

Whenever equation [4.3] does not hold for the two recognized inputs, we can account for any discrepancy in only two distinctly different ways. First, since equation [4.2] is simply a matter of definition, there is nothing keeping us from claiming, as suggested at the end of Chapter 3, that the discrepancy is due to the fact that the partial derivative is defined for properties of equilibrium states and that their meaning does not carry over to disequilibrium states. Most economists will probably not like this claim since it would certainly appear to put our standard method of explanation into serious question. Second, apart from a spurious question of whether X, L and K are correctly measured, or defined, the only other way is to attribute all the discrepancy to a missing variable, such as we did above with the implicit constraint J. This presents serious dangers of producing either tautological or circular lines of argument [see, once again, Samuelson, 1947/65, pp. 84–5]. At the very minimum it makes the analysis of the firm in a state of disequilibrium very mysterious – the mystery of the missing variable.

If a state of disequilibrium is ever going to be explained as a matter of choices made by a significant number of individuals, yet in a manner that does not explain the disequilibrium away (as discussed in Ch. 2), then either methodological individualism must be violated or the partial equilibrium analysis method of explanation needs to be critically examined starting with the concept of the partial derivative itself. To avoid the former, we are led to a critical examination of the foundations of calculus.

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Proofs vs Conjectures in Analytical Economics

> Zermelo [proved] in 1904 that every set can be well ordered and in doing so he called attention to the fact that he used the axiom of choice.... [The axiom of choice] is that, given any collection of sets, finite or infinite, one can [choose] one object from each set and form a new set.... It is used, for example, to prove that in a bounded infinite set one can select a sequence of numbers that converge to a limit point of the set....
>
> Zermelo's explicit use of the axiom of choice brought forth a storm of protest.... [According to] Zermelo's only staunch defender ... the assertion of the existence of objects did not require describing them. If the mere assertion of the existence enables mathematics to make progress, then the assertion is acceptable....
>
> The key issue with respect to the axiom of choice was what mathematics means by existence.... To some it covers any mental concept found useful that does not lead to contradictions.... To others, existence means a specific, clear-cut identification or example of the concept, one which would enable anyone to point to or at least describe it. The mere possibility of a choice is not enough.
>
> Morris Kline [1980, pp. 209–11]

> Leibniz, unlike Aristotle, seemed to feel that his position was to be justified by an appeal to the principle of sufficient reason to determine, in this connection, the transition from possibility to actuality.
>
> Carl Boyer [1949/59, p. 209]

> [Calculus is] the art of numbering and measuring exactly a thing whose existence cannot be conceived.
>
> Voltaire [1733]

Near the end of the last chapter we asked about the apparent contradiction between a single individual's marginal adjustments, which do not affect the equilibrium price, and the aggregation of many individuals' marginal adjustments, which does affect the equilibrium price. This apparent contradiction is not peculiar to economics. It has lurked in the halls of calculus for at least three centuries and is merely the relationship between the integral and the differential. To better understand the possible limitations of equilibrium analysis in the study of states of disequilibrium, we turn now to examine the apparent (but widely unrecognized) contradiction that is associated with the use of partial equilibrium analysis.

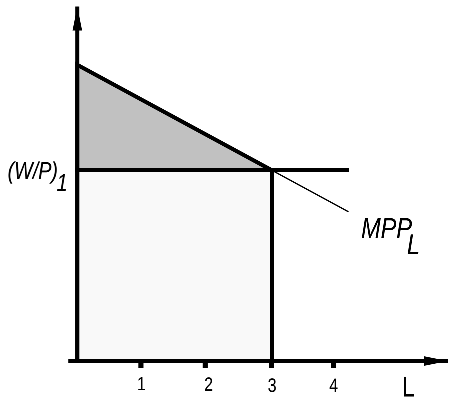

Figure 5.1. Marginal product of infinitessimal units of labor

1. The Problem of the Integral vs. the Differential

Historians can still argue about who invented calculus, Newton or Leibniz, but it does not really matter. What does matter is that many mathematicians have been concerned about whether the basic tools of calculus – the derivative, the partial derivative and the integral – make any real sense. The problem that concerns us is most apparent in the idea of an integral. Consider Figure 5.1, which represents the marginal product for infinitesimal variations in labor input, and Figure 5.2, which represents the marginal product for discrete units of labor. Supposedly, we can calculate the total output by integrating the function represented by the continuous marginal productivity curve from zero to the point of input in question – that is, by adding up the contributions of each unit of labor from zero to a specific level of input (as noted at the end of Ch. 3).

While integration will always make sense whenever we are calculating the total output for discrete units of input (Fig. 5.2), there is a

potential for significant discrepancies when compared to the calculated output for infinitely divisible units of labor (Fig. 5.1). The discrepancies in question would supposedly disappear if we were to make the finite differences in Figure 5.2 so small that for practical purposes the curve of Figure 5.1 would be indistinguishable from the line connecting the upper right-hand corners of the boxes in Figure 5.2.

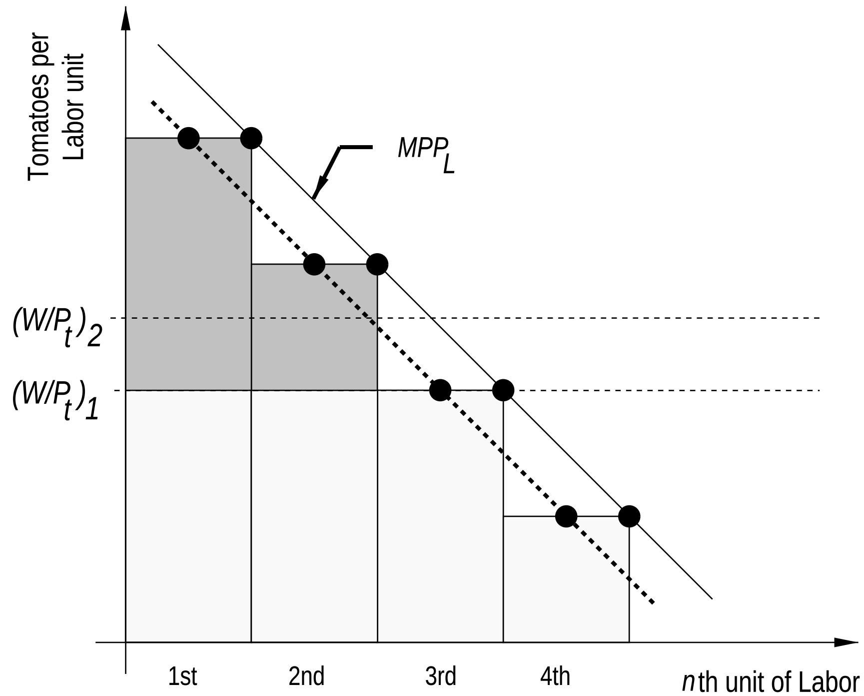

Figure 5.2. Marginal product of discrete units of labor

From a crude practical perspective it is difficult to see any problem here, but the logical basis for the alleged equivalence of these two figures is not very satisfactory. In Figure 5.2 we see that calculating the area as the sum of all the boxes (each representing the marginal contribution of the *n*th unit of labor) ignores the little triangle at the top and thus the calculated area is always less that of the area under the corresponding smooth curve representing the partial derivative. So the question is, why do we learn to ignore the obvious discrepancy illustrated in the comparison of these two diagrams?

The usual argument explains away the apparent discrepancy. One very special case is when the marginal productivity curve is a straight line that connects the midpoints of the tops of all the boxes. In this special case there appears to be no discrepancy since the two triangles between the marginal productivity curve and the top of any box are congruent triangles and hence the one that overestimates marginal productivity is cancelled out by the other one which is an underestimate – but this is a very special case and is only accurate for straight-line marginal productivity curves.

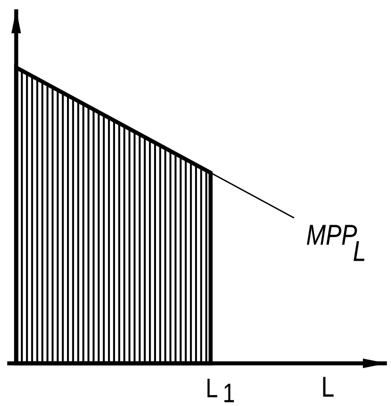

Figure 5.3. Apparent infinitessimal units of labor

The more common form of explanation would have us see that each unit of labor is extremely small, such that the width of each box in Figure 5.2 is less than what we could show by even a single vertical line, and thus would have us pretend that the apparent discrepancy disappears from sight. Consider Figure 5.3 where there is supposedly no space between the vertical lines. In this sense the vertical lines would fill the area under the curve. Unfortunately, this is more a commentary on printing technology than on the alleged equivalence of Figures 5.1 and 5.2. So long as labor is measured in discrete units there will always be an empty triangle uncounted at the top and the sum of the triangles will always be finite.

To avoid the discrepancy we are taught to believe in the idea of an infinitesimal. That is, we are to believe that it is logically possible to have the unit of labor be so small that it is *as if* it has a zero width so that the triangle at the top has a zero area (since its base would be zero), while simultaneously the area of the box (which also has a base of zero) is not zero. We cannot honestly avoid the contradiction here.

Many beginning students of economics are aware of this logical problem. Consider again Figure 5.2. If the real wage-rate were equal to $(W/P)_1$ then the third unit of labor produces neither surplus nor loss.

Students when faced with this situation ask, why would the third unit be hired? In Figure 5.1, where $(W/P)_1$ equals the marginal product of labor at three units of labor, the firm is said to hire three units of labor to maximize the net surplus (the area between the marginal product curve and the real-wage level line). But in Figure 5.2 there does not appear to be any incentive for the firm to hire the third unit. With a lot of hand waving teachers usually explain away the obvious discrepancy by claiming that, again, the unit of measure is so small that the difference between the second unit's marginal product and the third's does not matter or, if the teachers are really clever, they say the issue is only about why the firm does not hire the fourth unit.

Early critics of Newton's and Leibniz's calculus were quite aware of this logical problem – the sum of the areas of the boxes being positive yet the sum of the areas of the corresponding triangles being considered zero. Today, judging by calculus textbooks, it is widely believed that there is no problem here. The accepted proof that there is no problem resides in an argument that the area under a curve (such as that in Fig. 5.3) can be considered to be the 'limit' of the sum of an *infinite series* of units of labor as the unit of measure 'approaches zero' – or when the unit of labor is an infinitesimal. Now, this solves the logical problem *only* if we accept the idea of an infinite series or an infinitesimal – logically, these two options amount to the same thing. If we do not accept the idea of either a 'limit' or an 'infinitesimal', applications of calculus are left in a questionable state.

2. Equivalence of Set Theory and Calculus Analysis

Since the early 1950s economists have learned to look away from these potential problems of calculus by restating the familiar economic propositions in terms of set theory. For example, consider the usual indifference curve as shown in Figure 5.4 for goods *X* and *Y*. In the 1930s the indifference curve was viewed as a differentiable function and the slope of the curve was the partial derivative which Hicks and Allen [1934] called the 'marginal rate of substitution' or *MRS* for short. They proposed to argue that most of the usual propositions of demand theory could be shown to depend on the assumption that this *MRS*, or slope of the indifference curve diminishes (i.e. approaches zero) as points to the right along the curve are considered. At any consumer's chosen point the *MRS* equals the ratio of the respective prices since that ratio is the slope of the usual price-taker's budget line. The idea of diminishing *MRS* was supposed to be methodologically superior to the older assumption of diminishing marginal utility since the latter seemed to imply a cardinal measure of utility and the former did not. This was a bit misleading as the function representing indifference was just a special case of the multi-good utility function where the utility is held constant. How can we hold utility constant without being able to measure its cardinal value? Without answering this rhetorical question, in the 1950s we were taught to abandon calculus in favor of set theoretical interpretations of the familiar concepts such as indifference.

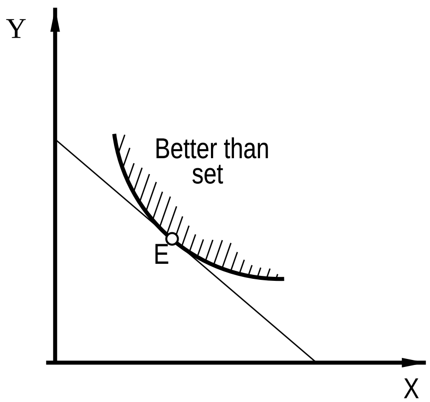

Figure 5.4. Strict convexity

In the set theory version of the indifference curve, the curve is a set of points between which the individual consumer is indifferent. And if the consumer is assumed to be spending all of his or her budget, the indifference curve drawn through the chosen point is also the boundary of two sets. On one side is the 'worse set' containing all points considered inferior to the chosen point. On the other side is a set of points all of which are considered better than the chosen point. The reason why the points in this 'better set' are not chosen is simply that they are all outside the set of affordable points which is represented by the area of the triangle whose hypotenuse is the budget line. The size of this triangle is determined by the size of the budget (or income) and the prices of the two goods in the usual way.

Defining these sets is still not enough of a description of the situation for the individual who is doing something like maximizing utility or, in the newer terminology, choosing the 'best bundle or point that is affordable'. What is needed to complete the description is an assumption that the 'no-worse set' (which is the 'better set' combined with its boundary, the indifference curve) forms a *convex* set. A convex set is a set of points which if any straight-line segment is drawn between any two points *in* the set, all points on the line are contained *in* the set. This is still not enough if the chosen point is the only point the individual would choose when facing the budget line in question. That is, if the chosen point is unique, the 'no-worse set' must be *strictly* convex. With a strictly convex set, only the two endpoints of the line segment connecting any two points are allowed to be points on the boundary; that is, all points between the end points of the line segment must be *in the set but not on the boundary*. This rules out such cases of convex sets as that illustrated in Figure 5.5, while Figure 5.4 illustrates a strictly convex 'no-worse set'.

We note that Figure 5.5 would not satisfy the Hicks-Allen assumption of diminishing *MRS* since between points *a* and *b* *MRS* is not diminishing. Furthermore, and more to the point, if the individual maximizer faced the indifference curve of Figure 5.5, we could not completely explain why point *E* was chosen rather than *a* or *b*, or any other point on the line segment between *a* and *b*. With respect to describing the unique choice option of *E*, either we assume that each indifference curve always displays a diminishing *MRS* or we assume that the 'no-worse set' is strictly convex. So long as we maintain that the individual must be sensitive to all price changes, the two supposedly different assumptions are logically equivalent.

If, as we will argue, the two assumptions – diminishing *MRS* and strictly convex 'no-worse set' – are equivalent, why would anyone bother going to the trouble of reinterpreting all the propositions of economics into the language of set theory? Obviously, it must be because the two assumptions are not considered equivalent in some important way.

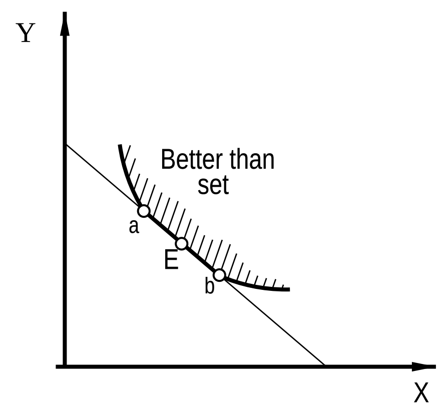

Figure 5.5. Non-strict convexity

3. Continuity vs Connectedness in Choice Theory

Just as cardinality of utility was once considered too strong a requirement for any realistic analysis of consumer demand, continuity of any indifference curve is sometimes considered to be more than what is necessary for a logically complete analysis of consumer demand. When we say that the consumer chooses the best point among those that he or she can afford, there is nothing obviously implied to indicate that the chosen point is on some continuum which allows for infinitesimal adjustments, as was implied by partial equilibrium analysis. For example, let us say that the individual considers the choice of how much to buy of a good that is available only in indivisible units. The question we must consider is about what we are going to do with a situation where the optimum bundle does not have integer values for the goods being purchased – that is, can an individual buy one half of a radio? Many responses are possible. The two obvious responses are that two individuals could choose to share one radio or one individual could rent half of a radio. Either way, the original choice problem is changed to create an effective continuum in the case of the rental or something close to a continuum in the case of the shares. But these responses are avoiding the original question [cf. Lloyd, 1979; Lloyd *et al.*, 1979].

What is being considered is a choice of a particular integer from a set of integers. Such a set is considered ‘connected’ rather than continuous. A connected set is one which can always be separated into two subsets such that there is no point in the set that is not in one of the subsets [see Chipman, 1960]. For example, the set of integers can be separated between those less than or equal to N and those greater than or equal to N+1. There is no integer in the set between N and N+1, by the usual definition of an integer. Now the critical question here is whether a set being ‘connected’ is in any important way different from being ‘continuous’. Surely, the mere idea of recognizing the concept of an integer presupposes some number which is conceived not to be an integer. If not, then there cannot be any difference between the boundary of a connected set and a continuous function such as an indifference curve.

For reasons unclear to us, it is still maintained that by discussing set theory, in the sense of a set of integers, we are in some way not discussing continuous functions and hence, not discussing something for which calculus methods would be applicable. Even when discussing such things as a textbook ‘kinked demand curve’ or any continuous function which has a sharp bend in it, all that is begged is the question of why there are holes in the curve representing the derivative of that continuous function (or representing the partial derivative when there are many arguments in the function). Of course, what is really questioned here is the definition of a ‘sharp bend’.

Consider Figure 5.6. If Figure 5.6(a) represents a continuous total revenue function, $f(X)$, that has a kink in it, then the usual idea is that the derivative appears as shown in Figure 5.6(b). The function representing the derivative, $f'(X)$, may be continuous with respect to X, in the sense

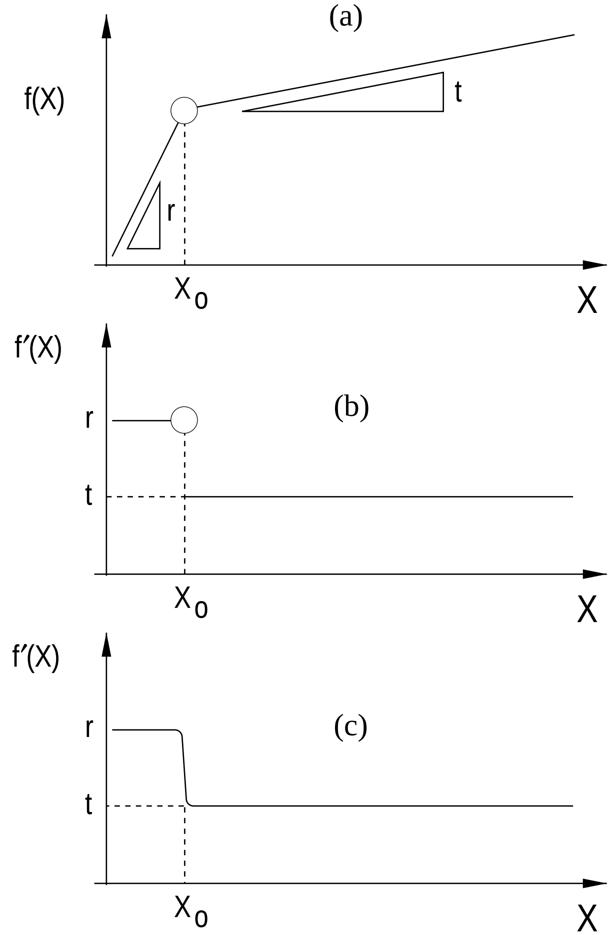

Figure 5.6. Apparent discontinuity

that there are no values of X for which the value of the derivative is not defined. However, while mathematicians are only concerned with whether the derivative is continuous over the values of X, the derivative is not continuous with respect to its own value as there are conceivable values (between r and t) which are not represented by the derivative-function. As economic theorists we want to give meaning to the value of the derivative, such as when we set the value of marginal revenue equal to the value of the marginal cost for profit maximization. Of course, analytically we can have any kind of function we can conceive. But the question that might be asked is whether Figure 5.6(b) can actually represent a realistic process as, in the case where the (partial) derivative represents marginal revenue. What Figure 5.6(b) implies is that as X increases value from that below X_0 to that above, somehow the derivative instantaneously changes from r to t at X_0. The term 'instantaneous' really means infinitely fast and since an infinite speed of change cannot be represented by a real world process, the realism of Figure 5.6(b) is questionable.

What concerns us here is what is meant by a 'real world process'. While we may be free to assume any analytical function we wish, we are just as free to say that anything requiring infinite speed or infinite time or space is something that is not of the real world. The case shown in Figure 5.6(b) is impossible but that in Figure 5.6(c) is possible. This is to say that the 'sharp bend' in the function of Figure 5.6(a) is one where the slope changes from r to t in a continuous way, such that there are no missing values between r and t as there were in Figure 5.6(b). We will have more to say about this view of 'realistic' functions in a later section of this chapter. For now all that we wish to establish is that we can always rule out any discontinuous functions as unrealistic functions and thereby say that any realistic boundary of a set of 'connected' points is also a continuous function. In this sense, there is nothing to be learned from set theory that cannot be discussed using calculus concepts.

4. Continuity, Convexity, Uniqueness and Choice Theory

Set theory has served as the medium for many sophisticated presentations of the logical foundations of the neoclassical theory of the consumer [e.g. Chipman *et al.*, 1971]. Virtually all the sophisticated analyses of consumer theory fail to restrict the conception of consumer choice to one that is appropriate for price theory. They usually present a consumer theory without a purpose other than theoretical analysis for its own sake – that is, without regard for how it fits with the needs of any methodological individualist explanation of prices. If our interest in consumer theory is only the mathematical rigor of our representation of the idea of maximization or optimization, then there may very well be no significant differences between calculus-type analysis and set-theoretical constructs. But, if we require both that the consumer's choice be completely explained and that our theory of the consumer be consistent with our theory of prices and of the economy as a whole, there is still a problem here. Particularly so, if we try to accommodate set-theoretical representations in the way explained above. We will discuss these two requirements in turn.

4.1. Completeness of Explanations

In Chapter 1 we discussed the methodological dilemma concerning complete explanations of an individual's behavior and the question of whether a successful explanation denies the possibility of the individual's exercising free will. Rather than worry again about 'free will', let us just focus on what constitutes a complete explanation of an individual's behavior (i.e. of an individual's choices or decisions).

The decisions or choices of direct relevance to an individual are his or her consumption decisions. Our usual neoclassical theory is conceptually rather simple in this regard. We say the individual consumer chooses the 'best' point that he or she 'can afford' with the given budget (or income) and prices. The individual provides the subjective criterion used to define what is 'best' and the objective criterion determining what the individual 'can afford' is merely a matter of arithmetic. What is to be explained is the specific choice or decision made by the individual in question. Put this way the choice is necessarily unique and any explanation should entail such uniqueness. The question of what the consumer can afford is essentially an objective matter since the prices are public events and under certain conditions the income or budget is revealed by inference from the choice made. The conditions, however, are not trivial. To be able to infer the consumer's budget for the choice made, we must assume that the individual is not completely satiated by the choice made and is not facing survival choices. In effect, neoclassical consumer theory is a theory that only applies to the middle class consumer! The consumer is assumed always to be facing scarce budgetary resources but is also assumed to have enough to give some freedom for matters of taste. The theorist need only conjecture what the individual's preferences or decision criteria are to complete the explanation of the consumer's unique choice.

The logic of explanation is as follows. Given the theorist's conjecture concerning the consumer's preferences (or subjective criteria) and given the objective prices and incomes, the chosen point can be shown logically to be the one and only 'best' point. If there were more than one affordable point that the individual would consider equivalently the 'best' given the theorist's conjectured preferences, then the question is immediately begged about why the individual chose the one 'best' point rather than any of the other 'best' points. In other words, unless the conjectured preferences lead to the conclusion that there is only one 'best' point and that one 'best' point is the one that was chosen, then the theorist's explanation of the individual consumer in question is clearly incomplete. To be complete the explanation must not only entail the chosen point but it must be the only point the individual would choose under the circumstances. We must explain why all other points are not chosen.

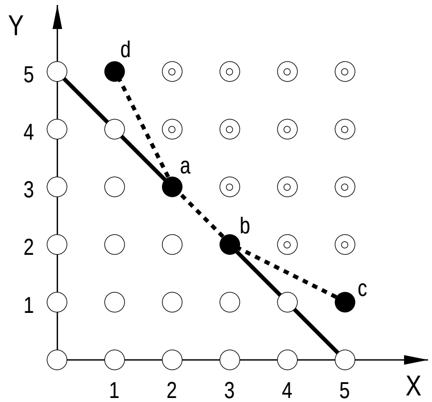

Figure 5.7. Indivisible goods and unexplained choice

Now what does this have to say about the differences between calculus and set-theoretic analysis? Consider Figure 5.7 where we are again having the consumer choose amounts of indivisible goods – that is, goods that must be purchased in integer amounts. Let us say that we are to explain why the individual chose point *a* – that is, to buy two units of good *X* and three units of good *Y* – given a budget for which the individual could buy either five units of *X* or five units of *Y* or any linear combination (implying that the ratio of the prices is one). If the four solid points are on a conjectured indifference 'curve' then the individual in question would be indifferent between the chosen point and the non-chosen point *b* (representing three units of *X* and two units of *Y*). Since both points lie on the budget line and both lie on the same indifference curve, they are equivalent according to both subjective and objective criteria. Thus, even though the 'no-worse set' is connected (i.e. there are no conceivable non-integer points) and is convex, the explanation is incomplete.

If we allow the calculus-type analysis to define the conjectured indifference curve to be over the non-integers as well as the integers so that it appears as a smooth curve exhibiting the usual assumption of diminishing *MRS* (see Fig. 5.8), the curve will be conjectured to be tangent to the budget line at only the chosen point. The other point, *b*, will be inferior. So, it would seem that in terms of explanatory completeness, calculus has a decided advantage over set-theoretic analysis.

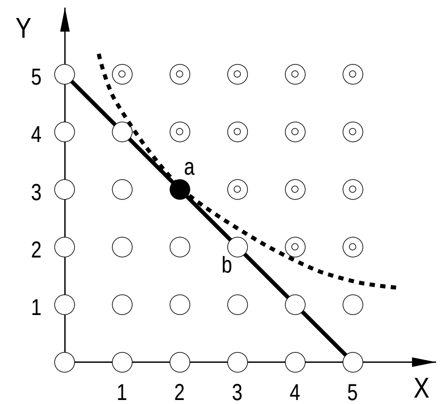

Figure 5.8. Indivisible goods with continuous preferences

4.2. The Wider Role of Choice Theories

Defenders of set theory will surely claim that the comparative advantage of calculus here is due only to our conjecturing the particular disadvantageous indifference curve of Figure 5.7 rather than one like that in Figure 5.9. In the latter figure the chosen point is unique in a manner similar to a calculus based explanation – i.e. point *a* in Figure 5.9 is preferred just as it was in Figure 5.8. So, what is wrong with Figure 5.9?

If we are constructing an explanation of the individual's decision behavior that is to be used only to explain the uniquely chosen point, there is no significant difference between Figures 5.8 and 5.9. However, if we are constructing an explanation that is intended to be more general in terms of the circumstances to which it is to apply, both versions of an explanation suffer methodological difficulties – even though these difficulties are different.

The issue that we have to face concerns the purpose of any explanation of any consumer's behavior. Again, every theorist is free to do whatever he or she wants. Nevertheless, the primary reason we discuss the consumer theory in the context of neoclassical economics has always been to see the consumer as a part of our larger theory of prices where the individual is conjectured to play a significant role. From the perspective of methodological individualism, the theory of the individual

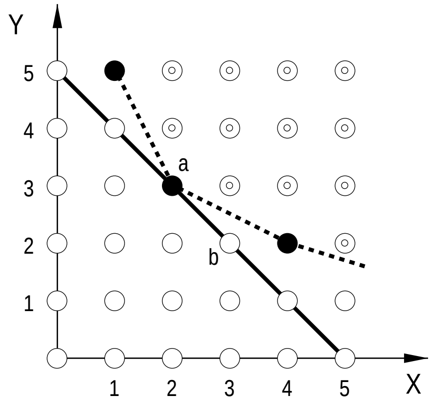

Figure 5.9. Indivisible goods and unique choice

consumer is the foundation for the theory of market demand which, when conjoined with the separate theory of market supply, explains the price in each market. Given this purpose, any representation of consumer theory must be adequate for this purpose. What is necessary for this adequacy? What limitations does the question of adequacy put on any analysis of an individual's choice behavior?

Uniqueness and price theory Since the individual consumer is almost always seen as having an infinitesimal effect on the equilibrium price and the total supply – that is, being a price-taker and being able to buy as much as wanted – any requirements for the individual's playing a part in a market equilibrium do not have an explicit role in the theory of the consumer's behavior. The most important requirement is uniqueness since it is a primary requirement for a consistent role for the individual. By uniqueness here, we are referring to the choice made by the individual whenever the individual faces the same circumstances. That is, if at two different points in time the individual faces the same set of prices, income and indifference map (or preferences), the individual would be expected to make the same choice. If any explanation of the individual does not entail such a unique choice, two different choices are possible and thus, when we aggregate the explained behavior of all consumers, there will not be a unique quantity demanded at any point in time. In other words, the question of explanatory completeness, even if ignored at the level of the individual consumer, will remain at the level of market demand, and by remaining there will remain at the level of explaining the market price.

Completeness and price theory It should be evident from our discussion in Chapters 3 and 4 that questions about the completeness of explanations will always present difficulties for any consideration of uniqueness. There is another requirement that is almost as important and it concerns the completeness of the conjectured preference ordering of the consumer. If we are to use the theory of the consumer as a foundation for price theory, then we must be able to explain the consumer's behavior no matter what prices are present in the market. This is because to explain prices we must not only explain why the price is what it is, but also why it is not what it is not [Nikaido, 1960/70, p. 268]. Thus, it is never enough to explain the individual's choice given just one budget line [e.g. Batra and Pattanaik, 1972]. Whatever the prices may be, the individual must be able to make a choice. This means that the conjectured preference ordering or indifference map must extend indefinitely in all directions. That is, the individual must be able to compare *any* two conceivable points, or be able to attach a specific level of utility to any conceivable point. While uniqueness is not always seen to be a requirement, many analytical consumer theorists do see the need to have a complete preference ordering [e.g. Chipman *et al.*, 1971]. Unfortunately, the analytical consumer theorists seldom, if ever, tell us why preference orderings must be complete.

5. Infinity and Induction in Analytical Economics

Certain questions are raised by these considerations. In effect, the conjectured indifference map or preference ordering must extend over an infinity of conceivable points. How does the individual learn what his or her preferences really are? Such knowledge might require an infinity of trials! But what is even worse, any sophisticated analysis of consumer preferences must also deal with preference orderings over an infinity of conceivable points regardless of how the individual learns. Some sophisticated consumer theorists rely on a so-called 'axiom of choice' to extend knowledge about the preferences from being over realistic finite subsets to being over infinite sets as is required for completeness [see Chipman *et al.*, 1971, p. 250]. This is the axiom often used by mathematicians (as noted in the above quotation from Kline [1980]) and is to be distinguished from the axiom of choice discussed by economists [e.g. Frisch, 1926/71; Samuelson, 1938]. We will discuss the uses of the mathematical axiom a little later. The important point is that the question of completeness of preference orderings too easily involves us in a discussion of infinite sets. This is a problem since, in realistic terms, the meaning of 'infinity' always refers to an impossibility.

The common ideas of continuity, completeness, infinity and infinitesimals are all closely related, even though this is not always obvious. The relationship between infinity and infinitesimals is the most obvious. Any ratio such as *A*/*X* is said to become an infinitesimal (i.e. approach zero) as *X* approaches infinity. We discussed above the direct relationship between completeness and infinite sets. What is probably in doubt is the relationship between continuity and completeness. Let us discuss this and then get to the real concern, which is the less obvious relationship between the complete preference orderings, infinite sets and inductive learning.

5.1. Continuity and Completeness

Continuity is very important for calculus considerations, as is well known. Nevertheless, establishing continuity always runs the risk of an infinite regression. We take for granted that Euclidian space (which is just ordinary rectilinear space named after the famous geometrician Euclid) can be represented by real numbers along each of the co-ordinates. For example, we can conceivably plot a consumer's choice point as being equal to one-half of a radio and two and one-third calculators, regardless of the question of whether such non-integer quantities make sense to us. Given the assumption that radios and calculators only come in whole units, the set of possible (as opposed to conceivable) choice points do not completely cover the Euclidian space representing quantities of radios and calculators. Now consider an indifference curve for radios and calculators such as the one in Figures 5.7 or 5.9. If one insists on using the Euclidian co-ordinates to represent quantities of these indivisible goods, when only integer points are possible in the eyes of the consumer, then the indifference curve will only be a sequence of points that are unconnected in Euclidian space – that is, points with large (Euclidian) spaces between them. The preferences represented by this integer indifference map will be neither continuous nor complete *with respect to the Euclidian space* that we commonly use as our co-ordinates. But from the viewpoint of the consumer, the non-integer points are irrelevant and thus the alleged discontinuities in the indifference map are misleading. This is why the question of viewing the set of possible choice options as a connected set rather than a continuous space can be important in any analytical treatment of consumer theory.

Switching from incomplete continuous-space indifference maps to connected sets of possible choice points solves the problem of misleading non-continuity but it may not ensure that all preference orderings of such connected sets are complete. What if the individual is, perhaps for mysterious psychological reasons, unable to evaluate the single point representing three radios and three calculators? The indifference map, whether for Euclidian space or the connected set of possible choice points, will have a hole in it at that point. On the one hand, if the prices and income facing the individual consumer are such that he or she cannot afford to buy three radios and three calculators, then the hole in the map would seem to be irrelevant for our theory of the consumer's behavior. On the other hand, if the consumer can afford this point, our explanation of why he or she bought any other point will be incomplete, since we cannot explain why the point representing three units of each good was not chosen. Inability to evaluate the point is not a sufficient reason, since the point is still possible and since a non-evaluation is not the same as an underevaluation.

The idea here is simple. A continuous indifference map must also be a complete map – whether we mean continuous in the Euclidian space or in the restricted terms of the set of connected possible points. Any discontinuity (or hole) in the map is also an instance of incompleteness.

5.2. Infinity and Completeness

Much of what we have been discussing has been the concern of analytical consumer theorists who have tried to prove that demand curves with certain specified mathematical properties can always be shown to be 'generated by the maximization of a utility function' [Hurwicz and Uzawa, 1971, p. 114]. More generally, they have been concerned with the problem of how much must we know about the demand curves to be able to deduce the utility function that is being maximized. Since a demand curve is the locus of utility maximization by all demanders, its calculus properties are those of the various relevant partial derivatives in the close neighborhood of the maximizing points. However, any demand curve (or demand function, if we wish to stress that more than one good is being simultaneously chosen) is just a line connecting a subset of singular points drawn from all the points on the indifference map. One demand curve cannot tell us much about the entire indifference map from which it was derived. To determine the underlying map or utility function we would need many observations of many demand curves. This problem of deducing the general nature of the utility function from the singular marginal properties of any particular set of demand curves (i.e. curves for many different choice situations) has been identified by many theorists as the 'problem of integrability'. But giving it a name does not make it solvable [see Wong, 1978].

For our purposes here, what is important is the following. All analytical theorems, which are 'proved' by the analytically sophisticated consumer theorists, involve some sort of infinity assumption. They do so either directly by referring to an infinite set or indirectly by referring to infinitesimals in the neighborhood of the consumer's chosen point. The irony of this is that infinities must be invoked to explain the finiteness (or discreteness) of the consumer's unique choice or the market's unique demand curve.

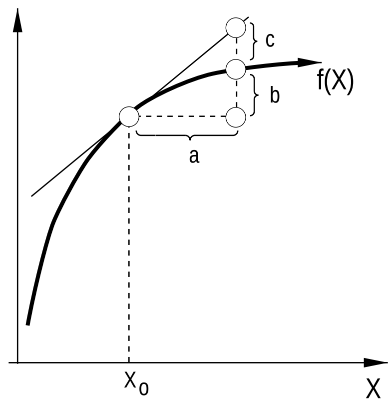

Figure 5.10. Slope as derivative

The use of infinitesimals is obvious in any analytical proof involving derivatives or differentiable functions. Even the most simple definition of a derivative – namely the slope of a function – relies on the infinitesimal. Consider Figure 5.10 which shows a non-linear function $f(X)$ and its slope at point X_0. The slope there is $(c+b)/a$ and if X changes by a finite amount a, the ratio of the change in $f(X)$ to the change in X is b/a. So long as a is not zero there is a difference between the slope and the ratio of the changes (or differences). The slope will equal the derivative if the latter is defined as the ratio of the changes when a paradoxically has the value of zero but not yielding the usual consequences of division by zero. Usually, dividing by zero is considered to yield an infinitely large ratio value. Printing technology notwithstanding, we are to think of the function as being complete in the neighborhood of X_0, in the sense that it is continuous, and no matter how small a gets there exists a value for $f(X_0+a)$. In effect, between X_0 and X_0+a there must be an infinity of points on the function between $f(X_0)$ and $f(X_0+a)$ and the function as a mapping from X-space to space defined by $f(X)$ must be complete between X_0 and X_0+a.

Historically, many students of calculus have been uneasy about relying on the mysterious and paradoxical concept of an infinitesimal which supposedly has a zero value but has properties of being non-zero. To avoid the use of such a concept, most textbooks today define the derivative in terms of what are called 'limits'. Rather than refer to infinitesimals, today the derivative of $f(X)$ shown in Figure 5.10 is defined as the limit of the ratio b/a as a approaches zero. A simple

definition of a limit is as follows:

> Let $f(y)$ be a function of y and let k be a constant. If there is a number L such that, in order to make the value of $f(y)$ as close to L as may be desired, it is sufficient to choose y close enough to k, but different from k, then we say that the limit of $f(y)$, as y approaches k, is L.

Now it is a mystery to many of us how defining a derivative in terms of the concept of a limit is in any significant way an improvement over an infinitesimal-based definition. Naive defenders of the limit-based definition will say that it is because the derivative is defined by a real quantity, namely L, but this only begs the question of how we know we are at L. Sophisticated defenders will enhance the definition by referring to the limit L as the ultimate value of an infinite sequence of points where each additional point lies between the last point and the point representing L. Again, we are no better off and maybe worse off since we are again referring to an impossibility – namely, an infinite sequence.

While the limit-based definition of a derivative is still widely accepted, some mathematicians have tried to express such definitions in terms of what they call the 'axiom of choice'. This axiom is stated in the above quote from Morris Kline '... given any collection of sets, finite or infinite, one can [choose] one object from each set and form a new set.' This axiom is trivial for any finite collection of finite sets, but there is no reason to accept it otherwise. Nevertheless, it can be used to define a limit along the lines of a paradox of Zeno [Boyer, 1949/59]. Namely, take the distance between the limit L and any point different from L, form a set of the points representing one-third the distance and two-thirds the distance and choose the point which is closer to L. Now repeat this process *ad infinitum*. Supposedly, we can use the 'axiom of choice' to prove that the ultimate result is to choose L. Of course, this in no way escapes the criticism of relying on definitions and proofs which are impossibilities since they depend on infinite sets which are impossible.

It would probably be wiser to avoid trying to prove that the derivative of a function is the slope of a curve representing the function and accept the claim as a conjecture and move on from there.

5.3. Completeness and Inductive Learning

While accepting complete preference orderings as conjectures about infinite sets would seem to satisfy the requirements of analytical proofs, there are still questions begged when we turn to consider the implications of such conjectures for the capabilities of the individual whose behavior is being explained. If we say the individual chooses the one best point out of the infinity of possible points, how does the individual know it is the best point unless he or she has knowledge of the infinite set? Again, the question arises because the concept of infinity is by definition an impossibility. Does this mean that such knowledge is impossible?

It would be only if we were to continue the neoclassical tradition of believing that all learning must be inductive. There is no need to do so. Unfortunately, most economic theorists still take inductive learning for granted and thus often ignore some difficulties involved in any assumption that the individual knows his or her preference ordering or indifference map.

Recall that inductive learning is based on the assumption that we learn with each new bit of information acquired. That is, with only singular observations of a particular instance of a general proposition, we are led to conclude that the general proposition is true. The typical illustration is that by repeatedly observing white swans flying south for the winter we are learning that all swans are white. Inductive learning is learning the truth of a general statement from observing numerous particular examples. It is in this sense that the individual might be conjectured to learn what his or her preferences are by merely tasting each conceivable point in the relevant goods-space. But, unfortunately this theory of learning fails for simple reasons of logic. No amount of finite evidence about the singular elements of a infinite set could ever prove that such a set has specific general properties [see further, Popper, 1972, Ch. 1 and Appendix] – the next swan to fly over may not be white.

These logical considerations raise doubts about all analytical models that presuppose that the individual consumer has sufficient knowledge. This not only criticizes the view that an individual could evaluate the point representing a million radios and a million calculators, it also criticizes the view that the consumer has the complete ordering needed to be able to evaluate a point representing one-millionth of a unit of tea and one-millionth of a unit of coffee. While it is easy to see that it would be difficult to learn about points approaching infinity, it should also be equally apparent that it is just as difficult to learn about the infinity of points in the neighborhood of the maximum of any constrained and differentiable utility function. And so, the use of (partial) derivatives to explain the shape of indifference curves or demand curves necessarily goes far beyond what is intellectually possible for the individual decision-maker. While this might not matter for analyzing the properties of a state of equilibrium, a disequilibrium analysis is predicated on at least one individual in some way being aware that he or she is not optimizing. If one insists on maintaining the common presumption of inductive learning, then disequilibrium analysis is impossible.

6. Proofs and Conjectures

If one rejects the idea that people learn inductively, one will find it difficult to appreciate the many published articles and papers which provide proofs of propositions about the general properties of preference orderings or about demand curves based on those general properties. It does not matter whether the proofs are based on calculus concepts or set-theoretic concepts, since the proofs must always deal with some form of completeness of the individual's preference ordering and thus must refer to either infinite sets or infinitesimally close neighborhoods of specific points. A way out is to treat the individual's preference ordering or utility function as a conjecture *on the part of the individual consumer*. What is the cost of such an approach?

By viewing all individuals as inductive learners, theorists have been able to rely on the observability of the individual's objective situation to ensure unique and consistent choices. For any given type of preference ordering (determined by specific assumptions on the part of the theorist), proofs could thus be reliably constructed. But what if one does not really learn inductively? Even if an individual still has a specific type of psychologically given preference ordering, the individual consumer does not know its true nature and thus has to conjecture about his or her preference ordering. Using a conjectured preference ordering may not always produce choices consistent with the true ordering. This is because there is no reason why, without reliable inductive learning, the individual has been successful in learning his or her true preference ordering. For now we will postpone discussion of the benefits of viewing the individual's preference ordering as a conjecture. Later we will see that such an approach clears up some difficulties in the analysis of the stability of market determined prices and clarifies the arguments concerning the role of competition in the determination of prices.

Part III

Limits of Equilibrium Methodology