

UNDERSTANDING RISKS IN A HYBRID PENSION PLAN WITH STOCHASTIC RATES OF RETURN

by

Tingting Wen

B.Sc., Sun Yat-sen University, 2011

A PROJECT SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in the

Department of Statistics and Actuarial Science
Faculty of Sciences

© Tingting Wen 2014

SIMON FRASER UNIVERSITY

Spring 2014

All rights reserved.

However, in accordance with the *Copyright Act of Canada*, this work may be reproduced without authorization under the conditions for “Fair Dealing.” Therefore, limited reproduction of this work for the purposes of private study, research, criticism, review and news reporting is likely to be in accordance with the law, particularly if cited appropriately.

APPROVAL

Name: Tingting Wen
Degree: Master of Science
Title of Project: UNDERSTANDING RISKS IN A HYBRID PENSION PLAN
WITH STOCHASTIC RATES OF RETURN

Examining Committee: Dr. Tim Swartz
Professor
Chair

Dr. Yi Lu
Associate Professor
Senior Supervisor
Simon Fraser University

Ms. Barbara Sanders
Assistant Professor
Supervisor
Simon Fraser University

Dr. Gary Parker
Associate Professor
External Examiner
Simon Fraser University

Date Approved: March 3rd, 2014

Partial Copyright Licence



The author, whose copyright is declared on the title page of this work, has granted to Simon Fraser University the non-exclusive, royalty-free right to include a digital copy of this thesis, project or extended essay[s] and associated supplemental files ("Work") (title[s] below) in Summit, the Institutional Research Repository at SFU. SFU may also make copies of the Work for purposes of a scholarly or research nature; for users of the SFU Library; or in response to a request from another library, or educational institution, on SFU's own behalf or for one of its users. Distribution may be in any form.

The author has further agreed that SFU may keep more than one copy of the Work for purposes of back-up and security; and that SFU may, without changing the content, translate, if technically possible, the Work to any medium or format for the purpose of preserving the Work and facilitating the exercise of SFU's rights under this licence.

It is understood that copying, publication, or public performance of the Work for commercial purposes shall not be allowed without the author's written permission.

While granting the above uses to SFU, the author retains copyright ownership and moral rights in the Work, and may deal with the copyright in the Work in any way consistent with the terms of this licence, including the right to change the Work for subsequent purposes, including editing and publishing the Work in whole or in part, and licensing the content to other parties as the author may desire.

The author represents and warrants that he/she has the right to grant the rights contained in this licence and that the Work does not, to the best of the author's knowledge, infringe upon anyone's copyright. The author has obtained written copyright permission, where required, for the use of any third-party copyrighted material contained in the Work. The author represents and warrants that the Work is his/her own original work and that he/she has not previously assigned or relinquished the rights conferred in this licence.

Simon Fraser University Library
Burnaby, British Columbia, Canada

revised Fall 2013

Abstract

The solvency risk, contribution rate risk, and benefit risk of a hybrid pension plan with stochastic investment returns are studied in this project. Gaussian, autoregressive and moving average processes are used to model the rate of return. The first two moments of the funding level, the contribution rate and the benefit payment are presented both at the stationary status and during evolution. Three investment strategies are considered and the risks generated in the hybrid pension plan are compared. Different sets of valuation rates of interest are used to understand the impact of regulative environmental change on the hybrid pension plan. The trade-off between the contribution and benefit risks and the optimum region of risk sharing are discussed to provide an insight of the relationship between plan sponsors and employees under a hybrid pension plan.

Keywords: Hybrid Pension Plan; Investment Risk; Gaussian Process; AR(1) Process; MA(1) Process; Risk Sharing

To my beloved parents.

“It is our choices that show what we truly are, far more than our abilities. ”
— *Albus Dumbledore, HARRY POTTER AND THE CHAMBER OF SECRETS, 1998*

Acknowledgments

I would like to take this opportunity to express my deepest gratitude to people who helped me throughout my graduate studies.

First of all, I would like to thank my supervisor Dr. Yi Lu for her endless support, generosity, and gentle guidance. I can hardly imagine completing the thesis project without her substantial advice and great patience.

I also owe a lot of thanks to the members of the examining committee, Ms. Barbara Sanders and Dr. Gary Parker, for sharing their expertise in the area and giving me extremely helpful comments and suggestions for this project.

I am also grateful to the Department of Statistics and Actuarial Science for providing an amazing curriculum and enormous support for my graduate studies. Moreover, I am thankful to the fellow graduate students and alumni for their friendship during the pursuit of the degree, in particular Lilian Xia, Zhenhua Lin, Li Chen and Ting Zhang.

I am indebted to my dearest colleagues at Munich Reinsurance Toronto for their generous mentorship and friendship. Working with this fascinating team is an incredible experience, which has simulated my strong passion in actuarial practice.

Last but not least, I would like to thank my family, especially my parents for their unconditional love and support.

Contents

Approval	ii
Partial Copyright License	iii
Abstract	iv
Dedication	v
Quotation	vi
Acknowledgments	vii
Contents	viii
List of Tables	xi
List of Figures	xii
1 Introduction	1
1.1 Background and Motivation	1
1.2 Literature Review	2
1.2.1 Hybrid Pension Plan	2
1.2.2 Stochastic Rate of Return	5
1.3 Outline	10
2 Hybrid Pension Plan and Its Modeling	11
2.1 The Hybrid Pension Plan	11
2.2 Assumptions	12

2.3	Modelling and Risk Measurement	12
2.4	Defined Benefit Plan as a Special Case	14
2.5	Pseudo Hybrid Plan as a Special Case	15
3	Model and Assumptions	16
3.1	Interest Rate Models	16
3.1.1	Gaussian Model	16
3.1.2	AR(1) Model	17
3.1.3	MA(1) Model	19
3.2	Risk Measurement of Hybrid Pension Plan	20
3.2.1	Gaussian Model	22
3.2.2	AR(1) Model	24
3.2.3	MA(1) Model	26
4	Numerical Illustrations: Gaussian Case	28
4.1	Assumptions	28
4.2	Pension Funding Method	30
4.3	Gaussian Model	31
4.3.1	Benchmark Case	31
4.3.2	Different Investment Strategies	34
4.3.3	Different Valuation Interest Rates	35
4.3.4	Plan Evolution	39
4.3.5	Some Relationships	40
5	Numerical Illustration: AR(1) Case	45
5.1	Benchmark Case	45
5.2	Varying Spread Periods and Model Parameters	49
5.3	Varying Investment Strategies	51
5.4	Varying Valuation Rate of Interest	55
5.5	Some Relationships	56
6	Numerical Illustration: MA(1) Case	59
6.1	Benchmark Case	59
6.2	Varying Spread Period and Model Parameter	63

6.3	Varying Investment Strategy	64
6.4	Varying Valuation Rate of Interest	64
6.5	Some Relationships	65
7	Conclusion	70
	Bibliography	73
	Appendix A Derivations	76
A.1	The Derivation of Equation (3.6)	76
A.2	The Derivation of Equation (3.7)	77
A.3	The Derivation of Equation (3.11)	78
A.4	The Derivation of Equation (3.12)	78
A.5	The Derivation of Equation (3.40)	79
	Appendix B Mortality Table	80

List of Tables

- 1.1 Major References and Project Outline 10
- 4.1 Parameters Used under Different Investment Strategies 29
- 4.2 Means of the Funding Level, Contribution Rate and Benefit Payment 32
- 6.1 Summary of Observations in Plan Evolutions 60

List of Figures

4.1	Coefficients of Variation under Neutral Investment Strategy with Different Spread Periods ($i_v = i$)	33
4.2	Coefficients of Variation of Funding Level with Different Investment Strategies and Spread Periods ($i_v = i$)	34
4.3	Coefficient of Variation of Annual Benefit Payment with Different Investment Strategies and Spread Periods ($i_v = i$)	35
4.4	Mean of Funding Level with Different Spread Periods ($i_v \neq i$)	36
4.5	Mean of Annual Contribution and Benefit Payment with Different Spread Periods ($i_v \neq i$)	37
4.6	Standard Deviation of Funding Level with Different Spread Periods ($i_v \neq i$)	38
4.7	Standard Deviation of Annual Contribution and Benefit payment with Different Spread Periods ($i_v \neq i$)	39
4.8	Coefficients of Variation of Funding level at Different Times with Different Spread Periods under Neutral Strategy ($i_v = i$)	40
4.9	Coefficients of Variation of Contribution and Benefit at Different Times with Different Spread Periods under Neutral Strategy ($i_v = i$)	41
4.10	Standard Deviations of Contribution and Benefit with Different Spread Parameters under Neutral Strategy ($i_v = i$)	42
4.11	Aggregate Risk with Different Spread Parameters under Neutral Strategy ($i_v = i$)	43
4.12	Aggregate Risk with Different Spread Parameters under Different Investment Strategies ($i_v = i$)	44
5.1	Simulated Forces of Interest Using Gaussian and AR(1) Processes	46
5.2	Evolution of Means of Funding, Contribution and Benefit ($i_v = i$)	47

5.3	Evolution of Coefficients of Variation of Funding, Contribution and Benefit ($i_v = i$)	48
5.4	Means of Funding, Contribution, and Benefit with Different Spread Periods and ϕ_2 ($i_v = i$)	50
5.5	Simulated Discrete Rates of Return Using AR(1) Process	51
5.6	Coefficients of Variation (CV) of Funding, Contribution, and Benefit with Different Spread Periods and ϕ_2 ($i_v = i$)	52
5.7	Means of Contribution, Benefit under Different Investment Strategies ($i_v = i$)	53
5.8	Standard Deviations of Contribution and Benefit under Different Investment Strategies ($i_v = i$)	54
5.9	Standard Deviations of Contribution and Benefit with Different Spread Peri- ods ($i_v \neq i$)	55
5.10	Standard Deviations of Contribution and Benefit with Different Spread Pa- rameters ($\phi_2 = 0.5, i_v = i$)	56
5.11	Standard Deviations of Contribution and Benefit with Different Spread Pa- rameters ($\phi_2 = 0.01, i_v = i$)	57
5.12	Aggregate Risk with Different Spread Parameters for $\phi_2 = 0.5$ (left) and $\phi_2 = 0.01$ (right) ($i_v = i$)	58
6.1	Evolution of the Means of the Contribution, Benefit, and Funding Level ($i_v = i$)	61
6.2	Evolution of the Standard Deviations of the Contribution, Benefit, and Fund- ing Level ($i_v = i$)	62
6.3	Coefficients of Variation of Contribution and Benefit with Different Spread Periods and ϕ_3 ($i_v = i$)	64
6.4	Coefficients of Variation of Benefit and Funding with Different Investment Strategies ($i_v = i$)	65
6.5	Standard Deviations of Contribution and Benefit with Different Means ($i_v \neq i$)	66
6.6	Standard Deviations of Contribution and Benefit with Different Spread Pa- rameters ($\phi_3 = -0.7, i_v = i$)	67
6.7	Standard Deviations of Contribution and Benefit with Different Spread Pa- rameters ($\phi_3 = -0.01, i_v = i$)	68
6.8	Aggregate Risk with Different Spread Parameters for $\phi_3 = -0.7$ (left) and $\phi_3 = -0.01$ (right) ($i_v = i$)	69

Chapter 1

Introduction

1.1 Background and Motivation

We live in a world full of uncertainties. When it comes to saving for retirement, the risks in the investment return of a pension fund have been a challenge for both employers and employees. The shortcomings of the two typical pension structures are well known. A defined benefit (DB) plan ensures the promised retirement benefit to employees, while shifting all the risks to plan sponsors. By predetermining the contribution rates required, a defined contribution (DC) plan leaves the participants to worry about the value of the accumulated savings at the time of retirement. Both plans have their disadvantages and attempts to share the risk among both the plan sponsors and employees have been around for a while. Hybrid pension plans which combine features of both DB plans and DC plans are a great solution.

In practice, there still exists lots of limitations of legislation and plan management challenges for hybrid pension plans. In addition, more theoretical study of hybrid pension plans is needed. This project tries to answer the questions below so as to add to the understanding of the plan dynamics.

1. How does the hybrid plan behave and evolve when the investment rate of return is stochastic?
2. How do different investment strategies affect the risks embedded in the plan?
3. How do different valuation rates of interest affect the risks embedded in the plan?

4. Is there an optimal risk sharing scheme that can help the negotiations between plan sponsors and participants?

1.2 Literature Review

In this section, we review some research papers in the literature that are closely related to the topic studied in this project. This review is composed of two parts. The first part covers different forms of hybrid pension plans that have been proposed in the actuarial science literature. The second part reviews the previous work conducted to study the stochastic rates of return in pension funds.

1.2.1 Hybrid Pension Plan

Since the 1980s, hybrid pension plans have existed and evolved into very different forms. Many pension plans are named ‘hybrid’ because they are neither a pure defined benefit plan nor a pure defined contribution plan. In a broad sense, as long as a pension plan has some of the characteristics of both plans, it can be called ‘hybrid’.

A comprehensive summary of hybrid pension plan practice over the world was presented by Wesbroom et al. (2005). In this paper, hybrid pension plan was defined as ‘private pension schemes which are neither pure Defined Benefit (DB) nor Defined Contribution (DC) arrangements, where pure DB arrangements are taken to mean final salary pension schemes’. Under this definition, they categorized the available plan schemes into these types: career average and career average revalued earnings plans, sequential hybrids where members can join a DB scheme after a period of DC membership, combination hybrids where both DB and DC benefits are accrued, final salary lump sum plans, self-annuitising plans where a DC plan offers an in-house annuity option (rather than an open market option), underpin arrangements where the benefit is calculated as the greater of a DB or a DC benefit (also known as a floor-offset plan), and cash balance or retirement balance plans. The advantages and disadvantages of these plans were compared and a vision of the development of these designs were enlisted.

The hybrid plans that are intensively studied in the actuarial literature are the floor-offset pension plan, the cash balance plan, and the pooled variable balance plan. We briefly

summarize them below.

i. Floor-Offset Pension Plan

A floor-offset pension plan usually provides a guaranteed minimum benefit with a top up tied to a DC account. It is also called ‘underpin arrangement’, or the ‘greater of’ benefit plan. The DC feature of the plan gives members the benefit of favourable investment performance. The guaranteed minimum benefit is predetermined to protect members from market downturns. The minimum benefit is calculated from a formula which accounts for age, years of service, etc.. If the DC account provides a benefit equal to or higher than the minimum benefit, the employees receive the DC benefit. If the DC account does not meet the minimum benefit, the DB plan fills the gap. An example of this scheme is the York University Pension Plan¹.

Sherris (1995) considered pricing the ‘greater of’ benefit plan. The ‘greater of’ benefits are assumed to be a function of two state variables, the rate of return, and the growth rate of salary. It applied a contingent claims valuation approach using options and obtained a partial differential equation for the benefit value. Simulation results showed that a traditional deterministic actuarial valuation understates the plan costs by as much as 35 percent. However, this paper did not consider funding or risk management issues.

Similar to Sherris (1995), Bacinello (2000) proposed a scheme with a ‘greater-of’ benefit option, which provides the option to exchange benefits calculated from a DB formula with benefits calculated from a DC formula, or vice versa. Bacinello (2000) presented a valuation model, in which nominal interest rates, retail price index, unit value of investment portfolio and individual salary are modeled as state-variables.

Blake (1998) described a DC plan with a DB underpin and used option pricing theory and different pension funding methods to price the plan, and proposed some approaches to smooth the guarantee costs.

Interested readers can also refer to Chen and Hardy (2009, 2010) for more recent work on this type of plan.

A variation on the ‘floor-offset’ idea is a ‘protected DC’ plan where the sponsor guarantees a minimum return on investments. Mody (2004) suggested a target retirement account assigned to each individual member. In the proposed plan, the investment strategy is under

¹http://www.yorku.ca/hr/documents/pension/York.University_Pension.Plan.pdf

the control of plan trustees, and the rate of accumulation is guaranteed at a minimum to protect members from adverse experience. In spite of the many advantages claimed for this plan design, there was no formulation or numerical proof of the idea.

ii. Cash Balance Plan

A cash balance plan is a defined benefit plan with some defined contribution features. The benefit of the plan is a promised lump-sum payment or annuity at retirement. Each member has a notional account. The employer promises to make contributions (usually a percentage of salary) to the account, and to credit the account with a specified rate of return. The employer invests the contributions, retains earnings and bears the investment losses.

The differences between a cash balance plan and a DB plan are:

- each member has a hypothetical individual account, which is similar to a DC plan;
- the benefit is a lump-sum amount in a cash balance plan, as opposed to a series of monthly payments for life in a DB plan. That is to say, the investment risk and longevity risk are transferred to the members at the time of retirement in a cash balance plan.

Cash balance plans differ from DC plans in the sense that the size of an individual account is not directly linked to the returns of investments. The returns in employees' accounts may be guaranteed, or smoothed by sponsors, or subject to some form of underwriting by the scheme. Often, benefits are subject to some manual adjustments and therefore are smoother and more predictable than the benefits under DC plans.

Hardy et al. (2013) analysed the cash balance plan to give a market consistent valuation of the costs of the liabilities of the plan, using the tools of financial economics.

iii. Pooled Variable Benefit Plan

Pooled variable benefit plans have fixed employer contributions, which are pooled together in a pension fund and invested. The plan usually sets a target benefit using a defined benefit plan formula. The target benefit acts as a goal, not a promise of benefit to the employees. Employers are not obliged to achieve the goal. When there is a deficit in the fund, the benefits are reduced. If there is a surplus in the fund, plan managers can save it

to smooth future deficits, or spend the surplus by increasing the benefit. These plans are also known as target benefit plans in Canada.

An early example of a pooled variable benefit plan is found in Khorasanee (1995), which explored the possibility of operating an integrated DB/DC plan, whose defined benefit scale has a fixed target rate of payment and is linked to career average salaries. This design distinguishes itself from a pure DB plan by varying actual benefit accruals based on the surplus or deficit of the fund.

A ‘Variable payment Plan’ was proposed in Khorasanee and Ng (2000). The contribution allocated to each participant is a fixed contribution plus the difference between the asset shares and benefit payment at the same time. The benefit paid is equal to the product of a constant and the sum of the contribution allocations. As a result, the benefit depends on past investment experience.

Sanders (2010) explored a target benefit plan design with fixed contributions and future benefit accruals being adjusted by reference to an aggregate valuation.

Khorasanee (2012) modified the traditional DB design by adding an adjustment parameter to the annual benefit payment. The annual benefit payment is the annual target benefit payment in together with some adjustments from unfunded liability, and the annual contribution level is the normal cost adjusted by unfunded liability. This hybrid plan is discussed in details in Section 2.

1.2.2 Stochastic Rate of Return

The investment risk in DB plans has been studied under a stochastic approach extensively since the 1980s. Here we briefly cover some popular topics in this area, and then discuss in detail the major references used in this project, followed by a summary of some further discussions on DB and other pension plans.

Markov processes have proved themselves to be a powerful modeling tool when it comes to describing possible changes in economic, financial and demographic environment. Balcer and Sahin (1983) used a semi-Markov reward process to characterize the ultimate benefits, which enables the investigation of many structural characteristics of a pension plan. The employment patterns are captured by a two-state semi-Markov process. Combined with a reward function, they derived the first two moments of the benefit and contribution. To be realistic and to reflect the non-homogeneous properties of the parameters in pension modeling, Janssen and Manca (1997) furthered the theoretical work of Janssen and De Dominicis

(1984) and presented a discrete time non-homogeneous semi-Markov pension fund model. This model enables the study of different scenarios of salary inflation, population growth, etc. Chang and Cheng (2002) modified the model to include changes in plan demographics, showcased a procedure for implementing the proposed mechanism into a monitoring system, and illustrated the risk management tools in Taiwan Public Employees Retirement System. Implementation of their work requires a powerful computer environment to conduct the simulations.

There are papers studying the problem under a continuous time, which is close to reality since the value of a pension fund does evolve with time continuously. Interested readers can refer to O'Brien (1986), Dufresne (1990), Cairns (1996, 1997). However, it is worth noting that the differential equations are hard to understand and popularize, and applying the model also requires a lot of computational work. In addition, pension valuation and adjustment decisions are usually carried out once a year or less frequently.

Dynamic optimization is a very popular approach to determine the optimal contribution level of DB plans. Haberman and Sung (1994) proposed a dynamic model of pension funding that measures 'contribution rate risk' and 'solvency risk'. Dynamic optimization method is used to minimize the two risks in an objective function. Chang (1999) integrated this dynamic programming approach with a stochastic simulation to determine the optimal contribution level. An improvement of these two papers was presented by Chang et al. (2003). The performance criterion function is revised to give more weight to under-funding and over-contribution risks, since these scenarios raise more concern in practice. Instead of using the assumption of independent identically distributed rates of return, this paper assumed autoregressive rates of return.

As was mentioned earlier, the purpose of this project is to study the mechanism of hybrid pension plans when rates of return are stochastic. We focus on modelling investment returns and observe the patterns we get from the illustration results. With similar purposes, a lot of work has already been done for DB plans.

In Dufresne (1986a), the evolution of funding level and contribution level were studied using a mathematical model. It firstly examined and compared traditional actuarial cost methods; then the model was modified to include stochastic rates of return and inflation, in which case the other actuarial assumptions are borne out by experience. Recursive formulas were derived for the first two moments of funding and contribution level, and an optimal region was found under a so-called spread method. Similar work was conducted in

continuous time.

As a special case of Dufresne (1986a), Dufresne (1988) studied the moments of pension contributions and fund levels when only the rate of return was assumed to be stochastic. To simplify the formulas, it is assumed that the population is stationary, the salary increase is nil, and the valuation rate of interest is equal to the mean of the earned rates of return. This approach was adopted by other authors who aimed to focus on studying the stochastic rate of return. From the plot of the standard deviations of funding level against the standard deviations of the contribution level in the stationary case, it identifies an ‘inadmissible’ region for the spread period, where both standard deviations are an increasing function of the spread period. Both pension sponsors and members would prefer to move to a different region so as to reduce the risks taken by both parties. An ‘admissible’ region is also recognized, where there is a trade-off between the standard deviation of funding level and the standard deviation of contribution level. For finite time periods, there may or may not exist an ‘inadmissible’ region, depending on the spread period and the properties of the stochastic return.

Haberman (1990, 1991, 1994) developed Dufresne’s work further assuming that the rate of return follows an autoregressive (AR) process. To simplify the numerical illustration, the limiting values of the first two moments of the funding level and the contribution level were approximated under certain conditions. It was shown that the approximation works well when the coefficients are not too close to ± 1 . Since the variance of the contribution rate is proportional to the actuarial liability, an attempt to minimize the variance could lead to a reduction in actual liability. The paper used the coefficient of variation as the main risk measurement, which divides the standard deviation by the mean. We also use the coefficient of variation as the main measurement of risks in this project. For some combinations of parameters, Haberman found a similar pattern of relationships between funding level and contributions as the one described in Dufresne (1988). However, in some cases, all choices of spread period lay in an ‘admissible’ region; in other cases, the optimal spread period is 1 because both coefficients of variation are monotonically increasing functions of the spread period.

Parallel to Haberman (1994), similar results were derived when assuming the rate of return follows a moving average (MA) process in Haberman and Wong (1997). They observed similar patterns to what was found in Haberman (1994). Since the coefficient ϕ_3 plays a different role in MA process than in AR process, the relationship between the spread period

and the coefficient is different. Bédard and Dufresne (2001) generalized this result to a moving average process of any order, by recognizing the actuarial losses as a bilinear time series process. They also noticed that when the valuation rate of interest is set different from the average rate of return, the funding level does not converge on average to the actuarial liability.

Khorasanee (2012) modified the model in Dufresne (1988) and proposed a hybrid pension model, assuming that investment returns are independent and identically distributed. It assumed that the surpluses and deficits are amortized by adjusting both the benefit outgo and the contribution, which enables risk sharing between plan sponsors and members. The concept of ‘Aggregate Risk’ was introduced as the sum of the standard deviations of contribution level and benefit level. The optimum spread periods for different investment portfolios were illustrated. The performance of this hybrid plan was compared with a DC plan providing the same expected retirement benefit. It is shown that the hybrid plan is more efficient in managing investment risk than a DC plan.

Some extensions on the above framework are summarized below.

1. Dufresne (1989) investigated the stability conditions based on the first order and second order stability of actuarial losses. Gerrard and Haberman (1996) extended the analysis to an autoregressive model.
2. Instead of focusing on minimizing the variance of the contribution level in DB plan, Haberman (1997) explored the situation where the objective is to minimize the variability in the present value of future contributions.
3. Since previous work assumed that the valuation rate of interest is equal to the mean of the distribution of the earned rates of return, Cairns and Parker (1997) investigated the cases where the valuation rate of interest was set at different levels. An ‘efficient frontier’ was introduced which captured the minimum variance of contribution, given the level of mean contribution rate. They also found bounds for the moments of the funding level when rates of return follow an autoregressive process, which are more precise than the approximations in Haberman (1994). The moments and distribution functions of funding given the initial funding level were studied in their paper.
4. Owadally and Haberman (1999) compared three different methods of adjusting the unfunded liability when rates of return are assumed to be random. They found that the

amortization method achieves greater fund security than the spread method, but it is less efficient because of the feedback of delayed information. Owadally and Haberman (2004) further examined the mathematical results in Dufresne (1989) and Owadally and Haberman (1999) with stochastic simulations in which AR(1) and MA(1) rates of return are assumed. The results supported the conclusion that the spread method is more efficient. A more realistic simulation using fitted equity returns, bond yields and price inflation in Canada, the United Kingdom, and the United States were also performed. Efficient spreading and amortization periods are suggested for the three countries.

5. Since the past work looked at the variability in pension funding and contribution separately, Haberman et al. (2000) introduced a performance criterion aiming at minimizing both risks at the same time.

Another important work on stochastic returns in pension plan was done by Sanders (2010) for a target benefit plan with fixed annual contributions and variable benefit payments. A recursive formula for the annual benefit payment was derived, assuming that the rate of return follows an AR(1) process. Due to the existence of a feedback loop in the model, a simulation study was conducted to estimate the distributional properties of the benefit payments and the total pension entitlement. It is observed that a near-stationary distribution exists over the 200-year horizon when asset returns are independent year to year. When the correlation between assets returns increases, it takes longer for the distributions to converge to a stationary status, if they converge at all. Compared with a DC plan with the same contributions, the target benefit plan provides less pension entitlement and performs worse in terms of intergenerational equity.

A very realistic study of the volatilities in pension plan projections was conducted by Yuen (2011). Time series models for key economic variables were selected and their parameters were estimated. The resulting economic variables were used in modeling a sample DB plan and a sample DC plan. The results highlight the high volatility of employer contributions in DB plans and the inadequacy of retirement income in DC plans.

1.3 Outline

The purpose of this project is to study the risks in a hybrid pension plan, using the model proposed by Khorasanee (2012). The first two moments of the funding level, annual contribution rate, and annual benefit payment are calculated, when the rate of return is assumed to follow a Gaussian process, AR(1) process, and MA(1) process. In addition to the illustration of the stationary status results, we also show the evolution of the hybrid pension plan, and the impact of different choices of valuation rates of interest. Table 1.1 below summarizes the major references used in this project and the structure of our study. The check marked cells represent the model scenarios that are studied in this project. Dufresne (1988) and Khorasanee (2012) assumed that the rates of return are independently and identically distributed, but we assume that they follow a Gaussian process in this scenario.

Table 1.1: Major References and Project Outline

Pension Plan	$i_v = E(i_t)$			$i_v \neq E(i_t)$		
	Gaussian	AR(1)	MA(1)	Gaussian	AR(1)	MA(1)
Hybrid	Khorasanee (2012)	✓	✓	✓	✓	✓
DB	Dufresne (1988)	Haberman (1994)	Haberman (1997)	Cairns and Parker (1997)	Cairns and Parker (1997)	✓
Pseudo Hybrid	✓	✓	✓	✓	✓	✓

This project is arranged in the following way. A hybrid plan and its modeling is presented in Chapter 2 to study the performance of Khorasanee's hybrid plan. The DB plan and the pseudo hybrid plan are considered as special cases. Chapter 3 presents the derivations of important results in order to measure the risk of hybrid pension plans. The first two moments of the contribution, benefit and funding level under different investment return assumptions are studied. In Chapter 4, numerical illustrations are presented to answer the questions set in the beginning of this report. Chapter 5 concludes the project and suggests future work that could be done.

Chapter 2

Hybrid Pension Plan and Its Modeling

In this chapter, we first present a hybrid plan structure, which was proposed by Khorasanee (2012). Some risk measures and criteria are laid out for convenience. Then we discuss the defined benefit plan and pseudo hybrid plan as special cases of the proposed hybrid plan.

2.1 The Hybrid Pension Plan

The hybrid pension plan studied in this project was proposed by Khorasanee (2012), following a similar mathematical structure to the DB plan studied in Dufresne (1988). It assumed that the surpluses and deficits are amortized by adjusting both the benefit outgo and the contribution, which enables risk sharing between plan sponsors and members.

The difference between the actuarial liability and the actual pension fund is called unfunded liability. When the actuarial liability is higher than the fund level, there is a deficiency; when the actuarial liability is lower than the fund level, there is a surplus. The plan adjusts the contribution and benefit in order to absorb the deficiency and surplus. The adjustment is equal to the unfunded liability divided by the present value of an annuity for a fixed term, calculated at the valuation rate of interest. This is called the ‘spread’ method.

Khorasanee (2012) has answered part of the questions we raised at the beginning of the report. It derived the first two moments in finite time and pointed out that the expected benefit outgo at finite time is different from its target value by some fraction of the difference

between the initial funding and actuarial liability. It illustrated that a riskier investment portfolio produces higher aggregate risk than a less risky portfolio. A suggestion was made to set the sum of the spread parameters equal to the value that minimize the aggregate risk.

Our project differs from Khorasanee (2012) in the following aspects.

1. Instead of setting the target benefit to one unit, we assume that the target benefit is a lump sum payment that equals the present value at retirement of a life annuity due with annual payments of $1/3$ of the final salary.
2. We loosen the assumption for the valuation interest rate and explore the impact of different valuation interest rates.
3. Khorasanee (2012) studied the case where the effective investment returns are independent and identically distributed. In this project, we model the forces of interest of the investment return using Gaussian, AR(1), and MA(1) processes.

2.2 Assumptions

We make the following assumptions when studying the risks in a hybrid pension plan.

1. All actuarial assumptions are realized exactly, except for investment returns.
2. The population is stationary.
3. There are no salary increases, whether from inflation or promotion. If we were to assume a deterministic salary increase rate, since the benefits are proportional to the salary, the benefits would increase at the same rate as salaries. For simplicity, we therefore consider all the variables to be in real terms and set the annual salary at entry to be one unit.
4. The valuations are carried out annually. The effective valuation interest rate is fixed at i_v per year.

2.3 Modelling and Risk Measurement

Having the hybrid plan of interest described, we introduce the notation and present the model in a mathematical form following Khorasanee (2012). The variables used in the model are defined below.

- F_t : the value of the pension fund at time t , and $t = 0, 1, 2, \dots$ unless otherwise specified. We assume that F_0 is a known constant.
- C_t : the total annual contribution paid by the plan sponsor at time t .
- B_t : the total actual annual benefit outgo at time t .
- i_t : the effective rate of return on the pension fund between time $t - 1$ and t ; the corresponding force of interest is δ_t .
- i_v : the valuation interest rate. It is treated as a constant given for the pension valuation.
- ${}^{ind}NC(i_v)$: the normal cost of each plan member; it is determined by the valuation interest rate i_v and the mortality assumption.
- $NC(i_v)$: the total normal cost of the plan; it is determined by the valuation interest rate i_v and the mortality assumption.
- $AL(i_v)$ is the actuarial liability of the plan; it is determined by the valuation interest rate i_v and the mortality assumption.
- TB : the total annual target benefit outgo.
- k : the spread parameter of the plan. It is equal to $1/\ddot{a}_{\overline{m}|}$, where m is the number of years that the unfunded liability is spread into. We call m the spread period.
- k_c is the spread parameter for contributions. It is equal to a proportion p_c of k , $k_c = p_c * k$. It means that p_c of the unfunded liability is adjusted into the contributions in m years. We let $0 \leq p_c \leq 1$.
- k_b is the spread parameter for benefit outgo. It is equal to a proportion p_b of k , $k_b = p_b * k$. It means that p_b of the unfunded liability is adjusted into the benefit in m years. We let $0 \leq p_b \leq 1$.

Note that $p_c + p_b = 1$, and $k = k_c + k_b$.

For $t = 0, 1, 2, \dots$, the following relationships exist:

$$F_{t+1} = (1 + i_{t+1})(F_t + C_t - B_t), \quad (2.1)$$

$$C_t = NC(i_v) + k_c(AL(i_v) - F_t), \quad (2.2)$$

$$B_t = TB - k_b(AL(i_v) - F_t). \quad (2.3)$$

The contribution at time t is equal to the sum of the normal cost and a proportion of the unfunded liability. The benefit at time t is the sum of the target benefit and a proportion of the unfunded liability.

Both the employer and the employees are concerned with whether the pension fund can meet the actuarial liability, which is sometimes called ‘solvency risk’. A reasonable objective of a pension scheme is to minimize the variation in the fund level, which can be measured by the variance of the funding level, $Var(F_t)$. At the same time, plan sponsors hope to maintain a stable contribution level so as to minimize the impact of the pension funding on the business. ‘Contribution rate risk’ is measured by the variance of the contribution level, $Var(C_t)$. For plan members, the stability and predictability of the benefit is crucial for financial planning. As a result, they may focus on keeping the variance of the annual benefit, $Var(B_t)$, in a reasonable range. We call this ‘benefit risk’. These measurements are good indicators of the risks borne by the plan sponsor, and participants, when contributions and benefits are expressed in the same unit, say Canadian dollars. However, as Haberman (1994) pointed out, a pure attempt to minimize the variances may lead to a lower contribution level and a lower benefit level. In this project, we use the coefficient of variation as the main criteria to determine the optimum spread period so that the level of the variable of interest is also taken into account.

2.4 Defined Benefit Plan as a Special Case

In a defined benefit plan, the benefits are predetermined and only the contributions are adjusted to fund the benefit. It can be considered as a special case of the hybrid plan presented above. Setting k_b to 0, there is no adjustment made to the benefits. The model becomes the DB model studied in Dufresne (1988).

2.5 Pseudo Hybrid Plan as a Special Case

Another special case of the hybrid plan is to set k_c to 0, meaning that the contribution level is fixed. The plan has a target benefit that employees and employers have agreed to aim for. When there is a deficit or surplus in the fund, the benefit is adjusted away from the target, but the contributions do not change. This plan is called a pseudo hybrid plan in Khorasaneh (2012).

Chapter 3

Model and Assumptions

This chapter presents the risk measurement framework of the hybrid pension plan. We start by introducing the interest rate models and reviewing their important results; then we state the main assumptions we made in order to derive the first two moments of the funding level. Finally, mathematical results for the risk measurement of the hybrid pension plan are presented.

3.1 Interest Rate Models

3.1.1 Gaussian Model

We denote by i_{t+1} the effective rate of return earned on the pension fund during the period $[t, t + 1)$. The corresponding force of interest is denoted by δ_{t+1} , and $1 + i_{t+1} = e^{\delta_{t+1}}$.

We assume that the forces of interest $\{\delta_{t+1} : t = 0, 1, 2, \dots\}$ are independent and identically distributed, and they follow a normal distribution with mean θ_1 and variance ν_1^2 . Under these assumptions, $\{\delta_{t+1} : t = 0, 1, 2, \dots\}$ follows a weakly stationary Gaussian time series process, since the first two moments are independent of time t . Unless specified otherwise, the times t and s take values in $\{0, 1, 2, \dots\}$.

Since $1 + i_{t+1} = e^{\delta_{t+1}}$, it is easy to conclude that $\{i_{t+1} : t = 0, 1, 2, \dots\}$ are independent and identically distributed as well, and are lognormally distributed. We denote the mean

and variance of i_{t+1} as i and σ^2 , which can be derived using the following relationships:

$$\begin{aligned} E(1 + i_t) &= 1 + i \\ &= E(e^{\delta t}) = \exp \left\{ \theta_1 + \frac{1}{2} \nu_1^2 \right\}, \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \text{Var}(1 + i_t) &= \text{Var}(i_t) = \sigma^2 \\ &= \text{Var}(e^{\delta t}) = \exp \{ 2\theta_1 + \nu_1^2 \} (\exp \{ \nu_1^2 \} - 1). \end{aligned} \quad (3.2)$$

3.1.2 AR(1) Model

Another model that is of great interest is the autoregressive process of order 1 (AR(1)). We assume that the force of interest follows a discrete stationary AR(1) process, namely,

$$\delta_t = \theta_2 + \phi_2(\delta_{t-1} - \theta_2) + \epsilon_t, \quad t = 1, 2, 3, \dots \quad (3.3)$$

where θ_2 is the long term mean of the forces of interest. The autoregressive coefficient ϕ_2 satisfies the stationary condition $|\phi_2| < 1$. The noises $\{\epsilon_t : t = 1, 2, \dots\}$ are independent and identically distributed normal random variables with mean 0 and variance γ_2^2 . We call γ_2^2 the local variance of δ_{t+1} given δ_t . The noise ϵ_t is also referred to as error, residual, white noise or random shock in the literature.

Equation (3.3) can be interpreted as follows: the distance between the force of interest δ_t and the long term mean θ_2 is a proportion ϕ_2 of the distance between the force of interest during the last period, δ_{t-1} , and the long term mean θ_2 , plus a white noise ϵ_t . Unlike the independent and identical assumption in the Gaussian model, the current force of interest is correlated with all the previous ones, although with decreasing coefficients. The name of the process, autoregressive, refers to the regression on the process itself.

The long term mean and variance of the process can be expressed as

$$\begin{aligned} E(\delta_t) &= \theta_2, \\ \text{Var}(\delta_t) &= \frac{\gamma_2^2}{1 - \phi_2^2} \triangleq \nu_2^2, \end{aligned} \quad (3.4)$$

and

$$\text{Cov}(\delta_t, \delta_s) = \frac{\gamma_2^2}{1 - \phi_2^2} \phi_2^{|t-s|} \triangleq \gamma_2(t, s).$$

The sum of forces of interest is denoted by $\Delta(t) = \sum_{u=1}^t \delta_u$ with $\Delta(0) = 0$. The following results are derived in Haberman (1994) and frequently used in the derivation of moments for the hybrid pension plan.

First, we have

$$E[\Delta(t)] = E\left[\sum_{u=1}^t \delta_u\right] = t\theta_2,$$

$$Var[\Delta(t)] = Var\left(\sum_{u=1}^t \delta_u\right) = 2\mu_2^2 G(t, 0),$$

where $G(t, s) = \frac{1+\phi_2}{2(1-\phi_2)}(t-s) - \frac{\phi_2(1-\phi_2^{t-s})}{(1-\phi_2)^2}$.

Equation (3.3) can be expressed as

$$\delta_t - \theta_2 = \phi_2^t(\delta_0 - \theta_2) + \sum_{i=1}^t \phi_2^i \epsilon_{t-i}. \quad (3.5)$$

Given δ_0 , δ_t is normally distributed, and hence $\Delta(t)$ also follows a normal distribution. The accumulation factor from time s to time t calculates the accumulated value at time t of 1 dollar at time s . It can be expressed as $e^{\Delta(t)-\Delta(s)}$ and it follows a lognormal distribution. For $0 \leq s < t$, the expected value of this accumulation factor is

$$E\left[e^{\Delta(t)-\Delta(s)}\right] = c^{t-s} e^{-z(1-\phi_2^{t-s})}, \quad (3.6)$$

where

$$\begin{cases} c = \exp\left\{\theta_2 + \frac{1+\phi_2}{2(1-\phi_2)}\nu_2^2\right\} \\ z = \nu_2^2\phi_2(1-\phi_2)^{-2} \end{cases}.$$

Furthermore, we state the following result for the cross-product expectation, for $0 \leq s < r < t$

$$E\left[e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)}\right] = \exp\left\{(t-s)\theta_2 + (t-r)\theta_2 + \nu_2^2 H(t, r, s)\right\}, \quad (3.7)$$

where

$$H(t, r, s) = \frac{1+\phi_2}{2(1-\phi_2)}[t-s+3(t-r)] - \frac{1}{(1-\phi_2)^2}[\phi_2(3-2\phi_2^{t-r}-2\phi_2^{t-s}+\phi_2^{r-s})].$$

The derivations of Equations (3.6) and (3.7) can be found in Appendix A.

3.1.3 MA(1) Model

In this section, we use a stationary MA(1) process to model the force of interest. We assume that the force of interest follows a moving average process in discrete time of order 1 (MA(1)):

$$\delta_t = \theta_3 + e_t - \phi_3 e_{t-1}, \quad t = 1, 2, 3, \dots \quad (3.8)$$

where $\{e_t : t = 1, 2, \dots\}$ are independent and identically distributed normal random variables with mean 0 and variance γ_3^2 . We assume that the moving average coefficient satisfies $|\phi_3| < 1$, which makes the process invertible, as was stated in Shumway and Stoffer (2000).

As we can tell from Equation (3.8), the force of interest δ_t is the sum of its long term mean θ_3 , a noise term e_t , and a proportion $-\phi_3$ of the noise term at the last time step, e_{t-1} . We assume that $\{e_t : t = 1, 2, \dots\}$ are independent and identically distributed.

The long term mean and variance of the process can be expressed as

$$E(\delta_t) = \theta_3, \quad (3.9)$$

$$Var(\delta_t) = (1 + \phi_3^2)\gamma_3^2 \triangleq \nu_3^2, \quad (3.10)$$

and the covariance is

$$Cov(\delta_t, \delta_s) = \begin{cases} -\phi_3\gamma_3^2 & |t - s| = 1 \\ 0, & |t - s| > 1 \end{cases}.$$

The following results are used in the derivation of moments for the hybrid pension plan.

Similar to the AR(1) process case, $\Delta(t) = \sum_{u=1}^t \delta_u$ is also normally distributed but with mean and variance

$$E[\Delta(t)] = E\left[\sum_{u=1}^t \delta_u\right] = t\theta_3,$$

$$Var[\Delta(t)] = Var\left(\sum_{u=1}^t \delta_u\right) = t\nu_3^2 - 2(t-1)\phi_3\gamma_3^2.$$

The lognormal random variable $e^{\Delta(t)}$ has mean

$$E\left[e^{\Delta(t)}\right] = \exp\left\{\left(\theta_3 + \frac{1}{2}\nu_3^2\right)t - (t-1)\phi_3\gamma_3^2\right\}.$$

Similarly, for $t, s = 0, 1, \dots$ and $t > s$

$$\begin{aligned} E \left[e^{\Delta(t) - \Delta(s)} \right] &= \exp \left\{ (t-s) \left(\theta_3 + \frac{1}{2} \nu_3^2 \right) - (t-s-1) \phi_3 \gamma_3^2 \right\} \\ &= f^{t-s} \exp \left\{ -(t-s-1) \phi_3 \gamma_3^2 \right\}, \end{aligned} \quad (3.11)$$

where $f = \exp \left\{ \theta_3 + \frac{1}{2} \nu_3^2 \right\}$.

In addition, for $0 \leq s < r < t$, the cross-product expectation is

$$E \left[e^{\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)} \right] = \alpha^{t-s} \beta^{t-r} g, \quad (3.12)$$

where $\alpha = f * \exp \left\{ -\phi_3 \gamma_3^2 \right\}$, $\beta = \exp \left\{ \theta_3 + \frac{3}{2} \nu_3^2 - 3\phi_3 \gamma_3^2 \right\}$, and $g = \exp \left\{ 3\phi_3 \gamma_3^2 \right\}$.

The derivation of Equations (3.11) and (3.12) can be found in Appendix A.

We make a remark here that when $r = s$, Equation (3.12) becomes

$$\begin{aligned} E \left[e^{2(\Delta(t) - \Delta(s))} \right] &= \exp \left\{ 2(t-s)\theta_3 + 2\text{Var} [\Delta(t) - \Delta(s)] \right\} \\ &= \exp \left\{ (t-s) \left(\theta_3 + \frac{1}{2} \nu_3^2 - \phi_3 \gamma_3^2 \right) + (t-s) \left(\theta_3 + \frac{3}{2} \nu_3^2 - 3\phi_3 \gamma_3^2 \right) + 4\phi_3 \gamma_3^2 \right\} \\ &= (\alpha\beta)^{t-s} g^{4/3}. \end{aligned}$$

3.2 Risk Measurement of Hybrid Pension Plan

We first point out how to derive the first two moments of the annual contribution rate and benefit payment, using the first two moments of the funding level. Next we present the first two moments of the funding level, under different assumptions for the rate of return. The main results for a Gaussian model are from Dufresne (1988) and Khorasanee (2012), the derivation for the AR(1) case can be found in Haberman (1994), and the results for the MA(1) case are from Haberman (1997).

From Equations (2.2) and (2.3), the following results hold, given the expectation of funding level, $E(F_t)$:

$$E(C_t) = NC(i_v) + k_c (AL(i_v) - E(F_t)), \quad (3.13)$$

$$E(B_t) = TB(i_v) - k_b (AL(i_v) - E(F_t)), \quad (3.14)$$

$$\text{Var}(C_t) = k_c^2 \text{Var}(F_t), \quad (3.15)$$

$$\text{Var}(B_t) = k_b^2 \text{Var}(F_t). \quad (3.16)$$

Taking the square root of Equations (3.15) and (3.16). Respectively, we have

$$SD(C_t) = k_c SD(F_t), \quad (3.17)$$

$$SD(B_t) = k_b SD(F_t). \quad (3.18)$$

Since $k_c + k_b = k$, the sum of the standard deviation of the annual contributions and benefits is equal to a proportion k of the standard deviation of the funding level. In addition, the ratio $SD(C_t)/SD(B_t)$ is a constant equal to k_c/k_b . As a special case, when the spread period is 1 year, $k = 1$, the sum of the standard deviation of the contributions and the benefits equals the standard deviation of the funding level.

In the following sections, we focus on deriving results for $E(F_t)$ and $Var(F_t)$. The first two moments of benefit payment and contribution can be derived accordingly, using the above relationships.

We further denote $AggR$ as the aggregate risk, which is the sum of the standard deviation of the contributions and the standard deviation of the benefits, namely,

$$AggR = SD(C_t) + SD(B_t) = kSD(F_t). \quad (3.19)$$

This is a convenient notation when we are mainly concerned about the total risk in the contribution and benefit levels. Again, it is worthwhile noting that this notation only has a good interpretation when contributions and benefits are measured in the same unit.

If we replace C_t and B_t in Equation (2.1) with their corresponding expressions in (2.2) and (2.3), we have, for $t = 0, 1, 2, \dots$,

$$\begin{aligned} F_{t+1} &= (1 + i_{t+1})(F_t + C_t - B_t) \\ &= (1 + i_{t+1})((1 - k)F_t + NC(i_v) - TB(i_v) + kAL(i_v)). \end{aligned} \quad (3.20)$$

When $t \rightarrow \infty$, the plan is said to reach its ultimate situation, or a stationary status. The stationary status captures the main characteristics of a mature pension plan, without specifying the time period. Since models with different parameters converge at different speeds, focusing on the ultimate status makes the comparison between different scenarios and rate of return models more convenient. To simplify the derivation, Haberman (1994) took $F_0 = 0$ when deriving the ultimate funding level and second moment, since the starting value does not affect the ultimate value when $t \rightarrow \infty$. We take the same approach in this project.

3.2.1 Gaussian Model

In Section 3.1.1, we have stated that $\{i_{t+1} : t = 0, 1, 2, \dots\}$ are independent and identically distributed, and follow a lognormal distribution with $E(i_t) = i$, and $Var(i_t) = \sigma^2$. Equation (3.20) can be re-expressed as

$$F_{t+1} = w_{t+1} [qF_t + R_v(1 + i)], \quad (3.21)$$

where $w_{t+1} = \frac{1+i_{t+1}}{1+i}$, $q = (1 + i)(1 - k)$, and

$$R_v = NC(i_v) - TB(i_v) + kAL(i_v). \quad (3.22)$$

1. Transient Situation

Taking expectations on both sides of Equation (3.21), we have the following recursive formula

$$E(F_{t+1}) = qE(F_t) + R_v(1 + i).$$

It is not difficult to have, for $t = 1, 2, \dots$,

$$E(F_t) = q^t F_0 + R_v(1 + i) \frac{1 - q^t}{1 - q}. \quad (3.23)$$

For the second moment,

$$\begin{aligned} E(F_{t+1}^2) &= E(w_{t+1}^2) E[q(F_t - E(F_t)) + qE(F_t) + R_v(1 + i)]^2 \\ &= E(w_{t+1}^2) [q^2 Var(F_t) + [E(F_{t+1})]^2], \end{aligned}$$

and since we have

$$E(w_{t+1}^2) = \frac{E[(1 + i_{t+1})^2]}{(1 + i)^2} = \frac{(1 + i)^2 + \sigma^2}{(1 + i)^2},$$

the following results hold

$$\begin{aligned} Var(F_{t+1}) &= E(F_{t+1}^2) - [E(F_{t+1})]^2 \\ &= E(w_{t+1}^2) q^2 Var(F_t) + [E(w_{t+1}^2) - 1] [E(F_{t+1})]^2 \\ &= (1 - k)^2 [(1 + i)^2 + \sigma^2] Var(F_t) + \frac{\sigma^2}{(1 + i)^2} [E(F_{t+1})]^2 \\ &= a Var(F_t) + b [E(F_{t+1})]^2, \end{aligned}$$

where $a = (1 - k)^2 [(1 + i)^2 + \sigma^2]$, and $b = \sigma^2(1 + i)^{-2}$.

Since F_0 is a known constant, $Var(F_0) = 0$, we have, for $t = 1, 2, \dots$, that

$$Var(F_t) = b \sum_{j=1}^t a^{t-j} [E(F_j)]^2. \quad (3.24)$$

2. Ultimate Situation

If we let $q < 1$, Equation (3.23) converges to

$$\lim_{t \rightarrow \infty} E(F_t) = E(F_\infty) = R_v \frac{1+i}{1-q}. \quad (3.25)$$

The convergence condition $q < 1$ is equivalent to $k > i/(1+i)$, which means that as long as the annual adjustment to the contributions and the benefits exceeds the interest earned on the unfunded liability, the funding level eventually converges to a stationary status.

Under the condition $a < 1$, which is equivalent to

$$k > 1 - \frac{1}{\sqrt{(1+i)^2 + \sigma^2}},$$

it has been proven in Dufresne (1988) that

$$\lim_{t \rightarrow \infty} Var(F_t) = \frac{b}{1-a} \lim_{t \rightarrow \infty} [E(F_t)]^2. \quad (3.26)$$

3. A special case

When we set $i_v = E(i_t) = i$, the above results have a simpler form. Since the equation of equilibrium is

$$AL(i_v) = (1+i_v) [AL(i_v) + NC(i_v) - TB(i_v)],$$

we have

$$\frac{i}{1+i} AL(i) = TB(i_v) - NC(i),$$

implying by (3.22) that

$$\frac{1-q}{1+i} AL(i) = R_v,$$

and Equations (3.25) and (3.26) become

$$\lim_{t \rightarrow \infty} E(F_t) = AL(i), \quad (3.27)$$

$$\lim_{t \rightarrow \infty} Var(F_t) = \frac{b}{1-a} [AL(i)]^2. \quad (3.28)$$

Accordingly, the aggregate risk given by (3.19) can be expressed as

$$\text{AggR} = k\sqrt{\text{Var}(F_t)} = k\sqrt{\frac{b}{1-a}}AL(i).$$

Given the mean and variance of i_t , AggR is a function of the spread parameter k . Taking the first derivative of AggR with respect to k , and noting that a is also a function of k , we can find the optimum k^* that minimizes the aggregate risk:

$$k^* = 1 - (\sigma^2 + (1+i)^2)^{-1}. \quad (3.29)$$

3.2.2 AR(1) Model

Recall that from Equation (3.20), $F_{t+1} = (1 + i_{t+1})[(1 - k)F_t + R_v]$. Our presentation is different from Haberman (1994) in that we allow $i_v \neq i$. Given i_v , $NC(i_v)$ and $AL(i_v)$ are constants, so is R_v given by (3.22). This difference would not affect the derivations in the AR(1) case, and we can safely replace R in Haberman (1994) by the counterpart in our report R_v .

We denote $Q = 1 - k$ for simplicity. Now F_t can be expressed as

$$F_t = (1 + i_t)(QF_{t-1} + R_v), \quad (3.30)$$

and we can write it recursively as

$$F_t = F_0Q^t e^{\Delta(t)} + R_v \sum_{s=0}^{t-1} Q^{t-s-1} e^{\Delta(t)-\Delta(s)}, \quad t = 1, 2, \dots \quad (3.31)$$

Taking expectations on both sides of Equation (3.31) and using Equation (3.6), we have

$$\begin{aligned} E(F_t) &= F_0Q^t E(e^{\Delta(t)}) + R_v \sum_{s=0}^{t-1} Q^{t-s-1} E[e^{\Delta(t)-\Delta(s)}] \\ &= F_0(Qc)^t e^{z(\phi_2^t - 1)} + \frac{R_v}{Q} \sum_{s=0}^{t-1} (Qc)^{t-s} e^{z(\phi_2^{t-s} - 1)}. \end{aligned} \quad (3.32)$$

Using the results from Haberman (1994), when $Qc < 1$, $E(F_t)$ converges to an ultimate value as $t \rightarrow \infty$. This value is approximated as

$$\lim_{t \rightarrow \infty} E(F_t) \cong e^{-z} \frac{R_v c}{1 - Qc}. \quad (3.33)$$

For the second moment, applying Equations (3.7) and (3.6), we have

$$\begin{aligned}
E(F_t^2) &= F_0^2 Q^{2t} E(e^{2\Delta(t)}) + 2F_0 Q^t R_v \sum_{s=0}^{t-1} Q^{t-s-1} E \left[e^{\Delta(t)} e^{\Delta(t)-\Delta(s)} \right] \\
&\quad + R_v^2 E \left[\sum_{s=0}^{t-1} \sum_{r=0}^{t-1} Q^{t-s-1} Q^{t-r-1} e^{\Delta(t)-\Delta(s)} e^{\Delta(t)-\Delta(r)} \right] \\
&= F_0^2 Q^{2t} E(e^{2\Delta(t)}) + 2F_0 Q^{2t-1} R_v E(e^{2\Delta(t)}) + 2F_0 Q^t R_v \sum_{s=1}^{t-1} Q^{t-s-1} E \left[e^{2\Delta(t)-\Delta(s)} \right] \\
&\quad + \frac{2R_v^2}{Q^2} \sum_{r=1}^{t-1} \sum_{s=0}^{r-1} Q^{t-s} Q^{t-r} E \left[e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)} \right] + \frac{R_v^2}{Q^2} \sum_{s=0}^{t-1} Q^{2(t-s)} E \left[e^{2\Delta(t)-2\Delta(s)} \right] \\
&= (F_0^2 Q^{2t} + 2F_0 Q^{2t-1} R_v) c^{2t} \exp \left\{ 2\nu^2 t \frac{1+\phi_2}{1-\phi_2} + 4z(\phi_2^t - 1) \right\} \\
&\quad + 2F_0 Q^t R_v \sum_{s=1}^{t-1} Q^{t-s-1} \exp \left\{ (2t-s)\theta_2 + \nu^2 H(t, s, 0) \right\} \\
&\quad + \frac{2R_v^2}{Q^2} \sum_{r=1}^{t-1} \sum_{s=0}^{r-1} Q^{2t-s-r} \exp \left\{ (2t-s-r)\theta_2 + \nu^2 H(t, r, s) \right\} \\
&\quad + \frac{R_v^2}{Q^2} \sum_{s=0}^{t-1} Q^{2(t-s)} c^{4(t-s)} \exp \left\{ -2(t-s)\theta_2 - 4z(1-\phi_2^{t-s}) \right\}. \tag{3.34}
\end{aligned}$$

Using the results from Haberman (1994), when $Qc < 1$ and $Q^2cp < 1$, $\lim_{t \rightarrow \infty} E(F_t^2)$ exists, and it is approximately equal to

$$\lim_{t \rightarrow \infty} E(F_t^2) \cong e^{-3z} \frac{2R_v^2 Q c^2 p}{(1-Qc)(1-Q^2cp)} + e^{-4z} \frac{R_v^2 cp}{1-Q^2cp},$$

where $p = \exp \left\{ \theta_2 + \frac{3}{2} \frac{1+\phi_2}{1-\phi_2} \nu_2^2 \right\}$, and

$$\lim_{t \rightarrow \infty} \text{Var}(F_t) \cong e^{-3z} \frac{2R_v^2 Q c^2 p}{(1-Qc)(1-Q^2cp)} + e^{-4z} \frac{R_v^2 cp}{1-Q^2cp} - e^{-2z} \frac{R_v^2 c^2}{(1-Qc)^2}. \tag{3.35}$$

Remark: It should be noted that when the spread period $m = 1$, it implies that $k = 1$ and hence $Q = 0$, and then (3.30) can be simplified to

$$F_t = R_v(1 + i_t) = R_v e^{\delta t}.$$

Recall that $e^{\delta t}$ follows a lognormal distribution and hence the first two moments of F_∞ are

$$E(F_\infty) = R_v \exp \left\{ \theta_2 + \frac{1}{2} \nu_2^2 \right\}, \tag{3.36}$$

$$\text{Var}(F_\infty) = R_v^2 e^{2\theta_2 + \nu_2^2} (e^{\nu_2^2} - 1). \tag{3.37}$$

3.2.3 MA(1) Model

Similar to the case of the AR(1) model, the result in Haberman and Wong (1997) can be applied by replacing R in the paper by R_v .

We start with $F_{t+1} = (1 + i_{t+1})[QF_t + R_v]$, which also gives

$$F_t = F_0 Q^t e^{\Delta(t)} + R_v \sum_{s=0}^{t-1} Q^{t-s-1} e^{\Delta(t)-\Delta(s)}, \quad t = 1, 2, \dots$$

Taking expectations on both sides of the equation, we get

$$\begin{aligned} E(F_t) &= F_0 Q^t E(e^{\Delta(t)}) + R_v \sum_{s=0}^{t-1} Q^{t-s-1} E[e^{\Delta(t)-\Delta(s)}] \\ &= f \left[F_0 Q (Q\alpha)^{t-1} + R_v \frac{1 - (Q\alpha)^t}{1 - Q\alpha} \right] \end{aligned} \quad (3.38)$$

using the result from Equation (3.11) where $f = \exp\{\theta_3 + \frac{1}{2}\nu_3^2\}$ and $\alpha = f * \exp\{-\phi_3\gamma_3^2\}$. According to Haberman and Wong (1997), when $Q\alpha < 1$, the ultimate value of the expectation is given by

$$\lim_{t \rightarrow \infty} E(F_t) = \frac{f R_v}{1 - Q\alpha}. \quad (3.39)$$

For the second moment, we assume $r > s$ and use Equations (3.11) and (3.12)

$$\begin{aligned} E(F_t^2) &= F_0^2 Q^{2t} E(e^{2\Delta(t)}) + 2F_0 Q^t R_v \sum_{s=0}^{t-1} Q^{t-s-1} E \left[e^{\Delta(t)} e^{\Delta(t)-\Delta(s)} \right] \\ &\quad + R_v^2 E \left[\sum_{s=0}^{t-1} \sum_{r=0}^{t-1} Q^{t-s-1} Q^{t-r-1} e^{\Delta(t)-\Delta(s)} e^{\Delta(t)-\Delta(r)} \right] \\ &= F_0 g^{\frac{4}{3}} \left(F_0 + \frac{2R_v}{Q} \right) (\alpha\beta Q^2)^t + 2F_0 g R_v (\beta Q)^t \frac{1 - (Q\alpha)^{t-1}}{1 - Q\alpha} \\ &\quad + \frac{2g R_v^2 \alpha^2 \beta Q}{1 - Q\alpha} \left[\frac{1 - (Q^2 \alpha \beta)^{t-1}}{1 - Q^2 \alpha \beta} - (Q\alpha)^{t-1} \frac{1 - (Q\beta)^{t-1}}{1 - Q\beta} \right] + R_v^2 g^{\frac{4}{3}} \alpha \beta \frac{1 - (Q^2 \alpha \beta)^t}{1 - Q^2 \alpha \beta}. \end{aligned} \quad (3.40)$$

Haberman (1997) has derived that when $Q\alpha < 1$ and $Q^2\alpha\beta < 1$, $\lim_{t \rightarrow \infty} E(F_t^2)$ exists and is equal to

$$\lim_{t \rightarrow \infty} E(F_t^2) = \frac{2R_v g \alpha}{Q(1 - Q\alpha)} \frac{Q^2 \alpha \beta}{1 - Q^2 \alpha \beta} + \frac{R_v^2 g Q^2 \alpha \beta}{Q^2 (1 - Q^2 \alpha \beta)} e^{\phi_3 \gamma_3^2},$$

and

$$\lim_{t \rightarrow \infty} Var(F_t) = \frac{2R_v^2 g \alpha^2 Q \beta}{(1 - Q\alpha)(1 - Q^2 \alpha \beta)} + \frac{R_v^2 g \alpha \beta}{1 - Q^2 \alpha \beta} e^{\phi_3^2} - \frac{R_v^2 \alpha^2}{(1 - Q\alpha)^2} e^{2\phi_3 \gamma_3^2}. \quad (3.41)$$

Remark: It should be noted that when the spread period $m = 1$, we can obtain similar results as the ones we have shown for the AR(1) process with $m = 1$, that is:

$$E(F_\infty) = R_v \exp \left\{ \theta_3 + \frac{1}{2} \nu_3^2 \right\},$$
$$\text{Var}(F_\infty) = R_v^2 e^{2\theta_3 + \nu_3^2} (e^{\nu_3^2} - 1).$$

These results indicate that with a spread period of 1, that is eliminating the unfunded liability each year, the first two moments of the funding level in the stationary case will be the same for both processes, AR(1) and MA(1), as long as they have the same long term means and variances.

Chapter 4

Numerical Illustrations: Gaussian Case

This chapter is composed of two parts. In the first two sections, we present the assumptions we make and the pension funding method used in the numerical illustrations in Section 4.3, Chapter 5 and 6. In Section 4.3, we illustrate the numerical calculation results for a hybrid pension plan using the Gaussian assumption. The first two moments of the funding level, contribution and benefit are shown, when investment strategies differ. We also analyze the impact of using different valuation interest rates. The trade-off between different spread parameters for the contributions and benefits is also studied.

4.1 Assumptions

We first lay out the assumptions used in the illustrations, in addition to those assumptions we have stated in Section 2.2.

- For valuing the retirement benefit, we use the UP94 male life table¹, projected to year 2020.
- Members join the plan at age 25 and retire at age 65. There is one active member at each age from 25 to 64 inclusive. There are no decrements prior to age 65, which leads to a stable workforce with 40 active members in the plan at all times.

¹See Appendix B

- The investment portfolio of the pension fund is composed of two parts: $(1 - x) * 100\%$ of the portfolio is invested in risk free government bonds earning 0% return in real terms, and the remaining $x * 100\%$ of the portfolio is invested in equities earning an effective real annual rate of return of 5% with a standard deviation of 20%.
- The investment strategy is implemented by choosing x , the proportion of the equity in the portfolio. We set $x = 0.1$ to be a conservative strategy, $x = 0.4$ to be a neutral strategy, and $x = 0.9$ to be an aggressive investment strategy. Since the three rate of return models we have chosen model the force of interest, we convert the parameters presented above in discrete terms into the equivalent means and variances of force of interest, using Equations (3.1) and (3.2). Table 4.1 summarizes the mean and standard deviation of the effective real annual rates of return, and the corresponding means and standard deviations of the force of interest.

Table 4.1: Parameters Used under Different Investment Strategies

Strategy	$E(i_t)$	$SD(i_t)$	$E(\delta_t)$	$SD(\delta_t)$
Aggressive	0.045	0.180	0.029	0.171
Neutral	0.020	0.080	0.017	0.078
Conservative	0.005	0.020	0.005	0.020

Remark 1: In this project, for the convenience of description, when we observe that the third digits of the value are the same for two successive time steps, we say that the plan has reached ‘plateau’.

Remark 2: As we have discussed in Section 1.2.2, we prefer to use the coefficient of variation as the main measurement of risk instead of using the standard deviation. However, we sometimes come across very small average values, which drives the coefficient of variation to an extremely big value. In this situation, we make our analysis and comparisons based on the standard deviation.

Remark 3: Since the population is stationary and the liabilities of the plan are mature, the initial funding level F_0 is set to be the actuarial liability (i.e., $F_0 = AL(i_v)$) for each analysis of plan evolution.

4.2 Pension Funding Method

We use the individual entry age normal method to calculate the actuarial liability and normal cost in the illustration. All the actuarial symbols in this section use the valuation interest rate i_v .

Since there is only one member retiring every year, the target benefit payout of the plan is

$$TB(i_v) = \frac{1}{3}\ddot{a}_{65}, \quad (4.1)$$

which is the present value of the annuity, evaluated at the current valuation interest rate. In reality, the discount rate of a life annuity at the time of retirement is unknown. A more realistic approach is to use the projected discount rate of an annuity product in the market at the time of retirement.

Normal cost is defined as the level amount needed to fund the benefit over the working life time of each employee. There are two common definitions of the actuarial liability. Prospectively, the actuarial liability is the difference between the actuarial present value of future benefits and the actuarial present value of future normal costs; retrospectively, the actuarial liability is the accumulated value of normal costs with adjustments due to interest and benefit payment.

For one employee, the actuarial present value of the future retirement benefit valued at entry (i.e. age 25) is

$$APV(\text{future benefit}) = \frac{1}{3}{}_{40|}\ddot{a}_{25}.$$

Since we assume no decrements before retirement, this becomes

$$APV(\text{future benefit}) = \frac{1}{3}(1 + i_v)^{-40}\ddot{a}_{65}. \quad (4.2)$$

Similarly, the actuarial present value of future salaries for each new member joining the plan at age 25 is

$$APV(\text{future salary}) = \ddot{a}_{25:\overline{40}}.$$

Under the assumption of no decrements before retirement, this is equal to

$$APV(\text{future salary}) = \ddot{a}_{\overline{40}}. \quad (4.3)$$

The individual normal cost is the ratio of Equations (4.2) and (4.3) and can be expressed as

$${}^{ind}NC(i_v) = \frac{APV(\text{future benefit})}{APV(\text{future salary})} = \frac{(1+i_v)^{-40}\ddot{a}_{65}}{3\ddot{a}_{40}}. \quad (4.4)$$

Since there are 40 active members in the plan, the total normal cost of the plan at any given time is

$$NC(i_v) = 40 \times {}^{ind}NC(i_v) = \frac{40(1+i_v)^{-40}\ddot{a}_{65}}{3\ddot{a}_{40}}. \quad (4.5)$$

Because we assume that the valuation assumptions are borne out by experience (Section 2.2), and because retiring members are paid out in full, the following equation of equilibrium holds

$$AL(i_v) = (1+i_v)(AL(i_v) + NC(i_v) - TB(i_v)).$$

We can therefore express the actuarial liability as

$$\begin{aligned} AL(i_v) &= \frac{(1+i_v)(TB(i_v) - NC(i_v))}{i_v} \\ &= \frac{1+i_v}{3i_v}\ddot{a}_{65} \left(1 - \frac{40v^{40}}{\ddot{a}_{40}} \right). \end{aligned} \quad (4.6)$$

4.3 Gaussian Model

In our numerical illustrations, we first set up a benchmark case, and then change one parameter at a time to study the impact through comparison. Then we discuss some relationships when two parameters are subject to change at the same time.

4.3.1 Benchmark Case

We set our benchmark case as follows:

- Investment strategy: neutral.
- Valuation rate of interest: $i_v = i$.
- Time: $t = \infty$. We focus on the ultimate values, that is the stationary case.

- Optimum spread period: 30 years. This is the spread period that minimizes both the coefficient of variation of the contribution and the coefficient of variation of the benefit.
- Spread Parameter: $k_c = 0.3k$. We arbitrarily choose $k_c = 0.3k$ in this illustration, so as to focus on analysing the aggregate risk and the funding level. The trade-off between the risks in contributions and benefits is addressed later.

1. First Moments

Table 4.2 lists the first moments of the funding level, the contributions and the benefit payments, respectively, calculated under the three investment strategies, using Equations (3.27), (3.13) and (3.14).

Table 4.2: Means of the Funding Level, Contribution Rate and Benefit Payment

Strategy	$E(F_\infty)$	$E(C_\infty)$	$E(B_\infty)$
Aggressive	62.12	1.49	4.16
Neutral	92.89	3.37	5.19
Conservative	119.75	5.44	6.04

As we have shown in Section 3.2.1, when $i_v = E(i_t) = i$ and in the stationary case, the average funding level converges to the actuarial liability, the expected benefit payment per year is the target benefit, and the average contribution per year converges to the normal cost. From Equations (4.1), (4.5) and (4.6), it is not difficult to conclude that a riskier investment strategy requires, on average, lower contributions and implies lower expected benefit amounts and expected funding levels than a less aggressive investment strategy.

2. Second Moments

We calculate the coefficient of variation for different values of the spread period, m , in 5-year increment, using Equation (3.28). Figure 4.1 shows the coefficients of variation of the funding level, the annual contribution rate and the benefit payment under the neutral investment strategy. We have similar observations as those in Dufresne (1988).

- Funding level

The variation in funding increases as the spread period increases. The longer the unfunded liability is smoothed into the future, the more uncertainty lies in the ultimate level of the funding level, because of the growing uncertainty of future investment rates of return.

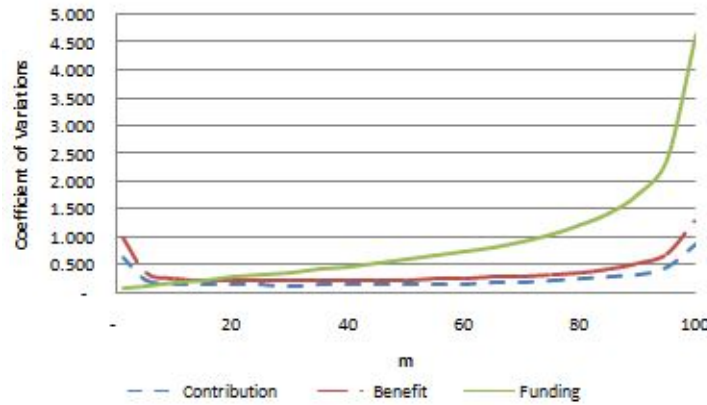


Figure 4.1: Coefficients of Variation under Neutral Investment Strategy with Different Spread Periods ($i_v = i$)

- Contribution rate

The graph of the coefficient of variation for contributions against the spread period has a ‘U’ shape. The coefficient of variation is minimized when the spread period is 30 years. We note that the annual contribution is composed of the normal cost and an adjustment due to the unfunded liability. When the adjustment takes place over a shorter period, it causes a bigger fluctuation in the contribution level. As a result, spreading the unfunded liability over a longer period can smooth out the variation in contributions. However, when the spread period gets too long, and the unfunded liability is deferred further into the future, it eventually results in more variation in contributions due to investment risks.

- Benefit Payment

For similar reasons as in the case of contributions, the graph of the coefficients of variation of the annual benefit payment has a ‘U’ shape and the coefficient of variation is minimized when the spread period is 30 years. A moderate choice of spread period finds a balance between smoothing out the fluctuations induced by the adjustment from the unfunded liability and controlling the exposure to investment risks.

- Comparison

The contributions show more variation relative to their average level than the benefit payments do. This is because the average annual benefit is about 1.5 times the mean

level of contribution.

The standard deviation of contributions is always $3/7$ of the standard deviation of benefits. This is because we set $k_c = 0.3 * k$ in the model assumption. Of the annual adjustment due to the unfunded liability, 30% goes to contributions, and 70% goes to benefits. This relationship is shown in Equations (3.17) and (3.18).

4.3.2 Different Investment Strategies

Now we compare the risks under different investment strategies. We plot in Figure 4.2 the coefficients of variation of the ultimate funding level with various choices of spread period, using different investment strategies.

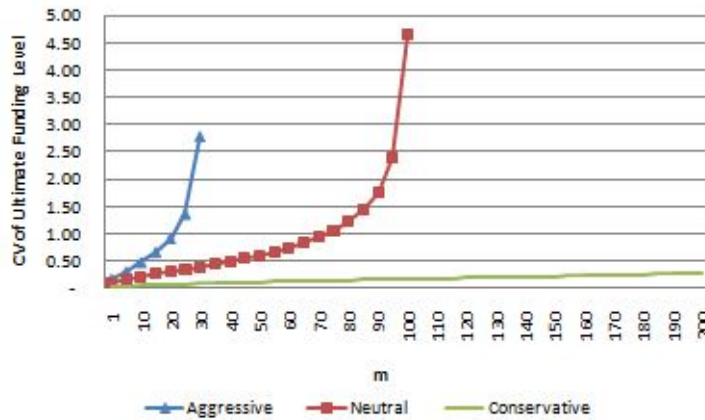


Figure 4.2: Coefficients of Variation of Funding Level with Different Investment Strategies and Spread Periods ($i_v = i$)

The curves of coefficients of variation are cut off when the plan fails to converge when using very long spread periods. We notice from Figure 4.2 that the riskier the portfolio is, the shorter the maximum spread period is allowed to be if we want to achieve a stationary status of the funding level. The cut off points are 30 and 100 years of spread period when using the aggressive strategy and the neutral strategy, respectively; however, when using the conservative strategy, the model has a stationary status even at $m = 200$. With a risky investment portfolio, when the unfunded liability is spread over a long period, the uncertainty of rates of return builds up in the funding value, which results in failure of

convergence. With a conservative strategy, the variation in rates of return is really low and we can look at the future as almost predictable. That is why the stationary status exists even though the unfunded liability is projected many years into the future.

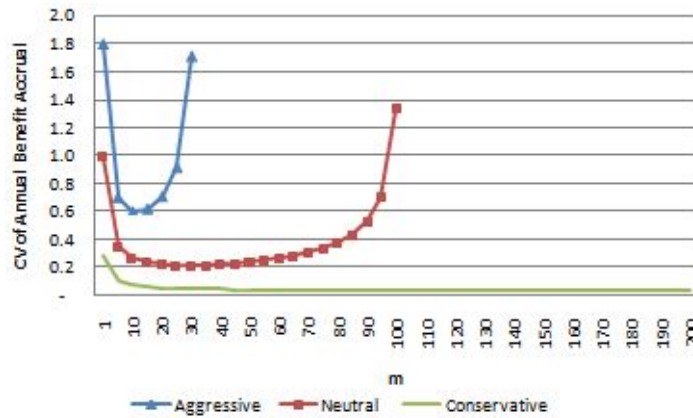


Figure 4.3: Coefficient of Variation of Annual Benefit Payment with Different Investment Strategies and Spread Periods ($i_v = i$)

Since the coefficients of variation for the contributions and the benefits behave in a similar fashion, in Figure 4.3 we only show the coefficients of variation for the ultimate annual benefit payment. We can see that the ‘U’ shape exists for all three investment scenarios, even though it is relatively flat when using a conservative strategy. The optimum spread period is 10, 30, and 130 years for the aggressive, neutral, and conservative strategies, respectively. The ultimate values of the coefficients of variation at the optimum spread period are higher for riskier investment strategies than that of less risky portfolios.

4.3.3 Different Valuation Interest Rates

In this section we look at how different valuation rates of interest affect the risks of the hybrid plan. We assume a neutral investment strategy since it is most common in practice. Recall from Table 4.1 that under this strategy, the expected real rate of return is $i = 0.02$ and the standard deviation is 0.08. Cairns and Parker (1997) have investigated this issue for a DB plan, and calculated the first moments of the contribution and funding level, using different valuation bases. We observe similar results in the hybrid plan. Since this project

uses a different valuation method from Cairns and Parker (1997), we do not compare the bases directly.

1. First Moment

The first moments are calculated using Equation (3.25). We notice that the plan cannot converge to an ultimate status when m is 100 years or longer, regardless of the valuation basis. This is because the convergence condition is only related to the investment rate of return and has nothing to do with the valuation rate of interest.

A valuation basis affects the plan by changing the normal costs and actuarial liabilities. From Equation (4.4), we know that a high valuation interest rate means a small present value for future liabilities, which results in a low normal cost and therefore a low actuarial liability.

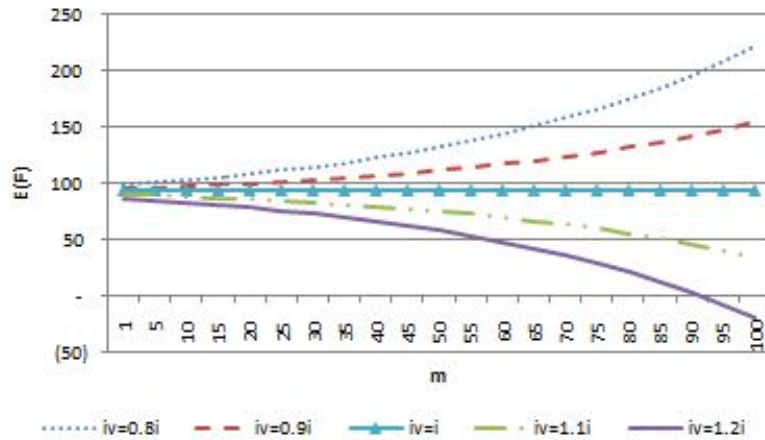


Figure 4.4: Mean of Funding Level with Different Spread Periods ($i_v \neq i$)

Figure 4.4 shows the mean of the ultimate funding level at different valuation rates of interest and spread periods. We notice that a higher valuation interest rate corresponds to a lower fund level. This is due to the lower normal costs that we project for higher valuation interest rates.

For valuation rates that are higher than the mean investment rate of return, there may be negative expected funding when the spread period is very long. For example, for $i_v = 1.2i$, when the spread period is 95 years or longer, there is a negative expected funding level,

which is undesirable. This is due to a lack of sufficient contributions from low normal costs and the accumulation of a deficit. When the deficit is further spread into the future, it is like avoiding the debt by continuing to borrow more money to spend more, and then the deficit gets worse. By contrast, for valuation rates that are lower than the mean investment rate of return, there is a high level of normal costs. The longer the spread period is, the higher the ultimate value of the expected funding level.

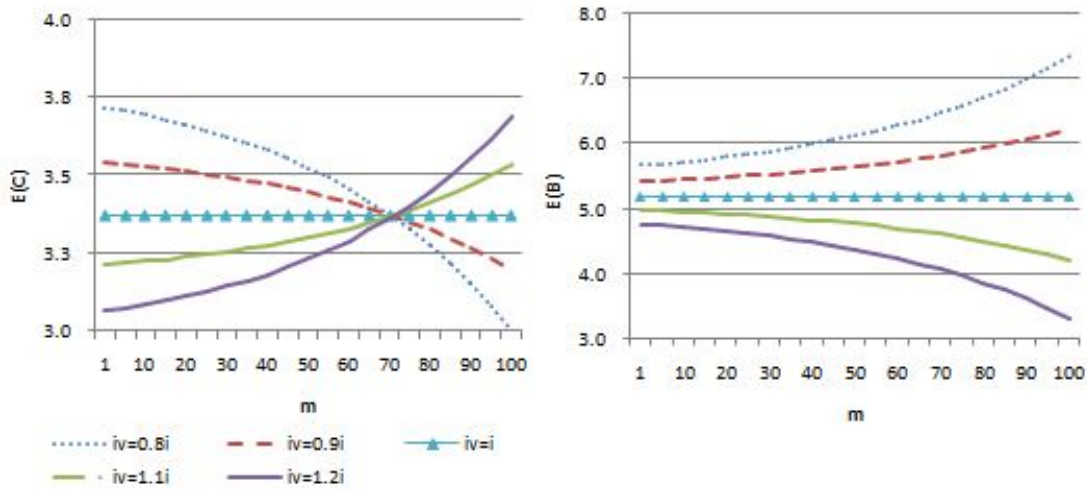


Figure 4.5: Mean of Annual Contribution and Benefit Payment with Different Spread Periods ($i_v \neq i$)

Figure 4.5 shows the first moments of the stationary value of the annual contribution (left) and the annual benefit payment (right), using different spread periods and valuation bases. We have three points to make:

- For short spread periods, $m \leq 70$ in our example, a lower valuation interest rate results in a higher ultimate contribution rate, due to the higher normal cost required to fund the pension plan.
- For longer spread periods, $m > 70$ in our example, a lower valuation interest rate leads to a lower ultimate contribution rate. This is because the expected funding level is higher when using a lower valuation interest rate (Figure 4.4), which creates a large surplus and is adjusted by decreasing the contribution rate.

- Without surprise, the mean annual benefit payment shown on the right in Figure 4.5 behaves the opposite of the mean contribution, but similarly to the funding level.

2. Second Moment

Next we study the second moments of funding, benefit and contribution rates with different spread periods using Equation (3.26).

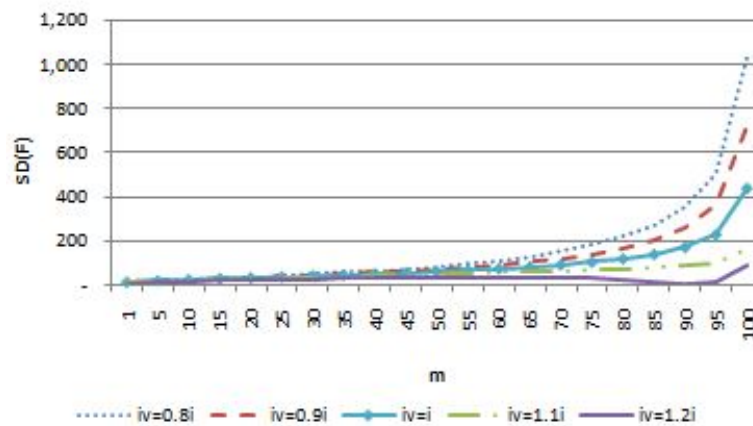


Figure 4.6: Standard Deviation of Funding Level with Different Spread Periods ($i_v \neq i$)

In Figure 4.6, we show the standard deviation of the funding level. From the earlier analysis, we know that the bigger the difference between the valuation rate of interest and the long term mean i , the larger the fund surplus or deficit, thus the more variation we expect in the funding level. A valuation basis with $i_v < i$ creates more variations in funding than a valuation basis with $i_v > i$ in general. This is a result of high normal costs and actuarial liabilities when $i_v < i$, and it scales up the variability of the funding level.

In Figure 4.7, we plot the standard deviations of the contribution and benefit with different spread periods, against their corresponding average levels. When $i_v = i$, the mean of the benefit payment converges to the target benefit and the mean contribution rate converges to the normal cost. Since the target benefit and the normal cost are not related to the spread period, the mean values of the contributions and the benefits are constant regardless of the spread period. This relationship shows up as a vertical line in the middle of the graphs.

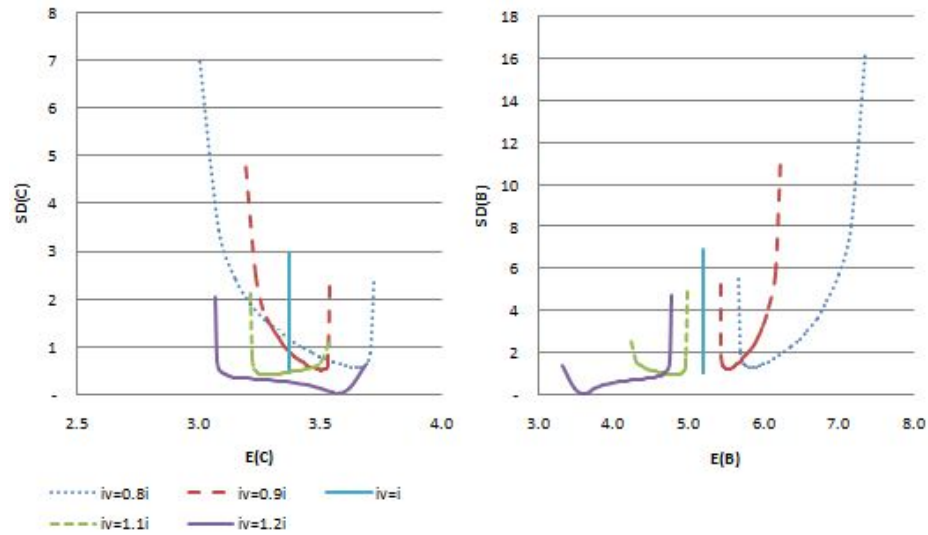


Figure 4.7: Standard Deviation of Annual Contribution and Benefit payment with Different Spread Periods ($i_v \neq i$)

All the other curves have a ‘U’ shape. The optimum spread periods are 25, 25, 30, 40, and 75 years, for $i_v = 0.8i, 0.9i, i, 1.1i,$ and $1.2i,$ respectively. On the right, we can see clearly an efficient frontier that defines the minimum standard deviation available for a given mean.

4.3.4 Plan Evolution

In this section, we investigate how the hybrid plan evolves with time in the benchmark case. We project the first two moments at the optimum spread period over a 200-year horizon. The formulas used for the illustrations are Equations (3.23) and (3.24). We choose spread periods of 10 years, 50 years, and the optimum period 30 years for this comparison. Recall that, in the case when $t = \infty,$ we have observed in Section 4.3.1 that the first moments are constant, the variance of funding level increases with the spread period, and the variations of the contribution and benefit are minimized at $m = 30.$

Since the initial funding level F_0 is set to be the actuarial liability, the first moments are already at their ultimate levels from the outset. The mean contribution equals the normal cost at all times and the mean benefit outgo is the target benefit outgo each year.

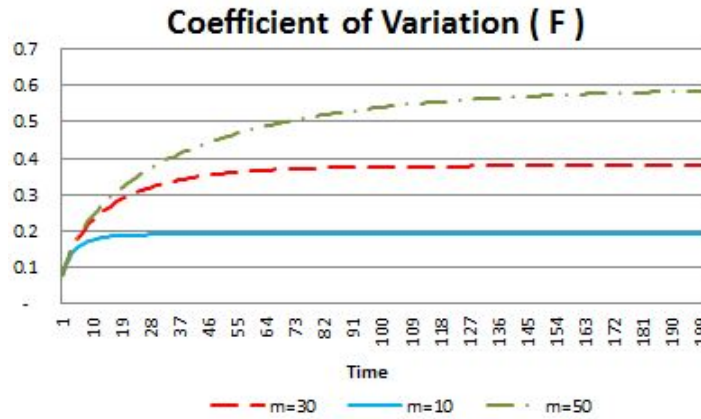


Figure 4.8: Coefficients of Variation of Funding level at Different Times with Different Spread Periods under Neutral Strategy ($i_v = i$)

The coefficients of variation of the funding level at different times are shown in Figure 4.8. Similar to the observation we have made in the stationary case, the variation of the funding level is an increasing function of the spread period, since spreading unfunded liability into the far future brings in more investment risks in the long run. The longer the spread period is, the longer it takes for the plan to converge in the second moment.

We plot the coefficients of variation of annual contributions and benefits in Figure 4.9. As usual, for short spread periods, we can see a quick pick up in the variation and an early arrival at stationary status. In our example, the benefit always has more variability than the contribution rate, due to the fact that we spread only 30% of the unfunded liability into contributions while 70% of it into the benefits.

4.3.5 Some Relationships

So far we have looked at the impact of different investment strategies, spread periods and valuation bases on the hybrid pension plan separately, holding other factors constant. We now study and show the interactions of two factors together with the help of three-dimensional surfaces.

1. Varying proportions of spread parameters and spread periods

In previous illustrations, we have taken $k_c = 0.3k$ for all the examples. For a given

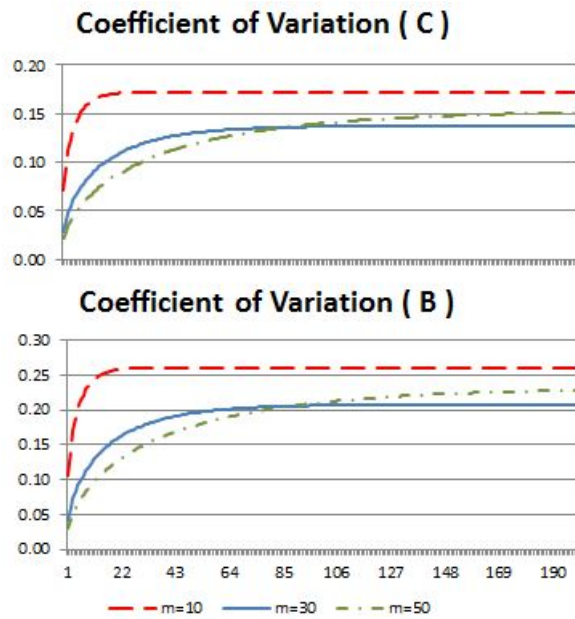


Figure 4.9: Coefficients of Variation of Contribution and Benefit at Different Times with Different Spread Periods under Neutral Strategy ($i_v = i$)

spread period, we now allow different proportions of the spread parameters.

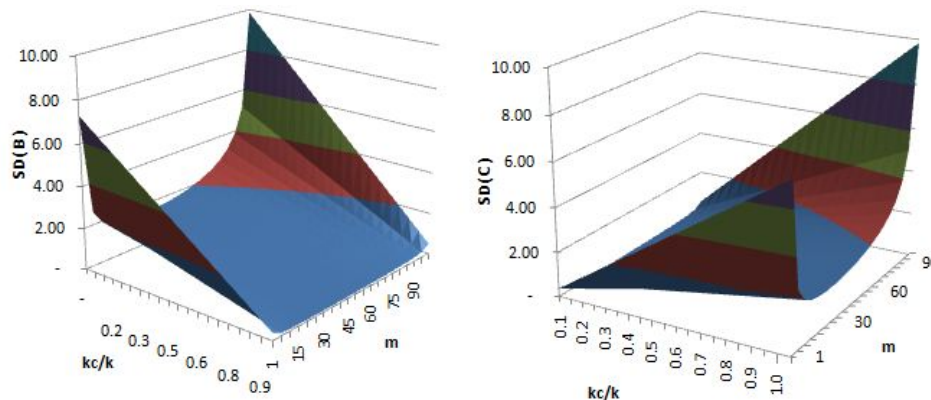


Figure 4.10: Standard Deviations of Contribution and Benefit with Different Spread Parameters under Neutral Strategy ($i_v = i$)

In Figure 4.10, we choose the proportion k_c/k as one variable and the spread period m as the other variable, and calculate the corresponding standard deviations of annual contribution rates and annual benefit payments in the stationary case. This gives us two ‘identical’ graphs, except that the k_c/k turns one to the opposite direction.

When $k_c = 0$, it is a special case of the hybrid plan, the pseudo hybrid plan, where the contribution is fixed at the normal cost and the unfunded liability is adjusted into the benefit only. When $k_b = 0$, it is a special case of the hybrid plan, the DB plan, where the annual benefit payment is fixed at the target benefit level and the unfunded liability is adjusted into the contributions only.

Whichever k_c/k we take, the optimum spread period is 30 years. This observation lines up with Equations (3.17) and (3.18). The standard deviation of contributions and benefits have a linear one-to-one trade off relationship. An increase in the standard deviation of contributions means the same amount of decrease in the standard deviation of benefits. The sum of the two standard deviations is fixed, for a given spread period.

2. Varying spread parameters for contribution and benefit

The relationship we have claimed earlier can be better illustrated without the constraint

of a given spread period. Recall that we define the sum of the standard deviation of contributions and benefits as the aggregate risk (Equation (3.19)). In Figure 4.11, we show the aggregate risk at the stationary status on the vertical axis with the parameters k_c and k_b each chosen between 0 and 1 on the floor surface. Each point in the graph represents the aggregate risk given k_c and k_b .

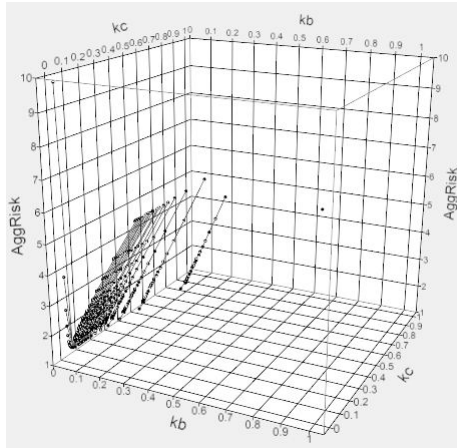


Figure 4.11: Aggregate Risk with Different Spread Parameters under Neutral Strategy ($i_v = i$)

The aggregate risk drops immediately as k_c and k_b increase from the lower-left corner to the upper-right corner, and rise after hitting the minimum. At the minimum level of the aggregate risk, the sum of k_c and k_b is constant and equals to 0.04, which in turn means an optimum spread period of 30, since $k = 1/\bar{a}\bar{m}$.

This observation is consistent with the findings in Khorasane (2012). The practical meaning of this observation is that, despite the fact that the risk in funding level is a monotonically increasing function of the spread period, we can pick the optimum spread period that minimizes the total risk borne in the contributions and the benefits. As for the choice of spread parameters k_c and k_b , the sponsor and members can negotiate, as long as the sum of the spread parameters is fixed at the optimum level.

3. Varying investment strategy and spread period

Last but not least, we take a look at the situation when decisions regarding investment strategies and the optimum spread period are made simultaneously. The spread period is

m , and the proportion of the risky assets in the portfolio is x . In Figure 4.12, we show the aggregate risk at stationary status when these two variables are changed at the same time.

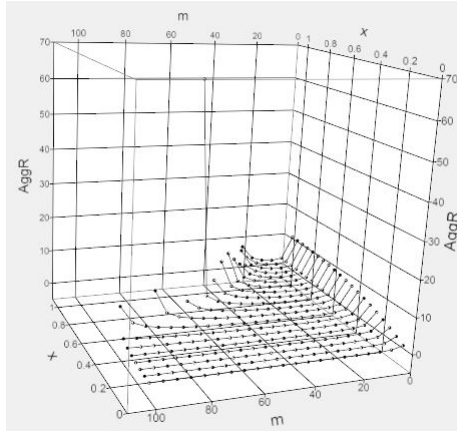


Figure 4.12: Aggregate Risk with Different Spread Parameters under Different Investment Strategies ($i_v = i$)

The upper-left corner of the ground surface is empty because when the portfolio is highly risky and the spread period is very long, the plan fails to converge to a stationary status.

For a given choice of x , the aggregate risk decreases quickly first and increases slowly as the choice of spread period increases. As x increases, the optimum m gets smaller, which shows as a valley in the middle of the graph.

Given the spread period m , the stationary aggregate risk is an increasing function of x .

Based on the above description, we can conclude that if we were to put a golf ball on the surface, it would roll down to the lowest point, which is the lower-left corner on the ground surface. If pension plan managers were to look for a global minimum of the aggregate risk on this three-dimensional surface, they would end up with the least risky investment strategy and the longest spread period.

Chapter 5

Numerical Illustration: AR(1) Case

In this chapter, we study the behavior of and the risks inherent in our hybrid pension plan, when the force of interest is assumed to follow an AR(1) process. Since there are many parameters involved in this model, we set up a benchmark case, and then change one parameter at a time to study the impact through comparison. Similar to the last chapter, we also discuss some relationships when two parameters are subject to change at the same time.

5.1 Benchmark Case

We set our benchmark case as follows.

- Investment strategy: neutral.
- Coefficient of AR(1) model: we arbitrarily set $\phi_2 = 0.5$, because in real life the forces of interest usually have a positive correlation with the previous period's rate of return.
- Valuation rate of interest: we set $i_v = i$.
- Optimum spread period: 13 years. The spread period that minimizes the coefficient of variation of the contributions is 12 years, compared to 14 years for the benefits. Since the spread period that minimizes the standard deviations of the contributions and the benefits is 13 years, we choose 13 years as the optimum spread period.
- The parameters are summarized in Table 4.1. The variance of the white noise, γ_2^2 , is obtained from Equation (3.4) given the long term variance and coefficient of the process.

The optimum spread period in the AR(1) case is shorter than the optimum spread period under the Gaussian assumption (30 years). Even though the mean and variance of the force of interest are set to be equivalent in these two processes, the two processes behave differently. To assist the explanation, we simulate one set of random forces of interest using a normal distribution, and another set of random forces of interest using an AR(1) process with $\phi_2 = 0.9$. Both simulations use the mean and variance of the neutral investment strategy, a starting point that is equal to the means of the processes. The simulation results are plotted in Figure 5.1 in a 200-year horizon.

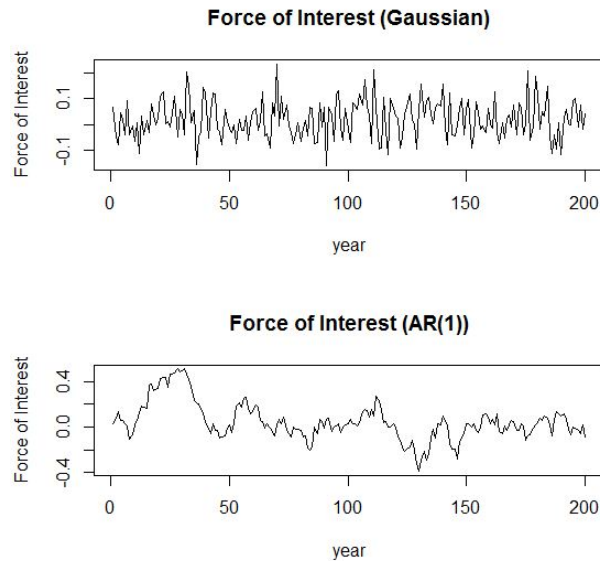


Figure 5.1: Simulated Forces of Interest Using Gaussian and AR(1) Processes

As we can see from Figure 5.1, the forces of interest under the Gaussian assumption fluctuate around the long term mean 0.029.

For an AR(1) process with $\phi_2 = 0.9$, it is very likely that the force of interest in the next year is around the level of the force of interest in the current year. This dependency in rates of return amplifies the level of surplus and deficiency. In order to achieve minimum variation, it is preferable to adjust the unfunded liability sooner than later, compared with a process with less memory.

After setting up the benchmark case, we show the evolution of the first two moments of the hybrid plan in Figure 5.2 and Figure 5.3.

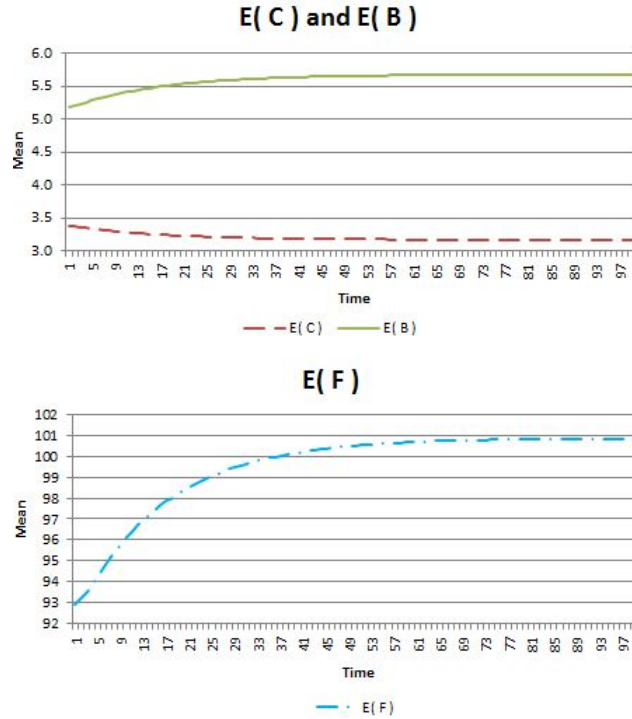


Figure 5.2: Evolution of Means of Funding, Contribution and Benefit ($i_v = i$)

Figure 5.2 shows the means of the funding level, annual contribution rate and benefit payment at different times, calculated from Equation (3.32). Unlike the Gaussian case, the stationary mean of the fund is higher than its target level, the actuarial liability. The mean of the funding level increases from 92.9 at time 0 and reaches a plateau status around year 105. The ultimate level of the mean benefit outgo is slightly higher than the target benefit and the ultimate level of the mean contributions is slightly lower than the normal cost. Haberman (1994) also made the comment that the ultimate value of the funding level is different from the actuarial liability, due to the exponentiation of forces of interest used in the AR(1) process.

We illustrate the evolution of the coefficients of variation of the funding level, annual contribution level, and annual benefit payment in Figure 5.3. The calculation is based on

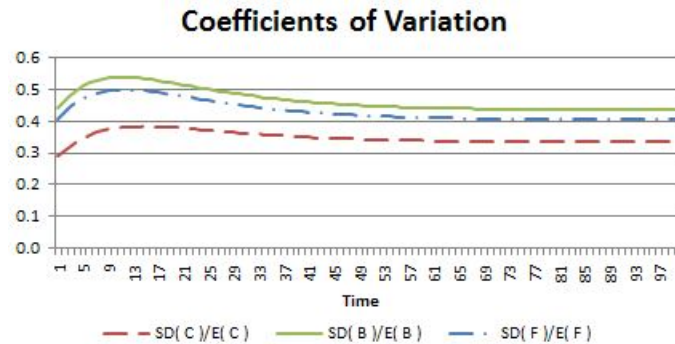


Figure 5.3: Evolution of Coefficients of Variation of Funding, Contribution and Benefit ($i_v = i$)

Equation (3.34). Since the initial funding is smaller than its stationary level, there is an increase in the variations in the first 13 years and a slow decrease afterwards. The plateau status is observed around year 50. The variation at year 300 is 7.8% higher than the level at year 1. This is very different from the consistent and steady increase we have seen in the Gaussian case.

Remark:

Haberman (1994) showed that for values of $\phi_2 < 0.7$ and under certain conditions on ν_2 , the relative errors in using the approximated Equations (3.33) and (3.35) are small. We also notice in the calculations that the approximation produces good results up until $\phi_2 = 0.6$. As ϕ_2 gets closer to 1, the approximation has bigger deviation from the true value. Therefore we set $\phi_2 = 0.5$ in the benchmark case.

In spite of the disadvantage of the approximation results, we continue using the approximated equations in the rest of the demonstration due to the following concerns. When we are conducting comparison analysis using different parameter sets, the model converges at different speeds and it is hard to decide how long the projection period needs to be if we hope to obtain the exact result. The approximations used in Haberman (1994) significantly reduces the calculation time, and enables the comparison of results from different parameter sets. In addition, we also check that the optimum spread period chosen by the approximation results is also the optimum choice while using exact values. However, we do not recommend the use of this approximation for other purposes.

5.2 Varying Spread Periods and Model Parameters

After showing the plan evolution using an optimum spread period of 13 years and autoregression parameter $\phi_2 = 0.5$, we study the stationary status of the pension plan when these two parameters change. Since it is unlikely for a real life rate of return to have negative correlation between time steps, we focus on the positive autoregression coefficients in this project. The parameters used in this section are:

m : integer values between 1 and 340 (steps of 1 up to 15, then in steps of 5 up to 50, then in steps of 10 up to 100, then in steps of 20);

ϕ_2 : 0.01, 0.1, 0.3, 0.5.

1. First Moment

We plot the first moments of the stationary contribution rate, annual benefit payment and the funding level with different choices of spread periods and ϕ_2 in Figure 5.4.

For small values of the spread period m , the first moments show a flat shape in the graphs as m gets bigger. This is because the stationary results are close to the target level, regardless of choice of the spread period and ϕ_2 .

For a given coefficient ϕ_2 , the expected annual contribution in the stationary case is a decreasing function of the spread period; the expected annual benefit and the expected funding level are increasing functions of the spread period. For a given spread period m , the expected annual contribution in the stationary case is a decreasing function of ϕ_2 ; the mean annual benefit and the mean funding level are increasing functions of ϕ_2 .

It is hard to tell the existence of these relationships from Equation (3.33). We simulate two series of forces of interest to help explain the phenomenon in 200 years in Figure 5.5. These two series have the same long term mean and variance, and follow an AR(1) process, but with $\phi_2 = 0.6$ and 0.01. Similar to the comparison in Figure 5.1, we see more local fluctuations for the series generated with $\phi_2 = 0.01$. For 1 unit at time 0, the values accumulated to time 200 are 127 and 107 for series with $\phi_2 = 0.6$ and 0.01, respectively.

Assuming that the force of interest follows an AR(1) process with given long term mean and variance, the level of a pension fund is more likely to accumulate to a large value when the coefficient ϕ_2 is high. This explains why a large ϕ_2 results in a higher expected funding

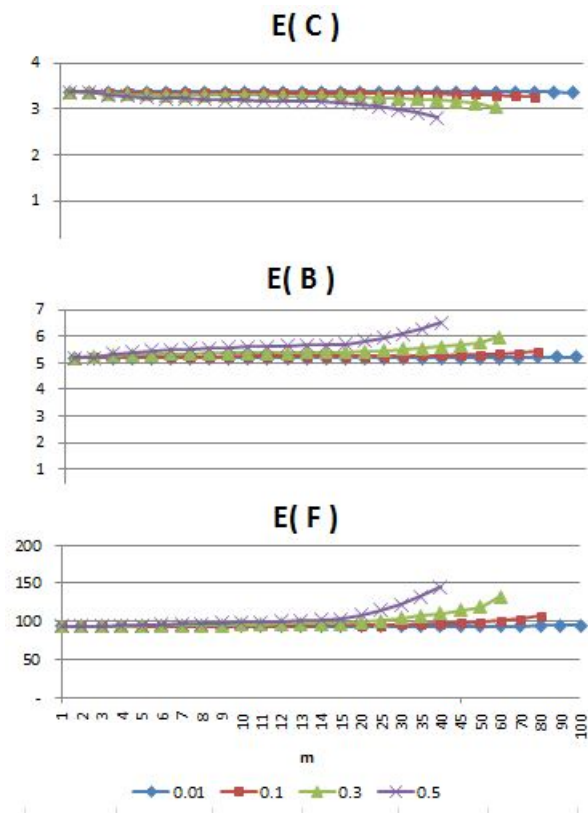


Figure 5.4: Means of Funding, Contribution, and Benefit with Different Spread Periods and ϕ_2 ($i_v = i$)

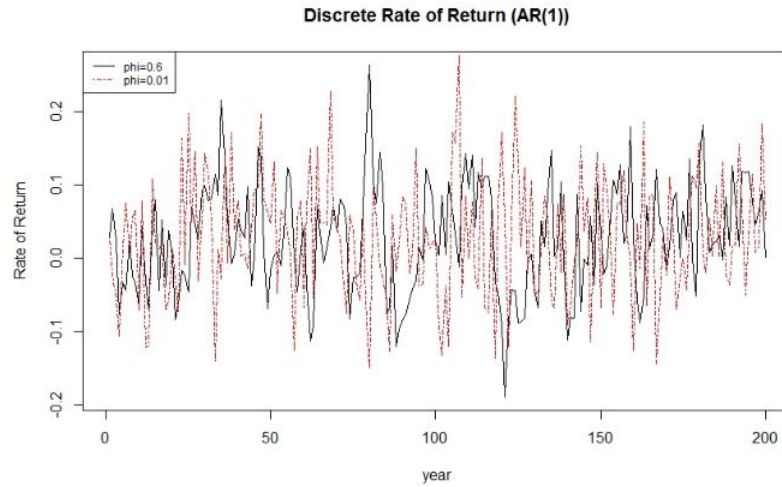


Figure 5.5: Simulated Discrete Rates of Return Using AR(1) Process

level and annual benefit, and therefore a lower contribution rate in the stationary situation.

2. Second Moment

We plot the coefficients of variation of the annual contribution rate, the annual benefit payment and the funding level in Figure 5.6.

When $\phi_2 = 0.01$, the AR(1) process is close to a Gaussian process. Similar to Figure 4.1 in the Gaussian case, the coefficients of variation of the contribution rate and the benefit payment have a ‘U’ shape. They are minimized at $m = 30, 25, 20,$ and 13 for $\phi_2 = 0.01, 0.1, 0.3, 0.5$ respectively. With higher coefficient ϕ_2 , the risks in the contribution, benefit and funding are higher.

5.3 Varying Investment Strategies

In this section, we compare the hybrid pension plan under different investment strategies.

It is hard to decide what the short term variations γ_2^2 and coefficients ϕ_2 are under each strategy, without an idea of the component in the investment portfolios in each strategy. For simplicity, we assume that $\phi_2 = 0.5$ under all investment strategies. We then use Equation (3.4) to calculate the short term variations γ^2 . The model parameters of the

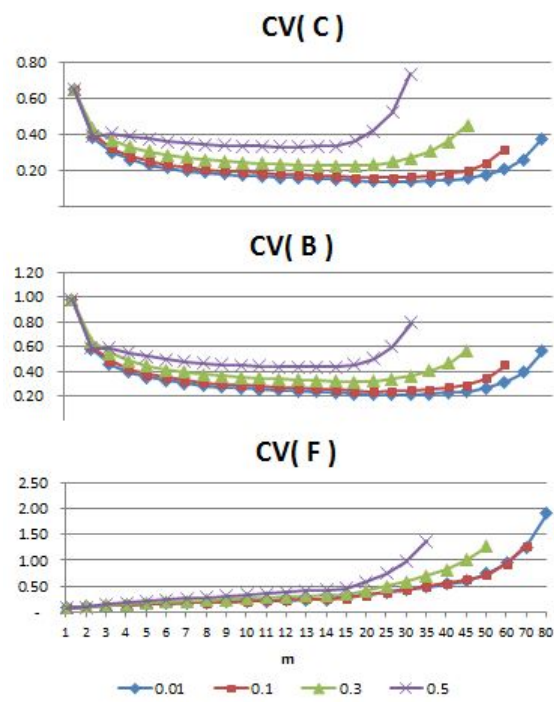


Figure 5.6: Coefficients of Variation (CV) of Funding, Contribution, and Benefit with Different Spread Periods and ϕ_2 ($i_v = i$)

AR(1) process used in this section are summarized in Table 4.1.

1. First Moments

We firstly compare the means of the plan funding, contribution, and benefit in the stationary case in Figure 5.7.

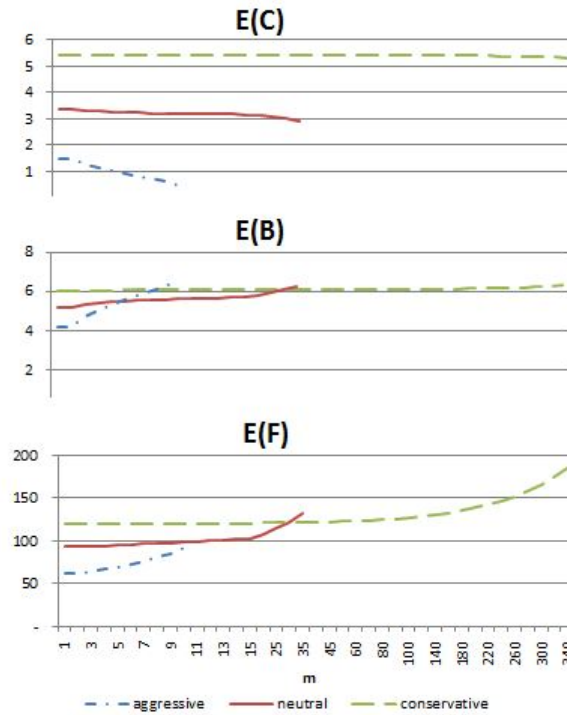


Figure 5.7: Means of Contribution, Benefit under Different Investment Strategies ($i_v = i$)

Overall, for a relatively riskier investment strategy, the maximum spread period the pension plan can use while having a stationary status is shorter, the mean contribution and funding level are lower, and the average annual benefit payment is higher.

2. Second Moments

The coefficient of variation is a great measure of volatility in the pension plan; however, we observe a large amount of fluctuation because the standard deviations are divided by some very small values of the first moments. The resulting graph can give us a biased idea

of the actual amount of risks involved. We therefore use standard deviations to compare the risks under different investment strategies in Figure 5.8.

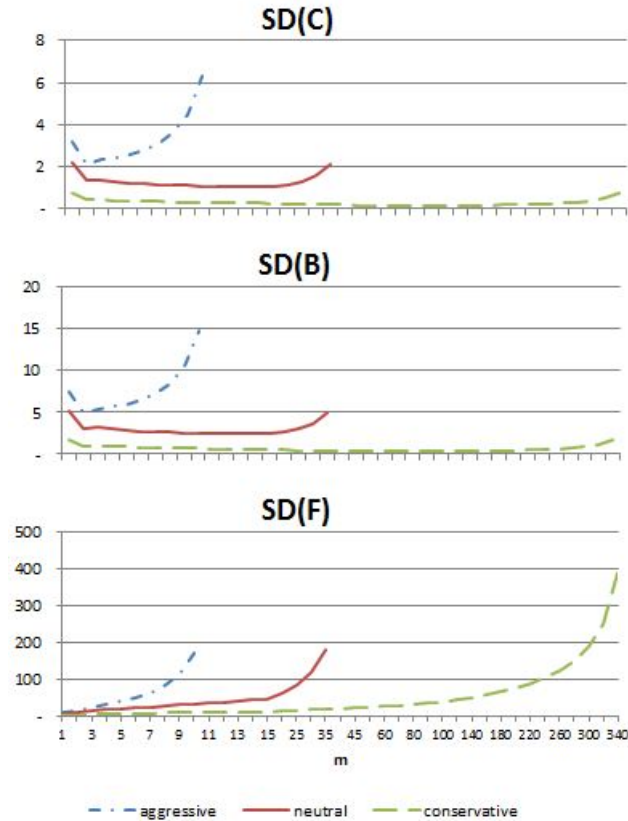


Figure 5.8: Standard Deviations of Contribution and Benefit under Different Investment Strategies ($i_v = i$)

Similar to what we have seen in the Gaussian case, for a relatively riskier investment strategy, the maximum spread period the pension plan can use while having a stationary status is shorter; the standard deviations for the contribution, benefit and funding are higher, due to the high variance of the force of interest. The spread periods that minimize the standard deviations of the contributions and the benefits are 2, 13, and 100 for the conservative, neutral and aggressive strategy respectively.

5.4 Varying Valuation Rate of Interest

Next we use different valuation rates of interest in the benchmark case. The means of ultimate funding level, contribution rate and benefit payment show a similar pattern as what we have observed in the Gaussian model. We do not discuss it again here. The standard deviations, however, as shown in Figure 5.9, are slightly different from the ones in the Gaussian case.

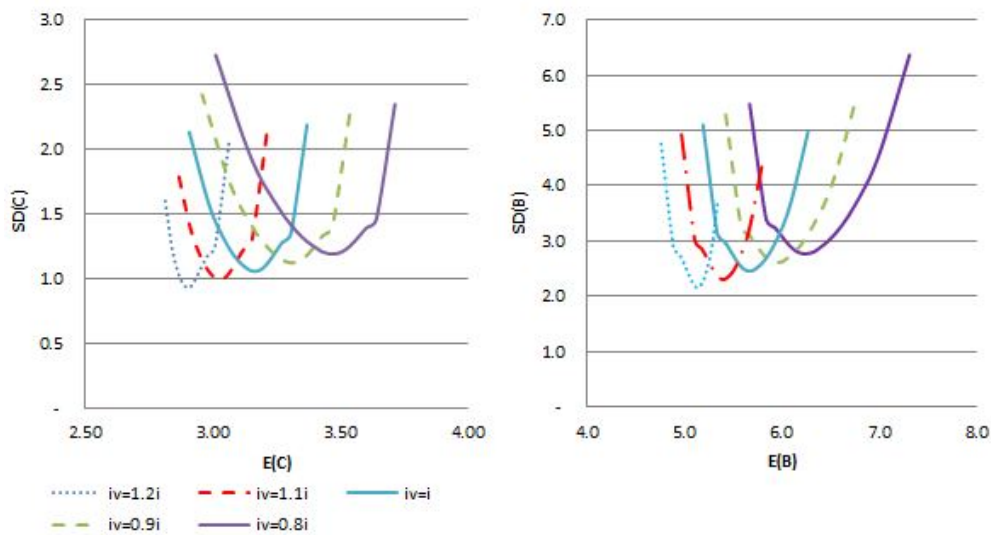


Figure 5.9: Standard Deviations of Contribution and Benefit with Different Spread Periods ($i_v \neq i$)

In Figure 5.9, we plot the standard deviations of the contributions and the benefits, given their average levels. The curves appear to be ‘hook’ shaped, as was described in Cairns and Parker (1997), and the ‘hooks’ for the contributions are placed in the opposite direction as those of the benefits. There appears to be an upward trending efficient frontier in both graphs. The optimum spread periods for all scenarios are 12, 12, 13, 14, 15 years with respect to the valuation rates of interest from the smallest to the largest.

5.5 Some Relationships

So far we have looked at the impact of autoregressive coefficients, spread periods, investment strategies and valuation bases on our hybrid pension plan separately. We now use the help of three-dimensional surfaces to study the interactions of two factors together.

1. Varying proportions of spread parameter and spread periods

In previous illustrations, we have set $k_c = 0.3k$ for all the examples. Here we look at the trade-off between the spread parameters of contributions and benefits.

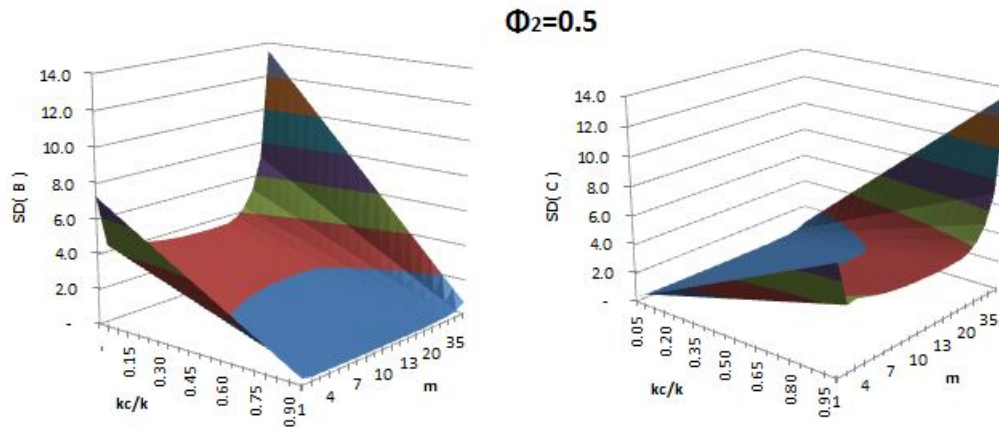


Figure 5.10: Standard Deviations of Contribution and Benefit with Different Spread Parameters ($\phi_2 = 0.5, i_v = i$)

In Figure 5.10, we choose the proportion k_c/k as one variable and the spread period m as the other variable, and calculate the standard deviations of the annual contribution rate and the annual benefit payment at stationary status. Similar to the Gaussian case, this gives us two ‘identical’ graphs, except that the k_c/k turns one to the opposite direction.

For any given k_c/k , the optimum spread period is always 13 years. For any given spread period, there is a one-to-one linear trade-off relationship between the standard deviation of the contribution and the standard deviation of the benefit. This is determined by Equations (3.17) and (3.18).

In Figure 5.11, we use $\phi_2 = 0.01$ instead. The process is close to a Gaussian process and the graph is similar to Figure 4.10, except that the maximum spread period allowed for the

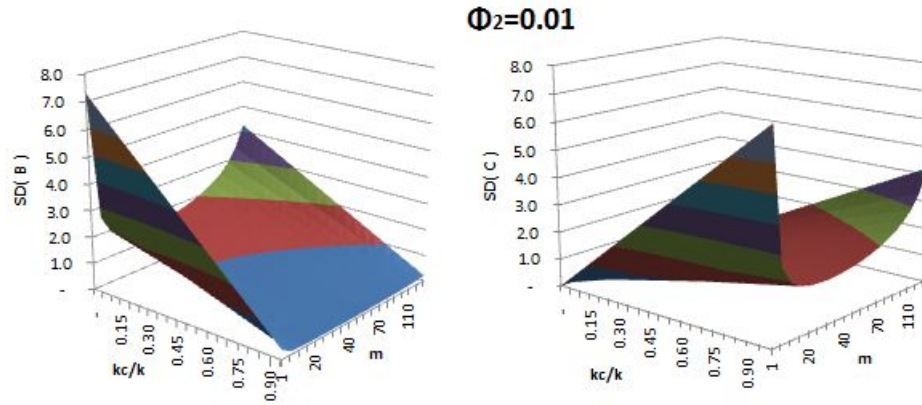


Figure 5.11: Standard Deviations of Contribution and Benefit with Different Spread Parameters ($\phi_2 = 0.01, i_v = i$)

existence of stationary status is shorter.

2. Varying spread parameters for contributions and benefits

The relationship claimed above can be better illustrated without the constraint of optimum spread period. Recall that we have defined the sum of the standard deviation of contribution and benefit as the aggregate risk. In Figure 5.12, we show the aggregate risk at stationary status with k_c and k_b chosen between 0 and 1.

The left part of Figure 5.12 corresponds to the case of $\phi_2 = 0.5$. For small values of k_c and k_b , the aggregate risk is high due to the long spread period. The aggregate risk drops when k_c and k_b start to increase, and then it slowly climbs up, and drops again until k_c and k_b are too large for the plan to converge. Unlike Figure 4.11 in the Gaussian case, the aggregate risk is not a linear function of k .

The right part of Figure 5.12 corresponds to the case of $\phi_2 = 0.01$. It is similar to Figure 4.11 in the Gaussian case, because the rates of return behave in a similar way.

A special case in the graph is when $k_c = 0, k_b = 1$ or $k_b = 0, k_c = 1$. In these cases, the hybrid plan is either a pseudo hybrid plan or a DB plan, and the unfunded liability is adjusted in one-year spread period. The calculation is based on Equations (3.36) and (3.37). The resulting aggregate risks stand out at the upper right corner of the graph.

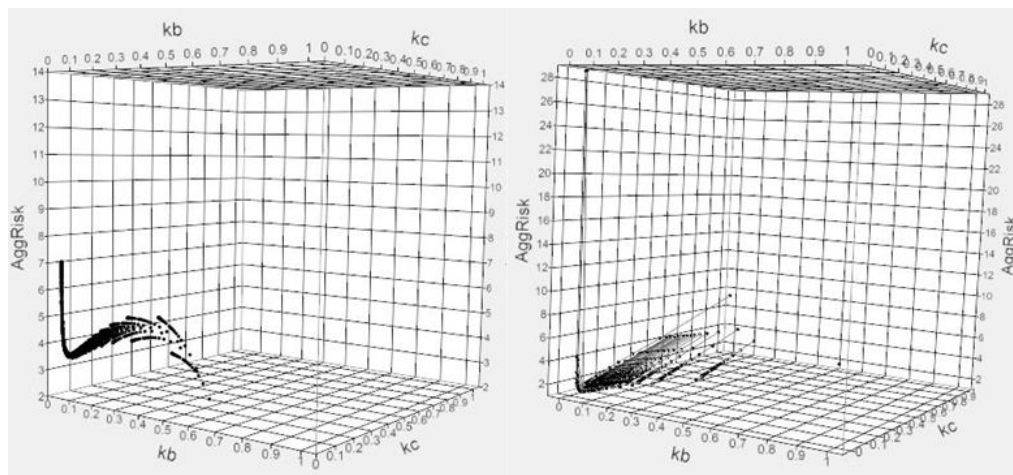


Figure 5.12: Aggregate Risk with Different Spread Parameters for $\phi_2 = 0.5$ (left) and $\phi_2 = 0.01$ (right) ($i_v = i$)

Chapter 6

Numerical Illustration: MA(1) Case

In this chapter, we study the behaviour and risks of the hybrid pension plan, when rates of return are assumed to follow an MA(1) process. Since there are many parameters involved in this model, we set up a benchmark case, and then change one parameter at a time to study the impact through comparison. Similar to the last two chapters, we also discuss some relationships when two parameters are subject to change at the same time.

6.1 Benchmark Case

We set our benchmark case as follows.

- Investment strategy: neutral.
- Coefficient of MA(1) process: we arbitrarily set $\phi_3 = -0.7$. The coefficient is negative in order to offset the negative sign in model (3.8), which delivers a positive dependence on the white noise from the last time step.
- Valuation rate of interest: we assume $i_v = i$.
- Optimum spread period: 20 years.
- The parameters are summarized in Table 4.1. The variance of the white noise γ_3^2 is obtained from Equation (3.10) given the long term variance and coefficient of the process.

We calculate the first two moments of the hybrid plan using Equations (3.38) and (3.40) under the benchmark case.

The first moments of the funding level, annual contribution rate and benefit payment show a similar pattern to the ones in the AR(1) case.

The second moments are increasing functions of time, and show a similar pattern to the ones in the AR(1) case. We therefore do not show the graphs here.

Table 6.1: Summary of Observations in Plan Evolutions

	Year	Gaussian	AR(1)	MA(1)
	ϕ_3	NA	0.5	-0.7
	Optimum Spread Period	30	12	20
	Plateau (Second Moment)	55	50	49

We can summarize the observations we have had in the three processes in Table 6.1. We now compare the evolution of the hybrid plan when assuming the force of interest follows these three processes and using the optimum spread period in each scenario. The investment strategy is neutral for all cases in Figure 6.1 and Figure 6.2.

As we can see from Figure 6.1, even though the three processes have equivalent long term means and variances of rates of return, the funding levels and the benefits generated in the plan are highest when using an AR(1) process, and lowest when using a Gaussian process. The expected benefits and funding levels in both AR(1) and MA(1) processes converge to values that are higher than their targets, but it takes a longer time for the benefits and funding levels to converge under the MA(1) case. The differences in stationary level and convergence speed are results of the differences in the nature of the process. The AR(1) process possesses the highest dependency on previous values.

Figure 6.2 shows obvious differences in the standard deviations of the contribution, the benefit and the funding level. The AR(1) process generates the highest risks in contributions and benefits, and the Gaussian process is the least risky among the three processes. Unlike the consistent, slow increase in variations in the Gaussian case, for the AR(1) process, the standard deviations increase during the first 13 years and then slowly decrease and reach a plateau. For the MA(1) process, the rise in standard deviations does not stop until year 56.

Even though the long term means and variances of the rate of return are set to be equivalent in these three processes, the realizations of the effective rates of return behave differently, which leads to the differences we observe here. Similar to the argument we had

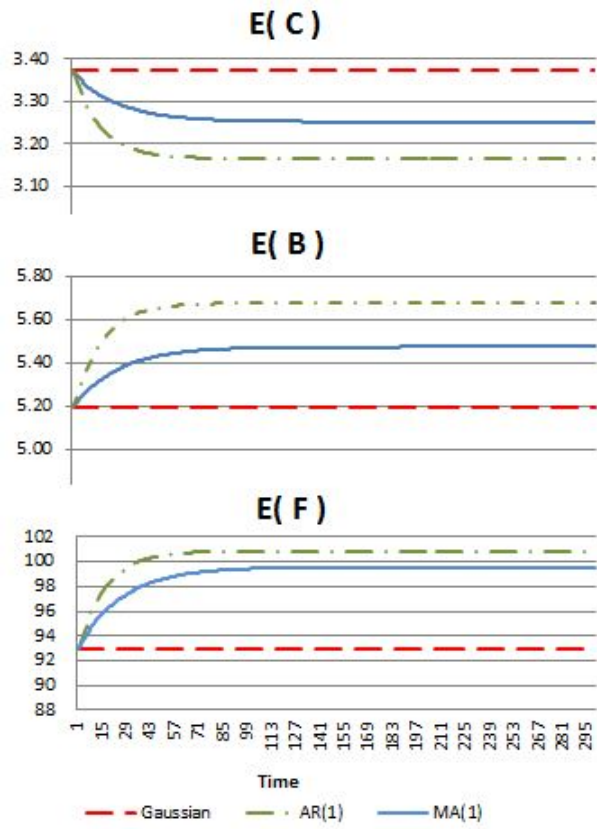


Figure 6.1: Evolution of the Means of the Contribution, Benefit, and Funding Level ($i_v = i$)

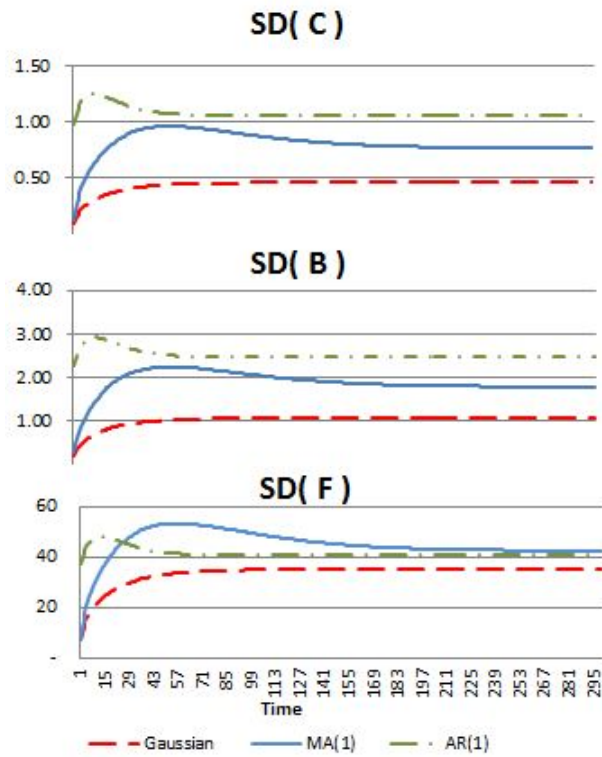


Figure 6.2: Evolution of the Standard Deviations of the Contribution, Benefit, and Funding Level ($i_v = i$)

for Figure 5.1, the dependency in rates of return amplifies the level of surplus and deficiency. The AR(1) process creates a high level of dependency between forces of interest, and the process has a long memory which carries all the information from past experience. Forces of interest that follow an MA(1) process are correlated with the past through the white noise from the last time step only. As for the Gaussian case, the rates of return at different time steps are independent of each other.

6.2 Varying Spread Period and Model Parameter

After showing the plan evolution using an optimum spread period of 20 years and a moving average coefficient $\phi_3 = -0.7$, we discuss the stationary status of the pension plan when these two variables vary, with the help of Equations (3.39) and (3.40). Recall that under an MA(1) process, the force of interest is the sum of the long term mean, a noise at current time, and a proportion of the noise at previous time. Since it is very unlikely for a real life rate of return to have negative correlation between time steps, we focus on the negative moving average coefficient in this project. The parameters used in this section are

m: integer values between 1 and 340 (steps of 1 up to 15, then in steps of 5 up to 50, then in steps of 10 up to 100, then in steps of 20);

ϕ_3 : $-0.1, -0.3, -0.5, -0.7, -0.9$.

1. First Moment

We calculate the values of the first moments of the stationary contribution rate, annual benefit payment and funding level with different choices of spread periods and ϕ_3 . The values and patterns are similar to the ones in Figure 5.4, so we do not present the graphs here.

2. Second Moment

We plot the coefficients of variation of the annual contribution rate and the annual benefit payment in Figure 6.3.

When $\phi_3 = -0.1$, the MA(1) process is close to a Gaussian process. The graph shows a similar pattern as what we have seen in Figure 4.1 and the optimum spread period is 25 years. As ϕ_3 increases, the optimum spread period decreases.

As ϕ_3 decrease to -0.9 , the ‘U’ shape gets steep at the end with longer spread period.

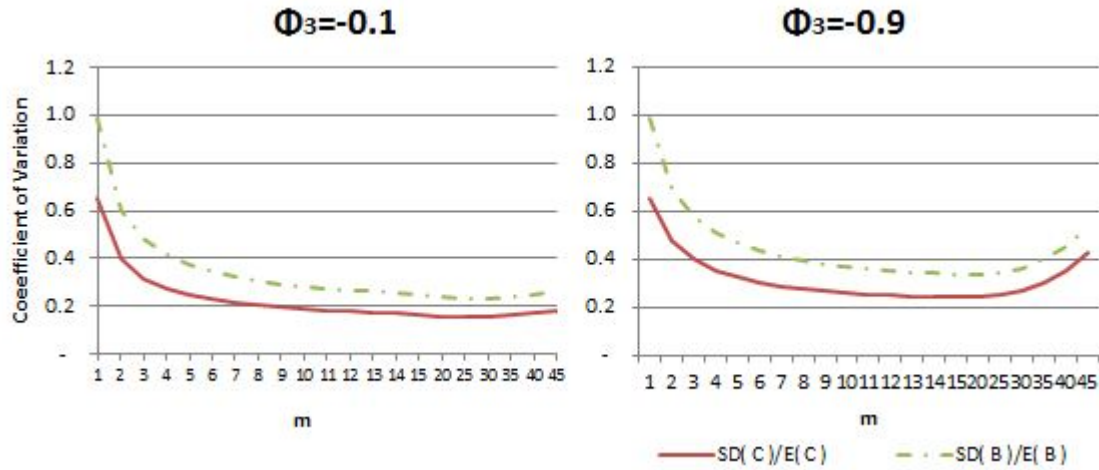


Figure 6.3: Coefficients of Variation of Contribution and Benefit with Different Spread Periods and ϕ_3 ($i_v = i$)

The optimum spread period shifts from 25 years to 20 years for the coefficient of variation of the contribution.

6.3 Varying Investment Strategy

Similar to the comparison we have performed in the AR(1) case, we compare the risks in the benchmark case with different investment strategies. We show the coefficients of variations of the annual benefit and the funding level using different spread periods in Figure 6.4. The optimum spread periods are 120, 20, and 6 years for the conservative, neutral, and aggressive strategies, respectively. The calculations are based on Equations (3.39) and (3.40)

6.4 Varying Valuation Rate of Interest

In this section we deviate from the benchmark case by using different valuation rates of interest. The first moments of the ultimate funding level, the contribution rate and the benefit payment show a similar pattern as what we have seen in the Gaussian model and AR(1) model. We do not discuss it again here. The standard deviations, however, as we can see from Figure 6.5, are slightly different from the Gaussian case and AR(1) case. Since

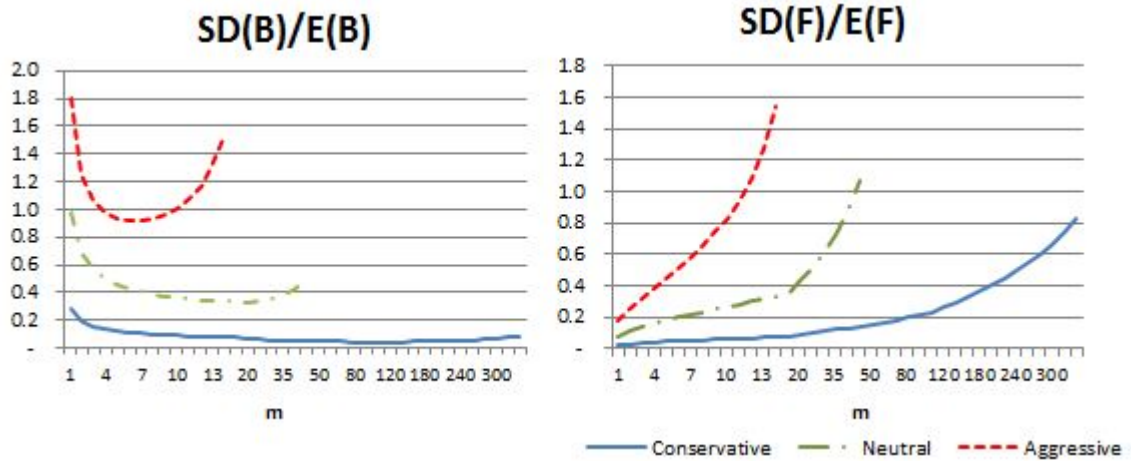


Figure 6.4: Coefficients of Variation of Benefit and Funding with Different Investment Strategies ($i_v = i$)

we are dealing with the stationary case, we find Equations (3.39) and (3.41) very helpful in the calculation.

In Figure 6.5, we plot the standard deviations of the contribution and the benefit, with their average levels on the horizontal axis. Recall that in the Gaussian case, the curve is a vertical line for $i_v = i$, due to constant first moments for all choices of spread periods. In the MA(1) case, the ultimate value of the mean benefit payment is an increasing function of the spread period; while the ultimate value of the mean contribution rate is a decreasing function of the spread period.

We can recognize an upward trending efficient frontier in both graphs in Figure 6.5. Equation (4.4) has shown that with lower valuation interest rates, the normal cost required per year is higher, and the actuarial liability is higher, which encourages the plan manager to spread the unfunded liability over a shorter period so as to prevent the investment risk from amplifying the unfunded liability.

6.5 Some Relationships

So far we have looked at the impact of moving average coefficients, spread periods, investment strategies and valuation bases on the hybrid pension plan. We now study the

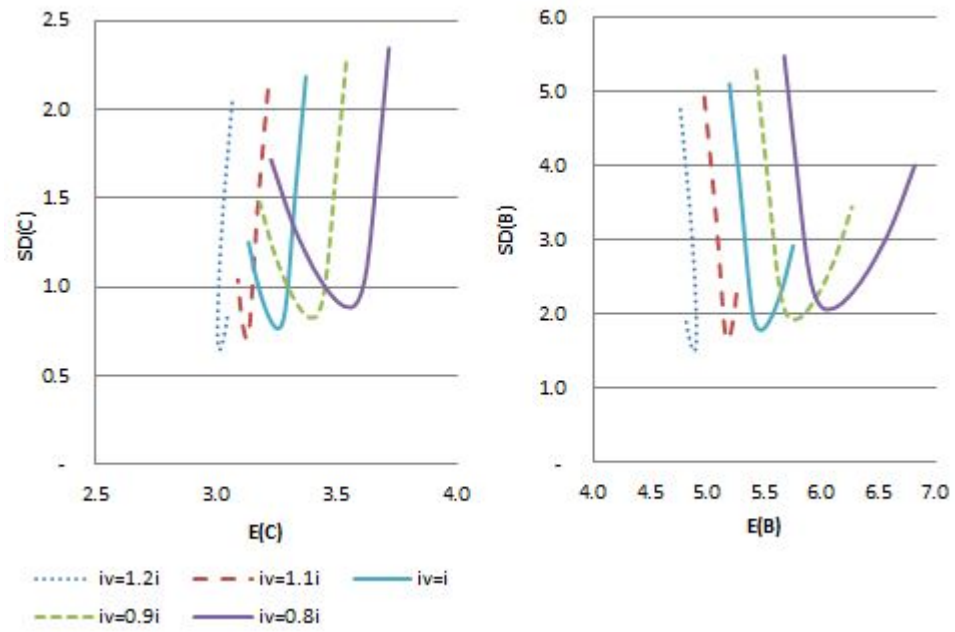


Figure 6.5: Standard Deviations of Contribution and Benefit with Different Means ($i_v \neq i$)

interactions of two plan factors together, using the help of three-dimensional surfaces.

1. Varying proportion of spread parameters and spread period

In previous illustrations, we set $k_c = 0.3k$ for all the examples. Here we discuss the trade-off between the spread parameters of contributions and benefits.

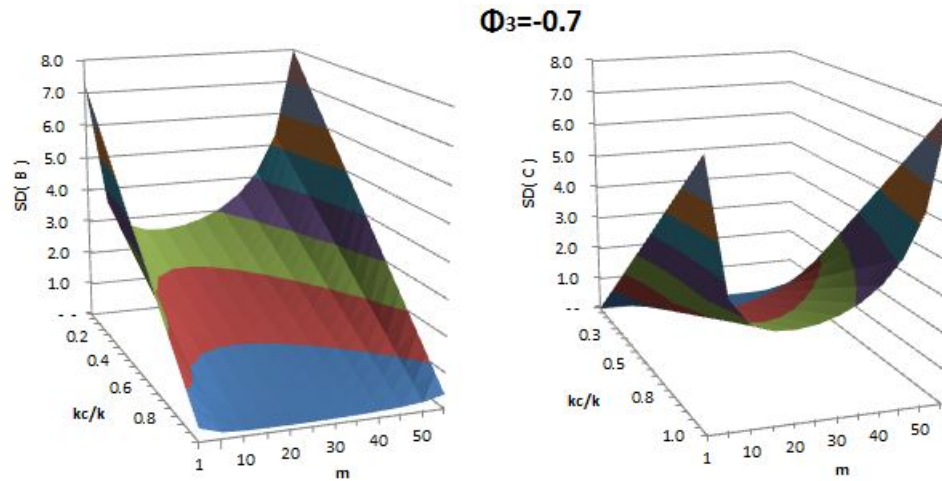


Figure 6.6: Standard Deviations of Contribution and Benefit with Different Spread Parameters ($\phi_3 = -0.7, i_v = i$)

In Figure 6.6, we choose the proportion k_c/k as one variable and the spread period m as the other variable, and calculate the standard deviations of the annual contribution rate and the annual benefit payment at the stationary status. Similar to the Gaussian case and the AR(1) case, this gives us two ‘identical’ graphs, except that the k_c/k turns one to the opposite direction.

For any given k_c/k , the optimum spread period is always 20 years. For any given spread period, there is a one-to-one linear trade off relationship between the standard deviation of the contribution and the standard deviation of the benefit. The other difference in the three processes is the maximum spread period that allows the existence of a stationary status, which is 50 in MA(1) case and 40 in AR(1) case.

In Figure 6.7, we use $\phi_3 = -0.01$ instead. The process is close to a Gaussian process

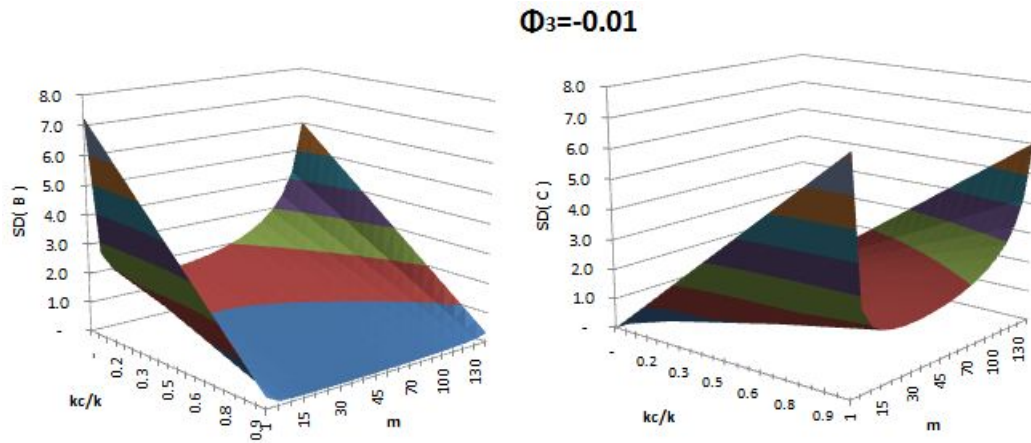


Figure 6.7: Standard Deviations of Contribution and Benefit with Different Spread Parameters ($\phi_3 = -0.01, i_v = i$)

and the graph is similar to Figure 4.10.

2. Varying spread parameters for contributions and benefits

Similar to the AR(1) case, we show the aggregate risk in the stationary status with k_c and k_b chosen between 0 and 1 in Figure 6.8 .

The left part of Figure 6.8 corresponds to the case of $\phi_3 = -0.7$, and the right part of Figure 6.8 corresponds to the case of $\phi_3 = -0.01$. The patterns of the graphs are similar to Figure 4.11 in the Gaussian case. As we have pointed out in the remark in Section 3.2.3, with the same means and variances of the force of interest, we have the same results in the special cases when $k_c = 0, k_b = 1$ or $k_b = 0, k_c = 1$ as the ones in the AR(1) process.

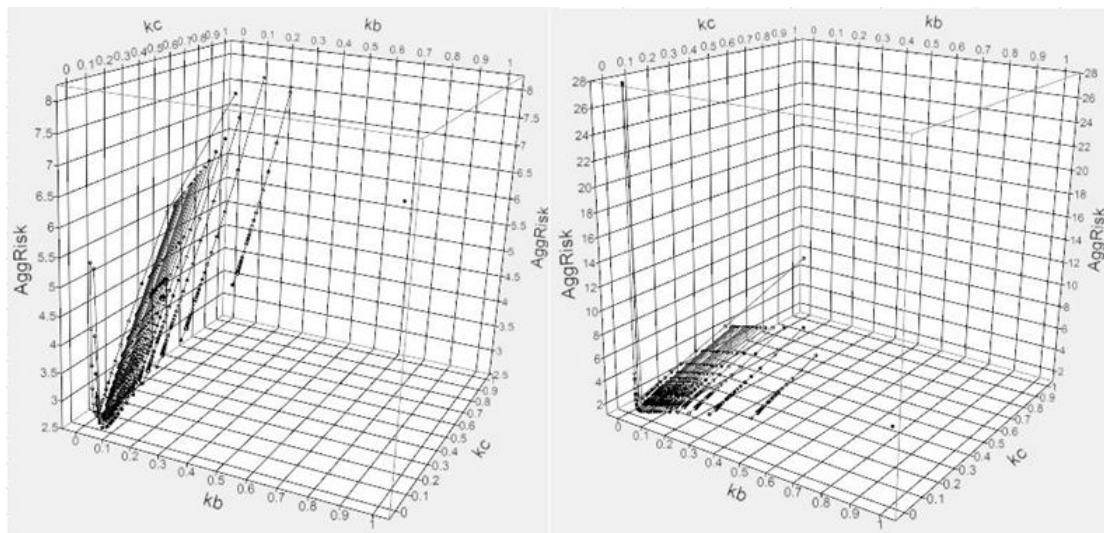


Figure 6.8: Aggregate Risk with Different Spread Parameters for $\phi_3 = -0.7$ (left) and $\phi_3 = -0.01$ (right) ($i_v = i$)

Chapter 7

Conclusion

The randomness of rates of return is an issue that actuaries deal with on a regular basis. In this project, we study the solvency risk, contribution rate risk, and the benefit risk of a hybrid pension plan with stochastic investment returns, by extending the methodologies presented in Dufresne (1988), Haberman (1994), Haberman (1997), and Khorasaneh (2012). The first two moments of the funding level, the annual contribution rate, and the benefit payment are calculated under three different models for the rate of return: the Gaussian process, the autoregressive process and the moving average process.

We have the following observations to answer the questions we have asked at the beginning of this report.

1. How does the hybrid plan behave and evolve when the investment rate of return is stochastic?

With an initial funding level of the same amount as the actuarial liability, the expected value of the fund starts off and stays at the stationary status when forces of interest follow a Gaussian process. For AR(1) and MA(1) processes, the ultimate values of the expected benefit and expected funding are higher than their targets. Variability increases with time in the Gaussian case, while it increases for a certain period and then drops down under the AR(1) and MA(1) processes.

2. How do different investment strategies affect the risks embedded in the plan?

In a mature plan, the conservative strategy requires the highest contribution rate, and

generates the lowest benefit payment. In terms of convergence speed, the riskier the investment portfolio is, the shorter the optimum spread period is, and the higher the convergence speed is. The aggressive strategy generates the highest coefficient of variation and the conservative strategy generates the lowest coefficient of variation, which may not be the case for standard deviations.

3. How do different valuation assumptions affect the risks embedded in the plan?

We observe that a higher valuation interest rate corresponds to a lower funding level. The further the valuation rate of interest is from the long term mean i , the more variation we expect in funding level. A valuation basis with $i_v < i$ creates more variations in funding than a valuation basis with $i_v > i$ in general. We also recognize an upward trending efficient frontier for the benefit payment, but the relationship is not obvious for the contributions. This observation may change when we set a different k_c/k rate. As the valuation rate of interest increases, the optimum spread period increases.

4. Is there an optimal risk sharing scheme that is ideal for both plan sponsors and participants?

As noted by Khorasane (2012), despite the fact that the risk in funding level is a monotonically increasing function of the spread period, we can minimize the total risk borne in the contribution and the benefit by choosing the optimum spread period k . As for the choice of spread parameters k_c and k_b , the sponsor and members can negotiate, as long as the sum of the spread parameters k is fixed at the optimum level.

As the British statistician George Box said, “Essentially, all models are wrong, but some are useful.” There is no known model that fits the real investment return perfectly. Even if there is, it would be too complicated to apply in practice. It is still worthwhile to make use of the characteristics of available stochastic models and understand their role in a pension plan. Some limitations of our project and possible areas of further work are listed below.

As we have pointed out in Section 5.1 under the AR(1) case, the approximation approach from Haberman (1994) does not work well for ϕ_2 that is close to 1. An improvement of the accuracy of our comparison analysis would be to use the upper and lower bound proposed by Cairns and Parker (1997).

In this project, we ignore inflation and assume that there are no salary increases. It would be interesting to consider the stochastic nature of inflation rates and salary increases.

The design of the hybrid pension plan we adopt here treats the plan surplus and deficiency in a same way, which leads to some unrealistic results, such as negative contribution rates. It was suggested by Khorasane (2012) that upper and lower bounds can be put on the benefit to limit the level of benefit to a realistic region. We suggest the same for the contribution rate.

We choose an optimum spread period and use it when illustrating the evolution of the plan. It would be interesting to consider a more dynamic approach in the choice of spread period and allow the plan manager to re-examine the position of the fund and adjust the spread period after a certain time period.

It would also be interesting to fit real life data to a more realistic stochastic investment return model, simulate multiple realizations based on the fitted model, and study the risks in the hybrid pension plan, as done in Sanders (2010) and Yuen (2011).

Bibliography

- Bacinello, A. R. (2000). Valuation of contingent-claims characterising particular pension schemes. *Insurance: Mathematics and Economics*, 27(2):177–188. 3
- Balcer, Y. and Sahin, I. (1983). A stochastic theory of pension dynamics. *Insurance: Mathematics and Economics*, 2(3):179–197. 5
- Bédard, D. and Dufresne, D. (2001). Pension funding with moving average rates of return. *Scandinavian Actuarial Journal*, 2001(1):1–17. 8
- Blake, D. (1998). Pension schemes as options on pension fund assets: implications for pension fund management. *Insurance: Mathematics and Economics*, 23(3):263–286. 3
- Cairns, A. J. (1996). Continuous-time pension-fund modelling. In *Proceedings of the 6th AFIR International Colloquium, Nuremberg*, pages 609–624. 6
- Cairns, A. J. (1997). A comparison of optimal and dynamic control strategies for continuous time pension plan models. In *Proceedings of the 7 AFIR International Colloquium, th Cairns*, pages 309–326. 6
- Cairns, A. J. and Parker, G. (1997). Stochastic pension fund modelling. *Insurance: Mathematics and Economics*, 21(1):43–79. 8, 10, 35, 36, 55, 71
- Chang, S., Tzeng, L. Y., and Miao, J. C. (2003). Pension funding incorporating downside risks. *Insurance: Mathematics and Economics*, 32(2):217–228. 6
- Chang, S.-C. (1999). Optimal pension funding through dynamic simulations: the case of taiwan public employees retirement system. *Insurance: Mathematics and Economics*, 24(3):187–199. 6
- Chang, S.-C. and Cheng, H.-Y. (2002). Pension valuation under uncertainties: implementation of a stochastic and dynamic monitoring system. *Journal of Risk and Insurance*, 69(2):171–192. 6
- Chen, K. and Hardy, M. (2010). Trade-off between benefits and guarantees: a DB underpin pension plan case. In *Proceedings of Demonstrations at the 15th Conference of the Asia-Pacific Risk and Insurance Association (APRIA)*, Tokyo, Japan. 3

- Chen, K. and Hardy, M. R. (2009). The DB underpin hybrid pension plan: fair valuation and funding. *North American Actuarial Journal*, 13(4):407–424. 3
- Dufresne, D. (1986a). *The dynamics of pension funding*. PhD thesis, City University. 6, 7
- Dufresne, D. (1986b). *The valuation and risk management of a DB underpin pension plan*. PhD thesis, City University.
- Dufresne, D. (1988). Movements of pension contributions and fund levels when rates of return are random. *Journal of the Institute of Actuaries*, 115(03):535–544. 7, 8, 10, 11, 14, 20, 23, 32, 70
- Dufresne, D. (1989). Stability of pension systems when rates of return are random. *Insurance: Mathematics and Economics*, 8(1):71–76. 8, 9
- Dufresne, D. (1990). The distribution of a perpetuity, with applications to risk theory and pension funding. *Scandinavian Actuarial Journal*, 1990(1):39–79. 6
- Gerrard, R. and Haberman, S. (1996). Stability of pension systems when gains/losses are amortized and rates of return are autoregressive. *Insurance: Mathematics and Economics*, 18(1):59–71. 8
- Haberman, S. (1990). Variability of pension contributions and fund levels with random and autoregressive rates of return. *Actuarial Research Clearing House*, pages 141–171. 7
- Haberman, S. (1991). Pension funding methods and autoregressive interest rate models. pages 181–204. Proceedings of the 2nd AFIR International Colloquium. 7
- Haberman, S. (1994). Autoregressive rates of return and the variability of pension contributions and fund levels for a defined benefit pension scheme. *Insurance: Mathematics and Economics*, 14(3):219–240. 7, 8, 10, 14, 18, 20, 21, 24, 25, 47, 48, 70, 71
- Haberman, S. (1997). Stochastic investment returns and contribution rate risk in a defined benefit pension scheme. *Insurance: Mathematics and Economics*, 19(2):127–139. 10, 20, 26, 70
- Haberman, S., Butt, Z., and Megaloudi, C. (2000). Contribution and solvency risk in a defined benefit pension scheme. *Insurance: Mathematics and Economics*, 27(2):237–259. 9
- Haberman, S. and Sung, J.-H. (1994). Dynamic approaches to pension funding. *Insurance: Mathematics and Economics*, 15(2):151–162. 6
- Haberman, S. and Wong, L. Y. P. (1997). Moving average rates of return and the variability of pension contributions and fund levels for a defined benefit pension scheme. *Insurance: Mathematics and Economics*, 20(2):115–135. 7, 26

- Hardy, M., Saunders, D., and Zhu, X. M. (2013). Market consistent valuation and funding of cash balance pensions. 4
- Janssen, J. and De Dominicis, R. (1984). Finite non-homogeneous semi-markov processes: Theoretical and computational aspects. *Insurance: Mathematics and Economics*, 3(3):157–165. 5
- Janssen, J. and Manca, R. (1997). A realistic non-homogeneous stochastic pension fund model on scenario basis. *Scandinavian Actuarial Journal*, 1997(2):113–137. 5
- Khorasanee, M. Z. (1995). A pension plan incorporating both defined benefit and defined contribution principles. *Journal of Actuarial Practice*, 3(2):269–300. 5
- Khorasanee, M. Z. and Ng, H. K. (2000). A retirement plan based on fixed accumulation and variable accrual. *North American Actuarial Journal*, 4(1):63–76. 5
- Khorasanee, Z. M. (2012). Risk-sharing and benefit smoothing in a hybrid pension plan. *North American Actuarial Journal*, 16(4):449–461. 5, 8, 10, 11, 12, 15, 20, 43, 70, 71, 72
- McLeod, C. C. (2009). *Standard of Practice*. Canadian Institute of Actuaries. Actuarial Standards Board (2009). 80
- Mody, R. (2004). Middle-ground pension plan benefit design and pension communication to members. *Pensions: An International Journal*, 10(1):25–36. 3
- O’Brien, T. (1986). A stochastic-dynamic approach to pension funding. *Insurance: Mathematics and Economics*, 5(2):141–146. 6
- Owadally, M. and Haberman, S. (1999). Pension fund dynamics and gains/losses due to random rates of investment return. *North American Actuarial Journal*, 3(3):105–117. 8, 9
- Owadally, M. and Haberman, S. (2004). Efficient gain and loss amortization and optimal funding in pension plans. *North American Actuarial Journal*, 8(1):21–36. 9
- Sanders, B. (2010). Analysis of variable benefit plans. Master’s thesis, Simon Fraser University. 5, 9, 72
- Sherris, M. (1995). The valuation of option features in retirement benefits. *Journal of Risk and Insurance*, pages 509–534. 3
- Shumway, Robert H., D. S. S. and Stoffer, D. S. (2000). *Time series analysis and its applications*. New York: Springer. 19
- Wesbroom, K., Reay, T., Britain, G., and Hewitt, B. (2005). *Hybrid pension plans: UK and international experience*. Corporate Document Services, Leeds. Research Report 271, Department for Work and Pensions. 2
- Yuen, H. (2011). Stochastic modeling of economic variables for pension plan projections. Master’s thesis, Simon Fraser University. 9, 72

Appendix A

Derivations

A.1 The Derivation of Equation (3.6)

Firstly, for $0 \leq s \leq t$, $\Delta(t) - \Delta(s)$ follows a normal distribution with mean

$$E[\Delta(t) - \Delta(s)] = (t - s)\theta_2,$$

and variance

$$\text{Var}[\Delta(t) - \Delta(s)] = \sum_{u=s+1}^t \sum_{w=s+1}^t \nu_2^2 \phi_2^{|u-w|} = 2\nu_2^2 G(t, s),$$

where

$$\begin{aligned} G(t, s) &= \frac{1}{2} \sum_{u=s+1}^t \sum_{w=s+1}^t \phi_2^{|u-w|} \\ &= \frac{1}{2}(t - s) + \sum_{u=s+1}^{t-1} \sum_{w=u+1}^t \phi_2^{w-u} \\ &= \frac{1}{2}(t - s) + \sum_{u=s+1}^{t-1} \sum_{x=1}^{t-u} \phi_2^x \\ &= \frac{1}{2}(t - s) + \sum_{x=1}^{t-s-1} (t - s - x)\phi_2^x \\ &= \frac{1 + \phi_2}{2(1 - \phi_2)}(t - s) - \frac{\phi_2(1 - \phi_2^{t-s})}{(1 - \phi_2)^2}, \end{aligned}$$

in which we set $x = w - u$ at the third step.

Therefore, by the fact that $e^{\Delta(t)-\Delta(s)}$ follows a lognormal distribution, we have

$$\begin{aligned} E \left[e^{\Delta(t)-\Delta(s)} \right] &= \exp \{ (t-s)\theta_2 + \nu_2^2 G(t, s) \} \\ &= c^{t-s} e^{-z(1-\phi_2^{t-s})}, \end{aligned}$$

where

$$\begin{cases} c = \exp \left\{ \theta_2 + \frac{1+\phi_2}{2(1-\phi_2)} \nu_2^2 \right\} \\ z = \nu_2^2 \phi_2 (1-\phi_2)^{-2} \end{cases}.$$

A.2 The Derivation of Equation (3.7)

Firstly, we have, for $0 \leq s < r < t$, that

$$E [\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)] = (t-s)\theta_2 + (t-r)\theta_2.$$

The variance can be derived as

$$\begin{aligned} &Var [\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)] \\ &= Var [\Delta(r) - \Delta(s) + 2(\Delta(t) - \Delta(r))] \\ &= Var [\Delta(r) - \Delta(s)] + 4Var [\Delta(t) - \Delta(r)] + 4Cov [\Delta(r) - \Delta(s), \Delta(t) - \Delta(r)] \\ &= \sum_{u=s+1}^r \sum_{w=s+1}^r \gamma_2(u, w) + 4 \sum_{u=r+1}^t \sum_{w=r+1}^t \gamma_2(u, w) + 4 \sum_{u=s+1}^r \sum_{w=r+1}^t \gamma_2(u, w) \\ &= \sum_{u=s+1}^r \sum_{w=s+1}^r \gamma_2(u, w) + 4 \sum_{u=s+1}^t \sum_{w=r+1}^t \gamma_2(u, w). \end{aligned}$$

Therefore,

$$\begin{aligned} &E \left[e^{\Delta(t)-\Delta(s)+\Delta(t)-\Delta(r)} \right] \\ &= \exp \left\{ (t-s)\theta_2 + (t-r)\theta_2 + \frac{1}{2} \sum_{u=s+1}^r \sum_{w=s+1}^r \gamma_2(u, w) + 2 \sum_{u=s+1}^t \sum_{w=r+1}^t \gamma_2(u, w) \right\} \\ &= \exp \{ (t-s)\theta_2 + (t-r)\theta_2 + \nu_2^2 H(t, r, s) \} \end{aligned}$$

where $H(t, r, s) = \frac{1+\phi_2}{2(1-\phi_2)} [t-s+3(t-r)] - \frac{1}{(1-\phi_2)^2} [\phi_2(3-2\phi_2^{t-r}-2\phi_2^{t-s}+\phi_2^{r-s})]$.

A.3 The Derivation of Equation (3.11)

Firstly, $\Delta(t) - \Delta(s)$ follows a normal distribution with mean

$$E[\Delta(t) - \Delta(s)] = (t - s)\theta_3,$$

and variance

$$\text{Var}[\Delta(t) - \Delta(s)] = (t - s)\nu_3^2 - 2(t - s - 1)\phi_3\gamma_3^2.$$

Therefore, by the fact that $e^{\Delta(t) - \Delta(s)}$ follows a lognormal distribution, we have

$$\begin{aligned} E[e^{\Delta(t) - \Delta(s)}] &= \exp\left\{(t - s)\left(\theta_3 + \frac{1}{2}\nu_3^2\right) - (t - s - 1)\phi_3\gamma_3^2\right\} \\ &= f^{t-s} \exp\left\{-(t - s - 1)\phi_3\gamma_3^2\right\}, \end{aligned}$$

where $f = \exp\left\{\theta_3 + \frac{1}{2}\nu_3^2\right\}$.

A.4 The Derivation of Equation (3.12)

Firstly, we have

$$E[\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)] = (t - s)\theta_3 + (t - r)\theta_3. \quad (\text{A.1})$$

For $0 \leq s < r < t$,

$$\begin{aligned} \text{Cov}[\Delta(r) - \Delta(s), \Delta(t) - \Delta(r)] &= \sum_{u=s+1}^r \sum_{w=s+1}^t \text{Cov}(\delta_u, \delta_w) \\ &= \text{Cov}(\delta_r, \delta_{r+1}) \\ &= -\phi_3\gamma_3^2. \end{aligned} \quad (\text{A.2})$$

Using (A.1) and (A.2), we get

$$\begin{aligned} &\text{Var}[\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)] \\ &= \text{Var}[\Delta(t) - \Delta(s)] + 4\text{Var}[\Delta(t) - \Delta(r)] + 4\text{Cov}[\Delta(r) - \Delta(s), \Delta(t) - \Delta(r)] \\ &= [(r - s)\nu_3^2 - 2(r - s - 1)\phi_3\gamma_3^2] + 4[(t - r)\nu_3^2 - 2(t - r - 1)\phi_3\gamma_3^2] - 4\phi_3\gamma_3^2 \\ &= [3(t - r) + (t - s)]\nu_3^2 - 2\phi_3\gamma_3^2[3(t - r) + (t - s) - 3]. \end{aligned} \quad (\text{A.3})$$

Therefore, we have

$$\begin{aligned}
& E \left[e^{\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)} \right] \\
&= \exp \left\{ E [\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)] + \frac{1}{2} \text{Var} [\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)] \right\} \\
&= \exp \left\{ (t-s) \left(\theta_3 + \frac{1}{2} \nu_3^2 - \phi_3 \gamma_3^2 \right) \right\} \exp \left\{ (t-r) \left(\theta_3 + \frac{3}{2} \nu_3^2 - 3\phi_3 \gamma_3^2 \right) \right\} \exp \{ 3\phi_3 \gamma_3^2 \} \\
&= \alpha^{t-s} \beta^{t-r} g,
\end{aligned}$$

where $\alpha = f * \exp \{ -\phi_3 \gamma_3^2 \}$, $\beta = \exp \{ \theta_3 + \frac{3}{2} \nu_3^2 - 3\phi_3 \gamma_3^2 \}$, and $g = \exp \{ 3\phi_3 \gamma_3^2 \}$.

A.5 The Derivation of Equation (3.40)

For the second moment, we assume $r > s$, then

$$\begin{aligned}
E(F_t^2) &= F_0^2 Q^{2t} E(e^{2\Delta(t)}) + 2F_0 Q^t R_v \sum_{s=0}^{t-1} Q^{t-s-1} E \left[e^{\Delta(t)} e^{\Delta(t) - \Delta(s)} \right] \\
&\quad + R_v^2 E \left[\sum_{s=0}^{t-1} \sum_{r=0}^{t-1} Q^{t-s-1} Q^{t-r-1} e^{\Delta(t) - \Delta(s)} e^{\Delta(t) - \Delta(r)} \right] \\
&= F_0^2 Q^{2t} E(e^{2\Delta(t)}) + 2F_0 Q^{2t-1} R_v E(e^{2\Delta(t)}) + 2F_0 Q^t R_v \sum_{s=1}^{t-1} Q^{t-s-1} E \left[e^{2\Delta(t) - \Delta(s)} \right] \\
&\quad + \frac{2R_v^2}{Q^2} \sum_{r=1}^{t-1} \sum_{s=0}^{r-1} Q^{t-s} Q^{t-r} E \left[e^{\Delta(t) - \Delta(s) + \Delta(t) - \Delta(r)} \right] + \frac{R_v^2}{Q^2} \sum_{s=0}^{t-1} Q^{2(t-s)} E \left[e^{2\Delta(t) - 2\Delta(s)} \right] \\
&= F_0 g^{\frac{4}{3}} \left(F_0 + \frac{2R_v}{Q} \right) (\alpha \beta Q^2)^t + 2F_0 g R_v (\beta Q)^t \frac{1 - (Q\alpha)^{t-1}}{1 - Q\alpha} \\
&\quad + \frac{2g R_v^2 \alpha^2 \beta Q}{1 - Q\alpha} \left[\frac{1 - (Q^2 \alpha \beta)^{t-1}}{1 - Q^2 \alpha \beta} - (Q\alpha)^{t-1} \frac{1 - (Q\beta)^{t-1}}{1 - Q\beta} \right] + R_v^2 g^{\frac{4}{3}} \alpha \beta \frac{1 - (Q^2 \alpha \beta)^t}{1 - Q^2 \alpha \beta}.
\end{aligned}$$

Appendix B

Mortality Table

UP94 projected to 2020, Males

Source: Memorandum for McLeod (2009), Practice-Specific Standards for Actuarial Evidence, Paragraph 4330.02, Mortality Table.

(<http://www.actuaries.ca/members/publications/2009/209001e.xls>).

Age	Male q	Age	Male q	Age	Male q
0	0	43	0.001091	86	0.094766
1	0.000377	44	0.001146	87	0.106362
2	0.000254	45	0.001208	88	0.119853
3	0.000211	46	0.001284	89	0.131627
4	0.000164	47	0.001378	90	0.148168
5	0.000151	48	0.001486	91	0.162051
6	0.000144	49	0.001601	92	0.181273
7	0.000138	50	0.001729	93	0.197295
8	0.000128	51	0.001875	94	0.214507
9	0.000124	52	0.002043	95	0.238449
10	0.000125	53	0.002279	96	0.256724
11	0.000132	54	0.002530	97	0.274387
12	0.000144	55	0.002889	98	0.298873
13	0.000163	56	0.003319	99	0.315657
14	0.000194	57	0.003843	100	0.332357
15	0.000225	58	0.004454	101	0.358560
16	0.000256	59	0.005012	102	0.376699
17	0.000281	60	0.005638	103	0.396884
18	0.000301	61	0.006523	104	0.418855
19	0.000316	62	0.007366	105	0.440585
20	0.000331	63	0.008549	106	0.460043
21	0.000355	64	0.009644	107	0.475200
22	0.000383	65	0.010833	108	0.485670
23	0.000427	66	0.012426	109	0.492807
24	0.000477	67	0.013799	110	0.497189
25	0.000548	68	0.014801	111	0.499394
26	0.000641	69	0.016194	112	0.500000
27	0.000686	70	0.017225	113	0.500000
28	0.000712	71	0.018838	114	0.500000
29	0.000736	72	0.020674	115	0.500000
30	0.000757	73	0.022648	116	0.500000
31	0.000775	74	0.024717	117	0.500000
32	0.000792	75	0.027733	118	0.500000
33	0.000801	76	0.030450	119	0.500000
34	0.000801	77	0.034563	120	1.000000
35	0.000803	78	0.039446		
36	0.000814	79	0.045054		
37	0.000841	80	0.051359		
38	0.000864	81	0.058325		
39	0.000896	82	0.065910		
40	0.000936	83	0.072000		
41	0.000983	84	0.080273		
42	0.001036	85	0.087105		