COUNTERPARTY CREDIT RISK FOR AMERICAN OPTIONS IN A REDUCED-FORM MODEL

by

Yiwen Yuan
B.Sc. of Honours in Mathematics and Computing Science
Simon Fraser University 2013

and

Qiuyue Qu
Bachelor of Economics in Economics and Finance
Tsinghua University 2013

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Name: **Yiwen Yuan and Qiuyue Qu**

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Supervisory Committee:

_____

Dr. Peter Klein
Senior Supervisor
Professor

_____

Dr. Phil Goddard
Second Reader
Instructor

Date Approved: _____

Abstract

This thesis follows the idea of Klein and Yang (2013) to study the effect of counterparty credit risk and valuation of Vulnerable American options. Most existing literatures use the structural model (Merton 1974) to study the vulnerable options. However, structural model uses the unrealistic assumptions for the corporate asset. In addition, calibration stochastic asset processes using public information in the structural model is some more difficult than anticipated (Wang 2009). This thesis uses the reduced-form (intensity) model to study the credit risk of vulnerable American put options and compare the results with Klein and Yang (2013). We conclude that counterparty credit risk will affect the vulnerable option value as Klein and Yang did in their paper. Throughout, we rely on binominal tree method to derive our numerical results.

Keywords: structural model; reduced-form (intensity) model; counterparty credit risk; Vulnerable American options

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1: Introduction

Over-the-Counter (OTC) options are one of the most widely used derivatives in the industry that can be affected by the counterparty credit risk among contracts. Many reasons such as not enough collateral, stress mark-to-market changes will lead to a huge credit exposure. There are many examples of vulnerable options, such as currency options, exchange, real estate options etc. They are privately written and the third party cannot guarantee their payoff (Johnson and Stulz 1987). "Poor management of credit risk has been widely recognized as one of the causes of the global financial crisis of 2008" (Klein and Yang 2013). Therefore, the study of effect of credit risk on OTC options is considered critical and valuable.

This thesis focuses on the effect of credit event on the early exercise policy of an American style option. We follow the same idea of Klein and Yang (2013), where they used the structural model (Merton 1974) to study the effect of counterparty credit risk on an American option. The structural model (Merton's model) has limitations in many places such as evaluating the default probabilities, implementation of models and data collecting issue. Leland stated that the structural model would under-predict short-term default probabilities (Leland 2004). As well, structural model used the unrealistic continuous tradability assumption for the corporate assets. In addition, calibrating stochastic asset processes using publicly available information is some more difficult than anticipated (Wang 2009). Therefore, people will benefit from using the reduced-form approach particularly in data collecting and model implementation. The contribution of this thesis is to use the reduced-form (Intensity) model to study the effect of counterparty credit risk on an American style option. In reduced-form (Intensity) model, we assume options' default intensity λ is a function of stock price S and time t. By using the λ, we are able to model the probability of default of an American option. In terms of the assumptions, we use Klein and

Yang's paper for reference. We set up one of the assumptions as the option writer can default not only at maturity but also at each time point prior to the maturity. We also follow the assumption in Klein and Yang's paper such that "In an event of default, the option holder's nominal claim is the value of the otherwise identical non-vulnerable options f" (Klein and Yang 2013). These are our main assumptions during the price calculation of the vulnerable options.

This thesis is organized as follows. Section 2 is our literature review, which lists some papers containing either the pricing process of the vulnerable options or the effect of credit risk on the early exercise policy. We also demonstrate our motivation for doing this thesis. Section 3 demonstrates models of pricing both vulnerable and non-vulnerable options. We elaborate on our modification to the non-vulnerable option model in order to consider the credit process for its vulnerable twin. In Section 4, we show our numerical results by demonstrating the parameter changing effects on the critical asset price and the property of our pricing results. Section 5 is our conclusion.

2: Literature Review

In this section, we review the literatures on the optimality of the early exercise policy of vulnerable options.

Johnson and Stulz (1987) first researched on the early exercise of vulnerable options. They made the following assumption in their model: the option value is the liability of the writer and the option holder will receive the entire option writer's asset if the writer defaults. By doing this, they demonstrated that "It is possible for the value of a vulnerable European option to fall with time to maturity, with the interest rate, and with the variance of the underlying asset". Overall, the effect of credit risk on either option value or the early exercise policy can be quite significant. Further, they concluded that it might be optimal to early exercise a vulnerable American call option on a non-dividend-paying asset. However, they assumed that the default occurs only at maturity. There were no numerical results demonstrating the property of the American style option.

Hull and White (1995) presented a model for valuing derivative securities when the derivative securities holder considers that the option writer will default. They made an assumption such that the default can occur at any time before the option expires. The probability of default and the size of the proportional recovery rate are random, meaning in their model, the default will occur as long as the value of the assets of the option writer falls below the value of a fixed default barrier. They drew the conclusion that default risk has larger effect on the European options than the American options.

Klein (1996) further applied what was concluded in Johnson and Stulz (1987). He allowed other liabilities of the option writer in addition to the potential claim under the option. He

also allowed that the payout ratio is dependent on the value of asset and liability of the option writer. Based on the assumptions he made, he derived a closed-form solution for pricing the vulnerable European options. This formula is based on the structural model (Merton 1974) of modelling the credit risk.

Klein and Inglis (2001) extended the results from Johnson and Stulz (1987) and Klein (1996). Their model allows a default boundary that depends on the potential liability of the written option and other liabilities of the option writer. They also linked the payout ratio when there is a financial distress to the value of option writer's assets, and modelled the correlation between the assets of the option writer and asset underlying. They concluded that the variable default boundary is important in evaluation of a vulnerable European option when the payoff of the option is greater than the value of the other liabilities of the option writer (Klein and Inglis 2001).

Klein and Yang (2010) studied the properties of the vulnerable American options. In their paper, they showed the probability of early exercise is higher for a vulnerable American option than its non-vulnerable American twin. They also demonstrated that the degree of credit risk of the option writer would affect the underlying asset price at which early exercise is optimal.

Klein and Yang (2013) further extended their idea in their previous paper, and studied the effect of counterpart credit risk on optimal early exercise policy. They found the optimal early exercise policy would become quite different when option holder considers the credit risk of the option writer. In their paper, they applied the structural model (Merton 1974) and priced the value of American option. Klein and Yang concluded the following: "The early exercise when a credit event is very likely can mitigate but not eliminate the effect of credit risk on the value of American options" (Klein and Yang 2013).

Those papers discussed above mainly deal with pricing the vulnerable European options. Additionally, most of the papers used the Structural model (Merton 1974) to either price the

options or consider the early exercise policy. There are limitations in using such a model as we have mentioned in the previous section. Thus, we would like to present the reduced-form (intensity) model for a further research.

3: The Model

3.1 Binomial Tree for Non-vulnerable American Put Option

By applying the CRR model, we are able to derive the following tree of the underlying asset price (Cox, Ross and Rubinstein 1979). Figure 1 shows an example of the binomial tree of the price for the underlying asset, which is stock.

$S_2{}^{uu}$

$S_1{}^{u}$

$S_1{}^{u} = u * S_0$

$S_1{}^{d} = d * S_0$

$S_2{}^{ud}$

$S_2{}^{uu} = u * u * S_0$

S_0

$S_2{}^{ud} = u * d * S_0$

$S_2{}^{dd} = d * d * S_0$

$S_1{}^{d}$

$S_2{}^{dd}$
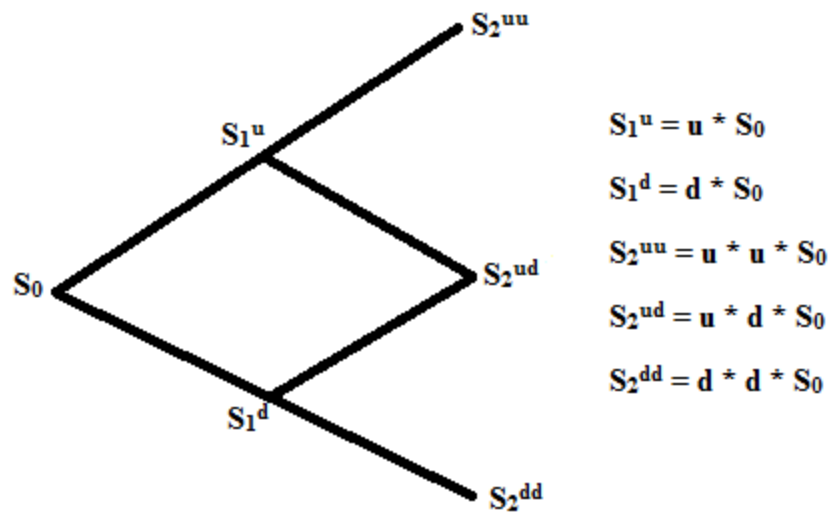
Figure 1: Binomial Tree for stock price

For a non-vulnerable American put option, we calculate the payoff at each node during the backward induction process.

$$V_n = \text{Max} [X - S_n, (P_u *V_u + (1 - P_u)*V_d) / R]$$

In above, $X - S_n$ stands for the intrinsic value at node n; V_u and V_d are the option payoffs at the up and down nodes in the next time step; R is the continuous risk free return from time t to t + dt, which equals e^{r*dt}. If the underlying asset generates a continuous dividend rate of q, the value of R becomes the following:

$$R = e^{(r-q)*dt}$$

We start with the notations we use in non-vulnerable options[1]. Let f (t) denote the value of non-vulnerable option at time t. The price of underlying asset at time t is denoted as S(t). We use S_c to represent the critical price of underlying asset when early exercise is optimal for the non-vulnerable option. That is to say, $f(S_c)$ stands for the non-vulnerable option value when early exercise is optimal. Take a two-time-step binomial tree as an example. We can see from Figure 1 how the stock price moves gradually in the binomial tree. Based on the stock price we already have, we can derive the following tree for the non-vulnerable put option.

At maturity:
$$f(2)^{uu} = Max[X - S_2{}^{uu}, 0]$$
$$f(2)^{ud} = Max[X - S_2{}^{ud}, 0]$$
$$f(2)^{dd} = Max[X - S_2{}^{dd}, 0]$$

At time t = 1:
$$f(1)^u = Max[X - S_1{}^u, (P_u * f(2)^{uu} + (1-P_u) * f(2)^{ud}) / R]$$
$$f(1)^d = Max[X - S_1{}^d, (P_u * f(2)^{ud} + (1-P_u) * f(2)^{dd}) / R]$$

At time = 0:
$$f(0) = Max[X - S_0, (P_u * f(1)^u + (1-P_u) * f(1)^d) / R]$$
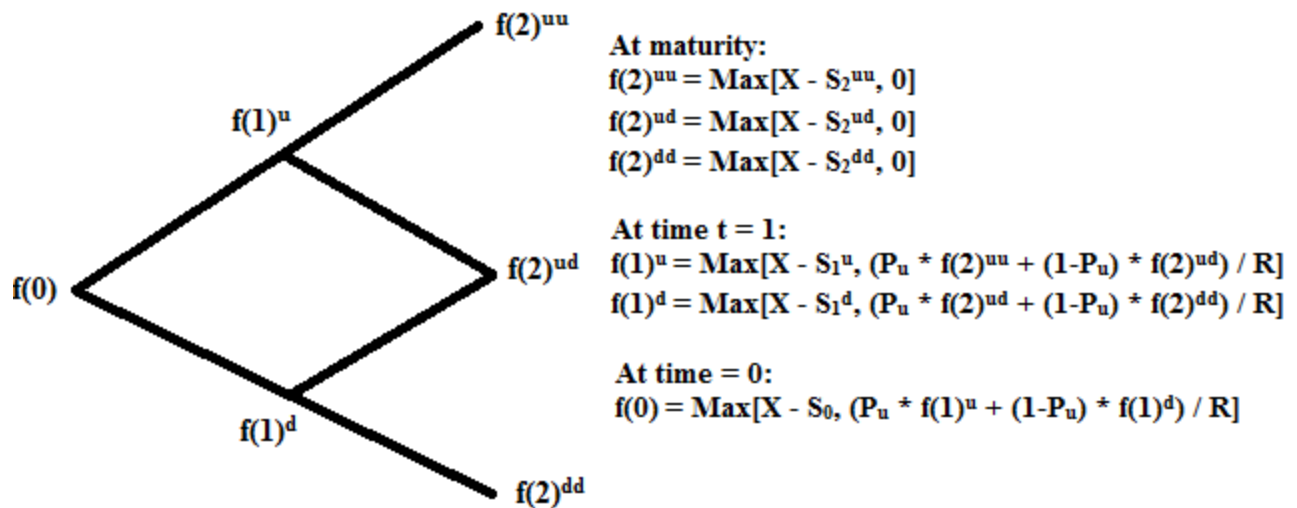
Figure 2: Binomial Tree for Non-vulnerable American put option

At each node, if the intrinsic value > calculated value, then early exercise is optimal and underlying asset price at this node is denoted as S_c.

Next, we have to create a similar two-time-step binomial tree for the vulnerable put option using intensity model to get probability of default in each time step of the tree.

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[1] Without further notification, we are discussing the American options in our thesis.

3.2 Binomial Tree for Vulnerable American Put Option

The vulnerable options use the similar notation. They are as follows: Let $f^{*\wedge}(t)$ denote the value of vulnerable option at time t. The price of underlying asset at time t which is S(t) is the same with it in the non-vulnerable options. We use $S_c^{\wedge}$ to represent the critical price of underlying asset when early exercise is optimal for a vulnerable option. That is to say, $f^{*\wedge}(S_c^{\wedge})$ stands for the vulnerable option value when early exercise is optimal. For vulnerable options, if a default happens, we need the recovery rate denoted as ω to calculate the compensation. The value of ω ranges from 0 to 1. For example, when $\omega = 1$, the option holder will lose nothing if the option writer defaults. Otherwise, when $\omega = 0$, the option holder will get nothing if the option writer defaults. P_d is the conditional probability of default in each time step, given there is no default prior to that step. We still use Figure 1 as a tree of underlying asset for the vulnerable option. Then we can derive the following tree for our vulnerable option value.

At maturity:
$$f^{*\wedge}(2)^{uu} = P_d*\omega*f(2)^{uu} + (1-P_d)*Max[(X-S_2^{uu}), 0]$$
$$f^{*\wedge}(2)^{ud} = P_d*\omega*f(2)^{ud} + (1-P_d)*Max[(X-S_2^{ud}), 0]$$
$$f^{*\wedge}(2)^{dd} = P_d*\omega*f(2)^{dd} + (1-P_d)*Max[(X-S_2^{dd}), 0]$$

At time t = 1:
$$f^{*\wedge}(1)^u = P_d*\omega*f(1)^u + (1-P_d)*Max[(X-S_1^u), (P_u * f^{*\wedge}(2)^{uu} + (1-P_u) * f^{*\wedge}(2)^{ud}) / R]$$
$$f^{*\wedge}(1)^d = P_d*\omega*f(1)^d + (1-P_d)*Max[(X-S_1^d), (P_u * f^{*\wedge}(2)^{ud} + (1-P_u) * f^{*\wedge}(2)^{dd}) / R]$$

At time t=0:
$$f^{*\wedge}(0) = P_d*\omega*f(0) + (1-P_d)*Max[(X-S_0), (P_u * f^{*\wedge}(1)^u + (1-P_u) * f^{*\wedge}(1)^d) / R]$$
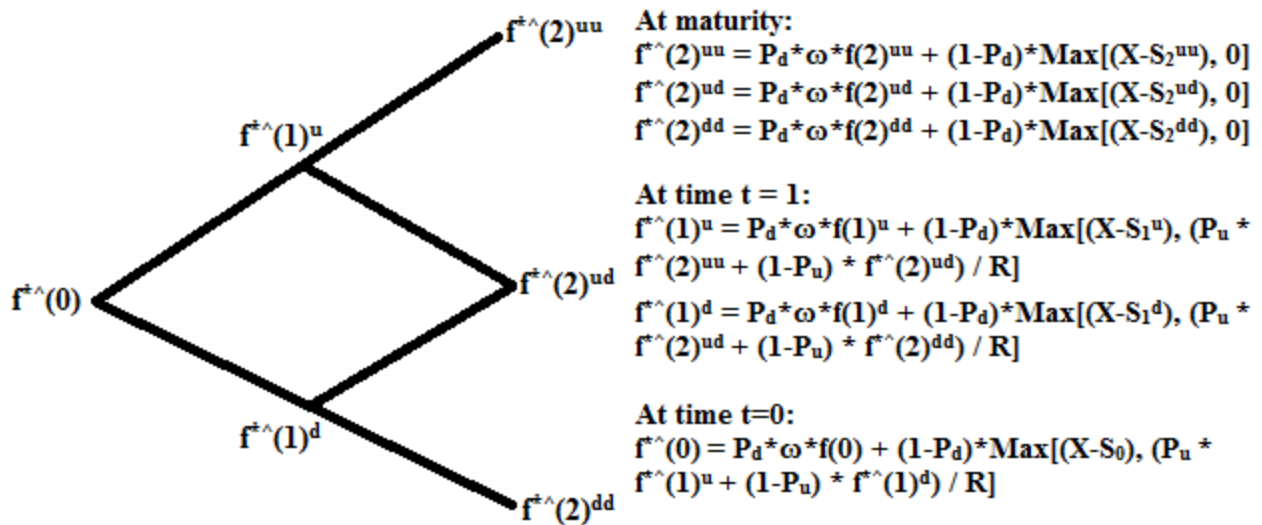
Figure 3: Binomial Tree for Vulnerable American put option

Before we discuss our formulas in detail, we present the following assumption at first:

Klein and Yang (2013) indicated, "In the event of default, the option holder's nominal claim is the value of the otherwise identical non-vulnerable option f" (Klein and Yang 2013).

At each node, we calculate the expected payoff, which equals to E [payoff when default happens, payoff when no default happens]. According to the assumption above, the payoff when default happens is ω * f. When there is no default happens, the payoff for American put at expiry is Max[X – S, 0]; while at other nodes before expiry, the payoff is Max[X - S, Calculated Value]. If the payoff when no default happens is equal to X – S, then we are able to conclude that the early exercise is optimal at this node, and we denote the underlying asset price at the node as $S_c^{\wedge}$.

Next, we introduce our intensity model for the calculation of probability of default. We set our intensity λ as a function of stock price S and time t, which is what Andersen did in his paper (Andersen 2002).

$$\lambda(t,S) = a + b / S^p,$$

In our base case, we set the value of a to be equal to 0 and p as a positive constant. This will lead us to see a more reasonable asymptotic behaviour such that $\lim_{s \to \infty} \lambda = 0$ and $\lim_{s \to 0} \lambda = \infty$ (Andersen 2002). This behaviour implies that it is likely to default when the underlying asset is at low price.

Based on the stock price we have, we are able to calculate the intensity by setting the parameters in the model above. Giesecke showed that the probability of default occurring by time t is $1 - e^{-\lambda * t}$ (Giesecke 2004). In our tree, we have multiple nodes in each time step. Since the stock price is different at each node, we can get different λ at each node in each time step. Thus, we can derive a cumulative probability of default at node n by time step i, which is denoted as CumuPd(n, i).

We claim that the cumulative probability of default we derived above is path independent. In a binomial tree, there are many paths to reach a certain node. However, no matter which path we choose, they have the same value of time t = i*dt and the underlying asset price at that node S(n, i) is fixed. In our model, CumuPd(n, i) at a certain node is a function of underlying asset price at that node S(n, i) and the time spent to reach that node, which is t. Therefore, we conclude that the cumulative probability of default is path independent in theory.

Further, we can use a two-time-step binomial tree to prove our claim above.
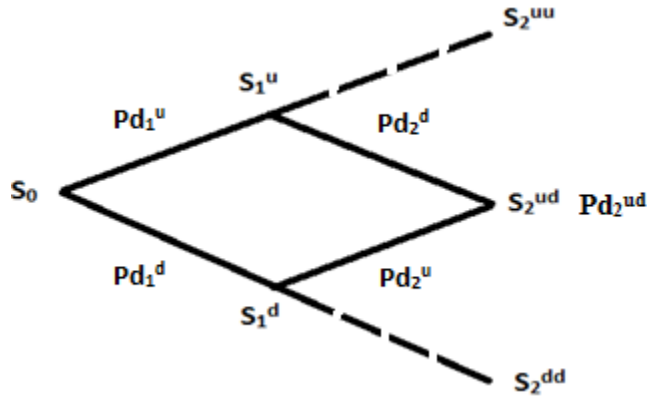
Figure 4: Binomial Tree for Probability of Default

As shown in Figure 4, we denote probability of default conditional on no early default before from node S_0 to $S_1{}^u$, S_0 to $S_1{}^d$, $S_1{}^u$ to $S_2{}^{ud}$, $S_1{}^d$ to $S_2{}^{ud}$ and S_0 to $S_2{}^{ud}$ as $Pd_1{}^u$, $Pd_1{}^d$, $Pd_2{}^d$, $Pd_2{}^u$ and $Pd_2{}^{ud}$ respectively. Based on our intensity model and the conditional probability of default formula, we can get the following equations:

$$Pd_1^u = 1 - e^{-(a + \frac{b}{(S_1^u)^p}) * dt};$$

$$Pd_1^d = 1 - e^{-(a + \frac{b}{(S_1^d)^p}) * dt};$$

$$Pd_2^d = \frac{\left(1 - e^{-(a+\frac{b}{(S_2^{ud})^p})*2*dt}\right) - \left(1 - e^{-(a+\frac{b}{(S_1^u)^p})*dt}\right)}{1 - \left(1 - e^{-(a+\frac{b}{(S_1^u)^p})*dt}\right)} = 1 - e^{\frac{b}{(S_1^u)^p}*dt - \frac{b}{(S_2^{ud})^p}*2*dt} \; ;$$

$$Pd_2^u = \frac{\left(1 - e^{-(a+\frac{b}{(S_2^{ud})^p})*2*dt}\right) - (1 - e^{-(a+\frac{b}{(S_1^d)^p})*dt})}{1 - (1 - e^{-(a+\frac{b}{(S_1^d)^p})*dt})} = 1 - e^{\frac{b}{(S_1^d)^p}*dt - \frac{b}{(S_2^{ud})^p}*2*dt} \; .$$

Actually, Pd_2^{ud} is the probability of defaulting at any point in time from node S_0 to S_2^{ud}, which is the cumulative probability of default at node S_2^{ud} in our model. We calculate cumulative probability of default at a specific node, using one minus the probability of no default until that node. Thus, when we choose path $S_0 \rightarrow S_1^u \rightarrow S_2^{ud}$, $Pd_2^{ud} = 1 - (1-Pd_1^u)*(1-Pd_2^d)$; when we choose path $S_0 \rightarrow S_1^d \rightarrow S_2^{ud}$, $Pd_2^{ud} = 1 - (1-Pd_1^d)*(1-Pd_2^u)$. After substituting each conditional probability of default calculated above into these two equations, we can find that Pd_2^{ud} got through the two paths are both equal to $1 - e^{-\frac{b}{(S_2^{ud})^p}*2*dt}$, which is consistent with the cumulative probability of default formula at node S_2^{ud} in our model. Therefore, you can see that the cumulative probability of default is path independent, which strongly proves our claim.

Next, we denote the unconditional probability of default at node n in time step i as UcPd(n, i). Thus, the unconditional probability of default at the first node (n = 1) in time step i is the difference between CumuPd(1, i) and CumuPd(1, i-1). The unconditional probability of default at the last node (n = i) in time step i is the difference between CumuPd(i, i) and CumuPd(i-1, i-1). The unconditional probability of default at the other node (1 < n < i) in time step i is the difference between CumuPd(n, i) and $\frac{CumuPd(n-1,i-1) + CumuPd(n,i-1)}{2}$:

$$\text{UcPd(n, i)} = \begin{array}{ll} CumuPd(n,i) - CumuPd(n,i-1) & (n=1) \\ CumuPd(n,i) - \dfrac{CumuPd(n-1,i-1)+CumuPd(n,i-1)}{2} & (1<n<i) \\ CumuPd(n,i) - CumuPd(n-1,i-1) & (n=i) \end{array}$$

The conditional probability of default at node n in time step i, given there is no early

default before, denoted as CondPd(n, i) is

$$\text{CondPd(n, i)} = \begin{array}{ll} UcPd(n,i)/(1 - CumuPd(n,i-1)) & (n=1) \\ UcPd(n,i)/(1 - \dfrac{CumuPd(n-1,i-1)+CumuPd(n,i-1)}{2}) & (1<n<i) \\ UcPd(n,i)/(1 - CumuPd(n-1,i-1)) & (n=i) \end{array}$$

By doing this, we are able to calculate the unique conditional probability at each node in

each time step.

Given the probability of default, we can go back to our payoff tree of the vulnerable

option and calculate the option value. As well, we keep track of S_c and $S_c^{\wedge}$ in order to compare the

early exercise policy in vulnerable and non-vulnerable options.

In order to calculate the early exercise price S_c and option value for the non-vulnerable

options, we simply set our parameters for the credit process equal to zero so that the default can

never occur. In our model, we set a = b = 0.

In the next session, we will show our numerical results based on the intensity model.

4: Numerical Results

We use the American put as an example. Based on our model described above, we set the base case parameters of the non-vulnerable American put as $S = 50$, $X = 50$, $\sigma = 0.2$, $T = 2$, time steps $= 100$, $r = 0.05$, $q = 0.05$, $a=0$, $b = 0$. The parameters are chosen in order to follow what Klein and Yang (2013) did in their paper, so that the base case parameters are similar to many business situations. For a vulnerable American put, since the option holder concerns if the option writer would default, we use the probability of default and recovery rate to calculate the vulnerable option value. In our model, we use the intensity model proposed in Andersen (2002) to derive the probability of default at each node in each time step, which is $\lambda(t, S) = a + b / S^p$. Therefore, the base case parameters of the vulnerable American put are added by $a=0$, $b = 0.657139$, $p = 0.06$, $\omega = 0.5$ in order to obtain the same value of vulnerable American put as demonstrated in Klein and Yang (2013). In order to check if the early exercise policy will be affected by default risk, we compare the differences between S_c for a non-vulnerable American put and $S_c^{\wedge}$ for its vulnerable American twin. We exhibit our detailed results in Table 1 and Table 3.

Next, we continue to do research on the effect of changing the values of parameters. At each time, we only change the value of one parameter and keep the rest fixed, indicated in the first column of Table 1. It clearly shows the different critical asset prices for both S_c and $S_c^{\wedge}$ and the percentage of difference between S_c and $S_c^{\wedge}$, which is denoted as Difference (%) in the last columns in Table 1. The value of S_c and $S_c^{\wedge}$ we are looking for is the value of underlying asset that would make it optimal to exercise the option immediately at the first node for both vulnerable and non-vulnerable put options.

Table 1 Parameter changing effects on the early exercise policies between the two types of puts

	Vulnerable American put	Non-vulnerable American put	Difference (%)
Base Case	$S_c^{\,\wedge} = 39.6$	$S_c = 33.2$	19.3
b = 1.0	$S_c^{\,\wedge} = 40.7$	$S_c = 33.2$	22.6
b = 0.8	$S_c^{\,\wedge} = 40.1$	$S_c = 33.2$	20.8
b = 0.4	$S_c^{\,\wedge} = 38.3$	$S_c = 33.2$	15.4
b = 0.2	$S_c^{\,\wedge} = 36.6$	$S_c = 33.2$	10.2
p = 0.02	$S_c^{\,\wedge} = 40.0$	$S_c = 33.2$	20.5
p = 0.04	$S_c^{\,\wedge} = 39.8$	$S_c = 33.2$	19.9
p = 0.08	$S_c^{\,\wedge} = 39.4$	$S_c = 33.2$	18.7
p = 0.1	$S_c^{\,\wedge} = 39.2$	$S_c = 33.2$	18.1
X = 55	$S_c^{\,\wedge} = 43.5$	$S_c = 36.6$	18.9
X = 45	$S_c^{\,\wedge} = 35.7$	$S_c = 29.9$	19.4
r = 0.075	$S_c^{\,\wedge} = 41.2$	$S_c = 37.7$	9.3
r = 0.025	$S_c^{\,\wedge} = 37.4$	$S_c = 21.4$	74.8
q = 0.075	$S_c^{\,\wedge} = 38.0$	$S_c = 27.7$	37.2
q = 0.025	$S_c^{\,\wedge} = 41.0$	$S_c = 36.9$	11.1
T = 3	$S_c^{\,\wedge} = 39.1$	$S_c = 31.9$	22.6
T = 1	$S_c^{\,\wedge} = 40.5$	$S_c = 35.6$	13.8
σ = 0.3	$S_c^{\,\wedge} = 35.3$	$S_c = 27.2$	29.8
σ = 0.1	$S_c^{\,\wedge} = 44.5$	$S_c = 40.7$	9.3
ω = 0	$S_c^{\,\wedge} = 42.2$	$S_c = 33.2$	27.1
ω = 1	$S_c^{\,\wedge} = 32.2$	$S_c = 33.2$	0
b = 0.4109 p = -0.06	$S_c^{\,\wedge} = 39.5$	$S_c = 33.2$	19.0
a = 1.0393 b = -0.657139	$S_c^{\,\wedge} = 39.5$	$S_c = 33.2$	19.0

Results of the structural model in Klein and Yang (2013) are shown in Figure 5 below.

Effect of Counterparty Credit Risk on Optimal Early Exercise Price

In the base case, the parameters are $S = 50$, $K = 50$, $\sigma = 0.2$, $T = 2$, $r = 0.05$, $q = 0.05$, $V = 5000$, $V = 5000$, $D* = 4,500$, $\sigma_V = 0.2$, $\rho = 0$, $\alpha = 0.5$. Values represent the price in dollars of the underlying asset at which early exercise is optimal.

	PUT		
	Vulnerable	Nonvulnerable	Difference
Base Case	35.1	33.2	5.7%
$V = 4500$	44.3	33.2	33.4%
$V = 4600$	44.3	33.2	33.4%
$V = 4800$	35.8	33.2	7.8%
$V = 5200$	34.6	33.2	4.2%
$V = 6000$	33.7	33.2	1.5%
$\sigma_V = 0.3$	36.1	33.2	8.7%
$\sigma_V = 0.1$	33.8	33.2	1.8%
$\alpha = 1$	35.3	33.2	6.3%
$K = 55$	38.6	36.6	5.5%
$K = 45$	31.6	29.9	5.7%
$r = 0.075$	39.2	37.7	4.0%
$r = 0.025$	22.4	21.4	4.7%
$q = 0.075$	29.1	27.7	5.1%
$q = 0.025$	38.7	36.9	4.9%
$T = 3$	33.9	31.9	6.3%
$T = 1$	36.8	35.6	3.4%
$\sigma_S = 0.3$	29.4	27.2	8.1%
$\sigma_S = 0.1$	41.9	40.7	2.9%
$\rho = 0.5$	37.1	33.2	11.7%
$\rho = -0.5$	33.7	33.2	1.5%

Figure 5: Exhibit 1 in Klein and Yang (2013)

The results in Table 1 clearly show that the critical asset price is different for the vulnerable option and its non-vulnerable twin. In the base case, the vulnerable put option is optimal to be exercised immediately if the price of underlying asset turns to 39.6. However, its non-vulnerable put twin will not be exercised immediately until the price of underlying asset becomes 33.2. The difference between them is approximately 19%. In other cases, the critical asset price for immediate exercise of the vulnerable put option is always larger than that of its non-vulnerable put twin, with an apparent difference.

Table 1 also provides us with the results for different values of probability of default, which are reflected by the changes of parameters b and p. When b is getting larger or p is getting smaller, the probability of default turns out to be larger. When b is smaller or p is larger, the critical asset prices for both vulnerable and non-vulnerable options become closer but not equal to each other. We conclude that if the probability of default is getting greater, the option holder will be likely to have a higher early exercise price.

The critical asset prices for both the vulnerable and non-vulnerable options are sensitive to changes of the other parameters as well. If the dividend yield decreases, the strike price increases, or the riskless interest rate increases, the critical asset prices for both the vulnerable and non-vulnerable put options will increase.

When we change the value of the continuous dividend yield, the result is quite different with what is shown in Klein and Yang's paper. For example, when q changes from 0.075 to 0.025, Klein and Yang showed 5.1% to 4.9%. Our result shows 37.2% to 11.1%. This is due to the critical asset price change in the vulnerable option. For instance, if q = 0.075, we have a value of 38.0 compared with 29.1 shown in Klein and Yang (2013). We conclude that the early exercise price in the intensity model is more sensitive to the value change of q than in the structural model in Klein and Yang (2013).

Changing the value of the riskless interest rate has the greatest effect on our results. For example, changing the value of riskless interest rate r from 0.05 (base case) to 0.025 increases the difference of the critical asset prices for vulnerable put and its non-vulnerable twin from 19% to 75%. This is not consistent with what is presented in Klein and Yang's paper. We further study this issue and test our model with a zero value of r. As a result, the non-vulnerable option will never be early exercised. The vulnerable option will be optimal to be exercised immediately with an underling asset price of 34.3. Table 2 shows a summary of this result.

Table 2: Effects of r changes on the early exercise policies between the two types of puts

	Vulnerable American put	Non-vulnerable American put
r = 0.075	$S_c^{\wedge} = 41.2$	$S_c = 37.7$
r = 0.025	$S_c^{\wedge} = 37.4$	$S_c = 21.4$
r = 0	$S_c^{\wedge} = 34.3$	No early exercise

We noticed that when the value of r decreases, the early exercise policy changes significantly in the non-vulnerable option case. For example, when r drops from 0.075 to 0.025, the early exercise price drops about 43% (= (37.7-21.4) / 37.7). However, when we consider the vulnerable option, the early exercise price drops only about 9% (= (41.2-37.4) / 41.2). By comparing these two numbers, we can see the early exercise price is not changing as that much as in the vulnerable option. This might because the effect of r on the early exercise price is partially offset by the effect of the credit process of our model. Additionally, by comparing to the results of Klein and Yang (Figure 5), when r drops from 0.075 to 0.025, $S_c^{\wedge}$ is 22.4 when we consider the vulnerable option, which generates about 43% (= (39.2-22.4) / 39.2) difference. By comparing the two differences 9% vs 43%, we conclude that the intensity model is less sensitive to the value change of r.

In contrast, if we change the strike price, the difference in the critical asset prices for the two types of puts is 18.9% vs 19.4%. In Klein and Yang, the difference is 5.5% vs 5.7%. Although the number is quite different, we can see the difference is not changing greatly when we change the strike. This is consistent with Klein and Yang's results.

Changing the time to maturity or the volatility of the underlying asset price affects the critical asset prices for the two types of puts as well. As we can see the results from Table 1, increasing the value of time to maturity (T) and volatility of the underlying asset (σ) will decrease the value of the early exercise price and increase the difference of critical asset prices between vulnerable put and its twin. This is due to the fact that these two parameters will have effects on the time value of the options. For the option holders, they would like to choose an earlier exercise to avoid the potential default of option writers. If they choose to exercise their options earlier, it means that they have to give up the time value of those options (Klein and Yang 2013). In other words, if their option's value increases later, they will lose the time value in the case of early exercise. As a result, counterparty credit risk will have a greater influence on the critical asset price for options that have potential high time value. These results are consistent with Klein and Yang's results.

At last, we want to test our model by setting the value of p or b to be negative as shown in the last two rows of Table 1, which replicate the cases of ρ in Klein and Yang (2013). After we make any of the changes above to our model, the intensity and the underlying asset price become positively correlated. This implies that if the underlying asset price increases, then the probability of default will also become greater. This result is comparable with the case that dZ_v and dZ_s have a negative correlation which is the case of $\rho = -0.5$ in Klein and Yang (2013). They show that if $\rho = -0.5$, then the immediate exercise price for the vulnerable put will become 33.7, which is lower than 35.1 in the base case. In our model, firstly, we change the value of p from 0.06 to -0.06. In order to achieve the same initial intensity in our base case, we multiply the value of b in our base

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case by $S^{2p} = 50^{2*(-0.06)} = 0.6254$ to get a new value of b as 0.4109. Next, instead of the

changes above, we let the value of b to be negative as -0.657139. In order to replicate the initial

intensity in our base case, we set the value of a as $\frac{-2b}{S^p} = \frac{-2*(-0.657139)}{50^{0.06}} = 1.0393$. In the two new

cases, both of the immediate exercise prices are 39.5 for the vulnerable put, which is also lower

than 39.6 in our base case. Therefore, our results are consistent with what Klein and Yang did.

In general, we found that the early exercise price for the vulnerable option is quite

different with those shown in Klein and Yang (2013). For example, the result for vulnerable put

in our base case is 39.6 and it is 35.1 in Klein and Yang's result. The early exercise price is 41.2

if change the value of the interest rate r to 0.075, while Klein and Yang showed 39.2 in their case

etc. Moreover, we find that our results are larger than those in Klein and Yang's paper for all the

cases. This is due to the reason that the credit process differs over time as compared to Klein and

Yang, meaning the intensity approach generates greater early exercise price than the structural

model. We suspect that the value of probability of default may be the cause of the differences.

Further, we consider the probability of default is greater than the ones in the structural model.

Therefore, the option holder is likely to early exercise the option at a higher price comparing with

the structural model. Leland studied the behavior of probability of default in structural model. He

calculated the distance to default, which is measured by the number of standard deviation

between the log ratio of the expected future asset and KMV default boundary at t. Then, he

mapped it to the default probabilities by using the standard normal distribution. He concluded that

structural model would under-predict the short-term default probabilities (Leland 2004). This

conclusion is consistent with our intuitions. We show our conditional and cumulative probability

of default through time in the following plots. In order to see that the probability is increasing, we

choose a larger value of T, which is 20 in this case. As we can see from Figure 6 and 7, the value

of probability of default increases very slowly through time. Figure 8 shows an example from

Leland's paper. The dotted line is the given probability of default from Moody 's . The solid line

plots the cumulative probability of default predicted by the Leland and Toft' model. The short-term probability defaults are underestimated (Leland 2004).
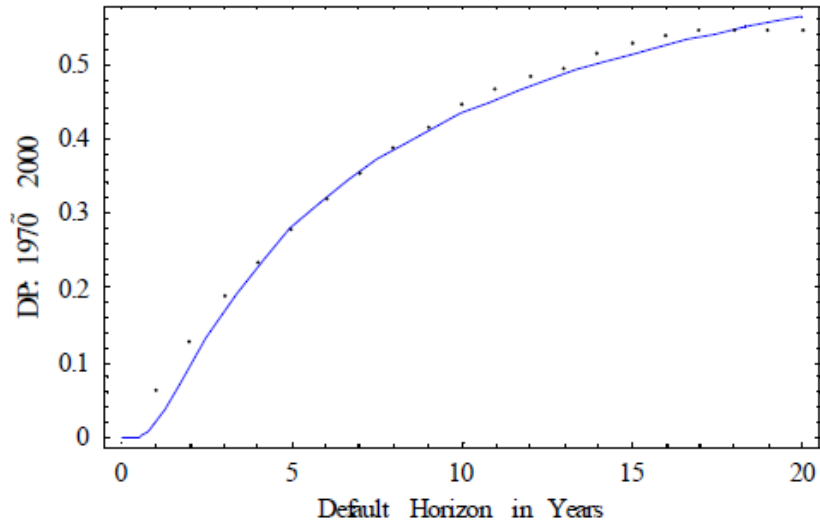

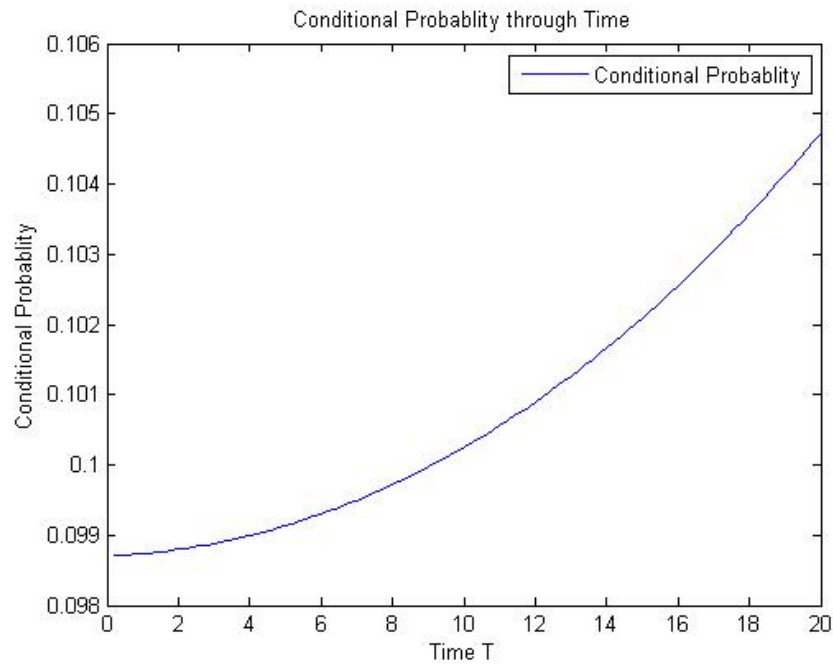*Figure 6: Conditional Probability through Time*

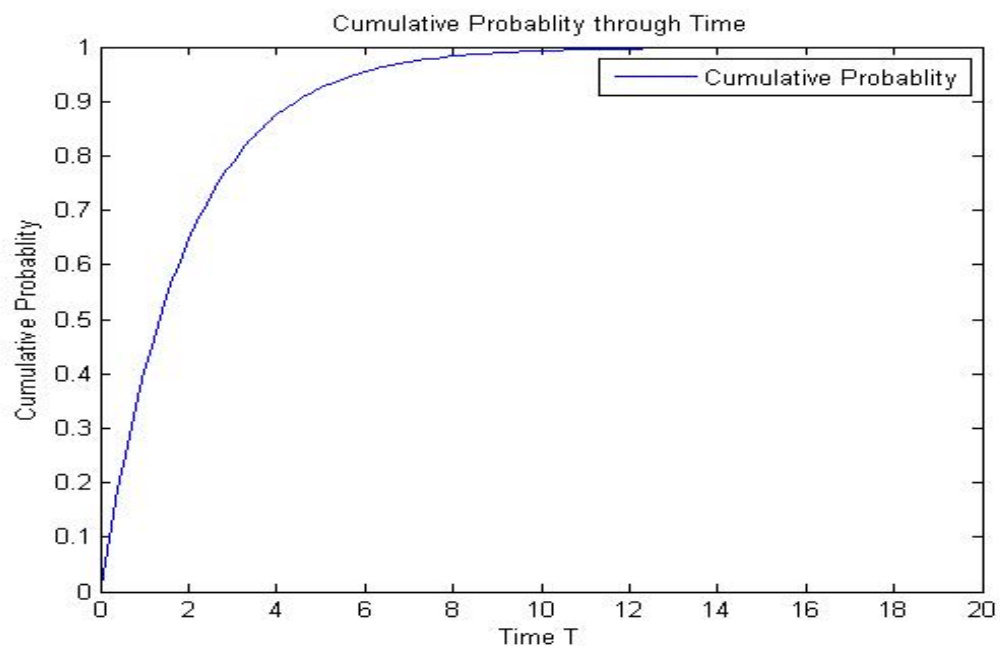*Figure 7: Cumulative Probability through Time*

Figure 8: Cumulative Default Probabilities through Time in Structural Model (Leland 2004)

Table 3: Parameter changing effects on the option prices between the two types of puts

	Non-vulnerable put price f	Vulnerable put price f$^{*\wedge}$	Percentage reduction vs. f (%)
Base Case	5.21	4.11	21.20
b = 1.0	5.21	3.82	26.74
b = 0.8	5.21	3.98	23.75
b = 0.4	5.21	4.42	15.25
b = 0.2	5.21	4.75	8.88
p = 0.02	5.21	4.00	23.20
p = 0.04	5.21	4.06	22.19
p = 0.08	5.21	4.16	20.22
p = 0.1	5.21	4.21	19.25
X = 55	8.18	6.79	17.04
X = 45	2.96	2.23	24.49
r = 0.075	4.35	3.54	18.52
r = 0.025	6.45	4.83	25.06
q = 0.075	6.15	4.68	23.83
q = 0.025	4.47	3.62	19.06
T = 3	6.17	4.58	25.81
T = 1	3.82	3.30	13.78
σ = 0.3	7.79	6.14	21.17
σ = 0.1	2.61	2.06	21.21
ω = 0	5.21	3.23	38.14
ω = 1	5.21	5.21	0

Table 3 demonstrates the effects of the same parameters' changes as in Table 1 on the

option prices for vulnerable put and its non-vulnerable twin, so the first column is the same with

it in Table 1. The second and third columns of Table 3 state the values of non-vulnerable

American put (f) and its vulnerable American twin (f$^{*\wedge}$). Additionally, the last column of Table 3

indicates the percentage reduction of the vulnerable put option value compared with its non-vulnerable twin.

Our base case is set up as before such that S = 50, X = 50, σ = 0.2, T = 2, time steps = 100, r = 0.05, q = 0.05, b = 0.657139, p = 0.06 and ω = 0.5. By doing this, we make sure our result is matched with what Klein and Yang (2013) did through the structural model (see Figure 9 below). Our base case result shows that the non-vulnerable option has the value 5.21 today and the vulnerable option has the value of 4.11. This generates 21.2% reduction in value in terms of the non-vulnerable option value. This result exactly matches with what Klein and Yang did in their research, as shown in Figure 9.

Effect of Counterparty Credit Risk on Value of American Put Options

In the base case, the parameters are $S = 50$, $K = 50$, $\sigma_c = 0.2$, $T = 2$, $r = 0.05$, $q = 0.05$, $V = 5000$, $D* = 4500$, $\sigma_V = 0.2$, $\rho = 0$, $\alpha = 0.5$. Non-vulnerable option values are in dollars.

	Nonvulnerable Price (f)	Percentage reduction vs. f f^{**}
Base Case	5.21	21.2%
$V = 4500$	5.21	40.7%
$V = 4600$	5.21	38.5%
$V = 4800$	5.21	28.9%
$V = 5200$	5.21	15.3%
$V = 6000$	5.22	3.7%
$\sigma_V = 0.3$	5.21	30.4%
$\sigma_V = 0.1$	5.21	5.2%
$\alpha = 1$	5.21	35.4%
$S = 55$	3.45	24.6%
$S = 45$	7.69	14.7%
$K = 55$	8.18	15.5%
$K = 45$	2.96	24.6%
$r = 0.075$	4.35	18.0%
$r = 0.025$	6.45	25.0%
$q = 0.075$	6.25	22.6%
$q = 0.025$	4.47	19.8%
$T = 3$	6.17	20.3%
$T = 1$	3.82	16.4%
$\sigma_S = 0.3$	7.79	21.2%
$\sigma_S = 0.1$	2.61	21.2%
$\rho = 0.5$	5.21	24.7%
$\rho = -0.5$	5.21	18.3%

Figure 9: Exhibit 2A in Klein and Yang (2013)

We further study the effect of changing our intensity parameters such as b and p in our model. Firstly, we change the value of b. Based on the intensity formula below, the intensity will become greater if we increase the value of b, which will further increase the probability of default. Based on our numerical results, we can find that the value of the vulnerable option decreases with an increasing value of b. This implies that greater the probability of default, less the value of the option. Next, we study the effect of changing value in p, which is another parameter in deciding the value of intensity. We can also find that the value of intensity will

decrease if we increase the value of p according to the formula below. From Table 3, we can see that when the value of p increases, the option value increases.

$$\lambda(t, S) = a + b / S^p,$$

In sum, we conclude that, the intensity value indeed has effect on the value of the options, which is that the option will have a greater value if the option is less likely to default. This conclusion appears to be the same result as what was concluded in Klein and Yang (2013).

The option value for both the non-vulnerable and vulnerable option can be affected by the changes in other parameters as well. We present them in the following.

From Table 3, we see the changes of values in both non-vulnerable and vulnerable put if we change the value of the strike price. When the strike price increases, the option value increases. The results in our table also show that non-vulnerable option value is less sensitive to the change of strike price than the vulnerable option value. We calculate the following percentage change in value: the non-vulnerable option value changes (8.18 (X = 55) – 5.21 (X =50))/5.21 (X =50) = 57%. However, the vulnerable option shows (6.79 (X = 55) – 4.11 (X =50))/4.11(X= 50) = 65% change in its value.

Next, we want to test the effect on the option value from changing the value of r, q and σ. From Table 3, we can see that the option value increases when we decrease the value of r, increase the value of q or increase the value of σ. As well, we test how sensitive the value changes of the two types of puts when we change the value of r, q and σ.

For example, in the non-vulnerable option, the value changes (6.15 (q = 0.075) – 5.21 (q = 0.05)) / 5.21 (q = 0.05) = 18%. However, the vulnerable options shows (4.68 (q = 0.075) – 4.11 (q=0.05)) / 4.11 (q=0.05) = 14% change in its values. These results support the conclusion that the two types of puts have similar sensitivity to the value change of q. Further, we see the percentage reduction values in Table 3 and Figure 9 are very close. 22.6% vs 23.8% (q = 0.075)

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and 19.8% vs 19.0% (q = 0.025). This implies that the effect of changing the value of q in structural model and intensity model are very similar.

The same conclusion can be derived if we change the value of r and the volatility of the underlying asset σ.

We test our results when we change the value of time to maturity T. From Table 3, we see that the value of both non-vulnerable and vulnerable option increase when the value of T increases. As we mentioned in the results discussion section of Table 1, greater the value of T is, greater the time value of the option is, thus greater the effect of credit risk is. We notice that Klein and Yang showed 20.3% and 16.4% change when the value of T becomes 3 and 1 respectively. Our results shows 25.8% and 13.8 % compared to the results in Klein and Yang (2013). Therefore, we have a higher value of percentage reduction vs. f when T is larger. This result is consistent with what is shown in Klein and Yang (2013).

Lastly, we test the effect of value change of the recovery rate. When the value of ω goes from 0 to 1, the vulnerable option will appear to be the non-vulnerable option. By definition, ω = 1 stands for if the option holder defaults the option holder will lose nothing, which is equivalent to a non-vulnerable option. Therefore, the value is exactly the same as the non-vulnerable case, which is 5.21. When ω = 0, the option holder will get northing and the option value drops if the option writer defaults.

In all, we test all the parameters in our model. The numerical results show that the value of vulnerable option is always lower than that of its non-vulnerable twins. This result is consistent with Klein and Yang's result. Further, according to Table 3, we show that in the intensity model, changing the value of r, q, σ and T has very similar effect in the option price compared to what Klein and Yang showed in their paper.

5: Conclusion

The main purpose of this thesis is to apply the intensity model to verify that the early exercise policy will be affected in a vulnerable American option. We follow what Klein and Yang did in their paper. We retain the assumptions hold by Klein and Yang (2013) that in the event of option writer defaults, the option holder's nominal claim is the value of the otherwise identical non-vulnerable option. We also follow the assumption that the default can occur any time before the expiry. Based on these assumptions, we come up with the critical asset price for both non-vulnerable and vulnerable options. We provide numerical examples to verify that the early exercise price will be affected by the counterparty credit risk. We further study the effect of the counterparty credit risk on the value of the options. Numerical results are also provided.

The main contribution of this thesis is to use the intensity approach to model the counterparty credit risk on OTC options. The intensity model has several advantages in modelling the credit process. For example, the data used in this model is more practical and easier to be obtained than what is used in the structural model. Lastly, the intensity model is easier to be implemented.

The numerical results in our thesis confirm that the intensity model can be applied to test the early exercise policy and price the vulnerable American options. The model can be employed to a general vulnerable American option with parameter adjustments.

Appendices

Appendix A Program: Function for Non-vulnerable American Put (matlab program)

```matlab
function [stockTree,valueTree,Sc] = ...
    NonvulnerableAmericanOption(S0,X,r,q,sig,dt,steps,drift)

% CRR model parameters
a = exp((r-q)*dt);
u = exp(drift*dt + sig*sqrt(dt));
d = exp(drift*dt - sig*sqrt(dt));
p = (a-d)/(u-d);

% Loop over each node and simulate the underlying stock prices
stockTree = nan(steps+1,steps+1);
Sc = zeros(steps+1,steps+1);

stockTree(1,1) = S0;
for idx = 2:steps+1
    stockTree(1:idx-1,idx) = stockTree(1:idx-1,idx-1)*u;
    stockTree(idx,idx) = stockTree(idx-1,idx-1)*d;
end

valueTree = nan(size(stockTree));

% Calculate the value at expiry

valueTree(:,end) = max(X-stockTree(:,end),0);

steps = size(stockTree,2)-1;

% Loop backwards to get option value
for idx = steps:-1:1

    valueTree(1:idx,idx) = exp(-r*dt)*(p*valueTree(1:idx,idx+1) +...
        (1-p)*valueTree(2:idx+1,idx+1));

    valueTree(1:idx,idx) = max(X-stockTree(1:idx,idx),valueTree(1:idx,idx));

    %   Keep track of the early exericse Price
    for jdx = 1: idx

        if (valueTree(jdx,idx) == X-stockTree(jdx,idx) )
            Sc(jdx,idx) = stockTree(jdx,idx);
        end

    end
end

end
```

Appendix B Program: Function for Vulnerable American Put

(matlab program)

```matlab
function [ valueTree_StarHat,Sc_Hat ]...
    = VulnerableAmericanOption( S0,X,r,q,sig,dt,steps,PdCond,omega,drift )

% CRR model parameters
a = exp((r-q)*dt);
u = exp(drift*dt + sig*sqrt(dt));
d = exp(drift*dt - sig*sqrt(dt));
p = (a-d)/(u-d);

% Loop over each node and simulate the underlying stock prices
stockTree = nan(steps+1,steps+1);

stockTree(1,1) = S0;
for idx = 2:steps+1
    stockTree(1:idx-1,idx) = stockTree(1:idx-1,idx-1)*u;
    stockTree(idx,idx) = stockTree(idx-1,idx-1)*d;
end

% Recalculate the option value in the non-vunlerable option
[~,valueTree,~] = NonvulnerableAmericanOption(S0,X,r,q,sig,dt,steps,drift);

% Preallocate the option value output
valueTree_StarHat = nan(size(stockTree));
Sc_Hat = zeros(steps+1,steps+1);

% Option value at expiry
valueTree_StarHat(:,end) = omega*PdCond(:,end).*valueTree(:,end)...
    +(1-PdCond(:,end)).*max(X-stockTree(:,end),0);

% Loop backwards to get option value
steps = size(stockTree,2)-1;

for idx = steps:-1:1

    valueTree_StarHat(1:idx,idx) =...
        exp(-r*dt)*(p*valueTree_StarHat(1:idx,idx+1) + (1-p)*valueTree_StarHat(2:idx+1,idx+1));

    valueTree_StarHat(1:idx,idx) = omega*PdCond(1:idx,idx).*valueTree(1:idx,idx)+(1-
PdCond(1:idx,idx))...
        .*max((X-stockTree(1:idx,idx)), valueTree_StarHat(1:idx,idx));

    %       Keep track of the early exercise price
    for jdx = 1: idx

        if  ( valueTree_StarHat(jdx,idx) == PdCond(jdx,idx)*omega*valueTree(jdx,idx)...
```

```
                +(1-PdCond(jdx,idx))*(X-stockTree(jdx,idx)));
            Sc_Hat(jdx,idx) = stockTree(jdx,idx);
        end

    end

end

end
```

Appendix C Program: Test Script

(matlab program)

```matlab
clear all
close all
clc

for S0 = 50:-0.1:0

    % Set S0 to be a fixed number and then calculate the option value

    %     S0 =50;
    X = 50;
    r = 0.05;
    q = 0.05;
    sig = 0.2;
    T = 2;
    steps = 100;
    dt = T/steps;

    a = 0;
    %     a = 1.0393;

    b = 0.657139;
    %     b = 0.4109;
    %     b = -0.657139;

    pi = 0.06;
    %     pi = -0.06;

    omega = 0.5;

    drift = 0; %Conside the the CRR model with Drift only.

    [stockTree,valueTree,Sc] = ...
        NonvulnerableAmericanOption(S0,X,r,q,sig,dt,steps,drift);

    tMat = repmat((0:dt:dt*steps),(steps+1),1);

    % Calculate intensityTree
    intensityTree = a + b./(stockTree.^pi);

    % Calculate Tree for probability of default by each step
    PdTree = 1-exp(-tMat.*intensityTree);

    PdUC = zeros(size(PdTree));
    PdCond = zeros(size(PdTree));
```

```matlab
% Calculat the unconditional probablity
PdUC(1,2:end)  = PdTree(1,2:end) - PdTree(1,1:end-1);
PdCond(1,2:end) = PdUC(1,2:end)./(1-PdTree(1,1:end-1));

%  Calculate the conditional probablity

for idx = 2 : 1:  size(PdTree,2)

    for jdx = 2: idx

       Ave = ( PdTree(jdx-1,idx-1) + PdTree(jdx,idx-1))/2;
       PdUC(jdx,idx) =PdTree(jdx,idx)- Ave;
       PdCond(jdx,idx) = PdUC(jdx,idx)./(1-Ave);
    end
    PdUC(idx,idx)  = PdTree(idx,idx) - PdTree(idx-1,idx-1);
    PdCond(idx,idx) = PdUC(idx,idx)./(1-PdTree(idx-1,idx-1));

end

[ valueTree_StarHat,Sc_Hat ] =...
    VulnerableAmericanOption( S0,X,r,q,sig,dt,steps,PdCond,omega,drift );

%    if Sc(1,1) ~= 0
%        Sc(1,1)
%        break
%    end

    if Sc_Hat(1,1) ~= 0
    Sc_Hat(1,1)
    break
    end
end
```

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