CONES, LATTICES AND HILBERT BASES OF CUTS

by

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Abstract

A Hilbert basis is defined to be a set of vectors S such that every vector in the cone and lattice generated by S can also be expressed as a non-negative integer combination of vectors in S. Goddyn (1991) conjectured that characteristic vectors of cuts of graphs form Hilbert basis. A counterexample to this conjecture was given by Laurent in 1996. We study the class of graphs whose cuts form a Hilbert basis and prove that the cuts of graphs formed by uncontractions of K_5 and those of $K_{3,3}$-free graphs form Hilbert bases. In addition, we repair an incorrect result of Laurent that says the cuts of all proper subgraphs of K_6 form Hilbert bases by proving that the cuts of $K_6 \setminus e$ do not form a Hilbert basis. We also study the cones, lattices and Hilbert bases of contractible circuits of projective planar graphs by looking at the cuts of their dual graphs.

This thesis is dedicated to my late father,

Mr. Saiprasad Deshpande.

His words of inspiration and encouragement
in pursuit of knowledge, still linger on.

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Chapter 1

Introduction

The concept of Hilbert basis was introduced to study the concept of total dual integrality of polyhedra. Several examples of Hilbert bases arising from combinatorial objects are introduced in [23]. In this thesis we study the Hilbert bases arising from the characteristic vectors of cut cones of certain graphs.

1.1 Hilbert bases

Let **S** be a finite set of vectors in $\mathbb{R}^n$. Then we can define the *convex hull*, *cone*, *lattice* and *integer cone* of **S** as follows.

$$\begin{aligned} \mathit{ConvexHull}(\mathbf{S}) &:= \left\{ \sum_{c \in \mathbf{S}} \alpha_c c : \alpha_c \in \mathbb{R}_{\geq 0}; \sum_{c \in \mathbf{S}} \alpha_c = 1 \right\} \\ \mathit{Cone}(\mathbf{S}) &:= \left\{ \sum_{c \in \mathbf{S}} \alpha_c c : \alpha_c \in \mathbb{R}_{\geq 0} \right\} \\ \mathit{Lattice}(\mathbf{S}) &:= \left\{ \sum_{c \in \mathbf{S}} \alpha_c c : \alpha_c \in \mathbb{Z} \right\} \\ \mathit{IntCone}(\mathbf{S}) &:= \left\{ \sum_{c \in \mathbf{S}} \alpha_c c : \alpha_c \in \mathbb{Z}_{\geq 0} \right\} \end{aligned}$$

Here, $\mathbb{R}, \mathbb{Z}, \mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$ denote the sets of real numbers, integers, non-negative real numbers and non-negative integers respectively.

It is clear that for any set of vectors $\mathbf{S} \subseteq \mathbb{R}^n$,

$$IntCone(\mathbf{S}) \subseteq Cone(\mathbf{S}) \cap Lattice(\mathbf{S}) \tag{1.1}$$

Definition 1.1.1. A set of vectors **S** is called a *Hilbert basis* if equality holds in (1.1).

Example 1.1.2. The set $S = \{(1,0),(0,1)\} \subseteq \mathbb{R}^2$ clearly forms a Hilbert basis for $\mathbb{R}^2$. More generally any orthogonal set will form a Hilbert basis in its vector space.

Example 1.1.3. The set $S = \{2,3\} \subseteq \mathbb{R}$ does *not* form a Hilbert basis for $\mathbb{R}$ as 1 is both in the cone and the lattice of S but not in the integer cone of S.

Example 1.1.4. The set $S = \{(1,2),(1,3),(2,1)\} \subseteq \mathbb{R}^2$ does not form a Hilbert basis as the vector $(1,1)$ is in the cone and the lattice of S but not in the integer cone of S.

Hilbert bases were first introduced in [12] to study the concept of total dual integrality.

Definition 1.1.5. A linear system $Ax \leq b$ is called *totally dual integral* if the minimum in the LP,

$$\max\{wx : Ax \leq b\} = \min\{yb : yA = w, y \leq 0\}$$

can be achieved by an integer vector y for each integral w for which the optimum exists.

It can be proved that $Ax \leq b$ is totally dual integral if and only if for each minimal nonempty face of $\{x : Ax \leq b\}$, the set of active rows of A generates the lattice $\mathbb{Z}^n$ and forms a Hilbert basis.

It was proved by Hilbert [15] that every finite set of rational vectors extends to a finite Hilbert basis. A cone C is *pointed* if C does not contain a 1-dimensional linear subspace. Schrijver in [24] shows that every Hilbert basis contains a unique minimal Hilbert basis if the cone generated by the vectors is pointed. Here the Hilbert basis is *minimal* if deleting an element changes its cone or its lattice or is no longer a Hilbert basis.

Let C be a cone in $\mathbb{R}^n$. As an abuse of terminology, we say that an inequality $v^T x \leq 0$ is a *facet* of C if C is in the half-space $\{x \in \mathbb{R}^n \mid v^T x \leq 0\}$ and $\{x \in C \mid v^T x \leq 0\}$ is a facet of C.

One motivation for the study of Hilbert bases comes from a family of polyhedra arising in discrete optimization. The problem of when the perfect matchings of a graph forms a Hilbert basis has been studied in [10], [18]. It can be proved that for the Petersen graph, the vector of all 1's belongs to the cone and the lattice of characteristic vectors of perfect matchings but it does not belong to the integer cone. Hence the characteristic vectors of perfect matchings of the Petersen graph do not form a Hilbert basis. It is conjectured in [18] that the perfect matchings of a graph form a Hilbert basis if it has no Petersen minor.

1.2 Cut cone and cut lattice

Let $G = (V, E)$ be a graph. For $F \subseteq E(G)$ we define the *characteristic vector* of F to be the vector $\chi(F) \in \mathbb{R}^{E(G)}$ such that $\chi(F)_e = 1$ if $e \in F$ and $\chi(F)_e = 0$ otherwise. We sometimes write F instead of $\chi(F)$ where no confusion results. We say that $\chi(F)$ *indicates* the set F.

For $S \subseteq V$, let $\delta(S)$ denote the set of edges with exactly one endpoint in S. We call $\delta(S) = \delta_G(S)$, the *cut* in G generated by S. Let C_G denote the set of cuts of G. The *cut cone* of G is defined as the cone generated by the characteristic vectors of cuts of G. To ease notation we write $Cone(G)$ instead of $Cone(\{\chi(c) : c \in C_G\})$. We define $ConvexHull(G)$, $Lattice(G)$ and $IntCone(G)$ similarly.

The problem of finding a cut with the maximum number of edges is known as the max-cut problem in combinatorial optimization. This problem is known to be NP-hard. More generally the problem of stating whether a vector is in $ConvexHull(G)$ is NP-hard. Accordingly for a given graph G, it is generally difficult to find a compact description of $ConvexHull(G)$ in terms of facets (or else we may maximize the weight function over the cut polytope using a linear programming to get the solution to the max cut problem in polynomial time).

For a large class of graphs however the max cut problem is solvable in polynomial time. It is solvable for planar graphs, [14]. More generally, Barahona in [2] proved that the problem is solvable in polynomial time for all graphs not having K_5 as a minor.

It has been proved that computing the facets of the cut cone in general is also NP-hard. (See for example [8], [9].) Characterising the cut lattice of a graph is quite simple however. [13]

Theorem 1.2.1. *Given a simple graph $G = (V, E)$ and $x \in \mathbb{R}^E$, $x \in Lattice(G)$ if and only if $x \in \mathbb{Z}^E$ and $x(C)$ is even for each circuit C of G.*

Here we have abbreviated $\sum_{e \in C} x_e$ to $x(C)$ for any $C \subseteq E(G)$.

Given a graph G we define the *deletion of an edge e of G* denoted by $G\backslash e$, to be the graph formed on the same vertex set as G with the edge set being $E(G) \setminus e$. The *contraction of an edge* $e = (u, v)$ *of* G, denoted by G/e, is defined as the graph on the vertex set $V(G) \setminus \{u, v\} \cup \{w\}$ and edge set $E(G) \setminus e \cup \{(w, x) \mid \text{either } (x, u) \in E(G) \text{ or } (x, v) \in E(G) \text{ for all } x \in V(G)\}$. We say that a graph G has a G'-minor if G' can be obtained from G by a sequence of contractions and deletions. Describing the cut cone of a graph G is easy if G does not contain a K_5-minor [25]. We define a *circuit* of a graph G to be the edge set of a simple cycle in G.

Theorem 1.2.2. *Let* $G = (V, E)$ *be a simple graph and* $x \in \mathbb{R}^{E(G)}$. *If* $x \in Cone(G)$, *then*

$$\begin{aligned} &x(e) \geq 0 && e \in E(G) \\ &x(e) \leq x(C \setminus e) && C \subseteq E(G);\ C \text{ is a circuit of } G\ ; e \in C. \end{aligned} \tag{1.2}$$

Furthermore, the constraints completely describe the cut cone of G if and only if G does not contain a K_5 *minor.*

We call the constraints of the form $x(e) \leq x(C \setminus e)$, *cycle constraints* of the cut cone of the graph. If C is a cycle in G, then we define *cycle constraints given by* C to be all inequalities of the form $x(e) \leq x(C \setminus e)$ where $e \in C$.

The question of whether the characteristic vectors of cuts of a graph form a Hilbert basis was asked in [13]. The conjecture was proved false in [17] where it is shown that the cuts of K_6 do not form a Hilbert basis. In fact it is proved that there exists an infinite family of graphs whose cuts do not form a Hilbert bases.

The dual problem of when the circuits of a graph form a Hilbert basis is completely solved in [1]. Specifically these are precisely all the graphs which do not contain a Petersen minor. Describing the cone and the lattice of cycles of graphs however is relatively "easy". The constraints in Theorem 1.2.1 and 1.2.2, where C varies over the cuts of the graphs instead of its circuits, are enough to completely describe the cycle cone and cycle lattice of any graph.

This thesis aims to study the Hilbert basis property of the sets of (characteristic vectors of) cuts of certain classes of graphs. We state some properties of the cut cones of K_n in Chapter 2. Also in Chapter 2, we consider the set of graphs that are contractible to K_5. We also study the class of graphs that do not have $K_{3,3}$ as a minor. Here we prove that the characteristic vectors of cuts of all graphs in these two classes form Hilbert bases and present a more general result for 2-sums of graphs.

In Chapter 3, we look at subgraphs of K_6 and prove that the cuts of K_6 and $K_6 \setminus e$ do not form a Hilbert basis. This corrects an error in [17].

Chapter 4 deals with projective planar graphs and the Hilbert bases of contractible cycles and some matroidal variations.

Chapter 2

Hilbert bases of cuts

In this chapter we study the class of graphs whose cuts form Hilbert basis. We prove that under a certain condition, the property of having cuts that form Hilbert bases is closed under 2-sums. As a corollary, we prove that the set of uncontractions of K_5 and the set of graphs with no $K_{3,3}$ minor have this condition and deduce that the characteristic vectors of their cuts form Hilbert bases.

2.1 Cut cones of complete graphs

In this section, we look at a class of facet defining inequalities for the cone of cuts of K_n. As noted earlier, the cycle inequalities are enough to completely describe the cone of K_n for $n \leq 4$. Now consider a vector $y \in \mathbb{Z}^{E(K_5)}$ with weight of $y_e = 1$ for all edges e in a subgraph of K_5 isomorphic to $K_{2,3}$ and 2 everywhere else. Clearly y satisfies the cycle constraints. To show that this vector is not in the cut cone of K_5, consider a vector s such that $s_e = -1$ if $x_e = 1$ and $s_e = 1$ otherwise. Then s has a positive inner product with x but has a non positive inner product with each of the cuts of K_5. Therefore we have found a hyperplane $s^T x \leq 0$ in $\mathbb{R}^{E(K_5)}$ that separates the cone of cuts of K_5 from y. Thus the cycle constraints are not sufficient to describe $Cone(K_5)$.

Definition 2.1.1. Let $b = (b_1, b_2, \ldots b_n) \in \mathbb{Z}^n$ be such that

$$\sum_{i=1}^{n} b_i = 1$$

Then the inequality,

$$\sum_{1\le i<j\le n} b_i b_j x_{ij} \le 0$$

is called the *hypermetric inequality* defined by b and is denoted by $Hyp_n(b)$.

Note that a Hypermetric inequality may be a cycle constraint. For example if $b = (1,1,-1,0,0,0)$, then $Hyp_6(b)$ is a cycle constraint corresponding to a triangle in K_6. For $n \ge 2$, the hypermetric inequalities are valid for $Cone(K_n)$. The hypermetric inequalities together with (1.2) are enough to completely describe the cut cones of K_5 and K_6 [8]. Below we give a complete description of the cones, $Cone(K_5)$ and $Cone(K_6)$.

- $Cone(K_5)$ has 40 facets given by
  - 30 facets coming from $Hyp_5(1,1,-1,0,0)$ along with all the permutation of the vertices
  - 10 facets coming from $Hyp_5(1,1,1,-1,-1)$ with its permutations
- $Cone(K_6)$ has 210 facets given by
  - 60 facets coming from $Hyp_6(1,1,-1,0,0,0)$ along with its permutations
  - 60 facets coming from $Hyp_6(1,1,1,-1,-1,0)$ along with its permutations
  - 30 facets coming from $Hyp_6(-2,-1,1,1,1,1)$ along with its permutations
  - 60 facets coming from $Hyp_6(2,1,1,-1,-1,-1)$ along with its permutations

From $n \ge 7$ onwards, the hypermetric inequalities are not enough to describe the cut cones of K_n.

2.2 Hilbert bases of cuts

In this section we study the class $\mathscr{H}$ of graphs whose cuts form Hilbert bases. We assume that all graphs in this section have no loops or parallel edges as loops do not belong to any cut in a graph and an edge belongs to a cut if and only if all its parallel edges belong to it. It follows that a given graph belongs to $\mathscr{H}$ if and only if the graph formed by deleting its loops and parallel edges belongs to $\mathscr{H}$.

First we state a few results about Hilbert bases of cuts of graphs.

Proposition 2.2.1. *If $G \in \mathscr{H}$, then $G/e \in \mathscr{H}$ for all edges e of G.*

The proof of the above statement follows from the fact that the cut cone of G/e is in fact a face of the cut cone of G [17]. The case for deletion minors is not so simple. It is not known whether edge deletion preserves the property $G \in \mathscr{H}$. We quote a sufficient condition of Laurent [17] without proof.

Proposition 2.2.2. *Let G be a graph whose cut cone is given by the inequalities $v_i^T x \leq 0$ such that each $v_i \in \{0, -1, 1\}^{E(G)}$ and $v_i^T \delta(S) \in 2\mathbb{Z}$ for all cuts $\delta(S)$ of G. If $G \in \mathscr{H}$, then $G \setminus e \in \mathscr{H}$ for all edges e of G.*

Proposition 2.2.3. *(Fu, Goddyn [11]) If G is a graph such that G does not contain a K_5 minor, then $G \in \mathscr{H}$.*

Definition 2.2.4. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Define $G_1 \cup G_2$ as the graph on $V_1 \cup V_2$ with edge set $E_1 \cup E_2$ and define $G_1 \cap G_2$ as the graph on $V_1 \cap V_2$ with edge set $E_1 \cap E_2$ respectively.

Definition 2.2.5. Let G_1 and G_2 be graphs such that $G_1 \cap G_2$ is a complete graph on k vertices such that $G_1 \cap G_2$ does not contain a non empty cut of either G_1 or G_2. Define the *k-sum* of G_1 and G_2 to be the graph obtained from $G_1 \cup G_2$ by deleting the edges of $G_1 \cap G_2$. If $G_1 \cap G_2$ consists of an edge f. We denote the two sum of G_1 and G_2 by $G_1 \oplus_f G_2$.

It can be proved that if G_1 and G_2 are planar and $k = 0, 1$ or 2, then the *k-sum* of G_1 and G_2 is also planar. We observe that a *2-sum* of a graph and a cycle C_n with respect to a particular edge is a subdivision of that edge.

The *k-clique sum* is defined by taking $G_1 \cup G_2$ without deleting the edges of $G_1 \cap G_2 \cong K_k$. We quote two theorems relating to the cones and Hilbert bases of cuts.

Theorem 2.2.6. *(Barahona [2]) Let G be the k-clique sum ($k = 0, 1, 2, 3$) of two graphs G_1 and G_2. Then a system of linear inequalities sufficient to describe $Cone(G)$ is obtained by juxtaposing the inequalities that define the cones of $Cone(G_1)$ and $Cone(G_2)$ and identifying the variables associated with the common edges of G_1 and G_2.*

Theorem 2.2.7. *(Laurent [17]) Let G be the k-clique sum ($k = 0, 1, 2, 3$) of two graphs G_1 and G_2. Then $G \in \mathscr{H}$ if and only if $G_1 \in \mathscr{H}$ and $G_2 \in \mathscr{H}$.*

The aim of this section is to prove an analogue of Laurent's result for k-sums under additional assumptions on G_1 and G_2.

Definition 2.2.8. Let G be a graph and let $f \in E(G)$. Define a vector $x \in \mathbb{R}^E$ to be *almost in the lattice of G with respect to f* if $x \in \mathbb{Z}^E$ and $x(C)$ is an even integer for all circuits C of G such that $f \notin C$. Given sets A and B, we define $A \triangle B := (A \setminus B) \cup (B \setminus A)$. We call this the *symmetric difference* of A and B.

Lemma 2.2.9. *Let G be a graph and $f \in E(G)$. Let a vector x be almost in the lattice of G with respect to f. If there is a circuit C of G such that $f \in C$ and $x(C)$ is an even integer, then x is in the lattice of cuts of G.*

Proof. Let C' be a circuit different from C such that $f \in C'$. Then,

$$x(C' \triangle C) = x(C) + x(C') - 2x(C_1 \cap C_2)$$

Now $(C' \triangle C)$ is a circuit in G such that $f \notin C' \triangle C$. So $x(C' \triangle C) \equiv 0 \pmod 2$ and since $x(C) = 0 \pmod 2$, we have, $x(C') = 0 \pmod 2$. i.e. x adds up to an even integer over all circuits of G. So $x \in Lattice(G)$. □

Definition 2.2.10. Let G be a graph and $x \in \mathbb{Z}^{E(G)}$. Define $E_{odd}(x)$ to be the set of edges e such that x_e is odd.

We state a result in [4] without proof.

Proposition 2.2.11. *Let G be a graph. Then $B \subseteq E(G)$ is an edge cut of G if and only if $|B \cap C|$ is even for all circuits C of G.*

Lemma 2.2.12. *Let G be a graph. Let $x \in \mathbb{Z}^{E(G)}$ be almost in the lattice of G with respect to a fixed edge f. Then*

(1) $x \in Lattice(G)$ if and only if $E_{odd}(x)$ is an edge cut of G.

(2) $x \notin Lattice(G)$ if and only if $E_{odd}(x) = B \triangle \{f\}$ where B is an edge cut of G.

Proof. We prove (1) first. Let $x \in \mathbb{Z}^{E(G)}$. Then $x \in Lattice(G)$ if and only if $|C \cap E_{odd}(x)|$ is even for every circuit C of G. Thus by Proposition 2.2.11, $E_{odd}(x)$ is a cut of G.
For (2), note that x is almost in the lattice with respect to f but not in the lattice if and only if $x + \chi(\{f\})$ is in the lattice. Equivalently $E_{odd}(x + \chi(\{f\}))$ is an edge cut of G whence $E_{odd}(x) = B \triangle \{f\}$ for some edge cut B. □

Definition 2.2.13. Let G be a graph with a fixed edge f. Let $x \in \mathbb{R}^{E(G)}$. Let $x(\gamma) \in \mathbb{R}^{E(G)}$ be a vector obtained by redefining the single entry $x_f = \gamma$. Define the *feasibility interval* $I(G, x, f)$ for G, x and f to be the (possibly empty) interval $[\gamma_{\min}, \gamma_{\max}]$ such that $x(\gamma) \in Cone(G)$ if and only if $\gamma \in [\gamma_{\min}, \gamma_{\max}]$.

Definition 2.2.14. Let $v \in \mathbb{R}^{E(G)}$. We say a vector y is *tight for the inequality* $v^T y \leq 0$ if $v^T y = 0$. We say a subset $B \subseteq E(G)$ is *tight for the inequality* $v^T y \leq 0$ if $v^T \chi(B) = 0$. We say that y (or B) is *v-tight* if y (or B) is tight for the inequality $v^T y \leq 0$.

Example 2.2.15. Consider K_5 with a fixed edge $f = (u, v)$. Let T be the triangle not containing either of u, v. Let x be the vector with $x_e = 6$ for $e \in E(T)$, and $x_e = 4$ for $e \in E(K_5) \setminus E(T)$. Then $I(G, x, f)$ is the interval $[0, 6]$ (Figure 2.1). The lower bound of the interval is met by taking thrice of each of the cuts $\delta(v)$ where $v \in V(T)$ plus the cut $\delta(u, v)$. The upper bound is met by summing six cuts $\delta(x, y)$ where $x \in V(T)$; $y \in \{u, v\}$ and the three cuts $\delta(v)$ where $v \in V(T)$. Here x is tight for the facet $v^T x \leq 0$ defined by $v_f = +1$, $v_e = +1$ for every $e \in E(T)$, and -1 everywhere else.

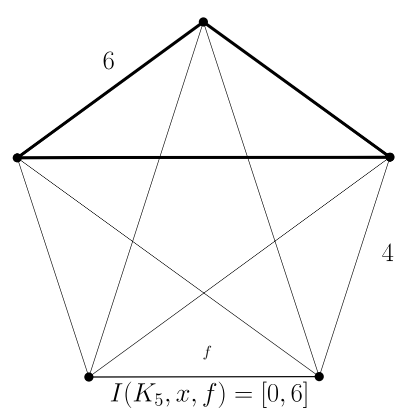


Figure 2.1: The interval $I(G, x, f)$

Definition 2.2.16. Let G be a graph with a fixed edge f. We say that G has the *lattice endpoint property for the edge* f if, for all $x \in \mathbb{R}^{E(G)}$ that is almost in the lattice with respect to f, we have, γ is an endpoint of $I(G, x, f)$ implies $x(\gamma)$ is in the lattice of G.

Proposition 2.2.17. *Planar graphs have the lattice endpoint property for all edges.*

Proof. Let G be a planar graph with a fixed edge f. Let x be a vector almost in the lattice of G with respect to f and x_f is equal to an endpoint of $I(G, x, f)$. Then x is tight to a constraint of the form $x_f \geq 0$ or a cycle constraint.
If x is tight to a constraint of the form $x_f \geq 0$, we have $x_f = 0$. So x is a non-negative linear combination of cuts that do not use the edge f. Consider a graph $G' = G/f$. Then the vector $x' \in \mathbb{R}^{E(G')}$ defined by $(x')_e = x_e$ for $e \neq f$ is in the cone and the lattice of G'. Since G' is planar, $G' \in \mathscr{H}$ and therefore $x' \in IntCone(G')$. So we can write x' as a non-negative integer combination of cuts of G'. Therefore we can write x as a non-negative combinations of cuts of G. In particular $x \in Lattice(G)$.
If x is tight to a cycle constraint in G, then there exists a circuit C such that $f \in C$ and $x_f = x(C \setminus f)$ for some edge $f \in C$. Since $x_f \in \mathbb{Z}$, we infer that $x(C)$ is an even integer. So by Lemma 2.2.9, $x \in Lattice(G)$.

□

Proposition 2.2.18. *K_5 has the lattice endpoint property for all edges.*

Proof. Let f be a fixed edge in K_5. Let $x \in \mathbb{R}^{E(K_5)}$ be a vector almost in the lattice of K_5 such that x_f is equal to an endpoint of $I(K_5, x, f)$. If x is tight to a constraint of the form $x_e \geq 0$ or a cycle constraint, we use the fact that K_5/f is planar and argue as the proof of Proposition 2.2.17 to conclude that $x \in Lattice(K_5)$. So assume x is tight to a hypermetric constraint say $v^T x \leq 0$ where $v \in \{\pm 1\}^{E(K_5)}$.
Then $v^T x \equiv 0 \pmod 2$. So $|E_{odd}(v) \cap E_{odd}(x)|$ is even. Since $E_{odd}(v) = E(G)$, it follows that $|E_{odd}(x)|$ is even.
Now by Lemma 2.2.12, $E_{odd}(x) = B$ or $E_{odd}(x) = B \triangle \{f\}$ for some edge cut B of K_5. But for all minimal[1] edge cuts B of K_5, $|B|$ is even and $|B \triangle \{f\}|$ is odd. So by Lemma 2.2.11 we have $E_{odd}(x) = B$, and $x \in Lattice(K_5)$. □

Lemma 2.2.19. *Let G be a graph with a fixed edge f. If $G \setminus f \in \mathscr{H}$, then for every x that is almost in the lattice with respect to f such that $I(G, x, f)$ is nonempty, there exists $\gamma \in I(G, x, f)$ such that $x(\gamma)$ is in the integer cone of G.*

Proof. Let x be almost in the lattice of G with respect to f. Now consider the graph $G' = G \setminus f$ and the vector $x' \in \mathbb{Z}^{E(G')}$ defined by $(x')_e = x_e$ for $e \neq f$. So $x' \in Cone(G') \cap Lattice(G')$.

[1] An edge cut is *minimal* if it is not properly contained in any other edge cut.

Since $G' \in \mathscr{H}$, we have $x' \in IntCone(G')$. Therefore there exist $\alpha_S \in \mathbb{Z}_{\geq 0}$ such that

$$x' = \sum_{S \subseteq V(G)} \alpha_S \delta(S). \tag{2.1}$$

Now consider $\mathcal{S} = \{S \subseteq V(G) \mid \mid S \cap \{u, v\} \mid = 1\}$. We define,

$$\gamma = \sum_{S \in \mathcal{S}} \alpha_S.$$

The sum in (2.1) shows that $x(\gamma) \in IntCone(G)$. In particular $\gamma \in I(G, x, f)$. □

Theorem 2.2.20. *Let G be a 2-sum of G_1 and G_2 with respect to the edge f. If each of G_1, G_2 and $G_2 \setminus f$ is in $\mathscr{H}$ and G_1 has the lattice endpoint property for f, then $G \in \mathscr{H}$.*

Proof. Let G be the 2-sum of G_1 and G_2 using $f = (u, v)$. Let $x \in Cone(G) \cap Lattice(G)$. We write

$$x = \sum_{S \subseteq V(G)} \beta_S \delta(S) \qquad\qquad \beta_S \in \mathbb{R}^E_{\geq 0}.$$

For $\gamma \in \mathbb{R}$ and $i \in \{1, 2\}$, we define $x_i(\gamma) \in \mathbb{R}^{E(G_i)}$ as $x_i(\gamma)_e = x_e$ if $e \neq f$ and $x_i(\gamma)_f = \gamma$. Note that $x_i(\gamma)$ is almost in the lattice of G_1 and G_2 with respect to f if $\gamma \in \mathbb{Z}$. Let

$$\gamma_0 := \sum_{S \subseteq V(G) \mid |\{u,v\} \cap S| = 1} \beta_S.$$

Now consider the intervals $I_1 = I(G_1, x_1(\gamma_0), f)$ and $I_2 = I(G_2, x_2(\gamma_0), f)$. First note that $I = I_1 \cap I_2$ is nonempty since $\gamma_0 \in I$. We observe that for all γ, we have $x_1(\gamma) \in Lattice(G_1)$ if and only if $x_2(\gamma) \in Lattice(G_2)$. This follows from Lemma 2.2.9, for if C_i is a circuit in G_i for $i \in \{1, 2\}$, then $C_1 \triangle C_2$ is a circuit in G. So $x_1(C_1) + x_2(C_2) = x(C_1 \triangle C_2)$ is even.

If $I_2 \subset I_1$, then by Lemma 2.2.19, there exists $\gamma \in I$ such that $x_2(\gamma) \in Lattice(G_2)$ for $i \in \{1, 2\}$. If not, then I contains an endpoint γ_1 of I_1. Because G_1 has the lattice endpoint property for f we have, $x_1(\gamma_1) \in Lattice(G_1)$.

In either case we have found a $\gamma_1 \in I$ such that $x_i(\gamma_1) \in Lattice(G_i)$ for $i \in \{1, 2\}$. Since $G_i \in \mathscr{H}$ we have $x_i(\gamma_1) \in IntCone(G_i)$ so there exist multisets[2] $\mathscr{A}$ and $\mathscr{B}$ of cuts of G_1 and G_2 respectively such that

$$x_1(\gamma_1) = \sum_{A \in \mathscr{A}} \chi(A)$$

[2] A multiset is a set in which the elements may appear more than once

$$x_2(\gamma_1) = \sum_{B\in\mathscr{B}} \chi(B).$$

Let

$$\begin{aligned}
\mathscr{A}_0 &:= \{\delta(S) \in \mathscr{A}; |S \cap \{u, v\}| = 1\}\\
\mathscr{A}_1 &:= \mathscr{A} \setminus \mathscr{A}_1\\
\mathscr{B}_0 &:= \{\delta(S) \in \mathscr{B}; |S \cap \{u, v\}| = 1\}\\
\mathscr{B}_1 &:= \mathscr{B} \setminus \mathscr{B}_1.
\end{aligned}$$

We have $\gamma_1 = |\mathscr{A}_0| = |\mathscr{B}_0|$, so we can list members of $\mathscr{A}_0$ and $\mathscr{B}_0$ as

$$\mathscr{A}_0 = \{A_1, A_2, \ldots, A_{\gamma_1}\}$$

$$\mathscr{B}_0 = \{B_1, B_2, \ldots, B_{\gamma_1}\}.$$

Therefore we have

$$x = \sum_{A\in\mathscr{A}_1} \chi(A) + \sum_{B\in\mathscr{B}_1} \chi(B) + \sum_{j=1}^{\gamma_1} \chi(A_j \triangle B_j)$$

where each $A_j \triangle B_j$ is an edge cut of G. Therefore $x \in IntCone(G)$ and $G \in \mathscr{H}$.

□

2.3 Uncontractions of K_5 and $K_{3,3}$-free graphs

In this section, we apply Theorem 2.2.20 to the set of $K_{3,3}$-free graphs and also to the set of graphs that can be contracted down to K_5.

Definition 2.3.1. Let G be a graph. Define an *uncontraction* of G to be a graph G' such that $G = G'/F$ for some $F \subseteq E(G')$.

We have verified using the Normaliz software ([5], [6], See Appendix A) that $K_5 \in \mathscr{H}$. We prove that every uncontraction of K_5 is in $\mathscr{H}$. First we deal with the relatively simple case of graphs having a cut-edge.

Theorem 2.3.2. *Set G' be an uncontraction of G where $E(G') \setminus E(G) = \{f\}$ and f is a cut-edge of G'. If $G \in \mathscr{H}$, then $G' \in \mathscr{H}$.*

Proof. Let S be the set of minimal cuts of G. Then the set of minimal cuts of G' is given by $S \cup \{\{f\}\}$. Hence the cut cone of G' will be generated by the characteristic vectors of the cuts of G along with the unit vector $\chi(\{f\})$. Let $x' \in Cone(G') \cap Lattice(G')$. Since the cuts of G form a Hilbert basis, there exist $\alpha_c's$ such that $\alpha_c \in \mathbb{Z}_{\geq 0}$ and $x' = \sum_c \alpha_c \chi(c) + x'_{n+1}\chi(\{f\})$ where c ranges over the cuts of G. Since $x'_{n+1} \in \mathbb{Z}_{\geq 0}$, $x' \in IntCone(G')$. Thus $G' \in \mathscr{H}$. □

Let G be a graph and let $v^T x \leq 0$ be a facet of $Cone(G)$. Let B be a tight cut of $v^T x \leq 0$. Then we define $w \in \mathbb{R}^E$ by $w_e = -v_e$ if $e \in B$ and $w_e = v_e$ otherwise. Laurent in [17] observed that the inequality $w^T x \leq 0$ also forms a facet of the cut cone of G. We say that $w^T x \leq 0$ is obtained from $v^T x \leq 0$ by *switching along the cut B*. Two facet defining inequalities are *switching equivalent* if one can be obtained from the other by a sequence of switchings along tight cuts.

Let H_6 be the unique 3-connected graph obtained by a single edge uncontraction of K_5 labeled as in Figure 2.2. Each edge $ij \in E(H_6)$ is assigned the variable x_{ij}. Then its cone of cuts is given as follows.

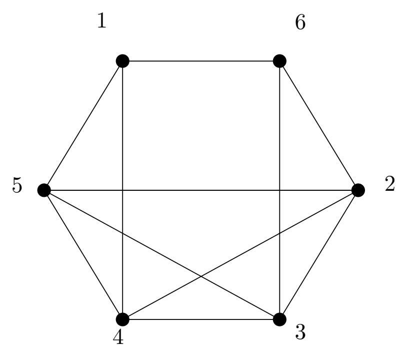


Figure 2.2: The Graph H_6

The cone of cuts of H_6 has 46 facets in all [3], [17]. They are classified by the following inequalities,

1. The inequality $x_{16} \geq 0$ forms a facet of H_6.

2. The 34 cycle inequalities given by Theorem 1.2.2 that involve the six triangles and the four 4-circuits in the graph using the edge 16.

3. The third class of constraints has 11 inequalities. They consist of the following inequality and all of its switching equivalent constraints.

$$w^T x := 2x_{16} + x_{23} + x_{45} - x_{26} - x_{36} - \sum_{i=1,2,3;j=4,5} x_{ij} \leq 0. \tag{2.2}$$

The vector w from (2.2) is illustrated in Figure 2.3. Each of the other ten vectors in its switching class is displayed in one of Figures 2.4, 2.5, or 2.6 (up to graph automorphism of H_6).

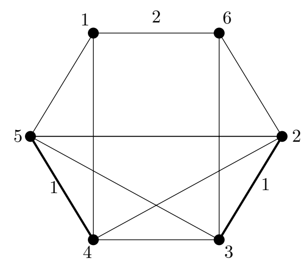


Figure 2.3: The vector w of (2.2). Unlabeled edges have weight -1.

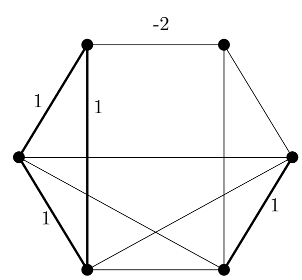


Figure 2.4: The vector obtained from w in (2.2) by switching on the w-tight cut $\delta\{1\}$. Unlabeled edges have weight -1.

Let w be the vector in (2.2). There are exactly 10 edge cuts in H_6 that are w-tight. Let B_0 be a w-tight cut of H_6 and let $16 \in B_0$. Switching on B_0 transforms w into one of the other 10 constraint vectors, say v, described in [3]. We have $(v)^{-1}(-2) = \{16\}$. Here $(v)^{-1}(-2)$ denotes the set of edges that have weight -2 in v. Let v' be the restriction of v to the edge set of the graph $H_6/16 \cong K_5$. Then v' is one of the 10 hypermetric inequalities of K_5 that are not cyclic. There are exactly 10 v-tight cuts in H_6 and B_0 is the only one among them

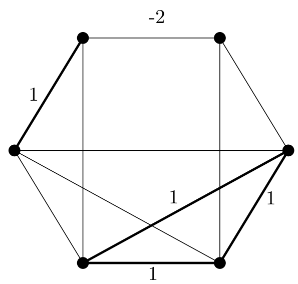


Figure 2.5: The vector obtained from w in (2.2) by switching on the w-tight cut $\delta\{1,4\}$. Unlabeled edges have weight -1.

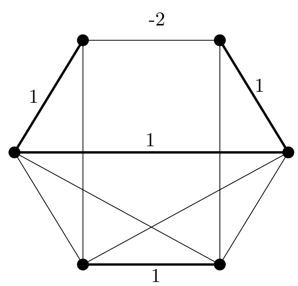


Figure 2.6: The vector obtained from w in (2.2) by swiching on the w-tight cut $\delta\{1,2,4\}$. Unlabeled edges have weight -1.

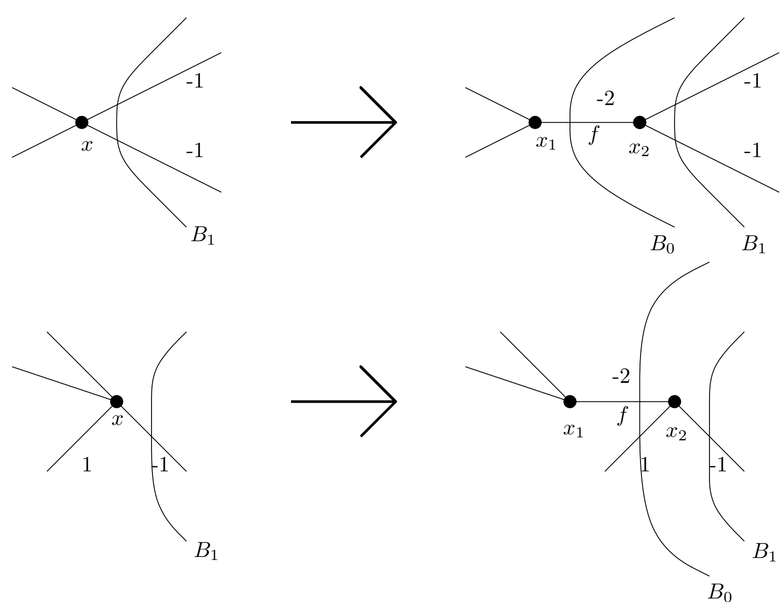


Figure 2.7: Modification of cuts

that contains the edge 16. Each of the other 9 v-tight cuts is a set of edges which forms one of the 9 v'-tight cuts in $H_6/16$.

We consider the two cuts in H_6 defined by $B_i = B_0 \triangle \delta(i)$ for $i \in \{1, 6\}$. Depending on the choice of v, either the transformation $B_0 \mapsto B_i$ either replaces an edge whose value in v is -2 with two edges with value -1 in v, or replaces two edges whose value in v is -2 and 1 in v with a single edge whose value in v is -1. The curved lines on the right side of Figure 2.7 depict these two possibilities for the pair (B_0, B_i). Therefore B_i is also a v-tight cut of H_6. The cut B_i is also a v-tight cut of $H_6/16$ which is represented by the curved lines on the left side of Figure 2.7.

We now view the process in reverse writing v^1 instead of v and v^0 instead of v'. That is, let v^0 be a noncylic hypermetric constraint vector for K_5. Let B^0 be any v^0-tight cut of K_5 and let $x^0 \in V(K_5)$ be such that $B^0 \cap \delta(x^0)$ consists of one or two edges whose value(s) in v^0 is -1. There is at least one edge $e \in \delta(x^0) \setminus B^0$ with $v^0_e = +1$. We now uncontract a new edge $f = x^0_1 x^0_2$ by partitioning $\delta(x^0)$ and replacing x^0 with x^0_1 and x^0_2. One end of f should be incident with either both edges in $B^0 \cap \delta(x^0)$ or both edges in $(B^0 \cap \delta(x^0)) \cup \{e\}$. This results in a new graph $G^1 \cong H_6$. We extend the vector v^0 to a vector $v^1 \in \mathbb{R}^{E(G^1)}$ by defining $v^1_f = -2$. We say that we have *extended* the pair (K_5, v^0) to (G^1, v^1) by uncontracting at vertex x.

We may now select a v^1-tight cut in G^1 say B^1, and another vertex of degree 4 in G, say x^1, for which $B^1 \cap \delta(x^1)$ consists of one or two edges whose value(s) in v^1 is -1, and repeat the above process with G^1 in place of K_5, x^1 in place of x^0, and B^1 in place of B^0. In this way we extend (G^1, v^1) to a new pair (G^2, v^2), where $(v^2)^{-1}(-2) = \{x^0_1 x^0_2, x^1_1 x^1_2\}$ and G^2 is an uncontraction of K_5 having 7 vertices. One can check that there is a unique v^2-tight cut in G^2 that uses the new edge in G^2, and it has the form $B^1 \triangle \delta(x^1_1)$, where x^1_1 is one of the two new vertices arising in the uncontraction $G^1 \mapsto G^2$.

We may repeat the extension process as above up to three more times, each time starting with a v^i-tight cut B^i and a vertex of degree 4, say x^i, in G^i, to define G^{i+1} and v^{i+1}. After $k \leq 5$ such steps, we obtain a pair (G^k, v^k) where G^k is a 3-connected uncontraction of K_5 and where v^k is a valid constraint vector for $Cone(G^k)$. We have that $G^k/F \cong K_5$, where $F := (v^k)^{-1}(-2)$. We may now select any one of the k new edges, say $f \in F$, and construct a new constraint vector from v^k by switching on the unique v^k-tight cut containing that edge. This results in a constraint vector w^f for $Cone(G^k)$ satisfying $(w^f)^{-1}(2) = \{f\}$. As with the case $G^1 = H_6$, every w^f-tight cut of G^k includes the edge f.

The following theorem states that every noncyclic facet-inducing constraint vector for G^k takes the form of either v^k or w^f, where $f \in F$ and $G/F \cong K_5$, and v^k restricts to a noncyclic hypermetric inequality of K_5, as described above.

Theorem 2.3.3. *Let G be a 3-connected uncontraction of K_5. The cone of cuts of G is given by the cycle constraints of G along with the following two families of inequalities.*

$$\sum_{e\in H} -2x_e + \sum_{f\notin H} v_f x_f \leq 0. \tag{2.3}$$

There is one such inequality for each pair (H, v) where $H \subseteq E(G)$ for which $G/H \cong K_5$, and where v is one of the ten vectors for which $v^T x \leq 0$ is a hypermetric inequality of G/H.

$$\sum_{e\in H'} -2x_e + \sum_{f\notin H'} w_f x_f \leq 0. \tag{2.4}$$

There is one such inequality for each $H' \subseteq E(G)$ for which $G/H' \cong H_6$, and where w is as given in (2.2).
Furthermore $G \in \mathscr{H}$.

Proof. We used Normaliz ([5], [6], See Appendix A) to verify with a computer the fact that $G \in \mathscr{H}$ for all 3-connected uncontractions of K_5. The polyhedral description of $Cone(G)$ is verified sorting the facets found by Normaliz. □

Example 2.3.4. The non cycle facets of the cut cone of Petersen graph P, belong to two families. The first have the form $v^T x \leq 0$ where v has weight -2 on a matching M, 1 on a fixed triangle of P/M and an edge not incident to it and -1 everywhere else (See Figure 2.8) for each of the six perfect matchings M. There are 18 other switching equivalent constraints that have weight 2 on one edge of M and -2 on the other 4 edges of M such that contracting the 4 edges of M of weight -2 gives the constraint (2.2) of H_6. We give a detailed description of the facets of $Cone(P)$ in Appendix A.

Proposition 2.3.5. *If G is formed by successive uncontractions of K_5, then $G \in \mathscr{H}$.*

Proof. Let G be a graph formed by successive uncontractions of K_5. Then G is obtained from a 3-connected uncontraction H of K_5 by subdividing edges and adding cut-edges. By

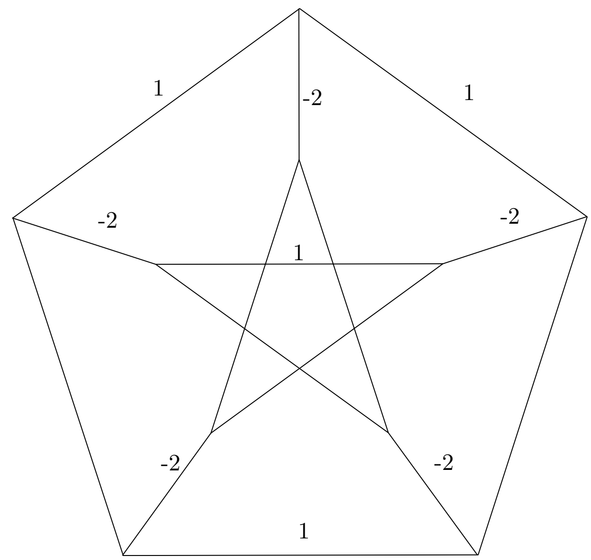


Figure 2.8: A facet of the cut cone of the Petersen graph

Theorem 2.3.3, we have $H \in \mathscr{H}$. Also addition of cut-edges to H does not affect the membership in the class $\mathscr{H}$. So we prove the result only for successive subdivisions of H. Let H' be a subdivision of H and let $f \in E(H')$. Then $H' \setminus f$ does not contain a K_5 minor, so $H' \setminus f \in \mathscr{H}$ by Proposition 2.2.3. Therefore by Theorem 2.2.20, the 2-sum of G and C_n on the edge f is in $\mathscr{H}$. □

We have not investigated the case of 2-sums of uncontractions of K_5 but we conjecture that they also belong to $\mathscr{H}$. However we do have a positive result for $K_{3,3}$-free graphs.

Theorem 2.3.6. *All $K_{3,3}$-free graphs are in $\mathscr{H}$.*

Proof. Note that $K_{3,3}$-free graphs can be formed by 2-sums of planar graphs with copies of K_5 (Wagner. 1937, [28]). Let G be a graph with no $K_{3,3}$-minor. We prove the result by induction on the number of edges of G. Let G be a 2-sum of G_1 and G_2 where G_1 is either planar or K_5. Now $G_1 \in \mathscr{H}$ since all planar graphs and K_5 are in $\mathscr{H}$. By induction hypothesis, $G_2 \setminus f \in \mathscr{H}$. Also by Lemmas 2.2.17 and 2.2.18, G_1 has the lattice endpoint property for all edges. Therefore by Theorem 2.2.20, we have, $G \in \mathscr{H}$. □

Chapter 3

Hilbert bases of cuts of subgraphs of K_6

In this chapter we look at the Hilbert basis of cuts of the complete graph on six vertices and its subgraphs. We shall prove that the cuts of K_6 and $K_6 \setminus e$ do not form a Hilbert basis.

3.1 Cuts of K_6

Definition 3.1.1. Let $S = \{x_1, x_2, \ldots, x_m\}$, $x_i \in \mathbb{R}^n$. Define the set of *quasi-Hilbert points of* S denoted by *quasi-H*(S) to be the unique set of vectors H such that the set $S \cup H$ forms a Hilbert basis and that H is the inclusion-wise minimal set with this property. Schrijver [24] shows that H is unique provided the $Cone(S)$ is pointed. For a graph G, we denote the set of quasi-Hilbert points of the set of characteristic vectors of its cuts by quasi-H(G). Note that $G \in \mathscr{H}$ if and only if quasi-H$(G) = \emptyset$.

Theorem 3.1.2. *(Laurent [17])* $K_6 \notin \mathscr{H}$ *and quasi-H*(K_6) *consists of the 15 vectors* $d^f \in \mathbb{R}^{E(K_6)}$ *defined by* $(d^f)_e = 2$ *if* $e \neq f$ *and* $(d^f)_f = 4$.

The vectors d^f are clearly in the lattice of cuts since they add up to an even integer over each circuit in K_6. It can be seen that d^f's are in the cone by adding all cuts that use f and dividing by 4. Now each of the d^f lie on exactly four triangular facets of the cut cone. We can check that $d^f - \chi(B) \notin Cone(K_6)$ for each cut B that is tight for all of these facets. These cuts B are precisely the cuts of K_6 that contain f. Therefore d^f cannot be in the integer cone of the cuts of K_6.

The next theorem corrects a statement of Laurent [17] where she incorrectly asserts that any proper subgraph of K_6 is in $\mathscr{H}$.

Theorem 3.1.3. *Let $G = K_6 \setminus e$ where $e = (5,6)$. Then quasi-$H(G)$ consists the eight vectors $d^f \in \mathbb{R}^{E(K_6\setminus e)}$ defined as follows. Let f be a fixed edge in $\delta(\{5,6\})$. If, say $f \in \delta(5)$, then define $d^f_f = 3$ and $d^f_e = 1$ for the remaining edges in $\delta(5)$ and $d^f_e = 2$ everywhere else.*

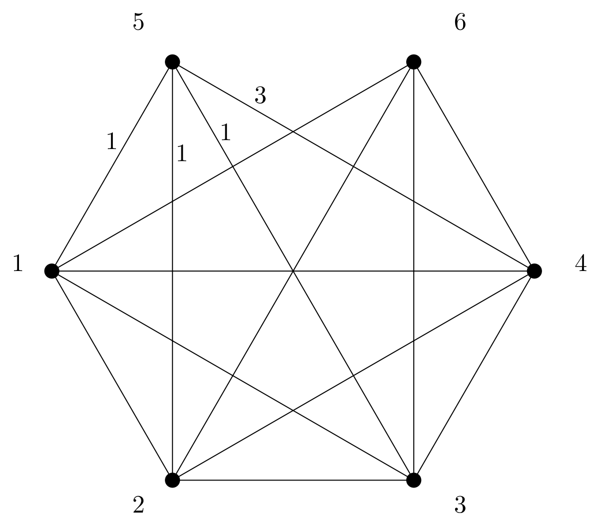


Figure 3.1: A quasi-Hilbert element of $Cone(K_6 \setminus e))$

Proof. Without loss of generality, we assume $f = (4,5)$. Then $d^{(4,5)}$ is in the cone since

$$d^{(4,5)} = \frac{1}{2}\sum_S(\delta(S))$$

where S ranges over the following collection of vertex sets. $\{1,4\}$, $\{2,4\}$, $\{3,4\}$, $\{1,4,6\}$, $\{2,4,6\}$, $\{3,4,6\}$ and $\{6\}$. By Lemma 2.2.12, the integer vector $d^{(4,5)}$ is in the lattice since $E_{odd}(d^{(4,5)}) = \delta(5)$.

The vector $d^{(4,5)}$ lies on exactly 7 facets of the cut cone:

- One facet given by $Hyp_6(1,1,1,-1,-1,0)$
- 3 triangular facets given by
    - $x_{45} - x_{15} - x_{14} \leq 0$

- $x_{45} - x_{25} - x_{24} \leq 0$
- $x_{45} - x_{35} - x_{34} \leq 0$

- 3 more triangular facets given by
  - $x_{12} - x_{15} - x_{25} \leq 0$
  - $x_{23} - x_{25} - x_{35} \leq 0$
  - $x_{31} - x_{15} - x_{35} \leq 0$

One can check that subtracting any cut that is tight for all of the above facets from $d^{(4,5)}$ takes us out of the cone. Therefore $d^{(4,5)}$ is not in the integer cone.

We have verified by computer (using Normaliz version 2.8 [5], [6], See Appendix A) that the 8 vectors described above are precisely the quasi-Hilbert points for cuts of $K_6 \backslash e$. □

The preceding theorem corrects an error in [17] in which it is claimed that all proper subsets of K_6 are in $\mathscr{H}$. The error appears in equation (10) on page 270 where 16 is erroneously assumed to be an edge of $K_6 \setminus 16$. We have verified using Normaliz that all other subgraphs of K_6 do belong to $\mathscr{H}$. So we can state the result as follows.

Theorem 3.1.4. *A simple graph of order ≤ 6 belongs to $\mathscr{H}$ if an only if it is a proper subgraph of $K_6 \setminus e$.*

3.2 Forbidden minors for the class $\mathscr{H}$

Since the property of having cuts that form a Hilbert Basis is not known to be preserved under deletion minors, we do not know whether there is a forbidden minor for the class $\mathscr{H}$. We do however have the following partial result. If H, G are graphs, we write $H \subseteq G$ if H is a subgraph of G.

Proposition 3.2.1. *Let G be a simple graph on n vertices such that $H \notin \mathscr{H}$ for every graph H with $G \subseteq H \subseteq K_n$. Then every graph G' with a G-minor does not belong to $\mathscr{H}$.*

Proof. Suppose $G = G'/S \setminus T$ where $S, T \subset E(G')$ and T contains no cut-edges of G. Note that G'/S may contain parallel edges or loops. Let H be the graph formed by deleting the loops and all but one edge of each parallel class of G'/S. Then $G'/S \in \mathscr{H}$ if and only if $H \in \mathscr{H}$. Since $G \subseteq H \subseteq K_n$, we have $H \notin \mathscr{H}$. Thus $G' \notin \mathscr{H}$. □

Corollary 3.2.2. *If G is a graph with a $(K_6 \setminus e)$-minor, then $G \notin \mathscr{H}$.*

Proof. By Theorems 3.1.2 and 3.1.3, we have, $K_6, K_6 \setminus e \notin \mathscr{H}$. So by the previous proposition, $G \notin \mathscr{H}$ if G has a $(K_6 \setminus e)$-minor. □

Chapter 4

Contractible circuits of projective planar graphs

4.1 Signed graphic matroids

A *signed graph* is a pair (G, σ) where G is a graph and σ is a function $\sigma : E(G) \to \{+1, -1\}$ which we call the *sign function*. Define $\Sigma := \sigma^{-1}(-1)$. We sometimes write (G, Σ) instead of (G, σ). We refer to the edges in Σ as *odd* edges. Define the *signature* of a circuit $C \subseteq E$ to be the product of signs of its edges. We say that C is *even (odd)* if $|C \cap \Sigma|$ is even (odd). Thus even circuits are precisely the circuits with signature $+1$.

Given a signed graph (G, Σ), we can define a matroid $M(G, \Sigma)$ on its edge set. We assume basic knowledge of matroid theory and follow the notation in [21]. A subset S of E is independent in M if it contains at most one circuit and that circuit is odd. The circuits of M are the even circuits of (G, Σ) together with sets $C_1 \cup C_2$, where C_1, C_2 are disjoint odd circuits in G satisfying $|V(C_1) \cap V(C_2)| \leq 1$. Thus $M(G, \Sigma)$ is the binary matroid represented by the matrix obtained from the vertex-edge incidence matrix of G by adding a new row which is the characteristic vector of Σ.

Given a signed graph (G, σ) and a vertex labeling $\eta : V(G) \to \{+1, -1\}$, we define a new signing σ^η of G by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$ where $e = (v, w)$. The function η can be represented by the edge cut $\delta(A)$ where $A = \eta^{-1}(+1)$. We will refer to the operation, $(G, \sigma) \mapsto (G, \sigma^\eta)$ as *resigning on the cut* $\delta(A)$. It is easy to see that (G, σ^η) has the same set of even circuits as (G, σ). Thus $M(G, \sigma^\eta) = M(G, \sigma)$. Conversely if $M(G, \sigma) = M(G, \sigma')$, then $\sigma' = \sigma^\eta$ for some $\eta : V(G) \to \pm 1$. Two signed graphs are *switching equivalent* if they have the same set of even circuits.

4.2 Graphs on a projective plane

The projective plane N_1 can be defined as a set homeomorphic to the closed unit disc D where antipodal points on its boundary $\delta(D)$ are identified. After this identification, $\delta(D)$ has the topology of a simple closed uncontractible curve on N_1 and it is called an *equator* of N_1. Any uncontractible simple curve in N_1 can serve as $\delta(D)$.

Given a graph G we define an embedding of G on N_1 to be a drawing of G on N_1 where vertices are represented by points on N_1 and edges are represented by arcs in N_1 that are pairwise disjoint except possibly at common endpoints. A graph G is called *projective planar* if such a drawing of G exists. By a *projective plane graph*, we mean a graph G along with a drawing on a projective plane. A projective plane graph is called *standard* if no vertex lies on $\delta(D)$ and no edge meets $\delta(D)$ in more than one point. Every projective planar graph has a standard embedding with respect to any given equator. The reader may consult [20] for more on embedding of graphs.

Definition 4.2.1. Let G be a graph with a standard embedding on N_1 and let Q be an equator of N_1. Define an *equitorial signing* of G to be the signed graph (G, Σ) where $\Sigma = \{e \in E(G) : e \cap Q \neq \emptyset\}$.

Definition 4.2.2. A circuit $C \subseteq E(G)$ is *contractible* if C is represented by a contractible curve on the projective plane. Thus if G is a standard embedding, then C is contractible if and only if C meets the equator of N_1 in an even number of points. Define a *bicycle* to be the edge set of two non contractible circuits having exactly one vertex in common.

Remark: If (G, Σ) is an equatorial signing of G on a projective plane, then a resigning $\Sigma' = \Sigma \triangle \delta(S)$, where $S \subseteq V(G)$ is not necessarily equitorial. For example if no edge incident to a vertex v crosses the equator, then $\Sigma \triangle \delta(v)$ is not equitorial.

4.3 Surface duals and dual matroids

Definition 4.3.1. The *cut matroid* $M^*(G)$ of a graph G is the matroid on $E(G)$ whose circuits are the minimal cuts of G.

Definition 4.3.2. The *surface dual* G^* of a projective plane graph G is the graph whose vertices are the connected regions of $N_1 \setminus G$ and each edge $e \in E(G)$ corresponds to an edge $e^* \in E(G^*)$ joining the two regions incident to e.

Proposition 4.3.3. *Let G be a projective plane graph and let Σ be an equitorial signing. Let G^* be its surface dual. Then the cut matorid of G^*, $M^*(G^*)$ is isomorphic to the signed graphic matroid $M(G, \Sigma)$. In particular the minimal cuts of G^* correspond bijectively to the set of contractible circuits and bicycles of G, which are precisely the circuits of $M(G, \Sigma)$.*

For a detailed proof of the above, refer [27]. The above proposition enables us to examine the cuts of a projective plane graph in the context of contractible circuits and bicycles of its surface dual. We list some observations regarding operations on graphs and their duals on a projective plane.

Let G be a projective plane graph. Let G^* be its surface dual. Let e^* denote the edge of G^* that correspond to the edge e of G. Then

- Deleting an edge e of G corresponds to contracting the dual edge e^* in G^* and vice versa.
- Subdividing an edge e of G corresponds to adding a parallel edge e^* in G^*.
- Adding a facial chord in G corresponds to uncontracting an edge in G^* and vice versa.

We now state a theorem of Seymour [25] which is a generalization of Theorem 1.2.2 to matroids.

Theorem 4.3.4. *(Sums of Circuits) Given a matroid M with set of circuits C,*

$$Cone(C) \subseteq \{v \in \mathbb{R}^E_{\geq 0} : v(e) \geq 0; v(e) \leq v(B \setminus e) \textit{ for all cocircuits } B \textit{ and all } e \in B\}$$

with equality if and only if M has no minor isomorphic to one of, $U_4^2, M^*(K_5), F_7^*, R_{10}$.

The complete set of representations of the above four minors as signed graphs is given here. [22]

- U_4^2 is not binary so it cannot be represented as a signed graph.
- Up to switching equivalence the unique representation of R_{10} is $R_{10} = M(K_5, \Sigma)$ where $\Sigma = E(K_5)$.
- $M^*(K_5)$: The four switching inequivalent representations of $M^*(K_5)$ are given in Figure 4.1.
- F_7^*: The two switching inequivalent representations of F_7^* are given in Figure 4.2.

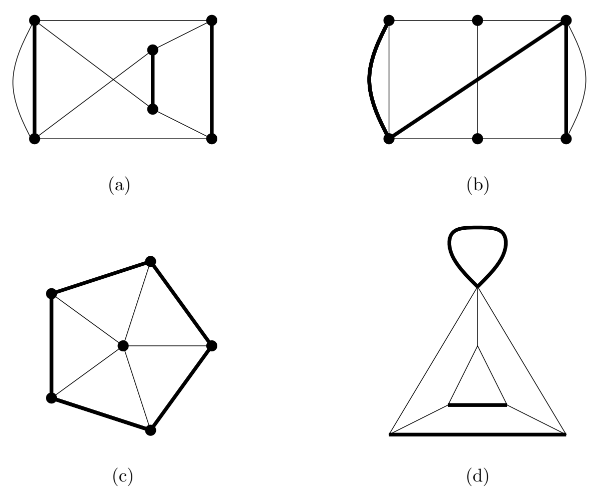
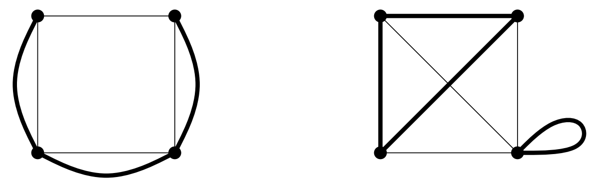


Figure 4.1: All signed graphic representations of $M^*(K_5)$ up to switching equivalence. Bold edges indicate Σ.

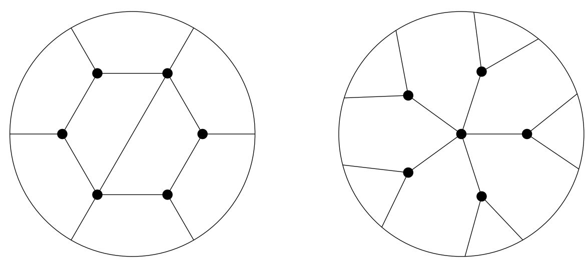
It is not hard to see that among the forbidden minors of Theorem 4.3.6, only $M^*(K_5)$ can be represented by an equitorial signing of a projective plane graph. In fact, there are exactly two such representations of $M^*(K_5)$, namely (a) and (c) of Figure 4.1, and they correspond to the two topologically distinct projective plane embeddings of K_5. The surface duals of these two embeddings are shown in Figure 4.3.

Definition 4.3.5. Let G be a projective plane graph. Let $\mathcal{D}(G)$ denote the set of (characteristic vectors of) contractible circuits and bicycles of G.



(a) (b)

Figure 4.2: All signed graphic representations of F_7^* up to switching equivalence. Bold edges indicate Σ.



(a) (b)

Figure 4.3: Duals of the two embeddings of K_5 on the projective plane

We can interpret the sums of circuits result for projective plane graphs.

Corollary 4.3.6. *Let G is a projective plane graph and let Σ be a corresponding equitorial signing. Then for every $x \in Cone(\mathcal{D}(G))$ we have*

$$\begin{aligned} x(e) &\geq 0 \\ x(e) &\leq x(B \setminus e) \end{aligned} \tag{4.1}$$

for every pair (e, B) where $e \in B$ and, for some edge cut $\delta(S)$ of G, either $B = \delta(S)$ or $B = \delta(S) \triangle \Sigma$. Furthermore (4.1) is a sufficient condition of x to be in $Cone(\mathcal{D}(G))$ if and only if G does not contain either of the two graphs given in Figure 4.3 as a surface minor.

Proof. Let $M(A)$ be the representation of $M(G, \Sigma)$ over over $GF(2)$ that is obtained from the vertex-edge incidence matrix of G by adding a row that indicates Σ. Then the binary sum of a subset rows takes the form either $\delta(S) \triangle \Sigma$ or $\delta(S)$, depending on whether or not the last row of A is used, and where $S \subseteq V(G)$ corresponds to the other rows of A appearing in the sum. Thus $\chi(B)$ in (4.1), ranges over the range space of A. The condition (4.1) holds for all such B if and only if it holds for the minimal nonempty sets B for which $\chi(B)$ is in the row space of A. These are precisely the cocircuits of $M^*(G,\Sigma)$.

Now by Theorem 4.3.4, we know that the cone of circuits of a matroid $M(G,\Sigma)$ is completely described by (4.1) if and only if $M(G,\Sigma)$ does not contain a minor isomorphic to one of, $U_4^2, M^*(K_5), F_7^*, R_{10}$. As discussed above, if either of the two graphs in Figure 4.3 do not appear as a surface minor of G, then $\mathcal{D}(G,\Sigma)$ will be completely described by (4.1). □

Theorem 1.2.1 was extended to binary matroids by Seymour [26].

Theorem 4.3.7. *(Lattice of Circuits) Given a binary matroid $M = (E, C)$, we have*

$$\begin{aligned} Lattice(C) \subseteq \{v \in \mathbb{Z}^E : & v(B) \text{ is even for all cocircuits } B, \\ & v_b = 0 \text{ if } b \text{ is a bridge }, \\ & v_e = v_f \text{ if } \{e, f\} \text{ is a cocircuit}\} \end{aligned} \tag{4.2}$$

with equality if M has no minor isomorphic to F_7^*.

A matroid is said to have the *lattice of circuits property* if equality holds in (4.2). Since F_7^* does not have a representation as a signed graph which embeds in the projective plane via a standard signing, we have the following result.

Corollary 4.3.8. *Even circuit matroids arising from equitorial signings of graphs on a projective plane have the lattice of circuits property.*

4.4 Computational results

In this section we report some of the findings of some additional computational experiments done regarding contractible circuits of projective plane graphs and its variations. We have used the Normaliz software [5], [6] for computing Hilbert bases. Refer to Appendix A for a more detailed description of the software.

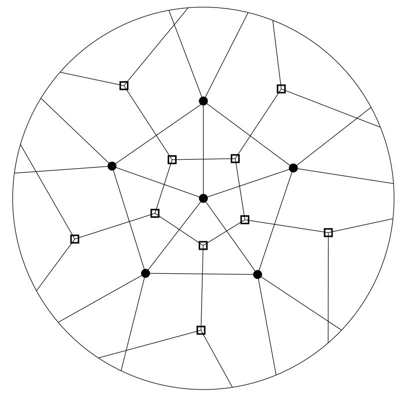


Figure 4.4: Petersen graph with K_6 as the dual graph

4.4.1 Signed Petersen graphs

Theorem 4.4.1. *(Zaslavsky [29]) Let P denote the Petersen graph. Up to switching equivalence, there are exactly six signings of P. (See Figure 4.5.)*

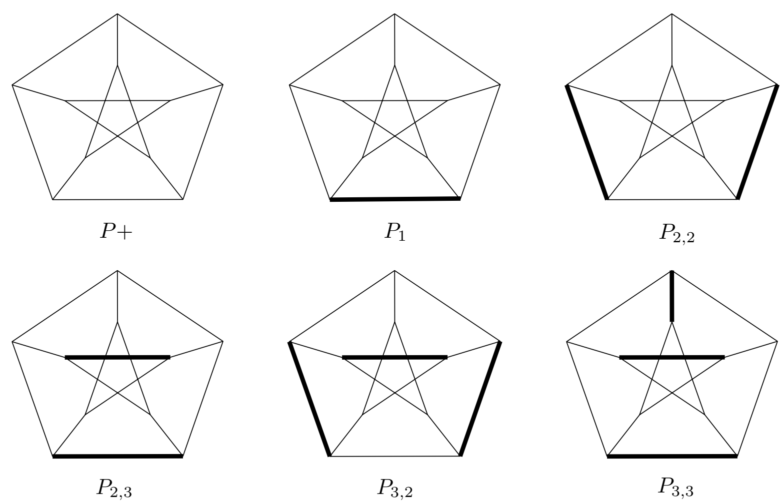


Figure 4.5: Non isomorphic signings of the Petersen Graph

Here $P+$ is the P with no odd edges. P_1 is the signing of P with one odd edge. $P_{i,j}$ is the signing of P with i odd edges at pairwise distance of j for $j = 2, 3$.

The motivation for studying the signings of Petersen graph is that P can be embedded on the projective plane with K_6 as its dual. This embedding is unique, so out of the six signings, only $P_{3,2}$ is equitorial.

Definition 4.4.2. Let (G, Σ) be a signed graph. Define $\mathcal{D}(G, \Sigma)$ to be the set of (characteristic vectors of) circuits of $M(G, \Sigma)$.

Note that this notation is compatible with Definition 4.3.5 since for any equitorial signing Σ of a projective plane graph G, we have $\mathcal{D}(G, \Sigma) = \mathcal{D}(G)$.

We state the Hilbert basis properties of $\mathcal{D}(P)$ of the six signings below.

- $\mathcal{D}(P+)$ does not form a Hilbert basis by the main result of [1].
- No even circuit of P_1 contains the unique negative edge f of P_1. Thus $\mathcal{D}(P_1)$ is the set of circuits of the unsigned graph $P \setminus f$. Since $P \setminus f$ does not contain a Petersen minor, $\mathcal{D}(P_1)$ is a Hilbert Basis by the main result of [1].
- $\mathcal{D}(P_{2,2})$ does not form a Hilbert basis.
- $\mathcal{D}(P_{2,3})$ does not form a Hilbert basis.
- $\mathcal{D}(P_{3,2})$ does not form a Hilbert basis. This is equivalent to the statement $K_6 \notin \mathscr{H}$.
- $\mathcal{D}(P_{3,3})$ forms a Hilbert basis.

4.4.2 Contractible circuits and non contractible circuits of projective plane graphs

Let G be a projective plane graph. Until now we have considered contractible circuits and bicycles of G since they correspond to cuts of G^*. So it is natural to want to examine the set $\mathcal{K}(G)$ of contractible circuits of G.

In general $\mathcal{K}(G)$ is not the set of circuits of any matroid. In Figure 4.6, the sets $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are contractible circuits. But it is not possible to find a circuits using, say e_5 and avoiding e_3 contradicting the circuit exchange axiom for matroids.[1]

We ran some computer tests regarding $Cone(\mathcal{K}(G))$. We have $\mathcal{K}(G) \subseteq \mathcal{D}(G)$. However even for relatively simple graphs the cone $Cone(\mathcal{K}(G))$ generally has more complicated facet structure than $Cone(\mathcal{D}(G))$. Consider for example K_6. The set $\mathcal{D}(K_6)$ has 191 elements

[1]The circuit exchange axiom states that if C_1, C_2 are circuits of a matroid and $C_1 \neq C_2$, and $e \in C_1 \cap C_2$ there exists a circuit C_3 of the matroid such that $C_3 \subset (C_1 \cup C_2) \setminus e$

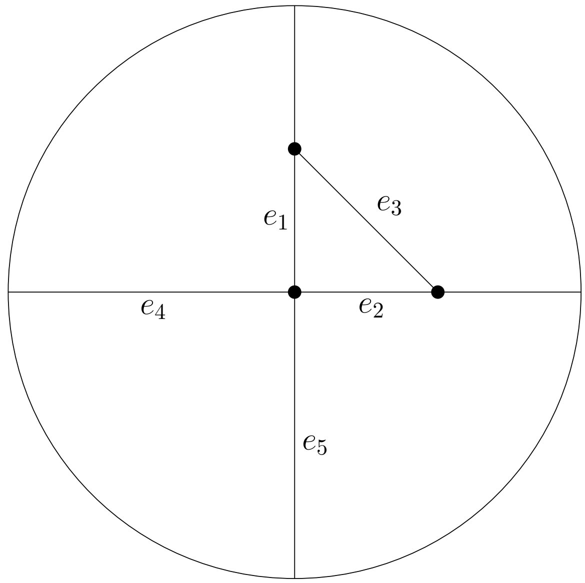


Figure 4.6: Contractible circuits

and $Cone(\mathcal{D}(K_6))$ has 225 support hyperplanes. In contrast $\mathcal{K}(K_6)$ has 96 elements and $Cone(\mathcal{K}(K_6))$ has as many as 920 support hyperplanes.

Let $\mathcal{L}(G)$ denote the set of non-contractible circuits of a projective plane graph G. By running further experiments, we found that the facet structure of $Cone(\mathcal{L}(G))$ appears to be more complicated than those of $Cone(\mathcal{D}(G))$ and $Cone(\mathcal{K}(G))$. Again considering K_6, we found that although $\mathcal{L}(K_6)$ has only 61 elements, $Cone(\mathcal{L}(K_6))$ has 6674 support hyperplanes.

Chapter 5

Conclusions

In this thesis, we studied the cones, lattices and Hilbert bases of characteristic vectors of cuts of graphs. This continued the research previously done by Laurent [17], Lauberthe et al. [16] in the same area. Our results can be summarized as follows.

We studied the class $\mathscr{H}$ of graphs whose cuts form Hilbert bases. We derived a sufficient condition for when a 2-sum of graphs in $\mathscr{H}$ also belongs to $\mathscr{H}$ namely the lattice endpoint property. As a corollary we proved that all $K_{3,3}$-free graphs belong to $\mathscr{H}$. We also applied a similar argument to conclude that all graphs contractible to K_5 belong to $\mathscr{H}$.

We also repaired a result of Laurent [17] that said all proper subgraphs of $K_6 \in \mathscr{H}$ by proving that the graph $K_6 \setminus e \notin \mathscr{H}$.

Finally we applied these results to the even circuit matroids of signed graphs on a projective plane and performed some computational experiments.

One of the more interesting results of our experiments was that the only potential excluded minor we found for the class $\mathscr{H}$ is in fact the graph $K_6 \setminus e$. A possible direction for future research is to investigate the matter to check if there exist any graphs without a $K_6 \setminus e$ minor that do not belong to $\mathscr{H}$.

Conjecture 1. *Let G be a graph such that G does not contain a ($K_6 \setminus e$)-minor. Then* $G \in \mathscr{H}$.

Another possible direction for research is to extend the main theorem of 2-sums of graphs and Hilbert bases to 3-sums. It seems likely that a mild sufficient condition exists for a 3-sum of graphs in $\mathscr{H}$ to also be in $\mathscr{H}$.

We believe that all 2-sums of copies of uncontractions of K_5 with planar graphs belong to the class $\mathscr{H}$. It is even possible that the 2-sum or 3-sum of any two graphs in $\mathscr{H}$ is also in $\mathscr{H}$.

Conjecture 2. *Let G be a graph formed by 2-sums of copies of uncontractions of K_5 and planar graphs. Then $G \in \mathscr{H}$.*

Appendix A

Computational Results

In this appendix we present miscellaneous computational results about cones of cuts of graphs and even circuits of signed graphs. For computations we have used the Normaliz software [5] [6]. It allows us to compute the Hilbert basis, extreme rays and support hyperplane of the cone generated by a set of vectors. The ambient lattice for the Hilbert basis can be chosen to be $\mathbb{Z}^n$ or the lattice generated by the vectors themselves. We first present a sample output of the program.

```
 //sample_1.in :
 //In this case we calculate the Hilbert basis
 //with respect to the lattice Z^n.

2 //number of rows
2 //number of columns
2 0 //list of vectors
0 2
0

// '0' in the last line represents the input needed for lattice Z^n
// while '1' represents the lattice generated by the vectors.

// sample_1.out : Output of the above file
```

```
2 Hilbert basis elements
2 Hilbert basis elements of degree 1
2 extreme rays
2 support hyperplanes

rank = 2 (maximal)
index = 4
original monoid is integrally closed

size of triangulation   = 0
resulting sum of |det|s = 0

grading:
1 1

degrees of extreme rays:
1: 2

Hilbert basis elements are of degree 1

*****************************************************************

//Vectors corresponding to the Hilbert basis of the input vectors
2 Hilbert basis elements:
 0 1
 1 0

//Extreme rays of the cone generated
2 extreme rays:
 1 0
 0 1

// Support hyperplanes interpreted as :  *.x1 + *.x2 >= 0
2 support hyperplanes:
```

```
 1 0
 0 1

2 Hilbert basis elements of degree 1:
 0 1
 1 0
```

We now present the results of several computational experiments that we performed. We first look at the cone and Hilbert basis for 3-connected uncontractions of K_5. Our algorithm was roughly as follows.

1. First list graphs by uncontracting the edges from K_5.
2. Use "Nauty" to separate the non isomorphic graphs. [19]
3. At this point we used C++ to generate all cuts of the graphs.
4. Insert data into Normaliz to generate cones and Hilbert basis of the vectors

We present the facets of the cut cone of the Petersen graph. To save space, we only present the support hyperplanes of the cut cone. Other 22 graphs followed a similar pattern.

```
Cuts of Pete,
which are the contractible circuits of K_6 on the projective plane

 1  0  0  0  0  0  0  0  0  0  0  0  0  0  0
 0  1  0  0  0  0  0  0  0  0  0  0  0  0  0
 0  0  1  0  0  0  0  0  0  0  0  0  0  0  0
 0  0  0  1  0  0  0  0  0  0  0  0  0  0  0
 0  0  0  0  1  0  0  0  0  0  0  0  0  0  0
 0  0  0  0  0  1  0  0  0  0  0  0  0  0  0
 0  0  0  0  0  0  1  0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  1  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0  1  0  0  0  0  0  0
 0  0  0  0  0  0  0  0  0  1  0  0  0  0  0
 0  0  0  0  0  0  0  0  0  0  1  0  0  0  0
```

```
  0  0  0  0  0  0  0  0  0  0  0  1  0  0  0
  0  0  0  0  0  0  0  0  0  0  0  0  1  0  0
  0  0  0  0  0  0  0  0  0  0  0  0  0  1  0
  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1
***************
 15 constraints*   x(e) \ge 0
***************

 -1  1  1  1  1  0  0  0  0  0  0  0  0  0  0
  1 -1  1  1  1  0  0  0  0  0  0  0  0  0  0
  1  1 -1  1  1  0  0  0  0  0  0  0  0  0  0
  1  1  1 -1  1  0  0  0  0  0  0  0  0  0  0
  1  1  1  1 -1  0  0  0  0  0  0  0  0  0  0

 -1  1  0  0  0  0  0  0  1  1  0  0  0  1  0
  1 -1  0  0  0  0  0  0  1  1  0  0  0  1  0
  1  1  0  0  0  0  0  0 -1  1  0  0  0  1  0
  1  1  0  0  0  0  0  0  1 -1  0  0  0  1  0
  1  1  0  0  0  0  0  0  1  1  0  0  0 -1  0

 -1  0  0  0  1  1  0  0  0  0  1  0  0  0  1
  1  0  0  0 -1  1  0  0  0  0  1  0  0  0  1
  1  0  0  0  1 -1  0  0  0  0  1  0  0  0  1
  1  0  0  0  1  1  0  0  0  0 -1  0  0  0  1
  1  0  0  0  1  1  0  0  0  0  1  0  0  0 -1

 -1  0  0  0  0  1  1  1  1  0  0  0  0  0  0
  1  0  0  0  0 -1  1  1  1  0  0  0  0  0  0
  1  0  0  0  0  1 -1  1  1  0  0  0  0  0  0
  1  0  0  0  0  1  1 -1  1  0  0  0  0  0  0
  1  0  0  0  0  1  1  1 -1  0  0  0  0  0  0

  0 -1  1  0  0  1  0  1  0  0  0  0  1  0  0
  0  1 -1  0  0  1  0  1  0  0  0  0  1  0  0
```

```
 0  1  1  0  0 -1  0  1  0  0  0  0  1  0  0
 0  1  1  0  0  1  0 -1  0  0  0  0  1  0  0
 0  1  1  0  0  1  0  1  0  0  0  0 -1  0  0

 0 -1  0  0  0  1  0  0  0  1  1  1  0  0  0
 0  1  0  0  0 -1  0  0  0  1  1  1  0  0  0
 0  1  0  0  0  1  0  0  0 -1  1  1  0  0  0
 0  1  0  0  0  1  0  0  0  1 -1  1  0  0  0
 0  1  0  0  0  1  0  0  0  1  1 -1  0  0  0

 0  0 -1  1  0  0  0  0  0  1  0  1  0  0  1
 0  0  1 -1  0  0  0  0  0  1  0  1  0  0  1
 0  0  1  1  0  0  0  0  0 -1  0  1  0  0  1
 0  0  1  1  0  0  0  0  0  1  0 -1  0  0  1
 0  0  1  1  0  0  0  0  0  1  0  1  0  0 -1

 0  0 -1  0  0  0  1  0  0  1  0  0  1  1  0
 0  0  1  0  0  0 -1  0  0  1  0  0  1  1  0
 0  0  1  0  0  0  1  0  0 -1  0  0  1  1  0
 0  0  1  0  0  0  1  0  0  1  0  0 -1  1  0
 0  0  1  0  0  0  1  0  0  1  0  0  1 -1  0

 0  0  0 -1  1  0  1  0  1  0  0  0  1  0  0
 0  0  0  1 -1  0  1  0  1  0  0  0  1  0  0
 0  0  0  1  1  0 -1  0  1  0  0  0  1  0  0
 0  0  0  1  1  0  1  0 -1  0  0  0  1  0  0
 0  0  0  1  1  0  1  0  1  0  0  0 -1  0  0

 0  0  0 -1  0  0  0  1  0  0  1  0  1  0  1
 0  0  0  1  0  0  0 -1  0  0  1  0  1  0  1
 0  0  0  1  0  0  0  1  0  0 -1  0  1  0  1
 0  0  0  1  0  0  0  1  0  0  1  0 -1  0  1
 0  0  0  1  0  0  0  1  0  0  1  0  1  0 -1
```

```
  0  0  0  0 -1  0  0  0  1  0  0  1  0  1  1
  0  0  0  0  1  0  0  0 -1  0  0  1  0  1  1
  0  0  0  0  1  0  0  0  1  0  0 -1  0  1  1
  0  0  0  0  1  0  0  0  1  0  0  1  0 -1  1
  0  0  0  0  1  0  0  0  1  0  0  1  0  1 -1

  0  0  0  0  0  0 -1  1  0  0  1  1  0  1  0
  0  0  0  0  0  0  1 -1  0  0  1  1  0  1  0
  0  0  0  0  0  0  1  1  0  0 -1  1  0  1  0
  0  0  0  0  0  0  1  1  0  0  1 -1  0  1  0
  0  0  0  0  0  0  1  1  0  0  1  1  0 -1  0
***************
60 constraints*   x(e) \le x(C-e) : C is a pentagon and e is in C
***************

 -1  1  0  0  1  0  0  0  0  1  0  1  0  0  1
  1 -1  0  0  1  0  0  0  0  1  0  1  0  0  1
  1  1  0  0 -1  0  0  0  0  1  0  1  0  0  1
  1  1  0  0  1  0  0  0  0 -1  0  1  0  0  1
  1  1  0  0  1  0  0  0  0  1  0 -1  0  0  1
  1  1  0  0  1  0  0  0  0  1  0  1  0  0 -1

 -1  1  1  0  0  0  1  0  1  0  0  0  1  0  0
  1 -1  1  0  0  0  1  0  1  0  0  0  1  0  0
  1  1 -1  0  0  0  1  0  1  0  0  0  1  0  0
  1  1  1  0  0  0 -1  0  1  0  0  0  1  0  0
  1  1  1  0  0  0  1  0 -1  0  0  0  1  0  0
  1  1  1  0  0  0  1  0  1  0  0  0 -1  0  0

 -1  0  0  1  1  1  0  1  0  0  0  0  1  0  0
  1  0  0 -1  1  1  0  1  0  0  0  0  1  0  0
  1  0  0  1 -1  1  0  1  0  0  0  0  1  0  0
  1  0  0  1  1 -1  0  1  0  0  0  0  1  0  0
  1  0  0  1  1  1  0 -1  0  0  0  0  1  0  0
```

```
 1  0  0  1  1  1  0  1  0  0  0  0 -1  0  0

-1  0  0  0  0  1  0  0  1  0  1  1  0  1  0
 1  0  0  0  0 -1  0  0  1  0  1  1  0  1  0
 1  0  0  0  0  1  0  0 -1  0  1  1  0  1  0
 1  0  0  0  0  1  0  0  1  0 -1  1  0  1  0
 1  0  0  0  0  1  0  0  1  0  1 -1  0  1  0
 1  0  0  0  0  1  0  0  1  0  1  1  0 -1  0

 0 -1  1  1  0  1  0  0  0  0  1  0  0  0  1
 0  1 -1  1  0  1  0  0  0  0  1  0  0  0  1
 0  1  1 -1  0  1  0  0  0  0  1  0  0  0  1
 0  1  1  1  0 -1  0  0  0  0  1  0  0  0  1
 0  1  1  1  0  1  0  0  0  0 -1  0  0  0  1
 0  1  1  1  0  1  0  0  0  0  1  0  0  0 -1

 0 -1  0  0  0  1  1  1  0  1  0  0  0  1  0
 0  1  0  0  0 -1  1  1  0  1  0  0  0  1  0
 0  1  0  0  0  1 -1  1  0  1  0  0  0  1  0
 0  1  0  0  0  1  1 -1  0  1  0  0  0  1  0
 0  1  0  0  0  1  1  1  0 -1  0  0  0  1  0
 0  1  0  0  0  1  1  1  0  1  0  0  0 -1  0

 0  0 -1  1  1  0  0  0  1  1  0  0  0  1  0
 0  0  1 -1  1  0  0  0  1  1  0  0  0  1  0
 0  0  1  1 -1  0  0  0  1  1  0  0  0  1  0
 0  0  1  1  1  0  0  0 -1  1  0  0  0  1  0
 0  0  1  1  1  0  0  0  1 -1  0  0  0  1  0
 0  0  1  1  1  0  0  0  1  1  0  0  0 -1  0

 0  0 -1  0  0  0  0  1  0  1  1  1  1  0  0
 0  0  1  0  0  0  0 -1  0  1  1  1  1  0  0
 0  0  1  0  0  0  0  1  0 -1  1  1  1  0  0
 0  0  1  0  0  0  0  1  0  1 -1  1  1  0  0
```

```
 0  0  1  0  0  0  0  1  0  1  1 -1  1  0  0
 0  0  1  0  0  0  0  1  0  1  1  1 -1  0  0

 0  0  0 -1  0  0  1  0  0  0  0  1  1  1  1
 0  0  0  1  0  0 -1  0  0  0  0  1  1  1  1
 0  0  0  1  0  0  1  0  0  0  0 -1  1  1  1
 0  0  0  1  0  0  1  0  0  0  0  1 -1  1  1
 0  0  0  1  0  0  1  0  0  0  0  1  1 -1  1
 0  0  0  1  0  0  1  0  0  0  0  1  1  1 -1

 0  0  0  0 -1  0  1  1  1  0  1  0  0  0  1
 0  0  0  0  1  0 -1  1  1  0  1  0  0  0  1
 0  0  0  0  1  0  1 -1  1  0  1  0  0  0  1
 0  0  0  0  1  0  1  1 -1  0  1  0  0  0  1
 0  0  0  0  1  0  1  1  1  0 -1  0  0  0  1
 0  0  0  0  1  0  1  1  1  0  1  0  0  0 -1
***************
60 constraints* x(e) \le x(C-e) : C is a hexagon and e is in C
***************

-2  1  2  1  1  1  1  2  1  1  1 -1 -1  2  2
 2  1 -2  1  1  1  1  2 -1  1 -1  1  1  2  2
 2  1  2  1 -1  1  1 -2  1 -1  1  1  1  2  2
 2  1  2 -1  1 -1  1  2  1  1  1  1  1 -2  2
 2 -1  2  1  1  1 -1  2  1  1  1  1  1  2 -2

 2 -1  1  2  1  1 -2  1  1  2  2  1  1  1 -1
 2  1 -1  2  1  1  2  1 -1  2 -2  1  1  1  1
 2  1  1 -2  1 -1  2  1  1  2  2  1  1 -1  1
-2  1  1  2  1  1  2  1  1  2  2 -1 -1  1  1
 2  1  1  2 -1  1  2 -1  1 -2  2  1  1  1  1

 1 -2  1  2  1  1 -1  2  2  1  1  2  1  1 -1
 1  2  1 -2  1 -1  1  2  2  1  1  2  1 -1  1
```

```
 1  2  1  2 -1  1  1 -2  2 -1  1  2  1  1  1
 1  2 -1  2  1  1  1  2 -2  1 -1  2  1  1  1
-1  2  1  2  1  1  1  2  2  1  1 -2 -1  1  1

-1  2  1  1  2  1  1  1  1  1  2 -1 -2  2  1
 1  2  1 -1  2 -1  1  1  1  1  2  1  2 -2  1
 1 -2  1  1  2  1 -1  1  1  1  2  1  2  2 -1
 1  2  1  1 -2  1  1 -1  1 -1  2  1  2  2  1
 1  2 -1  1  2  1  1  1 -1  1 -2  1  2  2  1

 1  1  2  1 -2  2  2 -1  1 -1  1  2  1  1  1
 1  1  2 -1  2 -2  2  1  1  1  1  2  1 -1  1
 1 -1  2  1  2  2 -2  1  1  1  1  2  1  1 -1
-1  1  2  1  2  2  2  1  1  1  1 -2 -1  1  1
 1  1 -2  1  2  2  2  1 -1  1 -1  2  1  1  1

 1  1 -1  1  1  2  1  1 -2  2 -1  1  2  1  2
-1  1  1  1  1  2  1  1  2  2  1 -1 -2  1  2
 1  1  1  1 -1  2  1 -1  2 -2  1  1  2  1  2
 1  1  1 -1  1 -2  1  1  2  2  1  1  2 -1  2
 1 -1  1  1  1  2 -1  1  2  2  1  1  2  1 -2
***************
30 constraints* corresponding to equation (2.3)
***************

 2  1  2 -1  1 -1 -1  2 -1  1  1  1  1  2  2
 2  1  2  1 -1  1 -1  2  1 -1  1  1 -1  2  2
 2 -1  2  1  1  1  1  2 -1 -1 -1  1  1  2  2
 2 -1  2 -1 -1  1 -1  2  1  1  1  1  1  2  2
 2  1  2 -1  1  1  1  2 -1  1 -1  1 -1  2  2
 2  1  2 -1  1 -1  1  2  1 -1  1 -1  1  2  2
 2 -1  2  1  1  1 -1  2  1  1 -1 -1  1  2  2
 2 -1  2  1  1 -1  1  2  1  1  1 -1 -1  2  2
 2  1  2  1 -1 -1  1  2  1 -1 -1  1  1  2  2
```

2	1	2	1	-1	1	1	2	-1	1	1	-1	-1	2	2

2	1	-1	2	1	-1	2	-1	-1	2	2	1	1	1	1
2	1	-1	2	1	1	2	1	-1	2	2	-1	1	1	-1
2	1	-1	2	1	-1	2	1	1	2	2	1	-1	-1	1
2	-1	1	2	1	1	2	-1	1	2	2	1	-1	1	-1
2	-1	1	2	1	1	2	1	-1	2	2	1	1	-1	-1
2	1	1	2	-1	-1	2	1	1	2	2	1	1	-1	-1
2	1	1	2	-1	1	2	1	-1	2	2	-1	-1	1	1
2	1	1	2	-1	1	2	-1	1	2	2	-1	1	-1	1
2	-1	-1	2	-1	1	2	-1	1	2	2	1	1	1	1
2	-1	1	2	1	-1	2	1	1	2	2	-1	-1	1	1

1	2	-1	2	1	-1	1	2	2	1	1	2	-1	-1	1
1	2	1	2	-1	1	-1	2	2	-1	1	2	-1	1	1
-1	2	1	2	1	1	1	2	2	1	-1	2	-1	1	-1
-1	2	-1	2	-1	1	1	2	2	1	-1	2	1	1	1
-1	2	1	2	1	-1	-1	2	2	1	1	2	1	1	-1
-1	2	1	2	1	1	1	2	2	-1	1	2	-1	-1	1
1	2	1	2	-1	-1	1	2	2	-1	-1	2	1	1	1
1	2	1	2	-1	-1	1	2	2	1	1	2	1	-1	-1
1	2	-1	2	1	1	-1	2	2	-1	1	2	1	1	-1
1	2	-1	2	1	1	-1	2	2	1	-1	2	1	-1	1

1	2	-1	1	2	1	1	1	-1	1	2	-1	2	2	-1
1	2	-1	1	2	1	-1	1	1	-1	2	1	2	2	-1
1	2	1	-1	2	-1	1	1	1	-1	2	-1	2	2	1
1	2	1	-1	2	1	1	-1	1	-1	2	1	2	2	-1
1	2	1	-1	2	-1	-1	1	-1	1	2	1	2	2	1
-1	2	1	1	2	1	-1	-1	1	1	2	-1	2	2	1
-1	2	-1	-1	2	1	1	1	1	1	2	-1	2	2	1
-1	2	1	1	2	1	1	-1	-1	-1	2	1	2	2	1
-1	2	1	1	2	-1	-1	1	1	1	2	1	2	2	-1
1	2	-1	1	2	-1	1	-1	-1	1	2	1	2	2	1

```
 1 -1  2  1  2  2  2  1 -1 -1 -1  2  1  1  1
 1 -1  2  1  2  2  2 -1  1  1  1  2 -1  1 -1
 1  1  2 -1  2  2  2  1 -1  1 -1  2 -1  1  1
 1 -1  2  1  2  2  2  1 -1  1  1  2  1 -1 -1
 1  1  2 -1  2  2  2 -1  1  1 -1  2  1 -1  1
-1  1  2  1  2  2  2  1  1  1 -1  2 -1  1 -1
-1 -1  2 -1  2  2  2  1  1  1  1  2  1 -1  1
-1  1  2  1  2  2  2  1  1 -1  1  2 -1 -1  1
-1  1  2  1  2  2  2 -1 -1 -1  1  2  1  1  1
 1  1  2 -1  2  2  2 -1  1 -1  1  2  1  1 -1

-1  1  1  1  1  2 -1 -1  2  2  1 -1  2  1  2
 1 -1  1  1  1  2 -1  1  2  2 -1 -1  2  1  2
 1  1 -1  1  1  2 -1  1  2  2 -1  1  2 -1  2
 1  1  1 -1  1  2  1 -1  2  2 -1  1  2 -1  2
 1  1  1  1 -1  2  1 -1  2  2  1 -1  2 -1  2
 1 -1  1 -1 -1  2 -1  1  2  2  1  1  2  1  2
 1 -1 -1  1 -1  2  1 -1  2  2  1  1  2  1  2
-1  1 -1 -1  1  2  1  1  2  2  1 -1  2  1  2
-1  1 -1  1 -1  2  1  1  2  2 -1  1  2  1  2
-1 -1  1 -1  1  2  1  1  2  2  1  1  2 -1  2
***************
60 constraints* corresponding to equation (2.4)
***************

Total =  225
```

We now look at the Hilbert basis of cuts of $K_6 \setminus e$ and present the complete output of Normaliz.

```
38 Hilbert basis elements
30 extreme rays
124 support hyperplanes
```

```
rank = 14 (maximal)
index = 1
original monoid is not integrally closed

size of triangulation   = 13107
resulting sum of |det|s = 13868

No implicit grading found

*******************************************************

38 Hilbert basis elements:
 1 1 1 1 1 0 0 0 0 0 0 0 0 0
 0 1 0 0 0 1 1 0 0 0 0 0 1 1
 0 0 1 0 0 0 1 1 0 1 1 0 0 0
 0 0 0 1 0 0 0 1 1 0 0 1 1 0
 1 0 0 0 0 1 0 0 0 0 1 1 0 0
 0 0 0 0 1 0 0 0 1 1 0 0 0 1
 1 1 1 1 0 0 0 0 1 1 0 0 0 1
 1 1 0 0 0 0 1 0 0 0 1 1 1 1
 1 0 1 0 0 1 1 1 0 1 0 1 0 0
 1 0 0 1 0 1 0 1 1 0 1 0 1 0
 0 1 1 1 1 1 0 0 0 0 1 1 0 0
 0 1 0 0 1 1 1 0 1 1 0 0 1 0
 0 0 1 0 1 0 1 1 1 0 1 0 0 1
 0 0 0 1 1 0 0 1 0 1 0 1 1 1
 1 1 1 0 1 0 0 1 1 0 0 1 1 0
 1 1 1 0 0 0 0 1 0 1 0 1 1 1
 1 1 0 1 1 0 1 1 0 1 1 0 0 0
 1 1 0 1 0 0 1 1 1 0 1 0 0 1
 1 0 1 1 1 1 1 0 0 0 0 0 1 1
 1 0 1 1 0 1 1 0 1 1 0 0 1 0
 0 1 1 0 1 1 0 1 1 0 1 0 1 0
 0 1 1 0 0 1 0 1 0 1 1 0 1 1
```

```
 0 1 0 1 1 1 1 1 0 1 0 1 0 0
 0 1 0 1 0 1 1 1 1 0 0 1 0 1
 0 0 1 1 1 0 1 0 0 0 1 1 1 1
 0 0 1 1 0 0 1 0 1 1 1 1 1 0
 1 1 0 0 1 0 1 0 1 1 1 1 1 0
 1 0 1 0 1 1 1 1 1 0 0 1 0 1
 1 0 0 1 1 1 0 1 0 1 1 0 1 1
 0 1 1 1 0 1 0 0 1 1 1 1 0 1
 1 2 2 2 2 1 2 2 2 2 1 3 2 2
 3 2 2 2 2 1 2 2 2 2 1 1 2 2
 2 2 2 2 3 2 2 2 1 1 2 2 2 1
 2 2 2 2 1 2 2 2 3 1 2 2 2 1
 2 2 2 2 1 2 2 2 1 3 2 2 2 1
 2 2 2 2 1 2 2 2 1 1 2 2 2 3
 1 2 2 2 2 1 2 2 2 2 3 1 2 2
 1 2 2 2 2 3 2 2 2 2 1 1 2 2

30 extreme rays:
 0 0 0 0 1 0 0 0 1 1 0 0 0 1
 0 0 0 1 0 0 0 1 1 0 0 1 1 0
 0 0 0 1 1 0 0 1 0 1 0 1 1 1
 0 0 1 0 0 0 1 1 0 1 1 0 0 0
 0 0 1 0 1 0 1 1 1 0 1 0 0 1
 0 0 1 1 0 0 1 0 1 1 1 1 1 0
 0 0 1 1 1 0 1 0 0 0 1 1 1 1
 0 1 0 0 0 1 1 0 0 0 0 0 1 1
 0 1 0 0 1 1 1 0 1 1 0 0 1 0
 0 1 0 1 0 1 1 1 1 0 0 1 0 1
 0 1 0 1 1 1 1 1 0 1 0 1 0 0
 0 1 1 0 0 1 0 1 0 1 1 0 1 1
 0 1 1 0 1 1 0 1 1 0 1 0 1 0
 0 1 1 1 0 1 0 0 1 1 1 1 0 1
 0 1 1 1 1 1 0 0 0 0 1 1 0 0
 1 0 0 0 0 1 0 0 0 0 1 1 0 0
```

```
 1 0 0 1 0 1 0 1 1 0 1 0 1 0
 1 0 0 1 1 1 0 1 0 1 1 0 1 1
 1 0 1 0 0 1 1 1 0 1 0 1 0 0
 1 0 1 0 1 1 1 1 1 0 0 1 0 1
 1 0 1 1 0 1 1 0 1 1 0 0 1 0
 1 0 1 1 1 1 1 0 0 0 0 0 1 1
 1 1 0 0 0 0 1 0 0 0 1 1 1 1
 1 1 0 0 1 0 1 0 1 1 1 1 1 0
 1 1 0 1 0 0 1 1 1 0 1 0 0 1
 1 1 0 1 1 0 1 1 0 1 1 0 0 0
 1 1 1 0 0 0 0 1 0 1 0 1 1 1
 1 1 1 0 1 0 0 1 1 0 0 1 1 0
 1 1 1 1 0 0 0 0 1 1 0 0 0 1
 1 1 1 1 1 0 0 0 0 0 0 0 0 0
```

```
124 support hyperplanes:
  0  0  0  1  1  0  0  0 -1  0  0  0  0  0
  0  0  1  0  1  0  0  0  0 -1  0  0  0  0
  0  0  1  1  0  0  0 -1  0  0  0  0  0  0
  0  1  0  0  1  0  0  0  0  0  0  0  0 -1
  0  1  0  1  0  0  0  0  0  0  0  0 -1  0
  0  1  1  0  0  0 -1  0  0  0  0  0  0  0
  1  0  0  1  0  0  0  0  0  0  0 -1  0  0
  1  0  1  0  0  0  0  0  0  0 -1  0  0  0
  1  1  0  0  0 -1  0  0  0  0  0  0  0  0
  0  0  0  0  0  0  0 -1  1  1  0  0  0  0
  0  0  0 -1  1  0  0  0  1  0  0  0  0  0
  0  0  0  0  0  0  0  0  1  0  0  0 -1  1
  0 -1  0  0  1  0  0  0  0  0  0  0  0  1
  0  0 -1  0  1  0  0  0  0  1  0  0  0  0
  0  1  1  1 -1  0 -1 -1  1  1  0  0 -1  1
  0  0  0  0  0  0 -1  0  0  1  0  0  0  1
  0  1  1 -1  1  0 -1  1  1 -1  0  0  1 -1
  0  0  0  0  0  0  0  0  1  0  0  0  1 -1
```

```
 0  0  0  1 -1  0  0  0  1  0  0  0  0  0
 0  0  0  0  0  0  0  1  1 -1  0  0  0  0
 0  1 -1  1  1  0  1  1 -1  1  0  0 -1 -1
 0 -1  1  1 -1  0  1 -1  1  1  0  0  1 -1
 0  0  0  0  0  0  1  0  0  1  0  0  0 -1
 0  0  0  0  0  0  0  1 -1  1  0  0  0  0
 0  0  1  0 -1  0  0  0  0  1  0  0  0  0
 0  0 -1  1  0  0  0  1  0  0  0  0  0  0
 0  0  0  0  0  0 -1  1  0  0  0  0  1  0
 0  0  0  0  0  0  0  1  0  0  1 -1  0  0
 0 -1 -1  1  1  0 -1  1 -1  1  0  0  1  1
 0  0  0  0  0  0  0  1  0  0 -1  1  0  0
 0  0  0  0  0  0  1  1  0  0  0  0 -1  0
 0  0  1 -1  0  0  0  1  0  0  0  0  0  0
 0 -1  1 -1  1  0  1  1  1 -1  0  0 -1  1
 0 -1  1  1  1  0  1 -1 -1 -1  0  0  1  1
 0  0  0  0  0  0  1  0  0 -1  0  0  0  1
 0  0  0  0  0  0  0  0 -1  0  0  0  1  1
 0  1  0  0 -1  0  0  0  0  0  0  0  0  1
 0  1 -1  1 -1  0  1  1  1 -1  0  0 -1  1
 0  1  1 -1 -1  0 -1  1 -1  1  0  0  1  1
-1  0  0  1  0  0  0  0  0  0  0  1  0  0
 0  0  0  0  0  1  0  0  0  0  0  1 -1  0
 0 -1  0  1  0  0  0  0  0  0  0  0  1  0
 0  0  0  0  0  1  0  0  0  0  0 -1  1  0
 1  1  0 -1  1 -1  0  0  1  1 -1  1  1 -1
 0  0  0  0  0 -1  0  0  0  0  0  1  1  0
 0  0  0  0  0  0  1 -1  0  0  0  0  1  0
 0  1  0 -1  0  0  0  0  0  0  0  0  1  0
 0  1 -1 -1  1  0  1 -1  1  1  0  0  1 -1
 0  0  0  0  0  0  0 -1  0  0  1  1  0  0
 1  0  0 -1  0  0  0  0  0  0  0  1  0  0
 1  1  1 -1  0 -1 -1  1  0  0 -1  1  1  0
 1  0  0  0  1 -1 -1  1 -1  1 -1  1  1  1
```

```
 1  0  1 -1  1 -1  0  1  1 -1 -1  1  0  1
 1  1  1  0 -1 -1 -1  0  1  1 -1  1  0  1
-1  0  1  0  0  0  0  0  0  0  1  0  0  0
-1  1  0  0  0  1  0  0  0  0  0  0  0  0
 0  0  0  0  0  1 -1  0  0  0  1  0  0  0
-1  1  1  1  0  1 -1 -1  0  0  1  1 -1  0
 1 -1  1  0  1  1  1  0  1 -1 -1 -1  0  1
 0 -1  1  0  0  0  1  0  0  0  0  0  0  0
 0  0  0  0  0  1  1  0  0  0 -1  0  0  0
 1 -1  0  0  0  1  0  0  0  0  0  0  0  0
 1 -1  1  1  0  1  1 -1  0  0 -1 -1  1  0
 1  0  0  0  1  1  1 -1  1  1 -1 -1  1 -1
 1  1 -1  0  1 -1  1  0  1  1  1 -1  0 -1
 0  0  0  0  0 -1  1  0  0  0  1  0  0  0
 0  1 -1  0  0  0  1  0  0  0  0  0  0  0
 1  0 -1  0  0  0  0  0  0  0  1  0  0  0
 1  1 -1  1  0 -1  1  1  0  0  1 -1 -1  0
 1  0  0  0  1 -1  1  1  1 -1  1 -1 -1  1
 1 -1  0  1  1  1  0  0 -1  1 -1 -1  1  1
 1  0  1  1 -1  1  0 -1  1  1 -1 -1  0  1
 1  0 -1  1  1 -1  0  1 -1  1  1 -1  0  1
 1  1  0  1 -1 -1  0  0  1  1  1 -1 -1  1
-1 -1  1  1  0 -1  1 -1  0  0  1  1  1  0
-1  1 -1  1  0  1  1  1  0  0 -1  1 -1  0
-1  1  1 -1  0  1 -1  1  0  0  1 -1  1  0
 1  0  0  0  1  1 -1  1  1 -1  1 -1  1 -1
 1 -1 -1  1  0  1 -1  1  0  0  1 -1  1  0
 1 -1  0  1  1  1  0  0 -1 -1  1 -1  1  1
 1  0 -1  1  1  1  0  1 -1  1  1 -1  0 -1
 1  0  0  0  1  1  1  1 -1  1 -1  1 -1 -1
 1 -1  1 -1  0  1  1  1  0  0 -1  1 -1  0
 1 -1  1  0  1  1  1  0 -1 -1 -1  1  0  1
 1  0  1 -1  1  1  0  1  1 -1 -1  1  0 -1
 1  0  0  0  1 -1  1 -1 -1 -1  1  1  1  1
```

```
 1  1  0 -1  1 -1  0  0  1 -1  1  1  1 -1
 1  1 -1  0  1 -1  1  0 -1  1  1  1  0 -1
 1  1 -1 -1  0 -1  1 -1  0  0  1  1  1  0
-1  1  1  0  1  1 -1  0  1 -1  1  1  0 -1
-1  1  0  1  1  1  0  0 -1  1  1  1 -1 -1
-1  0  1  1  1  1  0 -1 -1 -1  1  1  0  1
-1  0  0  0  1  1 -1  1 -1  1  1 -1  1  1
-1  0 -1  1  1  1  0  1 -1  1 -1  1  0  1
-1 -1  0  1  1 -1  0  0 -1  1  1  1  1  1
 1 -1 -1  0  1  1 -1  0 -1  1  1  1  0  1
-1  0  1 -1  1  1  0  1  1 -1  1 -1  0  1
-1  0  0  0  1  1  1  1  1 -1 -1  1 -1  1
-1 -1  1  0  1 -1  1  0  1 -1  1  1  0  1
 1 -1  0 -1  1  1  0  0  1 -1  1  1 -1  1
-1  1  0 -1  1  1  0  0  1  1  1 -1  1 -1
-1  1 -1  0  1  1  1  0  1  1 -1  1  0 -1
-1  0  0  0  1 -1  1 -1  1  1  1  1  1 -1
 1  0 -1 -1  1  1  0 -1  1  1  1  1  0 -1
 1 -1 -1  0  1  1 -1  0  1  1  1 -1  0  1
 1 -1  0  1 -1  1  0  0  1  1  1 -1  1 -1
 1  0 -1  1 -1  1  0  1  1 -1  1 -1  0  1
 1  0  0  0 -1  1 -1  1 -1  1  1 -1  1  1
 1 -1  0 -1  1  1  0  0  1  1 -1  1 -1  1
 1 -1  1  0 -1  1  1  0  1  1 -1  1  0 -1
 1  0  1 -1 -1  1  0  1 -1  1 -1  1  0  1
 1  0  0  0 -1  1  1  1  1 -1 -1  1 -1  1
 1  0  0  0 -1 -1  1 -1  1  1  1  1  1 -1
 1  0 -1 -1  1 -1  0 -1  1  1  1  1  0  1
 1  1  0 -1 -1 -1  0  0 -1  1  1  1  1  1
 1  1 -1  0 -1 -1  1  0  1 -1  1  1  0  1
 1  0  0  0 -1  1 -1 -1  1  1  1  1 -1  1
-1  0  1  1 -1 -1  0 -1  1  1  1  1  0  1
-1  1  0  1 -1  1  0  0  1  1 -1  1 -1  1
-1  1  1  0 -1  1 -1  0  1  1  1 -1  0  1
```

```
-1  1  1  0 -1  1 -1  0 -1  1  1  1  0  1
-1  1  0  1 -1  1  0  0  1 -1  1  1 -1  1
-1  0  1  1 -1  1  0 -1  1  1  1  1  0 -1
-1  0  0  0  1  1 -1 -1  1  1  1  1 -1  1
```

Next we look at the cones and Hilbert basis of even circuits of the six signed Petersen graphs. For this we have generated the list of even circuits in C++. For this we encoded the graph as a signed incidence matrix of the graph, i.e. the incidence matrix with an added row specifying the signature of each edge. Calling Normaliz we get the required information. Again, to save space, we only present the number of Hilbert basis elements, extreme rays and support hyperplanes of the cone.

```
P+
63 Hilbert basis elements
57 extreme rays
60 support hyperplanes

P1
29 Hilbert basis elements
29 extreme rays
32 support hyperplanes

P22
59 Hilbert basis elements
29 extreme rays
200 support hyperplanes

P23
63 Hilbert basis elements
31 extreme rays
480 support hyperplanes

P32
46 Hilbert basis elements
31 extreme rays
```

```
210 support hyperplanes

P33
31 Hilbert basis elements
31 extreme rays
165 support hyperplanes
```

Notice that since $P_{3,2}$ embeds on the projective plane with the appropriate signing and K_6 as a dual, the cone of even circuits is the same as the cone of cuts of K_6.

Finally, we present the results for the cones and Hilbert basis of the contractible circuits and bicycles, contractible circuits and non contractible circuits of K_6 on the projective plane.

```
Contractible circuits and bicycles of signed K6 on projective plane
-------------------------------------------------
191 Hilbert basis elements
191 extreme rays
225 support hyperplanes


Contractible circuits of K6 on projective plane
-------------------------------------------------
96 Hilbert basis elements
96 extreme rays
920 support hyperplanes


Non contractible circuits of K6 on projective plane
-------------------------------------------------
193 Hilbert basis elements
61 extreme rays
6674 support hyperplanes
```

Appendix B

3-connected uncontractions of K_5

In this appendix we list the 22 non isomorphic 3-connected uncontractions of K_5.

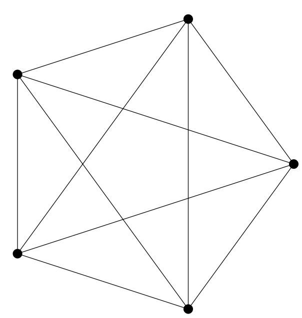


Figure B.1: The graph K_5

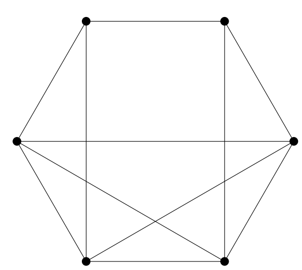


Figure B.2: The graph K_5 with one edge uncontraction

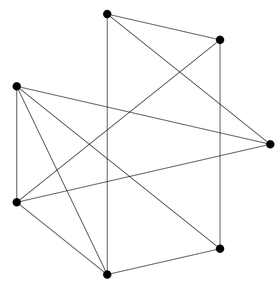


Figure B.3: Non isomorphic two-edge uncontractions of K_5

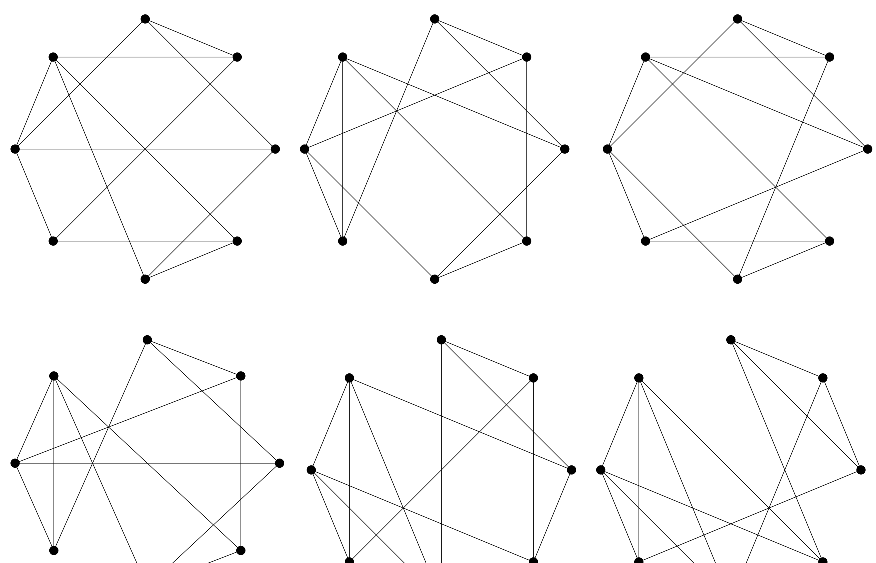


Figure B.4: Non isomorphic three-edge uncontractions of K_5

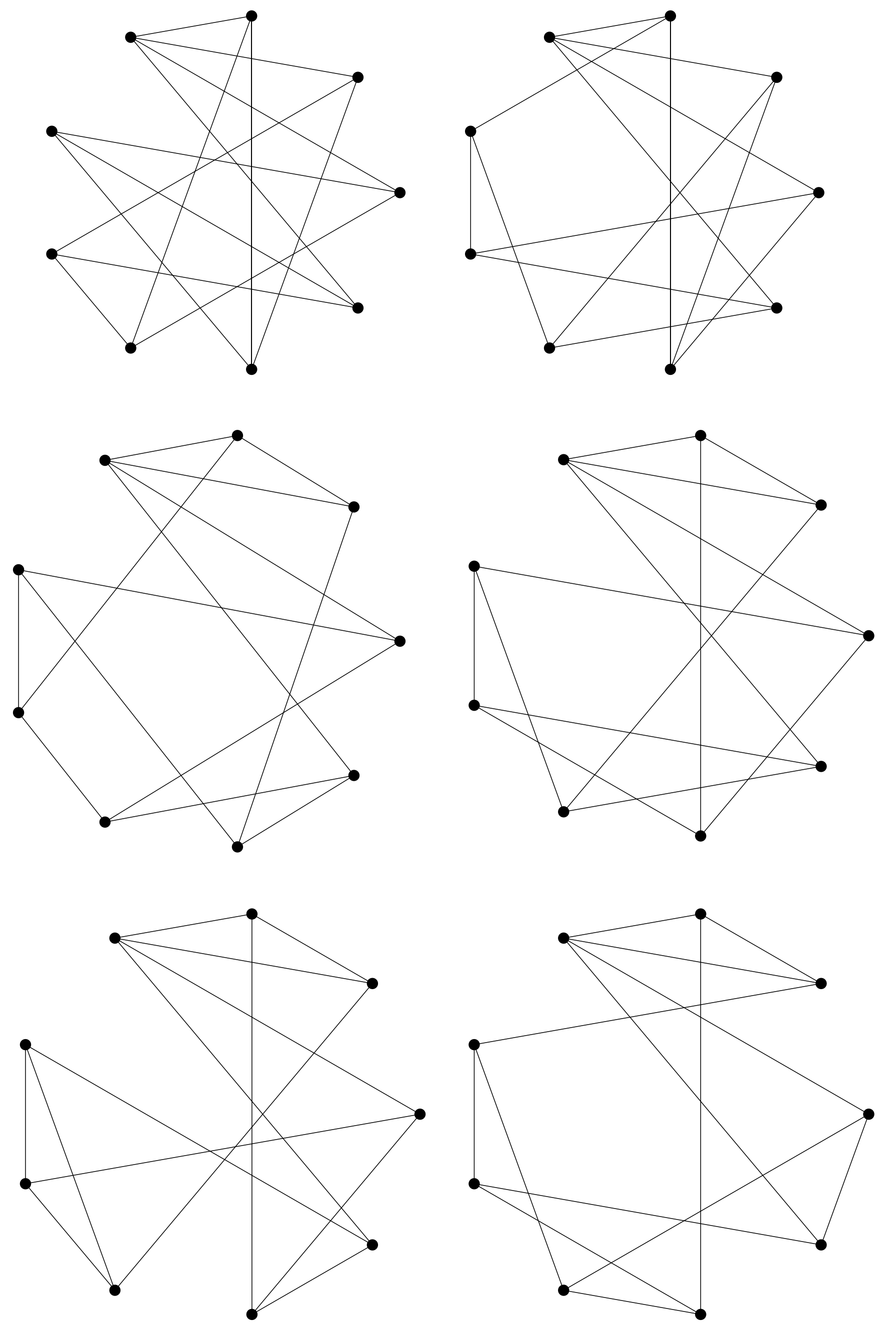


Figure B.5: Non isomorphic four-edge uncontractions of K_5

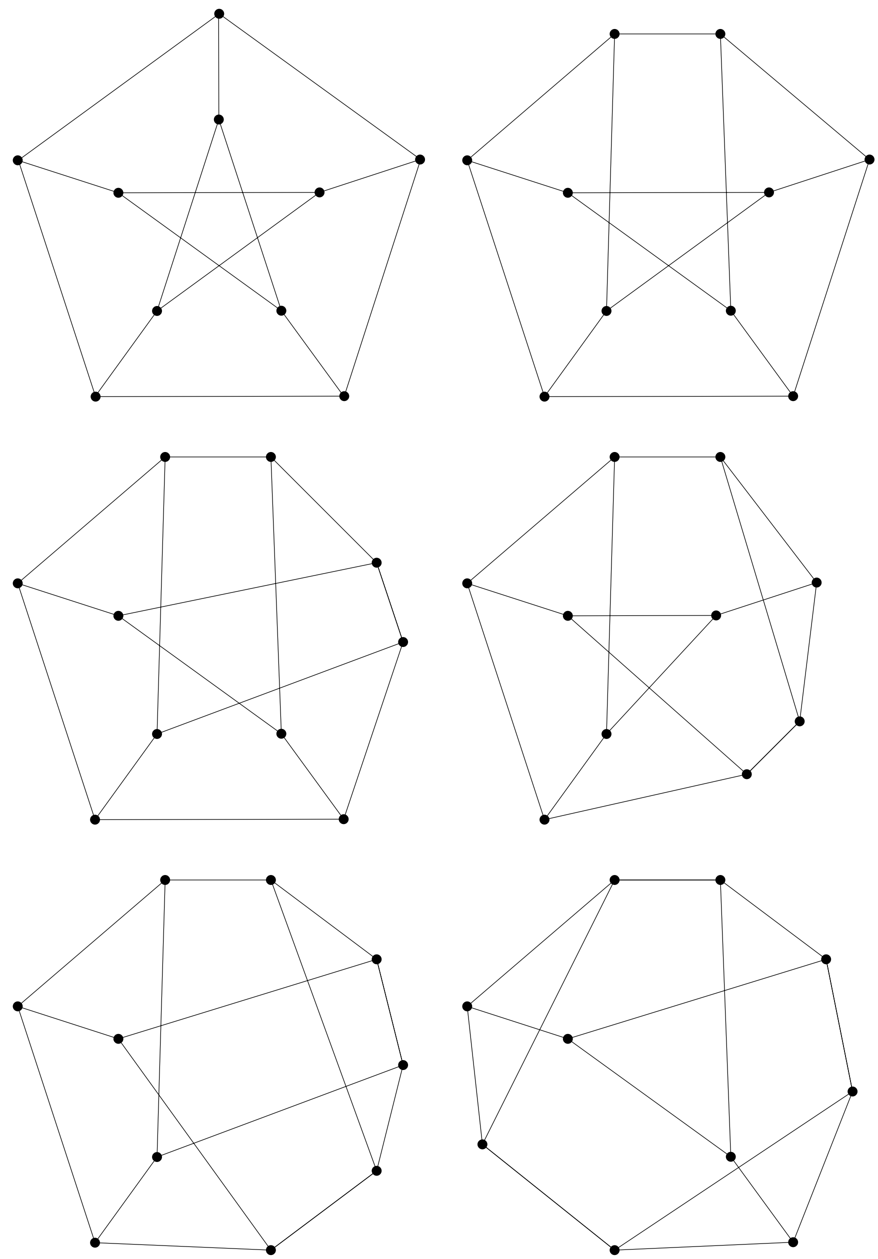


Figure B.6: Non isomorphic five-edge uncontractions of K_5

Bibliography

[1] B. Alspach, L. Goddyn, C.Q. Zhang. *Graphs with the circuit cover property*, Trans. Amer. Math. Soc. 344 (1994) 131-154

[2] F. Barahona. *The Max-cut Problem on Graphs Not Contractible to K_5*, Operations Research Letters. (1983 - 2) 107–111

[3] F. Barahona, A. R. Mahjoub. *On the cut polytope*, Mathematical Programming 36 (1986) 157–173

[4] J. A. Bondy, U. S. R. Murty. *Graph Theory* North-Holland (1976)

[5] W. Bruns, B. Ichim, C. Söger. *Normaliz: Algorithms for rational and affine monoids*, Software (2012)

[6] W. Bruns, B. Ichim, C. Söger. *The power of pyramid decompositions in Normaliz*, Preprint arXiv:1206.1916v1

[7] W. Cook, J. Fonlupt, A. Schrijver. *An Integer Analogue of the Caratheodory's Theorem*, Journal of Combinatorial Theory, Series B. Vol 40, No 1. February 1986.

[8] M. Deza, M. Laurent. *Facets for the cut cone I*, Mathematical Programming 56 (1992) 121–160

[9] M. Deza, M. Laurent. *Geometry of Cuts and Metrics*, Springer - 1997

[10] J. Edmonds. *Maximum matching and a polyhedron with (0,1)-vertices*, J. Res. Nat. Bur. Standards B 69(1965) 125–130

[11] X. Fu, L. A. Goddyn. *Matroids with the circuits cover property*, 1995

[12] R. Giles, W. Pulleybank. *Total dual integrality and integer polyhedra*, Linear Algebra and its applications 25, 191–196

[13] L. A. Goddyn. *Cones, Lattice and Hilbert Basis of Circuits and Perfect Matchings*, Contemporary Mathematics 147 American Mathematical Society. (1993) 419–440

[14] F.O. Hadiock. *Finding a maximum cut of planar graph in polynomial time*, SIAM Journal on Computing. 4, 221–225 (1975)

[15] D. Hilbert. *Uber die theorie der algebraischen formen*, Math Ann. 36. (1890) 473–534

[16] F. Laburthe, M. Deza, M. Laurent. *The Hilbert Basis of the Cut Cone over the Complete Graph on Six Vertices*, Laboratoire d'Informatique URA 1327 du CNRS, Ecole Normale Supérieure. LIENS - 95–7 April 1995

[17] M. Laurent. *Hilbert Bases of cuts*, Discrete Mathematics 150 Elsevier (1996) 257–279

[18] L. Lovasz. *Matching structure and the matching lattice*, Journal of Combinatorial Theory Ser. B 43 (1987) 187–222

[19] B. McKay. *Practical Graph Isomorphism*, Congressus Numerantium, 30 (1981) 45–87

[20] B. Mohar, C. Thomassen. *Graphs on Surfaces*, Johns Hopkins University Press (2001)

[21] J. Oxley. *Matroid Theory*, Oxford University Press (2011)

[22] I. Pivotto. *Even Cycle and Even Cut Matroids*, Ph.D. Thesis University of Waterloo (2011)

[23] A. Sebő. *Hilbert Bases, Caratheodory's Theorem and Combinatorial Optimization*, Integer Programming and Combinatorial Optimization Proceedings (IPCO) R. Kannan and W. Pulleybank Eds., University of Waterloo (1990) 431–456

[24] A. Schrijver. *On total dual integrality*, Linear Algebra Appl. 38 (1981) 27–32

[25] P. Seymour. *Sums of Circuits*, Graph Theory with Applications American Elsevier (1979) 341–355

[26] P. Seymour *Matroids and Multicommodity Flows*, European Journal of Combinatorics 2 (1981) 257–290

[27] D. Slilaty. *On Cographic Matroids and Signed-Graphic Matroids*, Discrete Mathematics (301) 2-3 (2005) 207–217

[28] K. Wagner. *Über eine Erweiterung des Satzes Von Kuratowski*, Deutsche Math. 2. (1937) 280–285

[29] T. Zaslavsky. *Six Signed Petersen Graphs, and their Automorphisms*, Discrete Mathematics, Elsevier (2012) Volume 312, Issue 9, 1558–1583