# BELIEF CHANGE AND BASE DEPENDENCE

by

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# Abstract

The *AGM* paradigm of *belief change* studies the dynamics of belief states in light of new information. For theoretical simplification, AGM idealizes a belief state as a *belief set*: a set of logical formulas that is closed under implication. A variant to the original AGM approach generalizes belief sets into *belief bases* which are not necessarily deductively closed. Many authors have argued that, compared to belief sets, belief bases are easier to represent in computers, more expressive and more inconsistency-tolerant.

A strong intuition for belief change operations, Gärdenfors suggests, is that formulas that are independent of a change should remain intact. Linking belief change and dependence is significant because, for example, it can narrow the number of formulas considered during a belief change operation. Then, based on Gärdenfors' intuition, Fariñas and Herzig axiomatize a *dependence* relation, and formalize the connection between dependence and belief change.

The work in this thesis is also based on Gärdenfors' intuition. We first introduce the notion of *base dependence* as a relation between formulas with respect to some belief base (instead of a belief set). After an axiomatization of base dependence, we present a formalization of the connection between base dependence and a particular belief base change operation, saturated kernel contraction.

We also prove that base dependence is a *reversible generalization* of Fariñas and Herzig's dependence. That is, in the special case when the underlying belief base is deductively closed (i.e., it is a belief set), base dependence reduces to dependence.

Finally, an intriguing feature of Fariñas and Herzig's formalism is that it meets other criteria for dependence, namely, Keynes' conjunction criterion for dependence (CCD) and Gärdenfors' conjunction criterion for independence (CCI). We show that our base dependence formalism also meets these criteria. More interestingly, we offer a new and more specific conjunction criterion for dependence that implies both CCD and CCI, and show our base dependence formalism also meets this new criterion.

**Keywords:** Belief change; belief revision; contraction; belief base; dependence; relevance

*To The Beacons Of Knowledge*

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# Chapter 1

# Introduction

## 1.1 General Setting

> "Philippa, a Greek nineteen year old student at Patras University, has just discovered that Nikos and Angela are not her true parents; she was adopted when she was six months old from an orphanage in Sao Paulo. The news really shook Philippa. Much of what she used to believe all her life about herself and her family was wrong. After recovering from the initial shock she started putting her thoughts back in order: so that means that Alexandros is not really her cousin, and she did not take her brown eyes from (who she used to believe was) her grandmother, and she no longer needs to worry about developing high blood pressure because of the bad family history from both Nikos' and Angela's side. Moreover, she probably has siblings somewhere in Brazil, and if she really looked into it, she might be entitled to a Brazilian citizenship which could come in handy for that long trip she always wanted to make to Latin America."

The above scenario offered by Peppas [Pep08], despite being dramatic, provides a simple and comprehensible real life example of *belief change*. Belief change is concerned with the dynamics of an agent changing its beliefs in light of new information becoming available, or in other words it attempts to model the dynamics of epistemic states of an intelligent agent.[1] The challenge is on how to incorporate a new belief into a set of beliefs to create a new consistent set of beliefs containing the new belief, and as many from the old beliefs as possible. To consider some of the different aspects involved in belief change, let us take a closer look at the Philippa example:

- Initially, she has a ***set of beliefs*** about "her life about herself and her family."
- Once she receives the rather shocking news that she has been adopted, she has to ***revise*** much of her beliefs. That is, she has to abandon (***contract***) some of her old beliefs in order to consistently accept (***expand***) some new facts and their implications about her life.
  - For example, she has to give up or contract her deeply ***entrenched*** belief that Nikos and Angela were her biological parents.
    - This means that she should also contract beliefs which were justified solely ***based*** on the old contracted beliefs; e.g. "she no longer needs to worry about developing high blood pressure because of the bad family history from both Nikos' and Angela's side."
    - This, in turn, means she no longer needs to be extraordinarily conscious about her salt consumption. That is, many of her ***implied*** beliefs, which were based on contracted beliefs, may also need to be contracted.
  - Philippa also has to expand her beliefs, adopting some new beliefs alongside their ***implications***; e.g. perhaps she has some family members in Brazil, and she quite possibly is entitled to a Brazilian citizenship.
- While revising her beliefs, Philippa always has a good intuition about ***dependence*** and ***independence*** of her beliefs upon one another. That is, during the revision of her beliefs, she only considers relevant facts, and does not bother with anything irrelevant to the change.
  - For example, being more susceptible to developing high blood pressure ***depends*** on who her biological parents are.

[1] As a research area, *Belief Change* is also known as *Belief Revision*. However, in the approach adopted in this work, the term "belief revision" only refers to a particular operation of belief change and not the whole research area.

– Yet, the politics in Greece or the geographical location of Brazil are ***independent*** and irrelevant to who her parents are. She can leave ***unchanged*** all of her previous beliefs which were independent of the revision.

This seemingly simple example helps to illustrate and highlight many of the important aspects of belief change and closely related research areas that we will look into more deeply in this work.

## Modeling Belief Change

Belief change scenarios, like the above, happen naturally and frequently in vastly diverse contexts from trivial day-to-day experiences to advanced research areas. If the size and complexity of the problem is not overwhelming, belief change can come very naturally to human beings. It would be quite desirable to be able to mimic belief change in an automated system that can deal with a large and complex set of facts. That is because "[a]part from being of interest in themselves," Gärdenfors states, the "solutions to these problems will be crucial for any attempt to use computers to handle *changes* of knowledge systems." [Gä05]

To achieve this, one may first wish to capture what constitutes *belief*, or what it means for human beings to *believe* something. Yet, this remains a colossal task having attracted many renowned scholars to this day. Similarly, how human beings review their beliefs or *revise* them are deep matters that merit studies in their own right. However, this is not exactly the focus of this work.

Here, instead of attempting to capture the meaning of beliefs as held by human beings, we turn our focus to the *representation of beliefs* and useful ways to *manipulate* this representation of beliefs that *resemble* belief change.

For many practical applications, it suffices to work with representation of beliefs. Quite commonly, for example, beliefs are *represented* using written natural languages, which undoubtedly have proven to be greatly useful throughout human history. Moreover, making use of some appropriate *formal language* to represent beliefs, instead of informal natural languages, paves the way for the employment of state-of-the-art

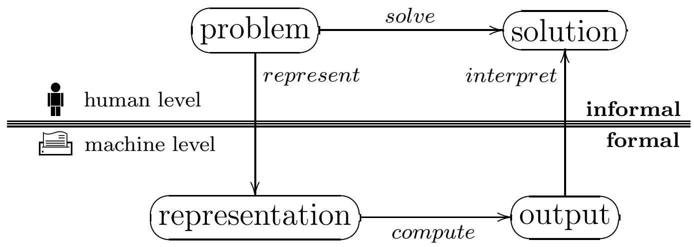



Figure 1.1: “The role of representations in solving problems” (adapted from [PM10])

methods from other well-established research areas such as *automated reasoning*. Automated reasoning can be of significant value to help with manipulation of sets of beliefs in a rationally acceptable manner, or in other words in ways that mimic natural belief change as much as possible. To this end, the sets of beliefs should be represented in some form of formal language that is suitable for the input to automated reasoners. Potentially this makes it possible to achieve, although partially[2], *automated belief change* at the representation level.

Indeed, this approach falls under a general framework, depicted in Figure 1.1, for solving problems by computer as stated by David Poole and Alan Mackworth [PM10]:

“To solve a problem, the designer of a system must

- flesh out the task and determine what constitutes a solution;
- represent the problem in a language with which a computer can reason;
- use the computer to compute an output, which is an answer presented to a user or a sequence of actions to be carried out in the environment;
- and interpret the output as a solution to the problem.”

[2]At least currently, there is no generally accepted framework to achieve automated belief change that closely mimics natural, rational belief change. As we will see in upcoming sections, in current models of belief change (e.g. AGM), there is always some *extra-logical* factors which are assumed to be provided to the system, for example, by some domain experts. While these extra-logical factors are not formally modeled and are merely assumed to be given, they significantly affect the final choices made by a belief change system. Therefore, any model that relies on receiving such extra-logical factors may not be considered to have achieved full automated belief change. That said, what can be achieved automatically is still quite helpful and desirable for many different theoretical and practical purposes.

There have been attempts throughout the last few decades to formalize belief change, with the most widely accepted theoretical framework being the *AGM* model. This model of belief change was developed by Peter Gärdenfors in collaboration[3] with Carlos Alchourrón and David Makinson – hence the acronym AGM [Gä84, AGM85, Gä88, GM88]. In AGM, a belief is simply represented using a *logical formula* from a given formal language (typically propositional logic), and a belief state is represented as a *belief set*, which is a set of formulas that is closed under logical implication. As usual, logical closure means that, for example, if $K$ is a belief set and it logically entails the formula $\alpha$, then $\alpha$ is an element in $K$. Being deductively closed, a belief set is sometimes also called a *theory*.

Belief sets model the *statics* of epistemic states. AGM next goes on to model the *dynamics* of epistemic states (i.e., belief change) by providing theoretically justified and constructive means of modifying belief sets. More specifically, to model rational belief change, AGM attempts both to *describe* what constitute operators for belief change and to specify how to *construct* such operators modifying belief sets. It is noteworthy though that, in a given situation, different intelligent agents could change their beliefs differently. Thus, it is not always possible to *uniquely* define belief change operators. Instead, AGM introduces some *constraints* on belief change operators that distinguish between operators that result in some rational changes to belief sets as opposed to the ones making invalid or unnecessary changes, which are rationally unacceptable.

**Example 1.1.** Bob and his wife Amy live in *Vancouver* ($v$). Bob knows that if it is *raining* ($r$), Amy always takes an *umbrella* ($u$). In an evening that was reportedly rainy ($r$), Amy comes back home, but, to Bob's surprise, not having her umbrella ($\neg u$).

In other words, Bob initially believes $v$, $r \to u$ and $r$, and by implication he also believes $u$. Nevertheless, he later observes that $\neg u$. There are a few different ways that Bob can deal with his initial beliefs in order to accommodate his new observation:

[3] According to Gärdenfors [Gä05], the main references for the AGM model were published by Gärdenfors in 1984 [Gä84], Alchourrón, Gärdenfors, and Makinson in 1985 [AGM85], Gärdenfors in 1988 [Gä88], and Gärdenfors and Makinson in 1988 [GM88].

Some *reasonable* modifications:

1. Stop believing $r \to u$.
   Maybe she does not *always* take an umbrella when it is raining.

2. Stop believing $r$.
   Or maybe it is not raining despite the report on weather conditions.

3. Stop believing $r \to u$ and $r$.
   Although stretching it a bit too far, Bob doubts them both.

And some *unreasonable* modifications to Bob's beliefs:

4. Keep all his old beliefs alongside with the added new observation $\neg u$ even though it contradicts $r \to u$ and $r$.

5. Abandon all of his beliefs altogether!

The AGM constraints are meant to reject any candidate *belief change operator* that make changes such as 4 or 5, which are normally considered unreasonable. On the other hand, these constraints would accept belief change operators that bring about any of the changes 1, 2 or 3 above. □

In other words, the AGM constraints narrow down the set of acceptable operators that can bring about *rational belief change*. In AGM, they are represented in the form of logical *postulates* or *axioms*, and are sometimes called *rationality postulates*. Also, this example shows that from a purely logical point of view, there cannot be a *uniquely* defined belief change operator because a rational agent may see fit to make any of the changes 1, 2 or 3 depending on other factors. Such factors are not formally represented and are sometimes referred to as *extra-logical* factors. The AGM model, including its rationality postulates, will be studied in some depth later in Chapter 2.

## Belief Bases and Belief Change

The original AGM model of belief change only deals with belief sets. As discussed earlier, a belief set is a set of formulas which is logically closed, containing all implications of its formulas. The deductive closure property of belief sets is merely an idealization assumption for theoretical simplification. This simplification comes with a price, making belief sets infinite in size, which negatively affects applicability of the AGM model in practice.

An important variant of the original AGM approach, instead of using belief sets, uses *belief bases* that are not necessarily deductively closed. A belief base can be understood as a set of formulas, which is usually *finite* because belief bases are not required to contain all of their logical consequences, and thus are more *practical* and suitable to be represented in finite machines.

Moreover, because of logical closure, belief sets are *syntax independent*, which makes them easier to work with in a formal setting. However, syntax independence also makes all logically equivalent formulas to have equal status. This can be problematic because no distinction is made between pieces of knowledge that are self-sustained, and pieces of knowledge that are merely consequences of them and have no independent standing. In contrast, a belief change model using belief bases is "based on the intuition that some of our beliefs have no independent standing but arise only as inferences from our more basic beliefs," Hansson explains [Han03]. That is, a belief base can contain the more basic beliefs that are held to be true regardless of the truth of other beliefs. Derived beliefs, on the other hand, are mere consequences of the basic beliefs and depend on them. In contrast, belief sets contain both basic beliefs and derived beliefs side by side, making the two indistinguishable.

Therefore, belief bases are more *expressive* compared to belief sets in the sense that they allow to distinguish between *explicit* and *implicit* beliefs. That is, beliefs that are explicitly stated in a belief base can be treated differently from those that are only implicitly derived from the explicit beliefs.

**Example 1.2** ([Han03])**.** I believe that Paris is the capital of France ($a$). I also believe that there is milk in my fridge ($b$). As a consequence of this, I believe that Paris is the capital of France if and only if there is milk in my fridge ($a \leftrightarrow b$). This is, however, a merely derivative belief. Its derivative character will be clearly seen when I find reasons to replace my belief in $b$ with belief in $\neg b$. I cannot then, on pain of inconsistency, retain both my belief in $a$ and my belief in $a \leftrightarrow b$. To retain the derived belief $a \leftrightarrow b$ and reject the basic belief $a$ is not, however, a serious possibility. □

Another important feature of belief bases is their tolerance for inconsistency [HW02]. While belief sets explode to include everything in the language in the presence of an inconsistency, belief bases give more room to try to handle inconsistencies, for example, by allowing for techniques to isolate an inconsistency in a belief base and still use the rest of the base.

Another shortcoming of the AGM model, some researchers have asserted [FPA05], is that, contrary to some models using belief bases, AGM cannot be applied to some practically important logics such as SHIF and SHOIN which are the underlying formal framework for OWL-Lite and OWL-DL of the Semantic Web [BLHL01].[4]

In summary, using belief bases for belief change have some important advantages over belief sets, namely: they are typically finite in size, more expressive, more tolerant of inconsistency, and potentially applicable to some more important practical logics. Therefore, belief bases can be more useful in practice than belief sets.

## Belief Change and Dependence

A long standing intuition concerning belief change is that formulas independent of a change should remain intact [Gä90]. Gärdenfors states this intuition in the form of the following preservation criterion:

> Gärdenfors' Preservation Criterion
>
> "If a belief state is revised by a sentence $A$, then all sentences in $K$ that are independent of the validity of $A$ should be retained in the revised state of belief." [Gä90]

Fariñas and Herzig in *Belief Change and Dependence* [FdCH96] attempt to ground this intuition by defining a particular *dependence* relation in a close relationship to belief change or more specifically belief contraction (or theory contraction). They

[4] Although this does not directly affect our research (as we will use propositional logic throughout this work which is discussed in §1.4 on page 12), our results are potentially more applicable for the above-mentioned logics because we focus on belief bases instead of belief sets.

show that, for closed belief sets, dependence is a "natural counterpart" for AGM postulates. That is, they specify how to define a dependence relation among formulas of a belief set, given a theory contraction operator, and vice versa.

To this end, they first assemble a collection of nine postulates that they say any dependence relation on formulas of a theory should always satisfy. Next, they state how to construct such a dependence relation given an AGM contraction operator, and they show that it indeed satisfies the nine dependence postulates. Conversely, they show that if a dependence relation, satisfying the nine postulates, is used to construct a theory contraction operator, it will indeed be a valid AGM contraction.

One interesting aspect of their work is that some of the important postulates that they use to capture the concept of dependence come from intuitions put forward previously. For example, Keynes in [Key21] (cited in [Gä78]) holds that there is an intuitive relationship between relevance (dependence) and logical conjunction that should stay valid for any reasonable definition of relevance. Fariñas and Herzig, calling it the *Conjunction Criterion for Dependence*, CCD, formulate it as follows:

If $\delta$ depends on $\alpha$ and $\delta$ depends on $\beta$ then $\delta$ depends on $\alpha \wedge \beta$. (CCD)

Moreover, Gärdenfors in [Gä78] puts forward another principle that he believes should hold for relevance/dependence relations, the *Conjunction Criterion for Independence*, CCI:

If $\delta$ is independent of $\alpha$ and $\delta$ is independent of $\beta$ then $\delta$ is independent of $\alpha \wedge \beta$. (CCI)

A more in depth study of CCD and CCI can be found in §3.2.2 on page 38. Here it suffices to emphasize that Fariñas and Herzig manage to successfully capture these appealing intuitions in their formalism. We will also see in Chapter 3 that what Fariñas and Herzig achieved in their work is an elegant addition to the theory of belief change that falls into place quite nicely with everything else in the AGM model. This deep integration into the AGM model sets apart their work from other works on relevance or dependence in the context of belief change.

## 1.2 Motivation and Main Ideas

Knowledge bases and ontologies (as in the Semantic Web [W3C01], for example) may be utilized to tackle many real life problems. Usually though, they need to be constantly updated and evolved to remain applicable to their respective problems. This process can be very labour intensive and error prone if carried out manually. This is an example where belief change using belief bases is expected to help. Belief base change can be used to incorporate a new piece of information (a new belief) into the knowledge base maintaining its consistency. This allows the initial knowledge base to evolve as more information becomes accessible.

Even though belief base change is expected to have a wide range of uses in diverse areas, there still remain some practical obstacles preventing its widespread use. At least, one important barrier to deploy belief base change operations for use in real life applications may be linked to the fact that the knowledge bases for such applications tend to be large and computationally too demanding to manipulate.

**Example 1.3.** In the morning, John realizes that he was wrong to think that he had some eggs in the fridge. As a result, the list of possibilities of what he might have for breakfast also changes, but this is unlikely to change his beliefs about how the sun works.

Indeed, a vast portion of John's beliefs need not change at all. Putting aside all the irrelevant information should be very helpful to John to quickly revise his beliefs, focusing only on the parts that could potentially change. □

One natural way to tackle this problem is to find ways to localize changes in a large belief base. Given the high computational complexity of reasoning and belief change operations, it would be of great help to be able to put aside all unrelated parts of a belief base and have the computationally demanding algorithms work locally only on the related parts. Finding intuitive and practical ways to formalize dependencies between formulas (or beliefs) can help to localize changes only to related formulas, as opposed to all of them.

Fariñas and Herzig's work is one helpful step forward, although their work is based on belief sets. Therefore, generalizing their work using belief bases is expected to be of significant practical value.

## 1.3 Belief Change and Base Dependence

As we saw in the previous sections, exploring the connection between belief change and dependence can be of great value because, for example, it can narrow the number of formulas that need to be considered during a belief change operation. Also as already discussed, one successful first step is the work of Fariñas and Herzig that focused on the relationship between dependence and AGM theory contraction.

A natural next step is to find a similar connection between dependence and *belief base contraction* that can have important practical consequences. We call such a dependence (or relevance) relation *base dependence* (or *base relevance*). In this work, we establish such a connection between belief base contraction and base dependence. That is, we provide an axiomatization of base dependence, and establish its relation to belief base contraction. Similar to the set of axioms suggested by Fariñas and Herzig, the base dependence axioms are also meant to capture the dependence among formulas, but base dependence involves formulas of belief bases as opposed to belief sets.

Since belief bases are a generalization of belief sets, their corresponding dependence relation, i.e. base dependence, can also be expected to be a generalization of Fariñas and Herzig's dependence relation. More interestingly, base dependence turns out to be a *reversible generalization* of dependence. That is, we prove that in the special case that a belief base is deductively closed (i.e., it is a belief set), the base dependence relation reduces to the original Fariñas and Herzig's dependence relation.

Quite notably, in this generalization, the formalism preserves some intriguing properties of Fariñas and Herzig's formalism, viz., the above-mentioned Keynes' conjunction criterion for dependence (CCD) and Gärdenfors' conjunction criterion for independence (CCI).

Furthermore, in Chapter 4, we offer a new and more specific criterion for dependence, which we call *Conjunction Criterion of Dependence Factoring*, CCDF. We next show that CCDF implies both Keynes' CCD and Gärdenfors' CCI, and that our base dependence formalism meets the three criteria: CCDF and so CCD and CCI.

Therefore, the contributions in this work include:

- An axiomatization of base dependence relation for belief base formulas.
- Characterization theorems relating base dependence to belief base contraction (similar to epistemic entrenchment in AGM).
- A result showing that the new base dependence relation is a reversible generalization of Fariñas and Herzig's dependence relation.
- A further result showing that, while generalizing the dependence relation, base dependence preserves some of the most interesting properties of dependence, particularly, Keynes' conjunction criterion of dependence, CCD, and Gärdenfors' conjunction criterion of independence, CCI.
- A new and more specific conjunction criterion of dependence, CCDF, that implies both Keynes' CCD and Gärdenfors' CCI, and show that this new criterion is also met by base dependence.

## 1.4 Preliminaries and Notation

We assume $\mathcal{L}$ to be a propositional language defined on a finite set of propositional variables or atoms $\mathcal{V}$ with the usual Boolean operators negation $\neg$, conjunction $\wedge$, disjunction $\vee$, and implication $\rightarrow$. We will use lower case Greek letters $\alpha$, $\beta$, $\delta$, etc. as meta variables over sentences in $\mathcal{L}$. For convenience, we introduce the sentential constants $\top$ and $\bot$ representing truth and falsity respectively.

A logical consequence $\alpha$ of a set of formulas $B$ is represented by $B \vdash \alpha$. Also Cn is a consequence operator, a total function taking sets of formulas to sets of formulas, which can be defined as $\mathrm{Cn}(B) = \{\alpha \mid B \vdash \alpha\}$. Also Cn is a Tarskian [Tar56] consequence operator satisfying:

$$B \subseteq \mathrm{Cn}(B) \qquad \text{(inclusion)}$$

$$\text{If } A \subseteq B \text{ then } \mathrm{Cn}(A) \subseteq \mathrm{Cn}(B) \qquad \text{(monotony)}$$

$$\mathrm{Cn}(B) = \mathrm{Cn}(\mathrm{Cn}(B)). \qquad \text{(iteration)}$$

The iteration axiom is also known as idempotency.

Furthermore, the Cn operator is assumed here to satisfy the following standard properties:

If $\alpha$ can be derived from $B$ by classical truth-functional logic then $\alpha \in \mathrm{Cn}(B)$ (supraclassicality)

$\beta \in \mathrm{Cn}(B \cup \{\alpha\})$ iff $(\alpha \rightarrow \beta) \in \mathrm{Cn}(B)$ (deduction)

If $\alpha \in \mathrm{Cn}(B)$ then $\alpha \in \mathrm{Cn}(B')$ for some finite subset $B' \subseteq B$. (compactness)

Throughout the document, $B$ denotes a usually finite set of formulas or a *belief base*, and $K$ denotes a logically closed set of formulas or a *belief set*. A set $K$ is said to be logically closed or to be closed under logical consequence if and only if it contains all its own logical consequences, $\mathrm{Cn}(K) \subseteq K$. We also know that $K \subseteq \mathrm{Cn}(K)$ by inclusion above. Therefore, the conventional formulation for the criterion for $K$ to be closed under logical consequence is: $K = \mathrm{Cn}(K)$ [Han99].

Finally, for the sake of simplicity, we may drop the curly brackets in some cases of using the consequence operator, e.g., $\mathrm{Cn}(\alpha, \beta)$ instead of $\mathrm{Cn}(\{\alpha, \beta\})$.

## 1.5 Structure of This Thesis

After the introduction in this chapter, we will discuss in Chapter 2 the AGM framework of belief change and some of its important extensions that are relevant to this research. Chapter 3 will be an elaborate review of the most related work to this research: Belief Change and Dependence by Fariñas and Herzig [FdCH96]. Chapter 4 will constitute the main body of this thesis which will extend and generalize Fariñas and Herzig's work. A summary of contributions in this thesis will be given in Chapter 5, which will also include an analysis of and comparison to some related work. We will also disscus in Chapter 5 some open problems regarding base dependence and some future research possibilities to extend the work in this thesis. Finally, all the proofs are moved to Appendix A.

# Chapter 2

# Belief Change

A brief introduction to belief change was provided in §1.1. To lay the foundation for the upcoming chapters, we now study belief change in more depth and offer several examples on the topic. Let us start off by assuming that there is an intelligent agent possessing a consistent set of beliefs about some domain. One interesting question is how this set of beliefs can be changed if needed. For example, say some credible new information about the domain becomes available. How should the agent change or revise its old beliefs to incorporate the new information, and end up with a new, consistent body of beliefs? Reconsidering old beliefs is not always easy, nor necessarily wise, thus the change should be as little as possible, and only as much as required to avoid any conflict between the new information and the old.

If the new piece of information happens to be consistent with the current set of beliefs, then it can simply be added to the set along with any implications this addition may have. A more challenging case, however, is when this credible new piece of information is in conflict with the current beliefs the agent holds. In this case, it first needs to dismiss some of its old beliefs, albeit as few as possible, to resolve any conflicts with the new information. In other words, it needs to give up on some of the old beliefs such that the remaining beliefs or their consequences do not include the negation of the new information. Then, the problem reduces to the previous case, and it can add the new information and accept any implications this addition may have.

Before getting into technical details of formalisms that attempt to address belief change, we consider another example below that should help to both further clarify and motivate the problem.

**Example 2.1** ([Gä03]). Let us assume that an agent believes the following statements:

$\alpha$: All European swans are white.

$\beta$: The bird caught in the trap is a swan.

$\gamma$: The bird caught in the trap comes from Sweden.

$\delta$: Sweden is part of Europe.

Given the above statements $\{\alpha, \beta, \gamma, \delta\}$, the following statement $\varepsilon$ can also be *derived*:

$\varepsilon$: The bird caught in the trap is white.

Thus, the agent should also believe, by implication, that $\varepsilon$ holds. Nevertheless, the agent is later assured of the following new fact:

$\varepsilon'$: The bird caught in the trap is not white (e.g. is black).

Clearly, $\varepsilon$ and $\varepsilon'$ are inconsistent. Hence, the only way to keep $\varepsilon'$ and maintain the consistency of the whole set of beliefs here is to retract at least one of the facts from $\{\alpha, \beta, \gamma, \delta\}$. That will ensure that $\varepsilon$ is not derived anymore, and thus $\varepsilon'$ can consistently be added to what remains. For example, assume the agent starts to doubt that the bird caught in the trap is a swan, and ceases to believe $\beta$. Then the remaining facts, $\{\alpha, \gamma, \delta\}$, no longer imply $\varepsilon$, allowing it to believe $\varepsilon'$ without being contradictory, with the new set of facts being $\{\alpha, \gamma, \delta, \varepsilon'\}$.

Note that it is possible to drop more than one of $\{\alpha, \beta, \gamma, \delta\}$. That too will allow $\varepsilon'$ to be added to the remaining facts without causing inconsistency. However, in this case more information than necessary will be lost. Intuitively, we want to maintain *Minimal Change* as a guiding principle, requiring that in the process of changing a set of beliefs, the change should be as small as possible.

Also note that from a logical point of view, it does not make a difference which one of $\alpha$, $\beta$, $\gamma$ and $\delta$ is dropped. Intuitively, however, we know that giving up on beliefs $\beta$ or $\gamma$ is much easier than on $\alpha$ or $\delta$. Yet, this idea is not easy to capture without counting on extra-logical considerations – something which we will explore in detail in the subsequent chapters.

## 2.1 AGM Paradigm

As mentioned in the previous chapter, the AGM theory has been the most influential work to formalize the dynamics of belief states of an intelligent agent. According to Peter Gärdenfors [Gä05], the AGM framework was developed by him in collaboration with Carlos E. Alchourrón and David Makinson [Gä84, AGM85, Gä88, GM88]. In AGM, logical sentences from a (propositional) language $\mathcal{L}$ are used to represent beliefs about a static and unchanging world or situation.[1] The belief state of an idealized intelligent agent, who knows all the consequences of its beliefs, is then represented using a formal theory, called a *belief set* – a set of beliefs alongside all its logical consequences, $K = \mathrm{Cn}(K)$.

For instance, let $K$ be the set of all logical consequences of $\{p, p \rightarrow q\}$, or in other words $K = \mathrm{Cn}(\{p, p \rightarrow q\})$. Then, for example, $p$, $p \rightarrow q$, $q$, $p \wedge q$, $p \vee q$, $p \vee \neg q$ are some statements or sentences in $K$. Sentences that are not entailed by $\{p, p \rightarrow q\}$ and thus are *not* in $K$ include $\neg q$, $p \rightarrow \neg q$, $\neg p \wedge q$. The logical closure property of belief sets is a simplifying assumption which may potentially be removed in later works (see §2.7 on page 25).

### Belief Change Operators

After representing a belief state using a belief set $K$, it is useful to know what are reasonable operations on $K$ that can resemble change of beliefs in rational agents. Example 2.1 above is an instance of belief revision, which is one of the three important ways in which a set of beliefs can change. The following lists three operations on $K$ alongside some simple examples for each.

[1] Changing beliefs because of changes in the situation is called *belief update* [KM92], which is outside the scope of this document.

Starting with the same $K$ as above, $K = \mathrm{Cn}(p,\, p \to q)$, we have:

**Contraction,** $K \div \alpha$**:** Retract $\alpha$ and other sentences in $K$ that imply $\alpha$.

**Ex.** $K \div p = \mathrm{Cn}(p \to q)$ (so $p \notin K \div p$, while $p \in K$)

**Ex.** $\begin{cases} K \div q = \mathrm{Cn}(p) & \text{(alternative 1)} \\ K \div q = \mathrm{Cn}(p \to q) & \text{(alternative 2)} \end{cases}$

i.e., it is not possible to uniquely determine $K \div q$, and it may take on different reasonable values

**Expansion,** $K + \alpha$**:** Add $\alpha$ to $K$ together with the logical consequences of its addition. The resulting belief set, $K + \alpha$, may or may not remain consistent.

**Ex.** $K + r = \mathrm{Cn}(p,\, p \to q,\, r)$

**Ex.** $K + q = \mathrm{Cn}(p,\, p \to q)$ ($q$ is redundant, making no difference)

**Ex.** $K + \neg q = \mathrm{Cn}(p,\, p \to q,\, \neg q)$ (inconsistent result, implying both $q$ and $\neg q$)

**Revision,** $K * \alpha$**:** Add $\alpha$ to $K$, but if $\alpha$ is inconsistent with $K$, drop some subset of $K$ before adding $\alpha$ to ensure consistency.

**Ex.** $K * r = \mathrm{Cn}(p,\, p \to q,\, r)$

**Ex.** $K * q = \mathrm{Cn}(p,\, p \to q)$ ($q$ is redundant but acceptable)

**Ex.** $\begin{cases} K * \neg q = \mathrm{Cn}(p,\, \neg q) & \text{(alternative 1)} \\ K * \neg q = \mathrm{Cn}(p \to q,\, \neg q) & \text{(alternative 2)} \end{cases}$

Out of the three operations above, expansion is the simplest one and can be uniquely defined as follows

$$K + \alpha = \mathrm{Cn}(K \cup \{\alpha\}).$$

It is quite apparent that expansion can bring about inconsistent results. On the other hand, revision and contraction (by a consistent formula) guarantee consistent results, but not unique results. Let us elaborate on the contraction example above, $K \div q$, to see why unique results are not always possible for these operations. Starting with $K = \mathrm{Cn}(p, p \to q)$, we know that $q \in K$, as it is a logical consequence of $\{p, p \to q\}$, but we are interested in another belief set that does not imply/contain $q$: $q \notin K \div q$. There is more than one way to obtain this: $K \div q = \mathrm{Cn}(p)$ or $K \div q = \mathrm{Cn}(p \to q)$. Therefore, in the general case, given a theory $K$ and a formula $\alpha$, we cannot expect to uniquely define a contraction operation $K \div \alpha$, nor is it possible to uniquely define a revision operation $K * \alpha$ for similar reasons.

Instead, the AGM framework introduces sets of constraints to determine classes of reasonable belief change (revision and contraction) operators. These constraints are presented in the form of logical *axioms* or *postulates* which are introduced in Sections 2.2 and 2.3.

### Interconnection between Belief Change Operators

Via the following two important identities, belief revision and contraction operators are related to each other in the sense that one can be obtained from the other.

**Levi Identity** [AM82] defines revision in terms of contraction:

$$K * \alpha = (K \div \neg\alpha) + \alpha$$

**Harper Identity** also known as 'Gärdenfors Identity' [Mak85], defines contraction in terms of revision:

$$K \div \alpha = K \bigcap (K * \neg\alpha)$$

These identities are important because they show that if we find a way to construct a contraction operator, we can also easily extend it to construct a revision operator via the Levi identity, and the other way around via the Harper identity.

## 2.2 Belief Revision

The AGM paradigm models belief revision for a theory $K$ as a function $*$ that given a sentence $\alpha$ produces a new theory $K * \alpha$. As we saw above, since there could be many rationally acceptable revision functions, to make sure that the change from $K$ to $K * \alpha$ resembles rational belief revision as closely as possible, certain constraints are imposed on the revision operator in the form of *rationality postulates*.

The following are the rationality postulates for revision in the AGM paradigm, which any AGM revision operation $*$ needs to satisfy.

($K*$1) $K * \alpha$ is a belief set (closure)

($K*$2) $\alpha \in K * \alpha$ (success)

($K*$3) $K * \alpha \subseteq K + \alpha$ (inclusion)

($K*$4) If $\neg\alpha \notin K$ then $K + \alpha \subseteq K * \alpha$ (preservation)

($K*$5) If $\alpha$ is consistent then $K * \alpha$ is consistent (consistency)

($K*$6) If $\alpha \leftrightarrow \beta$ then $K * \alpha = K * \beta$ (extensionality)

($K*$7) $K * (\alpha \wedge \beta) \subseteq (K * \alpha) + \beta$ (subexpansion)

($K*$8) If $\neg\beta \notin K * \alpha$ then $(K * \alpha) + \beta \subseteq K * (\alpha \wedge \beta)$ (superexpansion)

The axiom $K*$1 ensures that when a belief set is revised by some new piece of information the result is also a belief set. $K*$2 requires that the revision operation be successful in the sense that the revised belief set should contain the new information. $K*$3 limits what can be added to the revised belief set by requiring that revision never introduces anything that expansion does not. $K*$4 ensures that nothing is unnecessarily omitted from the revised belief set; i.e., revision is the same as expansion if new information is consistent with the original belief set. Indeed, $K*$3 and $K*$4 together mean that $K * \alpha = K + \alpha$ when $\neg\alpha \notin K$. $K*$5 is self-explanatory. $K*$6 means that a revision operation is syntax independent.

Any operator $*$ satisfying postulates $K*$1 through $K*$6 is called a basic AGM revision operator. Also, the *supplementary postulates* $K*$7 and $K*$8 specify properties of composite belief revision, which involve revision by conjunction of sentences.

## 2.3 Belief Contraction

Similarly, the AGM paradigm models belief contraction for theory $K$ as a function $\div$ that given a sentence $\alpha$ produces a new theory $K \div \alpha$. As in the case of revision, a contraction operator needs to satisfy a set of constraints or postulates.

The set of rationality postulates for contraction in the AGM paradigm are as follows:

- $(K\div 1)$ $K \div \alpha$ is a belief set (closure)
- $(K\div 2)$ $K \div \alpha \subseteq K$ (inclusion)
- $(K\div 3)$ If $\alpha \notin K$ then $K \div \alpha = K$ (vacuity)
- $(K\div 4)$ If $\nvdash \alpha$ then $\alpha \notin K \div \alpha$ (success)
- $(K\div 5)$ If $\alpha \in K$ then $K \subseteq (K \div \alpha) + \alpha$ (recovery)
- $(K\div 6)$ If $\vdash \alpha \leftrightarrow \beta$ then $K \div \alpha = K \div \beta$ (extensionality)
- $(K\div 7)$ $K \div \alpha \cap K \div \beta \subseteq K \div \alpha \wedge \beta$ (conjunctive overlap)
- $(K\div 8)$ If $\alpha \notin K \div \alpha \wedge \beta$ then $K \div \alpha \wedge \beta \subseteq K \div \alpha$ (conjunctive inclusion)

Similar to revision, the first axiom $K\div 1$ states the closure property of contraction operation – contracting a belief set always results in a belief set. $K\div 2$ ensures that contraction does not introduce any new formula, and it may only take away some of the existing ones. By $K\div 3$, if a formula is not already in a belief set, contraction leaves the belief set unchanged. No change should be made when not necessary. $K\div 4$ guarantees the resulting belief set from a contraction does not contain the contracted formula, given it is not a tautology. In other words, $\alpha \in K \div \alpha$ can happen only for a tautological $\alpha$. By $K\div 5$, the original belief set can be recovered after contraction by re-expanding the result by the original contracted formula.[2] $K\div 6$ states that contraction is syntax independent.

Any operator $\div$ on $K$ satisfying all these postulates is called a basic AGM contraction operator. Similar to belief revision, the *supplementary postulates* $K\div 7$ and $K\div 8$

[2]The *recovery* postulate has turned out to be the most controversial AGM contraction postulate [Mak87, Fer01].

specify properties of composite belief contraction by conjunction of formulas. Indeed, the AGM model provides a third postulate regarding contraction of conjunctions:

$$\begin{aligned} \text{Either } & B \div \alpha \wedge \beta = B \div \alpha, \text{ or} \\ & B \div \alpha \wedge \beta = B \div \beta, \text{ or} \\ & B \div \alpha \wedge \beta = B \div \alpha \cap B \div \beta. \end{aligned} \qquad \text{(conjunctive factoring)}$$

This axiom is closely related to AGM supplementary postulates $K\div 7$ and $K\div 8$. A basic AGM contraction operator that satisfies conjunctive factoring, also satisfies both $K\div 7$ and $K\div 8$, and vice versa. The relationship between these three axioms is formally stated in the following theorem.

**Theorem 2.2** ([AGM85]). *Let $\div$ be an operation on belief set $K$ that satisfies $K\div 1$ – $K\div 6$, then* conjunctive factoring *is satisfied if and only if both* conjunctive overlap *($K\div 7$) and* conjunctive inclusion *($K\div 8$) are satisfied.*

## 2.4 Constructing Belief Change Operators

So far we have seen postulates that in a way describe what can be considered an AGM revision or contraction operator. Such postulates provide a set of formal conditions that a belief change operator must satisfy. On the other hand, there may be different ways to *construct* belief change operators in a way that the AGM postulates are satisfied. Then we need to show that these two orthogonal approaches match each other. That is we need a *representation theorem* to prove that the construction of a belief change operator does indeed represent the axioms specified for it. Such a theorem may also be alternatively called an *axiomatic characterization*, since it provides a characterization of an operation in the form of postulates.

In the subsequent sections, we briefly introduce some construction methods for belief change operators which are in some way relevant to the present work. Our focus, however, will largely be on belief contraction as opposed to belief change. That is because, first, contraction is used in the upcoming chapters; second, as we saw in §2.1 on page 18, belief contraction and revision are interconnected via the Levi and Harper identities. The following theorems show that having either a contraction operator or a revision operator suffices to get the other one, using one of these two identities.

**Theorem 2.3** ([Gä88]). *If $\div$ is a construction function satisfying $K\div 1$ – $K\div 4$ and $K\div 6$, then its associated revision function via the Levi identity satisfies $K*1$ – $K*6$.*

**Theorem 2.4** ([Gä88]). *If $*$ is a revision function satisfying $K*1$ – $K*6$, then its associated contraction function via the Harper identity satisfies $K\div 1$ – $K\div 6$.*

The following theorems also use the Levi and Harper identities but they take into account the supplementary postulates as well.

**Theorem 2.5** ([AGM85, Fer01]). *If $\div$ is a construction function satisfying $K\div 1$ – $K\div 4$ and $K\div 6$, then its associated revision function via the Levi identity satisfies:*
*$K*7$ if $K\div 5$ and $K\div 7$ are also satisfied, and*
*$K*8$ if $K\div 8$ is also satisfied.*

**Theorem 2.6** ([Gä88]). *If $*$ is a revision function satisfying $K*1$ – $K*6$, then its associated contraction function via the Harper identity satisfies:*
*$K\div 7$ if $K*7$ is also satisfied, and*
*$K\div 8$ if $K*8$ is also satisfied.*

## 2.5 Partial Meet Contraction

### Remainder Sets

For constructing both revision and contraction operators it is useful to determine maximal subsets of a theory $K$ that do not entail a given sentence $\alpha$. Such a maximal non-implying subset of $K$ is called a *remainder*. Typically, for a given $K$ and $\alpha$, there is more than one remainder, and the collection of all such remainders, denoted by $K \perp \alpha$, is called a *remainder set*.

**Definition 2.7** ([AM82]). Let $K$ be a set of formulas and $\alpha$ a formula. A set $X \in K \perp \alpha$ is a *remainder* of $K$ and $\alpha$ if and only if:

1. $X \subseteq K$
2. $X \nvdash \alpha$
3. for all $X'$ such that $X \subset X' \subseteq K$, $X' \vdash \alpha$.

Since usually there is more than one remainder, a selection function is employed to choose between them.

**Definition 2.8** ([AGM85]). A *selection function for $K$ is a function $\gamma$ such that for all sentences $\alpha$:*

1. if $K \perp \alpha$ is non-empty then $\gamma(K \perp \alpha)$ is a non-empty subset of $K \perp \alpha$, and
2. if $K \perp \alpha$ is empty then $\gamma(K \perp \alpha) = \{K\}$.

## Partial Meet Contraction and Its Special Cases

The following are examples for contraction constructors using remainder sets. Given a remainder set $K \perp \alpha$:

**Partial Meet** is obtained with the intersection of *some* of the remainders picked by a selection function $\bigcap \gamma(K \perp \alpha)$

**Full Meet** is obtained by the intersection of *all* of the remainders $\bigcap(K \perp \alpha)$

**Maxichoice** is obtained when $\gamma(K \perp \alpha)$ selects *one* remainder from $K \perp \alpha$

Note that full meet and maxichoice are two special cases for partial meet because if a selection function $\gamma$ selects only one remainder then the constructed partial meet contraction is indeed a maxichoice contraction. Also, if $\gamma$ selects all the remainders then the produced contraction is full meet.

Here, we only give a formal definition for partial meet contraction, and mention the important results regarding this contraction.

**Definition 2.9** ([AGM85]). For any sentence $\alpha$, the operation of *partial meet contraction* over a belief set $K$ determined by the selection function $\gamma$ is given by:

$$K \div_\gamma \alpha = \bigcap \gamma(K \perp \alpha).$$

Alchourrón, Gärdenfors and Makinson [AGM85] show the following representation results, connecting the above construction to the AGM contraction postulates.

**Theorem 2.10** ([AGM85]). *Let $\div$ be a function taking a belief set $K$ and a sentence $\alpha$ and returning a new belief set $K \div \alpha$. For every theory $K$, $\div$ is a partial meet contraction operation over $K$ if and only if $\div$ satisfies the basic postulates for contraction, $K\div 1$ – $K\div 6$.*

### Partial Meet Revision via the Levi Identity

As discussed before, it is possible to construct a revision operator using a given contraction operator via the Levi identity:

$$K * \alpha = (K \div \neg\alpha) + \alpha.$$

The following definition combines the Levi identity with the construction for partial meet contraction to obtain a new construction for *partial meet revision*.

**Definition 2.11** ([AGM85]). Let $K$ be a belief set and $\gamma$ a selection function. For any sentence $\alpha$, the operation of *partial meet revision* over a belief set $K$ determined by $\gamma$ is given by:

$$K *_\gamma \alpha = \mathrm{Cn}(\bigcap \gamma(K \perp \alpha) \cup \{\alpha\}).$$

## 2.6 Epistemic Entrenchment

Some of our beliefs about the world are more important than others for being more informational and offering more explanatory power. Such beliefs are more epistemically entrenched than others, making them harder to give in a contraction or revision. Based on this intuition, Gärdenfors introduced *epistemic entrenchment*, and defined the properties of an order relation, $\leq$, between sentences as follows [Gä88]:

(EE1) If $\alpha \leq \beta$ and $\beta \leq \delta$ then $\alpha \leq \delta$ (transitivity)

(EE2) If $\alpha \vdash \beta$ then $\alpha \leq \beta$ (dominance)

(EE3) $\alpha \leq (\alpha \wedge \beta)$ or $\beta \leq (\alpha \wedge \beta)$ (conjunctiveness)

(EE4) If $K \nvdash \bot$ then $\alpha \notin K$ iff $\alpha \leq \beta$ for all $\beta$ (minimality)

(EE5) If $\beta \leq \alpha$ for all $\beta$ then $\vdash \alpha$ (maximality)

Gärdenfors and Makinson [GM88] then studied the relation between epistemic entrenchment ordering and belief contraction, and showed that the two are connected:

$$\alpha \leq \beta \text{ iff } \alpha \notin K \div (\alpha \wedge \beta) \text{ or } \vdash (\alpha \wedge \beta) \qquad (\mathsf{C}\leq)$$

$$\beta \in K \div \alpha \text{ iff } \beta \in K \text{ and either } \vdash \alpha \text{ or } \alpha < (\alpha \vee \beta). \qquad (\mathsf{C}\div_{\mathsf{G}})$$

As expected, the symbol $<$ denotes that $\alpha \leq \beta$ but $\beta \not\leq \alpha$. Based on their results, for a belief set $K$, given an epistemic entrenchment relation $\leq$ that satisfies postulates EE1–EE5, a contraction operator $\div$ can be constructed, via $\mathsf{C}\div_{\mathsf{G}}$, that satisfies $K\div 1$–$K\div 8$. Conversely, given a contraction operator $\div$ satisfying $K\div 1$–$K\div 8$, an epistemic entrenchment relation $\leq$ can be constructed, via $\mathsf{C}\leq$, that satisfies EE1–EE5. Figure 2.1 on the next page schematically demonstrates these results.

After establishing the connection between epistemic entrenchment and belief contraction, a similar connection between epistemic entrenchment and belief revision can be obtained using the Levi identity, just as in the case of partial meet revision in the previous section.

## 2.7 Changing Belief Bases: A Generalization of AGM

The original AGM paradigm of belief change studies the dynamics of belief states using belief sets. Requiring belief sets to be deductively closed, which makes them infinite in size, is a simplifying assumption of great theoretical value. However, it comes at the cost of reducing the practicality of the AGM model.

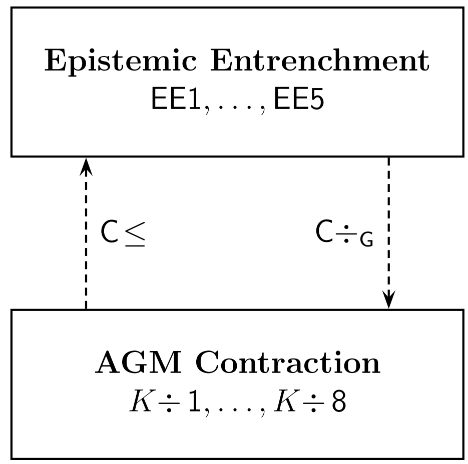



Figure 2.1: Gärdenfors and Makinson show that "the problem of constructing appropriate contraction and revision functions can be reduced to the problem of providing an appropriate ordering of epistemic entrenchment" [GM88].

As was discussed in Chapter 1, to address this shortcoming of AGM, belief bases are introduced which generalize belief sets by removing the deductive closure requirement. This means that for practical applications belief bases are typically finite in size and hence computationally more representable.

Belief sets and belief bases are closely related though. Given a belief base $B$, it is always possible to obtain the corresponding belief set $K$ using the consequence operator: $K = \text{Cn}(B)$. Also, there can be many different belief bases whose logical closure is the same belief set. This makes belief bases more expressive compared to belief sets. Belief bases can distinguish between explicit, or more basic beliefs, $B$, and implicit beliefs, $\text{Cn}(B) \setminus B$, which depend on the basic beliefs [Han03].

Belief bases also allow for the handling of inconsistencies [HW02]. For example, let $A = \{p, \neg p, q\}$ and $B = \{p, \neg p, \neg q\}$. Because both $A$ and $B$ are inconsistent, their corresponding belief set is the same: $\text{Cn}(A) = \text{Cn}(B) = \mathcal{L}$. Yet, $A \div p = \{\neg p, q\}$ and $B \div p = \{\neg p, \neg q\}$ are different and so are their closures, $\text{Cn}(A \div p) \neq \text{Cn}(B \div p)$.

## 2.8 Belief Base Contraction

The connection between contraction on belief bases and contraction on belief sets is well established. A contraction operator $-$ on a belief base $B$ gives rise to a base-generated operation $\div$ on the belief set $K = \mathrm{Cn}(B)$, such that $K \div p = \mathrm{Cn}(A - p)$ for all sentences $p$ [Han93, Han11].

Just as in the case of the AGM operators for belief sets, belief change operators for belief bases are also constrained by a set of postulates. Some examples of such postulates are listed below.

If $\nvdash \alpha$ then $\alpha \notin \mathrm{Cn}(B \div \alpha)$ (success)

$B \div \alpha \subseteq B$ (inclusion)

If $\mathrm{Cn}(B) \subseteq B$ then $\mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ (closure)

$B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ (relative closure)

If $\alpha \notin \mathrm{Cn}(B)$ then $B \div \alpha = B$ (vacuity)

If $\vdash \alpha$ then $B \div \alpha = B$ (failure)

If $\alpha \leftrightarrow \beta$ then $B \div \alpha = B \div \beta$ (extensionality)

If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $B \div \alpha = B \div \beta$ (uniformity)

If $\beta \in B$ and $\beta \notin B \div \alpha$ then
$\alpha \notin \mathrm{Cn}(B \div \alpha)$ and $\alpha \in \mathrm{Cn}((B \div \alpha) \cup \{\beta\})$ (fullness)

If $\beta \in B$ and $\beta \notin B \div \alpha$ then there is some $B'$ s. t.
$B \div \alpha \subseteq B' \subseteq B$ and $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ (relevance)

If $\beta \in B$ and $\beta \notin B \div \alpha$ then there is some $B'$ s. t.
$B' \subseteq B$ and $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ (core-retainment)

$\beta \in B \div \alpha$ iff $\beta \in B$ and there is no $B'$ s. t.
$B' \subseteq B$ and $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ (core identity)

These postulates will be used for different base contractions in the subsequent sections. Let us start by identifying the bare minimum axioms that need to be satisfied for base contraction. A contraction operation needs to at least satisfy success and inclusion. That is, the result of a contraction operation should be a subset of the original belief base that does not imply the sentence to be contracted if it is not a tautology.

**Definition 2.12** ([Han99])**.** An operator $\div$ for a set $B$ is an operator of contraction if and only if it satisfies success and inclusion.

### Partial Meet Base Contraction

Partial meet contraction is very well applicable to both belief bases and belief sets. Also for belief bases, just as it was the case for belief sets, maxichoice base contraction and full meet base contraction are special cases of the more general partial meet base contraction. Here are the axioms that each of these base contractions satisfy [Han99]:

**Partial Meet Base Contraction:** success, inclusion, uniformity and relevance.

**Maxichoice Base Contraction:** success, inclusion, uniformity and fullness.

**Full Meet Base Contraction:** core identity.

The core identity axiom alone suffices to guarantee full meet contraction.

Notice that with the above belief base version of the postulates, these operations are applicable to both belief bases and belief sets. This fact is implicitly reflected in the statement of the following theorem:

**Theorem 2.13** ([Han03])**.** *The operator $\div$ is an operator of* partial meet contraction *for a set $B$ if and only if it satisfies the postulates of* success*,* inclusion*,* relevance *and* uniformity*.*

## 2.9 Kernel Contraction

### Kernels

We saw in §2.5 on page 22 that remainders are maximal subsets of a set of beliefs that do not entail a given formula $\alpha$. Here we introduce another tool similar to remainders, namely *kernels*, which prove to be very useful in constructing belief change operators for belief bases. A kernel is a minimal subset of a base that implies a given formula. An $\alpha$-kernel is a minimal subset of a belief base that entails $\alpha$. A kernel set $B \perp\!\!\!\perp \alpha$ is the set of all possible $\alpha$-kernels for a belief base $B$, so a kernel set is a set of some subsets (i.e. kernels) of a base.

**Definition 2.14** ([Han95])**.** Let $B$ be a belief base and $\alpha$ a formula. A set $B \perp\!\!\!\perp \alpha$ is such that $X \in B \perp\!\!\!\perp \alpha$ if and only if:

1. $X \subseteq B$
2. $X \vdash \alpha$
3. if $X' \subset X$, then $X' \nvdash \alpha$.

Again, in the general case, there can be more than one minimal implying subset of $B$. Thus, an *incision function* $\sigma$ can be employed to make a decision when there are multiple options to choose from. The incision function $\sigma$ picks at least one element from each kernel.

**Definition 2.15** ([Han95])**.** An incision function $\sigma$ for $B$ is a function such that for all $\alpha$:

1. $\sigma(B \perp\!\!\!\perp \alpha) \subseteq \bigcup(B \perp\!\!\!\perp \alpha)$, and
2. if $\emptyset \neq X \in B \perp\!\!\!\perp \alpha$, then $X \cap \sigma(B \perp\!\!\!\perp \alpha) \neq \emptyset$

## Building Kernel Contraction Operators

Let us assume that base $B$ implies formula $\alpha$, $B \vdash \alpha$. Then, by definition, the kernel set $B \perp\!\!\!\perp \alpha$ contains all $\alpha$-kernels, minimal subsets $X \subseteq B$ each of which implies $\alpha$, $X \vdash \alpha$. Also, let $\sigma$ be an incision function such that $\sigma(B \perp\!\!\!\perp \alpha)$ will pick at least one formula from each kernel $X$. Now, if we remove all the formulas in $\sigma(B \perp\!\!\!\perp \alpha)$ from $B$, the remaining set will no longer imply $\alpha$, $[B \setminus \sigma(B \perp\!\!\!\perp \alpha)] \nvdash \alpha$. Therefore, given an incision function $\sigma$, a *kernel contraction* operator $\div_\sigma$ may be defined as follows:

**Definition 2.16** ([Han95])**.** Let $\sigma$ be an incision function for $B$. The kernel contraction $\div_\sigma$ for $B$ is defined as follows:

$$B \div_\sigma \alpha = B \setminus \sigma(B \perp\!\!\!\perp \alpha)$$

Kernel contraction satisfies success, inclusion, core-retainment and uniformity. It is more general than base partial meet contraction [Han99]. This is because partial meet contraction satisfies relevance while kernel contraction satisfies only core-retainment which is a much looser constraint, making kernel contraction more general than partial meet contraction.

**Theorem 2.17** ([Han95]). *The operator* $\div$ *for* $B$ *is a* kernel contraction *if and only if it satisfies* success, inclusion, core-retainment *and* uniformity.

### Saturated Kernel Contraction

Indeed kernel contraction constitutes a very general class of belief contraction operations [Han99], and there are some subsets of this class of contraction operations that exhibit some rather interesting properties. In particular, let us consider *saturated kernel contraction* which is basically a kernel contraction that additionally satisfies relative closure.

**Theorem 2.18** ([Han95]). *The operator* $\div$ *for* $B$ *is a saturated kernel contraction*[3] *if and only if it satisfies* success, inclusion, core-retainment, uniformity *and* relative closure.

Saturated kernel contraction is an "interesting generalization" [Han99] of partial meet contraction for belief bases because, in the special case where a belief base is closed (i.e. is a belief set), saturated kernel contraction is equivalent to partial meet contraction.

**Theorem 2.19** ([Han95]). *Let* $B$ *be a belief set. Then an operation is a saturated kernel contraction for* $B$ *if and only if it is a partial meet contraction for* $B$.

## 2.10 Conclusion

We introduced in some detail the AGM paradigm of belief change [AGM85] representing belief states as logically closed belief sets. We also reviewed a variant to the

[3] *The original theorem in [Han95] actually mentions* "smooth *kernel contraction" which is equivalent to* saturated *kernel contraction [Han99].*

original AGM approach that instead uses belief bases which generalize belief sets and are not necessarily deductively closed. Furthermore, we saw that many authors have argued that, compared to belief sets, belief bases are more practical as they are easier to represent in computers, more expressive and more inconsistency-tolerant.

# Chapter 3

# Belief Change and Dependence

## 3.1 Overview

A long standing intuition concerning belief change is that formulas independent of a change should remain intact [Gä90]. Fariñas and Herzig in *Belief Change and Dependence* [FdCH96] attempt to ground this intuition by defining a particular *dependence* relation in a close relationship to belief change, or more specifically, contraction. Since the present thesis is rooted in their work, this chapter is dedicated to a discussion and analysis of it.

### 3.1.1 Minimal Change: Crucial to Belief Change

As seen in Chapter 2, minimality of change is a guiding principle in the theory of belief change: a change in a belief state should preserve as many of the old beliefs as possible. As such, "what is meant by a minimal change of a state of belief," Gärdenfors asserts, is a "central problem for the theory of belief revision" [Gä90]. To partially address this problem, the AGM model uses a set of constraints, called *rationality postulates*, that any valid belief change operator is required to satisfy. However, by satisfying these postulates, it is only possible to prohibit operators that make *unreasonable* changes that rational agents would not make. Yet, there may still remain more than one reasonable change to choose from.

**Example 3.1.** Assume Mary believes that $p$, $q$ and $q \rightarrow r$ are true; i.e., $K = \mathrm{Cn}(\{p, q, q \rightarrow r\})$. Thus, she implicitly believes that $r$ is also true, as it is entailed by $q$ and $q \rightarrow r$. Now, say, for some reason, she starts to doubt that $r$ is true, so she wants to contract her beliefs by $r$. Consequently, this leads her to also cease believing either $q$, or $q \rightarrow r$, or even both. The rationality postulates allow her to make any such possible change as she sees fit. However, these postulates *disallow* some other changes that seem to be irrational. For example, it does not make sense that just because she stops believing $r$, she also forsakes all her (non-tautological) beliefs, with her new belief set shrinking to $K' = \mathrm{Cn}(\emptyset)$. □

As this example illustrates, although the rationality postulates logically limit what changes to a set of beliefs are considered "rational," there can still remain multiple reasonable changes in a given situation. As far as the AGM model is concerned, it makes no difference what choice Mary makes. Yet, there might be some domain-specific or case-dependent considerations to make her prefer one over another. Such considerations are clearly absent from this logical model of her belief state, namely $K$. Thus, to make a decision, she needs some *extra-logical* information.

### 3.1.2 Dependence: Crucial to Minimal Change

The criterion of minimality used in the AGM model has been "based on almost exclusively logical considerations. However, there are a number of non-logical factors that should be important when characterizing a process of belief revision," Gärdenfors confirms [Gä90]. He, thus, focuses on the notion of relevance and its relationship to belief change, introducing the following *preservation criterion*.

> Gärdenfors' Preservation Criterion
>
> "If a belief state is revised by a sentence $A$, then all sentences in $K$ that are independent of the validity of $A$ should be retained in the revised state of belief." [Gä90]

In other words, full capture of *minimal change* requires focusing only on the *related statements* and leaving out any independent statements from the change.

Here we use relevance as synonymous to dependence: "$\alpha$ is relevant to $\beta$" is equivalent to "$\beta$ depends on $\alpha$."

**Example 3.2.** Assume as in Example 3.1, Mary believes the same $K = \mathrm{Cn}(\{p,\, q,\, q \rightarrow r\})$, but this time she *also knows* that if she ever doubts $r$, then she would prefer doubting $q$, but not $q \rightarrow r$. Thus, for her, believing $q$ depends on $r$, but believing $q \rightarrow r$ does not. In other words, her belief $r$ is relevant to her belief $q$ but it is not relevant to her belief $q \rightarrow r$.

Knowing the dependencies between these statements allows full modeling of the dynamics of Mary's belief state in the event that she stops believing $r$. That is, in that case, she would also stop believing $q$, and according to Gärdenfors' preservation criterion above, she would retain the rest of her beliefs: $K' = \mathrm{Cn}(\{p,\, q \rightarrow r\})$. □

Note that, with the help of the extra-logical information available in this example, the outcome here is the only acceptable resulting belief state for Mary. This is in contrast with the situation in Example 3.1 where the outcome, $K'$, could be one of many different possibilities.

Nevertheless, the challenge to formulate extra-logical factors still remains. The preservation criterion only shifts the problem to that of determining the dependencies between statements (or equivalently the relevance amongst them). As Gärdenfors puts it "a criterion of this kind cannot be given a technical formulation in a model based on belief sets." Instead, to decide about the preference over different beliefs, belief change operators resort to exploiting some form of exogenous factors such as epistemic entrenchment ordering[1] $\leq$, selection function[2] $\gamma$, or incision function[3] $\sigma$.

Gärdenfors' preservation criterion, however, highlights the importance of the dependencies between statements to uphold the minimal change principle in belief change. This importance has also been recognized by many other researchers studying

[1]See §2.6 on page 24

[2]See Definition 2.8 on page 23

[3]See Definition 2.15 on page 29

relevance/dependence in the context of belief change, which we will discuss in more detail in the *Related Work* section on page 95.

### 3.1.3 Belief Change and Dependence

So far, we have seen that on the one hand *belief change* and *minimal change* are strongly related, and on the other hand *minimal change* and *dependence* are strongly related. Thus, it is only natural to expect that *belief change* and *dependence* are strongly connected as well.

Fariñas and Herzig explore this connection in *Belief Change and Dependence* [FdCH96]. Their work is particularly interesting and unique in the sense that it fits the original AGM model of belief change, adding a theoretical foundation for dependence. Their stated aim is both "to give a formal account of the notion of dependence, and to employ it in belief change."

As for the formal account, they offer an axiomatization of dependence, assembling a collection of nine postulates that a dependence relation should satisfy. Some of these postulates are from different authors working on the notion of relevance and dependence such as Keynes [Key21] and Gärdenfors [Gä78, Gä90]. We will study these postulates in §3.2.2 on page 38.

Next they employ their formal account of dependence into the AGM model of belief change. That is, they specify how to define a dependence relation with respect to formulas of a belief set given a contraction operator, and vice versa. More specifically, they show that a dependence relation, obtained from an AGM contraction operator, indeed satisfies the nine dependence postulates. Conversely, they show that if a dependence relation, satisfying the nine postulates, is used to obtain a contraction operator, it will be a valid AGM operator. They also point out that analogous results for belief revision can easily be achieved via the Levi identity. We will come back to this in more detail in §3.2.3 on page 42.

Using the above two stages, they basically provide an axiomatization of the intuitions behind the notion of dependence that interestingly corresponds directly to

AGM postulates. Apparently, Gärdenfors expected such a result before it was actually developed by Fariñas and Herzig. He states that he looks for a version of the preservation criterion above as an addition to the theory of belief change [Gä90]. He even lays out a plan for how to achieve this by providing two subgoals: first analyzing the concept of relevance and then incorporating the results of this analysis into belief change.

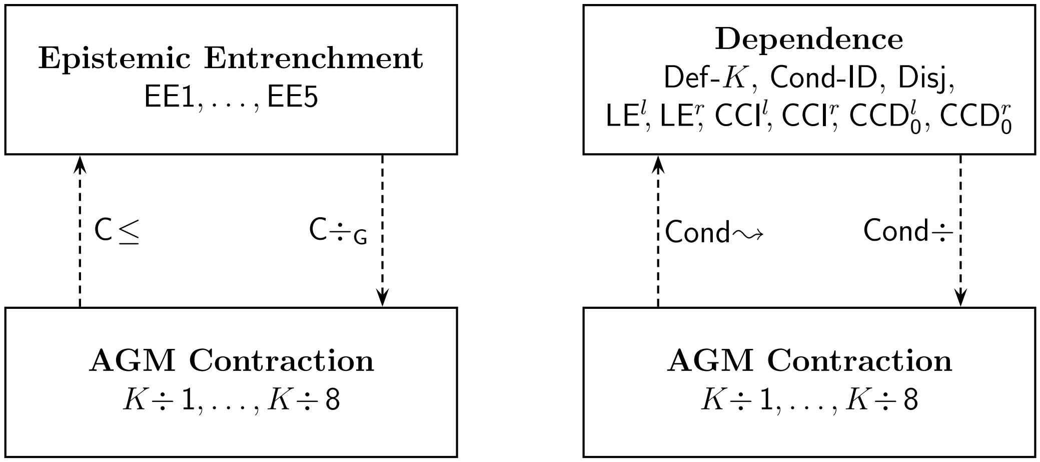
Moreover, as Fariñas and Herzig acknowledge [FdCH96], Gärdenfors provides some properties for relevance relations and he further shows that these properties hold for all contraction operations.

## 3.2 Formalization of Belief Change and Dependence

To formalize the relationship between belief change and dependence, Fariñas and Herzig establish a logic-based account of dependence, and show that it fits the original AGM model. One side note here is that Fariñas and Herzig first start by an attempt to give a formal account of (in)dependence from a probability theory perspective, a.k.a. *probabilistic independence*. However, they then argue that that approach is unsatisfactory, and turn their focus on the logic-based account of dependence, which is the account we consider here.

### 3.2.1 Approach: Resembling Epistemic Entrenchment

We saw in §2.6 on page 24 that epistemic entrenchment as a binary relation between sentences of a given belief set $K$ was studied by Gärdenfors and Makinson [Gä84, GM88, Gä88]. As depicted in Figure 2.1 on page 26, they show that for a belief set $K$, given an epistemic entrenchment relation $\leq$ that satisfies postulates $\mathsf{EE1}$ - $\mathsf{EE5}$, a contraction operator $\div$ can be constructed via $\mathsf{C}\div_{\mathsf{G}}$ that satisfies $K\div\mathsf{1}$–$K\div\mathsf{8}$. Also, given a contraction operator $\div$ satisfying $K\div\mathsf{1}$–$K\div\mathsf{8}$, an epistemic entrenchment relation $\leq$ can be constructed via $\mathsf{C}\leq$ that satisfies $\mathsf{EE1}$ - $\mathsf{EE5}$.




(a) Gärdenfors and Makinson show that "the problem of constructing appropriate contraction and revision functions can be reduced to the problem of providing an appropriate ordering of epistemic entrenchment" [GM88].

(b) Fariñas and Herzig also demonstrate that "the AGM-postulates have a natural counterpart in terms of dependence (just as they have one in terms of epistemic entrenchment)" [FdCH96].

Figure 3.1: The relationship between *Dependence* and *AGM contraction* is similar to the relationship between *Epistemic Entrenchment* and *AGM contraction*.

Gärdenfors and Makinson conclude:

> "From an epistemological point of view, these results suggest that the problem of constructing appropriate contraction and revision functions can be reduced to the problem of providing an appropriate ordering of epistemic entrenchment." [GM88]

Indeed, Fariñas and Herzig formalized the notion of dependence and its connection with belief change, "[p]roceeding exactly in the same way as Gärdenfors [and Makinson] (1988) did in the case of epistemic entrenchment" [FdCH96]. Figure 3.1 is a schematic illustration of the parallelism between these two works.

We now proceed to show in more detail how Fariñas and Herzig axiomatize the dependence relation in §3.2.2, and how they employ this dependence relation in belief change in §3.2.3.

### 3.2.2 A Formalization of Dependence

To formalize dependence, Fariñas and Herzig investigate a binary relation $\rightsquigarrow$ on formulas. $\alpha \rightsquigarrow \beta$ reads as "$\beta$ depends on $\alpha$" (or synonymously "$\alpha$ is relevant to $\beta$"). Independence, then, is denoted by $\not\rightsquigarrow$, which is the complement of $\rightsquigarrow$, so $\alpha \not\rightsquigarrow \beta$ reads as "$\beta$ is independent of $\alpha$" (or "$\alpha$ is irrelevant to $\beta$").

**Example 3.3.** In Example 3.2 on page 34, Mary is starting to doubt $r$ while her current belief state is modeled as $K = \mathrm{Cn}(\{p, q, q \rightarrow r\})$. The additional pieces of information that her belief $q$ depends on her belief $r$ and that her belief $q \rightarrow r$ is independent from $r$ are respectively denoted by $r \rightsquigarrow q$ and $r \not\rightsquigarrow q \rightarrow r$. □

Fariñas and Herzig did not provide a precise definition for dependence relations. Instead, much like how the AGM model constrains what can be considered as a belief change operator, Fariñas and Herzig put forward a framework for what can be considered as a dependence relation by providing a set of postulates that any such relation should satisfy. Some of these postulates are based on ideas explored in much earlier works of other authors, notably Keynes (1921) and Gärdenfors (1978).

Keynes in [Key21] holds that there is an intuitive relationship between relevance (dependence) and logical conjunction that should be valid for any reasonable definition of relevance. Fariñas and Herzig, calling it the *Conjunction Criterion for Dependence*, CCD, express it as follows:

If $\delta$ depends on $\alpha$ and $\delta$ depends on $\beta$ then $\delta$ depends on $\alpha \wedge \beta$. (CCD)

Using the $\rightsquigarrow$ notation, they formalize CCD as follows:

If $\alpha \rightsquigarrow \delta$ and $\beta \rightsquigarrow \delta$ then $\alpha \wedge \beta \rightsquigarrow \delta$. ($\mathsf{CCD}^{l}$)

They use superscript $\iota$ for expanding from *left*, allowing them to use superscript $r$ for *right* to name another closely related axiom:

If $\delta \rightsquigarrow \alpha$ and $\delta \rightsquigarrow \beta$ then $\delta \rightsquigarrow \alpha \wedge \beta$. ($\mathsf{CCD}^{r}$)

Fariñas and Herzig explain that because $\rightsquigarrow$ is not necessarily symmetric, $\mathsf{CCD}^{l}$ and its symmetric counterpart $\mathsf{CCD}^{r}$ are both needed.

Moreover, Gärdenfors in [Gä78] puts forward another principle that he believes should hold for any relevance relation, the *Conjunction Criterion for Independence*, CCI:

> If $\delta$ is independent of $\alpha$ and $\delta$ is independent of $\beta$ then $\delta$ is independent of $\alpha \wedge \beta$. (CCI)

Gärdenfors' CCI maintains its intuitive appeal in its contrapositive form:

If $\delta$ depends on $\alpha \wedge \beta$ then $\delta$ depends on $\alpha$ or $\delta$ depends on $\beta$.

Once again, using the $\rightsquigarrow$ notation, and expanding from both left and right, Fariñas and Herzig give the $\mathsf{CCI}^l$ and $\mathsf{CCI}^r$ postulates based on CCI:

If $\alpha \wedge \beta \rightsquigarrow \delta$ then $\alpha \rightsquigarrow \delta$ or $\beta \rightsquigarrow \delta$. ($\mathsf{CCI}^l$)

If $\delta \rightsquigarrow \alpha \wedge \beta$ then $\delta \rightsquigarrow \alpha$ or $\delta \rightsquigarrow \beta$. ($\mathsf{CCI}^r$)

Interestingly, looking a bit further ahead, Fariñas and Herzig eventually come up with a formalism of dependence in relation to contraction that adheres to both Keynes' CCD and Gärdenfors' CCI, or all four of $\mathsf{CCD}^l$, $\mathsf{CCD}^\mathsf{r}$, $\mathsf{CCI}^l$ and $\mathsf{CCI}^r$. While their formalism does not directly use $\mathsf{CCD}^l$ and $\mathsf{CCD}^\mathsf{r}$ as postulates, they nonetheless remain valid as derivable principles.

Going back to describing the rest of Fariñas and Herzig's postulates for dependence, the following postulates $\mathsf{LE}^l$ and $\mathsf{LE}^r$ are also symmetric counterparts, which are standard postulates for *syntax independence*:

If $\alpha \leftrightarrow \beta$ and $\alpha \rightsquigarrow \delta$ then $\beta \rightsquigarrow \delta$. ($\mathsf{LE}^l$)

If $\alpha \leftrightarrow \beta$ and $\delta \rightsquigarrow \alpha$ then $\delta \rightsquigarrow \beta$. ($\mathsf{LE}^r$)

A more intuitive equivalent form of these postulates is as follows:

If $\alpha \leftrightarrow \beta$ then $\alpha \rightsquigarrow \delta$ iff $\beta \rightsquigarrow \delta$.

If $\alpha \leftrightarrow \beta$ then $\delta \rightsquigarrow \alpha$ iff $\delta \rightsquigarrow \beta$.

The next postulate $\mathsf{Def}\text{-}K$ is the only one that involves $K$. This is important for providing the link between dependence and belief change, which was the second stated goal of Fariñas and Herzig's formalism as discussed above. Just as an AGM belief contraction operator $\div$ is with regards to some belief set $K$, a Fariñas and Herzig's dependence relation $\rightsquigarrow$ is also with regards to some $K$ (more details in §3.2.3 below).

$$\alpha \in K \quad \text{iff} \quad \text{either} \vdash \alpha \text{ or } \alpha \rightsquigarrow \beta \text{ for some } \beta. \tag{$\mathsf{Def}\text{-}K$}$$

Since $K$ is deductively closed, it contains all tautologies, so if $\vdash \alpha$ then obviously $\alpha \in K$. The more interesting case is when $\nvdash \alpha$ and $\alpha \in K$ which means $\alpha$ is a contingent truth in $K$. Then $K$ is contractable by $\alpha$; i.e., $K \div \alpha$ is smaller than $K$. That is, there is some $\beta$ such that $\beta \in K$ but $\beta \notin K \div \alpha$. A trivial case for that is when $\beta$ is $\alpha$ itself, so $\alpha \in K$ but $\alpha \notin K \div \alpha$, which is guaranteed by the success [p. 27] postulate of AGM contraction.

The next postulate in the Fariñas and Herzig's list is Cond-ID:

$$\text{If } \alpha \rightsquigarrow \beta \text{ then } \alpha \rightsquigarrow \alpha. \tag{Cond-ID}$$

This means that $\alpha$ depends on itself if there is anything depending on it. If any formula $\beta$ depends on $\alpha$, it means that $\alpha \in K$, and it also means that $\nvdash \alpha$. So, $\alpha$ is a contingent truth and it is contractable, thus $\alpha \rightsquigarrow \alpha$.

Their next postulate is Disj:

$$\text{If } \vdash \alpha \vee \beta \text{ then } \alpha \not\rightsquigarrow \beta. \tag{Disj}$$

This ensures that a formula and its negation are always independent. That is, $\alpha \not\rightsquigarrow \neg\alpha$. Also, for $\alpha \rightsquigarrow \beta$, the postulate Disj does not allow $\alpha$ or $\beta$ to be a tautology.

As it was mentioned earlier, Fariñas and Herzig's formalism of dependence does not use Keynes' CCD directly as postulates because both $\mathsf{CCD}^l$ and $\mathsf{CCD}^\mathsf{r}$ are derivable. They instead use $\mathsf{CCD}^l_0$ which is similar to but stronger than Keynes' CCD. Finally, $\mathsf{CCD}^r_0$ is the counterpart of the conjunctive inclusion [p. 20] postulate, $K\div 8$.

$$\text{If } \alpha \rightsquigarrow \delta \text{ and } \alpha \wedge \beta \rightsquigarrow \alpha \text{ then } \alpha \wedge \beta \rightsquigarrow \delta. \tag{$\mathsf{CCD}^l_0$}$$

$$\text{If } \delta \rightsquigarrow \alpha \text{ and } \beta \rightsquigarrow \beta \text{ then } \delta \rightsquigarrow \alpha \wedge \beta. \tag{$\mathsf{CCD}^r_0$}$$

In summary, the following is the final set of postulates that Fariñas and Herzig provide for the axiomatization of dependence.

If $\alpha \leftrightarrow \beta$ and $\alpha \rightsquigarrow \delta$ then $\beta \rightsquigarrow \delta$ $\quad$ ($\mathsf{LE}^l$)

If $\alpha \leftrightarrow \beta$ and $\delta \rightsquigarrow \alpha$ then $\delta \rightsquigarrow \beta$ $\quad$ ($\mathsf{LE}^r$)

If $\alpha \wedge \beta \rightsquigarrow \delta$ then $\alpha \rightsquigarrow \delta$ or $\beta \rightsquigarrow \delta$ $\quad$ ($\mathsf{CCI}^l$)

If $\delta \rightsquigarrow \alpha \wedge \beta$ then $\delta \rightsquigarrow \alpha$ or $\delta \rightsquigarrow \beta$ $\quad$ ($\mathsf{CCI}^r$)

$\alpha \in K$ iff either $\vdash \alpha$ or $\alpha \rightsquigarrow \beta$ for some $\beta$ $\quad$ (Def-$K$)

If $\alpha \rightsquigarrow \beta$ then $\alpha \rightsquigarrow \alpha$ $\quad$ (Cond-ID)

If $\vdash \alpha \vee \beta$ then $\alpha \not\rightsquigarrow \beta$ $\quad$ (Disj)

If $\alpha \rightsquigarrow \delta$ and $\alpha \wedge \beta \rightsquigarrow \alpha$ then $\alpha \wedge \beta \rightsquigarrow \delta$ $\quad$ ($\mathsf{CCD}_0^l$)

If $\delta \rightsquigarrow \alpha$ and $\beta \rightsquigarrow \beta$ then $\delta \rightsquigarrow \alpha \wedge \beta$ $\quad$ ($\mathsf{CCD}_0^r$)

The following are also derivable principles in their framework:

If $\alpha \rightsquigarrow \delta$ and $\beta \rightsquigarrow \delta$ then $\alpha \wedge \beta \rightsquigarrow \delta$ $\quad$ ($\mathsf{CCD}^l$)

If $\delta \rightsquigarrow \alpha$ and $\delta \rightsquigarrow \beta$ then $\delta \rightsquigarrow \alpha \wedge \beta$ $\quad$ ($\mathsf{CCD}^\mathsf{r}$)

Thus, using the axioms above, Fariñas and Herzig's dependence relation is defined as follows:

**Definition 3.4.** A relation $\rightsquigarrow$ is a *dependence* relation if and only if it satisfies the axioms $\mathsf{LE}^l$, $\mathsf{LE}^r$, $\mathsf{CCI}^l$, $\mathsf{CCI}^r$, Def-$K$, Cond-ID, Disj, $\mathsf{CCD}_0^l$ and $\mathsf{CCD}_0^r$.

The above definition is their main definition of dependence, which they show, using a characterization theorem, to correspond to AGM contraction satisfying $K\div 1$–$K\div 8$ (see Theorem 3.8 on page 44). They also use some subsets of these axioms for dependence relation. In particular, they show that a dependence relation that only satisfies $\mathsf{LE}^l$, $\mathsf{LE}^r$, $\mathsf{CCI}^r$, Def-$K$, Cond-ID, Disj and $\mathsf{CCD}_0^r$ corresponds to basic AGM contraction satisfying $K\div 1$–$K\div 6$ (see Theorem 3.9 on page 44).

### 3.2.3 Employment of Dependence in Belief Change

After axiomatizing dependence relations, Fariñas and Herzig study dependence in relationship to belief change. As their guiding principle, they use Gärdenfors' preservation criterion (see page 33) which basically requires that independent beliefs from a belief change should remain intact in the revised state of belief. For example, if $\beta \in K$ to begin with, but $\beta \notin K \div \alpha$, then we can say that $\beta$ depends on $\alpha$, or $\alpha \rightsquigarrow \beta$. They then proceed to show how to define a contraction operation from a given dependence relation, and, vice versa, how to obtain a dependence relation using a contraction operation. Finally, they complete the link between AGM contraction and dependence for closed belief sets through establishing characterization theorems.

To provide this connection between dependence and AGM contraction, Fariñas and Herzig introduce the following two conditions $\mathsf{Cond}\rightsquigarrow$ and $\mathsf{Cond}\div$ whose roles closely resemble those of $\mathsf{C}\leq$ and $\mathsf{C}\div_{\mathsf{G}}$ for epistemic entrenchment (see §2.6 on page 24).

$$\alpha \rightsquigarrow \beta \text{ iff } \beta \in K \text{ and } \beta \notin K \div \alpha. \tag{$\mathsf{Cond}\rightsquigarrow$}$$

This condition allows one to define $\rightsquigarrow$ based on a given AGM contraction operation $\div$ for belief set $K$. On the other hand, the next condition, $\mathsf{Cond}\div$, allows the defining an AGM contraction operation $\div$, given a dependence relation $\rightsquigarrow$.

$$\beta \in K \div \alpha \quad \text{iff} \quad \text{either } \vdash \beta \text{ or } \beta \rightsquigarrow \beta \text{ and } \alpha \not\rightsquigarrow \beta. \tag{$\mathsf{Cond}\div$}$$

One important note here is that an AGM contraction operation $\div$ is defined with respect to some belief set $K$. Thus, the contraction operation $\div$ obtained via condition $\mathsf{Cond}\div$ also requires an associated belief set $K$ to be specified. Fariñas and Herzig provide the following definition for the belief set $K_{\rightsquigarrow}$ associated with a given $\rightsquigarrow$ relation:

$$K_{\rightsquigarrow} \stackrel{def}{=} \{\alpha \mid \vdash \alpha \text{ or } \alpha \rightsquigarrow \beta \text{ for some } \beta\}.$$

That is, if $\alpha$ is a tautology, it is trivially part of $K_{\rightsquigarrow}$. On the other hand, $\alpha$ is a contingent truth if there is a sentence $\beta$ in $K_{\rightsquigarrow}$ such that it is removed once $\alpha$ is contracted. This is because contraction of $\alpha$ will have no effect if $\alpha$ is a tautology, by failure [p. 27], nor if $\alpha$ is not in the theory, by vacuity [p. 27]. Note that one special case for $\beta$ is that it be equal to $\alpha$ in which case we have $\alpha \leadsto \alpha$. This just means that $\alpha$ is contingent, as seen earlier in this chapter. For the sake of brevity, they simply use $K$ to refer to $K_{\leadsto}$ afterwards.

They also remark that in the presence of Def-$K$ [p. 40], Cond$\div$ can be re-written as the following equivalent form:

$$\beta \in K \div \alpha \text{ iff } \beta \in K \text{ and } \alpha \not\leadsto \beta.$$

It is noteworthy that this variant of Cond$\div$ is a formal expression of Gärdenfors' preservation criterion (on page 33) which requires a belief change operation to let beliefs that are independent of a change remain intact.

With all the building blocks in place, finally, Fariñas and Herzig provide formal results to actually show how dependence is a natural counterpart of AGM contraction.

The following theorem defines a dependence relation using a contraction operation.

**Theorem 3.5** ([FdCH96]). *Given two relations* $\leadsto$ *and* $\div$ *such that* Cond$\leadsto$ *holds, if* $\div$ *is an AGM contraction, then* $\leadsto$ *is a dependence relation.*

The next theorem expresses the converse. It defines a contraction operation, given a dependence relation.

**Theorem 3.6** ([FdCH96]). *Given two relations* $\leadsto$ *and* $\div$ *such that* Cond$\div$ *holds, if* $\leadsto$ *is a dependence relation then* $\div$ *is an AGM contraction.*

Once again as in the case of epistemic entrenchment, through establishing a characterization theorem, they complete the link between AGM contraction and dependence for closed belief sets. Such a characterization theorem would be expected to state that for any two arbitrary relations $\leadsto$ and $\div$ on formulas that satisfy Cond$\leadsto$ (or equivalently Cond$\div$), $\div$ is an AGM contraction iff $\leadsto$ is a dependence relation. To achieve this, however, it turns out that they first have to make the following assumption:

**Remark 3.7** ([FdCH96, page 158]). In order to establish an axiomatic characterization based on Cond$\leadsto$, it is assumed that the relation $\div$ satisfies inclusion, $K \div \alpha \subseteq K$.

Fariñas and Herzig briefly explain the rationale behind making this assumption, which we expand on as follows. In constructing the $\rightsquigarrow$ relation using a contraction operation via the condition $\mathsf{Cond}\rightsquigarrow$, the set of all $\beta$ such that $\alpha \rightsquigarrow \beta$ is equal to those $\beta \in K$ and $\beta \notin K \div \alpha$, or in set difference notation $\beta \in K \setminus (K \div \alpha)$. We know as a matter of fact that $K \div \alpha \subseteq K$ because an AGM contraction $\div$ is required to satisfy inclusion. However, for the sake of argument, let us assume that there could be some statements in $K \div \alpha$ that are not in $K$. In that case, such statements would have been lost in the set difference $\beta \in K \setminus (K \div \alpha)$. What that means is that while constructing a contraction $\div$ via $\mathsf{Cond}\rightsquigarrow$, we do not have enough information to prove that inclusion holds. Instead, we have to assume that $\div$ already satisfies inclusion.

The assumption stated in Remark 3.7 may be perceived by some to have a negative impact on the simplicity and elegance of Fariñas and Herzig's characterization theorem. Yet, since by Definition 2.12 on page 28 [Han99] all contraction operations satisfy inclusion, this assumption is not a serious loss of generality.

**Theorem 3.8** ([FdCH96]). *Let two relations* $\rightsquigarrow$ *and* $\div$ *be such that* $\div$ *satisfies* inclusion*,* $K \div \alpha \subseteq K$*, and that* $\mathsf{Cond}\rightsquigarrow$ *holds:* $\alpha \rightsquigarrow \beta$ iff $\beta \in K$ and $\beta \notin K \div \alpha$*. Then* $\div$ *is an AGM contraction if and only if* $\rightsquigarrow$ *is a dependence relation.*

This completes their work on providing the correspondence between AGM contraction and dependence as shown in Figure 3.1b on page 37. In addition, they provide two more weaker dependence relations which are characterized by a subset of their axioms. One particularly interesting weaker dependence corresponds to the *basic* postulates for AGM contraction: $K\div\mathsf{1}$–$K\div\mathsf{6}$, as stated in the following theorem.

**Theorem 3.9** ([FdCH96]). *Let two relations* $\rightsquigarrow$ *and* $\div$ *be such that* $\div$ *satisfies* inclusion*,* $K \div \alpha \subseteq K$*, and that* $\mathsf{Cond}\rightsquigarrow$ *holds:* $\alpha \rightsquigarrow \beta$ iff $\beta \in K$ and $\beta \notin K \div \alpha$*. Then* $\div$ *is a basic AGM contraction satisfying* $K\div\mathsf{1}$–$K\div\mathsf{6}$ *if and only if* $\rightsquigarrow$ *is a dependence relation satisfying* $\mathsf{LE}^l$*,* $\mathsf{LE}^r$*,* $\mathsf{CCI}^r$*,* $\mathsf{Def}\text{-}K$*,* $\mathsf{Cond\text{-}ID}$*,* $\mathsf{Disj}$ *and* $\mathsf{CCD}^r_0$*.*

## 3.3 Conclusion

We saw that Gärdenfors [Gä90] offered a strong intuition for belief change operations: formulas that are independent of a change should remain intact. We then discussed how Fariñas and Herzig [FdCH96] formalized Gärdenfors' intuition, offering an axiomatization of the dependence relation, and establishing the connection between dependence and belief change. All in all, Fariñas and Herzig's work is an elegant addition to the theory of belief change that falls into place with everything else in the AGM model. This deep integration into the AGM model is what sets apart Fariñas and Herzig's work from other works on relevance and dependence in the context of belief change. Their work is further analyzed in the Related Work section on page 95.

# Chapter 4

# Belief Change and Base Dependence

## 4.1 Overview

As discussed in Chapter 3 (particularly in §3.1.2), Gärdenfors states the intuition that while revising beliefs, formulas independent of a change should remain intact. As this intuition lays the foundation of the present thesis, once again we quote his statement below:

> Gärdenfors' Preservation Criterion
>
> "If a belief state is revised by a sentence $A$, then all sentences in $K$ that are independent of the validity of $A$ should be retained in the revised state of belief." [Gä90]

We also discussed in Chapter 3 that a few years later in order to ground this intuition, Fariñas and Herzig, in *Belief Change and Dependence* [FdCH96], define a *dependence* relation in a close relationship to belief contraction. More specifically, they put forward an axiomatization for a dependence relation between logical formulas with respect to a belief set. Then, based on Gärdenfors' preservation criterion, they show how to construct such a dependence relation that satisfies all of their axioms for dependence, given an AGM contraction operator. Conversely, they show how to use a dependence relation that satisfies all their axioms in order to construct an AGM contraction operator.

Finally it was noted that some of Fariñas and Herzig's axioms for dependence are based on intuitions stated by previous authors working on the notion of relevance and dependence such as Keynes [Key21] and Gärdenfors [Gä78, Gä90]. More specifically, their dependence axiomatization satisfies both Keynes' conjunction criterion for dependence, CCD [p. 38], and Gärdenfors' conjunction criterion for independence, CCI [p. 39].

### 4.1.1 Problem Definition: Belief Change and Base Dependence

Similar to Fariñas and Herzig's work, our work is another attempt to connect notions of dependence and belief change, but using belief bases instead of belief sets. In a sense, a *dependence relation* specifies what formulas are relevant to the formulas of a *belief set*, but a *base dependence* relation specifies what formulas are relevant to the formulas of a *belief base*. As discussed in Chapters 1 and 2, belief bases are advantageous over belief sets in a number of important ways such as being finite in size, more expressive, and more tolerant of inconsistency. Therefore, using belief bases can be much more desirable in practice than using belief sets.

As discussed, belief bases are a generalization of belief sets. Hence, it seems natural to anticipate that base dependence also be a generalization of Fariñas and Herzig's dependence relation. Furthermore, base dependence should ideally be a *reversible generalization* of dependence. That is, where a base dependence relation corresponds to a belief set, the base dependence relation should reduce to Fariñas and Herzig's dependence. This in turn means that, for closed sets, base dependence corresponds to AGM belief change because dependence corresponds to AGM belief change.

Finally, one significant achievement in Fariñas and Herzig's work is in how their axiomatic characterization of dependence accounts for conjunctions. Notably, their formalism, either in the form of axioms or derivable statements, adheres to principles put forward by earlier authors, namely, the conjunction criterion for dependence (CCD

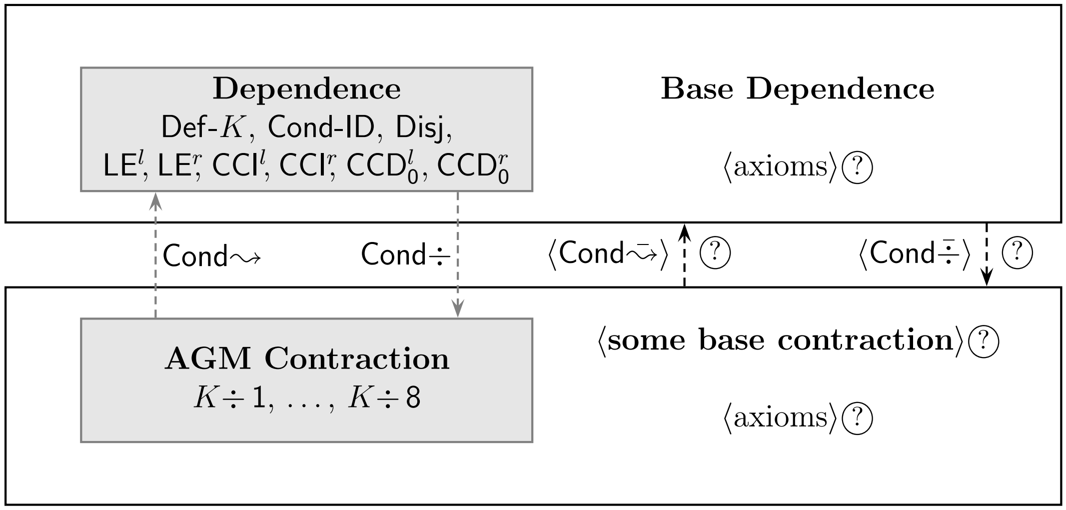



Figure 4.1: Schematic depiction of the anticipated results: *Belief Change and Base Dependence* as a reversible generalization of *Belief Change and Dependence*, which was shown in Figure 3.1b on page 37. The unknowns that need to be investigated, marked with ? above, include: axioms of base dependence, conditions Cond$\bar{\rightsquigarrow}$ and Cond$\bar{\div}$, and an appropriate corresponding base contraction along with its axioms.

[p. 38]) by Keynes [Key21] and the conjunction criterion for independence (CCI [p. 39]) by Gärdenfors [Gä78, Gä90]. Clearly, it is desirable to preserve this property of dependence while generalizing it to base dependence.

### 4.1.2 Characteristics of an Anticipated Solution

It is apparent from the outset that a formalization of Gärdenfors' preservation criterion that involves belief bases instead of belief sets should still retain the general scheme of Fariñas and Herzig's work. That is, we still need an axiomatization of dependence that is bidirectionally connected to an axiomatization of belief change; the only difference is that both of these axiomatizations need to be *generalized* to account for belief bases.

An anticipated generalization of Fariñas and Herzig's work to account for belief bases is given in Figure 4.1. The unknowns or the missing building blocks that need to be investigated to complete the picture are indicated with a circled question mark ?, which include:

**Base Dependence**: the intuition that base dependence is intended to capture; axiomatization of base dependence; it should be a reversible generalization of dependence.

**Base Contraction**: some appropriate base contraction to correspond to base dependence; axiomatization of the base contraction used; it should be a reversible generalization of AGM contraction.

$\mathsf{Cond}\bar{\rightsquigarrow}$: allowing to construct a base dependence relation, given a base contraction operator (corresponding to Fariñas and Herzig's $\mathsf{Cond}{\rightsquigarrow}$ [p. 57]).

$\mathsf{Cond}\bar{\div}$: allowing to construct a base contraction operator, given a base dependence relation. (corresponding to Fariñas and Herzig's $\mathsf{Cond}{\div}$ [p. 68]).

### 4.1.3 Organization of the Chapter

As usual, after axiomatization of base dependence, the obtained axioms will collectively *define* or *describe* what base dependence is. On the other hand, $\mathsf{Cond}\bar{\rightsquigarrow}$ will specify how to *construct* a base dependence relation by utilizing a given base contraction operator. Therefore, although it is not set in stone, it seems more natural to first define base dependence by putting forward an axiomatization for it, and then provide a method for constructing it. This is exactly how Fariñas and Herzig proceed in their work.

However, for the case of belief bases, we find it more suitable to start from construction of base dependence and then move on to its axiomatization. On the one hand, to come up with the right $\mathsf{Cond}\bar{\rightsquigarrow}$ condition, Gärdenfors' preservation criterion serves as the guiding principle. On the other hand, Gärdenfors' preservation criterion has originally been stated for belief sets. This has made Fariñas and Herzig's formalization of it for belief sets, $\mathsf{Cond}{\rightsquigarrow}$ [p. 57], relatively straightforward; there is no obvious way to offer an alternative formalization of it other than what they have suggested. However, when applying Gärdenfors' preservation criterion for the case of belief bases, there is more than one way to construct base dependence via base contraction. As it will be discussed in detail in §4.4 on page 54, one particular *kind* of base dependence constructed in this way is what we call *weak base dependence*, which we argue is implausible or of little use in practice. As such, we state the conditions under which weak base dependence may be avoided, and in order to manage the scope of research we focus on situations where there exist no weak base dependence. This in turn affects the axiomatization of base dependence. The set of axioms offered should be such that they disallow weak base dependence to occur. (Indeed, this turns out to be achievable by means of one axiom, redundancy [p. 74].)

There appears to be a natural flow for the other notions discussed in the chapter which more or less predetermines its organization. We start by stating the intuition behind dependence of logical formulas with respect to a base in §4.2. Next, as discussed in the previous subsection, there needs to be some appropriate base contraction to correspond to base dependence. In §4.3, we argue that a suitable candidate for this purpose can be saturated kernel contraction[1].

In §4.4, we put forward ways to construct base dependences using contraction. As discussed above, we also show how such construction can lead to weak base dependence, which we argue to be undesirable, state how it can be avoided, and assume its absence for the rest of the chapter. In §4.5, the opposite direction is studied: how to construct contraction using base dependence. Next, we move on to axiomatization of base dependence in §4.6, and give a characterization theorem showing the parallelism of base dependence and base contraction in §4.7. An enhanced version of the characterization theorem is presented in §4.8 that also accounts for conjunctions in base dependence. Finally, in §4.9, we show that base dependence is a reversible generalization of dependence. The last section, §4.10, contains closing and concluding remarks. All the proofs are moved to Appendix A starting on page 101.

## 4.2 Base Dependence

Keeping in mind our high-level goal, demonstrated in Figure 4.1, we first aim to refine our intuitive understanding of what base dependence is meant to represent.

[1]See §2.9 on page 30 for an introduction to saturated kernel contraction.

The meaning of “dependence” in base dependence is the same as what Fariñas and Herzig (and Gärdenfors) studied, which refers to the dependence or relevance of logical statements towards one another. Using their notation, we read $\alpha \rightsquigarrow \beta$ as “$\beta$ depends on $\alpha$”. With more elaboration, we can also read it as “doubting in $\alpha$ leads to doubting in $\beta$.” Syntactically, $\alpha$ and $\beta$ can be any grammatically correct logical statements: $\alpha, \beta \in \mathcal{L}$. However, Fariñas and Herzig's dependence relation $\rightsquigarrow$ is associated with some belief set $K \subseteq \mathcal{L}$, and the interesting cases arise when both formulas are from the belief set, $\alpha, \beta \in K$. If either of $\alpha$ or $\beta$ is not in $K$, then automatically $\alpha \not\rightsquigarrow \beta$. It is also possible that $\alpha, \beta \in K$ but still $\alpha \not\rightsquigarrow \beta$. For example, if either of $\alpha$ or $\beta$ is a tautology, then $\beta$ is independent of $\alpha$, $\alpha \not\rightsquigarrow \beta$. Of course, even if both $\alpha$ and $\beta$ are contingent but are simply irrelevant to each other, still we have $\alpha \not\rightsquigarrow \beta$. The important point here is that, in Fariñas and Herzig's study, dependence can only happen between (contingent) sentences from $K$:

If $\alpha \rightsquigarrow \beta$ then $\alpha \in K$ and $\beta \in K$.

Using belief base notation, if $B$ is a base for $K$, $K = \mathrm{Cn}(B)$, then we have:

$$\text{If } \alpha \rightsquigarrow \beta \text{ then } \alpha \in \mathrm{Cn}(B) \text{ and } \beta \in \mathrm{Cn}(B). \tag{4.1}$$

One way to generalize the dependence relation $\rightsquigarrow$ above is to make $\alpha$ or $\beta$ be from $B$ instead of $\mathrm{Cn}(B)$. Therefore, our sought-for *base dependence* should somehow involve formulas explicitly mentioned in a belief base, thus the name.

Using $\bar{\rightsquigarrow}$ to denote base dependence, we read $\alpha \bar{\rightsquigarrow} \beta$ as “$\beta$ *base-depends* on $\alpha$,” which is the same as $\alpha \rightsquigarrow \beta$ except that $\alpha \bar{\rightsquigarrow} \beta$ also implies that $\alpha$ or $\beta$ or both are formulas in the base. Now, we need to decide which one of these three alternatives should be the case.

$$\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \alpha \in B \text{ and } \beta \in \mathrm{Cn}(B) \tag{4.2a}$$

$$\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \alpha \in \mathrm{Cn}(B) \text{ and } \beta \in B \tag{4.2b}$$

$$\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \alpha \in B \text{ and } \beta \in B. \tag{4.2c}$$

Note that regardless of which alternative we eventually choose to proceed with, all of them are compatible with our goal that base dependence $\bar{\rightsquigarrow}$ should be a *reversible* generalization of dependence $\rightsquigarrow$, meaning that the $\bar{\rightsquigarrow}$ relation should reduce to $\rightsquigarrow$ in the special case where $B = \mathrm{Cn}(B)$. In other words, none of the alternatives (4.2a,b,c) is inconsistent with (4.1) when $B = \mathrm{Cn}(B)$.

The meaning of the base dependence relation $\bar{\rightsquigarrow}$ will slightly differ depending on which alternative we adopt, or which one of $\alpha$ or $\beta$ we require to be in $B$:

$\alpha \in B$ — Requiring that $\alpha$ be in the base in which case $\alpha \bar{\rightsquigarrow} \beta$ means "doubting in $\alpha$ from the base leads to doubting in $\beta$." That is, if $B$ is contracted by $\alpha$ which is already in $B$, then $\alpha \bar{\rightsquigarrow} \beta$ holds for all $\beta$ that are also retracted as a result of $\alpha$'s contraction. See also (4.4a).

$\beta \in B$ — Requiring that $\beta$ be in the base in which case $\alpha \bar{\rightsquigarrow} \beta$ means "doubting in $\alpha$ leads to doubting in $\beta$ from the base." That is, if $B$ is contracted by $\alpha$, then $\alpha \bar{\rightsquigarrow} \beta$ holds for all $\beta$ that are in $B$ but are retracted from it as a result of $\alpha$'s contraction. See also (4.4b).

$\alpha, \beta \in B$ — Requiring that both $\alpha$ and $\beta$ be in the base in which case $\alpha \bar{\rightsquigarrow} \beta$ means "doubting in $\alpha$ from the base leads to doubting in $\beta$ also from the base." That is, if $B$ is contracted by $\alpha$ which is already in $B$, then $\alpha \bar{\rightsquigarrow} \beta$ holds for all $\beta$ that are in $B$ but are retracted from it as a result of contracting $\alpha$. See also (4.4c).

To help make our final choice, we rewrite these options more formally. First, let us reconsider the dependence relation here. By (4.1), if $\alpha \rightsquigarrow \beta$ then $\beta \in \mathrm{Cn}(B)$. However, $\alpha \rightsquigarrow \beta$ can only hold for a $\beta$ that is retracted as a result of $\alpha$'s contraction. Thus, instead of just stating that $\beta \in \mathrm{Cn}(B)$, we can state more precisely that $\beta \in [\mathrm{Cn}(B) \setminus \mathrm{Cn}(B \div \alpha)]$. That is, we can make (4.1) slightly more specific as follows:

$$\text{If } \alpha \rightsquigarrow \beta \text{ then } \alpha \in \mathrm{Cn}(B) \text{ and } \beta \in [\mathrm{Cn}(B) \setminus \mathrm{Cn}(B \div \alpha)]. \tag{4.3}$$

By the same line of argument, and using the meanings of these options as listed above, we can also make each of the alternatives (4.2a,b,c) more precise as follows:

$$\text{If } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ then } \alpha \in B \text{ and } \beta \in [\mathrm{Cn}(B) \setminus \mathrm{Cn}(B \div \alpha)] \quad \text{(if requiring } \alpha \in B\text{)} \tag{4.4a}$$

$$\text{If } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ then } \alpha \in \mathrm{Cn}(B) \text{ and } \beta \in [B \setminus B \div \alpha] \quad \text{(if requiring } \beta \in B\text{)} \tag{4.4b}$$

$$\text{If } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ then } \alpha \in B \text{ and } \beta \in [B \setminus B \div \alpha]. \quad \text{(if requiring } \alpha, \beta \in B\text{)} \tag{4.4c}$$

We believe the third alternative (4.4c), requiring both $\alpha$ and $\beta$ to be in $B$, appears to be too strong to try first. The second alternative (4.4b) offers a more interesting semantics, compared to the first alternative (4.4a). As stated above, the second alternative means that "doubting in $\alpha$ leads to doubting in $\beta$ from the base." It allows us to study how the statements in the *base* depend on, or are susceptive to, changes of other statements. Stated in terms of belief change, it means contracting $B$ by any formula $\alpha$ from the *infinite* set $\mathrm{Cn}(B)$ can result in removal of $\beta$ from the *finite* set $B \setminus B \div \alpha$. On the other hand, the first alternative (4.4a) means contracting $B$ by any formula $\alpha$ from the usually *finite* set $B$ can result in removal of $\beta$ from the *infinite* set $\mathrm{Cn}(B) \setminus \mathrm{Cn}(B \div \alpha)$.

Therefore, we proceed with the second alternative; hence from now on, we assume $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ requires that $\beta \in B$:

$$\text{If } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ then } \beta \in B. \tag{4.5}$$

It turns out that statement (4.5) does not need to be explicitly specified as an axiom for base dependence. Rather, it will be implied by other axioms and conditions for base dependence which will be put forward in the upcoming sections (e.g., Def-$B$ [p. 73] and Cond$\bar{\rightsquigarrow}$ [p. 57]).

It goes without saying that the other two alternatives remain open for further exploration in future studies. (See open problem in §5.3.1 on page 98.)

## 4.3 A Candidate for Base Contraction: Saturated Kernel Contraction

Let us consider Figure 4.1 again to investigate one more of its building blocks. On the one hand, we need to come up with a generalization of Fariñas and Herzig's dependence relation, which we have called base dependence. On the other hand, this new base dependence relation needs to correspond to some kind of belief contraction in a way that fulfills Gärdenfors' preservation criterion. In this section, we come up with a candidate to fill this role.

As a starting point, let us consider the well known kernel contraction (see §2.9 on page 28) devised by Hansson [Han95] because it consists of a very general class of base contraction operations [Han99, p. 90]. Next, since we would like base dependence to be a reversible generalization of dependence, its corresponding base contraction should also be a reversible generalization of AGM contraction. However, kernel contraction does not meet this criterion. That is, for closed belief bases (i.e., belief sets), kernel contraction does *not* reduce to AGM contraction. Nevertheless, this is a property that a subclass of kernel contraction satisfies, viz., saturated kernel contraction. In other words, saturated kernel contraction is an interesting generalization of AGM contraction that also coincides with AGM contraction for belief sets [Han99, p. 92]. Thus, these three classes of contraction operations are subsets of one another:

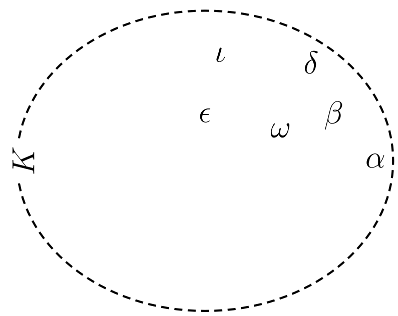
$$\{\text{AGM Contractions}\} \subseteq \{\text{Saturated Kernel Contractions}\} \subseteq \{\text{Kernel Contractions}\}.$$

Therefore, saturated kernel contraction is an interesting candidate to serve as the base contraction for our formalization of belief change and base dependence. The next steps are taken in §4.7 on page 79 and §4.8 on page 84 to actually *prove* that saturated kernel contraction and base dependence correspond to each other, fulfilling Gärdenfors' preservation criterion.

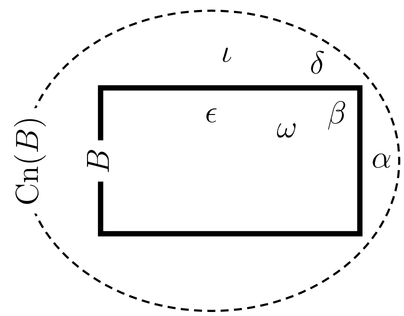
## 4.4 Constructing Base Dependences Using Contraction

Once again let us consider Figure 4.1 on page 48. Fariñas and Herzig connect the notions of dependence and belief contraction using Gärdenfors' preservation criterion as a guideline for their formalism. Similarly here, we aim to use the same guideline to connect base dependence and base contraction while adhering to Fariñas and Herzig's work as a special case.

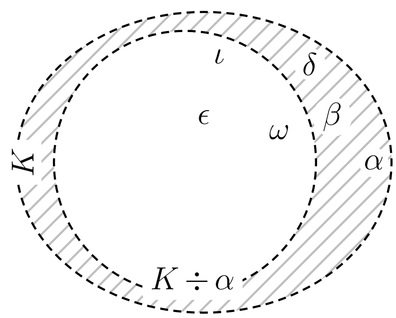
| **Dependence** | **Base Dependence** |
|---|---|



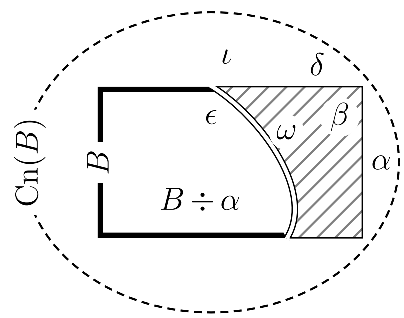

(a) Closed belief set $K$ with some of its sentences: $\alpha$, $\beta$, $\omega$, $\delta$, $\iota$ and $\epsilon$.



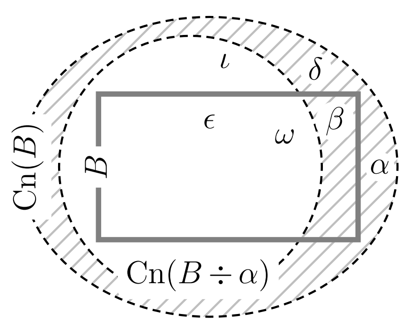

(d) A belief base $B$ and its logical closure $\mathrm{Cn}(B)$.



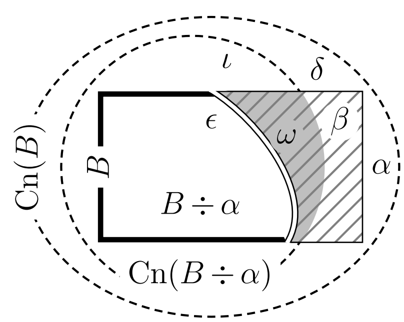

(b) Contracting $K$ by $\alpha$ leads to removal of $\beta$ and $\delta$: $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \delta$.




(e) Intermediary step: contracting $B$ by $\alpha$ leads to removal of $\beta$ and $\omega$ from $B$: $\alpha \,\bar{\rightsquigarrow}\, \beta$ and $\alpha \,\bar{\rightsquigarrow}\, \omega$.




(c) Base-generated contraction with $K = \mathrm{Cn}(B)$ produces same results: $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \delta$.




(f) But $\omega$ is still implied by the remainder of $B$, $\omega \in \mathrm{Cn}(B \div \alpha)$, so it only *weakly* depends on $\alpha$: $\alpha \,\check{\rightsquigarrow}\, \omega$.

| | | | | | | |
|---|---|---|---|---|---|---|
| By Cond$\rightsquigarrow$ [p. 57]: | $\alpha \rightsquigarrow \alpha$ | $\alpha \rightsquigarrow \beta$ | $\alpha \not\rightsquigarrow \omega$ | $\alpha \rightsquigarrow \delta$ | $\alpha \not\rightsquigarrow \iota$ | $\alpha \not\rightsquigarrow \epsilon$ |
| By Cond$\bar{\rightsquigarrow}$ [p. 57]: | $\alpha \not\bar{\rightsquigarrow} \alpha$ | $\alpha \bar{\rightsquigarrow} \beta$ | $\alpha \bar{\rightsquigarrow} \omega$ | $\alpha \not\bar{\rightsquigarrow} \delta$ | $\alpha \not\bar{\rightsquigarrow} \iota$ | $\alpha \not\bar{\rightsquigarrow} \epsilon$ |
| By Cond$\hat{\rightsquigarrow}$ [p. 58]: | $\alpha \not\hat{\rightsquigarrow} \alpha$ | $\alpha \hat{\rightsquigarrow} \beta$ | $\alpha \not\hat{\rightsquigarrow} \omega$ | $\alpha \not\hat{\rightsquigarrow} \delta$ | $\alpha \not\hat{\rightsquigarrow} \iota$ | $\alpha \not\hat{\rightsquigarrow} \epsilon$ |
| By Cond$\check{\rightsquigarrow}$ [p. 59]: | $\alpha \not\check{\rightsquigarrow} \alpha$ | $\alpha \not\check{\rightsquigarrow} \beta$ | $\alpha \check{\rightsquigarrow} \omega$ | $\alpha \not\check{\rightsquigarrow} \delta$ | $\alpha \not\check{\rightsquigarrow} \iota$ | $\alpha \not\check{\rightsquigarrow} \epsilon$ |

Figure 4.2: Comparing *Dependence* and various *Base Dependence* relations.

However, one challenge is that Gärdenfors' preservation criterion was originally stated for belief sets, and applying it to belief bases can bring about some complexities which need to be addressed. In particular, we will show in this section that there is more than one way to construct base dependence via base contraction. One such construction of base dependence leads to a situation where the dependence of formulas on one another is in a sense ambiguous: From one perspective there can exist base dependence between two formulas, but from another perspective there is no base dependence between the same two formulas. As such, we consider this *kind* of dependence to be *weak base dependence*. We also argue that weak base dependence is not as practically useful as the other kind of base dependence that persists irrespective of different perspectives, which we call *strong base dependence*. Next, in §4.4.2, we specify the conditions under which weak base dependence may be avoided. By guaranteeing that such conditions hold, we can focus on situations where there exist no weak base dependence in order to limit the scope of our research.

Throughout this section, Figure 4.2 will provide a running example to help illustrate different existing or new concepts discussed or introduced.

As discussed in Chapter 3, inspired by Gärdenfors' preservation criterion, Fariñas and Herzig use the following condition to construct a dependence relation $\rightsquigarrow$ via a given AGM contraction $\div$:

$$\alpha \rightsquigarrow \beta \quad \text{iff} \quad \beta \in K \text{ and } \beta \notin K \div \alpha. \qquad (\mathsf{Cond}\rightsquigarrow \text{ [p. 42]})$$

An example application of $\mathsf{Cond}\rightsquigarrow$ is depicted in Figures 4.2a and 4.2b. Figure 4.2a shows a belief set $K$ along with some arbitrary formulas: $\alpha$, $\beta$, $\omega$, $\delta$, $\iota$ and $\epsilon$. (As we proceed in this section and different kinds of dependence are differentiated, each of these formulas will fall into a different subarea.) Figure 4.2b shows how contracting $K$ by $\alpha$ also results in retraction of some other formulas, namely, $\beta$ and $\delta$. That is, $\beta$ and $\delta$ are in $K$ but are not in $K \div \alpha$ anymore, so we conclude by $\mathsf{Cond}\rightsquigarrow$ that they depend on $\alpha$: $\alpha \rightsquigarrow \beta$ and $\alpha \rightsquigarrow \delta$. By the same token, we conclude that the other named formulas from $K$ do not depend on $\alpha$: $\alpha \not\rightsquigarrow \omega$, $\alpha \not\rightsquigarrow \iota$ and $\alpha \not\rightsquigarrow \epsilon$. These results are summarized on the first row of the table at the bottom of Figure 4.2.

One last comment on $\mathsf{Cond}{\rightsquigarrow}$ is regarding its representation which can be straightforwardly transformed to an equivalent base-generated representation. If $B$ is a base for $K$ and $\div$ is a base-generated contraction, which exists for any given AGM contraction [Han93], then $\mathsf{Cond}{\rightsquigarrow}$ has the following representation:

$$\alpha \rightsquigarrow \beta \;\text{ iff }\; \beta \in \mathrm{Cn}(B) \text{ and } \beta \notin \mathrm{Cn}(B \div \alpha). \qquad (\mathsf{Cond}{\rightsquigarrow})$$

Note that the same name "$\mathsf{Cond}{\rightsquigarrow}$" has been reused here since this is only a notational change. As an example on how this new representation preserves the original meaning of $\mathsf{Cond}{\rightsquigarrow}$, compare Figure 4.2b, depicting the old representation, with Figure 4.2c, depicting the new one. In both figures the actual dependency between formulas is the same. As such, from now on, we use this new, base-generated representation because it makes it easier to compare and contrast $\mathsf{Cond}{\rightsquigarrow}$ with the corresponding conditions for belief bases that will be introduced shortly.

## 4.4.1 Different Kinds of Base Dependence

### Base Dependence

As we saw before, particularly in §4.2, we say that if $\beta$ depends on $\alpha$, $\alpha \rightsquigarrow \beta$, then $\beta$ is retracted as a result of $\alpha$'s contraction, so $\beta \in [\mathrm{Cn}(B) \setminus \mathrm{Cn}(B \div \alpha)]$. For a belief base oriented formalism of the same concept, we start by a decision we made in §4.2 that (4.5) should hold: if $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\beta \in B$. Then, it is intuitively appealing to say that if $\beta$ base-depends on $\alpha$, $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$, then $\beta$, originally from the base, has been retracted as a result of $\alpha$'s contraction, so $\beta \in [B \setminus B \div \alpha]$. With this intuition in mind, we offer the following as a condition to correspond to Fariñas and Herzig's $\mathsf{Cond}{\rightsquigarrow}$:

$$\boxed{\alpha \mathrel{\bar{\rightsquigarrow}} \beta \;\text{ iff }\; \beta \in B \text{ and } \beta \notin B \div \alpha.} \qquad (\mathsf{Cond}{\bar{\rightsquigarrow}})$$

This basically allows for a straightforward construction of base dependence via a given base contraction. For example, Figure 4.2d shows a belief base $B$ and its logical closure $\mathrm{Cn}(B)$, and in Figure 4.2e some formulas from base $B$ have been retracted, namely, $\beta$ and $\omega$, so that the remaining set, $B \div \alpha$, does not imply $\alpha$ anymore. By $\mathsf{Cond}{\bar{\rightsquigarrow}}$, then, we conclude that $\beta$ and $\omega$ base-depend on $\alpha$: $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ and $\alpha \mathrel{\bar{\rightsquigarrow}} \omega$. No other named formula in these figures have base dependence on $\alpha$: $\alpha \not\bar{\rightsquigarrow} \alpha$, $\alpha \not\bar{\rightsquigarrow} \delta$, $\alpha \not\bar{\rightsquigarrow} \iota$ and $\alpha \not\bar{\rightsquigarrow} \epsilon$. For $\alpha$, $\delta$ and $\iota$, this is because they are not originally in the base $B$ (even though they are implied by it). The reason for $\alpha \not\bar{\rightsquigarrow} \epsilon$ is that even though $\epsilon \in B$, it is not retracted as a result of $B$'s contraction by $\alpha$. In all these instances, $\mathsf{Cond}\bar{\rightsquigarrow}$ maintains its intuitive appeal as a reasonable formalization of Gärdenfors' preservation criterion. Nevertheless, as we will see next, this is not the only possible way to formulate this criterion for belief bases.

### Strong Base Dependence

Let us reconsider $\mathsf{Cond}\bar{\rightsquigarrow}$ to see if we can find any sensible variations of it. Indeed, another possible formalization of Gärdenfors' preservation criterion for belief bases can be proposed as follows:

$$\boxed{\alpha \mathrel{\hat{\rightsquigarrow}} \beta \;\text{ iff }\; \beta \in B \text{ and } \beta \notin \mathrm{Cn}(B \div \alpha).} \qquad (\mathsf{Cond}\hat{\rightsquigarrow})$$

This provides a stronger condition for base dependence in the following sense. $\mathsf{Cond}\hat{\rightsquigarrow}$ implies that $\beta$ has *strong base dependence* on $\alpha$, $\alpha \mathrel{\hat{\rightsquigarrow}} \beta$, when $\beta$ is originally in $B$, but it is not only retracted as a result of contraction by $\alpha$, $\beta \notin B \div \alpha$, but also guaranteed not to be implied by the remaining set, $\beta \notin \mathrm{Cn}(B \div \alpha)$. Note that $\mathsf{Cond}\hat{\rightsquigarrow}$ does not need to explicitly mention that $\beta \notin B \div \alpha$ as it is implied by $\beta \notin \mathrm{Cn}(B \div \alpha)$, given that $B \div \alpha \subseteq \mathrm{Cn}(B \div \alpha)$ by $\mathsf{inclusion}$ property of the Cn operator.

As seen above, in the example illustrated in Figure 4.2e, both $\beta$ and $\omega$ base-depend on $\alpha$ by $\mathsf{Cond}\bar{\rightsquigarrow}$: $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ and $\alpha \mathrel{\bar{\rightsquigarrow}} \omega$. There remains, however, a subtle difference between base dependence of $\beta$ on $\alpha$ and that of $\omega$ on $\alpha$. According to $\mathsf{Cond}\hat{\rightsquigarrow}$, $\beta$ has strong base dependence on $\alpha$, $\alpha \mathrel{\hat{\rightsquigarrow}} \beta$, and $\omega$ does not, $\alpha \not\hat{\rightsquigarrow} \omega$. The difference between $\beta$ and $\omega$ becomes more evident in Figure 4.2f. When contracting $B$ by $\alpha$, $\beta$ is retracted whether we consider the contraction remaining set, $B \div \alpha$, or its closure, $\mathrm{Cn}(B \div \alpha)$. This is not the case for $\omega$, which we will study next.

### Weak Base Dependence

In the example above, based on Figure 4.2f, we saw that $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ and $\alpha \mathrel{\bar{\rightsquigarrow}} \omega$ by $\mathsf{Cond}\bar{\rightsquigarrow}$, but $\alpha \mathrel{\hat{\rightsquigarrow}} \beta$ and $\alpha \not\hat{\rightsquigarrow} \omega$ by $\mathsf{Cond}\hat{\rightsquigarrow}$. The base dependence of $\beta$ on $\alpha$ is persistent, but that of $\omega$ is not. To further investigate the difference between $\mathsf{Cond}\bar{\rightsquigarrow}$ and $\mathsf{Cond}\hat{\rightsquigarrow}$, observe that $\omega \in B$ and $\omega \notin B \div \alpha$, but $\omega \in \mathrm{Cn}(B \div \alpha)$. In other words, $\omega$ is originally in $B$, and it is then retracted as a result of contracting $B$ by $\alpha$, $\omega \notin B \div \alpha$, but later it is reintroduced as a logical implication of the contracted set, $\omega \in \mathrm{Cn}(B \div \alpha)$. We refer to this non-persistent base dependence of $\omega$ on $\alpha$ as *weak base dependence*, denoted by $\alpha \check{\rightsquigarrow} \omega$. On the one hand, $\alpha \check{\rightsquigarrow} \omega$ refers to a kind of base dependence in the sense that $\omega$ is removed from the base as a result of contracting by $\alpha$. On the other hand, it does not fully capture the concept of dependence because $\omega$ is still implicitly present in the consequences of the contracted set. Thus even though it is a kind of base dependence, it is a *weak* dependence. Basically a base dependence which is not a strong base dependence is a weak base dependence. The following condition $\mathsf{Cond}\check{\rightsquigarrow}$ specifies how to construct weak base dependence via base contraction:

$$\boxed{\alpha \check{\rightsquigarrow} \beta \ \text{ iff } \ \beta \in B \text{ and } \beta \notin B \div \alpha \text{ and } \beta \in \mathrm{Cn}(B \div \alpha).} \qquad (\mathsf{Cond}\check{\rightsquigarrow})$$

### 4.4.2 Avoiding Weak Base Dependence

Even though such a contraction operation $\div$ as in the above example that fulfills $\mathsf{Cond}\check{\rightsquigarrow}$ can theoretically exist, it is *intuitively implausible*. That is because if $\omega$ has been given up and is not in $B \div \alpha$ while it is still implied by $B \div \alpha$, why should it have been excluded in the first place? This exclusion is "unnecessary, and violates the basic principle of minimality of belief change: nothing should be given up without reason." [Han99, p. 90]

Therefore, we aim to find ways to avoid weak base dependence in order to further narrow the scope of our research here. That is, we focus on the situation where there is no weak base dependence, $\alpha \check{\not\rightsquigarrow} \beta$, between any given pair of formulas $\alpha$ and $\beta$. Formally, we define absence of weak base dependence as follows.

**Definition 4.1** (Absence of Weak Base Dependence)**.** Given relations $\check{\rightsquigarrow}$ and $\div$ for base $B$ such that $\mathsf{Cond}\check{\rightsquigarrow}$ holds, we say that:

there is *no weak base dependence* if and only if $\alpha \check{\not\rightsquigarrow} \beta$ for all formulas $\alpha$ and $\beta$.

### Redundancy Gives Rise to Weak Base Dependence

In this section we show that there is a powerful correspondence between two seemingly different concepts: weak base dependence and redundancy of belief bases. More specifically, we will show that weak base dependence can only exist when its corresponding belief base contains some redundant formulas.

To this end, we start by clarifying what redundancy in a base means. For example, consider the base $B = \{p, q, p \to q\}$, and a subset of it $B' = \{p, p \to q\}$. It is clear that $q \notin B'$, yet it is also easy to see that $q \in \mathrm{Cn}(B')$. This happens because indeed $q$ is redundant in $B$ with respect to $B'$. The following provides a formal definition to capture this intuition regarding redundancy.

**Definition 4.2** (Redundancy)**.** $\beta$ is *redundant* in $B$ with respect to $B'$ if and only if $B' \subseteq B$ and $\beta \in B$ and $\beta \notin B'$ and $\beta \in \mathrm{Cn}(B')$.

As intuitively expected, we then say a base $B$ has *redundancy* if and only if it contains at least one redundant formula with respect to some $B' \subseteq B$.

As we will see in Corollary 4.4, if $B$ contains no redundancy, then there can be no weak base dependence for any of its formulas. However, the converse does not necessarily hold. It is possible for a base $B$ to have redundant statements, but none of its formulas has weak base dependence on any other formulas.

Indeed, we will show that weak base dependence $\tilde{\leadsto}$ is characterized by its corresponding base $B$ and its corresponding contraction operator $\div$: there needs to be some redundancy in base $B$ and the corresponding $\div$ does not avoid the redundant statements.

The following theorem shows that weak base dependence exists if and only if some of the contracted statements, $B \setminus B \div \alpha$, were redundant with respect to the remaining statements, $B \div \alpha$. This means the redundant contracted statements are still implied by the remaining statements.

**Theorem 4.3.** *Given relations* $\tilde{\leadsto}$ *and* $\div$ *for base* $B$*, where* inclusion *holds,* Cond$\tilde{\leadsto}$ *is equivalent to the following:*

$$\alpha \tilde{\leadsto} \beta \;\text{ iff }\; \beta \textit{ is redundant in } B \textit{ with respect to } B \div \alpha. \tag{4.6}$$

[Proof on page 101.]

One immediate and interesting implication of this theorem is that weak base dependence cannot occur in a belief base that contains no redundancy.

**Corollary 4.4.** *Given relations* $\tilde{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* $\mathsf{Cond}\tilde{\rightsquigarrow}$ *and* $\mathsf{inclusion}$ *hold, the following also holds:*

*if there is no redundan*cy in $B$, *then* $B$ *contains no weak base dependence.*

The proof for this corollary is not provided here as it trivially follows from Theorem 4.3.

To summarize the results so far, let us consider Definition 4.1 for Absence of Weak Base Dependence. Note that for any $\beta \notin B$, it trivially holds by $\mathsf{Cond}\tilde{\rightsquigarrow}$ [p. 59] that $\alpha \tilde{\not\rightsquigarrow} \beta$ for all $\alpha$. More interesting instances of absence of weak base dependence can occur when $\beta \in B$. It is a property of the belief base $B$ and/or the contraction operator $\div$ used that determines whether any weak base dependence can exist. By Corollary 4.4, if base $B$ does not contain any redundancy, then there will be no weak base dependence involving any of its formulas. Also by Theorem 4.3, neither will there be any weak base dependence via a contraction operation $\div$ using $\mathsf{Cond}\tilde{\rightsquigarrow}$ that can properly handle any redundancy that may exist in the base $B$. Next, we will determine what is the exact contraction property that needs to hold to avoid weak base dependence.

### How To Avoid Weak Base Dependence

So far we have seen that avoiding weak base dependence may be achieved through utilizing some properties of the contraction operator $\div$ used to construct it. Here we further explore to identify such property(ies).

As it was demonstrated in Theorem 4.3, weak base dependence can occur only when there is redundancy in a base with respect to the remaining formulas of a contraction:

$$\alpha \tilde{\rightsquigarrow} \beta \text{ iff } \beta \text{ is redundant in } B \text{ with respect to } B \div \alpha. \tag{4.6}$$

Thus in order for $\beta$ not to have weak base dependence on $\alpha$, $\alpha \not\overset{\sim}{\rightsquigarrow} \beta$, we can simply negate both sides of (4.6):

$\alpha \not\overset{\sim}{\rightsquigarrow} \beta$ iff $\beta$ is *not* redundant in $B$ with respect to $B \div \alpha$.

Replacing the right hand side using Definition 4.2 for redundancy, we get:

$\alpha \not\overset{\sim}{\rightsquigarrow} \beta$ iff $\neg\,[B \div \alpha \subseteq B$ and $\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \mathrm{Cn}(B \div \alpha)]$,

or

$\alpha \not\overset{\sim}{\rightsquigarrow} \beta$ iff $[B \div \alpha \not\subseteq B$ or $\beta \notin B$ or $\beta \in B \div \alpha$ or $\beta \notin \mathrm{Cn}(B \div \alpha)]$.

The first disjunct $B \div \alpha \not\subseteq B$ should be ignored as it violates inclusion, $B \div \alpha \subseteq B$, which is an indispensable property of any contraction operation. When the second disjunct, $\beta \notin B$, holds, no kind of base dependence can occur, including weak base dependence $\alpha \not\overset{\sim}{\rightsquigarrow} \beta$. A much more interesting scenario is when $\beta \in B$. This leaves only the last two disjuncts to determine whether $\alpha \not\overset{\sim}{\rightsquigarrow} \beta$, namely, $\beta \in B \div \alpha$ or $\beta \notin \mathrm{Cn}(B \div \alpha)$. To summarize, the sufficient and necessary condition for $\beta$ not to have weak base dependence on $\alpha$, $\alpha \not\overset{\sim}{\rightsquigarrow} \beta$, is as follows:

If $\beta \in B$ then $\beta \in B \div \alpha$ or $\beta \notin \mathrm{Cn}(B \div \alpha)$,

or

If $\beta \in B$ and $\beta \in \mathrm{Cn}(B \div \alpha)$ then $\beta \in B \div \alpha$.

Using set notation to represent this requirement for $\alpha \not\overset{\sim}{\rightsquigarrow} \beta$ reveals that it can be guaranteed to hold for any contraction operator that satisfies relative closure:

$B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha.$ (relative closure [p. 27])

The following theorem formally identifies relative closure as the condition under which there does not exist weak base dependence between any given pair of sentences.

**Theorem 4.5.** *Given relations $\overset{\sim}{\rightsquigarrow}$ and $\div$ for base $B$ such that* Cond$\overset{\sim}{\rightsquigarrow}$ *holds, there is no weak base dependence if and only if* relative closure *holds for $\div$.*

[Proof on page 102.]

Thus, avoidance of weak base dependence can be achieved solely based on the properties of the corresponding contraction operation. A base dependence constructed using a contraction operator that satisfies relative closure is guaranteed to avoid weak base dependence altogether.

Interestingly, this result further supports that saturated kernel contraction is a suitable candidate as base contraction to correspond to base dependence, as it was suggested in §4.3 on page 53. A saturated kernel contraction is a kernel contraction that satisfies relative closure.

### 4.4.3 Connections Among (Base) Dependence Constructions

The fact that we can avoid weak base dependence turns out to be quite helpful in our formalization of base dependence. That is because, as we will show in this section, after guaranteeing to avoid weak base dependence, base dependence and strong base dependence become equivalent: $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \hat{\rightsquigarrow} \beta$ for all formulas $\alpha$ and $\beta$ by Theorem 4.7. This allows us not to worry about *weak* and *strong* base dependence, and we can consider only *base dependence* in our formalism.

We start by showing that a base dependence between two formulas, $\alpha \bar{\rightsquigarrow} \beta$, either is a strong base dependence, $\alpha \hat{\rightsquigarrow} \beta$, or is a weak base dependence, $\alpha \check{\rightsquigarrow} \beta$.

**Theorem 4.6.** *Given relations* $\bar{\rightsquigarrow}$, $\hat{\rightsquigarrow}$, $\check{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\rightsquigarrow}$, Cond$\hat{\rightsquigarrow}$ *and* Cond$\check{\rightsquigarrow}$ *hold, the following also holds:*

$$\alpha \bar{\rightsquigarrow} \beta \text{ iff } \alpha \hat{\rightsquigarrow} \beta \text{ or } \alpha \check{\rightsquigarrow} \beta$$

[Proof on page 103.]

On a side note, it is easy enough to show that, by conditions Cond$\hat{\rightsquigarrow}$ and Cond$\check{\rightsquigarrow}$, strong and weak base dependence are mutually exclusive:

$$\neg[\alpha \hat{\rightsquigarrow} \beta \text{ and } \alpha \check{\rightsquigarrow} \beta].$$

Thus, the right hand side of the equivalence in Theorem 4.6 above can be shown to involve an exclusive-or operation (instead of its current inclusive-or):

$$\alpha \bar{\rightsquigarrow} \beta \text{ iff } ([\alpha \hat{\rightsquigarrow} \beta \text{ or } \alpha \check{\rightsquigarrow} \beta] \text{ and } \neg[\alpha \hat{\rightsquigarrow} \beta \text{ and } \alpha \check{\rightsquigarrow} \beta]).$$

However, it turns out that for our purposes here, it suffices to go with the simpler equivalence: $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \hat{\rightsquigarrow} \beta$ or $\alpha \check{\rightsquigarrow} \beta$, which was already established in Theorem 4.6.

Next, we show that base dependence and strong base dependence become equivalent when there is no weak base dependence, which is guaranteed in the following theorem by requiring that relative closure be satisfied. (See Theorem 4.5.)

**Theorem 4.7.** *Given relations* $\bar{\rightsquigarrow}$, $\hat{\rightsquigarrow}$, $\check{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* $\mathsf{Cond}\bar{\rightsquigarrow}$, $\mathsf{Cond}\hat{\rightsquigarrow}$ *and* $\mathsf{Cond}\check{\rightsquigarrow}$ *hold and* relative closure *is satisfied, the base dependence relation* $\bar{\rightsquigarrow}$ *and the strong base dependence relation* $\hat{\rightsquigarrow}$ *are equivalent:*

$$\alpha \bar{\rightsquigarrow} \beta \text{ iff } \alpha \hat{\rightsquigarrow} \beta.$$

[Proof on page 104.]

Therefore, from now on, we focus on base dependence and put aside the concepts of *weak* and *strong* base dependence altogether. All that is required is that when using a contraction operator $\div$ to construct a base dependence relation, $\div$ should satisfy relative closure. Then, no weak base dependence can exist by Theorem 4.5, and base dependence and strong base dependence become identical by Theorem 4.7.

### Dependence and Base Dependence

The final theorems in this section establish the connection between dependence and base dependence. We start with a lemma which offers a useful equivalent form for relative closure.

**Lemma 4.8.** *An operator* $\div$ *on base* $B$ *satisfies* relative closure *if and only if it satisfies the following:*

$$\text{If } \beta \in B \text{ then } \quad \beta \in B \div \alpha \text{ iff } \beta \in \mathrm{Cn}(B \div \alpha). \tag{4.7}$$

[Proof on page 104.]

The following theorem shows that, in the presence of relative closure or equivalently in the absence of weak base dependence, base dependence is equivalent to dependence for the formulas in the base.

**Theorem 4.9.** *Given relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* $B$ *such that* $\mathsf{Cond}\bar{\rightsquigarrow}$ *and* $\mathsf{Cond}\rightsquigarrow$ *hold and* relative closure *is satisfied, the following also holds:*

$$\alpha \mathbin{\bar{\rightsquigarrow}} \beta \quad \text{iff} \quad \beta \in B \text{ and } \alpha \rightsquigarrow \beta.$$

[Proof on page 105.]

This result makes it clear that when $B$ is logically closed, $B = \mathrm{Cn}(B)$, base dependence and dependence become equivalent. That is because, as given in the following theorem, $B$ being closed means that $\beta \in B$ in the expression above can be replaced with $\beta \in \mathrm{Cn}(B)$: $\alpha \mathbin{\bar{\rightsquigarrow}} \beta$ iff $\beta \in \mathrm{Cn}(B)$ and $\alpha \rightsquigarrow \beta$. Then, as $\beta \in \mathrm{Cn}(B)$ is already implied by $\alpha \rightsquigarrow \beta$, it may be omitted to obtain: $\alpha \mathbin{\bar{\rightsquigarrow}} \beta$ iff $\alpha \rightsquigarrow \beta$.

**Theorem 4.10.** *Given relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* $B$ *such that* $\mathsf{Cond}\bar{\rightsquigarrow}$ *and* $\mathsf{Cond}\rightsquigarrow$ *hold and* closure *is satisfied, in the special case where* $B$ *is logically closed,* $\bar{\rightsquigarrow}$ *reduces to* $\rightsquigarrow$*:*

$$\alpha \mathbin{\bar{\rightsquigarrow}} \beta \quad \text{iff} \quad \alpha \rightsquigarrow \beta.$$

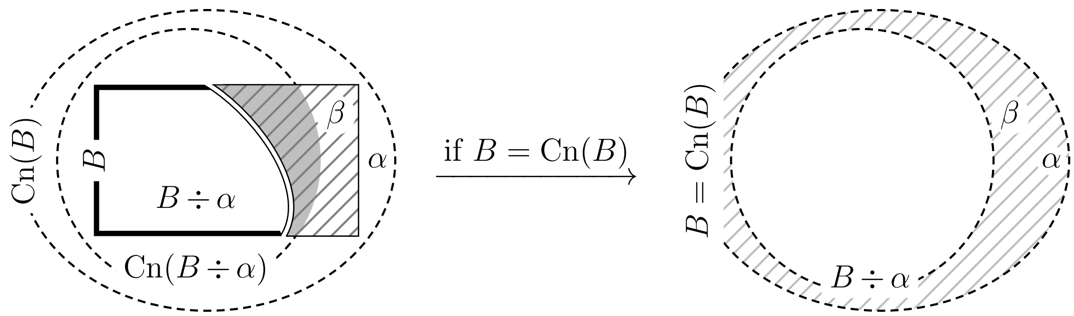
[Proof on page 106.]

Another way to demonstrate the fact that, for logically closed sets, base dependence and dependence become identical relations is to show that the conditions that make their construction possible, namely, $\mathsf{Cond}\bar{\rightsquigarrow}$ and $\mathsf{Cond}\rightsquigarrow$, become identical. This is depicted in Figure 4.3 on the following page.

## 4.5 Constructing Contraction Using Base Dependence

We first provide a simplifying notation $\bar{\vdash}$ to help represent *tautologies present in the base*. Although it is not necessary to introduce this notation, it will prove to be quite effective in simplifying representation of the axioms and shortening some of the proofs (by quite a few steps at times).

**Definition 4.11.** Given a base $B$ and an entailment relation $\vdash$, the *base entailment* relation $\bar{\vdash}$ is defined as follows: $A \mathrel{\bar{\vdash}} \beta$ if and only if $\beta \in B$ and $A \vdash \beta$.




(a) Base dependence for base $B$, which is generally not closed

(b) When $B$ is closed though, base dependence reduces to dependence

Figure 4.3: Dependence is a *special case* of base dependence

A useful special case is when $A = \emptyset$. For example, $\bar{\vdash}\ \beta$ means $\beta$ is a tautology in the base: $\beta \in B$ and $\vdash \beta$. One important usage is to help handling tautologies in base dependence axioms. Such axioms are primarily concerned with contingencies, but they have to also deal with tautologies, usually as exceptional cases.

As a side note, a more general usage of Definition 4.11 could be in a study of redundancy in belief bases: $A \mathrel{\bar{\vdash}} \beta$ signifies that $\beta$ is both *derived* and *already present* in the base. One special case of such a situation is when $A$ is the empty set $\emptyset$, so we have $\bar{\vdash}\ \beta$ which we saw above. From this perspective, $\beta$ is a tautology, which is *redundantly* in the base too. However, since we are deliberately avoiding weak base dependence, which is closely related to redundancy in the base (see §4.4.2 on page 59), the general form of base entailment, $A \mathrel{\bar{\vdash}} \beta$, is not as useful as the special case, $\bar{\vdash}\ \beta$, for our work here.

### 4.5.1 Using Base Dependence to Reconstruct Belief Bases

As we saw in Chapter 3, there is a belief set $K$ associated with a dependence relation $\rightsquigarrow$. Fariñas and Herzig provide the following definition to recreate a belief set $K_{\rightsquigarrow}$ given a dependence $\rightsquigarrow$ relation:

$$K_{\rightsquigarrow} \overset{def}{=} \{\alpha \mid \vdash \alpha \text{ or } \alpha \rightsquigarrow \beta \text{ for some } \beta\}.$$

That is, if $\alpha$ is a tautology, it is trivially part of $K_{\leadsto}$. On the other hand, $\alpha$ is a contingent truth if there is a sentence $\beta$ in $K_{\leadsto}$ such that it is removed once $\alpha$ is contracted.

One small note is that for their work we could swap the role of $\alpha$ and $\beta$ in the definition and obtain the same results (but this will be important in the case of base dependence):

$$K_{\leadsto} \stackrel{def}{=} \{\beta \mid\ \vdash \beta \text{ or } \alpha \leadsto \beta \ \text{ for some } \alpha\}.$$

Similarly, a base dependence relation $\bar{\leadsto}$ is associated with a belief base $B$. Thus, it should be possible to recreate the associated belief base $B$ via a given $\bar{\leadsto}$ relation:

$$B_{\bar{\leadsto}} = \{\beta \mid \alpha \,\bar{\leadsto}\, \beta \ \text{ for some } \alpha\}\,.$$

Note, however, that one caveat is that $B_{\bar{\leadsto}}$ will not contain any *tautologies* that may be in $B$. Still, we can say that $B$ and $B_{\bar{\leadsto}}$ are equivalent for most practical purposes. Also, their closure is obviously equivalent:

$$\mathrm{Cn}(B_{\bar{\leadsto}}) = \mathrm{Cn}(B).$$

If in addition to the base dependence relation $\bar{\leadsto}$, we are also given $\bar{\vdash}$ that identifies tautologies in the base, then we can have the following, which guarantees that $B_{\bar{\leadsto}} = B$:

$$B_{\bar{\leadsto}} = \left\{\beta \mid \bar{\vdash}\ \beta \text{ or } \alpha \,\bar{\leadsto}\, \beta \ \text{ for some } \alpha\right\}$$

To summarize, when $B$ is not given and it needs to be created, the following scenarios are possible:

- Only the base dependence relation is given $\bar{\leadsto}$. This is the worst case scenario in which $B_{\bar{\leadsto}}$ will not contain any *tautologies* that might have been in $B$. However, if there were no tautologies in $B$ to begin with, then $B_{\bar{\leadsto}} = B$.
- In addition to $\bar{\leadsto}$, the set of tautologies in the base $\{\beta \mid \bar{\vdash}\ \beta\}$ is also given. In this scenario, it is possible to find the exact original base; i.e., $B_{\bar{\leadsto}} = B$.

In the rest of this work, we assume $B_{\bar{\leadsto}} = B$. In the worst case scenario, there are tautologies in $B$ and only $\bar{\leadsto}$ is given, so the tautologies in $B$ are not present in $B_{\bar{\leadsto}}$.

### 4.5.2 Using Base Dependence to Construct Contraction Operators

Now for $\mathsf{Cond}\bar{\div}$, again we start with Fariñas and Herzig's $\mathsf{Cond}\div$:

$$\beta \in K \div \alpha \ \text{ iff } \ \text{either} \vdash \beta \text{ or } \beta \rightsquigarrow \beta \text{ and } \alpha \not\rightsquigarrow \beta. \qquad (\mathsf{Cond}\div \text{ [p. 42]})$$

As in the case of $\mathsf{Cond}\rightsquigarrow$, we present a straightforward transformation to the equivalent base-generated operation. Again, we reuse the equation's name, "$\mathsf{Cond}\div$":

$$\beta \in \mathrm{Cn}(B \div \alpha) \ \text{ iff } \ \text{either} \vdash \beta \text{ or } \beta \rightsquigarrow \beta \text{ and } \alpha \not\rightsquigarrow \beta. \qquad (\mathsf{Cond}\div)$$

$\mathsf{Cond}\div$ says $\beta \in \mathrm{Cn}(B \div \alpha)$ means either that $\beta$ is a tautology, or that $\beta$ is a contingent truth, $\beta \rightsquigarrow \beta$, but contraction by $\alpha$ does not lead to retraction of $\beta$, meaning $\alpha \not\rightsquigarrow \beta$.

To adapt this for belief bases, we need something along the line of the following: $\beta \in B \div \alpha$ means either that $\beta$ is a tautology in $B$, $\bar{\vdash} \beta$, or that $\beta$ is a contingent truth in $B$, $\beta \mathrel{\bar{\rightsquigarrow}} \beta$, but contraction by $\alpha$ does not lead to retraction of $\beta$ from $B$, $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$:

$$\beta \in B \div \alpha \ \text{ iff } \ \text{either} [\beta \in B \text{ and } \vdash \beta] \text{ or } [\beta \mathrel{\bar{\rightsquigarrow}} \beta \text{ and } \alpha \mathrel{\not\bar{\rightsquigarrow}} \beta]. \qquad (4.8)$$

Here, Definition 4.11 for $\bar{\vdash}$ helps to simplify the condition 4.8 into the following form (which resembles the original condition $\mathsf{Cond}\div$ for closed sets more closely):

$$\beta \in B \div \alpha \ \text{ iff } \ \text{either } \bar{\vdash} \beta \text{ or } \beta \mathrel{\bar{\rightsquigarrow}} \beta \text{ and } \alpha \mathrel{\not\bar{\rightsquigarrow}} \beta. \qquad (\mathsf{Cond}\bar{\div})$$

As it is apparent and will be discussed in depth in the upcoming sections, condition $\mathsf{Cond}\bar{\div}$ is a natural counterpart of the condition $\mathsf{Cond}\bar{\rightsquigarrow}$ [p. 57]. The former allows for the construction of a contraction operator given a base dependence relation, and the latter allows for the opposite, construction of a base dependence relation given a contraction operator.

Now, given that $\mathsf{Cond}\bar{\div}$ corresponds to $\mathsf{Cond}\bar{\rightsquigarrow}$, two other variations of it come to mind: one to correspond to $\mathsf{Cond}\hat{\rightsquigarrow}$ for strong base dependence $\hat{\rightsquigarrow}$, and another to correspond to $\mathsf{Cond}\breve{\rightsquigarrow}$ for weak base dependence $\breve{\rightsquigarrow}$. Fortunately, however, neither of these two conditions would be necessary here as we are avoiding weak base dependence in our study, as discussed in §4.4.2 on page 59. Also base dependence and strong base dependence become equivalent as a consequence of avoiding weak base dependence by Theorem 4.7. This means that we can content ourselves with Cond$\bar{\div}$ as the only condition needed in our formalism, and we need not investigate further to come up with conditions that could be labeled as Cond$\hat{\div}$ and Cond$\check{\div}$ to correspond to conditions Cond$\hat{\rightsquigarrow}$ and Cond$\breve{\rightsquigarrow}$, respectively.

Thus, to construct contractions using base dependence we only need Cond$\bar{\div}$, just as to construct base dependence using contraction we will only need Cond$\bar{\rightsquigarrow}$.

## 4.6 Base Dependence Postulates

A goal of this work is to provide an axiomatization of base dependence. As illustrated in our guiding Figure 4.1 on page 48, we also expect it to be a generalization of Fariñas and Herzig's axiomatization of dependence. A summary of their axioms and derivable principles is provided on page 41. It turns out that some of base dependence axioms closely resemble dependence axioms (e.g. Cond-ID$^B$), and some remain valid and derivable but are no longer needed as axioms (e.g. Disj$^B$). Yet, there are also some other axioms offered for base dependence (e.g. redundancy [p. 74]) that are not similar to any of the dependence axioms. The relationship between Fariñas and Herzig's axioms of dependence and the axioms of base dependence offered here is studied in §4.9 on page 87.

### 4.6.1 Basic Postulates

We start this section by offering a basic set of axioms for base dependence that does not make any explicit reference to base dependence on conjunctive statements. Next we enhance this axiomatization by further specifying characteristics of base dependence on conjunctions.

We begin with base dependence axioms closely resembling Fariñas and Herzig's dependence axioms. Let us consider dependence axiom Cond-ID [p. 40]:

If $\alpha \rightsquigarrow \beta$ then $\alpha \rightsquigarrow \alpha$ (Cond-ID [p. 40])

If there is any formula $\beta$ that depends on $\alpha$ then $\alpha$ depends on itself. Interpreting this axiom in light of Gärdenfors' preservation criterion, it means that the fact that removal of $\alpha$ results in removal of any formula is sufficient to show that it is possible to remove $\alpha$. Notice that there are two scenarios where it is not possible to contract by $\alpha$. One is when $\alpha$ is not in the belief set to begin with, $\alpha \notin \mathrm{Cn}(B)$. Then retracting $\alpha$ is impossible or even meaningless and no formula can depend on it, $\alpha \not\rightsquigarrow \beta$ for all $\beta$, including itself $\alpha \not\rightsquigarrow \alpha$. The second scenario is when $\alpha$ is in the belief set but it is a tautology. Then it cannot be contracted from the belief set, and again no formula can depend on it, $\alpha \not\rightsquigarrow \beta$ for all $\beta$, not even itself $\alpha \not\rightsquigarrow \alpha$. The contrapositive reading of Cond-ID could be helpful here. If $\alpha$ is not contractable at all $\alpha \not\rightsquigarrow \alpha$, then no $\beta$ is removed as a result of its (impossible) contraction, $\alpha \not\rightsquigarrow \beta$.

One observation here is that for dependence, which focuses only on the closed belief sets and both $\alpha, \beta \in \mathrm{Cn}(B)$, we can offer a dual axiom for Cond-ID that talks about self-dependence of $\beta$ instead of $\alpha$:

If $\alpha \rightsquigarrow \beta$ then $\beta \rightsquigarrow \beta$ (Cond-ID$^l$)

Let us consider the contrapositive: if $\beta \not\rightsquigarrow \beta$ then $\alpha \not\rightsquigarrow \beta$. Again $\beta \not\rightsquigarrow \beta$ means that it is impossible to contract $\beta$ either because it is not in the belief set $\beta \notin \mathrm{Cn}(B)$ to begin with or because $\beta$ is a tautology. In that case, contracting by any $\alpha$ will never result in retraction of $\beta$, $\alpha \not\rightsquigarrow \beta$ for all $\alpha$.

Because for dependence both $\alpha, \beta \in \mathrm{Cn}(B)$, it should be hypothetically possible to put forward other alternatives to Fariñas and Herzig's axiomatization of dependence where Cond-ID is substituted with Cond-ID$^l$. In the case of base dependence, though, this symmetry is absent because $\alpha \bar{\rightsquigarrow} \beta$ means that $\alpha \in \mathrm{Cn}(B)$ but $\beta \in B$. As such, there is no straightforward counterpart to Cond-ID for base dependence. In particular, the following is *wrong*:

If $\alpha \bar{\rightsquigarrow} \beta$ then $\alpha \bar{\rightsquigarrow} \alpha$.

This is incorrect simply because $\alpha \mathrel{\bar{\leadsto}} \alpha$ requires that $\alpha \in B$, but what $\alpha \mathrel{\bar{\leadsto}} \beta$ guarantees is only $\alpha \in \mathrm{Cn}(B)$. All that we could conclude from $\alpha \mathrel{\bar{\leadsto}} \beta$ for $\alpha$ is $\alpha \leadsto \alpha$:

If $\alpha \mathrel{\bar{\leadsto}} \beta$ then $\alpha \leadsto \alpha$.

On the contrary, it is quite straightforward to suggest a base dependence counterpart to Cond-ID$^l$:

$$\boxed{\text{If } \alpha \mathrel{\bar{\leadsto}} \beta \text{ then } \beta \mathrel{\bar{\leadsto}} \beta.} \qquad (\textsf{Cond-ID}^B)$$

This can be interpreted as follows. If retraction of some formula $\alpha$ results in retraction of $\beta$ from the base $B$, then first obviously $\beta$ must be originally in $B$ and second it cannot be a tautology. If $\beta$ is a contingent formula in the base $B$, then it can in principle be contracted from the base, so $\beta \mathrel{\bar{\leadsto}} \beta$. This confirms the expected result that a formula in a base is either tautological or contingent: for all $\beta \in B$ either $\bar{\vdash} \beta$ or $\beta \mathrel{\bar{\leadsto}} \beta$. Cond-ID$^B$ is boxed above to indicate that it turns out to be one of the axioms that we put forward for base dependence.

Let us now consider another one of Fariñas and Herzig's axioms of dependence:

If $\vdash \alpha \vee \beta$ then $\alpha \not\leadsto \beta$. (Disj [p. 40])

The base dependence counterpart of the above axiom is as follows:

If $\vdash \alpha \vee \beta$ then $\alpha \mathrel{\not\bar{\leadsto}} \beta$. (Disj$^B$)

For an interpretation of Disj$^B$ (or Disj), let us consider Gärdenfors' preservation criterion once more. No set can be contracted by tautologies, and thus no formulas are retracted because of contraction by tautologies. Therefore, $\beta$ cannot base-depend on $\alpha$, $\alpha \mathrel{\not\bar{\leadsto}} \beta$, if either one of them is a tautology. Tautologies are always independent of other formulas and no other formulas depend on them. Furthermore, contraction of a formula $\alpha$ does not retract its negation $\neg\alpha$, if present. Thus, a formula and its negation are always independent. This is also captured by Disj$^B$ (and Disj): since $\vdash \alpha \vee \neg\alpha$, we have $\alpha \mathrel{\not\bar{\leadsto}} \neg\alpha$.

It turns out that we do not need Disj$^B$ as an axiom because it will be derivable, as demonstrated in Theorem 4.13 on page 74, from another axiom that we will introduce shortly, viz., contribution [p. 74].

We now consider Def-$K$ as the next Fariñas and Herzig's axiom of dependence:

$$\alpha \in K \ \text{ iff } \ \text{either} \vdash \alpha \ \text{ or } \ \alpha \rightsquigarrow \beta \text{ for some } \beta. \qquad (\textsf{Def-}K \text{ [p. 40]})$$

We start with a straightforward notational change for Def-$K$ using belief base notation, $\alpha \in \mathrm{Cn}(B)$, instead of belief set notation, $\alpha \in K$. Because this is only a change in notation, we reuse the same name, Def-$K$:

$$\alpha \in \mathrm{Cn}(B) \ \text{ iff } \ \text{either} \vdash \alpha \text{ or } \alpha \rightsquigarrow \beta \text{ for some } \beta \qquad (\textsf{Def-}K)$$

The base dependence counterpart of Def-$K$ closely resembles it:

$$\alpha \in \mathrm{Cn}(B) \ \text{ iff } \ \text{either} \vdash \alpha \text{ or } \alpha \bar{\rightsquigarrow} \beta \text{ for some } \beta. \qquad (\textsf{Def-}B_{\mathrm{Cn}})$$

If $\alpha$ is a tautology, then it should be in the consequences of $B$, $\alpha \in \mathrm{Cn}(B)$. Also, if contracting by $\alpha$ leads to retracting some $\beta$ from the base, $\alpha \bar{\rightsquigarrow} \beta$, then $\alpha \in \mathrm{Cn}(B)$ again. Otherwise its contraction would have no affect on $B$. Conversely, if $\alpha \in \mathrm{Cn}(B)$, then either $\alpha$ is a tautology or it is a contingency which can be contracted and its contraction will retract some formula from $B$.

However, even though Def-$B_{\mathrm{Cn}}$ may be a sensible axiom, it does not turn out to be a sufficiently *informative* one. In this study, which focuses on belief bases, we are interested in the membership in the base, $\beta \in B$, more than the membership in the closure of the base, $\alpha \in \mathrm{Cn}(B)$. Thus, we would like to come up with an axiom in terms of $\beta \in B$ to correspond to Def-$K$ which is stated in terms of $\alpha \in K$.

In order to achieve this, the first observation is that there is a symmetry issue here resembling the case of Cond-ID and Cond-ID$^l$ above. Similarly, here we start by offering a symmetrical counterpart to Def-$K$ for belief sets:

$$\beta \in K \ \text{ iff } \ \text{either} \vdash \beta \ \text{ or } \ \alpha \rightsquigarrow \beta \text{ for some } \alpha \qquad (\textsf{Def-}K^l)$$

A formula $\beta \in K$ means that $\beta$ is either a tautology or a contingency, in which case it could be retracted as a result of contraction of $K$ by some $\alpha$ (where $\alpha$ could be $\beta$). It is now much easier to offer a base dependence axiom as a counterpart to Def-$K^l$:

$$\beta \in B \ \text{ iff } \ \text{either} \, [\beta \in B \text{ and } \vdash \beta] \text{ or } \alpha \bar{\rightsquigarrow} \beta \text{ for some } \alpha. \qquad (4.9)$$

To say that $\beta$ is in the base, $\beta \in B$, is equivalent to saying either that $\beta$ is a tautology in the base, $\bar{\vdash}\ \beta$, or that it is a contingency in the base, which would be retracted as a result of contraction of $B$ by some $\alpha$ (where $\alpha$ could be $\beta$). As one last notational simplification step, we substitute $[\beta \in B$ and $\vdash \beta]$ with $\bar{\vdash}\ \beta$ using Definition 4.11 to obtain the following:

$$\boxed{\beta \in B \text{ iff either } \bar{\vdash}\ \beta \text{ or } \alpha \mathrel{\bar{\rightsquigarrow}} \beta \text{ for some } \alpha.} \qquad (\mathsf{Def}\text{-}B)$$

Before carrying on, one last note about $\mathsf{Def}$-$B$ is that it makes the important connection between the base $B$ and the base dependence relation $\bar{\rightsquigarrow}$. As we will see later on in this section, there are other axioms making reference to $B$ or a subset $B' \subseteq B$, but every reference to $B$ in those axioms could be substituted using $\mathsf{Def}$-$B$. Thus, $\mathsf{Def}$-$B$ can be thought of as the only axiom containing irreplaceable explicit reference to the base $B$.

The next dependence axiom by Fariñas and Herzig we consider is $\mathsf{LE}^l$:

$$\text{If } \alpha \leftrightarrow \beta \text{ and } \alpha \rightsquigarrow \delta \text{ then } \beta \rightsquigarrow \delta. \qquad (\mathsf{LE}^l \text{ [p. 39]})$$

The base dependence counterpart of this axiom is as follows:

$$\text{If } \alpha \leftrightarrow \beta \text{ and } \alpha \mathrel{\bar{\rightsquigarrow}} \delta \text{ then } \beta \mathrel{\bar{\rightsquigarrow}} \delta. \qquad (\mathsf{LE}^B)$$

Notice that, similar to $\mathsf{LE}^l$, $\mathsf{LE}^B$ has the following equivalent form:

$$\text{If } \alpha \leftrightarrow \beta \text{ then } \alpha \mathrel{\bar{\rightsquigarrow}} \delta \text{ iff } \beta \mathrel{\bar{\rightsquigarrow}} \delta.$$

When $\alpha$ and $\beta$ are equivalent, $\alpha \leftrightarrow \beta$, whenever contracting by $\alpha$ leads to retraction of $\delta$ from the base, we can also say that contracting by $\beta$ also leads to retraction of $\delta$ from the base, and vice versa. This makes sense not only for $\alpha$ and $\beta$, but also for the formulas that are true *just because* $\alpha$ or $\beta$ are true. For example, assume that $\alpha \vee \theta$ holds just because $\alpha$ holds. Then we can say: If $\alpha \leftrightarrow \beta$ then $(\alpha \vee \theta) \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta$. To capture all such cases we propose the following:

$$\boxed{\text{If } \alpha \in \mathrm{Cn}(B') \text{ iff } \beta \in \mathrm{Cn}(B') \text{ for all } B' \subseteq B \text{ then } \alpha \mathrel{\bar{\rightsquigarrow}} \delta \text{ iff } \beta \mathrel{\bar{\rightsquigarrow}} \delta} \qquad (\mathsf{conjugation})$$

The following theorem shows that $\mathsf{conjugation}$ implies $\mathsf{LE}^B$.

**Theorem 4.12.** *If a relation* $\bar{\rightsquigarrow}$ *satisfies* conjugation*, then it also satisfies* $\mathsf{LE}^B$*.*

[Proof on page 107.]

The next base dependence axiom we offer, contribution, is not inspired by axioms of dependence, so it does not directly correspond to any of them.

$$\boxed{\text{If } \alpha \bar{\rightsquigarrow} \beta \text{ then } \alpha \notin \text{Cn}(B') \text{ and } \alpha \in \text{Cn}(B' \cup \{\beta\}) \text{ for some } B' \subseteq B}$$

(contribution)

This axiom says that $\beta$ must somehow contribute to the justification of $\alpha$ if $\beta$ base-depends on $\alpha$, or, in other words, if contracting $B$ by $\alpha$ requires retracting $\beta$ from the base. When discussing $\mathsf{Disj}^B$ [p. 71] earlier in this section, we said that it is possible to show that contribution implies $\mathsf{Disj}^B$. The following theorem does exactly that.

**Theorem 4.13.** *If a relation* $\bar{\rightsquigarrow}$ *satisfies* contribution*, then it also satisfies* $\mathsf{Disj}^B$*.*

[Proof on page 108.]

The following is the next base dependence axiom we study:

$$\boxed{\text{If } \alpha \in \text{Cn}(B') \text{ and } B' \subseteq B \text{ then either } \vdash \alpha \text{ or } \alpha \bar{\rightsquigarrow} \beta \text{ for some } \beta \in B'}$$

(modularity)

Consider a subset $B' \subseteq B$ that implies $\alpha$, $\alpha \in \text{Cn}(B')$. This could be because $\alpha$ is a tautology, but when $\nvdash \alpha$, there is some $\beta$ from the same subset $B'$ that base-depends on $\alpha$. Thus, this axiom allows considering base dependence of formulas in specific modules or compartments in $B$.

The last basic axiom for base dependence is redundancy:

$$\boxed{\text{If } \beta \in \text{Cn}(B') \text{ and } B' \subseteq B \text{ then either } \alpha \not\bar{\rightsquigarrow} \beta \text{ or } \alpha \bar{\rightsquigarrow} \delta \text{ for some } \delta \in B'}$$

(redundancy)

We have seen earlier that by Theorem 4.3 on page 60, the undesired weak base dependence can only arise from a redundant base $B$. This axiom ensures a proper handling of any redundancy that might exist in $B$. First notice that redundancy holds vacuously when $\alpha \not\bar{\rightsquigarrow} \beta$. This may be easier to see if we consider the following equivalent form of it:

If $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ and $\alpha \bar{\rightsquigarrow} \beta$ then $\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$.

To consider the principal case of redundancy, assume that: $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ and $\alpha \bar{\rightsquigarrow} \beta$. When $\beta \in B'$, $\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$ trivially holds because then $\delta$ could be $\beta$ which means $\alpha \bar{\rightsquigarrow} \beta$, which is assumed. When $\beta \notin B'$, there is some redundancy in the base $B$ because on the one hand $\beta \in B$ (as $\alpha \bar{\rightsquigarrow} \beta$), and on the other hand there is $B' \subseteq B$ such that $\beta \notin B'$ but $\beta \in \mathrm{Cn}(B')$. Thus, in order for $\alpha \bar{\rightsquigarrow} \beta$ to hold, $\alpha \bar{\rightsquigarrow} \delta$ should also hold at least for one formula $\delta \in B'$.

**Example 4.14.** The following are two examples of $\bar{\rightsquigarrow}$ violating redundancy:

- Assume $B = \{p,\, q,\, p \vee q\}$, $p \bar{\rightsquigarrow} p$, $p \not\bar{\rightsquigarrow} q$ and $p \bar{\rightsquigarrow} (p \vee q)$. Letting $B' = \{q\}$, we have $p \vee q \in \mathrm{Cn}(B')$ and $B' \subseteq B$. Since $p \bar{\rightsquigarrow} (p \vee q)$ by assumption, $p \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$ by redundancy. Yet, there is no $\delta \in B'$ such that $p \bar{\rightsquigarrow} \delta$.
- Assume $B = \{p\, p \leftrightarrow q,\, p \vee q\}$, $(p \wedge q) \bar{\rightsquigarrow} (p \vee q)$ and $(p \wedge q) \not\bar{\rightsquigarrow} p$. Let $B' = \{p\}$. Thus, $p \vee q \in \mathrm{Cn}(B')$ and $B' \subseteq B$ hold. Again, given that $(p \wedge q) \bar{\rightsquigarrow} (p \vee q)$, $(p \wedge q) \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$ by redundancy, but no such $\delta \in B'$ exists. □

### Summary

We have put forward a collection of basic properties for base dependence in this section. The definition below states that any relation that satisfies the following six basic axioms is a base dependence relation.

**Definition 4.15.** A relation $\bar{\rightsquigarrow}$ is a *base dependence* if and only if it satisfies the axioms Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity and redundancy.

Notice that so far we have not specified any criteria on how to handle conjunctions, which we will do next.

### 4.6.2 Conjunction Criterion of Dependence Factoring

As we have seen in §3.2.2 on page 38, Keynes in [Key21] holds that there is an intuitive relationship between relevance (dependence) and logical conjunction that should stay valid for any reasonable definition of relevance. Fariñas and Herzig, calling it the *Conjunction Criterion for Dependence*, CCD, formulate it as follows:

> If $\delta$ depends on $\alpha$ and $\delta$ depends on $\beta$ then $\delta$ depends on $\alpha \wedge \beta$. (CCD)

Using the $\rightsquigarrow$ notation, they then formalize CCD as:

If $\alpha \rightsquigarrow \delta$ and $\beta \rightsquigarrow \delta$ then $\alpha \wedge \beta \rightsquigarrow \delta$ (CCD$^l$ [p. 38])

Likewise, using the base dependence $\bar{\rightsquigarrow}$ notation, we have:

If $\alpha \bar{\rightsquigarrow} \delta$ and $\beta \bar{\rightsquigarrow} \delta$ then $\alpha \wedge \beta \bar{\rightsquigarrow} \delta$. (CCD$^B$)

Moreover, Gärdenfors in [Gä78] puts forward another principle that he believes should hold for any relevance relation, the *Conjunction Criterion for Independence*, CCI:

> If $\delta$ is independent of $\alpha$ and $\delta$ is independent of $\beta$ then $\delta$ is independent of $\alpha \wedge \beta$. (CCI)

As discussed, Fariñas and Herzig formalize CCI as follows:

If $\alpha \wedge \beta \rightsquigarrow \delta$ then $\alpha \rightsquigarrow \delta$ or $\beta \rightsquigarrow \delta$ (CCI$^l$ [p. 39])

Likewise, we have:

If $\alpha \wedge \beta \bar{\rightsquigarrow} \delta$ then $\alpha \bar{\rightsquigarrow} \delta$ or $\beta \bar{\rightsquigarrow} \delta$. (CCI$^B$)

CCD and CCI state intuitions regarding dependence on conjunctions in the form of *conditional* statements. One wonders whether it is possible to capture such intuitions regarding dependence on conjunctions using *equivalences*. Such a statement would have to capture different cases. That is, for any reasonable dependence relation, at least one of the following statements hold:

**Case 1:** The set of formulas that depend on $\alpha \wedge \beta$ is *the same as* the set of those that depend on $\alpha$

**Case 2:** The set of formulas that depend on $\alpha \wedge \beta$ is *the same as* the set of those that depend on $\beta$

**Case 3:** The set of formulas that depend on $\alpha \wedge \beta$ is *the same as* the set of those that depend on $\alpha$ *or* depend on $\beta$

Using set notation, these cases can be stated as follows:

$$\begin{aligned} \text{Either } & \{\delta \mid \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\delta \mid \alpha \mathrel{\bar{\rightsquigarrow}} \delta\}, \text{ or} \\ & \{\delta \mid \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\delta \mid \beta \mathrel{\bar{\rightsquigarrow}} \delta\}, \text{ or} \\ & \{\delta \mid \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\delta \mid \alpha \mathrel{\bar{\rightsquigarrow}} \delta\} \cup \{\delta \mid \beta \mathrel{\bar{\rightsquigarrow}} \delta\} \end{aligned} \tag{4.10}$$

or equivalently,

$$\begin{aligned} \text{Either } & [\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_1 \text{ iff } \alpha \mathrel{\bar{\rightsquigarrow}} \delta_1], \text{ or} \\ & [\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_2 \text{ iff } \beta \mathrel{\bar{\rightsquigarrow}} \delta_2], \text{ or} \\ & [\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_3 \text{ iff } \alpha \mathrel{\bar{\rightsquigarrow}} \delta_3 \text{ or } \beta \mathrel{\bar{\rightsquigarrow}} \delta_3]. \end{aligned} \tag{$\mathsf{CCDF}^B$}$$

Each line of $\mathsf{CCDF}^B$ needs to use a unique variable name $\delta_1$, $\delta_2$ and $\delta_3$ because, in each line of (4.10), $\{\delta \mid \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta\}$ refers to a different set.

$\mathsf{CCDF}^B$ is a formalization of the intuition expressed in the three cases above, which we restate more concisely as follows, calling it the *Conjunction Criterion of Dependence Factoring*, CCDF:

| The set of all formulas that depend on $\alpha \wedge \beta$ is the same as the set of all formulas that depend on $\alpha$, or on $\beta$, or on either of them. |
|---|

(CCDF)

Indeed, CCDF may be considered as a third maxim for dependence of conjunctions in addition to Keynes' CCD and Gärdenfors' CCI.

As a side note, although it seems that the third clause of $\mathsf{CCDF}^B$ should be redundant, in light of the first two, in fact it isn't.

**Example 4.16.** Assume $\alpha$, $\beta$, $\theta_1$, $\theta_2$ and $\theta_3$ are formulas and $\bar{\rightsquigarrow}$ is a relation such that

$$\begin{array}{lll} \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \theta_1 & \alpha \mathrel{\bar{\rightsquigarrow}} \theta_1 & \beta \mathrel{\not\bar{\rightsquigarrow}} \theta_1 \\ \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \theta_2 & \alpha \mathrel{\not\bar{\rightsquigarrow}} \theta_2 & \beta \mathrel{\bar{\rightsquigarrow}} \theta_2 \\ \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \theta_3 & \alpha \mathrel{\bar{\rightsquigarrow}} \theta_3 & \beta \mathrel{\bar{\rightsquigarrow}} \theta_3. \end{array}$$

Clearly, $\bar{\rightsquigarrow}$ violates the first two clauses of $\mathsf{CCDF}^B$, but not the third one. This may be easier to see using (4.10). Note that $\{\delta \mid \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\theta_1, \theta_2, \theta_3\}$, $\{\delta \mid \alpha \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\theta_1, \theta_3\}$ and $\{\delta \mid \beta \mathrel{\bar{\rightsquigarrow}} \delta\} = \{\theta_2, \theta_3\}$, which satisfy the third clause of (4.10) but not the first two. □

As a second note, $\mathsf{CCDF}^B$ is only one way of formalizing $\mathsf{CCDF}$, using base dependence relation; of course, it can also be formalized using Fariñas and Herzig's dependence relation as shown below, which we call $\mathsf{CCDF}^l$:

$$\begin{array}{ll} \text{Either} & [\alpha \wedge \beta \rightsquigarrow \delta_1 \ \text{ iff } \ \alpha \rightsquigarrow \delta_1], \text{ or} \\ & [\alpha \wedge \beta \rightsquigarrow \delta_2 \ \text{ iff } \ \beta \rightsquigarrow \delta_2], \text{ or} \\ & [\alpha \wedge \beta \rightsquigarrow \delta_3 \ \text{ iff } \ \alpha \rightsquigarrow \delta_3 \text{ or } \beta \rightsquigarrow \delta_3]. \end{array} \tag{$\mathsf{CCDF}^l$}$$

Finally, an important observation here is that $\mathsf{CCDF}^B$ is a more specific criterion than $\mathsf{CCD}^B$ and $\mathsf{CCI}^B$, and it implies both of them, as stated in the following theorem.

**Theorem 4.17.** *If a relation* $\bar{\rightsquigarrow}$ *satisfies* $\mathsf{CCDF}^B$*, then it also satisfies both* $\mathsf{CCD}^B$ *and* $\mathsf{CCI}^B$*:*

$$\text{If } \alpha \mathrel{\bar{\rightsquigarrow}} \delta \text{ and } \beta \mathrel{\bar{\rightsquigarrow}} \delta \text{ then } \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta \tag{$\mathsf{CCD}^B$}$$

$$\text{If } \alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta \text{ then } \alpha \mathrel{\bar{\rightsquigarrow}} \delta \text{ or } \beta \mathrel{\bar{\rightsquigarrow}} \delta \tag{$\mathsf{CCI}^B$}$$

[Proof on page 109.]

Notice that although Theorem 4.17 is stated in terms of base dependence $\bar{\rightsquigarrow}$, it does not have to be. Indeed, the theorem (and its proof) may straightforwardly be restated in terms of $\mathsf{CCDF}$ that implies both $\mathsf{CCD}$ and $\mathsf{CCI}$. As such, the dependence version of $\mathsf{CCDF}$, i.e. $\mathsf{CCDF}^l$, also implies both Fariñas and Herzig's $\mathsf{CCD}^l$ and $\mathsf{CCI}^l$.

The following are all the conjunction criteria and related axioms we have discussed:

| Conjunction Criterion | Dependence Axioms | Base Dependence Axioms |
|---|---|---|
| CCD (Keynes) | $\mathsf{CCD}^l$ (Fariñas and Herzig) | $\mathsf{CCD}^B$ |
| CCI (Gärdenfors) | $\mathsf{CCI}^l$ (Fariñas and Herzig) | $\mathsf{CCI}^B$ |
| CCDF | $\mathsf{CCDF}^l$ | $\mathsf{CCDF}^B$ |

Also, please see §5.1 on page 93 for a listing of all base dependence axioms and related conditions.

## 4.7 Saturated Kernel Contraction and Base Dependence

### 4.7.1 Overview

Given our high-level goal depicted in Figure 4.1 on page 48, we need a suitable base contraction that corresponds to our base dependence relation, for which saturated kernel contraction was proposed as a candidate in §4.3. One interesting property of saturated kernel contraction is that it is a reversible generalization of AGM contraction: it can handle closed and non-closed belief bases, and it reduces to AGM contraction in the special case when the belief base is closed. Moreover, it satisfies relative closure which is important to avoid weak base dependence. (See Theorem 4.5 on page 62.)

Now, we need to actually *prove* the correspondence between saturated kernel contraction and base dependence. One note, however, is that standard saturated kernel contraction has no axioms for conjunctions, and such axioms need to be added as necessary. To make this task more manageable, we aim to achieve our final goal in two steps. In this section, we prove the correspondence between saturated kernel contraction and base dependence without considering any axioms for conjunctions. This step is illustrated in Figure 4.4a on page 83. In contrast, Figure 4.4b shows the next step explored in §4.8 where conjunction axioms are augmented both to base dependence and to saturated kernel contraction.

Thus, for this section, to prove the correspondence between base dependence and saturated kernel contraction with no axioms for conjunctions, we need to show that under the right conditions (viz., Cond$\bar{\rightsquigarrow}$ and inclusion by Characterization Theorem 4.23 on page 83) the following two sets of axioms are equivalent:

**Saturated Kernel Contraction Axioms:**
success, inclusion, core-retainment, uniformity and relative closure. (See §2.9.)

**Base Dependence Axioms:**
Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity and redundancy. (See §4.6.)

### 4.7.2 From Base Dependence to Contraction

To construct a contraction operator $\div$, assume all the following are present:

- A base dependence relation $\bar{\rightsquigarrow}$ (Definition 4.15).
- A list of tautologies present in the base $T \subseteq B$ where $T = \{\beta \mid \bar{\vdash}\ \beta\}$.
- The Cond$\bar{\div}$:

$$\beta \in B \div \alpha \ \text{ iff } \ \text{either } \bar{\vdash}\ \beta \text{ or } \beta \bar{\rightsquigarrow} \beta \text{ and } \alpha \not\bar{\rightsquigarrow} \beta. \qquad (\text{Cond}\bar{\div} \text{ [p. 68]})$$

We do *not* need to assume that $B$ is provided because it can be obtained using $\bar{\rightsquigarrow}$ and $\bar{\vdash}$, as discussed in §4.5.1 on page 66.

To show that the obtained contraction operator $\div$ is indeed a saturated kernel contraction, in Theorem 4.19, we show that all of the axioms of saturated kernel contraction hold.

Before that though we need one small and handy lemma, basically showing that

$$\alpha \not\bar{\rightsquigarrow} \beta \text{ for all } \beta \in B \div \alpha.$$

This can equivalently be expressed as, if $\beta \in B \div \alpha$ then $\alpha \not\bar{\rightsquigarrow} \beta$, meaning that if $\beta$ is not affected by the contraction of $\alpha$ then it does not depend on it. Indeed, it is a rather trivial principle, specially in the presence of Cond$\bar{\rightsquigarrow}$ [p. 57]. However, it may be less obvious to see how this also holds when Cond$\bar{\div}$ is given instead of Cond$\bar{\rightsquigarrow}$, which will be the case in Theorem 4.19. The following lemma is meant to help with that.

**Lemma 4.18.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\div}$ *holds and* contribution *is satisfied, it also holds that* $\alpha \not\bar{\rightsquigarrow} \beta$ for all $\beta \in B \div \alpha$.

[Proof on page 110.]

**Theorem 4.19** (Base Dependence to Contraction)**.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\div}$ *holds, if* $\bar{\rightsquigarrow}$ *is a base dependence, then* $\div$ *is a saturated kernel contraction.*

[Proof on page 110.]

### 4.7.3 From Contraction to Base Dependence

This section shows how to obtain a base dependence $\bar{\rightsquigarrow}$ relation given a saturated kernel contraction operator $\div$. We assume all the following are present:

- A saturated kernel contraction operator $\div$.
- The Cond$\bar{\rightsquigarrow}$:

$$\alpha \bar{\rightsquigarrow} \beta \quad \text{iff} \quad \beta \in B \text{ and } \beta \notin B \div \alpha. \qquad (\text{Cond}\bar{\rightsquigarrow} \text{ [p. 57]})$$

Theorem 4.20 states that, given the above assumptions, all axioms of base dependence $\bar{\rightsquigarrow}$ relation are satisfied.

**Theorem 4.20** (Contraction to Base Dependence)**.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\rightsquigarrow}$ *holds, if* $\div$ *is a saturated kernel contraction, then* $\bar{\rightsquigarrow}$ *is a base dependence.*

[Proof on page 115.]

### 4.7.4 Axiomatic Characterization

We now need an axiomatic characterization theorem to establish that there is a mutual relationship between base dependence and belief contraction. We would like our characterization theorem to state something along the line of the following:

> *Let the relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *be such that* Cond$\bar{\rightsquigarrow}$ *holds. Then,* $\div$ *is a saturated kernel contraction if and only if* $\bar{\rightsquigarrow}$ *is a base dependence.*

Unfortunately, just as it was the case for Fariñas and Herzig (see Remark 3.7 on page 43), there remains one obstacle to establish an axiomatic characterization in the above-mentioned simple form. The left to right direction has already been established in Theorem 4.20. The problem surfaces when attempting to achieve the right to left direction. That is, it is *not* possible to say that $\div$ is a saturated kernel contraction if $\bar{\rightsquigarrow}$ is a base dependence where $\div$ and $\bar{\rightsquigarrow}$ are such that $\mathsf{Cond}\bar{\rightsquigarrow}$ holds. To be able to say this we need to also assume that $\div$ satisfies $\mathsf{inclusion}$, $B \div \alpha \subseteq B$. Thus, in the following, we adopt the Fariñas and Herzig assumption in Remark 3.7 on page 43 with slight modifications to make it suitable for base dependence:

**Remark 4.21.** In order to establish an axiomatic characterization based on $\mathsf{Cond}\bar{\rightsquigarrow}$, it is assumed that the relation $\div$ satisfies $\mathsf{inclusion}$, $B \div \alpha \subseteq B$.

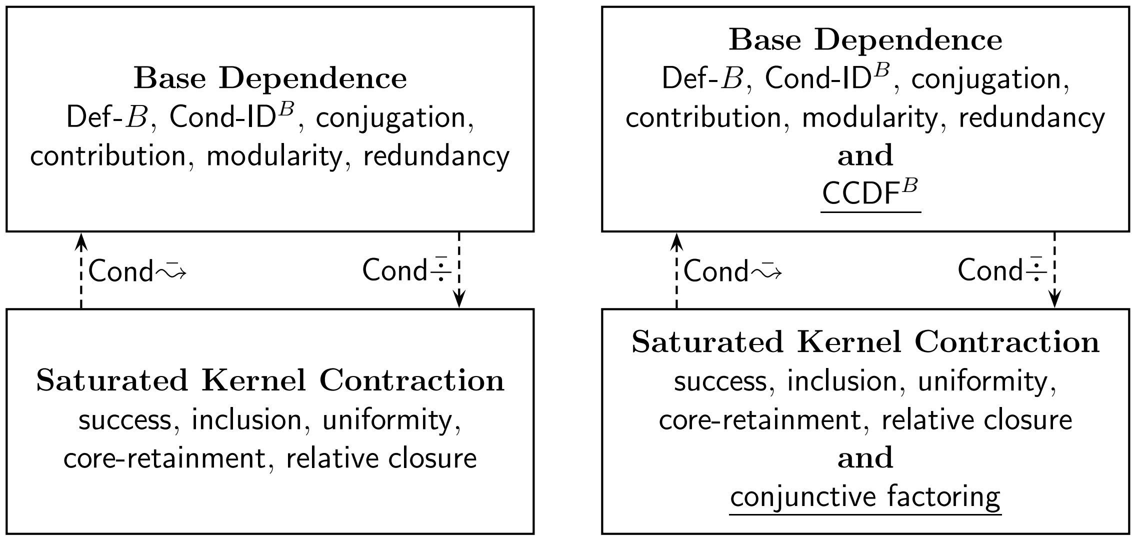
The rationale for this assumption is as follows. When constructing the $\bar{\rightsquigarrow}$ relation using a contraction operation via $\mathsf{Cond}\bar{\rightsquigarrow}$, the set of all $\beta$ such that $\alpha \bar{\rightsquigarrow} \beta$ is equal to those $\beta \in B$ and $\beta \notin B \div \alpha$, or using set difference notation $\beta \in B \setminus (B \div \alpha)$. We know as a matter of fact that $B \div \alpha \subseteq B$ holds because, by Definition 2.12 on page 28, any contraction operator satisfies $\mathsf{inclusion}$. However, even if, for the sake of argument, $\div$ did not satisfy $\mathsf{inclusion}$ and there were some statements in $B \div \alpha$ that were not in $B$, such statements would have been lost in the set difference $\beta \in B \setminus (B \div \alpha)$. That, in turn, means that to use $\bar{\rightsquigarrow}$ to construct a contraction $\div$ via $\mathsf{Cond}\bar{\rightsquigarrow}$, we do not have enough information to prove or disprove $\mathsf{inclusion}$. Instead, we have to assume that $\div$ already satisfies $\mathsf{inclusion}$. Since all contraction operations satisfy $\mathsf{inclusion}$, this assumption is not a serious loss of generality.

In summary, the characterization theorem needs to assume $\mathsf{inclusion}$. Before the characterization theorem though, we offer the following lemma that simplifies its proof:

**Lemma 4.22.** *In the presence of* $\mathsf{Def}\text{-}B$, $\mathsf{Cond}\text{-}\mathsf{ID}^B$ *and* $\mathsf{contribution}$, *the following is equivalent to* $\mathsf{Cond}\bar{\div}$*:*

$$\beta \in B \div \alpha \ \text{ iff } \ \beta \in B \text{ and } \alpha \not\bar{\rightsquigarrow} \beta.$$

[Proof on page 120.]




(a) Correspondence between *Saturated Kernel Contraction* and *Base Dependence* (without any specific criterion for conjunction).

(b) Adding *Conjunction Criterion for Dependence Factoring*, CCDF$^B$, which also implies CCD$^B$ and CCI$^B$. Added axioms are underlined.

Figure 4.4: Belief Change and Base Dependence (a) without or (b) with conjunction.

**Theorem 4.23** (Characterization)**.** *Let the relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *be such that* $\div$ *satisfies* inclusion, $B \div \alpha \subseteq B$, *and that* Cond$\bar{\rightsquigarrow}$ *holds:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$. *Then,*

$\div$ *is a saturated kernel contraction if and only if* $\bar{\rightsquigarrow}$ *is a base dependence.*

[Proof on page 121.]

The completion of the parallelism between belief change and base dependence, depicted in Figure 4.4a, concludes this section. In the next section, this parallelism is further enhanced by also considering conjunctions, as shown in Figure 4.4b.

## 4.8 Enhancements for Handling Conjunctions

### 4.8.1 Overview

Comparing Figure 4.1 on page 48 as a guide for our final goal with Figure 4.4a, which summarizes what has been achieved so far, helps to show how much progress we have made and what needs our attention now. In §4.7, we established the correspondence between saturated kernel contraction and base dependence. However, neither saturated kernel contraction nor base dependence specify how to appropriately handle conjunctions. In this section, we set to enhance the model developed so far by adding conjunction axioms both to base dependence and to saturated kernel contraction.

The results of this enhancement is schematically depicted in Figure 4.4b, which basically shows that the base dependence axioms are augmented by $\mathsf{CCDF}^B$ [p. 77] and the saturated kernel contraction axioms by conjunctive factoring [p. 21]. These supplementary axioms are selected considering a set of guiding criteria that more or less dictate what the supplementary axioms can be or at least narrow down the set of possibilities significantly. The following is a list of such criteria along with explanations why $\mathsf{CCDF}^B$ and/or conjunctive factoring axioms can fulfill each criterion.

***Criterion 1:*** As stated in the problem definition in §4.1.1 on page 47, while base dependence generalizes Fariñas and Herzig's dependence, it is desirable that base dependence preserves this interesting characteristic of dependence in that it satisfies both Keynes' conjunction criterion for dependence, CCD [p. 38], and Gärdenfors' conjunction criterion for independence, CCI [p. 39].

*Fulfillment:* For a base dependence relation $\tilde{\leadsto}$, $\mathsf{CCD}^B$ [p. 76] represents CCD, and $\mathsf{CCI}^B$ [p. 76] represents CCI. Theorem 4.17 on page 78 shows that if $\tilde{\leadsto}$ satisfies $\mathsf{CCDF}^B$, then it also satisfies both $\mathsf{CCD}^B$ and $\mathsf{CCI}^B$.

***Criterion 2:*** Based on Figure 4.1 on page 48, there needs to be a mutual correspondence between the two sets of supplementary axioms for base dependence and for saturated kernel contraction.

*Fulfillment:* Theorems 4.24 and 4.26 on the following page establish the mutual correspondence between $\mathsf{CCDF}^B$ [p. 77] and conjunctive factoring [p. 21].

***Criterion 3:*** As seen in §4.3, saturated kernel contraction is a reversible generalization of the basic AGM contraction (satisfying $K\div 1$–$K\div 6$) and they coincide with each other in the special case of closed belief bases (belief sets). This should continue to be the case for the augmented contractions. That is, that augmented saturated kernel contraction (with the new conjunction axioms) should also coincide with the full AGM contraction satisfying all $K\div 1$–$K\div 6$, $K\div 7$ and $K\div 8$. After all, Fariñas and Herzig's main characterization theorem uses the full AGM contraction, $K\div 1$–$K\div 8$.

*Fulfillment:* Indeed by Theorem 2.2 on page 21 [AGM85], in the presence of $K\div 1$–$K\div 6$ and for logically closed sets, the two axioms $K\div 7$ and $K\div 8$ (also known as conjunctive overlap and conjunctive inclusion, respectively) hold if and only if conjunctive factoring holds. Based on this theorem, for logically closed sets, a full AGM contraction satisfying $K\div 1$–$K\div 8$ is equivalent to a saturated kernel contraction that satisfies conjunctive factoring (see Lemma 4.30).

Therefore, $\mathsf{CCDF}^B$ and conjunctive factoring meet all our major criteria for enhancing the basic model achieved in the previous section. As such, this section establishes the correspondence between base dependence and saturated kernel contraction both augmented with conjunction axioms. More specifically, we show that under the right conditions the following two sets of axioms are equivalent:

**Saturated Kernel Contraction Axioms, Augmented for Conjunction:**
success, inclusion, core-retainment, uniformity, relative closure and conjunctive factoring.

**Base Dependence Axioms, Augmented for Conjunction:**
Def-$B$, Cond-$\mathsf{ID}^B$, conjugation, contribution, modularity redundancy and $\mathsf{CCDF}^B$.

### 4.8.2 From Base Dependence to Contraction

We start by showing that when $\mathsf{Cond}\bar{\div}$ holds, $\mathsf{CCDF}^B$ for base dependence leads to conjunctive factoring for base contraction. This paves the way for Theorem 4.25 which is the main theorem for this subsection.

**Theorem 4.24.** *Given relations $\bar{\rightsquigarrow}$ and $\div$ for base $B$ such that* $\mathsf{Cond}\bar{\div}$ *holds, if $\bar{\rightsquigarrow}$ satisfies* $\mathsf{CCDF}^B$ *then $\div$ satisfies* conjunctive factoring.

[Proof on page 122.]

**Theorem 4.25** (Base Dependence to Contraction)**.** *Given relations $\bar{\rightsquigarrow}$ and $\div$ for base $B$ such that* $\mathsf{Cond}\bar{\div}$ *holds, if $\bar{\rightsquigarrow}$ is a base dependence that satisfies* $\mathsf{CCDF}^B$*, then $\div$ is a saturated kernel contraction that satisfies* conjunctive factoring.

[Proof on page 123.]

### 4.8.3 From Contraction to Base Dependence

For the opposite direction, we show that when $\mathsf{Cond}\bar{\rightsquigarrow}$ holds, conjunctive factoring for base contraction implies $\mathsf{CCDF}^B$ for base dependence. Again, this paves the way for Theorem 4.27 which is the main theorem for this subsection.

**Theorem 4.26.** *Given relations $\bar{\rightsquigarrow}$ and $\div$ for base $B$ such that* $\mathsf{Cond}\bar{\rightsquigarrow}$ *holds, if $\div$ satisfies* conjunctive factoring *then $\bar{\rightsquigarrow}$ satisfies* $\mathsf{CCDF}^B$.

[Proof on page 124.]

**Theorem 4.27** (Contraction to Base Dependence)**.** *Given relations $\bar{\rightsquigarrow}$ and $\div$ for base $B$ such that* $\mathsf{Cond}\bar{\rightsquigarrow}$ *holds, if $\div$ is a saturated kernel contraction that satisfies* conjunctive factoring*, then $\bar{\rightsquigarrow}$ is a base dependence that satisfies* $\mathsf{CCDF}^B$.

[Proof on page 125.]

As it was discussed at the beginning of the section, it is noteworthy here that, for logically closed sets, conjunctive factoring holds if and only if conjunctive overlap ($K\div\mathsf{7}$) and conjunctive inclusion ($K\div\mathsf{8}$) hold, by Theorem 2.2 on page 21 [AGM85].

### 4.8.4 The Main Characterization Theorem

Finally, we can present the main characterization theorem here. The introduction of axioms for conjunctions is the only difference between this theorem and the previous Characterization Theorem 4.23 in §4.7.4. Figure 4.4 highlights this difference, making a side by side comparison.

One note is that Remark 4.21 on page 82 for Characterization Theorem 4.23 is also applicable here. That is, for the same reasons elaborated in §4.7.4, and just as it was the case for Fariñas and Herzig (see Remark 3.7 on page 43), we have to assume that the $\div$ relation satisfies inclusion, $B \div \alpha \subseteq B$, in order to state a characterization theorem based on $\mathsf{Cond}\bar{\rightsquigarrow}$.

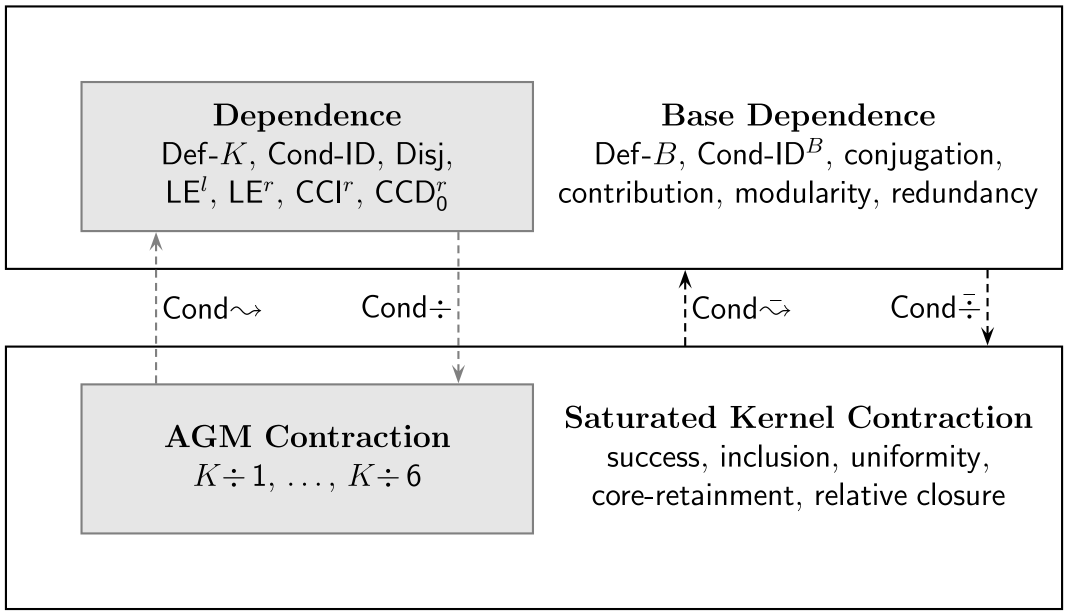
**Theorem 4.28** (Main Characterization)**.** *Let the relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *be such that* $\div$ *satisfies* inclusion, $B \div \alpha \subseteq B$, *and that* $\mathsf{Cond}\bar{\rightsquigarrow}$ *holds:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$. *Then,*

$\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring *if and only if* $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* $\mathsf{CCDF}^B$.

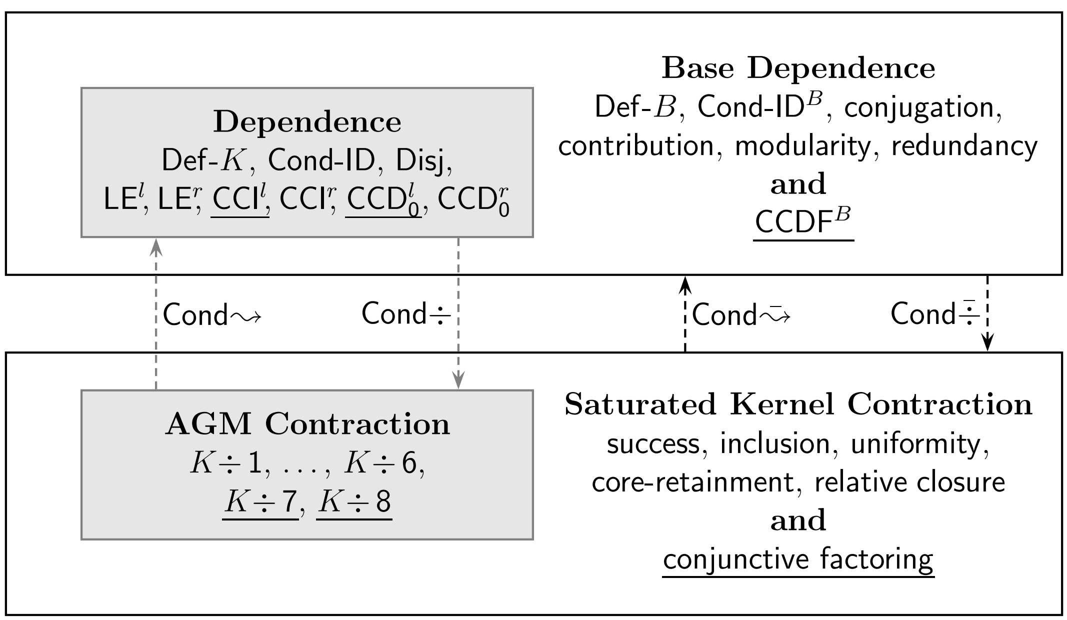
[Proof on page 125.]

## 4.9 Base Dependence as a Reversible Generalization of Dependence

Our final goal is to generalize Fariñas and Herzig's dependence relation for closed sets into a new relation, base dependence, for both closed and non-closed sets. More specifically, we would like base dependence to be a reversible generalization of dependence. This generalization happens simultaneously from both descriptive and constructive perspectives. The former means that the set of axioms for base dependence is a reversible generalization of the set of axioms for dependence. The latter means that the construction method of base dependence is also a reversible generalization of the construction of dependence. In other words, in the special case when a belief base is logically closed, two things will happen: the set of axioms for base dependence is




(a) Generalization with no specific criterion for conjunctions.




(b) Generalization satisfying Conjunction Criterion for Dependence Factoring, $\mathsf{CCDF}^B$ (which also implies $\mathsf{CCD}^B$ and $\mathsf{CCI}^B$). The added axioms are underlined.

Figure 4.5: Belief Change and Base Dependence: A Generalization of the relationship between Belief Change and Dependence, (a) without or (b) with conjunction criteria.

equivalent to the set of axioms for dependence, and also the construction method for base dependence produces the same relation as the construction method for dependence.

Let us for one last time go back to our guiding Figure 4.1 on page 48, as well as what has been achieved so far demonstrated in Figure 4.4 on page 83. Using Figure 4.4b to fill in all the blanks in the guiding Figure 4.1, it is not hard to come up with Figure 4.5b. Indeed, Figure 4.5b summarizes what we are after, which was just reviewed above, and if hypothetically it turns out to be accurate, then it will conclude our search for a generalization solution. Figure 4.5a represents another hypothetical solution similar to Figure 4.5b but without considering conjunctions.

To show that both of the hypothetical solutions represented in Figure 4.5 are actually true, we need to show for each one that for logically closed belief bases the outer boxes coincide with the inner boxes. For instance, in Figure 4.5a, for closed sets, saturated kernel contraction needs to be equivalent to basic AGM contraction, and base dependence needs to be equivalent to dependence. On the other hand, one of the characterization theorems offered by Fariñas and Herzig, Theorem 3.9 on page 44, specifies the conditions under which basic AGM contraction is logically equivalent to dependence. Similarly, a corresponding characterization theorem studied earlier in this chapter, Theorem 4.23, specifies the conditions under which saturated kernel contraction is logically equivalent to base dependence.

To summarize, in order to prove that what is represented in Figure 4.5a is a solution for our generalization problem, we need to show that for closed sets and under the right conditions all the following are logically equivalent: saturated kernel contraction, basic AGM contraction, base dependence and dependence. The following Theorem 4.29 establishes this without any specific criterion for conjunction. Generalization satisfying criteria for conjunctions is proved next by Theorem 4.31.

One last note is that, interestingly, all the building blocks necessary for these theorems and lemma are already provided either by other authors or in this work. In particular, we make heavy use of the valuable contributions in other works, viz., Theorems 2.2 ([AGM85]), 2.19 ([Han95]), 3.8 and 3.9 ([FdCH96]).

**Theorem 4.29** (Dependence Generalization). *Let relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* $B$ *be such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* inclusion *is satisfied. In the special case where* $B$ *is logically closed,*

(1) *the following are logically equivalent:*

- a) $\bar{\rightsquigarrow}$ *is a base dependence, which satisfies* Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity *and* redundancy
- b) $\rightsquigarrow$ *is a dependence that satisfies* Def-$K$, Cond-ID, Disj, LE$^l$, LE$^r$, CCI$^r$ *and* CCD$^r_0$
- c) $\div$ *is a saturated kernel contraction, which satisfies* success, inclusion, core-retainment, uniformity *and* relative closure
- d) $\div$ *is a basic AGM contraction, which satisfies* $K\div 1$ – $K\div 6$

(2) *if any one of* 1.a–1.d *above hold, then* $\bar{\rightsquigarrow}$ *reduces to* $\rightsquigarrow$*:*

$$\alpha \bar{\rightsquigarrow} \beta \text{ iff } \alpha \rightsquigarrow \beta.$$

[Proof on page 126.]

In the theorem above, the given list of axioms for the dependence relation $\rightsquigarrow$ does not include all the 9 axioms that Fariñas and Herzig have put forward for dependence. Rather, it is a subset of their axioms that corresponds to basic AGM contraction satisfying $K\div 1$–$K\div 6$ (see Theorem 3.9). To be able to account for $K\div 7$ and $K\div 8$ as well, we need to use all of their 9 axioms that correspond to full AGM contraction, $K\div 1$–$K\div 8$. This in turn means that we also need a base contraction that, for closed sets, is equivalent to full AGM contraction. This is shown to be the case, in the following lemma, for saturated kernel contraction that satisfies conjunctive factoring.

**Lemma 4.30.** *In the special case where base* $B$ *is logically closed, an operator* $\div$ *on* $B$ *is an AGM contraction satisfying* $K\div 1$ – $K\div 6$, $K\div 7$ and $K\div 8$ *if and only if* $\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring.

[Proof on page 127.]

Now everything is in place to extend the formalism for a base dependence relation $\bar{\rightsquigarrow}$ that satisfies CCDF$^B$. This is significant because, as it was shown in Theorem 4.17, satisfying CCDF$^B$ allows $\bar{\rightsquigarrow}$ to meet both Keynes' conjunction criterion for dependence

(CCD) and Gärdenfors' conjunction criterion for independence (CCI) just it was the case for Fariñas and Herzig's dependence relation $\leadsto$.

**Theorem 4.31** (Dependence Generalization with Conjunction)**.** *Let relations* $\bar{\leadsto}$, $\leadsto$ *and* $\div$ *for base* $B$ *be such that* Cond$\bar{\leadsto}$ *and* Cond$\leadsto$ *hold and* inclusion *is satisfied. In the special case where* $B$ *is logically closed,*

(1) *the following are logically equivalent:*

- a) $\bar{\leadsto}$ *is a base dependence that satisfies* Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity, redundancy *and* CCDF$^B$
- b) $\leadsto$ *is a dependence that satisfies* Def-$K$, Cond-ID, Disj, LE$^l$, LE$^r$, CCI$^l$, CCI$^r$, CCD$_0^l$ *and* CCD$_0^r$
- c) $\div$ *is a saturated kernel contraction that satisfies* success, inclusion, core-retainment, uniformity, relative closure *and* conjunctive factoring
- d) $\div$ *is an AGM contraction, which satisfies* $K\div 1$ – $K\div 6$, $K\div 7$ *and* $K\div 8$

(2) *if any one of* 1.a–1.d *above hold, then* $\bar{\leadsto}$ *reduces to* $\leadsto$*:*

$\alpha \bar{\leadsto} \beta$ iff $\alpha \leadsto \beta$.

[Proof on page 128.]

## 4.10 Conclusion

In this chapter we have achieved our general aim for this work: To provide a formalism of Gärdenfors' preservation criterion such that it generalizes the dependence formalism studied by Fariñas and Herzig so that

- it works for belief bases (and belief sets), and
- in the special case when a belief base is closed, the generalized dependence is equivalent to the original Fariñas and Herzig dependence relation, and
- while generalizing Fariñas and Herzig's work, it preserves some of the important characteristics of their study such as Keynes' conjunction criterion for dependence (CCD) and Gärdenfors' conjunction criterion for independence (CCI).

In the next chapter, we provide a more elaborate summary of the work and a listing of new contributions of this research.

# Chapter 5

# Conclusion and Future Work

## 5.1 Summary

Linking belief change and dependence can be of great value because, for example, it can narrow the number of formulas that need to be considered during a belief change operation. This, in turn, can greatly improve the performance of the operation. Gärdenfors' preservation criterion suggests a particularly interesting way of establishing this link. One successful work of great *theoretical* value based on Gärdenfors' preservation criterion is that of Fariñas and Herzig that focuses on the relationship between dependence and AGM theory contraction.

In the present work, we take a natural next step of finding a similar connection between dependence and *belief base contraction* that can have important *practical* consequences. We call such a dependence relation *base dependence*. Since, belief bases, which can be closed or non-closed, are a generalization of belief sets, it would be nice if their corresponding dependence relation, base dependence, also turns out to be a generalization of Fariñas and Herzig's dependence relation.

In this work, we establish such a connection between belief base contraction and base dependence. That is, we provide an axiomatization of base dependence, and establish its relation to belief base contraction. Similar to the set of axioms suggested by Fariñas and Herzig, the base dependence axioms are also meant to capture the dependence among formulas, only for base dependence the formulas are from the base, which may or may not be closed. Thus base dependence generalizes dependence.

More interestingly, base dependence turns out to be a *reversible generalization* of dependence. That is, we prove that in the special case that a belief base is deductively closed (i.e., it is a belief set), the base dependence relation reduces to the original Fariñas and Herzig's dependence relation.

What sets apart Fariñas and Herzig's approach from that of other authors (see the Related Work below) is its integration into the AGM model, being closely intertwined with AGM contraction. This in turn means that their work provides a theoretical limit for other approaches trying to capture or approximate concepts of relevance and dependence in the context of belief change. By generalizing their work, our approach inherits this useful property for both belief bases and belief sets.

On a separate note, another interesting characteristic of Fariñas and Herzig's axioms for dependence is that some of them are based on intuitions stated by previous authors working on the notion of relevance and dependence such as Keynes [Key21] and Gärdenfors [Gä78, Gä90]. More specifically, their dependence axiomatization meets both Keynes' conjunction criterion for dependence, CCD [p. 38], and Gärdenfors' conjunction criterion for independence, CCI [p. 39].

Not only does our base dependence generalization preserves this characteristic of dependence, but we also go one step further and provide a more specific intuition called *conjunction criterion of dependence factoring*, CCDF, that encompasses both Keynes' CCD and Gärdenfors' CCI intuitions.

## List of Postulates and Conditions

The following are axioms of base dependence and conditionals for mutual construction of base dependence and base contraction.

Base Dependence:

$$\alpha \bar{\rightsquigarrow} \beta \quad \text{iff} \quad \beta \in B \text{ and } \beta \notin B \div \alpha \qquad (\mathsf{Cond}\bar{\rightsquigarrow})$$

Strong Base Dependence:

$\alpha \mathrel{\hat{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ (Cond$\hat{\rightsquigarrow}$)

Weak Base Dependence:

$\alpha \mathrel{\breve{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \mathrm{Cn}(B \div \alpha)$ (Cond$\breve{\rightsquigarrow}$)

Contraction using Base Dependence:

$\beta \in B \div \alpha$ iff either $\bar{\vdash} \beta$ or $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ and $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ (Cond$\bar{\div}$)

Base Dependence Axioms:

$\beta \in B$ iff either $\bar{\vdash} \beta$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\alpha$ (Def-$B$)

If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ (Cond-ID$^B$)

If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta$ (conjugation)

If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ (contribution)

If $\alpha \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then
either $\vdash \alpha$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\beta \in B'$ (modularity)

If $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then
either $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ for some $\delta \in B'$ (redundancy)

Base Dependence Axioms with Conjunctions:

If $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ and $\beta \mathrel{\bar{\rightsquigarrow}} \delta$ then $\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta$ (CCD$^B$)

If $\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta$ then $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ or $\beta \mathrel{\bar{\rightsquigarrow}} \delta$ (CCI$^B$)

Either $[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_1$ iff $\alpha \mathrel{\bar{\rightsquigarrow}} \delta_1]$, or
$[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_2$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta_2]$, or (CCDF$^B$)
$[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_3$ iff $\alpha \mathrel{\bar{\rightsquigarrow}} \delta_3$ or $\beta \mathrel{\bar{\rightsquigarrow}} \delta_3]$

### Contributions

In summary, the new contributions in this work include:

- Provided an axiomatization of base dependence relation for belief base formulas.
- Provided representation theorems to construct base dependence via belief base contraction and vice versa, similar to epistemic entrenchment in AGM.
- Proved that the new base dependence relation is a reversible generalization of Fariñas and Herzig's dependence relation.
- Shown, while generalizing the dependence relation, how base dependence preserves some of the most interesting properties of dependence, particularly, Keynes' conjunction criterion of dependence, CCD, and Gärdenfors' conjunction criterion of independence, CCI.
- Introduced conjunction criterion of dependence factoring, CCDF, which is more specific than CCD and CCI, and entails both of them.

## 5.2 Related Work

There are several works that define the concepts of relevance and dependence of formulas. Hansson and Wassermann propose that these can be classified into two groups [HW02]. Some authors capture relevance/dependence of formulas through syntactical means such as variable sharing and language splitting, including [Par99, CP00, MK06, KM07, Mak07, JQH08, SQJH08, WZZ11, PMZ11, FGKIS11]. Other authors have focused on inferential dependency of formulas, or, in other words, how some formulas deductively contribute to inference of other formulas. Examples of this approach include [FdCH96, HW02, CGHWK07], as well as the work reported in the present thesis. Typically, syntactical approaches are simpler and computationally more efficient compared to inferential approaches. However, the latter usually provide a more accurate and tighter definition of relevance and dependence than syntactical approaches. We now analyze a selection of these studies and discuss how they relate and differ from our research here.

The most related work to our study is that of Fariñas and Herzig in *Belief Change* *and Dependence* [FdCH96]. As such, we reviewed this work in some detail in Chapter 3. Here, we provide more analysis of some high-level properties that set their work apart form other studies, which are also inherited in our generalization of their work.

There is one comment that some may want to make on Fariñas and Herzig's approach in studying the notion of dependence in the context of belief change. An expected high level goal of such a study is to use dependence or relevance to reduce the number of candidate belief statements that can potentially be affected by a particular change. This should significantly improve tractability of belief change operations. Therefore, quite naturally, Fariñas and Herzig expect that dependence "will be a useful tool in the practical implementation of contraction and revision operations." On the other hand, the dependence relation in their work is constructed using a given AGM contraction operator. Thus, this way of *constructing* the dependence relation may seem to some as a deviation from that original goal. There appears to be a circularity: dependence was meant to *help* with the process of belief change operations, but now its construction is *based* on such operations.

Nevertheless, there are quite a few important and sometimes unique benefits to their approach. First, even though their dependence relation is constructed based on AGM contraction, their *axiomatization* of dependence is mostly separate from the AGM concepts. Indeed, as seen in §3.1 and §3.2.2, many of the postulates in Fariñas and Herzig's axiomatization are based on concepts introduced long before the AGM model was developed, for example CCD [p. 38] [Key21] and CCI [p. 39] [Gä78].

Moreover, construction of a dependence relation based on AGM contraction has an important implication: it provides the most theoretically accurate definition of dependence in the context of belief change. This is because Fariñas and Herzig construct their dependence relation using AGM contraction. Therefore, any other definition of dependence that is put forward to be used in relation to AGM contraction is either as good as Fariñas and Herzig's dependence relation or less accurate in capturing dependence of formulas compared to their dependence relation. To help clarify this point, in the following we look at another similar relationship for comparison.

As an example, consider the relevance relation provided by Riani and Wassermann in [RW04]. They define a kind of syntactical relevance $R$ such that, given two formulas

$\alpha$ and $\beta$, $R(\alpha, \beta)$ if and only if the formulas $\alpha$ and $\beta$ share an atom. Simply put, they consider formulas that share atoms as related (a.k.a. variable sharing). *Computational tractability* is one appealing benefit of such an approach in defining relevance. Also, this definition is clearly not constructed based on belief change operations in any way. Thus, it does not suffer from circularity if it is used in the context of belief change. However, they also acknowledge that it "can be argued that this notion [of relevance] is very simplistic" and an approximation that "gives us a 'quick and dirty' method for retrieving the most relevant elements of a set of formulas."

This is to be put in contrast with Fariñas and Herzig's dependence in belief change, which is completely precise in the sense that for every contraction operator there is one dependence relation and vice versa. That is because this dependence relation has been constructed using AGM contraction. This is merely a (desirable) implication of the above mentioned circularity in this definition. As discussed before, our base dependence formalism preserves all these benefits of Fariñas and Herzig's dependence, and additionally it can handle belief bases.

An example of studies considering inferential dependency (as opposed to considering syntactical means) is that of Hansson and Wassermann in Local Change [HW02]. They "consider relevant to a formula $\alpha$ the formulas that appear in a minimal derivation of $\alpha$ or its negation." They use belief bases and interestingly they also use kernels to come up with minimal derivations for formulas. Thus, there is a significant amount of common ground between their study and base dependence which corresponds to saturated kernel contraction (which is also based on kernels). Nevertheless, their formalism and concept of dependence (relevance) differs from that of Fariñas and Herzig's and from ours in some important ways. First, for them, anything that depends on $\alpha$ also depends on $\neg\alpha$. In contrast, $\alpha$ and $\neg\alpha$ can never depend nor base-depend on each other: $\alpha \not\rightsquigarrow \neg\alpha$ and $\alpha \not\bar{\rightsquigarrow} \neg\alpha$ by $\mathsf{Disj}$ [p. 40] and $\mathsf{Disj}^B$ [p. 71], respectively. This turns out to be important for mutuality of dependence and belief change or similarly base dependence and belief change. More specifically, they can construct contraction operators from given dependence relations (similar to $\mathsf{Cond}\div$ [p. 68] or $\mathsf{Cond}\bar{\div}$ [p. 68]), but they cannot construct dependence relations if given contraction operators (lacking anything similar to $\mathsf{Cond}\rightsquigarrow$ [p. 57] or $\mathsf{Cond}\bar{\rightsquigarrow}$ [p. 57]).

## 5.3 Future Work

There are a number of future research paths from this study. Here we provide some examples of open questions and possible research directions.

### 5.3.1 Base Dependence and Formulas of the Base

In §4.2, we made the decision to axiomatize base dependence relation $\bar{\rightsquigarrow}$ in a way to guarantee statement (4.5) on page 53:

If $\alpha \bar{\rightsquigarrow} \beta$ then $\beta \in B$.

This was not needed to be explicitly stated as an axiom because it is implied by the set of axioms offered in our study for base dependence.

There remain two other alternative approaches which could be explored in other studies. First, instead of requiring $\beta \in B$ as above, require that $\alpha \in B$.

If $\alpha \bar{\rightsquigarrow} \beta$ then $\alpha \in B$.

This alternative can be useful when we are interested on the effect of changing the base on other statements. The next alternative requires both $\alpha, \beta \in B$.

If $\alpha \bar{\rightsquigarrow} \beta$ then $\alpha \in B$ and $\beta \in B$.

This alternative may be particularly useful in a study of redundancy in the base. That is, exploring how removal of statements from the base requires removal of other statements in the base which can happen in the presence of redundant statements.

### 5.3.2 Direct Proof of Base Dependence Generalizing Dependence

In §4.9 on page 87, we showed indirectly that the set of *base dependence relations* is a superset of the set of *dependence relations*. In principle, this could be shown directly using the *axioms* of base dependence and dependence.

For some of the base dependence axioms it is straightforward to see their connection with their counterpart dependence axioms; e.g. $\mathsf{Disj}^B$, $\mathsf{Cond\text{-}ID}^B$ and $\mathsf{Def}\text{-}B$. Also Theorem 4.12 on page 74 supports this conjecture by showing that conjugation implies $\mathsf{LE}^B$.

### 5.3.3 Weak Base Dependence

In this work we have argued that weak base dependence is generally not desirable and have focused on ways to avoid it throughout the work (see §4.4.2 on page 59). One could, however, come up with cases to study where weak base dependence may be beneficial.

#### Weak Base Dependence and Redundancy

As shown in Theorem 4.3 on page 60, there is a strong relationship between weak base dependence and redundancy.

Therefore, the fact that weak base dependence captures redundancy may be exploited for various purposes. For example, one may use weak base dependence to distinguish between redundant and informative formulas in a belief base.

#### Axiomatization of Weak Base Dependence

Our axiomatization of base dependence here intentionally avoided weak base dependence. Thus, providing an alternative axiomatization that embraces weak base dependence could be beneficial to a study which involves this kind of dependence.

### 5.3.4 Dependence and Epistemic Entrenchment

As illustrated in Figure 3.1 on page 37, both dependence and epistemic entrenchment are counterparts of AGM contraction. Thus, it is only natural to expect to find a strong and more direct connection between them. Interestingly, Gärdenfors [Gä03] states that the condition $\mathsf{C}\div_{\mathsf{G}}$ [p. 25] "gives an explicit answer to which sentences are included in the contracted belief set, given the initial belief set and an ordering of

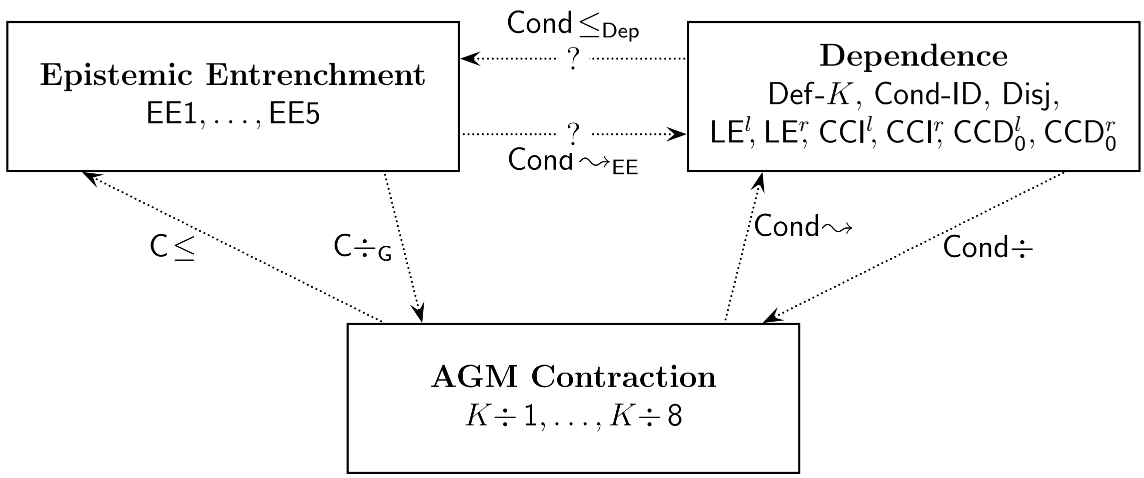



Figure 5.1: Open Problem: Provide the conditions $\mathsf{Cond}\rightsquigarrow_{\mathsf{EE}}$ and $\mathsf{Cond}\leq_{\mathsf{Dep}}$ and representation theorems to directly connect *Dependence* to *Epistemic Entrenchment*.

epistemic entrenchment." As we saw earlier in Chapter 3 indeed dependence is also concerned with specifying which sentences stay in the contracted belief set and which ones do not.

This hypothetical relation is shown in Figure 5.1. The unknown conditions $\mathsf{Cond}\rightsquigarrow_{\mathsf{EE}}$ and $\mathsf{Cond}\leq_{\mathsf{Dep}}$ can help to provide a more direct connection between dependence and epistemic entrenchment.

### 5.3.5 Base Dependence and Ensconcement

On the one hand, there is a potential relationship between dependence and epistemic entrenchment as shown in Figure 5.1. On the other hand, it is known that an epistemic entrenchment formalism for belief bases is not possible. "Probably the most successful application to belief bases of the ideas behind entrenchment is the theory of ensconcement relations that has been developed by Mary-Anne Williams [Wil94]," Hansson states in [Han99].

Therefore, one interesting hypothesis to explore is to find the relationship between base dependence and ensconcement.

# Appendix A

# Proofs

All theorems and their proofs are gathered in this appendix so that it is easier to review them, and to help avoid clutter.

## A.1 Proofs for §4.4.2: Avoiding Weak Base Dependence

**Theorem 4.3.** *Given relations* $\tilde{\rightsquigarrow}$ *and* $\div$ *for base* $B$*, where* inclusion *holds,* Cond$\tilde{\rightsquigarrow}$ *is equivalent to the following:*

$$\alpha \tilde{\rightsquigarrow} \beta \text{ iff } \beta \textit{ is redundant in } B \textit{ with respect to } B \div \alpha. \tag{4.6}$$

***Proof (theorem originally on page 60).***

| | | |
|---|---|---|
| 1 | $B \div \alpha \subseteq B$ | inclusion [p. 27] |
| 2 | $\alpha \tilde{\rightsquigarrow} \beta$ iff<br>$\beta$ is redundant in $B$ with respect to $B \div \alpha$ | Assume (4.6) holds |
| 3 | $\alpha \tilde{\rightsquigarrow} \beta$ iff<br>$B \div \alpha \subseteq B$ and $\beta \in B$ and<br>$\beta \notin B \div \alpha$ and $\beta \in \text{Cn}(B \div \alpha)$ | 2 and Definition 4.2 [p. 60]<br>(letting $B' = B \div \alpha$) |
| 4 | $\alpha \tilde{\rightsquigarrow} \beta$ iff<br>$\top$ and $\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \text{Cn}(B \div \alpha)$ | 1, 3 (replacing $B \div \alpha \subseteq B$<br>with $\top$ as it is true by 1) |
| 5 | $\alpha \tilde{\rightsquigarrow} \beta$ iff<br>$\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \text{Cn}(B \div \alpha)$ | 4 (removing conjunct $\top$);<br>Cond$\tilde{\rightsquigarrow}$ [p. 59] derived |
| 6 | Since lines 2 through 5 are logically equivalent, the reverse order also holds | |

□

**Theorem 4.5.** *Given relations* $\breve{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\breve{\rightsquigarrow}$ *holds, there is no weak base dependence if and only if* relative closure *holds for* $\div$.

***Proof (theorem originally on page 62).***
Based on Definition 4.1 on page 59, we know that there is *no weak base dependence* if and only if $\alpha \not\breve{\rightsquigarrow} \beta$ for all formulas $\alpha$ and $\beta$. In the following, we show that indeed relative closure holds if and only if $\alpha \not\breve{\rightsquigarrow} \beta$ for all $\alpha$ and $\beta$.

| | | |
|---|---|---|
| 1 | $\alpha \breve{\rightsquigarrow} \beta$ iff<br>$\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \mathrm{Cn}(B \div \alpha)$ | Cond$\breve{\rightsquigarrow}$ [p. 59] |
| 2 | $\alpha \not\breve{\rightsquigarrow} \beta$ iff<br>$\beta \notin B$ or $\beta \in B \div \alpha$ or $\beta \notin \mathrm{Cn}(B \div \alpha)$ | 1 (negating both sides) |
| 3 | $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | Assume relative closure [p. 27] |
| 4 | If $\beta \in B$ and $\beta \in \mathrm{Cn}(B \div \alpha)$ then $\beta \in B \div \alpha$ | 3 (by the set theory) |
| 5 | $\neg [\beta \in B$ and $\beta \in \mathrm{Cn}(B \div \alpha)]$ or $\beta \in B \div \alpha$ | 4 |
| 6 | $\beta \notin B$ or $\beta \notin \mathrm{Cn}(B \div \alpha)$ or $\beta \in B \div \alpha$ | 5 |
| 7 | $\alpha \not\breve{\rightsquigarrow} \beta$ | 2, 6; so no weak base dep. exists |
| 8 | Since lines 3 through 7 are logically equivalent, the reverse order also holds | |

□

## A.2 Proofs for §4.4.3: Connections Among (Base) Dependence Constructions

**Theorem 4.6.** *Given relations* $\bar{\leadsto}$, $\hat{\leadsto}$, $\check{\leadsto}$ *and* $\div$ *for base* $B$ *such that* $\mathsf{Cond}\bar{\leadsto}$, $\mathsf{Cond}\hat{\leadsto}$ *and* $\mathsf{Cond}\check{\leadsto}$ *hold, the following also holds:*

$$\alpha \bar{\leadsto} \beta \text{ iff } \alpha \hat{\leadsto} \beta \text{ or } \alpha \check{\leadsto} \beta$$

***Proof (theorem originally on page 63).***

| | | |
|---|---|---|
| 1 | $B \div \alpha \subseteq \text{Cn}(B \div \alpha)$ | By $\mathsf{inclusion}$ [p. 12] for Cn |
| 2 | If $\beta \notin \text{Cn}(B \div \alpha)$ then $\beta \notin B \div \alpha$ | 1 (by the set theory) |
| 3 | $\alpha \bar{\leadsto} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | $\mathsf{Cond}\bar{\leadsto}$ [p. 57] |
| 4 | $\alpha \hat{\leadsto} \beta$ iff $\beta \in B$ and $\beta \notin \text{Cn}(B \div \alpha)$ | $\mathsf{Cond}\hat{\leadsto}$ [p. 58] |
| 5 | $\alpha \check{\leadsto} \beta$ iff<br>$\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \text{Cn}(B \div \alpha)$ | $\mathsf{Cond}\check{\leadsto}$ [p. 59] |
| 6 | $\alpha \bar{\leadsto} \beta$ iff $\alpha \hat{\leadsto} \beta$ or $\alpha \check{\leadsto} \beta$ | Assumed to be verified |
| 7 | $[\beta \in B$ and $\beta \notin B \div \alpha]$ iff<br>$[\beta \in B$ and $\beta \notin \text{Cn}(B \div \alpha)]$ or<br>$[\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \text{Cn}(B \div \alpha)]$ | 3, 4, 5, 6 (substituting each term in 6 with its equivalent) |
| 8 | $[\beta \in B$ and $\beta \notin B \div \alpha]$ iff<br>$[\beta \in B$ and $[\beta \notin B \div \alpha$ and $\beta \notin \text{Cn}(B \div \alpha)]]$ or<br>$[\beta \in B$ and $\beta \notin B \div \alpha$ and $\beta \in \text{Cn}(B \div \alpha)]$ | 2, 7 (adding redundant conjunct $\beta \notin B \div \alpha$) |
| 9 | $[\beta \in B$ and $\beta \notin B \div \alpha]$ iff<br>$[[\beta \in B$ and $\beta \notin B \div \alpha]$ and $\beta \notin \text{Cn}(B \div \alpha)]$ or<br>$[[\beta \in B$ and $\beta \notin B \div \alpha]$ and $\beta \in \text{Cn}(B \div \alpha)]$ | 8 (regrouping conjuncts) |
| 10 | $[\beta \in B$ and $\beta \notin B \div \alpha]$ iff<br>$[\beta \in B$ and $\beta \notin B \div \alpha]$ and<br>$[\beta \notin \text{Cn}(B \div \alpha)$ or $\beta \in \text{Cn}(B \div \alpha)]$ | 9 (factoring out the common term) |
| 11 | $[\beta \in B$ and $\beta \notin B \div \alpha]$ iff<br>$[\beta \in B$ and $\beta \notin B \div \alpha]$ | 10 (omitting tautological conjunct) |
| 12 | $\top$ (i.e., reached a tautology) | 11; assumption 6 verified |

□

**Theorem 4.7.** *Given relations* $\bar{\rightsquigarrow}$, $\hat{\rightsquigarrow}$, $\check{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\rightsquigarrow}$, Cond$\hat{\rightsquigarrow}$ *and* Cond$\check{\rightsquigarrow}$ *hold and* relative closure *is satisfied, the base dependence relation* $\bar{\rightsquigarrow}$ *and the strong base dependence relation* $\hat{\rightsquigarrow}$ *are equivalent:*

$$\alpha \bar{\rightsquigarrow} \beta \text{ iff } \alpha \hat{\rightsquigarrow} \beta.$$

***Proof (theorem originally on page 64).***
By Theorem 4.5 (proof on page 102), when relative closure holds, there is no weak base dependence, meaning that $\alpha \not\check{\rightsquigarrow} \beta$ for all $\alpha$ and $\beta$ by Definition 4.1 on page 59. Thus the proof straightforwardly follows from Theorem 4.6:

| | | |
|---|---|---|
| 1 | $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | relative closure [p. 27] |
| 2 | $\alpha \not\check{\rightsquigarrow} \beta$ for all $\alpha$ and $\beta$ | 1 and Thm. 4.5 |
| 3 | $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \hat{\rightsquigarrow} \beta$ or $\alpha \check{\rightsquigarrow} \beta$ | By Thm. 4.6 |
| 4 | $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \hat{\rightsquigarrow} \beta$ | 2, 3 |

□

## Dependence and Base Dependence

**Lemma 4.8.** *An operator* $\div$ *on base* $B$ *satisfies* relative closure *if and only if it satisfies the following:*

$$\text{If } \beta \in B \text{ then } \quad \beta \in B \div \alpha \text{ iff } \beta \in \mathrm{Cn}(B \div \alpha). \tag{4.7}$$

***Proof (lemma originally on page 64).***

| | | |
|---|---|---|
| 1 | $B \div \alpha \subseteq \mathrm{Cn}(B \div \alpha)$ | By inclusion [p. 12] for Cn |
| 2 | If $\beta \in B \div \alpha$ then $\beta \in \mathrm{Cn}(B \div \alpha)$ | 1 |
| 3 | $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | Assume relative closure [p. 27] |
| 4 | If $\beta \in B$ and $\beta \in \mathrm{Cn}(B \div \alpha)$ then $\beta \in B \div \alpha$ | 3 |
| 5 | If $\beta \in B$ then $[\beta \notin \mathrm{Cn}(B \div \alpha)$ or $\beta \in B \div \alpha]$ | 4 |
| 6 | If $\beta \in B$ then [if $\beta \in \mathrm{Cn}(B \div \alpha)$ then $\beta \in B \div \alpha]$ | 5 |
| 7 | If $\beta \in B$ then $[\beta \in \mathrm{Cn}(B \div \alpha)$ iff $\beta \in B \div \alpha]$ | 2, 6; so (4.7) is derived |
| 8 | Since lines 3 through 7 are logically equivalent, the reverse order also holds | |

□

**Theorem 4.9.** *Given relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* relative closure *is satisfied, the following also holds:*

$$\alpha \mathrel{\bar{\rightsquigarrow}} \beta \;\text{ iff }\; \beta \in B \text{ and } \alpha \rightsquigarrow \beta.$$

***Proof (theorem originally on page 65).***

| | | |
|---|---|---|
| 1 | If $\beta \in B$ then $\beta \in \mathrm{Cn}(B)$ | By inclusion [p. 12] for Cn |
| 2 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 3 | $\alpha \rightsquigarrow \beta$ iff $\beta \in \mathrm{Cn}(B)$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ | Cond$\rightsquigarrow$ [p. 57] |
| 4 | $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | relative closure [p. 27] |
| 5 | If $\beta \in B$ then $[\beta \in B \div \alpha$ iff $\beta \in \mathrm{Cn}(B \div \alpha)]$ | 4 and Lemma 4.8 |
| 6 | If $\beta \in B$ then $[\beta \notin B \div \alpha$ iff $\beta \notin \mathrm{Cn}(B \div \alpha)]$ | 5 (negating both sides of iff) |
| 7 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ | Assumption |
| 8 | $\beta \in B$ and $\beta \notin B \div \alpha$ | 2, 7 |
| 9 | $\beta \in B$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ | 6, 8 |
| 10 | $[\beta \in B$ and $\beta \in \mathrm{Cn}(B)]$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ | 1, 9 (adding redundant conjunct $\beta \in \mathrm{Cn}(B)$) |
| 11 | $\beta \in B$ and $[\beta \in \mathrm{Cn}(B)$ and $\beta \notin \mathrm{Cn}(B \div \alpha)]$ | 10 |
| 12 | $\beta \in B$ and $\alpha \rightsquigarrow \beta$ | 3, 11 |
| 13 | If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\beta \in B$ and $\alpha \rightsquigarrow \beta$ | 7, 12 |
| 14 | Since lines 7 through 12 are logically equivalent, the reverse order also holds | |
| 15 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\alpha \rightsquigarrow \beta$ | 13, 14 |

□

**Theorem 4.10.** *Given relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* closure *is satisfied, in the special case where* $B$ *is logically closed,* $\bar{\rightsquigarrow}$ *reduces to* $\rightsquigarrow$*:*

$$\alpha \mathrel{\bar{\rightsquigarrow}} \beta \quad \text{iff} \quad \alpha \rightsquigarrow \beta.$$

***Proof (theorem originally on page 65).***

| | | |
|---|---|---|
| 1 | $B \div \alpha \subseteq \mathrm{Cn}(B \div \alpha)$ | By inclusion [p. 12] for Cn |
| 2 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 3 | $\alpha \rightsquigarrow \beta$ iff $\beta \in \mathrm{Cn}(B)$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ | Cond$\rightsquigarrow$ [p. 57] |
| 4 | If $\mathrm{Cn}(B) \subseteq B$ then $\mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | closure [p. 27] |
| 5 | $B = \mathrm{Cn}(B)$ | Logical Closure |
| 6 | $\mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | 4, 5 |
| 7 | $\beta \in B \div \alpha$ iff $\beta \in \mathrm{Cn}(B \div \alpha)$ | 1, 6 |
| 8 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ | Assumption |
| 9 | $\beta \in B$ and $\beta \notin B \div \alpha$ | 2, 8 |
| 10 | $\beta \in \mathrm{Cn}(B)$ and $\beta \notin B \div \alpha$ | 5, 9 |
| 11 | $\beta \in \mathrm{Cn}(B)$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ | 7, 10 |
| 12 | $\alpha \rightsquigarrow \beta$ | 3, 11 |
| 13 | If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\alpha \rightsquigarrow \beta$ | 8, 12 |
| 14 | Since lines 8 through 12 are logically equivalent, the reverse order also holds | |
| 15 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\alpha \rightsquigarrow \beta$ | 13, 14 |

□

## A.3 Proofs for §4.6: Base Dependence Postulates

**Theorem 4.12.** *If a relation* $\bar{\rightsquigarrow}$ *satisfies* conjugation, *then it also satisfies* $\mathsf{LE}^B$.

***Proof (theorem originally on page 74).***

| | | |
|---|---|---|
| 1 | If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $\alpha \bar{\rightsquigarrow} \delta$ iff $\beta \bar{\rightsquigarrow} \delta$ | conjugation [p. 73] |
| 2 | $\alpha \leftrightarrow \beta$ | Assumption |
| 3 | $\alpha \rightarrow \beta$ | 2 |
| 4 | $\beta \rightarrow \alpha$ | 2 |
| 5 | If $\alpha \in \mathrm{Cn}(C)$ then $\beta \in \mathrm{Cn}(C)$ for all $C$ | 3, supraclassicality [p. 13] for Cn |
| 6 | If $\beta \in \mathrm{Cn}(C)$ then $\alpha \in \mathrm{Cn}(C)$ for all $C$ | 4, supraclassicality for Cn |
| 7 | $\alpha \in \mathrm{Cn}(C)$ iff $\beta \in \mathrm{Cn}(C)$ for all $C$ | 5, 6 |
| 8 | $\alpha \bar{\rightsquigarrow} \delta$ iff $\beta \bar{\rightsquigarrow} \delta$ | 1, 7 |
| 9 | If $\alpha \leftrightarrow \beta$ then $\alpha \bar{\rightsquigarrow} \delta$ iff $\beta \bar{\rightsquigarrow} \delta$ | 2, 8 |
| 10 | If $\alpha \leftrightarrow \beta$ and $\alpha \bar{\rightsquigarrow} \delta$ then $\beta \bar{\rightsquigarrow} \delta$ | 9; $\mathsf{LE}^B$ [p. 73] derived |

□

**Theorem 4.13.** *If a relation* $\bar{\rightsquigarrow}$ *satisfies* contribution*, then it also satisfies* $\mathsf{Disj}^B$.

***Proof (theorem originally on page 74).***

| | | |
|---|---|---|
| 1 | If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then | contribution [p. 74] |
| | $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ | |
| | for some $B' \subseteq B$ | |
| 2 | If $\alpha \in \mathrm{Cn}(B')$ or $\alpha \notin \mathrm{Cn}(B' \cup \{\beta\})$ | 1 (contrapositive) |
| | for all $B' \subseteq B$ then $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ | |
| 3 | $\vdash \alpha \vee \beta$ | Assumption |
| 4 | $\alpha \vee \beta \in \mathrm{Cn}(\emptyset)$ | 3 |
| 5 | $\alpha \vee \beta \in \mathrm{Cn}(C)$ (for all $\emptyset \subseteq C$) | 4, monotony [p. 12] for Cn |
| 6 | $\alpha \in \mathrm{Cn}(C \cup \{\beta\})$ | Assumption |
| 7 | $(\beta \rightarrow \alpha) \in \mathrm{Cn}(C)$ | 6, deduction [p. 13] for Cn |
| 8 | $(\alpha \vee \neg\beta) \in \mathrm{Cn}(C)$ | 7, supraclassicality [p. 13] for Cn |
| 9 | $((\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)) \in \mathrm{Cn}(C)$ | 5, 8, supraclassicality for Cn |
| 10 | $\alpha \in \mathrm{Cn}(C)$ | 9, supraclassicality for Cn |
| | | (using the resolution rule) |
| 11 | If $\alpha \in \mathrm{Cn}(C \cup \{\beta\})$ then $\alpha \in \mathrm{Cn}(C)$ | 6, 10 |
| 12 | $[\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ or $\alpha \notin \mathrm{Cn}(B' \cup \{\beta\})]$ | Tautological truth |
| | for all $B' \subseteq B$ | |
| 13 | $[\alpha \in \mathrm{Cn}(B')$ or $\alpha \notin \mathrm{Cn}(B' \cup \{\beta\})]$ | 11, 12 |
| | for all $B' \subseteq B$ | |
| 14 | $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ | 2, 13 |
| 15 | If $\vdash \alpha \vee \beta$ then $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ | 3, 14; $\mathsf{Disj}^B$ [p. 71] derived |

□

**Theorem 4.17.** *If a relation* $\bar{\leadsto}$ *satisfies* $\mathsf{CCDF}^B$*, then it also satisfies both* $\mathsf{CCD}^B$ *and* $\mathsf{CCI}^B$*:*

$$\text{If } \alpha \mathrel{\bar{\leadsto}} \delta \text{ and } \beta \mathrel{\bar{\leadsto}} \delta \text{ then } \alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta \tag{$\mathsf{CCD}^B$}$$

$$\text{If } \alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta \text{ then } \alpha \mathrel{\bar{\leadsto}} \delta \text{ or } \beta \mathrel{\bar{\leadsto}} \delta \tag{$\mathsf{CCI}^B$}$$

***Proof (theorem originally on page 78).***

From $\mathsf{CCDF}^B$ to $\mathsf{CCD}^B$:

| | | |
|---|---|---|
| 1 | Either $[\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta_1$ iff $\alpha \mathrel{\bar{\leadsto}} \delta_1]$, or<br>$[\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta_2$ iff $\beta \mathrel{\bar{\leadsto}} \delta_2]$, or<br>$[\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta_3$ iff $\alpha \mathrel{\bar{\leadsto}} \delta_3$ or $\beta \mathrel{\bar{\leadsto}} \delta_3]$ | $\mathsf{CCDF}^B$ [p. 77] |
| 2 | $\alpha \mathrel{\bar{\leadsto}} \delta$ and $\beta \mathrel{\bar{\leadsto}} \delta$ | Assumption |
| 3 | $\alpha \mathrel{\bar{\leadsto}} \delta$ | 2 |
| 4 | $\beta \mathrel{\bar{\leadsto}} \delta$ | 2 |
| 5 | $\alpha \mathrel{\bar{\leadsto}} \delta$ or $\beta \mathrel{\bar{\leadsto}} \delta$ | 3, 4 |
| 6 | $\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta$ | 1, 3, 4, 5 |
| 7 | If $\alpha \mathrel{\bar{\leadsto}} \delta$ and $\beta \mathrel{\bar{\leadsto}} \delta$ then $\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta$ | 2, 6; $\mathsf{CCD}^B$ [p. 76] derived |

From $\mathsf{CCDF}^B$ to $\mathsf{CCI}^B$:

| | | |
|---|---|---|
| 1 | Either $[\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta_1$ iff $\alpha \mathrel{\bar{\leadsto}} \delta_1]$, or<br>$[\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta_2$ iff $\beta \mathrel{\bar{\leadsto}} \delta_2]$, or<br>$[\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta_3$ iff $\alpha \mathrel{\bar{\leadsto}} \delta_3$ or $\beta \mathrel{\bar{\leadsto}} \delta_3]$ | $\mathsf{CCDF}^B$ [p. 77] |
| 2 | $\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta$ | Assumption |
| 3 | $[\alpha \mathrel{\bar{\leadsto}} \delta]$ or $[\beta \mathrel{\bar{\leadsto}} \delta]$ or $[\alpha \mathrel{\bar{\leadsto}} \delta$ or $\beta \mathrel{\bar{\leadsto}} \delta]$ | 1, 2 |
| 4 | $\alpha \mathrel{\bar{\leadsto}} \delta$ or $\beta \mathrel{\bar{\leadsto}} \delta$ | 3 |
| 5 | If $\alpha \wedge \beta \mathrel{\bar{\leadsto}} \delta$ then $\alpha \mathrel{\bar{\leadsto}} \delta$ or $\beta \mathrel{\bar{\leadsto}} \delta$ | 2, 4; $\mathsf{CCI}^B$ [p. 76] derived |

□

## A.4 Proofs for §4.7: Saturated Kernel Contraction and Base Dependence

**Lemma 4.18.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\bar{\div}$ *for base* $B$ *such that* Cond$\bar{\div}$ *holds and* contribution *is satisfied, it also holds that* $\alpha \not\bar{\rightsquigarrow} \beta$ for all $\beta \in B \bar{\div} \alpha$.

***Proof (lemma originally on page 81).***

| | | |
|---|---|---|
| 1 | If $\vdash \alpha \vee \beta$ then $\alpha \not\bar{\rightsquigarrow} \beta$ | By Thm. 4.13 and contribution |
| 2 | $\beta \in B \bar{\div} \alpha$ iff either $\bar{\vdash} \beta$ or $\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta$ | Cond$\bar{\div}$ [p. 68] |
| 3 | $\beta \in B \bar{\div} \alpha$ | Assumption |
| 4 | $\bar{\vdash} \beta$ or $[\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta]$ | 2, 3 |
| 5 | $\bar{\nvdash} \beta$ | **Case1** Assumption |
| 6 | $\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta$ | 4, 5 |
| 7 | $\alpha \not\bar{\rightsquigarrow} \beta$ | 6 |
| 8 | $\bar{\vdash} \beta$ | **Case2** Assumption |
| 9 | $\vdash \beta$ | 8 and Definition 4.11 [p. 65] |
| 10 | $\alpha \not\bar{\rightsquigarrow} \beta$ | 1, 9 |
| 11 | $\alpha \not\bar{\rightsquigarrow} \beta$ | By **Case1** and **Case2** |
| 12 | If $\beta \in B \bar{\div} \alpha$ then $\alpha \not\bar{\rightsquigarrow} \beta$ | 3, 11 |
| 13 | $\alpha \not\bar{\rightsquigarrow} \beta$ for all $\beta \in B \bar{\div} \alpha$ | 12 |

□

**Theorem 4.19** (Base Dependence to Contraction)**.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\bar{\div}$ *for base* $B$ *such that* Cond$\bar{\div}$ *holds, if* $\bar{\rightsquigarrow}$ *is a base dependence, then* $\bar{\div}$ *is a saturated kernel contraction.*

***Proof (theorem originally on page 81).***
Assume all the following hold: Cond$\bar{\div}$ and postulates of base dependence, namely, Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity and redundancy (see Definition 4.15 on page 75). We show that then the postulates of saturated kernel contraction, viz., inclusion, success, uniformity, core-retainment and relative closure (see §2.9 on page 30) also hold:

$B \div \alpha \subseteq B$ (inclusion [p. 27])

| | | |
|---|---|---|
| 1 | $\beta \in B \div \alpha$ iff either $\bar{\vdash}\ \beta$ or $\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta$ | Cond$\bar{\div}$ [p. 68] |
| 2 | $\beta \in B$ iff either $\bar{\vdash}\ \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$ | Def-$B$ [p. 73] |
| 3 | $\beta \in B \div \alpha$ | Assumption |
| 4 | $\bar{\vdash}\ \beta$ or $[\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta]$ | 1, 3 |
| 5 | $\bar{\nvdash} \beta$ | **Case1** Assumption |
| 6 | $[\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta]$ | 4, 5 |
| 7 | $\beta \bar{\rightsquigarrow} \beta$ | 6 (letting $\alpha$ be $\beta$) |
| 8 | $\beta \in B$ | 2, 7 |
| 9 | $\bar{\vdash}\ \beta$ | **Case2** Assumption |
| 10 | $\beta \in B$ and $\vdash \beta$ | 9 and Definition 4.11 [p. 65] |
| 11 | $\beta \in B$ | 10 |
| 12 | $\beta \in B$ | By **Case1** and **Case2** |
| 13 | If $\beta \in B \div \alpha$ then $\beta \in B$ | 3, 12 |
| 14 | $B \div \alpha \subseteq B$ | 13; inclusion derived |

If $\nvdash \alpha$ then $\alpha \notin \mathrm{Cn}(B \div \alpha)$ (success [p. 27])

| | | |
|---|---|---|
| 1 | $\alpha \not\bar{\rightsquigarrow} \beta$ for all $\beta \in B \div \alpha$ | By Lemma 4.18, contribution and Cond$\bar{\div}$ |
| 2 | $B \div \alpha \subseteq B$ | inclusion [p. 27] (proved above) |
| 3 | If $\alpha \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then either $\vdash \alpha$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\beta \in B'$ | modularity [p. 74] |
| 4 | $\alpha \in \mathrm{Cn}(B \div \alpha)$ | Assumption |
| 5 | $\alpha \in \mathrm{Cn}(B \div \alpha)$ and $B \div \alpha \subseteq B$ | 2, 4 |
| 6 | $\vdash \alpha$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\beta \in B \div \alpha$ | 3, 5 (letting $B' = B \div \alpha$) |
| 7 | $\vdash \alpha$ | 1, 6 |
| 8 | If $\alpha \in \mathrm{Cn}(B \div \alpha)$ then $\vdash \alpha$ | 4, 7 |
| 9 | If $\nvdash \alpha$ then $\alpha \notin \mathrm{Cn}(B \div \alpha)$ | 8; success derived |

If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $B \div \alpha = B \div \beta$

(uniformity [p. 27])

| | | |
|---|---|---|
| 1 | $\delta \in B \div \theta$ iff either $\bar{\vdash}\, \delta$ or $\delta \mathrel{\bar{\rightsquigarrow}} \delta$ and $\theta \not\rightsquigarrow \delta$ | Cond$\bar{\div}$ [p. 68] |
| 2 | If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta$ | conjugation [p. 73] |
| 3 | $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ | Assumption |
| 4 | $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta$ | 2, 3 |
| 5 | $\delta \in B \div \alpha$ | Assumption |
| 6 | $\bar{\vdash}\, \delta$ or $[\delta \mathrel{\bar{\rightsquigarrow}} \delta$ and $\alpha \not\rightsquigarrow \delta]$ | 1, 5 |
| 7 | $\bar{\nvdash}\, \delta$ | **Case1** Assumption |
| 8 | $[\delta \mathrel{\bar{\rightsquigarrow}} \delta$ and $\alpha \not\rightsquigarrow \delta]$ | 6, 7 |
| 9 | $\delta \mathrel{\bar{\rightsquigarrow}} \delta$ | 8 |
| 10 | $\alpha \not\rightsquigarrow \delta$ | 8 |
| 11 | $\beta \not\rightsquigarrow \delta$ | 4, 10 |
| 12 | $\delta \mathrel{\bar{\rightsquigarrow}} \delta$ and $\beta \not\rightsquigarrow \delta$ | 9, 11 |
| 13 | $\delta \in B \div \beta$ | 1, 12 |
| 14 | $\bar{\vdash}\, \delta$ | **Case2** Assumption |
| 15 | $\delta \in B \div \beta$ | 1, 14 |
| 16 | $\delta \in B \div \beta$ | By **Case1** and **Case2** |
| 17 | If $\delta \in B \div \alpha$ then $\delta \in B \div \beta$ | 5, 16 |
| 18 | $B \div \alpha \subseteq B \div \beta$ | 17 |
| 19 | $B \div \beta \subseteq B \div \alpha$ | Also by symmetry (steps 5-18) |
| 20 | $B \div \alpha = B \div \beta$ | 18, 19 |
| 21 | If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $B \div \alpha = B \div \beta$ | 3, 20; uniformity derived |

If $\beta \in B$ and $\beta \notin B \div \alpha$ then there is some $B'$ s. t. $B' \subseteq B$ and $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ (core-retainment [p. 27])

| | | |
|---|---|---|
| 1 | $\beta \in B \div \alpha$ iff either $\bar{\vdash} \beta$ or $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ and $\alpha \mathrel{\bar{\not\rightsquigarrow}} \beta$ | Cond$\div$ [p. 68] |
| 2 | $\beta \notin B \div \alpha$ iff $\bar{\nvdash} \beta$ and $[\beta \mathrel{\bar{\not\rightsquigarrow}} \beta$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta]$ | 1 (contrapositive) |
| 3 | $\beta \in B$ iff either $\bar{\vdash} \beta$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\alpha$ | Def-$B$ [p. 73] |
| 4 | If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ | Cond-ID$^B$ [p. 71] |
| 5 | If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then | contribution [p. 74] |
| | $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ | |
| 6 | $\beta \in B$ | Assumption |
| 7 | $\beta \notin B \div \alpha$ | Assumption |
| 8 | $\bar{\vdash} \beta$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\alpha$ | 3, 6 |
| 9 | $\bar{\nvdash} \beta$ and $[\beta \mathrel{\bar{\not\rightsquigarrow}} \beta$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta]$ | 2, 7 |
| 10 | $\bar{\nvdash} \beta$ | 9 |
| 11 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\alpha$ | 8, 10 |
| 12 | $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ | 4, 11 |
| 13 | $[\beta \mathrel{\bar{\not\rightsquigarrow}} \beta$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta]$ | 9 |
| 14 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ | 12, 13 |
| 15 | $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ | 5, 14 |
| 16 | If $\beta \in B$ and $\beta \notin B \div \alpha$ then | 6, 7, 15; core-retainment |
| | $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ | derived |

$B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ (relative closure [p. 27])

| | | |
|---|---|---|
| 1 | $\alpha \not\bar{\rightsquigarrow} \delta$ for all $\delta \in B \div \alpha$ | By Lemma 4.18, contribution and Cond$\bar{\div}$ |
| 2 | $B \div \alpha \subseteq B$ | inclusion [p. 27] (proved above) |
| 3 | $\beta \in B \div \alpha$ iff either $\bar{\vdash}\ \beta$ or $\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta$ | Cond$\bar{\div}$ [p. 68] |
| 4 | $\beta \in B$ iff either $\bar{\vdash}\ \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$ | Def-$B$ [p. 73] |
| 5 | If $\alpha \bar{\rightsquigarrow} \beta$ then $\beta \bar{\rightsquigarrow} \beta$ | Cond-ID$^B$ [p. 71] |
| 6 | If $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then either $\alpha \not\bar{\rightsquigarrow} \beta$ or $\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$ | redundancy [p. 74] |
| 7 | If $\vdash \alpha \vee \beta$ then $\alpha \not\bar{\rightsquigarrow} \beta$ | By Thm. 4.13 and contribution |
| 8 | If $\vdash \beta$ then $\alpha \not\bar{\rightsquigarrow} \beta$ | 7 |
| 9 | If $\vdash \beta$ and $\beta \in B$ then $\alpha \not\bar{\rightsquigarrow} \beta$ | 8 (introducing extra conjunct $\beta \in B$ to antecedent) |
| 10 | If $\bar{\vdash}\ \beta$ then $\alpha \not\bar{\rightsquigarrow} \beta$ | 9 and Definition 4.11 [p. 65] |
| 11 | $\beta \in B$ | Assumption |
| 12 | $\bar{\vdash}\ \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$ | 4, 11 |
| 13 | $\bar{\vdash}\ \beta$ or $\beta \bar{\rightsquigarrow} \beta$ | 5, 12 |
| 14 | $\beta \in \mathrm{Cn}(B \div \alpha)$ | Assumption |
| 15 | $\beta \in \mathrm{Cn}(B \div \alpha)$ and $B \div \alpha \subseteq B$ | 2, 14 |
| 16 | $\alpha \not\bar{\rightsquigarrow} \beta$ or $\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B \div \alpha$ | 6, 15 (letting $B' = B \div \alpha$) |
| 17 | $\alpha \not\bar{\rightsquigarrow} \beta$ | 1, 16 |
| 18 | $[\bar{\vdash}\ \beta$ or $\beta \bar{\rightsquigarrow} \beta]$ and $[\alpha \not\bar{\rightsquigarrow} \beta]$ | 13, 17 |
| 19 | $[\bar{\vdash}\ \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta]$ or $[\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta]$ | 18 (distributing conjunction) |
| 20 | $[\bar{\vdash}\ \beta]$ or $[\beta \bar{\rightsquigarrow} \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta]$ | 10, 19 (omitting redundant conjunct $\alpha \not\bar{\rightsquigarrow} \beta$) |
| 21 | $\beta \in B \div \alpha$ | 3, 20 |
| 22 | If $\beta \in B$ and $\beta \in \mathrm{Cn}(B \div \alpha)$ then $\beta \in B \div \alpha$ | 11, 14, 21 |
| 23 | $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | 22; relative closure derived |

□

**Theorem 4.20** (Contraction to Base Dependence)**.** *Given relations $\bar{\rightsquigarrow}$ and $\div$ for base $B$ such that* Cond$\bar{\rightsquigarrow}$ *holds, if $\div$ is a saturated kernel contraction, then $\bar{\rightsquigarrow}$ is a base dependence.*

***Proof (theorem originally on page 81).***
Assume all the following hold: Cond$\bar{\rightsquigarrow}$ and the postulates for saturated kernel contraction, namely, inclusion, success, uniformity, core-retainment and relative closure (see §2.9 on page 30). We show that then the postulates of base dependence, viz., Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity and redundancy (see Definition 4.15 on page 75) also hold:

If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ (Cond-ID$^B$ [p. 71])

| | | |
|---|---|---|
| 1 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ |
| 2 | $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ iff $\beta \notin B$ or $\beta \in B \div \alpha$ | 1 (negating both sides) |
| 3 | If $\beta \in \mathrm{Cn}(B \div \beta)$ then $\beta \in \mathrm{Cn}(\emptyset)$ | success [p. 27] (contrapositive) |
| 4 | $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | relative closure [p. 27] |
| 5 | $\beta \mathrel{\not\bar{\rightsquigarrow}} \beta$ | Assumption |
| 6 | $\beta \notin B$ or $\beta \in B \div \beta$ | 2, 5 |
| 7 | $\beta \in B$ | **Case1** Assumption |
| 8 | $\beta \in B \div \beta$ | 6, 7 |
| 9 | $\beta \in \mathrm{Cn}(B \div \beta)$ | 8, inclusion [p. 12] for Cn |
| 10 | $\beta \in \mathrm{Cn}(\emptyset)$ | 3, 9 |
| 11 | $\beta \in \mathrm{Cn}(B \div \alpha)$ | 10, monotony [p. 12] for Cn |
| 12 | $\beta \in B \div \alpha$ | 4, 7, 11 |
| 13 | $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ | 2, 12 |
| 14 | $\beta \notin B$ | **Case2** Assumption |
| 15 | $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ | 2, 14 |
| 16 | $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ | By **Case1** and **Case2** |
| 17 | If $\beta \mathrel{\not\bar{\rightsquigarrow}} \beta$ then $\alpha \mathrel{\not\bar{\rightsquigarrow}} \beta$ | 5, 16 |
| 18 | If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ | 17; Cond-ID$^B$ derived |

$\beta \in B$ iff either $\bar{\vdash}\, \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$ (Def-$B$ [p. 73])

| | | |
|---|---|---|
| 1 | $B \div \alpha \subseteq \mathrm{Cn}(B \div \alpha)$ | By inclusion [p. 12] for Cn |
| 2 | If $\beta \notin \mathrm{Cn}(B \div \alpha)$ then $\beta \notin B \div \alpha$ | 1 |
| 3 | $\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 4 | If $\nvdash \beta$ then $\beta \notin \mathrm{Cn}(B \div \beta)$ | success [p. 27] |
| 5 | $\beta \in B$ | Assumption, for *left to right* |
| 6 | $\vdash \beta \vee \nvdash \beta$ | Tautological truth |
| 7 | $\vdash \beta$ | **Case1** Assumption |
| 8 | $\beta \in B$ and $\vdash \beta$ | 5, 7 |
| 9 | $\bar{\vdash}\, \beta$ | 8 and Definition 4.11 [p. 65] |
| 10 | $\nvdash \beta$ | **Case2** Assumption |
| 11 | $\beta \notin \mathrm{Cn}(B \div \beta)$ | 4, 10 |
| 12 | $\beta \notin B \div \beta$ | 2, 11 |
| 13 | $\beta \in B$ and $\beta \notin B \div \beta$ | 5, 12 |
| 14 | $\beta \bar{\rightsquigarrow} \beta$ | 3, 13 |
| 15 | $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$ | 14 (e.g., let $\alpha$ be equal to $\beta$) |
| 16 | $\bar{\vdash}\, \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$ | By 6, **Case1** and **Case2** |
| 17 | If $\beta \in B$ then [$\bar{\vdash}\, \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$] | 5, 16 |
| 18 | $\beta \notin B$ | Assumption, for *right to left* |
| 19 | $\bar{\nvdash}\, \beta$ | 18 and Definition 4.11 [p. 65] |
| 20 | $\alpha \not\bar{\rightsquigarrow} \beta$ for all $\alpha$ | 3, 18 |
| 21 | $\bar{\nvdash}\, \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta$ for all $\alpha$ | 19, 20 |
| 22 | If $\beta \notin B$ then [$\bar{\nvdash}\, \beta$ and $\alpha \not\bar{\rightsquigarrow} \beta$ for all $\alpha$] | 18, 21 |
| 23 | If [$\bar{\vdash}\, \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$] then $\beta \in B$ | 22 (contrapositive) |
| 24 | $\beta \in B$ iff either $\bar{\vdash}\, \beta$ or $\alpha \bar{\rightsquigarrow} \beta$ for some $\alpha$ | 17, 23; Def-$B$ derived |

If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta$

(conjugation [p. 73])

| | | |
|---|---|---|
| 1 | $\theta \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\delta \in B$ and $\delta \notin B \div \theta$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 2 | If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $B \div \alpha = B \div \beta$ | uniformity [p. 27] |
| 3 | $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ | Assumption |
| 4 | $B \div \alpha = B \div \beta$ | 2, 3 |
| 5 | $\delta \in B \div \alpha$ iff $\delta \in B \div \beta$ | 4 |
| 6 | $\delta \notin B \div \alpha$ iff $\delta \notin B \div \beta$ | 5 |
| 7 | $[\delta \in B$ and $\delta \notin B \div \alpha]$ iff $[\delta \in B$ and $\delta \notin B \div \beta]$ | 6 (adding conjunct $\delta \in B$ to both sides) |
| 8 | $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta$ | 1, 7 |
| 9 | If $\alpha \in \mathrm{Cn}(B')$ iff $\beta \in \mathrm{Cn}(B')$ for all $B' \subseteq B$ then $\alpha \mathrel{\bar{\rightsquigarrow}} \delta$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta$ | 3, 8; conjugation derived |

If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$

(contribution [p. 74])

| | | |
|---|---|---|
| 1 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 2 | If $\beta \in B$ and $\beta \notin B \div \alpha$ then $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ | core-retainment [p. 27] |
| 3 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ | Assumption |
| 4 | $\beta \in B$ and $\beta \notin B \div \alpha$ | 1, 3 |
| 5 | $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ | 2, 4 |
| 6 | If $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ then $\alpha \notin \mathrm{Cn}(B')$ and $\alpha \in \mathrm{Cn}(B' \cup \{\beta\})$ for some $B' \subseteq B$ | 3, 5; contribution derived |

If $\alpha \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then
either $\vdash \alpha$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\beta \in B'$ (modularity [p. 74])

| | | |
|---|---|---|
| 1 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 2 | $\alpha \mathrel{\bar{\not\rightsquigarrow}} \beta$ iff $\beta \notin B$ or $\beta \in B \div \alpha$ | 1 |
| 3 | If $\nvdash \alpha$ then $\alpha \notin \mathrm{Cn}(B \div \alpha)$ | success [p. 27] |
| 4 | If $\alpha \in \mathrm{Cn}(B \div \alpha)$ then $\vdash \alpha$ | 3 |
| 5 | $\alpha \in \mathrm{Cn}(B')$ | Assumption |
| 6 | $B' \subseteq B$ | Assumption |
| 7 | $\beta \in B$ for all $\beta \in B'$ | 6 (by the set theory) |
| 8 | $\neg\,[\vdash \alpha$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\beta \in B']$ | Assume for the sake of contradiction |
| 9 | $\not\vdash \alpha$ and $\alpha \mathrel{\bar{\not\rightsquigarrow}} \beta$ for all $\beta \in B'$ | 8 |
| 10 | $\not\vdash \alpha$ | 9 |
| 11 | $\alpha \mathrel{\bar{\not\rightsquigarrow}} \beta$ for all $\beta \in B'$ | 9 |
| 12 | $[\beta \notin B$ or $\beta \in B \div \alpha]$ for all $\beta \in B'$ | 2, 11 |
| 13 | $[\beta \in B \div \alpha]$ for all $\beta \in B'$ | 7, 12 (i.e., $\beta \notin B$ is false by 7) |
| 14 | $B' \subseteq B \div \alpha$ | 13 (by the set theory) |
| 15 | $\mathrm{Cn}(B') \subseteq \mathrm{Cn}(B \div \alpha)$ | 14, monotony [p. 12] for Cn |
| 16 | $\alpha \in \mathrm{Cn}(B \div \alpha)$ | 5, 15 |
| 17 | $\vdash \alpha$ | 3, 14 |
| 18 | $\not\vdash \alpha$ and $\vdash \alpha$ | 10, 17 |
| 19 | $\bot$ (i.e., reached a contradiction) | 18 |
| 20 | $\vdash \alpha$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\beta \in B'$ | 8, 19 |
| 21 | If $\alpha \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then either $\vdash \alpha$ or $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\beta \in B'$ | 5, 6, 20; modularity derived |

If $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then
either $\alpha \not\bar{\rightsquigarrow} \beta$ or $\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$ (redundancy [p. 74])

| | | |
|---|---|---|
| 1 | $\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 2 | $\alpha \not\bar{\rightsquigarrow} \beta$ iff $\beta \notin B$ or $\beta \in B \div \alpha$ | 1 |
| 3 | $B \cap \mathrm{Cn}(B \div \alpha) \subseteq B \div \alpha$ | relative closure [p. 27] |
| 4 | $\beta \in \mathrm{Cn}(B')$ | Assumption |
| 5 | $B' \subseteq B$ | Assumption |
| 6 | $\delta \in B$ for all $\delta \in B'$ | 5 (by the set theory) |
| 7 | $\neg\,[\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B']$ | Assumption |
| 8 | $\alpha \not\bar{\rightsquigarrow} \delta$ for all $\delta \in B'$ | 7 |
| 9 | $[\delta \notin B$ or $\delta \in B \div \alpha]$ for all $\delta \in B'$ | 2, 8 |
| 10 | $[\delta \in B \div \alpha]$ for all $\delta \in B'$ | 6, 9 |
| 11 | $B' \subseteq B \div \alpha$ | 10 (by the set theory) |
| 12 | $\mathrm{Cn}(B') \subseteq \mathrm{Cn}(B \div \alpha)$ | 11, monotony [p. 12] for Cn |
| 13 | $\beta \in \mathrm{Cn}(B \div \alpha)$ | 4, 12 |
| 14 | $\beta \in B$ | **Case1** Assumption |
| 15 | $\beta \in B \div \alpha$ | 3, 13, 14 |
| 16 | $\alpha \not\bar{\rightsquigarrow} \beta$ | 2, 15 |
| 17 | $\beta \notin B$ | **Case2** Assumption |
| 18 | $\alpha \not\bar{\rightsquigarrow} \beta$ | 2, 17 |
| 19 | $\alpha \not\bar{\rightsquigarrow} \beta$ | By **Case1** and **Case2** |
| 20 | If $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ and $\neg\,[\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B']$ then $\alpha \not\bar{\rightsquigarrow} \beta$ | 4, 5, 7, 19 |
| 21 | If $\beta \in \mathrm{Cn}(B')$ and $B' \subseteq B$ then $\alpha \not\bar{\rightsquigarrow} \beta$ or $\alpha \bar{\rightsquigarrow} \delta$ for some $\delta \in B'$ | 20; redundancy derived |

$\square$

**Lemma 4.22.** *In the presence of* Def-$B$, Cond-ID$^B$ *and* contribution, *the following is equivalent to* Cond$\bar{\div}$*:*

$$\beta \in B \div \alpha \ \text{ iff } \ \beta \in B \text{ and } \alpha \not\rightsquigarrow \beta.$$

***Proof (lemma originally on page 82).***

| | | |
|---|---|---|
| 1 | If $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ then $\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta$ | Trivially holds: e.g., let $\delta$ be $\beta$ |
| 2 | If $\delta \mathrel{\bar{\rightsquigarrow}} \beta$ then $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ | Cond-ID$^B$ [p. 71] |
| 3 | $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta$ | 1, 2 |
| 4 | $\beta \in B$ iff either $\bar{\vdash} \beta$ or $\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta$ | Def-$B$ [p. 73] |
| 5 | If $\vdash \alpha \vee \beta$ then $\alpha \not\rightsquigarrow \beta$ | By Thm. 4.13 and contribution |
| 6 | If $\vdash \beta$ then $\alpha \not\rightsquigarrow \beta$ | 5 |
| 7 | $\beta \in B \div \alpha$ iff either $\bar{\vdash} \beta$ or $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ and $\alpha \not\rightsquigarrow \beta$ | Assume Cond$\bar{\div}$ [p. 68] |
| 8 | $\beta \in B \div \alpha$ iff $[\beta \in B$ and $\vdash \beta]$ or $[\beta \mathrel{\bar{\rightsquigarrow}} \beta$ and $\alpha \not\rightsquigarrow \beta]$ | 7 and Definition 4.11 [p. 65] |
| 9 | $\beta \in B \div \alpha$ iff $[\beta \in B$ and $\vdash \beta]$ or $[[\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta]$ and $\alpha \not\rightsquigarrow \beta]$ | 3, 8 (replacing $\beta \mathrel{\bar{\rightsquigarrow}} \beta$ with its equivalent $\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta$) |
| 10 | $\beta \in B \div \alpha$ iff $[[\beta \in B$ and $\vdash \beta]$ and $\alpha \not\rightsquigarrow \beta]$ or $[[\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta]$ and $\alpha \not\rightsquigarrow \beta]$ | 6, 9 (adding redundant conjunct $\alpha \not\rightsquigarrow \beta$ since $\vdash \beta$) |
| 11 | $\beta \in B \div \alpha$ iff $[[\beta \in B$ and $\vdash \beta]$ or $[\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta]]$ and $\alpha \not\rightsquigarrow \beta$ | 10 |
| 12 | $\beta \in B \div \alpha$ iff $[\bar{\vdash} \beta$ or $[\delta \mathrel{\bar{\rightsquigarrow}} \beta$ for some $\delta]]$ and $\alpha \not\rightsquigarrow \beta$ | 11 and Definition 4.11 [p. 65] |
| 13 | $\beta \in B \div \alpha$ iff $\beta \in B$ and $\alpha \not\rightsquigarrow \beta$ | 4, 12 |
| 14 | Since lines 7 through 13 are logically equivalent, the reverse order also holds | |

□

**Theorem 4.23** (Characterization)**.** *Let the relations* $\overset{\div}{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *be such that* $\div$ *satisfies* inclusion, $B \div \alpha \subseteq B$, *and that* Cond$\overset{\div}{\rightsquigarrow}$ *holds:* $\alpha \overset{\div}{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$. *Then,*

$\div$ *is a saturated kernel contraction if and only if* $\overset{\div}{\rightsquigarrow}$ *is a base dependence.*

***Proof (theorem originally on page 83).***
Given that Cond$\overset{\div}{\rightsquigarrow}$ [p. 57] holds by assumption, the left to right direction is already proved in theorem 4.20 (proof on page 115).

Similarly, for the right to left direction, theorem 4.19 (proof on page 110) can be used, provided that Cond$\overline{\div}$ [p. 68] holds. That is, to construct a saturated kernel contraction relation $\div$, given a base dependence relation $\overset{\div}{\rightsquigarrow}$, it suffices to show that Cond$\overline{\div}$ holds. This is achieved below after first assuming all the following hold:

- three of the base dependence postulates (see Definition 4.15 on page 75), namely, Def-$B$, Cond-ID$^B$ and contribution,
- Cond$\overset{\div}{\rightsquigarrow}$, and
- inclusion (see Remark 4.21 on page 82).

| | | |
|---|---|---|
| 1 | $B \div \alpha \subseteq B$ | inclusion [p. 27] |
| 2 | $\alpha \overset{\div}{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\overset{\div}{\rightsquigarrow}$ [p. 57] |
| 3 | $\beta \in B \div \alpha$ | Assumption |
| 4 | $\beta \in B$ | 1, 3 |
| 5 | $\alpha \not\overset{\div}{\rightsquigarrow} \beta$ | 2, 3 |
| 6 | If $\beta \in B \div \alpha$ then $\beta \in B$ and $\alpha \not\overset{\div}{\rightsquigarrow} \beta$ | 3, 4, 5 |
| 7 | If $\beta \in B$ and $\beta \notin B \div \alpha$ then $\alpha \overset{\div}{\rightsquigarrow} \beta$ | 2 (using right to left) |
| 8 | If $\beta \notin B \div \alpha$ then $\beta \notin B$ or $\alpha \overset{\div}{\rightsquigarrow} \beta$ | 7 |
| 9 | If $\beta \in B$ and $\alpha \not\overset{\div}{\rightsquigarrow} \beta$ then $\beta \in B \div \alpha$ | 8 (contrapositive) |
| 10 | $\beta \in B \div \alpha$ iff $\beta \in B$ and $\alpha \not\overset{\div}{\rightsquigarrow} \beta$ | 6, 9 |
| 11 | $\beta \in B \div \alpha$ iff either $\overline{\vdash} \beta$ or $\beta \overset{\div}{\rightsquigarrow} \beta$ and $\alpha \not\overset{\div}{\rightsquigarrow} \beta$ | 10 and Lemma 4.22, applicable as Def-$B$, Cond-ID$^B$ and contribution hold |

□

## A.5 Proofs for §4.8: Enhancements for Handling Conjunctions

**Theorem 4.24.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* $\mathsf{Cond}\bar{\div}$ *holds, if* $\bar{\rightsquigarrow}$ *satisfies* $\mathsf{CCDF}^B$ *then* $\div$ *satisfies* conjunctive factoring*.*

***Proof (theorem originally on page 86).***

| | | |
|---|---|---|
| 1 | $\delta \in B \div \theta$ iff either $\bar{\vdash}\ \delta$ or $\delta \mathrel{\bar{\rightsquigarrow}} \delta$ and $\theta \mathrel{\not\bar{\rightsquigarrow}} \delta$ | $\mathsf{Cond}\bar{\div}$ [p. 68] |
| 2 | $[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_1$ iff $\alpha \mathrel{\bar{\rightsquigarrow}} \delta_1]$, or<br>$[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_2$ iff $\beta \mathrel{\bar{\rightsquigarrow}} \delta_2]$, or<br>$[\alpha \wedge \beta \mathrel{\bar{\rightsquigarrow}} \delta_3$ iff $\alpha \mathrel{\bar{\rightsquigarrow}} \delta_3$ or $\beta \mathrel{\bar{\rightsquigarrow}} \delta_3]$ | Assume $\mathsf{CCDF}^B$ [p. 77]<br>(The word "Either" is omitted to save space) |
| 3 | $[\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_1$ iff $\alpha \mathrel{\not\bar{\rightsquigarrow}} \delta_1]$, or<br>$[\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_2$ iff $\beta \mathrel{\not\bar{\rightsquigarrow}} \delta_2]$, or<br>$[\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_3$ iff $\alpha \mathrel{\not\bar{\rightsquigarrow}} \delta_3$ and $\beta \mathrel{\not\bar{\rightsquigarrow}} \delta_3]$ | 2 (negating both sides of each iff statement) |
| 4 | $[[\delta_1 \mathrel{\bar{\rightsquigarrow}} \delta_1$ and $\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_1]$ iff $[\delta_1 \mathrel{\bar{\rightsquigarrow}} \delta_1$ and $\alpha \mathrel{\not\bar{\rightsquigarrow}} \delta_1]]$, or<br>$[[\delta_2 \mathrel{\bar{\rightsquigarrow}} \delta_2$ and $\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_2]$ iff $[\delta_2 \mathrel{\bar{\rightsquigarrow}} \delta_2$ and $\beta \mathrel{\not\bar{\rightsquigarrow}} \delta_2]]$, or<br>$[[\delta_3 \mathrel{\bar{\rightsquigarrow}} \delta_3$ and $\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_3]$ iff<br>$[\delta_3 \mathrel{\bar{\rightsquigarrow}} \delta_3$ and $\alpha \mathrel{\not\bar{\rightsquigarrow}} \delta_3]$ and $[\delta_3 \mathrel{\bar{\rightsquigarrow}} \delta_3$ and $\beta \mathrel{\not\bar{\rightsquigarrow}} \delta_3]]$ | 3 (adding the conjunct $\delta_i \mathrel{\bar{\rightsquigarrow}} \delta_i$ to both sides of each iff statement) |
| 5 | $[(\bar{\vdash}\ \delta_1$ or $[\delta_1 \mathrel{\bar{\rightsquigarrow}} \delta_1$ and $\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_1])$ iff<br>$(\bar{\vdash}\ \delta_1$ or $[\delta_1 \mathrel{\bar{\rightsquigarrow}} \delta_1$ and $\alpha \mathrel{\not\bar{\rightsquigarrow}} \delta_1])]$, or<br>$[(\bar{\vdash}\ \delta_2$ or $[\delta_2 \mathrel{\bar{\rightsquigarrow}} \delta_2$ and $\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta])$ iff<br>$(\bar{\vdash}\ \delta_2$ or $[\delta_2 \mathrel{\bar{\rightsquigarrow}} \delta_2$ and $\beta \mathrel{\not\bar{\rightsquigarrow}} \delta_2])]$, or<br>$[(\bar{\vdash}\ \delta_3$ or $[\delta_3 \mathrel{\bar{\rightsquigarrow}} \delta_3$ and $\alpha \wedge \beta \mathrel{\not\bar{\rightsquigarrow}} \delta_3])$ iff<br>$(\bar{\vdash}\ \delta_3$ or $[\delta_3 \mathrel{\bar{\rightsquigarrow}} \delta_3$ and $\alpha \mathrel{\not\bar{\rightsquigarrow}} \delta_3])$, and<br>$(\bar{\vdash}\ \delta_3$ or $[\delta_3 \mathrel{\bar{\rightsquigarrow}} \delta_3$ and $\beta \mathrel{\not\bar{\rightsquigarrow}} \delta_3])]$ | 4 (adding the disjunct $\bar{\vdash}\ \delta_i$ to both sides of each iff statement) |
| 6 | $[\delta_1 \in B \div \alpha \wedge \beta$ iff $\delta_1 \in B \div \alpha]$, or<br>$[\delta_2 \in B \div \alpha \wedge \beta$ iff $\delta_2 \in B \div \beta]$, or<br>$[\delta_3 \in B \div \alpha \wedge \beta$ iff $\delta_3 \in B \div \alpha$ and $\delta_3 \in B \div \beta]$ | 1, 5 |

| | | |
|---|---|---|
| 7 | $B \div \alpha \wedge \beta = B \div \alpha$, or<br>$B \div \alpha \wedge \beta = B \div \beta$, or<br>$B \div \alpha \wedge \beta = B \div \alpha \cap B \div \beta$ | 6 (by the set theory);<br>conjunctive factoring<br>derived |

□

**Theorem 4.25** (Base Dependence to Contraction)**.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\div}$ *holds, if* $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* CCDF$^B$*, then* $\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring*.*

***Proof (theorem originally on page 86).***
Assume that the relations $\bar{\rightsquigarrow}$ and $\div$ for base $B$ are such that Cond$\bar{\div}$ holds and that $\bar{\rightsquigarrow}$ is a base dependence relation also satisfying CCDF$^B$. Then, by Theorem 4.19 (proof on page 110), $\div$ is a saturated kernel contraction since $\bar{\rightsquigarrow}$ is a base dependence relation. Moreover, because $\bar{\rightsquigarrow}$ also satisfies CCDF$^B$, $\div$ further satisfies conjunctive factoring by Theorem 4.24. □

**Theorem 4.26.** *Given relations* $\bar{\leadsto}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\leadsto}$ *holds, if* $\div$ *satisfies* conjunctive factoring *then* $\bar{\leadsto}$ *satisfies* CCDF$^B$.

***Proof (theorem originally on page 86).***

| | | |
|---|---|---|
| 1 | $\theta \bar{\leadsto} \delta$ iff $\delta \in B$ and $\delta \notin B \div \theta$ | Cond$\bar{\leadsto}$ [p. 57] |
| 2 | $B \div \alpha \wedge \beta = B \div \alpha$, or<br>$B \div \alpha \wedge \beta = B \div \beta$, or<br>$B \div \alpha \wedge \beta = B \div \alpha \cap B \div \beta$ | Assume conjunctive factoring<br>(The word "Either" is omitted to save space) |
| 3 | $[\delta_1 \in B \div \alpha \wedge \beta$ iff $\delta_1 \in B \div \alpha]$, or<br>$[\delta_2 \in B \div \alpha \wedge \beta$ iff $\delta_2 \in B \div \beta]$, or<br>$[\delta_3 \in B \div \alpha \wedge \beta$ iff $\delta_3 \in B \div \alpha$ and $\delta_3 \in B \div \beta]$ | 2 (by the set theory) |
| 4 | $[\delta_1 \notin B \div \alpha \wedge \beta$ iff $\delta_1 \notin B \div \alpha]$, or<br>$[\delta_2 \notin B \div \alpha \wedge \beta$ iff $\delta_2 \notin B \div \beta]$, or<br>$[\delta_3 \notin B \div \alpha \wedge \beta$ iff $\delta_3 \notin B \div \alpha$ or $\delta_3 \notin B \div \beta]$ | 3 (negating both sides of each iff statement) |
| 5 | $[[\delta_1 \in B$ and $\delta_1 \notin B \div \alpha \wedge \beta]$ iff<br>$[\delta_1 \in B$ and $\delta_1 \notin B \div \alpha]]$, or<br>$[[\delta_2 \in B$ and $\delta_2 \notin B \div \alpha \wedge \beta]$ iff<br>$[\delta_2 \in B$ and $\delta_2 \notin B \div \beta]]$, or<br>$[[\delta_3 \in B$ and $\delta_3 \notin B \div \alpha \wedge \beta]$ iff<br>$[\delta_3 \in B$ and $\delta_3 \notin B \div \alpha]$ or<br>$[\delta_3 \in B$ and $\delta_3 \notin B \div \beta]]$ | 4 (adding the conjunct $\delta_i \in B$ to both sides of each iff statement) |
| 6 | $[\alpha \wedge \beta \bar{\leadsto} \delta_1$ iff $\alpha \bar{\leadsto} \delta_1]$, or<br>$[\alpha \wedge \beta \bar{\leadsto} \delta_2$ iff $\beta \bar{\leadsto} \delta_2]$, or<br>$[\alpha \wedge \beta \bar{\leadsto} \delta_3$ iff $\alpha \bar{\leadsto} \delta_3$ or $\beta \bar{\leadsto} \delta_3]$ | 1, 5; CCDF$^B$ derived |

□

**Theorem 4.27** (Contraction to Base Dependence)**.** *Given relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *such that* Cond$\bar{\rightsquigarrow}$ *holds, if* $\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring*, then* $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* $\mathsf{CCDF}^B$*.*

***Proof (theorem originally on page 86).***
Assume that the relations $\bar{\rightsquigarrow}$ and $\div$ for base $B$ are such that Cond$\bar{\rightsquigarrow}$ holds and that $\div$ is a saturated kernel contraction also satisfying conjunctive factoring. Then, by Theorem 4.20 (proof on page 115), $\bar{\rightsquigarrow}$ is a base dependence since $\div$ is a saturated kernel contraction. Moreover, because $\div$ also satisfies conjunctive factoring, $\bar{\rightsquigarrow}$ further satisfies $\mathsf{CCDF}^B$ by Theorem 4.26. □

**Theorem 4.28** (Main Characterization)**.** *Let the relations* $\bar{\rightsquigarrow}$ *and* $\div$ *for base* $B$ *be such that* $\div$ *satisfies* inclusion*,* $B \div \alpha \subseteq B$*, and that* Cond$\bar{\rightsquigarrow}$ *holds:* $\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$*. Then,*

$\div$ *is a saturated kernel contraction that satisfies* conjunctive factoring *if and only if* $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* $\mathsf{CCDF}^B$*.*

***Proof (theorem originally on page 87).***
This representation theorem proof closely resembles (with only a few changes) the proof of the Representation Theorem 4.23 (proof on page 121) which did not have any criteria for conjunctions (whereas here conjunctive factoring and $\mathsf{CCDF}^B$ are present).

Given that Cond$\bar{\rightsquigarrow}$ [p. 57] holds by assumption, the left to right direction is already proved in Theorem 4.27.

For the right to left direction, Theorem 4.25 (proof on page 123) can be used, provided that Cond$\bar{\div}$ [p. 68] holds. That is, it suffices to show that Cond$\bar{\div}$ holds in order to construct a saturated kernel contraction relation $\div$ that also satisfies conjunctive factoring, using a base dependence relation $\bar{\rightsquigarrow}$ that also satisfies $\mathsf{CCDF}^B$. Therefore, the last step needed for this proof is to show that Cond$\bar{\div}$ holds, which can be achieved exactly in the same way as in Representation Theorem 4.23 (proof on page 121). Thus, it is omitted for brevity here. □

## A.6 Proofs for §4.9: Base Dependence as a Reversible Generalization of Dependence

**Theorem 4.29** (Dependence Generalization)**.** *Let relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* $B$ *be such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* inclusion *is satisfied. In the special case where* $B$ *is logically closed,*

(1) *the following are logically equivalent:*

a) $\bar{\rightsquigarrow}$ *is a base dependence, which satisfies* Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity *and* redundancy

b) $\rightsquigarrow$ *is a dependence that satisfies* Def-$K$, Cond-ID, Disj, LE$^l$, LE$^r$, CCI$^r$ *and* CCD$_0^r$

c) $\div$ *is a saturated kernel contraction, which satisfies* success, inclusion, core-retainment, uniformity *and* relative closure

d) $\div$ *is a basic AGM contraction, which satisfies* $K{\div}1$ – $K{\div}6$

(2) *if any one of* 1.a–1.d *above hold, then* $\bar{\rightsquigarrow}$ *reduces to* $\rightsquigarrow$*:*

$\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\alpha \rightsquigarrow \beta$.

***Proof (theorem originally on page 89).***
Part (1): We show 1.a–1.d are logically equivalent using Theorems 2.19, 3.9 and 4.23.

| | | |
|---|---|---|
| 1 | $B = \mathrm{Cn}(B)$ | Logical Closure |
| 2 | $B \div \alpha \subseteq B$ | inclusion [p. 27] |
| 3 | $\alpha \rightsquigarrow \beta$ iff $\beta \in \mathrm{Cn}(B)$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ | Cond$\rightsquigarrow$ [p. 57] |
| 4 | $\alpha \mathrel{\bar{\rightsquigarrow}} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 5 | $\bar{\rightsquigarrow}$ is a base dependence relation (1.a) | Assumption |
| 6 | $\div$ is a saturated kernel contraction (1.c) | 2, 4, 5 and Thm. 4.23 |
| 7 | $\div$ is a basic AGM contraction which satisfies $K{\div}1$–$K{\div}6$ (1.d) | 1, 6 and Thm. 2.19 ([Han95]) |
| 8 | $\rightsquigarrow$ is a dependence relation (1.b) | 1, 2, 3, 7 and Thm. 3.9 ([FdCH96]) |
| 9 | Lines 5 through 8 (corresponding to 1.a–1.d) are logically equivalent as all Theorems 2.19, 3.9 and 4.23 connecting these lines use logical equivalence | |

Part (2): We start by assuming one of 1.a–1.d holds, and by part (1) it means all of them hold. Thus, $K\div 1$ holds and by Thm. 4.10 $\bar{\rightsquigarrow}$ and $\rightsquigarrow$ are equivalent.

| | | |
|---|---|---|
| 10 | One of the lines 5 to 8 (or 1.a–1.d) holds | Assumption for part (2) |
| 11 | Line 7 holds (so $K\div 1 - K\div 6$ hold) | 9, 10 (because all lines 5 to 8 hold) |
| 12 | $\div$ satisfies $K\div 1$ (a.k.a. closure) | 11 |
| 13 | $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \rightsquigarrow \beta$ | 1, 3, 4, 12 and Thm. 4.10 |

□

**Lemma 4.30.** *In the special case where base $B$ is logically closed, an operator $\div$ on $B$ is an AGM contraction satisfying $K\div 1 - K\div 6$, $K\div 7$ and $K\div 8$ if and only if $\div$ is a saturated kernel contraction that satisfies* conjunctive factoring.

***Proof (lemma originally on page 90).***

| | | |
|---|---|---|
| 1 | $B = \mathrm{Cn}(B)$ | Logical Closure |
| 2 | Let $\div$ be a saturated kernel contraction | Assumption |
| 3 | Let $\div$ also satisfy conjunctive factoring | Assumption |
| 4 | $\div$ satisfies $K\div 1 - K\div 6$ | 1, 2 and Thm. 2.19 ([Han95]) |
| 5 | $\div$ satisfies $K\div 7$ and $K\div 8$ | 1, 3, 4 and Thm. 2.2 ([AGM85]) |
| 6 | $\div$ satisfies $K\div 1 - K\div 8$ | 4, 5 |
| 7 | Let $\div$ satisfy $K\div 1 - K\div 6$ | Assumption |
| 8 | Let $\div$ also satisfy $K\div 7$ and $K\div 8$ | Assumption |
| 9 | $\div$ is a saturated kernel contraction | 1, 7 and Thm. 2.19 ([Han95]) |
| 10 | $\div$ satisfies conjunctive factoring | 1, 7, 8 and Thm. 2.2 ([AGM85]) |
| 11 | $\div$ satisfies $K\div 1 - K\div 8$ if and only if $\div$ is a saturated kernel contraction that also satisfies conjunctive factoring | 2-6, 7-10 |

□

**Theorem 4.31** (Dependence Generalization with Conjunction)**.** *Let relations* $\bar{\rightsquigarrow}$, $\rightsquigarrow$ *and* $\div$ *for base* $B$ *be such that* Cond$\bar{\rightsquigarrow}$ *and* Cond$\rightsquigarrow$ *hold and* inclusion *is satisfied. In the special case where* $B$ *is logically closed,*

(1) *the following are logically equivalent:*

a) $\bar{\rightsquigarrow}$ *is a base dependence that satisfies* Def-$B$, Cond-ID$^B$, conjugation, contribution, modularity, redundancy *and* CCDF$^B$

b) $\rightsquigarrow$ *is a dependence that satisfies* Def-$K$, Cond-ID, Disj, LE$^l$, LE$^r$, CCI$^l$, CCI$^r$, CCD$_0^l$ *and* CCD$_0^r$

c) $\div$ *is a saturated kernel contraction that satisfies* success, inclusion, core-retainment, uniformity, relative closure *and* conjunctive factoring

d) $\div$ *is an AGM contraction, which satisfies* $K\div 1$ – $K\div 6$, $K\div 7$ *and* $K\div 8$

(2) *if any one of* 1.a–1.d *above hold, then* $\bar{\rightsquigarrow}$ *reduces to* $\rightsquigarrow$*:*

$\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \rightsquigarrow \beta$.

***Proof (theorem originally on page 91)****.*

Part (1): We show 1.a–1.d are logically equivalent using Theorems 3.8, 4.28 and 4.30.

| | | |
|---|---|---|
| 1 | $B = \mathrm{Cn}(B)$ | Logical Closure |
| 2 | $B \div \alpha \subseteq B$ | inclusion [p. 27] |
| 3 | $\alpha \rightsquigarrow \beta$ iff $\beta \in \mathrm{Cn}(B)$ and $\beta \notin \mathrm{Cn}(B \div \alpha)$ | Cond$\rightsquigarrow$ [p. 57] |
| 4 | $\alpha \bar{\rightsquigarrow} \beta$ iff $\beta \in B$ and $\beta \notin B \div \alpha$ | Cond$\bar{\rightsquigarrow}$ [p. 57] |
| 5 | $\bar{\rightsquigarrow}$ is a base dependence relation that also satisfies CCDF$^B$ (1.a) | Assumption |
| 6 | $\div$ is a saturated kernel contraction that also satisfies conjunctive factoring (1.c) | 2, 4, 5 and Thm. 4.28 |
| 7 | $\div$ is an AGM contraction which satisfies $K\div 1$–$K\div 8$ (1.d) | 1, 6 and Lemma 4.30 |
| 8 | $\rightsquigarrow$ is a dependence relation (1.b) | 1, 2, 3, 7 and Thm. 3.8 ([FdCH96]) |
| 9 | Lines 5 through 8 (corresponding to 1.a–1.d) are logically equivalent as all Theorems 3.8, 4.28 and 4.30 connecting these lines use logical equivalence | |

Part (2): We start by assuming one of 1.a–1.d holds, and by part (1) it means all of them hold. Thus, $K\div\mathsf{1}$ holds and by Thm. 4.10 $\bar{\rightsquigarrow}$ and $\rightsquigarrow$ are equivalent.

| | | |
|---|---|---|
| 10 | One of the lines 5 to 8 (or 1.a–1.d) holds | Assumption for part (2) |
| 11 | Line 7 holds (so $K\div\mathsf{1}$–$K\div\mathsf{8}$ hold) | 9, 10 (because all lines 5 to 8 hold) |
| 12 | $\div$ satisfies $K\div\mathsf{1}$ (a.k.a. closure) | 11 |
| 13 | $\alpha \bar{\rightsquigarrow} \beta$ iff $\alpha \rightsquigarrow \beta$ | 1, 3, 4, 12 and Thm. 4.10 |

□

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