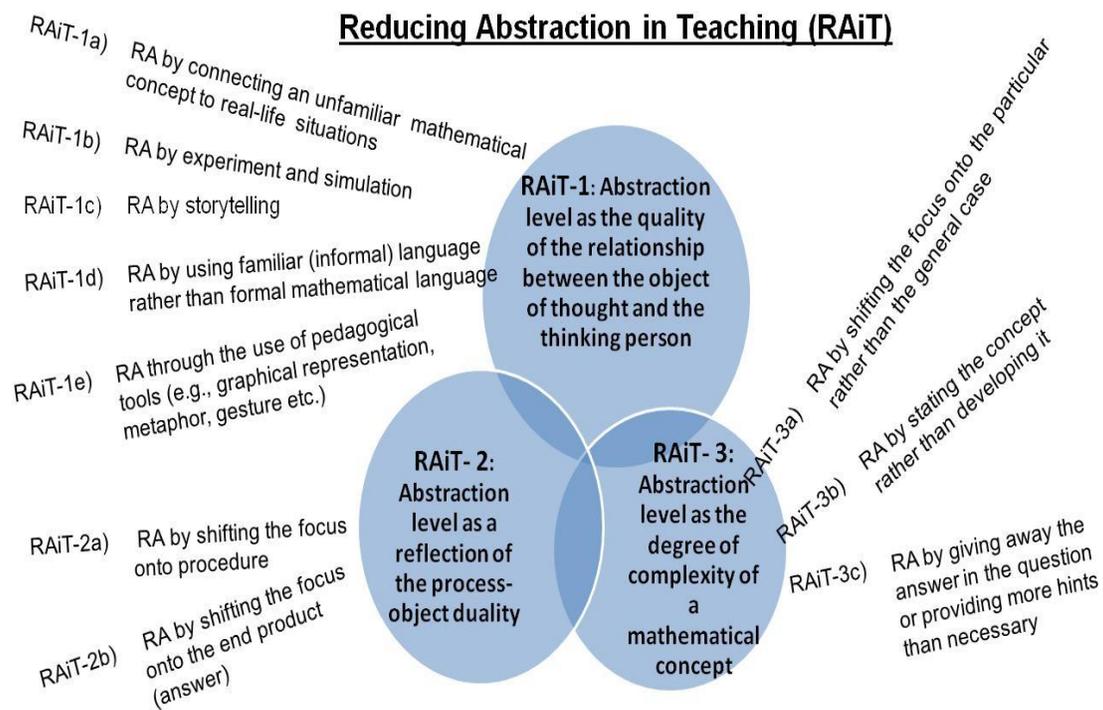


Figure 11. Reducing Abstraction in Teaching (RAiT) Framework

Of note is that these three categories and various subcategories of reducing abstraction in teaching are not an exhaustive list. Neither these categories or their subcategories are disjointed, but rather, they intersect. Depending on the perspective one takes, reducing abstraction behaviour in one category/subcategory can be interpreted as reducing abstraction in another category/subcategory. For example, a

teacher trying to implement task making it less complex for students either by focusing on a particular case (third category) or focusing on the process (second category) can be thought of as him trying to make the concept more familiar (first category) for his or her students. However, I put them in the category/subcategory that I deem they fit best, acknowledging that there are other possible interpretations of reducing abstraction behaviours and that the list of categories/subcategories may grow longer. Further, as stated earlier, this study aimed to explore 'how' and 'what' aspects of teachers' task implementation behaviour, rather than 'how many times' teachers use any particular approach. As such, the quantitative data regarding the frequency of different approaches of reducing abstraction is out of the scope of this study.

In the next chapter (Chapter 6), using the RAIT framework developed in this chapter, I look into some teaching episodes that came from the part of my primary data collection.

6. Teaching Episodes: My Classroom Observation

In chapter 5, I provided detailed description of how the Reducing Abstraction in Teaching (RAiT) framework emerged in the process of my analysis of the TIMSS 1999 public release video lessons. In particular, I described, characterized and contrasted teachers' task implementation behaviour in terms of how mathematical abstraction is dealt with in the classroom. In so doing, various approaches of teachers' task implementation behaviours were identified that seemed to fit in one of the three categories of level of abstraction as described in Chapter 4. As a result Reducing Abstraction in Teaching (RAiT) framework emerged with three categories and various subcategories. In this chapter, I now turn to examining additional teaching episodes, this time observed through primary data collection, through the RAiT framework and describe how these exemplify some instances of Reducing Abstraction in Teaching. Further, I also present teachers' justification and clarification of some of their task implementation behaviours through the informal interviews I conducted with these teachers.

As mentioned earlier, these particular teaching episodes come from my observation of the natural classroom settings of a university preparatory mathematics course taught at local universities and colleges. I observed a total of nine classes (three each) taught by three different teachers, who were all experienced in teaching this course, as well as being professionally trained mathematics instructors. I would like to restate that, my motivation for this part of data (primary data) is related to the fact that it provided me with an opportunity to ask teachers to explain/ justify their choices and approaches of dealing with abstraction in teaching, an opportunity that was not available from the first part of data (secondary data).

As stated earlier in chapter 4, I collected the data as a non-participant observer. A non-participant observation is that with no interaction or very limited interaction with

the people one observes. I recorded audio of the classroom interaction but avoided video recording to minimize the risk of influencing the natural classroom situation. In addition, as much as possible, I noted down the phrases, statements or sentences the instructors used to explain a concept, including some observable behaviours such as gesture, as well as students' responses that I found relevant for the study. Below, I present three examples (one from each instructor) and analyse through the RAiT framework. The reason behind the choice of these three examples is that I found them illustrating teachers' approaches of reducing abstraction more clearly than the other examples.

6.1. Teaching Episodes: RAiT Implemented

6.1.1. *The Fraction Problem*

This example comes from a lesson at a university preparation course, the focus of which is on fraction and word problems related to fractions. After discussing some basic concepts of fractions, the instructor posed the following problem to the class.

John spent a quarter of his life as a boy growing up, one sixth of his life in college, and one half of his life as a teacher. He spent his last six years in retirement. How old was he when he died?

Discussing the problem and considering the total years of his life as 1, the teacher wrote the different stages of John's ages in fractions as $\frac{1}{4} + \frac{1}{6} + \frac{1}{2} = \frac{3+2+6}{12} = \frac{11}{12}$. Then, she subtracted the sum of the fractions from 1 in order to get the fraction of John's life spent in retirement (*i. e.* $1 - \frac{11}{12} = \frac{1}{12}$). Finally, she solved the problem by equating the fraction with the given amount of years (in this case, six). The teacher wrote: $\frac{1}{12}$ of life = 6 years; $\frac{12}{12}$ of life = $12 \times 6 = 72$ years is the total age. This is how the problem was solved and the answer obtained.

Some of the students expressed their frustration with this method, as they could not make sense of what was going on in the solution process. Some of the immediate reactions of the students to the way their teacher solved or implemented the problem

were as follows: Pointing to $1 - \frac{11}{12} = \frac{1}{12}$, “where does that 1 come from? Why did you subtract $\frac{11}{12}$ from 1?”, “Why did you suppose his total age equals 1... it does not make sense to me!” Such student reactions clearly show that some of the students struggled to understand the mathematics behind the task as it was implemented.

An alternative approach:

As an alternative approach, the teacher explained the problem to the students as follows: First, they were told that since one quarter, one-sixth and one half are the fractions used in the problem, the best number to choose to set up the number line is 12 (for the sake of convenience), because it is the lowest common multiple of the denominator of those fractions. Then they were shown how to do this through the use of an appropriate diagramming technique (see Figure 12).

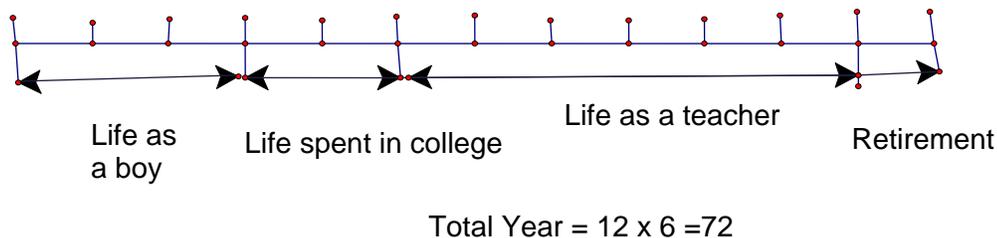


Figure 12. Fraction Problem

The idea here was to allow the students to see the relationship between the fractions of John’s age in concrete terms through the use of a diagram including a “time as a number in a number line” (source domain) metaphor (Lakoff and Núñez, 2000). They were then able to understand much more concretely that representing the part of John’s life as a boy (which is a quarter of his life $=\frac{1}{4}$ of 12) suggests moving John’s age-point “to the right by three” on the time line. Similarly, life spent in college (which is one-sixth of his life $=\frac{1}{6}$ of 12) suggests moving the age-point “two units to the right” and so on. Finally, one partition out of twelve partitions in the time line was left for retirement. Since the retirement period is given as six years, it could easily be seen that each partition in the time-line represents six years (time is a quantity metaphor). Hence students could

see that John was actually $6 \times 12 = 72$ years old when he died. Following is the conversation that followed after the problem was solved by this method:

T: As you can see from the diagram, how old John is?

S: 72.

T: Do you have any question?

S: This method makes a whole lot more sense than the other one.

(Many students nodded their heads in agreement.)

One of the reasons to which students' difficulties can be attributed in the first case is that manipulation of the fractions without any concrete referent might be too abstract for some of the students. In other words, while implementing the task, the teacher started from the idea of fractions inherent in the problem and stayed at the same level of abstraction. Noticeable reduction in abstraction did not occur for the students even though she solved the problem and obtained the answer. But in the second approach, the use of pedagogical tools such as graphical representation, familiar objects and contexts such as a number line and John's age, and the metaphor such as "time as a number in a number line" provided students something to hang onto and allowed them to work in their familiar context and situation.

The benefit of the use of pedagogical tools in teaching has long been discussed in the literature (Lakoff & Núñez, 2000; Sfard, 1991; Edwards, Radford & Arzarello, 2009). As stated earlier (Chapter 5), the reason behind the benefit of pedagogical tools is that these tools act as a bridge between the abstract mathematical concepts (target domain) and the familiar and concrete ideas (source domain).

In the case discussed here, the use of pedagogical tools in the teaching activity reduced the abstraction level of the concepts for students in a number of ways. First, graphical representation (diagram) helped students to see the concept of the fraction and its operation concretely. Second, the incorporation of metaphor, such as "time as a number in a number line", helped students to relate the concept of fractions and time to their familiar objects and situations, such as numbers and number lines. Third, the whole (or 1) is represented by number 12 (12 pieces), and as a result, the need to deal with the concept and operation of fractions is transformed to the concept of a whole number and

its operation, which is more familiar. The student's comment that "this method makes a whole lot more sense than the previous methods" also supports the fact that the second approach to task implementation behaviour reduced the abstraction level of the concept for the students. This example illustrates the teaching activity of reducing abstraction in subcategory RAiT-1e.

6.1.2. Graphing Linear Equation Problem

This example comes from one of the college/ university preparatory mathematics lessons that I observed as a non participatory observer. The focus of the lesson was graphing linear equations in two variables. The teacher posed the following question:

1. Consider the following equations:

$$2x + 4y = 16; 4x - 3y = 6 \text{ and } 3x + y = -2.$$

Graph the three lines and label them. Do they form a triangle?

After putting the question up in the overhead projector, the teacher let the students copy the question during which she walked around the classroom. After about one minute the teaching began:

(Note: T= teacher, SS= more than one student, S= an individual student)

- T: How can you graph these equations? (Paused about four seconds) Let me show you how. I choose the first equation (pointing to $2x + 4y = 16$) first and show you how to graph it, ok...? Use the cover method. I cover $4y$ (she covers $4y$ with her hand and completely hide it from the scene). Now tell me what is the value of x ?
- SS: Eight (group response)
- T: So, We have one point (8, 0).
- T: Now if I cover $2x$. (She covers $2x$ with her hand). What is the value of y ?
- SS: Four (group response).
- T: So, the other point is (0, 2). Now we have two points (8, 0) and (0, 2). Let me plot these points on the graph and draw the straight line.
- S: Oh, I see. That's easy!

Understanding the relationship between graphical representation and algebraic representation of a linear function is one of the most important concepts in the university preparatory mathematics course. Usually, these concepts are introduced in the textbooks by three methods:

1) Slope- Intercept method: First, equation is transformed in to slope intercept form $y = mx + b$ where ' m ' is the slope and ' b ' is the y-intercept. Then using the slope and y-intercept, the graph of the function is drawn.

2) T- table Method: Making a T-table for ' x ' and ' y ', and randomly plugging in few values (usually 3-5 values) for independent variable x and calculating corresponding values for dependent variable y . Then the coordinates are plotted in the xy - plane and the graph of the function is drawn.

3) Intercepts Method: By finding x - intercept and y - intercept in which case students should know that on the x -axis, y -coordinate is zero and vice versa.

The 'cover up' method as employed by the teacher was not fundamentally different from the method (3) above. On the x - intercept, y - coordinate is zero. Hiding $4y$ with her hand (gesture) while finding the value of x , the teacher was using "Zero is the lack of an object" metaphor (Lakoff & Núñez, 2000, p. 372). Her gesture and the use of metaphor significantly reduce the level of abstraction of the concept for the student. This is an attempt from the teacher's part to make the unfamiliar 'intercepts' concept more familiar with the use of gesture and "zero is the lack of an object metaphor". From this perspective, this task implementation act can be interpreted as reducing abstraction in category RAiT- 1e (use of pedagogical tools).

Viewed from the other perspective, it can be put in the process-object duality because the cover up method emphasize the process conception how to do it but not what it means. This can be interpreted as reducing abstraction in RAiT- 2a (focus on procedure).

It should be noted however that the teacher's intention might be to use this method to make the process easier while keeping the concept meaningful to the student.

However, many students' responses in the second question revealed that they did not understand the underlying concept.

- T: To draw the line for the second equation (points to the second equation which is $4x - 3y = 6$) we need to find any two points, yeah! Let's find them. (After an instance of mental calculation, the teacher writes (0, -2) and (3, 2) as two points).
- S: How did you get (3, 2)? It has to be (1.5, 0). That's what I got.
- SS: I also got (1.5, 0).

At this moment, there was confusion and bewilderment among most of the students as to how the teacher arrived with points (0, -2) and (3, 2). It was evident that the student could find the correct points on the line mechanically but with no meaningful understanding (RAiT- 2a). They doubted whether (3, 2) could be a point on the line. Later, by plugging in the values that yielded whole numbers rather than fractions for the points to be plotted in the graph, the instructor successfully helped student understand that there were, in fact, many (infinitely many) points in a line but for the sake of simplicity, a 'nice' number (non-fractional number) was chosen.

One of the interesting questions to ask here would be why the teacher shifted the focus on process conception at the beginning? Informal interview with the teacher after the class revealed the fact that this strategy of reducing abstraction is associated with her learning experience as a student herself and her previous teaching experience. She said:

I kind of liked this method even when I was a student. This method was so easy and worked well for me. Later, I used this strategies in my teaching and I found that my student loved this strategy. With this method, even the weakest student in my class is able to find x and y intercepts without much difficulty. But, you know, this strategy also has its limitation. As you saw in the class, given a linear equation, most of the students knew how to find the two points of a line mechanically without understanding the principle behind it. But I think it's OK to know 'how' without knowing 'what' and 'why' initially. This gives them some kind of self confidence to do math. Once the students are into it, you can always extend their understanding to 'what' and 'why' aspect of concept.

(Beth [pseudonym], personal communication, November 21, 2011)

Notice that the use of the 'covering up' strategies not only associated with her own experience as a students and later as a teacher, it is also associated with her belief that process conception is easy to understand for students and the benefit of it is that it promotes motivation and self confidence of students to do mathematics. As in our example above, all the students were able to find the x and y intercepts of the equation as two points in the line and draw the line in the graph. But it was not clear whether they understood the principle behind it until some of the students raised the question for the second equation that their answer (1.5, 0) did not match with the teacher's answer (3, 2). As evident in the data above, this situation created some confusion and obstacle for the students. As the teacher stated, this strategy of uplifting students' thinking from lower to higher level is also a planed and intended act which is related to her previous experience:

I think there is nothing wrong with the difficulty they encountered when their answer did not match with mine. I think it is even helpful for them to understand the underlying principle behind it. In my own experience as a student, when I find something counter-intuitive along my learning process, something that do not match with my intuition, I had to struggle to learn it at first but once I grasp the concept, such experience used to be very rewarding. I could retain that information for a longer period of time.

(Beth [pseudonym], personal communication, November 21, 2011)

This idea is in line with many other researchers and educators (Vygostky, 1986; Piaget, 1965). Vygotsky for example, maintains that it is beneficial to establish obstacles and difficulties in teaching, at the same time providing students with ways and means for the solution of the tasks posed. Rubinshtein (1989) also asserts that "the thinking usually starts from a problem or question, from surprise or bewilderment, from a contradiction" (cited in Safuanov, 2004, p. 4). It is similar to Piaget's phenomenon of equilibration in the process of cognitive development which usually occurs from the violation of balance between assimilation and accommodation.

6.1.3. *The Box Problem*

This was the first lesson on the concept of Polynomial functions and their graphs. In the earlier lesson I observed, the concept of quadratic functions and the method of

sketching them were introduced. The objectives of the lesson were to introduce the concept of polynomial function of degree three or greater and to sketch their graphs using the leading coefficient, x -intercepts and the multiplicity of the zeros. The class began with a very brief review of quadratic functions. The teacher then put up the following problem on the board and read the question out loud for the students. She instructed them to discuss the problem in groups of three or four and to try to find the equation to model the volume of the box.

Question: *Suppose that you are a manager of a packaging company that manufactures identical rectangular boxes from square sheets of cardboard, each sheet having the dimension 8 inches by 8 inches. To save money, you want to manufacture boxes of maximum volume by cutting out a square of x inches by x inches from each corner of a sheet and then folding their sides up to make an open topped box. What length should you select for x in order for the maximum possible volume of the box?*

All of the students discussed the problem with their neighbours. Each group seemed to agree on the fact that the height of the box would be x . A source of debate, however, was the dimension of the base of the box, particularly whether each dimension of the box's base would be $8 - x$ or $8 - 2x$. After about two minutes or so, while the students were still working on the problem, the teacher handed out cardboard, a pair of scissors and masking tape to each group. Each group cut the square corner and folded the side to make a box. At this point everyone seemed to be convinced that each dimension of the base of the box was actually $8 - 2x$.

This problem was challenging for students at this level (university and college preparatory mathematics courses). However, when provided with concrete objects, they were able to figure out each of the dimension of the box in terms of x and, consequently, the function representing the volume of the box by using their familiar formula: volume = length \times width \times height. In other words, the teacher's act of allowing her students to use concrete objects seemed to reduce the abstraction level of the problem for them, thus allowing them to find the function for the volume of the box once they could see the bases as well as height of the box visually.

The teacher then put the following figure (see Figure 13) on the board:

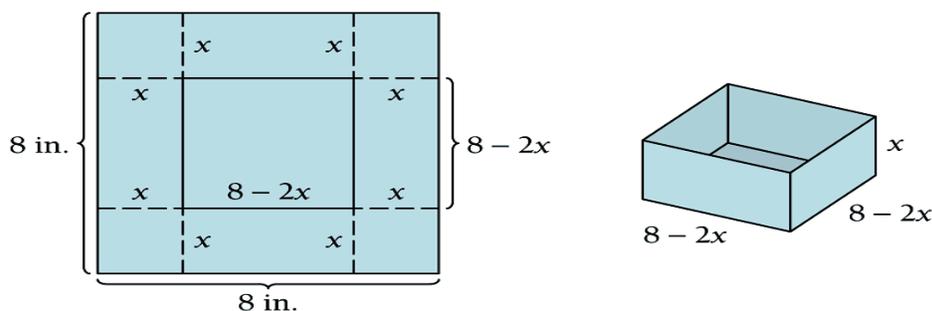


Figure 13. Volume of open top box

The class as a whole found the volume of the open top box as:

Volume (V) = $(8 - 2x)(8 - 2x)x$, which yielded,

$$V(x) = 4x^3 - 32x^2 + 64x$$

The teacher mentioned that this was a third degree polynomial function and wrote the definition of the general form of polynomial functions as follows:

Let n be a nonnegative integer and let $a_n, a_{n-1}, a_{n-2} \dots a_2, a_1, a_0$ be real number with $a_n \neq 0$. The function defined by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

is called a Polynomial function of x of degree n .

At this point, there was a realization among the students that polynomial functions are not an abstract entity but something that relates to real world situations. Further, using dynamic geometry software, the teacher showed the graph of the equation. She then told the students, using gestures to indicate that the shape of Polynomial functions of this type (referring to the graph of $V(x)$) always become like this  (gesture- left hand down and right hand up).

After a brief discussion about the (theoretical as well as practical) domain for this particular problem, the teacher asked the students to refer to the graph and to predict the value of x that gives maximum volume to the box. All students were able to find the value of x that maximizes the volume, the dimensions of the box and the maximum

volume. This activity (particularly the real world connection (RAiT-1a), the pedagogical tools such as gesture and the use of concrete objects/materials (RAiT-1d) in implementing the box problem seemed to reduce the level of abstraction of the task, as students could see and locate the value of x and the volume in the graph, which was otherwise difficult and abstract for them to understand. A detailed discussion of these types of task implementation behaviour was given in Chapter 5.

The teacher then shifted the focus of the lesson to sketching the graph using the leading term, x -intercept and multiplicity of the zeros of a function. Considering the function $V(x) = 4x^3 - 32x^2 + 64x$, she told students that the term with highest degree is called the leading term and, hence in this example, the leading term is $4x^3$ and the leading coefficient is 4. She also explained the meaning of zeros of a function, which are in fact x -intercepts of the graph of a function. In this example, the zeros of $V(x)$ are $x = 0$ and $x = 4$, as can be seen in the graph visually. The teacher then informed the students that a discussion about the multiplicity of the zeros would follow in the latter part of the class. At this point, the discussion primarily focused on the relationship between the leading term and the *end behaviour* of the graph.

She described the *end behaviour* of the graph as the nature of the graph (if it goes up or down) of a function to the far right as $x \rightarrow \infty$ and to the far left as $x \rightarrow -\infty$. She also told the class that the *end behaviour* of any polynomial function is the same as that of the leading term, that is, the term with highest degree. The dialogue continued as follows:

- T: OK guys, as you have seen in the last example (pointing to the function $V(x) = 4x^3 - 32x^2 + 64x$ that the graph of third degree polynomial is like this  (gesture- with left hand down and right hand up)
- I will tell you one thing right now. To find the end behaviour of the graph, what you need is just the leading term, ok, the term with highest degree. In this function (pointing to $(V(x) = 4x^3 - 32x^2 + 64x)$ what we need to look at is just the leading term- four x cube ($4x^3$). We don't care about the other terms. Ok.
- S: Ms. I don't get it. Why you don't care the other term?
- T: I mean, when x goes to positive or negative infinity (writes in the board, $x \rightarrow \infty$ or as $x \rightarrow -\infty$) the function $V(x) = 4x^3 -$

$32x^2 + 64x$ and the function $f(x) = 4x^3$ behaves the same way. There is a theorem that tells that (she writes in the board) if $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ is an n degree polynomial function of x , then the "end behavior" of $P(x)$ is the same as that of $y = a_nx^n$. Ok.

T: Now I will tell you how to figure out the shape or the end behaviour of different type of polynomial functions. What you have to look is the degree of the leading term and the sign of the leading coefficient. OK. If the degree is odd, then think of "odd" as meaning "different". OK. That means the end behaviours on the left and the right must be different from each other. OK. Either this  (left hand down and right hand up) or this  (Left hand up and right hand down)

T: And if the degree is even, think of "even" as meaning "the same". For example, if two players are "even", in a game, what would you think? They have the same score, right. This will help you to remember that if n is even, both ends of the graph must go in the same direction like this  (both hands up) or this  (both hands down).

T: But how to find which one is which? It is also easy. First let's do the odd degree polynomial, even degree later, ok. To find the direction, just look at the sign of leading coefficient. If it is positive, the right end goes up and the left end goes down like this  (left hand down and right hand up) and if it is negative the right end of the graph goes down and left end goes up like this  (left hand up and right hand down)

The teacher then wrote two functions with an odd degree on the board and asked the students to figure out the end behaviour of the graph:

$f(x) = -2x^3 + 2x - 3$ and $g(x) = 3x^5 - 4x^2 + 6$. The class seemed to be really engaged at this point, as everyone was using his or her hands to model the end behaviour of the graph. The teacher moved closer to one group:

T: Ben, can you tell me what the end behaviour of the first function is?

Ben: It's like this  (left hand down and right hand up). Oh, wait a minute; it's like this  (left hand up and right hand down). Umh, I forgot. I really don't get it.

When she realized that students were still struggling to remember the end behaviour, she used another strategy to explain it:

T: Alright guys, think this way. Your leading term is the leader or manager of a company, OK. If the leader of a company is positive or say has a positive attitude, the business finally goes up even though it was down earlier, right. And if the leader is negative or say has a negative attitude, the business goes down even though it was up earlier, right.

S: Oh, I see!

T: Alright. Ben, do you want to try the second function?

Ben: Ok. So the leading term is $-3x^5$. So, 5 is odd and 3 is negative. So the guy (the guy here refers to the leader or manager) has negative attitude. So, the business finally goes down. The graph is like this  (left hand up and right hand down), right?

T: Yes Ben, you are absolutely right!

Ben: Oh, I get it!

T: Remember that the positive and negative leader metaphor works for even degree of function as well. OK. If the leading coefficient is positive, umm, both ends go up like this (raised her both hands up) and if the leading coefficient is negative, both ends go down like this (raised down her both hands). Ok, now let's try few more functions.

The teacher wrote a few more functions with odd degrees and even degrees, and with both positive and negative leading coefficients. At this point, every student seemed to be able to identify the end behaviour of the functions correctly. Then the teacher moved to the concept of multiplicity.

T: Alright guys, now let's move to the other concepts that we need to know to sketch the graph. They are zeros and the multiplicity. Ok. The zeros mean the value of x when we equate the function to 0. And "Multiplicity" means the number of times a factor appears in a polynomial function. Multiplicity tells you um... if the graph crosses the x -axis at a zero or turns around at the zero. If r is a zero of even multiplicity, then the graph touches the x -axis and

turns around at r . alright! If r is a zero of odd multiplicity, then the graph crosses the x -axis at r . Ok. You must memorize this ok.

T: Let's see one example. Consider a function $g(x) = x^4 + 2x^3 - 4x^2 - 8x$. To find the zeros and multiplicity, your first step is to factorize the equation by equating it to zero like this $x^4 + 2x^3 - 4x^2 - 8x = 0$, Ok. You know how to factorize it, right?

The teacher then factorized the equation on the whiteboard, which yielded $x(x+2)^2(x-2) = 0$. Upon solving the equation, it yielded three zeros: $x = 0$, $x = 2$ and $x = -2$. The teacher then described the relationship between the multiplicity of the zeros and the nature of the graph, i.e., whether the graph crosses the x -axis or touches and turns around (tangent) at each zeros. She went on to say:

T: Now we have three zeros, right. Let's see what is the multiplicity of each of zeros. For this, just look at exponent or power of each zero. Here x has power or exponent 1, $x+2$ has power 2 and $x-2$ has also power 1. Now remember the rule: if the multiplicity or power of zeros is even, the graph touches the x axis and turns around, and if the multiplicity is odd, then the graph crosses the x axis at that zero. Ok. Just remember this rule. Ok.

No further explanation was provided as to why the multiplicity of the zeros acts like that. And surprisingly, no one in the class raised the question regarding this rule.

This activity offered a wealth of opportunities to connect the pattern found in the table, graphs and polynomial functions, and helped the students see that real life situations can be modeled by polynomial functions. The domain is one of the important concepts that any student is expected to understand in this level. Task implementation for the concept of polynomial through the box problem provided an opportunity to discuss the idea of practical domain and theoretical domain. For example, since we are dealing with the length and width of cardboard (real life situation), the value of x cannot be negative, and since each dimension of the cardboard is 8, the value of x cannot exceed 4. Therefore, the practical domain for this function is $(0, 4)$, whereas the theoretical domain is $(-\infty, \infty)$. Here, connection of the task to real life situation helped students understand the abstract idea of theoretical and practical domain of a function and the distinction between them (RAiT- 1a; real life connection).

Further, the teacher's use of the leader of a company as a leading term of a polynomial metaphor as well as her use of gesture reduced the abstraction of the unfamiliar and abstract concept of end behaviour of polynomial functions, thereby making it a familiar idea via descriptive hand motions (RAiT-1e; pedagogical tools - metaphor and gesture).

One point to note, however, is that in response to the student's query regarding the end behaviour of polynomials and the end behaviour of the leading term, the teacher did not develop the concept but rather stated it by saying if $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ is an n degree polynomial function of x , then the "end behaviour" of $P(x)$ is the same as that of $y = a_n x^n$ (RAiT-3b). As a result, some of the students developed a rather vague and inappropriate rule, as I observed in the work of one of the students who was sitting next to me. In response to my query about his work with regards to the function $g(x) = 3x^5 - 4x^2 + 6$, he mentioned his approach using the following rule: as $x \rightarrow \infty$, $g(\infty) = 3\infty^5 - 4\infty^2 + 6$ becomes positive infinity because any positive quantity to the power odd becomes positive and the first term has a bigger infinitive. Therefore $x \rightarrow \infty, g(x)$ equals positive infinity. A similar reason was given for $g(x)$ as $x \rightarrow -\infty$, and he concluded that $x \rightarrow -\infty, g(x)$ equals negative infinity, which is not correct.

Similar to what Hazzan (1999) reported, the student seemed to reduce the abstraction level here by using his familiar ways of finding functional values when x tends to some finite value. In fact, if x tends to some finite value, say, $x \rightarrow 2$, the functional value of $g(2)$ can be found by plugging in 2 for x in the function, such as $g(2) = 3(2)^5 - 4(2)^2 + 6 = 86$. But for $x \rightarrow \infty$, we know that $3x^5 \rightarrow \infty$ and $4x^2 \rightarrow \infty$. However, we cannot simply add these two limits together to find the limit of $g(x)$, since the limit of the form $(\infty - \infty)$ is indeterminate.

This misconception could have been avoided if the concept was developed instead of stated. As discussed earlier (section 5.4.3), given the mathematics level of students, rules development can be complicated in some situation and therefore formulas are given. In this case however, rules development is not beyond students' level of mathematics. For example, the Polynomial function $P(x) = a_n x^n + a_{n-1} x^{n-1} +$

$\dots + a_2x^2 + a_1x + a_0$ could easily be shown to equal a_nx^n for a very large $|x|$ by using simple algebra as follows:

$P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$ can be rewritten as

$$P(x) = a_nx^n \left(1 + \frac{a_{n-1}}{a_n} \frac{1}{x} + \frac{a_{n-2}}{a_n} \frac{1}{x^2} + \dots + \frac{a_0}{a_n} \frac{1}{x^n} \right) \quad [\text{by taking } a_nx^n \text{ common}]$$

For a very large x , $P(x) = a_nx^n \left(1 + \frac{a_{n-1}}{a_n} \frac{1}{x} + \frac{a_{n-2}}{a_n} \frac{1}{x^2} + \dots + \frac{a_0}{a_n} \frac{1}{x^n} \right) = a_nx^n$, because all the terms inside the brackets except 1 become zero for $\frac{1}{x^k} \rightarrow 0$ when $x \rightarrow \infty$. Similar results hold for $x \rightarrow -\infty$. Hence for $|x|$ large, $P(x) \approx a_nx^n$

Using limit, this can be written as:

$$\lim_{x \rightarrow \infty} P(x) = \lim_{x \rightarrow \infty} a_nx^n \left(1 + \frac{a_{n-1}}{a_n} \frac{1}{x} + \frac{a_{n-2}}{a_n} \frac{1}{x^2} + \dots + \frac{a_0}{a_n} \frac{1}{x^n} \right) = \lim_{x \rightarrow \infty} a_nx^n$$

To understand the reason for stating the concept rather than developing it, I decided to talk to the teacher after the lesson. This informal interview shed some light on this issue. In response to my curiosity about her approach to task implementation with regard to stating the concept rather than developing it, she replied that the time allotted for the course was too short to allow her to work on the concept in detail. Further, the concept was a complex or difficult one for students at this level. She also pointed out the fact that students would not be tested on the proof behind the theory, saying:

You know, the time allotted for this course is too short. I have to cover so many things and I always run out of time in this course and so I decided not to go in detail of the concept. Moreover, the theorem that you are referring, the proof involves the limit concept and they do not have that concept yet. The limit concept generally introduced in calculus I course. If I had imposed the concept on them, many of them would find it too difficult to understand and lose their interest. I did not want them to develop any anxiety for the rest of the class just because of that theorem. After all, I told them that they won't be tested on the proof of the theorem in the exam. So, it was not that important to dig deep in to that concept.

(Cathy [pseudonym], personal communication, Jan 17, 2012)

This teaching activity, which involved stating the concept to reduce the complexity or difficulty of the concept for students, illustrates this teacher's reducing

abstraction behaviour in RAIT-3c. The views she expressed regarding reducing abstraction in task implementation are related to various factors such as time constraints, mismatch between the complexity of the problem and students' level of thinking, and content that would actually be tested in the exam.

6.2. Insights from the interviews

As can be seen from the data (and its analysis) that most teaching strategies for reducing abstraction that fit into the first category seemed to support the ideologies and goals of reformed teaching, and were often carried out by the teacher as a pedagogical move. Concerns arise, however, when it came to the strategies for reducing abstraction that does not seem to align with the reformed teaching ideas and these strategies mostly seem to fit in the second and third categories.

One of the reasons given by teachers to explain the more problematic approaches to reducing abstraction as identified in this study was associated with teaching time constraints. For example, one of the teacher participants, Cathy, mentioned in her interview (see Section 6.1.3) that the time for the course she is teaching is too short to be adequate and, therefore, going into detail for each of the concepts she's teaching is not possible. Her interview revealed that when the amount of time is insufficient for the completion of required tasks, some of the strategies teachers employ to help students finish the tasks include stating the concept rather than developing it, stating the procedures to follow or simply focusing on the end product or answer at the expense of justification and meaningful learning. This view was also supported by other participant instructors during my personal communication with them.

The other possible reason identified for reducing abstraction, concerning the problematic approaches, is related to mismatch between the task and the students' interest and prior knowledge. Tasks which are not closely related to students' interest and prior knowledge are often perceived by the students as difficult, complex and/or uninteresting, which in turn results in these students' low engagement. To make the situation more amicable to the students, teachers tend to reduce the abstraction of the task, which often results in reducing abstraction in such a way to correspond with the

second and third categories. For example, with regard to the theorem on end behaviour of a polynomial, Cathy stated the theorem (concept) that the end behaviour of $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ is the same as that of $y = a_n x^n$, without offering further discussion on how and why they have the same end behaviour (see Section 6.1.3). In her interview, Cathy said that the development of the theorem is beyond the level of the students' mathematical thinking and, hence, she preferred to simply state the concept without further discussion.

Further, the word "imposed", as used by Cathy in her interview (see section 6.1.3) in response to her approach of reducing abstraction, strongly suggests that, in Cathy's view, developing the concept of "the end behaviour of a polynomial is the same as the end behaviour of the leading term" is something beyond her students' level of thinking, and that further discussion of this topic might have negative consequences on students' attitudes towards mathematics for the rest of the class.

Similar view was expressed by Beth (another teacher participant). In her view, motivation and self confidence are important factors in learning and that the approaches the teacher takes in implementing a task largely influence students' motivation and self confidence towards the material they are trying to learn. For example, she used the 'cover up' strategy to teach how to find two points of a line given a linear equation (see Section 6.1.2). Although initially this approach seemed to reduce abstraction by focusing on process conception and closely related to the instrumental understanding (Skemp, 1976), Beth did not leave students' understanding at that level. Stepping on this lower level (process conception), she extended students' thinking to higher level (relational understating).

Another reason for such behaviour can be attributed to mismatch between the task specification and assessment. In other words, if the assessment does not emphasise the theorem, proof and conceptual understanding, teachers tend to shift the focus onto procedure, the answer or stating the concept. In our example, as Cathy mentioned, the theorem on the end behaviour of polynomials would not be tested and, hence, concept development or the proof of the theorem was not deemed important.

This study supports the findings of other studies in which researchers found that some of the factors related to lowering the cognitive level of tasks during implementation were time constraint and the complexity of the problem. Doyle (1988), for example, found that high-level tasks were perceived by teachers (and students) as risky and ambiguous, and therefore there was a tendency to reduce the complexity of the task so as to manage the accompanying anxiety. Doyle's study revealed several approaches to reducing the complexity of the problem by teachers. Among them, and often cited, were shifting the focus towards knowledge of procedure rather than conceptual understanding, stating the concept rather than developing it, and allotting inappropriate time (too much or too little) for the task. Looking this from the RAI_T framework, when this is done, the complexity (or the abstraction) of the problem is reduced for the students. The problem with such implementation behaviour is that it may deny students the opportunity to maximize their learning.

Another study pointed out that the way teachers implement tasks is also related to the nature of assessment. Bennett and Desforges (1988), for example, found that even though a task might have asked for higher order thinking, if (formative and summative) assessment emphasised the procedures and correctness, teachers in the process of task implementation often tended to lower the cognitive level of the task by shifting the focus to the procedures, correctness and completeness of the task or stating the concept rather than developing it.

6.3. Summary

In this chapter, I presented three representative examples with their analysis through the Reducing Abstraction in Teaching (RAIT) framework and explained how these exemplify some instances of reducing abstraction in teaching. These three examples came from primary data collection, which consisted of my observation of the natural classroom setting at nine university/college preparatory courses in mathematics taught by three different instructors. The RAI_T framework proved to be very helpful in analysing the data in that it allowed me to describe, characterize and contrast teacher task implementation behaviours involving how mathematical abstraction is dealt with in the classroom. The instructors in these teaching situations also exhibited patterns of

reducing abstraction behaviour very similar to those observed in the case of secondary data (TIMSS 1999 Public Release video lessons) as described in Chapter 5. Thus, the applicability of the RAIT framework in both school and university level mathematics courses is demonstrated.

Further, I also presented teachers' justification for and clarification of some of their task implementation behaviours through the informal interviews I conducted with these teachers after observing their classes. This section of the data shed light not only on how teachers deal with abstraction while implementing tasks, but also why they chose a particular approach to reduce abstraction. Even though questions about the factors associated with various kinds of behaviour intended to reduce abstraction are not the main questions under exploration in this study, this subject came up along the way and I found it relevant to briefly discuss it in this chapter.

Oftentimes, approaches to reducing abstraction in the first category (quality of relationship between the learner and the concept) were carried out consciously by the instructors as a pedagogical move, which is in line with the movement towards reformed education ideology. However, in some instances, the choice of approach in reducing abstraction—particularly in the third (avoidance of the complexity or complication of implementing a task or concept) categories—were related to factors such as time constraints, mismatch between the level of students' mathematical thinking and the task/concept, and mismatch between the task/concept and the nature and content of assessment. When the amount of time was insufficient or the instructor perceived a concept/task to be above the students' ability to understand, or if a concept/task was assumed to not be tested in the exam, instructors tended to shift the focus of their lesson onto the completion of the task, stating rather than developing the concept or simply focusing on the answer or end product at the expense of justification and meaningful learning. The data also revealed the fact that reducing abstraction approaches are sometimes associated with the teacher's experience of teaching those concepts as well as her learning experience as a learner herself.

In the following chapter, I present a summary and discussion of the study, the implications of its results and avenues for further research, followed by the limitations of the study and final reflection.

7. Discussion and Conclusion

It is widely believed that mathematics is a subject that deals with abstract objects. The nature of mathematical tasks and students' engagement with these is likewise widely recognized as a central component of mathematics education at all levels. Although the nature of the task plays an important role in the way students engage in doing mathematics, the task itself does not guarantee the enhancement of learning. Rather, the way in which a mathematical task is implemented and worked on during a lesson largely determines the quality of teaching and learning, and these issues are at the heart of teaching mathematics. In this regard, the purpose of this study was to find out how teachers deal with abstraction while implementing mathematical tasks. In this section, I provide a summary and discussion of the research questions explored in this study. Further, I describe the findings as well as their implications for how mathematics is taught and for mathematics education in general. Finally, I offer suggestions of avenues for further research and the potential continuing impact of these on my practice as a mathematics instructor and educator.

7.1. Summary and Discussion

One of the questions that I addressed in this study was related to mathematical abstraction and how teachers deal with this in teaching. In particular, as stated earlier, my study aimed “to know not how many or how well, but simply how” (Shulman, 1981, p. 7) teachers deal with mathematical abstraction in teaching.

With regard to mathematical abstraction, my literature review (see Chapter 2) suggested that even though researchers and educators in the field of mathematics education agree that mathematical abstraction is one of the important features of mathematics, and that some concepts in mathematics are more abstract than others, they, however, do not agree on a single meaning for abstraction. Nevertheless, the

literature—particularly the work of Wilensky (1991), Sfard (1991) and (Hazzan, 1999)—revealed that abstraction in mathematics education has been interpreted mainly from three perspectives:

- a. Abstraction as the lack of a concrete referent for a mathematical object
- b. Abstraction as the quality of relationship between the mathematical concept and the learner
- c. Abstraction as the level of difficulty and complexity of mathematical concepts

Hazzan's (1999) reducing abstraction framework was initially developed to examine the learner's coping strategies while dealing with unfamiliar concepts and has been used in different areas of mathematics and computer science (Hazzan, 1999, 2003; Raychaudhuri, 2014; Hazzan & Zazkis, 2005). As evidenced in the literature, all the researchers who conducted their studies using the reducing abstraction framework examined learners' tendencies with regard to dealing with mathematical abstraction (Hazzan, 1999; Raychaudhuri, 2001; Hazzan & Zazkis, 2005). In my literature review, I was not aware of any study that specifically aimed to look at reducing abstraction from the perspective of teaching activity. With the intention of complementing and expanding upon the body of knowledge in this regard, my study aimed to explore how teachers deal with mathematical abstraction in teaching. Do teachers reduce the abstraction level of concepts or tasks while implementing them? And if they do, what ways of reducing abstraction are used? Further, how successful are each of these methods when it comes to achieving meaningful learning? It is with these questions in mind that I approached the data samples of this study.

My literature review (see Chapters 2 and 3), particularly the work of Hazzan (1999), Wilensky (1991), Sfard (1991) and others from the constructivist camp suggested that both students and teachers reduce abstraction level in learning and teaching respectively, but that they do this for different reasons. Reducing abstraction as proposed by Hazzan (1999) concerns with students' behaviour in dealing with mathematical abstraction when learning new concepts. She posits that since learners usually do not have the mental constructs or resources '*to hang on to*' or cope with the same abstraction level of the task as intended by the authorities (e.g., teacher,

textbook), they tend to reduce the level of abstraction *'unconsciously'* in order to make the concept more accessible, which often results in inappropriate reduction of the abstraction.

As stated earlier, from a teacher's perspective, the choice of words and phrases such as *'unconscious'*, *'lacks the mental construct'*, and *'to hang on to'* are problematic. The assumption here is that teachers are the experts and usually do have sufficient mental resources to deal with the abstraction level of the mathematical concept at the same or even at a higher level than that intended by the authorities (e.g., textbook). Therefore, the working hypothesis here is that teachers will reduce abstraction appropriately as compared with their students, who are more likely to do so inappropriately, and thus, reducing abstraction in teaching has pedagogical value.

This shift in perspective from students' activity to teachers' activity necessitated a different interpretation of reducing abstraction. Hence, in this study, I borrowed ideas from the literature—particularly from the work of Hazzan (1999), Wilensky (1991) and Sfard (1991)—and have redefined and reinterpreted the Hazzan's three folds interpretation of abstraction from the perspective of teachers' approaches to task implementation as follows (also stated in chapter 4).

1) *Abstraction level as the quality of the relationships between the object of thought and the thinking person*

On the basis of this perspective, the level of abstraction is measured by the relationship between the learners and the concept (mathematical object). From a teaching perspective, this refers to the situation where an attempt has been made by the teacher to establish a *right relationship* (in the sense of Wilensky) between the students and the abstract mathematical concept.

2) *Abstraction level as a reflection of the process-object duality*

Reducing abstraction in this category refers to the task implementation behaviour in which there is an emphasis on the process or the correct answer rather than the concept, object, meaning or understanding.

3) Degree of complexity of mathematical task or concept

In this category, abstraction level is measured by the degree of complexity involved. Here, this refers to teachers' task implementation behavior in which teachers tend to make the problem less complex in various ways. These include focusing on the particular rather than the general case, stating rather than explaining the concept, and various other methods intended to avoid complication of or the complexity involved in implementing a task or concept.

Figure 14 illustrates the three categories of Reducing Abstraction in Teaching (RAiT)

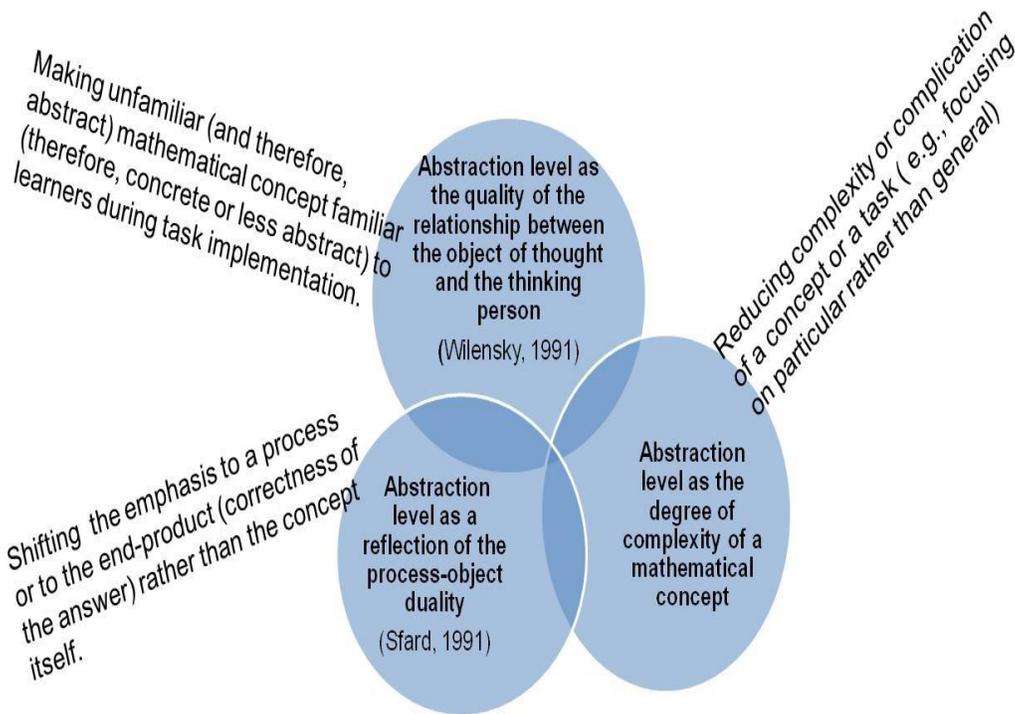


Figure 14. Three categories of Reducing Abstraction as Teaching Activity

In light of these three interpretations of abstraction, I analysed the TIMSS (1999) Public Release video lessons. In so doing, various teaching practices with regard to reducing abstraction were identified under each thematic category. Using the methodology as described in Chapter 4, I listed these as sub-categories and assigned each to the thematic category. I would like to reiterate here that these categories or their

subcategories are not disjointed but rather, they intersect. Depending on the perspective one takes, reducing abstraction behaviour in one category/subcategory can be interpreted as reducing abstraction in another category/subcategory. However, I put them in the category where they were deemed to fit best, acknowledging that there are other possible interpretations of these reducing abstraction behaviours and that the list of categories/subcategories may indeed grow longer (over time). As a result, a new framework that examine teachers' approaches of dealing with mathematical abstraction in teaching has emerged, which I call Reducing Abstraction in Teaching (RAiT). A detailed discussion of this was given in Chapter 5. Below, I however, present the framework of RAiT that emerged from the analysis of TIMSS 1999 Public Release video lessons.

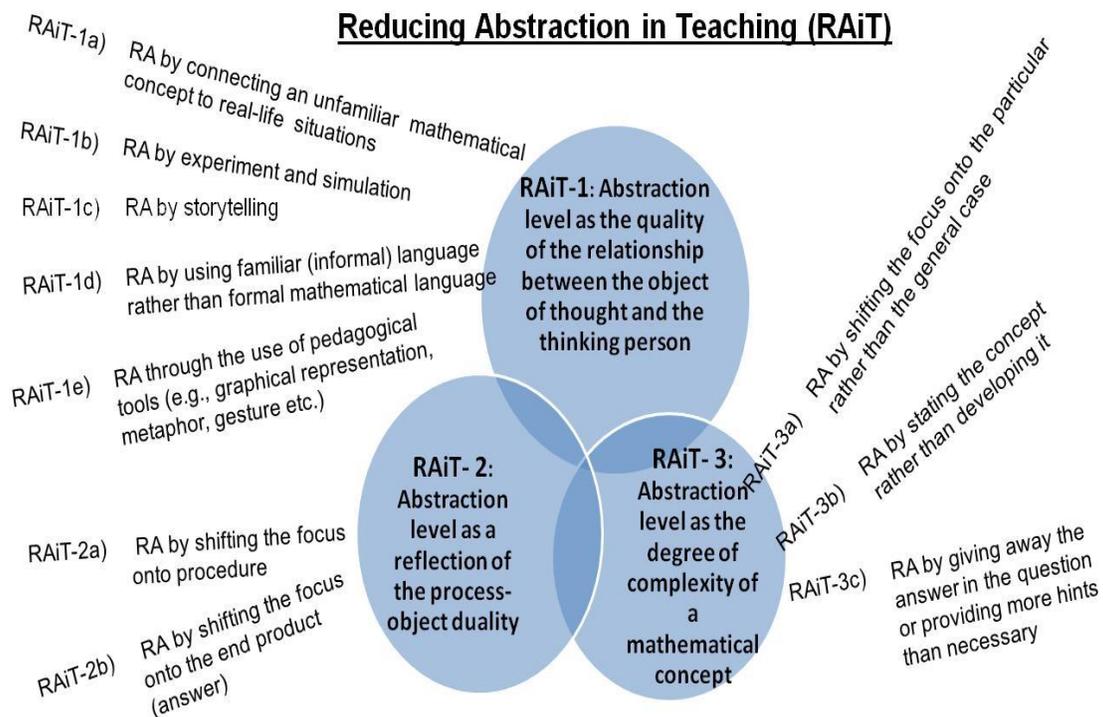


Figure 15. Reducing Abstraction in Teaching (RAiT) Framework (repeat of Figure 11)

Of note here is that the TIMSS 1999 video lessons were already analysed by the TIMSS 1999 video study team itself, as well as by various researchers, but from different theoretical perspectives. The TIMSS 1999 video study team, for example, analysed the

data from the perspective of whether the task is presented as ‘making connection’ or ‘non-making connection’.

The TIMSS 1999 video team found that teachers most often led ‘making connections’ discussions by drawing conceptual connections. The team also suggested that tasks implemented as ‘making connection’ closely aligned with the intent of the mathematics reform camp and promoted students’ mathematical understanding. In contrast, ‘non-making connection’ problems often align with traditional teaching strategies, which consist mainly of teacher explanations and demonstrations of procedures, followed by student practice of those procedures, with an emphasis on basic facts and skills. Similar to these findings, my study revealed that most of the former kinds of teaching activities which the TIMSS 1999 video team called “making connection” seem to be associated with Category 1 in RAIT, whereas the latter kinds of task implementation behaviours mostly seemed to be associated with either second or third categories in RAIT.

Having observed various approaches to reducing abstraction in teaching as captured in the TIMSS 1999 Public Release video lesson, I was interested to know more about “why teachers dealt with mathematical abstraction in teaching the way they did”, and to learn the reasons and intentions behind each of those implementation behaviours. Though this question is intriguing and worthy of further examination, it was not possible for me to explore this question using the first part of data (TIMSS 1999 Public Release video lessons) because follow-up interviews with the teachers were not included, nor is it possible for researchers to go back and try to obtain this kind of data retroactively for this sample. Hence, this question remained unanswered in relation to this data set. Using the second part of my data collection and analysis (the primary data, i.e. my classroom observation and informal interviews with teachers), however, I have been able to shed some light on this issue.

As stated earlier, most teaching strategies for reducing abstraction that fit into the first category seemed to support the ideologies and goals of reformed teaching, and were often carried out by the teacher as a pedagogical move. Concerns arise, however, when it came to the strategies for reducing abstraction that does not seem to align with

the reformed teaching ideas and these strategies mostly seem to fit in the second and third categories.

One of the reasons given by teachers to explain the more problematic approaches to reducing abstraction as identified in this study was associated with teaching time constraints. If the amount of time is insufficient for the completion of required tasks, some of the strategies teachers employ to help students finish the tasks include stating the concept rather than developing it, stating the procedures to follow or simply focusing on the end product or answer at the expense of justification and meaningful learning (see Section 6.1.2 and 6.1.3).

The other possible reason identified for reducing abstraction, concerning the problematic approaches, is related to mismatch between the task and the students' interest and prior knowledge. For example, Cathy used the word "imposed" in her interview which suggests that in Cathy's view, developing the concept of "the end behaviour of a polynomial is the same as the end behaviour of the leading term" is something beyond her students' level of thinking, and that further discussion of this topic might have negative consequences on students' attitudes towards mathematics for the rest of the class (see Section 6.1.3). Similar view was expressed by Beth (another teacher participant). In Beth's view, motivation and self confidence are important factors in learning and that the approaches the teacher takes in implementing a task closely related to the students' motivation and self confidence towards the material they are trying to learn.

Another reason for such behaviour can be attributed to mismatch between the task specification and assessment. For example, Cathy mentioned, the theorem on the end behaviour of polynomials would not be tested and, hence, concept development or the proof of the theorem was not deemed important (see Section 6.1.3).

Table 1 provides the summary of the factors associated to the choice of teachers' reducing abstraction behaviour in teaching as identified in this study.

Table 1: Factors associated to the teachers' choice of the approaches of Reducing Abstraction in Teaching

a) Pedagogical Choice
b) Time factor "the time allotted for this course is too short [...] so, I decided not to go in detail of that concept". (Cathy [pseudonym], personal communication, Jan 17, 2012)
c) Experience as a learner herself/ himself and later as a teacher "I kind of liked this method even when I was a student. Later, I used this strategies in my teaching and I found that my student loved this strategy". (Beth [pseudonym], personal communication, November 21, 2011)
d) Students' existing level of knowledge as perceived by the teacher "the proof involves the limit concept and they do not have that concept yet [...] . If I had imposed the concept on them [...] they will lose their interest". (Cathy [pseudonym], personal communication, Jan 17, 2012)
e) Whether the concept will be tested in the exam or not "[...]they won't be tested on the proof of that theorem in the exam". (Cathy [pseudonym], personal communication, Jan 17, 2012)

This study supports the findings of other studies closely related to teachers' task implementation behaviour. Those studies, however, looked at task implementation from the perspective of the cognitive level of the task, i.e. whether cognitive level is maintained or reduced while implementing the task (see QUASAR, Silver & Stein, 1996; Henningsen & Stein, 1997; Charalambous, 2008). As stated in Section 3.4, I want to point out an important point of clarification; this is that the cognitive level of a task or concept should not be confused with the abstraction level of the task or concept. One of the important distinctions is that abstraction level as interpreted in this study involves subjective phenomenon. For example, a concept that is too abstract for one learner may be less abstract for another, depending on one's existing knowledge, experience and familiarity with the concept. On the contrary, cognitive level as used in the literature (Henningsen & Stein's, 1997) describe an objective aspect of the level of students'

thinking without taking into account the subjective aspect, such as the students' familiarity with the concept.

The QUASAR research project, for example, found that teachers reduce the cognitive level of the task by telling students how to do it or solving the problem for the students when insufficient time is available to complete the task (Henningsen & Stein, 1997). Other studies have identified various factors such as teachers' experience, knowledge, and beliefs that play key roles for raising, maintaining or reducing the cognitive level of the task during implementation (Charalambous, 2008; Ben-Peretz, 1990; Henningsen & Stein, 1997). Charalambous's (2008) study, for example, found that teachers with high MKT (Mathematical Knowledge for Teaching) "largely maintained the cognitive demand of curriculum tasks at their intended level during task presentation and enactment" and that teachers with low MKT "proceduralized even the intellectually demanding tasks she was using and placed more emphasis on students' remembering and applying rules and formulas" (p. 287).

Putnam et al. (1992) also argued that teachers' knowledge and beliefs about learners, mathematical content knowledge, and knowledge and beliefs about mathematics (i.e. how they, themselves, view mathematics) significantly influence their tendency to change the nature of the tasks (i.e. maintain or reduce the cognitive level) and the way they are implemented. However, in my study, since data regarding teachers' knowledge, experience and beliefs was not available, the issue related to the effect of these factors on task implementation behaviour was not clear and therefore lies outside of the scope of this study. Future studies are needed to shed more light on this connection.

7.2. Contribution and Implication

As stated earlier, Hazzan (1999) provided three interpretations of abstraction, and based on these interpretations she proposed a framework of reducing abstraction to examine how learners deal with abstraction (learners activity) while learning new mathematical concepts. But my study aimed to explore how teachers deal with abstraction while implementing mathematical tasks. Hence, for my particular purpose,

Hazzan's framework of reducing abstraction fell short in many ways. For this reason, I redefined and reinterpreted the notion of reducing abstraction and extended its scope to examine teachers' task implementation behaviour with regard to dealing with mathematical abstraction.

With these three (re)interpretation (three thematic categories) of abstraction in mind, I turned my theoretical perspective towards analysing the TIMSS (1999) Public Release video lessons with regard to teachers' approaches to dealing with abstraction while implementing mathematical tasks. In so doing, various strategies for reducing abstraction in teaching were identified under each thematic category. Using the methodology as described in Chapter 4, I listed these as sub-categories and ordered each in the thematic category where they were deemed to fit best. As a result, Reducing Abstraction in Teaching (RAiT) has taken shape in a new and improved form with three main categories and various subcategories; a detailed discussion of each was given in Chapter 5.

Using the RAiT framework, I analyzed three representative examples obtained from my primary data collection. Of note here is that primary data collection included my observation of nine lessons (three lessons by each instructor) from a university preparatory mathematics course at local universities/colleges. This part of the data analysis (see Chapter 6) confirmed the usefulness and applicability of the RAiT framework in examining teaching activities with regard to dealing with abstraction not only in high school level mathematics, but also in college/university settings. I believe that the framework has "the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice" (Dreyfus & Gray, 2002, p. 113).

This study suggested that while dealing with mathematical abstraction in teaching, teachers reduce the abstraction level of a concept or a task with the primary goal of making the concept or the task more accessible to their students. However, the findings suggested that not all approaches to reducing abstraction are created equal. Different strategies for reducing abstraction in teaching provide different learning opportunities and, likewise, afford different kinds of outcomes for learners. For example, as evident from this study, approaches to reducing abstraction in teaching within the data sample of this study from the first category seemed to share much with the values

and goals of the reformed teaching movement, and were found to be pedagogically beneficial; by contrast, the strategies of reducing abstraction that fell under the second and third categories oftentimes seemed to align more with traditional teaching methods.

As stated earlier, traditional teaching methods refers to the teacher centered approach in which teachers serve as the source of knowledge while learners serve as passive receivers. The activity consists mainly of teacher explanations and demonstrations of procedures, followed by student practice of those procedures, with an emphasis on basic facts and skills. In contrast, reformed teaching refers to the student-centered approach which encourages inquiry-based activities and challenges students to make sense of the mathematical ideas through exploration and projects (often in real life context). In reformed classroom, teachers draw on a range of representations and tools (such as graphs, diagrams, models, images, stories, technology, everyday language etc.) to support students' mathematical development (National Research Council, 2001; Silver and Mesa, 2011).

Having said this, it does not follow that the use of RAiT in the second and third categories is never pedagogically beneficial. For example, a teacher can implement abstract concepts or tasks by initially focusing on the process conception (RAiT-2a), and once the students become familiar with this, the teacher can then introduce the object conception utilizing the process conception students have already mastered (as in the case of Beth's 'cover up' approach). With regard to RAiT-2b, focusing on the correct answer regardless of what method or process is used (in many cases, there is more than one method available to solve a problem) to arrive at that answer can provide freedom for students in selecting the method or process already familiar to them.

Similarly, reducing abstraction in the third category, particularly in RAiT-3a (focus on a particular case rather than the general case), for example, allow teachers to focus on the lead element rather than the general one, and once the concept is understood in a particular case, it can then be extended to other contexts and situations and build a general concept.

In the case of RAiT-3b (stating concept rather than developing it), in some instances stating the concept may be pedagogically beneficial if the development of the

concept is too complicated or lies too far beyond the students' level of understanding. For example, a basic statistics course involves finding the correlation between two variables such as the relationship between students' scores in a mathematics class and the number of hours they study. The problem can be solved by using the Person's correlation coefficient formula $r = \frac{n(\sum xy) - (\sum x)(\sum y)}{\sqrt{[n\sum x^2 - (\sum x)^2][n\sum y^2 - (\sum y)^2]}}$, but developing the formula is beyond the level of students, and in most cases, this formula is stated rather than developed.

Similarly, with regard to RAiT-3c, giving more hints than necessary in teaching can deny students the opportunity of meaningful learning. However, in some cases, it can also be used as a motivational factor to get students to move forward when they get frustrated or anxious with a task, or when there is a danger of losing the students' interest for the rest of the lesson.

Therefore, whether the reducing abstraction behaviour in teaching is pedagogically beneficial or not largely depends on various factors such as content, context, students' level of understanding and the teachers' plan and intention. However, in the data sample of this study, reducing abstraction as per the first category (RAiT-1) seemed to align with recent trends and reformed ideas in teaching and learning mathematics (RAiT-1), while strategies that fell under the second and third categories (RAiT-2 and RAiT-3) oftentimes seemed to fit more with ideas central to traditional teaching practices.

Hence, this study suggests that it is important for teachers to pay attention not only to the type of problem they present in the classroom, but also to how they implement the problem and how abstraction is reduced during task implementation because different strategies for reducing abstraction in teaching provide different learning opportunities and, likewise, afford different kinds of outcomes for learners. By the same token, it is important for researchers to examine how teachers reduce abstraction in teaching and what kinds of learning opportunities are made available to the students.

7.3. Further Research

The results of this study suggest several lines of research in the teaching (and learning) of mathematics. First, studies are needed to examine and link the various impacts upon student learning of each of the approaches, as identified here, to reducing abstraction employed by teachers while implementing mathematical tasks. This study primarily focused on the teaching practices of teachers as seen in the TIMSS 1999 Public Release videos and my observation of teaching in nine university mathematics preparatory classrooms in local universities/colleges. Data relating to students' actual mathematics learning with respect to problem solving, conceptual understanding, and mathematical reasoning was not available and, hence, the impact of each approach to reducing abstraction on student achievement is not clear. Thus, future research is needed to shed more light in this area.

Second, since interviews with the teachers in the first data set (the TIMSS 1999 Public Release video lessons) were not available (or possible), the intention and motivation of the teachers with regard to each act of reducing abstraction remains unknown. The second data set (my own classroom observation), however, did reveal some information about the intention and motivation of the teachers that shed light on their chosen strategies for reducing abstraction. That said, the population was rather small and the results, therefore, are not generalizable. Hence, a study with a larger population would greatly increase understanding on this issue.

Although various frameworks do exist in the mathematics education literature that can be used to examine mathematical tasks and task implementation, these were developed with the goal of examining different aspects of teaching and learning from different perspectives— chiefly, that of the learner. This new RAiT framework is the first of its kind that aims to examine teachers' task implementation behaviour with regard to dealing with mathematical abstraction. One of the important results of this study is the emergence of the RAiT theoretical framework, and, hence, it is in its preliminary stages. Further research would identify the strengths and weaknesses of this framework.

Further research is also required to determine whether this (RAiT) framework is applicable to other areas of education (such as physics, chemistry, biology, social

studies, etc.) and within various educational levels and settings (e.g., elementary, middle school, high school, college/university).

7.4. Limitations

A number of caveats must be noted regarding the present study. The first of these is that the study was conducted from a North American perspective even though the data used for this study comes from seven different countries. Since teaching is a cultural activity (Stigler & Hiebert, 1998, 2009) and the study is limited by a lack of knowledge about specific approaches used in different countries and within different cultures, it is quite possible that some important teaching behaviours may have been overlooked.

Another limitation is that I do not know whether or not the data samples used in this study (both the TIMSS 1999 Public Release video lessons and my own classroom observation) were representative of those teachers' consistent task implementation behaviour over the entire school year. Also, it is quite possible that the awareness of the fact that the lesson (teaching) was being recorded or observed might have had some effect on the way teachers implemented tasks.

Additionally, factors such as teachers' beliefs, knowledge, level of education, teaching experience, departmental and university cultures, classroom norms and other contextual factors that might influence how teachers deal with abstraction in the classroom were not within the scope of this study. Similarly, factors that influence the engagement level of students with mathematics such as students' background and skills, classroom norms and culture of the university were not considered.

Another limitation is that since the study was limited to an examination of a small number of eighth grade mathematics classrooms and a small number of university preparatory mathematics classrooms, the findings may not be generalizable to other countries and other levels (e.g., elementary school, higher level of university mathematics courses).

Despite these limitations, this study does shed light on important issues related to teachers' task implementation behaviour with regard to dealing with abstraction in teaching. In particular, this study provided mathematics educators and researchers with information about how teachers deal with mathematical abstraction in teaching and the various approaches to reducing abstraction while implementing tasks in the classroom, as well as possible outcomes and effects of these strategies.

7.5. Final Reflection

On a personal note, this research has greatly affected the way I look at how I teach mathematics. I now realize that the approaches I choose in implementing tasks in the classroom represent all my efforts to establish a right relationship (in the sense of Wilensky) between a mathematical concept/task and my students. This activity often results in reducing the abstraction level of the concept or task while presenting to my students. Nonetheless, the goal is to move them to a higher level of abstraction by first stepping on a lower rung level. I also now understand that there is no single best way of dealing with abstraction; different approaches in turn afford different kinds of learning opportunities for students. I cautiously anticipate that my awareness of these various approaches to dealing with mathematical abstraction in teaching and the kinds of learning opportunities they provide for students will empower me to explore a greater range of possibilities in my future teaching career.

Although I have learned a great deal about the various strategies available to deal with abstraction in teaching, as well as the different kinds of learning opportunities each of these approaches has to offer, at the same time, I am also aware of the various constraints (e.g., time allotted for the course, fixed curriculum, available resources) that must be taken into account when making choices about how best to go about teaching specific concepts. Based on the new knowledge and insights I have gained herein, however, I will continue to adjust my task implementation approaches in response to my students and to reflect on the impact of such choices on my students' learning outcomes.

In closing

While the findings of this study and the perspectives I gained through my literature review have been of benefit to me personally, I hope that this study is also beneficial to others in the field of mathematics education and particularly in teacher education. Finally, I would like to reiterate my belief that the RAIT framework has “the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice” (Dreyfus & Gray, 2002, p. 113).

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