Modeling the Variance of Variance through a Constant Elasticity of Variance Generalized Autoregressive Conditional Heteroskedasticity Model

by

Mehdi Saedi

Bachelor of Science, Simon Fraser University, 2007

Jared Wolk

Bachelor of Arts, University of Calgary, 2004

RESEARCH PROJECT SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE IN FINANCE

BEEDIE SCHOOL OF BUSINESS

SIMON FRASER UNIVERSITY
Summer 2012

Approval

Name: **Mehdi Saedi and Jared Wolk**

Degree: **Master of Science in Finance**

Title of Project: **Measuring the Variance of Variance through a Constant Elasticity of Variance Generalized Autoregressive Conditional Heteroskedasticity Model**

Supervisory Committee:

____

Dr. David Peterson
Senior Supervisor
Adjunct Professor, Faculty of Business Administration

____

Dr. Christina Atanasova
Second Reader
Professor, Faculty of Business Administration

Date Approved: ____

Acknowledgements

We wish to thank Drs. Peterson and Atanasova for their support and input with this paper and throughout the year; as well as the entire faculty of the MSc program who filled our time at SFU with amazement. We thank Markit Analytics for presenting us the opportunity and technology to work through this fascinating subject matter. And to our families whose unwavering and unconditional support is inspiring and deeply appreciated.

Abstract

This paper compares a standard GARCH model with a Constant Elasticity of Variance GARCH model across three major currency pairs and the S&P 500 index. We discuss the advantages and disadvantages of using a more sophisticated model designed to estimate the variance of variance instead of assuming it to be a linear function of the conditional variance. The current stochastic volatility and GARCH analogues rest upon this linear assumption. We are able to confirm through empirical estimation that for equity returns and for some currency crosses the variance of variance does in fact grow at a rate which exceeds the standard linear expectations.

Table of Contents

Introduction: ... 6
Literature Review: ... 7
The Models: ... 10
Data & Methodology: ... 12
  Forecast Evaluation: ... 21
  Simulation: ... 22
Results: ... 24
Conclusion ... 38
References ... 39

Introduction:

This paper examines the effectiveness of the econometric modeling tools available to measure the variance of variance in univariate stochastic time series. Benjamin Graham, in his seminal work *The Intelligent Investor*, quotes legendary financer J.P. Morgan's response to a reporter's question, "what will the stock market do?" His response: "It will fluctuate." These fundamental fluctuations are the residuals of price discovery in a liquid market, they are the risk, the volatility, the uncertainty. The fundamental theories of finance are built off the foundation that returns in excess of the risk free rate can only be achieved in the presence of variance. Risk is the necessary condition for reward was the conclusion of Markowitz (1952) and economists in the 1950's. It soon became apparent the fluctuations JP Morgan referenced were themselves fluctuating, the volatility was itself volatile, risk was proving to be risky.

Mandelbrot (1963) was the one of the first empirical investigations into the newly observed phenomena of volatility clustering through his study of cotton price data. Mandelbrot initiated a whole field of financial study by analyzing the fractal behavior of cotton returns over multiple timeframes. One conclusion he drew was that after suitable renormalization the same distribution is maintained for all price changes over all changes in time. The draws from this distribution enforced randomness or difference into the equation. Mandelbrot was also able to demonstrate that price movements did not respect the Bachelier, and later Black-Scholes diffusion models which assumed Gaussian distributions. He noted volatility came in bursts and demonstrated varying degrees of dispersion. Heteroskedasticity, for the Greek 'hetero'-different 'skedasis'-dispersion, refers to the non-constant dispersion of asset returns. Because of the diffusive nature of price evolution the study of heteroskedasticity is a non-trivial pursuit in derivative and asset pricing theory.

Literature Review:

Bachelier (1900) laid the groundwork for option pricing theory by investigating speculation in his doctoral dissertation which lay dormant until the late 1950's. Paul Samuelson a professor at MIT was introduced to Bachelier's work and further developed the mathematics. Two of his notable students were Myron Scholes and Robert Merton. Scholes collaborated with Fischer Black to develop a hedging formula to accurately price an option contract. The Black Scholes (1973) equations transformed the financial world, but they rest on a base assumption of uniform dispersion, homoskedasticity. Merton (1973) derived the Black Scholes formulas by employing tools from stochastic calculus. This initial attempt at option pricing models attributed all randomness to a single source, the underlying asset price. However, market crashes like that of 1987 and the ensuing aftermath displayed with ferocity the fact that asset returns are not homoskedastic, and the need for more advanced volatility models was self-evident. The variance of variance was being investigated as a second source of uncertainty in the process. This work was pioneered by Cox, Ingersoll & Ross (1985) and Hull & White (1987).

Heston (1993) developed one of the first closed form stochastic volatility models using a stochastic variable correlated to the stochastic process driving the innovation in prices. An appealing feature of Heston's SV model allowed users to extract model parameters from sample time series of option data, and then use these parameters to compute out of sample option prices and hedging ratios with closed form solutions to compute delta and vega. One limitation for these stochastic volatility models is their dependence on the geometric Brownian diffusion process. This allows for pricing in a risk neutral measure, but does limit objective measure modeling and risk analysis.

Engle (1982) approached volatility modeling with the introduction of the Auto Regressive Conditional Heteroskedstic (ARCH) model. Bollerslev (1986) improved the model by introducing a generalized parameter and published the GARCH model. Engle and Bollerslev (1986) collaborated and introduced nonlinear versions of the model which would later be developed into our Constant Elasticity of Variance Generalized Auto Regressive Conditional Heteroskedasticity model (CEVGARCH). Engle and Bollerslev are responsible for the genesis of a vast body of literature on time varying volatility. Both authors have had their respective articles referenced by over 12,000 subsequent articles. An overview of the variations has yielded over 50 different published modifications with some of the most prolific being EGARCH Nelson (1991), GJRGARCH Glosten et al. (1993), AGARCH Engle (1990), NAGARCH Engle & Ng (1993) , APARCH Ding et al. (1993) and HNGARCH, a closed form option pricing equation developed by Heston & Nandi (2000)

Most of the modifications to the models attempt to accurately model the empirically observed asymmetry of asset return variance. The phenomena of asymmetry in stock return volatility has been documented since Nelson (1990b) and Pagan & Schewert (1990). Market price declines have been more highly correlated to higher future volatility than an equivalent market price increase. There is still much debate on precisely why this is observable; one of the prevailing theories attributes it to the "leverage effect", describing the increase in leverage functionally experienced when a firm's debt level remains constant while the value of the firm's equity experiences a large sudden decline. The theory suggests that such a firm is currently more leveraged then before, so the equity is now riskier, hence more volatility. Black (1976), Christie (1982) and Koutmos & Saidi (1995) are proponents that the firm's leverage ratios are the cause, while others like Hens & Steude (2009) suggest the phenomena is not related to a firm's fixed financing costs and can be observed in firms with no leverage at all. Whatever the reason, the phenomena exists but is not captured in standard GARCH models which parameterize conditional volatility using the squared innovations. EGARCH, NAGARCH and CEVGARCH are examples of modified GARCH models which have modifications to allow for varying quadratic impacts on future conditional variance from either positive or negative market moves.

We conclude our literature review with a discussion of Ishida and Engle (2001) who developed the CEVGARCH model. They borrowed from the work of Cox (1975) who first described the constant elasticity of variance while describing the interest rate process. Engle and Lee (1996) developed a Constant Elasticity of Variance (CEV) Stochastic volatility model to model variance as the process instead of interest rates. The bridge from a continuous time CEV stochastic volatility model to a discrete CEVGARCH model was explored by Ishida & Engle five years later. The value of the CEVGARCH model has been applied to the development of volatility derivatives like variance swaps. Javaheri et. al. (2004) demonstrates how GARCH models can be used to model the volatility process for the valuation and hedging of volatility swaps. In his PhD dissertation Javaheri (2004) confirmed the conclusions of Ishida & Engle when measuring the CEV exponent for the S&P 500. Ishida & Engle measure the precise value at 1.71 whereas Javaheri constrained his measurements to 0.5, 1.0 and 1.5 and reports 1.5 to give the better fit. We seek to confirm the findings of these researchers by analyzing the CEVGARCH and GARCH models using current risk factor data which covers periods of high volatility like 2008 and 2010.

The Models:

We are investigating the benefit of estimating the variance of variance in a GARCH process, and comparing our results to a standard model. The control model we are testing against is the standard GARCH(1,1) due to Bollerslev (1986). This basic symmetric model assumes that the response to a shock to the conditional variance is symmetric for both positive and negative returns. This assumption has been challenged by Nelson (1991), which has led to the development of family of asymmetric GARCH models such as EGARCH, NAGARCH, AGARCH and others mentioned above. Following the model elaborated by Bollerslev, we define $\mu_t \coloneqq E_{t-1}[R_t]$,

$$R_t = \mu_t + \epsilon_t \ , \tag{1.1}$$

$$h_t = \sigma_t^2 \ , \tag{1.2}$$

$$\epsilon_t = \sqrt{h_t} z_t \ , \tag{1.3}$$

$$h_{t+1} = \omega + \beta h_t + \alpha \epsilon_t^2 \ , \tag{1.4}$$

Where the z_ts are i.i.d., with zero mean and unit variance. We can rearrange the conditional variance (Eq. 1.4) as follows:

$$h_{t+1} = \omega + (\alpha + \beta) h_t + \alpha(\epsilon_t^2 - h_t) \ , \tag{1.5}$$

$$= \omega + \gamma h_t + \alpha h_t \eta_t \ . \tag{1.6}$$

Here $\gamma \coloneqq \alpha + \beta$, $\eta_t \coloneqq z_t^2 - 1$, and ω, α, β are positive constants, we require $\alpha + \beta < 1$ for covariance stationarity of $\{\epsilon_t\}$. Eq.1.6 is the standard GARCH model of Bollerslev. To follow this model through to the development of CEV GARCH we need to first recall the stochastic differential equation Cox, Ingersoll and Ross presented in 1985:

$$dY_t = [\xi\mathrm{Y} + \zeta]\mathrm{dt} + \nu\sqrt{\mathrm{Y}}\, dW_t \, . \quad (1.7)$$

By comparing the drift portion of this equation to Eq. 1.6, we define the rate at which the conditional variance reverts to its mean $\theta \coloneqq E[h_t]$ as $\varphi \coloneqq 1 - \gamma$ if $\gamma < 1$. Changing labels slightly, the stochastic process can then be written as:

$$d\nu_t = \varphi(\theta - \nu_t)dt + \sigma_\nu\sqrt{\nu_t}dW_t. \quad (1.8)$$

Equation 1.8 is the continuous time formulation used by Heston (1993). It can be discretized from continuous time into the GARCH analogue:

$$h_{t+1} - h_t = \varphi(\theta - h_t) + \alpha\sqrt{h_t}\eta_t, \quad (1.9)$$

$$h_{t+1} = \omega + \gamma h_t + \alpha\sqrt{h_t}\eta_t \, , \quad (1.10)$$

$$h_{t+1} = \omega + \gamma h_t + \alpha h_t^{\frac{1}{2}}\eta_t \, . \quad (1.11)$$

When we compare Eq. 1.6 with Eq. 1.11 we can see the only difference lies in the exponent, 1/2, associated with the conditional variance in the ARCH component of the equation. The standard GARCH model, Eq. 1.6 has an exponent value of 1 which is not displayed. To the best of our knowledge there is no theoretical or practical reason to restrict this exponent to be either 1 or 1/2. Once we accept a possible range of positive values for the exponent we arrive at the Constant Elasticity of Variance GARCH model:

$$h_{t+1} = \omega + \gamma h_t + \alpha h_t^p\eta_t. \quad (1.12)$$

The p exponent on the ARCH conditional variance modifies the way the variance of variance evolves with the current level of variance. The asymmetry of variance can be captured by defining $\eta_t \coloneqq z_t^2 + \phi z_t - 1$ instead of $\eta_t = z_t^2 - 1$ with $\phi \in \mathbb{R}$. We can easily extend this CEVGARCH(1,1) model to allow for optimal fitting at the appropriate lags by defining a CEVGARCH(P,Q) model:

$$h_{t+1} = \omega + \sum_{i=1}^{P} \gamma_i h_t + \sum_{i=1}^{Q} \alpha_i h_t^p \eta_t. \tag{1.13}$$

Data & Methodology:

The foreign exchange (FX) markets are the largest markets in the world with $3.98 trillion in notional volume trading hands daily according to the Bank of International Settlements. With such a large liquidity pool and twenty four hour access the FX markets are the closest to a continuous market available on the planet. We have selected four risk factors from the FX markets and have also chosen to investigate two equity indices to verify the research of others. We have selected to analyze the following risk factors: USD/CAD, GBP/USD, EUR/USD the American S&P 500 and the Canadian S&P TSX composite index. Time series data for each of the risk factors has been provided by the Economic Research department of the Federal Reserve Bank of St. Louis and Bloomberg. The sample time series for each of the risk factors extends from Jan 2000 through to Dec 2011 for a complete 12 years of daily prices. Our sample period was selected on the smallest sample available, the EUR/USD exchange rate. The other risk factors have extended time series extending back to Jan 1971, these extended series are used for only one of the calculations which required more data points then were available in the sample series.

We find evidence for heteroskedasticity in the sample time series by applying a Ljung Box test to the demeaned residuals. The test statistic from the test is defined as:

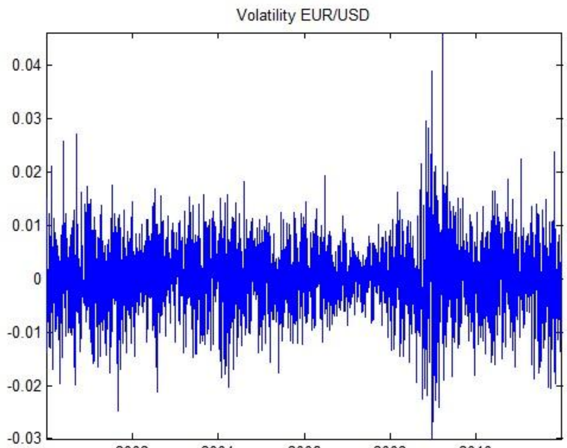
$$Q = T(T+2)\sum_{k=1}^{L}\frac{\rho_k^2}{T-k}\,. \tag{2.1}$$

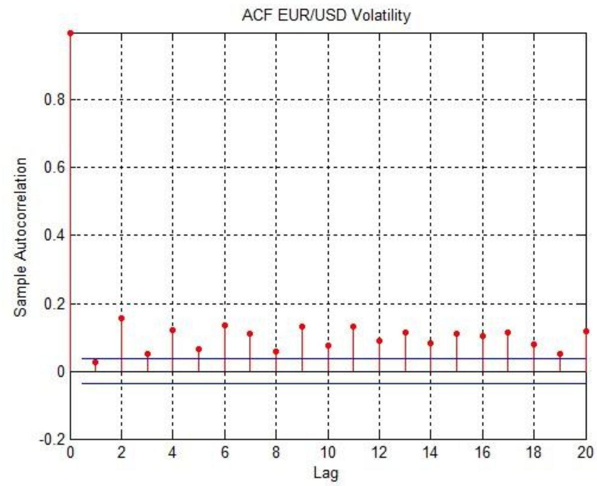
Here ρ_k is the sample autocorrelation at lag k. The asymptotic distribution of Q is chi-square with L degrees of freedom under the null hypothesis. Each of our time series tests rejected the null hypothesis indicating serial correlation exists in the squared residuals demonstrating heteroskedasticity. The results of these tests can be verified in table 1. We can also infer heteroskedasticity in the sample time series by visually inspecting the autocorrelation functions. We can clearly see autocorrelation present in excess of the confidence interval at almost all lags in all time series in figure 1 - 5. The volatility graphs adjacent to the ACF graphs display the clustering described by Mandelbrot.

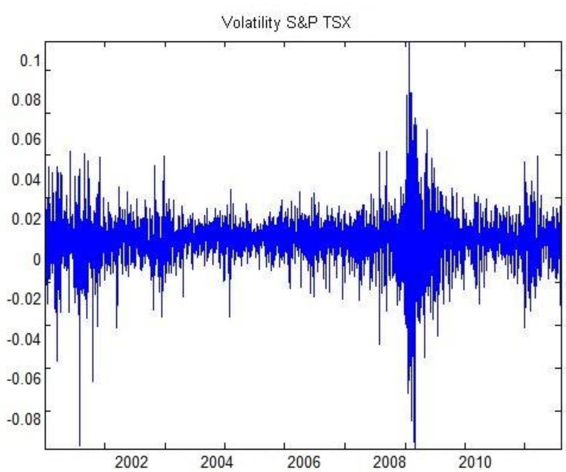
Table 1

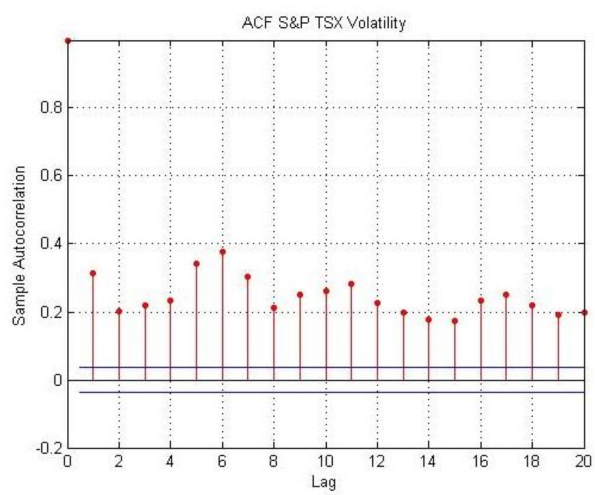
	Q	P-Value	Null
EUR/USD	645.4271	0	Reject
GBP/USD	3.38E+03	0	Reject
USD/CAD	1.08E+04	0	Reject
S&P500	7.45E+03	0	Reject
S&P TSX	5.06E +04	0	Reject

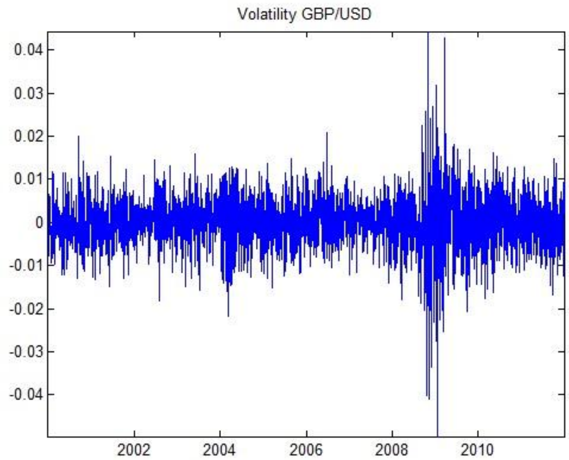
Figures 1 - 5

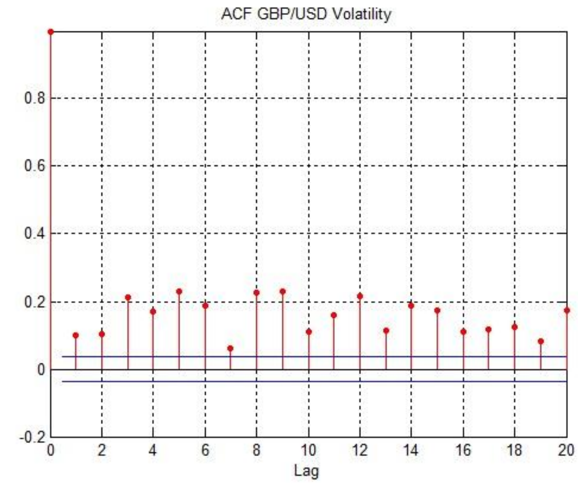

Volatility EUR/USD
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
2002
2004
2006
2008
2010

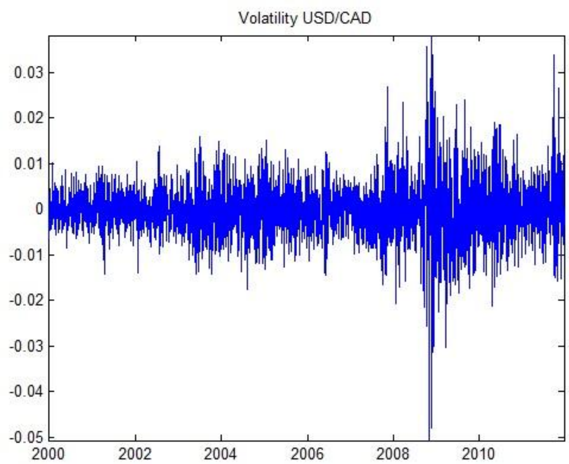

ACF EUR/USD Volatility
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

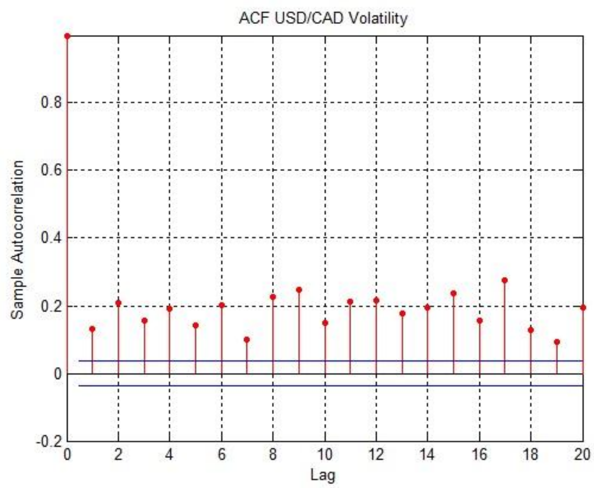

Volatility S&P TSX
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
2002
2004
2006
2008
2010

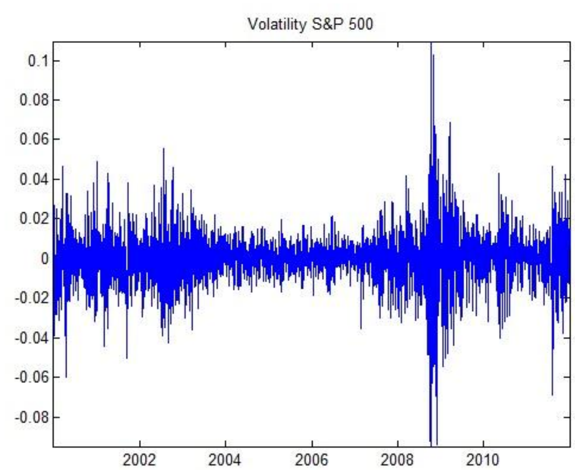

ACF S&P TSX Volatility
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

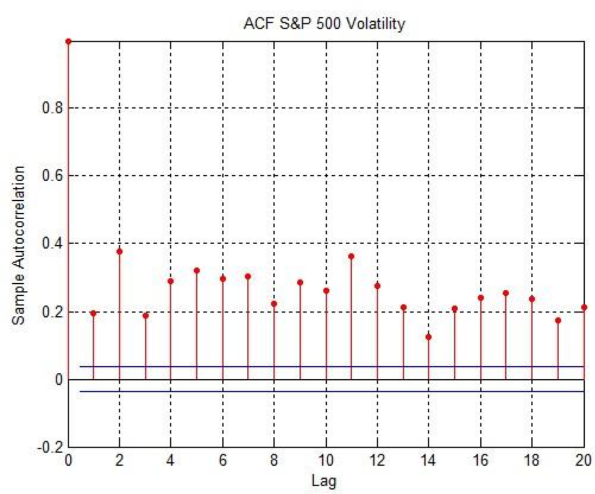

Volatility GBP/USD
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
2002
2004
2006
2008
2010


ACF GBP/USD Volatility
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

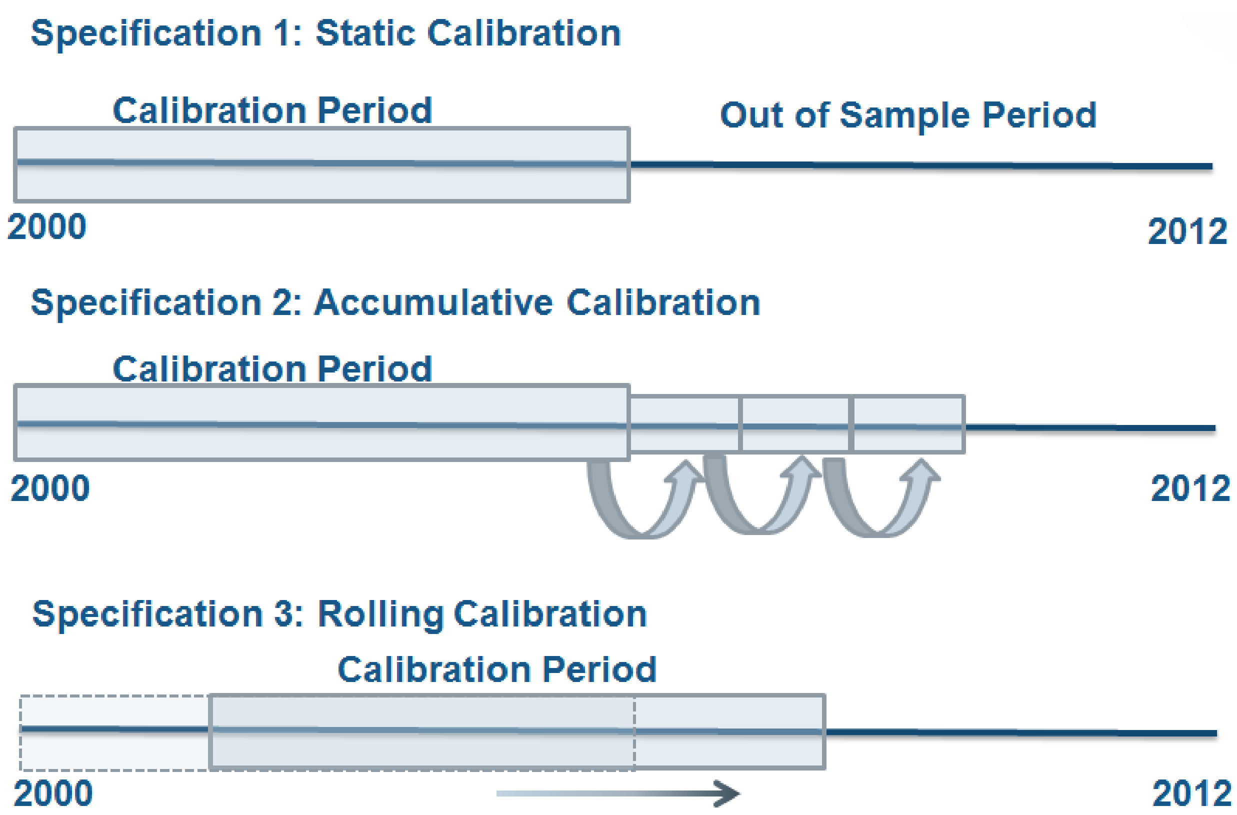

Volatility USD/CAD
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
2000
2002
2004
2006
2008
2010


ACF USD/CAD Volatility
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag


Volatility S&P 500
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
2002
2004
2006
2008
2010


ACF S&P 500 Volatility
Sample Autocorrelation
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
14
16
18
20
Lag

Now that we have evidence for heteroskedasticity, we can apply the GARCH and CEVGARCH models. The initial step is to calibrate the models. Model calibration occurs over 3 different specifications. First, a static calibration is performed on a portion of the time series called the calibration period; second, an accumulative calibration period and finally, a rolling calibration period. The accumulative calibration begins at $T - x$ periods through to T. As the time series is processed recalibration occurs at Y period intervals. The data for the recalibration now includes $T - x$ periods through to $T + Y$ periods. The calibration period grows linearly along with the processed data to mimic the data accumulation practitioners experience in the world. The third calibration environment has a rolling calibration period which is limited to a X period window. This window rolls forward through time discarding the most distant past X period of data as every new recalibration takes place at X period intervals.



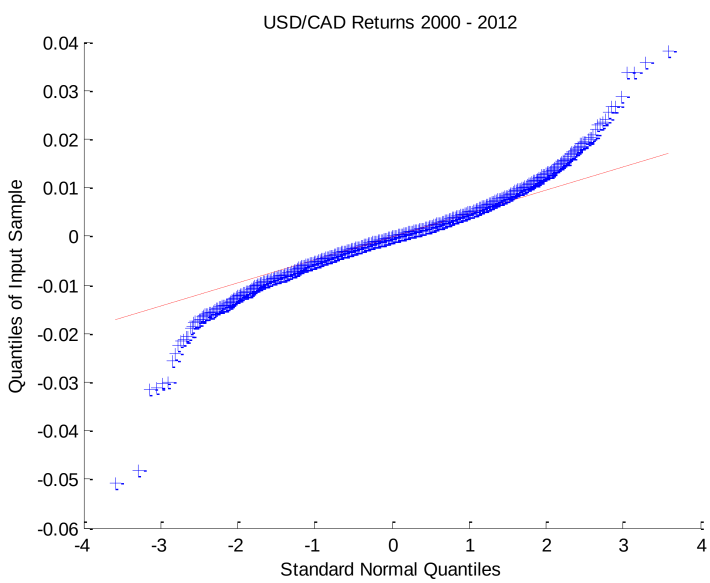
We employed two likelihood functions to estimate the parameters for both the GARCH and CEVGARCH models to capture the distributional assumptions of ϵ_t. Under the Gaussian distribution assumption the likelihood function we seek to maximize is

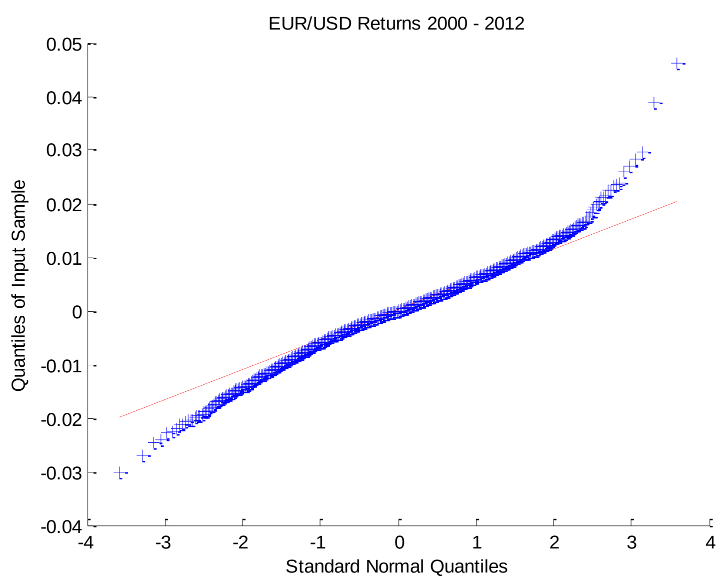
$$L = \prod \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{\epsilon_t^2}{2\sigma_t^2}}. \tag{2.2}$$

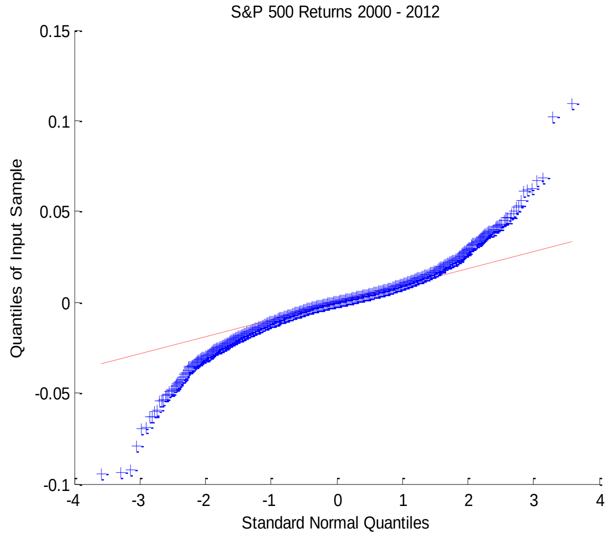
In practice we use an algorithm to minimize the inverse logarithm of the likelihood equation. The log likelihood function is

$$l = \sum_{t=m+1}^{T} \left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma_t^2) - \frac{1}{2}\frac{\epsilon_t^2}{\sigma_t^2} \right]. \tag{2.3}$$

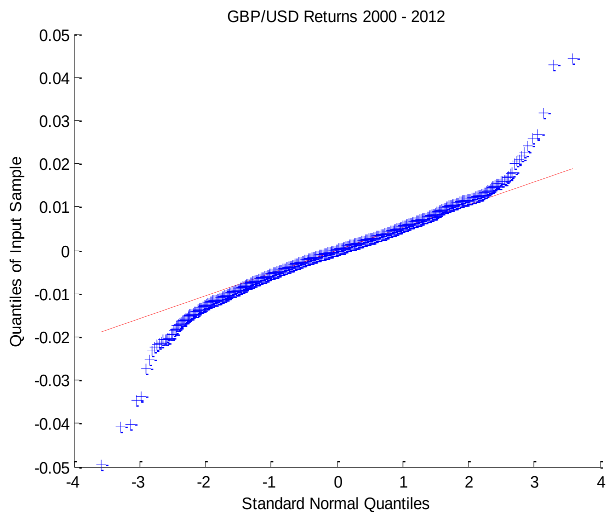
Figures 5-9 below demonstrate that our risk factors returns are not normally distributed.



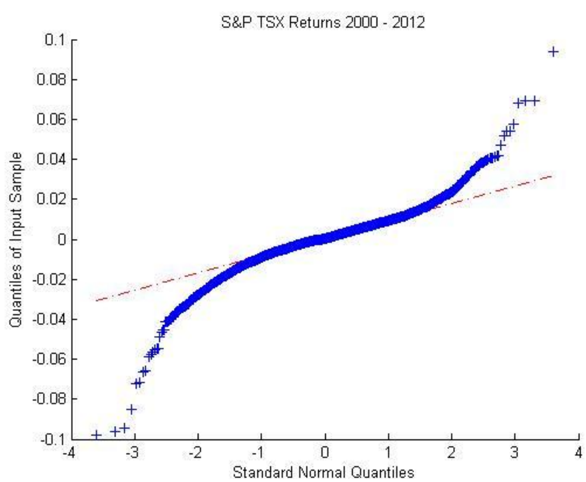




S&P 500 Returns 2000 - 2012
Quantiles of Input Sample
Standard Normal Quantiles



GBP/USD Returns 2000 - 2012
Quantiles of Input Sample
Standard Normal Quantiles



S&P TSX Returns 2000 - 2012
Quantiles of Input Sample
Standard Normal Quantiles

Due to the leptokurtic nature of the return distributions for our risk factors we performed our parameterization for both models using a standardized student-*t* distribution. If we assume ϵ_t is from a heavier tailed student-t distribution, the probability density function is:

$$f(\epsilon_t|\nu) = \left(\frac{\Gamma\left[\frac{\nu+1}{2}\right]}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu-2)\pi}}\right)\left(1+\frac{\epsilon_t^2}{\nu-2}\right)^{-\frac{\nu+1}{2}}, \ \nu > 2\,, \tag{2.4}$$

$$\text{Where } \ \Gamma(x) = \int_0^\infty y^{x-1}e^{-y}dy. \tag{2.5}$$

We estimate ν jointly with the other parameters, so the conditional log likelihood function we are minimizing is:

$$l = -\sum_{i=1}^{T}\left(\ln(\sigma) + \left(\frac{\nu+1}{2}\right)\ln\left(1+(\nu-2)^{-1}\left(\frac{\epsilon_t}{\sigma_t}\right)^2\right)\right) + Tln\left[\left((\nu-2)\pi\right)^{-\frac{1}{2}}\Gamma\left(\frac{\nu}{2}\right)^{-1}\Gamma\left(\frac{\nu+1}{2}\right)\right]. \tag{2.6}$$

In order to determine the significance of our model parameters we also measure the standard errors of our parameter estimates. The standard errors are the variance of the parameter estimates, so we require the variance-covariance matrix of our parameter estimates. We arrive here by inverting the information matrix, defined as $I[\theta] = -E[H(\theta)]$, as per Eq. 2.7.

$$var(\theta) = [I(\theta)]^{-1} \tag{2.7}$$

The H in the information matrix expectation is the hessian matrix which is composed of the second derivatives of the likelihood with respect to the estimated parameters.

$$H(\theta) = \frac{\partial^2 ln\mathcal{L}(\theta)}{\partial\theta\partial\theta'} \tag{2.8}$$

We can now calculate the variance by computing the square root of the diagonal terms in the variance-covariance matrix.

$$var(\theta) = \left(-E\left[\frac{\partial^2 ln\mathcal{L}(\theta)}{\partial\theta\partial\theta'}\right]\right)^{-1} \quad (2.9)$$

Our method of choice to calculate our hessian matrix is the Outer Product of Gradients (OPG) method. In this method, we first evaluate the log likelihood objective function at the MLE parameter estimates. In contrast to the optimization, which is interested in the single scalar objective function value logL, here we are interested in the log likelihoods for each observation of y(t), which sum to –logL:

Next, we calculate the OPG scores matrix:

$$S = [S_1\ S_2 \dots S_m]$$

Where m is the number of parameter estimates and

$$S_i = \begin{bmatrix} \dfrac{(loglikelihood_1 - \Delta loglikelihood_1)}{d\theta_i} \\ \dfrac{(loglikelihood_2 - \Delta loglikelihood_2)}{d\theta_i} \\ \dots \\ \dfrac{(loglikelihood_n - \Delta loglikelihood_n)}{d\theta_i} \end{bmatrix}$$

n is the number of observations (t_i), log likelihood$_i$ is the likelihood for the observation y(t_i) using the original estimates θ, Δlog likelihood$_i$ is the likelihood for the observation y(t_i) using

$$\Delta\theta = \theta \times (1 + \varepsilon) \quad (2.10)$$

And

$$d\theta_i = \theta \times \varepsilon \quad (2.11)$$

Finally, the covariance matrix of the MLE parameters is approximated by inverting the OPG scores matrix:

$$var(\theta) = [S' \times S]^{-1} \quad (2.12)$$

Forecast Evaluation:

After calibrating the model over a given period, we use the parameterized model to forecast the next period ahead conditional variance. The estimate is compared with the realized variance of the period returns to determine the accuracy of the models. We compare our results using the following two statistical metrics:

1. R^2 , the coefficient of determination from a linear regression:

$$y_i = \alpha + \beta x_i + \epsilon_i \,, \tag{3.1}$$

$$f_i = \alpha + \beta q_i \,, \tag{3.2}$$

$$R^2 = 1 - \frac{\sum_i (y_i - f_i)^2}{\sum_i (y_i - \bar{y})^2} . \tag{3.3}$$

2. Breaches of the confidence intervals, with confidence for i^{th} interval defined as:

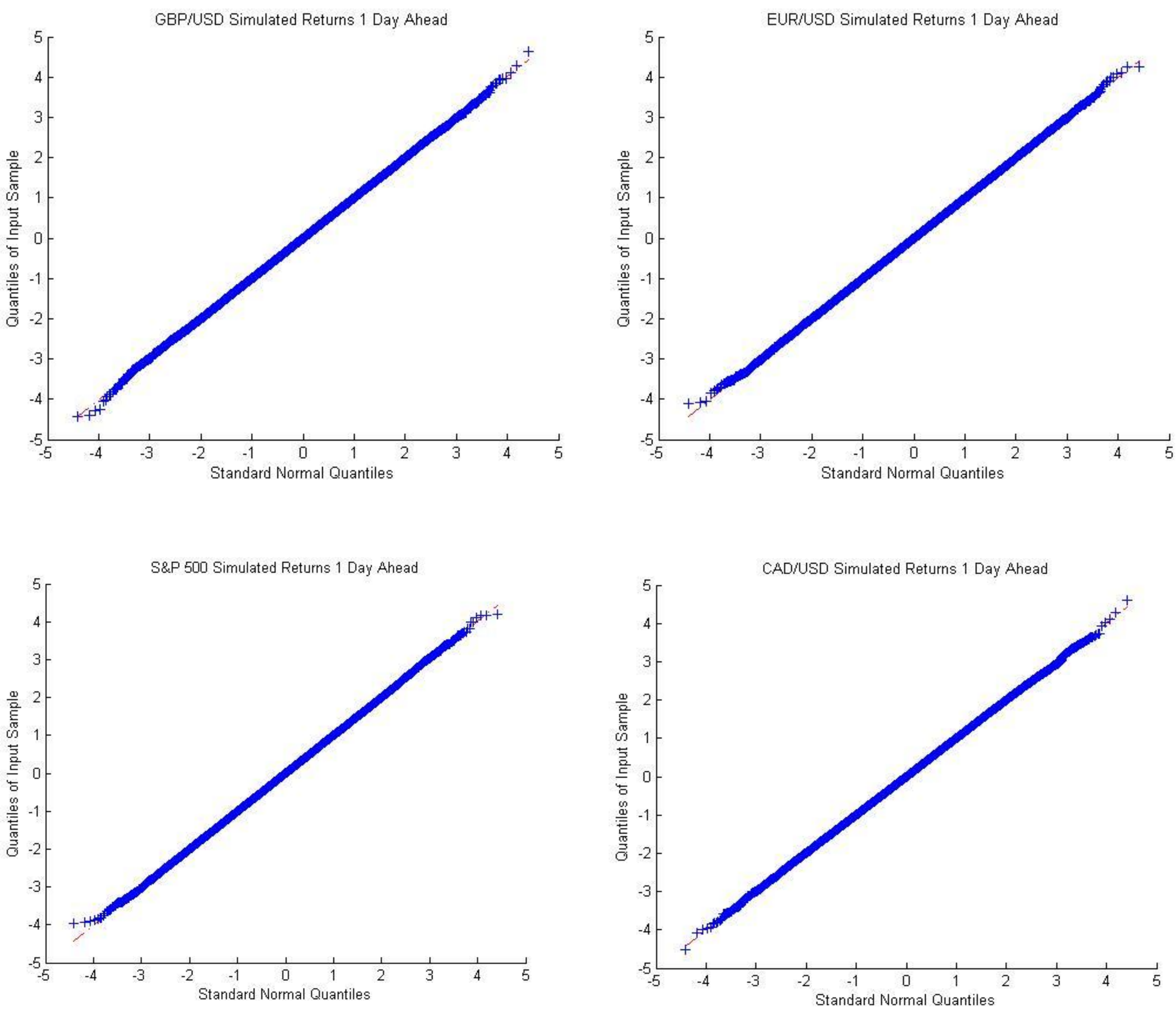
$$\pm C_i = \Phi^{-1}\left(\frac{i+1}{2}\right) . \tag{3.4}$$

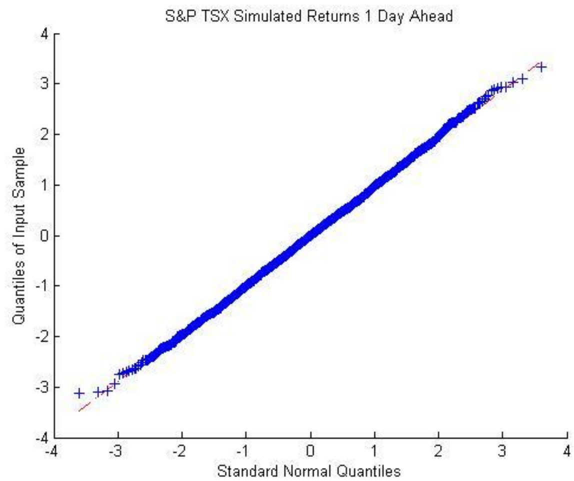
A breach is defined as $|r| > \pm C_i$.

Simulation:

Once we have a calibrated a CEVGARCH or GARCH model, we employ Monte Carlo simulation methods to simulate the risk factor process forward over selected time intervals. Returns simulated over 1 day demonstrate a Gaussian distribution. Indeed, we can see in Fig 9-12 that for all risk factors, both models exhibit the expected distribution. Q-Q plots demonstrate visually the variation from the Gaussian distribution.

Figures 9 - 12





If our models are functioning properly we would expect to see leptokurtic simulated returns as we move further and further through simulated time. We measure the kurtosis of the simulated distributions as the standardized fourth moment.

$$k = \frac{E[R_t^4]}{\{E[R_t^2]\}^2} \ . \tag{3.5}$$

We simulate the risk factors evolving according to the GARCH and CEVGARCH models across time frames of 10 days, 20 days, 60 days, 90 days and 126 days. The simulated asset paths follow the process described in Eq. 3.6 with the assumptions $Z_i \approx \mathcal{N}(0,1)$ and $\forall i \ \in [0, T-1]$:

$$\log S_{t+1} = \log S_t + \left(\mu - \frac{1}{2}\sigma_t^2\right)\Delta t + \left(\sqrt{\sigma_t^2 \times \Delta t}\right) Z_t \ . \tag{3.6}$$

Results:

Our initial calibration results are what we use as simulation parameters for the Monte Carlo simulations. The calibration period for all of the risk factors is the same 6 year period spanning January 2000 through to December 2005. The calibrated parameters for both models are listed in tables 2 - 4.

Tables 2 – 4

Risk Factor	**EUR**				**GBP**			
	GARCH		**CEV GARCH**		**GARCH**		**CEV GARCH**	
Parameter	Estimate	(S.E)	Estimate	(S.E)	Estimate	(S.E)	Estimate	(S.E)
μ	**0.000142**	**0.00005**	**0.00012**	**0.000051**	**8.87E-05**	**0.000038**	**6.00E-05**	**0.000025**
ω	**4.69E-07**	**<1.0E-10**	**5.12E-08**	**<1.0E-10**	**8.59E-07**	**<1.0E-10**	**6.72E-07**	**<1.0E-10**
γ	**0.988399**	**0.451523**	**1.06114**	**0.366900**	**0.96948**	**0.427712**	**0.997**	**0.407357**
α	**0.018733**	**0.007483**	**0.29734**	**0.128051**	**0.05174**	**0.021862**	**0.010992**	**0.005555**
ϕ			**-1.64458**	**0.62449**			**-0.43884**	**0.18843**
p			**1.17913**	**0.508746**			**0.838571**	**0.324688**
ν	**12.344**	**1.037398**	**14.5255**	**2.067867**	**14.396**	**6.504646**	**15.5134**	**1.992637**
H_0	**0.006417**	**0.002193**	**5.54E-05**	**0.000024**	**0.00528**	**0.002704**	**2.44E-05**	**0.00001**
Log Likelihood	**5481.71**		**6342.68**		**5793.53**			**6659.65**

Risk Factor	**CAD**				**S&P 500**			
	GARCH		**CEV GARCH**		**GARCH**		**CEV GARCH**	
Parameter	Estimate	(S.E)	Estimate	(S.E)	Estimate	(S.E)	Estimate	(S.E)
μ	**-6.62E-05**	**-0.000027**	**-0.00013**	**--0.000044**	**0.00021**	**0.00008**	**-6.00E-05**	**-0.000034**
ω	**1.13E-07**	**<1.0E-10**	**3.27E-08**	**~0.0**	**7.21E-07**	**~0.0**	**8.40E-07**	**<1.0E-10**
γ	**0.994956**	**0.466796**	**1.00554**	**0.518187**	**0.99556**	**0.361068**	**0.989468**	**0.414689**
α	**0.033564**	**0.015116**	**0.13057**	**0.052482**	**0.07462**	**0.031638**	**0.463255**	**0.160782**
ϕ			**-0.24817**	**0.09461**			**-0.084525**	**0.03464**
p			**1.13233**	**0.493900**			**1.20573**	**0.411746**
ν	**12.246**	**3.345225**	**22.503**	**4.433275**	**7.1165**	**3.082957**	**18.4793**	**2.601551**
H_0	**0.00464**	**0.001966**	**9.93E-06**	**0.000005**	**0.01193**	**0.004996**	**0.00036**	**0.0002**
Log likelihood	**6037.75**		**6904.09**		**4736.2**		**5604.21**	

Risk Factor	**TSX**			
	GARCH		**CEV GARCH**	
Parameter	Estimate	(S.E)	Estimate	(S.E)
μ	**0.000634**	**0.00025**	**0.00023**	**0.00011**
ω	**3.94E-07**	**<1.0E-10**	**2.66E-07**	**<1.0E-10**
γ	**0.997581**	**0.45846**	**1.05184**	**0.41782**
α	**0.059422**	**0.02634**	**0.3196**	**0.11605**
ϕ			**-0.77**	**0.33905**
p			**1.13693**	**0.47088**
ν	**12.344**	**4.77136**	**7.08336**	**2.63477**
H_0	**0.010532**	**0.00561**	**4.70E-04**	**0.00019**
Log Likelihood	**5051.14**		**5963.67**	

The estimated CEV parameter confirms the previous conclusions of prior researchers, Ishida & Engle as well as Javaheri, who obtained exponent estimates greater than 1 for the CEV specification of the conditional variance. Although we obtain different parameter estimates, we also obtain a CEV exponent greater than 1 for the S&P 500 as well as for the TSX. The FX risk factors all had a CEV exponent greater than 1 except the GBP.USD which had an estimated value at 0.86.

We did not include all of the parameter estimates for each of the periods, which require multiple calibrations. Instead, results from multiple calibration periods were tested using the out of sample testing measure previously described. Results in tables 4-8 demonstrate the test statistics following a static period calibration, an accumulative calibration period with the accumulation period equal to 126 or 252 trading days, then a rolling calibration where the size of the calibration window stays constant but moves along adding new and subtracting old 126 and 252 periods.

	Table 4	**Confidence 95%**	**R2**	**S&P500 Confidence 99%**	**R2**	**Confidence 99.9%**	**R2**
Static	GARCH	6.25%	28.83%	2.21%	28.83%	0.37%	19.98%
	CEV	6.07%	29.22%	2.39%	29.22%	0.31%	20.63%
A. 126	GARCH	5.96%	27.09%	2.34%	27.09%	0.43%	20.25%
	CEV	5.95%	28.87%	2.21%	28.87%	0.39%	20.35%
A. 252	GARCH	6.19%	28.56%	2.39%	28.56%	0.39%	20.40%
	CEV	5.95%	27.23%	2.39%	27.23%	0.42%	20.00%
Roll 126	GARCH	6.19%	28.30%	2.57%	28.30%	0.41%	17.45%
	CEV	5.89%	26.40%	2.43%	26.43%	0.38%	19.73%
Roll 252	GARCH	6.37%	27.37%	2.57%	27.37%	0.39%	19.91%
	CEV	5.43%	28.08%	2.39%	28.08%	0.43%	16.29%

Table 5		EUR Confidence 95%	R2	Confidence 99%	R2	Confidence 99.9%	R2
Static	GARCH	5.39%	10.11%	1.35%	10.11%	N/A	6.23%
	CEV	5.10%	9.25%	1.46%	9.25%	N/A	5.66%
A. 126	GARCH	5.27%	8.19%	1.41%	8.19%	N/A	2.60%
	CEV	5.16%	8.88%	1.29%	8.88%	N/A	2.60%
A. 252	GARCH	5.33%	8.92%	1.23%	8.92%	N/A	3.07%
	CEV	5.10%	9.69%	1.17%	9.69%	N/A	2.75%
Roll 126	GARCH	5.33%	8.19%	1.23%	8.19%	N/A	2.67%
	CEV	5.51%	8.85%	1.29%	8.85%	N/A	2.53%
Roll 252	GARCH	5.33%	9.08%	1.17%	9.08%	N/A	2.86%
	CEV	5.21%	9.39%	1.11%	9.39%	N/A	2.69%

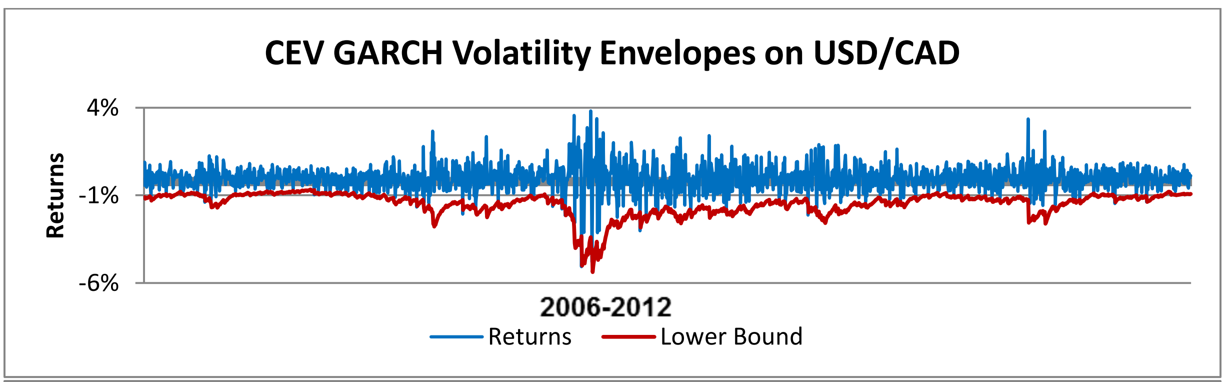
Table 6		GBP Confidence 95%	R2	Confidence 99%	R2	Confidence 99.9%	R2
Static	GARCH	5.98%	16.75%	1.87%	16.75%	0.63%	13.92%
	CEV	6.09%	17.22%	1.82%	17.22%	0.59%	13.75%
A. 126	GARCH	5.86%	11.43%	1.82%	11.43%	0.75%	13.94%
	CEV	6.21%	11.79%	1.70%	11.79%	0.41%	14.34%
A. 252	GARCH	5.80%	12.38%	1.82%	12.38%	0.75%	13.57%
	CEV	5.86%	13.13%	1.58%	13.13%	0.40%	13.83%
Roll 126	GARCH	5.68%	11.25%	1.64%	11.25%	0.70%	14.28%
	CEV	5.39%	9.28%	1.64%	9.28%	0.42%	4.50%
Roll 252	GARCH	5.62%	12.38%	1.70%	12.38%	0.73%	13.78%
	CEV	5.27%	11.13%	1.64%	11.13%	0.44%	10.03%

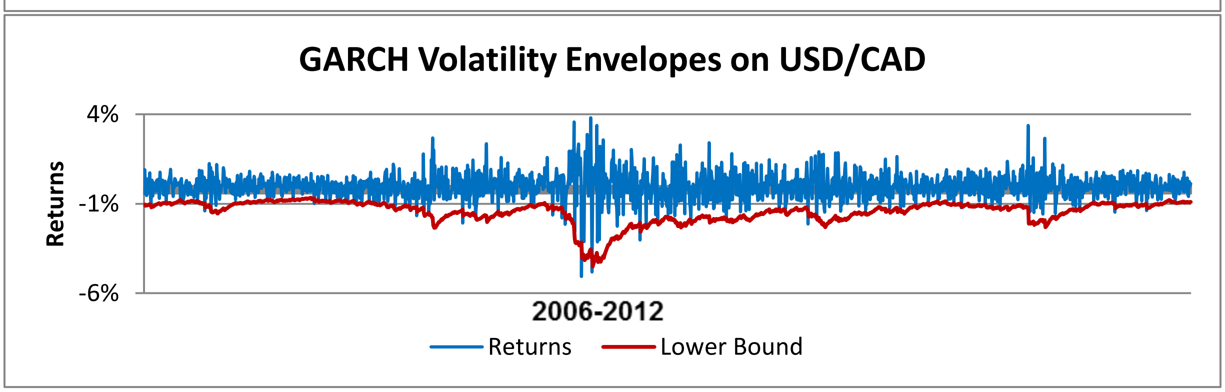
	Table 7	CAD					
		Confidence 95%	**R2**	**Confidence 99%**	**R2**	**Confidence 99.9%**	**R2**
Static	GARCH	6.32%	13.76%	1.35%	13.76%	0.27%	26.55%
	CEV	6.26%	13.95%	1.28%	13.96%	0.29%	26.43%
A. 126	GARCH	6.33%	11.85%	1.58%	11.85%	0.46%	24.61%
	CEV	6.09%	13.10%	1.52%	13.10%	0.37%	26.70%
A. 252	GARCH	6.20%	13.08%	1.41%	13.07%	0.45%	26.10%
	CEV	5.92%	13.54%	1.28%	13.55%	0.36%	26.55%
Roll 126	GARCH	6.09%	12.30%	1.34%	12.30%	0.43%	26.16%
	CEV	5.09%	13.03%	1.05%	13.04%	0.49%	26.87%
Roll 252	GARCH	6.04%	13.24%	1.28%	13.24%	0.43%	26.46%
	CEV	5.91%	12.75%	1.40%	12.76%	0.45%	26.85%

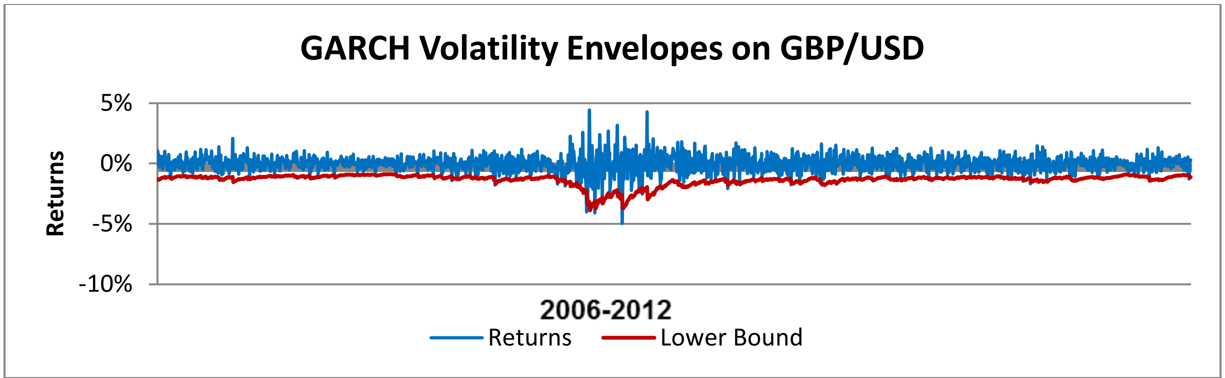
	Table 8	TSX					
		Confidence 95%	**R2**	**Confidence 99%**	**R2**	**Confidence 99.9%**	**R2**
Static	GARCH	6.55%	28.20%	2.62%	28.20%	0.51%	23.54%
	CEV	7.18%	28.18%	2.50%	28.18%	0.54%	23.66%
A. 126	GARCH	7.05%	18.25%	2.56%	18.25%	0.51%	12.37%
	CEV	6.99%	18.60%	2.31%	18.60%	0.48%	13.25%
A. 252	GARCH	6.94%	21.70%	2.56%	21.70%	0.54%	13.53%
	CEV	6.99%	21.78%	2.31%	21.78%	0.67%	12.92%
Roll 126	GARCH	7.18%	18.93%	2.75%	18.93%	0.54%	12.45%
	CEV	7.62%	18.17%	2.68%	18.17%	0.61%	12.77%
Roll 252	GARCH	7.12%	22.10%	2.62%	22.10%	0.51%	13.52%
	CEV	7.24%	23.89%	2.62%	23.89%	0.67%	13.67%

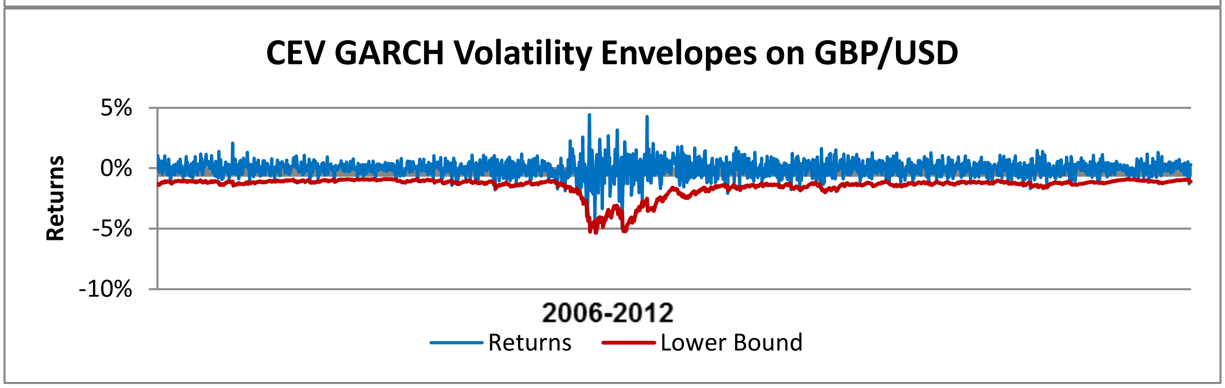
We are seeking a higher value for the R^2 to determine the quality of the out of sample fit. For each confidence interval measurement we can comment on the accuracy of the model by analyzing the breach measurement's spread to the i^{th} confidence interval in Eq. 3.4. The results demonstrate consistent superior performance along the accumulative calibration specification with a 126 period modifier. We can see in tables 4 -8 the A.126 tests demonstrate that the CEVGARCH produces a higher quality out of sample fit with a higher R^2 as well as a breach percentage which is closer to the confidence interval measured. The breaches along the single tailed distribution at the given confidence levels can be visually inspected in Figures 13-22. We observe that when the current variance is large the two models produce differing estimates for the next period conditional variance, which reinforces the purpose and validity of modeling the variance of variance.

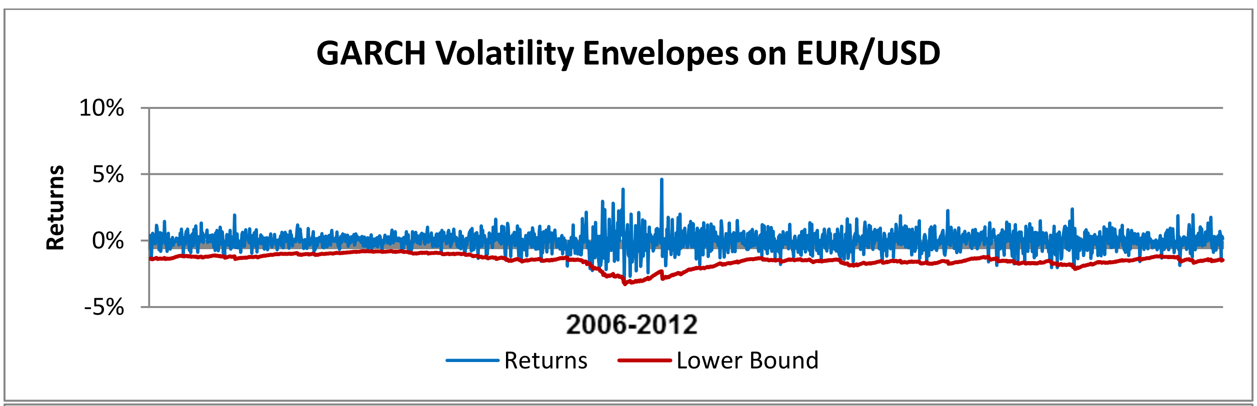
FIGURES 13 - 22

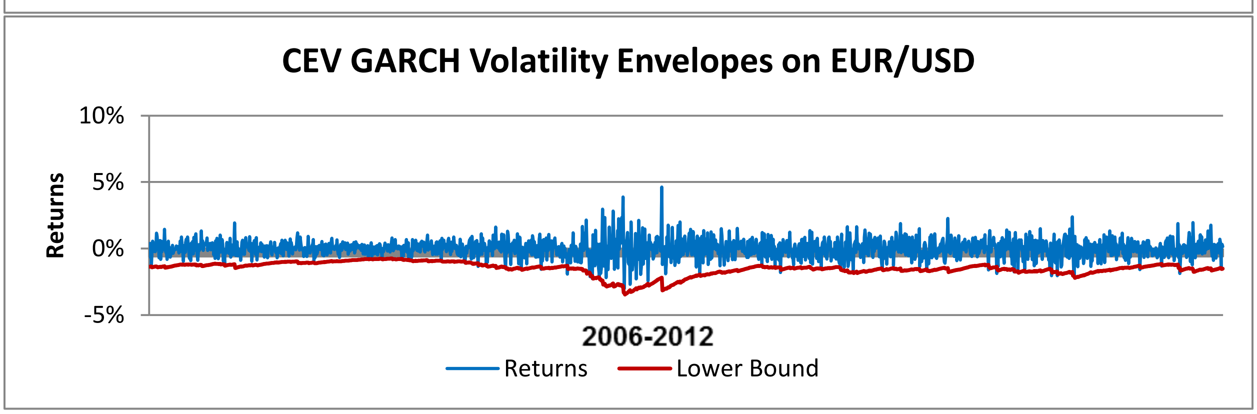


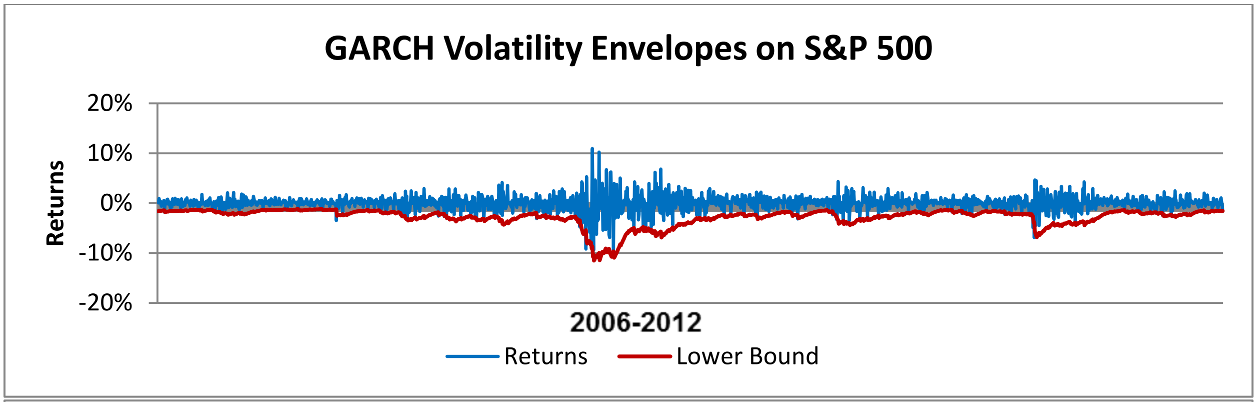


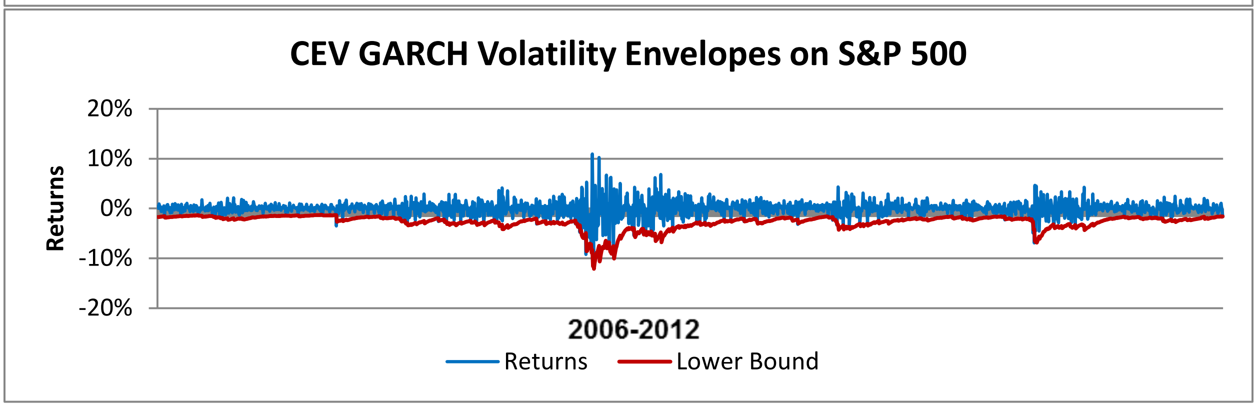

GARCH Volatility Envelopes on GBP/USD
5%
0%
-5%
-10%
Returns
2006-2012
Returns
Lower Bound

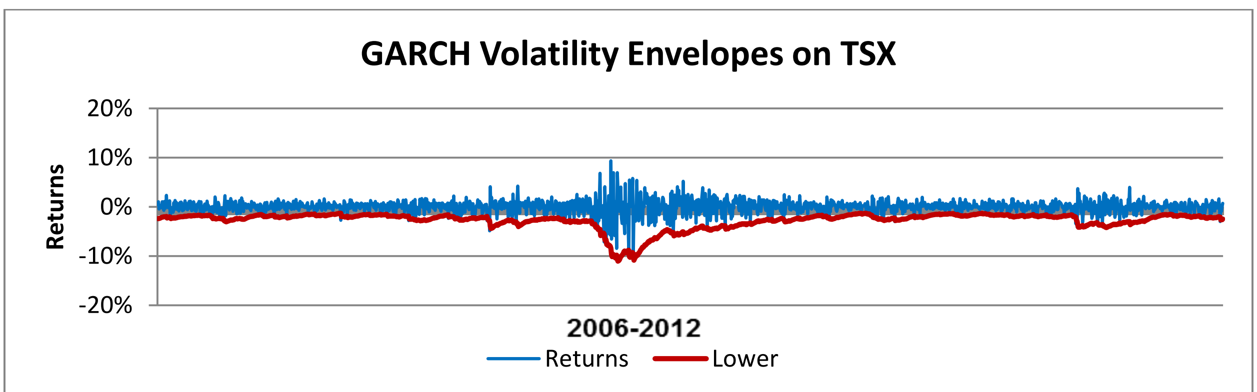

CEV GARCH Volatility Envelopes on GBP/USD
5%
0%
-5%
-10%
Returns
2006-2012
Returns
Lower Bound

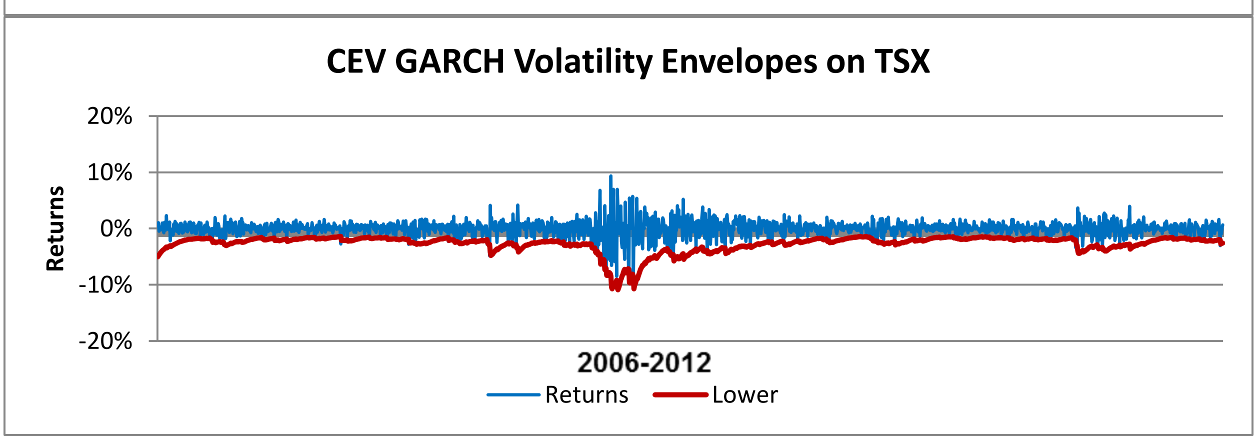

GARCH Volatility Envelopes on EUR/USD
10%
5%
0%
-5%
Returns
2006-2012
Returns
Lower Bound


CEV GARCH Volatility Envelopes on EUR/USD
10%
5%
0%
-5%
Returns
2006-2012
Returns
Lower Bound


GARCH Volatility Envelopes on S&P 500
20%
10%
0%
-10%
-20%
Returns
2006-2012
Returns
Lower Bound


CEV GARCH Volatility Envelopes on S&P 500
20%
10%
0%
-10%
-20%
Returns
2006-2012
Returns
Lower Bound


GARCH Volatility Envelopes on TSX
20%
10%
0%
-10%
-20%
Returns
2006-2012
Returns
Lower


CEV GARCH Volatility Envelopes on TSX
20%
10%
0%
-10%
-20%
Returns
2006-2012
Returns
Lower

A result which we feel warrants further investigation is the negative ϕ values which are consistent across the risk factors. This parameter acts like a degree of correlation between the variance shocks and returns. Ishida comments that a negative correlation may cause the smirk observed when graphing Black Scholes implied volatility. As discussed earlier there are well debated theories regarding the asymmetric persistence of variance in equity markets, but we do not see any economic validity to transferring the conclusion into foreign exchange markets. Our results for FX risk factors do display an asymmetry biased towards negative returns independent of the funding currency.

By comparing the Monte Carlo results of the GARCH model and the CEVGARCH model for the same risk factor we do see an increasing deviation from normality as we move farther into simulated time, Fig 21 - 36. It is significant to notice that when comparing the GARCH model to the CEVGARCH model for the same simulated risk factor the CEVGARCH model produces a higher kurtosis sooner and exhibits the higher peaked fatter tailed distribution typical of the market. This suggests that the sensitivity of the CEVGARCH model could be more accurate than the standard GARCH model in modeling the short term evolution of risk factors. The GARCH model has earned popularity for its ability to capture this aspect of market observed return dynamics, but we can see the CEVGARCH is clearly more sensitive as the distributions deviate from normality at earlier time steps and with increasing excess kurtosis. Our results for the calculated kurtosis are listed in table 9.

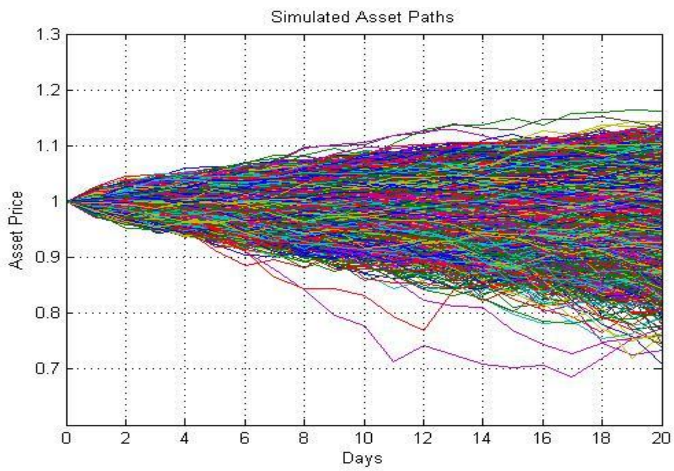
Table 9

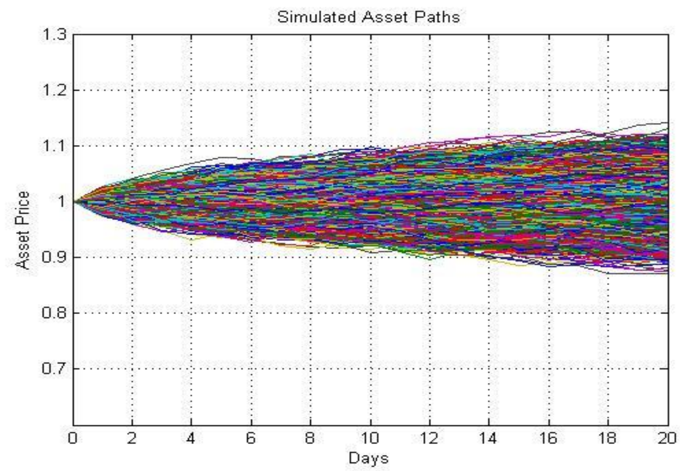
Days Ahead	EUR/USD GARCH	EUR/USD CEVGARCH	GBP/USD GARCH	GBP/USD CEVGARCH	USD/CAD GARCH	USD/CAD CEVGARCH	S&P500 GARCH	S&P500 CEVGARCH	TSX GARCH	TSX CEVGARCH
10	3.08	3.67	3.28	4.26	3.17	3.61	3.46	4.53	3.35	4.18
20	3.09	4.09	3.3	4.87	3.19	3.85	3.57	5.8	3.42	4.99
60	3.15	14.6	3.37	10.63	3.35	7.24	4.18	232.02	3.85	57.41
90	3.16	993.1	3.33	32.33	3.4	138.59	4.47	23660.03	4.03	19239.55
126	3.16	2186.44	3.28	263.94	3.45	2048.74	5.07	11275.06	4.38	19386.1

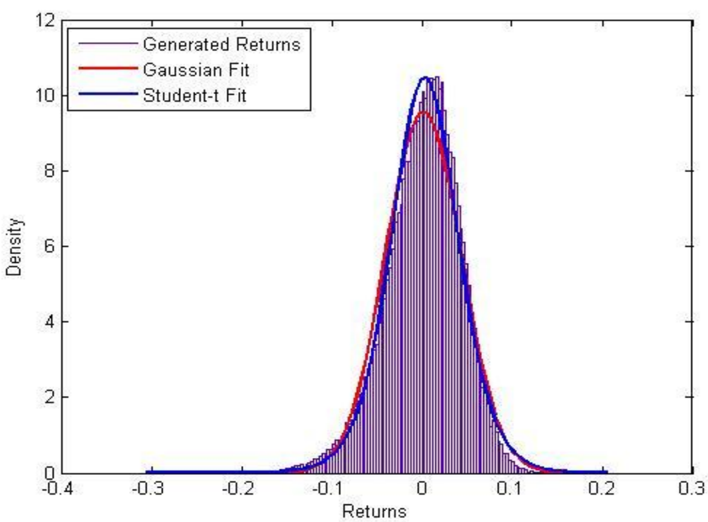
EUR/USD

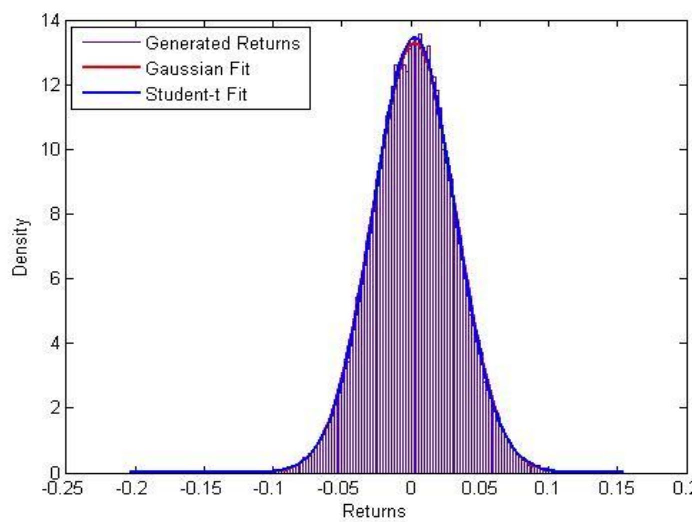
CEVGARCH

GARCH





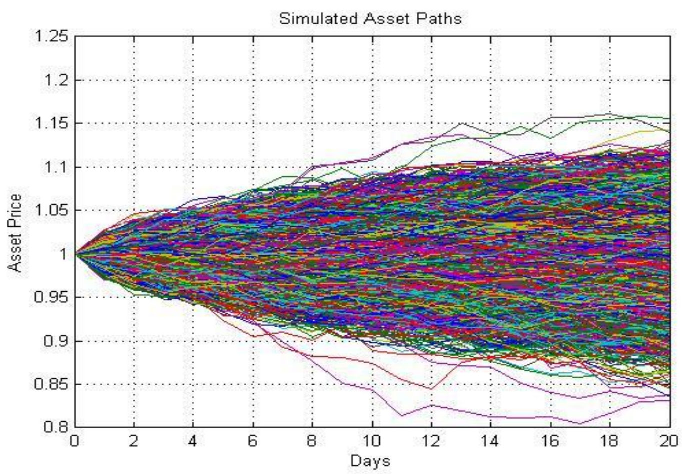


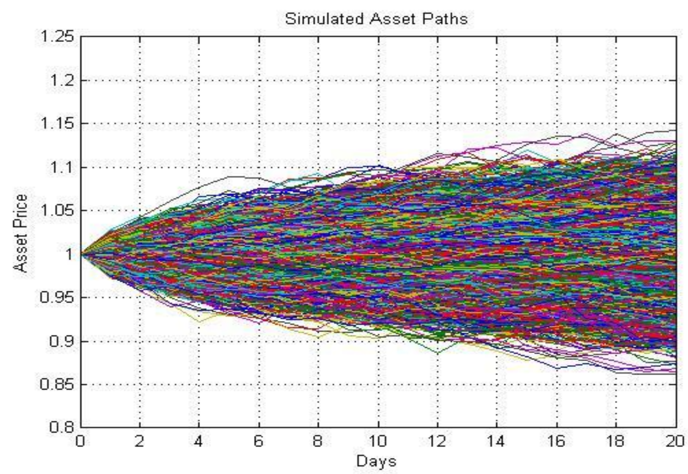


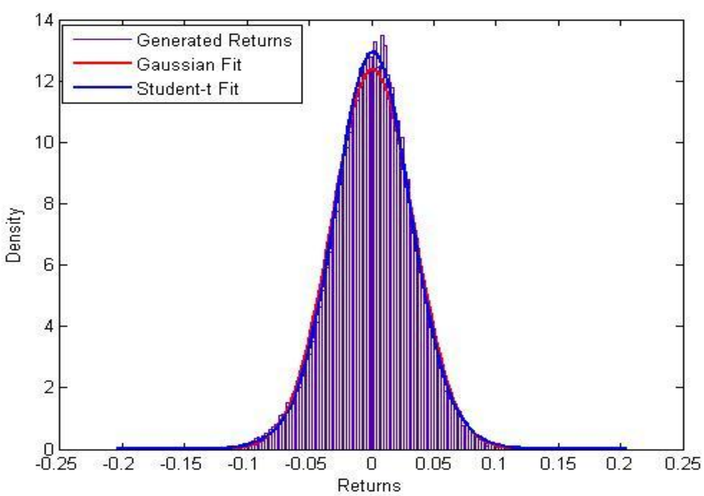
GBP/USD

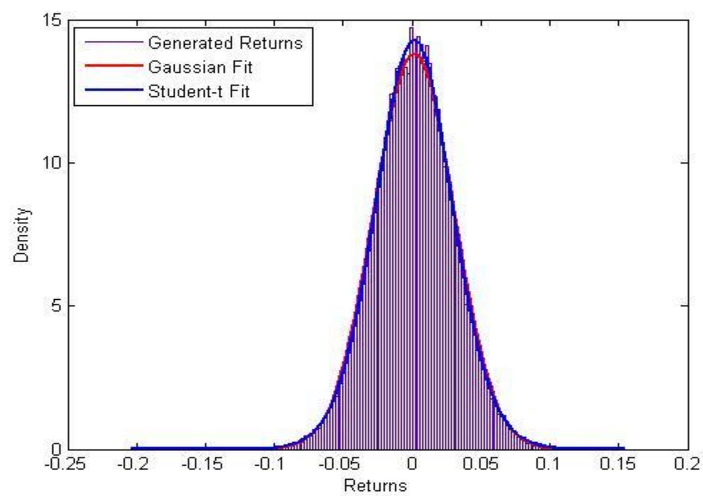
CEVGARCH

GARCH



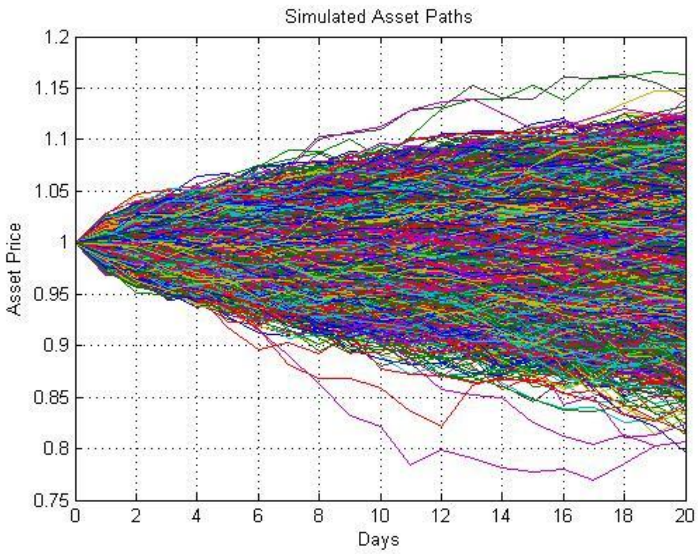


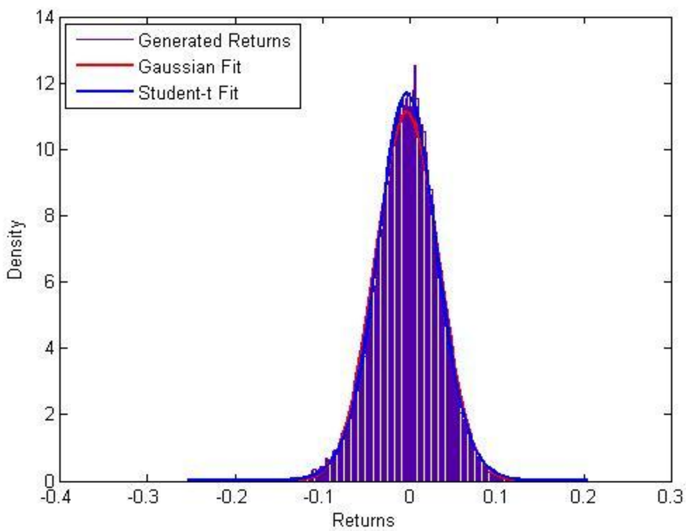




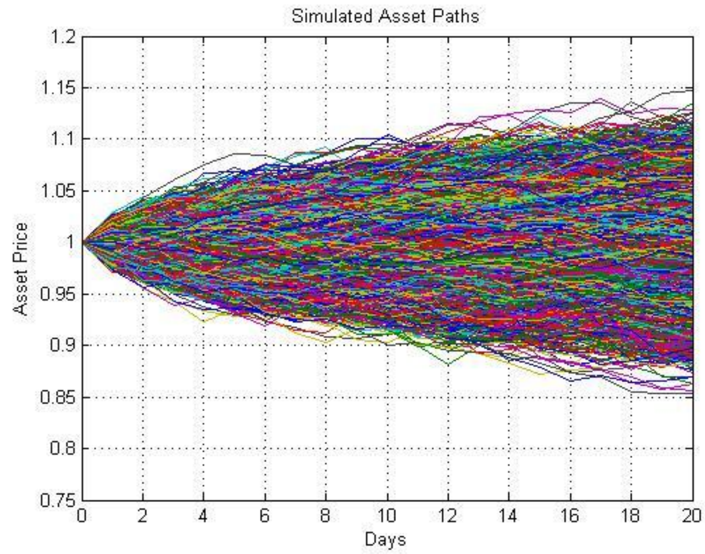
USD/CAD

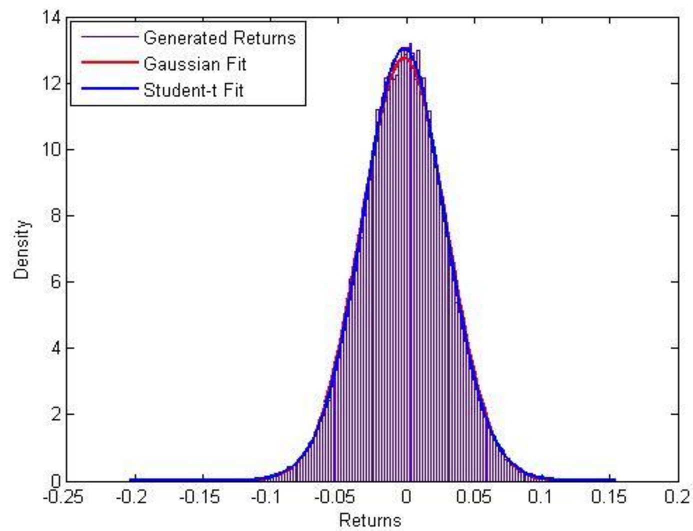
CEVGARCH





GARCH

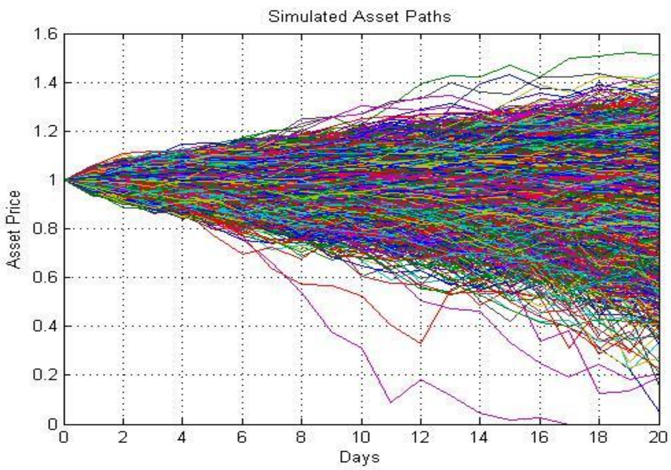


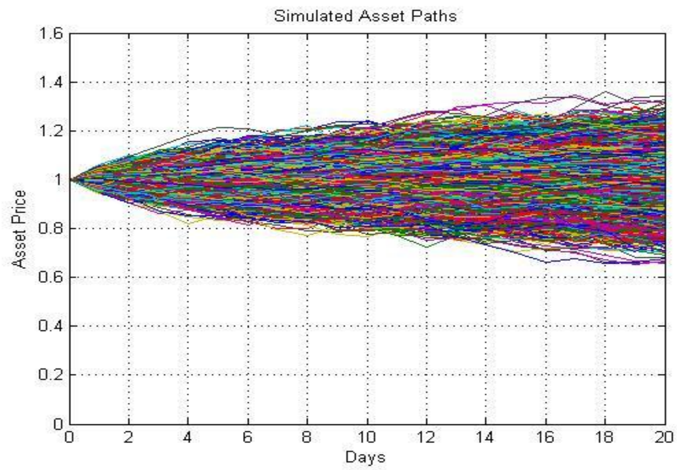


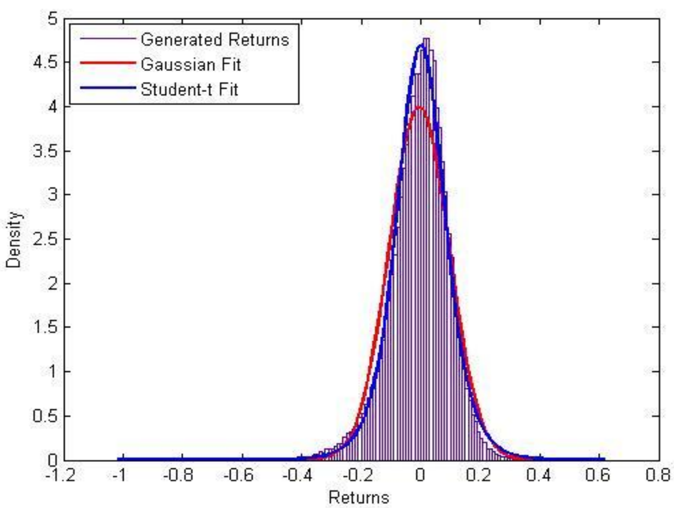
S&P 500

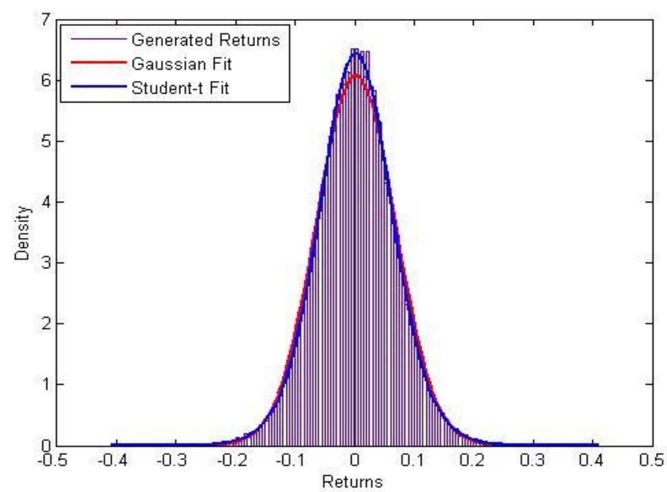
CEVGARCH

GARCH



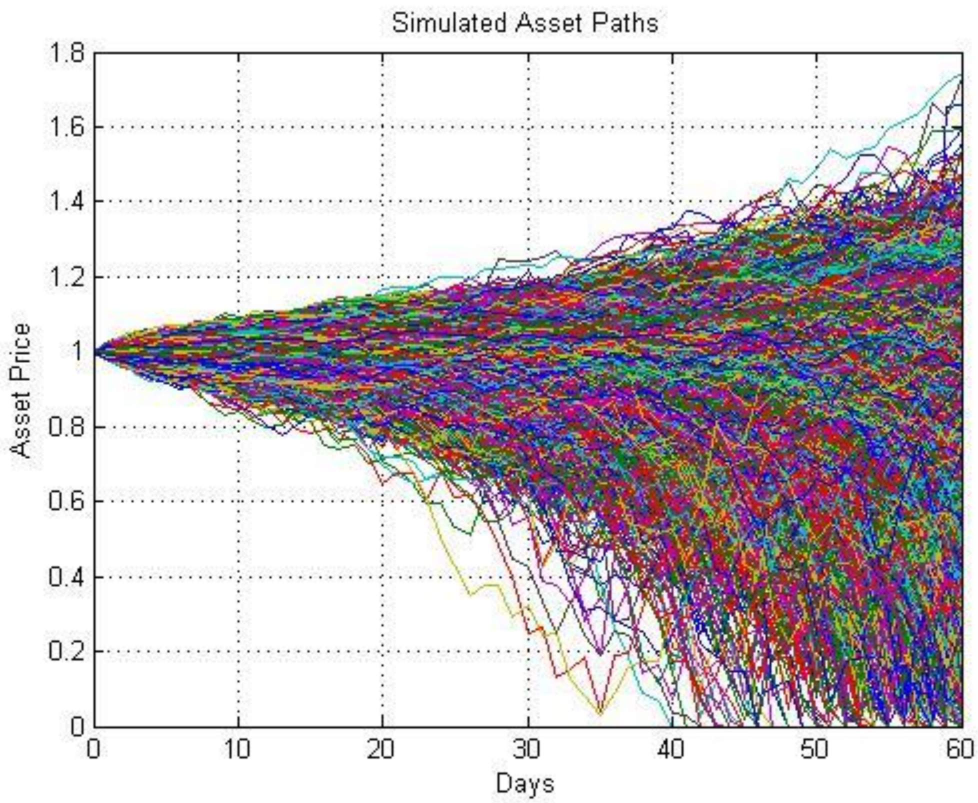






As the simulation horizon extends further out we see an increasing sensitivity to variance. Figure 29 display the simulated asset paths of the Monte Carlo trials for varying time frames. The visualization across time frames is useful to see how some paths can exhibit explosive conditions.

EUR/USD 60 days CEVGARCH (Fig. 29)



Conclusion

In this paper we have investigated one of the first discrete time models which attempts to model the sensitivity of variance at varying levels of variance. Ishida & Engle's CEVGARCH models the variance of variance as a linear function of the conditional variance. Our tests demonstrate the CEVGARCH model does capture the leptokurtic nature of risk factor returns sooner and with a greater degree of sensitivity when compared to the standard GARCH model. This benefit comes at the cost of the model being overly sensitive and unstable during asset path simulations with medium to long term durations. When using variance models to evolve risk factors for instrument pricing, the time frame selected for the model has a significant impact on the validity of the results. Our Monte Carlo simulations demonstrate an inherent weakness of the CEVGARCH model's stability in long term simulation. The model was developed with the intention of modeling the next period ahead variance. When allowing a risk factor to evolve without the economic constraints in actual markets, like central bank interventions, some of the asset paths take on extreme values along larger time frames.

When investigating equity returns our results are consistent with Ishida & Engle as well as Javaheri which all suggest a CEV exponent greater than 1 contrary to the standard GARCH model. Chacko and Viceira (1999) used spectral GMM to estimate the CEV process at 1.10 for the SP500 using weekly data. This suggests to us that further study is needed to investigate the assumptions of stochastic volatility models like Heston's, which use the square root of variance as a fixed measure of elasticity. Having an accurate model for the variance of variance will allow for better hedging and ultimately better pricing of financial derivative instruments.

References

[1] Bachelier, L., 1900, Théorie de la spéculation. *Ann. Éc. Norm. Super*. **17**, 21–86

[2] Black, F., 1976, Studies of stock market volatility changes, *Proceedings of the American Statistical Association, Business and Economics Statistics Section,* 177-181.

[3] Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy*, **81**, 637–654.

[4] Bollerslev, T., 1986, Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics*, **31**, 307-327.

[5] Christie, A.A., 1982, The stochastic behavior of common stock variance: Value, Leverage and interest rate effects, *Journal of Financial Economics*, **10**, 407-432.

[6] Cox, J., 1975, Notes on option pricing I: Constant elasticity of variance diffusion, Working Paper, Stanford University.

[7] Cox, J., J. Ingersoll and S. Ross, 1985, A theory of the term structure of interest rates, *Econometrica*, **53,** 385-408.

[8] Ding, Z., C.W.J. Granger, and R.F. Engle, 1993, A long memory property of stock market returns and a new model. *Journal of Empirical Finance*, **1**, 83-106.

[9] Engle, R.F., 1982, Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation, *Econometrica*, **50**, 987-1008.

[10] Engle, R.F., 1990, Discussion: Stock market volatility and the crash of '87, *Review of Financial Studies*, **3**, 103-106.

[11] Engle, R.F. and T. Bollerslev, 1986, Modelling the persistence of conditional variances. *Econometric Reviews*, **5(1),** 1-50.

[12] Engle, R.F. and V. Ng, 1993, Measuring and testing the impact of news on volatility, *Journal of Finance* **48**, 1749-1778.

[13] Engle, R.F., and G. Lee, 1996, Estimating diffusion models of stochastic volatility, in P. Rossi, ed., *Modelling Stock Market Volatility,* San Diego: Academic Press.

[14] Glosten, L.R., R. Jagannathan, and D.E. Runkle, 1993, On the relation between the expected value and the volatility of the nominal excess return on stock, *Journal of Finance* **48**, 1779-1801.

[15] Hens, T., and S. Steude, 2009, The leverage effect without leverage, *Finance Research Letters*, **6(2),** 83-94.

[16] Heston, S.L., 1993, A closed form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies*, **6**, 327-343.

[17] Heston S.L., and S. Nandi, 2000, A closed form GARCH option pricing model, *Review of Financial Studies*, **13**, 585-625

[18] Hull, J. and A. White, 1987, The pricing of options on assets with stochastic volatility, *Journal of Finance* **42**, 281-300.

[19] Itô, K., 1951, On stochastic differential equations. *Memoirs, American Mathematical Society* **4**, 1-51.

[20] Ishida, I., and Engle, R., 2002, Modelling the variance of variance: The square root, the affine and the CEV GARCH models, NYU working paper.

[21] Javaheri, A., 2004, The volatility process: A study of stock market dynamics via parametric stochastic volatility models and a comparison to the information embedded in option prices. PhD dissertation.

[22]Javaheri, A., P. Wilmott, and E.G. Haug, 2004, GARCH and volatility swaps, *Quantitative Finance*, **4(5),** 589-595.

[23] Koutmos, G., and R. Saidi, 1995, The leverage effect in individual stocks and the debt to equity ratio, *Journal of Business Finance & Accounting*,**22,** 1063-1075.

[24] Merton, R.C., 1973, Rational theory of option pricing, *Bell Journal of Economics and Management Science*, **4(1),** 141-183.

[25] Mandelbrot, B., 1963, The variation of certain speculative prices, *Journal of Business* **36**, 394-419.

[26] Mandelbrot, B., 1967, On the distribution of stock price differences, *Operational Research*, **15**,1057-1062.

[27] Markowitz, H., 1952, Portfolio Selection, *The Journal of Finance*,**7**,77-91.

[28] Nelson, D., 1990(b), ARCH models as diffusion approximations, *Journalof Econometrics*, **45**, 7-38.

[29] Nelson, D., 1991, Conditional heteroskedasticity in asset returns: A new approach. *Econometrica*, **59**, 347-370.

[30] Pagan, A.R., and G.W. Schwert, 1990, Alternative models for conditional stock volatility, *Journal of Econometrics*,**45**, 267-290.