ALGEBRAIC THINKING IN THE ELEMENTARY CLASSROOM

by

Deanna Berg
B.Ed., Simon Fraser University, 2006
B.A., Simon Fraser University, 2004

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

In the
Individual Program
Faculty of Education

SIMON FRASER UNIVERSITY
Fall 2012

APPROVAL

Name: **Deanna Berg**

Degree: **Master of Arts**

Title of Thesis: **Algebraic Thinking in the Elementary Classroom**

Examining Committee:

Chair: **Elina Birmingham, Assistant Professor**

Rina Zazkis, Professor
Senior Supervisor

Steven Campbell, Associate Professor
Committee Member

Peter Liljedahl, Associate Professor
Faculty of Education, Simon Fraser University
External Examiner

Date Defended/Approved: December 3, 2012

Ethics Statement

The author, whose name appears on the title page of this work, has obtained, for the research described in this work, either:

a. human research ethics approval from the Simon Fraser University Office of Research Ethics,

or

b. advance approval of the animal care protocol from the University Animal Care Committee of Simon Fraser University;

or has conducted the research

c. as a co-investigator, collaborator or research assistant in a research project approved in advance,

or

d. as a member of a course approved in advance for minimal risk human research, by the Office of Research Ethics.

A copy of the approval letter has been filed at the Theses Office of the University Library at the time of submission of this thesis or project.

The original application for approval and letter of approval are filed with the relevant offices. Inquiries may be directed to those authorities.

ABSTRACT

My interest in early algebra led me to seek out the definitions of algebra and early algebra according to various researchers. Algebra can be thought of as a set of tools for solving problems, where letters are used when making generalizations, finding unknowns, or working with variables. Early algebra is about providing children with opportunities to use their algebraic thinking in order to develop an in-depth understanding of arithmetic. This led me to wonder if elementary school students display evidence of algebraic thinking, and if so, how their algebraic thinking is evident. On a number of occasions, I presented mathematical tasks to students in a grade two/three class and made observations on how they approached the problems. I found that there was evidence of algebraic thinking in elementary school students; they were able to find the unknown, they were able to work with variables, and they were able to make generalizations.

Keywords: algebra; early algebra; algebraic thinking; making generalizations; finding the unknown; using variables

This thesis is dedicated to my mom, Judy Berg. Thank you for sharing your strong belief in the value of receiving a higher education, and for instilling in me a desire to continually expand my knowledge.

ACKNOWLEDGEMENTS

Thank you to Professor Rina Zazkis for all of your support and encouragement. Thank you as well to Professor Stephen Campbell, Professor Peter Liljedahl and Professor Tom O'Shea. I feel that I have learned so much from all four of you.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

1: INTRODUCTION

When I embarked on this journey of writing a thesis, I was a teacher-on-call in the Burnaby School District. Since that time, I spent a year teaching grade six/seven, grade one French immersion, and English as a second language. I am currently teaching grade four for my second full year of teaching.

I completed my undergraduate work in French; however, I regretted not studying mathematics. When I decided to pursue a master's degree, I knew that I wanted it to be related to mathematics.

I sought information on the Masters programs that were offered at SFU and I had two options: a secondary mathematics program or a numeracy program. Both programs sounded interesting but neither of them felt like the perfect fit. I was going to settle for the numeracy program; however, it was offered in another part of BC, so I contacted Rina Zazkis, one of the program coordinators, to see if it would be offered in Burnaby as well. I met with Rina and we decided that I would enrol in an individual program with her as my supervisor. It was the best of both worlds since I would be taking some courses that were in each of the two previously mentioned programs.

The courses that I took in this graduate program allowed me to explore different topics and they assisted me in narrowing down the topic of study for my thesis. I was quite interested in algebra, and since I am an elementary school teacher, I decided to examine algebraic thinking in the elementary classroom.

In my research, I look at how algebra and early algebra are defined. I conduct a study in which I present a number of tasks to a grade two/three class and observe for evidence of algebraic thinking.

In the second chapter of this thesis, I seek the definition of algebra according to a few different researchers. The two core aspects are explained as well as the three stages through which algebra has progressed. Finally, I examine the six topics which are regarded as the core of school algebra.

In the third chapter, I examine how early algebra is defined by various researchers and why early algebra would be beneficial. I also look at the powers that students have, and how the use of these powers, in relation to numbers and relationships, leads to algebraic thinking. Finally, I examine the three purposes of algebra and how they tie in with early algebra.

In the fourth chapter, I explain my study, starting with the location and the participants, and then I explain what each of the tasks are, how I came up with them, and how the data is collected.

In the fifth chapter, I look more closely at the tasks that I did in my study. I explain how I presented the tasks to the students, the work that the students did, and how I saw evidence of algebraic thinking in each of the tasks.

In the final chapter, I summarize my results, state the limitations of this study, what I envision as a next step and how I have developed throughout this study.

2: ALGEBRA

In this chapter, I investigate how algebra is defined according to a few different researchers. As well, the two core aspects and the three stages through which algebra has progressed are explained. Finally, I look at the six topics which are regarded as the core of school algebra. There is a vast amount of research on this topic, so I give a brief summary of the information most relevant to my thesis.

2.1 What is algebra?

In the ninth century, Al-Khwarizmi, a Persian mathematician, wrote a book entitled *Hisab al-jabr w'al-muqabala,* or *Calculation by Restoration and Reduction*. It is from this book, which was a mixture of mathematical traditions such as Babylonian, Indian and Greek, that we have the word "algebra" (Suggate, Davis, & Goulding, 2006). Among the many solutions in this book are those for the problems of inheritance and property division. At the root of algebra is the use of mathematical knowledge to solve practical problems. Algebra "can be thought of as a highly compact and efficient set of tools for solving problems (practical and theoretical)" (Suggate, Davis, & Goulding, 2006, p. 127).

The content of what is referred to as algebra has changed over the years. Kieran (1992), states how algebra has evolved through three stages:

1. The Rhetorical Stage, the period before Diophantus (c. 250AD), is characterized by the use of sentences (ordinary language) for solving problems. There were no symbols or signs to represent unknowns.
2. The Syncopated Algebra Stage was initiated by Diophantus, who introduced the use of letters for unknown quantities. During this stage, from the third to the 16^{th} centuries, the main concern of algebraists was to find the unknown number rather than to express the general. Diophantus' "Arithmetica" was a step-by-step description of how to solve 189 different problems. His work was translated into Latin which allowed Vieta to read it, and this led to Vieta's use of a letter to stand for a given as well as an unknown quantity.
3. During the Symbolic Algebra Stage, it became possible to express general solutions, and algebra became used as a tool for proving rules governing numerical relations.

Kieran (1992) notes how algebraic symbolism permitted a change from a procedural to a structural perspective on algebra. The procedural perspective involves arithmetic operations carried out on numbers to yield numbers. The objects that are operated on are not the expressions but their numerical instantiations. The operations are computational, they yield a numerical result.

For example: $3x + y$

$x = 4, y = 5$

$3(4) + (5)$

The answer is 17

The structural perspective involves a different set of operations that are carried out, not on numbers, but on algebraic expressions. The objects that are operated on are the algebraic expressions, not some numerical instantiation. The operations are not computational. The implicit objectives of school algebra are structural.

For example: $5x + 5 = 2x – 4$

$5x + 5 – 2x = 2x – 4 – 2x$

$3x + 5 = - 4$

With the changes that algebra has gone through, we see the growth and development of a form of mathematics, as well as a range of opinions on what algebra represents.

Kaput (2008) recognizes that there are two perspectives on algebra: one that sees algebra as a stand-alone body of knowledge, and the other that sees it as human activity. With the first perspective, algebra is thought of by its rules and not relating to the students. Algebra is seen to evolve "as a cultural artifact in terms of the symbol systems it embodies (most recently due to electronic technologies)" (Kaput, 2008, p. 9). While with the second perspective, the way that students do, think and talk about mathematics is fundamental. "Algebra evolves as a human activity as students learn and develop" (Kaput, 2008, p. 9).

Kaput expands on his explanation of algebra, and moves from the two perspectives of algebra to what he calls the two core aspects of algebra. The first core aspect is “algebra as systematically symbolizing generalizations of regularities and constraints” and the second core aspect is “algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems” (Kaput, 2008, p. 11). Determining which of the two core aspects is more central to defining algebra would be difficult since even mathematicians and mathematics educators have different views about that. Those who favour the first core aspect believe that algebra is characterized by generalizations made using whatever resources are available, especially drawings or natural language. Those who favour the second core aspect believe that rule-based actions on symbols are the distinguishing feature of algebra.

Kaput (2008) lists three strands of algebra, each of which has the two core aspects of algebra embodied within.

1. The first strand involves generalizing, and expressing those generalizations explicitly and systematically. Carraher and Schliemann (2007), in reference to Kaput’s three strands, state that this strand “accentuates the use of algebra to capture mathematical structures” (p. 676).

2. The second strand that Kaput mentions involves generalizing toward the idea of function.

3. The third strand involves modelling. Carraher and Schliemann (2007) state that this approach refers to “using mathematics to describe real-world data, relations and processes” (p. 676).

According to Bass (1998), algebra in school, which is the root of all algebra, is about the following:

- The basic *number systems* – the *integers* and the *real numbers* and those derived from them, such as the rational and complex numbers.
- The *arithmetic operations* (+, -, x, ÷) on these number systems.
- The *linear ordering* and resulting *geometric structure* defined on the real line. By this I mean the notions of size (whether one number is larger or smaller than another) and of distance between numbers.
- The study of the *algebraic equations* that arise naturally in these systems (p. 9).

Mason (1996) adds to this definition of school algebra by stating that it is about manipulating generalities in number contexts and doing so with symbols. Algebra is currently seen as the part of mathematics that deals with symbolizing general numerical relationships and mathematical structures, and with operating on these structures. In algebra, letters are used to represent *unknowns* in equation solving and they are used to represent *givens* in expressing general solutions. As Porteous (2008) mentions, the fact that a letter can represent a particular unknown or it can represent a variable, can be confusing to some students, while others might be able to cope with the idea that a letter has more than one meaning. At first, algebraic representations are treated as generalized statements of the operations carried out in arithmetic. After this introduction is over, algebraic representations begin immediately to be treated as mathematical objects on which certain structural operations can be carried out, for example, combining like terms, factoring, or subtracting the same term from both sides of an equation. Note that a term is a number, letter, or combination of the two, an expression is a term or combination of terms, separated by an operational sign, and an equation is two expressions separated by an equal sign.

2.2 Teaching and learning algebra

The literature tells us that when teaching algebra, it is important to remember that students start out procedurally, this is the first step in the acquisition of new mathematical notions. It is then a lengthy process to move from a “process” conception (procedural) to an “object” conception (structural). Sfard (1991) has created a three-phase model of conceptual development.

1. During the first phase, interiorization, students are becoming familiar with processes which will later lead to new concepts. “These processes are operations performed on lower-level mathematical objects” (p. 18).

2. During the second phase, condensation, students are able to think of a process as a whole, which leads to a new concept.

3. In the third and final phase, reification, the student no longer sees the processes, but instead, sees the new concept. As Kieran (1992) puts it, the student has “the sudden ability to see something familiar in a new light” (p. 392).

If teachers remember these three phases, this should be very helpful in the teaching of algebra. Blanton (2008) also has some advice for teachers on how to build a classroom that develops children's algebraic thinking. She provides a list of some things for teachers to keep in mind:

1. The ability to make and defend generalizations is at the heart of mathematical power and can be learned by all children
2. Children can deepen their understanding and skill with any particular mathematical topic by generalizing and building arguments for their generalizations.
3. In order for the mathematical power of generalizing to develop, the classroom environment must foster meaningful inquiry and communication.
4. Teachers must be able to find, extend, and exploit their instructional resources to serve their evolving needs.
5. Communities of teachers can learn from and teach each other, even when the ideas are new to everyone involved (pp. 7–8).

According to Kieran (1992), the learning and teaching of algebra can be discussed according to the six topics which are generally accepted as the core of school algebra: literal terms and expressions, simplifying expressions, equations, solving equations, word problems, and functions and graphs.

2.2.1 Literal terms and expressions

There are some algebraic expressions that are similar to arithmetic expressions. If students have a good understanding of these arithmetic expressions and can understand what is going on, then they will have an easier time with algebra. For example, if students cannot understand that a set containing four items and a set containing seven items can be written as 4 + 7, instead of 11, then they will have difficulty understanding that $a + b$ represents two sets containing a and b items respectively. Collis (1974, 1975 as cited in Kieran, 1992) gave students a task to find the value of ☐ in expressions such as (436 + ☐) + (791 – 285) = (436 + 791). He used small numbers, large numbers and letters. Collis found that young children were successful only with the small numbers because the only tool they had was to compute. “Collis described the ability to work with expressions without reducing them by calculating as ‘Acceptance of Lack of Closure’” (Kieran, 1992, p. 396).

2.2.2 Simplifying expressions

When simplifying expressions, at first students can relate them to simple, procedural conceptions of expressions, but this cannot last for long. Students must develop a sense of operating on the algebraic expression as a mathematical expression in its own right. According to Greeno (1982, as cited in Kieran, 1992) there was no consistency in the way that beginning algebra students approached problems or in the way they carried out the operations. On various occasions, students would perform different operations on similar tasks, therefore displaying that they were not always sure of when to use each piece of knowledge. This was also evident in students with advanced knowledge of algebra who were able to successfully complete tasks in a given area but who were not able to transfer their knowledge to the next topic.

2.2.3 Equations

In order to properly understand equations, students must understand the symmetric and transitive character of equality. Beginning algebra students mistakenly see the equal sign as a “do something” symbol rather than as a symbol of equivalence between the left and right sides of an equation. This incorrect perception of the equal sign continues with older algebra students who use the equal sign to show steps in solving an equation and this is even evident with college students who, despite being able to correctly solve problems, still have a poor understanding of the equal sign.

2.2.4 Solving equations

Solving an equation involves performing the same operation on both sides of the equation. There have been many studies that have investigated student equation solving and the various methods used by algebra students have been classified into the following types (Kieran, 1992, p. 400):

1. use of number facts
2. use of counting techniques
3. cover-up
4. undoing (or working backwards)
5. trial and error substitution
6. transposing (that is, change side – change sign)
7. performing the same operation on both sides

The last two types are referred to as the formal methods and the first two are not taught since those are things with which students come to algebra. Whitman (1976, as cited in Kieran, 1992) looked into the relationship between the cover-up method and the formal procedure of performing the same operation on both sides of the equal sign. She found that students who learned the cover-up method performed better than students who learned both methods close together, and both of these groups of students performed better than students who only learned the formal method. The trial and error method is time consuming and unless it is performed and recorded in a structured way, it requires an extremely good working memory. Once students learn a formal method, they usually forget about the trial and error method, however, the trial and error method is good for verifying that an answer is correct. Even though there are formal methods of equation solving, it is the students who have experience with multiple methods, rather than just the formal methods, who have a better understanding of equation solving. Students are unable to distinguish structural features of equations. Errors such as the switching addends error and the redistribution error are commonly made among beginning algebra students. For example, students believe the following sets of equations to be equal:

Switching Addends Error:

$x + 37 = 150$ is judged to be the same as $x = 37 + 150$

Redistribution Error:

$x + 37 = 150$ is judged to be the same as $x + 37 - 10 = 150 + 10$

Greeno (1982, as cited in Kieran, 1992) states that beginner algebra students are not able to show that an incorrect solution is wrong except to re-solve the given equation. Again, as mentioned before, students are forgetting that the trial and error method is good for checking whether or not an answer is correct. Also, they are not realizing that when a solution is substituted back into the original equation, it will yield equivalent values on both sides if it is correct.

2.2.5 Word problems

Research literature on algebra word problems is divided into three parts: traditional word problems that are found in many textbooks, problems that have been approached from a functional perspective and open-ended generalization problems that are seldom found in American algebra textbooks.

Generating equations to go with traditional word problems that are found in many textbooks is one of the most difficult things for high school algebra students. To represent a problem, students use either a direct-translation or a principle-driven approach. A direct translation is a phrase-by-phrase translation of the word problem into an equation containing numbers, variables, and operations. A principle-driven approach uses a mathematical principle to organize the variables and constants of a problem. The technique most commonly used to teach algebra classes how to solve word problems is to put together an equation that involves unknowns and operations according to a mathematical relation and to then use a process of algebraic manipulation to isolate the unknown term and find the solution.

By approaching problems from a functional perspective, researchers have attempted to provide an alternative avenue for students to understand unknowns and variables. Kieran (1992) states that studies have been done, by Kieran, Boileau and Garançon (1989) among others, that provide evidence that a computer environment can be helpful for and can support a procedural approach. Students are able to enter their problems into the program using regular language and can then input trial values and see what the corresponding output is.

Open-ended generalization problems showed how students passed through the same three stages of development through which the content of algebra evolved:

1. In the rhetorical method, the student uses words to describe a general procedure, but does not represent that with a letter.
2. In the Diophantine method, students use a letter (or more than one letter) to represent an unknown quantity and to say that the method can be applied to any number but does not use symbols for a general "given" quantity.
3. In the Vietan method, the student uses letters for both unknown and given quantities. Students find it difficult to first formulate an algebraic generalization then, once they do have that, they find it difficult to use it and appreciate it as a general statement.

2.2.6 Functions and graphs

In most algebra textbooks, a function is defined as a relation between members of two sets or members of the same set, such that each member of the domain has only one image. This means that the teaching of functions in algebra classes emphasizes structural rather than procedural interpretations. Verstappen (1982, as stated in Kieran, 1992, p. 408), lists three categories for recording functional relations using mathematical language:

1. geometric – schemes, diagrams, histograms, graphs, drawings
2. arithmetical – numbers, tables, ordered pairs
3. algebraic – letter symbols, formulas, mappings

Functional thinking requires children to pay attention to change and growth, and to look “for patterns in how quantities vary in relation to each other. A *function* is a way to express that variation.” (Blanton, 2008,) “A *function* is a mathematical statement that describes how two (or more) quantities vary in relation to each other” (ibid., p. 31). Functional thinking provides students with the opportunity to work with a rich set of tools: tables, graphs, function machines, input/output charts, etc.

With the knowledge of what has been said about algebra, let us now consider what is said about early algebra.

3: EARLY ALGEBRA

In this chapter, I look at early algebra as defined by various researchers and I examine why early algebra would be beneficial. I also look at the powers that students have and how the use of these powers, in relation to numbers and relationships, leads to algebraic thinking. Finally, I look at what the three purposes of algebra are and how they tie in with early algebra.

Kieran (1992) mentions that the introductory chapter of most textbooks emphasizes links to arithmetic. This idea could be tied in with the ideas of Porteous (2008) since he believes that algebra concepts should be taught at a much earlier age. In his article, Porteous (2008) makes a bold statement by saying that algebra should not be taught, and that instead, arithmetic should be taught really well. By the end of his article, however, he has adjusted his statement and states that algebra should just be taught earlier and should not be called algebra since that term and the idea of moving on to a different topic is what scares students.

Porteous (2008) is not alone in his belief that algebra should be introduced earlier than is currently happening in North America. Kaput (1995) and Mason (1996), among others, also suggest that algebra be introduced earlier in the current curricula. As well, there are Soviet studies that show that children can be taught algebra at an earlier age than is currently happening in North American classes and that the way we currently teach algebra should be changed (Davydov, 1991, as cited in Schliemann, 2007).

3.1 What is early algebra?

In defining early algebra, it is important to note that it is not the same as pre-algebra. Pre-algebra is usually taught to students in intermediate grades and reviews some arithmetic concepts with the idea of emphasizing the ties to algebra. Pre-algebra is focused on making it a smooth transition for students to move from arithmetic to algebra.

Early algebra is the algebraic reasoning among and the algebra-related instruction of elementary school-aged children. Bastable and Schifter (2008) state that early algebra is about exploring generalizations that have to do with arithmetic operations and that it involves making and expressing generalizations.

Carpenter, Franke, and Levi, (2003) clarify that early algebra is not just implementing the high school algebra curriculum in elementary school. The goal is for elementary schools to support the development of students' thinking so that they are thinking about arithmetic in the same way that they would eventually have to think in order to be successful with algebra. Carpenter, Franke, and Levi (2003) note that "the fundamental properties that children use in carrying out arithmetic calculations provide the basis for most of the symbolic manipulation in algebra" (p. 2).

Kaput (2008) also believes that early algebra is not simply starting algebra early. Certain things must be considered, such as reconstructing the curriculum, changing classroom practice and assessment, and changing teacher education. Kaput (2008) notes that the “Principles and Standards of School Mathematics” has been encouraging a longitudinal view of algebra, one that encompasses thinking and problem solving beginning in elementary school and progressing all the way through mathematics education.

According to Carraher, Schliemann, and Schwartz (2008), early algebra has three distinguishing characteristics. It “builds on background contexts of problems,” “formal notation is introduced only gradually,” and it “tightly interweaves existing topics of early mathematics” (pp. 236-237).

3.2 Why early algebra?

Schoenfeld (2008) asserts that “the purpose of early algebra is to provide students with the kinds of sense-making experiences that will enable then [*sic*] to engage appropriately in algebraic thinking” (p. 482). Blanton (2008) states that early algebra is meant to encourage students to see and describe mathematical structure and relationships for which they have constructed meaning. Early algebra is encouraging students to “think” about mathematics, not just “do” mathematics. Early algebra is about providing students with opportunities to use their algebraic thinking to help them to really develop an in-depth understanding of arithmetic.

Tierney and Monk (2008) believe that elementary school students are able to think in a way that underlies algebra. It appears that the National Council of Teachers of Mathematics (NCTM) would agree with Tierney and Monk because, as Carraher and Schliemann (2007) took note of, the NCTM is in support of early algebra and recommends that algebra be woven into the preK-12 curriculum. Having an algebraic strand in the elementary school would help students to have a better understanding of algebraic symbolization (Bastable & Schifter, 2008).

Kaput (2008) states that there are a few beneficial reasons why algebra in early grades mathematics should be considered:

1. To add a degree of coherence, depth, and power typically missing in K-8 mathematics.
2. To ameliorate, if not eliminated, the most pernicious and alienating curricular element of today's school mathematics: late, abrupt, isolated, and superficial high school algebra courses.
3. To democratize access to powerful ideas by transforming algebra from an inadvertent engine of inequity to a deliberate engine of mathematical power.
4. To build conceptual and institutional capacity and open curricular space for new 21st-century mathematics desperately needed at the secondary level, space locked up by the 19th-century high school curriculum now in place (p. 6).

3.3 Children's powers

Children have incredible powers for making sense of the world around them. When they use their powers in the context of number and relationships, algebraic thinking is what happens (Mason, 2008). Early algebra gives children the opportunity to use and develop these powers. Mason (2008) lists five powers that students have that are relevant to mathematics and, in particular, algebraic thinking.

1. Imagining and Expressing. Being able to imagine things that are not physically present while still remaining physically present in the material world is a power that is important to the development of mathematical thinking. “The power to imagine and to express needs to be invoked in the classroom, frequently and effectively, by encouraging students to express perceived generalities, relationships, connections, properties, and so on” (Mason, 2008, p. 60).

2. Focusing and De-Focusing. Children are able to focus their attention on certain details, to shift their attention from one detail to another, and to de-focus their attention in order to take in a broader view. With teaching, the concern is getting students to focus on the details that the teacher knows are important to notice.

3. Specializing and Generalizing. In language, there are words that are used to generalize, such as *cup* or *chair*. To make these words specific, we add adjectives, making the words: *tea cup* or *rocking chair*. Mathematics is similar in that students should know how to look at a question from a general point of view, but should also possess the ability to focus in on a specific part when necessary.

4. Conjecturing and Convincing. In life, children make conjectures and then change them when they come to counterexamples. Teachers need to provide children with opportunities to do this in class since this power is essential to algebraic thinking. It helps if teachers

display examples of this power themselves so that students can learn by example, instead of trying to make conjectures for the students. Mason (2008) states that “mathematicians work best in a conjecturing atmosphere in which conjectures are articulated in order to try them out, see how they sound and feel, to test them and so to see how to modify them as and when necessary” (p.65). Mathematics thrives in a classroom where students are able to have moments to do individual work as well as opportunities to work with others to come up with conjectures that do not have counterexamples.

5. Classifying and Characterizing. Sorting involves stressing some features and ignoring others, which requires the ability to discriminate these features. Young children organize objects such as blocks according to their colour, size, shape, thickness, or a combination of those, allowing them to focus on some attributes while ignoring others. Older children might sort numbers based on their divisibility.

Even though each child has these powers, it is not until they have exercised and developed them and can use them intentionally that they are actually thinking mathematically. It is possible to believe that over the years, people may not have been able to fully develop their powers and therefore might not succeeded at algebra. These people believe that arithmetic is all they need to know, and do not see the importance of teaching algebra to future generations.

Algebra should be taught to everyone because it comes from the natural development of their powers. Being able to use their algebraic thinking allows people to more fully participate in a democratic society since this involves: “recognition and use of general methods, testing and challenging of generalizations, and questioning the modelling assumptions on which such generalizations are often made” (Mason, 2008, p.79).

Some teachers may think that the curriculum is crowded enough already and that there is no time to let students develop their powers. Mason (2008) believes the opposite, that it is a waste of time to not promote students to use and develop their powers. Upfront, there may be some extra time spent on the development of students’ powers but this greatly helps the students since this will help them to be more efficient and effective learners (Mason, 2008).

Mason (2008) gives some examples of how “algebra, like arithmetic, emerges from the use of children’s natural powers” (p. 80).

1. Awareness of and Expressing Generality. When a teacher notices an opportunity for students to express a generality and to give a specific example for this generality, time should be taken to allow for this to happen. Algebra is like a new language and students need to be given opportunities to express themselves in this new language.

2. Working With the As-Yet-Unknown. When students have solved a problem, they can be asked to find other problems of the same type

that would produce the same answer, thereby getting them to think of the as-yet-unknown values that will give them the same answer.

3. Freedom and Constraint. The aspects of variables (expressing generality and referring to as-yet-unknown numbers) are present at a very young age and are brought together by looking at situations in terms of freedom and constraint. Freedom can be evident when the teacher says that she is thinking of a number; constraint is put on this when the teacher then says that the number is between five and ten.

4. Toward Manipulation: Multiple Expressions for the Same Thing. This is when students use different ways to express how they solved a problem. From the different expressions, we can see how the students were thinking about the problem.

5. Guessing and Testing. Guess and test is a good tool for mathematics and can progress to more sophisticated versions such as “try and improve” or “spot and check.”

3.4 Algebraic thinking in arithmetic

In order for arithmetic to be fully experienced, algebraic thinking must be present, which involves the use of students’ powers relating to generalizing. Mason (2008) gives a few examples of when students’ powers are used to produce algebraic thinking.

1. Rhythm and Sequence. Students create a sequence such as blue, red, blue, red, blue, red, etc. and from this sequence they come up with clapping instructions such as hard on blue and soft on red, and from there, they can count at the same time and eventually predict what colour a certain number will be.
2. Dealing With Not Knowing. When there are, for example, three cows in one field and five cows in another field, the total number of cows is unknown. In mathematics, acknowledging ignorance, that there is an unknown, allows students to work with the unknown and treat it as though it were known.
3. Methods as Implicit Generalization. In learning arithmetic, the point is to learn methods for determining sums or products, etc. not to memorize all possible results. In order to learn these methods, students experience different samples of these methods in hopes that they will recognize for themselves what the method is and be able to apply it in different situations. To help students do this, teachers should ask questions that make students think about the method, such as “What is the method that was used?”, “What if something was changed?”, or “In what other situations would this method work?”
4. Tracking Arithmetic. Even though arithmetic is presented to children as being the production of a single number answer, it is so much more than that. Zazkis (2001) suggests using activities such as

Think of a Number since it will display the role of the number in a calculation, which is also known as the algebraic use of numbers. In this activity, students think of a number, perform a number of operations on their number, and then end up with the same number that they started with. Ideally, in order to figure out how the students end up with the same number they started with, they will look at the question from a general point of view and not use specific number examples. For the students who do insist on using specific numbers, Zazkis (2001) suggests using very large numbers, such as 683 157 249, so that the attention of the students is turned towards the structure.

5. Implicit Awareness of Arithmetic Structure. Children are aware of how numbers can be manipulated, even though they may not be aware of the specific rules that they are using. Mason (2008) states that "algebraic thinking is the thinking that results in general statements about number (in school) and about structure (in university)" (p.76). Kaput, Blanton and Moreno (2008) put it another way by saying that when the labels that are given to objects are manipulated, you have algebra.

3.5 Like a language

Mason (2008) states that the most useful view of algebra is to see it as a concise, pliable language where generalities, as well as constraints on these generalities, can be expressed. Algebra is like a new language for students to learn, and the use of letters as symbols is a clear sign that you are “operating in the mathematical language of algebra” (Osborne & Wilson, 1992, p. 422). If children conquer these symbols early, they will learn faster and they will have a greater in-depth understanding (Osborne & Wilson, 1992). Since algebra, and all of mathematics, is like a new language, it is important for teachers to use correct terminology in mathematics class so that students will become familiar with the new vocabulary (Brizuela & Earnest, 2008). As well, it is important for students to be able to practice this language by having mathematical discussions with their peers or with the teacher.

3.6 The three purposes of algebra and how they appear in early algebra

Algebra can be used for three different purposes: making generalizations, finding the unknown, and using variables. All three of these purposes should be present in elementary school mathematics.

3.6.1 Generalizations

In mathematics, a generalization is a statement that expresses a general truth about a set of mathematical data (Blanton, 2008). Expressing generalities is a holistic approach to mathematics and should be present in every mathematics lesson (Mason, 2008).

Generalized arithmetic, which refers to making generalizations in regards to the operations on and properties of numbers (Blanton, 2008), is usually the first type of algebra that students experience (Osborne & Wilson, 1992). It is not necessary to have arithmetic as a prerequisite to generalized arithmetic; instead, both can result from the use of students' powers to make sense of the world that they live in. Children are able to express generalities independent of number, so it is possible for algebraic thinking to be present before arithmetic is mastered (Mason, 2008). Osborne and Wilson (1992) note that algebraic concepts and skills help students to generalize arithmetic concepts.

Mathematics educators have come to see working with generalities as one of the characteristics of algebra, and therefore consider the recognition and description of general rules for describing patterns in elementary school to be algebra (Stacey & MacGregor, 2001). "Algebra is a very powerful way of expressing patterns concisely. It is concerned with generalities and finding equivalences among expressions" (Suggate, Davis, & Goulding, 2006, p. 129).

3.6.2 The unknown

Algebra uses letters to represent unknowns but arithmetic uses letters to represent something different. In arithmetic, a letter could be an abbreviation of a unit of measure (15m means 15 meters) or a letter could represent an object like drinks or pizza (5d + 3p means five drinks plus three pizzas). This use of letters in arithmetic can cause students to be confused when learning algebra, since in algebra, 15m would mean 15 times a number, and 5d + 3p would mean five times one number plus three times another number. Since there is this possibility for confusion, it is best to use a different symbol, for example, a question mark or a box, to represent the unknown when first introducing it to students: 5 + ? = 9 or 5 + ☐ = 9. (Suggate, Davis & Goulding, 2006)

3.6.3 Variables

Letters in algebra are not only used to represent unknowns, but also to represent variables. For example, in $y = 7 + x$, x and y are called variables because x could take many values and depending on the value that x takes, we can figure out y (Suggate, Davis & Goulding, 2006). Osborne and Wilson (1992) state that "the use of variables is the single characteristic that determines whether you are doing algebra or doing arithmetic" (p. 422). They also note that it is important for students to develop an understanding of variable when getting ready for algebra and that this is possible through numerical experiences. One such experience could be making a table, which is a good way to emphasize the use of variables as the summary of many cases.

Küchemann, (as cited in Osborne & Wilson, 1992), has come up with six stages that students progress through in their journey to understand the concept of variable.

1. In the first stage, Letter Evaluated, with a question such as: if $3 + 2x = 11$, then what is x?, students might give an answer for x such as 4 or 5. In this stage, there is no manipulation of the letter, students just recall a number that seems to "fit", and this could result in an incorrect response.

2. In the second stage, Letter Not Used, the student is able to respond to a question without ever considering the letters.

3. In the third stage, Letter Used as an Object, the letter is seen to stand for an object. In other words, $6a$ means six apples.

4. In the fourth stage, Letter Used as a Specific Unknown, students assign a value to a letter even though it is not known. For example, in the case of $R = S + T$ and $R + S + T = 30$, students said that $R = 10$.
5. In the fifth stage, Letter Used as a Generalized Number, if students are asked to list all of the values of *A* when $A + B = 12$, at first they may only list a couple of possibilities and not realise that all of the possible values for *A* are needed.
6. In the sixth stage, Letter Used as a Variable, the letter is seen as representing a range of possible values. For the question: Which is larger: $3K$ or $K + 3$?, if students find the answer by testing one or just a few numbers, then they are still at stage five, but if they consider this relationship in terms of all numbers, then they are at this stage.

These three purposes of algebra: making generalizations, finding the unknown, and using variables, are what I use as my main focus when examining the students' work on each of the tasks. As well, I look to see if the students display any of the powers that Mason (2008) has mentioned.

4: THE STUDY

The literature made me curious about the presence of algebra and algebraic thinking among elementary school students. It made me want to know how students are solving problems in mathematics and whether they are showing evidence of algebraic thinking. This led me to my research questions: Can elementary school students show evidence of algebraic thinking? If so, in what ways is their algebraic thinking evident?

In this chapter, I describe where I conducted my study and who the participants are. I reveal each of the tasks that I presented to the participants and explain how those tasks were chosen or designed.

4.1 Setting

The data collection took place at an elementary school in Burnaby in the spring of 2011. I was a teacher-on-call in Burnaby and had been called to this school many times. Both the staff and students all knew who I was, so it made for a comfortable environment when I began presenting mathematics tasks to one of the classes. For three months, I visited the class on Mondays and Tuesdays after lunch for 30 - 40 minutes each day.

4.2 Participants

The participants were 22 grade two/three students, 10 girls and 12 boys. Included in these 22 students are five students who had irregular participation in the tasks: one student was usually in ESL class, one student had trouble focusing in school, and three students received reading support during the time that I worked on mathematical tasks with the class.

4.3 The tasks

The following tasks were chosen for specific reasons. They are accessible to the various abilities of the students, while at the same time providing the possibility for the task to be extended. They involve topics that the students would find interesting and can relate to. As well, the tasks could be solved either arithmetically or algebraically depending on how the students choose to approach the problem.

1. ***Think of a number:*** Create a series of operations (addition and subtraction) to perform on any number, and that would result in the same number that was started with.

While reading a chapter from "Mathematical knowledge for primary teachers" (Suggate, Davis & Goulding, 2006), I was reminded of this task. I have done this type of task in the past where the operations included multiplication, division, addition and subtraction, but for this grade two/three class I used only addition and subtraction.

2. ***Guess my rule:*** Partner #1: think of an operation that will be performed on any given number (input) and then give the result

(output). Partner #2: guess what the operation is, based on the input and the output.

I adapted this task from problems that I found in “Beginning algebra thinking for grades 5 – 6” (Goodnow, 1994). This type of task was also mentioned in “Mathematical knowledge for primary teachers” (Suggate, Davis & Goulding, 2006).

3. ***Handshakes:*** There were five people at a party. How many handshakes would there be if all of the people shook hands with each other once. What if the number of people was six, seven, eight, nine, or ten?

This is a well-known task and I do not recall where I first encountered it.

4. ***Popsicle stick pattern:*** Given the number of Popsicle sticks in the first, second, and third figures of a pattern, determine the number of Popsicle sticks in the tenth and 25th figures.

I found this task in “Curriculum reform and approaches to algebra” (Stacey & MacGregor, 2001).

5. ***The number of 7s:*** How many times does the digit seven appear from zero to 500?

This task is adapted from a problem that was introduced in EDUC 847 in the spring 2009 semester.

6. ***Cows and chickens:*** At a farm with only cows and chickens, if there are 100 legs all together, how many cows and chickens could there be?

The idea for this task came from one of my university courses.

7. ***Venn diagrams:*** Given some information about a Venn diagram, identify the numbers that represent each part of the Venn diagram.

I found this task in “Beginning algebra thinking for grades 3 – 4” (Hoogeboom & Goodnow, 1994).

8. ***Balloons in a box:*** Given the colours of balloons and the total number of balloons in a box, figure out how many balloons of each colour there could be in the box.

I adapted this task from problems in “Beginning algebra thinking for grades 5 – 6” (Goodnow, 1994).

9. ***Magic Square:*** Fill in a 3 x 3 grid using nine specified numbers once only. Each row, column, and diagonal must have the same, given sum.

I found this task in “Beginning algebra thinking for grades 3 – 4” (Hoogeboom & Goodnow, 1994).

4.4 Data collection

The data collected consists of the students’ work that they handed in at the end of each session and the notes that I wrote down immediately following each session. In the notes, I was able to record my observations as well as the conversations that I had with the students.

5: RESULTS AND ANALYSIS

This chapter consists of an examination of the students' work on the tasks that I presented. I explain the algebraic thinking that is evident in the students' work on these tasks.

5.1 Think of a number

5.1.1 The task

Create a series of operations (addition and subtraction) to perform on any number, and that would result in the same number that was started with.

5.1.2 How the task was presented

I asked each of the students to think of a number. I then asked them to add four, subtract two, add five, and then subtract seven. Most students ended up with the number that they originally started with, which is what was expected. A few students made a mistake along the way and did not end up with the same number that they started with. Some students noticed that the number that they ended up with was the same number that they started with, but not many students seemed to find that significant. We did another example, this time they thought of a number and then added ten, subtracted one, added five, subtracted eight, added one and subtracted seven. Again, most students got the same number that they started with and this time they seemed impressed by this. We did a third example and before we even started, the students said that they thought the answer was going to be the same number that they started with, so we tried it. They thought of a number then added six, subtracted five, added one, subtracted two, added four, subtracted one, added two, added one, and subtracted six. They were all satisfied that they got the number that they started with. Now, the task for the students was to create a series of operations (addition and subtraction) to perform on any number that would result in the same number that they started with. The students sat with a partner, and when they had created their own series of operations that they thought would result in the same number that they started with, they presented it to their partner.

5.1.3 Students' work

Twenty students worked on this task. All of the students approached this task in a similar fashion. They thought of a number and then added and subtracted other numbers until they ended up with the number that they started with. When the students were ready to present the problem to their partner, they removed their number and asked their partner to think of a number. Figures 5-1 and 5-2 exemplify a typical student's work.

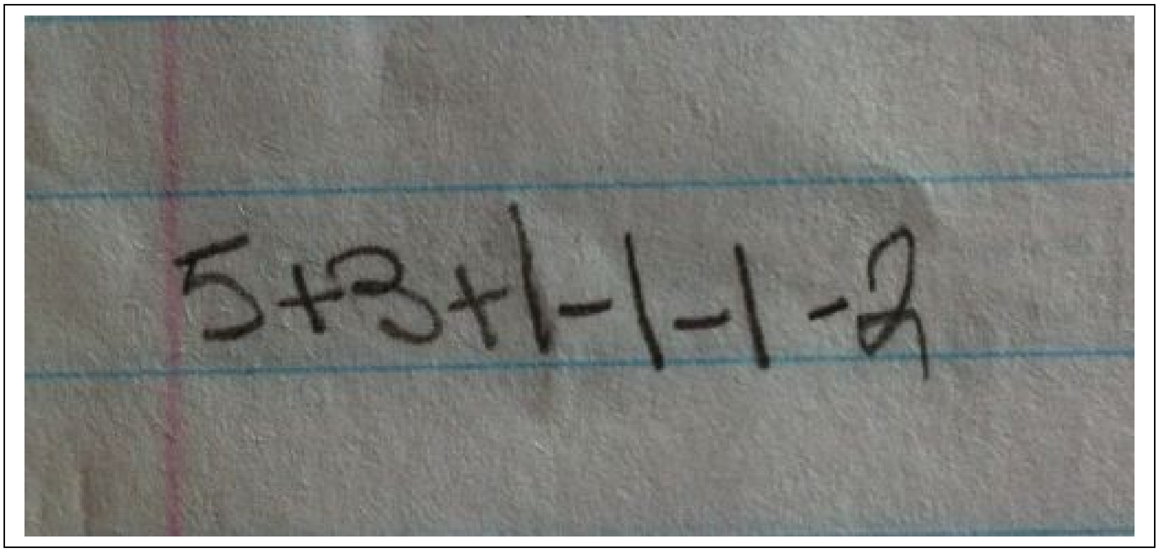

Figure 5-1 Kelsey's work on task 1

Kelsey (Figure 5-1) began with the number five and then added and subtracted numbers until she ended up with five again.

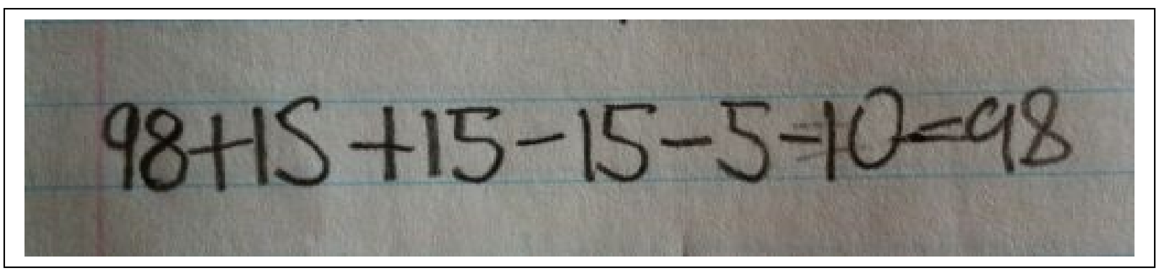

Figure 5-2 Diego's work on task 1

Diego (Figure 5-2) began with the number 98 and, like Kelsey, added and subtracted numbers until he ended up with 98 again.

5.1.4 Algebraic thinking

The students are showing the beginning stages of algebraic thinking. They are not yet using letters to represent generalities but they are recognizing that if the same amount is added and subtracted from a number, the answer will be that number.

When students find a series of operations that results in the same number that they started with, they notice that no matter what number they start with, they are still ending up with that same number as the result. In this way, students are making the generalization that their series of operations will hold true for any number.

The students are displaying evidence of algebraic thinking according to the definition of Kaput's (2008) first core aspect of algebra. They are making generalizations using any resources available, which, for them at the moment, is natural language.

The students are displaying one of the powers that Mason (2008) talks about: focusing and defocusing. When students look at their whole series of operations, they are displaying their ability to defocus. When they look at an individual operation in their series, the students are displaying their ability to focus.

Of the 20 students, 18 were beginning to demonstrate their awareness of the inverse nature of addition and subtraction. Some examples of what they included in their series of operations are: +1, -1, or +15, -15 (see Figures 5-1 and 5-2).

5.2 Guess my rule

5.2.1 The task

Partner #1: think of an operation that will be performed on any given number (input) and then give the result (output). Partner #2: guess what the operation is, based on the input and the output.

5.2.2 How the task was presented

I drew a cloud on the board and told the class that it was a magic cloud. If we put a number into the cloud, it did something magical to the number and then gave us back a different number. I asked the students for numbers to put into the cloud. We put three into the magic cloud and out came six; some students suggested that the magic cloud added three to the number that we put in. We put 10 into the magic cloud, and out came 13, now the students seemed sure that the magic cloud added three. I told the class that it was always a good idea to check our guess three times before coming to a conclusion on what the magic cloud did, so we tried a third number. We put four into the magic cloud and out came seven. The class concluded that this magic cloud added three to any number that was put into it. We tried a second magic cloud. This time we put in 10 and out came nine, we put in six and out came five, we put in five and out came four. After the second number that we put into the magic cloud, some students were ready to state what the magic cloud did, but they were reminded by the rest of the class that they should make three guesses before saying what the magic cloud did. After our third guess, the class concluded that the magic cloud subtracted one from any number that was put into it. Now that we had done a couple of examples as a class, it was the students' turn to create their own magic clouds. Students worked in pairs and each pair of students received a piece of paper to record their work.

5.2.3 Students' work

Eleven pairs of students worked on this task. In pairs, students took turns making up magic clouds and guessing what the magic clouds did. Once their partner had successfully guessed what the magic cloud did, students could write the answer on the magic cloud.

In the examples that we did as a class, I tried to stress the importance of choosing at least three different numbers as inputs. My reasoning for this was to prepare the students for the future when they might encounter this type of problem when multiplication and division were possibilities as well as addition and subtraction. None of the students remembered to use three inputs; however, for our purposes one input could suffice since we were only dealing with the addition and subtraction of positive integers. Figure 5-3 shows how the operation of the magic cloud, which is minus 85, was figured out after only one input, 90.

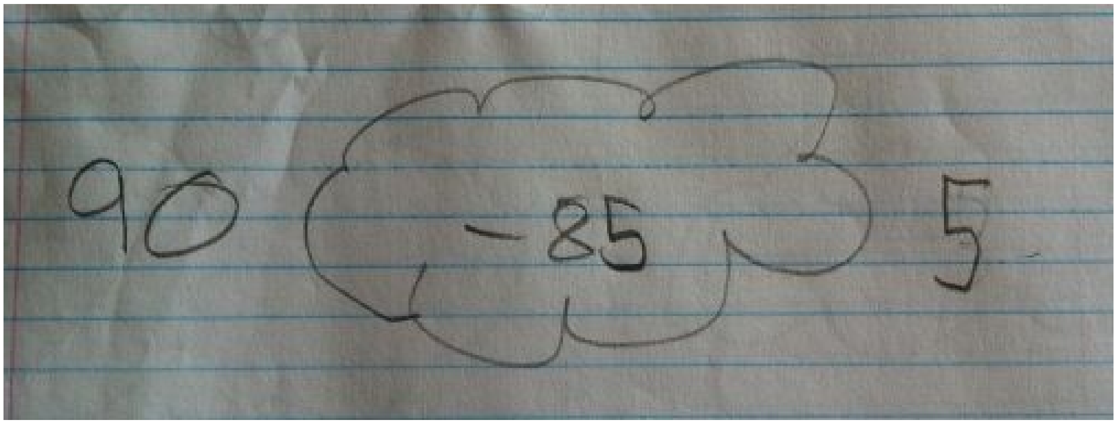

Figure 5-3 Jaymes and Michael's work on task 2

Yvette and Andrea, after figuring out what each other's magic cloud did, asked if they could try using two magic clouds. I was curious as to what they might come up with so I said that they could. The rest of the class heard this and immediately got excited. I could hear students saying things like, "We get to try it with TWO magic clouds!", or "After we try it with two magic clouds, let's try it with THREE!"

By using more than one cloud, the number of possible answers was increased, potentially making it more difficult for students to guess the operations that their partner chose. Some students, like Yvette and Andrea (Figure 5-4), suspiciously displayed that they only made one guess to correctly get the answer. Upon circulating around the class, I observed that students were giving each other hints and were only recording the correct guess and not the incorrect ones. Amanda and Kelsey also showed that they were able to only make one guess before getting the correct answer; however, in their case, it seems believable (Figure 5-5).

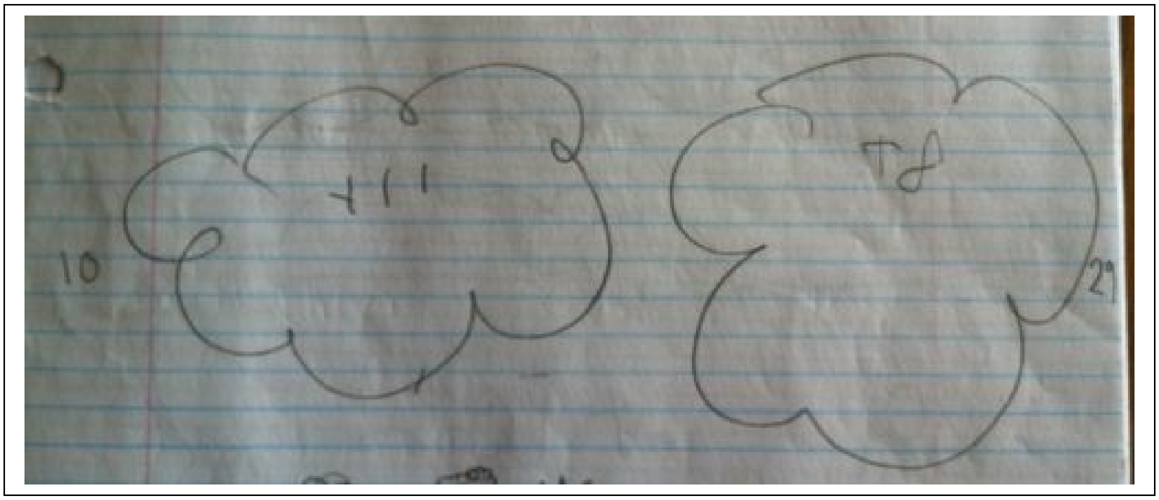

Figure 5-4 Yvette and Andrea's work on task 2

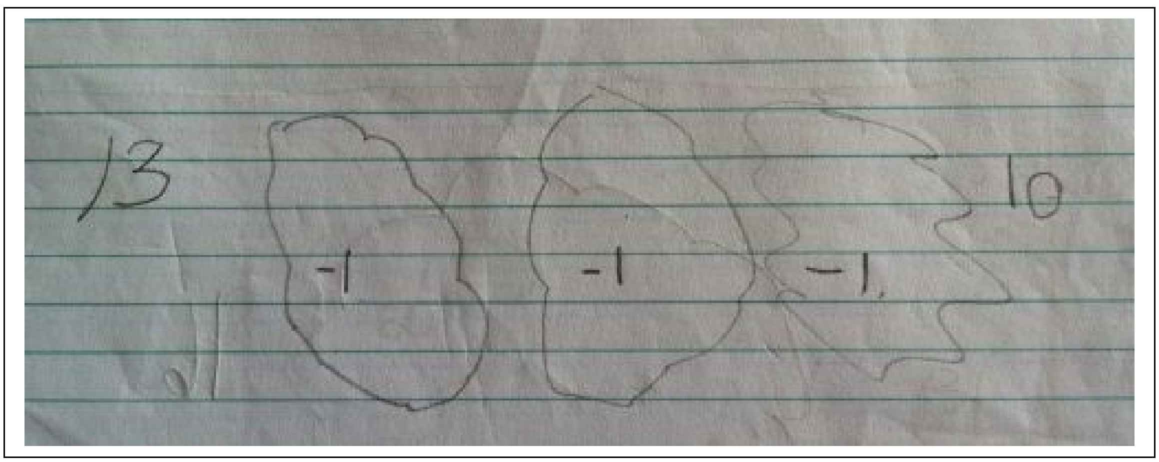

Figure 5-5 Amanda and Kelsey's work on task 2

5.2.4 Algebraic thinking

With this task, students were able to figure out which operation, when performed on a given number, would result in the given answer.

In students' work on this task, there is evidence of finding the unknown. Students are given a starting number and an ending number, and their job is to find out what the addition or subtraction is. For example, with Jaymes and

Michael's magic cloud (see Figure 5-3), they are first noticing that to go from 90 to 5, you must subtract, then they are asking themselves "90 minus what equals five?" This becomes, 90 - ☐ = 5, which eventually becomes 90 – x = 5.

One of the powers that Mason (2008) talks about is being displayed in this task. The students are imagining and expressing; they are using their imagination to see a magic cloud that performs some kind of operation on a number to produce another number. The students are then able to take this mental image and write on paper what the magic cloud does.

Guessing and testing, one of the examples that Mason (2008) gives of how algebraic thinking is derived from children using their powers, is being displayed by the students in this task. Students are making guesses in their head as to what the magic cloud does and they are then testing their guesses to see if they are correct.

When the students used two magic clouds, or two operations, they were learning and accepting that there was more than one correct combination of operations that would result in the given answer.

5.3 Handshakes

5.3.1 The task

There were five people at a party. How many handshakes would there be if all of the people shook hands with each other once. What if the number of people was six, seven, eight, nine, or ten?

5.3.2 How the task was presented

I started with getting three students to come up to the front of the class. I explained that they were going to shake hands with each other and that we were going to count how many handshakes there were. I also clarified that a handshake between two people counted as one handshake, not two. Once the three students finished shaking hands, the rest of the class agreed that there were three handshakes.

I told the class that, in pairs, they were going to figure out the number of handshakes that there would be if five people shook hands with each other. Each group received a piece of paper to record their work.

5.3.3 Students' work

Eleven pairs of students worked on this task. There were different ways that the pairs approached this problem. Four pairs counted the number of handshakes one at a time either by drawing a picture, counting in their head or enlisting the help of five classmates to act out the handshakes. This proved successful for some groups, but other groups lost track of their counting and ended up with an incorrect answer. Another approach that two pairs, such as Zachary and Thomas in Figure 5-6, used was drawing five people in a circle and then drawing a line between the people to represent a handshake. This proved to be quite successful and the students who used this approach felt confident that their answer was correct. Three pairs, such as Leroy and Diego in Figure 5-7, noticed that of the five people, the first person had to shake hands with four other people, because they did not shake their own hand. These students then jumped to the conclusion that each person had to shake hands with four other people in order to shake hands with everyone. This incorrectly gave them the answer of 20 handshakes. Two pairs, including Rita and Jessica in Figure 5-8, noticed the same thing, that of the five people, the first person had to shake hands with four other people. They also noticed that the second person only had to shake hands with three other people because he had already shaken hands with the first person. The third person shook hands with two people, the fourth person shook hands with one person and the fifth person had already shaken everyone's hand. This way of thinking led the students to discover the equation 4 + 3 + 2 + 1 = 10.

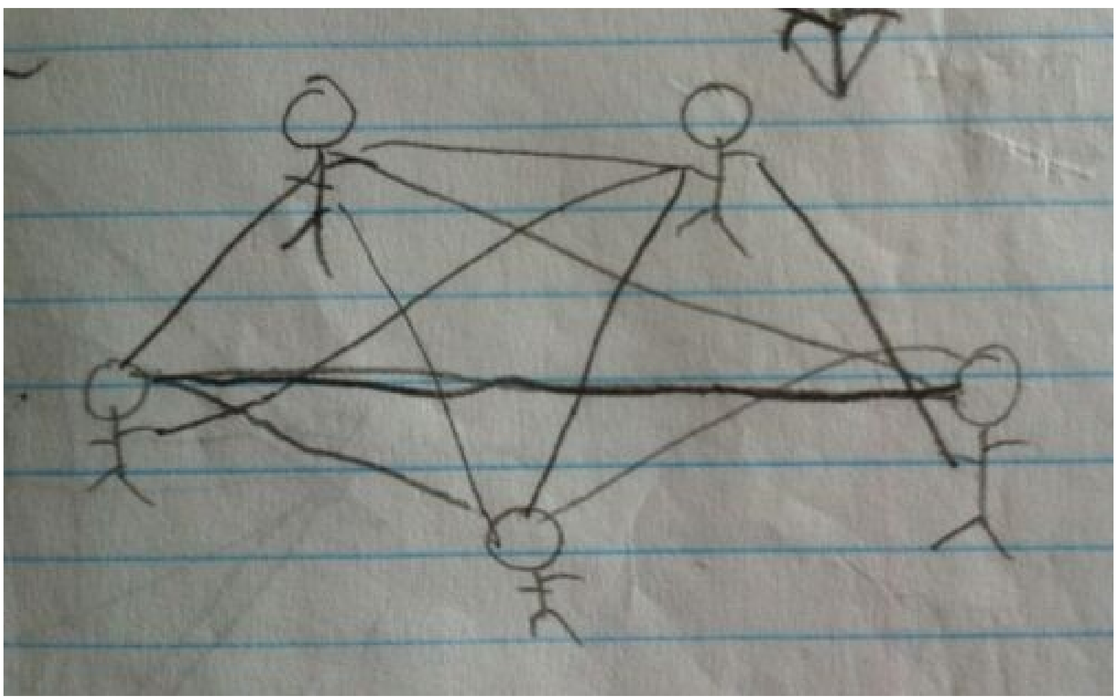

Figure 5-6 Zachary and Thomas' work on task 3

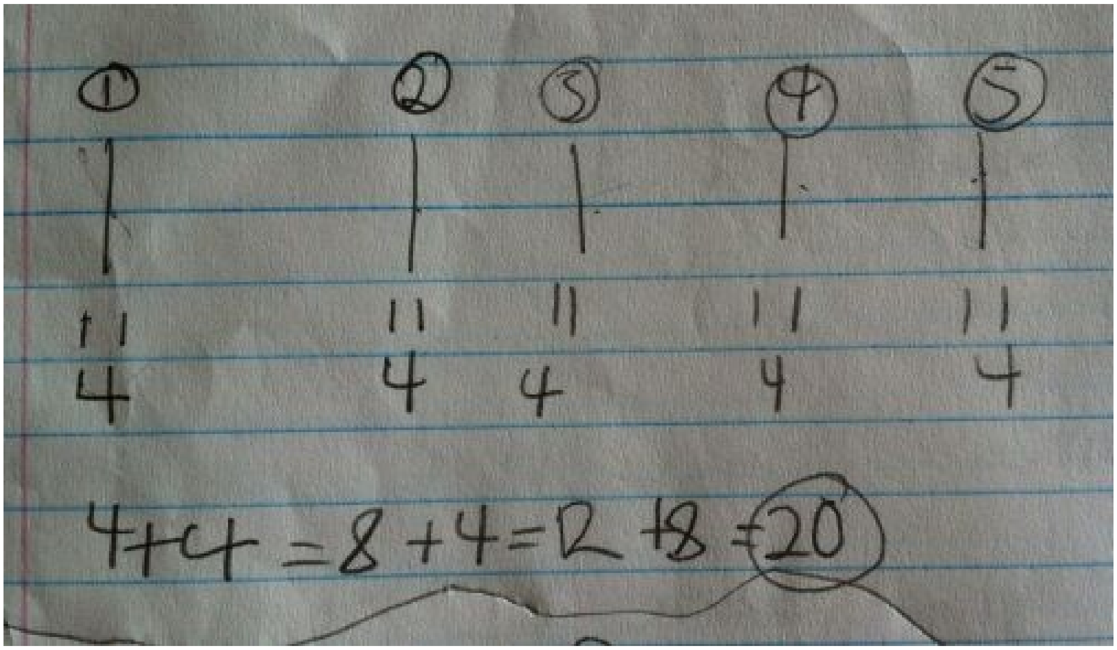

Figure 5-7 Leroy and Diego's work on task 3

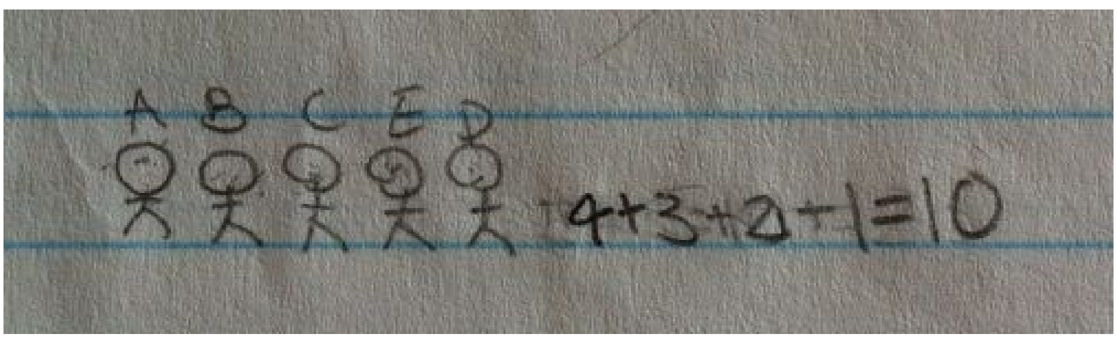

Figure 5-8 Rita and Jessica's work on task 3

As students thought that they had found the number of handshakes for five people, they brought their page to me to show me their work. If I saw that they made a mistake in their work, I would ask them a question about it, which usually led to them noticing their own mistake without me having to say that something was incorrect. If a group had correctly found the number of handshakes for five people, I then asked them to find the number of handshakes for eight people. The students who used the equation that was mentioned above (see Figure 5-8), were able to easily find the answer and even move on to finding the number of handshakes for ten people. The students who had drawn a circle with lines between the people found it more challenging to keep track of the lines (handshakes) when there were more people because their drawing became messy. They tried putting the people in different formations to resolve this problem, however, they still lost track of their lines and ended up with an incorrect answer (see Figure 5-9).

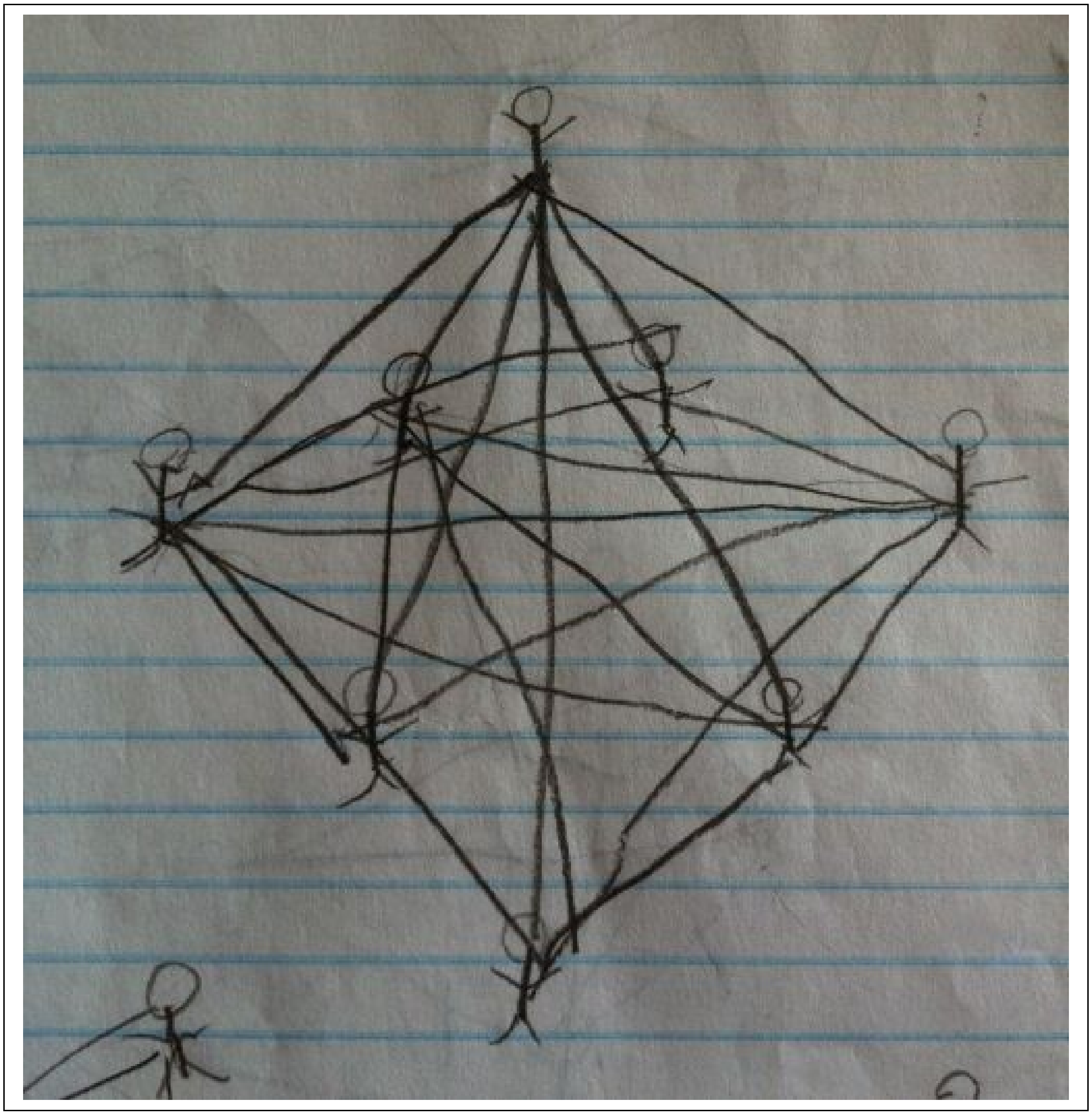

Figure 5-9 Zachary and Thomas' work on task 3

At the end of this class, students shared with each other the different ways that they had figured out the number of handshakes. The class showed interest in the circle drawing with the lines drawn between the people to represent handshakes. Also, once the students explained how they got the number sentence, the rest of the class found it really easy to understand how you would figure out the number of handshakes for other numbers of people. To check their understanding, I asked how they would figure out the number of handshakes for eight people. They all replied with “7 + 6 + 5 + 4 + 3 + 2 + 1.” Then I asked how they would find the number of handshakes for 25 people, and they replied with “24 + 23 + 22 + . . . + 3 + 2 + 1.”

On the second day that we worked on this task, I asked the students to fill in a chart that I wrote on the board (see Figure 5-10). Nine pairs of students worked on this part of the task.

The number of people	**The number of handshakes**
2	
3	3
4	
5	10
6	
7	
8	28
9	
10	

Figure 5-10 The chart that I wrote on the board for task 3

As a class, we filled in the numbers that we had already figured out for three, five and eight people. Then, in pairs, the students copied the chart and got to work on filling in the blanks in the chart.

The equation that some students shared last class proved to be a memorable thing, because this time everyone seemed to be using equations to figure out the number of handshakes. One pair correctly filled in the whole chart. Four pairs correctly filled in the chart except for one mistake. One pair had two mistakes but the rest of the chart was filled in correctly. Two pairs saw that there were ten handshakes for five and made the assumption that for the rest of the answers, they just had to double the number of people to get the number of handshakes. One pair, consisting of students who have trouble focusing, did not complete the chart.

5.3.4 Algebraic thinking

Just over half of the pairs are making generalizations using available resources, which consists of drawings and natural language. This is mentioned in Kaput's (2008) first core aspect of algebra.

The two pairs of students who drew a circle of people and then drew lines between all of the people to represent handshakes, are displaying their ability to use a model, as seen in the work of Zachary and Thomas in Figure 5-6. This is the third in Kaput's (2008) three strands of algebra. The students have noticed that for any number of people, if there is a line drawn between each of the people, the total number of lines will equal the total number of handshakes. They have found a way to help themselves figure out the unknown.

The two pairs of students who came up with the equation 4 + 3 + 2 + 1 = 10 are displaying their ability to generalize, as seen in the work of Rita and Jessica in Figure 5-8. First, they were able to correctly come up with that equation, and then they noticed that it was a sum of consecutive descending numbers. They also noticed that this sum of consecutive descending numbers started one number below the number of people that there were. In other words, for five people shaking hands, the equation started at four (five minus one). The students were able to use their knowledge to help them easily find the number of handshakes for other numbers of people.

When the students who came up with the equation shared their idea with the rest of the class, the other students were able to see how the equations were found. In this situation, the whole class was able to see the pattern of a sum made from consecutive decreasing numbers starting at the number that is one less than the number of people who are shaking hands.

On the second day, six out of eleven pairs correctly (some with minor mistakes) filled in the chart. As I circulated around the classroom, I could hear them talk about noticing that as the number of people increased, the number of handshakes increased as well. This displays the beginning of understanding dependant and independent variables. In this case, the independent variable would be the number of people who are shaking hands. The number of handshakes is the dependent variable since it is determined entirely by the number of people who are shaking hands.

In this task, the students have displayed some of the powers that Mason (2008) has mentioned. They are imagining and expressing when they picture people shaking hands, and then turn this image into something written on paper or something spoken aloud. They are conjecturing and convincing while they are working with their partner, coming up with possibilities and then either proving it or finding a counterexample.

In the students' work on this task, we can see their ability to generalize and then apply their generalization to help them solve other problems. By the end of our work on this task, most students understood that in order to find the number of handshakes for five people, for example, they needed to find the sum of 4 + 3 + 2 + 1. They were then able to apply this knowledge to find the number of handshakes for other numbers of people. For example, this led them to be able to say with confidence that the number of handshakes that would occur between 25 people would be the sum of 24 + 23 + 22 + . . . + 3 + 2 + 1.

5.4 Popsicle stick Pattern

5.4.1 The task

Given the number of Popsicle sticks in the first, second, and third figures of a pattern, determine the number of Popsicle sticks in the tenth and 25th figures.

5.4.2 How the task was presented

I told the students that we were going to look at a pattern, and then on the board, I drew the first three figures of the pattern (see Figure 5-11). The first figure had four lines (I called them Popsicle sticks) in the shape of a square but the Popsicle sticks did not touch. The second figure had seven Popsicle sticks in total, arranged to look like two squares beside each other. The third figure had ten Popsicle sticks in total that were arranged to look like three squares beside each other. I asked the students if they had an idea of what the next figure in the pattern would be. Many hands went up and the students correctly described what the next figure would look like. I then told the students that with a partner, they were going to figure out how many Popsicle sticks there were in the tenth figure. I told the students that there was a shortcut to find the answer. By telling the students this, I was hoping that instead of just drawing out the figures and counting the Popsicle sticks, they would find a different way of getting the answer that would make it easier to find the number of Popsicle sticks in any figure. When a pair of students had correctly figured out the number of Popsicle sticks in the tenth figure, I asked them to find the number of Popsicle sticks in the 25th figure, and I reminded them that there was a short cut. Each pair was given a piece of paper to record their work.

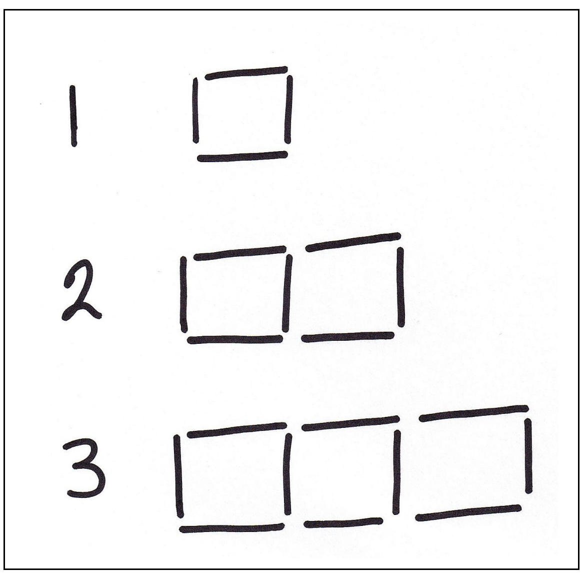

Figure 5-11 The first three figures for task 4

5.4.3 Students' work

Ten pairs of students worked on this task. One pair noticed that from one figure to the next, you had to add three, however they started with three, giving them the pattern 3, 6, 9, 12, 15, 18, etc. and therefore resulting in an incorrect response. Another group found the number of Popsicle sticks for the fifth figure and then thought that they would double it to find the answer for the tenth figure. I pointed out their mistake and they corrected it, but it was clear that they did not really understand where they went wrong, because they did the same thing again when they were finding the number of Popsicle sticks for the 25th figure. They added 16 (the number of Popsicle sticks in the fifth figure) five times to find the number of Popsicle sticks in the 25th figure. All of the other groups managed to come up with the correct number of Popsicle sticks for the tenth figure, but they found different ways of doing so. Four pairs drew the fifth through tenth figures and then counted the number of Popsicle sticks in the tenth figure. Jack and Paul (see Figure 5-12) then noticed that they were adding three each time and to figure out the number of Popsicle sticks in the 25th figure, they wrote the number of Popsicle sticks for the 11th through 25th figures, but did not draw the figures this time.

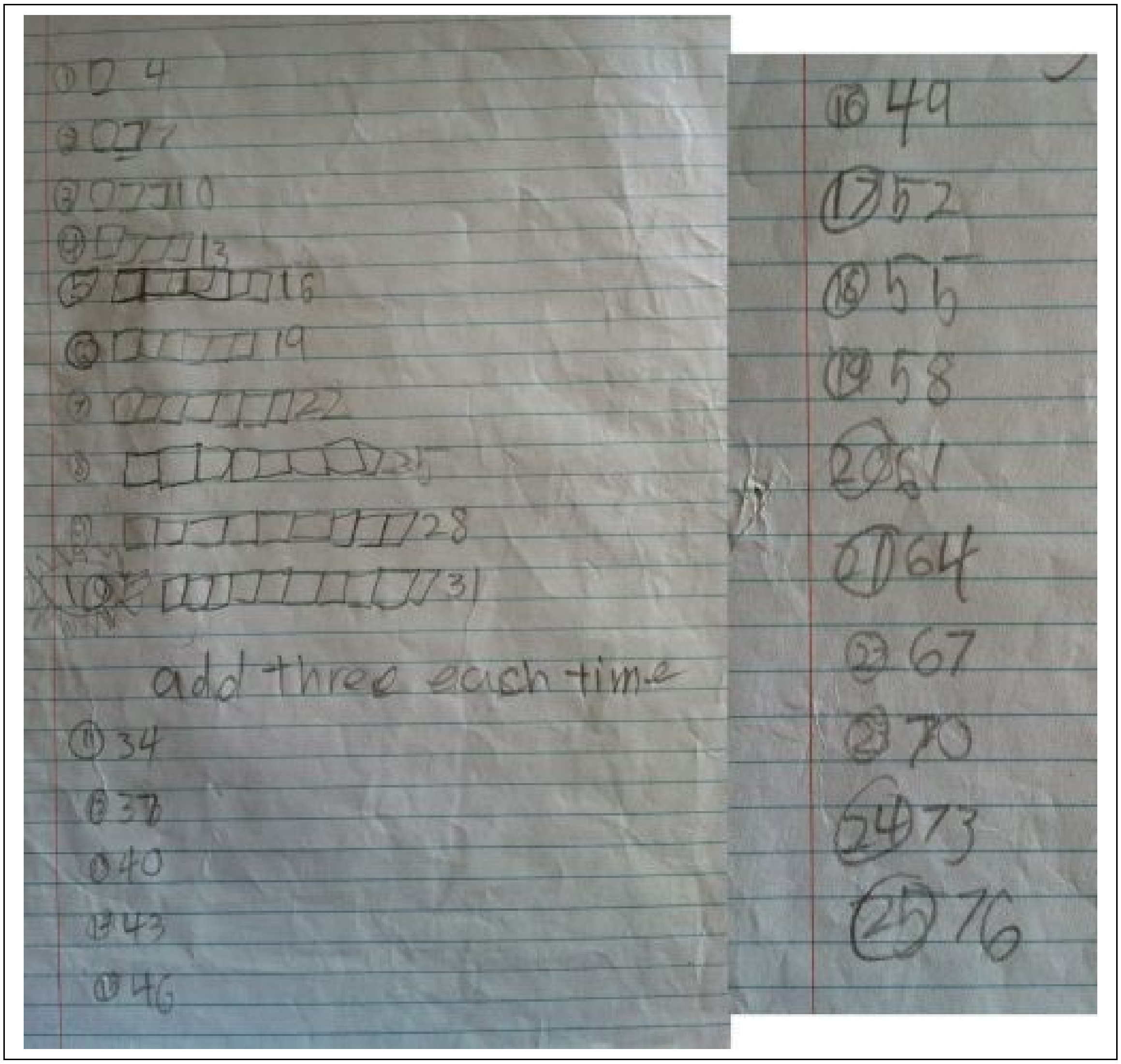

Figure 5-12 Jack and Paul's work on task 4

Two pairs noticed right away that they were adding three each time, so they listed the number of Popsicle sticks for the fifth through tenth figures without drawing the figures. Jordana and Nicole (Figure 5-13) noticed that they started with four and added three each time, so they wrote the number of Popsicle sticks for the tenth figure right away and drew the tenth figure as well. They also tried to write a sentence to explain how to find the answer.

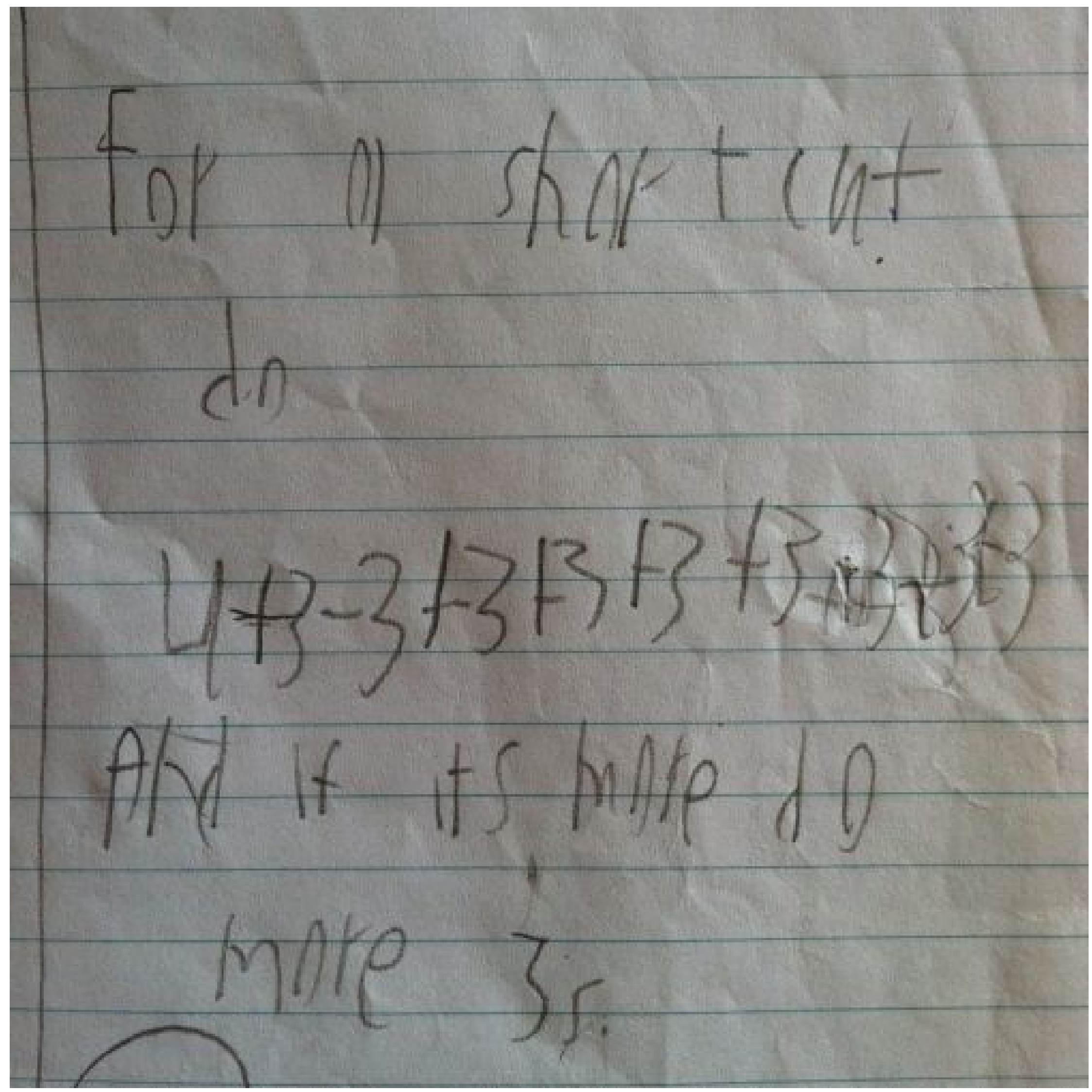

Figure 5-13 Jordana and Nicole's work on task 4

Andrew and Diego seemed to internalize the problem and drew a completely different figure to help them find the answer (see Figure 5-14). They recognized that the first figure had four Popsicle sticks and they also recognized that for each additional figure, three more Popsicle sticks were added. They were able to use their own drawing to correctly figure out the number of Popsicle sticks for the tenth and 25th figures. When they tried to write it up in a sentence, they did mistakenly say that they were counting by fours, but their drawing clearly displays counting by threes.

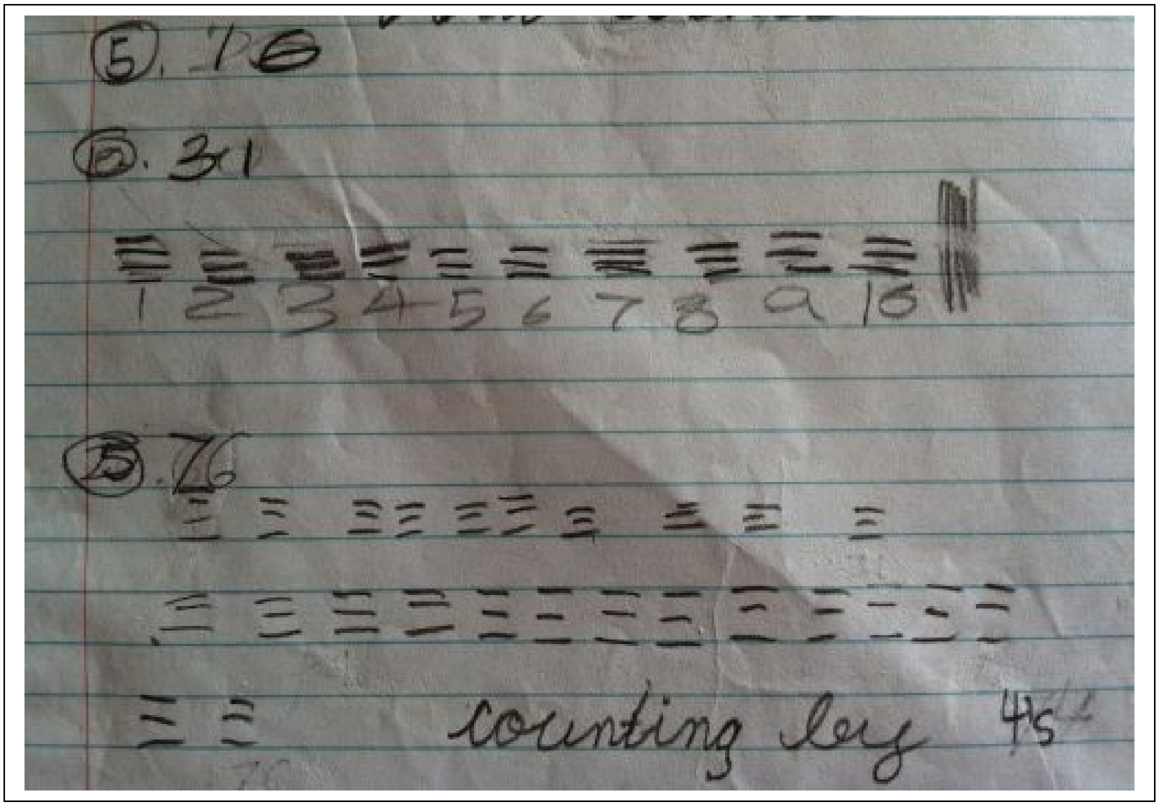

Figure 5-14 Andrew and Diego's work on task 4

5.4.4 Algebraic thinking

In this task, Jordana and Nicole, as shown in Figure 5-13, noticed, after drawing a few figures and counting the Popsicle sticks, that from one figure to the next, the number of Popsicle sticks increased by three. These students recognized the additive invariant.

Another pair also noticed that from one figure to the next, the number of Popsicle sticks increased by three. They took this one step further and were able to say that to find the number of Popsicle sticks in any figure, you take the number of the figure, multiply it by three and then add one. These students were able to see both the multiplicative and additive invariants. The students have not yet used variables, but even though they are not yet able to correctly use formal algebraic notation, they are still displaying evidence of algebraic thinking. They are making generalizations and expressing them using regular language since they have not yet become fluent in the language of algebra.

Imagining and expressing, one of the powers mentioned by Mason (2008), is present here. The students are using whatever means they have available to them to express what they are thinking. Some students drew the pattern, some made a list with the number of Popsicle sticks in each figure, and others tried to write a sentence to describe what was happening in the pattern.

Another power, conjecturing and convincing, is evident as well. The students are making conjectures about the pattern and are changing or modifying them as necessary.

Andrew and Diego were able to internalize this problem. Even though the Popsicle sticks were drawn in the shape of squares, they represented them as stacks of Popsicle sticks. They were able to take the information that was given and express it in a different way, therefore displaying evidence of algebraic thinking.

5.5 The number of 7s

5.5.1 The task

> How many times does the digit seven appear in numbers from zero to 500?

5.5.2 How the task was presented

I told the students that they were going to figure out the number of times that the digit seven appeared in numbers from zero to 500. I gave a couple of examples such as 27 and 70, and I made sure that they noticed that 77, for example, counted as two sevens. The students were sitting at their desks, which were in groups of four so I told them that they would be working by themselves, but that they were allowed to chat with the other people in their group if they would like. Just as I had mentioned in the previous task, I told them that this problem also had a shortcut, which they were excited to hear. Each student received a piece of paper to record their work.

5.5.3 Students' work

Sixteen students worked on this task. One student had trouble getting started and focusing on the problem. Another student appeared to interpret the question a different way, and was repeatedly adding seven until he passed 100. Nine students, such as Aaron (Figure 5-15), took the approach of trying to list all of the numbers between zero and 500 that had one or more sevens in it. These students had missed numbers while they were making their list so none of them ended up with the correct answer.

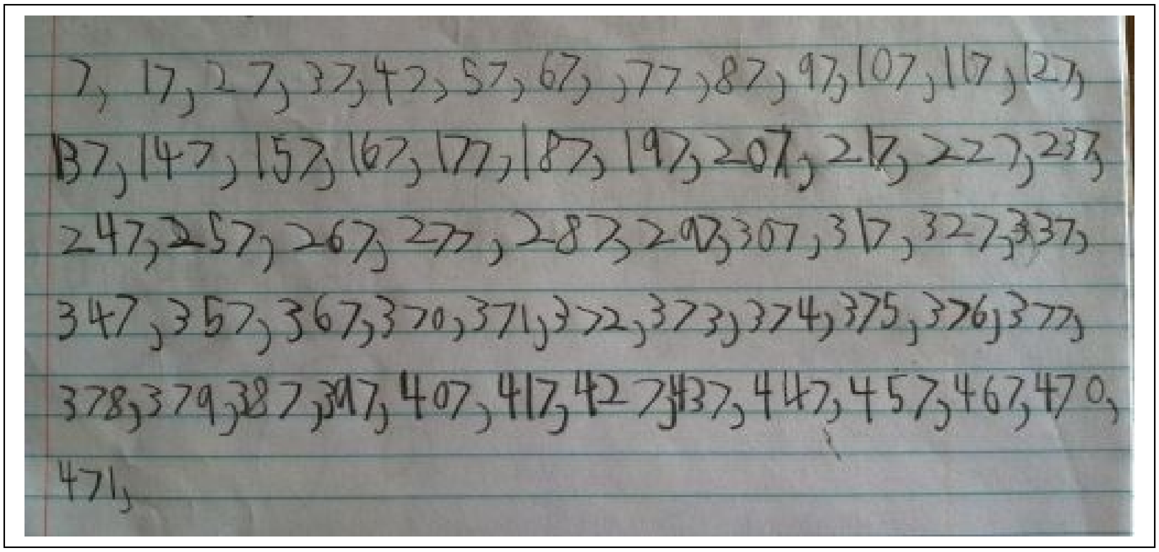

Figure 5-15 Aaron's work on task 5

Five students, such as Rita (Figure 5-16) found a shortcut and correctly found the number of sevens from zero – 500. These students first counted the number of sevens between zero and 100, resulting in 20. Then, they either added 20 five times or multiplied 20 by five to end up with the answer of 100 sevens between zero and 500.

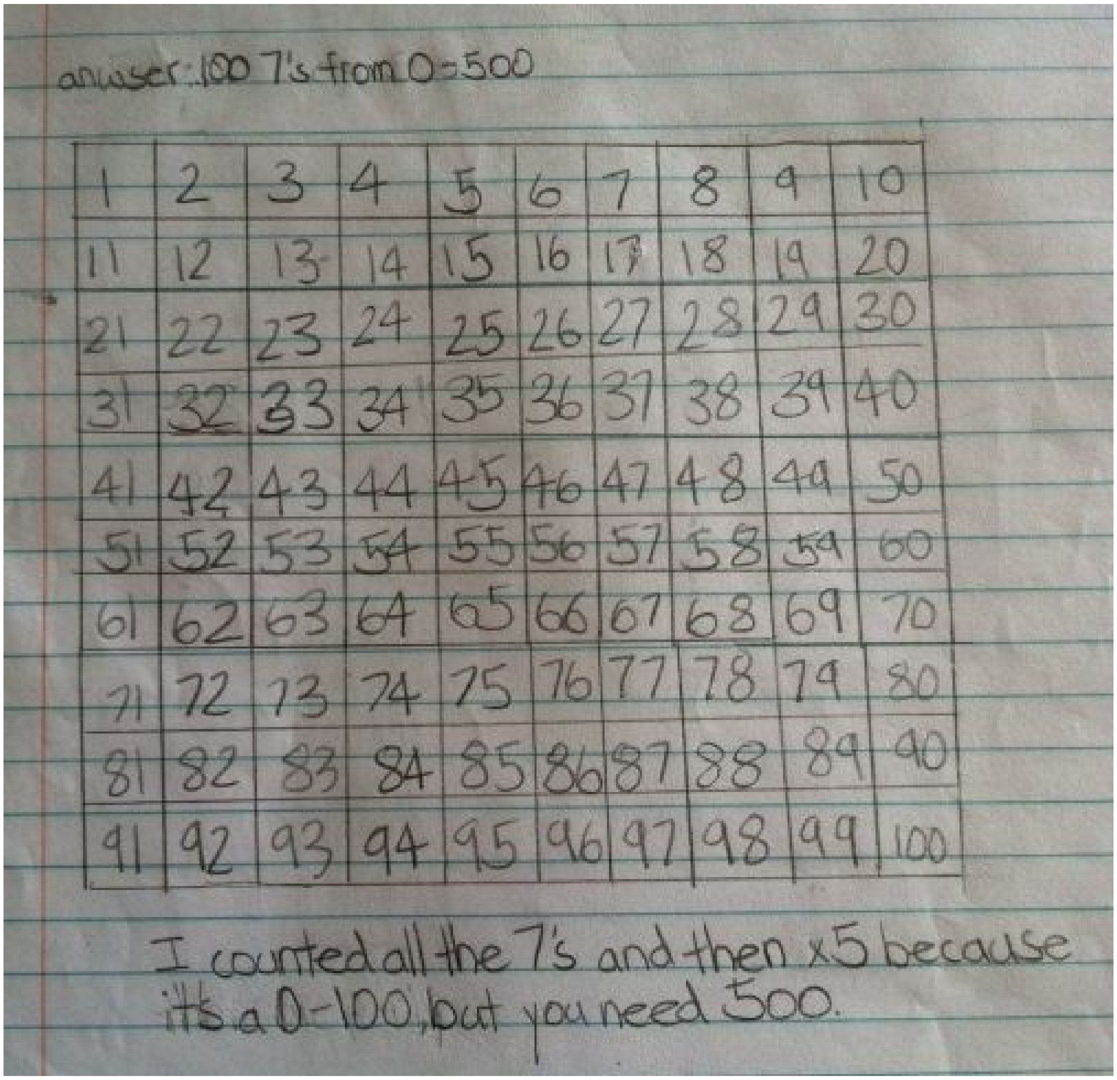

Figure 5-16 Rita's work on task 5

5.5.4 Algebraic thinking

The five students who found the shortcut had similar ways of approaching the problem (see Figure 5-16). Rita explains what she did:

Rita: “The answer is 100! There are 100 sevens between zero and 500.”

Miss B: “How did you find your answer?”

Rita: “I did five times 20.”

Miss B: “Why did you multiply five by 20?”

Rita: “First I counted the number of sevens from 0 – 100, that was 20.
Then, since the question is to find the number of sevens up
to 500, I multiplied the 20 by five since there are five
hundreds, and that’s how I got 100!”

When these five students each figured out that there were 20 sevens from zero to 100, they displayed expansive generalization when they figured out the number of sevens from 101 – 500 (Harel and Tall, 1989). The students used their method for finding the number of sevens from zero to 100 and applied it to the other groups of one hundred as well, such as 101-200. This resulted in their cognitive structure being extended without making changes to the current idea.

Rita (Figure 5-16) and the other four students who approached the problem in the same way, recognized that the total number of sevens that appeared in numbers from zero to 500 was equal to five times the number of sevens in numbers from zero to 100. The students were not able to use an equation or letters to express how they found the unknown, but they were able to show their work in a hundred-grid and express their thinking in a sentence.

In this task there was a lot of evidence of conjecturing and convincing, one of the powers mentioned by Mason (2008). Students were constantly coming up with answers that they thought were correct until a counterexample was brought up. In Figure 5-15, Aaron was convinced that he had created a list of all of the numbers from zero to 500 that contained the digit 7, however, simple counterexamples could be 72, or 175 which have both been left out.

5.6 Cows and chickens

5.6.1 The task

At a farm with only cows and chickens, if there are 100 legs all together, how many cows and chickens could there be?

5.6.2 How the task was presented

I told the students a story to present this task. “One day, Little Billy and his classmates went on a fieldtrip to a farm. They had tons of fun and saw many cool things. Later that day, when Little Billy got home from school, his mother asked him what he had seen at the farm. He said that he had seen cows and chickens. When his mother asked him how many cows and how many chickens there were, Little Billy said that he could not remember but that he did remember that he had counted the legs and that the cows and chickens had 100 legs all together.” I then told the students that their job was to figure out how many cows and how many chickens there were at the farm. I told the students that they were going to work on this problem by themselves but that they were allowed to chat with the other people sitting at their group. Each student received a piece of paper to record their work. On the second day, I told students that they could finish trying to find an answer if they had not already, and that the goal was to find a second possible answer.

5.6.3 Students’ work

Twenty students worked on this activity. Three students did not understand the task so I worked through it with them step by step.

One student did not seem to grasp the idea that there were exactly 100 legs, not more. He made a list of 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 cows and then listed the corresponding number of legs beside, and then did the same thing for chickens. This is where he got stuck and did not see how to find an answer from his work. It took him a while, but after some hints, he was eventually able to come up with an answer, and even found a second answer.

Three students, including Vincent (Figure 5-17) started by counting the number of chickens that there would be for 50 legs and then they counted the number of cows that there would be for the remaining 50 legs. They found that there would be 25 chickens, which would give 50 legs, and 12 cows, which would give 48 legs. To take up the remaining two legs, they added another chicken. This gave them a final count of 26 chickens and 12 cows.

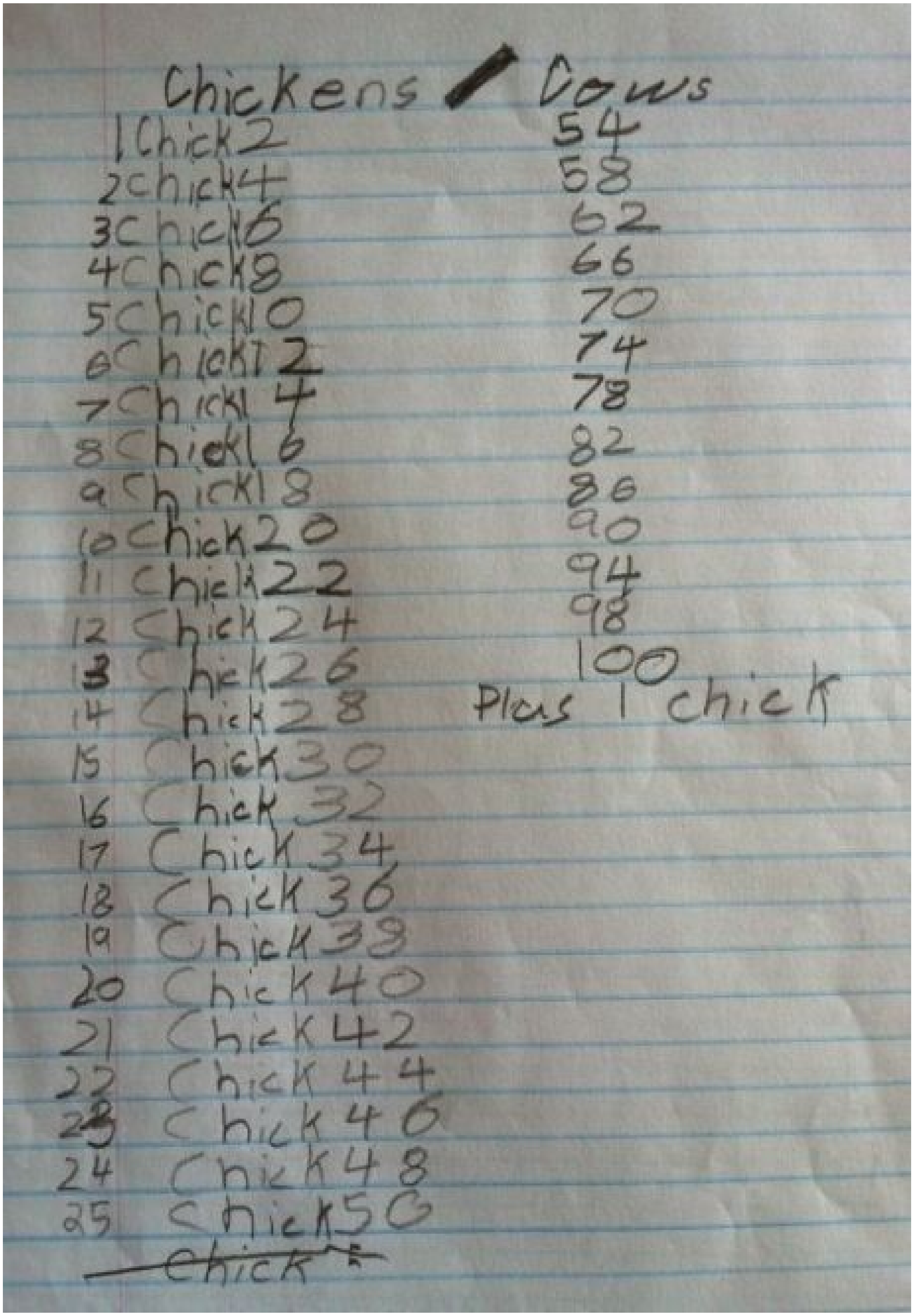

Figure 5-17 Vincent's work on task 6

Four students, including Paul (Figure 5-18), performed repeated addition with four and two, representing cow legs and chicken legs, respectively, to make a total of 100 legs. They then counted the number of times that they added four and two to find the number of cows and chickens. Two of these students were able to find a second possible answer using this method.

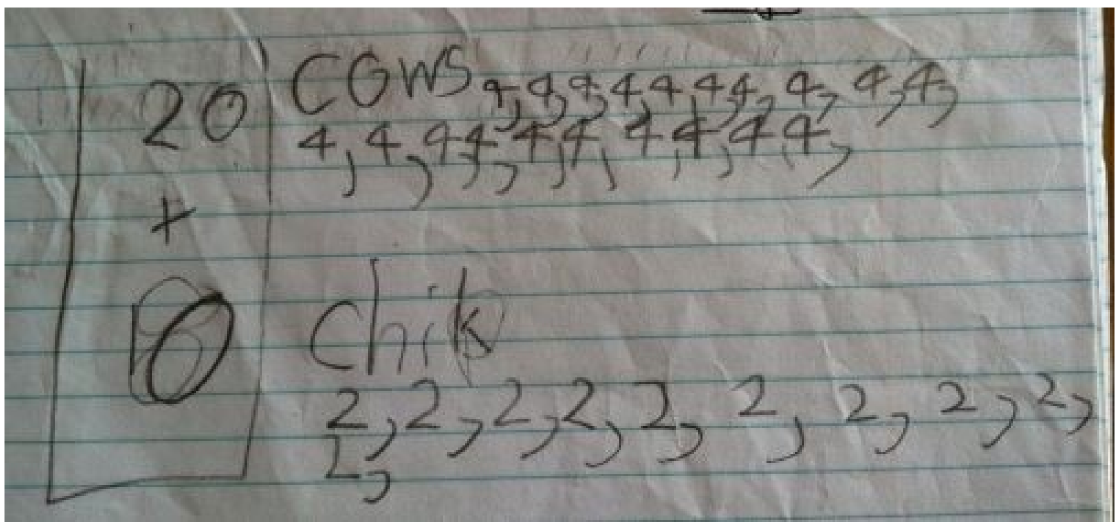

Figure 5-18 Paul's work on task 6

Five students, such as Jessica in Figure 5-19, added four cow legs and two chicken legs to get a total of six legs. Then they counted by six sixteen times until they came to 96, which was the closest to 100 without going over. This meant that there were 16 cows and 16 chickens, but they still needed four more legs, so they added one more cow, making a total of 17 cows and 16 chickens. Four of these students were also able to find other answers using this method.

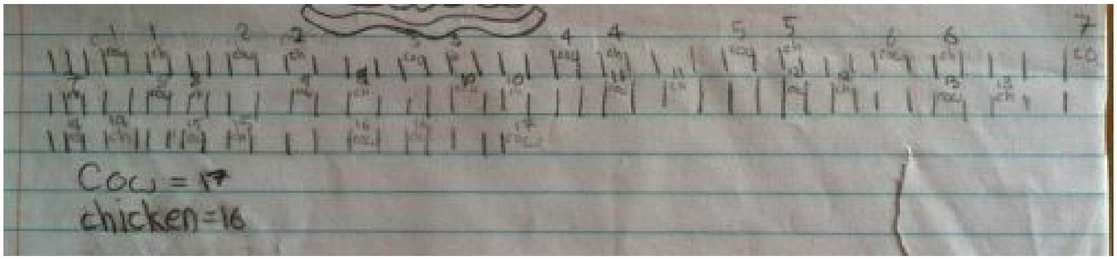

Figure 5-19 Jessica's work on task 6

Three students, like Andrea in Figure 5-20, made sums of 100 and then made each of the addends represent either cows or chickens. They divided the addend by either four or two to get the number of cows and chickens (see Figures 5-20 and 5-21). All of these students were able to find more answers using this method.

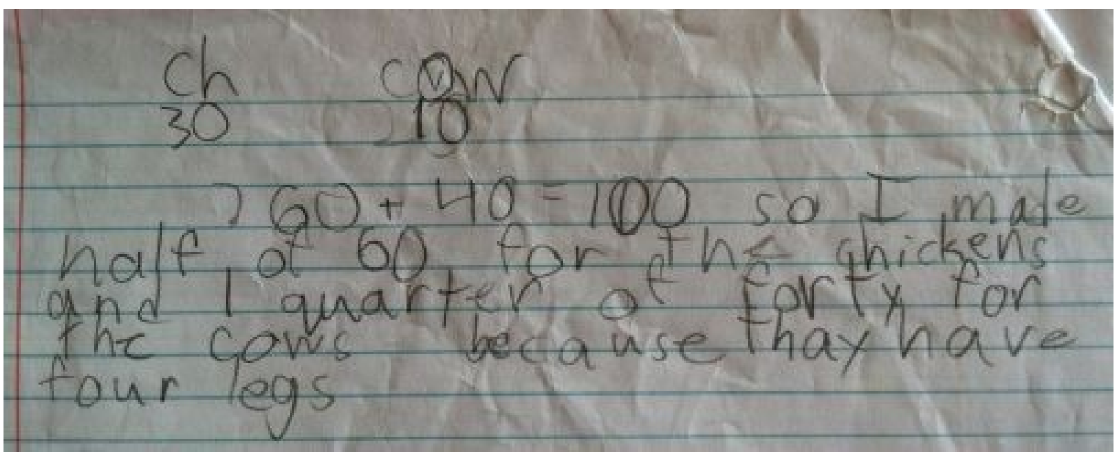

Figure 5-20 Andrea's work on task 6

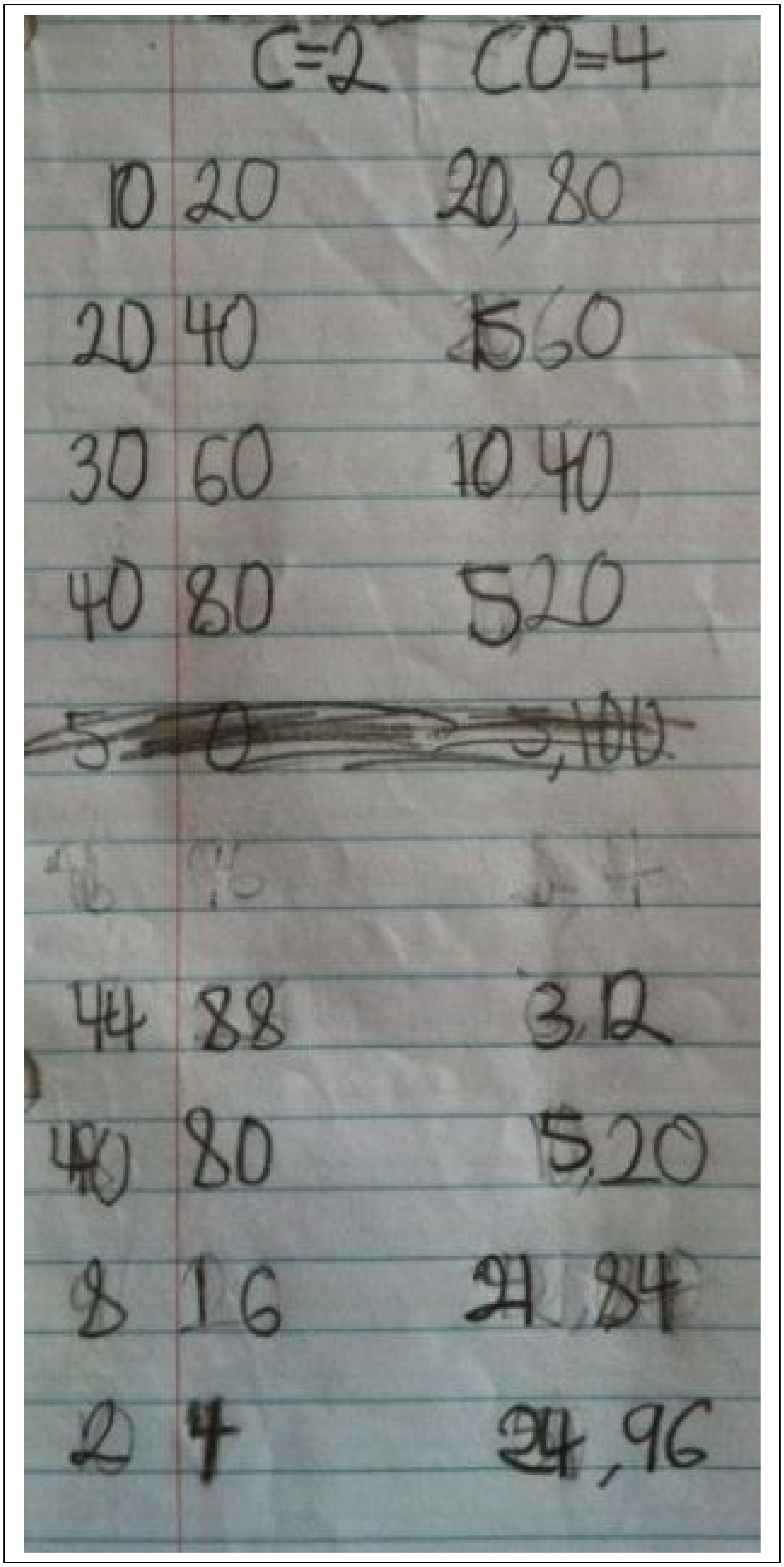

Figure 5-21 Diego's work on task 6

One student found out how many cows there would be for 100 legs, which is 25, then subtracted four cows, 16 legs, and replaced them with eight chickens. This resulted in 21 cows and eight chickens.

5.6.4 Algebraic thinking

In this task, there is evidence of generalization leading to using variables. One student (see Figure 5-20) had an interesting way of solving this problem that displayed the start of using variables. This is a grade three student who had an understanding of multiplication and division. First, she found two numbers that added up to 100; these were the number of legs. Then she took one of these numbers of legs and divided it by two to get the number of chickens. She then took the other number of legs and divided it by four to get the number of cows. Conveniently, the numbers that this student chose for the number of legs were divisible by both four and two. Using variables, one could express this student's method in the following way:

Insert any numbers for x and y to make the following statement true:

$x + y = 100$ (x is the number of chicken legs, and is a multiple of two, y is the number of cow legs, and is a multiple of four)

Then,

$x \div 2 = a$ (a is the number of chickens)

$y \div 4 = b$ (b is the number of cows)

Another student (see Figure 5-21) had an interesting way of figuring out the number of cows and chickens. His method also displayed thinking that would lead to using variables. He started with a number to represent the number of chickens and then he multiplied it by two to get the number of chicken legs. Next he chose a number to represent the number of cows and then he multiplied this number by four to get the number of cow legs. He made sure that the number of chicken legs plus the number of cow legs equalled 100. We could write out this student's method in the following way:

$2x + 4y = 100$ (x is the number of chickens and y is the number of cows)

Some of the powers that Mason (2008) mentions are present here. The students are using their power to imagine and express. In their minds, they are able to picture varying numbers of cows with four legs and chickens with two legs, and then they are able to express this on paper. Students also used their power to conjecture and convince. As I circulated around the classroom, I could hear students sharing their ideas with each other. In some cases both students would have the same answer so they would be satisfied with that. In other cases, students had different answers and this is where the good discussions were present. One student would ask the other how he found his answer which would allow that student the opportunity to explain his process. This process either made sense to the other student who accepted the answer, or if the process did not make sense, that student would pose questions for clarification. With these discussions, there was the possibility that the student who was explaining his answer might catch a mistake that was made, therefore ending up with a different answer. Another possibility was that students would recognize that both their answer and their partner's answer were correct, even though the answers were different, which would allow them to acknowledge that there was more than one possible answer.

Guessing and testing was something that was present with this task, although perhaps in a more sophisticated version, such as “try and improve.” The students would try a certain number of cows and a certain number of chickens and see if that produced 100 legs in total. Usually an incorrect answer would still prove helpful since the student would improve upon that method.

With this task, many students realized that performing repeated addition would be very helpful in finding an answer. These students were in the beginning stages of algebraic thinking. They were aware of what needed to be done, however, they were not yet able to express their repeated addition more concisely. Some students were able to take their answer for the number of cows and chickens and adjust it to come up with a different answer. For example, they started with 17 cows and 16 chickens and then adjusted their numbers to get a second answer of 16 cows and 18 chickens. The fact that they are able to change their numbers to come up with a different answer shows that they had a good understanding of the problem as well as a good understanding of the way that they solved the problem. These students seem to be ready to take the next step and learn how to express their ideas using variables.

5.7 Venn diagrams

5.7.1 The task

Given some information about a Venn diagram, identify the numbers that represent each part of the Venn diagram.

5.7.2 How the task was presented

On the board, I drew two circles that overlapped in the middle. I asked the students if they knew what it was. Most students had forgotten that it was called a Venn diagram but they all knew how to use it, and one student even gave me an example: “You could have everything that I like in one circle, then everything that he likes in the other circle and the space in the middle is for things that we both like.”

I gave each student a piece of paper with four Venn diagram problems on it, and told them that they were going to be working on the problems by themselves but if they needed to, they could chat with the other people sitting in their group.

5.7.3 Students' work

Twenty-one students worked on this task. Two students did not complete this task; they are both students who have trouble focusing. Three students completed three of the problems and made a mistake on the fourth one. The rest of the class, sixteen students, easily figured out all four problems. I then asked them to choose one of the problems and write a few sentences about how they figured it out.

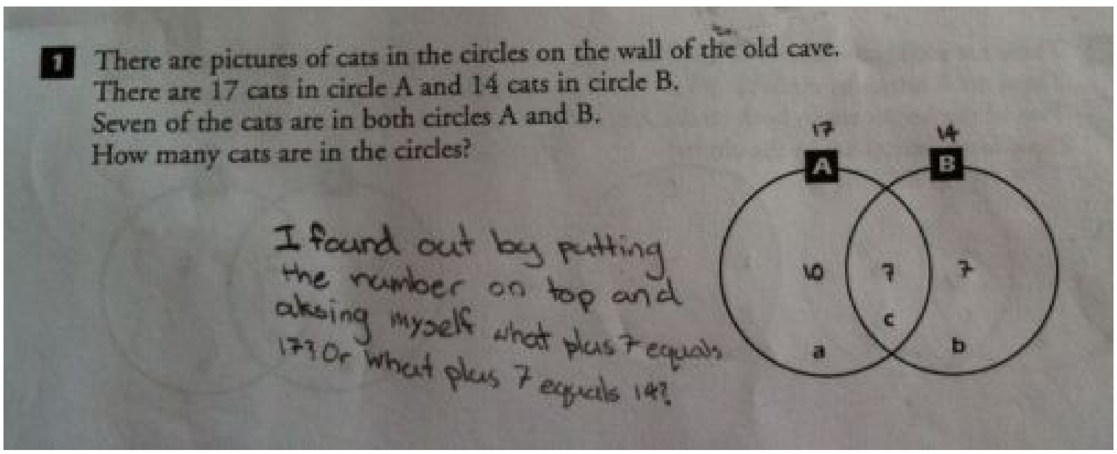

Figure 5-22 Jessica's work on task 7

When it seemed like a good portion of the class was finished, I asked them what they thought of these questions, and they all replied that they were easy. So, I told them that if they were finished, they could come to the back carpet and I would give them another, harder question to work on. This time I gave them a problem with three circles: "There are pictures of bears in the circles on the wall of the old cave. There are eight bears in circle A, six bears in circle B, and eight bears in circle C. Three of the bears are in both circles A and B. One of the bears is in both circles B and C. How many bears are in the circles?” All seven of the students who were working on this problem correctly figured it out.

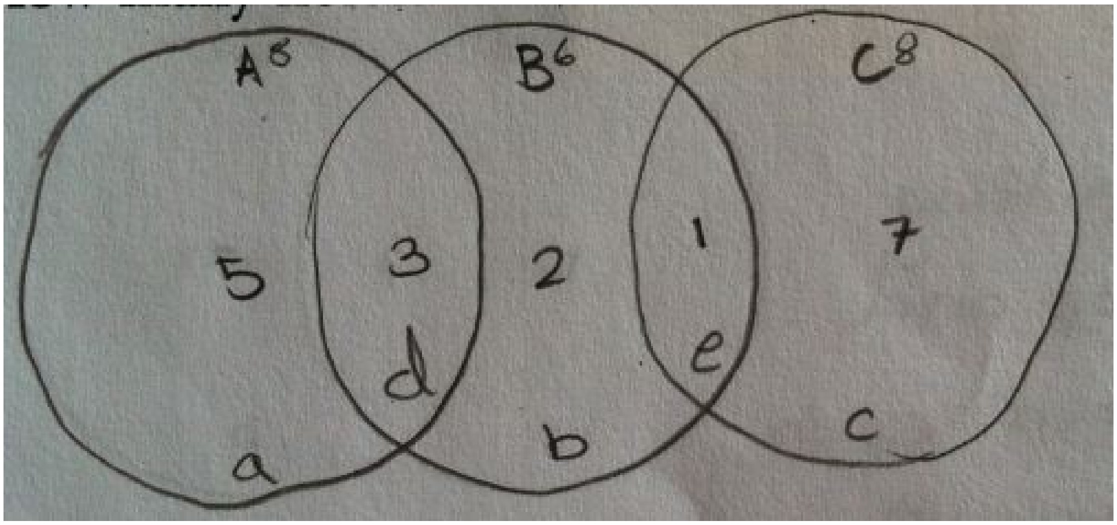

Figure 5-23 Jessica's work on task 7

5.7.4 Algebraic thinking

In this task there is evidence of finding the unknown. Jessica (see Figure 5-22) was one of the students who was able to correctly fill in all four Venn diagrams. Here is how she described the way that she solved one of the questions: “I found out by putting the number on top and asking myself what plus seven equals 17 and what plus seven equals 14?” What Jessica was doing in her head can be expressed in two equations that have a missing addend:

☐ + 7 = 17 which eventually becomes $x + 7 = 17$

☐ + 7 = 14 which eventually becomes $x + 7 = 14$

Since there was a slight story aspect to this task, the students were able to use their power of imagining and expressing (Mason, 2008). For the task in

Figure 5-22, for example, the students were imagining pictures of cats in the circles on the wall of an old cave. They were then able to take the image that was in their head and use it to help them figure out the problem on paper.

There is evidence of expansive generalization (Harel &Tall, 1989) with this task, and it came about unexpectedly. The four questions that I gave the students were worded exactly the same except that the numbers as well as the animal that was being talked about were different for each question. As I circulated around the class, I noticed that the students spent most of their time on the first question, but once they figured that out, they breezed through the other three. This was because the students saw the similarity in the way that the questions were worded and correctly made the assumption that if their method worked for the first question, it would work for the rest of the questions as well.

The students are dealing with unknowns; however, they are not yet using letters to represent the unknowns. They are using their powers of imagining and expressing (Mason, 2008) and are either counting in their head or picturing the task in their head to help them figure out the answer.

5.8 Balloons in a box

5.8.1 The task

Given the colours of balloons and the total number of balloons in a box, figure out how many balloons of each colour there could be in the box.

5.8.2 How the task was presented

On the board, I drew a box and wrote “8 balloons” inside of it. I told the students that in the box there were some red balloons and some yellow balloons. Beside the box, I wrote “red balloons” with a red marker and “yellow balloons” with a yellow marker. Then, I told the students that in pairs, they were going to figure out how many balloons of each colour could be in the box. Each pair received a piece of paper to record their work.

After a little while, when I noticed some students were close to finishing, I called the class’ attention and introduced the next problem. I drew a box and wrote “12 balloons” inside. I told the students that inside the box there were some blue balloons, some white balloons and some pink balloons. I wrote the colours of balloons on the board like before. Then I told the students to continue working on the first question and that when they finished that question, they could start working on the second question.

5.8.3 Students’ work

Eleven pairs of students worked on this task. Two pairs came up with only one possibility, that there were four red balloons and four yellow balloons in the box. Two pairs made random guesses that gave them some correct answers. Four pairs also made random guesses but kept doing this until they had all of the possible combinations. Two pairs, such as Whitney and Lorraine (Figure 5-24), made a chart and listed the numbers one through seven down one side and then on the other side, listed the corresponding number that would make a sum of eight.

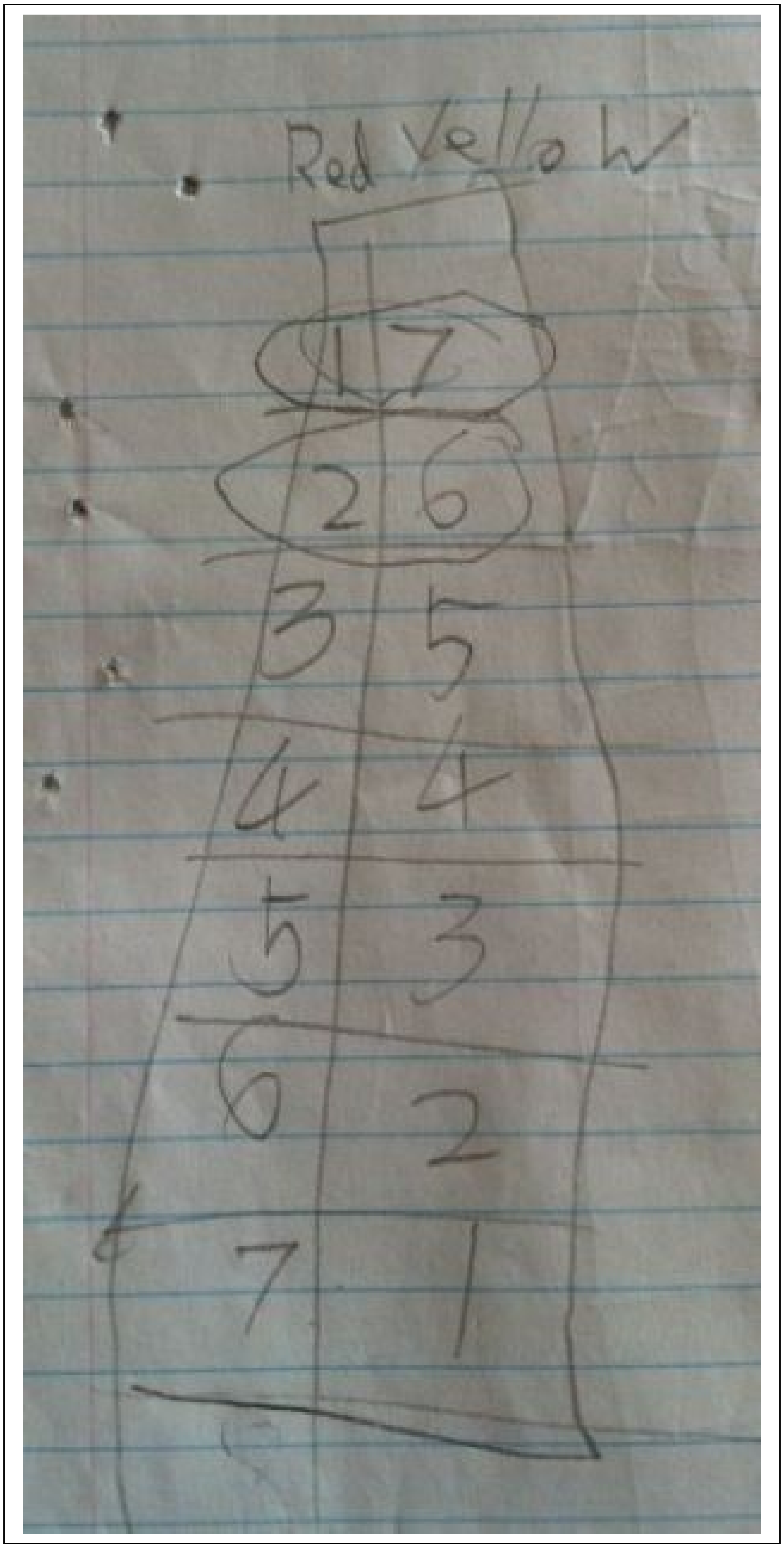

Figure 5-24 Whitney and Lorraine's work on task 8

Most of the pairs who attempted the question with three colours of balloons, continually sought out three different numbers to sum up to 12. Andrea and Yvette had an interesting way of approaching it. They found three numbers that added up to twelve, and then they shifted those same numbers around to get a few other possible answers (see Figure 5-25).

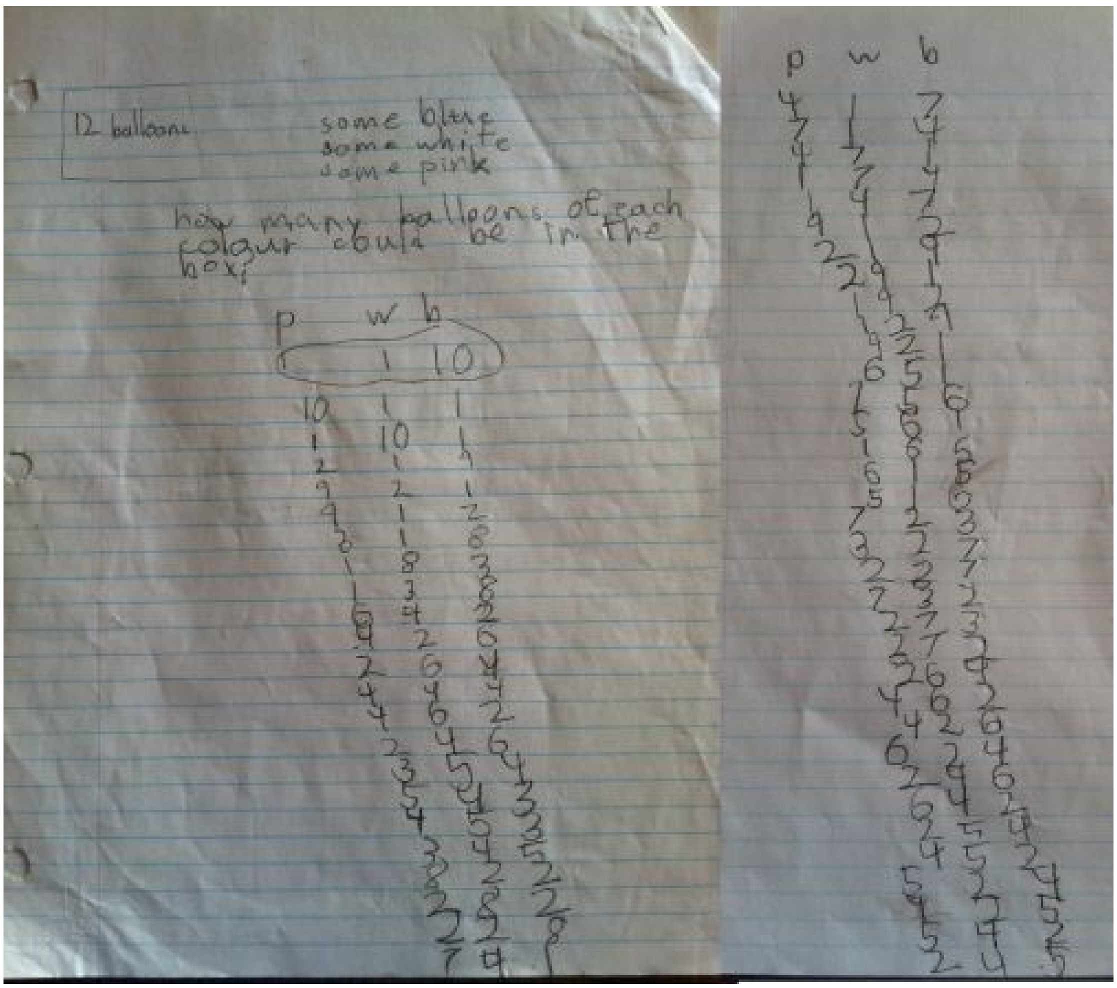

Figure 5-25 Andrea and Yvette's work on task 8

5.8.4 Algebraic thinking

In this task, there is evidence of using variables. Students are finding all possible solutions for the equation $r + y = 8$, where r and y are positive integers.

They are holding one variable independent and calculating the dependent variable. As well, they are displaying their knowledge of the commutative property of arithmetic when they use numbers twice such as 1 + 7 and 7 + 1 (see Figure 5-24).

In this task, students are also using their power of imagining and expressing when they picture balloons in a box, and use that image to help them solve the problem.

For the second part of this task, we can use letters to express what the students were thinking: $a + b + c = 12$, where a, b, and c are positive integers.

Andrea and Yvette (see Figure 5-25) worked on the second question and had an interesting way of approaching it. They came up with three numbers that made a sum of 12 and then they moved those three numbers around, clearly displaying their understanding of the commutative property of addition. They continued this method for numerous other sums of 12, and had the rest of the class beat by a large margin when it came to the number of answers that were found for the second question. Andrea and Yvette have figured out that they can move the three addends around and still have the same sum; they no longer have to think specifically about each number that they move. They can see this process as a whole, and are beginning to see a new concept: the commutative property of addition.

5.9 Magic square

5.9.1 The task

Fill in a 3 x 3 grid using nine specified numbers once only. Each row, column, and diagonal must have the same, given sum.

5.9.2 How the task was presented

I showed the class the grid and told everyone that they were going to fill it in a certain way. Using the numbers one to nine only once each, they were going to fill in the grid so that each row, column and diagonal added up to 15. I reviewed what rows, columns and diagonals were. Each piece of paper had four grids on it and I told the students that if they needed to start over they could decide if they wanted to erase their numbers or if they wanted to use a different grid. I gave each student a piece of paper with grids on it and they began. They worked by themselves but they were allowed to chat with the other people in their group.

I wrote a second and third question on the board for those who finished early. The second one was using the numbers two to 10 and making a sum of 18. The third one was using the numbers zero to eight and making a sum of 12.

5.9.3 Students' work

Nineteen students worked on this task. As soon as the students had their page, they went straight to work and the room was silent for a long time. The students worked hard for the entire 30-minute class, and when it was time to hand in the papers, they all moaned and said that they needed more time. I told them that we could continue working on this task the following class and they were all excited to hear this.

The next class, the students got their papers back and immediately got to work. Everyone was hard at work but this time there was a little more chatting as students were sharing ideas. As students came to show me their work, I either pointed out where they made a mistake or, if they filled in the grid correctly, I told them they could try filling in a grid according to the next set of rules on the board.

In the end, four students correctly filled in the grid according to the first set of rules and one student filled two grids according to the first and second sets of rules. The other students came very close to completing the first grid, usually there was just one set of three numbers in their grid that did not add up correctly.

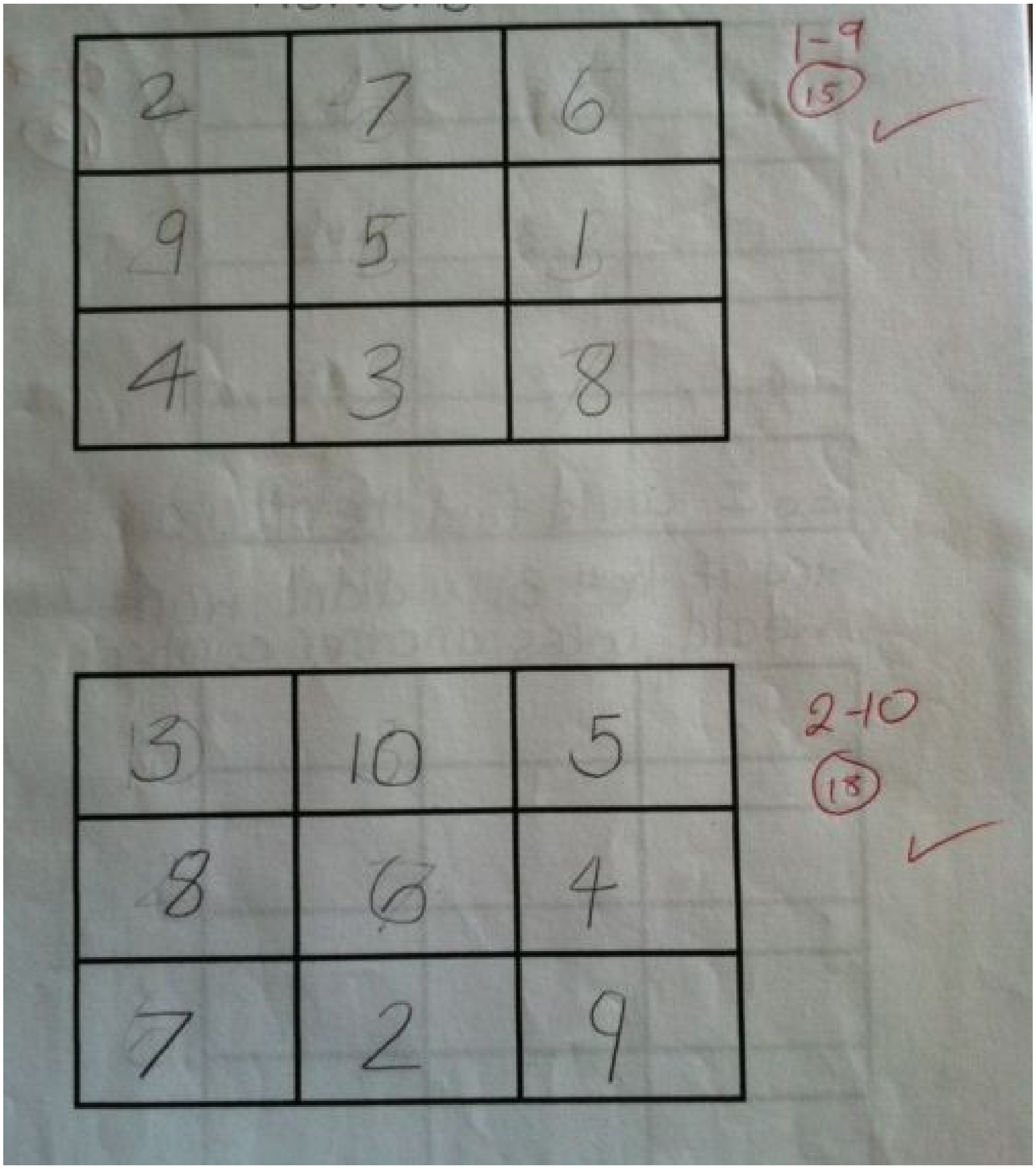

Figure 5-26 Rowena's work on task 9

5.9.4 Algebraic thinking

In this task, there is evidence of using variables, but with restrictions. For the first question, students are solving the equation $a + b + c = 15$, however, one restriction is that they can only use the numbers 1 – 9. Another restriction is that each number is also part of a second equation and some numbers are also part of a third equation of the same form. Students had to take into consideration all of this information at the same time when choosing numbers to fill in the grid.

In this task, the students are displaying their power to focus and de-focus (Mason, 2008). They can focus on finding three numbers with the given sum, but they can also de-focus and look at the whole magic square to see how the numbers that they have already written relate to each other and to the numbers that they are about to write. Conjecturing and convincing is another power that is present. Students came up with a possible layout for the nine numbers that they were allowed to use in their grid and were happy with it until they came across a counterexample, a set of three numbers in a column, row or diagonal that did not give the desired sum. After coming across that counterexample, the students would either try to make a few changes or they would start over.

The students are becoming familiar with having to come up with three numbers that give a certain sum and then carefully placing those three numbers into a grid so that they work with the numbers that are already there. The students have not yet mastered this skill, but they are becoming more comfortable with it.

6: CONCLUSION

6.1 Summary

This study has helped me to answer my research questions: Can elementary school students show evidence of algebraic thinking? If so, in what ways is their algebraic thinking evident? After giving students a number of tasks to complete, I found that they did display evidence of algebraic thinking even though they were not yet able to use letters to represent their thinking. The students were able to make generalizations: in the fourth task, for example, Popsicle Stick Pattern, the students were able to generalize when they recognized and explained that they were starting with four and adding three each time. The students were able to find the unknown, which, in the third task, Handshakes, was the number of handshakes for a given number of people. As well, the students were able to work with variables: in the eighth task, for example, Balloons in a Box, the students were working with variables when they found different groups of numbers that each answered the question. The students were able to orally explain their work that led them to their answer. They also used drawings and written language to explain their work. This shows that even though the students did not use formal algebraic notation when making generalizations, finding the unknown or using variables, they are still able to display evidence of algebraic thinking.

Table 1 shows a summary of the purposes of algebra and the powers, according to Mason, that are evident in each task. The purposes that are seen in each task are predetermined by the task itself and not how the students approach the task. The students were not yet familiar with using letters to make

	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8	Task 9
The three purposes of algebra									
1. Making generalizations	✓		✓	✓		✓		✓	
2. Finding the unknown (not using letters yet)		✓	✓	✓	✓	✓	✓	✓	
3. Using variables (not using letters yet)			✓	✓		✓		✓	✓
The five powers according to Mason (2008)									
1. Imagining and expressing		✓	✓	✓		✓	✓	✓	
2. Focusing and de-focusing	✓		✓	✓	✓				✓
3. Specializing and generalizing	✓		✓	✓		✓		✓	
4. Conjecturing and convincing	✓	✓	✓	✓	✓	✓	✓	✓	✓
5. Classifying and characterizing									

Table 6-1 Summary of Results

generalizations, find the unknown or use variables; however, they still understood these concepts. Instead of using formal notation, the students worked with the tools that they had available to them, which consisted of using normal language and drawing pictures to explain the work that they were doing.

Of the five powers that Mason (2008) has mentioned, the last power was not evident in these tasks. The nature of these tasks makes it more likely for students to use the first four powers over the last one. Children use imagination daily, most notably in their play time, so it makes sense that they would use this power when it comes to mathematics. Children have such great imaginations that when they are given a mathematics problem to solve, that allows them the possibility to use their imagination, they achieve great success. Children are able to focus in on certain details of a problem but also de-focus and look at the problem as a whole. Children can make generalizations and they can also look specifically at a problem. Conjecturing and convincing is another power that is present in the daily lives of children, since they would believe that a certain story was true until there was a counterexample to prove otherwise.

I found that students were usually successful at generalizing. There were instances when some students had difficulty with this, and this difficulty occurred when the students were too focused on the numbers that they were working with instead of looking at the question as a whole and focusing on the operations. For example, in the fifth task, The Number of 7s, instead of trying to generalize, some students focused on listing all of the numbers between zero and 500 that they could think of that had a seven in it. At this level, before students are using letters to represent their thinking, they are able to express their generalizations using normal language or drawings; they are using the tools that they have available to them.

For most of the tasks, I told the students that there was a shortcut, which encouraged many students to shift their approach form purely arithmetic to one that involved algebraic thinking. They were very receptive to the idea of trying to find a shortcut.

The students were able to develop a climate that was focused on thinking about mathematics and not as focused on the written presentation of their thinking. This can be seen by looking at their work; their pages are wrinkled and their writing is not done in their best printing.

6.2 Limitations

There were limitations to this study. There were only 22 students and one teacher involved. As well, there is no evidence on how early algebra tasks assist with algebra in high school. If I were to redesign this study with no time or funding restrictions, there are a number of things that I would change. I would get all of the teachers and students from one district to participate and it would last 13 years, following one group of students from kindergarten to grade 12. All of the teachers would meet prior to the start of the study to make sure that they understood how the study would work. Each year, the teachers participating in the study would change as the students go through the various grades. In the first year, all kindergarten teachers would meet and be presented with tasks that would assist to develop the algebraic thinking of the students. As a group, the kindergarten teachers would look over the tasks and the people leading the study would make sure that the teachers understood the tasks and how to deliver them. There would be follow-up meetings throughout the school year so that teachers could share their experiences and have discussions with the other teachers. The following year, the same thing would happen with all of the grade one teachers, the year after that, it would be with the grade two teachers, and so on, until the group of students was in grade 12.

There would be a group of researchers who would be leading the study and examining the development of algebraic thinking among the group of students. There would also be a control group of students from another school district of a similar size. This group of students would not be participating in the tasks that the other students would be doing but their algebraic thinking and ability would still be assessed.

At the end of the study, the abilities of both the study group and the control group would be compared. My hypothesis is that the study group would display a far more advanced ability to display evidence of algebraic thinking compared to the control group. As well, I would imagine that there would be greater participation in mainstream mathematics courses in high school because students would have the ability and confidence to handle the harder courses. Even if students do not go on to take mathematics courses past grade 12, I believe that these early algebra tasks could be creating a group of people who would be able to function in society with a higher level of mathematical and algebraic thinking.

6.3 What next

If I were to continue with my research and pursue a PhD, I would be interested in collecting and designing tasks for each of the different grade levels. The tasks would cover all of the prescribed learning outcomes, they would be accessible to students at any level and they would promote algebraic thinking. I know that there are a number of tasks already out there, so I would collect those tasks as well as the ones I design and categorize them into booklets according to grade level so that teachers could use them as a resource.

Once these tasks have been created and administered to students, a good next step would be for a group of researchers to follow the mathematical development of these students through elementary school and into high school, comparing the results to the mathematical development of students who are not completing the set of tasks.

6.4 My journey

In this study, I feel that I have assisted the grade two/three class to develop their algebraic thinking and that I have enabled them to experience mathematical problems that they may not have experienced otherwise. As well, I consider this experience to be one that will aid them with their future work in mathematics.

Throughout this study, I believe that I have developed as a teacher. I have taken the information that I have learned and have applied it to the math lessons that I teach. With my grade one class, when we learned about addition and subtraction, I made sure that they saw the equal sign as an indication that both sides of the equation were equal and that they did not see the equal sign as a “do something” symbol. As well, any time that the students made a generalization, we took some time to talk about what they noticed. With my grade 6/7 class, I wanted to encourage them to develop their algebraic thinking and to make their own generalizations. To do this, instead of teaching them concepts, I presented them with problems that they worked on in groups. These problems enabled the students to discover for themselves the topic to be learned and gave them the opportunity to collaborate with classmates and increase their ability to think mathematically and to orally express their thoughts.

For my own future as a teacher, I want to continually develop the way that I teach mathematics. I want to present the students with tasks that are accessible to students of all levels and that allow them to use their mathematical thinking to solve problems that are engaging. I want my students to discover new topics and ideas as a result of their own hard work and investigation instead of having me present the ideas to them. My goal is to help my students to become the best mathematical thinkers that they can be.

I enjoy mathematics and would never turn down the opportunity to work on a mathematical problem or mind game. As a teacher, I take pleasure in teaching mathematics and always strive to present the curriculum in a fun and engaging way that will encourage students to develop their mathematical thinking. As well, I enjoy taking advantage of opportunities to educate myself on different aspects of mathematics education and look forward to continuing to grow as a mathematics teacher.

REFERENCE LIST

Bass, H. (1998). Algebra with integrity and reality. *The nature and role of algebra in the K-14 curriculum: Proceedings of a National Symposium (pp.9-15).* Washington, DC: National Academy Press.

Bastable, V. & Schifter, D. (2008). Classroom stories: Examples of elementary students engaged in early algebra. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 165-184).* New York: Taylor and Francis Group.

Blanton, M. L. (2008). *Algebra and the elementary classroom: Transforming thinking, transforming practice.* Portsmouth, NH: Heinemann.

Brizuela, B. M. & Earnest, D. (2008). Multiple notational systems and algebraic understandings: The case of the “best deal” problem. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 273-302).* New York: Taylor and Francis Group.

Carpenter, T., Franke, M. & Levi, L (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

Carraher, D. W. & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester (Ed.) *Second handbook of research on mathematics teaching and learning.* Charlotte, N.C.: Information Age Publishing & National Council of Teachers of Mathematics.

Carraher, D. W., Schliemann, A. D. & Schwartz, J. L. (2008). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 235-272).* New York: Taylor and Francis Group.

Goodnow, J. (1994). *Beginning algebra thinking for grades 5-6.* Alsip, Illinois: Ideal School Supply Company.

Harel, G. & Tall, D. (1989). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, 11(1), 38-42.

Hoogeboom, S. & Goodnow, J. (1994). *Beginning algebra thinking for grades 3-4.* Grand Rapids, MI: Ideal.

Kaput, J. J. (1995). A research base supporting long term algebra reform? In D. T. Owens, M. K. Reed & G. M. Millsaps (Eds.) *Proceedings of PME-NA XVII (pp. 71-94)*. Retrieved from ERIC database. (ED389539)

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 5-18).* New York: Taylor and Francis Group.

Kaput, J. J., Blanton, M. L., & Moreno, L. (2008) Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 19-56).* New York: Taylor and Francis Group.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.) *Handbook of research on mathematics teaching and learning (pp. 390- 419).* London: Simon & Schuster and Prentice Hall International.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.) *Approaches to algebra (pp. 65-86)*. Dordrecht: Kluwer Academic Publishers.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 57-94).* New York: Taylor and Francis Group.

Osborne, A. & Wilson, P. S. (1992). Moving to algebraic thought. In T. R. Post (Ed.) *Teaching mathematics in grades k-8: Research-based methods (pp. 421-442).* Boston: Allyn and Bacon.

Porteous, K., (2008). How to not teach algebra. *Mathematics in School, 37(*3), 9-13.

Schoenfeld, A. H. (2008). Early algebra as mathematical sense making. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 479-510).* New York: Taylor and Francis Group.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36. Retrieved from http://www.jstor.org.proxy.lib.sfu.ca/stable/3482237

Stacey, K. & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Sutherland et al. (Eds.) *Perspectives on school algebra.* Netherlands: Kluwer Academic Publishers.

Suggate, J., Davis, A., & Goulding, M. (2006). *Mathematical knowledge for primary teachers* (pp. 127-141). Great Britain: David Fulton Publishers.

Tierney, C. & Monk, S. (2008). Children's reasoning about change over time. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.) *Algebra in the early grades (pp. 185-200).* New York: Taylor and Francis Group.

Zazkis, R. (2001). From arithmetic to algebra via big numbers. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra (pp. 676-681)*. Melbourne, Australia: University of Melbourne Press.