CENTRALIZERS IN ASSOCIATIVE ALGEBRAS

by

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Abstract

This thesis is divided into two parts. The subject of the first part is the structure of centralizers in associative algebras. We prove that over an algebraically closed field of characteristic zero, the centralizer of a nonconstant element in the second Weyl algebra has Gelfand-Kirillov (GK for short) dimension one, two or three. Those centralizers of GK dimension one or two are commutative and those of GK dimension three contain a finitely generated subalgebra which does not satisfy a polynomial identity. We show that for each $n \in \{1, 2, 3\}$ there exists a centralizer of GK dimension n. We also give explicit forms of centralizers for some elements of the second Weyl algebra and a connection between the problem of finite generation of centralizers in the second Weyl algebra and Dixmier's Fourth Problem.

Some algebras such as the first Weyl algebra, quantum planes and finitely generated graded algebras of GK dimension two can be viewed as subalgebras of some skew Laurent polynomial algebra over a field. We prove that if K is a field, $\sigma \in \mathrm{Aut}(K)$ and the fixed field of σ is algebraically closed, then the centralizer of a nonconstant element of a subalgebra of $K[x, x^{-1}; \sigma]$ is commutative and a free module of finite rank over some polynomial algebra in one variable.

In the last chapter, which is the second part of this thesis, we first prove a new version of Shirshov's theorem. We then use this theorem to prove an analogue of Kaplansky's theorem, i.e. if D is a central division k-algebra which is left algebraic of bounded degree d over some subfield, which is not necessarily central, then $[D : k] \leq d^2$.

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Contents

Introduction

This thesis is divided into two parts. The subject of the first part is the structure of centralizers in associative algebras. In the second part, which is chapter 4, division algebras that are left algebraic over a subfield are studied.

Centralizers in Associative Algebras

Except for Chapter 1 which gives the background needed throughout the thesis, every chapter of this thesis begins with an introductory section. For a ring R and an element $a \in R$, let $C(a; R)$ denote the centralizer of a in R.

Chapter 2 is divided into two parts. The first part consists of sections 2 to 5. This part reviews some of well-known results on centralizers in associative algebras. In the second part of this chapter, which is section 6, we give our results on centralizers in semiprime PI-rings. Let C be a commutative ring. A C-algebra A is said to *satisfy a polynomial identity* (PI for short) if there exists an integer n and a monic polynomial $f \in C\langle x_1, \ldots, x_n \rangle$ such that $f(r_1, r_2, \cdots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$. If every finitely generated subalgebra of an algebra is PI, then the algebra is called *locally* PI.

In section 2 of chapter 2, centralizers in the first Weyl algebra are studied. The first notable appearance of centralizers is probably in Schur's 1905 paper [57]. He considered the $\mathbb{C}$-algebra R consisting of ordinary differential operators over complex-valued functions which are infinitely differentiable. He proved that if P is an element of degree at least one in R, then $C(P; R)$ is a commutative $\mathbb{C}$-algebra.

Another result regarding the centralizer of an ordinary differential operator is due to Burchnall and Chaundy [20]. They proved that two ordinary differential operators P and Q of coprime degrees m and n commute if and only if there exists a polynomial $f \in \mathbb{C}[u, v]$ of the form $f = u^n - v^m + \cdots$ such that $f(P, Q) = 0$.

Schur's result was rediscovered and generalized by Flanders [27] and Amitsur [1] a few decades later. Let R be a field of characteristic zero and suppose that there exists a linear map $\delta : R \longrightarrow R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. This map is called a *derivation* of R. Now let $k = \{r \in R : \ \delta(r) = 0\}$. Consider the ring $S := R[y;\delta]$ of differential polynomials $f = \sum_{i=0}^{n} r_i y^i$, where multiplication is defined by $yr = ry + \delta(r)$ for all $r \in R$. Amitsur proved that if $n \geq 1$, then $C(f;S)$ is a commutative k-algebra and also a free module of finite rank over $k[f]$. In fact, Amitsur's proof of this result works for a more general setting. Suppose that R is a commutative domain of characteristic zero. We extend δ to $Q(R)$, the quotient field of R. If $k := \{q \in Q(R) : \ \delta(q) = 0\}$ is a subfield of R, then centralizers are again commutative and free modules of finite rank over some polynomial ring in one variable. We give a proof of this result in this section. If we choose $R = k[x]$, then $S \cong A_1(k)$, the first Weyl algebra over k. Now let $k := \mathbb{C}$ and $R := \mathbb{C}(x)$. Let $E := \mathrm{End}_{\mathbb{C}}(R)$ and let $L \in E$ be the left multiplication by x. Let A be the $\mathbb{C}$-subalgebra of E generated by L and d/dx. Then $R[y; d/dx] \cong A$ and we recover Schur's result.

In section 3 of chapter 2, we look at centralizers in the algebras of formal series and polynomials over a field. Let k be a field and let X be a set of noncommuting variables, which may or may not be finite. We denote by $k\langle\langle X\rangle\rangle$ the k-algebra of formal series. Cohn proved that if $f \in k\langle\langle X\rangle\rangle$ is not a constant, then $C(f;k\langle\langle X\rangle\rangle) = k[[g]]$, for some formal series g. Here $k[[g]]$ is the ring of formal power series in g. This is known as *Cohn's centralizer theorem* and we give a proof of this result in this section. By Cohn's centralizer theorem, the centralizer of every nonconstant element in $k\langle\langle X\rangle\rangle$ is commutative. Thus, since $k\langle X\rangle$ is a k-subalgebra of $k\langle\langle X\rangle\rangle$, the centralizer of a nonconstant element of $k\langle X\rangle$ is also commutative. Bergman proved that if $f \in k\langle X\rangle$ is not constant, then $C(f;k\langle X\rangle)$ is integrally closed. He used this result to prove that $C(f;k\langle X\rangle) = k[g]$ for some $g \in k\langle X\rangle$. This is called *Bergman's centralizer theorem*. We do not prove this theorem but we give a nice application of it.

In section 4 of chapter 2, we take a different approach to study centralizers. We use the GK dimension theory to obtain some information about centralizers. The GK dimension of an algebra over a field was first introduced by Gelfand and Kirillov in 1966 [29]. The GK dimension measures the rate of the growth of an algebra in terms of any generating set of the algebra. Let k be a field and let A be a finitely generated k-algebra. Let V be a generating subspace of A which contains k. The GK *dimension* of A is defined by $\mathrm{GKdim}(A) := \limsup_{n\to\infty} \log_n(\dim V^n)$. The definition does not depend on V.
The results in this section are due to Bell and Small. The first result that we prove is this:

if A is a finitely generated domain of GK dimension two over some algebraically closed field k and if A is not PI and $a \in A \setminus k$, then $C(a;A)$ is a commutative domain of GK dimension one. We also prove that if $a \in A$ is not algebraic over the center of the quotient division algebra of A, then $C(a;A)$ is PI. This result is a consequence of a lemma which is very useful for studying centralizers in algebras of low GK dimension. The lemma states that if k is a field and A is a finitely generated k-algebra which is a domain of finite GK dimension, then $\mathrm{GKdim}(C(a;A)) \leq \mathrm{GKdim}(A) - 1$ provided that a is not algebraic over the center of the quotient division algebra of A.

In section 5 of chapter 2, we give Makar-Limanov's result on centralizers in quantum planes. Let $q \in \mathbb{C}$. The *quantum plane* $A := k_q[x,y]$ is the $\mathbb{C}$-algebra generated by x and y subject to the relation $yx = qxy$. Using a pretty argument, Makar-Limanov proved that if q is not a root of unity and $a \in A \setminus \mathbb{C}$, then $C(a;A) \subseteq \mathbb{C}[b]$, for some $b \in A$. In particular, $C(a;A)$ is commutative.

Section 6 of chapter 2 presents our results on centralizers in semiprime PI-rings. Let k be a field and let R be a finite dimensional central simple k-algebra. Let $a \in R \setminus k$. By a result of Werner [65], the center of $C(a;R)$ is $k[a]$. Now let R be any semiprime PI-ring. One of the properties of the maximal left quotient ring $Q := Q_{\max}(R)$ of R is that the localization Q_M at any central maximal ideal M of Q is a finite dimensional central simple algebra. Then using Werner's result and some other properties of Q, we find the center of the centralizer of a noncentral element of R (Theorem 2.6.6). We also characterize semiprime PI-rings in which the centralizer of every noncentral element is commutative. The characterization is in terms of finite dimensional central division algebras in which the centralizer of every noncentral element is commutative (Theorem 2.6.12).

In chapter 3, we give our results on centralizers in two algebras both of which contain the first Weyl algebra as a subalgebra, i.e. the second Weyl algebra and the algebra of skew Laurent polynomials.

If k is an algebraically closed field, then Amitsur's result on centralizers in $A_1(k)$ becomes a trivial result of two facts. The first one is that, by a result of Bell (Theorem 2.4.6), centralizers must have GK dimension one. The second fact is that a domain of GK dimension one over an algebraically closed field is both commutative and a free module of finite rank over some polynomial algebra in one variable. Of course, these facts were not known to Amitsur when he was writing his paper in the 1950s.

It is natural now to ask about centralizers in the second Weyl algebras. By the first fact that we just mentioned, the GK dimension of centralizers is at most 3. The second fact gives the form of centralizers of GK dimension one. There is no algebra of GK dimension strictly between one and two, by Bergman's gap theorem. So we only need to study centralizers of GK dimension between 2 and 3 and the following theorem is the result of our study.

Theorem 3.2.11. *Let k be an algebraically closed field of characteristic zero. Let C denote the centralizer of a nonscalar element a in $A_2(k)$, the second Weyl algebra over k. Then* $\mathrm{GKdim}(C) \in \{1,2,3\}$. *If* $\mathrm{GKdim}(C) \in \{1,2\}$, *then C is commutative and if* $\mathrm{GKdim}(C) = 3$, *then C is not locally PI. Furthermore, for each $n \in \{1,2,3\}$ there exists an element of $A_2(k)$ whose centralizer has* GK *dimension n.*

In the second subsection of section 2 of chapter 3, we find explicit form of centralizers of some elements of the second Weyl algebra (Theorem 3.2.12) and we use it to show that it is not always the case that centralizers of GK dimension three contain a copy of the first Weyl algebra. However, our counter-example contains a subalgebra of GK dimension two which is isomorphic to some subalgebra of the first Weyl algebra. In the third subsection of this section, we give a necessary condition for a centralizer in the second Weyl algebra to contain a nontrivial simple subalgebra.

In section 3 of chapter 3, we consider the algebra of skew Laurent polynomials over fields of characteristic zero. Some of important algebras such as the first Weyl algebra, quantum planes and finitely generated graded algebras of GK dimension two can be embedded into certain skew Laurent polynomial algebras over fields. The main result of this section is the following.

Theorem 3.3.7. *Let K be a field and let $\sigma \in \mathrm{Aut(K)}$. Let k be the fixed field of σ and suppose that k is algebraically closed. Let A be a subalgebra of $K[x, x^{-1}; \sigma]$. Let $f \in A \setminus k$ and let C denote the centralizer of f in A. If $f \in K$, then $C = A \cap K$. If $f \notin K$, then C is commutative and a free module of finite rank over $k[u]$ for some $u \in C$.*

A few problems are posed in section 4 of chapter 3 and a connection between the problem of finite generation of centralizers in the second Weyl algebra and Dixmier's Fourth Problem is given.

Left Algebraic Division Algebras of Bounded Degree

Chapter 4 is not directly related to centralizers and so we can look at this chapter as the second part of this thesis. The subject of this chapter is division algebras that are (left) algebraic over a subfield which may or may not be central. Bell and Rogalski [15] proved that if D is a division algebra over an uncountable field of characteristic zero and if D is not left or right algebraic over the centralizer of a nonzero element, then D contains a free subalgebra on two generators. This may be considered as a link between centralizers and the (left and right) algebraic property in division algebras.

Let C be a commutative ring. It is one of Jacobson's results that a C-algebra which is integral of bounded degree over C is PI. By Kaplansky's theorem, a (left) primitive PI-algebra is a finite dimensional central simple algebra. So a division algebra which is algebraic of bounded degree over its center is a finite dimensional central division algebra. It is natural now to ask whether or not we have the same result if a division algebra D is algebraic of bounded degree over a subfield K which is not inside the center. But first, since K is not central, we need to explain what we mean by *algebraic*.

We say that an element $x \in D$ is *left algebraic* over K if $x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1}x + \alpha_n = 0$ for some integer n and $\alpha_i \in K$. If the coefficients are on the right-hand side of powers of x, then we say that x is right algebraic over K. We prove the following result which is an analogue of Kaplansky's theorem. This is joint work with Bell and Drensky [17].

Theorem 4.1.2. *Let d be a natural number, let D be a division ring with center $Z(D)$ and let K be any subfield of D. If D is left algebraic of bounded degree d over K, then $[D : Z(D)] \leq d^2$.*

The crucial step in the proof of the above theorem is to show that every finitely generated subalgebra of D satisfies some standard polynomial identity S_n where n depends only on the number of generators of the subalgebra. It turns out that in order to prove this, we need to strengthen the well-known Shirshov's theorem.

Let m, p and q be natural numbers and let M be a free monoid generated by m elements $x_1, \ldots, x_m$. For $w \in M$, let $|w|$ be the length of w. We say that w is q-decomposable if there exist $w_1, \ldots, w_q \in M$ such that $w = w_1 w_2 \cdots w_q$ and for all permutations $\sigma \in \mathrm{Sym}(q)$ with $\sigma \neq \mathrm{id}$ we have $w_1 w_2 \cdots w_q \succ w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(q)}$, where $\succ$ is degree lexicographic order. If in addition, we can choose $w_1, \ldots, w_q$ such that $(q-1)|w_i| < |w|$ for all $i \in \{1, \ldots, q\}$, then we say that w is strongly q-decomposable. Shirshov [58] proved that there exists a positive integer $N(m,p,q)$, depending on m, p, and q, such that every word on $x_1,\ldots,x_m$ of length greater than $N(m,p,q)$ has either a q-decomposable subword or has a nontrivial subword of the form w^p. We prove the following stronger result.

Theorem 4.2.4. *Let m, p, and q be natural numbers and let M be a free monoid generated by m elements $x_1,\ldots,x_m$. Then there exists a positive integer $N(m,p,q)$, depending on m, p, and q, such that every word on $x_1,\ldots,x_m$ of length greater than $N(m,p,q)$ has either a strongly q-decomposable subword or has a nontrivial subword of the form t^p.*

Chapter 1

Preliminaries

This chapter provides the reader with necessary background to follow the thesis easily. All rings throughout this thesis are associative with multiplicative identity. If R is a ring, the *center* of R is $Z(R) := \{r \in R : \ rx = xr, \text{ for all } x \in R\}$. We denote by $M_n(R)$ the ring of $n \times n$ matrices with entries in R. All R-modules, if not specified, are left R-modules. Let M be an R-module and let X be a subset of M. We denote by $\mathrm{r.ann}_R(X)$ and $\mathrm{l.ann_R(X)}$ the right and the left annihilator of X in R, respectively. So $\mathrm{r.ann}_R(X) := \{r \in R : \ Xr = 0\}$ and $\mathrm{l.ann}_R(X) := \{r \in R : \ rX = 0\}$.

1.1 The Jacobson Radical

We begin this section with a quick review of the definition of the Jacobson radical of a ring.

Definition 1.1.1. The *Jacobson radical* $J(R)$ of a ring R is the intersection of the left maximal ideals of R.

Before giving some of characterizations of $J(R)$, let us recall some definitions.

Definition 1.1.2. An R-module M is called *simple* if M has no nonzero proper submodule. A nonzero ring R is called simple if R has no nonzero proper ideal.

Definition 1.1.3. An ideal P of R is called *left primitive* if $P = \mathrm{l.ann}_R(M)$ for some simple left R-module M. If the zero ideal is left primitive, then R is called a *left primitive ring.* Right primitive ideals and rings are defined analogously.

The following proposition gives two characterizations of $J(R)$.

Proposition 1.1.4 ([39], Lemma 4.1)**.** *Let R be a ring and let $U(R)$ be the set of invertible elements of R.*

(1) The Jacobson radical $J(R)$ is the intersection of the left primitive ideals of R.

(2) We have $J(R) = \{x \in R : \ 1 - rx \in U(R), \forall r \in R\}$.

Definition 1.1.5. A ring R is called *semiprimitive* if $J(R) = (0)$.

Let $S = R[x]$ be the ring of polynomials in the central variable x. Amitsur [39, Theorem 5.10] proved that $J(S) = (R \cap J(S))[x]$. In fact, Amitsur's result holds for polynomial rings in any number of variables, finite or infinite. We will not need this theorem in this thesis and so we will not prove it. What we prove is that the leading coefficient of every element of $J(S)$ is nilpotent.

Proposition 1.1.6. *Let $S = R[x]$ be the ring of polynomials in the central variable x with coefficients in R. The leading coefficient of every element of $J(S)$ is nilpotent. In particular, $R \cap J(S)$ is a nil ideal of R and so if R has no nonzero nil ideals, then S is semiprimitive.*

Proof. Let $f = \sum_{i=0}^{n} a_i x^i$ be a nonzero element of $J(S)$. Then $xf \in J(S)$ and thus, by Proposition 1.1.4, there exists some unit $g = \sum_{i=0}^{m} b_i x^i \in S$ such that $(1 - xf)g = 1$. Thus $g = 1 + xfg$ and then an easy induction shows that for all positive integers k

$$g = x^k f^k g + \sum_{i=0}^{k-1} x^i f^i. \tag{1.1}$$

So, choosing k large enough, for every $0 \leq i \leq m$ the coefficient of $x^{(n+1)k+i}$ on the left hand-side of (1.1) is zero. Thus $a_n^k b_i = 0$ for all i and hence $a_n^k g = 0$. Therefore $a_n^k = 0$ because g is a unit of S. □

Lemma 1.1.7 ([39], Theorem 4.12)**.** *If R is left artinian, then $J(R)$ is nilpotent.*

Theorem 1.1.8. (Amitsur, [39, Theorem 4.20]) *Let k be a field and let A be a k-algebra. If $\dim_k A < |k|$, as cardinal numbers, then $J(A)$ is nil.*

Proof. If k is finite, then $\dim_k A < \infty$ and so A is artinian. Thus $J(A)$ is nilpotent, by Lemma 1.1.7, and hence nil. Suppose now that k is infinite. Let $a \in J(A)$. Then, by Proposition 1.1.4, $a - \gamma$ is invertible for all $0 \neq \gamma \in k$.

Now consider the set $S = \{(a-\gamma)^{-1} : 0 \neq \gamma \in k\}$. Since k is infinite, we have $|S| = |k| > \dim_k A$. Thus the elements of S cannot be k-linearly independent. So there exist an integer $n \geq 1$ and nonzero elements $\beta_i, \gamma_i \in k$ such that

$$\sum_{i=1}^{n} \beta_i (a-\gamma_i)^{-1} = 0. \tag{1.2}$$

Obviously all $a - \gamma_i$ commute with each other and with all β_i. Multiplying (1.2) through by $\prod_{i=1}^{n}(a-\gamma_i)$ clears denominators and gives $\sum_{i=1}^{n} \beta_i \prod_{j \neq i}(a-\gamma_j) = 0$ and so if we let $p(x) = \sum_{i=1}^{n} \beta_i \prod_{j \neq i}(x-\gamma_j) \in k[x]$, then $p(a) = 0$. Also, $p(x)$ is not identically zero because, for example, $p(\gamma_1) \neq 0$. So a is algebraic over k and thus

$$a^m + \alpha_1 a^{m-1} + \cdots + \alpha_m = 0,$$

for some integer $m \geq 1$ and $\alpha_i \in k$. We have $\alpha_m = 0$ because a is not invertible. If $\alpha_i = 0$ for all $1 \leq i \leq m$, then $a^m = 0$ and we are done. Thus we may assume that there exists $1 \leq \ell < m$ such that $\alpha_\ell \neq 0$ and

$$a^m + \alpha_1 a^{m-1} + \cdots + \alpha_\ell a^{m-\ell} = 0.$$

Let $c = -\alpha_\ell^{-1}(a^{\ell-1} + \alpha_1 a^{\ell-2} + \cdots + \alpha_{\ell-1})$. Then $\alpha_\ell(1-ca)a^{m-\ell} = 0$ and hence $a^{m-\ell} = 0$, by Proposition 1.1.4. □

Remark 1.1.9. If A is finitely generated and k is uncountable, then $\dim_k A$ is countable, because A is finitely generated, and thus $\dim_k A < |k|$. So, by the theorem, $J(A)$ is nil.

1.2 Ore Localization

Throughout this section R is a ring and S is a multiplicatively closed subset of R such that $1 \in S$ and $0 \notin S$. We know from commutative algebra that if R is commutative, then we can always localize R at S and find the quotient ring $S^{-1}R$. An element $q \in S^{-1}R$ is in the form $s^{-1}r$, where $s \in S$ and $r \in R$. This quotient ring contains R if S consists of elements which are not zero-divisors in R. Also, every element of S is a unit in $S^{-1}R$. Suppose now that R is not commutative and we want to construct a left quotient ring $S^{-1}R$ with the same properties as the commutative case. Let $r \in R$ and $s \in S$. Then rs^{-1} would be an element of $S^{-1}R$ and so $rs^{-1} = s'^{-1}r'$ for some $s' \in S$ and $r' \in R$. Thus $s'r = r's$, i.e. $Rs \cap Sr \neq \emptyset$. So $Rs \cap Sr \neq \emptyset$ for all $r \in R, s \in S$. This property is called the *Ore condition* and S is called a *left Ore* set. So we have shown that a necessary condition for a left quotient ring $S^{-1}R$ to exist is that S is left Ore and we will prove later in this section that this is also a sufficient condition in many important cases, e.g. when S is central.

1.2.1 Definitions and Basic Results

We begin with giving the definition of a left (resp. right) quotient ring with respect to a multiplicatively closed subset. Then we investigate the existence and uniqueness and also the common denominator property in a quotient ring.

Definition 1.2.1. A ring Q is called a *left quotient ring* of R (with respect to S) if there exists a ring homomorphism $f : R \longrightarrow Q$ such that the following conditions are satisfied.

(1) $f(s)$ is a unit in Q for all $s \in S$.

(2) Every element of Q is in the form $(f(s))^{-1}f(r)$ for some $r \in R$ and $s \in S$.

(3) $\ker f = \{r \in R : \ sr = 0 \text{ for some } s \in S\}$.

A *right quotient ring* of R is defined symmetrically.

We now show that if a left or right quotient ring exists, then up to isomorphism it is unique. This is an immediate result of the following lemma.

Lemma 1.2.2. *Suppose that $g : R \longrightarrow R_1$ is a ring homomorphism and Q is a left or right quotient ring of R with respect to S. If $g(s)$ is a unit in R_1 for every $s \in S$, then there exists a unique homomorphism $h : Q \longrightarrow R_1$ which extends g.*

Proof. Assuming that f is the map in Definition 1.2.1 we define h by

$$h(f(s)^{-1}f(r)) = (g(s))^{-1}g(r)$$

for all $r \in R$ and $s \in S$. We only prove that h is well-defined. Suppose that $(f(s))^{-1}f(r) = (f(s'))^{-1}f(r')$, for some $r, r' \in R$ and $s, s' \in S$. Then $f(s')(f(s))^{-1}f(r) = f(r')$. We also have $(f(t))^{-1}f(r'') = f(s')(f(s))^{-1}$ for some $t \in S$ and $r'' \in R$. Hence $f(r''r) = f(tr')$ and $f(r''s) = f(ts')$. Thus $r''r - tr', ts' - r''s \in \ker f$ and so $ur''r = utr'$, $vts' = vr''s$, for some $u, v \in S$. Therefore $g(r'')g(r) = g(t)g(r')$ and $g(t)g(s') = g(r'')g(s)$. It follows that $g(t)g(r') = g(r'')g(r) = g(t)g(s')(g(s))^{-1}g(r)$ and so $(g(s))^{-1}g(r) = (g(s'))^{-1}g(r')$ because $g(t)$ is a unit in R_1. $\square$

Theorem 1.2.3. *If a left (resp. right) quotient ring R exists, then it is unique up to isomorphism. If R has a left quotient ring Q and a right quotient ring Q' with respect to S, then $Q \cong Q'$.*

Proof. An easy result of the lemma. $\square$

Definition 1.2.4. The left (resp. right) quotient ring of R with respect to S, if it exists, is also called the *left (resp. right) localization of* R *at* S and is denoted by $S^{-1}R$ (resp. RS^{-1}).

The question now is that under what conditions the left (resp. right) quotient ring of R exists. Our next goal is to find an answer for this question.

Definition 1.2.5. A multiplicatively closed subset S of R is called *left Ore* if it satisfies the *Ore condition*, i.e. $Rs \cap Sr \neq \emptyset$ for all $r \in R$ and $s \in S$. Similarly, if $sR \cap rS \neq \emptyset$ for all $r \in R,\ s \in S$, then S is called *right Ore*. We call S *Ore* if S is both left and right Ore.

Definition 1.2.6. A left Ore set S is called a *left denominator set* if for every $r \in R$ and $s \in S$ with $rs = 0$, there exists $s' \in S$ such that $s'r = 0$. Similarly, A right Ore set S is called a *right denominator set* if for every $r \in R$ and $s \in S$ with $sr = 0$, there exists $s' \in S$ such that $rs' = 0$. An Ore set S is called a *denominator set* if S is both left and right denominator.

Theorem 1.2.7 ([55], Proposition 3.1.3 and Theorem 3.1.4)**.** *The left (resp. right) quotient ring of R with respect to S exists if and only if S is a left (resp. right) denominator set.*

Proof. Suppose that the left quotient ring of R with respect to S exists and let $Q = S^{-1}R$. Let f be the homomorphism in Definition 1.2.1. Let $r \in R$ and $s \in S$. Then, since $f(r)(f(s))^{-1} \in Q$, there exist $r' \in R$ and $s' \in S$ such that $f(r)(f(s))^{-1} = (f(s'))^{-1}f(r')$. Thus $f(s'r) = f(r's)$ and hence $s'r - r's \in \ker f$. So $s''(s'r - r's) = 0$ for some $s'' \in S$ giving us $s''s'r = s''r's \in Rs \cap Sr$. So S is left Ore. Also, if $rs = 0$ for some $r \in R$ and $s \in S$, then $f(r)f(s) = f(rs) = 0$ and hence $f(r) = 0$ because $f(s)$ is a unit in Q. Thus $r \in \ker f$ and hence $s'r = 0$ for some $s' \in S$. This proves that S is a denominator set.

Conversely, assuming that S is a left denominator set, we construct the left quotient ring of R by first defining a relation on $S \times R$. We say $(s_1, r_1) \sim (s_2, r_2)$ if and only if there exist $r_1', r_2' \in R$ such that $r_1'r_1 = r_2'r_2$ and $r_1's_1 = r_2's_2 \in S$. It turns out that the relation $\sim$ is an equivalence relation. The equivalence class of $(s, r) \in S \times R$ is then denoted by $s^{-1}r$ and we let Q be the set of all $s^{-1}r$.

We are going to put a ring structure on Q. Let $\alpha = s_1^{-1}r_1,\ \beta = s_2^{-1}r_2$ be two elements of Q. By the left Ore condition, $Rs_1 \cap Ss_2 \neq \emptyset$ and thus there exist $r \in R$ and $s \in S$ such that $rs_1 = ss_2 = t \in S$. Now define $\alpha + \beta = t^{-1}(rr_1 + sr_2)$. Also, since $Rs_2 \cap Sr_1 \neq \emptyset$, there exist some $r' \in R$ and $s' \in S$ such that $r's_2 = s'r_1$. Let $s's_1 = t' \in S$ and define $\alpha\beta = t'^{-1}r'r_2$. It is straightforward to show that the addition and the multiplication that we have defined are well-defined and satisfy all the conditions needed to make Q a ring. Let $1 = 1_R$. Then $1^{-1}0 = 0_Q$ and $1^{-1}1 = 1_Q$. Finally define $f : R \longrightarrow Q$ by $f(r) = 1^{-1}r$. We see that f is a ring homomorphism. Now $r \in \ker f$ if and only if $(1, r) \sim (1, 0)$ if and only if there exist some $r_1, r_2 \in R$ such that $r_1 r = 0$ and $r_1 = r_2 = s \in S$. Thus $\ker f = \{r \in R : \ sr = 0, \text{for some } s \in S\}$. Therefore $Q = S^{-1}R$ is the left quotient ring of R with respect to S. □

We also have the common denominator property in quotient rings as the next result shows.

Proposition 1.2.8 ([55], Lemma 3.1.10)**.** *Every finite subset of* $S^{-1}R$ *can be written as* $\{s^{-1}x_1, \cdots, s^{-1}x_n\}$.

Proof. We first show that if $s_1, \cdots, s_n \in S$, then there exist $r_1, \ldots, r_n \in R$ such that $r_1s_1 = \cdots = r_ns_n \in S$. The proof is by induction on n. We choose $r_1 = 1$ if $n = 1$. Suppose that $n > 1$ and that the claim is true for $n - 1$. Choose $r'_1, \ldots, r'_{n-1}$ so that $r'_1s_1 = \cdots = r'_{n-1}s_{n-1} = s \in S$. Also there exist $r_n \in R$ and $t \in S$ such that $r_ns_n = ts$, since $Rs_n \cap Ss \neq \emptyset$. Let $r_j := tr'_j$ for $j = 1, \ldots, n-1$. Then for all $1 \leq j \leq n-1$ we have $r_js_j = tr'_js_j = ts = r_ns_n$ and $ts \in S$.

Now, let $\{s_1^{-1}y_1, \ldots, s_n^{-1}y_n\} \subset S^{-1}R$. As we just proved, there exist $r_1, \ldots, r_n \in R$ such that $r_1s_1 = \cdots = r_ns_n = s \in S$. Let $x_j = r_jy_j,\ 1 \leq j \leq n$. Then $s_j^{-1}y_j = s^{-1}x_j$ for all j. □

1.2.2 Localization at Regular Submonoids

The left or right quotient ring of a ring is more useful if it contains the ring, i.e. $\ker f = (0)$ where f is the map defined in Definition 1.2.1. This leads us to the following definition.

Definition 1.2.9. An element $s \in R$ is called *left regular* if $\text{l.ann}_R(s) = (0)$ and it is called *right regular* if $\text{r.ann}_R(s) = (0)$. If s is both left and right regular, then we say that s is *regular*. A subset S of R is called a *regular submonoid* if it is multiplicatively closed, $1 \in S$ and every element of S is regular.

Clearly a regular submonoid is a left (resp. right) denominator set if and only if it is left (resp. right) Ore. Thus if S is a regular submonoid, then $S^{-1}R$ exists if and only if S is left (resp. right) Ore. In this case, the map $f : R \longrightarrow S^{-1}R$ (resp. $f : R \longrightarrow RS^{-1}$) defined by $f(r) = 1^{-1}r$ (resp. $f(r) = r1^{-1}$) would be injective because if $r \in \ker f$, then $sr = 0$ (resp. $rs = 0$) for some $s \in S$ and hence $r = 0$. That means R can be viewed as a subring of $S^{-1}R$ (resp. RS^{-1}). Note that if S is a regular submonoid of R contained in the center of R, then S is Ore.

Definition 1.2.10. Let S be the set of all regular elements of R. If S is left Ore, then $Q(R) := S^{-1}R$ is called the *classical left quotient ring* of R. Similarly, if S is right Ore, then $Q(R) := RS^{-1}$ is called the *classical right quotient ring* of R. If $Q(R)$ is the left or right classical quotient ring of R, then R is called an *order* in $Q(R)$.

Note that localization at a regular submonoid S does not always give us a larger ring. For example if R is left artinian, then $S^{-1}R = R$. This is easy to see: for any $s \in S$ the chain $Rs \supseteq Rs^2 \supseteq \cdots$ must terminate, i.e. there exists some integer $n \geq 1$ such that $Rs^n = Rs^{n+1}$. Then $s^n = rs^{n+1}$ for some $r \in R$ and thus $(1 - rs)s^n = 0$ which implies $rs = 1$. That means $Rs = R$ and hence $Rs \cap Sr' = Sr' \neq \emptyset$ for all $s \in S,\ r' \in R$. So S is left Ore and $S^{-1}R = R$.

The following easy proposition gives a relationship between the ideals of R and $Q(R)$. This result will be used in the proof of Goldie's theorem in section 3 of this chapter. We assume that S, the regular submonoid of R, is left Ore and so $Q(R) = S^{-1}R$ exists.

Proposition 1.2.11. *Let I, J be left ideals of $Q(R)$. Then $I = Q(R)(I \cap R)$ and $I \cap R = J \cap R$ if and only if $I = J$.*

Proof. Straightforward. □

An important case of localization at regular submonoids is when R is a domain.

Definition 1.2.12. Let R be a domain and let $S = R \setminus \{0\}$. If S is left Ore, then R is called a *left Ore domain*. Similarly, if S is right Ore, then R is called a *right Ore domain*. If S is Ore, then R is called an *Ore domain*.

Clearly a domain R is a left (resp. right) Ore domain if and only if $Rr_1 \cap Rr_2 \neq (0)$ (resp. $r_1R \cap r_2R \neq (0)$) for all nonzero elements $r_1, r_2 \in R$. It is also obvious that if R is a left (resp. right) Ore domain, then $Q(R)$ is a division ring. Every division ring is an Ore domain. The following result gives an important class of Ore domains.

Proposition 1.2.13 ([48], Theorem 2.1.15)**.** *A left (resp. right) noetherian domain R is a left (resp. right) Ore domain.*

Proof. Let $r_1, r_2 \in R$ be nonzero. We need to show that $Rr_1 \cap Rr_2 \neq (0)$. So suppose, to the contrary, that $Rr_1 \cap Rr_2 = (0)$. We show that the sum $\sum_{n=0}^{\infty} Rr_1 r_2^n$ is direct and thus R cannot be noetherian. Suppose that the sum is not direct and choose n to be the smallest positive integer for which there exist $a_j \in R, \ j = 0, \cdots, n$, not all zero, such that $\sum_{j=0}^{n} a_j r_1 r_2^j = 0$. Then $n \geq 1$ because R is a domain and thus

$$-a_0 r_1 = \left(\sum_{j=0}^{n-1} a_{j+1} r_1 r_2^j \right) r_2 \in Rr_1 \cap Rr_2 = (0).$$

Hence $a_0 = 0$ and $\sum_{j=0}^{n-1} a_{j+1} r_1 r_2^j = 0$, contradicting the minimality of n. □

Let X be a set of noncommuting variables and let C be a commutative ring. We denote by $C\langle X \rangle$ the C-algebra of polynomials in variables from X. If $X = \{x_1, \ldots, x_n\}$ is finite, then we write $C\langle x_1, \ldots, x_n \rangle$ for $C\langle X \rangle$. Note that if C is a domain and $|X| \geq 2$, then $R := C\langle X \rangle$ is not Ore because then $Rx_1 \cap Rx_2 = (0)$ for any two distinct elements $x_1, x_2 \in X$.

We are now going to prove that those domains which are not Ore have one thing in common: all of them contain a polynomial ring in two noncommuting variables.

Lemma 1.2.14. (Jategaonkar, [40, Lemma 9.21]) *Let R be a ring and suppose that $a, b \in R$ are left or right linearly independent over R. Let C be a central subring of R. Then the C-subalgebra of R generated by a and b is isomorphic to $C\langle x, y \rangle$ for some noncommuting variables x and y.*

Proof. We assume that a, b are left linearly independent over R. We need to prove that the set of all monomials in a and b is a C-basis for the algebra generated by a and b over C. So suppose that the claim is false. Then there exists a nonzero $f \in C\langle x, y \rangle$ of minimum total degree such that $f(a, b) = 0$. Write

$$f(x, y) = u + g(x, y)x + h(x, y)y,$$

where $u \in C$ and $g, \ h \in C\langle x, y \rangle$ with $g \neq 0$. Now

$$0 = bf(a, b) = ub + bg(a, b)a + bh(a, b)b = bg(a, b)a + (u + bh(a, b))b.$$

Thus $bg(a,b) = 0$ because a and b are left linearly independent over R. Again, we can write

$$g(x,y) = u' + g'(x,y)x + h'(x,y)y,$$

where $u' \in C$ and $g', h' \in C\langle x,y\rangle$. Then

$$0 = bg(a,b) = bg'(a,b)a + (u' + bh'(a,b))b$$

and so $bg'(a,b) = 0$. But that contradicts the minimality of $\deg f$ because $\deg yg'(x,y) \leq \deg g(x,y) < \deg f(x,y)$. □

The converse of Lemma 1.2.14 is also true because clearly $Rx \cap Ry = (0)$.

Proposition 1.2.15. *Let R be a C-algebra which is a domain. Then R is Ore if and only if it does not contain a polynomial C-algebra in two noncommuting variables.*

Proof. Choose $r_1, r_2 \neq 0$ such that $Rr_1 \cap Rr_2 = (0)$. Then r_1, r_2 are left linearly independent over R and thus Lemma 1.2.14 can be applied. A similar argument shows that $r_1 R \cap r_2 R \neq (0)$ for all nonzero elements r_1, r_2 of R. □

Definition 1.2.16. Let S be the set of regular elements of R contained in $Z := Z(R)$. Then S is clearly Ore and thus $Q_Z(R) := S^{-1}R$ exists. The ring $Q_Z(R)$ is called the *central localization* of R.

Definition 1.2.17. A ring R is called *prime* if $IJ \neq (0)$ for any nonzero ideals I, J of R. If R has no nonzero nilpotent ideal, then R is called *semiprime*.

Clearly every prime ring is semiprime and it is easy to see that R is prime (resp. semiprime) if and only if $aRb = \{0\}$ (resp. $aRa = \{0\}$) implies $a = 0$ or $b = 0$ (resp. $a = 0$) for all $a, b \in R$.

Proposition 1.2.18 ([54], Propositions 1.7.4 and 1.7.5)**.** *Let $Z := Z(R)$ be the center of R and let S be the set of regular elements of R contained in Z.*

(1) $Z(Q_Z(R)) = S^{-1}Z$.

(2) If R is prime (resp. semiprime), then $Q_Z(R)$ is prime (resp. semiprime).

Proof. Clearly $S^{-1}Z \subseteq Z(Q_Z(R))$. Now let $q = s^{-1}a \in Z(Q_Z(R))$ and $r \in R$. Then

$$s^{-1}(ra - ar) = r(s^{-1}a) - (s^{-1}a)r = rq - qr = 0$$

and hence $ra = ar$, i.e. $a \in Z$. Thus $Z(Q_Z(R)) \subseteq S^{-1}Z$. The proof of the second part is straightforward. □

1.3 Goldie's Theorem

The idea of localization is to embed a ring into a larger ring with a nicer structure. If the larger ring has a more complicated structure, then there is no point of localizing it. So it is important to find rings which have nice classical quotient rings. During the 1950s, Alfred Goldie, an English mathematician, proved that the classical quotient ring of a semiprime Noetherian ring is a semisimple ring. A semisimple ring is a nice ring because, by the Artin-Wedderburn theorem, it is a finite direct product of matrix rings over division rings. Goldie proved his theorem for a larger class of rings. To state his theorem we need the following definition.

Definition 1.3.1. A ring R is called *left Goldie* if R satisfies the ascending chain condition on left annihilators and R does not contain an infinite direct sum of left ideals.

Theorem 1.3.2. (Goldie, [48, Theorem 2.3.6]) *Let R be a ring and let $Q(R)$ be the left classical quotient ring of R. Then $Q(R)$ is semisimple (resp. simple artinian) if and only if R is semiprime left Goldie (resp. prime left Goldie).*

Every left noetherian ring is clearly left Goldie. Also, every commutative domain is left Goldie because the intersection of any two nonzero ideals of a commutative domain is nonzero. So if C is a commutative domain and $\{x_i : i \in \mathbb{N}\}$ is a set of commuting variables over C, then the polynomial ring $C[x_1, x_2, \ldots]$ is a nonnoetherian Goldie ring.

In this section, the goal is to prove one side of Goldie's theorem, i.e. if R is semiprime left Goldie, then $Q(R)$ is semisimple. The other side of the theorem is much easier and not useful for our purpose. Note that we also need to prove that $Q(R)$ basically exists. We first recall the definition of semisimple rings. Let R be a ring. A left R-module M is called semisimple if every submodule of M is a direct summand of M. The ring R is called *semisimple* if it is semisimple as a left R-module, i.e. every left ideal of R is a direct summand of R. We now have the following result.

Proposition 1.3.3 ([39], Theorem 4.14)**.** *A ring is semisimple if and only if it is left artinian and semiprimitive.*

Semisimple rings are characterized by the celebrated Artin-Wedderburn theorem.

Theorem 1.3.4. (Artin-Wedderburn, [39, Theorem 3.5]) *A ring R is simple and Artinian if and only if $R \cong M_n(D)$ for a unique integer $n \geq 1$ and, up to isomorphism, a unique* *division ring D. More generally, R is semisimple if and only if*

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

for a unique integer $k \geq 1$, some division rings D_i and some integers $n_i \geq 1$. The pairs (n_i, D_i), up to permutation and isomorphism of D_i, are uniquely determined.

Assumption. For the rest of this section, R is a semiprime left Goldie ring and S is the submonoid of regular elements of R. Also, by $\mathrm{ann}(a)$ we mean the left annihilator of a in R.

We now begin the proof of Goldie's theorem with a definition.

Definition 1.3.5. A left ideal I of R is called *essential* if $I \cap J \neq (0)$ for every nonzero left ideal J of R.

It is clear that a left ideal I is essential if and only if $I \cap Ra \neq (0)$ for all $0 \neq a \in R$. Also, if $I_1 \subseteq I_2$ are left ideals of R and I_1 is essential, then I_2 is essential too.

Lemma 1.3.6. *If I is a left ideal of R and $I \cap S \neq \emptyset$, then I is essential.*

Proof. Let $a \in I \cap S$. So $a \in I$ and $\mathrm{ann}(a) = 0$. If we show that Ra is essential, then we are done because $Ra \subseteq I$. So suppose, to the contrary, that J is a nonzero left ideal of R and $Ra \cap J = (0)$. Let $0 \neq b \in J$. Then $Ra \cap Rb = (0)$ and so the sum $\sum_{n=0}^{\infty} Rab^n$ is direct (see the proof of Proposition 1.2.13). Thus R is not left Goldie, contradiction! $\square$

The converse of Lemma 1.3.6 is also true and it is the key to the proof of Goldie's theorem. In order to prove the converse, we need a few lemmas.

Lemma 1.3.7. *If $a \in R$ is left regular, then a is right regular too and hence $a \in S$.*

Proof. We need to show that if $\mathrm{ann}(a) = 0$, then $\{b \in R : \ ab = 0\} = (0)$. So suppose, to the contrary, that $ab = 0$ for some $0 \neq b \in R$. Then $\mathrm{ann}(b)$ is a proper left ideal of R and $a \in \mathrm{ann}(b) \cap S$. Thus, by Lemma 1.3.6, $\mathrm{ann}(b)$ is essential. Since R is left Goldie, the set

$$A = \{\mathrm{ann}(c) : \ 0 \neq c \in R, \ \mathrm{ann}(b) \subseteq \mathrm{ann}(c)\}$$

has a maximal element, say $\mathrm{ann}(u)$. Hence $uRu \neq (0)$ because R is semiprime. Thus $uvu \neq 0$ for some $v \in R$. Since $\mathrm{ann}(u) \subseteq \mathrm{ann}(uvu)$, we have

$$\mathrm{ann}(u) = \mathrm{ann}(uvu),$$

by the maximality of $\mathrm{ann}(u)$ in A. It follows that $\mathrm{ann}(u) \cap Ruv = (0)$ and, since $\mathrm{ann}(b) \subseteq \mathrm{ann}(u)$, we have $\mathrm{ann}(b) \cap Ruv = (0)$. Therefore $\mathrm{ann}(b)$ is not essential, contradiction! $\square$

Lemma 1.3.8. *Every nil left ideal of R is zero.*

Proof. Suppose, to the contrary, that I is a nonzero nil left ideal of R and $0 \neq a \in I$. Let

$$A = \{\mathrm{ann}(ar) : \ r \in R, \ ar \neq 0\}.$$

Since R is left Goldie, A has a maximal element, say $\mathrm{ann}(au)$. Let $b = au$. Since R is semiprime, $bRb \neq 0$ and so $bvb \neq 0$ for some $v \in R$. Since I is nil, $uva \in I$ is nilpotent. Thus $bv = auv$ is nilpotent too. Let n be the smallest integer such that $(bv)^n = 0$. Note that since $bv \neq 0$, we have $n \geq 2$. Thus $(bv)^{n-1} \neq 0$. We also have $(bv)^{n-1} \in aR$ because $bv = auv \in aR$. Hence $\mathrm{ann}((bv)^{n-1}) \in A$. Now, $\mathrm{ann}(b) \subseteq \mathrm{ann}((bv)^{n-1})$ and so $\mathrm{ann}(b) = \mathrm{ann}((bv)^{n-1})$, by the maximality of $\mathrm{ann}(b)$ in A. But $bv \in \mathrm{ann}((bv)^{n-1})$ and hence $bv \in \mathrm{ann}(b)$, i.e. $bvb = 0$, contradiction! $\square$

Lemma 1.3.9. *If I is a nonzero ideal of R, then $Ra \cap \mathrm{ann}(a) = (0)$ for some $0 \neq a \in I$.*

Proof. Since I is nonzero, I has an element b which is not nilpotent, by Lemma 1.3.8. Since R is left Goldie, the ascending chain

$$\mathrm{ann}(b) \subseteq \mathrm{ann}(b^2) \subseteq \mathrm{ann}(b^3) \subseteq \cdots,$$

stops at some point. So $\mathrm{ann}(b^n) = \mathrm{ann}(b^{2n})$, for some n, and hence $Rb^n \cap \mathrm{ann}(b^n) = (0)$. $\square$

We are now ready to prove the converse of Lemma 1.3.6, which is the heart of the proof of Goldie's theorem.

Lemma 1.3.10. *If a left ideal I of R is essential, then $I \cap S \neq \emptyset$.*

Proof. Suppose, to the contrary, that $I \cap S = \emptyset$. We claim that there exists a sequence $a_1, a_2, \ldots$ of nonzero elements of I such that $a_{n+1} \in \mathrm{ann}(a_1, \ldots, a_n)$ and $Ra_n \cap \mathrm{ann}(a_n) = (0)$ for all n. If we prove this claim, then the sum $\sum_{n=1}^{\infty} Ra_n$ will be direct and so R will not be Goldie.

Now, by Lemma 1.3.9, there exists $0 \neq a_1 \in I$ such that $Ra_1 \cap \mathrm{ann}(a_1) = (0)$. Let $n \geq 1$ and suppose that we have found nonzero elements $a_1, \ldots, a_n$ in I such that $a_{k+1} \in \mathrm{ann}(a_1, \ldots, a_k)$ for all $1 \leq k \leq n-1$ and $Ra_k \cap \mathrm{ann}(a_k) = (0)$ for all $1 \leq k \leq n$. Let $I_n := \mathrm{ann}(a_1, \ldots, a_n)$. Clearly $I_n \subseteq \mathrm{ann}(a_1 + \cdots + a_n)$. Now, if $r \in \mathrm{ann}(a_1 + \cdots + a_n)$, then $\sum_{i=1}^{n} ra_i = 0$ and so $ra_1^2 = 0$ because $a_2, \ldots, a_n \in \mathrm{ann}(a_1)$. But then $ra_1 \in Ra_1 \cap \mathrm{ann}(a_1)$ and so $ra_1 = 0$. Similarly $ra_2 = \cdots = ra_n = 0$ and so $r \in \mathrm{ann}(a_1, \ldots, a_n)$. Thus $I_n =$

$\operatorname{ann}(a_1 + \cdots + a_n)$ and hence $I_n \neq (0)$ because if $I_n = (0)$, then $a_1 + \cdots + a_n \in S$, by Lemma 1.3.7, contradicting $I \cap S = \emptyset$. So $I_n \cap I \neq (0)$, because I is essential, and hence there exists $0 \neq a_{n+1} \in I_n \cap I$ such that $Ra_{n+1} \cap \operatorname{ann}(a_{n+1}) = (0)$ by Lemma 1.3.9. □

Proposition 1.3.11. *The classical left quotient ring $Q(R)$ exists.*

Proof. We only need to prove that S is left Ore. Let $s \in S,\ r \in R$ and define the left ideal $K := \{a \in R : \ ar \in Rs\}$. Note that Rs is essential, by Lemma 1.3.6, because $s \in Rs \cap S$. We now show that K is also essential. So suppose that J is a nonzero left ideal of R. If $Jr = (0)$, then $J \subseteq K$ and thus $J \cap K = J \neq (0)$. If $Jr \neq (0)$, then $Rs \cap Jr \neq (0)$, because Rs is essential. So we can choose $as = br \neq 0$, for some $a \in R$ and $b \in J$. But then $br \in Rs$ and hence $0 \neq b \in J \cap K$. So K is essential and hence $K \cap S \neq \emptyset$, by Lemma 1.3.10. Thus there exists $a \in S$ such that $ar \in Rs$. That means $ar \in Sr \cap Rs$ and so $Sr \cap Rs \neq \emptyset$, i.e. S is left Ore. □

The last step of the proof of Goldie's theorem is to prove that $Q(R)$ is semisimple.

Theorem 1.3.12. *The classical left quotient ring $Q(R)$ is semisimple.*

Proof. By Proposition 1.3.3, we need to prove that $Q(R)$ is semiprimitive and left artinian. We first prove that $Q(R)$ is left artinian. So suppose, to the contrary, that $I_1 \supset I_2 \supset I_3 \supset \cdots$ is a strictly descending chain of left ideals of $Q(R)$. Let $J_i = I_i \cap R,\ i \geq 1$. Then

$$J_1 \supset J_2 \supset J_3 \supset \cdots$$

is a strictly descending chain of left ideals of R, by Proposition 1.2.11. Fix an integer $i \geq 1$. Since $J_i \supset J_{i+1}$, there exists $a_i \in J_i \setminus J_{i+1}$. Let

$$K_i = \{r \in R : \ ra_i \in J_{i+1}\},$$

which is a left ideal of R. Suppose that $s \in K_i \cap S$. Then $sa_i \in J_{i+1}$ and so, by Proposition 1.2.11, $a_i \in s^{-1}J_{i+1} \subseteq Q(R)J_{i+1} = Q(R)(I_{i+1} \cap R) = I_{i+1}$. Hence $a_i \in I_{i+1} \cap R = J_{i+1}$, which is false. So we have proved that $K_i \cap S = \emptyset$. Therefore K_i is not essential, by Lemma 1.3.10. So there exists a nonzero left ideal L_i of R such that $K_i \cap L_i = (0)$. Since $a_i \in J_i$, we have $L_i a_i \subseteq J_i$. We also have $L_i a_i \cap J_{i+1} = 0$ because if $ba_i \in J_{i+1}$ for some $b \in L_i$, then $b \in K_i \cap L_i = (0)$. Let $T_i = L_i a_i$. So we have proved that $T_i \subseteq J_i$ and $T_i \cap J_{i+1} = (0)$. Hence the sum $\sum_{i=1}^{\infty} T_i$ is direct, which is impossible because R is left Goldie. This contradiction proves that $Q(R)$ is left artinian.

We now prove that $Q(R)$ is semiprimitive, i.e. $J(Q(R)) = (0)$. Since $Q(R)$ is artinian, $J(Q(R))$ is nilpotent by Lemma 1.1.7. Hence $J(Q(R)) \cap R$ is a nilpotent ideal of R. But R is semiprime and so $J(Q(R)) \cap R = (0)$. Therefore $J(Q(R)) = Q(R)(J(Q(R)) \cap R) = (0)$, by Proposition 1.2.11. $\square$

1.4 Polynomial Identity Rings

In section 2 of this chapter, we used the concept of localization in commutative rings to define localization in noncommutative rings. In this section, we generalize another side of commutative rings to noncommutative rings. Let $\mathbb{Z}\langle x_1, \ldots, x_n \rangle$ denote the ring of polynomials in noncommuting variables $x_1, x_2, \ldots x_n$ with coefficients in $\mathbb{Z}$. If R_1 is a commutative ring and $f_1(x_1, x_2) := x_1x_2 - x_2x_1 \in \mathbb{Z}\langle x_1, x_2 \rangle$, then $f_1(a, b) = 0$ for all elements $a, b \in R_1$. Now let C be a commutative ring and let $R_2 := M_2(C)$. Let $f_2(x_1, x_2, x_3) := (x_1x_2 - x_2x_1)^2x_3 - x_3(x_1x_2 - x_2x_1)^2 \in \mathbb{Z}\langle x_1, x_2, x_3 \rangle$. Then $f_2(a, b, c) = 0$ for all $a, b, c \in R_2$ (see Example 1.4.3). Both R_1 and R_2 *satisfy* some polynomial in $\mathbb{Z}\langle x_1, \ldots, x_n \rangle$. We say that R_1 and R_2 are PI. Here, PI stands for Polynomial Identity. So the class of PI-rings contains the class of commutative rings properly.

In this section, we look at primitive and prime rings which are PI. The goal is to prove two theorems. First, a theorem due to Kaplansky and Amitsur. The theorem states that a primitive ring is PI if and only if it is a finite-dimensional central simple algebra. There is another central theorem in the theory of PI-rings which is due to Posner. By Posner's theorem, a prime ring is PI if and only if it is an order in a finite-dimensional central simple algebra. To be more precise, Posner's theorem says that a prime ring is PI if and only if its central localization is a finite-dimensional central simple algebra.

Throughout this section, C is a commutative ring with 1 and $C\langle x_1, \ldots, x_n \rangle$ is the C-algebra of polynomials in noncommuting variables $x_1, \ldots, x_n$ with coefficients in C.

1.4.1 Definitions and Basic Results

This subsection is an introduction to subsections 4 and 5. We first give the definition and some examples and properties of PI-algebras. Then we prove that the polynomial ring in one variable over a semiprime PI-ring is a semiprimitive PI-ring. This result will be used in subsection 5 of this section to prove the celebrated Posner's theorem.

Definition 1.4.1. Let R be a C-algebra. We say that R *satisfies a polynomial identity* if there exists an integer $n \geq 1$ and a monic polynomial $f \in C\langle x_1, \dots, x_n\rangle$, i.e. the coefficient of at least one of the monomials of the highest degree in f is 1, such that $f(r_1, r_2, \cdots, r_n) = 0$ for all $r_1, \dots, r_n \in R$. The algebra R is also called a *PI-algebra.* A *PI-ring* is a PI-algebra over $C = \mathbb{Z}$. An algebra R is called *locally PI* if every finitely generated subalgebra of R is PI.

Example 1.4.2. Commutative algebras are PI because they satisfy $x_1x_2 - x_2x_1$.

Example 1.4.3. Consider the C-algebra $R := M_2(C)$ and let $\mathrm{Tr}(r)$ be the trace of $r \in R$. Then for any $r_1, r_2 \in R$ we have $\mathrm{Tr}(r_1r_2 - r_2r_1) = 0$. Thus, by the Cayley-Hamilton theorem, $(r_1r_2 - r_2r_1)^2$ is a scalar multiple of the identity element of R and so it commutes with all elements of R. Thus R satisfies the polynomial $f = (x_1x_2 - x_2x_1)^2x_3 - x_3(x_1x_2 - x_2x_1)^2$. The identity f is called Wagner's identity.

Remark 1.4.4. If R satisfies a polynomial f, then subalgebras and homomorphic images of R satisfy f too.

Definition 1.4.5. A polynomial $f \in C\langle x_1, \dots, x_n\rangle$ is called *multilinear* if for every i the degree of x_i in every monomial occurring in f is one, i.e. $f = \sum_{\sigma \in \mathrm{Sym}(n)} c_\sigma x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}$, where Sym(n) is the set of all permutations of $1, 2, \dots, n$ and $c_\sigma \in C$. The multilinear polynomial $S_n = S_n(x_1, \dots, x_n) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}$, where $\mathrm{sgn}(\sigma)$ is the signature of σ, is called the *standard polynomial* of degree n.

Remark 1.4.6. If a C-algebra R satisfies a multilinear polynomial of degree one, then $R = (0)$. To see this, let $f = \sum_{i=1}^{n} c_i x_i,\ c_i \in C$ be an identity for R. We may assume that $c_1 = 1$. If in f we let $x_i = 0$ for all $i \neq 1$, then we see that the polynomial $g = x_1$ is also an identity for R and hence $R = (0)$.

Proposition 1.4.7 ([48], Corollary 13.1.13 (i))**.** *A C-algebra R which is finitely generated as C-module, is PI.*

Proof. It is easy to see that if R is generated by n elements, then R satisfies the standard polynomial S_{n+1}. The point here is that $S_n(a_1, \dots, a_n) = 0$ if $a_i = a_j$ for some $i \neq j$. $\square$

Example 1.4.8. By Proposition 1.4.7, $R = M_n(C)$ satisfies S_{n^2+1}, because R, as C-module, is generated by n^2 elements.

Proposition 1.4.7 is also a result of the following important theorem.

Theorem 1.4.9. (Jacobson, [35, page 14]) *Let R be a C-algebra which is integral of bounded degree over C, i.e. there exists an integer $n \geq 1$ such that for every $r \in R$ there exists a monic polynomial $g \in C[x]$ of degree n with $g(r) = 0$. Then R is a PI-algebra.*

Proof. Choose two elements $r_1, r_2 \in R$ and a polynomial $g_1(x) = x^n + c_1x^{n-1} + \cdots + c_n$ in $C[x]$ with $g_1(r_1) = 0$. For every $a, b \in R$ let $[a, b] = ab - ba$. We have

$$0 = [g_1(r_1), r_2] = [r_1^n, r_2] + c_1[r_1^{n-1}, r_2] + \cdots + c_{n-1}[r_1, r_2].$$

Now let $g_2(x) = [g_1(x), r_2]$. Then

$$0 = [g_2(r_1), [r_1, r_2]] = [[r_1^n, r_2], [r_1, r_2]] + c_1[[r_1^{n-1}, r_2], [r_1, r_2]] + \cdots + c_{n-2}[[r_1^2, r_2], [r_1, r_2]]$$

Then we let $g_3(x) = [g_2(x), [r_1, r_2]$ and consider $[g_3(r_1), [[r_1^2, r_2], [r_1, r_2]]]$ to get rid of c_{n-2}. If we continue in this manner until all the coefficients c_i are gone, we will end up with a polynomial $f \in C\langle x_1, x_2\rangle$ such that $f(r_1, r_2) = 0$. Then f has a monomial of highest degree with coefficient 1. □

We will use the following obvious fact later in this section.

Remark 1.4.10. Let R be a C-algebra and let K be a commutative C-algebra. If R satisfies a multilinear polynomial f, then $R \otimes_C K$ satisfies f too.

The above is not necessarily true if f is not multilinear. For example, let $\mathbb{F}_q$ denote the field of order q. Let $f(x) = x^2 - x$ and $R = C = \mathbb{F}_2$. Then f is an identity for R. Now let $K = \mathbb{F}_4$. Clearly K is a C-algebra, $R \otimes_C K \cong K$ and f is not an identity for K. The following proposition considers this situation.

Proposition 1.4.11 ([24], Part B, Theorem 1.9)**.** *Let k be an infinite field, K a commutative k-algebra and R a k-algebra satisfying a polynomial $f \in k\langle x_1, \ldots, x_n\rangle$. Then $R \otimes_k K$ satisfies f too.*

Proof. Let $A = \{r_j : j \in J\}$ be a basis for R over k and let $B = \{y_{ij} : 1 \leq i \leq n, j \in J\}$ be a set of commuting variables over K. For each $1 \leq i \leq n$, let $z_i = y_{i\alpha_1}r_{\alpha_{i1}} + \cdots + y_{i\alpha_s}r_{\alpha_{is}}$, where s can be any integer for each i. Then $f(z_1, \ldots, z_n) = u_1v_1 + \cdots + u_tv_t$, where each u_i is a monomial in $k[B]$ and each v_i is a monomial in $k\langle A\rangle \subseteq R$. So we can write each v_i

as a finite k-linear combination of elements of A because A is a basis for R over k. Thus $f(z_1,\ldots,z_n) = w_{j_1}r_{j_1} + \cdots + w_{j_m}r_{j_m}$, where each w_{j_p} is an element of $k[B]$. Let B_1 be the set of all elements of B which appear in $f(z_1,\ldots,z_n)$. We give to each element of B_1 a value in k. Then $z_i \in R$ for all i and thus $f(z_1,\ldots,z_n) = 0$. Hence if the variables in each polynomial w_{j_p} have values in k, then $w_{j_p} = 0$. But k is infinite and therefore each w_{j_p} must be identically zero and so $f(z_1,\ldots,z_n) = 0$ if elements of B_1 have values in K. $\square$

Proposition 1.4.12 ([48], Proposition 13.1.9)**.** *If a C-algebra R satisfies a polynomial f of degree n, then R satisfies a multilinear polynomial of degree at most n.*

Proof. Suppose that $f \in C\langle x_1,\cdots,x_n\rangle$. First note that for any variable, say x_1 for the sake of simplicity, we can write $f = g(x_1,\ldots,x_n) + h(x_2,\ldots,x_n)$, where h consists of those monomials of f in which x_1 does not occur. Now if we let $x_1 = 0$ and let $x_2,\cdots,x_n$ be any elements of R, then h, and hence $g = f - h$, is an identity of R. So we may assume that x_1 occurs in every monomial of f. Continuing this process, eventually we may assume that every x_i occurs in every monomial of f.

Now, if f is not multilinear itself, then the monomials of highest degree occurring in f cannot be multilinear. Thus, say x_1, occurs in those monomials with maximal degree $k \geq 2$. Let

$$g(x_1,\ldots,x_n,x_{n+1}) = f(x_1 + x_{n+1}, x_2,\ldots,x_n) - f(x_1,x_2,\ldots,x_n) - f(x_{n+1},x_2,\ldots,x_n).$$

Clearly R satisfies g and g is monic of degree at most n and the degree of x_1 in each monomial of g is at most $k-1$. An induction now completes the proof. $\square$

Corollary 1.4.13 ([24], Part B, Corollary 1.15)**.** *Let R be a C-algebra. If R is PI and x is a central variable, then the polynomial algebra $R[x]$ is PI too.*

Proof. By Proposition 1.4.12, R satisfies some multilinear polynomial $f(x_1,\cdots,x_m)$. Now if $g_j(x) \in R[x]$, $j = 1,2,\ldots,m$, then there exist an integer n and some elements $r_{ij} \in R$ such that

$$f(g_1,\ldots,g_m) = \sum_{i\leq k} f(r_{i1},\ldots,r_{im})x^i$$

and thus $f(g_1,\ldots,g_m) = 0$. $\square$

Lemma 1.4.14 ([48], Proposition 13.3.2)**.** *If R is a prime PI-ring, then R does not contain any nonzero nil ideal.*

Proof. The proof is by induction on n, the degree of the multilinear polynomial satisfied by R. By Remark 1.4.6, there is nothing to prove if $n = 1$. Suppose that R is a prime PI-ring which satisfies a multilinear polynomial $f(x_1, \dots, x_n)$ of degree $n \geq 2$ and any prime ring, which may or may not have an identity element, which satisfies a multilinear polynomial of degree less than n has no nonzero nil ideal.

Now, to get a contradiction, suppose that I is a nonzero nil ideal of R and choose $0 \neq a \in I$ such that $a^2 = 0$. Let $J = Ra$ and $S := J/(J \cap \text{l.ann}_R(J))$. Then S is a prime ring. Now, after permuting the indices of $x_1, \dots, x_n$ if necessary, we can write

$$f(x_1, \dots, x_n) = x_1 g(x_2, \dots, x_n) + h(x_1, \dots, x_n),$$

where both g, h are multilinear, no monomial in h begins with x_1 and g is not identically zero. Let w be a monomial in h. Since no monomial in h begins with x_1, there exists some $i \neq 1$ such that the term $x_i x_1$ appears in w. So if r, r' are any elements of R, then the value of w at $x_i = ra,\ x_1 = ar'$ is zero because $x_i x_1 = ra^2 r' = 0$. Thus if $r_1, r_2, \dots, r_n \in R$, then $h(ar_1, r_2 a, \dots, r_n a) = 0$ and hence $ar_1 g(r_2 a, \dots, r_n a) = 0$. Therefore $aRg(r_2 a, \dots, r_n a) = 0$ and so $g(r_2 a, \dots, r_n a) = 0$ because R is prime and $a \neq 0$. Thus J, and hence S, satisfies g. But the degree of g is less than n and so, by the induction hypothesis, S has no nonzero nil ideal. On the other hand, $J \subseteq I$ and so J is nil because I is nil. Thus S is a nil ideal of itself and so $S = (0)$. Therefore $J^2 = (0)$, contradicting the hypothesis that R is prime. $\square$

Corollary 1.4.15 ([24], Part B, Lemma 6.1)**.** *If R is a semiprime PI-ring and x is a central variable, then $R[x]$ is a semiprimitive PI-ring.*

Proof. By Corollary 1.4.13, $R[x]$ is a PI-ring. Let $\{P_i : \ i \in I\}$ be the set of prime ideals of R. Then $\bigcap_{i \in I} P_i = (0)$, because R is semiprime. Let $R_i = R/P_i,\ i \in I$, and define the map $\nu : R[x] \longrightarrow \prod_{i \in I} R_i[x]$ by $\nu(\sum r_j x^j) = (\sum (r_j + P_i) x^j)$. Then ν is an injective ring homomorphism. Now, R_i is a prime PI-ring for every $i \in I$ and hence $R_i[x]$ is a semiprimitive ring, by Lemma 1.4.14 and Proposition 1.1.6. If $p(x) = \sum r_j x^j \in J(R[x])$, then $\sum (r_j + P_i) x^j \in J(R_i[x]) = 0$ for all $i \in I$. Thus $r_j \in P_i$ for all j and $i \in I$ and hence $r_j = 0$ for all j. $\square$

For the next result we refer the reader to Definition 1.2.16.

Corollary 1.4.16. *Let R be a C-algebra. If R is PI, then $Q_Z(R)$ is PI too.*

Proof. By Proposition 1.4.12, R satisfies some multilinear polynomial $f(x_1, \ldots, x_m)$. Now if $q_i = s_i^{-1} r_i \in Q_Z(R)$, then, since f is multilinear and s_i are central, we have $f(q_1, \ldots, q_m) = (s_1 \ldots s_m)^{-1} f(r_1, \ldots, r_m) = 0$. $\square$

Corollary 1.4.17 ([48], Proposition 13.3.2)**.** *The C-algebra $R = M_n(C)$ does not satisfy any polynomial of degree less than $2n$.*

Proof. Suppose, to the contrary, that R satisfies a polynomial of degree at most $2n - 1$. Then, by Proposition 1.4.12, R satisfies a multilinear polynomial

$$g(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)},$$

for some $k \leq 2n - 1$ and $c_\sigma \in C$. Renaming the variables, if necessary, we may assume that $c = c_{\text{id}} \neq 0$. Let $\{e_{ij}\}$ be the standard basis of R over C. Then $g(e_{11}, e_{12}, e_{22}, e_{23}, \cdots) = ce_{1\ell} = 0$, for some ℓ. Therefore $ce_{ij} = e_{i1}(ce_{1\ell})e_{\ell j} = 0$ for all i, j. Thus $c = 0$, which is a contradiction. $\square$

Theorem 1.4.18. (Amitsur-Levitzki, [51]) *The C-algebra $M_n(C)$ satisfies S_{2n}.*

1.4.2 Kaplansky's Theorem

It is easy to show that a primitive ring is commutative if and only if it is a field. Now, which primitive rings are PI? The answer is that a primitive ring R is PI if and only if $R = M_n(D)$ for some division ring which is finite-dimensional over its center. The goal in this subsection is to prove this important result. This result is due to Kaplansky and Amitsur and it is often called the Kaplansky's theorem. We begin with recalling a few facts about primitive rings. Given a left R-module M, let $\text{End}_R(M)$ denote the ring of R-homomorphisms from M into M.

Lemma 1.4.19. (Schur's Lemma, [54, Lemma 1.5.1]) *If M is a simple R-module, then $\text{End}_R(M)$ is a division ring.*

Proof. For any nonzero element $f \in \text{End}_R(M)$, the kernel and the image of f are R-submodules of M. Therefore, since M is simple and $f \neq 0$, we have $\ker f = 0$ and $f(M) = M$, i.e. f is an isomorphism and so it is an invertible element of $\text{End}_R(M)$. $\square$

We have already defined primitive rings in Definition 1.1.3. We now rephrase the definition.

Definition 1.4.20. A ring R is called *left primitive* if there exists a simple left R-module M which is faithful, i.e. $\mathrm{l.ann}_R(M) = \{r \in R : \ rM = (0)\} = (0)$.

Example 1.4.21. Every simple ring R is left primitive because if I is a maximal left ideal of ring R, then $M = R/I$ is a faithful simple left R-module.

Theorem 1.4.22. *Let D be a division ring with the center Z and let K be a subfield of D. Let $A = D \otimes_Z K$.*

(1) A is a simple ring and thus left primitive.

(2) D is a faithful simple left A-module and if K is a maximal subfield, then $\mathrm{End}_A(D) \cong K$.

Proof. Let $\{k_j : \ j \in J\}$ be a Z-basis for K. Suppose that A is not simple. Let I be a nonzero ideal of A. Choose n to be the smallest integer for which there exists $0 \neq x = \sum_{j=1}^{n} d_j \otimes k_{t_j} \in I$. Then $d_1 \neq 0$ and so, by replacing x with $(d_1^{-1} \otimes 1)x$ if necessary, we may assume that $d_1 = 1$. Now, for any $d \in D$ we have

$$\sum_{j=2}^{n}(dd_j - d_j d) \otimes k_{t_j} = (d \otimes 1)x - x(d \otimes 1) \in I,$$

which gives $dd_j = d_j d$, by the minimality of n. So $d_j \in Z$ for all j. Thus $x = 1 \otimes k$, for some $0 \neq k \in K$. But then $1_A = 1 \otimes 1 = (1 \otimes k^{-1})x \in I$ and so $I = A$.

For the second part of the theorem, define $(d_1 \otimes k)d_2 = d_1 d_2 k$ for all $d_1, d_2 \in D, \ k \in K$ and extend it linearly. Then D becomes a faithful left A-module because $\mathrm{l.ann}_A(D) \neq A$ is an ideal of A and A is a simple ring by the first part of the theorem. To prove that D is a simple A-module, let $d_1 \neq 0$ and $d_2 \in D$. Then $(d_2 d_1^{-1} \otimes 1)d_1 = d_2$ and hence $Ad_1 = D$. To prove that $\mathrm{End}_A(D) \cong K$ we define the map $\varphi : \mathrm{End}_A(D) \longrightarrow K$ by $\varphi(f) = f(1)$. It follows that if K is a maximal subfield, then φ is a ring isomorphism. The reason is that the centralizer of K in D is K itself. $\square$

The following theorem gives a structure theorem for left primitive rings.

Theorem 1.4.23 ([39], Theorem 11.19)**.** *Let R be a left primitive ring with a faithful simple left R-module M. Let $D = \mathrm{End}_R(M)$.*

(1) If $\dim_D M = n < \infty$, then $R \cong M_n(D)$.

(2) If $\dim_D M = \infty$, then for any integer $n \geq 1$, $M_n(D)$ is the homomorphic image of some subring of R.

Lemma 1.4.24. *Let R be a left primitive C-algebra. Let M be a faithful simple left R-module and let $D = \mathrm{End}_R(M)$. If R satisfies a polynomial f of degree d, then $\dim_D M = n \leq \lfloor d/2 \rfloor$ and $R \cong M_n(D)$.*

Proof. Suppose $\dim_D M > \lfloor d/2 \rfloor$. Then, by Theorem 1.4.23, there exists some $k > [d/2]$ such that either $R \cong M_k(D)$ or $M_k(D)$ is a homomorphic image of some subring of R. In either case, $M_k(D)$, and hence $M_k(Z(D))$, satisfies f by Remark 1.4.4. Thus by Lemma 1.4.17

$$d \geq 2k \geq 2(\lfloor d/2 \rfloor + 1) > d,$$

which is absurd. □

Theorem 1.4.25. (Kaplansky's theorem, [54, Theorem 1.5.16]) *Let R be a left primitive C-algebra and let M be a faithful simple left R-module. Let $D = \mathrm{End}_R(M)$. Suppose that R satisfies a polynomial of degree d.*

(1) $R \cong M_n(D)$, where $n = \dim_D M \leq \lfloor d/2 \rfloor$ and so $k = Z(R) \cong Z(D)$ is a field.

(2) $\dim_k D = \dim_k R \leq (\lfloor d/2 \rfloor)^2$.

Proof. We have already proved the first part in Lemma 1.4.24. So we only need to prove the second part of the theorem. By Proposition 1.4.12, R satisfies some multilinear polynomial f of degree at most d. Clearly D satisfies f too because it is a subring of $M_n(D) \cong R$. Let K be a maximal subfield of the division ring D and let $A := D \otimes_k K$. By the first part of Theorem 1.4.22, A is simple and thus left primitive. By Remark 1.4.10, A satisfies f. Now, if in Lemma 1.4.24 we let $R = D \otimes_k K$ and $M = D$ and apply the second part of Theorem 1.4.22, then we get $A \cong M_m(K)$, for some integer $m \geq 1$. Therefore

$$R \otimes_k K \cong M_n(D) \otimes_k K \cong M_n(A) \cong M_{mn}(K)$$

and hence

$$\dim_k R = \dim_K R \otimes_k K = (mn)^2.$$

On the other hand $R \otimes_k K$ satisfies f and thus $d \geq \deg f \geq 2mn$. So $mn \leq d/2$ and therefore $mn \leq \lfloor d/2 \rfloor$. Finally we have $\dim_k R \leq \dim_k R \otimes_k K = (mn)^2 \leq (\lfloor d/2 \rfloor)^2$. □

If R is simple (resp. division) ring, then we say R is a *central* simple (resp. division) k-algebra if $Z(R) = k$.

Corollary 1.4.26. *Let D be a finite-dimensional central division k-algebra and $R = M_m(D)$. Let K be a maximal subfield of D. Then R is PI, $R \otimes_k K \cong M_s(K)$ and $\dim_k R = s^2$ for some integer $s \geq 1$.*

Proof. By Proposition 1.4.7, R is PI. Clearly R is left primitive because it is simple. Thus, as we showed in the proof of Theorem 1.4.25, $R \otimes_k K \cong M_s(K)$ for some integer s and $\dim_k R = s^2$. $\square$

Definition 1.4.27. If R is a finite-dimensional central simple k-algebra, then the integer $\sqrt{\dim_k R}$ is called the *degree* of R.

So, by Kaplansky's theorem, a primitive ring R satisfies a polynomial identity if and only if R is a finite-dimensional central division algebra. What can we say about a prime ring that satisfies a polynomial identity? We will answer this question in the next section.

1.5 Posner's Theorem

A prime ring is commutative if and only if it is a commutative domain. Now, which prime rings are PI? The answer is that a prime ring R is PI if and only if $Q_Z(R) = M_n(D)$, where $Q_Z(R)$ is the central localization of R and D is a division ring which is finite-dimensional over its center. The goal in this subsection is to prove this result which is due to Posner.

Let R be a prime ring. Since $Z(R)$, the center of R, is a domain, the set of nonzero elements of $Z(R)$ is a regular submonoid which is obviously Ore. Thus $Q_Z(R)$, the localization of R at $S = Z(R) \setminus \{0\}$, exists and contains R as a subring (see Definition 1.2.16). By Proposition 1.2.18, the center of $Q_Z(R)$ is the localization of $Z(R)$ at S, i.e. the quotient field of $Z(R)$.

Before proving Posner's theorem we need some preparation.

Definition 1.5.1. Let R be a C-algebra. A polynomial $f(x_1, \ldots, x_n) \in C\langle x_1, \ldots, x_n\rangle$ is called a *central polynomial* for R if the constant term of f is zero, f is not an identity for R and $f(r_1, \ldots, r_n) \in Z(R)$ for all $r_1, \ldots, r_n \in R$.

Example 1.5.2. Let k be a field. By Example 1.4.3, $f(x_1, x_2) = (x_1x_2 - x_2x_1)^2$ is a central polynomial for $M_2(C)$.

Next, We prove that central polynomials always exist for $M_n(k)$ if k is a field. But first we give a definition.

Definition 1.5.3. Let $n \geq 2$ be an integer, k a field and $k[z_1, \dots, z_{n+1}]$ the polynomial algebra in $n+1$ commuting variables z_i. Define the map

$$\mu : k[z_1, \dots, z_{n+1}] \longrightarrow k\langle x, y_1, \dots, y_n\rangle$$

on monomials by

$$\mu(z_1^{m_1} z_2^{m_2} \cdots z_{n+1}^{m_{n+1}}) = x^{m_1} y_1 x^{m_2} y_2 \cdots x^{m_n} y_n x^{m_{n+1}}$$

and extend the definition k-linearly. Now let

$$g(z_1, \dots, z_{n+1}) = \prod_{2 \leq i \leq n} (z_1 - z_i)(z_{n+1} - z_i) \prod_{2 \leq i < j \leq n} (z_i - z_j)^2 \in k[z_1, \dots, z_{n+1}]$$

and define $\overline{g}(x, y_1, \dots, y_n) = \mu(g(z_1, \dots, z_{n+1}))$. Finally, define

$$F(x, y_1, \dots, y_n) = \overline{g}(x, y_1, \dots, y_n) + \overline{g}(x, y_2, \dots, y_n, y_1) + \cdots + \overline{g}(x, y_n, y_1, \dots, y_{n-1}).$$

The polynomial $F(x, y_1, \dots, y_n)$ is called the *Formanek polynomial.*

For example, if $n = 2$, then $g(z_1, z_2, z_3) = (z_1 - z_2)(z_3 - z_2) = z_1 z_3 - z_1 z_2 - z_2 z_3 + z_2^2$. Thus $\overline{g}(x, y_1, y_2) = x y_1 y_2 x - x y_1 x y_2 - y_1 x y_2 x + y_1 x^2 y_2$ and the Formanek polynomial is $F(x, y_1, y_2) = \overline{g}(x, y_1, y_2) + \overline{g}(x, y_2, y_1)$.

Theorem 1.5.4. (Formanek, [24, Part B, Theorem 3.4]) *Let k be a field and let $n \geq 2$ be an integer. The Formanek polynomial $F(x, y_1, \dots, y_n)$ is a central polynomial for $M_n(k)$.*

We are now going to prove Formanek's theorem for $n = 3$. The proof for the general case is similar. We begin with a useful lemma.

Lemma 1.5.5. *Let k be a field and let $n \geq 2$ be an integer. Let $F(x, y_1, \dots, y_n)$ be the Formanek polynomial. If $F(X, Y_1, \dots, Y_n)$ is a scalar matrix for all diagonal matrices $X \in M_n(k)$ and all matrix units $Y_1, \dots, Y_n \in M_n(k)$, then all valuations of $F(x, y_1, \dots, y_n)$ on $M_n(k)$ are scalar matrices.*

Proof. The Formanek polynomial is multilinear in $y_1, \dots, y_n$ and the matrix units e_{ij} span $M_n(k)$. Thus if $F(a, b_1, \dots, b_n)$ is a scalar matrix for all elements $X \in M_n(k)$ and all matrix units $Y_1, \dots, Y_n \in M_n(k)$, then all valuations of $F(x, y_1, \dots, y_n)$ on $M_n(k)$ are scalar matrices.

Now, Let $x_{ij},\ 1 \le i, j \le n$ be n^2 commuting independent variables over k and let K be the field generated by k and these variables. Let $\overline{K}$ denote the algebraic closure of K. Let X_0 be the element of $M_n(K)$ whose (i, j)-entry is x_{ij}. In order to show that all valuations of F on $M_n(k)$ are scalar matrices, we only need to show that $F(X_0, Y_1, \dots, Y_n) \in M_n(K)$ is a scalar matrix for all $Y_1, \dots, Y_n \in M_n(k)$. Note that X_0 has n distinct eigenvalues because the diagonal matrix $\sum_{i=1}^{n} x_{ii}e_{ij}$ has n distinct eigenvalues. Thus X_0 is diagonalizable, i.e. there exist invertible matrix $P \in M_n(\overline{K})$ such that PX_0P^{-1} is diagonal. Now, $PF(X_0, Y_1, \dots, Y_n)P^{-1} = F(PX_0P^{-1}, PY_1P^{-1}, \dots, PY_nP^{-1})$ and the fact that $F(X_0, Y_1, \dots, Y_n)$ is a scalar matrix if and only if $PF(X_0, Y_1, \dots, Y_n)P^{-1}$ is a scalar matrix, completes the proof. □

Proof of Theorem 1.5.4 for $n = 3$. . Let $X = a_1e_{11} + a_2e_{22} + a_3e_{33},\ a_i \in k$ be a diagonal matrix and let $Y_i = e_{p_iq_i},\ i = 1, 2, 3$ be any matrix units of $M_3(k)$. By Lemma 1.5.5, we only need to show that $F(X, Y_1, Y_2, Y_3)$ is a scalar matrix and also F is not identically zero on $M_3(k)$.

Let g and $\overline{g}$ be the functions as defined in Definition 1.5.3. We have

$$g(z_1, z_2, z_3, z_4) = (z_1 - z_2)(z_4 - z_2)(z_1 - z_3)(z_4 - z_3)(z_2 - z_3)^2. \tag{1.3}$$

Let r, s, t be any permutation of $1, 2, 3$. Since $Xe_{ij} = a_ie_{ij}$ and $e_{ij}X = a_je_{ij}$ for all i, j, we have $X^m e_{ij} = a_i^m e_{ij}$ and $e_{ij}X^m = a_j^m e_{ij}$. Thus

$$X^{m_1}Y_rX^{m_2}Y_sX^{m_3}Y_tX^{m_4} = a_{p_r}^{m_1}a_{p_s}^{m_2}a_{p_t}^{m_3}a_{q_t}^{m_4}e_{p_rq_r}e_{p_sq_s}e_{p_tq_t}$$

. Hence

$$\overline{g}(X, Y_r, Y_s, Y_t) = g(a_{p_r}, a_{p_s}, a_{p_t}, a_{q_t})e_{p_rq_r}e_{p_sq_s}e_{p_tq_t}. \tag{1.4}$$

By (1.3), we also have

$$g(a_{p_r}, a_{p_s}, a_{p_t}, a_{q_t}) = (a_{p_r} - a_{p_s})(a_{q_t} - a_{p_s})(a_{p_r} - a_{p_t})(a_{q_t} - a_{p_t})(a_{p_s} - a_{p_t})^2. \tag{1.5}$$

By (1.4), if $e_{p_rq_r}e_{p_sq_s}e_{p_tq_t} = 0$ or $g(a_{p_r}, a_{p_s}, a_{p_t}, a_{q_t}) = 0$, then $\overline{g}(X, Y_1, Y_2, Y_3) = 0$. Now, $e_{p_rq_r}e_{p_sq_s}e_{p_tq_t} = 0$ unless $q_r = p_s$ and $q_s = p_t$. Also, by (1.5), $g(a_{p_r}, a_{p_s}, a_{p_t}, a_{q_t}) = 0$ unless p_r, p_s, p_t is a permutation of $1, 2, 3$ and $q_t = p_r$. Note that if $q_t = p_r$, then by (1.5),

$$g(a_{p_r}, a_{p_s}, a_{p_t}, a_{q_t}) = (a_{p_r} - a_{p_s})^2(a_{p_r} - a_{p_t})^2(a_{p_s} - a_{p_t})^2.$$

Let

$$\Delta(a_1, a_2, a_3) := (a_{p_r} - a_{p_s})^2 (a_{p_r} - a_{p_t})^2 (a_{p_s} - a_{p_t})^2.$$

Note that since p_r, p_s, p_t is a permutation of $1, 2, 3$, we have

$$\Delta(a_1, a_2, a_3) = (a_1 - a_2)^2 (a_2 - a_3)^2 (a_1 - a_3)^2.$$

So $\overline{g}(X, Y_r, Y_s, Y_t) = 0$ unless p_r, p_s, p_t is a permutation of $1, 2, 3$ and $q_r = p_s,\ q_s = p_t,\ q_t = p_r$ and in this case $\overline{g}(X, Y_1, Y_2, Y_3) = \Delta(a_1, a_2, a_3) e_{p_r p_r}$. Thus $F(X, Y_r, Y_s, Y_t) = 0$ unless p_r, p_s, p_t is a permutation of $1, 2, 3$ and $q_r = p_s,\ q_s = p_t,\ q_t = p_r$ and in this case

$$\begin{aligned} F(X, Y_r, Y_s, Y_t) &= \overline{g}(X, Y_1, Y_2, Y_3) + \overline{g}(X, Y_2, Y_3, Y_1) + \overline{g}(X, Y_3, Y_1, Y_2) \\ &= \Delta(a_1, a_2, a_3)(e_{p_r p_r} + e_{p_s p_s} + e_{p_t p_t}) = \Delta(a_1, a_2, a_3) I. \end{aligned} \tag{1.6}$$

To complete the proof of the theorem, we show that F is not identically zero on $M_3(k)$. This is clear if k is infinite because then we can choose three distinct elements $a_1, a_2, a_3 \in k$. Then $\Delta(a_1, a_2, a_3) \neq 0$ and the result follows from (1.6).

If k is finite, let $\overline{k}$ be the algebraic closure of k. Let z be a variable over k and let $f \in k[z]$ be an irreducible polynomial of degree three. Let $X \in M_3(k)$ be its companion matrix. Then X has distinct eigenvalues in $\overline{k}$ and so it is diagonalizable. Hence there exist an invertible element $P \in M_3(\overline{k})$ such that PXP^{-1} is diagonal and its diagonal entries are distinct. So, by (1.6), there exist matrix units $E_1, E_2, E_3 \in M_3(k)$ such that $F(PXP^{-1}, Y_1, Y_2, Y_3) \neq 0$. Let $P^{-1} E_i P = U_i,\ i = 1, 2, 3$. Then $U_i \in M_3(\overline{k})$ and

$$PF(X, U_1, U_2, U_3)P^{-1} = F(PXP^{-1}, E_1, E_2, E_3).$$

Thus $F(X, U_1, U_2, U_3) \neq 0$. Finally, since F is multilinear in y_i and each U_i is a linear combination of matrix units and $F(X, U_1, U_2, U_3) \neq 0$, it follows that there exist matrix units $Y_1, Y_2, Y_3 \in M_3(k)$ such that $F(X, Y_1, Y_2, Y_3) \neq 0$ and the proof is complete. $\square$

The following lemma extends Formanek's theorem to finite-dimensional central simple algebras.

Lemma 1.5.6 ([24], Part B, Lemma 4.14 (d)). *Let R be a finite-dimensional central simple k-algebra of degree n.*

(1) The standard polynomial S_{2n} is an identity for R and R does not satisfy any polynomial of degree less than $2n$.

(2) The Formanek polynomial $F(x, y_1, \ldots, y_n)$ is a central polynomial for R.

Proof. So $R \cong M_m(D)$ for some finite-dimensional central division k-algebra. Let K be a maximal subfield of D. Then, by Corollary 1.4.26, $R \otimes_k K \cong M_n(K)$. By Remark 1.4.10, R and $M_n(K)$ satisfy the same multilinear polynomials. Thus the standard polynomial S_{2n} is an identity for R and no polynomial of degree less than $2n$ is an identity for R, by Corollary 1.4.17 and Theorem 1.4.18.

To prove the second part of the lemma, we consider two cases. If k is finite, then D is a field and we are done by Theorem 1.5.4. So we may assume that k is infinite. Let z be a variable and consider the polynomial $h = [F, z]$. Clearly h is an identity for $M_n(K) \cong R \otimes_k K$ because F is a central polynomial for $M_n(K)$. Therefore h is an identity for R too. Also, by Proposition 1.4.11, F is not an identity for R because it is not an identity for $M_n(K)$. Thus F is a central polynomial for R. □

Lemma 1.5.7. *Let R be a finite-dimensional central simple k-algebra of degree m. The Formanek polynomial $F(x, y_1, \ldots, y_n)$, $n > m$, is an identity for R.*

Proof. By Corollary 1.4.26, there exists a field extension K/k such that $R \otimes_k K \cong M_m(K)$. So we only need to prove that F is an identity for $M_m(K)$. Now, there is an embedding $M_m(K) \longrightarrow M_n(K)$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $a, b_1, \ldots, b_n \in M_m(K)$. Then $c = F(a, b_1, \ldots, b_n) \in M_m(K)$ is in the center of $M_n(K)$, i.e. $c = \alpha I$, where $\alpha \in K$ and I is the identity matrix of $M_n(K)$. But the only scalar multiple of I which is in $M_m(K)$ is the zero matrix. Thus $c = 0$. □

Theorem 1.5.8. (Rowen, [53]) *Let R be a semiprime PI-ring. If $J \neq (0)$ is an ideal of R, then $J \cap Z(R) \neq (0)$.*

Proof. Suppose first that we have proved the theorem for semiprimitive PI-rings and let J be a nonzero ideal of R. Let $S = R[x]$, the polynomial ring in the variable x. By Corollary 1.4.15, S is a semiprimitive PI-ring and clearly $J[x]$ is a nonzero ideal of S. Thus, since the center of S is $Z(R)[x]$, we have $(J \cap Z(R))[x] = J[x] \cap Z(R)[x] \neq (0)$ and hence $J \cap Z(R) \neq (0)$.

So we may assume that R is a semiprimitive PI-ring. Let J be a nonzero ideal of R and suppose that R satisfies a polynomial identity of degree d. Let $\{P_i : \ i \in I\}$ be the set of left primitive ideals of R. Then $\bigcap_{i\in I} P_i = (0)$ and $R_i = R/P_i$ is a primitive PI-ring for all $i \in I$. Thus, by Theorem 1.4.25, each R_i is a finite-dimensional central simple algebra of degree, say, n_i. Let $\pi_i : R \longrightarrow R_i,\ i \in I$ be the natural projection defined by $\pi_i(r) = r + P_i,\ r \in R$. We also have an injective ring homomorphism $\nu : R \longrightarrow \prod_{i\in I} R_i$ defined by $\nu(r) = (\pi_i(r))_{i\in I}$.

Clearly every polynomial identity for R is also an identity for each R_i. Also, by Lemma 1.5.6, R_i does not satisfy any polynomial of degree less than $2n_i$. Thus $d \geq 2n_i$ for all i. Therefore the set $\{n_i : \ i \in I\}$ is bounded above by $d/2$. Now, $\pi_i(J) = J_i,\ i \in I$, is an ideal of R_i. Since R_i is a simple ring, we have $J_i = (0)$ or $J_i = R_i$ for each i. We cannot have $J_i = (0)$ for all i because then $J = (0)$. So there exists some $s \in I$ such that $J_s = R_s$. Let $n = \max\{n_s : \ J_s = R_s\}$. Let $F(x, y_1, \ldots, y_n)$ be the Formanek polynomial and let

$$A = \{F(a, b_1, \ldots, b_n) : \ a, b_1, \ldots, b_n \in J\} \subseteq J.$$

For each i let $\pi_i(A) = A_i \subseteq J_i$. If $n_i > n$, then $A_i = (0) \subseteq Z(R_i)$ because $J_i = (0)$. If $n_i < n$, then again $A_i = (0) \subseteq Z(R_i)$ because F is an identity for R_i, by Lemma 1.5.7. If $n_i = n$, then F is a central polynomial for R_i, by the second part of Lemma 1.5.6, and thus $A_i \subseteq Z(R_i)$. So each A_i is central in R_i and hence $\nu(A)$ is central in $\prod_{i\in I} R_i$. Therefore A is central in R because ν is injective. Thus $A \subseteq J \cap Z(R)$.

The only thing left is to show that $A \neq (0)$. To prove this, choose an $i \in I$ such that $n_i = n$. Then, $J_i = R_i$ and F is a central polynomial for R_i and so it is not an identity for R_i. Hence there exist $u, v_1, \ldots, v_n \in R_i$ such that $F(u_1, v_1, \ldots, v_n) \neq 0$. Now, since $\pi_i(J) = J_i = R_i$, there exist $a, b_1, \ldots, b_n \in J$ such that $\pi_i(a) = u$ and $\pi_t(b_t) = v_t,\ t = 1, \ldots, n$. Therefore $\pi_i(F(a, b_1, \ldots, b_n)) = F(u, v_1, \ldots, v_n) \neq 0$ and so $0 \neq F(a, b_1, \ldots, b_n) \in A$. $\square$

Corollary 1.5.9 ([24], Part B, Corollary 6.3)**.** *If the center of a semiprime PI-ring R is a field, then R is a finite-dimensional central simple algebra.*

Proof. Let J be an ideal of R. If $J \neq (0)$, then there exists a nonzero element $a \in J \cap Z(R)$, by Theorem 1.5.8. Since $Z(R)$ is a field, a is invertible and thus $1 \in J$, i.e. $J = R$. So R is a simple ring and we are done by Theorem 1.4.25. $\square$

Theorem 1.5.10. (Posner, [50]) *Let R be a prime ring and let $Q_Z(R)$ be the central localization of R.*

(1) The ring $Q_Z(R)$ is prime and $Z(Q_Z(R))$ is the quotient field of $Z(R)$.

(2) The ring R is PI if and only if $Q_Z(R)$ is a finite-dimensional central simple algebra.

Proof. The first part follows from Proposition 1.2.18 and the fact that the center of a prime ring is a domain. We now prove the second part of the theorem. If $Q_Z(R)$ is a finite-dimensional central simple algebra, then it is PI and so R is PI too because $R \subseteq Q_Z(R)$. Conversely, if R is PI, then $Q_Z(R)$ is PI by Corollary 1.4.16. Thus, by the first part of the theorem, $Q_Z(R)$ is a prime PI-ring whose center is a field and so it is a finite-dimensional central simple algebra by Corollary 1.5.9. □

Corollary 1.5.11. *If R is a PI-domain, then $Q(R) = Q_Z(R)$ and thus $Q(R)$ is PI too.*

Proof. If R is a PI-domain, then R is Ore and hence $Q(R)$ exists and it is a division ring. On the other hand, $Q_Z(R)$ is a domain and thus, by Theorem 1.5.10, $Q_Z(R)$ is a division ring too. The result now follows from $R \subseteq Q_Z(R) \subseteq Q(R)$ and the fact that $Q(R)$ is the smallest division ring containing R. Now Corollary 1.4.16 completes the proof. □

1.6 Gelfand-Kirillov Dimension

Throughout this section, k is a field. Let A be a finitely generated k-algebra. The Gelfand Kirillov dimension, or GK dimension, of A measures the rate of the growth of A in terms of any generating set of A. The smallest possible value of the GK dimension of A is zero and this happens if and only if $\dim_k A < \infty$. On the other hand, if A contains a free algebra on two variables, then the GK dimension of A is infinity. If A is commutative, then the GK dimension of A is just the Krull dimension of A. We begin with the definition and some basic facts about the GK dimension of algebras.

1.6.1 Definitions and Basic Results

Let A be a k-algebra. Let V be a k-vector subspace of A spanned by the set $\{a_1, \ldots, a_m\}$. For any integer $n \geq 1$ we denote by V^n the k-subspace of A spanned by all monomials of length n in $a_1, \ldots, a_m$. We also define $V^0 = k$. If A, as a k-algebra, is generated by $a_1, \cdots, a_n$, then V is called a *generating subspace* of A. We denote by V_n the union $\bigcup_{i=0}^n V^i$. Note that $A = \bigcup_{n=0}^\infty V^n = \bigcup_{n=0}^\infty V_n$, for any generating subspace of A. A generating subspace of A which contains 1 is called a *frame* of A. If V is a frame of A, then clearly $V_n = V^n$ for all n.

For example, let $A = k[x_1, \cdots, x_n]$, the polynomial algebra over k in n variables $x_1, \cdots, x_n$. Then $V = kx_1 + \cdots + kx_n$ is a generating subspace but not a frame of A. The vector space $k + V$ is a frame of A.

Lemma 1.6.1. *Let A be a finitely generated k-algebra and let V and W be two generating subspaces of A. Then $\limsup_{n\to\infty} \log_n(\dim V_n) = \limsup_{n\to\infty} \log_n(\dim W_n)$.*

Proof. We have $A = \bigcup_{n=0}^{\infty} V_n = \bigcup_{n=0}^{\infty} W_n$. Since both V and W are finite-dimensional, there exist integers $r \geq 1$ and $s \geq 1$ such that $V \subseteq W_r$ and $W \subseteq V_s$. Thus $V_n \subseteq W_{rn}$ and $W_n \subseteq V_{sn}$ for all integers $n \geq 0$. Now, $\dim V_n \leq \dim W_{rn}$ implies that

$$\log_n(\dim V_n) \leq \log_n(\dim W_{rn}) = (1 + \log_n r) \log_{rn}(\dim W_{rn}).$$

Taking limsup of both sides of the above inequality gives

$$\limsup_{n\to\infty} \log_n(\dim V_n) \leq \limsup_{n\to\infty} \log_n(\dim W_n),$$

because $\lim_{n\to\infty} (1 + \log_n r) = 1$ and

$$\limsup_{n\to\infty} \log_{rn}(\dim W_{rn}) \leq \limsup_{n\to\infty} \log_n(\dim W_n).$$

Similarly $\dim W_n \leq \dim V_{sn}$ implies

$$\limsup_{n\to\infty} \log_n(\dim W_n) \leq \limsup_{n\to\infty} \log_n(\dim V_n),$$

which completes the proof. $\square$

By the above lemma, $\limsup_{n\to\infty} \log_n(\dim V_n)$ does not depend on the generating subspace V. So the following definition makes sense.

Definition 1.6.2. Let A be a finitely generated k-algebra and let V be a generating subspace of A. The *Gelfand-Kirillov dimension*, or GK dimension, of A is defined by $\text{GKdim}(A) := \limsup_{n\to\infty} \log_n(\dim V_n)$. If V is a frame of A, then

$$\text{GKdim}(A) = \limsup_{n\to\infty} \log_n(\dim V^n).$$

We will see later in this section that if A is commutative, then the GK dimension of A and the Krull dimension of A are equal.

The following simple result allows us to extend the definition of GK dimension to arbitrary algebras, as given in Definition 1.6.4.

Proposition 1.6.3. *Let A be a finitely generated k-algebra and let B be a finitely generated subalgebra of A. Then* $\mathrm{GKdim}(B) \geq \mathrm{GKdim}(A)$.

Proof. Let W be a frame of B and extend W to a frame V of A. Then $W^n \subseteq V^n$ for all n. Thus $\dim W^n \leq \dim V^n$ for all n and the result follows. $\square$

So by the above Proposition, if A is a finitely generated k-algebra, then $\sup_B \mathrm{GKdim}(B) = \mathrm{GKdim}(A)$, where sup runs over all finitely generated k-subalgebras B of A. Now we can define the GK dimension of an arbitrary algebra.

Definition 1.6.4. Let A be a k-algebra. We define $\mathrm{GKdim}(A) := \sup_B \mathrm{GKdim}(B)$, where sup runs over all finitely generated k-subalgebras B of A.

Proposition 1.6.5 ([48], Proposition 8.2.2)**.** *Let A be a k-algebra, B a subalgebra of A and I an ideal of A. Then* $\mathrm{GKdim}(A) \geq \max\{\mathrm{GKdim}(B), \mathrm{GKdim}(A/I)\}$.

Proof. The inequality $\mathrm{GKdim}(A) \geq \mathrm{GKdim}(B)$ is obvious by Definition 1.6.4 and the fact that every finitely generated subalgebra of B is a finitely generated subalgebra of A.

Now let A_1 be a finitely generated subalgebra of A/I. Let $\pi : A \longrightarrow A/I$ be the natural homomorphism. Then $A_0 = \pi^{-1}(A_1)$ is a finitely generated subalgebra of A. Let W be a frame of A_1. Then $V = \pi^{-1}(W)$ is a frame of A_0 and clearly $\dim_k V^n \geq \dim_k W^n$ for all n. Thus $\mathrm{GKdim}(A) \geq \mathrm{GKdim}(A_0) \geq \mathrm{GKdim}(A_1)$. Taking supremum over all finitely generated subalgebras A_1 gives $\mathrm{GKdim}(A) \geq \mathrm{GKdim}(A/I)$. $\square$

If I in Proposition 1.6.5 contains a left or right regular element, e.g. if A is a domain, then we have a stronger upper bound for the GK dimension of A/I, as the next result shows.

Proposition 1.6.6 ([56], Proposition 6.2.24)**.** *Let A be a k-algebra and let I be an ideal of A. If* $\mathrm{l.ann}_A(a) = (0)$ *for some $a \in I$, then* $\mathrm{GKdim}(A/I) \leq \mathrm{GKdim}(A) - 1$.

Proof. Let B be any finitely generated subalgebra of A and let V be a frame of $B' := B[a]$ which contains a. Let $\overline{V} = (V + I)/I$. Clearly $\overline{V}$ is a frame of $\overline{B'} = (B' + I)/I = (B + I)/I$. If n is an integer, then, as k-vector spaces, $V^n = (V^n \cap I) \oplus W_n$ for some finite-dimensional k-vector space W_n. Note that $W_n \cong V^n/(V^n \cap I) \cong (V^n + I)/I = \overline{V}^n$. Also, since $Aa \cap W_n \subseteq I \cap V^n$ and $Aa \cap W_n \subseteq W_n$, we have $Aa \cap W_n = (0)$ for all n. Therefore, since $\mathrm{l.ann}_A(a) = (0)$, the sum $\sum_{i=0}^{n} W_n a^i$ is direct for all n. Clearly $\sum_{i=0}^{n} W_n a^i \subseteq V^{2n}$ for all n because both a

and W_n are in V^n. Thus

$$\dim_k V^n \geq \dim_k \sum_{i=0}^{n} W_n a^i = \sum_{i=0}^{n} \dim_k W_n a^i = (n+1)\dim_k W_n > n \dim_k W_n = n \dim_k \overline{V}^n.$$

Hence

$$\mathrm{GKdim}(A) \geq \mathrm{GKdim}(B') \geq 1 + \mathrm{GKdim}(\overline{B'}).$$

Since every finitely generated subalgebra of A/I is in the form $(B+I)/I$ for some finitely generated subalgebra B of A, the above inequality holds for any finitely generated subalgebra of A/I. Thus $\mathrm{GKdim}(A) \geq 1 + \mathrm{GKdim}(A/I)$. □

And here is a nice little application of the proposition.

Corollary 1.6.7. *Let A be k-algebra which is a domain. Let B be a simple subalgebra of A. If $\mathrm{GKdim}(A) < \mathrm{GKdim}(B) + 1$, then A is simple too.*

Proof. Let I be a nonzero ideal of A. If $I \cap B \neq (0)$, then $I \cap B = B$, because B is simple, and so $I = A$. Suppose now that $I \cap B = (0)$. Then the natural homomorphism $B \longrightarrow A/I$ would be injective and so $\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A/I) \leq \mathrm{GKdim}(A) - 1$, by Proposition 1.6.6, which is a contradiction. □

Now, we are going to look at algebras with the smallest and the largest possible GK dimension, i.e. zero and infinity. First, we show that free algebras have infinite GK dimension and then we characterize algebras of GK dimension zero.

Proposition 1.6.8 ([48], Proposition 8.1.15 (iv))**.** *If X is a set of noncommuting variables with $|X| \geq 2$ and $A := k\langle X\rangle$, then $\mathrm{GKdim}(A) = \infty$.*

Proof. Let $x, y \in X$ and put $B = k\langle x, y\rangle$. Let $V = k + kx + ky$. Then V is a frame of B and $\dim V^n = 1 + 2 + \cdots + 2^n \geq 2^n$. Thus $\log_n(\dim V^n) \geq n \log_n 2$. Hence $\mathrm{GKdim}(A) = \limsup_{n\to\infty} \log_n(\dim V^n) \geq \lim_{n\to\infty} n \log_n 2 = \infty$ and so $\mathrm{GKdim}(A) = \infty$, by Definition 1.6.4. □

Corollary 1.6.9 ([48], Corollary 8.1.21)**.** *Let A be a k-algebra which is a domain. If $\mathrm{GKdim}(A) < \infty$, then A is Ore.*

Proof. If A contains a copy of $k\langle x, y\rangle$, where x and y are noncommuting variables, then $\mathrm{GKdim}(A) = \infty$, by Proposition 1.6.8. Thus A does not contain such a subalgebra and hence A is Ore by Proposition 1.2.15. □

Proposition 1.6.10 ([48], Proposition 8.1.17 (i))**.** *Let A be a k-algebra. Then* $\mathrm{GKdim}(A) = 0$ *if and only if A is locally finite, i.e. every finitely generated subalgebra of A is finite-dimensional over k.*

Proof. Suppose first that A is locally finite and let B be a finitely generated subalgebra of A. Then B is finite-dimensional over k and so $V = A$ is a frame of B. Clearly $V^n = B$ and thus $\mathrm{GKdim}(B) = \limsup_{n\to\infty} \log_n(\dim B) = 0$, because $\dim B$ does not depend on n. Conversely, suppose that $\mathrm{GKdim}(A) = 0$ and let B be a finitely generated subalgebra of A. Let V be a frame of B and suppose for now that $V^n \subset V^{n+1}$ for all n. Then $k \subset V \subset V^2 \subset \cdots$ and thus $1 < \dim V < \dim V^2 < \cdots$. Hence $\dim V > 1,\ \dim V^2 > 2$ and in general $\dim V^n > n$. Therefore

$$0 = \mathrm{GKdim}(B) = \limsup_{n\to\infty} \log_n(\dim V^n) \geq \lim_{n\to\infty} \log_n n = 1,$$

which is absurd. So our assumption that $V^n \subset V^{n+1}$ for all n, is false. Hence $V^n = V^{n+1}$, for some integer $n \geq 0$ and so $B = \bigcup_{i=0}^{\infty} V^i = V^n$. Thus B is finite-dimensional and so A is locally finite. □

Remark 1.6.11. If A is a domain of GK dimension zero, then A is a division ring. To see this, let $a \in A$. Then, by Proposition 1.6.10, the k-subalgebra generated by a is finite-dimensional and hence a is algebraic over k. Thus A is algebraic over k and we know that an algebraic k-algebra which is a domain is a division ring.

Assumption. For the rest of this section, all algebras have finite GK dimension.

There is no algebra whose GK dimension is strictly between zero and one, as the next result shows.

Proposition 1.6.12 ([48], Proposition 8.1.17 (ii))**.** *Let A be a k-algebra. If* $\mathrm{GKdim}(A) \neq 0$, *then* $\mathrm{GKdim}(A) \geq 1$.

Proof. Since $\mathrm{GKdim}(A) \neq 0$, there exists a finitely generated k-subalgebra B of A such that $\mathrm{GKdim}(B) \neq 0$. Let V be a frame of B. If $V^n = V^{n+1}$, for some integer $n \geq 0$, then $B = \bigcup_{i=0}^{\infty} V^i = V^n$ and so B is finite-dimensional. But then $\mathrm{GKdim}(B) = 0$, by Proposition 1.6.10, which is false. Thus $k \subset V \subset V^2 \subset \cdots$ and hence $\dim V^n > n$ for all $n \geq 0$. Therefore $\mathrm{GKdim}(B) = \limsup_{n\to\infty} \log_n(\dim V^n) \geq 1$ and so $\mathrm{GKdim}(A) \geq \mathrm{GKdim}(B) \geq 1$. □

So if $0 < \alpha < 1$, then there is no algebra A with $\mathrm{GKdim}(A) = \alpha$. We will see in the next subsection that for any integer $m \geq 1$ there exists an algebra A such that $\mathrm{GKdim}(A) = m$.

1.6.2 GK Dimension of Polynomial and Weyl Algebras

In this subsection, we find the GK dimension of a commutative polynomial k-algebra in n variables and the n-th Weyl algebra over k. We begin by evaluating the GK dimension of polynomial algebras.

Proposition 1.6.13 ([48], Proposition 8.2.7 (iii))**.** *Let A be a k-algebra. Then* $\mathrm{GKdim}(A[x]) = 1 + \mathrm{GKdim}(A)$.

Proof. Let B_0 be a finitely generated subalgebra of $A[x]$ generated by $f_1, \dots, f_m \in A[x]$. Let B be the subalgebra of A generated by the coefficients of $f_i,\ i = 1, \dots, m$. Then clearly B is a finitely generated subalgebra of A and $B_0 \subseteq B[x]$. Now, let W be a frame of B. Let $V = W + kx$. Then V is a frame of $B[x]$ and clearly $V^n = (W + kx)^n \subseteq \bigoplus_{i=0}^n W^n x^i$ for all integers $n \geq 0$. Hence $\dim V^n \leq (n+1) \dim W^n$ and so

$$\mathrm{GKdim}(B_0) \leq \mathrm{GKdim}(B[x]) \leq \lim_{n\to\infty} \log_n(n+1) + \mathrm{GKdim}(B) = 1 + \mathrm{GKdim}(B) \leq 1 + \mathrm{GKdim}(A).$$

Therefore $\mathrm{GKdim}(A[x]) \leq 1 + \mathrm{GKdim}(A)$. It is also clear that $V^{2n} = (W + kx)^{2n} \supseteq \bigoplus_{i=0}^n W^n x^i$ for all integers $n \geq 0$. Thus $\dim V^{2n} \geq (n+1) \dim W^n$ and so

$$\mathrm{GKdim}(A[x]) \geq \mathrm{GKdim}(B[x]) \geq \lim_{n\to\infty} \log_n(n+1) + \mathrm{GKdim}(B) = 1 + \mathrm{GKdim}(B).$$

Hence $\mathrm{GKdim}(A[x]) \geq 1 + \mathrm{GKdim}(A)$ and the result follows. $\square$

Corollary 1.6.14. *Let A be a k-algebra. Then* $\mathrm{GKdim}(A[x_1, \dots, x_m]) = m + \mathrm{GKdim}(A)$. *In particular,* $\mathrm{GKdim}(k[x_1, \dots, x_m]) = m$.

So, by the above corollary, for any integer $m \geq 1$ there exists a finitely generated algebra A such that $\mathrm{GKdim}(A) = m$. Thus, by Definition 1.6.4, if X is an infinite set of commuting variables, then $\mathrm{GKdim}(k[X]) = \infty$. The following theorems together give the possible values of the GK dimension of an algebra.

Theorem 1.6.15. (Bergman's gap theorem, [37, Theorem 2.5]) *There is no algebra A with* $1 < \mathrm{GKdim}(A) < 2$.

Theorem 1.6.16. (Warfield, [48, Proposition 8.1.18]) *For any real number $\alpha \geq 2$ there exists a finitely generated k-algebra of* GK *dimension α.*

Proof. We only need to show that for every $2 \leq q < 3$ there exists a finitely generated k-algebra of GK dimension q. The reason is that if $r \geq 3$ is any real number, then $r = q+m$ for some $2 \leq q < 3$ and some positive integer m. So if A is a finitely generated k-algebra of GK dimension q, then, by Corollarty 1.6.14, the polynomial algebra $A[x_1, \ldots, x_m]$ is a finitely generated algebra of GK dimension $q + m = r$.

Now, fix a real number $2 \leq q < 3$ and let $R := k\langle X, Y\rangle$. Let I be the two-sided ideal of R generated by Y. It is easy to see that $\text{GKdim}(R/I^2) = 2$ and $\text{GKdim}(R/I^3) = 3$. Now, consider the algebras of the form R/J where $I^3 \subset J \subset I^2$. It turns out that we can choose J somehow that $\text{GKdim}(R/J) = q$. We define J as follows. First, for every positive integer n let $\alpha_n := \lfloor n^{(q-1)/2} \rfloor$. Let

$$J := I^3 + L,$$

where L is the two-sided ideal of R generated by all monomials $X^r Y X^s Y X^t$ of length n, where n is any positive integer and $s < n - \alpha_n$. The claim is that $\text{GKdim}(R/J) = q$. To prove the claim, let x, y be the images of X, Y in R/J, respectively, and consider the generating subspace $V = kx + ky$ of R/J. Let n be a positive integer. By the definition of J, every monomial in V^n whose degree in y is ≥ 3 is zero. Also, every monomial $x^r y x^s y x^t \in V^n$, where $s < n - \alpha_n$, is zero. So the set

$$\{x^n, x^u y x^v, x^r y x^s y x^t : \ u + v + 1 = r + s + t + 2 = n, \ s \geq n - \alpha_n\}$$

is a k-basis for V^n and hence

$$\dim V^n = 1 + n + \frac{\alpha_n(\alpha_n - 1)}{2}.$$

The rest of the proof is just simple calculus: using the fact that $n^{(q-1)/2} - 1 < \alpha_n \leq n^{(q-1)/2}$, we see that there exists an integer N and positive constants β, γ such that

$$\beta n^{q-1} \leq \dim V^n \leq \gamma n^{q-1}$$

for all $n \geq N$. Thus if we estimate $\dim \bigcup_{i=0}^n V^i = \sum_{i=0}^n \dim V^i$ with a definite integral, we get $\text{GKdim}(R/J) = \lim\limits_{n\to\infty} \log_n \sum\limits_{i=0}^{n} \dim V^i = q$. $\square$

We now give the definition of the Weyl algebras and then we find their GK dimension.

Definition 1.6.17. Let R be a ring and let $n \geq 0$ be an integer. The n-th *Weyl algebra* over R is defined as follows. First we define $A_0(R) = R$ and for $n \geq 1$, $A_n(R)$ is defined to be the ring of polynomials in $2n$ variables $x_i, y_i,\ 1 \le i \le n$, with coefficients in R and subject to the relations $x_i x_j = x_j x_i,\ y_i y_j = y_j y_i$, and $y_i x_j = x_j y_i + \delta_{ij}$ for all i, j, where δ_{ij} is the Kronecker delta. We will assume that every element of R commutes with all $2n$ variables x_i and y_i.

So, for example, $A_1(R)$ is the ring of polynomials generated by x, y with coefficients in R and subject to the relation $yx = xy + 1$. An element of $A_1(R)$ is in the form $\sum r_{ij} x^i y^j,\ r_{ij} \in R$. It is not hard to prove that the set of monomials in the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_n^{\beta_n}$ is an R-basis for $A_n(R)$. We also note that $A_n(R) = A_1(A_{n-1}(R))$. If R is a domain, then $A_n(R)$ is a domain too. It is well-known that if $\mathrm{char}(k) = 0$, then $A_n(k)$ is a simple noetherian domain [48, Theorem 1.3.5].

We are now going to prove that if R is a k-algebra, then $\mathrm{GKdim}(A_n(R)) = 2n + \mathrm{GKdim}(R)$, a result similar to Corollary 1.6.14.

Proposition 1.6.18 ([48], Proposition 8.1.15 (ii))**.** *If R is a k-algebra, then* $\mathrm{GKdim}(A_1(R)) = 2 + \mathrm{GKdim}(R)$.

Proof. Suppose first that R is finitely generated and let V be a frame of R. Let $U = k + kx + ky$. Since $yx = xy + 1$, we have

$$\dim_k U^n = \frac{(n+1)(n+2)}{2}. \tag{1.7}$$

Let $W = U + V$. Clearly W is a frame of $A_1(R)$. We have $W^n = \sum_{i+j=n} U^i V^j$ for all n, because every element of V commutes with every element of U. Therefore, since $V^j \subseteq V^n$ and $U^i \subseteq U^n$ for all $i, j \le n$, we have $W^n \subseteq U^n V^n$ and $W^{2n} \supseteq U^n V^n$. Thus $W^n \subseteq U^n V^n \subseteq W^{2n}$. Hence $\log_n \dim_k W^n \le \log_n \dim_k U^n + \log_n \dim_k V^n \le \log_n \dim_k W^{2n}$ and so

$$\mathrm{GKdim}(A_1(R)) \le 2 + \mathrm{GKdim}(R) \le \mathrm{GKdim}(A_1(R)),$$

by (1.7), and we are done.

For the general case, let R_0 be any finitely generated k- subalgebra of R. Then, by what we just proved, $2 + \mathrm{GKdim}(R_0) = \mathrm{GKdim}(A_1(R_0)) \le \mathrm{GKdim}(A_1(R))$ and hence

$$2 + \mathrm{GKdim}(R) \le \mathrm{GKdim}(A_1(R)).$$

Now, let A_0 be a k-subalgebra of $A_1(R)$ generated by a finite set $\{f_1, \ldots, f_m\}$. Let R_0 be the k-subalgebra of R generated by all the coefficients of $f_1, \ldots, f_m$. Then $A_0 \subseteq A_1(R_0)$ and so $\mathrm{GKdim}(A_0) \le \mathrm{GKdim}(A_1(R_0)) = 2 + \mathrm{GKdim}(R_0) \le 2 + \mathrm{GKdim}(R)$. Thus $\mathrm{GKdim}(A_1(R)) \le 2 + \mathrm{GKdim}(R)$ and the proof is complete. $\square$

Corollary 1.6.19. *If R is a k-algebra, then* $\text{GKdim}(A_n(R)) = 2n + \text{GKdim}(R)$ *for all n. In particular,* $\text{GKdim}(A_n(k)) = 2n$.

Proof. It follows from Proposition 1.6.18 and the identity $A_n(R) = A_1(A_{n-1}(R))$. □

1.6.3 GK Dimension of Extensions of Algebras

Let A be a k-algebra. In this subsection, we study the behavior of the GK dimension under some extensions of A. We first show that the GK dimension of any algebra which is a finite module over A is equal to the GK dimension of A. This extends Proposition 1.6.10. We also prove that the GK dimension of a central localization of A is equal to the GK dimension of A. These fundamental results have two important consequences. The first one is that the GK dimension and the Krull dimension of a finitely generated commutative algebra are equal. The second consequence is that, over an algebraically closed field k, every finitely generated k-algebra which is a domain of GK dimension at most 1 is commutative. We begin with a lemma.

Lemma 1.6.20 ([48], Proposition 8.2.3)**.** *If A and B are k-algebras and* $\text{GKdim}(B) = 0$, *then* $\text{GKdim}(A \otimes_k B) = \text{GKdim}(A)$.

Proof. Since $A \cong A \otimes_k 1 \subseteq A \otimes_k B$, we have $\text{GKdim}(A) \leq \text{GKdim}(A \otimes_k B)$. Now, let C be a finitely generated subalgebra of $A \otimes_k B$ with a frame W. Since $\dim_k W < \infty$, there exist finite-dimensional subspaces U, V of A, B, respectively, such that $1_A \in U$, $1_B \in V$ and $W \subseteq U \otimes_k V$. Let A_0, B_0 be the algebras generated by U, V respectively. Now, $W^n \subseteq U^n \otimes_k V^n$ for all n, and hence $\dim_k W^n \leq (\dim_k U^n)(\dim_k V^n)$. Therefore $\log_n \dim_k W^n \leq \log_n \dim_k U^n + \log_n \dim_k V^n$ and hence, taking limsup, gives

$$\text{GKdim}(C) \leq \text{GKdim}(A_0) + \text{GKdim}(B_0) = \text{GKdim}(A_0) \leq \text{GKdim}(A).$$

Since the above holds for any finitely generated subalgebra C of $A \otimes_k B$, we have $\text{GKdim}(A \otimes_k B) \leq \text{GKdim}(A)$. □

Corollary 1.6.21 ([48], Proposition 8.2.17 (i))**.** *If B is a k-algebra, then* $\text{GKdim}(M_n(B)) = \text{GKdim}(B)$.

Proof. We have $M_n(B) \cong B \otimes_k M_n(k)$ and $\text{GKdim}(M_n(k)) = 0$, by Proposition 1.6.10. Thus $\text{GKdim}(M_n(B)) = \text{GKdim}(B)$, by Lemma 1.6.20. □

Let A be a k-algebra and let B be a subalgebra of A. If A, as a B-module, is both finitely generated and free over B, then $\mathrm{GKdim}(\mathrm{End}_B(A)) = \mathrm{GKdim}(B)$, by Corollary 1.6.21. If A is finitely generated but not necessarily free over B, then we have the following result.

Lemma 1.6.22 ([48], Proposition 8.2.9 (i))**.** *Let B be a subalgebra of a k-algebra A. Suppose that, as a left module, A is finitely generated over B. Then* $\mathrm{GKdim}(\mathrm{End}_B(A)) \leq \mathrm{GKdim}(B)$.

Proof. So $A = \sum_{i=1}^{n} Ba_i$ for some $a_i \in A$. Define $\varphi : B^n \longrightarrow A$ by $\varphi(b_1, \ldots, b_n) = \sum_{i=1}^{n} b_i a_i$ and let $I = \ker \varphi$. Let $C = \{f \in \mathrm{End}_B(B^n) : \ f(I) \subseteq I\}$. Clearly C is a subalgebra of $\mathrm{End}_B(B^n) \cong M_n(B)$. Now, given $f \in C$ define $\overline{f} : A \longrightarrow A$ by $\overline{f}(a) = \varphi f(u)$, where u is any element of B^n with $\varphi(u) = a$. Note that $\overline{f}$ is well-defined because if $\varphi(v) = a$ for some other $v \in B^n$, then $u - v \in I$ and thus $f(u - v) \in I$. Hence $0 = \varphi f(u - v) = \varphi f(u) - \varphi f(v)$ and so $\varphi f(u) = \varphi f(v)$. It is easy to see that $\overline{f} \in \mathrm{End}_B(A)$. Finally, define $\psi : C \longrightarrow \mathrm{End}_B(A)$ by $\psi(f) = \overline{f}$. Then ψ is a k-algebra onto homomorphism and hence

$$\mathrm{GKdim}(\mathrm{End}_B(A)) \leq \mathrm{GKdim}(C) \leq \mathrm{GKdim}(M_n(B)) = \mathrm{GKdim}(B),$$

by Proposition 1.6.5 and Corollary 1.6.21. □

Proposition 1.6.23 ([48], Proposition 8.2.9 (ii))**.** *Let B be a subalgebra of a k-algebra A. If, as a left module, A is finitely generated over B, then* $\mathrm{GKdim}(A) = \mathrm{GKdim}(B)$.

Proof. The algebra A has a natural embedding into $\mathrm{End}_B(A)$ and so

$$\mathrm{GKdim}(A) \leq \mathrm{GKdim}(\mathrm{End}_B(A)).$$

Thus $\mathrm{GKdim}(A) \leq \mathrm{GKdim}(B)$, by Lemma 1.6.22. □

We recall that $\mathrm{Kdim}(A)$, the Krull dimension of a commutative algera A, is the largest integer $n \geq 0$ for which there exist prime ideals $P_i, \ 0 \leq i \leq n$, of A such that $P_0 \subset P_1 \subset \ldots \subset P_n$. If there is no such integer, then we define $\mathrm{Kdim}(A) = \infty$. A consequence of Proposition 1.6.23 is that $\mathrm{GKdim}(A) = \mathrm{Kdim}(A)$ for any finitely generated commutative k-algebras A. This is a simple consequence of the following well-known result.

Theorem 1.6.24. (Noether normalization theorem, [25, Theorem A1, p. 221]) *Let A be a finitely generated commutative k-algebra of Krull dimension m. There exists a k-subalgebra B of A such that $B \cong k[x_1, \ldots, x_m]$, the polynomial algebra in m variables $x_1, \ldots, x_m$, and A is a finitely generated B-module.*

Corollary 1.6.25 ([48], Theorem 8.2.14 (i)). *If A is a finitely generated commutative k-algebra, then* $\mathrm{GKdim}(A) = \mathrm{Kdim}(A)$.

Proof. Let $m = \mathrm{Kdim}(A)$. Then A contains a polynomial k-algebra $B = k[x_1, \ldots, x_m]$ and A is a finitely generated B-module, by Theorem 1.6.24. Thus, by Corollary 1.6.14 and Proposition 1.6.23, $\mathrm{GKdim}(A) = \mathrm{GKdim}(B) = m$. □

The GK dimension is also well-behaved under central localization, as the next result shows.

Proposition 1.6.26 ([48], Proposition 8.2.13). *Let A be a k-algebra. Suppose that S is a regular submonoid of A contained in the center of A. Then* $\mathrm{GKdim}(S^{-1}A) = \mathrm{GKdim}(A)$.

Proof. Let T be a finitely generated k-subalgebra of $S^{-1}A$ and suppose that

$$W = \{w_1 = 1, \ldots, w_m\}$$

is a frame of T. Then, by Proposition 1.2.8, there exit $s \in S$ and $a_1, \ldots, a_m \in A$ such that $w_i = s^{-1}a_i$ for all i. Let B be the k-subalgebra of A generated by a_i and let V be the k-subspace generated by 1 and a_i. Now, since S is in the center of A, we have $s^n W^n \subseteq V^n$. Thus $\dim W^n = \dim s^n W^n \leq \dim V^n$. Therefore

$$\mathrm{GKdim}(T) \leq \mathrm{GKdim}(B) \leq \mathrm{GKdim}(A),$$

for every finitely generated k-subalgebra of T of $S^{-1}A$, and so $\mathrm{GKdim}(S^{-1}A) \leq \mathrm{GKdim}(A)$. On the other hand, $A \subseteq S^{-1}A$, because S is regular, and thus $\mathrm{GKdim}(A) \leq \mathrm{GKdim}(S^{-1}A)$. □

As an application of the above proposition, we can find the GK dimension of a Laurent polynomial ring.

Corollary 1.6.27 ([48], Corollary 8.2.15). *Let A be a k-algebra. Then* $\mathrm{GKdim}(A[x, x^{-1}]) = 1 + \mathrm{GKdim}(A)$.

Proof. Since $A[x, x^{-1}]$ is the localization of $A[x]$ at the central regular submonoid $S = \{1, x, x^2, \ldots\}$, we have $\mathrm{GKdim}(A[x, x^{-1}]) = \mathrm{GKdim}(A[x])$. The result now follows from Proposition 1.6.13. □

We are now able to prove an important result, i.e. Proposition 1.6.32, that will be used frequently in chapters two and three. We have already seen that $\text{GKdim}(A) = 0$ if and only if A is locally finite and that there is no algebra of GK dimension strictly between 0 and one. What can we say about the case $\text{GKdim}(A) = 1$? The answer is that in many important cases they are finitely generated modules over some polynomial algebra in one variable. In order to prove this result, we need the following two theorems.

Theorem 1.6.28. (Artin-Tate, [25, p. 143]) *Let $A \subseteq B \subseteq C$ be commutative k-algebras. Suppose that A is Noetherian and C is a finitely generated A-algebra. If C is a finitely generated B-module, then B is a finitely generated A-algebra.*

Theorem 1.6.29. (Small, Stafford and Warfield, [59, Theorem 1.6]) *Let A be a finitely generated semiprime k-algebra. If* $\text{GKdim}(A) = 1$*, then A is finitely generated over its center $Z(A)$.*

Proposition 1.6.30 ([56], Recapitulation 6.2.34)**.** *If A is a finitely generated semiprime k-algebra, then* $\text{GKdim}(A) = 1$ *if and only if A is finitely generated as a module over some polynomial algebra $k[x]$.*

Proof. If A is finitely generated as a module over some polynomial algebra $k[x]$, then $\text{GKdim}(A) = \text{GKdim}(k[x]) = 1$, by Proposition 1.6.23 and Corollary 1.6.14. Conversely, if $\text{GKdim}(A) = 1$, then by Theorem 1.6.29, A is finitely generated $Z(A)$ and thus $\text{GKdim}(Z(A)) = 1$, by Proposition 1.6.23. We also have $k \subseteq Z(A) \subseteq A$ and we know that A is both a finitely generated k-algebra and a finitely generated $Z(A)$-module. Thus, by Theorem 1.6.28, $Z(A)$ is a finitely generated k-algebra. Therefore $\text{Kdim}(Z(A)) = 1$, by Corollary 1.6.25, and so $Z(A)$ is a finitely generated module over some polynomial algebra $k[x]$, by Theorem 1.6.24. The result now follows because A is a finitely generated $Z(A)$-module. □

An important consequence of Proposition 1.6.30 together with Tsen's theorem is that domains of GK dimension one over algebraically closed fields are commutative. We first state Tsen's theorem.

Theorem 1.6.31. (Tsen, [64], see also [41]) *Let D be a finite-dimensional division K-algebra and suppose that K is a finitely generated extension of a field k of transcendence degree one. If k is algebraically closed, then D is commutative.*

Proposition 1.6.32. *Let A be a k-algebra which is a domain and* $\mathrm{GKdim}(A) \leq 1$. *If k is algebraically closed, then A is commutative.*

Proof. First note that if $a, b \in A$, then the k-subalgebra generated by a, b has GK dimension at most one too and so we may assume that A is finitely generated. The case $\mathrm{GKdim}(A) = 0$ easily follows because then A would be finite-dimensional, and hence algebraic, over k and therefore $A = k$ because k is algebraically closed. Now, suppose that $\mathrm{GKdim}(A) = 1$. The algebra A is PI, by Proposition 1.6.30 and Proposition 1.4.7, and thus $Q_Z(A)$, the central localization of A, is a finite-dimensional central simple algebra by Theorem 1.5.10. Since A is a domain, $Q_Z(A)$ is a domain and hence $Q_Z(A) = D$ is a finite-dimensional division algebra over its center F, which is the quotient field of $Z(A)$. Thus $\mathrm{Kdim}(F) = \mathrm{Kdim}(Z(A)) = 1$, by Corollary 1.6.25. Hence, by Theorem 1.6.31, $Q_Z(A) = F$. Thus $Q_Z(A)$, and so A itself, is commutative. □

We close this section with two fundamental theorems in GK dimension theory. The first one gives a lower bound for the GK dimension of an algebra which is not locally PI. This result is due to Smith and Zhang. A special case of the second theorem, which is due to W. Borho and H. Kraft, gives some information about any subalgebra of a finitely generated algebra whose GK dimension is equal to the GK dimension of the algebra. We begin with a lemma.

Lemma 1.6.33. *If A is a k-algebra which is a domain, then either A is locally PI or* $\mathrm{GKdim}(A) \geq 2$.

Proof. Let B be a finitely generated k-subalgebra of A. If $\mathrm{GKdim}(B) = 0$, then B is finite-dimensional over its center and hence PI. If $\mathrm{GKdim}(B) = 1$, then B is again PI by [59]. Thus if B is not PI, then we must have $\mathrm{GKdim}(B) \geq 2$. □

Theorem 1.6.34. (S. Smith and J. Zhang, [60]) *Let A a k-algebra and let $B \subseteq Z(A)$ be a k-subalgebra such that $S = B \setminus \{0\}$ is a regular subset of A. If A is not locally PI, then* $\mathrm{GKdim}(A) \geq 2 + \mathrm{GKdim}(B)$.

Proof. We know from Lemma 1.6.33 that $\mathrm{GKdim}(A) \geq 2$. So there is nothing to prove if $\mathrm{GKdim}(B) = 0$. Thus we may assume that $\mathrm{GKdim}(B) \geq 1$. Let B_0 be any finitely generated k-subalgebra of B. By Corollary 1.6.25, B_0 contains a polynomial k-algebra B_1 in d variables B_1 such that $\mathrm{GKdim}(B_1) = \mathrm{GKdim}(B_0)$ and by Corollary 1.6.14, $\mathrm{GKdim}(B_1) = d$. We only need to prove that $\mathrm{GKdim}(A) \geq 2 + \mathrm{GKdim}(B_1)$. Let S_1 be the set of nonzero elements of B_1 and let $Q := S_1^{-1}A$. Let $F := S_1^{-1}B_1$. Since B_1 is a domain, F is the quotient field of B_1. By Proposition 1.6.26, $\mathrm{GKdim}(Q) = \mathrm{GKdim}(A) \geq 2$ and $\mathrm{GKdim}(F) = \mathrm{GKdim}(B_1) \geq 1$. Let $0 \leq d < \mathrm{GKdim}(F)$ and $0 \leq e < \mathrm{GKdim}_F(Q)$. Then there exist a finite-dimensional k-vector subspace V of F which contains 1 and

$$\dim_k V^n \geq n^d$$

for all large enough integers n. Also, there exists a finite-dimensional F-vector subspace W of Q which contains 1 and

$$\dim_F W^n \geq n^e$$

for all large enough integers n. Clearly $V \subset W$ and hence, for large enough integers n we have

$$\dim_k W^{2n} \geq \dim_k(W^n V^n) \geq (\dim_F W^n)(\dim_k V^n) \geq n^{e+d}.$$

Thus

$$\mathrm{GKdim}(Q) \geq e + d.$$

Since the above inequality holds for all real numbers $0 \leq d < \mathrm{GKdim}(F)$ and $0 \leq e < \mathrm{GKdim}_F(Q)$, we have

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(Q) \geq \mathrm{GKdim}_F(Q) + \mathrm{GKdim}(F) = \mathrm{GKdim}_F(Q) + \mathrm{GKdim}(B_1).$$

Now, Q is not locally PI over F because A is not locally PI over k. Thus $\mathrm{GKdim}_F(Q) \geq 2$, by Lemma 1.6.33, and the proof is complete. □

Theorem 1.6.35. (Borho and Kraft, [19]) *Let A be a finitely generated k-algebra which is a domain. Let B be a k-subalgebra of A and suppose that $\mathrm{GKdim}(A) < \mathrm{GKdim}(B) + 1$. Let $S := B \setminus \{0\}$. Then S is an Ore subset of A and $S^{-1}A = Q(A)$. Also, $Q(A)$ is finite-dimensional as a (left or right) vector space over $Q(B)$.*

Proof. First note that, by Corollary 1.6.9, A is an Ore domain and hence both $Q(A)$ and $Q(B)$ exist and they are division algebras. Now, suppose, to the contrary, that S is not left Ore. Then there exist $x \in S$ and $y \in A$ such that $Sy \cap Ax = \emptyset$. This implies that the sum $By + Byx + \cdots + Byx^m$ is direct for any integer m. Let W be a frame of a finitely generated subalgebra of B. Let $V = W + kx + ky$. Then for any positive n we have

$$V^{2n} \supseteq W^n(kx + ky)^n \supseteq W^n y + W^n yx + \cdots + W^n yx^{n-1}$$

and thus $\dim_k V^{2n} \geq n \dim_k W^n$ because the sum is direct. Therefore $\log_n \dim_k V^{2n} \geq 1 + \log_n \dim_k W^n$ and hence $\mathrm{GKdim}(A) \geq 1 + \mathrm{GKdim}(B)$, a contradiction. A similar argument shows that S is right Ore. So we have proved that S is an Ore subset of A.

Before we show that $S^{-1}(A) = Q(A)$, we prove that $Q(B)A = S^{-1}A$ is finite-dimensional as a left vector space over $Q(B)$. So let V be a frame of A. For any positive integer n, let

$$r(n) = \dim_{Q(B)} Q(B)V^n.$$

Clearly $Q(B)V^n \subseteq Q(B)V^{n+1}$ for all n and $\bigcup_{n=0}^{\infty} Q(B)V^n = Q(B)A$ because $\bigcup_{n=0}^{\infty} V^n = A$. So we have two possibilities: either $Q(B)V^n = Q(B)A$ for some n or the sequence $\{r(n)\}$ is strictly increasing. If $Q(B)V^n = Q(B)A$, then we are done because V^n is finite-dimensional over k and hence $Q(B)V^n$ is finite-dimensional over $Q(B)$. Now suppose that the sequence $\{r(n)\}$ is strictly increasing. Then $r(n) > n$ because $r(0) = \dim_{Q(B)} Q(B) = 1$. Fix an integer n and let $e_1, \ldots, e_{r(n)}$ be a $Q(B)$-basis for $Q(B)V^n$. Clearly we may assume that $e_i \in V^n$ for all i. Let W be a frame of a finitely generated subalgebra of B. Then

$$(V + W)^{2n} \supseteq W^n V^n \supseteq W^n e_1 + \cdots + W^n e_{r(n)},$$

which gives

$$\dim_k (V + W)^{2n} \geq r(n) \dim_k W^n > n \dim_k W^n,$$

because the sum $W^n e_1 + \cdots + W^n e_{r(n)}$ is direct. Therefore $\mathrm{GKdim}(A) \geq 1 + \mathrm{GKdim}(B)$, which is a contradiction. So we have proved that the second possibility is in fact impossible and hence $Q(B)A$ is finite-dimensional over $Q(B)$.

Finally, since, as we just proved, $\dim_{Q(B)} Q(B)A < \infty$, the algebra $Q(B)A$ is algebraic over $Q(B)$ and thus it is a division algebra. Hence $Q(B)A = Q(A)$ because $A \subseteq Q(B)A \subseteq Q(A)$ and $Q(A)$ is the smallest division algebra containing A. $\square$

Chapter 2

Centralizers

2.1 Introduction

Except for the last section, which gives our results on centralizers in semiprime PI-rings, the rest of this chapter reviews some of well-known results on the structure of centralizers in associative algebras. For a ring R and a subset $X \subseteq R$, we denote by $C(X;R)$ the set of all elements of R which commute with every element of X. We say that $C(X;R)$ is the *centralizer* of X in R. That is,

$$C(X;R) = \{r \in R : \ rx = xr, \ \forall x \in X\}.$$

If $X = \{a\}$, then we simply write $C(a;R)$ instead of $C(\{a\};R)$. Clearly $C(X;R)$ is a subring of R and it contains the center $Z(R)$. It is also clear that $C(X;R) = R$ if and only if $X \subseteq Z(R)$. We are only interested in $C(a;R)$ where $a \notin Z(R)$.

2.2 Centralizers in Differential Polynomial Rings

In this section, we give Amitsur's results on centralizers in differential polynomial rings [1]. Let k be a field and let L be a k-vector space. Suppose that $[-,-] : L \times L \longrightarrow L$ is a k-bilinear map such that $[a,a] = 0$ and

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$$

for all $a,b,c \in L$. Then L is called a *Lie algebra*. For example, $\mathbb{R}^3$ with the vector cross product is a Lie algebra over $\mathbb{R}$. Any associative algebra is a Lie algebra because we can define $[a, b] = ab - ba$ for all $a, b \in A$. Now, let L be a Lie k-algebra and define $U(L)$ to be the associative k-algebra generated by the elements of L with the relation

$$ab - ba = [a, b]$$

for all $a, b \in L$. The algebra $U(L)$ is called the *universal enveloping algebra* of L.

For example, let L be a two-dimensional k-vector space with a basis $\{a, b\}$. Define $[-, -]$ by $[a, a] = [b, b] = 0,\ [a, b] = b$ and extend the definition to L by linearity. It is easy to see that L is a Lie algebra. So $U(L)$ is the algebra $k[a, b]$ with the relation $ab - ba = b$. Thus $ab = ba + b$ and an easy induction shows that $af = fa + bf'$ for all $f \in k[b]$, where f' is the derivation of f with respect to b. The map $\delta := b\frac{d}{db} : k[b] \longrightarrow k[b]$ is a *derivation* of $k[b]$, i.e. δ is k-linear and $\delta(fg) = \delta(f)g + f\delta(g)$ for all $f, g \in k[b]$. So an element of $U(L)$ is in the form $\sum_{i=0}^{n} f_i a^i,\ f_i \in k[b]$, and when we multiply two elements of $U(L)$ we need to apply the rule

$$af = fa + \delta(f),\ f \in k[b].$$

We say that $U(L)$ is a *differential polynomial ring* and we write $U(L) = R[a; \delta]$ where $R = k[b]$.

Definition 2.2.1. Let R be a ring and let σ be an endomorphism of R. A *σ-derivation* of R is a linear map $\delta : R \longrightarrow R$ such that

$$\delta(r_1 r_2) = \sigma(r_1)\delta(r_2) + \delta(r_1)r_2$$

for all $r_1, r_2 \in R$. If σ is the identity map, then δ is called a *derivation* of R.

Definition 2.2.2. Let R be a ring and let σ be an endomorphism of R. Let δ be a σ-derivation of R. A ring S is called a *skew polynomial ring* over R if S satisfies the following conditions.

(1) S contains R as a subring.

(2) There exists $x \in S$ such that S is a free left R-module with basis $\{1, x, x^2, \ldots\}$.

(3) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

In this case, we write $S = R[x; \sigma, \delta]$. If $\delta = 0$ (resp. σ is the identity map), we write $S = R[x; \sigma]$ (resp. $S = R[x; \delta]$.).

So an element of S is in the form $\sum_{i=0}^{n} r_i x^i$ and elements of S are added and multiplied just like ordinary polynomials with this exception that if $r \in R$, then we have the rule $xr = \sigma(r)x + \delta(r)$. To show that the ring S exists, let t be an independent variable over R and let E be the endomorphism ring of the polynomial ring $R[t]$. Clearly E contains a copy of R. Define $x \in E$ by

$$x(rt^n) = \sigma(r)t^{n+1} + \delta(r)t^n$$

for all $r \in R$ and $n \geq 0$. It follows that $xr = \sigma(r)x + \delta(r)$ and that $\sum_{n=0}^{\infty} Rx^n$ is a ring. Hence $S = \sum_{n=0}^{\infty} Rx^n$. Note that if R is a domain and σ is injective, then S is a domain too.

In this section, we will only deal with skew polynomial rings in the form $R[x;\delta]$, i.e. we assume that σ is the identity map. The ring $R[x;\delta]$ is called a *differential polynomial ring.*

We note that if δ is a derivation of R, then the set $\{r \in R : \ \delta(r) = 0\}$ is a subring of R which contains the identity element of R. Also, if R is a field, then $\{r \in R : \ \delta(r) = 0\}$ is a subfield of R. To see this, let $0 \neq r \in R$, then $0 = \delta(1) = \delta(rr^{-1}) = \delta(r)r^{-1} + r\delta(r^{-1}) = r\delta(r^{-1})$. Thus $\delta(r^{-1}) = 0$.

Remark 2.2.3. Let δ be a derivation of a commutative domain R and let $Q(R)$ be the field of fractions of R. Let $p = a/b \in Q(R)$ where $a, b \in R$ and $b \neq 0$. We define $\tilde{\delta} : Q(R) \to Q(R)$ by

$$\tilde{\delta}(p) = \frac{\delta(a)b - a\delta(b)}{b^2}.$$

It is straightforward to see that $\tilde{\delta}$ is a derivation of $Q(R)$. For the sake of simplicity, we write δ for $\tilde{\delta}$.

Assumption. For the rest of this section, we assume that R is a commutative domain, δ is a derivation of R and $k := \{p \in Q(R) : \ \delta(p) = 0\}$ is a subfield of characteristic zero of R. If $f = r_n y^n + r_{n-1}y^{n-1} + \cdots + r_0 \in R[y;\delta], \ r_n \neq 0$, then we call n the *degree* of f and we write $\deg f = n$.

Remark 2.2.4. An easy induction on n shows that in $R[y;\delta]$ we have

$$y^n r = \sum_{i=0}^{n} \binom{n}{i} \delta^i(r) y^{n-i} \tag{2.1}$$

for all $r \in R$ and integer $n \geq 1$. In particular, since R is a domain, $\deg fg = \deg f + \deg g$ for all $0 \neq f, g \in R[y;\delta]$ and so $R[y;\delta]$ is a domain too.

Remark 2.2.5. Let $S := R[y;\delta]$. If $r \in R \setminus k$, then $C(r;S) = R$. This is easy to see: clearly $R \subseteq C(r;S)$ because R is commutative. Conversely, if $g = r_n y^n + \cdots + r_0 \in S$, $r_n \neq 0$, commutes with r and $n \geq 1$, then comparing the coefficients y^{n-1} in both sides of $rg = gr$ gives $nr_n\delta(r) = 0$. This is a contradiction because R is a domain and the characteristic of k is zero. Thus $n = 0$ and hence $g \in R$.

Lemma 2.2.6. *Let $S := R[y;\delta]$ and suppose that $f = a_n y^n + \cdots + a_0 \in S$, $n \geq 1$, $a_n \neq 0$. Let $g = b_m y^m + \cdots + b_0$, $b_m \neq 0$, and $h = c_m y^m + \cdots + c_0$, $c_m \neq 0$, be two elements of $C(f;S)$. Then $b_m = \alpha c_m$, for some $\alpha \in k$.*

Proof. By (2.1) $y^\ell r = ry^\ell + \ell\delta(r)y^{\ell-1} + \cdots$, for any integer $\ell \geq 1$ and $r \in R$. Therefore the coefficient of y^{n+m-1} in fg and gf are $na_n\delta(b_m) + a_n b_{m-1} + a_{n-1}b_m$ and $mb_m\delta(a_n) + b_m a_{n-1} + b_{m-1}a_n$, respectively. Thus, since $fg = gf$, we must have

$$na_n\delta(b_m) + a_n b_{m-1} + a_{n-1}b_m = mb_m\delta(a_n) + b_m a_{n-1} + b_{m-1}a_n.$$

Hence, since R is commutative, we have

$$na_n\delta(b_m) = m\delta(a_n)b_m. \tag{2.2}$$

A similar argument shows that $fh = hf$ implies

$$na_n\delta(c_m) = m\delta(a_n)c_m. \tag{2.3}$$

Now, multiplying both sides of (2.2) by c_m and both sides of (2.3) by b_m and then subtracting the resulting identities gives $na_n(c_m\delta(b_m) - b_m\delta(c_m)) = 0$. Thus

$$c_m\delta(b_m) - b_m\delta(c_m) = 0, \tag{2.4}$$

because R is a domain, $a_n \neq 0$ and the characteristic of k is zero. So, in $Q(R)$, we have $\delta(b_m/c_m) = 0$, by (2.4), and hence $b_m/c_m \in k$. □

Theorem 2.2.7. *Let $S := R[y;\delta]$ and let $f \in S$ with $\deg f = n \geq 1$. Then $C := C(f;S)$ is a free $k[f]$-module of rank d, where d is a divisor of $\deg f$.*

Proof. Suppose that N is the set of all integers $m \geq 0$ for which there exists some $g \in C$ such that $\deg g = m$. Clearly N is a submonoid of $\mathbb{Z}$: $0 \in N$ because $k \subset C$ and N is closed under addition because C is a subring of S. For any $m \in N$ let $\overline{m}$ be the image of m in

$\mathbb{Z}/n\mathbb{Z}$ and put $\overline{N} = \{\overline{m} : m \in N\}$. Since $\overline{N}$ is a submonoid of a finite cyclic group, it is a (cyclic) subgroup and hence $d = |\overline{N}|$ divides $|\mathbb{Z}/n\mathbb{Z}| = n$. Let $m_1 = 0$ and, in general, for every $1 \leq i \leq d$, choose $m_i \in N$ to be the smallest member of its class $\overline{m_i}$. That means if $m \equiv m_i \mod n$ and $m \in N$, then $m \geq m_i$. For any $1 \leq i \leq d$, choose $g_i \in C$ with $\deg g_i = m_i$. So g_1 can be any nonzero constant (degree zero) in C. We choose $g_1 = 1$. To complete the proof of the theorem, we are going to show that, as a $k[f]$-module, $g_1, \ldots, g_d$ generate C and $g_1, \ldots, g_d$ are linearly independent over $k[f]$.

We first show that $C = \sum_{i=1}^{d} g_i k[f]$. Clearly $\sum_{i=1}^{d} g_i k[f] \subseteq C$ because $f, g_i \in C$ for all $1 \leq i \leq d$. Now let $g \in C$ and suppose that $\deg g = \ell$. If $\ell = 0$, then $\deg g = \deg g_1$ and hence, by Lemma 2.2.6, $g \in k \subset g_1 k[f] \subseteq \sum_{i=1}^{d} g_i k[f]$. If $\ell \geq 1$, then $\overline{\ell} = \overline{m_j}$, for some j. We also have $\ell \geq m_j$ by the minimality of m_j. Thus

$$\ell = m_j + nu$$

for some integer $u \geq 0$. Therefore $\deg g = \ell = m_j + nu = \deg g_j f^u$. Both g and $g_j f^u$ are obviously in C. So if b and c are the leading coefficients of g and $g_j f^u$, respectively, then by Lemma 2.2.6, $b = \alpha c$ for some $\alpha \in k$. Therefore $\deg(g - \alpha g_i f^u) \leq \ell - 1$ and, since $g - \alpha g_i f^u \in C$, we can apply induction on $\deg g$ to get $g - \alpha g_j f^u \in \sum_{i=1}^{d} g_i k[f]$. Thus $g \in \sum_{i=1}^{d} g_i k[f]$.

It remains to show that $g_1, \ldots, g_d$ are linearly independent over $k[f]$. Suppose, to the contrary, that

$$g_1\mu_1(f) + \cdots + g_d\mu_d(f) = 0, \tag{2.5}$$

for some $\mu_i(f) \in k[f]$ and not all $\mu_i(f)$ are zero. Note that if $i \neq j$ and $\mu_i(f), \mu_j(f) \neq 0$, then $\deg(g_i\mu_i(f)) \equiv m_i \mod n$ and $\deg(g_j\mu_j(f)) \equiv m_j \mod n$. Since $i \neq j$, we have $m_i \not\equiv m_j \mod n$ and hence $\deg(g_i\mu_i(f)) \neq \deg(g_j\mu_j(f))$. Thus the left hand side of (2.5) is a polynomial of degree $\max\{\deg(g_i\mu_i(f)) : g_i\mu_i(f) \neq 0\}$ and so it cannot be equal to zero. This completes the proof of the theorem. $\square$

Now, we are going to prove that $C(f; R[y;\delta])$ is commutative if $f \in R[y;\delta]$ and $\deg f \geq 1$.

Lemma 2.2.8. *Let $S := R[y;\delta]$ and let $f \in S$ with $\deg f \geq 1$. If $m \geq 0$ is an integer, then the set V_m consisting of all elements of $C(f;S)$ of degree at most m is a finite-dimensional k-vector space.*

Proof. It is clear that V_m is a k-vector space. The proof of finite dimensionality of V_m is by induction on m. If $m = 0$, then $V_m = k$ and there is nothing to prove. So suppose that $m \geq 1$

and fix an element $g \in V_m$ with $\deg g = m$. If there is no such g, then $V_m = V_{m-1}$ and we are done by induction. Now, let $h \in V_m$. If $\deg h < m$, then $h \in V_{m-1}$ and if $\deg h = m$, then there exists some $\alpha \in k$ such that $h - \alpha g \in V_{m-1}$, by Lemma 2.2.6. Thus $V_m = kg + V_{m-1}$ and hence $\dim_k V_m = \deg_k V_{m-1} + 1$ and we are done again by induction. $\square$

Theorem 2.2.9. *Let $S := R[y;\delta]$ and let $f \in S$ with $\deg f = n \geq 1$. Then $C := C(f;S)$ is commutative.*

Proof. Let N and $\overline{N}$ be the sets defined in the proof of Theorem 2.2.7. As we mentioned in there, $\overline{N}$ is a cyclic subgroup of $\mathbb{Z}/n\mathbb{Z}$ of order d, for some divisor d of n. Let $\overline{m}$, $m > 0$, be a generator of $\overline{N}$ and choose $g \in C$ such that $\deg g = m$. Now let

$$A = k[f] + gk[f] + \cdots + g^{d-1}k[f].$$

Clearly $A \subseteq C$. Let

$$M = \{mi + nj : \ 0 \leq i \leq d-1, \ j \geq 0, \ i, j \in \mathbb{Z}\}.$$

So basically M is the set of all nonnegative integers which appear as the degree of some element of A. Let $p \in N$. Then $p \equiv mi \mod n$, for some integer $0 \leq i \leq d-1$ because $\overline{m}$ is a generator of $\overline{N}$. Hence $p = mi + nj$, for some integer j. If $j \geq 0$, then $p \in M$ and if $j < 0$, then $0 \leq p \leq mi \leq m(d-1)$. Thus if $h \in C$ and $\deg h > m(d-1)$, then $\deg h \in M$. Let V be the set of all elements of C of degree at most $m(d-1)$. By Lemma 2.2.8, V is k-vector space and

$$\dim_k V = v < \infty.$$

The claim is that

$$C = A + V. \tag{2.6}$$

Clearly $A + V \subseteq C$ because both A and V are in C. To prove that $C \subseteq A + V$, let $h \in C$. We use induction on $\deg h$. If $\deg h = 0$, then $h \in k$, by Lemma 2.2.6. If $\deg h \leq m(d-1)$, then $h \in V$ and we are done. Otherwise, $\deg h \in M$ and hence there exists some $h_1 \in A$ such that $\deg h = \deg h_1$. Thus, by Lemma 2.2.6, there exists some $\alpha \in k$ such that $\deg(h - \alpha h_1) < \deg h$. Therefore by induction $h - \alpha h_1 \in A + V$ and hence $h \in A + V$ because $\alpha h_1 \in A$. This completes the proof of (2.6).

Now let $h \in C$ and let $0 \leq i \leq v = \dim_k V$. Clearly $f^i h \in C$ and so

$$f^i h - h_i \in A, \tag{2.7}$$

for some $h_i \in V$. Since $\dim_k V = v$, the elements $h_0, \ldots, h_v$ are k-linearly dependent and so $\sum_{i=0}^{v} \alpha_i h_i = 0$ for some $\alpha_i \in k$ which are not all zero. It now follows from (2.7) that $\mu(f)h \in A$, where $0 \neq \mu(f) = \sum_{i=0}^{v} \alpha_i f^i \in k[f]$. So we have proved that for every $h \in C$ there exists some $0 \neq \mu(f) \in k[f]$ such that $\mu(f)h \in B$. Let $h_1, h_2 \in C$ and let $0 \neq \mu_1(f), \mu_2(f) \in k[f]$ be such that $\mu_1(f)h_1 \in A$ and $\mu_2(f)h_2 \in A$. Then, since A is clearly commutative, we have $\mu_1(f)h_1\mu_2(f)h_2 = \mu_2(f)h_2\mu_1(f)h_1$. Therefore, since $k[f]$ is commutative and h_1 and h_2 commute with f, we have

$$\mu_1(f)\mu_2(f)h_1h_2 = \mu_1(f)\mu_2(f)h_2h_1.$$

Thus, since S is a domain and $\mu_1(f), \mu_2(f) \neq 0$, we have $h_1h_2 = h_2h_1$. Hence C is commutative. $\square$

So we have proved that if k is a field of characteristic zero and $f \in R[y;\delta]$ with $\deg f \geq 1$, then $C(f; R[y;\delta])$ is a commutative domain and a free module of finite rank over $k[f]$. What can we say about the field of fractions Q of $C(f; R[y;\delta])$? The next theorem shows that Q has a very simple form.

Theorem 2.2.10. *Let $S := R[y;\delta]$ and let $f \in S$ with $\deg f \geq 1$. Let Q and $k(f)$ be the field of fractions of $C := C(f;S)$ and $k[f]$ respectively. Then Q is an algebraic extension of $k(f)$ and $Q = k(f)[g]$, for some $g \in C$.*

Proof. Let g, d and A be as defined in the proof of Theorem 2.2.9. We proved that for every $h \in C$ there exists some $0 \neq \mu(f) \in k[f]$ such that

$$\mu(f)h \in A = k[f] + gk[f] + \cdots + g^{d-1}k[f]. \tag{2.8}$$

If in (2.8) we choose $h = g^d$, then $g^d \in k(f) + gk(f) + \cdots + g^{d-1}k(f)$. So g is algebraic over $k(f)$ and thus $k(f)[g]$ is a subfield of Q. Also (2.8) shows that $h \in k(f)[g]$ for all $h \in C$ and thus $C \subseteq k(f)[g]$. Therefore $C \subseteq k(f)[g] \subseteq Q$ and hence $Q = k(f)[g]$. $\square$

Now let $R = k[x]$, the polynomial ring in the variable x. Clearly $\delta = d/dx$ is a derivation of R and $\{p \in Q(R) : \ \delta(p) = 0\} = \{p \in k(x) : \ dp/dx = 0\} = k$. Let $S := R[y;\delta]$. In S we have $yx = xy + \delta(x) = xy + 1$. So the generators of S and $A_1(k)$ satisfy the same relation. Thus there exists an onto k-algebra homomorphism $\varphi : A_1(k) \longrightarrow S$. Since $A_1(k)$ is simple, because $\text{char}(k) = 0$, we have $\ker\varphi = (0)$ and thus $S \cong A_1(k)$.

Corollary 2.2.11. *Let $f \in A_1(k)\backslash k$. Then $C := C(f; A_1(k))$ is commutative and a free $k[f]$-module of rank d, where d is a divisor of $\deg f$. Also, if Q and $k(f)$ are the field of fractions of C and $k[f]$, respectively, then Q is an algebraic extension of $k(f)$ and $Q = k(f)[g]$, for some $g \in C$.*

Proof. Write $f = \sum_{i=0}^{n} r_i y^i,\ r_i \in k[x],\ r_n \neq 0$. If $\deg f \geq 1$, then the result follows from Theorems 2.2.7, 2.2.9 and 2.2.10. If $n = 0$, then $f \in k[x]$ and thus $C = k[x]$, as we proved in Remark 2.2.5. Now, Theorems 2.2.7, 2.2.9 and 2.2.10 with $R = k$ and $\delta = 0$ complete the proof of the corollary in this case because $k[y] \cong k[x]$. $\square$

We close this section by giving another form of $A_1(k)$, the first Weyl algebra over k. Let $E := \mathrm{End}_k(k[x])$. Define $D, L \in E$ by $D(u) = du/dx$ and $L(u) = xu$ for all $u \in k[x]$. Now,

$$DL(u) = D(xu) = D(x)u + xD(u) = u + LD(u)$$

and hence $DL - LD = 1$, where 1 is the identity element of E. Let A be the k-subalgebra of E generated by D and L. An element of A is called a *differential operator*. Since the generators of A and $A_1(k)$ satisfy the same relation, there exists a k-algebra homomorphism from $A_1(k)$ onto A and that is in fact an isomorphism because $A_1(k)$ is simple. Note that if $\mathrm{char}(k) = p > 0$, then A and $A_1(k)$ are not isomorphic. The reason is that in this case $D^p = 0$ and so A is not a domain but $A_1(k)$ is a domain. To prove $D^p = 0$, we only need to show that $D^p(x^m) = 0$ for all integers $m \geq 0$, because D is k-linear. Now, if $m < p$, then the p-th derivative of x^m is zero and if $m \geq p$, then $D^p(x^m) = m(m-1)\cdots(m-p+1)x^{m-p} = p!\binom{m}{p}x^{m-p} = 0$.

2.3 Centralizers in Free Associative Algebras

In this section, we look at two well-known results on centralizers in free associative algebras, i.e. Cohn's centralizer theorem and Bergman's centralizer theorem. A nice application of Bergman's centralizer theorem is given at the end of this section.

Throughout this section, X is a set of noncommuting variables, which may or may not be finite, and k is a field. Let X^* denote the free monoid generated by X. An element of X (resp. X^*) is also called a letter (resp. word) and X is called an alphabet. Let $k\langle\langle X\rangle\rangle$ and $k\langle X\rangle$ denote the k-algebra of formal series and polynomials in X, respectively. So an element of $k\langle\langle X\rangle\rangle$ is in the form $a = \sum_{w\in X^*} a_w w$, where $a_w \in k$ is the coefficient of the word w in a. The *length* $|w|$ of $w \in X^*$ is the number of letters appearing in w. For example, if $X = \{x_i\}$ and $w = x_1 x_2^2 x_1 x_3$, then $|w| = 5$. Now, we define the valuation

$$\nu : k\langle\langle X\rangle\rangle \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

as follows: $\nu(0) = \infty$ and if $a = \sum_{w \in X^*} a_w w \neq 0$, then $\nu(a) = \min\{|w| : \ a_w \neq 0\}$. Note that if w is constant, then $\nu(w) = 0$ and $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in k\langle\langle X\rangle\rangle$. The following fact is easy to prove.

Lemma 2.3.1. (Levi's Lemma) *Let $w_1, w_2, w_3, w_4 \in X^*$ be nonzero with $|w_2| \geq |w_4|$. If $w_1 w_2 = w_3 w_4$, then $w_2 = w w_4$ for some $w \in X^*$.*

Proof. The proof is by induction on $|w_2|$. Let x be the last letter of w_2 and w_4. Then $w_2 = w_2' x$ and $w_4 = w_4' x$ and thus $w_1 w_2' = w_3 w_4'$. The result now follows by induction. □

Next lemma extends Levi's lemma to $k\langle\langle X\rangle\rangle$.

Lemma 2.3.2 ([46], Lemma 9.1.2)**.** *Let $a, b, c, d \in k\langle\langle X\rangle\rangle$ be nonzero. If $\nu(a) \geq \nu(c)$ and $ab = cd$, then $a = cq$ for some $q \in k\langle\langle X\rangle\rangle$.*

Proof. Fix a word u that appears in b and $|u| = \nu(b)$. So if v is any (nonzero) word appearing in d, then

$$|v| \geq \nu(d) = \nu(b) + \nu(a) - \nu(c) \geq \nu(b) = |u|. \tag{2.9}$$

Let w be any element of X^*. The coefficient of wu in ab is $\sum_{rs=wu} a_r b_s$, where a_r and b_s are the coefficients of the words r, s which appear in a, b respectively. Similarly, the coefficient of wu in cd is $\sum_{yz=wu} c_y d_z$. Since $ab = cd$, we have

$$\sum_{rs=wu} a_r b_s = \sum_{yz=wu} c_y d_z, \tag{2.10}$$

where the sums are over r, s and y, z. So $|z| \geq |u|$, by (2.9), and $|s| \geq |u|$ by the definition of u. Thus $rs = wu$ and $yz = wu$ imply $s = s_1 u$ and $z = z_1 u$ for some $s_1, z_1 \in X^*$, by Levi's lemma. Hence $rs_1 = yz_1 = w$. Therefore (2.10) can be written as

$$\sum_{rs_1=w} a_r b_{s_1 u} = \sum_{yz_1=w} c_y d_{z_1 u}, \tag{2.11}$$

where the sums are over r, s_1 and y, z_1. Let $b' = \sum_{s_1 \in X^*} b_{s_1 u} s_1$ and $d' = \sum_{z_1 \in X^*} d_{z_1 u} z_1$. Then (2.11) gives $ab' = cd'$. The constant term of b' is $b_u \neq 0$ and hence b' is invertible in $k\langle\langle X\rangle\rangle$. Hence if we let $q = d' b'^{-1}$, then $a = cq$. □

An interesting consequence of Lemma 2.3.2 is the following result that will be used at the end of this section.

Corollary 2.3.3. *Let $a \in k\langle\langle X\rangle\rangle$. Then $b \in C(a; k\langle\langle X\rangle\rangle)$ if and only if a, b are not free, i.e. $f(a,b) = 0$ for some nonzero series $f \in k\langle\langle x, y\rangle\rangle$.*

Proof. If $ab = ba$, then $f(a,b) = 0$ for $f = xy - yx$. Conversely, suppose that there exists a nonzero series $f \in k\langle\langle x, y\rangle\rangle$ such that $f(a,b) = 0$. Let $n = \nu(ab - ba)$. The proof is by induction on n. First note that the constant term of $ab - ba$ is zero and thus $n = 0$ if and only if $ab = ba$. Clearly we may assume that the constant terms of a and b are zero because if $a = \alpha + a_1$ and $b = \beta + b_1$, where α, β are the constant terms of a, b respectively, then $ab - ba = a_1b_1 - b_1a_1$. So we may assume that $\nu(a) \geq \nu(b) \geq 1$. We have $f = xg + yh + \gamma$, for some $g, h \in k\langle\langle x, y\rangle\rangle$ and $\gamma \in k$. Now, since $0 = f(a,b) = ag(a,b) + bh(a,b) + \gamma$ and the constant terms of a, b are zero, we must have $\gamma = 0$. Thus $ag(a,b) = -bh(a,b) = 0$. Lemma 2.3.2 now gives some $q \in k\langle\langle x, y\rangle\rangle$ such that $a = bq$. So $ab - ba = b(qb - bq)$ and thus $\nu(qb - bq) < \nu(ab - ba) = n$. We also have that b, q are not free because $a = bq$ and b are not free. Thus, by induction, $bq = qb$ and therefore $ab = ba$. $\square$

Lemma 2.3.4. *Suppose that the constant term of an element $a \in k\langle\langle X\rangle\rangle$ is zero and $b, c \in C(a; k\langle\langle X\rangle\rangle) \setminus \{0\}$. If $\nu(c) \geq \nu(b)$, then $c = bd$ for some $d \in C(a; k\langle\langle X\rangle\rangle)$.*

Proof. Since the constant term of a is zero, we have $\nu(a) \geq 1$. Thus, for n large enough, we have $\nu(a^n) = n\nu(a) \geq \nu(c)$. We also have $a^n c = ca^n$ because $c \in C(a; k\langle\langle X\rangle\rangle)$. Thus, by Lemma 2.3.2, $a^n = cq$ for some $q \in k\langle\langle X\rangle\rangle)$. Hence $cqb = a^n b = ba^n$ and since $\nu(c) \geq \nu(b)$, we have $c = bd$, for some $d \in k\langle\langle X\rangle\rangle)$, by Lemma 2.3.2. Finally,

$$bad = abd = ac = ca = bda,$$

which gives $ad = da$, i.e. $d \in C(a; k\langle\langle X\rangle\rangle)$. $\square$

Theorem 2.3.5. (Cohn's centralizer theorem, [46, Theorem 9.1.1]) *If $a \in k\langle\langle X\rangle\rangle$ is not constant, then*

$$C(a; k\langle\langle X\rangle\rangle) \cong k[[x]],$$

where $k[[x]]$ is the algebra of formal power series in the variable x.

Proof. Let $C := C(a; k\langle\langle X\rangle\rangle)$. If $\alpha_0 \in k$ is the constant term of a, then clearly $C = C(a - \alpha_0; k\langle\langle X\rangle\rangle)$ and so we may assume that the constant term of a is zero. Thus the set

$$A = \{c \in C : \ \nu(c) > 0\}$$

is not empty because $a \in C$ and so there exists $b \in A$ such that $\nu(b)$ is minimal. To show that $k[[b]] \cong k[[x]]$, suppose that $\sum_{i \geq m} \beta_i b^i = 0, \ \beta_i \in k, \beta_m \neq 0$. Then we must have $\infty = \nu(\sum_{i \geq m} \beta_i b^i) = \nu(b^m) = m\nu(b)$, which is absurd. So, to complete the proof of the theorem, we now need to prove that $C = k[[b]]$. Let $c \in C$. If c is constant, then obviously $c \in k[[b]]$. So we assume that c is not constant. The claim is that there exist $\beta_i \in k$ such that

$$\nu(c - \sum_{i=0}^{n} \beta_i b^i) \geq (n+1)\nu(b). \tag{2.12}$$

If we prove that, then we are done because then $\nu(c - \sum_{i \geq 0} \beta_i b^i) = \infty$ and so $c = \sum_{i \geq 0} \beta_i b^i \in k[[b]]$. The proof is by induction on n. Let β_0 be the constant term of c. Then $c - \beta \in A$ and thus $\nu(c - \beta_0) \geq \nu(b)$, by the minimality of b. This proves (2.12) for $n = 0$. Suppose now that we have found $\beta_0, \ldots, \beta_n \in k$ such that $\nu(c - \sum_{i=0}^{n} \beta_i b^i) \geq (n+1)\nu(b)$. Then, since $(n+1)\nu(b) = \nu(b^{n+1})$, we have

$$c - \sum_{i=0}^{n} \beta_i b^i = b^{n+1} d$$

for some $d \in C$, by Lemma 2.3.4. If d is constant, we are done because then $c \in k[b] \subset k[[b]]$. Otherwise, let β_{n+1} be the constant term of d. Then $d - \beta_{n+1} \in A$ and hence $\nu(d - \beta_{n+1}) \geq \nu(b)$, by the minimality of b. Therefore, by Lemma 2.3.4, $d - \beta_{n+1} = bd'$ for some $d' \in C$. Hence

$$c - \sum_{i=0}^{n} \beta_i b^i = b^{n+1} d = b^{n+1}(bd' + \beta_{n+1}) = b^{n+2} d' + \beta_{n+1} b^{n+1},$$

which gives $c - \sum_{i=0}^{n+1} \beta_i b^i = b^{n+2} d'$. Hence

$$\nu\left(c - \sum_{i=0}^{n+1} \beta_i b^i\right) = \nu(b^{n+2} d') = \nu(b^{n+2}) + \nu(d') = (n+2)\nu(b) + \nu(d') \geq (n+2)\nu(b),$$

which completes the induction and the proof of the theorem. $\square$

Now, since $k\langle X\rangle \subset k\langle\langle X\rangle\rangle$, it follows from the above theorem that if $a \in k\langle X\rangle$ is not constant, then $C(a; k\langle X\rangle)$ is commutative because $C(a; k\langle\langle X\rangle\rangle)$ is commutative. The next theorem shows that a result similar to Theorem 2.3.5 holds for $C(a; k\langle X\rangle)$.

Theorem 2.3.6. (Bergman's Centralizer Theorem, [18] or [46, Theorem 9.5.1]) *If $a \in k\langle X\rangle$ is not constant, then $C(a; k\langle X\rangle) \cong k[x]$, the polynomial algebra in one variable x.*

We close this section by an application of Theorem 2.3.6. We first recall the structure of the free product $R * k[t]$, where R is a k-algebra and $k[t]$ is the polynomial algebra in the variable t. If $\mathcal{B}$ is a k-basis for R, then the set $\{b_1tb_2t\cdots tb_n : \ n \geq 1, b_i \in \mathcal{B}\}$ is a k-basis for $R * k[t]$. The multiplication in $R * k[t]$ is done just like multiplication in the free associative algebra $k\langle X\rangle$. There is one point here. If $b, b' \in \mathcal{B}$, then $bb' \in R$ and so we can write $bb' = \sum \beta_i b_i, \ b_i \in \mathcal{B}, \beta_i \in k$. So in the product of two elements of $R * k[t]$, we must replace bb' with $\sum_i \beta_i b_i$.

Example 2.3.7. If x, y are two variables, then $k[x] * k[y] \cong k\langle x, y\rangle$.

Proposition 2.3.8. (Drensky, [23]) *Let R be a k-algebra. If $R*k[t] \cong k\langle x, y\rangle$, as k-algebras, then $R \cong k[z]$, as k-algebras. Here we are assuming that x, y, z, t are variables.*

Proof. Since $R * k[t] \cong k\langle x, y\rangle$, as k-algebras, $R * k[t]$ is generated by two elements as a k-algebra. Let $\langle t\rangle$ be the two-sided ideal of $R * k[t]$ generated by t. Then $(R * k[t])/\langle t\rangle \cong R$ and hence R is also generated by two elements, say u, v, as a k-algebra. If u, v are free, then $R \cong k\langle u, v\rangle$ and thus $k\langle x, y\rangle \cong R * k[t] \cong k\langle u, v, t\rangle$, which is absurd. Thus u, v are not free and hence $uv = vu$, by Corollary 2.3.3, and R is commutative because, as a k-algebra, R is generated by u, v. On the other hand, since $R \subset R * k[t] \cong k\langle x, y\rangle$, we have $R \subseteq C(u; R * k[t]) = k[w]$, for some $w \in R * k[t]$, by Theorem 2.3.6. From the structure of $R * k[t]$, we know that $w = w_0 + w_1$, where $w_0 \in R$ and t appears in each term of w_1. Let $f \in R$. Then $f = g(w) \in k[w]$ and thus $f = g(w_0 + w_1)$. Since f is independent of t, we may let $t = 0$ to get $f = g(w_0)$. So we have proved that $R \subseteq k[w_0]$, which gives $R = k[w_0]$ because $w_0 \in R$. $\square$

2.4 Centralizers in Domains of Finite GK Dimension

In this section, we give Bell and Small's results on centralizers in domains of finite GK dimension. Let k be a field and let A be a finitely generated k-algebra which is also a domain. Let $a \in A$. If $\mathrm{GKdim}(A) = 0$, then A is finite-dimensional over k, by Proposition 1.6.10, and hence PI, by Proposition 1.4.7. So $C(a; A)$ is PI too. If $\mathrm{GKdim}(A) = 1$, then A is PI, by [59, Theorem 1.6]. Thus $C(a; A)$ is PI too. Also, in this case, if k is algebraically closed, then A is commutative, by Proposition 1.6.32, and so $C(a; A)$ is commutative. There are no algebras of GK dimension between 1 and 2 by Bergman's gap theorem (Theorem 1.6.15).

In this section, we consider the cases $\operatorname{GKdim}(A) = 2$ and $\operatorname{GKdim}(A) = 3$. We first prove that if $\operatorname{GKdim}(A) = 2$ and k is algebraically closed, then $C(a; A)$ is PI. Also, in this case, if A is not PI, then $C(a; A)$ is commutative. We prove that if k is uncountable and algebraically closed, A is noetherian with $\operatorname{GKdim}(A) = 3$ and a is not algebraic over $Z(Q(A))$, the center of the quotient division algebra of A, then $C(a; A)$ is PI again and $\operatorname{GKdim}(C(a; A)) \leq 2$.

Now suppose that A is a finitely generated simple noetherian domain of GK dimension larger than 3. Suppose also that $a \in A$ is not algebraic over $Z(Q(A))$. Would $C(a; A)$ be PI? The answer is no and here is an example.

Example 2.4.1. [See Definition 1.6.17] Consider the n-th Weyl algebra $A_n(k), n \geq 2$, where k is a field of characteristic zero. It is easy to see that $C(x_1; A_n(k))$ is the subalgebra generated by $x_1, \ldots, x_n, y_2, \ldots, y_n$. So $C(a; A_n(k))$ contains the subalgebra generated by x_2, y_2 which is isomorphic to $A_1(k)$. Thus if $C(a; A_n(k))$ is PI, then $A_1(k)$ must be PI too. But $A_1(k)$ is simple and hence, by Theorem 1.4.25, $A_1(k)$ has to be finite-dimensional over k, which is false.

Theorem 2.4.2. (Bell and Small, [14]) *Let k be an algebraically closed field. Let A be a finitely generated k-algebra which is a domain and* $\operatorname{GKdim}(A) = 2$. *If A is not PI and $a \notin Z(A)$, then $C(a; A)$ is a commutative domain and* $\operatorname{GKdim}(C(a; A)) = 1$.

Proof. First note that, since k is algebraically closed and a is noncentral, a is transcendental over k and hence $\operatorname{GKdim}(k[a]) = 1$. Now let B be any finitely generated k-subalgebra of $C(a; A)$ which contains a. Then $k[a] \subseteq B$ and thus

$$\operatorname{GKdim}(B) \geq 1. \tag{2.13}$$

In fact, $k[a] \subseteq Z(B)$ and hence

$$\operatorname{GKdim}(Z(B)) \geq 1. \tag{2.14}$$

If B is not PI, then by (2.14) and Theorem 1.6.34, $\operatorname{GKdim}(B) \geq 3$, which is absurd because B is a subalgebra of A and hence $\operatorname{GKdim}(B) \leq 2$. Thus B is PI. Now, let $Q(A)$ and $Q(B)$ be the quotient division algebras of A and B respectively. Suppose that

$$\operatorname{GKdim}(B) > 1 = \operatorname{GKdim}(A) - 1.$$

Then, by Theorem 1.6.35, $Q(A)$ is a finite-dimensional vector space over $Q(B)$. It follows that $Q(A)$ is PI because $Q(B)$ is PI. But then A would be PI too, which is a contradiction. Thus $\text{GKdim}(B) \leq 1$ and so, by (2.13), $\text{GKdim}(B) = 1$. So we have proved that every finitely generated k-subalgebra of $C(a; A)$ which contains a has GK dimension one. Hence $\text{GKdim}(C(a; A)) = 1$ and every finitely generated k-subalgebra of $C(a; A)$ is commutative, by Proposition 1.6.32. Thus $C(a; A)$ is a commutative domain. $\square$

Now let A be a finitely generated k-algebra of finite GK dimension. How large could $\text{GKdim}(C(a; A))$ be if $a \notin Z(A)$? Bell [13] has proved that if A is a domain and a is not algebraic over $Z(Q(A))$, the center of the quotient division algebra of A, then $\text{GKdim}(C(a; A))$ is $\leq \text{GKdim}(A) - 1$. He then uses this result to show that if k is uncountable and algebraically closed, A is a finitely generated noetherian domain over k and $\text{GKdim}(A) = 3$, then $C(a; A)$ is a PI-domain of GK dimension at most 2 for every $a \in A$ which is not algebraic over $Z(Q(A))$. We need a few lemmas before proving these two results.

Lemma 2.4.3. *Let R and S be noetherian rings and suppose that, as a left and right R-module, S is finitely generated and free. If S is a simple ring, then R is simple too.*

Proof. Suppose that R is not simple. Then R has a proper nonzero two-sided ideal I. Let $J := IS$. Then J is a right ideal of S and, since S is a free left R-module, $J \cap R = I$. Let $M := S/J$. Clearly M is an (R, S)-bimodule and, since S is a finitely generated left R-module, M is a finitely generated left R-module too. Since M is also a torsion right S-module, M has a nonzero annihilator [43, Theorem 2.1], which is a two-sided ideal of S, a contradiction. $\square$

Lemma 2.4.4. *Let k be a field and let C be a commutative domain over k. Let $k(x)$ be the field of rational functions in one variable x. If C is not algebraic over k, then $R := C \otimes_k k(x)$ is not a field.*

Proof. Clearly R, as a k-algebra, is isomorphic to the k-algebra $T := \{p(t)/q(t) : \; p(t) \in C[t], 0 \neq q(t) \in k[t]\}$, the localization of $C[t]$ at $k[t]$. So we only need to prove that T is not a field. Let a be an element of C which is not algebraic over k. We claim that $a - t \in T$ is not invertible and hence T is not a field. So suppose, to the contrary, that $a - t$ is invertible. Then $(a - t)p(t) = q(t)$ for some $p(t) \in C[t], 0 \neq q(t) \in k[t]$. But then if we choose $t = a$, we get $q(a) = 0$ and so a is algebraic over k, contradiction. $\square$

Lemma 2.4.5. *Let A be a noetherian ring and let J be an ideal of A. If J is nil, then J is nilpotent.*

Proof. The proof still works if A is left or right noetherian and J is a one-sided ideal. Since A is noetherian, the set of nilpotent ideals of A has a maximal element, say M. Let $R := A/M$. Then R is a semiprime noetherian ring and $(I+M)/M$ is a nil ideal of R. Thus $(I+M)/M = 0$, by Lemma 1.3.8, and hence $I \subseteq M$ proving that I is nilpotent. □

Next result is especially useful for studying centralizers in algebras of low GK dimension.

Theorem 2.4.6. (Bell, [13]) *Let k be a field and let A be a finitely generated k-algebra of finite GK dimension. If A is a domain and $a \in A$ is not algebraic over $Z(Q(A))$, the center of the quotient division algebra of A, then* $\text{GKdim}(C(a;A)) \leq \text{GKdim}(A) - 1$.

Proof. Let $B := C(a;A)$. First note that $Q(A)$ and $Q(B)$ exist by Corollary 1.6.9. Now, we may replace A by $Z(Q(A))A$ if necessary and so we may assume that $k = Z(Q(A))$. Suppose, to the contrary, that $\text{GKdim}(B) > \text{GKdim}(A) - 1$. Then, by Theorem 1.6.35, $Q(A)$ is finite-dimensional as both a left and a right vector space over $Q(B)$. Let $R = Q(B) \otimes_k k(x)$ and $S = Q(A) \otimes_k k(x)$, where $k(x)$ is the field of rational functions in one variable x. Clearly S is free and finitely generated as a left R-module because $Q(A)$ is a finite-dimensional vector space over $Q(B)$. Also, S is simple because $Z(Q(A)) = k$. Finally, R is noetherian because R is isomorphic to the localization of $Q(B)[x]$ at $k[x]$. Similarly, S is noetherian. So by Lemma 2.4.3, R is simple too. In particular, $Z(R)$ is a field. But $Z(R) = Z(Q(B)) \otimes_k k(x)$ and $a \in Z(Q(B))$. Thus $Z(Q(B))$ is not algebraic over k because a is not algebraic over k. Hence $Z(R)$ is not a field by Lemma 2.4.4, contradiction. □

We mentioned in Example 2.4.1 that $C(x_1, A_n(k))$ is the subalgebra generated by

$$x_1, \ldots x_n, y_2, \ldots, y_n.$$

It is easy to see that this subalgebra has GK dimension $2n-1$. So it is possible to have equality in the above theorem.

Theorem 2.4.7. (Bell, [13]) *Let k be an uncountable algebraically closed field and let A be a finitely generated noetherian k-algebra of GK dimension three. If A is a domain and $a \in A$ is not algebraic over $Z(Q(A))$, then $C(a;A)$ is PI.*

Proof. Let $B := C(a; A)$. We have $\mathrm{GKdim}(B) \leq 2$, by Theorem 2.4.6. Since $k[a] \subseteq B$ and a is not algebraic over k, we have $\mathrm{GKdim}(B) \geq 1$ and thus $\mathrm{GKdim}(B) = 1$ or 2, by Theorem 1.6.15. If $\mathrm{GKdim}(B) = 1$, then B is PI, by [59, Theorem 1.6]. In fact, in this case B is commutative, by Proposition 1.6.32. So we may assume that $\mathrm{GKdim}(B) = 2$.

We now show that B is locally PI. Let B' be any finitely generated k-subalgebra of B. Then $B'[a] \subseteq B$ is also finitely generated and $k[a] \subseteq Z(B'[a])$. Thus $\mathrm{GKdim}(Z(B'[a])) \geq 1$. So if $B'[a]$ is not PI, then

$$\mathrm{GKdim}(B'[a]) \geq \mathrm{GKdim}(Z(B'[a])) + 2 \geq 3,$$

by Theorem 1.6.34, which is false because $\mathrm{GKdim}(B'[a]) \leq \mathrm{GKdim}(B) = 2$. Therefore $B'[a]$, and hence B', is PI. Now let B_0 be any finitely generated subalgebra of $Q(B)$. Suppose that B_0 is generated by $c_i = s_i^{-1}b_i,\ i = 1, \ldots, r$. Let B' be the subalgebra of B generated by $s_i, b_i,\ i = 1, \ldots, r$. Clearly

$$B_0 \subseteq Q(B'). \tag{2.15}$$

Since B is locally PI, B' is PI and hence, by Corollary 1.5.11, $Q(B')$ is PI too. Thus B_0 is PI, by (2.15), and so $Q(B)$ is locally PI. Also, (2.15) implies that $\mathrm{GKdim}(Q(B)) = 2$ and hence $\mathrm{GKdim}_{k(a)}(Q(B)) = 1$.

Let K be an algebraically closed field extension of $k(a)$ with

$$|K| > \dim_{k(a)} Q(B), \tag{2.16}$$

as cardinal numbers, and let $R := Q(B) \otimes_{k(a)} K$. Since A is noetherian, $A \otimes_k K$ is noetherian too by [16, Theorem 1.2]. Therefore $Q(A) \otimes_k K$ is noetherian because it is a localization of $A \otimes_k K$. Thus $Q(B) \otimes_k K$ is noetherian because $Q(A) \otimes_k K$ is free over $Q(B) \otimes_k K$. Hence R is noetherian because R is a factor of $Q(B) \otimes_k K$. Also, (2.15) implies that R is locally PI and $\mathrm{GKdim}_K(R) = \mathrm{GKdim}_{k(a)}(Q(B)) = 1$. Let $J := J(R)$, the Jacobson radical of R. Then, by (2.16) and Theorem 1.1.8, J is nil and thus nilpotent by Lemma 2.4.5. Therefore R/J is a semiprime noetherian ring and so, by Theorem 1.3.12, $Q(R/J)$ is semisimple. Thus, by Theorem 1.3.4,

$$Q(R/J) = \prod_{i=1}^{s} M_{n_i}(D_i),$$

for some division K-algebras D_i. Each D_i is locally PI and has GK dimension at most 1 as a K-algebras because R has this property. Since K is algebraically closed, each D_i is commutative by Proposition 1.6.32. Thus $Q(R/J)$, and hence R/J, is PI. Let f be a polynomial identity of R/J. Since J is nilpotent, $J^n = (0)$ for some n and thus f^n is an identity of R. So R is PI and hence B is PI too because B is a subalgebra of R. $\square$

2.5 Centralizers in Quantum Planes

This section gives Makar-Limanov's result on centralizers in quantum planes. Let k be an algebraically closed field and let q be a nonzero element of k. The *quantum plane* $k_q[x, y]$ is a k-algebra generated by x and y and with the relation $yx = qxy$. Clearly the set $\{x^i y^j : i, j \geq 0\}$ is a basis for $k_q[x, y]$ as a k-vector space. It is easily seen that $k_q[x, y]$ is a domain. If we choose $V = k + kx + ky$, then the relation $yx = qxy$ gives

$$\dim_k V^n = \frac{(n+2)(n+1)}{2}$$

and thus

$$\mathrm{GKdim}(k_q[x, y]) = 2.$$

If q is a root of unity, i.e. $q^n = 1$ for some n, then it follows from $yx = qxy$ that x^n and y^n are central. Thus, in this case, $k_q[x, y]$ is a finite module over the commutative algebra $k[x^n, y^n]$ and therefore it is PI, by Proposition 1.4.7.

Now suppose that q is not a root of unity. Then the center of $k_q[x, y]$ is just k. To see this, let $f = \sum a_{ij} x^i y^j$ be a central element. Then, since $y^j x = q^j x y^j$ and $xf = fx$, we have

$$\sum a_{ij} x^{i+1} y^j = \sum q^j a_{ij} x^{i+1} y^j.$$

Therefore $a_{ij}(q^j - 1) = 0$ for all i, j. Hence, since q is not a root of unity, we have $a_{ij} = 0$ for all i and $j \geq 1$. Similarly $yf = fy$ gives $a_{ij} = 0$ for all $i \geq 1$ and j. That means $a_{ij} = 0$ for all $(i, j) \neq (0, 0)$ and so $f \in k$. We now show that if q is not a root of unity, then $k_q[x, y]$ is not PI. To prove this, suppose, to the contrary, that $k_q[x, y]$ is PI. Let Q be the central localization of $k_q[x, y]$. Then, by Theorem 1.5.10, the center of Q is the quotient field of the center of $k_q[x, y]$, which is k, and Q is finite-dimensional over k. Therefore $k_q[x, y]$, which is a subalgebra of Q, is also finite-dimensional over k. This is of course absurd. Thus $k_q[x, y]$ is PI if and only if q is a root of unity.

So if q is not a root of unity, $k_q[x, y]$ satisfies all the conditions in Theorem 2.4.2 and thus $C(f; k_q[x, y])$ is a commutative domain of GK dimension one for every $f \in k_q[x, y] \setminus k$. Makar-Limanov [44] used a different approach to prove a stronger result. He proved that centralizers are in fact a subalgebra of a polynomial k-algebra in one variable. We are now going to give his proof of this result. We begin with a simple lemma.

Lemma 2.5.1. *If $q \in k$ is not a root of unity and $f(x,y) \in k_q[x,y] \setminus k$ such that $f(x,0) \notin k$, then there exists a k-algebra injective homomorphism from $C(f, k_q[x,y])$ into $k[x]$. Similarly, if $f(0,y) \notin k$, then there exists a k-algebra injective homomorphism from $C(f, k_q[x,y])$ into $k[y]$.*

Proof. We prove the lemma for the case $f(x,0) \notin k$. The proof for the case $f(0,y) \notin k$ is similar. We have $f(x,y) = \sum_{i=0}^{n} f_i(x)y^i$ and we are given that $f_0(x) = f(x,0) \notin k$. Let

$$0 \neq g(x,y) = \sum_{i=0}^{m} g_i(x)y^i \in C(f, k_q[x,y]).$$

We claim that $g(x,0) = g_0(x) \neq 0$. So suppose, to the contrary, that $g_0(x) = 0$. Then $g(x,y) = \sum_{i=r}^{m} g_i(x)y^i,\ r \geq 1,\ g_r(x) \neq 0$. Let $f_0(x) = \sum_{i=0}^{p} \alpha_i x^i,\ \alpha_i \in k$. Note that $y^s x^t = q^{st} x^t y^s$ for all integers $s, t \geq 0$. Thus equating the coefficients of y^r in both sides of $fg = gf$ gives

$$g_r(x) \sum_{i=0}^{p} \alpha_i x^i = g_r(x) \sum_{i=0}^{p} \alpha_i q^{ri} x^i.$$

Thus, since $k_q[x,y]$ is a domain and $g_r(x) \neq 0$, we have $\alpha_i(q^{ri} - 1) = 0$ for all $0 \leq i \leq p$. Therefore, since q is not a root of unity and $r \geq 1$, we have $\alpha_i = 0$ for all $i \geq 1$. Hence $f_0(x) = \alpha_0 \in k$, which is a contradiction. This completes the proof of the claim.

Now define the map $\varphi : C(f; k_q[x,y]) \longrightarrow k[x]$ by $\varphi(g(x,y)) = g(x,0)$. Obviously φ is a well-defined k-algebra homomorphism. If $g(x,0) = 0$, then, by the claim we just proved, $g(x,y) = 0$. Thus $\ker \varphi = 0$ and so φ is injective. □

Theorem 2.5.2. (Makar-Limanov, [44]) *If $q \in k$ is not a root of unity and $f \in k_q[x,y] \setminus k$, then $C(f; k_q[x,y]) \subseteq k[u]$, for some $u \in k_q[x,y]$.*

Proof. Let A be the set of all ordered pairs $(i,j) \in \mathbb{Z}^2$ such that $\alpha x^i y^j,\ \alpha \in k^{\times}$ is a term of f. By Lemma 2.5.1 we may assume that if $(i,j) \in A$, then $i \geq 1$ and $j \geq 1$. Now look at the elements of A as a finite set of points on the plane. Clearly there exists a line L which goes through both the origin and at least one of the points in A, say (p,q), such that each point in A lies either on L or on the left side of L. The equation of L is obviously $qx - py = 0$. Dividing by $\gcd(p,q)$, we can write the equation of L as $\lambda x - \mu y = 0$, where $\lambda, \mu \geq 1$ are some positive integers with $\gcd(\lambda, \mu) = 1$. The fact that each point in A lies either on L or on the left side of L is equivalent to saying that $\lambda i - \mu j \geq 0$ for all $(i, j) \in A$.

Now, define the *weight* of $\alpha x^i y^j \in k_q[x, y],\ \alpha \in k^\times$, by

$$w(\alpha x^i y^j) = \lambda i - \mu j.$$

So every monomial of f has a nonnegative weight and at least one monomial of f has weight zero (those that lie on L). Let $u = x^\mu y^\lambda$. Then $w(u) = 0$ and if $v = \alpha x^i y^j$ with $w(v) = 0$, then $\lambda i = \mu j$ and thus, since $\gcd(\lambda, \mu) = 1$, we must have $i = \mu c$ and $j = \lambda c$. Then $v = \alpha x^{\mu c} y^{\lambda c} = \alpha q^{-\binom{c}{2}} u^c$. So $\{\alpha u^c :\ \alpha \in k^\times,\ c \geq 0\}$ is the set of all monomials of weight zero. A similar argument shows that if $r \geq 0$ is an integer and $w(v) = r$, then $\{\alpha u^c v :\ \alpha \in k^\times,\ c \geq 0\}$ is the set of all monomials of weight r. So if $g \in k_q[x, y]$, then we can group all terms of g which have the same weight and find a unique presentation

$$g = \sum_{i=-r}^{s} g_i(u) G_i,$$

where G_i is a monomial of weight i.

We next show that $C(g(u); k_q[x, y]) = k[u]$ for any nonconstant polynomial $g(u) \in k[u]$. So suppose that $h \in C(g(u); k_q[x, y])$. By what we just showed we can write $h = \sum_{i=-r}^{s} h_i(u) H_i$, where $r, s \geq 0$ and H_i is a monomial of weight i. Now $g(u)h = hg(u)$ gives

$$\sum_{i=-r}^{s} h_i(u) g(u) H_i = \sum_{i=-r}^{s} h_i(u) H_i g(u).$$

Since the weight of each monomial in $g(u)H_i$ is i, the above identity holds if and only if

$$g(u) H_i = H_i g(u)$$

for all i. It is easy to see that only monomials of weight zero commute with a nonconstant element of $k[u]$. Thus $r = s = 0$ and hence $h \in k[u]$. This proves that

$$C(g(u); k_q[x, y]) = k[u], \tag{2.17}$$

for all nonconstant $g(u) \in k[u]$.

Finally, let $g \in C(f; k_q[x, y])$ and write $g = \sum_{i=-r}^{s} g_i(u) G_i$, where $r, s \geq 0$ and G_i is a monomial of weight zero. We also have $f = \sum_{i=0}^{n} f_i(u) F_i$ because, as we mentioned at the beginning of the proof, every monomial of f has a nonnegative weight and at least one of the monomials has wight zero. We also mentioned that in each monomial of f both x and y occur. So, if we choose $F_0 = 1$, then $f_0(u)$ cannot be a constant. Now looking at the monomials of minimum weight in both sides of $fg = gf$ we see that G_{-r} and $h_0(u)$ must commute, i.e. $G_{-r} \in C(h_0(u); k_q[x, y])$. Thus $G_{-r} \in k[u]$, by (2.17), and so $-r = w(G_{-r}) = 0$. Hence $r = 0$ and so every element of $C(f; k_q[x, y])$ is in the form $\sum_{i=0}^{s} g_i(u)G_i$. Therefore the map

$$\varphi : C(f; k_q[x, y]) \longrightarrow k[u]$$

defined by $\varphi(g) = g_0(u)$ is an injective k-algebra homomorphism. $\square$

Makar-Limanov then extended his idea to prove a similar result for quantum spaces. Let k be an algebraically closed field and let $n \geq 1$ be an integer. Let $\{q_{ij} : 1 \leq i, j \leq n\} \subset k^{\times}$ be such that $q_{ii} = 1$, $q_{ji} = q_{ij}^{-1}$ and the set $\{q_{ij} : i < j\}$ is a free basis for some abelian group. The *quantum n-space* $k_{\mathbf{q}}[x_1, \ldots, x_n]$ is the algebra generated by $x_1, \ldots, x_n$ subject to the relations $x_j x_i = q_{ij} x_i x_j$ for all i, j. Makar-Limanov [44] proved that the centralizer of a noncentral element of $k_{\mathbf{q}}[x_1, \ldots, x_n]$ is again contained in $k[u]$, for some $u \in k_{\mathbf{q}}[x_1, \ldots, x_n]$.

2.6 Centralizers in Semiprime PI-Rings

In this section, we give our results on centralizers in a semiprime PI-ring R. We first find the center of a centralizer in R and then we characterize those semiprime PI-rings in which the centralizer of every noncentral element is commutative.

We have already seen a few examples of rings in which the centralizer of every noncentral element is commutative, e.g. the first Weyl algebra and quantum planes. A ring R is called *CT* if the centralizer of every noncentral element of R is commutative. In the definition, CT is short for *commutative transitive*. The reason that we call such rings CT is this simple fact that the centralizer of every noncentral element of a ring R is commutative if and only if the property *x commutes with y* is transitive over noncentral elements. There are many examples of rings which are not CT. For example, the matrix algebra $M_n(\mathbb{C}), n \geq 3$ or the Weyl algebra $A_n(\mathbb{C}), n \geq 2$. In fact, it is easy to see that $M_n(R)$, $n \geq 2$, is *CT* if and only if $n = 2$ and R is a commutative domain. Therefore, by Theorem 1.5.10, a prime PI-ring R is CT if and only if $Q_Z(R) = M_n(D)$, where $Q_Z(R)$ is the central localization of R and $n = 1$ or 2. If $n = 1$, then D is a finite-dimensional central division algebra which is CT and if $n = 2$, then D is a field.

In this section, we characterize semiprime PI-rings which are CT. We show that a semiprime PI-ring R is CT if and only if $Q_{\max}(R) = C \oplus M_n(D)$, where $Q_{\max}(R)$ is the *largest left quotient ring* of R, C is a commutative ring (or the zero ring) and $n = 1$ or 2. If $n = 1$, then D is a finite-dimensional central division algebra which is CT and D is a field if $n = 2$. But what is this *largest left quotient ring* of R?

2.6.1 Maximal Left Quotient Rings

The proof of results mentioned in this paragraph can be found in [40, section 13B]. Fix a ring R. Let M be a left R-module. We say that a submodule N of M is *dense* and we write $N \subseteq_d M$ if for every $x, y \in M$, with $x \neq 0$, there exists $r \in R$ such that $rx \neq 0$ and $ry \in N$. Clearly every dense submodule is essential. A ring S is called a *general left quotient ring* of R if $R \subseteq_d S$. For example, if R has a classical left quotient ring $Q(R)$, then $Q(R)$ is a general left quotient of R. The reason is that if $x, y \in Q(R)$, with $x \neq 0$, then we can write $y = t^{-1}u$ for some $t, u \in R$. So $tx \neq 0$, because $x \neq 0$ and t is a unit in $Q(R)$, and $ty = u \in R$. Now, consider R as a left R-module and let $E := E(R)$ be the *injective hull* of R, i.e. the smallest injective R-module containing R. Let $H := \mathrm{End}_R(E)$ and $Q := \mathrm{End}_H(E)$. Then Q is a general left quotient ring of R and every general left quotient ring of R can be embedded into Q. The ring Q is called the *maximal left quotient ring* of R and we write $Q_{\max}(R)$. If R is commutative, then $Q_{\max}(R) \cong Z(H)$, the center of H, and hence $Q_{\max}(R)$ is commutative too. Also, if the classical left quotient ring $Q(R)$ of R exists and if every dense left ideal of R contains a regular element, then $Q_{\max}(R) = Q(R)$. In particular, if R is a semiprime left Goldie ring, then $Q_{\max}(R) = Q(R)$. So if D is a division ring, then $Q_{\max}(M_n(D)) = M_n(D)$. If R is the subring of $M_n(D)$ consisting of upper triangular matrices, then $Q(R) = R$ but $Q_{\max}(R) = M_n(D)$.

There is a useful characterization of the maximal left quotient ring of a semiprime PI-ring given in the following theorem.

Theorem 2.6.1. (Rowen, [52]) *If R is a semiprime PI-ring, then $Q := Q_{\max}(R)$ is characterized by the following properties.*

(1) There is a canonical injection $R \hookrightarrow Q$.

(2) For any essential ideal J of $Z(R)$, the center of R, and any $f \in \mathrm{Hom}_{Z(R)}(J, R)$, there exists $q \in Q$ such that $xq = f(x)$ for all $x \in J$.

(3) For any $q \in Q, Jq \subseteq R$ for some essential ideal J of $Z(R)$.

(4) $q = 0$ if and only if $Jq = (0)$ for some essential ideal J of $Z(R)$.

Note that parts three and four of the theorem show that $Q_{\max}(R)$ is both an essential extension of R and a semiprime PI-ring.

We recall that R is called left *nonsingular* if $\{r \in R : \text{l.ann}_R(r) \subseteq_e R\} = (0)$. Also, R is called *von Neumann regular* or just *regular* if for every $r \in R$ there exists $s \in R$ such that $r = rsr$. A regular ring is called *strongly regular* if it is *reduced*, i.e. it has no nonzero nilpotent element. It is known that $Q_{\max}(R)$ is regular if and only if R is nonsingular [40, Theorem 13.36] and in this case $Q_{\max}(R) \cong E(R)$. It is an easy consequence of Theorem 1.5.8 that semiprime PI-rings are nonsingular. So if R is a semiprime PI-ring, then $Q_{\max}(R)$ is regular and $Q_{\max}(R) \cong E(R)$. The following proposition gives more properties of $Q_{\max}(R)$ when R is a semiprime PI-ring.

Proposition 2.6.2. *Let R be a semiprime PI-ring.*

(1) $Z(Q_{\max}(R)) = Q_{\max}(Z(R))$.

(2) $Q_{\max}(R)$ is a finite module over its center.

(3) If M is a maximal ideal of $Z(Q_{\max}(R))$, then $Q_{\max}(R)_M$, the localization of $Q_{\max}(R)$ at M, is a finite-dimensional central simple algebra.

Proof. See [52, Corollary 3] for the proof of the first part. For the proof of the second and the third part of the theorem see [6, Theorem 3.7] and [4, Corollary 9], respectively. $\square$

By Proposition 2.6.2, $Q_{\max}(R)$ is finitely generated over its center for any semiprime PI-ring R. The following example shows that even being CT does not necessarily imply that a semiprime PI-ring is finite over its center.

Example 2.6.3. Let $A = \mathbb{Z}[x_1, x_2, \ldots]$, the polynomial algebra in an infinite set of commuting variables $\{x_1, x_2, \ldots\}$, and let k be the field of fractions of A. Let $I = \sum_{i \geq 1} Ax_i$ and

$$R := \begin{pmatrix} A & I \\ A & A \end{pmatrix}.$$

We show that $M_2(k) = kR$. It is clear that $kR \subseteq M_2(k)$. To prove $M_2(k) \subseteq kR$, we only need to show that kR contains the standard basis $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ of $M_2(k)$. This is clear because $e_{11}, e_{21}, e_{22} \in R$ and $e_{12} = x_1^{-1}(x_1 e_{12}) \in kR$.

Now, $M_2(k) = kR$ implies that R is semiprime, PI and CT because $M_2(k)$ is so. But R is not finitely generated over its center because I is not finitely generated over A.

2.6.2 The Double Centralizer Property in Semiprime PI-Rings

In this section we use Theorem 2.6.1 to find the center of the centralizer of an element in a semiprime PI-ring. This, in particular, gives the form of centralizers in semiprime PI-rings which are CT.

We know from linear algebra that if k is a field and $a \in M_n(k)$, then $b \in M_n(k)$ commutes with every matrix which commutes with a if and only if $b \in k[a]$. In other words, $Z(C(a; M_n(k))) = k[a]$ or equivalently we have the *double centralizer property*

$$C(C(a; M_n(k)); M_n(k)) = k[a].$$

This result has the following extension.

Lemma 2.6.4. (Werner, [65]) *If A is a finite-dimensional central simple k-algebra, then $C(C(a; A); A) = k[a]$.*

Armendariz [7] extended Werner's result by proving that if D is any central division k-algebra, then $a \in M_n(D)$ satisfies the double centralizer property if and only if a is algebraic over k. The following example shows that in general the double centralizer property does not hold in semiprime PI-rings even for elements which are integral over the center.

Example 2.6.5. Let $R = M_2(\mathbb{Z})$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $C(C(a; R); R) = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$ and, for example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \mathbb{Z}[a]$.

The next theorem shows that an element a in a semiprime PI-ring R satisfies the double centralizer property if and only if $Z(Q)[a] \cap R = Z(R)[a]$, where $Q := Q_{\max}(R)$. In particular, a semiprime PI-ring whose center is self-injective satisfies the double centralizer property.

Theorem 2.6.6. *Let R be a semiprime PI-ring and let $Q := Q_{\max}(R)$. Then*

$$Z(C(a; R)) = Z(Q)[a] \cap R$$

for all $a \in R$.

Proof. We first show that

$$Z(C(a;R)) \subseteq Z(C(a;Q)). \tag{2.18}$$

Let $b \in Z(C(a;R))$ and $q \in C(a;Q)$. By Theorem 2.6.1, there exists an essential ideal $J \subseteq Z(R)$ such that $Jq \subseteq R$. Then $Jq \subseteq C(a;R)$ and so $xqb = bxq = xbq$ for all $x \in J$. Therefore $J(qb - bq) = (0)$ and so $qb = bq$.. Hence $b \in Z(C(a;Q))$, by Theorem 2.6.1.

Next, we show that $Z(C(q;Q)) = Z(Q)[q]$ for all $q \in Q$. To see this, we suppose, to the contrary, that there exists some $p \in Z(C(q;Q)) \setminus Z(Q)[q]$. Let

$$I := \{x \in Z(Q) : \ xp \in Z(Q)[q]\}.$$

Clearly I is a proper ideal of $Z(Q)$. Let M be a maximal ideal of $Z(Q)$ which contains I. By Proposition 2.6.2, Q_M is a finite-dimensional central simple algebra. Therefore

$$Z(C(q/1;Q_M)) = Z(Q_M)[q/1],$$

by Lemma 2.6.4. Hence $p/1 \in Z(Q_M)[q/1]$ and so $p/1 = \sum_{k=0}^{n}(x_j/s)q^j$, for some integer n and $x_j/s \in Z(Q_M)$. Therefore there exists $u \in Z(Q) \setminus M$ such that

$$sup = \sum_{j=0}^{n} ux_j q^j. \tag{2.19}$$

By Proposition 2.6.2, Q is finitely generated over $Z(Q)$. Let

$$Q = \sum_{i=1}^{m} y_i Z(Q).$$

Since x_j/s commutes with $y_i/1$ for all i, there exists some $s_i \in Z(Q) \setminus M$ such that $s_i x_j$ commutes with y_i. Let $t_j = s_1 s_2 \cdots s_m$. Then $t_j x_j$ commutes with each y_i and hence $t_j x_j \in Z(Q)$. Let $t = t_1 t_2 \cdots t_n$. Then $tx_j \in Z(Q)$ for all j and hence by (2.19)

$$tsup = \sum_{j=0}^{n} utx_j q^j \in Z(Q)[q].$$

That means $tsu \in I \subseteq M$ which is absurd. Thus we have proved that for all $q \in Q$

$$Z(C(q;Q)) = Z(Q)[q]. \tag{2.20}$$

Now it is easy to prove the theorem. By (2.18) and (2.20) we have

$$Z(C(a;R)) \subseteq Z(C(a;Q)) = Z(Q)[a].$$

Therefore $Z(C(a;R)) \subseteq Z(Q)[a] \cap R$. The inclusion $Z(Q)[a] \cap R \subseteq Z(C(a;R))$ is trivial. $\square$

2.6.3 Characterizing Semiprime PI-Rings which are CT

We are now going to characterize semiprime PI-rings which are CT. First let us consider the case for division algebras. Let D be a finite-dimensional central division k-algebra and suppose that D is CT. Let K be a subfield of D which properly contains k and let $a \in K \setminus k$. Let L be any subfield of D which contains K. By Theorem 2.6.6

$$k[a] = Z(C(a; D)) = C(a; D).$$

We also have $k[a] \subseteq K \subseteq C(a; D) = k[a]$ and hence $K = k[a]$. Similarly, $L = k[a]$ and thus $K = L$. Therefore every subfield of D which properly contains k is maximal. Conversely, suppose that every subfield of D which properly contains k is maximal and let $a \in D \setminus k$. Let $b \in C(a; D)$. Then $k[a]$ and $k[a, b]$ are both subfields of D and so $b \in k[a]$. Hence $C(a; R) = k[a]$ and so D is CT. So we have proved that D is CT if and only if every subfield of D which properly contains k is maximal. For example, if p is a prime number and $[D : k] = p^2$, then D is CT. Now suppose that D is CT. We claim that $[D : k]$ is a prime power. To see this, we recall that D has a primary decomposition [26, Theorem 4.19], i.e. if $p_1, \ldots, p_n$ are the prime divisors of $[D : k]$, then

$$D = D_1 \otimes_k D_2 \otimes_k \cdots \otimes_k D_n$$

for some division algebras D_i such that p_i is the only prime divisor of $[D_i : Z(D_i)]$. If $n > 1$, then a subfield of D_1 which properly contains k would be a subfield of D which is not maximal in D and so D would not be CT. Thus a finite-dimensional central division k-algebra is CT if and only if D satisfies these properties: $[D : Z(D)]$ is a prime power and every subfield of D which properly contains k is a maximal subfield. It is clear that if D is CT, then D cannot be a crossed product, i.e. D cannot have a maximal subfield which is Galois over the center, unless $[D : k] = p^2$ for some prime p. This is a trivial result of the Galois correspondence theorem.

We see in this section that characterizing semiprime PI-rings which are CT is eventually reduced to the same problem for finite-dimensional central division algebras which are CT. We begin with two simple observations.

Lemma 2.6.7. *Let $R_i,\ i \in I$, be a family of rings and let $R := \prod_{i \in I} R_i$. Suppose that the ring R is not commutative. Then R is CT if and only if there exists $j \in I$ such that R_j is both noncommutative and CT and R_i is commutative for all $i \neq j$.*

Proof. Suppose that R is CT. Since R is not commutative, there exists $j \in I$ such that R_j is not commutative. Let $r := (r_i)_{i\in I} \in R$ where r_j is any noncentral element of R_j and $r_i = 0$ for all $i \neq j$. Then r is a noncentral element of R and hence $C(r; R) = \prod_{i\in I} C(r_i; R_i)$ has to be commutative. So $C(r_i; R_i)$ is commutative for all $i \in I$. Thus R_j is CT and R_i is commutative for all $i \neq j$ because $C(r_i; R_i) = C(0; R_i) = R_i$. Conversely, suppose that there exists $j \in I$ such that R_j is both noncommutative and CT and R_i is commutative for all $i \neq j$. Let $a := (a_i)_{i\in I}$ be any noncentral element of R. Then a_j is a noncentral element of R_j and hence $C(a_j; R_j)$ is commutative. We also have $C(a_i; R_i) = R_i$ for all $i \neq j$. Thus $C(a; R)$ is commutative. □

Lemma 2.6.8. *Let R be a semiprime PI-ring. Then R is CT if and only if $Q_{\max}(R)$ is CT.*

Proof. Let $Q := Q_{\max}(R)$. The if part of the lemma follows immediately from the fact that $Z(Q) \cap R = Z(R)$. Conversely, suppose that R is CT and let q_0 be a noncentral element of Q. Let $q_1, q_2 \in C(q_0; Q)$. We need to prove that $q_1q_2 = q_2q_1$. Let J be an essential ideal of $Z(R)$ such that $Jq_i \subseteq R$ for $i = 0, 1, 2$. If $Jq_0 \subseteq Z(R)$, then $q_0 \in Z(Q)$, which is not true. So there exists some $\alpha \in J$ such that $\alpha q_0 \in R \setminus Z(R)$. Thus $C(\alpha q_0, R)$ is commutative. Now the result follows from the fact that $Jq_i \subseteq C(\alpha q_0; R)$ for $i = 1, 2$. □

Another fact that we need is that in semiprime rings, commutative ideals are central. This is easy to prove.

Lemma 2.6.9 ([47], Lemma 1)**.** *Let R be a semiprime ring and let J be a left or right ideal of R. Considering J as a ring, we have $Z(J) = J \cap Z(R)$.*

Recall that the *index* of a nilpotent element a in a ring R is the smallest integer $n \geq 1$ such that $a^n = 0$. Let $i(a)$ denote the index of a. The index of a ring R is $i(R) = \sup\{i(a) : a \in R\}$. A ring R is said to have *bounded index* if $i(R) < \infty$.

For example, matrix rings over commutative rings have bounded index. More generally, every semiprime PI-ring has bounded index.

Lemma 2.6.10 ([48], Theorem 13.4.2)**.** *Every semiprime PI-ring can be embedded into some matrix ring $M_n(C)$, where C is a direct product of fields.*

Regular self-injective rings of bounded index have a nice form.

Lemma 2.6.11 ([33], Theorem 7.20)**.** *A regular self-injective ring has bounded index if and only if it is a finite direct product of full matrix rings over strongly regular rings.*

We are now ready to prove the main result of this section.

Theorem 2.6.12. *A noncommutative semiprime PI-ring R is CT if and only if*

$$Q_{\max}(R) = C \oplus M_n(D),\ n \le 2,$$

where C is either zero or a commutative ring and if $n = 1$ (resp. $n = 2$), then D is a finite-dimensional central division algebra which is CT (resp. field).

Proof. The if part follows from Lemma 2.6.8 and the fact that $M_2(k)$ is CT for any field k. Conversely, suppose that a semiprime PI-ring R is CT. Then $Q_{\max}(R)$ is also CT by Lemma 2.6.8. Since $Q_{\max}(R)$ is a semiprime PI-ring, it has a bounded index by Lemma 2.6.10, and so, by Lemma 2.6.11, $Q_{\max}(R)$ is a finite direct product of full matrix rings over strongly regular rings. Therefore, by Lemma 2.6.7,

$$Q_{\max}(R) = S \oplus M_n(T),\ n \le 2,$$

where S is either zero or a commutative ring and T is both strongly regular and CT.

If $n = 2$, then T is a field because a commutative domain is regular if and only if it is a field. If $n = 1$, then T cannot be commutative because R is not commutative. So, to complete the proof of the theorem, we only need to show that if a noncommutative strongly regular ring T is CT, then T is a direct product of a commutative (or zero) ring and a division ring, which clearly has to be CT and finite-dimensional over its center by Theorem 1.4.25. We now show that if a is a noncentral element of T, then $\text{l.ann}(a)$ is commutative. To see this, note that $T = Ta \oplus \text{l.ann}(a)$ because T is strongly regular. Thus, since T is CT, either $\text{l.ann}(a)$ or Ta must be commutative. Since a is not central, $\text{l.ann}(a)$ is commutative and hence central by Lemma 2.6.9. Now, let I be the sum of all commutative ideals of T. We show that I is a maximal left ideal of T. To see this, let $a \notin I$. Then $\text{l.ann}(a) \subseteq I$ and hence $T = Ta \oplus \text{l.ann}(a) \subseteq Ta + I \subseteq T$, which proves that I is a maximal ideal.

Finally, we have

$$\{0\} \neq \{xy - yx :\ x, y \in T\} \subseteq \text{l.ann}(I)$$

because T is not commutative and $I \subseteq Z(T)$. It follows, by the maximality of I, that $T = I \oplus \text{l.ann}(I)$ and hence $\text{l.ann}(I) \cong T/I$ is a division ring. $\square$

Corollary 2.6.13. *If R is a semiprime PI-ring which is CT, then $Q_{\max}(R)$ is the central extension of R.*

Proof. Let Z denote the center of $Q_{\max}(R)$. By Theorem 2.6.12 there exists a commutative ring C, a field k and a finite-dimensional central simple k-algebra V such that $Q_{\max}(R) = C \oplus V$. Thus $RZ = C \oplus Rk$ and since $C \oplus Rk$ is finitely generated over $Z = C \oplus k$, the result follows from [5, Theorem 5]. $\square$

Let R be a semiprime PI-ring and let S be a nil multiplicatively closed subset of R. We know from [54, Corollaries 1.6.23 and 1.6.26] that S is nilpotent. If R is CT, then $S^2 = \{0\}$ by Theorem 2.6.12. We also have the following result.

Proposition 2.6.14. *Let R be a ring and suppose that $Z(R)$ is semiprime. Let S be a nil multiplicatively closed subset of R. If $C(x; R)$ is commutative for every noncentral element $x \in S$, then S is commutative and hence locally nilpotent.*

Proof. Since $Z(R)$ is semiprime, noncentral elements of S are exactly nonzero elements of S. For $a, b \in R$ let $[a, b] := ab - ba$. Let $0 \neq x, y \in S$. Let $n \geq 2$ be the smallest integer such that $x^n = 0$. For any $z \in R$ and integer $j \geq 1$ we have $[x^{n-1}, xzx^j] = [x^{n-1}, x] = 0$. So $xzx^j, x \in C(x^{n-1}; R)$ and hence $x^2zx^j = xzx^{j+1}$. Thus if $i \geq 2$ and $j \geq 1$, then

$$x^i z x^j = xzx^{i+j-1}, \ y^i z y^j = yzy^{i+j-1}.$$

So every monomial in x, y has one of the following forms

$$x(yx)^r y^s x^t, \ y(xy)^r x^s y^t, \ (xy)^r x^s y^t, \ (yx)^r y^s x^t.$$

Hence there exists an integer N such that every monomial of degree N in x, y is zero. Let r be the smallest integer such that every monomial of degree r in x, y is zero. Let $w \neq 0$ be a monomial of degree $r - 1$ in x, y. Then $[w, x] = [w, y] = 0$. Thus $x, y \in C(w; R)$ and so $[x, y] = 0$. $\square$

Chapter 3

Centralizers in $A_2(k)$ and $K[x, x^{-1}; \sigma]$

3.1 Introduction

The structure of centralizers in a differential polynomial ring $S := R[y; \delta]$ has been studied by many authors. Amitsur proved that if R is a field of characteristic zero and if $k = \{r \in R : \delta(r) = 0\}$, then the centralizer of a nonconstant element $f \in S$ is a commutative k-algebra and also a free module of finite rank over $k[f]$. In fact, Amitsur's proof of this result works for a more general setting. That is, let R be a commutative domain of characteristic zero. We extend δ to $Q(R)$, the quotient field of R. If $k := \{q \in Q(R) : \delta(q) = 0\}$ is a subfield of R, then again the centralizer of a nonconstant element $f \in S$ is commutative and a free module of finite rank over $k[f]$. We gave the proof of this result in section 2 of chapter 2. In particular, if $R = k[x]$ and $\delta = d/dx$, then we have Amitsur's result for centralizers in $A_1(k)$, the first Weyl algebra. K. Goodearl [32] proved a similar result for S when R is a semiprime commutative ring. He proved that if $k := \{r \in R : \delta(r) = 0\}$ is a subfield of R, then the centralizer of an element of $f = \sum_{i=0}^{n} a_i y^i \in S$, where n is invertible in k and a_n is invertible in R, is a commutative domain and a free module of finite rank over $k[f]$.

Let k be a field of characteristic zero. Dixmier [22] gave explicit form of centralizers of some elements of $A_1(k)$. J. Guccione and others [34] proved that if $[q, p] = 1$ for some elements $p, q \in A_1(k)$, then $C(p; A_1(k)) = k[p]$. His proof is long and computational. V. Bavula [11] gave a shorter and more elegant proof of this result. A derivation δ of a k-algebra A is called *locally nilpotent* if for every $u \in A$ there exists an integer n such that $\delta^n(u) = 0$. Bavula proved that the centralizer C of a nonconstant element of $A_1(k)$ admits a locally nilpotent derivation δ if and only if $C = k[u]$ for some $u \in C$ and $\delta = d/du$. He used this result to give his proof of Guccione's result.

In section 2 of this chapter, we study centralizers in the second Weyl algebra $A_2(k)$. We will assume that k is an algebraically closed field of characteristic zero and we prove that the GK dimension of a centralizer in $A_2(k)$ has three possible values one, two and three. Those of GK dimension one or two are commutative and those of GK dimension three are not locally PI. We show that $A_2(k)$ has centralizers of GK dimension one, two or three. We also give explicit forms of centralizers of some elements of $A_2(k)$.

In section 3, we study the structure of centralizers in subalgebras of skew Laurent polynomial rings. These algebras contain $A_1(k)$ as well as some other algebras as subalgebras. So our result in this section is a generalization of Amitsur's result on centralizers in $A_1(k)$. We prove that, under some conditions, centralizers in subalgebras of skew Laurent polynomial rings are commutative and free modules of finite rank over some polynomial algebra in one variable. In section 4, a few problems are posed and a connection between Dixmier's Fourth Problem and the problem of finite generation of centralizers in $A_2(k)$ is given.

3.2 Centralizers in the Second Weyl Algebra

Throughout this section, k is an algebraically closed field of characteristic zero and $A_n(k), n \geq 1$, is the n-th Weyl algebra over k as defined in Definition 1.6.17. We assume that $x_1, \ldots, x_n, y_1, \ldots, y_n$ are the generators of $A_n(k)$ with the relations

$$[x_i, x_j] = [y_i, y_j] = 0, \ [y_i, x_j] = \delta_{ij},$$

for all i, j, where δ_{ij} is the Kronecker delta.

If $a \in A_1(k) \setminus k$, then, by Theorems 2.2.7 and 2.2.9, $C(a; A_1(k))$ is commutative and, as a $k[f]$-module, free and finitely generated. As we saw in Example 2.4.1, if $n \geq 2$, then centralizers in $A_n(k)$ may not even be PI.

In this section, we investigate the structure of centralizers in $A_2(k)$. In the first subsection, we prove that the GK dimension of a centralizer in $A_2(k)$ is one, two or three. Those centralizers of GK dimension one or two are commutative and those of GK dimension three are not locally PI. We also prove that for each integer $n = 1, 2, 3$ there exists an element of $A_2(k)$ whose centralizer has GK dimension n.

In the second subsection, we find the centralizer of any element of $A_2(k)$ in which exactly two of the four generators x_1, x_2, y_1, y_2 occur.

3.2.1 GK Dimension of Centralizers

We begin this subsection with stating two important results about commutative subalgebras of $A_n(k)$.

Theorem 3.2.1. (Makar-Limanov, [45]) *Let B be a commutative subalgebra of $A_n(k)$. If* $\mathrm{GKdim}(B) = n$, *then $C(B; A_n(k))$ is commutative.*

Theorem 3.2.2 ([10], Corollary 1.6)**.** *If B is a commutative subalgebra of $A_n(k)$, then* $\mathrm{GKdim}(B) \leq n$.

We also need the following simple fact.

Lemma 3.2.3. *Let B be a finitely generated k-algebra which is a domain of finite* GK *dimension. If B is PI, then* $\mathrm{GKdim}(B) = \mathrm{GKdim}(Z(B))$.

Proof. Since B is a PI-domain, the quotient division algebra $Q(B)$ and the central localization $Q_Z(B)$ are equal, by Corollary 1.5.11. By Theorem 1.5.10, $Q(B)$ is finite-dimensional over its center and $Z(Q(B)) = Q(Z(B))$. Thus $\mathrm{GKdim}(Q(B)) = \mathrm{GKdim}(Z(Q(B)))$, by Proposition 1.6.23. Hence

$$\mathrm{GKdim}(Z(B)) = \mathrm{GKdim}(Z(Q(B))) = \mathrm{GKdim}(Q(B)) = \mathrm{GKdim}(B),$$

because, by Proposition 1.6.26, the GK dimension of the central localization of an algebra is equal to the GK dimension of the algebra. □

We are now ready to prove the first half of the main result of this subsection.

Proposition 3.2.4. *Let $A := A_2(k), a \in A \setminus k$ and $C := C(a; A)$. Then* $\mathrm{GKdim}(C) \in \{1, 2, 3\}$. *If* $\mathrm{GKdim}(C) \in \{1, 2\}$, *then C is commutative and if* $\mathrm{GKdim}(C) = 3$, *then C is not locally PI.*

Proof. Since $k[a] \subseteq C$, we have $\mathrm{GKdim}(C) \geq 1$. We also have

$$\mathrm{GKdim}(C) \leq \mathrm{GKdim}(A) - 1 = 3,$$

by Theorem 2.4.6 and Corollary 1.6.19. If $\mathrm{GKdim}(C) = 1$, then C is commutative by Proposition 1.6.32. There is no algebra whose GK dimension is strictly between 1 and 2, by Theorem 1.6.15. Now, suppose that $\mathrm{GKdim}(C) = 2$. In order to prove that C is commutative, we only need to show that every finitely generated subalgebra B of C is commutative.

Clearly we may assume that $a \in B$ and so $\mathrm{GKdim}(B) \in \{1, 2\}$. If $\mathrm{GKdim}(B) = 1$, then B is commutative by Proposition 1.6.32. We claim that if $\mathrm{GKdim}(B) = 2$, then B is PI. So suppose, to the contrary, that B is not PI. Then Lemma 1.6.34 and the fact that $k[a] \subseteq Z(B)$ gives

$$2 = \mathrm{GKdim}(B) \geq 2 + \mathrm{GKdim}(Z(B)) \geq 2 + \mathrm{GKdim}(k[a]) = 3,$$

which is nonsense. Hence B is PI and so $\mathrm{GKdim}(Z(B)) = \mathrm{GKdim}(B) = 2$, by Lemma 3.2.3. Therefore $C(Z(B); A)$ is commutative, by Theorem 3.2.1, and so B is commutative because $B \subseteq C(Z(B); A)$. We now claim that if B is a finitely generated subalgebra of C such that $a \in B$ and $\mathrm{GKdim}(B) > 2$, then $\mathrm{GKdim}(B) = 3$. To see this, suppose first that B is PI. Then

$$\mathrm{GKdim}(Z(B)) = \mathrm{GKdim}(B) > 2,$$

by Lemma 3.2.3, contradicting Theorem 3.2.2. So B is not PI and hence $\mathrm{GKdim}(B) \geq 3$, by Theorem 1.6.34. Since $B \subseteq C$, we also have $\mathrm{GKdim}(B) \leq 3$ and the claim follows.

An immediate result of the claim is that if $\mathrm{GKdim}(B) > 2$, then $\mathrm{GKdim}(B) = 3$. In this case, C is not locally PI because if B is a finitely generated subalgebra of C with $\mathrm{GKdim}(B) > 2$, then, as we showed in the proof of the claim, B is not PI. □

We now show that for each $n \in \{1, 2, 3\}$ there exists some element in $A_2(k)$ whose centralizer has GK dimension n. We begin with centralizers of GK dimension three. It is easy to find examples of centralizers of GK dimension three, e.g. if $a \in k[x_1]$, then $C(a; A_2(k)) = k[x_1, x_2, y_2]$. More generally, we have the following result.

Proposition 3.2.5. *Let $a \in A_2(k) \setminus k$ and $C := C(a; A_2(k))$. If at most two of the four generators x_1, y_1, x_2, y_2 occur in a, then* $\mathrm{GKdim}(C) = 3$.

Proof. By Proposition 3.2.4, we only need to show that C is not commutative. If $a \in k[x_1, y_1]$, then $x_2, y_2 \in C$ and so C is not commutative. A similar argument holds if $a \in k[x_2, y_2]$. If $a \in k[x_1, x_2]$, let

$$u := [y_2, a]y_1 - [y_1, a]y_2. \tag{3.1}$$

Then $[u, a] = 0$ and thus $u \in C$. We also have $[u, x_1] = \frac{\partial a}{\partial x_2}$ and $[u, x_2] = -\frac{\partial a}{\partial x_1}$. Hence, since a is not a constant, either $[u, x_1] \neq 0$ or $[u, x_2] \neq 0$. Thus C is not commutative because $x_1, x_2 \in C$. A similar argument holds if a is an element of $k[x_1, y_2], k[y_1, x_2]$ or $k[y_1, y_2]$. □

Corollary 3.2.6. *Let $a \in A_2(k)$ and $C := C(a; A_2(k))$. Let $\deg(a)$ denote the total degree of a. If $\deg(a) = 1$, then $\mathrm{GKdim}(C) = 3$ and if $\deg(a) = 2$, then $\mathrm{GKdim}(C) = 2$ or 3.*

Proof. Suppose first that $\deg(a) = 1$ and let $a = \alpha x_1 + \beta y_1 + \gamma x_2 + \delta y_2$ where $\alpha, \beta, \gamma, \delta \in k$. If $\alpha \neq 0$, define $f \in \mathrm{Aut}(A_2(k))$ by

$$f(x_1) = \beta x_1 + \alpha^{-1} y_1, \; f(y_1) = -\alpha x_1, \; f(x_2) = x_2, \; f(y_2) = y_2.$$

Then $f(a) = y_1 + \gamma x_2 + \delta y_2$. If $\gamma \neq 0$, then we can also eliminate x_2 in $f(a)$ in a similar way. Thus we may assume that only two of the four generators x_1, y_1, x_2, y_2 occur in a and we are done by Proposition 3.2.5. If $\deg(a) = 2$, then one can find an automorphism f of $A_2(k)$ such that $f(a) = \alpha(x_1^2 + y_1^2) + \beta(x_2^2 + y_2^2) + \gamma$, for some $\alpha, \beta, \gamma \in k$ [21, Exercise 7.6.11]. Thus $k[x_1^2 + y_1^2, x_2^2 + y_2^2] \subseteq C$ and so $\mathrm{GKdim}(C) \geq 2$. Therefore $\mathrm{GKdim}(C) = 2$ or 3, by Proposition 3.2.4. □

The fact that the centralizer of every nonconstant element of $A_1(k)$ is commutative implies that a subalgebra C of $A_1(k)$ is a maximal commutative subalgebra if and only if C is the centralizer of some nonconstant element of $A_1(k)$. This is not true in $A_2(k)$ as the next corollary, which is an immediate result of Proposition 3.2.5, shows.

Corollary 3.2.7. *The algebra $k[x_1, x_2]$ is a maximal commutative subalgebra of $A_2(k)$ and $k[x_1, x_2] \neq C(a; A_2(k))$ for all $a \in A_2(k)$.*

Remark 3.2.8. By Proposition 3.2.5, if at most two of the four generators x_1, x_2, y_1, y_2 occur in a, then $\mathrm{GKdim}(C(a; A_2(k))) = 3$. Now, suppose that at most three of the four generators x_1, x_2, y_1, y_2 occur in $a \in A_2(k) \setminus k$ and let B be the k-algebra generated by those three generators. Let $C := C(a; A_2(k))$ and $C_0 := C(a; B)$. Then $\mathrm{GKdim}(C) = 2$ or 3 and $\mathrm{GKdim}(C) = 2$ if and only if $a \notin Z(B)$ and $C = C_0$. To see this, we assume that $B = k[x_1, x_2, y_2]$ and so $Z(B) = k[x_1]$. If $a \in k[x_1]$, then $B = C$ and so $\mathrm{GKdim}(C) = 3$. Otherwise, $k[x_1, a] \subseteq C$ and so $\mathrm{GKdim}(C) = 2$ or 3. Now, if $\mathrm{GKdim}(C) = 2$, then C is commutative, by Proposition 3.2.4, and so $\frac{\partial b}{\partial y_1} = [b, x_1] = 0$ for all $b \in C$. Hence $b \in C_0$ and so $C = C_0$. Conversely, suppose that $a \notin k[x_1]$ and $C = C_0$. Then a is not algebraic over $k(x_1)$, the field of fractions of $k[x_1]$, and hence

$$\mathrm{GKdim}(C) = \mathrm{GKdim}(C_0) \leq \mathrm{GKdim}(B) - 1 = 2$$

by Theorem 2.4.6. Alternatively, we can argue that since $B \cong A_1(k[x_1]) \subset A_1(k(x_1))$, we have $C_0 \subseteq C(a; A_1(k(x_1)))$ and so C_0 is commutative by Theorem 2.2.9. Thus C is commutative and hence $\mathrm{GKdim}(C) = 2$. Notice that the condition $C = C_0$ is equivalent to this condition: if $b_1 \frac{\partial a}{\partial x_1} = [a, b_0]$ for some $b_1 \in C_0$ and $b_0 \in B$, then $b_1 = 0$. The reason is that $C \neq C_0$ if and only if there exists $b \in C$ whose degree with respect to y_1 is at least one. Then, since $x_1 \in C$, we have $\frac{\partial b}{\partial y_1} = [b, x_1] \in C$ and so we may assume that the degree of b with respect to y_1 is one. Let $b = b_1 y_1 + b_0,\ b_1, b_0 \in B$. Then $[b, a] = 0$ if and only if $[b_1, a] = 0$ and $b_1 \frac{\partial a}{\partial x_1} = [a, b_0]$.

The next example gives a centralizer of GK dimension two in $A_2(k)$.

Example 3.2.9. Let $a := x_1 y_1 + \alpha x_2 y_2,\ \alpha \in k$. If $\alpha \notin \mathbb{Q}$, then $C(a; A_2(k)) = k[x_1 y_1, x_2 y_2]$ and so $\mathrm{GKdim}(C(a; A_2(k))) = 2$.

Proof. It is clear that both $x_1 y_1$ and $x_2 y_2$ belong to $C(a; A_2(k))$ and so

$$k[x_1 y_1, x_2 y_2] \subseteq C(a; A_2(k)).$$

Conversely, suppose that

$$b = \sum_{i,j,u,v} \beta_{ijuv} x_1^i y_1^j x_2^u y_2^v \in C(a; A_2(k)).$$

Then, since $[x_1^i y_1^j, x_1 y_1] = (j - i) x_1^i y_1^j$ and $[x_2^i y_2^j, x_2 y_2] = (j - i) x_2^i y_2^j$, our hypothesis that $[b, a] = 0$ gives

$$\sum \beta_{ijuv}((j - i) + \alpha(v - u)) x_1^i y_1^j x_2^u y_2^v = 0.$$

Thus $(j - i) + \alpha(v - u) = 0$ for all i, j, u, v and, since $\alpha \notin \mathbb{Q}$, we have $i = j$ and $u = v$. Therefore

$$b = \sum \beta_{iu} x_1^i y_1^i x_2^u y_2^u.$$

An easy induction shows that for every positive integer m there exist $\gamma_i \in k$ such that

$$x_1^m y_1^m = (x_1 y_1)^m + \gamma_1 (x_1 y_1)^{m-1} + \cdots + \gamma_{m-1} x_1 y_1.$$

Thus $x_1^m y_1^m \in k[x_1 y_1]$. Similarly, $x_2^m y_2^m \in k[x_2 y_2]$ and so $b \in k[x_1 y_1, x_2 y_2]$. □

We now give an example of a centralizer of GK dimension one. We begin with an element of the form $a = y_1 + a_1$ where $a_1 \in k[x_1, x_2, y_2]$. The idea is to find a_1 somehow that the leading coefficient of every element of $C(a; A_2(k))$ becomes constant. This implies that $C(a; A_2(k)) = k[a]$ and so the GK dimension of $C(a; A_2(k))$ is one.

Example 3.2.10. Let $a := y_1 + (x_1x_2 + 1)y_2$. Then $C(a; A_2(k)) = k[a]$ and therefore $\text{GKdim}(C(a; A_2(k))) = 1$.

Proof. Let $a_1 := (x_1x_2 + 1)y_2$. So $a = y_1 + a_1$ and if $b = \sum_{i=0}^{m} b_i y_1^i,\ b_i \in k[x_1, x_2, y_2]$, is an element of $C(a; A_2(k))$, then

$$ab = b_m y_1^{m+1} + \left(\frac{\partial b_m}{\partial x_1} + b_{m-1} + a_1 b_m\right) y_1^m + \cdots$$

and

$$ba = b_m y_1^{m+1} + (b_m a_1 + b_{m-1}) y_1^m + \cdots .$$

equating the coefficients of y_1^m in both sides of $ab = ba$ gives $\frac{\partial b_m}{\partial x_1} = [b_m, a_1]$. So

$$\frac{\partial b_m}{\partial x_1} = [b_m, (x_1x_2 + 1)y_2]. \tag{3.2}$$

We claim that $b_m \in k$ which, in particular, implies $C(a; A_2(k)) \cap k[x_1, x_2, y_2] = k$. There is nothing to prove if $b_m = 0$. Let

$$b_m = \sum_{i=0}^{n} c_i y_2^i,\ \ c_i \in k[x_1, x_2],\ \ c_n \neq 0.$$

We now find the coefficients of y_2^n in both sides of (3.2). Clearly the coefficient of y_2^n in $\frac{\partial b_m}{\partial x_1}$ is $\frac{\partial c_n}{\partial x_1}$. We also have

$$[b_m, (x_1x_2+1)y_2] = \sum_{i=0}^{n} [c_i y_2^i, (x_1x_2+1)y_2] = \sum_{i=0}^{n} \left((ix_1c_i - (x_1x_2 + 1)\frac{\partial c_i}{\partial x_2}) y_2^i + \cdots \right). \tag{3.3}$$

So the coefficient of y_2^n in $[b_m, (x_1x_2 + 1)y_2]$ is $nx_1c_n - (x_1x_2 + 1)\frac{\partial c_n}{\partial x_2}$. Thus (3.2) gives

$$\frac{\partial c_n}{\partial x_1} = nx_1c_n - (x_1x_2 + 1)\frac{\partial c_n}{\partial x_2}. \tag{3.4}$$

Let $c_n = \sum_{i=0}^{s} u_i x_2^i,\ u_i \in k[x_1],\ u_s \neq 0$. Then equating the coefficients of x_2^s and x_2^{s-1} in both sides of (3.4) gives

$$\frac{\mathrm{d}u_s}{\mathrm{d}x_1} = (n - s)x_1u_s \tag{3.5}$$

and

$$\frac{\mathrm{d}u_{s-1}}{\mathrm{d}x_1} = (n - s + 1)x_1u_{s-1} - su_s. \tag{3.6}$$

Comparing the degrees in both sides of (3.5) gives $n = s$ and $u_s \in k$. It then follows from (3.6) that $n = 0$ and so $b_m = c_0 = u_0 \in k$. Thus $b - b_m a^m$ is an element of $C(a; A_2(k))$ whose degree with respect to y_1 is smaller than m. An induction now shows that $b \in k[a]$ and so $C(a; A_2(k)) = k[a]$. $\square$

So we have proved the following result.

Theorem 3.2.11. *Let $a \in A_2(k) \setminus k$ and $C := C(a; A_2(k))$. Then* $\mathrm{GKdim}(C) \in \{1, 2, 3\}$. *If* $\mathrm{GKdim}(C) \in \{1, 2\}$, *then C is commutative and if* $\mathrm{GKdim}(C) = 3$, *then C is not locally PI. Furthermore, for each $n \in \{1, 2, 3\}$ there exists an element of $A_2(k)$ whose centralizer has* GK *dimension n.*

3.2.2 Centralizers of Elements of $k[x_1, x_2]$

We have already seen, in Proposition 3.2.5, that if at most two of the four generators x_1, x_2, y_1, y_2 occur in $a \in A_2(k)$, then $\mathrm{GKdim}(C(a; A_2(k))) = 3$. In this subsection, we would like to find $C(a; A_2(k))$. We consider two cases. First, suppose that the generators occurring in a do not commute with each other, i.e. $a \in k[x_1, y_1]$ or $a \in k[x_2, y_2]$. This case is trivial. If $a \in k[x_1, y_1]$, then $C(a; A_2(k)) = C_0[x_2, y_2]$ where C_0 is the centralizer of a in $k[x_1, y_1] \cong A_1(k)$. So in this case $C(a; A_2(k)) \cong A_1(k) \otimes_k C_0$ and the problem is reduced to centralizers in $A_1(k)$. A similar result holds if $a \in k[x_2, y_2]$.

The second case, which is not trivial, is when the generators occurring in a commute with each other. Theorem 3.2.12 solves this case for $a \in k[x_1, x_2]$. A similar argument can be used to find $C(a; A_2(k))$ when a is an element of $k[x_1, y_2], k[y_1, y_2]$ or $k[y_1, x_2]$. The key in the proof of Theorem 3.2.12 is the element introduced in (3.1). We note that if $a \in k[x_1]$ or $a \in k[x_2]$, then we are back to the first case and so we may assume that $\frac{\partial a}{\partial x_1} \frac{\partial a}{\partial x_2} \neq 0$.

We then give an example of an element of $k[x_1, x_2]$ whose centralizer has no subalgebra isomorphic to $A_1(k)$. At the end of this subsection, we prove a necessary condition for a centralizer in $A_2(k)$ to contain a nontrivial simple subalgebra.

Theorem 3.2.12. *Let $a \in k[x_1, x_2]$ and suppose that $\frac{\partial a}{\partial x_1} \frac{\partial a}{\partial x_2} \neq 0$. Let $d := \gcd(\frac{\partial a}{\partial x_1}, \frac{\partial a}{\partial x_2})$. Let $\frac{\partial a}{\partial x_2} = pd$, $\frac{\partial a}{\partial x_1} = qd$ and $u := py_1 - qy_2$. Then $C(a; A_2(k)) = k[x_1, x_2, u]$.*

Proof. Let $C := C(a; A_2(k))$. We have

$$[u, a] = p[y_1, a] - q[y_2, a] = p\frac{\partial a}{\partial x_1} - q\frac{\partial a}{\partial x_2} = 0$$

and so $u \in C$. We note that the set $\{1, u, u^2, \ldots\}$ is linearly independent over $k[x_1, x_2]$. To see this, suppose to the contrary that n is the smallest integer for which there exist $v_0, \ldots, v_n \in k[x_1, x_2]$ with $v_n \neq 0$ such that $\sum_{i=0}^{n} v_i u^i = 0$. Then

$$0 = \sum_{i=0}^{n} v_i [u^i, x_1] = npv_n u^{n-1} + (\text{terms of lower degree in u}).$$

This contradicts the minimality of n because $p \neq 0$.

Now, let

$$b := \sum_{i=0}^{m} b_i y_1^i, \ b_i \in k[x_1, x_2, y_2],$$

be any element of C. We claim that $b_m = vp^m$ for some $v \in k[x_1, x_2]$. To prove this claim, we equate the coefficients of powers of y_1 in both sides of $ab = ba$. That gives

$$[b_m, a] = 0 \tag{3.7}$$

and

$$[b_{m-i}, a] + \sum_{j=0}^{i-1} \binom{m-j}{i-j} b_{m-j} \frac{\partial^{i-j} a}{\partial x_1^{i-j}} = 0, \ i = 1, \ldots, m. \tag{3.8}$$

We note that since $a \in k[x_1, x_2]$, for every $c = \sum_{i=0}^{r} c_i y_2^i, \ c_i \in k[x_1, x_2]$, we have $[c, a] = rc_r \frac{\partial a}{\partial x_2} y_2^{r-1} + \cdots$. Thus $\deg[c, a] = \deg c - 1$ if $\deg c \geq 1$, where deg is with respect to y_2. So (3.7) and (3.8) imply that $\deg b_{m-i} = i, \ i = 0, \ldots, m$. In particular $b_m \in k[x_1, x_2]$ and hence

$$C \cap k[x_1, x_2, y_2] = k[x_1, x_2]. \tag{3.9}$$

Now, let

$$b_{m-i} = \sum_{r=0}^{i} \beta_{r,m-i} y_2^r, \ i = 0, \ldots, m, \ \beta_{r,m-i} \in k[x_1, x_2]. \tag{3.10}$$

Given $i = 1, \ldots, m$, the coefficient of y_2^{i-1} on the left-hand side of (3.8) is the sum of the coefficients of y_2^{i-1} in $[b_{m-i}, a]$ and $(m - i + 1) b_{m-i+1} \frac{\partial a}{\partial x_1}$. Thus applying (3.10) gives

$$i\beta_{i,m-i} \frac{\partial a}{\partial x_2} + (m - i + 1)\beta_{i-1,m-i+1} \frac{\partial a}{\partial x_1} = 0, \ i = 1, \ldots, m. \tag{3.11}$$

Multiplying both sides of (3.11) by q and applying the identity $p \frac{\partial a}{\partial x_1} = q \frac{\partial a}{\partial x_2}$ gives

$$i\beta_{i,m-i} p + (m - i + 1)\beta_{i-1,m-i+1} q = 0, \ i = 1, \ldots, m. \tag{3.12}$$

It follows from (3.12) that

$$\beta_{0,m} q^m = (-1)^m \beta_{m,0} p^m$$

and hence p^m divides $\beta_{0,m} = b_m$ because $\gcd(p, q) = 1$. So we have proved the claim.

We are now ready to prove that $b \in k[x_1, x_2, u]$. Let $b_m = vp^m$, where v is some element of $k[x_1, x_2]$. Since $(py_1)^m = p^m y_1^m + w_0$ for some $w_0 \in A_2(k)$ whose degree with respect to y_1 is less than m, we have

$$u^m = (py_1 - qy_2)^m = p^m y_1^m + u_0$$

for some $u_0 \in A_2(k)$ whose degree with respect to y_1 is less than m. Hence

$$b = \sum_{i=0}^{m} b_i y_1^i = vp^m y_1^m + \sum_{i=0}^{m-1} b_i y_1^i = vu^m + u_1$$

for some $u_1 \in A_2(k)$ whose degree with respect to y_1 is less than m. Since $u_1 = b - vu^m \in C$, an induction together with (3.9) show that $b \in k[x_1, x_2, u]$ and so $C = k[x_1, x_2, u]$. □

It is not true that every centralizer of GK dimension three in $A_2(k)$ contains a subalgebra isomorphic to $A_1(k)$. Theorem 3.2.12 gives the following counter-example.

Example 3.2.13. The algebra $C := C(x_1x_2; A_2(k))$ does not contain any subalgebra isomorphic to $A_1(k)$ but it contains a subalgebra of GK dimension two which is isomorphic to some subalgebra of $A_1(k)$.

Proof. By Theorem 3.2.12, $C = k[x_1, x_2, u]$ where $u := x_1y_1 - x_2y_2$. For every $v \in k[x_1, x_2]$ let $\delta(v) := [u, v] = x_1 \frac{\partial v}{\partial x_1} - x_2 \frac{\partial v}{\partial x_2}$. Let $b := \sum_{i=0}^{n} b_i u^i$ and $c = \sum_{j=0}^{m} c_j u^j$, where $b_i, c_j \in k[x_1, x_2]$, be two elements of C. Then the constant term of $[b, c]$, with respect to u, is

$$v_0 = \sum_{i=1}^{n} b_i \delta^i(c_0) - \sum_{j=1}^{m} c_j \delta^j(b_0).$$

Clearly $v_0 \neq 1$ because the constant term of $\delta^r(v)$, with respect to x_1, x_2, is zero for all $v \in k[x_1, x_2]$ and all integers $r \geq 1$.

For the second part, consider the subalgebra $k[x_1, u]$ of C. Since $[u, x_1] = x_1$, we can embed $k[x_1, u]$ into $A_1(k) = \frac{k\langle s,t \rangle}{(ts - st - 1)}$ via the map $x_1 \mapsto s$ and $u \mapsto st$. □

We now give a necessary condition for a centralizer in $A_2(k)$ to contain a simple subalgebra $B \neq k$. We first need two lemmas.

Lemma 3.2.14. ([37, Proposition 3.12]) *Let A and B be k-algebras. If* $\text{GKdim}(B) \leq 2$, *then* $\text{GKdim}(A \otimes_k B) = \text{GKdim}(A) + \text{GKdim}(B)$.

Lemma 3.2.15. *Let A be a k-algebra and let B be a central simple k-subalgebra of A. If C is a k-subalgebra of $C(B; A)$, then $B \otimes_k C \cong BC$.*

Proof. Define the k-algebra homomorphism $\varphi : B \otimes_k C \longrightarrow BC$ by $\varphi(b \otimes_k c) = bc$, $b \in B, c \in C$, which is clearly surjective. Suppose that $\ker \varphi \neq (0)$ and let n be the smallest integer for which there exist nonzero elements $b_i \in B$ and k-linearly independent elements

$c_i \in C$ such that $\sum_{i=1}^{n} b_i c_i = 0$. Since B is simple, there exist $x_j, y_j \in B$ and integer m such that $\sum_{j=1}^{m} x_j b_1 y_j = 1$. For each i let $b_i' := \sum_{j=1}^{m} x_j b_i y_j$. Then

$$c_1 + \sum_{i=2}^{n} b_i' c_i = 0. \tag{3.13}$$

Now let $b \in B$. Then (3.13) gives $\sum_{i=2}^{n} (bb_i' - b_i' b) c_i = 0$ and so $bb_i' = b_i' b$ for all i, by the minimality of n. Therefore $b_i' \in Z(B) = k$ for all i and hence, by (3.13), the elements $c_1, \ldots, c_n$ are k-linearly dependent, contradiction! □

Proposition 3.2.16. *Let $a \in A_2(k) \setminus k$ and $C := C(a; A_2(k))$. If $B \neq k$ is a simple subalgebra of C, then* $\text{GKdim}(C) = 3$, $\text{GKdim}(B) = 2$ *and $J \cap k[z] \neq (0)$ for all nonzero ideals J of C and all $z \in Z(C) \setminus k$.*

Proof. If $\text{GKdim}(C) \neq 3$, then C is commutative, by Proposition 3.2.4, and hence B is a field which gives the contradiction $B = k$. Thus $\text{GKdim}(C) = 3$ and so

$$3 = \text{GKdim}(C) \geq \text{GKdim}(k[a]B) = \text{GKdim}(k[a]) + \text{GKdim}(B) = 1 + \text{GKdim}(B),$$

by Lemma 3.2.15 and Lemma 3.2.14. Hence $\text{GKdim}(B) \leq 2$. If $\text{GKdim}(B) \leq 1$, then B is a field, by Proposition 1.6.32, and so we have the contradiction $B = k$. Therefore $\text{GKdim}(B) > 1$ and so $\text{GKdim}(B) = 2$ by Theorem 1.6.15.

For the second part, suppose to the contrary that $J \cap k[z] = (0)$ for some nonzero ideal J of C and some $z \in Z(C) \setminus k$. Since B is simple, we have $B \cap J = (0)$ and so the natural homomorphisms $k[z] \longrightarrow C/J$ and $B \longrightarrow C/J$ are injective. So we may assume that C/J contains $k[z]B$. By Proposition 1.6.6,

$$\text{GKdim}(k[z]B) \leq \text{GKdim}(C/J) \leq \text{GKdim}(C) - 1 = 2$$

and hence, by Lemma 3.2.15 and Lemma 3.2.14,

$$2 \geq \text{GKdim}(k[z]B) = \text{GKdim}(k[z]) + \text{GKdim}(B) = 3,$$

which is absurd. □

3.3 Centralizers in Skew Laurent Polynomial Algebras

Let R be a ring and let $\alpha \in \text{Aut}(R)$. We defined the skew polynomial ring $S = R[x; \alpha]$ in Definition 2.2.2. Let

$$X = \{1, x, x^2, \cdots\}.$$

Clearly X is a multiplicatively closed subset of S. The claim is that X is a denominator set and thus the localization of S at X exists. It is clear that all elements of X are regular. So we only need to show that X is Ore. Let $f = \sum_{i=0}^{n} r_i x^i \in S$ and $x^m \in X$. Let $g = \sum_{i=0}^{n} \sigma^m(r_i)x^i$ and $h = \sum_{i=0}^{n} \sigma^{-m}(r_i)x^i$. Then $x^m f = gx^m$ and $fx^m = x^m h$. Thus X is Ore and hence $X^{-1}S = SX^{-1}$ exists. It is clear that an element of SX^{-1} is in the form $\sum_{i=m}^{n} r_i x^i$ where $m \leq n$ are integers and $r_i \in R$. Of course, we still have the right multiplication rule, i.e. $xr = \sigma(r)x$ because SX^{-1} contains $R[x; \sigma]$. Since σ is an automorphism, it is invertible and so the right multiplication rule implies that $x^n r = \sigma^n(r)x^n$ for all $n \in \mathbb{Z}$. We call SX^{-1} a *skew Laurent polynomial ring* over R and we write $SX^{-1} = R[x, x^{-1}; \sigma]$. Note that if R is a domain, then $R[x, x^{-1}; \sigma]$ is a domain too.

Now, let K be a field and $\sigma \in \mathrm{Aut}(K)$. An argument similar to the standard proof of the Hilbert basis theorem shows that $B := K[x, x^{-1}; \sigma]$ is noetherian [48, Theorem 1.4.5]. Therefore B is an Ore domain, by Proposition 1.2.13, and hence it has a quotient division algebra. In this section, the goal is to show that if the fixed field k of σ is algebraically closed and if A is a k-subalgebra of B, then for every $f \in A \setminus K$, the centralizer $C(f; A)$ is a commutative algebra and a free module of finite rank over a polynomial algebra $k[u]$ for some $u \in C(f; A)$. But first we give three examples of algebras which are subalgebras of B for some field K.

Quantum planes. Let k be a field. Recall that the quantum plane $k_q[x, y]$ is the k-algebra generated by x, y subject to the relation $yx = qxy$ where $0 \neq q \in k$. Let $B = k(u)[v, v^{-1}; \sigma]$ where the automorphism σ is defined by $\sigma(u) = qu$. The algebra $k_q[x, y]$ has the obvious embedding $x \mapsto u,\ y \mapsto v$, into B.

The first Weyl algebra. Let k be a field of characteristic zero. Recall that the first Weyl algebra $A_1(k)$ is the k-algebra generated by x, y subject to the relation $yx = xy + 1$. Let $B = k(u)[v, v^{-1}; \sigma]$ where the automorphism σ is defined by $\sigma(u) = u + 1$. Define

$$\varphi : k\langle x, y\rangle \longrightarrow B$$

by $\varphi(x) = v^{-1}u$ and $\varphi(y) = v$ and then extend φ homomorphically to $k\langle x, y\rangle$. Then

$$\varphi(yx - xy - 1) = u - v^{-1}uv - 1 = u - \sigma^{-1}(u)v^{-1}v - 1 = u - (u - 1) - 1 = 0.$$

Thus $(yx - xy - 1) \subseteq \ker \varphi$ and so φ induces a k-algebra homomorphism

$$A_1(k) \cong \frac{k\langle x, y\rangle}{(yx - xy - 1)} \longrightarrow B.$$

The above homomorphism is injective because $A_1(k)$ is simple.

Finitely generated graded algebras of GK *dimension two.* Let k be a field and let A be a k-algebra. Suppose that there exist k-vector subspaces $A_i,\ i \geq 0$ of A such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$, and $A = \bigoplus_{i \geq 0} A_i$, as k-vector spaces. The algebra A is called a *graded k-algebra.* If $A_0 = k$, then A is called *connected.* An element $a \in A$ is called *homogeneous* if $a \in A_i$ for some $i \geq 0$. If $0 \neq a \in A$, then $a = \sum_{i=0}^n a_i$ for some $a_i \in A_i$ with $a_n \neq 0$. Now suppose that A is a domain of finite GK dimension. Then A is an Ore domain by Corollary 1.6.9. Let $Q(A)$ be the quotient division algebra of A. There is another quotient ring related to A that we are now going to define. Let

$$S = \bigcup_{i \geq 0} A_i \setminus \{0\},$$

which is clearly multiplicatively closed. The claim is that S is Ore. Suppose first that $a, s \in S$. Then, since A is Ore, there exist $b, c \in A$ such that $ba = cs$. Let $b = \sum_{i=0}^r b_i, c = \sum_{j=0}^m c_j$, where $b_i \in A_i, c_j \in A_j$ and $b_r c_m \neq 0$. Then $ba = cs$ implies that $b_r a = c_m s$ and so $As \cap Ss \neq (0)$. For the general case, let $0 \neq a = \sum_{i=0}^n a_i \in A,\ a_n \neq 0$ and $s \in S$. By what we have just proved, there exists $s_0 \in S$ such that $s_0 a_0 \in As$. In general, for every $0 \leq i \leq n$ there exists some $s_i \in S$ such that $s_i s_{i-1} \cdots s_1 s_0 a_i \in As$. So if we let $t = s_n s_{n-1} \cdots s_0 \in S$, then $ta \in As$ and hence $As \cap Sa \neq (0)$.

Therefore we can localize A at S and we call $S^{-1}A$ the *graded quotient ring* of A. Let $Q_{\mathrm{gr}}(A)$ denote the graded quotient ring of A. Clearly $A \subseteq Q_{\mathrm{gr}}(A) \subseteq Q(A)$. Now let

$$D = \{a^{-1}b : \ a, b \in A_n, a \neq 0,\ n \geq 0\}.$$

It is easy to see that D is a division k-subalgebra of $Q_{\mathrm{gr}}(A)$ and obviously $A_0 \subseteq D$. Choose and fix an element $0 \neq c \in A_1$ and let $x = c^{-1}$. If $d = a^{-1}b \in D$, then $xdx^{-1} = (ac)^{-1}bc \in D$ and so we have $\sigma \in \mathrm{Aut}(D)$ defined by

$$\sigma(d) = xdx^{-1}$$

and so $xd = \sigma(d)x$ for all $d \in D$. Note that if $\alpha \in k$, then $\sigma(\alpha d) = \alpha\sigma(d)$ because k is in the center of A. Now let $q = a^{-1}b$, where $0 \neq a \in A_m, b \in A_n$. If $n \geq m$, then $c^{n-m}a \in A_n$ and hence $(ac^{n-m})^{-1}b \in D$. Thus $q \in Dx^{m-n}$. If $m \geq n$, then $a^{-1}bc^{m-n} \in D$ and hence $q \in Dx^{m-n}$. So we have proved that $Q_{\mathrm{gr}}(A) = D[x, x^{-1}; \sigma]$. It is known that if k is algebraically closed and A is a connected finitely generated graded algebra of GK dimension two, then D is a field [8] and so $Q_{\mathrm{gr}}(A) = K[x, x^{-1}; \sigma]$ for some field K.

Assumption. For the rest of this section, K is a field and $\sigma \in \mathrm{Aut(K)}$. We assume that k is the fixed field of σ and that k is algebraically closed. If $K = k$, then $K[x, x^{-1}; \sigma]$ is commutative and so not interesting for our purpose. So we assume that $K \neq k$. An element $g \in K[x, x^{-1}; \sigma]$ is in the form $g = \sum_{i=m}^{n} a_i x^i,\ a_i,\ m, n \in \mathbb{Z},\ m \leq n$. We assume that $a_m a_n \neq 0$.

Definition 3.3.1. If $g = \sum_{i=m}^{n} a_i x^i \in K[x, x^{-1}; \sigma]$, then the integers $o(g) := m$ and $\deg g := n$ are called the *order* and the *degree* of g, respectively.

We are now going to prove our main result in this section. We begin with a lemma.

Lemma 3.3.2. *If $\sigma^m(c) = c$ for some $c \in K$ and some integer $m \neq 0$, then $c \in k$.*

Proof. Clearly we may assume that $m > 0$. Let

$$p(x) = \prod_{i=0}^{m-1} (x - \sigma^i(c)) \in K[x].$$

Then $p(c) = 0$ and each coefficient of $p(x)$ is invariant under σ. Thus $p(x) \in k[x]$ and so c is algebraic over k. Hence $c \in k$ because k is algebraically closed. □

It is easy to see that the center of $K[x, x^{-1}; \sigma]$ is k. A nontrivial result is the following.

Corollary 3.3.3. *The center of the quotient division algebra of $K[x, x^{-1}; \sigma]$ is k.*

Proof. Let Q be the quotient division algebra of $K[x, x^{-1}; \sigma]$ and let $0 \neq q = a^{-1}b \in Z(Q)$, the center of Q. Let

$$a = \sum_{i=m}^{n} a_i x^i,\ \ b = \sum_{i=r}^{s} b_i x^i,\ \ a_n b_s \neq 0.$$

Since $q \in Z(Q)$, we have $paq = qpa$ for all $p \in Q$ and thus $apb = bpa$. Let $p_1 = x^u,\ u \in \mathbb{Z}$ and $p_2 = \gamma \in K$. Then equating the coefficients of monomials of highest degree in both sides of $ap_1b = bp_1a$ and $ap_2b = bp_2a$ gives

$$a_n \sigma^{n+u}(b_s) = b_s \sigma^{s+u}(a_n),\ \ a_n \sigma^n(\gamma b_s) = b_s \sigma^s(\gamma a_n). \tag{3.14}$$

We claim that $n = s$ and $b_n = \alpha a_n$ for some $\alpha \in k$. So suppose that $n \neq s$ and let $u = -s$. Then the first equation in (3.14) gives $\sigma^{n-s}(b_s) = b_s$ and, since $n - s \neq 0$, we have $b_s \in k$ by Lemma 3.3.2. The same equation then gives $\sigma^{s+u}(a_n) = a_n$ for all integers u and hence

$a_n \in k$ by Lemma 3.3.2. But then the second equation in (3.14) becomes $\sigma^{n-s}(\gamma) = \gamma$ for all $\gamma \in K$. Thus, since $n - s \neq 0$, Lemma 3.3.2 gives the contradiction $K = k$. So $n = s$.

Now, the first equation in (3.14), with $u = 1 - n$, gives $\sigma(a_n^{-1}b_n) = a_n^{-1}b_n$ and hence $a_n^{-1}b_n \in k$. So $b_n = \alpha a_n$ for some $\alpha \in k$. We now show that $q = \alpha$, which completes the proof. Suppose, to the contrary, that $q - \alpha \neq 0$. Clearly $q - \alpha \in Z(Q)$ because $q, \alpha \in Z(Q)$. Thus $a^{-1}(b - \alpha a) = q - \alpha \in Z(Q)$ and so $\deg a = \deg(b - \alpha a)$. But this is not possible because $b_n = \alpha a_n$ and so $\deg(b - \alpha a) < n = \deg a$. □

Corollary 3.3.4. *Let A be a k-subalgebra of $K[x, x^{-1}; \sigma]$ and let $f = \sum_{i=m}^{n} a_i x^i \in A \setminus k$. Let $g = \sum_{i=r}^{s} b_i x^i$ and $h = \sum_{i=t}^{u} c_i x^i$ be two elements of $C(f; A)$. If $r = t$ and either $m \neq 0$ or $r \neq 0$, then $b_r c_r^{-1} \in k$. Similarly, if $s = u$ and either $n \neq 0$ or $s \neq 0$, then $b_s c_s^{-1} \in k$.*

Proof. We only prove the first part because the proof of the second part is similar. Equating The coefficients of x^{m+r} in both sides of $fg = gf$ and $fh = hf$ gives

$$a_m \sigma^m(b_r) = b_r \sigma^r(a_m)$$

and

$$a_m \sigma^m(c_r) = c_r \sigma^r(a_m).$$

Therefore $b_r c_r^{-1} = \sigma^m(b_r c_r^{-1})$. Thus, by Lemma 3.3.2, either $m = 0$ or $b_r c_r^{-1} \in k$. But if $m = 0$, then $r \neq 0$ and $\sigma^r(a_0) = a_0$. Hence $a_0 \in k$ and so we can replace f with $f - a_0$ because $C(f; A) = C(f - a_0; A)$. □

Lemma 3.3.5. *Let F be an algebraically closed field and let A be a domain and an F-algebra. If A is a finite module over some polynomial algebra $F[a]$, $a \in A$, then A is commutative and a free module of finite rank over $F[a]$.*

Proof. By Proposition 1.6.23, $\operatorname{GKdim}(A) = \operatorname{GKdim}(F[a]) = 1$ and hence A is commutative by Proposition 1.6.32. Since A is a domain, it is torsion free as $F[a]$-module and the result now follows from the fundamental theorem for finitely generated modules over a principal ideal domain. □

Lemma 3.3.6. *Let A be a k-subalgebra of $K[x, x^{-1}; \sigma]$ and let $f \in A \setminus K$. Let $C := C(f; A)$ and $C_0 := C \cap K[x; \sigma]$. Then $C \cap K = k$ and if $u \in C_0$ with $\deg u \geq 1$, then C_0 is commutative and a free module of finite rank over $k[u]$.*

Proof. Let $c \in C \cap K$ and let $f = \sum_{i=m}^{n} a_i x^i$. Then $fc = cf$ gives $\sigma^m(c) = \sigma^n(c) = c$. Thus, by Lemma 3.3.2, either $m = n = 0$, which is not possible because $f \notin K$, or $c \in k$. Hence $C \cap K = k$.

Now, for any remainder i modulo $m = \deg u$ let $u_i \in C_0$, if it exists, be such that $\deg u_i \equiv i \mod m$ and $\deg u_i$ is as small as possible. We choose $u_0 = 1$. Now let $v \in C_0$. If $\deg v = 0$, then $v \in C \cap K = k$. If $\deg v > 0$, then $\deg v \equiv \deg u_i \mod n$ for some i and so $\deg v - \deg u_i = rm$ for some integer $r \geq 0$. Hence we can apply Corollary 3.3.4 to get some $\alpha \in k$ such that $\deg(v - \alpha u^r u_i) < \deg u$. We can continue this process to eventually have $C_0 = \sum k[u]u_i$. The result now follows from Lemma 3.3.5. $\square$

We are now ready to prove the main result of this section. Let A be a subalgebra of $K[x, x^{-1}; \sigma]$ and let $f \in A \setminus k$. Suppose that $g(x) = \sum_{i=m}^{n} a_i x^i$ and $h(x) = \sum_{i=r}^{s} b_i x^i$ are in $C(f; A)$. The crucial step in the proof of Theorem 3.3.7 is to show that if $m = r$ (resp. $n = s$), then $a_m b_m^{-1} \in k$ (resp. $a_n b_n^{-1} \in k$). After proving this, we look at the degree and the order of elements of $C(f; A)$ modulo the degree and the order of f.

Theorem 3.3.7. *Let A be a subalgebra of $K[x, x^{-1}; \sigma]$. Let $f = \sum_{i=m}^{n} a_i x^i \in A \setminus k$ and $C := C(f; A)$. If $f \in K$, then $C = A \cap K$. If $f \notin K$, then C is commutative and a free module of finite rank over $k[u]$ for some $u \in C$.*

Proof. If $f = a_0 \in K$ and $g = \sum_{i=r}^{s} b_i x^i \in C$, then $fg = gf$ gives $\sigma^r(a_0) = \sigma^s(a_0) = a_0$ and hence, since $f \notin k$, we must have $r = s = 0$ by Lemma 3.3.2. Thus $g \in K$ and so $C = A \cap K$. Now, suppose that $f \notin K$. We consider two cases.
Case 1. $C \cap K[x; \sigma] = C$: this case follows from Lemma 3.3.6.
Case 2. $C \cap K[x; \sigma] \neq C$: so there exists some $h \in C$ such that $o(h) = p < 0$. For any remainder i modulo p let $h_i \in C$, if it exists of course, be such that $o(h_i) \leq 0$, $o(h_i) \equiv i \mod p$ and $o(h_i)$ is as large as possible. We choose $h_0 = 1$. We claim that

$$C = \sum k[h]h_j + C \cap K[x; \sigma]. \tag{3.15}$$

To prove the claim, let $w \in C$. If $o(w) = 0$, then $w \in C \cap K[x; \sigma]$. If $o(w) < 0$, then $o(w) \equiv o(h_i) \mod p$ for some i. Let $o(w) - o(h_j) = tp$. Then $t \geq 0$ and, since $o(w) \neq 0$, there exists some $\beta \in k$ such that $o(w - \beta h^t h_j) > o(w)$ by Corollary 3.3.4. Continuing in this manner, we will eventually get (3.15). Now, if $C \cap K[x; \sigma] = k$, then $C = \sum k[h]h_j$, because $h_0 = 1$ and so $k \subseteq k[h]h_0$, and we are done by Lemma 3.3.5.

So we may assume that $C \cap K[x;\sigma] \neq k$ and hence we can choose

$$g = b_0 + \cdots + b_s x^s \in C \cap K[x;\sigma]$$

with $s \geq 1$. Suppose that $b_0 \neq 0$. If $m = o(f) < 0$, then $gf = fg$ implies that $a_m \sigma^m(b_0) = a_m b_0$ and thus $\sigma^m(b_0) = b_0$. Therefore, by Lemma 3.3.2, $b_0 \in k$. So $g - b_0 \in C$ and $o(g - b_0) \geq 1$. If $m = 0$, then $hf = fh$ implies $a_0 \in k$. So $f - a_0 \in C$ and $o(f - a_0) \geq 1$. Thus $C \cap K[x;\sigma]$ contains an element of order at least one. So we may assume that $b_0 = 0$. By Lemma 3.3.6,

$$C \cap K[x;\sigma] = \sum k[g]g_i \tag{3.16}$$

for some $g_i \in C \cap K[x;\sigma]$, and $C \cap K[x;\sigma]$ is commutative. Now, let $z \in C$. Since $o(g) \geq 1$, there exists an integer $\ell \geq 0$ such that

$$zg^{\ell} \in C \cap K[x;\sigma] = \sum k[g]g_j.$$

Therefore, in the quotient division algebra of $K[x, x^{-1};\sigma]$, we have $z \in \sum k[g, g^{-1}]g_i$ and hence

$$C \subseteq \sum k[g, g^{-1}]g_i.$$

Thus C is commutative and finitely generated, as a k-algebra, by (3.15) and (3.16). Also, $\mathrm{GKdim}(C) = 1$ by Corollary 1.6.27 and Proposition 1.6.23. Thus C is a finite module over $k[u]$ for some $u \in C$ by Theorem 1.6.24 or Proposition 1.6.30. Now, Lemma 3.3.5 completes the proof the theorem. □

Remark 3.3.8. The element u in Theorem 3.3.7 is not always f. It is easy to see that $C := C(x; K[x, x^{-1};\sigma]) = k[x, x^{-1}]$. Let $f_1, \ldots, f_n \in C$. Clearly the set

$$\{o(g) : \ g \in \sum_{i=1}^{n} k[x]f_i\}$$

is bounded from below and thus $C \neq \sum_{i=1}^{n} k[x]f_i$. However, if we choose $u = x + x^{-1}$, then an induction shows that $x^n \in k[u]x + k[u]$ for all $n \in \mathbb{Z}$ and hence $C = k[u]x + k[u]$.

In fact, in the above remark, the set $\{1, x\}$ is a basis for the $k[u]$-module C. To see this, suppose that $1, x$ are $k[u]$-linearly dependent. Then $fx = g$ for some $f, g \in k[u]$. Let $f = \sum_{i=0}^{n} \alpha_i (x + x^{-1})^i$ and $g = \sum_{i=0}^{m} \beta_i (x + x^{-1})^i$, where $\alpha_i, \beta_i \in k$ and $\alpha_n \beta_m \neq 0$. If $n \geq m$, then multiplying $fx = g$ by x^n and equating the x-degree of both sides gives

$m = n + 1$, which is false. If $n \le m$, then multiplying $fx = g$ by x^m and equating the constant coefficients of both sides gives $\beta_m = 0$, which is false again. Thus the set $\{1, x\}$ is $k[u]$-linearly independent.

Remark 3.3.9. The result in Theorem 3.3.7 does not hold if k is not algebraically closed even if $|\sigma| = \infty$. To see this, let p_n be the n-th prime number and let $\zeta_n := \exp(2\pi i/p_n)$, the p_n-th primitive root of unity. Let $K = \mathbb{Q}(\zeta_1, \zeta_2, \ldots)$. The set

$$\{\zeta_{n_1}^{\alpha_1}\zeta_{n_2}^{\alpha_2}\cdots\zeta_{n_s}^{\alpha_s},\ s \ge 1,\ 0 \le \alpha_i \le p_{n_i} - 1\}$$

is a $\mathbb{Q}$-basis for K. Define the $\mathbb{Q}$-automorphism $\sigma \in \mathrm{Aut}(K)$ by $\sigma(\zeta_n) = \zeta_n^2,\ n \ge 1$. Clearly $|\sigma| = \infty$ and the fixed field k of σ is not algebraically closed. Now, in $K[x, x^{-1}; \sigma]$, the elements ζ_2 and x do not commute with each other but they both commute with x^2. So the centralizer of x^2 is not commutative. Also, note that x^2 is not central because, for example, it does not commute with ζ_3.

3.4 Problems

Let k be a field of characteristic zero. By Amitsur's theorem, the centralizer of every element of $A_1(k)$ is a finitely generated algebra. We proved in the second section of this chapter that if $a \in A_2(k)$ and if at most two generators of four generators x_1, x_2, y_1, y_2 of $A_2(k)$ appear in a, then the centralizer of a in $A_2(k)$ is a finitely generated k-algebra. It is natural to ask the following question.

Question 1. *Is it true that the centralizer of every element of $A_2(k)$ is a finitely generated k-algebra?*

There is an interesting connection between the problem of finite generation of centralizers in $A_2(k)$ and a weak version of Dixmier's Fourth Problem [22], which is still open [11]. We first recall the notions of a filtered algebra and its associated graded algebra. This is basically a generalization of the notion of a graded algebra. Let A be a k-algebra and suppose that there exists a sequence $B_0 \subseteq B_1 \subseteq \cdots$ of k-subspaces of A such that $A = \bigcup_{i \ge 0} B_i$ and $B_iB_j \subseteq B_{i+j}$ for all i, j. The algebra A is called a *filtered algebra*. Now define the k-vector space of $\mathrm{gr}(A) := \bigoplus_{i \ge 0} C_i$, where $C_i = B_i/B_{i-1}$ for all $i \ge 0$ and $B_{-1} := (0)$. The multiplication in $\mathrm{gr}(A)$ is defined by $(u+B_{i-1})(v+B_{j-1}) = uv+B_{i+j-1}$ for all $u \in B_i, v \in B_j$ and all i, j. It is easy to see that this multiplication is well-defined and it gives $\mathrm{gr}(A)$ the structure of a graded algebra. The algebra $\mathrm{gr}(A)$ is called *the associated graded algebra of A*.

Now let $a \in A_1(k)$ and let $\delta : A_1(k) \to A_1(k)$ be the map defined by $\delta(u) = [u, a]$ for every $u \in A_1(k)$. For any integer $i \geq 0$ let $N(a, i) := \ker \delta^{i+1}$. Let $N(a) := \bigcup_{i \geq 0} N(a, i)$. It is easy to see that $C(a; A_1(k)) = N(a, 0) \subseteq N(a, 1) \subseteq \cdots$ and $N(a, i)N(a, j) \subseteq N(a, i + j)$ for all i, j. So $N(a)$ is a filtered algebra.

Dixmier's Fourth Problem. Is $\mathrm{gr}(N(a))$ a finitely generated k-algebra?

We note that if the answer to Dixmier's Fourth Problem is positive, then $N(a)$ would be finitely generated too. So the problem of finite generation of $N(a)$ is a weak version of Dixmier's Fourth Problem. V. Bavula [9] proved that if a is a homogeneous element of $A_1(k)$, then $N(a)$ is finitely generated. The general case is still open.

The connection between this problem and the problem of finite generation of centralizers in $A_2(k)$ is given in the following proposition.

Proposition 3.4.1. *Let $a = y_1 + a_1 \in A_2(k)$, where $a_1 \in k[x_2, y_2] \cong A_1(k)$. Let $B = C(a; k[x_1, x_2, y_2])$. Then $C(a; A_2(k)) = B[a]$ and $B \cong N(a_1)$. Hence $C(a; A_2(k))$ is finitely generated if and only if $N(a_1)$ is finitely generated.*

Proof. Let $c = \sum_{i=0}^{m} c_i y_1^i \in A_2(k)$ where $c_i \in k[x_1, x_2, y_2]$. Let δ be the map $k[x_2, y_2] \to k[x_2, y_2]$ defined by $\delta(u) = [u, a_1]$ for every $u \in k[x_2, y_2]$. Then $c \in C(a; A_2(k))$ if and only if $c_i \in B$, for all i, because $y_1 \in C(a; A_2(k))$. Thus $C(a; A_2(k)) = B[y_1] = B[a]$. We are now going to prove that $B \cong N(a_1)$. Let $b = \sum_{j=0}^{r} \beta_j x_1^j \in k[x_1, x_2, y_2]$, where $\beta_j \in k[x_2, y_2]$. Then $b \in B$ if and only if $[b, a] = 0$ if and only if $\frac{\partial b}{\partial x_1} = [b, a_1]$ if and only if $\sum_{j=0}^{r} j\beta_j x_1^{j-1} = \sum_{j=0}^{r} \delta(\beta_j) x_1^j$ if and only if $\delta(\beta_r) = 0$ and $\delta(\beta_j) = (j + 1)\beta_{j+1}$ for all $j \leq r - 1$. It now follows that $b \in B$ if and only if $\beta_j = \frac{1}{j!}\delta^j(\beta_0)$ for all j and $\delta^{r+1}(\beta_0) = 0$. So $b = \sum_{j=0}^{r} \frac{1}{j!}\delta^j(\beta_0)x_1^j$ where β_0 is any element of $\ker \delta^{r+1}$.

We now define the map $\varphi : B \to N(a_1)$ by

$$\varphi\left(\sum_{j=0}^{r} \frac{1}{j!}\delta^j(\beta_0)x_1^j\right) = \beta_0$$

and we claim that φ is a k-algebra isomorphism. Let

$$b_1 = \sum_{j=0}^{r} \frac{1}{j!}\delta^j(\beta_0)x_1^j, \ b_2 = \sum_{j=0}^{s} \frac{1}{j!}\delta^j(\gamma_0)x_1^j$$

be two elements of B with $r \leq s$. We first show that φ is k-linear. Since $\delta^{r+1}(\beta_0) = 0$, we have $\delta^j(\beta_0) = 0$ for all $j > r$ and hence $b_1 = \sum_{j=0}^{s} \frac{1}{j!}\delta^j(\beta_0)x_1^j$. Let $\alpha \in k$. We have $\delta^{s+1}(\beta_0 + \alpha\gamma_0) = 0$. Thus

$$\varphi(b_1 + \alpha b_2) = \varphi\left(\sum_{j=0}^{s} \frac{1}{j!}\delta^j(\beta_0 + \alpha\gamma_0)x_1^j\right) = \beta_0 + \alpha\gamma_0 = \varphi(b_1) + \alpha\varphi(b_2).$$

Clearly φ is one-to-one and onto.

So we only need to show that $\varphi(b_1b_2) = \varphi(b_1)\varphi(b_2)$. Since δ is a derivation, we have $\delta^t(\beta_0\gamma_0) = \sum_{i+j=t} \binom{t}{i}\delta^i(\beta_0)\delta^j(\gamma_0)$. Hence $\delta^{r+s+1}(\beta_0\gamma_0) = 0$ and

$$\frac{1}{t!}\delta^t(\beta_0\gamma_0) = \sum_{i+j=t} \frac{1}{i!j!}\delta^i(\beta_0)\delta^j(\gamma_0)$$

On the other hand,

$$b_1b_2 = \sum_{t=0}^{r+s} \sum_{i+j=t} \frac{1}{i!j!}\delta^i(\beta_0)\delta^j(\gamma_0)x_1^t.$$

Thus $b_1b_2 = \sum_{t=0}^{r+s} \frac{1}{t!}\delta^t(\beta_0\gamma_0)x_1^t$ and hence $\varphi(b_1b_2) = \beta_0\gamma_0 = \varphi(b_1)\varphi(b_2)$. □

Question 2. *Let k be a field of characteristic zero. Is it true that* $\mathrm{GKdim}(C(a; A_n(k)))$ *is an integer for all n and all $a \in A_n(k)$?*

The last question is related to Theorem 3.2.12 and Example 3.2.13.

Question 3. *Is it true that if $a \in k[x_1, x_2]$, then $C(a; A_2(k))$ always contains a subalgebra of* GK *dimension two which is isomorphic to some subalgebra of $A_1(k)$?*

Chapter 4

Division Rings that are Left Algebraic over a Subfield

4.1 Introduction

Kurosch [38], see also [56, Problem 6.2.6] asked whether or not an algebra that is both finitely generated and algebraic over a field k is necessarily finite-dimensional over k. Kurosch's problem is a ring-theoretic analogue of Burnside's problem for groups. Both problem were shown to have a negative answer by Golod and Shafarevich [31]. In fact, Golod [30] used their construction to give an example of a finitely generated infinite group G with the property that every element in G has finite order, giving a negative answer to Burnside's problem. As Rowen [56, p. 116] points out, there are two special cases of Kurosch's problem: the case that the algebra we consider is a division ring and the case that it is a nil ring.

Many examples of finitely generated algebraic algebras that are not finite-dimensional over their base fields now exist [12, 63, 61, 62, 42]. The strangest of these examples are due to Smoktunowicz, who showed a simple nil algebra (without 1) exists [61] and that there is a nil algebra (without 1) with the property that the polynomial ring over it contains a free algebra on two generators [62]. Lenagan and Smoktunowicz [42] also showed that there is a finitely generated nil algebra (without 1) that is infinite-dimensional over its base field but has finite Gelfand-Kirillov dimension [42]. Despite the large number of pathological examples of nil rings, there are no similar pathological examples of algebraic division rings. At the moment, all known examples of algebraic division rings have the property that every finitely generated subalgebra is finite-dimensional over its center. Kaplansky considered algebraic algebras that have the stronger property that there is a natural number d such that every element of the algebra is algebraic of degree at most d. With this stronger property, one avoids the pathologies that arise when one considers algebras that are algebraic. Such algebras are PI by Theorem 1.4.9 and if they are primitive, then they are finite-dimensional over their centers by Theorem 1.4.25. In fact, a primitive algebra that is finite-dimensional over its center is a matrix ring over a division ring by Theorem 1.3.4.

We consider an analogue of Kaplansky's result for division rings that are *left algebraic* over some subfield.

Definition 4.1.1. Let A be a ring and let B be a subring of A such that A is a free left B-module. We say that A is *left algebraic* over B if for every $a \in A$ there is some natural number n and some elements $\alpha_0, \ldots, \alpha_n \in B$ such that α_n is regular and

$$\sum_{j=0}^{n} \alpha_j a^j \;=\; 0.$$

The left algebraic property has been used by Bell and Rogalski in investigating the existence of free subalgebras of division rings [15]. In section 3, we give an analogue of Kaplansky's result in which we replace the algebraic property with being left algebraic over a subfield.

Theorem 4.1.2. *Let d be a natural number, let D be a division ring with center $Z(D)$, and let K be a (not necessarily central) subfield of D. If D is left algebraic of bounded degree d over K then $[D : Z(D)] \le d^2$.*

We note that the bound of d^2 in the conclusion of the statement of Theorem 4.1.2 is the best possible. For example, let k be a field and let σ be an automorphism of k with $|\sigma| = d$. Let F be the fixed field of σ. Let D be the ring of formal skew Laurent series in x, i.e. $D = \{\sum_{i=n}^{\infty} \alpha_i x^i,\ n \in \mathbb{Z}, \alpha_i \in k\}$, where we define multiplication in D by $x\alpha = \sigma(\alpha)x,\ \alpha \in k$. It is easy to see that D is a division ring and $Z(D)$ is the field of formal Laurent series in x^d over F, i.e. $Z(D) = \{\sum_{i=n}^{\infty} \alpha_i x^{di},\ n \in \mathbb{Z}, \alpha_i \in F\}$. Let F_1 be the field of formal Laurent series in x^d over k. Then $\{1, x, \ldots, x^{d-1}\}$ is a basis for D/F_1 and $[F_1 : Z(D)] = |\sigma| = d$. Thus $[D : Z(D)] = d^2$. In particular, D is algebraic over $Z(D)$ and since $x^d \in Z(D)$, every element of D is algebraic of degree at most d over $Z(D)$.

The fact that K in Theorem 4.1.2 is not necessarily central complicates matters and as a result our proof is considerably different from Kaplansky's proof. We rely heavily on combinatorial results on semigroups due to A. Shirshov [58]. Usually Shirshov's theorem is applied to finitely generated PI-algebras R. It gives that the sufficiently long words on the generators contain either a q-decomposable subword or a high power of a nontrivial subword. The existence of a multilinear polynomial identity replaces the q-decomposable subword with a linear combination of words which are lower in the degree lexicographic order and the algebra R is spanned by words which behave like monomials in a finite number of commuting variables.

In section 2, we establish a new version of Shirshov's theorem which states that the factors in the q-decomposition may be chosen to be of almost the same length. Using these combinatorial results, we are able to prove that every finitely generated subalgebra of D satisfies a polynomial identity. Then we use classical results of structure theory of PI-algebras to complete the proof of Theorem 4.1.2.

4.2 A New Version of Shirshov's Theorem

In this section, we recall some of the basic facts from combinatorics on words and use them to give a strengthening of Shirshov's theorem.

Let M be the free monoid consisting of all words over a finite alphabet $\{x_1, \ldots, x_m\}$. Let $|w|$ denote the length of $w \in M$. We put a degree lexicographic order on all words in M by declaring that

$$x_1 \succ x_2 \succ \cdots \succ x_m.$$

Given a word $w \in M$ and a natural number q, we say that w is *q-decomposable* if there exist $w_1, \ldots, w_q \in M$ such that $w = w_1 w_2 \cdots w_q$ and for all permutations $\sigma \in \mathrm{Sym}(q)$ with $\sigma \neq \mathrm{id}$ we have

$$w_1 w_2 \cdots w_q \succ w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(q)}.$$

If in addition, we can choose $w_1, \ldots, w_q$ such that $(q-1)|w_i| < |w|$ for all $i \in \{1, \ldots, q\}$, we say that w is *strongly q-decomposable*. Shirshov proved the following famous combinatorial theorem.

Theorem 4.2.1. (Shirshov, [58], see also [54, Lemma 4.2.5]) *Let m, p, and q be natural numbers and let M be the free monoid generated by m elements $x_1, \ldots, x_m$. Then there exists a positive integer $N(m, p, q)$, depending on m, p, and q, such that every word on*

$x_1, \ldots, x_m$ of length greater than $N(m,p,q)$ has either a q-decomposable subword or has a nontrivial subword of the form w^p.

By following the proof of Pirillo [49], we are able to give a strengthened version of Shirshov's theorem. We first give some of the basic background from combinatorics on words.

Let $\Sigma = \{x_1, \ldots, x_d\}$ be a finite alphabet. We say that w is a *right infinite* word over the alphabet Σ if there is some map $f : \mathbb{N} \to \Sigma$ such that

$$w = f(1)f(2)f(3)\cdots .$$

We say that v is a subword of the right infinite word w if there exist natural numbers i and j with $i < j$ such that

$$v = f(i)f(i+1)\cdots f(j).$$

We say that the right infinite word w is *uniformly recurrent* if for each subword v of w there exists some natural number $N = N(v)$ such that the word $f(i)f(i+1)\cdots f(i+N)$ contains v as a subword for all $i \geq 1$. Given $i < j$, we let $v[i,j]$ denote the subword of v that starts at position i and ends at position j.

We recall two classical results in the theory of combinatorics of words. The first one is a consequence of König's infinity lemma in graph theory [36] which gives a sufficient condition for an infinite graph to have an infinitely long path, see e.g. [3, p. 28, Exercise 41].

Theorem 4.2.2. (König) *Let Σ be a finite alphabet and let S be an infinite subset of the free monoid Σ^* generated by Σ. Then there is a right infinite word w over Σ such that every subword of w is a subword of some word in S.*

Theorem 4.2.3. (Furstenberg, [28], see also [3, p. 337, Exercise 22]) *Let Σ be a finite alphabet and let w be a right infinite word over Σ. Then there is a right infinite uniformly recurrent word u over Σ such that every subword of u is also a subword of w.*

Using these results, we are able to prove the following result.

Theorem 4.2.4. *Let m, p, and q be natural numbers and let M be a free monoid generated by m elements $x_1, \ldots, x_m$. Then there exists a positive integer $N(m,p,q)$, depending on m, p, and q, such that every word on $x_1, \ldots, x_m$ of length greater than $N(m,p,q)$ has either a strongly q-decomposable subword or has a nontrivial subword of the form t^p.*

Proof. Suppose to the contrary that there are arbitrarily long words in M that do not have a subword of the form t^p or a strongly q-decomposable subword. Clearly $q \geq 2$. Then by König's theorem there is a right infinite word w over $\{x_1, \ldots, x_m\}$ such that each finite subword v of w has the property that it does not have a subword of the form t^p or a strongly q-decomposable subword. By Furstenberg's theorem, there is a right infinite uniformly recurrent word u such that each subword of u has the property that it does not have a subword of the form t^p or a strongly q-decomposable subword. Let $\omega(n)$ denote the number of distinct subwords of u of length n. Then $\omega(n)$ is not $\mathrm{O}(1)$, since otherwise we would have u is eventually periodic and thus it would have a subword of the form t^p. Hence there is some natural number N such that there are at least q distinct subwords of u of length N. Let $w_1 \succ w_2 \succ \cdots \succ w_q$ be q such words of length N. Since $w_1, \ldots, w_q$ are uniformly recurrent in u, there is some natural number L such that $w_1, \ldots, w_q$ occur in the interval $u[i, i + L]$ for each i. Then there is an occurrence of w_1 somewhere in $u[1, 1 + L]$. We let $j_1 \in \{1, \ldots, 1 + L\}$ denote the position of the first letter of w_1 in some occurrence in $u[1, 1 + L]$. Then there is an occurrence of w_2 somewhere in $u[2Lq + 1, 2Lq + L + 1]$; we let j_2 denote its starting position. Continuing in this manner, we define natural numbers $j_1, \ldots, j_q$ such that $j_i \in [2Lq(i - 1) + 1, 2Lq(i - 1) + L + 1]$ for $1 \leq i \leq q$ and such that $w_i = u[j_i, j_i + N - 1]$. We define $u_i := u[j_i, j_{i+1} - 1]$ for $i \in \{1, \ldots, q - 1\}$ and we define $u_q := u[j_q, j_q + 2Lq]$. Then by construction, $|u_i| < L(2q + 1)$ for all i and w_i is the initial subword of length N of u_i for all i. In particular, $u_1 \cdots u_q \succ u_{\sigma(1)} \cdots u_{\sigma(q)}$ for all $\sigma \neq \mathrm{id}$. Finally, note that $j_1 \leq L + 1$, $j_q \geq 2Lq(q - 1) + 1$ and hence

$$|u_1 \cdots u_q| = 2Lq + j_q - j_1 + 1 \geq L(2q^2 - 1) + 1 > (q - 1)L(2q + 1) > (q - 1)|u_i|$$

for $i \in \{1, \ldots, q\}$, which contradicts the assumption that u does not contain strongly q-decomposable subwords. □

4.3 An Analogue of Kaplansky's Theorem

In this section we prove Theorem 4.1.2. Let D be a division ring with center k. The proof is done by a series of reductions. We first prove that if D is left algebraic of bounded degree over a subfield K, then every finitely generated k-subalgebra satisfies a standard polynomial identity. We then use a theorem of Albert, i.e. Theorem 4.3.1, to prove that D must satisfy a standard identity. From there, we prove the main theorem by embedding D in a matrix ring and looking at degrees of minimal polynomials. We begin the proof of Theorem 4.1.2 with stating a theorem of Albert.

Theorem 4.3.1. (Albert, [2]) *A finite-dimensional central division k-algebra is generated by two elements as a k-algebra.*

We now prove the first step in our reduction. For the definition of the standard polynomial identity S_C see Definition 1.4.5.

Lemma 4.3.2. *Let the division algebra D be left algebraic of bounded degree d over a (not necessarily central) subfield K. If m is a natural number, then there is a positive integer $C = C(m,d)$, depending only on d and m, such that every k-subalgebra of D that is generated by m elements satisfies the standard polynomial identity S_C.*

Proof. Let $x_1, \ldots, x_m$ be m elements of D. Consider the k-subalgebra A of D generated by $x_1, \ldots, x_m$. We put a degree lexicographic order on all words over $\{x_1, \ldots, x_m\}$ by declaring that

$$x_1 \succ x_2 \succ \cdots \succ x_m.$$

Let $N = N(m,d,d)$ be a positive integer satisfying the conclusion of the statement of Theorem 4.2.1 in which we take $p = q = d$. We claim that the left K-vector space $V := KA$ is spanned by all words in $x_1, \ldots, x_m$ of length at most N. To see this, suppose that the claim is false and let w be the smallest degree lexicographically word with the property that it is not in the left K-span of all words of length at most N. Then w must have length strictly greater than N and so by Theorem 4.2.4, either w has a strongly d-decomposable subword or w has a nontrivial subword of the form u^d. If w has a nontrivial subword of the form u^d then we can write $w = w_1 u^d w_2$. Notice that conjugation by w_1 gives an automorphism of D and so D must also be left algebraic of bounded degree d over the subfield $F := w_1^{-1} K w_1$. Notice that the sum

$$Fu^d + Fu^{d-1} + \cdots + F$$

is not direct and thus we can find $\alpha_0, \ldots, \alpha_{d-1} \in K$ such that

$$u^d = w_1^{-1}\alpha_{d-1}w_1u^{d-1} + \cdots + w_1^{-1}\alpha_0 w_1.$$

Thus

$$\begin{aligned} w &= w_1 u^d w_2 \\ &= w_1 \left(w_1^{-1}\alpha_{d-1} w_1 u^{d-1} + \cdots + w_1^{-1}\alpha_0 w_1 \right) w_2 \\ &= \alpha_{d-1} w_1 u^{d-1} w_2 + \cdots + \alpha_0 w_1 w_2 \\ &\in \sum_{v \prec w} Kv. \end{aligned}$$

By the minimality of w, we get an immediate contradiction. Similarly, if w has a strongly d-decomposable subword, then we can write

$$w = w_1 u_1 \cdots u_d w_2$$

where we have

$$u_1 \cdots u_d \succ u_{\sigma(1)} \cdots u_{\sigma(d)}$$

for all $\mathrm{id} \neq \sigma \in \mathrm{Sym}(d)$ and such that $(d-1)|u_i| < |u_1 \cdots u_d|$ for each i. As before, we let $F = w_1^{-1} K w_1$. Given a subset $S \subseteq \{1, \ldots, d\}$, we let $u_S = \sum_{j \in S} u_j$. Then for each subset S of $\{1, \ldots, d\}$, we can find $\alpha_{0,S}, \ldots, \alpha_{d-1,S} \in K$ such that

$$u_S^d = w_1^{-1}\alpha_{d-1} w_1 u_S^{d-1} + \cdots + w_1^{-1}\alpha_0 w_1.$$

The condition $(d-1)|u_i| < |u_1 \cdots u_d|$ implies that if $k < d$, then

$$|u_{i_1} \cdots u_{i_k}| < |u_1 \cdots u_d|$$

and hence $u_{i_1} \cdots u_{i_k} \prec u_1 \cdots u_d$ for all summands of u_S^k, $k < d$. Notice that

$$\sum_{S \subseteq \{1,\ldots,d\}} (-1)^{d-|S|} u_S^d = u_1 \cdots u_d + \sum_{\substack{\sigma \in \mathrm{Sym}(d) \\ \sigma \neq \mathrm{id}}} u_{\sigma(1)} \cdots u_{\sigma(d)},$$

and so

$$\begin{aligned} w &= w_1 u_1 \cdots u_d w_2 \\ &= -\sum_{\substack{\sigma \in \mathrm{Sym}(d) \\ \sigma \neq \mathrm{id}}} w_1 u_{\sigma(1)} \cdots u_{\sigma(d)} w_2 + \sum_{S \subseteq \{1,\ldots,d\}} \sum_{j=0}^{d-1} (-1)^{d-|S|} \alpha_{j,S} w_1 u_S^j w_2 \\ &\in \sum_{v \prec w} Kv. \end{aligned}$$

By the minimality of w, we get a contradiction. Thus $V = KA$ is indeed spanned by all words over $\{x_1, \ldots, x_m\}$ of length at most N. Consequently, V is at most $(1 + m + m^2 + \cdots + m^N)$-dimensional as a left K-vector space. The right multiplication r_a by $a \in A$ of the elements of V commutes with the left multiplication by elements of K. Hence r_a acts as a linear operator on the left K-vector space V and A embeds in the opposite algebra $\mathrm{End}_K(V)^{\mathrm{op}}$ of $\mathrm{End}_K(V)$. In this way A embeds in the ring of $n \times n$ matrices over K for some $n \leq 1 + m + m^2 + \cdots + m^N$. Thus taking $C = 2(1 + m + m^2 + \cdots + m^N)$ and invoking the Theorem 1.4.18, we obtain the desired result. $\square$

Lemma 4.3.3. *Let D be a division algebra which is left algebraic of bounded degree over a subfield K. Then every finitely generated division k-subalgebra E of D is finite-dimensional over its center.*

Proof. Let E be generated (as a division k-algebra) by $x_1, \ldots, x_m$, and let A be the k-subalgebra of E generated by these elements, i.e., A is the k-vector space spanned by all words over $\{x_1, \ldots, x_m\}$. By Lemma 4.3.2 the algebra A satisfies a standard identity $S_C = 0$ of degree $C = C(m, d)$. Since A is a prime PI-algebra, $Q := Q_Z(A)$ is a finite-dimensional central simple algebra by Theorem 1.5.10. Since A is a subalgebra of E, the natural embedding $\iota : A \to E$ extends to an injection $\iota : Q \to E$. Since $\iota(Q)$ is a subring of the division ring E, it is a central simple algebra without zero-divisors, i.e. it is a division algebra. As a division k-algebra $\iota(Q)$ is generated by the same elements $x_1 \ldots, x_m$ as the division k-algebra E. Hence we obtain that $\iota(Q) = E$ and E is isomorphic to Q and so E is finite-dimensional over its center. $\square$

Proposition 4.3.4. *Let D be a division algebra that is left algebraic of bounded degree d over a maximal subfield K. Then D satisfies the standard polynomial identity S_C, where $C = C(2, d)$ is a constant satisfying the conclusion of the statement of Lemma 4.3.2.*

Proof. Let k be the center of D. If D does not satisfy the standard identity $S_C = 0$, then there exists a finitely generated division k-subalgebra E of D such that E does not satisfy the identity $S_C = 0$. By Lemma 4.3.3, E is finite-dimensional over its center $Z(E)$. By Theorem 4.3.1, E is generated by two elements as a $Z(E)$-algebra. Let a and b be the generators of the $Z(E)$-algebra E. By Lemma 4.3.2, the k-algebra A generated by a and b satisfies the standard identity of degree $C = C(2, d)$. Since the center k of D is contained in the center $Z(E)$ of E and $a, b \in E$, we have that $Z(E)A \subseteq E$. Since E is generated as a $Z(E)$-algebra by a and b we conclude that $E = Z(E)A$. Thus we have a surjective ring homomorphism

$$Z(E) \otimes_k A \to E$$

and since A satisfies the standard identity of degree C, the same holds for $Z(E) \otimes_k A$ and E, a contradiction. Thus D satisfies the standard polynomial identity of degree C. $\square$

We are now ready to prove our main result. We have already shown that if a division ring D is left algebraic of bounded degree over a subfield K, then D satisfies a polynomial identity and hence is finite-dimensional over its center. The only thing that remains is to get the upper bound that is claimed in the statement of Theorem 4.1.2. This is not difficult if the subfield K is separable over k as one can use a theorem of Brauer and Albert [39, Theorem 15.16]. The inseparable case presents greater difficulty.

Proof of Theorem 4.1.2. It is no loss of generality to assume that K is a maximal subfield of D. Let k denote the center of D. By Proposition 4.3.4, D satisfies a polynomial identity and hence it is finite-dimensional over k by Theorem 1.4.25. Let $n = \sqrt{[D:k]}$. Then $[D : K] = n$ and we must show that $d \geq n$. We note that D has a separable maximal subfield $L = k(x)$ and D is a faithful simple left $D \otimes_k L$-module, via the rule

$$(\alpha \otimes x^j)(\beta) \mapsto \alpha\beta x^j$$

for $j \geq 0$ and $\alpha, \beta \in D$ (see [39, Theorem 15.12]). We let $T \in \mathrm{End}_K(D)$ be defined by $T(\alpha) = \alpha x$. If $c_0, \ldots, c_{n-1} \in K$ then

$$\left(c_0\mathrm{id} + \cdots + c_{n-1}T^{n-1}\right)(\alpha) = \left(\sum_{i=0}^{n} c_i \otimes x^i\right)(\alpha).$$

Since D is a faithful $D \otimes_k L$-module, we see that if

$$c_0\mathrm{id} + \cdots + c_{n-1}T^{n-1} = 0$$

then $c_0 = \ldots = c_{n-1} = 0$ and so the operators $\mathrm{id}, T, \ldots, T^{n-1}$ are (left) linearly independent over K. We claim that there exists some $y \in D$ such that the sum

$$K + KT(y) + \cdots + KT^{n-1}(y)$$

is direct. To see this, we regard D as a left $K[X]$-module, with action given by $f(X) \cdot \alpha \mapsto f(T)(\alpha)$ for $f(X) \in K[X]$ and $\alpha \in D$. Let $g(X)$ denote the minimal polynomial of T over

k. Then $g(X)$ annihilates D and thus D is a finitely generated torsion $K[X]$-module. By the fundamental theorem for finitely generated modules over a principal ideal domain, there exists some $y \in D$ such that

$$\{f(X) \in K[X] \;:\; f(X) \cdot y = 0\} = \{f(X) \in K[X] \;:\; f(X) \cdot \alpha = 0 \text{ for all } \alpha \in D\}. \quad (4.1)$$

If the sum $K + KT(y) + \cdots + KT^{n-1}(y)$ is not direct, then we can find a polynomial $f(X) \in K[X]$ of degree at most $n-1$ such that $f(T) \cdot y = 0$. Thus $f(T) \cdot \alpha = 0$ for all $\alpha \in D$ by Equation (4.1), which contradicts the fact that the operators $\mathrm{id}, T, \ldots, T^{n-1}$ are (left) linearly independent over K. Hence the sum

$$K + KT(y) + \cdots + KT^{n-1}(y) = K + Kyx + \cdots + Kyx^{n-1}$$

is direct. Let $u = yxy^{-1}$. Then $K + Ku + \cdots + Ku^{n-1}$ is direct. But by assumption, every element of D is left algebraic over K of degree at most d and thus $n \leq d$. □

4.4 Problems

Unlike the algebraic property, which has been extensively studied in rings, the left algebraic property appears to be new. Many of the important open problems for algebraic algebras have analogues in which the algebraic property is replaced by being left algebraic. We pose a few problems.

Question 4. *Is it true that a division ring that is finitely generated over its center and left algebraic over some subfield is finite-dimensional over its center?*

Question 5. *Let k be an algebraically closed field and let A be a finitely generated noetherian k-algebra that does not satisfy a polynomial identity. Is it possible for the quotient division algebra of A to be left algebraic over some subfield?*

We note that the right algebraic property can be defined analogously.

Question 6. *If a division ring D is left algebraic over a subfield K must D also be right algebraic over K?*

We believe that the last question has probably been posed before, but we are unaware of a reference.

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