Partitions of Generalized Split Graphs

by

Oren Shklarsky

Hon. B.Sc. (Computer Science), University of Toronto, 2010

Thesis Submitted in Partial Fulfillment
of the requirements of the Degree of
Master of Science
in the
School of Computing Science
Faculty of Applied Sciences

Simon Fraser University
Summer 2012

Approval

Name: Oren Shklarsky

Degree: Master of Science

Title of Thesis: Partitions of Generalized Split Graphs

Examining Committee:

Dr. Joseph G. Peters
Professor, Computing Science
Chair

Dr. Pavol Hell
Professor, Computing Science
Senior Supervisor

Dr. Arthur L. Liestman
Professor, Computing Science
Supervisor

Dr. Ladislav Stacho
Associate Professor, Mathematics
Examiner

Date Approved:

Abstract

We discuss matrix partition problems for graphs that admit a partition into k independent sets and ℓ cliques. We show that when $k + \ell \leqslant 2$, any matrix M has finitely many (k, ℓ) minimal obstructions and hence all of these problems are polynomial time solvable. We provide upper bounds for the size of any (k, ℓ) minimal obstruction when $k = \ell = 1$ (split graphs), when $k = 2, \ell = 0$ (bipartite graphs), and when $k = 0, \ell = 2$ (co-bipartite graphs). When $k = \ell = 1$, we construct an exponential size split minimal obstruction for a particular matrix M, obtaining the first known exponential lower bound for any minimal obstruction. The construction also shows that the upper bounds are "nearly" tight.

Keywords: generalized graph colouring, matrix partition, split graphs, minimal obstructions, forbidden subgraphs, polynomial time algorithms, np-complete problems

Contents

List of Figures

List of Definitions

Chapter 1

Introduction

1.1 Preliminary Definitions

For natural numbers $m, n \in \mathbb{N}$, denote the set $\{1, ..., m\}$ by $[m]$; denote the set consisting of all sets of size n whose members are elements from $[m]$ by $\binom{[m]}{n}$.

A *digraph* $D = (V, E)$ consists of a *vertex set* $V(D)$ and a binary relation $E(D) \subseteq V(D) \times V(D)$ known as the *edge set of* D. A digraph G is an *undirected graph*, or simply a graph, if the edge relation is symmetric. That is, for any $u, v \in V(G)$, we have $(u, v) \in E(G)$ if and only if $(v, u) \in E(G)$. In an undirected graph we refer to the pairs (u, v) and (v, u) as the edge between u and v, and write $uv \in E(G)$. Further, when $uv \in E(G)$, say that u and v are *adjacent* and write $u \sim v$. For a vertex $u \in V(G)$, a loop is the edge uu. A (di)graph is *simple* if it has no loops. Note that as $E(G) \subseteq V(G) \times V(G)$, multiple edges are not permitted. In this thesis we restrict our attention to simple undirected graphs. The *open neighbourhood* of a vertex v in G, written $N_G(u)$, or simply $N(u)$ when the context is clear, is the set of vertices that are adjacent to v. The *closed neighbourhood* of u is then defined to be $N_G[u] = N_G(u) \cup \{u\}$.

A *homomorphism* of a graph G to a graph H is a mapping $h : V(G) \to V(H)$ such that $h(u)h(v) \in E(H)$ whenever $uv \in E(G)$. An *isomorphism* of G to H is a bijective homomorphism of G to H. If G has an isomorphism to H, we say that G and H are *isomorphic* and write $G \simeq H$.

A *subgraph* H of a graph G is a graph for which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that H is an *induced subgraph* of G, if H can be obtained from G by deleting a set of vertices from G, together with any edges that include these vertices. For a set $A \subseteq V(G)$, denote by $G - A$ the subgraph obtained by deleting the vertices of A. When A is a single vertex a, write $G - a$. For a set $B \subseteq V(G)$ of vertices, define $G[B] = G - (V(G) - B)$ to be the graph obtained from G by deleting all vertices of $V(G) - B$, and say that $G[B]$ is the *graph induced by* B. When $B = \{v_1, ..., v_n\}$, we omit the braces and write $G[v_1, ..., v_n]$ to mean $G[B]$. The *complement* $\overline{G}$ of a graph G is the graph with the same vertex set $V(\overline{G}) = V(G)$, in which two vertices u, v of $\overline{G}$ are adjacent if and only if u and v are not adjacent in G. If

$\mathcal{P}$ is a class of graphs, define co-$\mathcal{P}$ as the class of graphs that are complements of graphs in $\mathcal{P}$. That is, co-$\mathcal{P} = \{\overline{G}|G \in \mathcal{P}\}$.

For $m \in \mathbb{N}$, and given graphs $G_1, G_2, ..., G_m$ with disjoint vertex sets, let $G_1 \cup G_2 ... \cup G_m$ be the graph obtained by taking $V(G_1) \cup V(G_2) \cup ... \cup V(G_m)$ and $E(G_1) \cup E(G_2) \cup ... \cup E(G_m)$ as its vertex set and edge set, respectively. Then we say that $G_1 \cup G_2 ... \cup G_m$ is the *disjoint union* of $G_1, ..., G_m$. For a single graph G, let mG be the disjoint union of m copies of G, with the vertices renamed appropriately. Let G_1 and G_2 be graphs. Define $G_1 \cap G_2$ as the largest subgraph of G_1 that is isomorphic to a subgraph of G_2. Define the *join* of two graphs and denote it by $G_1 \oplus G_2$, as the graph obtained from adding to $G_1 \cup G_2$, every edge uv where $u \in V(G_1)$ and $v \in V(G_2)$.

For $n \in \mathbb{N}$, a *path* is a graph on n vertices for which an ordering $x_1, , , , x_n$ exists such that two vertices are adjacent if and only if they appear consecutively in the order. The path on n vertices and $n-1$ edges is denoted P_n and said to be of length $n-1$. For a pair of vertices u, v in a graph G, a *u, v-path* is an induced path in G whose vertices of degree 1 are u and v. Define the *distance* from u to v, denoted $d(u, v)$, as the length of a shortest u, v-path. If there is no u, v-path, let $d(u, v) = \infty$. A *cycle* C_n is the graph obtained from the path $x_1, ..., x_n$ by adding the edge x_1x_n. Consequently, the cycle is said to be of length n. An *odd cycle* is a cycle of odd length; we identify two special odd cycles that frequently occur - a *triangle* is a cycle of length 3 and a *pentagon* is a cycle of length 5.

For $n \in \mathbb{N}$, a graph on n vertices in which every two vertices are adjacent is known as a *complete graph*, and denoted K_n. A *clique* in a graph G is a set of vertices of which every two vertices are adjacent. An *independent set* or *stable set* in a graph G, is the complement of a clique - a set in which no two vertices are adjacent. Here we mention another type of graph that appears frequently in this thesis - for $m, n \in \mathbb{N}$, define mK_n as the disjoint union of m cliques on n vertices. A *maximal clique* (respectively stable set) of a graph G is a set of vertices of G which is a clique (stable set) and to which no vertex of G can be added to induce a larger clique (stable set). A *maximum clique* (stable set) of a graph G is a clique (stable set) in G with the largest number of vertices among all other cliques (stable sets) of G. Denote the size of the maximum clique in a graph G by $\omega(G)$, and the size of the maximum independent set in G by $\alpha(G)$. Note that each maximum clique is a maximal clique, but not vice versa.

A *colouring* of a graph G using $k \in \mathbb{N}$ colours is a labeling of the vertex set of the graph, $c : V(G) \to [k]$. A *proper colouring* of G is a colouring in which any two adjacent vertices receive different colours. The smallest number of colours needed to properly colour G is the *chromatic number* of G, and is denoted $\chi(G)$.

1.2 Graph Classes

In any graph G, we have $\omega(G) \leqslant \chi(G)$, since each of the $\omega(G)$ vertices of a maximum clique must receive a unique colour; of interest are the graphs for which the two invariants are equal, for every induced subgraph of the original graph. A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. The NP-complete decision versions of many graph problems admit polynomial time algorithms, when the input is restricted to perfect graphs. Of note are k-colouring, clique number, and maximum independent set [28].

Claude Berge, in 1961, proposed two conjectures underlining the importance of perfect graphs [2]. The first of these, now known as the *Weak Perfect Graph Theorem* was proven in 1972 by Lovász, and states that a graph is perfect if and only if its complement is perfect [35]. More recently, in 2006, Chudnovsky, Robertson, Seymour, and Thomas proved the second conjecture of Berge, now known as the Strong Perfect Graph Theorem, which states that a graph is perfect if and only if it does not contain an induced odd cycle of length at least five, or the complement of an induced odd cycle of length at least 5 [7]. These graphs are now known as *Berge graphs* and were shown separately by Chudnovsky et al. to be recognizable in polynomial time [6].

We now discuss some families of graphs which are perfect and are of interest in this thesis. A *chordal* graph is a graph in which the only induced cycles are triangles. A *simplicial* vertex v of a graph G, is a vertex whose closed neighbourhood $N_G[v]$ is a clique. A *simplicial elimination ordering*, sometimes referred to as a *perfect elimination ordering*, of a graph G is an ordering of the vertices $v_1, ..., v_n$ in which vertex v_i is a simplicial vertex in $G[v_i, ..., v_n]$, for $1 \leqslant i \leqslant n$. It is well known that a graph G is chordal if and only if it has a simplicial elimination ordering, and that finding such an ordering can be done in linear time [38]. To illustrate one of the many applications of the simplicial elimination ordering consider the following result, proved by Berge; the proof also serves to demonstrate the greedy coloring algorithm.

Theorem 1.2.1. *[1] Chordal graphs are perfect.*

Proof. Let G be a chordal graph on n vertices. Deleting vertices from G cannot create chordless cycles. Thus, every induced subgraph of G is chordal. It therefore suffices to prove that $\chi(G) = \omega(G)$. Given a simplicial elimination ordering $v_1, v_2, ..., v_n$, of the vertices of G, we apply greedy colouring to the vertices in reverse order. That is, for the vertices $v_n, ..., v_2, v_1$, assign to vertex v_i the least color not used by any of its neighbours that came before it in the reverse ordering. Suppose the largest colour used is $k \in \mathbb{N}$, and that some vertex v_i received color k. Then v_i has a neighbour in $v_n, v_{n-1}, ..., v_{i+1}$ of each colour $1, ..., k-1$. Since v_i is simplicial in $G[v_{i+1}, ..., v_n]$, together with v_i, these vertices form a clique of size k, giving $k \leqslant \omega(G)$. Since $\omega(G) \leqslant \chi(G) \leqslant k$, we conclude that $\chi(G) = \omega(G)$. □

There are graph families that are defined by a partition that their members admit. A bipartite graph is a graph whose vertices can be partitioned into two independent sets. For

$m, n \in \mathbb{N}$, a *complete bipartite* graph, denoted $K_{m,n}$, is a bipartite graph whose partite sets X and Y are of size m and n, respectively, and such that every $x \in X$ is adjacent to every $y \in Y$. A *star* graph on $n+1$ vertices is the graph $K_{1,n}$ for some $n \in \mathbb{N}$. A split graph is a graph whose vertex set can be partitioned into an independent set and a clique. More generally, for $k, \ell \in \mathbb{N}$, a *(k,ℓ)-graph* is a graph whose vertices can be partitioned into k independent sets and ℓ cliques. Thus a bipartite graph is a $(2,0)$-graph, and a split graph is a $(1,1)$-graph. We write (k,ℓ) to denote the class of all (k,ℓ)-graphs. Since every induced subgraph of a $(2,0)$-graph is a $(2,0)$-graph, and every $(2,0)$-graph has clique number at most 2, we have that $(2,0)$-graphs are perfect. In Section 1.3 we show that the class of split graphs is a subclass of the class of chordal graphs [26]. Hence split graphs are also perfect.

One more class of graphs discussed in this thesis is the class of *complement reducible graphs,* or *cographs*. This class is defined recursively as containing the single vertex graph K_1, and the graphs obtained from cographs G_1 and G_2 by taking either their disjoint union $G_1 \cup G_2$, or their join $G_1 \oplus G_2$.

From the definition it follows that the class of cographs is closed under complementation, and that the join of two cographs is equivalent to complementing the disjoint union of their complements. This information, together with the fact that the complement of every nontrivial connected component of a cograph is disconnected [9], has been used by the authors in [39, 9] to characterize the class of cographs by the absence of a particular induced subgraph, namely, the path on four vertices.

Cographs can be recognized in linear time [8], and can be represented by a special tree, which can be constructed, also in linear time [8]. The *cotree* T of a cograph G is a tree representing the recursive structure of G. The leaves of T are the vertices of G, while the internal nodes are marked either 0 or 1. The root of T is labeled 1, the children of a node labeled 1 are labeled 0 and the children of a node labeled 0 are labeled 1. The 1-nodes represent the join operation, while the 0-nodes represent the disjoint union. To be consistent with [9], we assume that the root node has exactly one child if and only if G is disconnected. For an internal node x of T, the graph G_x is the graph corresponding to the subtree of T rooted at x. Thus if the tree is rooted at a node r, then $G = G_r$. In Section 2.2 we show that cographs are perfect [9].

Aside from being perfect, the graph families described above share another interesting property. Each family can be characterized by the absence of certain configurations. Formally, let H be a fixed graph. A graph G is said to be *H-free* if no induced subgraph of G is isomorphic to H. Now fix a family of graphs $\mathcal{H}$. We say that G is *$\mathcal{H}$-free* if it is H-free for every $H \in \mathcal{H}$. Define $Forb(\mathcal{H}) = \{G | G \text{ is } \mathcal{H}\text{-free }\}$; say that every member of $\mathcal{H}$ is a *forbidden subgraph* of $Forb(\mathcal{H})$. We call $H \in \mathcal{H}$ a *minimal forbidden subgraph of $Forb(\mathcal{H})$* of H if it is a forbidden subgraph of $Forb(\mathcal{H})$, but every proper induced subgraph of H is not a forbidden subgraph of $Forb(\mathcal{H})$.

For example, the class of chordal graphs is defined as $Forb(\mathcal{C})$ where $\mathcal{C}$ is the set of all cycles of length at least four; for the set $\mathcal{C}_{odd} = \{C_{2n+1} | n \in \mathbb{N}\}$ of odd cycles, $Forb(\mathcal{C}_{odd})$ is the family of bipartite graphs, by a result of König [34], and by the Strong Perfect Graph Theorem, the class of perfect graphs is the class $Forb(\mathcal{H} \cup co\text{-}\mathcal{H})$ where $\mathcal{H} = \mathcal{C}_{odd} - C_3$. As was shown in [26], $Forb(\{2K_2, C_4, C_5\})$ defines the family of split graphs. Finally, as already mentioned, the class of cographs is defined by $Forb(\{P_4\})$.

The first three examples can be distinguished from the last two by an important characteristic. Namely, the infinite families $\mathcal{C}$, $\mathcal{C}_{odd}$ and $\mathcal{C}_{odd} - C_3$ are needed to describe the chordal, bipartite, and perfect graphs respectively. On the other hand, a finite number of minimal forbidden subgraphs suffices to describe split graphs and cographs.

A characterization of a graph family $\mathcal{G}$ by a finite set of forbidden subgraphs is desirable since it allows for a polynomial time recognition algorithm for $\mathcal{G}$. Note, however, that such a characterization isn't necessary for polynomial time recognition to be possible, as exemplified by the bipartite graphs. Further, characterization by a finite family of minimal forbidden subgraphs, while guaranteeing a polynomial time recognition algorithm, might not provide the best lower bound for such recognition. For example, linear time algorithms for the recognition of split graphs can be found in [29], and [31]. These algorithms do not rely on forbidden subgraphs. On the other hand, naively searching for the minimal forbidden subgraphs takes time $O(n^5)$, to determine whether an induced C_5 exists in the input graph.

1.3 Partitions of Graphs into Cliques and Independent Sets

Here we consider recognition and characterizations of (k, ℓ)-graphs for various values of k and ℓ.

Polynomial time recognition and a characterization of bipartite graphs is easily obtained from the following well-known Theorem by König.

Theorem 1.3.1. *[34] A graph is bipartite if and only it has no odd cycle.*

Proof. $\Rightarrow$ Suppose $G = (X, Y, E)$ is a bipartite graph, and let C be a cycle in G of length m with vertices $a_1, ...a_m$. Suppose without loss of generality that $a_1 \in X$. Then for $1 \leqslant i \leqslant m$, if $a_i \in X$, then i is odd, and if $a_i \in Y$, then i is even. Thus, as $a_m \in Y$, m is even.

$\Leftarrow$ Suppose G has no odd cycles. Assume without loss of generality that G is connected; otherwise, we apply what follows to each component of G. We construct a bipartition X, Y for G. Let $u \in V(G)$, and partition the vertices of G by their distance from u. Let $X = \{v \in V(G) | d(u, v) \text{ is even}\}$, and $Y = \{v \in V(G) | d(u, v) \text{ is odd}\}$. If v, v' are adjacent and both belong to X, then we can find an odd closed walk using the u, v-path, the edge vv', and the v', u-path. Since every closed odd walk contains an odd cycle [40], this contradicts the assumption that G has no odd cycles. Thus X is independent. By a similar argument, Y is independent as well, and so G is bipartite. $\square$

For split graphs, we have the following well-known characterization. We follow the proof presented in [27].

Note first that if G is a split graph, then $\overline{G}$ is a split graph. This follows from the fact that for a partition of G into a clique C and independent I, the complements $\overline{C}$ and $\overline{I}$ are a stable set and a clique, respectively.

Theorem 1.3.2. *[26] The following conditions are equivalent for a graph G:*

(i) G is a split graph.

(ii) G and $\overline{G}$ are both chordal.

(iii) G does not have $2K_2, C_4$, and C_5 as induced subgraphs.

Proof. $(i) \implies (ii)$ Suppose G is a split graph. Let $C \cup I$ be a partition of G into a clique C and independent set I. Then every vertex of I is simplicial in G and every vertex of C is simplicial in $G - I$. Hence, picking any ordering for the vertices I, followed by any ordering for the vertices of C produces a simplicial elimination ordering, so that G is chordal. Since $\overline{G}$ is also a split graph, the same argument shows that $\overline{G}$ is chordal.

$(ii) \implies (iii)$ Since G is chordal, it does not contain an induced C_4 or C_5. Since $\overline{G}$ is chordal, $\overline{G}$ does not contain an induced C_4. If G were to contain an induced $2K_2$, then $\overline{G}$ would contain an induced C_4, as the complement of C_4 is isomorphic to $2K_2$.

$(iii) \implies (i)$ Let C be a maximum clique in G, chosen so that the number of edges in $G-C$ is minimized, over all maximum cliques in G. Suppose for contradiction that $G-C$ is not an independent set, and let u, v be two adjacent vertices of $G - C$. Since C is maximal, u has a non-neighbour w in C and v has a non-neighbour x in C. Further, since C is of maximum size, we may choose w and x so that $w \neq x$. To see this, suppose otherwise, that u and v have the same non-neighbour w in C and that u and v are adjacent to every other vertex of C. Then the clique $(C-\{w\})\cup\{u, v\}$ is larger than C, contradicting the maximality of C. Now, if both edges ux and vw are not present in G, then $G[u, v, w, x] \simeq 2K_2$, and if both these edges are present, then $G[u, v, w, x] \simeq C_4$, but both these graphs are not induced subgraphs of G by assumption. Thus exactly one of the edges ux or vw is present in G. Say without loss of generality that $vw \in E(G)$. Then v is adjacent to every vertex of $C - \{w, x\}$, for if some vertex $y \in C - \{w, x\}$ is not adjacent to v, then if y is also not adjacent to u, we have $G[u, v, x, y] \simeq 2K_2$, and if y is adjacent to u, then $G[u, v, w, y] \simeq C_4$. Therefore the graph induced by $C' = C - \{x\} \cup \{v\}$ is a clique of maximum size. In $G - C$, the edge uv is present, while in $G-C'$ it is not. Since $G-C'$ has at least as many edges as $G-C$, by choice of C, there must be some edge to replace uv in the edge count for $G - C'$. That is, there must be a vertex $z \in G - C$ adjacent to x. Further, z cannot be adjacent to v, otherwise together with the edge uv, $G - C$ has at least one more edge than $G - C'$. Now z must be adjacent to u, otherwise $G[z, x, u, v] \simeq 2K_2$. This implies that z is not adjacent to w, because then we would have $G[z, w, v, u] \simeq C_4$. But then $G[z, x, w, v, u] \simeq C_5$, contradicting the assumption that C_5 is not an induced subgraph of G. $\square$

The split, bipartite, and co-bipartite graphs constitute those (k, ℓ)-graphs for which $k + \ell < 3$. In [23] a general technique is presented for recognizing graphs that admit certain partitioning schemes. Of interest here, the technique unifies the recognition of split, bipartite and co-bipartite graphs. Further, the same technique may be used for the recognition of graphs in $(1, 2) \cup (2, 1) \cup (2, 2)$.

Let $\mathcal{S}$ and $\mathcal{D}$ be two classes of graphs. We say that $\mathcal{S}$ is a class of sparse graphs and that $\mathcal{D}$ is a class of dense graphs, if $\mathcal{S}$ and $\mathcal{D}$ satisfy the following:

- Both $\mathcal{S}$ and $\mathcal{D}$ are closed under taking induced subgraphs.
- There is a constant c such that for every $S \in \mathcal{S}$ and $D \in \mathcal{D}$, we have $|S \cap D| \leqslant c$.

In a given graph G, say that a set of vertices A is sparse if $G[A] \in \mathcal{S}$. Say that A is dense if $G[A] \in \mathcal{D}$. A *Sparse-Dense Partition* of a graph G, with respect to classes $\mathcal{S}$ and $\mathcal{D}$ of sparse and dense graphs, is a partition of the vertex set $V(G)$ into two parts $V(G) = S \cup D$ such that $G[S] \in \mathcal{S}$ and $G[D] \in \mathcal{D}$.

Sparse-Dense partitions are a generalization of split graphs. If we take $\mathcal{S}$ to be the set of all independent sets, $\mathcal{D}$ to be the set of all cliques, and $c = 1$, then since any $S \in \mathcal{S}$ and $D \in \mathcal{D}$ have at most one vertex in common, a graph has a sparse-dense partition if and only if it can be partitioned into an independent set and a clique. That is, if and only if the graph is a split graph.

Given $\mathcal{S}$ and $\mathcal{D}$, it turns out that any graph has a polynomial number of sparse-dense partitions with respect to $\mathcal{S}$ and $\mathcal{D}$. Additionally, if $\mathcal{S}$ and $\mathcal{D}$ can be recognized in polynomial time, then this allows us to determine the existence of a sparse-dense partition in a given input graph, in polynomial time.

Theorem 1.3.3. *[23] Let $\mathcal{S}$ and $\mathcal{D}$ be classes of sparse and dense graphs, respectively. Let $c \in \mathbb{N}$ such that $|S \cap D| \leqslant c$ for every $S \in \mathcal{S}$ and $D \in \mathcal{D}$.*

(i) A graph on n vertices has at most n^{2c} different sparse-dense partitions with respect to $\mathcal{S}$ and $\mathcal{D}$.

(ii) Further, all n^{2c} partitions can be found in time proportional to $n^{2c+2} \cdot T(n)$, where $T(n)$ is the time required for recognizing members of $\mathcal{S}$ and $\mathcal{D}$.

Proof. (i) Suppose G has at least one sparse-dense partition. Fix a sparse-dense partition $V(G) = S \cup D$. Then any other sparse-dense partition $V(G) = S' \cup D'$ satisfies $|S' \cap D| \leqslant c$ and $|S \cap D'| \leqslant c$. Thus S' is obtained from S by deleting at most c vertices and inserting at most c vertices. Each of these at most $2c$ operations can be made in at most n ways, giving at most n^{2c} different partitions, and all of the partitions can be found in time $n^{2c} \cdot 2T(n)$, since at each change we must check whether the new partition is sparse-dense.

(ii) Having (i), it remains to find the first sparse-dense partition, if one exists. Suppose G has a sparse-dense partition, and let $V(G) = S \cup D$ be a sparse-dense partition maximizing the size of S. We show how to find $S \cup D$, starting with any sparse set S' such that $|S'| < |S|$. Since S' is sparse, $|S' \cap D| = c' \leqslant c$. Then there are at least $c'+1$ vertices in S that are not in S'. To see this, suppose otherwise, that $|S - S'| \leqslant c'$. Then $S = (S' - (S' \cap D)) \cup (S - S')$ and so $|S'| < |S| = |(S' - (S' \cap D))| + |(S - S')| \leqslant |S'|$. Further, if A is a set of $c'+1$ vertices of $S - S'$, then $(S' - (S' \cap D) \cup A$ is a subset of S and so must be sparse. Therefore starting with any sparse set S', say $S' = \emptyset$, we can obtain a larger set by deleting some selection of $c' = |S' \cap D|$ vertices and inserting some $c'+1$ other vertices. This can be done in time at most $n^{2c+1} \cdot T(n)$. After at most n such enlargements we obtain a sparse set S' of the same size as S.

With $|S'| = |S|$, the set S can be obtained from S' by a deletion of at most c vertices and an insertion of at most c vertices. Thus we can test all n^{2c} possible new sets for sparseness, and the rest of $V(G)$ for denseness. If no sparse-dense partition is found, then none exists. □

A variation on the problem of recognizing (k, ℓ)-graphs, defined more generally in Section 1.4 is that of recognizing (k, ℓ)-graphs with lists. In this version, the k stable sets and ℓ cliques are numbered using elements from $[(k+\ell)]$, with the stable sets numbered $1, ..., k$ and the cliques numbered $k+1, ..., k+\ell$. An input graph is provided with lists $L(v) \subseteq [(k+\ell)]$ for each v, and we ask whether G admits a (k, ℓ)-partition such every vertex v is placed in a part from $L(v)$. In this case, we say that the partition respects the given lists. Theorem 1.3.3 yields a recognition algorithm for (k, ℓ)-graphs where $k \leqslant 2$ and $\ell \leqslant 2$, even with lists.

Corollary 1.3.4. *[23] Let $k, \ell \in \mathbb{N}$. If $k \leqslant 2$ and $\ell \leqslant 2$, then recognizing (k, ℓ)-graphs with lists can be done in polynomial time. Otherwise, the recognition problem is NP-complete.*

Proof. Suppose first that $k \geqslant 3$. We show that 3-colourability is reducible to (k, ℓ)-recognition. That is, given a graph$G$, we construct a graph G' with lists such that G is 3-colourable if and only if G' admits a (k, ℓ)-partition respecting the lists. Let $G' = G$, but with lists $L(v) = \{1, 2, 3\}$ for each vertex of G'. Then G' is a (k, ℓ)-graph with respected lists if and only if G is 3-colourable. A similar argument is made when $\ell \geqslant 3$.

Suppose now that both $k \leqslant 2$ and $\ell \leqslant 2$. Let $\mathcal{S}$ be the class of all k-colourable graphs, and $\mathcal{D}$ be the class of all graphs whose complement is ℓ-colourable. As $k \leqslant 2$ and $\ell \leqslant 2$ both of these classes can be recognized in polynomial time. By Theorem 1.3.3 we generate all sparse-dense partitions of any input graph G, if they exist. If G has no lists, we are done.

Suppose G is an input graph with lists L. Generate all sparse-dense partitions of G, and for each sparse-dense partition update the lists of the vertices as follows. Let $S = (I_1 \cup I_2 \cup ... \cup I_k)$ and $D = (C_{k+1} \cup C_{k+2} \cup ... \cup C_{k+\ell})$ be a particular sparse-dense partition. If a vertex v belongs to S, remove from $L(v)$ any element $i \geqslant k+1$. If v belongs to D, remove from $L(v)$ any element $i \leqslant k$. The resulting instance of the graph has a set of lists L' all of size at most 2, and hence can be solved using 2-satisfiability (cf. [23] Proposition 2.1).

Further, G admits a (k, ℓ)-partition respecting the original lists L if and only if it admits a (k, ℓ)-partition respecting at least one of modified sets of lists L'. $\square$

1.4 Matrix Partition

Here we discuss graph partitions in terms of *patterns.* Up until now we have described different partition schemes of graphs into independent sets or cliques, while not placing any restrictions on how the different partite sets, or parts, interact. Some of the sets may be restricted to have no adjacency, while with others we may require that all edges be present between two given parts. To describe these additional requirements, we use a symmetric $m \times m$ matrix M whose entries range over $\{0, 1, *\}$. The requirements for the different parts of a graph partition are given by the diagonal entries of M, while the constraint on interaction between part i and part j are given by entry $M(i, j)$.

Formally, an *M-partition* of a graph G is a partition $P_1, ..., P_m$ of the vertex set $V(G)$ such that if u is a vertex of P_i and v is a vertex of P_j, with $u \neq v$, then

- If $M(i, j) = 1$ then u and v are adjacent
- If $M(i, j) = 0$ then u and v are non-adjacent
- If $M(i, j) = *$ there is no restriction on the adjacency of u and v.

Additionally, given parts P, P' of an M-partition, we define $M(P, P')$ as $M(i, j)$ where i and j are the diagonal indices of M that correspond to P and P', respectively. Note that $M(i, i) = 0$ signifies a stable set, and $M(i, i) = 1$ a clique. The *M-partition problem* asks whether or not an input graph G admits an M-partition. Note that if for some i, we have $M(i, i) = *$, then the M-partition problem is trivial. However there are variations in which a diagonal asterisk does not make the problem trivial. cf. the end of this Section for an example.

Thus bipartite graphs are the $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$-partitionable graphs and in general, if C_m is the matrix containing 0s on the main diagonal, and asterisks in all off diagonal entries, then the k-colourable graphs are the C_m-partitionable graphs. Similarly, split graphs are precisely the $\left(\begin{smallmatrix} 0 & * \\ * & 1 \end{smallmatrix}\right)$-partitionable graphs, and the matrix with k diagonal 0s, ℓ diagonal 1s and asterisks in every other entry captures the (k, ℓ)-graphs.

Given an $m \times m$ matrix M, a *principal matrix* of M is a matrix M' obtained from M by deleting any number of rows from M, together with their corresponding columns. Throughout this thesis, when M is symmetric and has no diagonal asterisks, we assume that it has k diagonal 0s and ℓ diagonal 1s. In this case M can be described in so called (A, B, C)-*block* form, in which the matrix A is the $k \times k$ principal matrix of M having only diagonal 0s, the matrix B is the $\ell \times \ell$ principal matrix of M having only diagonal 1s, and $C = M(1...k, k+1...\ell)$ is the $k \times \ell$ submatrix of M describing the interaction of independent parts with clique parts.

If all off diagonal entries of A are the same value, say a, and similarly all of diagonal entries of B are the same value b, we refer to M as an (a, b, C)-block matrix. Further, if the entries in C are all equal to c, we say that M is an (a, b, c)-block matrix and also call it a *constant matrix.*

For a matrix M, define the complementary matrix $\overline{M}$ as the matrix obtained from M by replacing 0 entries of M with 1s, and 1 entries with 0s. Clearly we have -

Fact 1.4.1. *A graph G is M-partitionable if and only $\overline{G}$ is $\overline{M}$-partitionable.* $\square$

In addition to colouring problems, matrix partitions generalize another well known problem. The *H-homomorphism problem* asks whether the input graph G admits a homomorphism to a fixed graph H. Note that a homomorphism of G to H can be viewed as a partition of $V(G)$ into sets P_v with $v \in V(H)$ such that $P_v = f^{-1}(v)$ and P_v is independent if v has no loop, and for two distinct vertices $u, v \in V(H)$, there are no edges between P_u and P_v whenever $uv \notin E(H)$.

Let M_H be the matrix obtained from the adjacency matrix of H by replacing each 1 entry with an asterisk. Then a homomorphism of G to H corresponds to an M_H-partition of G, and an M_H-partition of G corresponds to a homomorphism of G to H. In fact, if we restrict our attention to matrices M whose entries are not equal to 1, there always exists a graph H_M for which the H-homomorphism problem is equivalent to the M-partition problem: given M, we obtain the adjacency matrix of the graph H by replacing each asterisk entry of M by a 1. Thus we have the following equivalence.

Theorem 1.4.2. *[30] Let M be a matrix with no* 1*s. The M-partition problem is polynomial time equivalent to the H-homomorphism problem.*

Lastly, we describe a variation of the M-partition problem. While most of this thesis discusses the basic M-partition problem, this variation is touched upon as well. Given a matrix M, we may index parts of an M-partition using the numbers $\{1, ..., m\}$. Define the *list M-partition* problem to be the M-partition problem in which the input graph G is equipped with lists $L(v) \subseteq \{1, ..., m\}$ for each vertex $v \in V(G)$, where v may only be placed in the parts whose index is contained in $L(v)$. Note that in this variation, diagonal asterisks do not make the problem trivial.

Of course, the result of Theorem 1.4.2 extends to encompass list M-partition problems and *list H-homomorphism problems.* The list H-homomorphism problem asks whether an input graph G, together with lists $L(v) \subseteq V(H)$ for each $v \in V(G)$, admits a homomorphism of G to H that respects the lists.

The matrix partition problem, as formulated above, allows us discuss graph partition problems in which the vertices of the graph are partitioned into independent sets or cliques, and distinct parts are either fully connected, fully disconnected, or are unknown. For graph partition problems with other types of restrictions on the parts or the interconnectivity of parts, see [3, 4, 13, 36].

Chapter 2

Survey of Matrix Partition

Here we discuss the current state of the M-partition problem. The topics discussed in this survey may be coloured by two broad strokes. On the one hand, as always, we are interested in the complexity of an M-partition problem for a given matrix M, or perhaps for a restricted set of input graphs. On the other hand, as mentioned in Section 1.2, the notion of minimal forbidden subgraphs plays an interesting role in describing families of M-partitionable graphs, again, restricted to some M, or to some set of input graphs.

In the context of matrix partitions, we discuss minimal forbidden subgraphs by another name, capturing the idea that the existence of a minimal forbidden subgraph in an input graph *obstructs* us from partitioning the graph. That is, given a matrix M and a graph G, we say that G is a *minimal M-obstruction* if G does not admit an M-partition, but every proper induced subgraph of G admits an M-partition. If the context is clear, we omit the M and simply call G a *minimal obstruction.*

Formally, for an $m \times m$ matrix M, and a family of graphs $\mathcal{G}$, let $f_{M,\mathcal{G}}(m)$ denote the size of the largest minimal M-obstruction in $\mathcal{G}$. If there are infinitely many minimal M-obstructions in $\mathcal{G}$ for some matrix in M, say $f_{M,\mathcal{G}}(m) = \infty$. Note that the parameter m for $f_{M,\mathcal{G}}$ is implied by the matrix M. However as bounds for $f_{M,\mathcal{G}}$ tend to be given in terms of m, we include it in the definition explicitly. For a class of matrices $\mathcal{M}$, we are interested in the smallest upper bound for $f_{M,\mathcal{G}}(m)$ for all $M \in \mathcal{M}$. When the context is clear, we write $f_{\mathcal{M},\mathcal{G}}(m) = sup\{f_{M,\mathcal{G}}(m) | M \in \mathcal{M}\}$.

2.1 General Graphs

2.1.1 Complexity

We begin by examining list matrix partition problems, unrestricted to any graph class.

In Section 2.1.2 we describe a result from [19] which asserts that for any matrix M without asterisk entries, the class of M-partitionable graphs can be characterized by finitely many minimal obstructions. As a result, the decision problem for M-partitionability in this case can be solved in polynomial time. It turns out that this is true even with lists.

Theorem 2.1.1. *[23] If M has no asterisk entries then the list M-partition problem is solvable in polynomial time.*

Let M be a matrix with no 1s. By Theorem 1.4.2, the list M-partition problem is polynomially equivalent to the list H-homomorphism problem. In the list version of M-partition diagonal asterisks are permitted, and these correspond to loops in the graph H obtained from the matrix M. Feder et al. ([16, 21]) gave classifications of list M-partition problems for the case when all or none of the diagonal entries of M are asterisks, into those matrices M for which the list M-partition problem is NP-complete, and those matrices M for which the list M-partition problem is polynomial time solvable.

Let H_M be the graph obtained by replacing every asterisk entry in M with a one, and regarding M as an adjacency matrix for H_M. Note that if all of the diagonal entries of M are asterisks, then H_M is a reflexive graph; if no diagonal entries are asterisks, then H_M is irreflexive. An *interval graph* is the intersection graph of a family of intervals on a line.

Theorem 2.1.2. *[16] Let M be a matrix with no 1 entries. Suppose all diagonal entries of M are asterisks. If H_M is an interval graph, then the list M-partition problem is polynomial time solvable.*

Otherwise the list M-partition problem is NP-complete.

A *circular arc* graph is the intersection graph of a family of arcs of a circle.

Theorem 2.1.3. *[21] Let M be a matrix with no 1 entries. Suppose M has no diagonal asterisks. If H_M is bipartite and $\overline{H_M}$ is a circular arc graph, then the list M-partition problem is polynomial time solvable.*

Otherwise the list M-partition problem is NP-complete.

In fact in [22], the authors define the class of *bi-arc* graphs, which encompasses the reflexive interval graphs and the bipartite complements of circular arc graphs from Theorems 2.1.2 and 2.1.3, and show that, in terms of patterns, the list M-partition problem is polynomial time solvable if H_M is a bi-arc graph, and NP-complete otherwise. This provides dichotomy for the list matrix partition problem, when the patterns have no 1s. If M is allowed 1s, then while it is not known whether a dichotomy still holds, it is shown in [17] that each list M-partition problem is either NP-complete or solvable in time $n^{O(\log n)}$. Say an algorithm is *quasi-polynomial* if its running time on input of size n is bounded by $n^{O(log^p n)}$ for some $p \in \mathbb{N}$.

So far we have restricted the entries of M by disallowing some element from $\{0, 1, *\}$. Removing this restriction, a full dichotomy is known when M is small, specifically of size at most four. We start with the following Claim.

Claim 2.1.4. [30] If M is an $m \times m$ matrix, and M' or $\overline{M'}$ is a principal submatrix of M such that the list M'-partition is NP-complete, then the list M-partition problem is NP-complete.

Proof. Suppose M' is a principal submatrix of the matrix M, and that the list M'-partition problem is NP-complete. Then an input to the list M'-partition problem, a graph G with lists, can be passed unchanged to an algorithm for list M-partition. Such an algorithm accepts without using any parts of M that are not in M', if and only if G admits a list M'-partition. Thus if list M'-partition is NP-complete, then so is list M-partition. It follows by complementation, that if M' is a matrix for which list M'-partition is NP-complete, and M is a matrix containing M' or $\overline{M'}$ as a principal submatrix, then list M-partition is NP-complete. $\square$

Now the first four patterns in Figure 2.1.1 correspond to NP-complete list matrix partition problems. These are (i) the 3-*colouring* problem, well-known to be NP-complete, even without lists, (ii) the *stable cutset* problem, shown NP-complete in [33], (iii) *the stable cutset pair* [33], and (iv) the *reflexive four-cycle [16].*

$$\underset{(i)}{\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}} \quad \underset{(ii)}{\begin{pmatrix} 0 & * & * \\ * & * & 0 \\ * & 0 & * \end{pmatrix}} \quad \underset{(iii)}{\begin{pmatrix} * & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & * \end{pmatrix}}$$

$$\underset{(iv)}{\begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{pmatrix}} \quad \underset{\substack{(v) \\ \text{The Stubborn Problem}}}{\begin{pmatrix} 0 & * & 0 & * \\ * & 0 & * & * \\ 0 & * & * & * \\ * & * & * & 1 \end{pmatrix}}$$

Figure 2.1.1 – NP-complete small patterns

If M contains any of the patterns $(i) - (iv)$ or the complement of one of $(i) - (iv)$, then the list M-partition problem is NP-complete. All other patterns of size at most four, were shown to be either polynomial or quasi-polynomial time solvable in [23]. In [5] all patterns of size at most four with quasi-polynomial algorithms, except for pattern (v), were shown to be polynomial time solvable. This latter problem was recently shown to be polynomial in [10], so that if M is of size at most four and M or $\overline{M}$ contains one of the four matrices in Figure 2.1.1 as a principal submatrix, then list M-partition is NP-complete, and otherwise list M-partition is polynomial time solvable [23, 5, 10].

We turn now to the basic matrix partition problem, in which the input graph is presented without accompanying lists. When M is larger than four, it is not clear that the existence in M of a principal submatrix corresponding to an NP-complete problem implies that M-partition is NP-complete. Nonetheless, in [23] the authors were able to show that when M is of size at most four, if M or $\overline{M}$ contain matrix (i), then M-partition is NP-complete, and otherwise it is polynomial time solvable.

In this case diagonal asterisks make the problem trivial, so of the four patterns described in Figure 2.1.1 only pattern (i), corresponding to the 3-colouring problem, is relevant to the discussion of small matrices M. If M does not contain the matrix (i) or its complement, then the list M-partition algorithms (cf. [23] Theorem 6.2) can be applied with each list containing all parts of M, so the M-partition problem is polynomial time solvable in this case. On the other hand, if M contains (i) or its complement, then if M is of size 3×3, $M = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$ and M-partition is NP-complete. If M is of size 4×4, [23] provides the following.

Claim 2.1.5. Let M be a 4×4 matrix. If M contains the matrix $\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$ as a principal submatrix, then the M-partition problem is NP-complete.

Proof. Say the fourth column of M is (x_1, x_2, x_3, y), and that M contains the matrix $\begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}$ as a principal submatrix.

Case 1. Suppose first that $y = 0$. Then if $x_1 = x_2 = x_3 = *$, a graph G is M-partitionable if and only if it is 4-colourable. If some $x_i = 0$, then the union of the ith part and the fourth part is an independent set so an input graph G admits an M-partition if and only if it is 3-colourable. If some $x_i = 1$, let $G' = 2G$. If G is 3-colourable, both copies of G in G' can be placed in the first three parts, ignoring the fourth. If G' is M-partitionable, then if some vertex of the first copy of G in G' is placed in the ith part, no vertex of the second copy may be placed in the fourth part, so one of the copies is 3-coloured.

Case 2. It is left to deal with the case $y = 1$, and we use the same graph G'. Here however, the argument is that an M-partition of G' can place at most one copy of G in the clique part y. Therefore, G' is M-partitionable if and only if G is 3-colourable.

□

Finally, suppose M is of size $m \times m$ and contains no 1s. As there are no diagonal asterisks, H_M is irreflexive, and Theorem 2.1.3 asserts that if H_M is bipartite and the complement of a circular arc graph, then M-partition can be solved in polynomial time. However, without lists more problems can be solved in polynomial time.

Theorem 2.1.6. *[32] Let M be a matrix with diagonal entries all 0. If H_M is bipartite then the M-partition problem is polynomial time solvable.*

Otherwise the M-partition problem is NP-complete.

2.1.2 Obstructions

When discussing obstructions, we are dealing with the basic matrix partition problem, without lists. Hence we assume all matrices in this Section have no diagonal asterisks. For the function $f_{M,\mathcal{G}}(m)$, in this case $\mathcal{G}$ is the set of all graphs, and we examine different families of matrices, starting with matrices having asterisks in their A or B blocks. Say that a matrix M is *unfriendly* if for some i and j, the entries $M(i,i) = M(j,j) \neq *$, but $M(i,j) = M(j,i) = *$. A matrix that is not unfriendly is called *friendly*. Note that since M is symmetric, a matrix is friendly if and only if both blocks A and B have no asterisks. For a graph G, define the *girth* as the length of a shortest cycle in G, and denote it $g(G)$. If the graph is acyclic, define $g(G) = \infty$. Define the *circumference* of G as the length of a longest cycle in G, if one exists. Otherwise, say the circumference of G is ∞.

Theorem 2.1.7. *[25] If M is unfriendly, then there are infinitely many minimal M-obstructions.*

Proof. Let M be an (A,B,C)-block unfriendly matrix, with k diagonal 0s and ℓ diagonal 1s. Assume without loss of generality that M contains the submatrix $S = \left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$. Otherwise we may consider $\overline{M}$, the complement of M and use Fact 1.4.1.

We start by constructing an infinite family of minimal obstructions for the matrix A, and then from these construct an infinite family of minimal M-obstructions. If $k = 2$, then $A = S$, and a graph G is A-partitionable if and only if it has no odd cycle, so that the odd cycles form an infinite family of minimal A-obstructions. Suppose now that $k > 2$ and define an infinite family of minimal A-obstructions recursively as follows. Recall that Erdös [12] proved the existence of graphs with arbitrarily large chromatic number $\chi(G)$ and odd girth $g(G)$.

Let G_0 be any graph with chromatic number greater than k. Then G_0 is not M-partitionable, so it must contain a minimal A-obstruction G'_0. Since G'_0 has chromatic number larger than $k > 2$, it must have an odd cycle. Let c_0 be the odd circumference of G'_0 - the maximum length of an odd cycle in G'_0.

Assume that $G'_0, G'_1, ..., G'_i$ minimal A-obstructions have been constructed, with associated odd circumference $c_0 < c_1 < ... < c_i$, for $i \geqslant 0$. Let G_{i+1} be a graph with chromatic number greater than k, and odd girth $g(G_{i+1}) > c_i$. Since G_{i+1} is not M-partitionable it contains a minimal obstruction G'_{i+1}, and we have that the maximum length, c_{i+1}, of an odd cycle in G'_{i+1} satisfies $c_{i+1} \geqslant g(G'_{i+1}) \geqslant g(G_{i+1}) > c_i$, since removing a vertex from a graph does not create a new cycle.

Thus the graphs $G'_0, ..., G'_i, ...$ form a sequence of non-isomorphic minimal A-obstructions. Now for each i, the graph obtained by taking the disjoint union of $\ell + 1$ copies of G'_i is not M-partitionable, since at most one copy of G'_i can be placed in a clique part of M. Hence each such union of $\ell + 1$ copies of G'_i contains a minimal M-obstruction G''_i. Further, since removal of a vertex in a graph does not create a cycle, the odd girth of G''_i is at least that of G'_i, and the odd circumference is at most that of G''_i. Thus $\{G''_i\}_0^\infty$ is an infinite family of non-isomorphic minimal obstructions to M-partition. $\square$

Therefore, when dealing with the set of all simple graphs, with the goal of describing matrices M for which the M-partitionable graphs can be characterized by finitely many obstructions, we restrict ourselves to the friendly matrices. In other words, the submatrices A and B have no asterisks, and we focus on C. The first one of these matrix classes is the class of matrices that have no asterisk entries at all. A *homogeneous* set in a graph G is a subset H of the vertex set $V(G)$ such that every vertex not in H is either adjacent to every vertex in H, or not adjacent to every vertex in H. Trivially, a single vertex and the entire graph are homogeneous sets. For two disjoint subsets A and B of $V(G)$, say that A is *homogeneous with respect* to B if A is a homogeneous set in $G[A \cup B]$.

For the rest of this Section, we assume without loss of generality that $k \geqslant \ell$.

Theorem 2.1.8. *[25, 37] If M is an $m \times m$ matrix with no asterisk entries. Then any minimal obstruction for M has at most $2(k+\ell)(k+1)+1$ vertices.*

Proof. Suppose for contradiction that a minimal M-obstruction G has at least $2(k+\ell)(k+1)+2$ vertices, let v be an arbitrary vertex of G and consider an M-partition of $G-v$. Since there are $k+\ell$ parts, some part P must contain at least $2(k+1)+1 = 2k+3$ vertices. Further, since M has no asterisks, P is a homogeneous set in $G-v$, because either all edges or no edges are present between P and any other part of the partition. Therefore, since every vertex is either adjacent or non-adjacent to v, by the pigeonhole principle, G has a homogeneous set H of size at least $k+2$. Note that as $H \subseteq P$, H is either an independent set or a clique. Let $h \in H$ and consider now a partition of $G-h$. If H is an independent set, then with $k \geqslant \ell$, of the $k+1$ remaining vertices of $H-\{h\}$ at least one vertex h' is placed in an independent part P'. With h non adjacent to h' and h having the same neighbourhood in G as h', h can be placed in P' to obtain a partition of G. Similarly, if H is a clique, then at least one of the $k+1$ vertices of $H-\{h\}$ is placed in a clique part and we arrive at a similar contradiction. $\square$

With considerably more work, the bound in Theorem 2.1.8 can be improved.

Theorem 2.1.9. *[19] If M has no asterisk entries, then any minimal obstruction for M has at most $(k+1)(\ell+1)$ vertices. Further, there are at most two minimal M-obstructions with exactly $(k+1)(\ell+1)$ vertices.*

If C is particularly simple, namely has only asterisk entries or no asterisk entries, the converse of Theorem 2.1.7 also holds.

Theorem 2.1.10. *[25] If M is an (A, B, C)-block matrix, and C either has no asterisks or only asterisks, then M has finitely many obstructions if and only if M is friendly.*

Proof. If C has no asterisk entries and M is friendly, this follows from Theorem 2.1.8 or 2.1.9. So suppose M is a matrix in which C has only asterisks. For convenience, say a *labeled* graph G is a graph in which each vertex has a label A or B (but not both), with a vertex labeled A (respectively B) meaning that it must be placed in a part from A (respectively B). A *labeled M-partition* of G is an M-partition of G respecting the labels, and a *minimal labeled M-obstruction* is a labeled graph G that has no labeled M-partition, but for which any proper subgraph, with inherited labels, has a labeled M-partition.

With a labeled graph G as input, let G_1 be the subgraph of G induced on the vertices labeled A and G_2 be the subgraph of G on the vertices labeled B. Note that $V(G_1) \cap V(G_2) = \emptyset$. When C has only asterisk entries, G has no labeled M-partition if and only if G_1 has no A-partition, or G_2 has no B-partition (without labels). Say without loss of generality that G_1 is A-partitionable, so that G_2 does not have B-partition. Since G is a minimal labeled M-obstruction, removing any vertex of G produces a labeled M-partition of G and hence a B-partition for G_2. In particular, removing a vertex labeled A produces such a partition. But this is impossible, and so no vertex of G is labeled A, and we have $G_2 = G$, and G_2 is a minimal B-obstruction.

Now if M is friendly, then A and B have no asterisk entries, and Theorem 2.1.9 then gives us finitely many minimal A-obstructions, of size at most $k+1$, and finitely many minimal B-obstructions, each of size at most $\ell+1$, allowing us to conclude that each minimal labeled M-obstruction is of size $p = k+1$, since $k \geqslant \ell$ by assumption. It is shown in [18] that if a minimal labeled M-obstruction has p vertices, then a minimal M-obstruction has at most $2p^{2r+1}$ vertices, where r is defined as an integer such that any graph that admits both an A-partition and a B-partition has at most r vertices. To see that r exists, let H be a graph that is both A-partitionable and B-partitionable. Then H is k-colourable, and so has no clique of size $k+1$, and its complement is ℓ-colourable and so has no clique of size $\ell+1$. By Ramsey's Theorem, r is the integer for which any graph on at least r vertices contains a $k+1$ clique or the complement of an $\ell+1$ clique.

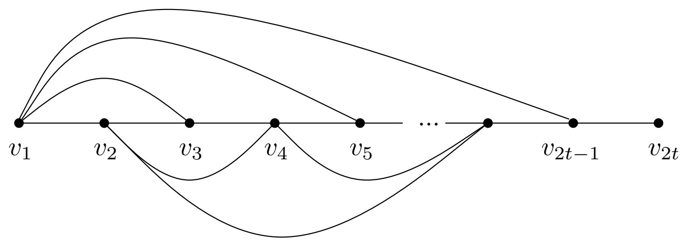
Hence, if $\mathcal{G}$ is the set of all graphs, and $\mathcal{M}_1$ is the set of all friendly matrices whose matrix C has no asterisks, we have $f_{M,\mathcal{G}}(k+\ell) = (k+1)(\ell+1)$ for all $M \in \mathcal{M}_1$ and if $\mathcal{M}_2$ is the set of all friendly matrices whose matrix C only has asterisk entries, we have $f_{M,\mathcal{G}}(k+\ell) \leqslant 2 \cdot (k+1)^{2r+1}$, for all $M \in \mathcal{M}_2$. $\square$

The last class of friendly matrices mentioned here is that of "small" matrices. Specifically, those matrices of size $m \leqslant 5$. For these matrices it is again the case that M has finitely many obstructions if and only if M is friendly [25].

However, in general, it is not true that any friendly matrix M yields a family of M-partitionable graphs that is characterized by finitely many obstructions. Figure 2.1.2 displays a matrix M with an infinite family of M obstructions $G(t)$ [25, 41]. For each natural number t, the graph $G(t)$ consists of a path on $2t$ vertices, $v_1, ..., v_{2t}$. The vertex v_1 is adjacent to every vertex v_i with odd $i > 1$, and all even numbered vertices less than $2t$ form a clique.

$$\begin{pmatrix} 0 & 1 & 0 & * & 0 & 0 \\ 1 & 0 & 1 & * & 0 & 0 \\ 0 & 1 & 0 & 0 & * & * \\ * & * & 0 & 1 & 0 & 1 \\ 0 & 0 & * & 0 & 1 & 0 \\ 0 & 0 & * & 1 & 0 & 1 \end{pmatrix}$$

M



$G(t)$

Figure 2.1.2 – A friendly matrix M and infinitely many min M-obstructions $G(t)$

Finally, as a special case and to highlight their connection to homomorphism problems, consider the set of patterns M with no 1s. Then by Theorem 1.4.2 we are dealing with the homomorphism problem, for which we have the following.

Corollary 2.1.11. *Let M be a matrix with no 1s. If H_M is an independent set, then M has finitely many minimal obstructions.*

Otherwise, M has infinitely many minimal obstructions.

Proof. Here M is in fact the matrix A. If H_M is an independent set, then M has no asterisks. Otherwise, M contains at least one asterisk and is thus unfriendly. □

2.2 Cographs

By examining the cotree representation of a cograph, many combinatorial optimization problems, the decision version of which is NP-complete for general graphs, may be computed efficiently when the input is a cograph. These include counting the number of cliques, the number of maximum cliques, the number of transitive orientations and, of interest here, the chromatic number of a cograph [9]. Since each level of the cotree represents a collection of subgraphs of the graph G, the technique used for the problems mentioned is that of dynamic programming. As this technique is also used for some of the matrix partition results presented later in this Section, we demonstrate the approach here on the colouring problem for cographs.

Say that two vertices of G *meet* at a node x if the paths from their respective leaf representations in the cotree intersect at x for the first time. This gives the following simple fact.

Fact 2.2.1. *[9] Let G be a cograph with cotree T. Two vertices of G are adjacent if and only if they meet at a* 1*-node in T.*

With a cograph G represented by cotree T, recall that for a node x of T, the graph G_x is the subgraph of G whose cotree representation is the subtree of T rooted at x. We collect at each node x, the minimum number of colours k needed to colour the graph G_x. Start by assigning the value 1 to every leaf of T. For a node with r children assume the number of colours needed for each of the children of a given node, $k_1, ..., k_r$, has been computed. For a 1-node, let $k = \sum_i^k k_i$, since all vertices that meet at a 1-node are adjacent in G. For a 0-node, take $k = \max_i^k\{k_i\}$, as vertices meeting at a 0-node are non-adjacent.

Note that the same computation gives the size of the maximal clique in the input graph G. Hence $\chi(G) = \omega(G)$ when G is cograph. Since every subgraph of a cograph is a cograph, cographs are perfect.

2.2.1 Complexity

For the class of cographs, the list M-partition problem for any M can be solved in time linear in n. Specifically,

Theorem 2.2.2. *[20] For any matrix M the list M-partition problem can be solved in time $2^{O(m)}n$, linear in n.*

The proof follows from a key observation about disconnected graphs. For an $m \times m$ matrix M, refer to the integers $1, ..., m$ as *parts,* and given two sets or parts $P, Q \subseteq \{1, ..., m\}$, define $M_{P,Q}$ to be the submatrix of M obtained by taking the rows of P and the columns of Q. Let $M_P = M_{P,P}$ and note that for any $P, Q \subseteq \{1, ..., m\}$, the matrices M_P and M_Q are principal matrices of M, and $M_{P,Q}$ consists of the constraints between parts in P and parts in Q and, if a part appears in $P \cap Q$, a constraint on the part.

The observation needed in proving Theorem 2.2.2 is given in the following Lemma.

Lemma 2.2.3. *[20] Let M be an $m \times m$ matrix, and $G = G_1 \cup G_2$ be a disconnected graph, with lists.*

Then G is an M-obstruction if and only if for any $P, Q \subseteq \{1, ..., m\}$, if $M_{P,Q}$ does not contain a 1, then G_1 (with lists) is an M_P-obstruction, or G_2 (with lists) is an M_Q-obstruction.

Proof. We prove the contrapositive, that G is M-partitionable if and only if there are $P, Q \subseteq \{1, ..., m\}$ such that $M_{P,Q}$ has no 1s, G_1 is M_P-partitionable and G_2 is M_Q-partitionable. Let G be a disconnected graph $G = G_1 \cup G_2$. If G is M-partitionable, let P be the set of nonempty parts containing vertices from G_1 and Q be the set of nonempty parts containing vertices from G_2, so that G_1 is M_P-partitionable, and G_2 is M_Q-partitionable. Further, if a part belongs to $P \cap Q$ it must be a 0 part, since it is not empty, and no vertex of G_1 is adjacent to a vertex of G_2. By similar reasoning $M(p, q) \neq 1$ for any $p \in P$ and $q \in Q$.

Conversely, if G_1 is M_P-partitionable and G_2 is M_Q-partitionable for some $P, Q \subseteq \{1, ..., m\}$ such that $M_{P,Q}$ has no 1s, then G is M-partitionable. Now if G also comes equipped with lists, the argument is mostly unchanged, except that M_P and M_Q partitions conform to the restrictions in the lists, which are assumed to be inherited by G_1 and G_2. □

This allows us to prove Theorem 2.2.2, by working with the cotree T. Associate with each node t of T a set of matrices $\mathscr{M}_t$. This set consists of all matrices M_X, where $X \subseteq \{1, ..., m\}$, such that the graph G_t, corresponding to the graph constructed by taking the subtree of T rooted at t, is an M_X-obstruction.

Proof of Theorem 2.2.2. As the leaves of T represent single vertex cographs, we have $\mathscr{M}_t = \emptyset$ for each leaf t. If t is a 0-node, with children t_1, t_2, then $G_t = G_{t_1} \cup G_{t_2}$ and for a fixed $X \subseteq \{1, ..., m\}$, by Lemma 2.2.3, G_t is M_X-obstruction if and only if for every $P.Q \subseteq X$, such that $M_{P,Q}$ has no 1s, $M_P \in \mathscr{M}_{t_1}$ or $M_Q \in \mathscr{M}_{t_2}$. If t is a 1-node, and $G_t = G_{t_1} \oplus G_{t_2}$, then G_t may also be thought of as $\overline{(\overline{G_{t_1}} \cup \overline{G_{t_2}})}$, so $\overline{G_t} = (\overline{G_{t_1}} \cup \overline{G_{t_2}})$ and we may compute $\overline{\mathcal{M}}_t$ for $\overline{G_t}$ and use Theorem 1.4.1. The graph G is at the root r of T, and $G = G_r$, so that G is M-partitionable if and only $M \notin \mathscr{M}_r$. Each $\mathscr{M}_t$ has at most 2^m members, since there are at most 2^m subsets of $\{1, ..., m\}$, so the running time of this algorithm is $2^{O(m)} \cdot n$. □

2.2.2 Obstructions

For a set of matrices $\mathscr{M}$, say that a graph G is an *$\mathscr{M}$-obstruction* if G is an M-obstruction for every $M \in \mathscr{M}$, and say that G is a *minimal $\mathscr{M}$-obstruction* if G is an $\mathscr{M}$-obstruction but every proper induced subgraph of G admits an M-partition for some $M \in \mathscr{M}$. Now for a fixed matrix M, and a disconnected graph $G = G_1 \cup G_2$, let $\mathscr{M}_1$ be the set of matrices M_P such that $P \subseteq \{1, ..., m\}$ and G_1 is an M_P-obstruction. Similarly, let $\mathscr{M}_2$ be the set of matrices M_Q such that $Q \subseteq \{1, ..., m\}$ and G_2 is an M_Q-obstruction. By definition, G_i is an $\mathscr{M}_i$-obstruction, for $i \in \{1, 2\}$. Suppose that for all choices of $P, Q \subseteq \{1, ..., m\}$ for which $M_{P,Q}$ contains no 1s, either $M_P \in \mathscr{M}_1$ or $M_Q \in \mathscr{M}_2$. Then the Lemma ensures that G is an M-obstruction. Now for $i \in \{1, 2\}$, by definition, G_i is an $\mathscr{M}_i$-obstruction, and so if G_i' is a subgraph of G_i that is also an $\mathscr{M}_i$-obstruction, then the graph $G' = G_1' \cup G_2'$ is an $\mathscr{M}$-obstruction. By contrapositive, we obtain the following corollary to Lemma 2.2.3.

Corollary 2.2.4. *[20] if $G = G_1 \cup G_2$ is a minimal M-obstruction, then G_1 is a minimal $\mathscr{M}_1$-obstruction and G_2 is a minimal $\mathscr{M}_2$-obstruction.*

Note also that if G is a minimal M-obstruction, then M is not in $\mathscr{M}_1$, otherwise G_1 is a proper subgraph of G which is an M-obstruction, contradicting minimality, and similarly $M \notin \mathscr{M}_2$. This allows us to prove an upper bound on $f_{M,\mathcal{G}}(m)$ for all matrices $M \in \mathcal{M}$ and the class $\mathcal{G}$ of cographs. For the remainder of this Section, $\mathcal{M}$ is the set of *all* matrices.

Theorem 2.2.5. *[20] Let $a = \frac{1}{ln(3/2)}$. For all $M \in \mathcal{M}$,*

$$f_{M,\mathcal{G}}(m) \leqslant a^m m!$$

Proof. For convenience let $f(m) = f_{\mathcal{M},\mathcal{G}}(m)$ where $\mathcal{M}$ and $\mathcal{G}$ are as defined. Fix $M \in \mathcal{M}$ and suppose G is a cograph minimal M-obstruction. Since G is a cograph, assume without loss of generality that G is disconnected and $G = G_1 \cup G_2$ (otherwise, $\overline{G}$ is disconnected and we may proceed with $\overline{M}$). By corollary 2.2.4, G_1 is a minimal $\mathscr{M}_1$-obstruction and G_2 is a minimal $\mathscr{M}_2$-obstruction, and the value of $f(m)$ is at most the sum of the sizes of minimal M_1'-obstructions and minimal M_2'-obstruction, with $M_1' \in \mathscr{M}_1$ and $M_2' \in \mathscr{M}_2$. For $j \in \{1, 2\}$, with $|\mathscr{M}_j| \leqslant \sum_{i<m} \binom{m}{i}$, and each $i \times i$ member of $\mathscr{M}_j$ having an obstruction of size at most $f(i)$, we have that

$$f(m) \leqslant 2 \sum_{i<m} \binom{m}{i} f(i)$$

Now, $f(1) \leqslant 2$ and $a^1 \cdot 1! > 2$, so assume inductively that $f(i) \leqslant a^i \cdot i!$. Then,

$$\begin{aligned}
f(m) \leqslant 2 \sum_{i<m} \binom{m}{i} f(i) &= 2 \sum_{i<m} \frac{m!}{i! \cdot (m-i)!} f(i) \\
&\leqslant 2 \sum_{i<m} \frac{m!}{i! \cdot (m-i)!} \cdot a^i \cdot i! \quad \text{by induction hypothesis} \\
&= 2m! \cdot \sum_{i<m} \frac{a^i}{(m-i)!} \cdot \frac{a^{m-i}}{a^{m-i}} \\
&= 2m! \cdot a^m \cdot \sum_{i<m} \frac{1}{(m-i)! \cdot a^{m-i}} \\
&= 2m! \cdot a^m \cdot \sum_{i<m} \frac{1}{i! a^i} \\
&\leqslant 2m! \cdot a^m (e^{1/a} - 1) \\
&= a^m m!
\end{aligned}$$

□

The class of cographs also admits lower bounds on the size of a largest minimal M-obstruction, for $M \in \mathcal{M}$. Consider the colouring matrix $M = C_m$ with m diagonal 0s, no diagonal 1s and asterisks in every other entry.

Theorem 2.2.6. *[20] For every $m \in \mathbb{N}$, there exists a cograph G, with lists, that is a minimal M-obstruction of size $(e - 1 - \epsilon(m)) \cdot m!$ where $1 \geqslant \epsilon(m) \in o(1)$.*

Proof. We construct G recursively. For each $K \subseteq \{1, ..., m\}$, of size k, define a graph $G(K)$, with lists from $\{1, ..., m\}$ such that

(i) For any $S \subseteq \{1, ..., m\}$, $G(K)$ is list M_S-partitionable if and only if $|S| \geqslant |K|$ and $S \neq K$.

(ii) For each $v \in V(G)$, the subgraph $G(K) - v$ is list M_K-partitionable.

For sets K containing a single element $i \in \{1, ..., m\}$, define $G(\{i\})$ as a single vertex v with $L(v) = \{1, ..., m\} - \{i\}$. Then clearly $G(\{i\})$ is M_S-partitionable if and only if $|S| \geqslant |K|$ and $S \neq K$ for any $S \subseteq \{1, ..., m\}$ and the second condition holds trivially.

For $k \geqslant 2$, assume $G(K')$ for $|K| \leqslant k-1$ has already been defined to satisfy properties (i) and (ii), and define $G(K)$ for $|K| = k$ as the disjoint union of all graphs $G(K - \{j\})$ for all $j \in K$, together with an additional vertex v_K with $L(v_K) = \{1, ..., m\}$, and v_K is adjacent to all of the other vertices. That is

$$G(K) = \left(\bigcup_{j \in K} G(K - \{j\}) \right) \oplus \{v_K\}$$

Assuming inductively that each $G(K - \{j\})$ is a cograph shows that $G(K)$ is a cograph.

If $S \subseteq \{1, ...m\}$, then $G(K)$ has a list M_S-partition, with v_K placed in part $j_0 \in S$, if and only if each $G(K - \{j\})$ has a list $M_{S-\{j_0\}}$-partition, since v_K is adjacent to every vertex of $G(K)$. This holds, inductively, if and only if each $|S - \{j_0\}| \geqslant |K - \{j\}|$ and $S - \{j_0\} \neq K - \{j\}$ so that $|S| \geqslant |K|$ and $S \neq K$.

Now if v_K is removed then by induction, each $G(K - \{j\})$ is M_K-partitionable by (i), and if $v \in G(K - \{j_1\})$, for some specified part $j_1 \in K$, is removed, then by (ii) $G(K - \{j_1\}) - v$ is list $M_{K-\{j_1\}}$-partitionable, and by (i) any other $G(K - \{j\})$ is $M_{K-\{j_1\}}$-partitionable, since $K - \{j_1\} \neq K - \{j\}$ when $j \neq j_1$. Finally, v_K may placed in part j_1. Therefore, any proper induced subgraph of $G(\{1, ..., m\})$ with the lists inherited is M-partitionable. Also, as $G = G(\{1, ..., m\})$ satisfies (i), G is not M-partitionable.

If $g(k)$ is number of vertices in $G(K)$, then $g(1) = 1$ and $g(k) = 1 + kg(k-1)$. Therefore,

$$g(m) = \sum_{i=0}^{m-1} \frac{m!}{(m-i)!} = m! \sum_{i=1}^{m} \frac{1}{i!} = m!(e - 1 - \epsilon(m))$$

with $1 \geqslant \epsilon(m) = \sum_{m+1}^{\infty} \frac{1}{i!}$. □

Note that as m grows, $\epsilon(m)$ tends to 0. We have previously defined $f_{\mathcal{M},\mathcal{G}}(m)$ to discuss the size of the largest M-obstruction in $\mathcal{G}$ for a matrix $M \in \mathcal{M}$, without lists. Letting $g(m)$ denote the size of the largest cograph minimal M-obstruction for a matrix $M \in \mathcal{M}$ with accompanying lists, we have

$$\frac{7}{10} m! < (e - 1 - \epsilon(m))m! \leqslant g(m) \leqslant a^m \cdot m!$$

In the case of the M-partition problem without lists, we obtain the following bounds for $f_{\mathcal{M},\mathcal{G}}$. However, we do not have an exponential lower bound for $f_{\mathcal{M},\mathcal{G}}$.

Theorem 2.2.7. *[20] Let $m \in \mathbb{N}$.*

$$\frac{m^2}{4} \leqslant f_{\mathcal{M},\mathcal{G}}(m) \leqslant O(\frac{8^m}{\sqrt{m}})$$

The lower bound is given by a particular subclass of the class of cographs, those graphs that are a disjoint union of cliques. While cographs are the graphs having no induced P_4, the graphs that are a disjoint union of cliques are precisely those graphs with no induced P_3.

In Section 2.1 we have seen that the minimal obstructions for matrices without asterisk entries have at most $(k+1)(\ell+1)$ vertices. For matrices with k 0s, ℓ 1s and asterisks in all other entries (i.e. the matrices describing (k,ℓ)-graphs), the authors in [20] show that any cograph minimal obstruction, for any k and ℓ, has exactly $(k+1)(\ell+1)$ vertices. Further, of these minimal M-obstructions of size $(k+1)(\ell+1)$, exactly one is a disjoint union of cliques.

Theorem 2.2.8. *[20] Let M be an $(*,*,*)$-block matrix, with k diagonal 0s and ℓ diagonal 1s. Then each cograph minimal M-obstruction admits a C_{k+1}-partition as well as a $\overline{C}_{\ell+1}$-partition.*

Proof. If $\ell = 0$, then any minimal M-obstruction is a minimal cograph that is not k-colourable. Since cographs are perfect, the only obstruction is K_{k+1} which admits a C_{k+1}-partition, and is partitionable into $(0+1)$ cliques and so admits a $\overline{C_1}$-partition. Since the complements of cographs are cographs, the case where $k = 0$ follows by complementation. Now suppose inductively that for an $(*,*,*)$-block matrix M' with k' diagonal 0s and ℓ' diagonal 1s, if $k' + \ell' \leqslant k + \ell$, then any minimal M'-obstruction admits a $C_{k'+1}$ and $\overline{C}_{\ell'+1}$ partitions.

We assume that $G = G_1 \cup G_2$ is disconnected, otherwise we can use the complement of G and M. The goal is to construct sets of matrices $\mathscr{M}_1$ and $\mathscr{M}_2$ for corollary 2.2.4. Let j be the smallest natural number such that G_1 has a partition into k independent sets and j cliques. Since G is a minimal M-obstruction and G_2 is not empty, $0 \leqslant j \leqslant \ell$. Since G_1 has a partition into k independent sets and j cliques, G_2 does not have a partition into k independent sets and $\ell - j$ cliques, otherwise G would have an M-partition. Let M_1 be the $(*,*,*)$-block matrix with k diagonal 0s and $j-1$ diagonal 1s, and define $\mathscr{M}_1$ to contain M_1 and all of its submatrices. Similarly, let M_2 be the $(*,*,*)$-block matrix with k diagonal 0s and $\ell - j$ diagonal 1s, and define $\mathscr{M}_2$ to contain M_2 and all of its submatrices.

Now if $P, Q \subseteq \{1, ..., m\}$ are sets of parts such that $M_P \notin \mathscr{M}_1$ and $M_Q \notin \mathscr{M}_2$, then M_P has at least j diagonal 1s, M_Q has at least $\ell - j + 1$ diagonal 1s, so some part $i > k$ lies in both P and Q, and so $M_{P,Q}$ contains a 1. Thus by corollary 2.2.4, G's minimality implies that G_1 is a minimal $\mathscr{M}_1$-obstruction and G_2 is a minimal $\mathscr{M}_2$-obstruction. In particular, G_1 is a minimal M_1-obstruction and G_2 is a minimal M_2-obstruction. By induction hypothesis, G_1 admits a C_{k+1}-partition as well as a $\overline{C}_j$-partition and G_2 admits a C_{k+1}-partition as well as a $\overline{C}_{\ell-j+1}$-partition. Therefore G admits a C_{k+1}-partition and a $\overline{C}_{\ell+1}$-partition. □

Thus every minimal M-obstruction is a collection of $(\ell+1)$ non empty cliques. Now since a clique and independent set can have at most one vertex in common, each clique is of size at most $(k+1)$. In fact, if one of the cliques has at most k vertices, then its members may be placed in the k independent parts of M, and every other clique can be partitioned by placing k of its vertices in the k independent parts of M and the single remaining vertex in a clique part of M. Thus each of the $(\ell + 1)$ cliques consists of exactly $(k + 1)$ vertices and we have the following corollary.

Corollary 2.2.9. *Let M be an $(*, *, *)$-block matrix. Then each cograph minimal M-obstruction has exactly $(k + 1)(\ell + 1)$ vertices.*

Let a cograph G be a minimal M-obstruction. For $i \in \{1, ..., k+1\}$, $j \in \{1, ..., \ell+1\}$, let $v_{i,j}$ be the ith vertex of the jth clique. So any two $v_{i,j}$, $v_{i',j}$ are adjacent. Further, since G is partitionable into $(k+1)$ independent sets, we may arrange the vertices so that no two $v_{i,j}$, $v_{i,j'}$ are adjacent. Every other adjacency $v_{i,j}, v_{i',j'}$ is left unspecified. It is shown in [11] that G is a minimal M-obstruction, regardless of the unspecified adjacencies. In particular, note that the graph $(\ell + 1)K_{k+1}$ is a minimal M-obstruction that is a disjoint union of cliques.

2.3 Chordal Graphs

Another subclass of perfect graphs that has received some attention is the class of chordal graphs. Unlike the class of cographs, discussed in Section 2.2, and the split graphs discussed in Chapter 3, studying chordal graphs yields matrices M for which the list M-partition problem is NP-complete, matrices for which the M-partition problem without lists is NP-complete, and small matrices M for which the M-partitionable chordal graphs cannot be characterized by finitely many minimal obstructions.

2.3.1 Complexity

The following two results describe families of matrices M for which the M-partition problem with lists is NP-complete. Theorem 2.3.2 provides a dichotomy for a family of matrices, when the input is restricted to chordal graphs. It turns out that the constructions involved in Theorems 2.3.1 and 2.3.2 are in fact split graphs. As such, we delay the presentation of the proofs of these results until Section 3.5 where they appear as Theorems 3.5.1 and 3.5.3. For a matrix M and a graph H, say that M *corresponds* to H if M can be obtained from the adjacency matrix of H by replacing every 1 entry with an asterisk.

Theorem 2.3.1. *[14] Let M be an (A, B, C)-block matrix. Let H be a bipartite graph that is not the complement of a circular arc graph, and let Z be the matrix corresponding to H.*

If A does not contain any 1s, B does not contain any 0s, and C is the matrix Z or its complement, then the chordal list matrix partition problem is NP-complete.

Further restricting the matrices A and B provides a dichotomy classification on these types of matrices for chordal graphs. Recall that given a graph H, the matrix M_H corresponds to H.

Theorem 2.3.2. *[14] Let M_H be a $(0,1,C)$-block matrix in which C or its complement corresponds to a bipartite graph H. If H is the complement of a circular arc graph, then the chordal list M_H-partition problem is polynomial. Otherwise, it is NP-complete.*

Even without lists, there are matrices M for which the M-partition problem is NP-complete, for chordal graphs. In [14], Feder, Hell, Klein, and Nogueria prove that there are infinitely many matrices M for which the M-partition problem is NP-complete, without lists, for chordal graphs. Given a bipartite graph H, define the *H-retraction* problem to be a restriction of the list H-colouring (edge preserving mapping with lists) problem to instances G containing H as a subgraph, with lists $L(v) = v$ for $v \in V(H)$, and $L(v) = V(H)$ otherwise. It is known that if H is an even cycle of length at least 6, the H-retraction problem is NP-complete [21]. Concretely, we demonstrate how to obtain a particular matrix for which matrix partition without lists is NP-complete. Fix H to be the cycle on 6 vertices. We now explicitly construct a matrix for which chordal matrix partition is NP-complete. To avoid conflict with existing notation, denote this matrix by $\widehat{M_H}$

Extend H to a larger bipartite graph H', by attaching to each white vertex of H a path of length five and to each black vertex a path of length four. Note that all the leaves (vertices of degree 1) of H' are black. Denote the leaves of H' by L. Denote the black vertices of H' by $V_B(H')$, and the white vertices of H' by $V_W(H')$; let $k := |V_W(H')|$ and $\ell := |V_B(H')|$. When $H \simeq C_6$, the number of vertices in H' is $|V(H')| = 33$, and $|V_B(H')| = k = 18$. See Figure 2.3.1

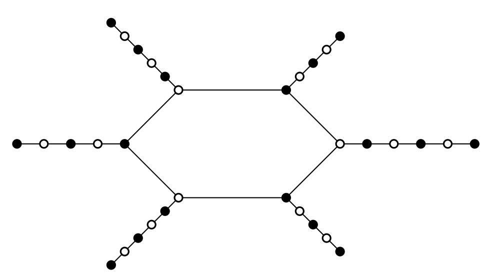


Figure 2.3.1 – The graph H'

To show that $\widehat{M_H}$ is NP-complete, a reduction is performed from the following problem, shown to be NP-complete in [14]. Define the *weak H' retraction* problem as follows: Given a bipartite graph G' containing H', together with a set $X \subseteq V_B(G')$ of size k and satisfying the property that a vertex of G' not in X is adjacent to at most one vertex in X, does G' have an edge-preserving, colour-preserving, mapping of the vertices of G' to the vertices H' such that X is mapped bijectively to $V_B(H')$.

We are ready to define $\widehat{M_H}$. Starting with the 6-cycle H, extend it to H' as above. Let $\widehat{M_H}$ be the (A,B,C)-block matrix in which A is a $k \times k$ matrix, every entry of which is 0; B is an $\ell \times \ell$ matrix with 1s on the main diagonal, and asterisks everywhere else; C is the $k \times \ell$ bipartite adjacency matrix of H'. Thus $\widehat{M_H}$ is a 33×33 matrix.

Theorem 2.3.3. *[14] Let H be the cycle on 6 vertices. The $\widehat{M_H}$-partition problem is NP-complete.*

Proof. Given H, extend it to H'. Then we have that the resulting weak H'-retraction problem is NP-complete, so it suffices to reduce from that. Note that the white vertices of H' correspond to parts of A in $\widehat{M_H}$ and the black vertices of H' correspond to parts of B.

If (G', X) is an instance to the weak H'-retraction problem, with $V_B(H') = \ell$ and $V_W(H') = k$, construct the following graph G'': For each white vertex of G' add an independent set $I(a)$ of size $\ell + 1$. For each black vertex in G' add a clique $K(b)$ of size 2. Whenever a and b are adjacent in G', add edges from all of $I(a)$ to all of $K(b)$. If b, b' are black vertices, not both in X, connect the vertices of $K(b)$ and $K(b')$.

If f' is a weak H-retraction, let a be a white vertex. $f'(a)$ is mapped to a white vertex v_i since f' is colour preserving. Add $I(a)$ to the part V_i. Similarly, a black vertex b is mapped to a black vertex v_j, so that $K(b)$ can be added to the corresponding V_j. Since f' is edge preserving, the relationship between each V_i, V_j is maintained in M_H: a and b are adjacent in G', if and only if $f'(a)$ and $f'(b)$ are adjacent in H' and the corresponding parts of M_H have all the edges between them.

On the other hand, if G'' admits an M_H-partition, then for any black vertex b at least one vertex of $K(b)$ must be mapped into a part V in B since A is an all-zero matrix. Define $f'(b)$ to be the vertex of H' corresponding to V. Further, since each independent set corresponding to a white vertex a has $\ell + 1$ vertices, but there are ℓ clique parts in B, at least one vertex from $I(a)$ is placed in a part V' of A. Define $f'(a)$ to be the vertex of H' corresponding to V'.

If two black vertices b, b' are in X then since $K(b), K(b')$ have no edges between them, those vertices of $K(b)$ and $K(b')$ placed in parts of B were placed in distinct parts of B. So f' maps X bijectively to $V_B(H')$. If a and b are adjacent vertices then $I(a)$ and $K(b)$ have all edges between them, so that the parts corresponding to $f'(a)$ and $f'(b)$ have all edges between them. i.e. the corresponding (a, b) entry of M_H is 1 and since this is the adjacency matrix of H', $f(a)$ is adjacent to $f(b)$. Replace non-adjacent with adjacent in the preceding argument to obtain that f' is edge preserving.

To show that G'' is chordal, consider $I(a)$ for a white vertex a. The vertices in $I(a)$ are adjacent to at most one $K(b)$ with $b \in X$ (by definition of X), so that for all $b_1, ...b_r$ adjacent to a, the cliques $K(b_i)$ are all mutually adjacent and so form a clique, with $1 \leqslant i \leqslant r$. Ordering the vertices of $I(a)$ in any order creates a partial simplicial elimination ordering. Doing this for every $I(a)$ leaves a union of cliques which clearly has a simplicial elimination ordering. □

Remark. For k at least 2, X has at least two vertices so that G'' contains a copy of $2K_2$. Thus G'' is not a split graph.

On the other hand, if M has a constant diagonal, or the C block of M has a special structure, then the chordal M-partition problem is polynomial time solvable.

Theorem 2.3.4. *[14] Let M be an $m \times m$ matrix. If $k = 0$ or $\ell = 0$ then the chordal list M-partition problem can be solved in polynomial time.*

In particular, if $\ell = 0$, the chordal list M-partition problem can be solved in time $O(nk(2k)^k)$.

If $k = 0$, the chordal list M-partition problem can be solved in time $O(n^{2\ell+d})$ for some constant d.

Say a matrix C is *crossed* if every non-asterisk entry in C belongs to a row or column of non-asterisk entries. Say that an (A, B, C)-block matrix M is *crossed* if its C matrix is crossed. Examples include any matrix in which C has no asterisk entries and

$$\left(\begin{array}{cc} \begin{matrix}0 & * \\ * & 0\end{matrix} & \overset{C}{\begin{bmatrix} * & 0 \\ 0 & 1\end{bmatrix}} \\ \underset{C^T}{\begin{bmatrix} * & 0 \\ 0 & 1\end{bmatrix}} & \begin{matrix}1 & 0 \\ 0 & 1\end{matrix}\end{array}\right)$$

When C is crossed, the sparse-dense technique introduced in Theorem 1.3.3 yields a polynomial time algorithm.

Theorem 2.3.5. *[14] Suppose M is a crossed (A, B, C)-block matrix. Then the chordal list M-partition problem can be solved in time $O(n^{k\ell})$.*

Proof. Let $\mathcal{A}$ be the class of chordal graphs that admit and A-partition, and $\mathcal{B}$ be the class of chordal graphs that admit a B partition. Both $\mathcal{A}$ and $\mathcal{B}$ are closed under taking induced subgraphs. Further, If $G_1 \in \mathcal{A}$ and $G_2 \in \mathcal{B}$, then $G_1 \cap G_2 \in \mathcal{A} \cap \mathcal{B}$. Therefore, $G_1 \cap G_2$ is coverable by ℓ cliques, each of size at most k. It follows that $G_1 \cap G_2$ has at most $c := k \cdot \ell$ vertices. By Theorem 1.3.3, there are at most n^{2c} choices for a partition of G into induced subgraphs G_A, G_B such that G_A admits an A-partition and G_B admits a B-partition. As in corollary 1.3.4, for each such partition we modify the lists $L(v)$ into lists $L'(v)$ for every $v \in V(G)$ such that G with the original lists is M-partitionable if and only if G with the new lists is M^*-partitionable for a related matrix M^*, defined below. Specifically, the modified lists will be used to represent the constraints of the crossed matrix C. Given one partition G_A and G_B, for each non-empty part of A, choose at most one vertex of G_A, and for each non-empty part of B choose one vertex of G_B. Call these vertices *representative* vertices, and note that there are at most $(n+1)^{k+\ell}$ of them. We perform the following modification for each choice of representative. Assume the parts of M are numbered $\{1, ..., k, k+1, ..., k+\ell\}$, and let $1 \leqslant i \leqslant k + \ell$. For each vertex $v \in V(G)$, call the modified list $L'(v)$.

If part i is empty, delete it from $L(v)$ for every $v \in V(G)$. Otherwise, If $i < k+1$, delete part i from the list $L(v)$ for every $v \in V(G_B)$. If $i \geqslant k+1$, delete part i from every list $L(v)$

such that $v \in V(G_A)$. Let x_i be the representative of part i. Delete from $L(x_i)$ all parts other than i. For every neighbour y of x_i, delete from $L(y)$ every part j (possibly $i = j$) such that $M(i,j) = 0$; similarly, for every non-neighbour y of x_i, delete from $L(y)$ every part j (possibly $i = j$) such that $M(i,j) = 1$. It is left to represent to constraints of C.

If all entries of some row i in C are non-asterisks, define for $r \in \{0,1\}$ the sets J_r of parts j in B such that $C(i,j) = r$. Let $y \in V(G_B)$. If y is a neighbour of x_i, delete all of J_0 from $L(y)$. Otherwise, delete all of J_1 from $L(y)$. Note that since row i of C is assumed to contain only non-asterisk entries, then if y is a neighbour of x_i, $L'(y)$ is contained in J_1; if y is a non-neighbour of x_i, then $L'(y) \subseteq L_0$. Now suppose $z \in V(G_A)$. If some neighbour $y \in V(G_B)$ of z has $L(y) \subseteq J_0$, delete part i from $L(z)$. If some non-neighbour $y \in V(G_B)$ of z has $L(y) \subseteq J_1$, delete part i from $L(z)$. If all entries of some column j in C are non-asterisks, we proceed in a fashion similar to the case above, exchanging the roles of A and B, and of rows and columns.

Let M^* be the matrix obtained from M by replacing all entries of C with asterisks. Then G admits an M-partition respecting the original lists, if and only if G admits an M^*-partition with the modified lists, for some choice of representatives. Now G admits an M^*-partition respecting the modified lists if and only if it can be partitioned into two subgraphs $G_A \in \mathcal{A}$ and $G_B \in \mathcal{B}$. This can be determined in polynomial time due to Theorems 2.3.4 and 1.3.3. $\square$

2.3.2 Obstructions

When all off diagonal entries of M are asterisks, we are dealing with recognition of (k,ℓ)-graphs. For this we have the following.

Theorem 2.3.6. *[31] A chordal graph G is a (k,ℓ)-graph if and only if it does not contain $(\ell+1)K_{k+1}$.*

Thus when $\mathcal{M}$ is the class of all square matrices whose off-diagonal entries are all asterisks, and $\mathcal{G}$ is the class of cographs, we have $f_{\mathcal{M},\mathcal{G}}(k+\ell) = (k+1)(\ell+1)$

However, even for small matrices there exist infinite families of chordal minimal obstructions. Let $M_1 = \begin{pmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 1 \end{pmatrix}$. For $t \geqslant 3$, let $G_1(t)$ consist of a path on $2t$ vertices, together with one more special vertex adjacent to all but the endpoint vertices of the path. Let $G_2(t)$ consist of a path on $2t$ vertices, together with two special vertices labeled $2t+1$ and $2t+2$. Add an edge in $G_2(t)$ from every $2t+1$ to every other vertex by $2t$, and an edge from $2t+2$ to every other vertex but 1. See Figure 2.3.2. In [24], the authors prove that each $G_1(t)$ for $t \geqslant 3$ is a chordal minimal obstruction to both M_1 and M_2. In [15], all chordal minimal obstructions for M_1 and M_2 are described, and it is proved that each $G_2(t)$ is a chordal minimal obstruction to M_2.

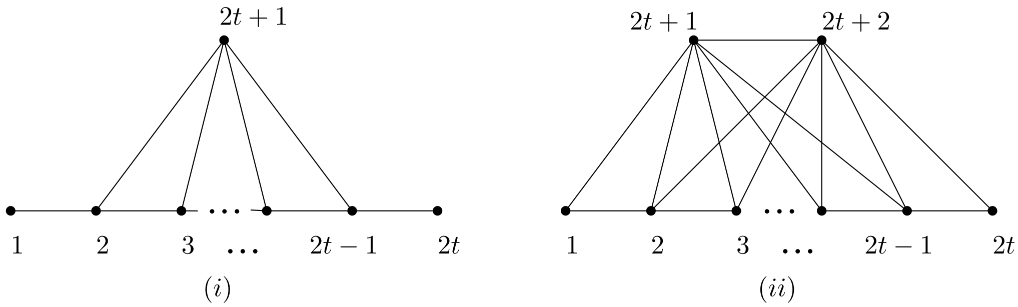


Figure 2.3.2 – Two infinite families of chordal minimal obstructions.

Chapter 3

New Results

This Chapter focuses on the matrix partition problem when the input graph is already known to admit some other partition. For example, if M is an arbitrary matrix, we discuss the M-partitionability of graphs that admit a split partition, $\left(\begin{smallmatrix} 0 & * \\ * & 1 \end{smallmatrix}\right)$, graphs that admit a bipartition, $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$, etc.

In Sections 3.1 to 3.3 we focus on split graphs. There are two main results. Firstly, we show that every matrix M has finitely many split minimal obstructions in Section 3.1, and present an efficient algorithm for deciding the M-partition problem when M is any matrix and the input graph is a split graph. Secondly, we exhibit matrices which have exponentially large split obstructions in Section 3.3. At the time of writing, this is the first known minimal obstruction of exponential size, in any graph class.

Section 3.4 focuses on graphs that admit other types of partitions, such as bipartite graphs and (k, ℓ)-graphs. Here we argue that any matrix has finitely many bipartite and co-bipartite minimal obstructions. Further, when lists are considered, it is possible to find matrices for which the list matrix partition problem is NP-complete, even for split graphs. These aspects are discussed in Section 3.5.

3.1 Matrix Partitions of Split Graphs.

Here we prove that for any matrix M, there are finitely many split minimal obstructions. This is surprising, as the only other graph class for which this is known to be true is the class of cographs. In contrast, we have seen in Section 2.3.2 that for the class of chordal graphs, which strictly contains the class of split graphs, there exist small matrices for which there are infinitely many chordal minimal obstructions. We then present an efficient algorithm for solving the M-partition problem for a fixed matrix M. This algorithm builds upon the crossed matrix algorithm of Theorem 2.3.5.

Theorem 2.1.8 from Section 2.1, restated below for convenience, gives a class of matrices having finitely many minimal obstructions, in the class of all graphs. For the remainder of this Section, we assume that the matrix M has k diagonal 0s, ℓ diagonal 1s, and that $k \geqslant \ell$.

In the case where $k < \ell$, we may interchange the roles of k and ℓ in what follows.

Theorem 2.1.8 *[19] If M is an $m \times m$ matrix with no asterisk entries and $k \geqslant \ell$, then any minimal obstruction for M has at most $2(k+\ell)(k+1)+1$ vertices.*

Some of the ideas used in the proof of Theorem 2.1.8 can be modified to accommodate split graphs. Recall that a subset H of the vertex set $V(G)$ is *homogeneous* if every vertex not in H is either adjacent to every vertex in H, or non-adjacent to every vertex of H. The argument of Theorem 2.1.8 uses the existence of a large enough homogeneous set in any graph that admits an M-partition. More precisely, vertices placed in any part of M form an independent or clique homogeneous set, so a large enough M-partitionable graph will have a large homogeneous set. This follows from the fact M has no asterisks. We shall use a similar idea to prove the following main result of this Section, noting that we may assume that the matrix C has no asterisks.

Theorem 3.1.1. *For any matrix M, there are finitely many split minimal M-obstructions.*

Specifically, we shall prove an upper bound on the function $f_{M,\mathcal{G}}(k+\ell)$, where M is any matrix, $\mathcal{G}$ is the class of split graphs, and k, ℓ denote the number of diagonal 0s and 1s, respectively, in M. Recall that $f_{M,\mathcal{G}}(k+\ell)$ is defined as the size of the largest minimal M-obstruction in $\mathcal{G}$. We begin with the following fact.

Fact 3.1.2. *Let M be an (A, B, C)-block matrix and let G be a split graph. If C has an asterisk entry, then G admits an M-partition.*

Proof. If C has an asterisk, then M contains the matrix $\left(\begin{smallmatrix} 0 & * \\ * & 1 \end{smallmatrix}\right)$ as a principal submatrix. Thus G admits this partition by definition. □

Thus, we may assume that C contains no asterisks when proving Theorem 3.1.1. In this case, every part in A is homogeneous with respect to every part in B, but we may have an asterisk between two parts in A or two parts in B. We prove the following for parts of A, with the result for parts of B following by complementation.

Proposition 3.1.3. *Let A be a $k \times k$ matrix whose diagonal entries are all zero. Let G_A be a split graph that admits an A-partition. Then every part P of an A-partition of G_A contains a homogeneous set in G_A of size at least $\frac{|P|-1}{2^{k-1}}$.*

Proof. Suppose the parts of the A-partition of G_A are $P_1, \ldots P_k$. Let $C \cup I$ be a partition of $V(G_A)$ into a clique C and independent set I. Note that for $1 \leqslant i \leqslant k$, we have that $|P_i \cap C| \leqslant 1$, since each P_i is an independent set. Now, the vertices in the set $P_1 \cap I$ are non-adjacent to all but at most $k-1$ vertices, one in each $P_i \cap C$, for $2 \leqslant i \leqslant k$ (see Figure 3.1.1). Assume without loss of generality that $|P_i \cap C| = 1$ and let $u_i \in P_i \cap C$, for $2 \leqslant i \leqslant k$. As each u_i is either adjacent to at least half of the vertices of $P_1 \cap I$, or non-adjacent to at least half of the vertices of $P_1 \cap I$, a homogeneous set of size at least $\frac{|P_1|-1}{2^{k-1}}$ can be found in $|P_1|$. Since this argument may be repeated for any other part in the partition, we have the desired conclusion.

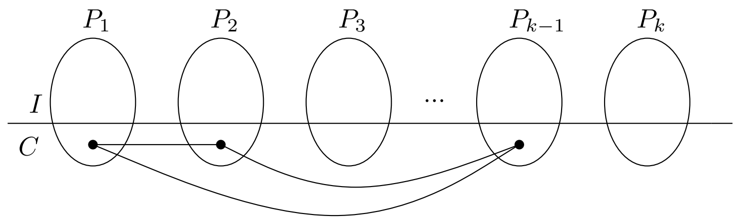


Figure 3.1.1 – Structure of a k-partite split graph

□

Corollary 3.1.4. *Let B be an $\ell \times \ell$ matrix whose diagonal entries are all 1. Let G_B be a split graph that admits a B-partition. Then every part P of a B-partition of G_B contains a homogeneous set in G_B of size at least $\frac{|P|-1}{2^{\ell-1}}$.*

Proof. Recall that the graph G_B admits an M-partition if and only if $\overline{G_B}$ admits an $\overline{M}$-partition. The matrix $\overline{M}$ is an $\ell \times \ell$ matrix with diagonal entries all 0, and $\overline{G_B}$ is a split graph, by the note above Theorem 1.3.2. Letting P be a part of $\overline{M}$, we have that by Proposition 3.1.3, $\overline{G_B}$ contains a homogeneous set of size at least $\frac{|P|-1}{2^{\ell-1}}$. As the complement of a homogeneous set is homogeneous, this set is homogeneous in G_B as well. □

Proposition 3.1.3 allows us to bound the value of $f_{M,\mathcal{G}}(m)$, proving Theorem 3.1.1.

Theorem 3.1.5. *Let $\mathcal{G}$ be the class of split graphs. For $k, \ell \in \mathbb{N}$, suppose $k \geqslant \ell$. Then for any matrix M without diagonal asterisks we have,*

$$f_{M,\mathcal{G}}(k+\ell) \leqslant 2^{k-1}(k+\ell)(2k+3)+1.$$

Proof. Let $M \in \mathcal{M}$ be an (A, B, C)-block matrix and assume without loss of generality that C contains no asterisks. We claim that any minimal obstruction has at most $2^{k-1}(k+\ell)(2k+3)+1$ vertices. Assume for contradiction that G is a minimal obstruction with at least $2^{k-1}(k+\ell)(2k+3)+2$ vertices. Pick an arbitrary vertex v and consider a partition of the graph $G-v$ on at least $2^{k-1}(k+\ell)(2k+3)+1$ vertices. As there are $k+\ell$ parts in the partition, by the pigeonhole principle there is a part, call it P, of size at least $2^{k-1}(2k+3)+1$. This part P is either an independent set or a clique, and each of these cases will be considered separately below. Either way, by Proposition 3.1.3 P contains a homogeneous set in A or B (depending on whether P is an independent set or a clique) of size at least $\frac{|P|-1}{2^{k-1}} \geqslant 2k+3$, and since C has no asterisks, this set is homogeneous in G. Thus $G-v$ has a homogeneous set of size at least $2k+3$, and so G has a homogeneous set H of size at least $k+2$, since by the pigeonhole principle at least $k+2$ of the vertices of P agree on v. Now let $w \in H$, consider a partition of $G-w$, and recall that P is either an independent set or a clique.

Case 1. If P is an independent set, then so is H; hence, there are at least $k+1$ independent vertices in $G - w$. As there are $\ell \leqslant k$ clique parts in the partition of $G - w$, and no two independent vertices of H may be placed in the same clique part, at least one vertex $w' \in H - \{w\}$ must be placed in an independent part P'. Since w is not adjacent to w' and both vertices belong to H, w can be added to P', contradicting the minimality of G.

Case 2. If P is a clique then $H - w$ is a clique of size at least $k + 1$, and so in the partition of $G - w$, at least one vertex of $H - w$ falls in a clique part P'. As in Case 1, w can be added to P', contradicting minimality. □

Having shown that every matrix M has finitely many split minimal obstructions, we obtain a polynomial time algorithm for the M-partition problem. However, a more efficient algorithm is obtained in what follows. Recall that a matrix M is *crossed* if each non-asterisk entry in its block C belongs to a row or column in C of non-asterisk entries. For chordal graphs, Theorem 2.3.5 tells us that if M is a crossed matrix, then the list M-partition for chordal graphs can be solved in polynomial time. Since split graphs are chordal, the same result applies for split graphs, and we can use this to solve the M-partition problem for split graphs in polynomial time.

Theorem 3.1.6. *If G is a split graph, then for any matrix M the M-partition problem can be solved in time $O(n^{k\ell})$*

Proof. Let M be an (A, B, C)-block matrix, and G be a split graph. If C contains an asterisk, then the matrix $\left(\begin{smallmatrix} 0 & * \\ * & 1 \end{smallmatrix}\right)$ is a submatrix of M so G admits an M-partition. Otherwise, as there are no asterisks in C, every non-asterisk entry belongs to a row or column of non-asterisk entries. That is, C is trivially crossed, and since split graphs are chordal, we apply Theorem 2.3.5, with the list $L(v)$ containing all parts of M, for every vertex v of G. The bound $O(n^{k\ell})$ is obtained from Theorem 2.3.5. □

3.2 Matrices with Constant Diagonal

Examining an arbitrary matrix M, as in the previous Section, provides a rough upper bound on the size of a largest split minimal obstruction for M. On the other hand, by restricting the matrices so that all diagonal entries have the same value $d \in \{0, 1\}$, we expose some of the structure found in a split minimal obstruction.

Let $\mathcal{M}_0$ be the class of matrices having all diagonal entries zero, and let $\mathcal{M}_1$ be the class of matrices having all diagonal entries one. Define $\mathcal{M}_c = \mathcal{M}_0 \cup \mathcal{M}_1$ as the class of matrices with *constant* diagonal. When $M \in \mathcal{M}_c$, we can describe the structure of minimal M-obstructions in more detail. Since the complements of split graphs are split graphs, and a matrix $M \in \mathcal{M}_0$ if and only if its complement $\overline{M} \in \mathcal{M}_1$, we may focus on $\mathcal{M}_0$, obtaining complementary results for the obstructions to matrices in $\mathcal{M}_1$.

Suppose $M \in \mathcal{M}_0$, so that $m = k$ and $\ell = 0$. If all of the off diagonal entries of M are asterisks then M is the colouring matrix C_k. Since split graphs are perfect, the only split minimal M-obstruction is K_{k+1}. If $M \neq C_k$, then for some $r \leqslant k+1$, K_r is a minimal obstruction (the value of r depends on the number and arrangement of off diagonal 0s in M), and any other split minimal obstruction G has clique number $k' \leqslant r-1 \leqslant k$.

Let K be a clique in a graph G. For a selection of $s \leqslant |K|$ vertices $k_1, ..., k_s$ from K, define the satellite set $S(\{k_1, ..., k_s\})$ as the set of vertices of $G - K$ whose neighbourhood in K consists exactly of the vertices $k_1, ..., k_s$. For example, the satellite $S(\emptyset)$ consists of vertices not adjacent to any vertex in K, the satellite $S(\{k_1\})$ each consists of vertices only adjacent to the vertex k_1, and so on. Note that the satellites of K partition $V(G)$, and that if K is of size $\omega(G)$, then $S(K)$ must be empty. We are now ready to state the main Theorem of this Section.

Theorem 3.2.1. *Let $M \in \mathcal{M}_0$ and G be a split graph. Let K be a maximum clique in G. If G is a minimal M-obstruction, then for any $X \subsetneq K$, the set $S(X)$ contains at most one vertex.*

The main idea of the proof to follow is that two vertices u and v, belonging to the same satellite set of a maximum clique in a split minimal obstruction must have a vertex that distinguishes them. Therefore they cannot be placed in the same part of any M-partition. However if u and v are distinguished in such a way, then the neighbourhood of one of them, say u, is small enough that, in a partition if $G - u$, we can find at least one other part in which to place u, contradicting the minimality of G. The details of this approach follow.

We first show that distinct vertices belonging to (possibly distinct) satellite sets interact in a very limited way.

Lemma 3.2.2. *Let G be a split graph with maximum clique K. Suppose $X, Y \subsetneq K$ and let $x \in S(X)$ and $y \in S(Y)$. If $x \sim y$, then $X \subsetneq Y$ or $Y \subsetneq X$. Further, the larger of X and Y is of size $|K| - 1$.*

Proof. Suppose $x \sim y$. Then $X \neq Y$. Suppose otherwise, and note that either $|X| = |K|-1$, or $|X| \leqslant |K| - 2$. If $|X| = |K| - 1$ then $\{x, y\} \cup K$ forms a clique of size $|K| + 1$, but $|K|$ is assumed to be a maximum clique. If $|X| \leqslant |K|-2$, then there are two vertices $w, z \in K - X$ such that $\{x, y\} \not\sim \{w, x\}$. But then $G[x, y, w, z] \simeq 2K_2$. Thus $X \neq Y$. Now if $Y \nsubseteq X$, then Y is not empty, and if $X \nsubseteq Y$, then X is not empty. Let $w \in X - Y$ and $z \in Y - X$. Then $w \not\sim y$ and $z \not\sim x$. But this gives a C_4 in $G[x, y, z, w]$. Thus $X \subsetneq Y$ or $Y \subsetneq X$. Assume without loss of generality that $Y \subsetneq X$. Further, assume for contradiction that $|X| \leqslant |K|-2$. Then there are at least two vertices $w, z \in K - X$, so that $x \not\sim \{w, z\}$. If both $w \notin Y$ and $z \notin Y$, then $y \not\sim \{w, z\}$ and so $G[x, y, w, z] \simeq 2K_2$. Thus at least one of w, z must be in Y. But since $w \notin X$ and $z \notin X$, this contradicts the assumption that $Y \subsetneq X$. $\square$

Note that the Lemma states in particular, that no two vertices of any $S(X)$ are adjacent.

We now continue with the proof of the main Theorem, in two large steps described broadly as follows. First suppose u and v are two members of some satellite set $S(X)$ for a maximum clique in a split graph G. Lemma 3.2.2 allows us to argue that if v has a neighbour that is a non-neighbour of u, then u has a small neighbourhood: u is not adjacent to any vertex in G, other than the vertices of A. Using this, in the second step we show that if $G - u$ admits an M-partition, so does G.

Proposition 3.2.3. *Let G be a split graph with maximum clique K. Suppose $W \subsetneq K$ such that there are two vertices $u, v \in S(W)$ with the property that v has a neighbour in $V(G) - K$ that is not a neighbour of u. Then $N_G(u) = W$.*

Proof. Let u, v, W be as in the statement of the Theorem. Let x be the vertex in $V(G) - K$ that is adjacent to v but non-adjacent to u, and suppose $x \in S(X)$ for some $X \subsetneq K$. By definition, we have that $W \subseteq N_G(u)$. Assume for contradiction that $W \neq N_G(u)$. Then there is a vertex y of $V(G) - K$ such that $y \sim u$. Say $y \in S(Y)$ for some $Y \subsetneq K$. Then G contains an induced cycle of length at least four. To show this, we must consider two cases . Note that by Lemma 3.2.2, u cannot be adjacent to v.

Case 1. Suppose $|W| = |K| - 1$. Let w be a vertex of W, adjacent to u and v but non-adjacent to y. The existence of w follows from the assumption that $u \sim y$ since by Lemma 3.2.2 the size of $|W|$ implies that $Y \subsetneq W$. Then v cannot be adjacent to y, or else $G[u, w, v, y] \simeq C_4$. Therefore $x \sim y$, otherwise $G[v, x, u, y] \simeq 2K_2$. Thus $u \not\sim \{v, x\}$ and $y \not\sim \{v, w\}$ and so G contains an induced C_4 if $w \sim x$ or an induced C_5 otherwise.

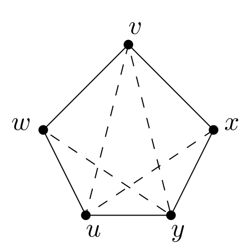


Figure 3.2.1 – Proposition 3.2.3, Case 1

Case 2. Suppose $|W| \leqslant |K| - 2$. Then $x \not\sim y$ since, by Lemma 3.2.2 and the assumption that $x \sim v$ and $y \sim u$, we have $|X| = |Y| = |K| - 1$. Thus $|Y| = |X|$ or $Y \not\subseteq X$ and $X \not\subseteq Y$. Either way, by Lemma 3.2.2 $x \not\sim y$ and therefore $v \sim y$, else $G[v, x, u, y] \simeq 2K_2$. Further, Lemma 3.2.2 also gives $W \subsetneq X$ and $W \subsetneq Y$ so that $W \subseteq X \cap Y$. In fact, $W = X \cap Y$. Otherwise, suppose there exists a vertex $w \in (X \cap Y) - W$. Then w is adjacent to both x and y but not to v. But then $G[x, w, y, v] \simeq C_4$. As this is impossible, $W = X \cap Y$. Now suppose $w_1 \in X - W$. Then w_1 cannot be adjacent to y, otherwise w_1 belongs to Y and so $w_1 \in X \cap Y = W$. Similarly, if $w_2 \in Y - W$, then w_2 is not adjacent to x. Therefore $w_1 \sim w_2$, otherwise $G[x, w_1, y, w_2] \simeq 2K_2$. Finally, as w_1, w_2 are both not in W, $v \not\sim \{w_1, w_2\}$. Thus $G[x, w_1, w_2, y, v] \simeq C_5$.

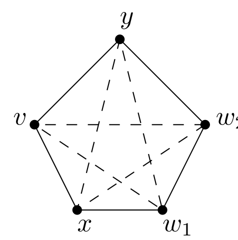


Figure 3.2.2 – Proposition 3.2.3, Case 2

Since G is a split graph, it is chordal, and thus cannot contain an induced cycle of length greater than three. Thus u cannot be adjacent to vertices in $V(G)-K$, so $N_G(u)=W$. $\square$

The fact that the vertex u of Proposition 3.2.3 is so limited in its neighbourhood, allows us to distinguish a part P in a matrix partition of $G-u$, so that u can be placed in P. Recall that given a numbering of the parts of a matrix M,$P_1,P_2,...,P_m$, we refer to the entry at row i and column j of M by $M(P_i,P_j)$.

Proposition 3.2.4. *Let G be a split graph with maximum clique K. Suppose $W \subsetneq K$ such that there are two vertices $u,v \in S(W)$ with the property that v has a neighbour in $V(G)-K$ that is not a neighbour of u.*

Suppose M is a matrix whose diagonal entries are all 0. If $G-u$ is M-partitionable, then G is M-partitionable.

Proof. Suppose $x \in S(X)$ for some $X \subsetneq K$. By Lemma 3.2.2 and the fact that $x \sim v$, either $X \subsetneq W \subsetneq K$, or $W \subsetneq X \subsetneq K$. Either way, there exists a vertex $r \in K-(X \cup W)$, non-adjacent to all three vertices u,v, and x. Suppose $G-u$ is M-partitionable and let P_r be the part containing r in an M-partition of $G-u$. We claim that u can be placed in P_r. Indeed, u is not adjacent to any vertex in P_r, since by Proposition 3.2.3 u is adjacent only to members of W, while $r \notin W$, and r is the only vertex of K in P_k. Further, there is no part P such that $M(P,P_r)=0$ and P contains a neighbour of u, since all the neighbours of u belong to $W \subsetneq K$, and no member of K can be placed in P as these are all adjacent to r. Finally, we might be prevented from placing u in P_r if there is a part P for which $M(P,P_r)=1$ but P contains a vertex y that is a non-neighbour of u. Suppose that this is the case, and say that $y \in S(Y)$ for some $Y \subsetneq K$. We show that G contains an induced subgraph isomorphic to C_5.

Start by noting that y must be adjacent to r, as $M(P,P_r)=1$; hence, $r \in Y$. Recall that by Lemma 3.2.2, we have either $W \subsetneq X$ or $X \subsetneq W$. Say without loss of generality that $X \subsetneq W$ (otherwise, exchange the roles of x and v in what follows). Thus $|X| \leqslant |K|-2$ and $|W|=|K|-1$ so that $W \nsubseteq Y$. Now since $r \in Y-W$, it must be that $Y \nsubseteq W$. Therefore, by Lemma 3.2.2, $v \nsim y$. This in turn requires that $x \sim y$; otherwise, $G[v,x,r,y] \simeq 2K_2$. Thus we have that $G[v,x,y,r]$ is an induced path on four vertices. We now find a vertex r' that is adjacent to $\{v,r\}$ and non-adjacent to $\{x,y\}$. This vertex r' can be found in $W-Y$. To see this, note that $x \in X$ and $y \in Y$ are adjacent, while $|X| \leqslant |K| - 2$. Lemma 3.2.2 then gives $|Y| = |K| - 1$ and $X \subsetneq Y$. As X is also a proper subset of W, we have $X \subseteq W \cap Y$. Now with $|W| = |Y| = |K| - 1$, and $W \neq Y$, these sets differ by exactly one vertex. That is, there is a vertex $r' \in W - Y$ such that $Y = (W - r') \cup \{r\}$. Thus r' is not adjacent to y, and since $X \subseteq W \cap Y$, r' cannot be in X and is thus not adjacent to x. As r' belongs to W, it is adjacent to v and since both r and r' are members of the clique K, they are adjacent. This gives an induced cycle on five vertices in $G[r', v, x, y, r]$. □

Theorem 3.2.1 now follows from Proposition 3.2.4.

Proof of Theorem 3.2.1. Suppose G is a split minimal obstruction containing a clique K of maximum size, and for contradiction suppose $u, v \in S(A)$ for some $A \subsetneq K$. Then $G - u$ is M-partitionable but u cannot be added to the part P containing v. Since by Lemma 3.2.2(c) $u \not\sim v$, and P is an independent set, this must be because u and v disagree on some vertex $x \in V(G) - K$. Say $u \not\sim x$ and $v \sim x$. Then by Proposition 3.2.4, since $G - u$ is M-partitionable, G is M-partitionable, contradicting the minimality of G. □

With $\mathcal{M}_0$, defined earlier as the class of matrices with all diagonal entries equal to zero, we obtain the following corollary of Theorem 3.2.1.

Corollary 3.2.5. *Let $\mathcal{G}$ is the class of split graphs. For any $M \in \mathcal{M}_0$, we have*

$$f_{M,\mathcal{G}}(k) \leqslant 2^k - 1 + k.$$

Proof. Let $M \in \mathcal{M}_0$, and G be a split minimal M-obstruction. Let K be a maximum clique in G. By Theorem 3.2.1, each satellite of K contains at most one vertex. The satellite $S(K) = \emptyset$, since K is a maximum clique. Thus there are at most $2^k - 1$ satellites, each containing at most one vertex. As $|K| \leqslant k$, we have that $|V(G)| \leqslant 2^k - 1 + k$. □

Recall that from Section 3.1 we have that for every matrix M from the class of all matrices $\mathcal{M}$, having size $m \times m$ with $m = k + \ell$ and $k \geqslant \ell$, the size of a minimal M-obstruction is at most $2^{k-1}(k + \ell)(2k + 3) + 1$. Thus, the upper bound for $f_{M,\mathcal{G}}(m)$ with $M \in \mathcal{M}_0$ is strictly smaller.

3.3 A Special Class of Matrices

Further restricting the entries of the matrix allows for an explicit construction of a particularly interesting split minimal obstruction. The obstruction obtained in this way is of exponential size in the dimensions of the matrix. As seen in Chapter 2, most known upper bounds for the the size of minimal obstructions are exponential (e.g. for cographs it is $a^m \cdot m!$, where $a > 2$ and m is the size of the matrix). Further, we note that Theorem 2.2.6 provides an exponential lower bound for the size of the largest cograph minimal obstruction to the list M-partition problem. The exponential size split construction of this Section thus provides the first known exponential lower bound for the matrix partition problem, without lists. In contrast, we show in [15] that the largest cograph minimal obstruction for the same matrix M, is of quadratic size in the dimensions of M.

Let M be a $k \times k$ matrix with all diagonal entries zero, and all off diagonal entries of in $\{*, 1\}$. Note that the graph representation of M can be considered as a simple graph in this case, with asterisk entries interpreted as non-edges. Formally, let M' be the matrix obtained from M by replacing every asterisk entry of M' by a zero, and let H_M be the graph with adjacency matrix M'. We characterize some of the split minimal M-obstructions for matrices M of this form. When H_M is particularly simple, this characterization provides a method of constructing split minimal M-obstructions. Specifically, if H_M has exactly one edge, all M-obstructions are explicitly constructed; if H_M consists of star and a stable set, an M-obstruction is constructed that is of exponential size in k. This is the first known minimal obstruction of exponential size, for any class of graphs for which there are finitely many minimal M-obstructions, without lists. We start with some preliminary information on M-partitionable split graphs. For any $m \times m$ matrix M, say that part i is an indpendent part if $M(i,i) = 0$ and that part i is a *clique part* if $M(i,i) = 1$, with $1 \leqslant i \leqslant m$.

Fact 3.3.1. *If two independent parts i and j of a matrix M have $M(i,j) = 1$, then in any M-partition of a chordal graph G, at least one of these parts is of size at most* 1.

Proof. If both parts have at least two vertices, a, b in part i and c, d in part j, then $G[a, c, b, d] \simeq C_4$. □

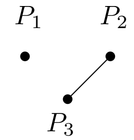
Lemma 3.3.2. *Let G be a split graph with $\omega(G) = m$. Then there is a clique K of size m in G for which $\bigcup_{X \subseteq K} S(X)$ is an independent set.*

Proof. Let K be a clique in G. If $\bigcup_{X \subseteq K} S(X)$ is already an independent set there is nothing to prove, so suppose $u, v \in \bigcup_{X \subseteq K} S(X)$ and $u \sim v$. By Lemma 3.2.2, $u \in S(Y)$, $v \in S(X)$ with $|X| = m - 1$ and $Y \subsetneq X \subsetneq K$, so that u is adjacent to at most $m - 2$ vertices of K. Now in a partition of G into a clique C and independent set S, $|K \cap S| \leqslant 1$. If u were placed in C, however, at least two vertices of K must be placed in S, so u must be in S implying that v belongs in C, as u and v are adjacent. Therefore the single vertex k of K that v is not adjacent to must be in S and C forms a clique of size m whose union of satellite sets is equal to S. □

We now restrict our attention even further, to matrices M for which the graph H_M is partitionable into a star graph and an independent set. For $k, t, \mathbb{N}$, with $1 \leqslant t \leqslant k - 1$, let $M_{k,t}$ be a $k \times k$ matrix with diagonal entries all zero, t 1s in row k, symmetrically, t 1s in column k and asterisks everywhere else. By permuting the rows and columns of $M_{k,t}$ we assume without loss of generality that the 1 entries of row k are in columns $m - t, ..., m - 1$ and symmetrically, that the 1 entries of column k are in rows $k - t, ..., m - 1$. See Figure 3.3.1 for example matrices.

$$\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{pmatrix} 0 & \ast & \ast \\ \ast & 0 & 1 \\ \ast & 1 & 0 \end{pmatrix} \end{array}$$

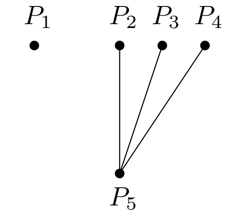
$M_{3,1}$



$H_{M_{3,1}}$

$$\begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{pmatrix} 0 & \ast & \ast & \ast & \ast \\ \ast & 0 & \ast & \ast & 1 \\ \ast & \ast & 0 & \ast & 1 \\ \ast & \ast & \ast & 0 & 1 \\ \ast & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

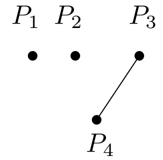
$M_{5,3}$



$H_{M_{5,3}}$

$$\begin{array}{c} \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ \begin{pmatrix} 0 & \ast & \ast & \ast \\ \ast & 0 & \ast & \ast \\ \ast & \ast & 0 & 1 \\ \ast & \ast & 1 & 0 \end{pmatrix} \end{array}$$

$M_{4,1}$



$H_{M_{4,1}}$

Figure 3.3.1 – Matrices $M_{k,t}$ and graph representations, $k \in \{3, 4, 5\}, t \in \{1, 3\}$

Of special interest are the matrices $M_{k,1}$, for $k \in \mathbb{N}$. When $k = 3$, there are infinitely many chordal minimal $M_{3,1}$-obstructions [24]. In contrast, the largest split minimal $M_{k,1}$-obstruction is of quadratic size in k, for any $k \in \mathbb{N}$. Specifically, we have the following.

Theorem 3.3.3. *If $\mathcal{G}$ is the class of split graphs then*

$$f_{M_{k,1},\mathcal{G}}(k) \leqslant \tfrac{k^2+k+2}{2}.$$

The proof of Theorem 3.3.3 follows from a characterization of $M_{k,1}$-partitionable split graphs.

Theorem 3.3.4. *Let G be a split graph with maximum clique K of size k.*

Then G is $M_{k,1}$-partitionable if and only if there exist vertices $u, v \in K$ such that $S(K - \{u\}) = S(K - \{u, v\}) = \emptyset$.

Proof. $\Rightarrow$ If G is $M_{k,1}$-partitionable then every part of the partition contains exactly one vertex of K. Let P and P' be the parts of the partition such that $M_{k,1}(P, P') = 1$. Then one of these parts, say P, is of size exactly one containing a single vertex u of K. It must be then that $S(K - \{u\}) = \emptyset$. Further, if v is the vertex of K placed in P', then a vertex in $S(K - \{u, v\})$ must be placed in either P or P'. As $M_{k,1}(P, P') = 1$, this is impossible, so $S(K - \{u, v\}) = \emptyset$.

$\Leftarrow$ Let K be a clique as in Lemma 3.3.2, let u, v be members of K satisfying $S(K - \{u\}) = S(K - \{u, v)\} = \emptyset$, and let P_u and P_v be the parts of $M_{k,1}$ for which $M_{k,1}(P_u, P_v) = 1$. We produce an $M_{k,1}$-partition of G. Start by placing u into P_u and v into P_v; distribute the remaining $m - 2$ vertices of K into the remaining parts of M arbitrarily, one vertex into each part. It is left to deal with vertices of G that are not in K. For $w \in K$, refer to the part containing w as P_w, and consider $x \in S(X)$ for some $X \subsetneq K$. If $\{u, v\} \subseteq X$, then there is at least one vertex $w \in K - A$. Place x in P_w. If $u \in X$ but $v \notin X$, place x in P_v, and if $u \notin X$ but $v \in X$, then since $S(K - \{u\}) = \emptyset$, we have $X \neq K - \{u\}$ so there is at least one other vertex $w \in K$ non adjacent to x, and x can be placed in P_w. Finally, if u, v both do not belong to X, then as $S(K-\{u,v\})=\emptyset$, we have $A \neq K-\{u,v\}$ so a vertex $w \in K$ exists that x is not adjacent to. $\square$

We state the negation of Theorem 3.3.4 as a corollary.

Corollary 3.3.5. *A split graph G is an $M_{k,1}$-obstruction if and only if for every pair of vertices $u,v \in K$, either $S(K-\{u\}) \neq \emptyset$ or $S(K-\{u,v\}) \neq \emptyset$.*

This allows us to find an upper bound for the largest split minimal $M_{k,1}$-obstruction.

Proof of Theorem 3.3.3: Fix $k \in \mathbb{N}$. A split graph G of maximum clique size at most $k-1$ is $M_{k,1}$-partitionable and so cannot be an obstruction. On the other hand, the graph K_{k+1} is a split minimal $M_{k,1}$-obstruction having $k+1 \leqslant \frac{k^2+k+2}{2}$ vertices. Thus we consider minimal $M_{k,1}$-obstructions of maximum clique size k. For $s \geqslant 0$, let G_s be a split graph containing a clique K of size $\omega(G_s)=k$, such that K has s vertices $u_1, ...u_s$ for which $S(K-\{u_1\})=S(K-\{u_2\})=...=S(K-\{u_s\})=\emptyset$. If G_s is a minimal $M_{k,1}$-obstruction, then by Corollary 3.3.5, G_s must have exactly $k-s$ vertices $v_1,...,v_{m-s}$ for which each $S(K-\{v_i\})$ is not empty, together with $\binom{s}{2}$ pairs of vertices for which $S(K-\{u_i,u_j\}) \neq \emptyset$, and $s(k-s)$ pairs of vertices for which $S(K-\{u_i,v_j\}) \neq \emptyset$. This gives a total of $k+(k-s)+\binom{s}{2}+s(k-s)$ vertices. This value is maximized when $s=k-2$:

- If $s=k$, then
$$k+(k-s)+\binom{s}{2}+s(k-s)=\frac{k(k+1)}{2}$$
- If $s=k-1$, then
$$k+(k-s)+\binom{s}{2}+s(k-s)=\frac{k^2+k+2}{2}$$
- If $s=k-2$ then
$$k+(k-s)+\binom{s}{2}+s(k-s)=\frac{k^2+k+2}{2}$$
- If $s=k-r$ where $3 \leqslant r \leqslant m$, then
$$k+(k-s)+\binom{s}{2}+s(k-s)=\frac{k^2+k-r(r-3)}{2} \leqslant \frac{k(k+1)}{2}$$

$\square$

Figure 3.3.2 lists the graphs G_s for different values of $s \in \{0,...,4\}$, when $m=4$. Note that G_3 is not a minimal $M_{4,1}$-obstruction.

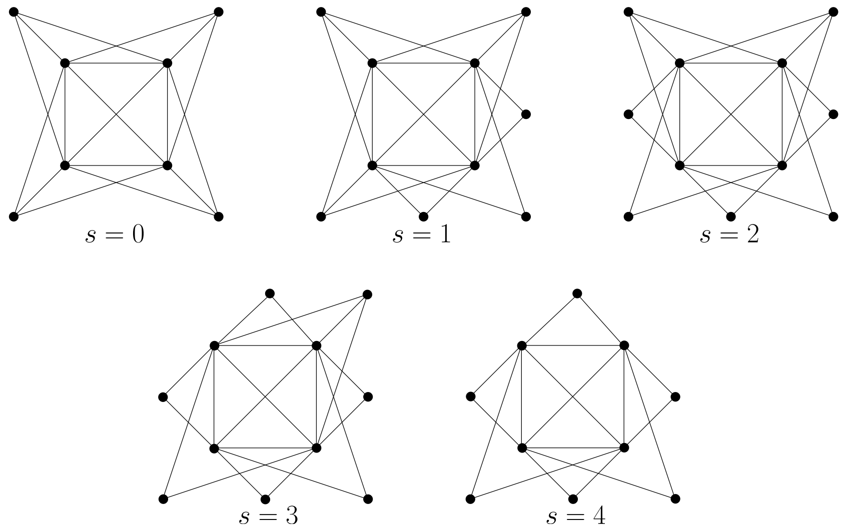


Figure 3.3.2 – Obstructions for $M_{k,1}$, with $m = 4$.

We now exhibit certain split minimal obstructions for a matrix in the family matrices $M_{k,t}$, for a particular value of t. These obstructions will be of exponential size, in the dimensions of the matrix.

Theorem 3.3.6. *There exist $k, t \in \mathbb{N}$ such that for the matrix $M = M_{k,t}$,*

$$f_{M,\mathcal{G}}(k) \geqslant \left(\pi \tfrac{k-1}{2}\right)^{-\frac{1}{2}} \cdot 2^{k-1} + 2k - 1$$

Proof. We choose values of k and t so that the matrix $M_{k,t}$ has a minimal split obstruction of the desired size. Let $k = 2n+1$ for some $n \in \mathbb{N}$, so that the matrix $M_{k,t}$ has $2n+1$ parts. Let $t = n$, place 1s in row $2n+1$ and columns $n, n+1, ...2n$ and symmetrically in column $2n+1$ and rows $n, n+1, ..., 2n$. The part in row and column $2n+1$ is designated P, and designate the n parts that have a 1 to P as *restricted parts*, $R_1, ..., R_n$ and the remaining n parts as *unrestricted parts*, $U_1, ..., U_n$. See Figure 3.3.3.

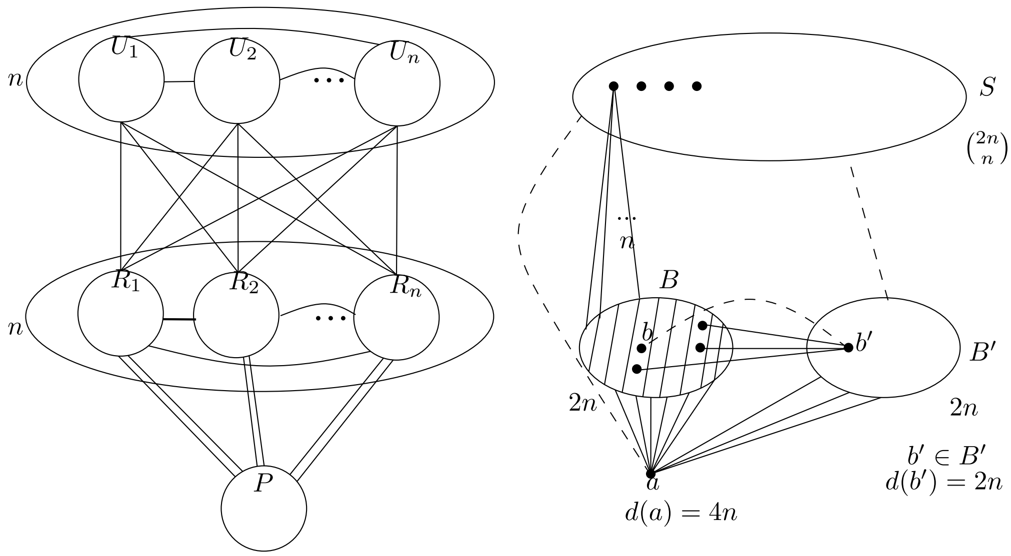


Figure 3.3.3 – The matrix $M_{2n+1,n}$ (left) and an obstruction G (right)

The minimal obstruction G, depicted in Figure 3.3.3, has a *special vertex* a, and $2n$ vertices forming a clique B, all of which are adjacent to a (so that $B \cup \{a\}$ is a clique of size $2n+1$); another $2n$ vertices forming an independent set B' such that for each $b \in B$ there is a $b' \in B'$ that is not adjacent to b but adjacent to every other vertex of $B \cup \{a\}$. Call b and b' *mates*. Finally, G has an independent set S of size $\binom{2n}{n}$ such that for every subset $\tilde{B}$ of B of size n, there is exactly one vertex $s \in S$ adjacent to exactly the vertices of $\tilde{B}$. Note that G is a split graph since $B \cup \{a\}$ is a clique and $B' \cup S$ is an independent set, as seen in Figure 3.3.4.

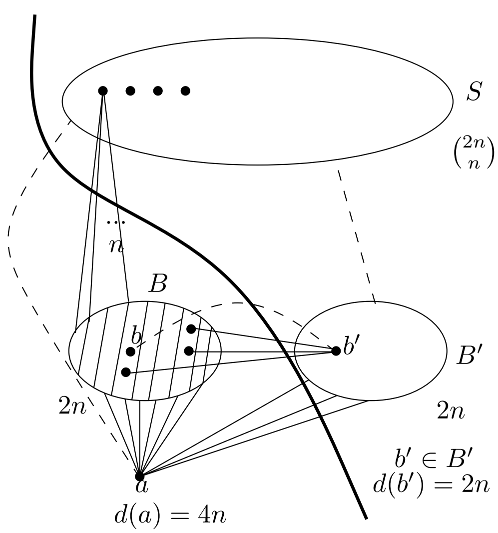


Figure 3.3.4 – A split partition for G.

To see that G is indeed an obstruction, suppose otherwise, and note that $B \cup \{a\}$ is a clique of size $2n+1$, so each of these vertices must be placed in a different part. Since each vertex of B has a mate in B' that is adjacent to a and all of the other vertices in B, all parts other than the part containing a have size at least two in any $M_{k,t}$-partition of G. Thus only the part containing a may be a singleton. Further P must be the only singleton part, otherwise all of the restricted parts must be singletons. Therefore $a \in P$. Now whichever n vertices of B are placed in the unrestricted parts, as in Figure 3.3.5, there is a vertex $s \in S$ adjacent to exactly these vertices, and so must be placed into one of the restricted parts. But as s is not adjacent to a, it cannot be placed in a restricted part, and s can't be added to P; hence, G is not $M_{k,t}$-partitionable.

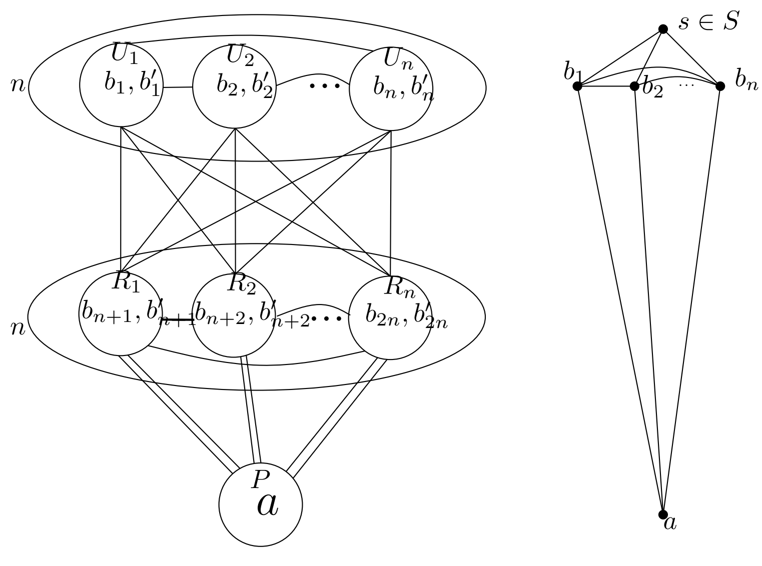


Figure 3.3.5 – An attempt to partition G.

To argue that G is a minimal obstruction, we show that removing a vertex from one of S, B, B', or $\{a\}$ allows a partition for the resulting graph:

(i) For $s \in S$ partition $G - s$ as follows: map a to P, place each $b \in B$, together with its mate $b \in B'$, in some part, taking care that neighbours of the missing s are placed in unrestricted parts. Now each remaining vertex of S has an unrestricted part to go to.

(ii) We consider $b \in B$ together with its mate $b' \in B'$. For $G - b$, place a in P, place b's mate b' in an unrestricted part $P_{b'}$, and place all of S and all of B' in $P_{b'}$. This is possible since $B' \cup S$ form an independent set. place the remaining $2n - 1$ vertices of B to the remaining $2n - 1$ parts arbitrarily. To partition $G - b'$, place b in P, and place a together with all of the vertices of S in an unrestricted part P_a, and place each other pair of mates v, v' from B and B' into a part, different from P and P_a.

(iii) Finally, $G - a$ can be partitioned using the restricted and unrestricted parts only, not placing anything in P. Place each b and its mate b' into a part. Each $s \in S$ is only forbidden from n out of the 2n parts and so can be placed somewhere.

Now G has $2k-1+\binom{k-1}{\frac{k-1}{2}} = 4n+1+\binom{2n}{n}$ vertices, and using Stirling's approximation, we get

$$\frac{2^{k-1}}{\sqrt{\pi\frac{k-1}{2}}} = \frac{2^{2n}}{\sqrt{\pi n}} \leqslant \binom{2n}{n} \leqslant \frac{2^{2n}}{\sqrt{\pi n}}\left(1-\frac{c}{n}\right) = \frac{2^{k-1}}{\sqrt{\pi\frac{k-1}{2}}}\left(1-\frac{c}{n}\right), \text{where } \frac{1}{9} < c < \frac{1}{8}$$

So that G is of size exponential in m. □

Recall that for a class of matrices $\mathcal{M}$, we define $f_{\mathcal{M},G}(m) = sup\{f_{M,\mathcal{G}}(m)|M \in \mathcal{M}\}$. In Section 3.1, we found that $f_{\mathcal{M},\mathcal{G}}(k+\ell) \leqslant 2^{max\{k,\ell\}-1}\cdot(k+\ell)\cdot(2\cdot max\{k,\ell)+3)+1$, where $\mathcal{M}$ is the class of all matrices with no diagonal asterisks. Since for any $k,t \in \mathbb{N}$ the matrix $M_{k,t}$ is a member of $\mathcal{M}$ we obtain nearly tight bounds in the following Corollary.

Corollary 3.3.7. *Let $\mathcal{M}$ be the class of all matrices with no diagonal asterisks, and $\mathcal{G}$ be the class of split graphs. For any $m \in \mathbb{N}$, with $m = k+\ell$, and $k \geqslant \ell$. Then,*

$$\left(\pi\frac{k-1}{2}\right)^{-\frac{1}{2}}\cdot 2^{k-1}+2k-1 \leqslant f_{\mathcal{M},\mathcal{G}}(m) \leqslant 2^{k-1}\cdot(k+\ell)\cdot(2\cdot k+3)+1$$

Finally, we contrast these results for split graphs with the fact that, for the class of cographs, the same matrices $M_{k,t}$ with $k,t \in \mathbb{N}$ have minimal obstructions of fairly simple structure and small size. We state the main results here; see [15] for the details.

Theorem 3.3.8. *The cograph minimal obstructions for $M_{k,t}$ are either K_{k+1}, or have the form $(K_{r_1} \cup K_{k-t}) \oplus (K_{r_2} \cup K_{k-t}) \oplus ... \oplus (K_{r_\ell} + K_{k-t})$ where $r_1 + ... + r_\ell = k$ and each $r_1 \geqslant k-t+1$ for $i \leqslant i \leqslant \ell$.*

Therefore, the largest cograph minimal obstruction to $M_{k,t}$ has size $O(k^2)$.

3.4 Generalized Split Graphs.

In this Section we examine some (k,ℓ) graphs, other then $(1,1)$-graphs. We start by discussing $(2,0)$ and $(0,2)$ graphs, of which any matrix M has only finitely many minimal obstructions. We then examine (k,ℓ) graphs when $k+\ell \geqslant 3$, exhibiting matrices for which there are infinitely many (k,ℓ) minimal obstructions. The families of graphs in these constructions are chordal, except for the cases in which $k \leqslant 1$. It is therefore natural to discuss (k,ℓ)-chordal graphs, when $k \leqslant 1$.

3.4.1 Bipartite or Co-bipartite Graphs

Let G be a bipartite graph. In the language of matrices, G admits an $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$-partition. Thus any matrix containing $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$ as a submatrix does not have any bipartite minimal obstructions. For the remainder of this Section, let M be an (A,B,C)-block matrix in which A has no asterisks. We proceed with an approach similar in nature to that used in Section 3.1. That is, given an M-partition for a bipartite graph G, we argue for the existence of a large homogeneous set in one of the parts of the M-partition, and use this set to bound the size of the largest bipartite minimal M-obstruction. The details follow.

Proposition 3.4.1. *Let M be an (A, B, C)-block matrix, with A of size $k \times k$ and B of size $\ell \times \ell$. Suppose the block A has no asterisk entries. If G is an M-partitionable bipartite graph, then any part P of A in an M-partition of G, contains a homogeneous set of size at least $\frac{|P|}{2^{2l}}$.*

Proof. Fix a bipartition of G and let P be a part of A in an M-partition of G. As A has no asterisks, the vertices of P all have the same adjacency relation to vertices in other parts of A. Let P' be a part of B. Since G is bipartite, P' can have at most two vertices, one from each part of the bipartition of G. Let these vertices be x and y. Then by the pigeonhole principle x is either adjacent or non-adjacent to at least half of the vertices of P', call the set of these vertices P'', and y is either adjacent or non-adjacent to at least half of the vertices of P' and we call the set of these vertices P'''. Note that $|P'''| \geqslant \frac{|P|}{2^2}$. Now there are $\ell - 1$ more clique parts, each of size at most two. Inductively, we obtain a homogeneous set in P of size at least $\frac{|P|}{2^{2\ell}}$. □

Theorem 3.4.2. *Let $m \geqslant 1$, and M be an (A, B, C)-block, $m \times m$ matrix. M has finitely many bipartite minimal obstructions*

Proof. As discussed, we may assume that A contains only entries in $\{0, 1\}$. We show that any bipartite minimal obstruction has at most $2^{2\ell}(k+\ell)(2\ell+3)$ vertices. Suppose otherwise, and let G be a minimal obstruction with at least $2^{2\ell}(k + \ell)(2k + 3) + 1$ vertices. For an arbitrary v, $G - v$ is M-partitionable, and so some part P in an M-partition of $G - v$ contains at least $2^{2\ell}(2\ell+3)$ vertices. Since $2^{2\ell}(2\ell+3) \geqslant 3$ for $\ell \geqslant 0$, and no clique part of M may contain more than two vertices, P must be an independent set. Thus by Proposition 3.4.1, P contains a homogeneous set of size at least $\frac{|P|}{2^{2\ell}} \geqslant 2\ell + 3$, and so by the pigeonhole principle, G has an (independent) homogeneous set H of size at least $\ell + 2$. Let $h \in H$, and consider a partition of $G - h$. As there are only ℓ cliques and $\ell + 1$ vertices in $H - \{h\}$, there must be a part P' of A that contains a vertex h' of $H - \{h\}$. But since H is an independent set and h has the same neighbourhood as h', we may add h to P', thus obtaining a partition for G, a contradiction. □

By complementation, a similar result can be proved for $(0, 2)$-graphs.

Theorem 3.4.3. *Let $m \geqslant 1$, and M be an (A, B, C)-block, $m \times m$ matrix. M has finitely many co-bipartite minimal obstructions.*

Proof. If the block B of M contains asterisk entries, then any co-bipartite graph admits an M-partition. Assume that B contains no asterisks, and let G be a co-bipartite graph. The complement matrix $\overline{M}$ is a $(\overline{B}, \overline{A}, \overline{C})$-block matrix in which $\overline{B}$ only has entries in $\{0, 1\}$. As $\overline{M}$ has finitely many bipartite minimal obstructions, M has finitely many co-bipartite minimal obstructions. □

Remark. Interestingly, if a matrix M does contain $\left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)$ as a submatrix, it is unfriendly; hence, there are infinitely many minimal obstructions to M-partition, unrestricted to any graph class [25]. By Theorem 3.4.2, none of these obstructions is bipartite.

3.4.2 (k,ℓ)-graphs with $k+\ell \geqslant 3$

For the matrix $M = \left(\begin{smallmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{smallmatrix}\right)$, the family of graphs depicted in Figure 2.3.2 is an infinite family of chordal minimal obstructions [24]. We redefine $G(t)$ here for convenience. Let $t \geqslant 3$. The graph $G(t)$ consists of a path on $2t$ vertices, and a special vertex u. The vertex u is adjacent to all vertices of the path except its endpoints. Note that $G(t)$ is not bipartite.

Proposition 3.4.4. *Let $t \in \mathbb{N}$. Each $G(t)$ is 3-colourable and is partitionable into a bipartite graph and a clique.*

Proof. The graph $G(t)$ consists of an even path on $2t$ vertices, which can be coloured using two colours. The special vertex u of $G(t)$ may be coloured using a third colour. Further, this last colour class contains a single vertex and so may be considered a clique. □

Corollary 3.4.5. *The matrix $M = \left(\begin{smallmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{smallmatrix}\right)$ has infinitely many $(2,1) \cap (3,0)$ minimal obstructions. The matrix $\overline{M}$ has infinitely many $(1,2) \cap (0,3)$ minimal obstructions.*

This allows us to prove the following.

Theorem 3.4.6. *If $k, \ell \in \mathbb{N}$ such that $k+\ell \geqslant 3$, then there exists a matrix M that has infinitely many (k,ℓ) minimal obstructions.*

Proof. Let $k, \ell \in \mathbb{N}$ such that $k+\ell \geqslant 3$. Let $M = \left(\begin{smallmatrix} 0 & * & * \\ * & 0 & 1 \\ * & 1 & 0 \end{smallmatrix}\right)$. If $k \leqslant 1$ then $\ell \geqslant 2$, and every member of the family of graphs $\{\overline{G(t)} | t \geqslant 3\}$ is a minimal $\overline{M}$-obstruction that is also a (k,ℓ)-graph. If $k \geqslant 2$, then every member of the family of graphs $\{G(t) | t \geqslant 3\}$ is a minimal M-obstruction that is also a (k,ℓ)-graph. □

Note that as shown in [24], the graphs $G(t)$ are chordal. This gives the following Corollary.

Corollary 3.4.7. *Let $k, \ell \in \mathbb{N}$, such that $k+\ell \geqslant 3$. If $k \geqslant 2$, then there exists a matrix M that has infinitely many (k,ℓ)-chordal minimal obstructions.*

On the other hand, the complement of any $G(t)$, with $t \geqslant 3$ is not a chordal graph, as the edges at either end point of the path on $2t$ vertices in $G(t)$ form a $2K_2$. Thus the graph $\overline{G(t)}$ contains an induced subgraph isomorphic to C_4. While Theorem 3.4.6 gives infinitely many (k,ℓ) minimal obstructions when $k \leqslant 1$, we do not know whether there are infinitely many (k,ℓ)-chordal minimal obstructions when $k \leqslant 1$.

Conjecture 3.4.8. *Let $k, \ell \in \mathbb{N}$, such that $k+\ell \geqslant 3$. If $k \leqslant 1$, then there exists a matrix M that has infinitely many (k,ℓ)-chordal minimal obstructions.*

3.5 Algorithmic Aspects

In Section 3.1 it is shown that every split M-partition problem is polynomial time solvable. However, when dealing with list M-partition problems, we exhibit in this Section matrices M for which the list M-partition problem is NP-complete, even for split graphs.

In [14], the authors described how to obtain infinitely many matrices M for which the list M-partition problem is NP-complete even for chordal graphs. We shall review these proofs and conclude that for the matrices M provided, list M-partition is NP-complete even for split graphs. In addition, an infinite family of matrices M' is provided for which M'-partition (without lists) is NP-complete for chordal graphs. We reproduce the results here in further detail, noting that many of the algorithmic results for split graphs follow from them.

Let H be a fixed bipartite graph with bipartition into white and black vertices, and let M be obtained from the adjacency matrix of H, by replacing each occurrence of a 1 by an asterisk. The matrix obtained in this way is an (X, Y, Z)-block matrix, with the X and Y blocks consisting entirely of 0s and corresponding to the white vertices and black vertices of H, respectively, and the off diagonal matrix Z describing adjacency between the white and black vertices of H - the (i, j) entry of C is $*$ if vertices i and j are adjacent in H, and is 0 otherwise. Say that Z *corresponds* to H. For a fixed H, the proof constructs a particular matrix M_H related to H and provides reduction from the list H-colouring problem to the list M_H-partition problem. Thus a suitable graph H for which the list H-colouring problem is NP-complete will produce a matrix M_H for which list M_H-partition is NP-complete. The graph constructed in the reduction is a split graph, and is therefore chordal.

Theorem 3.5.1. *[14] If H is a bipartite graph for which list H-colouring is NP-complete, then there is a matrix M_H for which list M_H-partition is NP-complete, even for split graphs.*

Proof. Let M_H be an (A, B, C)-block matrix with diagonal asterisks allowed such that entries of A are restricted to $\{0, *\}$, the entries of B are restricted to $\{1, *\}$ and C is the matrix Z corresponding to H. Note that each part P_v of M_H corresponds to vertex v of H. Further, since H is bipartite, we may assume that any input G is bipartite, and that the white vertices of G have lists corresponding to white vertices of H, and that the black vertices of G have lists corresponding to black vertices of H. Given a bipartite graph G as input to the list H-colouring problem, let G' be the graph obtained by from G by adding all edges between black vertices, thus producing a clique. The lists of G' are the same as the lists of G. If G admits a list H-homomorphism f, let $P_v = f^{-1}(v)$ for every $v \in H$. Now white vertices are mapped to white parts and A has no 1s. Similarly, black vertices are mapped to black parts and B has no 0s. Thus $P_v = f^{-1}(v)$ is an M_H-partition if every part of A satisfies the appropriate C constraint to every part of B. Because f is a homomorphism, this is in fact the case. On the other hand, a list M_H partition of G' describes a list homomorphism of G to H by letting $f^{-1}(v) = P_v$ for every $v \in H$, and the same argument applies. □

Note that the graph G' produced in the construction is a split graph, since the black vertices of G' form a clique and the white vertices of G' form an independent set.

It is known that if a bipartite graph H is the complement of a circular arc graph, then the list H-colouring problem is polynomial time solvable, and otherwise it isNP-complete [21]. Further, if we define the bipartite complement of G as the graph obtained by only exchanging edges and non-edges between the white and black vertices of G, then we may obtain the same result for the matrix M'_H defined as $(A, B, \overline{C})$, where $\overline{C}$ is the complement of the matrix Z. In other words, we have the following Corollary -

Corollary 3.5.2. *Let M be an (A, B, C)-block matrix, with diagonal asterisks allowed. Let H be a bipartite graph that is not the complement of a circular arc graph, and let Z be the matrix corresponding to H.*

If A does not contain any 1s, B does not contain any 0s, and C is the matrix Z or its complement, then the split (and hence chordal) list matrix partition problem is NP-complete.

Further restricting the matrices A and B provides a dichotomy on these types of matrices for chordal graphs.

Theorem 3.5.3. *[14] Let M_H be an $(0, 1, C)$-block matrix in which C or its complement corresponds to a bipartite graph H. If H is the complement of a circular arc graph, then the chordal list M_H partition problem is polynomial. Otherwise, it is NP-complete.*

Proof. Assume without loss of generality that C, rather than $\overline{C}$, corresponds to H. If H is not cocircular arc, then by Corollary 3.5.2 the chordal list M_H-partition problem is NP-complete. Otherwise, let G be an input chordal graph to list M_H-partition. If G is not a split graph, then G has an induced $2K_2$. In any M_H-partition of G, no edge may placed in A and no non edge may be placed in B. But then the two edges of the $2K_2$ must be placed in parts of B and so the four non edges of the $2K_2$ must also be placed in B, a contradiction. Thus if G is chordal but not split, then G does not admit an M_H-partition. Suppose now that G is a split graph, partitioned into a clique C and independent set I. We apply the reduction from the proof of Theorem 3.5.1. That is, for each of the $O(n^2)$ partitions of G into a clique C and independent set S, let $G'_{C,S}$ be the bipartite graph obtained from G by deleting all edges of C. Since H is a co-circular arc graph, the list H-colouring problem is solvable in polynomial time for $G'_{C,S}$. $\square$

Bibliography

[1] C. Berge. Les Problémes de Coloration en Théorie de Graphes. *Publ. Inst. Statist. Univ. Paris (As cited by West 2001)*, 9:123–160, 1960.

[2] C. Berge. Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 10:114, 1961.

[3] M Borowiecki, I Broere, M Frick, P Mihók, and G Semanivšin. A Survey of Hereditary Properties of Graphs. *Discuss. Math. Graph ...*, 17:5–50, 1997.

[4] J.I. Brown and D.G. Corneil. On Generalized Graph Colorings. *Journal of Graph Theory*, 11:87–99, 1987.

[5] K. Cameron, E. M. Eschen, C. T. Hoang, and R. Sritharan. The Complexity of the List Partition Problem for Graphs. *SIAM Journal on Discrete Mathematics*, 21(4):900–929, December 2007.

[6] M. Chudnovsky, G. Cornuéjols, L. Xinming, P. Seymour, and K. Vušković. Recognizing Berge Graphs. *Combinatorica*, 25:143–186, 2005. 10.1007/s00493-005-0012-8.

[7] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The Strong Perfect Graph Theorem. *Annals of Mathematics*, 164(1):51–229, July 2006.

[8] D. G. Corneil, Y. Perl, and L. K. Stewart. A Linear Recognition Algorithm for Cographs. *SIAM Journal on Computing*, 14(4):926–934, October 1985.

[9] D.G. Corneil, H. Lerchs, and L. S. Burlingham. Complement Reducible Graphs. *Discrete Applied Mathematics*, 3(3):163–174, July 1981.

[10] M. Cygan, M. Pilipczuk, M. Pilipczuk, and J. O. Wojtaszczy. The Stubborn Problem Is Stubborn No More (A Polynomial Algorithm for 3-Compatible Colouring and the Stubborn List Partition Problem). In *SODA*, pages 1666–1674, 2011.

[11] R. de Souza Francisco, S. Klein, and L.T. Nogueira. Characterizing (k, ℓ)-partitionable cographs. *Electronic Notes in Discrete Mathematics*, 22(0):277 – 280, 2005.

[12] P. Erdös. Graph Theory and Probability. *Canad. J. Math.*, 11:34–38, 1959.

[13] Alastair Farrugia, Peter Mihók, R. Bruce Richter, and Gabriel Semanivšin. Factorizations and Characterizations of Induced-hereditary and Compositive Properties. *Journal of Graph Theory*, 49(1):11–27, May 2005.

[14] T. Feder, , P. Hell, S. Klein, L. T. Nogueira, and F. Protti. List Matrix Partitions of Chordal Graphs. *Theoretical Computer Science*, 349:52–66, 2005.

[15] T. Feder, D. Epple, P. Hell, and O. Shklarsky. Obstructions to Partitions of Some Perfect Graph Classes, to appear.

[16] T. Feder and P. Hell. List Homomorphisms to Reflexive Graphs. *Journal of Combinatorial Theory Series B*, 72(2):236–250, 1998.

[17] T. Feder and P. Hell. Full Constraint Satisfaction Problems. *SIAM Journal on Computing*, 36(1):230, November 2006.

[18] T. Feder and P. Hell. Matrix Partitions of Perfect Graphs. *Discrete Mathematics*, 306(19-20):2450–2460, October 2006.

[19] T. Feder and P. Hell. On Realizations of Point Determining Graphs, and Obstructions to Full Homomorphisms. *Discrete Mathematics*, 308(9):1639–1652, 2008.

[20] T. Feder, P Hell, and W. Hochstättler. Generalized Colourings (Matrix Partitions) of Cographs. In *Graph Theory in Paris*, pages 149–167. Birkhauser Verlag, 2006.

[21] T. Feder, P. Hell, and J. Huang. List Homomorphisms and Circular Arc Graphs. *Combinatorica*, 19:487–505, 1999.

[22] T. Feder, P. Hell, and J. Huang. Bi-Arc Graphs and the Complexity of List Homomorphisms. *Journal of Graph Theory*, 42(1):61–80, 2003.

[23] T. Feder, P. Hell, Sulamita Klein, and Rajeev Motwani. List Partitions. *SIAM Journal on Discrete Mathematics*, 16(3):449, November 2003.

[24] T. Feder, P. Hell, and S. Rizi. Obstructions to Partitions of Chordal Graphs, to appear.

[25] T. Feder, P. Hell, and W. Xie. Matrix Partitions with Finitely Many Obstructions. *The Electronic Journal of Combinatorics*, 14, 2007.

[26] S. Földes and P.L. Hammer. Split Graphs. *Proc. 8th Southeastern Conf. on Combinatorics, Graph Theory and Computing (F. Hoffman et al., eds.), Louisiana State Univ., Baton Rouge, Louisiana. (As cited in Golumbic 2004)*, pages 311–315, 1977.

[27] M.C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Elsevier, 2004.

[28] M. Grötschel, L. Lovász, and A. Schrijver. The Ellipsoid Method and its Consequences in Combinatorial Optimization. *Combinatorica*, 1(2):169–197, 1981.

[29] P. L. Hammer and B. Simeone. The Splittance of a Graph. *Combinatorica*, 1(3):275–284, September 1981.

[30] P. Hell. Graph Partitions with Prescribed Patterns. *Manuscript*, 2012.

[31] P. Hell, S. Klein, L. T. Nogueira, and F. Protti. Partitioning Chordal Graphs Into Independent Sets and Cliques. *Discrete Applied Mathematics*, 141(1-3):185–194, May 2004.

[32] P. Hell and J. Nešetřil. On the Complexity of H-Coloring. *Journal of Combinatorial Theory, Series B*, 48(1):92–110, February 1990.

[33] S. Klein and C. M. H. De Figueiredo. The NP-Completeness of Multi-Partite Cutset Testing. *Congr. Numer*, 119:217–222, 1996.

[34] D. König. Theorie der Endlichen und Unendlichen Graphen (as cited in West, 2001). 1936.

[35] L. Lovász. Normal Hypergraphs and the Perfect Graph Conjecture. *Discrete Mathematics.*, 2:253–267, 1972.

[36] Gary MacGillivray and Min-Li Yu. Generalized Partitions of Graphs. *Discrete Applied Mathematics*, 91(1-3):143–153, January 1999.

[37] S. N. Rizi. Matrix Partition of Chordal Graphs. M.Sc. Thesis, School of Computing Science, Simon Fraser University, 2010.

[38] D. Rose, G. Lueker, and R. E. Tarjan. Algorithmic Aspects of Vertex Elimination on Graphs. *SIAM Journal on Computing*, 5 (2):266–283, 1976.

[39] D. Seinsche. On a Property of the Class of n-Colorable Graphs. *Journal of Combinatorial Theory, Series B*, 16(2):191 – 193, 1974.

[40] D. West. *Indroduction to Graph Theory.* Prentice Hall, 2001.

[41] W. Xie. Obstructions to Trigraph Homomorphisms. M.Sc. Thesis, School of Computing Science, Simon Fraser University, 2006.