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DOUBLY STOCHASTIC RIGHT MULTIPLIERS

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ABSTRACT. Let $P(G)$ be the set of normalized regular Borel measures on a compact group G. Let D_r be the set of doubly stochastic (d.s.) measures λ on $G\times G$ such that $\lambda(As\times Bs) = \lambda(A\times B)$, where $s \in G$, and A and B are Borel subsets of G. We show that there exists a bijection $\mu \leftrightarrow \lambda$ between $P(G)$ and D_r such that $\lambda\phi^{-1} = m \times \mu$, where m is normalized Haar measure on G, and $\phi(x,y) = (x,xy^{-1})$ for $x,y \in G$. Further, we show that there exists a bijection between D_r and M_r, the set of d.s. right multipliers of $L_1(G)$. It follows from these results that the mapping $\mu \to T_\mu$ defined by $T_\mu f = \mu * f$ is a topological isomorphism of the compact convex semigroups $P(G)$ and M_r. It is shown that M_r is the closed convex hull of left translation operators in the strong operator topology of $B[L_2(G)]$.

KEY WORDS AND PHRASES. *Compact group, regular Borel measures, doubly stochastic measures, multipliers.*
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1. INTRODUCTION.

Let G be an arbitrary compact Hausdorff group, let $B(G)$ be the σ-algebra of Borel subsets of G, and let m be normalized Haar measure on G. Let $P(G)$ be the set of regular Borel measures μ on G such that $\mu(G) = 1$. Assume that $1 \le p \le \infty$. Let $L_p(G)$ denote the complex Banach space $L_p(G,B(G),m)$, and let $B[L_p(G)]$ denote the complex Banach space of bounded linear operators from $L_p(G)$ into itself. An operator T in $B[L_p(G)]$ is called a positive contraction on $L_p(G)$ if $Tf \ge 0$ for each nonnegative f in $L_p(G)$ and $\|T\|_p \le 1$.

For each positive contraction T on $L_1(G)$, the adjoint T^* determined by the equation $\int_G (Tf)g\, dm = \int_G f(T^*g)dm$ for $f \in L_1(G)$ and $g \in L_\infty(G)$ is a positive contraction on $L_\infty(G)$. A positive contraction T on $L_1(G)$ such that $T1 = 1$, or equivalently, $\int_G T\, g\, dm = \int_G g dm$ for $g \in L_\infty(G)$, is called a doubly stochastic (d.s.)

operator. Let D be the set of d.s. operators. Note that $T \in D$ if and only if $T^* \in D$. By the Riesz convexity theorem, each d.s. operator T is also a positive contraction on $L_p(G)$, $1 < p < \infty$, with $\|T\|_p = 1$. Let Φ be the class of measure-preserving maps ϕ from $(G,B(G),m)$ onto itself, and let Φ_1 be the class of maps ϕ in Φ that are invertible and $\phi^{-1} \in \Phi$. Then each ϕ in Φ gives rise to a d.s. operator T_ϕ that is defined by $T_\phi f(x) = f(\phi(x))$. For brevity we also write Φ for $\{T_\phi : \phi \in \Phi\}$ and Φ_1 for $\{T_\phi : \phi \in \Phi_1\}$. For $s \in G$, let L_s and R_s be the left and the right translation operators defined by $L_s f(x) = f(s^{-1}x)$ and $R_s f(x) = f(xs^{-1})$. Then both L_s and R_s are in Φ_1.

An operator T in D is called a right multiplier (centralizer) of $L_1(G)$ if T commutes with right translation operators, that is, $TR_s = R_sT$ for each $s \in G$. Let M_r be the set of d.s. right multipliers of $L_1(G)$. The set M_ℓ of d.s. left multipliers of $L_1(G)$ is defined in an analogous fashion. We see readily that $L_s \in M_r$ and $R_s \in M_\ell$ for each $s \in G$. Let $M = M_r \cap M_\ell$. It is plain that $L_s = R_s \in M$ for each $s \in G^z$, the center of G. Note that if G is Abelian, then $M = M_r = M_\ell$.

For a semigroup S, the set of elements x in S such that $xy = yx$ for each y in S is called the center of S and is denoted by S^z. For topological semigroups S_1 and S_2, an (algebraic) isomorphism of S_1 into S_2 which is also a homeomorphism is called a topological isomorphism of S_1 into S_2. All functions on G are Borel mesurable and will always be considered up to m-equivalence. For two functions f and g on G, $f = g$, f g mean that the equality and the inequality, respectively, are satisfied in the almost everywhere (a.e.) sense with respect to m.

It follows from Theorem 1 of Wendel [1] (see also Edwards [2]; Hewitt and Ross [3], 35.5) that for each T in M_r, there exists a unique μ in $P(G)$ such that $Tf = \mu * f$ for each $f \in L_1(G)$. Using a d.s. measure, Brown [4] gave an alternate proof of the above result when the underlying group G is Abelian and compact. The purpose of this paper is to extend Brown's work [4] for an arbitrary compact group.

Certain preliminary results on $P(G)$ and d.s. measures are given in Section 2. In Section 3 we show that there exists a bijection between M_r and D_r (Proposition 5), and that there exists a bijection between D_r and $P(G)$ (Proposition 6). Using Propositions 5 and 6, we prove (Theorem 1) that the mapping $\mu \to T_\mu$ defined by $T_\mu f = \mu * f$ is a topological isomorphism of the compact convex semigroups $P(G)$ and M_r. In Section 4 we show that $T \in M_r$ is an isometry of $L_1(G)$ if and only if T is a left translation operator (Theorem 3), and that the set of extreme points of M_r is the group $\underline{G}$ of left translation operators, and M_r is the closed convex hull of G in the strong operator topology of $B[L_2(G)]$ (Theorem 4). By minor modifications of our arguments we obtain analogous results for M_ℓ.

2. PRELIMINARIES.

Let $C(G)$ be the Banach space of complex continuous functions on G. For μ, $\nu \in P(G)$, there exists, by the Riesz representation theorem, a unique measure $\mu * \nu$

in $P(G)$ such that $\int_G f(z)d\mu*\nu(z) = \int\int_{GG} f(xy)\, d\mu(x)\, d\nu(y)$ for each $f \in C(G)$. Thus $P(G)$ is a semigroup under the convolution operation. It follows from Theorem 2 of Stromberg [5] that $\mu*\nu(A) = \int_G \mu(Ay^{-1})d\nu(y) = \int_G \nu(x^{-1}A)d\mu(x)$ for each $A \in B(G)$. As usual we shall identify $P(G)$ with a subset of $C(G)^*$, the dual space of $C(G)$. We show readily that $P(G)$ is convex and is compact in the weak* topology. It is well-known (Rosenblatt [6]) that for μ,ν in $P(G)$, the convolution operation $\mu*\nu$ is jointly continuous in μ and ν with respect to the weak* topology. Therefore $P(G)$ with the convolution operation and the weak* topology, is a compact, convex, Hausdorff semigroup.

We state without proof the following result of Stromberg [5].

LEMMA 1. (Stromberg). Let X be a compact Hausdorff space and f a continuous mapping of X into itself. If μ is a regular Borel measure on X, then so is the measure μf^{-1}.

For $\mu \in P(G)$, the adjoint μ' defined by $\mu'(A) = \mu(A^{-1})$ is an element of $P(G)$ and $(\mu')' = \mu$. For each $x \in G$, ε_x be the probability measure such that $\varepsilon_x(A) = \chi_A(x)$ for $A \subset G$, where χ_A is the characteristic function of A. Note that $\varepsilon_x' = \varepsilon_{x^{-1}}$. It is easily seen that the mapping $x \to \varepsilon_x$ is a topological isomorphism of G into the compact semigroup $P(G)$.

A characterization of the center $P^z(G)$ of the semigroup $P(G)$ is given by Stromberg [7]. It is straightforward to prove the following proposition.

PROPOSITION 1. $P^z(G)$ is a compact, convex, Abelian subsemigroup of $P(G)$.

Let $P^i(G)$ be the set of idempotents μ of $P(G)$, that is, $\mu*\mu = \mu$. For any compact subgroup H of G, let m_H be normalized Haar measure on H. We shall always extend the measure m_H in $P(H)$ to a unique measure in $P(G)$, denoted also by m_H, as follows: $m_H(A) = m_H(A\cap H)$ for $A \in B(G)$. Then $m_H \in P^i(G)$. Theorem 1 of Wendel [8] states that $\mu \in P^i(G)$ if and only if there exists a unique compact subgroup H of G such that $\mu = m_H$. It is routine to verify that $P^i(G)$ is a compact subset of $P(G)$. Since, for arbitrary compact subgroups H and K of G, the set HK is not always a subgroup of G and $m_H*m_K = m_{HK}$, the set $P^i(G)$ is not necessarily a subsemigroup of $P(G)$. Let e denote the identity of G. Observe that

$$\tfrac{1}{2}(m + \varepsilon_e)* \tfrac{1}{2}(m + \varepsilon_e) = (3m + \varepsilon_e)/4 \neq \tfrac{1}{2}(m + \varepsilon_e).$$

Therefore $P^i(G)$ is not a convex set.

The set $P^i(G) \cap P^z(G)$ contains Haar measure m and the point mass ε_e and is a compact subset of $P(G)$. For $\mu \in P(G)$, let $S(\mu)$ be the support of μ.

PROPOSITION 2. Let μ be in $P(G)$. The following assertions are equivalent:

(i) μ is in $P^i(G) \cap P^z(G)$;

(ii) there exists a unique compact subgroup H of G such that $\mu = m_H$ and $\varepsilon_s*m_H*\varepsilon_s' = m_H$ for each $s \in G$;

(iii) there exists a unique compact normal subgroup H of G such that $\mu = m_H$.

PROOF. It is known (Stromberg [7]; Wendel [8]) that (i) and (ii) are equivalent.

Suppose that (ii) holds. Then we have, for each $s \in G$, $H = S(m_H) = S(\varepsilon_s * m_H * \varepsilon_s') = S(\varepsilon_s)S(m_H)S(\varepsilon_s') = sHs^{-1}$, so that H is a compact normal subgroup of G. Thus (ii) implies (iii).

Suppose that $\mu = m_H$, where H is a unique compact normal subgroup of G. We shall show that $\varepsilon_s * m_H * \varepsilon_s' = m_H$ for each $s \in G$, or equivalently, $m_H(s^{-1}Es) = m_H(E)$ for $s \in G$ and $E \in B(G)$. Let $\phi_s(x) = sxs^{-1}$. Since the inner automorphism ϕ_s of G is a homeomorphism of G onto itself, and $\phi_s(H) = H$, we have, from Lemma 1, $m_H\phi_s^{-1} \in P(G)$, and $S(m_H\phi_s^{-1}) = H$. For $s \in G$, $a \in H$, and $E \in B(G) \cap H$, we have $h = s^{-1}a^{-1}s \in H$, $\phi_s(x)a^{-1} = \phi_s(xh)$, and

$$m_H\phi_s^{-1}(Ea) = \int_H \chi_E(\phi_s(x)a^{-1})\, dm_H(x) = \int_H \chi_E(\phi_s(xh)dm_H(x) = \int_H \chi_E(\phi_s(x))dm_H(x) = m_H\phi^{-1}(E),$$

so that by the uniqueness of Haar measure on H, $m_H = m_H\phi_s^{-1}$. That is, $m_H(E) = m_H\phi_s^{-1}(E) = m(s^{-1}Es)$. Therefore (iii) implies (ii). □

It is easy to verify that for $\mu \in P^i(G)$ and $\nu \in P^i(G) \cap P^z(G)$, $\mu*\nu = \nu*\mu \in P^i(G)$. We also have

PROPOSITION 3. $P^i(G) \cap P^z(G)$ is a compact, non-convex subsemigroup of $P(G)$.

We omit the elementary proof of this proposition.

For $\mu \in P(G)$, let $P\colon G\times B(G) \to [0,1]$ be such that

$$P(x,A) = \mu' * \varepsilon_x(A) = \mu(xA^{-1}).$$

Then $P(x,A)$ is a transition (probability) function which is a regular probability measure on $B(G)$ for each $x \in G$ and a Borel function of x for each $A \in B(G)$. It follows easily that

$$\int_G P(x,A)dm(x) = m(A) \tag{2.1}$$

for $A \in B(G)$, and

$$P(xs,As) = P(x,A) \tag{2.2}$$

for $s,x \in G$ and $A \in B(G)$. Note that the transition function $P(x,A)$ has an invariant measure m and is invariant under right translations. The transition function $P(x,A) = \mu(xA^{-1})$ gives rise to the Markov operator $P = P_\mu$ from $C(G)$ into itself by the formula $Pf(x) = \int_G P(x,dy)f(y) = \int_G f(y^{-1}x)d\mu(y)$. It follows from (2.1) and (2.2) that $\int_G Pfdm = \int_G fdm$ for $f \in C(G)$, $P1 = 1$, and $PR_s = R_sP$ for $s \in G$. See Rosenblatt [6] for details. By the usual argument the Markov operator P_μ is uniquely extended to an operator T_μ in M_r. We shall see in Proposition 7 that each $T \in M_r$ is induced by a unique Markov operator P_μ on $C(G)$.

Let $P(G\times G)$ be the set of regular probability measures on $(G\times G, B(G\times G))$, where $B(G\times G)$ denotes the σ-algebra of Borel subsets of $G\times G$. A measure λ in $P(G\times G)$ is called a d.s. measure on $G\times G$ if

$$\lambda(A\times G) = \lambda(G\times A) = m(A) \tag{2.3}$$

for each $A \in B(G)$. Let $\mathbf{D}$ be the set of d.s. measures on $G\times G$. A probability measure λ on the product measurable space $(G\times G, B(G)\times B(G))$ satisfying (2.3) is called doubly stochastic by Brown [4]. Let $\{X_n : n \geq 1\}$ be the right random walk on G generated by m and $\mu \in P(G)$. That is, $\{X_n\}$ is the Markov process with state space G, initial distribution m, and stationary transition function $P(x,A) = \mu(xA^{-1})$. If λ denotes the joint distribution of X_1 and X_2, then λ is a d.s. measure on $(G\times G, B(G)\times B(G))$. If G is metrizable, then $B(G)\times B(G) = B(G\times G)$, so that these two definitions are equivalent. However we have $B(G)\times B(G) \subsetneq B(G\times G)$ in general, so that a d.s. measure in the sense of Brown [4] is not an element of $\mathbf{D}$. Let $B_0(G)$ be the σ-algebra of Baire subsets of G, and let $B_0(G\times G)$ be the σ-algebra of Baire subsets of $G\times G$. Note that $B_0(G\times G) = B_0(G)\times B_0(G)$.

LEMMA 2. For each probability measure σ on $(G\times G, B(G)\times B(G))$ satisfying (2.3), there exists a unique λ in $\mathbf{D}$ such that $\lambda(E) = \sigma(E)$ for each E in $B(G)\times B(G)$.

PROOF. Let σ_0 be the Baire restriction of σ, that is, $\sigma_0(E) = \sigma(E)$ for each $E \in B_0(G\times G)$. Then σ_0 is a Baire measure on $(G\times G, B_0(G\times G))$ such that $\sigma_0(A\times G) = \sigma_0(G\times A) = m(A)$ for each $A \in B_0(G)$. Let λ be the unique, regular Borel measure on $(G\times G, B(G\times G))$ which extends σ_0 (see Halmos [9], 54.D). We shall show that λ also extends σ. It is enough to show that $\lambda(C_1\times C_2) = \sigma(C_1\times C_2)$ for all compact sets C_1 and C_2 in G. By the regularity of Haar measure m there exist compact G_δ's, A_1 and A_2, such that $C_j \subset A_j$ and $m(A_j - C_j) = 0$ for $j = 1,2$. Since m is completion regular by a theorem of Kakutani-Kodaira [10] (see also, Halmos [9], 64.H), there exist B_1 and B_2 in $B_0(G)$ such that $A_j - C_j \subset B_j$ and $m(B_j) = 0$ for $j = 1,2$. Note that

$$A_1\times A_2 - C_1\times C_2 = [(A_1-C_1)\times A_2] \cup [C_1\times(A_2-C_2)] \subset (B_1\times G)\cup(G\times B_2).$$

Then we have

$$\lambda(A_1\times A_2 - C_1\times C_2) \leq \lambda(B_1\times G) + \lambda(G\times B_2) = \sigma_0(B_1\times G) + \sigma_0(G\times B_2) = m(B_1) + m(B_2) = 0,$$

so that $\lambda(C_1\times C_2) = \lambda(A_1\times A_2) = \sigma_0(A_1\times A_2)$. Similarly we also have $\sigma(C_1\times C_2) = \sigma(A_1\times A_2) = \sigma_0(A_1\times A_2)$, and so our assertion is proved.

If λ_1 and λ_2 are any two measures in $\mathbf{D}$ both of which extend σ, then they also extend the Baire measure σ_0, so that $\lambda_1 = \lambda_2$. □

We obtain readily from Corollary of Brown [11], together with Lemma 2, the following proposition.

PROPOSITION 4. There exists a bijection $T \leftrightarrow \lambda$ between D and $\mathbf{D}$ such that

$$\lambda(A\times B) = \int_G \chi_A T\chi_B \, dm \tag{2.4}$$

for all $A, B \in B(G)$.

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Let $\mathbb{D}_r$ be the set of measures λ in $\mathbb{D}$ such that $\lambda(As\times Bs) = \lambda(A\times B)$ for all $A,B \in B(G)$, $s \in G$. For each $s \in G$, let $\tau_s(x,y) = (xs^{-1}, ys^{-1})$ for $x,y \in G$. It is easy to see that for each $\lambda \in \mathbb{D}$, $\lambda \in \mathbb{D}_r$ if and only if $\lambda\tau_s^{-1} = \lambda$ for each $s \in G$.

PROPOSITION 5. There exists a bijection $T \leftrightarrow \lambda$ between M_r and $\mathbb{D}_r$ satisfying relation (2.4).

PROOF. Let T and λ be the associated d.s. operator and d.s. measure as in Proposition 4. Then $T \in M_r$ iff $T = R_{s^{-1}}TR_s$ for all $s \in G$ iff $T\chi_B = R_{s^{-1}}TR_s\chi_B$ for all $B \in B(G)$, $s \in G$ iff $\lambda(A\times B) = (As\times Bs)$ for all $A,B \in B(G)$, $s \in G$. □

For $\mu,\nu \in P(G)$, there exists, by the Riesz representation theorem, a unique regular Borel measure $\mu\tilde{\times}\nu$ in $P(G\times G)$ such that

$$\int_{G\times G} f(x,y)d\mu\tilde{\times}\nu(x,y) = \int_G(\int_G f(x,y)d\mu(x))d\nu(y) = \int_G(\int_G f(x,y)d\nu(y))d\mu(x)$$

for all continuous functions $f(x,y)$ on $G\times G$. Note that $\mu\tilde{\times}\nu$ is the unique regular Borel measure on $G\times G$ which extends the product measure $\mu\times\nu$ on $(G\times G, B(G)\times B(G))$. Note also that Haar measure $m\tilde{\times}m$ on the compact group $G\times G$ is an element of $\mathbb{D}_r$. Let ξ and η be the mappings from $G\times G$ onto G defined by $\xi(x,y) = y^{-1}x$ and $\eta(x,y) = xy^{-1}$. Then both ξ and η are continuous surjections. Define ϕ and ψ on $G\times G$ by

$$\phi(x,y) = (x,xy^{-1}),\ \psi(x,y) = (x,y^{-1}x).$$

Then ϕ is a homeomorphism of $G\times G$ onto itself with $\phi^{-1} = \psi$ and is $m\tilde{\times}m$ measure-preserving.

PROPOSITION 6. There exists a bijection $\lambda \leftrightarrow \mu$ between $\mathbb{D}_r$ and $P(G)$ such that $\lambda\phi^{-1} = m\tilde{\times}\mu$.

LEMMA 3. There exists an injection $\lambda \to \mu$ from $\mathbb{D}_r$ into $P(G)$ such that $\lambda\phi^{-1} = m\tilde{\times}\mu$, and in this case we have $\mu = \lambda\eta^{-1}$.

PROOF. For each $\lambda \in \mathbb{D}_r$ and for each $B \in B(G)$, let θ_B be the Borel measure on G defined by $\theta_B(A) = \lambda\phi^{-1}(A\times B)$. It follows from the regularity of Haar measure m that, for $\varepsilon > 0$ and $A \in B(G)$, there exist a compact set C and an open set U such that $C \subset A \subset U$ and $m(U-C) < \varepsilon$. Then we have $\theta_B(U-C) \leq m(U-C) < \varepsilon$, so that θ_B is a regular Borel measure. Since, for each $s \in G$, the mapping $\tau_s(x,y) = (xs^{-1}, ys^{-1})$ is λ measure-preserving,

$$\theta_B(A) = \lambda\phi^{-1}(A\times B) = \lambda\tau_s^{-1}\circ\phi^{-1}(A\times B) = \lambda\phi^{-1}(As\times B) = \theta_B(As)$$

for all $s \in G$, $A \in B(G)$, so that by the uniqueness of Haar measure there exists $c \geq 0$ such that $\theta_B = cm$. It follows that $c = \theta_B(G) = \lambda\eta^{-1}(B)$ and $\lambda\phi^{-1}(A\times B) = m(A)\lambda\eta^{-1}(B)$ for all $A, B \in B(G)$. By Lemma 1 we have $\lambda\phi^{-1} \in P(G\times G)$ and $\lambda\eta^{-1} \in P(G)$. Since two regular Borel measures $\lambda\phi^{-1}$ and $m\tilde{\times}\lambda\eta^{-1}$ agree on all Baire sets of $G\times G$, we have $\lambda\phi^{-1} = m\tilde{\times}\lambda\eta^{-1}$. If $\lambda\phi^{-1} = m\tilde{\times}\mu$ for some $\mu \in P(G)$, then $\mu(A) = m\tilde{\times}\mu(G\times A) = \lambda\phi^{-1}(G\times A) = \lambda\eta^{-1}(A)$ for $A \in B(G)$, so that $\mu = \lambda\eta^{-1}$. It is clear that the mapping $\lambda \to \mu$ is injective. □

PROOF OF PROPOSITION 6. It remains to show that the injection defined in Lemma 3 is surjective. For each $\mu \in P(G)$, let $\lambda = (m\otimes\mu)\psi^{-1}$. By Lemma 1 we have $\lambda \in P(G\times G)$. It follows that, for A and B in $B(G)$,

$$\lambda(A\times B) = (m\otimes\mu)\psi^{-1}(A\times B) = \int_G m(A\cap yB)d\mu(y),$$

so that $\lambda(A\times G) = \lambda(G\times A) = m(A)$. Since $m(As\cap yBs) = m(A\cap yB)$ for all $A,B \in B(G)$, $s,y \in G$, we obtain $\lambda(A\times B) = \lambda(As\times Bs)$. Therefore $\lambda \in \mathbb{D}_r$. Using $\phi = \psi^{-1}$ we also have $\lambda\phi^{-1} = m\otimes\mu$. □

PROPOSITION 7. There exists a bijection $\mu \leftrightarrow T_\mu$ between $P(G)$ and M_r such that $T_\mu f = \mu * f$ for each $f \in L_1(G)$.

PROOF. Let $\mu \in P(G)$, $\lambda \in \mathbb{D}_r$, and $T \in M_r$ be the associated elements determined by Propositions 5 and 6. Clearly the mapping $\mu \to T_\mu = T$ is a bijection from $p(G)$ onto M_r. Then we have

$$\int_G \chi_A T\chi_B dm = \lambda(A\times B) = (m\otimes\mu)\psi^{-1}(A\times B) = \int_G \chi_A(x) \ (\int_G \chi_B(y^{-1}x)d\mu(y))dm(x)$$

for A and B in $B(G)$, so that $T\chi_B(x) = \int_G \chi_B(y^{-1}x)d\mu(y)$. Therefore we have $Tf = \mu * f$ for each $f \in L_1(G)$. □

It is shown by Brown [11] that D with the weak operator topology of $B[L_2(G)]$ is a compact, convex Hausdorff semigroup, and that on the set D the weak operator topologies of $B[L_p(G)]$, $1 \leq p < \infty$, coincide. An elementary argument shows that M_r is a compact, convex subsemigroup of D.

THEOREM 1. The mapping $\mu \to T_\mu$ of Proposition 7 is a topological isomorphism between the compact convex semigroups $P(G)$ and M_r.

PROOF. It is straightforward to show that $T_\mu T_\nu = T_{\mu*\nu}$, $T_{\mu'} = T_\mu^*$, and $T_{t\mu+(1-t)\nu} = tT_\mu + (1-t)T_\nu$ for $\mu,\nu \in P(G)$ and $t \in [0,1]$. By Proposition 7 the mapping $\mu \to T_\mu$ is an isomorphism of $P(G)$ onto M_r. Note that the mapping is a regular representation of $P(G)$ (see Hewitt and Ross [12], 22.11).

To prove that the mapping $\mu \to T_\mu$ is a homeomorphism it is enough to show that the mapping is continuous, or equivalently, whenever a net(μ_α) converges to μ in $P(G)$, the net (T_{μ_α}) converges to T_μ in M_r. Let f and g be real continuous functions on G such that $|f(x)| \leq 1$, $|g(x)| \leq 1$ for $x \in G$. Since $f\circ\xi(x,y) = f(y^{-1}x)$ is right uniformly continuous on $G\times G$, there exists, for each $\varepsilon > 0$, a neighbourhood U of the identity e of G such that for all $y \in G$,

$$|f(y^{-1}x) - f(y^{-1}x')| < \varepsilon/8 \quad \text{for} \quad x'x^{-1} \in U .$$

Since G is compact, an open covering $\{Ux: x \in G\}$ of G has a finite subcovering $\{U_j: j = 1,2,\dots,n\}$, where $U_j = Ux_j$. Then we have that for all $y \in G$,

$$\sup_{x,x' \in U_j} |f(y^{-1}x) - f(y^{-1}x')| < \varepsilon/4, \ j = 1,2,\dots,n,$$

and that

$$\sup_{x,x' \in U_j} |P_\nu f(x) - P_\nu f(x')| < \varepsilon/4, \; j = 1,2,\ldots,n,$$

for all $\nu \in P(G)$, where $P_\nu f(x) = \int_G f(y^{-1}x)d\nu(y)$. Let $h_\alpha(x) = P_{\mu_\alpha} f(x) - P_\mu f(x)$. It follows that for all α,

$$\sup_{x,x' \in U_j} |h_\alpha(x) - h_\alpha(x')| < \varepsilon/2, \; j = 1,2,\ldots,n.$$

Define a finite partition $\{E_j : j = 1,2,\ldots,n\}$ of G by $E_1 = U_1$ and $E_j = U_j - \bigcup_{i=1}^{j-1} U_i$ for $j \geq 2$. Choose a point a_j in E_j, $j = 1,2,\ldots,n$. Then we obtain that for all α,

$$\sup_{x \in E_j} |h_\alpha(x) - h_\alpha(a_j)| < \varepsilon/2, \; j = 1,2,\ldots,n.$$

A simple calculation yields

$$|\int_G (x)(P_{\mu_\alpha} f(x) - P_\mu f(x))dm(x)| \leq \int_G |h_\alpha(x)|dm(x) = \sum_{j=1}^n \int_{E_j} |h_\alpha(x)|dm(x)$$

$$\leq \sum_{j=1}^n \int_{E_j} |h_\alpha(x) - h_\alpha(a_j)|dm(x) + \sum_{j=1}^n |h_\alpha(a_j)|$$

$$< \varepsilon/2 + \sum_{j=1}^n |P_{\mu_\alpha} f(a_j) - P_\mu f(a_j)|$$

for all α. Since $\mu_\alpha \to \mu$, there exists α_0 such that for all $\alpha \geq \alpha_0$,

$$|P_{\mu_\alpha} f(a_j) - P_\mu f(a_j)| < \varepsilon/2n, \; j = 1,2,\ldots,n,$$

so that

$$|\int_G g(x)(P_{\mu_\alpha} f(x) - P_\mu f(x))dm(x)| < \varepsilon \quad \text{for} \quad \alpha \geq \alpha_0 .$$

Thus $T_{\mu_\alpha} \to T_\mu$. □

REMARK 1. We may show readily that there exists a homeomorphism $\mu \to T_\mu$ between the compact convex semigroups $P(G)$ and M_ℓ such that $T_\mu f = f*\mu$ for $f \in L_2(G)$, and that this mapping is not an isomorphism. (See Hewitt and Ross [12], 22.21)

COROLLARY 1. The mapping $\mu \to T_\mu$ of Theorem 1 carries $P^z(G)$ onto M and is a topological isomorphism between the compact convex Abelian semigroups $P^z(G)$ and M.

PROOF. In view of Proposition 1 and Theorem 1, it is enough to show that the mapping $\mu \to T_\mu$ is a bijection between $P^z(G)$ and M. Note that $L_s = T_{\varepsilon_s}$ for $s \in G$. Then we have from Theorem 1, together with Theorem 2.5.1 of Stromberg [7], that $\mu \in P^z(G)$ iff $\mu*\varepsilon_s = \varepsilon_s*\mu$ for $s \in G$ iff $T_\mu L_s = T_{\mu*\varepsilon_s} = T_{\varepsilon_s*\mu} = L_s T_\mu$ for $s \in G$ iff $T_\mu \in M$. □

We also have from Theorem 1, the following corollary.

COROLLARY 2. M is the center of the semigroup M_r.

As an immediate consequence of Proposition 3 and Corollary 1 we have

COROLLARY 3. The mapping $\mu \to T_\mu$ of Theorem 1 induces a topological isomorphism between the compact semigroups $P^i(G) \cap P^z(G)$ and $\{T \in M: T^2 = T\}$.

It is easy to see that the mapping $\mu \to T_\mu$ of Theorem 1 induces a homeomorphism between the compact sets $P^i(G)$ and $\{T \in M_r: T^2 = T\}$.

PROPOSITION 8. Let $\mu \in P(G)$ and $T \in M_r$ be the associated elements as in Theorem 1. The following assertions are equivalent:

(i) $\mu * \mu = \mu$;

(ii) $T^2 = T$;

(iii) $T(fTg) = TfTg$ for $f, g \in L_\infty(G)$.

PROOF. The equivalence of (i) and (ii) is obvious. If we put $f = 1$ in (iii), then $T^2 g = Tg$ for all $g \in L_\infty(G)$, so that $T^2 = T$. Therefore (iii) implies (ii).

Suppose that (i) holds. Then $\mu = m_H$, where H is a compact subgroup of G. It follows that for $f, g \in L_\infty$ and $x \in G$,

$$\begin{aligned} T(fTg)(x) &= \int_H f(y^{-1}x)Tg(y^{-1}x)d\mu(y) = \int_H f(yx)Tg(yx)d\mu(y) \\ &= \int_H f(yx)(\int_H g(z^{-1}yx)d\mu(z))d\mu(y) = \int_H f(yx)(\int_H g(zyx)d\mu(z))d\mu(y) \\ &= \int_H f(yx)(\int_H g(zx)d\mu(z))d\mu(y) = Tf(x)Tg(x). \end{aligned}$$

Thus (i) implies (iii). □

4. LEFT TRANSLATION OPERATORS.

Let $\underline{G}$ be the set of left translation operators, $\underline{G} = \{L_s: s \in G\}$. Then it is plain that $\underline{G}$ is a subgroup of the compact semigroup M_r.

THEOREM 2. The mapping $s \to L_s$ is a continuous injection of G into M_r and is a topological isomorphism of the compact groups G and $\underline{G}$.

PROOF. We shall show that the mapping $s \to L_s$ is a continuous map from G into M_r. For $s \in G$, let

$$U(L_s: f,g,\varepsilon) = \{T: T \in M_r, |\langle g,(L_s - T)f\rangle| < \varepsilon\},$$

where $f, g \in C(G)$ and $\varepsilon > 0$. Since $h(x,s) = g(x)\bar{f}(s^{-1}x)$ is right uniformly continuous on $G \times G$, there exists a neighborhood V of e such that $|h(x,s) - h(x,t)| < \varepsilon$ for all t in Vs and for all x in G, so that $L_t \in U(L_s: f,g,\varepsilon)$ for all t in Vs. This proves that the mapping is continuous.

We verify easily that the mapping is an algebraic isomorphism of the groups G and $\underline{G}$. It follows that, since G is compact, $\underline{G}$ is a compact subgroup of M_r, and so the assertion follows. □

As an immediate consequence of Theorem 2 we obtain

COROLLARY 4. The mapping $s \to L_s$ of Theorem 2 carries the center G^z of G onto the center $\underline{G}^z$ of $\underline{G}$ and is a topological isomorphism between the compact Abelian subgroups G^z and $\underline{G}^z$.

We prove the following characterization of a left translation on G.

LEMMA 4. Let ϕ be a mapping from G into itself. The following assertions are equivalent:

(i) $\phi(xy) = \phi(x)y$ for all $x,y \in G$;

(ii) there exists an element s in G such that $\phi(x) = sx$ for all $x \in G$.

PROOF. If (i) holds, we put $s = \phi(e)$, so that $\phi(x) = \phi(ex) = \phi(e)x = sx$ for all $x \in G$. Clearly (ii) implies (i). □

Similarly we prove that a mapping ϕ from G into G is a right translation iff $\phi(xy) = x\phi(y)$ for all $x,y \in G$.

LEMMA 5. $\Phi_1 \cap M_r = \Phi \cap M_r = \underline{G}$.

PROOF. Since $\underline{G} \subset \Phi_1 \cap M_r \subset \Phi \cap M_r$, it suffices to show that $\Phi \cap M_r \subset \underline{G}$. If $T_\phi \in \Phi \cap M_r$, then $R_{y^{-1}} T_\phi = T_\phi R_{y^{-1}}$ for each $y \in G$, so that $\phi(xy) = \phi(x)y$ for $x,y \in G$. By Lemma 4 there exists $s \in G$ such that $\phi(x) = sx$ for all $x \in G$, and so $T_\phi = L_{s^{-1}} \in \underline{G}$. □

THEOREM 3. Let T be in M_r. The following assertions are equivalent:

(i) T is an isometry of $L_p(G)$ into itself for all $p \in [1,\infty)$;

(ii) T is an isometry of $P_p(G)$ into itself for some $p \in [1,\infty)$;

(iii) T is in $\underline{G}$.

PROOF. Clearly (i) implies (ii). Suppose that (ii) holds. Let $T \in M_r$ and $\mu \in P(G)$ be the associated elements as in Proposition 7. Let P be the Markov operator on $C(G)$ defined by $Pf(x) = \int_G P(x,dy)f(y)$, where $P(x,A) = \mu(xA^{-1})$. It follows from (ii), together with Jensen's inequality, that, for each nonegative f in $C(G)$, $|Pf|^p(x) = P|f|^p(x)$ m-a.e. If the equality holds at a point $x \in G$, then the measure $P(x,.)$ is a unit mass at some point $\sigma(x) \in G$, that is, $P(x,.) = \varepsilon_{\sigma(x)}(.)$. Thus we have $P(x,.) = \varepsilon_{\sigma(x)}(.)$ m-a.e. The mapping $\sigma(x)$ is defined on G m-a.e., but it can be defined everywhere on G in the usual manner. We show readily that $T = T_\sigma \in \Phi \cap M_r$, and so by Lemma 5 T is in $\underline{G}$. Therefore (iii) holds. Clearly (iii) implies (i). □

COROLLARY 5. Let $T \in M_r$. The following assertions are equivalent:

(i) T is an isometric, algebra isomorphism of $L_1(G)$;

(ii) T is an isometry on $L_1(G)$;

(iii) T is in $\underline{G}$.

COROLLARY 6. Let $T \in M_r$. The following assertions are equivalent:

(i) T is a unitary operator on $L_2(G)$;

(ii) T is an isometry on $L_2(G)$;

(iii) T is in $\underline{G}$.

By a measure-preserving set isomorphism ψ on $(G,B(G),m)$ we shall mean a mapping ψ of the measure algebra $\langle B(G),m\rangle$ into itself such that $\psi(G) = G$, $\psi(G-A) = G - \psi(A)$, $\psi(\bigcup_{j=1}^{\infty} A_j) = \bigcup_{j=1}^{\infty} \psi(A_j)$, and $m(\psi(A)) = m(A)$. Let Ψ be the family of such set mappings ψ. Each $\psi \in \Psi$ defines a unique operator $T_\psi \in D$ such that $T_\psi \chi_A = \chi_{\psi(A)}$. We also write Ψ for $\{T_\psi : \psi \in \Psi\}$. In particular, if G is a compact metrizable group, then each set-mapping $\psi \in \Psi$ is induced by a point-mapping

$\phi \in \Phi$, so that $T_\psi \chi_A = T_\phi \chi_A$, that is, $\Psi = \Phi$. See Lamperti [13]; Royden [14] for details. We prove the following analogue of Theorem 3 for D.

PROPOSITION 9. Let T be in D. The following assertions are equivalent:
(i) T is an isometry of $L_p(G)$ into itself for all $p \in [1,\infty)$;
(ii) T is an isometry of $L_p(G)$ into itself for some $p \in [1,\infty)$;
(iii) T is in Ψ.

PROOF. Clearly (i) implies (ii). We next show that (ii) implies (iii). If T is an isometry of $L_p(G)$ for some p: $1 \leq p < \infty$, $p \neq 2$, then by Theorem 3.1 of Lamperti [13], we have $T \in \Psi$. If T is an isometry of $L_2(G)$, then, by the argument of Brown [11], pages 22, 23, we obtain $T \in \Psi$. It is straightforward to show that (iii) implies (i). □

The following corollary follows from Theorem 3 and Proposition 9.

COROLLARY 7. $\Psi \cap M_r = \underline{G}$.

REMARK 2. By the Kawada-Wendel theorem (Kawada [15]; Wendel [1,16]), we have that for each T in D, T is an isometric, algebra isomorphism of $L_1(G)$ onto itself if and only if $T = T_\phi \in \Phi_1$, where ϕ is a homeomorphic automorphism of G.

For a convex subset K of a real or complex vector space, let ext K be the set of extreme points of K. We write G and G^z for the sets $\{\varepsilon_x: x \in G\}$ and $\{\varepsilon_x: x \in G^z\}$. We verify easily that $\text{ext}P(G) = G$ and

$$G^z = P^z(G) \cap \text{ext } P(G) \subset \text{ext } P^z(G).$$

Example 1 will show that $\text{ext } P^z(G) \neq G^z$ in general. It is known (Stromberg [7]) that for each $\mu \in P(G)$, the measure $\tilde{\mu}$ defined by $\tilde{\mu}(E) = \int_G \mu(xEx^{-1})dm(x)$ is an element of $P^z(G)$.

LEMMA 6. If μ is a measure in $P^z(G) - G^z$ such that $e \in S(\mu)$, then it is not an extreme point of $P^z(G)$.

PROOF. Let $S(\mu) = H$. Then there exists a neighborhood U of e such that $0 < t = \mu(U) < 1$. Define the measures μ_1 and μ_2 in P(G) by $\mu_1(E) = \mu(E\cap U)/t$ and $\mu_2(E) = \mu(E\cap(H-U))/s$, where $s = 1 - t$. It follows that $\mu_1 \neq \mu_2$, $\mu = t\mu_1 + s\mu_2$, and $\mu = t\tilde{\mu}_1 + s\tilde{\mu}_2$. Then there exists a neighbourhood V of e such that $xVx^{-1} \subset U$ for all $x \in G$. Observe that $\tilde{\mu}_2(V) = \frac{1}{s}\int_G \mu(xVx^{-1}\cap(H-U))dm(x) = 0$. On the other hand, since $f(x) = \mu(xVx^{-1})$ is lower semicontinuous on G, and $f(e) = \mu(V) > 0$, there exists a neighborhood W of e such that $f(x) > f(e)/2$ for each $x \in W$. Thus $\tilde{\mu}_1(V) = \frac{1}{t}\int_G f(x)dm(x) \geq \frac{1}{t}\int_W f(x)dm(x) > f(e)m(W)/2t > 0$. Accordingly $\tilde{\mu}_1 \neq \tilde{\mu}_2$ so that μ is not an extreme point of $P^z(G)$. □

EXAMPLE 1. Let $G = S_3$, the symmetric group on three symbols. Let $G = \{a_1,a_2,\ldots,a_6\}$, where $a_1 = e$, $a_2 = (1,2)$, $a_3 = (1,3)$, $a_4 = (2,3)$, $a_5 = (1,2,3)$, $a_6 = (1,3,2)$. For $a_j \in G$, let $[a_j]$ be the conjugacy class of G which contains a_j. Then $[a_1] = \{e\}$, $[a_j] = \{a_2,a_3,a_4\}$ for $j = 2,3,4$, and $[a_j] = \{a_5,a_6\}$ for $j = 5,6$. Let $A = [a_2]$ and $B = [a_5]$. Note that $G^z = \{e\}$.

Let μ be the uniform probability measure on A, that is, $\mu(a_j) = 1/3$ for $j = 2,3,4$. Then $\mu \in P^z(G)$ and $e \notin A = s(\mu)$. Suppose that $\mu = t\mu_1 + (1-t)\mu_2$, where $\mu_1, \mu_2 \in P^z(G)$ and $0 < t < 1$. We show readily that $S(\mu_j) = A$ for $j = 1,2$, and that $\mu = \mu_1 = \mu_2$. Thus $\mu \in \text{ext } P^z(G) - G^z$.

Let ν be the uniform probability measure on B. Clearly $\mu \neq \nu$. Define $\lambda = (\mu+\nu)/2$. It follows that ν and λ are elements of $P^z(G)$, $e \notin S(\lambda) = A \cup B$, and $\lambda \notin \text{ext } P^z(G)$.

THEOREM 4. ext $M_r = \underline{G}$, and M_r is the closed convex hull of $\underline{G}$ in the strong operator topology of $B[L_2(G)]$.

PROOF. Since ext $P(G) = G$, we have at once from Theorem 1 that ext $M_r = \underline{G}$. It follows from the Krein-Milman theorem, together with Theorem 1, that M_r is the closed convex hull of $\underline{G}$ in the weak operator topology of $B[L_2(G)]$. Since the convex hull of $\underline{G}$ has the same closure in both the weak operator and the strong operator topologies of $B[L_2(G)]$ (Dunford and Schwartz [17]), the assertion follows. □

Since $G^z \subset \text{ext } P^z(G)$, or equivalently, $\underline{G}^z \subset \text{ext } M$, we have from the Krein-Milman theorem, together with Corollary 1, that M contains the closed convex hull of $\underline{G}^z$ in the weak operator topology of $B[L_2(G)]$.

REMARK 3. Let $T \in M_r$ and $\mu \in P(G)$ be the associated elements as in Theorem 1. Since G and $\underline{G}$ are topologically isomorphic, we may view the probability measure μ as a probability measure on M_r supported by $\underline{G} = \text{ext } M_r$. It follows that for $f,g \in L_2(G)$,

$$\langle f,Tg\rangle = \int_G\int_G f(x)\bar{g}(y^{-1}x)dm(x)d\mu(y) = \int_G \langle f,L_y g\rangle d\mu(y) = \int_{\underline{G}} \langle f,Lg\rangle d\mu(L),$$

so that μ is the only probability measure on M_r which represents T and which is supported by ext M_r. Thus a sharper form of the Choquet-Bishop-de Leeuw theorem (see Phelps [18], page 24) holds for M_r.

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