The Importance of Parameter Estimates for Stock-REIT-Bond Optimal Asset Allocation

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Abstract

This study is an extension of the research done by Waggle & Agrrawal (2006), which assesses the marginal effects of changes in optimal portfolio weights with respect to changes in the REIT-stock risk premium and correlation coefficients under a three-asset setting. We also consider two time periods from 1988-2011 and from 2000-2011. The results show that the sensitivity of changes in the REIT-stock risk premium on optimal portfolio weights is significantly higher than the effect of changes in correlation coefficients. Although the findings of the study do not provide the optimal portfolio composition for asset allocation, it provides compelling evidence of the importance in forecasting expected parameters and choosing appropriate historical time periods for a mean-variance optimization.

Keywords: Real Estate Investment Trusts (REITs), Asset Allocation, Optimal Portfolio, Investments, Mean-Variance Optimization, Correlation, Diversification, Forward-Looking Parameter Estimates

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Introduction

Real estate has long been established as an important asset class to be included in an investment portfolio. From a traditional point of view, real estate investment provides three main advantages. First, real estate provides a diversification benefit as it has low correlation with stocks and bonds. Secondly, it exhibits historically high risk-adjusted returns, and lastly, it has positive correlation with inflation from stable income streams and negotiable lease renewals. There are two types of real estate investments including direct and securitized. Securitization comes in the form of Real Estate Investment Trusts (REITs), closed-end mutual funds governed by the Real Estate Investment Trust Act of 1960 where over 75% of the funds are invested directly in real estate. The outperformance of Real Estate Investment Trusts (REITs) relative to stocks on a risk-adjusted basis has considerable implications for the future asset allocations of all investors.

According to the Modern Portfolio Theory by Harry Markowitz dating back to the 1950's, an optimal portfolio maximizes the expected return at a given level of portfolio risk or equivalently to minimize portfolio risk at a given level of expected return. The most common approach employed to generate optimal portfolio composition is Mean-Variance Optimization (MVO), a quantitative tool designed to seek portfolio weights which maximize the tradeoff between expected return and risk. The MVO algorithm constructs optimal portfolios that maximize the risk-adjusted expected returns or Sharpe ratios. The Sharpe ratio is a measure of the excess return over the riskless rate per unit of standard deviation, where a higher Sharpe ratio is preferred. The MVO portfolios are able to achieve higher Sharpe ratios than individual asset classes due to the diversification benefits. When the assets are less

than perfectly correlated, the combined volatility of the total portfolio is less than the sum of its parts. This is the fundamental concept of diversification benefit.

The results of the MVO are heavily dependent on the inputs of the expected returns and risk estimates. Assuming "history repeats itself", the historical returns and the variance-covariance matrix are often used as static proxies of the forward-looking parameters. However, when back testing the expected parameters estimated from the historical data, the results show a significant deviation from the realized returns moving forward. Thus, the validity of historical values as inputs to the MVO raises questions of doubt. In addition, the historical data also shows that the correlations among asset classes vary considerably over time rather than being stationary.

Waggle & Agrrawal (2006) conducted a preliminary research in an attempt to assess the marginal impact of estimation errors of expected returns and the correlation matrix on optimal portfolio weights. Their work was based on three asset classes including stocks, bonds, and REITs covering the period from 1988-2002. The data from the National Association of Real Estate Investment Trusts (NAREIT) is available starting from 1972; however, Waggle & Agrrawal (2006) determined that the data from the beginning of 1972 to the end of 1987 was less relevant due to the limited number of REITs available during this time period. The initiation of the Tax Reform Act of 1986 has substantially boosted the growth of REITs from 12 REITs in 1972 with $330 million market capitalization to over 50 REITs by the end of 1987 with total market capitalization approaching $5 billion. Building on top of their research, we extended the data from 1988-2011. We examine two different time periods, from 1988-2011 and from 2000-2011. The first period captures the effect of the modern REIT era up to the most recent data available. The second period

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concentrates on the effect of the recent market downturns and their impact on the REIT-stock risk premium.

The purpose of Waggle & Agrrawal's work is to develop an understanding of the effect of the mean-variance inputs on optimal portfolio asset allocation in order to help investors make more informed investment decisions. They focus on the sensitivity analysis of changes in excess returns between REITs and stocks, and changes in the correlation coefficient. Due to the recent market downturns including the dotcom bubble and the 2008 financial turmoil, historical excess returns between REITs and stock have significantly changed. Specifically, we see that REITs have outperformed stocks in the recent years. We broaden the scope of our study by concentrating on the reversal in the relationship of the historical excess REIT returns and the dominance of REITs in asset allocation from the recent time data.

Literature Review

Our work is primarily an extension of the work done by Waggle & Agrrawal (2006). They were under the assumption that historical returns were no longer an acceptable method of determining optimal asset allocation. This led them to look into the sensitivities of portfolio weights in REITs with respect to changes in the correlation between REITs and stocks, and with respect to the expected excess return of REITs over stocks. They used a MVO to model the efficient frontier from a very aggressive investor denoted by A = 1 to the most risk averse investor denoted by A = 10 from a common utility function.

Waggle & Agrrawal do not attempt to forecast asset returns, correlations, volatilities, or optimal asset allocations. However, they show the effects of how incorrect expectations of correlations and returns can affect the optimal asset

allocation. As mentioned, the paper observed data from three-asset classes, stocks, bonds, and REITs over various periods from 1972-2002 but adopted only the data after 1988, which consists of a better representation of the REITs universe. They determined that the excess returns among asset classes dictate the optimal asset composition rather than the individual expected return for each asset class. Subsequently, they formed a base scenario where they utilized the excess return from 1988-2002 for REITs over stocks of -1.5% and a 2% premium of stocks over bonds to determine the optimal portfolios. Then, they provided sensitivities from those initial values. It is important to note that historical returns, volatilities, and correlations were used here to assign a base scenario for their sensitivity analysis.

They found that a 1% change in an investors expected returns of REITs had a 10.1% effect on the optimal portfolio weight of REITs for an average investor with risk aversion of 4. Alternatively, a 0.1 change in the correlation between REITs and stock had only a -2.2% effect on the optimal portfolio weight of REITs for the same investor. This relationship is consistent among investors of all risk levels. The sensitivity of REIT weights to the expected return increases significantly as investor risk tolerance increases while the opposite is observed for the correlation sensitivity for REIT weights. Erroneously estimating expected excess returns can have a significant effect on an investors' investment performance. Another significant finding is that as the expected return of bonds decreases, the portfolio allocation towards REITs increases. This is due to a reduction in the comparative advantage of higher risk-adjusted return for bonds.

Fogler (1984) is one of the early reference papers about real estate as an asset class utilizing a three-pronged approach to test if real estate should be considered as a viable asset class in an average investors' optimal portfolio. He observes data on the

Consumer Price Index (CPI), unanticipated inflation, stock and bond returns, and rents, from 1915 to 1978. As there was no reliable real estate data during these early years he used the average rent data to proxy real estate. He found that real estate performed well during times of high unanticipated inflation while stocks and bonds had rates of return below inflation.[1] Therefore, real estate should be included in an optimal portfolio as an inflation hedge.

Fogler also performed a MVO of four asset classes, T-bills, bonds, stocks, and real estate. His benchmark portfolio consists of 52% stocks, 21% bonds, 27% T-bills and 0% real estate, yielding 10.1% return and 2.3% standard deviation. By including 15% to 20% weight in real estate, the portfolio had lower standard deviation than the benchmark for same or higher returns. Therefore, real estate provides a diversification benefit for an average investor.

Finally, Fogler discusses the significance of the innate illiquidity issues present in real estate investments. He concludes that if 15% to 20% real estate is included at all times in the portfolio the illiquidity issues are "moot" since the inclusion of real estate is a long-term strategic choice for diversification and inflation hedging purposes.

Feldman (2003) questions why actual investment allocation in real estate is less than the optimal level when investment in direct real estate provides a tax savings and has the benefits of insulation from equity markets. He uses a MVO to determine the optimal asset allocation for direct and securitized real estate including six asset classes; domestic large cap stock, international stock, domestic small cap stock, direct real estate, REIT, bond and money market from 1987-2002. Feldman found that the maximum allocation to real estate (30% in direct, 15% in REITs) yields a 10.6%

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[1] Unanticipated inflation is not taken into account in the short term interest rate

return with 5.9% standard deviation opposed to the 50% stock and 50% bond portfolio which gave a 10.4% return and 6% standard deviation. Investment in real estate provides a slight diversification benefit. The analysis concludes that direct and securitized real estate are complementary investments as REITs prove to be more volatile assets favoured by more aggressive investors. From an overall view of the REIT space, the investment class is supply constrained in the short-term until they become more popular and securitization ramps up. This makes REITs a suboptimal investment for institutional investors but gives the average investor access to real estate, where they would otherwise not have the upfront investment requirement.

Georgiev, Gupta & Kunkel (2003) look for an explanation as to how investors can overcome the innate obstacles present in real estate. These include the lack of liquidity, large upfront investment requirements, and high transaction costs, with potential asymmetric information. In their analysis they discover that securitized REIT investments exhibit high correlation with stock returns, therefore providing little diversification benefit when added to a portfolio of stocks and bonds. However, the diversification benefit becomes apparent if direct real estate is added to the same portfolio. To examine this unique difference they hedged out the equity component of large cap stocks from REIT returns and unsmoothed the direct real estate index using a first-order autoregressive process. The result is the hedged-REIT returns fell below the returns of stocks, and the unsmoothed direct real estate returns dropped while the volatility increased by more than two-times. The MVO concludes that adding REITs does not increase the portfolio Sharpe ratio significantly due to high correlation to stocks. In addition, when hedge funds and commodities are added to the portfolio, real estate becomes a redundant asset class as hedge funds and commodities provide similar correlation and better risk-adjusted returns than real estate.

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When breaking down the real estate asset class into apartments, industrial, offices, and retail, Georgiev, Gupta & Kunkel find a significant increase in risk-adjusted returns from apartments. There is a benefit from selecting heterogeneous direct real estate over REITs. REITs are not good diversifiers for stock and bond portfolios due to their historically high correlation.

Direct real estate has many drawbacks including project-specific risk, high management and information costs, and illiquidity risk. Goetzmann & Ibbotson (1990) show evidence that direct real estate should account for a significant proportion of the optimal portfolios through comparison of the returns, standard deviations and correlations between different direct real estate investments with stocks and bonds. In addition, they ran a regression analysis to assess the co-movements of real estate returns with interest rates and inflation.

Firstly, Goetzmann & Ibbotson find that real estate provides comparable returns and lower correlations to stocks and bonds, therefore providing a valuable diversification benefit. Secondly, the regression shows that real estate is negatively correlated with interest rates. Rising interest rates increase the cost of financing for residential real estate investments, this leads to decreasing demand and declining real estate prices. Lastly, real estate exhibits a positive correlation with inflation, providing a hedge against inflation.

The time period that Booth & Broussard (2002) examined exhibits poor performance of the stock markets, similar to the current economic environment. They study the underrepresentation of real estate, comparing a MVO with Extreme Value Theorem (EVT) and Lower Partial Moments Analysis (LPM). EVT and LPM are used to estimate the downside risk on the portfolio performance. Only two assets are used, stocks and REITs. Both return distributions are observed to be non-Gaussian

with fat tails and are skewed to the left, though stock returns exhibit significantly more skewness and much fatter tails. This initial observation foreshadows the conclusion.

Based on their MVO the highest Sharpe ratio is obtained with a 10% allocation to REITs. The expected shortfall suggests a significantly higher level of REIT allocation between 40% and 100%. As the allocation to REITs increases, the amount of shortfall decreases for all levels of target shortfall, indicating REIT investment is safer with less downside risk than stocks.

Hudson-Wilson, Gordon, Fabozzi, Anson & Giliberto (2005) reassess real estate as an investment strategy to be included in an investor's portfolio through MVO and net operating income analysis. Anecdotally, they have found that real estate is becoming more accessible for an average investor. And empirically real estate generates better cash yields than stocks and bonds. Although real estate does not produce the highest absolute returns, it provides higher risk-adjusted returns than stocks and bonds. Now that publicly traded REITs have become an integral part of the investment universe it can no longer be overlooked in strategic asset allocation, otherwise the investor would be taking a huge bet against the market portfolio.

Idzorek, Barad & Meier (2007) attempt to expand the investment universe beyond the classic three-asset case of stocks, bonds, and money market by looking at commercial real estate on a global scale. They agree with Hudson-Wilson, Gordon, Fabozzi, Anson & Giliberto (2005) in that real estate is a fully accessible investment option for even an average investor through REITs. REITs are included in most stock indices and therefore must be included in the market portfolio. Unlike Feldman (2003), Idzorek, Barad & Meier conclude that REITS are a reasonable proxy for both direct and securitized real estate. They build on the traditional MVO approach as an

MVO using historical optimization leads to poor forward-looking asset allocations, particularly when short-term historical returns are used as input parameters. As historical returns will typically not generate robust results, the authors use a resampling method based on MVO through Monte Carlo simulations. The result of this optimization allocates 16% to real estate for an average investor. They also used the Black-Litterman model in a resampling MVO through a combination of the Capital Asset Pricing Model's (CAPM) expected returns and historical returns. This approach results in 23% allocation in real estate for an average investor. There are many methods for developing appropriate expected returns, variances, and correlations for forward-looking asset allocations; one method is not necessarily better than another. It is important that the method used generates decent expected parameters to formulate an investment strategy with sufficient diversification and hedging properties.

Chopra & Ziemba (1993) assess the effects of errors in the estimation of expected returns, variances, and covariances for optimal asset allocation. MVO is extremely sensitive to the parameters used, where for an investor with average risk tolerance, the expected returns are 11 times more important than variance estimates and over 20 times as important as covariance estimates. The magnitude of the importance is amplified as the investors risk tolerance increases. They conclude that if investors have limited resources to obtain appropriate estimates of parameters used to determine risk and reward they should concentrate their resources on finding good returns expectations. If investors are not confident in their expected returns estimates, investors can focus purely on optimizing only on portfolio risk by setting returns for all assets to zero.

Direct real estate investment traditionally has significant illiquidity issues when dealing with short-term investment horizons. Amihud & Mendelson (1991) claim that the illiquidity discount increases with increasing transaction costs, and decreases with increasing average holding period. So over longer time horizons illiquidity issues are minimized. Liquidity measures how quickly an asset can be converted into cash without causing a significant loss in value. The illiquidity discount accounts for the transaction cost, bid-ask spread, and timing of execution. Illiquidity can negatively impact asset values and unfavorably affect the realized return. In Amihud & Mendelson's empirical study, they examined the bid-ask spreads between treasury bills and notes by bundling the treasury bills together to match the maturity with the note. They also studied the bid-ask spreads among 49 portfolios of stocks grouped by beta coefficient, a sensitivity measure of systematic risk. They find that short-term investors tend to hold more liquid assets while long-term investors hold less liquid assets. In addition, they observed the effect of liquidity during the stock market crash of October 1987, in which the market liquidity deteriorated and investors experienced considerable delays in execution and substantial decline in asset prices. They concluded that illiquid assets result in higher return.

Description of the Data

As a continuation of Waggle & Agrrawal's research, we focus on the marginal effect of the estimated returns and the correlation matrix rather than formulating a predictive optimization model. They adopted returns, variances and correlations from the historical time period of 1988-2002 to set the base scenario for their sensitivity analysis. The data employed in this study consists of three asset classes including stocks, bonds, and REITs. Specifically, data for large-company stocks, long-term government bonds, and equity REITs represent each asset class, respectively.

Monthly data for the three asset classes are used to estimate the correlation coefficients and annualized data are used to estimate the mean returns.

The time period in this study spans from January 1988 to December 2011. The returns data for REITs is retrieved from NAREIT, which publishes publicly traded performance data for U.S. REITs. The total return data for stocks and bonds is obtained from the Ibbotson SBBI 2012 Classic Yearbook. To examine the financial conditions in the recent economic setting, we conduct a separate optimization from January 2000 to December 2011, named "recent-time period". The following shows the descriptive statistics for the precedent study and our two time periods.

	REITs	Stocks	Bonds
Panel A: 1988 - 2002			
Mean Return	11.50%	13.00%	11.00%
Standard Deviation	15.80%	18.60%	11.20%
Risk-adjusted Return	0.73	0.70	0.98
Correlations			
REITs	1.00		
Stocks	0.36	1.00	
Bonds	0.14	0.13	1.00
Panel B: 1988 - 2011			
Mean Return	12.48%	11.15%	10.15%
Standard Deviation	19.58%	18.61%	11.69%
Risk-adjusted Return	0.64	0.60	0.87
Correlations			
REITs	1.00		
Stocks	0.56	1.00	
Bonds	-0.01	-0.02	1.00
Panel C: 2000 - 2011			
Mean Return	14.24%	2.44%	10.10%
Standard Deviation	22.42%	19.51%	12.04%
Risk-adjusted Return	0.64	0.13	0.84
Correlations			
REITs	1.00		
Stocks	0.64	1.00	
Bonds	-0.11	-0.25	1.00

Note: The risk-adjusted return is the excess return over riskless rate per unit of standard deviation while assuming the riskless rate is zero

Exhibit 1: Descriptive Statistics: Mean and standard deviations are based on annualized returns. Correlation matrices are based on monthly returns.

We first examine the changes in correlation and in excess returns between stock and REITs over time. Exhibit 2 illustrates the change in the relationship over time with one-year non-overlapping time periods. To investigate the marginal effect of input assumptions, Waggle & Agrrawal's research centered on the premium returns that stocks earned relative to REITs. Prior to 2000, the returns on stocks generally outperformed REITs. However, contrary to the traditional belief, REITs have exhibited higher returns than stocks in the recent-time period, with the exception of 2007. The underperformance for stocks has followed due to the dotcom bubble and financial crisis. The capital markets experienced significant volatility with minimal return.

Also shown in Exhibit 2 is that the correlation between REITs and stocks varies over time. This implies a significant weakness in the MVO models' use of stationary correlation inputs. Another observation is the increasing correlation between stocks and REITs especially in the recent-time period. This increased co-movement is attributable to the recent shared macroeconomic uncertainties in the financial markets caused by global debt crises.
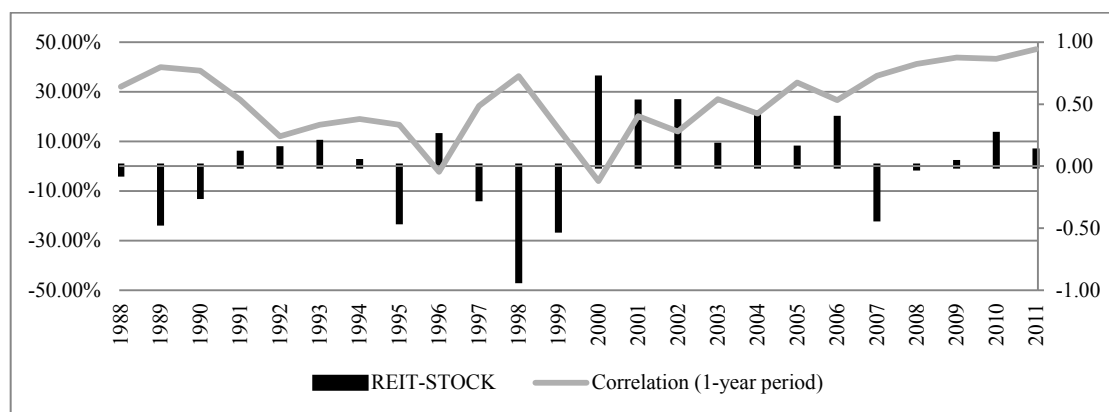

Exhibit 2: REITs vs. Stocks: Excess Return and Correlation, 1988-2011 with 1-Year Non-Overlapping Periods

Methodology

Since our study is an extension on the research done by Waggle & Agrrawal (2006), we first verified their methodologies using the same data to ensure that we could replicate their results. After successfully replicating their findings, we proceeded with extending the time period of their study comparing our parameter results to theirs and contrasting the differences in our sensitivity findings.

Mean-Variance and Utility

Like Waggle & Agrrawal (2006), the investors are assumed to be risk-averse and prefer to choose the portfolio weights that maximize their utility. The utility function is as follows:

$$U = r_p - \frac{1}{2}A\sigma_p^2 \tag{1}$$

Different investors have different levels of risk tolerance and these are captured by the levels of risk aversion, A. When A is small, the investor has a lower level of risk aversion and is considered to be a more aggressive investor. The risk aversion levels in our optimal portfolio allocations range from 1 to 10.

Under the three-asset case, the portfolio return and the portfolio variance as provided by Waggle & Agrrawal are calculated as:

$$r_p = w_R r_R + w_S r_S + w_B r_B \tag{2}$$

$$\sigma_p^2 = w_R^2\sigma_R^2 + w_S^2\sigma_S^2 + w_B^2\sigma_B^2 + 2w_R w_S \rho_{RS}\sigma_R\sigma_S + 2w_R w_B \rho_{RB}\sigma_R\sigma_B$$
$$+ 2w_S w_B \rho_{SB}\sigma_S\sigma_B \tag{3}$$

where the weights and standard deviations are denoted as w and σ, respectively. The subscripts R, S, and B denote for REITs, Stocks, and Bonds, respectively. The REIT-

Stock correlation is ρ_{RS}, the REIT-Bond correlation is ρ_{RB}, and the Stock-Bond correlation is ρ_{SB}.

Two constraints are imposed to ensure positive weights for all asset classes and assure portfolio completeness.

$$w_R, w_S, w_B \geq 0 \tag{4}$$

and

$$w_R + w_S + w_B = 1 \tag{5}$$

These equations effectively constrain an investor's ability to short-sell, as the average investor does not have access to short selling. Thus, an investor with a lower level of risk aversion is expected to allocate a greater amount of their optimal portfolio to the higher return REITs and stocks while decreasing their holdings in bonds.

Optimal Portfolio Weights

The closed-form solutions for optimal portfolio weights can be obtained by setting $\frac{\partial U}{\partial w_R} = 0$ and solving for w_R. The mathematics for two- and three-asset cases is provided by Waggle & Agrrawal and the three-asset case is reproduced in the appendix. As the investor becomes less risk averse, the optimal portfolio converges into a two-asset scenario, in which the optimal portfolio weight is allocated between REITs and stocks only. Since REITs and stocks comprise the whole portfolio, the weight in stock consists of the remainder of the positively weighted portfolio not invested in REITs. Under the two-asset case, the following optimal portfolio weight equation provided by Waggle & Agrrawal is generated as a function of the portfolio return and variance as the following:

$$r_p = w_R r_R + (1 - w_R) r_S \tag{6}$$

$$\sigma_p^2 = w_R^2 \sigma_R^2 + (1 - w_R)^2 \sigma_S^2 + 2w_R(1 - w_R)\rho_{RS}\sigma_R\sigma_S \tag{7}$$

$$w_R^* = \frac{r_R - r_S + A(\sigma_S^2 - \rho_{RS}\sigma_R\sigma_S)}{A(\sigma_S^2 + \sigma_R^2 - 2\rho_{RS}\sigma_R\sigma_S)} \tag{8}$$

The * indicates solutions for optimal portfolios and the optimal portfolio weight in stock is $1-w_R^*$.

Marginal Effects

To conduct the sensitivity analysis on the optimal portfolio weight from the estimated inputs, Waggle & Agrrawal derive the partial differential equation of w_R^* with respect to changes in return, r_R, and changes in correlation, ρ_{RS}.

$$\frac{\partial w_R^*}{\partial r_R} = \frac{1}{A(\sigma_S^2 + \sigma_R^2 - 2\rho_{RS}\sigma_R\sigma_S)} > 0 \tag{9}$$

$$\frac{\partial w_R^*}{\partial \rho_{RS}} = \frac{\sigma_R\sigma_S}{\sigma_S^2 + \sigma_R^2 - 2\rho_{RS}\sigma_R\sigma_S}\left[\frac{2\left(r_R - r_S + A(\sigma_S^2 - \rho_{RS}\sigma_R\sigma_S)\right)}{A(\sigma_S^2 + \sigma_R^2 - 2\rho_{RS}\sigma_R\sigma_S)} - 1\right] \tag{10}$$

Equation 9 and 10 illustrates that the marginal effects on optimal portfolio weights are dependent on the variance and covariance of both assets in the two asset portfolio. The complete partial differentiations of the optimal portfolio weight with respect to the return and correlation under the three-asset case can be found in the appendix.

To be comparable with Waggle & Agrrawal's findings, we adopted the same scaling system. The results for $\frac{\partial w_R^*}{\partial r_R}$ and $\frac{\partial w_S^*}{\partial r_R}$ are further divided by 100 to approximate the changes in the respective optimal portfolio weights of REITs and stocks per 1% change in expected return for REITs. As for the $\frac{\partial w_R^*}{\partial \rho_{RS}}$ and $\frac{\partial w_S^*}{\partial \rho_{RS}}$, we scaled the results by a factor of 10 to account for the change in the respective optimal portfolio weights per 0.1 change in the correlation between REITs and stocks.

Analysis

Using the closed-form solutions provided by Waggle & Agrrawal (2006), we constructed the optimal portfolio weights under the three-asset case along with the marginal effect of changes in REIT expected returns and changes in the REITs-stock correlation. The optimal asset weights and sensitivity results for investors with risk aversion measures of 1, 2, 3, 4, 5, 8, and 10 are shown in the following exhibits. To satisfy the positive weight constraint, when an asset weight results in a negative value, it is dropped from the portfolio and the remaining two assets are used for the optimization. This is consistent with Waggle & Agrrawal's approach and is demonstrated in boldface type.

Data from 1988-2002: Precedent Period

To verify the methodology in Waggle & Agrrawal (2006), we replicated the results of their study period, 1988-2002, in Exhibit 3. The stock-bond risk premium was 2% and REIT-stock risk premium was -1.5%. During Waggle & Agrrawal's research time, stock generated the highest absolute return of the three asset classes.

For an investor with risk aversion of 4 at -1.5% expected REIT-stock risk premium, the optimal weight in REITs is 46 times more sensitive to a 1% change in expected REIT-stock risk premium than a 1% change in correlation on an absolute scale. For the same investor, the sensitivity of the optimal weight in stock to a 1% change in expected REIT-stock risk premium is 86 times more sensitive than a 1% change in correlation on an absolute scale.

$r_B - r_S$		-2.0%	A (More aggressive an investor is will have lower A value)					
$r_R - r_S$		1	2	3	4	5	8	10
-1.5%	W^*_R	0.090	0.163	0.187	0.199	0.206	0.217	0.221
	W^*_S	0.676	0.406	0.315	0.270	0.243	0.202	0.189
	W^*_B	0.234	0.432	0.498	0.531	0.551	0.581	0.591
	dW^*_R/dr_R	0.403	0.201	0.134	0.101	0.081	0.050	0.040
	dW^*_S/dr_R	-0.173	-0.087	-0.058	-0.043	-0.035	-0.022	-0.017
	$dW^*_R/d\rho_{RS}$	-0.075	-0.040	-0.028	-0.022	-0.018	-0.013	-0.011
	$dW^*_S/d\rho_{RS}$	0.026	0.006	-0.001	-0.005	-0.007	-0.010	-0.011
-0.5%	W^*_R	0.493	0.364	0.321	0.299	0.287	0.267	0.261
	W^*_S	0.503	0.319	0.258	0.227	0.208	0.181	0.172
	W^*_B	0.004	0.317	0.422	0.474	0.505	0.552	0.568
	dW^*_R/dr_R	0.403	0.201	0.134	0.101	0.081	0.050	0.040
	dW^*_S/dr_R	-0.173	-0.087	-0.058	-0.043	-0.035	-0.022	-0.017
	$dW^*_R/d\rho_{RS}$	-0.034	-0.019	-0.014	-0.012	-0.010	-0.008	-0.007
	$dW^*_S/d\rho_{RS}$	-0.020	-0.017	-0.017	-0.016	-0.016	-0.015	-0.015
0.5%	W^*_R	**0.756**	0.565	0.455	0.400	0.367	0.318	0.301
	W^*_S	**0.244**	0.232	0.200	0.184	0.174	0.159	0.154
	W^*_B	**0.000**	0.202	0.345	0.416	0.459	0.523	0.545
	dW^*_R/dr_R	**0.260**	0.201	0.134	0.101	0.081	0.050	0.040
	dW^*_S/dr_R	**-0.260**	-0.087	-0.058	-0.043	-0.035	-0.022	-0.017
	$dW^*_R/d\rho_{RS}$	**0.039**	0.001	0.000	-0.001	-0.002	-0.003	-0.003
	$dW^*_S/d\rho_{RS}$	**-0.039**	-0.040	-0.032	-0.028	-0.025	-0.021	-0.020

Note: Assumes standard deviations and correlations are all at their historical levels based on the 1988-2002 time period as shown in Exhibit 1 Panel A.

dW^*_R/dr_R and dW^*_S/dr_R are divided by 100 to show approximate change per a 1% change in return. $dW^*_R/d\rho_{RS}$ and $dW^*_S/d\rho_{RS}$ are divided by 10 to show the approximate change per a 0.1 change in correlation.

Boldface type indicates optimization results from less than three assets.

Exhibit 3: Impact on Changes in REIT-Stock Correlation and Expected REIT Returns with $r_S - r_B$ =2.0%

Data from 1988-2011: Extended Period

The stock-bond risk premium ($r_S - r_B$) is 1% and the REIT-stock risk premium ($r_R - r_S$) is 0.8% from 1988-2011. Implying REITs outperformed the other two asset classes during this time. In comparison to Waggle & Agrrawal (2006), the stock-bond risk premium was 2% and REIT-stock risk premium was -1.5% from 1988-2002.

Another observation is the change of correlations over time. From the original time period of 1988-2002, the REIT-stocks correlation was 0.36, the REIT-bond correlation was 0.14 and the stock-bond correlation was 0.13. From 1988-2011, the

correlation was 0.56, -0.01, and -0.02, respectively. REITs and stocks exhibited increasing correlation while their correlations with respect to bonds decreased.

Assuming that the stock-bond risk premium is at the 1% level as observed from 1988-2011, we performed the optimization for 1.3% REIT-stock risk premium with analysis for risk premiums of ±0.5%. Under the 1.8% REIT-stock risk premium case for an investor with a risk aversion measure of one, A = 1, the entire asset weight is allocated to REITs. This situation arises because the historical return in REITs outperforms stock during this period and REITs has completely replaced the traditional role of stock in asset allocation.

Due to the excess return of REITs relative to stocks observed during the 1988-2011 time period, Exhibit 4 shows higher weight allocated to REITs relative to the optimization findings by Waggle & Agrrawal for all investors. Similarly, our optimization results illustrate that the portfolio asset weight is more sensitive to changes in the expected risk premium than to changes in correlations. A 1% fluctuation in the expected REIT-stock risk premium will have a dramatic impact on the optimal portfolio composition than a 1% movement in expected REIT-stock correlation coefficient. For example, an investor with risk aversion of 4 with a 1.3% REIT-stock risk premium, the optimal weight in REITs is 113 times more sensitive to a 1% change in expected REIT-stock risk premium than a 1% change in correlation on an absolute scale. For the same investor, the sensitivity of the optimal weight in stock to a 1% change in expected REIT-stock risk premium is 23 times more sensitive than a 1% change in correlation on an absolute scale.

$r_B - r_S$		-1.0%	A (More aggressive an investor is will have lower A value)						
$r_R - r_S$			1	2	3	4	5	8	10
1.3%	W^*_R		**0.709**	0.426	0.331	0.283	0.255	0.212	0.197
	W^*_S		**0.000**	0.088	0.121	0.138	0.148	0.163	0.168
	W^*_B		**0.291**	0.486	0.548	0.579	0.598	0.625	0.635
	dW^*_R/ dr_R		**0.191**	0.179	0.119	0.090	0.072	0.045	0.036
	dW^*_S/ dr_R			-0.127	-0.084	-0.063	-0.051	-0.032	-0.025
	$dW^*_R/ d\rho_{RS}$			0.028	0.015	0.008	0.004	-0.002	-0.004
	$dW^*_S/ d\rho_{RS}$			-0.051	-0.035	-0.027	-0.022	-0.014	-0.012
0.8%	W^*_R		0.533	0.336	0.271	0.238	0.219	0.189	0.180
	W^*_S		0.114	0.151	0.163	0.169	0.173	0.179	0.180
	W^*_B		0.353	0.513	0.566	0.592	0.608	0.632	0.640
	dW^*_R/ dr_R		0.358	0.179	0.119	0.090	0.072	0.045	0.036
	dW^*_S/ dr_R		-0.253	-0.127	-0.084	-0.063	-0.051	-0.032	-0.025
	$dW^*_R/ d\rho_{RS}$		0.034	0.011	0.004	0.000	-0.002	-0.006	-0.007
	$dW^*_S/ d\rho_{RS}$		-0.117	-0.059	-0.040	-0.031	-0.025	-0.017	-0.014
1.8%	W^*_R		**1.000**	0.516	0.391	0.328	0.290	0.234	0.215
	W^*_S		**0.000**	0.024	0.079	0.106	0.122	0.147	0.155
	W^*_B		**0.000**	0.460	0.531	0.566	0.587	0.619	0.630
	dW^*_R/ dr_R		**0.010**	0.179	0.119	0.090	0.072	0.045	0.036
	dW^*_S/ dr_R			-0.127	-0.084	-0.063	-0.051	-0.032	-0.025
	$dW^*_R/ d\rho_{RS}$			0.044	0.026	0.016	0.011	0.002	0.000
	$dW^*_S/ d\rho_{RS}$			-0.096	-0.065	-0.049	-0.040	-0.026	-0.021

Note: Assumes standard deviations and correlations are all at their historical levels based on the 1988-2011 time period as shown in Exhibit 1 Panel B.

dW^*_R/dr_R and dW^*_S/dr_R are divided by 100 to show approximate change per a 1% change in return.
$dW^*_R/d\rho_{RS}$ and $dW^*_S/d\rho_{RS}$ are divided by 10 to show the approximate change per a 0.1 change in correlation.

Boldface type indicates optimization results from less than three assets.

Exhibit 4: Impact on Changes in REIT-Stock Correlation and Expected REIT Returns with $r_S - r_B = 1\%$

The marginal changes of optimal weights in REITs with respect to changes in the REIT-stock risk premium is lower than Waggle & Agrrawal (2006)'s study. This marginal change is a function of all volatilities and correlations. Under our study period, there is an increase in correlation between REITs and stocks and an increase in volatility of REITs, causing a higher covariance between REITs and stocks. Consequently, the additional co-movement of these two assets reduces the diversification benefit of trading one asset for another. We see a lower sensitivity of the REIT asset weight to a change in the expected return.

Using all the available stock and bond return data from the Ibbotson SBBI 2012 Classic Yearbook, we determined that the stock-bond risk premium is at 6.5% from 1926 to 2011. To formulate optimal compositions from this long-term perspective, we conduct another optimization with the same constraint that all asset weights must be greater than or equal to zero. The results are presented in Exhibit 5 based on the observed REIT-stock risk premium, standard deviations, and correlation coefficients.

$r_B - r_S$		-6.5%	A (More aggressive an investor is will have lower A value)					
$r_R - r_S$			1	2	3	4	5	8	10
1.3%	W^*_R		**0.847**	**0.645**	0.524	0.428	0.371	0.284	0.255
	W^*_S		**0.153**	**0.355**	0.357	0.315	0.289	0.251	0.239
	W^*_B		**0.000**	**0.000**	0.119	0.257	0.340	0.465	0.506
	dW^*_R/ dr_R		**0.311**	**0.156**	0.119	0.090	0.072	0.045	0.036
	dW^*_S/ dr_R		**-0.311**	**-0.156**	-0.084	-0.063	-0.051	-0.032	-0.025
	$dW^*_R/ d\rho_{RS}$		**0.787**	**0.328**	0.002	-0.002	-0.004	-0.007	-0.008
	$dW^*_S/ d\rho_{RS}$		**-0.787**	**-0.328**	-0.040	-0.031	-0.025	-0.016	-0.014
0.8%	W^*_R		**0.692**	**0.567**	0.464	0.383	0.335	0.262	0.238
	W^*_S		**0.308**	**0.433**	0.399	0.346	0.315	0.267	0.251
	W^*_B		**0.000**	**0.000**	0.136	0.270	0.351	0.471	0.511
	dW^*_R/ dr_R		**0.311**	**0.156**	0.119	0.090	0.072	0.045	0.036
	dW^*_S/ dr_R		**-0.311**	**-0.156**	-0.084	-0.063	-0.051	-0.032	-0.025
	$dW^*_R/ d\rho_{RS}$		**0.434**	**0.152**	-0.009	-0.010	-0.010	-0.011	-0.011
	$dW^*_S/ d\rho_{RS}$		**-0.434**	**-0.152**	0.052	0.038	0.030	0.018	0.014
1.8%	W^*_R		**1.000**	**0.723**	0.584	0.473	0.406	0.307	0.273
	W^*_S		**0.000**	**0.277**	0.315	0.283	0.264	0.235	0.226
	W^*_B		**0.000**	**0.000**	0.101	0.244	0.330	0.458	0.501
	dW^*_R/ dr_R		**0.010**	**0.156**	0.119	0.090	0.072	0.045	0.036
	dW^*_S/ dr_R			**-0.156**	-0.084	-0.063	-0.051	-0.032	-0.025
	$dW^*_R/ d\rho_{RS}$			**0.505**	0.013	0.007	0.003	-0.002	-0.004
	$dW^*_S/ d\rho_{RS}$			**-0.505**	0.027	0.020	0.016	0.009	0.007

Note: Assumes standard deviations and correlations are all at their historical levels based on the 1988-2011 time period as shown in Exhibit 1 Panel B.

dW^*_R/dr_R and dW^*_S/dr_R are divided by 100 to show approximate change per a 1% change in return.
$dW^*_R/d\rho_{RS}$ and $dW^*_S/d\rho_{RS}$ are divided by 10 to show the approximate change per a 0.1 change in correlation.

Boldface type indicates optimization results from less than three assets.

Exhibit 5: Impact of Changes in REIT-Stock Correlation and Expected REIT Returns with $r_S - r_B = 6.5\%$

Exhibit 5 demonstrates similar finding to the previous optimizations in which the asset weights are more sensitive to changes in REIT-stock risk premium than in changes in REIT-stock correlation coefficient. Taking an investor with risk aversion of 4 in the 1.3% REIT-stock risk premium case, the optimal weight in REITs is 450 times more sensitive with respect to a 1% change in REIT-stock risk premium than to a 1% change in REIT-stock correlation coefficient. The change in optimal weight in stock is even more sensitive at 20 times. When the relative risk-adjusted expected returns of the assets diverge significantly, as in this case where stock-bond risk premium increases from 1% to 6.5% with unchanging volatility, the marginal effects of risk premiums becomes the most important factor and the effect of correlation becomes miniscule.

Data from 2000-2011

Taking into account the financial condition in the recent time period, we performed an optimization based on the data from 2000-2011. We scrutinize this time period because the REIT-stock risk premium drastically changed as a result of the dotcom bubble and the global financial crisis. The optimization results are displayed in Exhibit 6.

Using an 11-year window, the descriptive data from Exhibit 1 Panel B indicates that stocks significantly underperform the other two asset classes on a risk-adjusted basis. This is consistent with our optimization results in Exhibit 6, in which all portfolios do not contain any positive weight in stock. Comparable to the previous optimizations, the change in optimal weights is only 44 times as sensitive to the change in risk premium as correlation coefficient. A critical implication from Exhibit 6 is the choice of the historical time frame in predicting expected returns. MVO is simply a computing algorithm that optimizes investors' perspectives of expected

returns, variances and covariances. For example, an investor with a risk aversion of 4 in the 4.1% REIT-bond risk premium case will incur approximately 3.5% change in optimal weight in REITs for 1% estimation error in expected return.

$r_B - r_S$		8.0%	A (More aggressive an investor is will have lower A value)						
$r_R - r_S$			1	2	3	4	5	8	10
4.1%	W^*_R		0.833	0.540	0.442	0.393	0.364	0.320	0.306
	W^*_S		0.000	0.000	0.000	0.000	0.000	0.000	0.000
	W^*_B		0.167	0.460	0.558	0.607	0.636	0.680	0.694
	dW^*_R/ dr_R		0.142	0.071	0.047	0.035	0.028	0.018	0.014
	dW^*_B/ dr_R		-0.142	-0.071	-0.047	-0.035	-0.028	-0.018	-0.014
	$dW^*_R/ d\rho_{RS}$		0.025	0.003	-0.004	-0.008	-0.010	-0.014	-0.015
	$dW^*_B/ d\rho_{RS}$		-0.025	-0.003	0.004	0.008	0.010	0.014	0.015
3.6%	W^*_R		0.762	0.504	0.419	0.376	0.350	0.311	0.299
	W^*_S		0.000	0.000	0.000	0.000	0.000	0.000	0.000
	W^*_B		0.238	0.496	0.581	0.624	0.650	0.689	0.701
	dW^*_R/ dr_R		0.142	0.071	0.047	0.035	0.028	0.018	0.014
	dW^*_B/ dr_R		-0.142	-0.071	-0.047	-0.035	-0.028	-0.018	-0.014
	$dW^*_R/ d\rho_{RS}$		0.020	0.000	-0.006	-0.009	-0.011	-0.014	-0.015
	$dW^*_B/ d\rho_{RS}$		-0.020	0.000	0.006	0.009	0.011	0.014	0.015
4.6%	W^*_R		0.903	0.575	0.466	0.411	0.378	0.329	0.313
	W^*_S		0.000	0.000	0.000	0.000	0.000	0.000	0.000
	W^*_B		0.097	0.425	0.534	0.589	0.622	0.671	0.687
	dW^*_R/ dr_R		0.142	0.071	0.047	0.035	0.028	0.018	0.014
	dW^*_B/ dr_R		-0.142	-0.071	-0.047	-0.035	-0.028	-0.018	-0.014
	$dW^*_R/ d\rho_{RS}$		0.031	0.006	-0.003	-0.007	-0.009	-0.013	-0.014
	$dW^*_B/ d\rho_{RS}$		-0.031	-0.006	0.003	0.007	0.009	0.013	0.014

Note: Assumes standard deviations and correlations are all at their historical levels based on the 2000-2011 time period as shown in Exhibit 1 Panel C.
dW^*_R/dr_R and dW^*_B/dr_R are divided by 100 to show approximate change per a 1% change in return.
$dW^*_R/d\rho_{RS}$ and $dW^*_B/d\rho_{RS}$ are divided by 10 to show the approximate change per a 0.1 change in correlation.
Boldface type indicates optimization results for two assets.

Exhibit 6: Impact of Changes in REIT-Bond Correlation and Expected REIT Returns with $r_S - r_B = -8.0\%$

Contrary to the traditional financial conventions, our optimization over the recent-time period shows zero asset allocation in stocks. This again proves the importance of forward-looking parameter estimates in MVO and the role of selecting the appropriate time horizon in estimating the parameters. Investors must be cautious when formulating their expectations and scrutinize the validity of their inputs prior to finalizing their strategic asset allocations.

Critique

Waggle & Agrrawal successfully derived the optimal portfolio weights and partial differential equations for the two and three asset cases. However, a potential weakness presented in their study relates to the partial differentiation when imposing the all-positive weights constraint. When an asset consists of a negative optimal weight, Waggle & Agrrawal removed the asset from the optimization and formed the optimal composition over the remaining assets. This approach results in the partial differentiation only considering the remaining assets that are used in the optimal allocation. Specifically, when only stocks and REITs are considered for an aggressive investor the sensitivity measures with respect to the risk premium and correlation are based on only these two assets. This is inappropriate because when there are incremental changes in the risk premium or correlation the asset that was originally at negative weight may become positive. As such, the sensitivity measure should take into account this change in optimal portfolio weight from negative to positive.

Discussions

In the precedent paper, one of the constraints imposed in Waggle & Agrrawal (2006) is the all-positive-weights constraint which prevents investors from short selling. When investors enter long and short positions for highly correlated assets the result generates lower volatility than a long only portfolio. With the restriction on short-selling, the marginal effect from changes in correlation may be understated especially during our study period, where we see increasing positive correlation between stocks and REITs.

Critique

A large positive correlation coefficient indicates a more linear relationship between the two variables. If short selling is made possible, the marginal impact on optimal portfolio weights from the incremental changes in correlation will increase from our current observations. The result of our study shows significantly greater impact from changes in expected returns than changes in correlations; therefore, it is unlikely that removing the short selling restriction will prove our conclusions inaccurate, but this methodology will better capture the underlying marginal effects.

There are always going to be uncertainties when determining forward-looking parameters as the future can never be forecast with 100% accuracy. Historical data can be an important component for the approximations of an average investor's expectations. All investors have the same market data, however the model used has a major effect on the selection of data periods, quantity of data generated, and the method of calculation.
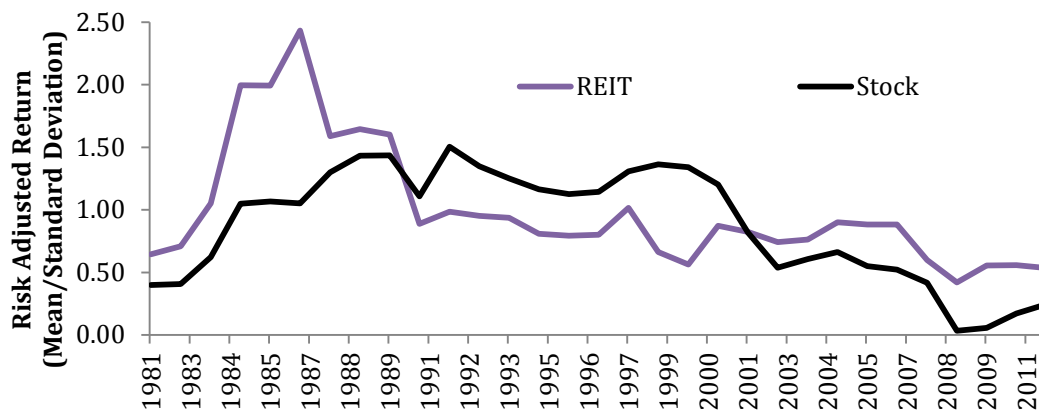

Exhibit 7: REIT vs Stock 10 Year Overlapping Risk Adjusted Returns

The importance of the time period used to estimate the excess risk premium of REITs over stocks is vital in determining the appropriate expected return and to avoid data-mining. Using overlapping 10-year returns and standard deviations we find that 10-year overlapping risk-adjusted return windows from the *1981-1990* period through

to the *1992-2001* period showed stocks outperforming REITs (Exhibit 7). However, in all other 10-year windows between 1972 and 2011, REITs had the better risk-adjusted returns. Based on this finding, Waggle and Agrrawal (2006) may have coincidentally selected the only period where stock's risk-adjusted returns exceeded REIT risk-adjusted returns, since the inception of the REIT data.

In the case of REITs, significant data is only available back to 1972 from NAREIT, whereas stock and bond data can be retrieved from 1926 to 2011. This large inconsistency in available data limits the reliability of the forward-looking parameters. Extensive historical data is critical for determining statistically appropriate real estate returns in evaluating relevant sensitivity analysis.

Different models utilize data in different ways such as arithmetic versus geometric returns. The frequency of the data such as daily, monthly, quarterly, and annually may also have a minor effect. The method of data sampling can have a major effect. Some Monte Carlo models sample returns with replacement and can generate a large number of sample windows using limited data. These model specification differences lead to large discrepancies in the forward-looking parameters.

The inclusion of real estate as an asset class is critical to the average investor as the major indices include a significant component in real estate. There are two methods for investment, the traditional direct investment in real estate and the now more popular REIT investment. The illiquidity of direct real estate investment is an important factor when assessing the choice between direct or securitized real estate. In a market where there are relatively few buyers and seller, illiquidity occurs when the asking price from a seller is significantly higher than the bidding price of a buyer, leading to possible reductions in the final transaction price. For an investor determining the appropriate expected return, the illiquidity of direct real estate may

cause a lower total return in the short-run. In the long-run, the loss of return from illiquidity can be averaged away. Securitized real estate is another solution as exchange traded securities are marked-to-market daily, there is no issue with liquidity of obtaining efficient prices to base expected return.

The advent of real estate as a necessary asset class has come to pass, where it comprises 15% of the global financial asset[2]. The FTSE EPRA/NAREIT Global REITS marked the global REIT market capitalization at $695 billion in 2012. This is the beginning of the rapidly increasing securitization of the over $24 Trillion of global commercial real estate estimated at the end of 2010.[3] It has broadened the investment universe for the average investor. With the onset of more securitization the historical supply-constraint will relax, lowering the expected returns investor should estimate when determining the appropriate parameters to build an optimal portfolio.

The importance of looking at the sensitivities with respect to REITs rather than other asset classes, such as hedge funds or commodities, comes primarily from the anecdotal evidence showing the evolution of REITs as an asset class. Securitization of real estate into REITs is expanding rapidly and has become an integral component of the average investors' portfolio. To a lesser extent, hedge funds and commodities are also part of an average investor's portfolio. However, further study into the effect of adding other asset classes can be done to further enhance the literature on this topic.

Concluding Remarks

The findings of our research confirm Waggle & Agrrawal (2006), in which the optimal weight is more sensitive to changes in expected risk premiums than to

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2 EPRA Annual Report 2012
3 EPRA Monthly Bulletin April 2012

changes in correlation coefficients; however the effects have been diminished due to the role reversal of REITs and stocks in the optimal portfolio. REITs have once again become the highest return and highest risk asset class out of the three we observed. The effectiveness of MVO asset allocation is only as good as the estimated forward-looking parameter estimates. Therefore, the validity of the historical data and the particular time frame chosen in estimating the input parameters is imperative to investors' asset allocation decisions.

Appendix

According to Waggle & Agrrawal (2006), under the three-asset case, the closed-form derivations for optimal portfolio weights and marginal effects to the weights due to changes in REITs expected return and REITs-stock correlations are as follows:

$$r_p = w_R r_R + (1 - w_R - w_B) r_S + w_B r_B \quad (A1)$$

$$\sigma_p^2 = w_R^2 \sigma_R^2 + (1 - w_R - w_B)^2 \sigma_S^2 + w_B^2 \sigma_B^2 + 2 w_R (1 - w_R - w_B) \sigma_{RS} + 2 w_R w_B \sigma_{RB}$$

$$+ 2(1 - w_R - w_B) w_B \sigma_{SB} \quad (A2)$$

The covariance between two assets is denoted as σ_{IJ}, which is determined by the correlation between two assets and their dispersions to their mean, $\sigma_{IJ} = \rho_{IJ} \sigma_I \sigma_J$.

Setting $\frac{\partial U}{\partial w_R} = 0$ and solving for w_R results in:

$$w_R^* = \frac{r_R - r_S + A(\sigma_S^2 - \sigma_{RS}) - w_B A(\sigma_S^2 - \sigma_{RS} - \sigma_{RB} - \sigma_{SB})}{A(\sigma_R^2 + \sigma_S^2 - 2\sigma_{RS})} \quad (A3)$$

Setting $\frac{\partial U}{\partial w_R} = 0$ and solving for w_B results in:

$$w_B^* = \frac{r_B - r_S + A(\sigma_S^2 - \sigma_{SB}) - w_R A(\sigma_S^2 - \sigma_{RS} - \sigma_{RB} - \sigma_{SB})}{A(\sigma_B^2 + \sigma_S^2 - 2\sigma_{RS})} \quad (A4)$$

Substituting A4 into A3 and rearranging the equation leads to:

$$w_R^* = \frac{[r_R - r_S + A(\sigma_S^2 - \sigma_{RS})]Y - [r_B - r_S + A(\sigma_S^2 - \sigma_{SB})]Z}{A[XY - Z^2]} \quad (A5)$$

$$w_B^* = \frac{[r_B - r_S + A(\sigma_S^2 - \sigma_{BS})]X - [r_R - r_S + A(\sigma_S^2 - \sigma_{RS})]Z}{A[XY - Z^2]} \quad (A6)$$

$$X = (\sigma_R^2 + \sigma_S^2 - 2\sigma_{RS})$$

$$Y = (\sigma_S^2 + \sigma_B^2 - 2\sigma_{SB})$$

$$Z = (\sigma_S^2 - \sigma_{RS} + \sigma_{RB} - \sigma_{SB})$$

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Solving for $\frac{\partial w_R^*}{\partial r}$ results in:

$$\frac{\partial w_R^*}{\partial r_R} = \frac{Y}{A[XY - Z^2]} \quad (A7)$$

$$\frac{\partial w_S^*}{\partial r_R} = \frac{Z - Y}{A[XY - Z^2]} \quad (A8)$$

Solving for $\frac{\partial w_R^*}{\partial \rho}$ results in:

Using the quotient rule $= \frac{\partial w_R^*}{\partial \rho_{RS}} = \frac{f'h - fh'}{[h]^2}$

$$f = [r_R - r_S + A(\sigma_S^2 - \sigma_{RS})]Y - [r_B - r_S + A(\sigma_S^2 - \sigma_{SB})]Z$$

$$f' = \sigma_R \sigma_S [(r_B - r_S + A(\sigma_S^2 - \sigma_{SB}) - AY]$$

$$h = A(XY - Z^2)$$

$$h' = -2\sigma_R \sigma_S AY + 2A(\sigma_R \sigma_S \sigma_S^2 - \sigma_{RS}\sigma_R \sigma_S + \sigma_R \sigma_S \sigma_{RB} - \sigma_R \sigma_S \sigma_{SB})$$

$w_S^* = 1 - w_R^* - w_B^*$, where $\frac{\partial w_S^*}{\partial \rho_{RS}} = -\frac{\partial w_R^*}{\partial \rho_{RS}} - \frac{\partial w_B^*}{\partial \rho_{RS}}$

Using the quotient rule $= \frac{\partial w_B^*}{\partial \rho_{RS}} = \frac{g'h - gh'}{[h]^2}$

$$g = [r_B - r_S + A(\sigma_S^2 - \sigma_{BS})]X - [r_R - r_S + A(\sigma_S^2 - \sigma_{RS})]Z$$

$$g' = -2\sigma_R \sigma_S [(r_B - r_S + A(\sigma_S^2 - \sigma_{SB})] + \sigma_R \sigma_S [AZ + (r_R - r_S + A(\sigma_S^2 - \sigma_{RS}))]$$

$$h = A(XY - Z^2)$$

$$h' = -2\sigma_R \sigma_S AY + 2A(\sigma_R \sigma_S \sigma_S^2 - \sigma_{RS}\sigma_R \sigma_S + \sigma_R \sigma_S \sigma_{RB} - \sigma_R \sigma_S \sigma_{SB})$$

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