ROOTED MINORS AND DELTA-WYE TRANSFORMATIONS

by

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Abstract

In this thesis, we study terminal minors and delta-wye reducibility. The concept of terminal minors extends the notion of graph minors to the case where we have a distinguished set of vertices T in our graph G that must correspond to a distinguished set of vertices Y in the minor. Delta-wye reducibility concerns the study of how graphs can be reduced under a set of six operations: the four series-parallel reductions, delta-wye, and wye-delta transformations.

For terminal minors, we completely characterize when, given a planar graph with four terminals, we can find a minor of $K_{2,4}$ in that graph with the four terminal vertices forming the larger part of the bipartition. This is an extension of a result due to Robertson and Seymour for the case when a graph contains three terminals. For delta-wye reducibility, we study the problem of reducibility for the class of graphs consisting of four-terminal planar graphs. Using the results about rooted $K_{2,4}$ minors, we are able to characterize when 3-connected graphs in this class are reducible.

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Chapter 1

Preliminaries

1.1 Definitions and Notation

In this section we provide the main definitions and terminology used in the thesis. We use standard terminology consistent with [17] unless otherwise noted.

We start with a few definitions. A *graph* $G = (V, E)$ consists of a set V of *vertices* and a set E of *edges*, where each edge consists of two vertices called its *endpoints*. We use the notation uv for an edge joining vertices u and v. When such an edge uv exists, the vertices u and v are said to be *adjacent* and are *incident* with the edge uv. A *loop* is an edge $vv \in E$ from a vertex v to itself. *Multiple edges* or *parallel edges* are edges having the same pair of endpoints. A graph is *simple* if it has no loops or parallel edges. Graphs in this thesis are assumed to be simple, except for those in Chapter 3, or where otherwise noted. The *degree* of a vertex v, denoted $deg(v)$, is the number of edges incident with v, with loops counted twice. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subset E(H)$. We denote this by $H \subseteq G$.

A *path* is a sequence of distinct vertices with each consecutive pair joined by an edge. The first and last vertices in the sequence are the *endpoints* of the path. A *cycle* is a path together with an edge between the endpoints. Two paths are *internally disjoint* if neither contains a non-endpoint vertex of the other. A graph G is *connected* if a path exists between each pair of vertices of G. A *component* of a graph is a maximal connected subgraph.

For $e = uv \in E(G)$, *deletion* of e is the operation of removing the edge e from $E(G)$. This is denoted $G - e$ or $G \backslash e$. *Contraction* of the edge uv is an operation that replaces the vertices u and v by a single vertex incident with each edge that was previously incident to u or v and deleting the edge uv. This is denoted by G/e. For a vertex v, $G - v$ is the graph obtained by deleting the vertex v and all edges incident with the vertex v. For a set of vertices, we define $G - \{v_1, \ldots, , v_k\}$ in the obvious manner. A graph H is said to be a *minor* of a graph G if H can be obtained from G by a sequence of edge contractions and deletions. We denote this $H \leq_M G$. A *model* of a minor of H in G is a map ϕ from H to G where vertices of H map to disjoint connected subgraphs of G; edges of H map to internally disjoint paths of G; $\phi(uv)$ is a path between a vertex in $\phi(v)$ and a vertex in $\phi(u)$ and no other vertex of $\phi(uv)$ is in $\phi(w)$ for any $w \in V(H)$.

For our purposes, a *terminal graph* (G, Y) consists of a graph G and a set $Y \subseteq V(G)$ whose elements are called *terminals*. If the set of terminals is clear from the context, then we can omit them. We say a terminal graph (H, Z) is a *terminal minor* or *rooted minor* of a terminal graph (G, Y) if $|Z| = |Y|$ and we can find a model of H in G such that $\phi(z) \cap Y \neq \emptyset$ for each $z \in Z$. A rooted $K_{2,4}$ is the graph on 6 vertices with 4 terminal vertices and an edge between each terminal vertex and each non-terminal vertex. When searching for a rooted $K_{2,4}$ minor in a graph G, we label the terminals of the minor $t_1, \ldots, t_4$ and we label the corresponding subgraphs of G in the model $T_1, \ldots, T_4$. The non-terminal vertices we will refer to as *big vertices*. We label the subgraphs for these S_1 and S_2.

A *vertex cut* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more components than G. We will refer to this as a *cut*. We say a graph is *k-connected* if every cut has at least k vertices or it is a complete graph on $k+1$ vertices. We say a graph is *internally k-connected* if any cut of size $< k$ gives precisely 2 components, one of which is a single vertex. We call a pair $\{A, B\}$ a *k-separation* of a graph G or a terminal graph (G, Y) if $A \cup B = V(G)$, $|A \cap B| = k$, $A \setminus B \neq \emptyset, B \setminus A \neq \emptyset$, and there are no edges from $A \setminus B$ to $B \setminus A$. We call a set W of k vertices a *k-cut* if there exists a k-separation $\{A, B\}$ with $A \cap B = W$. We say such a cut is *tight* if no subset of W of size $1 \leq \ell \leq k-1$ is an ℓ-cut. We say a separation $\{A, B\}$ of a terminal graph (G, Y) *isolates* a terminal t if $(A \setminus B) \cap Y = \{t\}$ or $(B \setminus A) \cap Y = \{t\}$.

Given a graph, an *embedding* of the graph in the plane is a drawing of the graph in $\mathbf{R}^2$ with points representing vertices and arcs representing edges such that arcs are pairwise internally disjoint and intersect vertices only at their endpoints. For a 2-connected graph, a *face* of an embedding is a region of the plane bounded by a cycle C in the graph, such that all vertices and edges of G not in C are drawn on the other side of the cycle. This is called the *bounding cycle* for the face F. Vertices and edges contained in the bounding cycle of F are said to be *incident* with F and F is said to be *incident* with those vertices and edges. Each face of an embedding induces two cyclic orderings of the vertices of that face, we will use the ordering which is clockwise with respect to the face when looking from a point inside that face. The *facial neighbourhood* of a vertex v is the cyclic ordering of the vertices on the face created after deleting the vertex v. The facial neighbourhood of a path is defined similarly on the face created by deleting the path. A *facial walk* is a sequence of vertices and edges occurring consecutively on a face of the given embedding.

For a facial walk F and vertices $v, w \in V(F)$ we define $P_F[v, w]$ to be the path from v to w clockwise along the face F. We similarly define $P_F(v, w), P_F(v, w],$ and $P_F[v, w)$ where ")" means we do not include the vertex w and the edge incident to it, and "(" means we do not include the vertex v and the edge incident to it. When $v = w$, we mean the path around the whole face and not the trivial 1-vertex path. We drop the subscript F when the face is clear from the context. To avoid any ambiguity, clockwise is always taken with respect to the face the path is on, so $P[c_1, c_3]$ on the Double Face diagram in Figure 2.7 will be along P_2. Note that clockwise in the infinite face looks like counter-clockwise with respect to the rest of the graph. Observe that if G is 3-connected with a, b cofacial and ab is not an edge, then a, b are on a unique face. If ab is an edge, $P[a, b]$ is understood to not be that edge, but rather the rest of one of the faces containing a, b. F_I will always denote the infinite face.

In all diagrams, black dots represent terminal vertices, and white dots represent non-terminal vertices. White regions are faces (this convention also applies to the outer face) and shaded regions are patches of the graph which may contain vertices and edges.

1.2 Overview of the Thesis

The rest of the thesis is organized as follows. In Chapter 2, we discuss rooted minors and prove a structural characterization for rooted $K_{2,4}$ minors in planar graphs. In chapter 3, we discuss delta-wye reducibility and prove a result on the reducibility of 4-terminal planar graphs as well as some other minor results. In Chapter 4 we discuss future work and propose some conjectures for related problems.

1.3 Basic Results

1.3.1 Planar Graphs

In most places in this thesis, we will be working with planar graphs. An excellent introduction to the topic can be found in Chapter 2 of Graphs on Surfaces, by Mohar and Thomassen [8]. We mention two important basic definitions. A graph is *planar* if it can be embedded in the Euclidean plane with no edges crossing. A *plane* graph is a planar graph which is embedded in the plane. We will require the use of an important result from Whitney [18].

Lemma 1.3.1 ([18])**.** *If G is 3-connected then it has an essentially unique planar embedding up to the choice of the infinite face and the orientation. For any embedding, the set of cycles which determine the facial boundaries will be the same, and at any vertex the order of the neighbours around that vertex will be the same.*

The uniqueness of Lemma 1.3.1 is taken up to homotopy. We also include a few observations about planar graphs that we will use throughout the thesis.

Observation 1.3.2. *If G is a simple 3-connected planar graph, then any pair of vertices u, v which are joined by an edge occur on exactly two common faces. Any pair not joined by an edge are on at most one common face.*

Observation 1.3.3. *If G is a simple 2-connected planar graph, then any pair of vertices u, v which do not form a 2-cut also satisfy Observation 1.3.2. If u, v form a 2-cut then the number of common faces they are on is equal to the number of components in $G - \{u, v\}$ (+1 if they are adjacent).*

Observation 1.3.4. *In a planar graph G, a set of vertices $v_1, \dots v_k$ will be a tight k-cut if $k \geq 3$ and there is a cyclic ordering of $v_1, \dots, v_k$ such that consecutive pairs of vertices are cofacial and other pairs are not cofacial. If $k = 3$ we must not have v_1, v_2, v_3 on a single face.*

1.3.2 Rooted Minors

Rooted minors appear as an important tool of Robertson and Seymour in their study of Graph Minors [10]. There have been some recent papers studying graph minor problems dealing mostly with extremal type results [6], [7], [19], [20]. In [11], Robertson and Seymour prove an important result about when a $K_{2,3}$ minor exists in a graph with three terminal vertices which are to form the large side of the bipartition.

Theorem 1.3.5 ([11]). *For distinct vertices a, b, c in a 3-connected graph G, there is a rooted $K_{2,3}$ minor using a, b, c unless G is planar with a, b, c on a common face.*

There have also been structural results characterizing when a graph has a rooted K_3 minor [21], or K_4 minor [4]. We include the main results from these papers here.

Lemma 1.3.6 ([21]). *For distinct vertices a, b, c in a graph G, there is a rooted K_3 minor on a, b, c unless for some vertex $v \in V(G)$ at most one of a, b, c are in each component of $G - v$.*

Theorem 1.3.7 ([4]). *For distinct vertices, a, b, c, d in a 4-connected graph G, there is a rooted K_4 minor on a, b, c, d unless G is planar with a, b, c, d on a common face.*

Theorem 1.3.8 ([4]). *For distinct vertices, a, b, c, d in a 3-connected planar graph G, there is a rooted K_4 minor on a, b, c, d unless a, b, c, d are on a common face.*

1.3.3 Disjoint Paths

A problem very closely related to rooted minors is the disjoint paths problem. Given a graph G and distinct vertices $s_1, \ldots s_k$ and $t_1, \ldots t_k$, when is it possible to find disjoint paths $P_1, P_2, \ldots P_k \in G$ such that P_i joins s_i to t_i for $i = 1, \ldots, k$? The result for $k = 2$ was given independently by Seymour [13] and Thomassen [14]. When we consider only 3-connected planar graphs, the result can be stated as follows.

Theorem 1.3.9. *Given distinct vertices s_1, s_2, t_1, t_2 in a 3-connected planar graph G, we can find disjoint paths from s_1 to t_1 and from s_2 to t_2 respectively unless the vertices lie on a common face in the order s_1, s_2, t_1, t_2.*

Theorem 1.3.9 has an algorithmic counterpart and the general k-linkage problem is polynomial time solvable.

Chapter 2

Rooted $K_{2,4}$ Minors in 4-Terminal Planar Graphs

2.1 Introduction

Our goal in this chapter is to prove a result of independent interest about existence of rooted $K_{2,4}$ minors in planar graphs. The proof is both long and complicated, and the result will be critical for proving our results about 4-terminal delta-wye reducibility. For a graph of low connectivity, we show how to reduce to an equivalent problem where the graph is 3-connected. When the graph is 3-connected, we provide a list of structures, such that the graph either has one of these structures or it has a rooted $K_{2,4}$ minor. This result provides a good characterization about rooted $K_{2,4}$ minors because no graph possessing a structure in the list can have a rooted $K_{2,4}$ minor. Checking for these structures requires determining if certain vertices are cofacial and determining if certain sets of vertices are 3-cuts that isolate terminals.

2.2 Low Connectivity Reductions

In this section we look at graphs which are not 3-connected. We begin by showing that we can easily reduce disconnected graphs to connected graphs (Section 2.2.1) and reduce graphs with cut vertices to 2-connected graphs (Section 2.2.2). For graphs which are 2-connected but not 3-connected, we consider cases depending on how many terminals are on each side of a 2-separation (Section 2.2.3). The cases where there is 0 or 2 terminals on one side

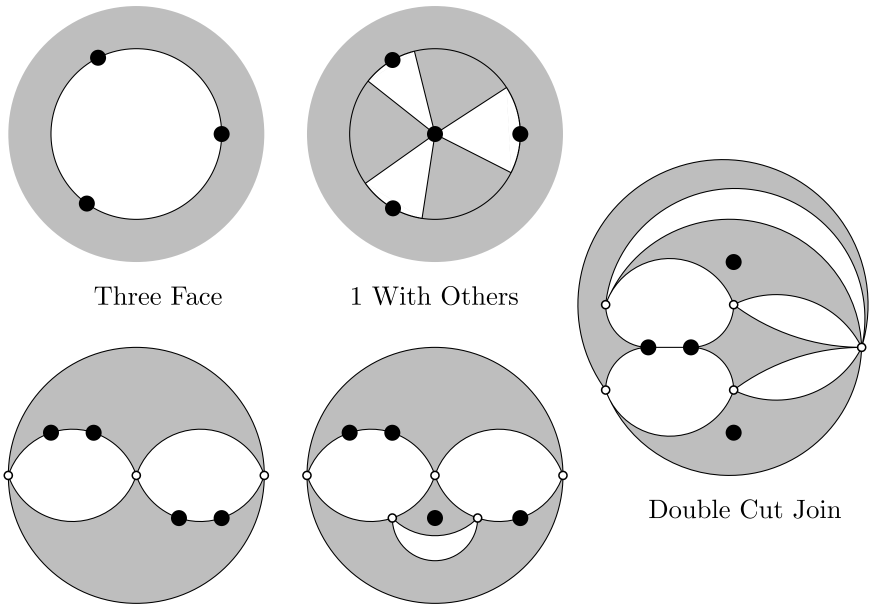


Figure 2.1: The five obstructions for the existence of a $K_{2,4}$ minor

of the separation are simple to check. For the case of a single terminal on one side of the separation, we show through a sequence of lemmas that an easy to describe minor of G, with fewer 2-separations, has a rooted $K_{2,4}$ if and only if G does.

Given a 4-terminal planar graph (G, Y), we would like to determine whether G has a rooted $K_{2,4}$ minor or not. Graphs with higher connectivity have more structure that can be exploited so we first deal with graphs which have small vertex cuts. If G has a separation $\{A, B\}$ of small order with cut set W, and $|W| \leq 3$, we will show that either we can find a rooted $K_{2,4}$ minor in G, show that no rooted $K_{2,4}$ minor exists, or find G' such that G' has a rooted $K_{2,4}$ minor if and only if G' does and G' is formed from G by minor operations and potentially some of the following reductions:

(R1) If there exists a tight 3-separation $\{C_1, C_2\}$ of G with $C_1 \cap C_2 = \{v_1, v_2, v_3\}$ and $(C_1 \backslash C_2) \cap Y = \emptyset$, we replace G by the subgraph induced by C_2 and add the edges v_1v_2, v_2v_3, v_3v_1 if they are not already present in G.

(R2) If there exists a tight 3-separation $\{C_1, C_2\}$ of G with $C_1 \cap C_2 = \{v_1, v_2, v_3\}$ and $(C_1 \backslash C_2) \cap Y = \{t\}$, we replace G by the subgraph induced by C_2 and the terminal t joined by edges to v_1, v_2, v_3. We also add the edge v_1v_2, unless v_1, v_2, t were on a common face in an embedding of G (or the edge was already present in G), and similarly for the edges v_2v_3 and v_3v_1.

Moreover, we assume for an (R2) reduction that $C_1 \backslash C_2$ has at least two vertices, since otherwise the reduction does not change the graph.

(R3) If a terminal t has exactly two non-terminal neighbours v_1, v_2, then if the edge v_1v_2 is present, it is deleted.

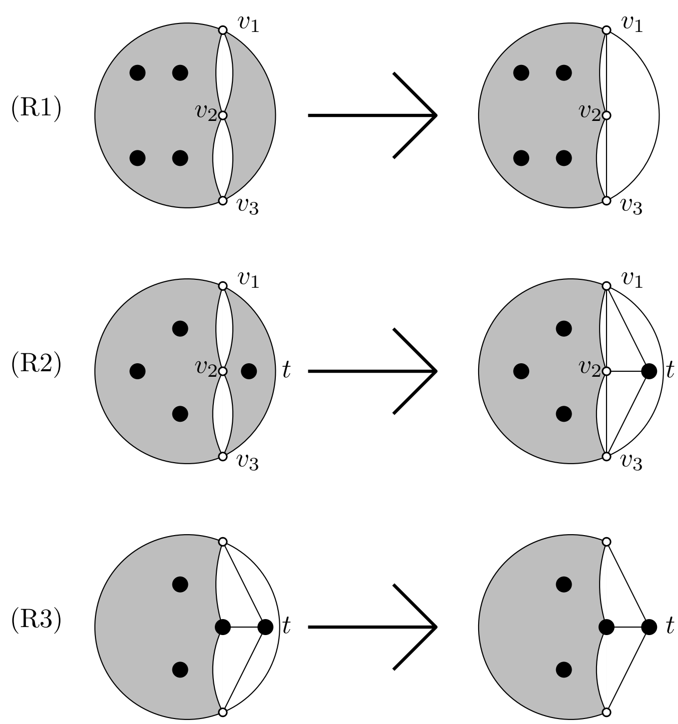


Figure 2.2: Low Connectivity Reductions

Figure 2.2 illustrates the three reductions. Before we proceed with the low connectivity cases, we prove a lemma which shows that performing these reductions does not affect the existence of a rooted $K_{2,4}$ minor.

Lemma 2.2.1. *Let G' be formed from G be performing one of the above reductions. Then G has a rooted $K_{2,4}$ minor if and only if G' does.*

Proof. Suppose we perform an (R1) on a graph G which has a rooted $K_{2,4}$ minor obtaining the graph G'. Let $T_1, T_2, T_3, T_4, S_1, S_2$ form a model of the minor in G. Suppose $v \in C_1$ is in T_1. Since $C_1 \cap Y = \emptyset$ and T_1 is connected, at least one of v_1, v_2, v_3 is in T_1. Suppose $w \in V(C_1)$ is in S_1. Since S_1 is connected and $C_1 \cap Y = \emptyset$ and at most 3 of the T_i are incident with a vertex in C_1, in order for the minor to exist in G, at least one of v_1, v_2, v_3 is in S_1. To construct subgraphs in G' we take the intersection of the subgraphs in G with C_2. If this disconnected a subgraph T_i or S_j, it must have used a path in C_1, so we may assume that v_1, v_2 are in that subgraph. In this case, we add the edge v_1v_2 to this subgraph and it is no longer disconnected (unless v_3 is also in the subgraph, in which case we add v_1v_3 to the subgraph as well). If two subgraphs that were joined by an edge are no longer joined by an edge, the connection must have used a vertex in C_1. This means that the two subgraphs both had vertices in $\{v_1, v_2, v_3\}$, and so we may assume one uses v_1 and the other uses v_2. But v_1, v_2 are joined by an edge, so these subgraphs are joined by an edge in G' as well.

Suppose we perform an (R1) on a graph G that does not have a rooted $K_{2,4}$ minor obtaining the graph G'. If we contract C_1 to a single vertex v, then we get a graph H that is a minor of G, so H also does not have a rooted $K_{2,4}$ minor. Suppose we have a rooted $K_{2,4}$ in G' composed of subgraphs $T_1, T_2, T_3, T_4, S_1, S_2$. Since $K_{2,4}$ is triangle-free we may assume that the minor uses at most 2 edges of the triangle $\{v_1, v_2, v_3\}$. Say it does not use the edge v_1, v_2, so $G' - v_1v_2$ also contains a rooted $K_{2,4}$ minor. However, contracting the edge vv_3 in H gives either G' or $G' - v_1v_2$. Thus, we can get $G' - v_1v_2$ from G by minor operations and so it cannot have a rooted minor when G does not.

Suppose we perform an (R2) reduction on a graph G which has a rooted $K_{2,4}$ minor obtaining the graph G'. Let $T_1, T_2, T_3, T_4, S_1, S_2$ form a model of the minor in G and assume that the terminal $t_1 = t$. We let $T_i' = T_i$ restricted to G' and S_i' similarly. If these do not give a model in G' then either some subgraph is no longer connected or some pair that was previously connected is no longer. Notice that up to relabelling, we must have either $v_1 \in S_1$ and $v_2 \in S_2$ or $v_1 \in T_1$ in order for T_1 to be joined to S_1 and S_2. Also, any subgraph other than T_1 which had a vertex in C_1 must contain at least one of v_1, v_2, v_3.

If $v_1 \in S_1$ and $v_2 \in S_2$ then we have a rooted $K_{2,4}$ minor in G' unless we have disconnected either S_1 or S_2, or if $v_3 \in T_2$ and T_2 is not connected to one of S_1 or S_2. If we have disconnected S_1 (or similarly S_2) then $v_3 \in S_1$ and the edge v_1v_3 is not present, but the subgraphs were joined in C_1 in G. This cannot occur, since if the edge is not present, v_1, v_3, t_1 are on a common face and so connecting S_1 in C_1 would prevent T_1 from connecting to S_2. Similarly, we cannot disconnect T_2 from S_1 or S_2, since if the edge v_1v_3 is not present, t_1 was on that face and the t_1 to v_2 path in G would disconnect t_2 from S_1.

If $v_1 \in T_1$ then we may assume that v_2 and v_3 are not in S_1 and S_2. Thus, restricting to G' in this instance can only disconnect some subgraph that was using both v_2 and v_3. However, this cannot occur, since if the edge v_2v_3 is present, the subgraph would be connected, and if it is not present, then v_2 and v_3 were not joined in the same subgraph in C_1.

Suppose we perform an (R2) on a graph G that does not have a rooted $K_{2,4}$ minor obtaining the graph G'. If there are no 2-cuts in C_1, then there are disjoint paths from t to v_1, v_2, v_3. We can contract these paths to single edges and contract the three regions in C_1 bounded by these paths to single edges, obtaining G' as a minor of G. Note that if t is on a face of the cut $\{v_1, v_2, v_3\}$ that we only get two regions, but still obtain G' as a minor.

If there is a 2-cut in C_1 then it must isolate t, so we may assume that t is of degree 2. If all three of the edges $\{v_1v_2, v_2v_3, v_3v_1\}$ are used in the model, then either they are all in a common subgraph, or they are used to pairwise join three of the vertices of the model. However, notice that we can find a model that only uses two, since if they are in a common subgraph, eliminating the edge will keep the subgraph connected, and there are no triangles in $K_{2,4}$ so we need not pairwise join three subgraphs. Thus, it suffices to show that for each of the three edges $\{v_1v_2, v_2v_3, v_3v_1\}$, if we delete that edge from G' we get a minor of G. If we can find paths from t to v_1 and v_2 without going through v_3 then we proceed as we did in the 3-connected case. If we cannot find such paths, then v_1, v_3 or v_2 v_3 is a 2-cut isolating t. This cannot occur, since then v_1, v_2, v_3 instead gives an (R1) reduction or t is the only vertex in C_1 and so there is no (R2) reduction to perform. The same argument holds for the other two pairs of cut vertices.

Suppose we perform an (R3) reduction on G. Since G' is a subgraph of G, if G' has a rooted $K_{2,4}$ minor then so has G. If G has a rooted $K_{2,4}$ minor but G' does not, then deleting the edge v_1v_2 must have either disconnected a subgraph or caused two subgraphs that need to be adjacent to no longer be. If deleting the edge caused a subgraph to be disconnected, then v_1, v_2 are in the same subgraph. If this subgraph is not T_1, then T_1 is only the vertex t_1 and so it cannot be connected by an edge to both S_1 and S_2 in G. Thus we may assume that the subgraph is T_1, but in this case the subgraph is still connected after removing v_1v_2 since v_1, v_2 are each connected by an edge to t_1. If deleting the edge caused two subgraphs to no longer be adjacent, then we may assume that v_1 is in S_1 and v_2 is in one of the terminal subgraphs. If v_2 was in a terminal subgraph other than T_1, then T_1 would not be connected to S_2 in G. Thus we may assume that v_2 is in T_1, but in this case S_1 and T_1 are joined by the edge v_1t_1. Thus we see that performing an (R3) reduction never creates or destroys a rooted $K_{2,4}$ minor. $\square$

2.2.1 Disconnected Graphs

Suppose that G is disconnected. Since $K_{2,4}$ is connected, G can have a rooted $K_{2,4}$ minor only if the terminal vertices are all in the same component. Given a disconnected graph, we can either find a connected component H which contains all the terminals, or we cannot. If such an H exists, then G has a rooted $K_{2,4}$ minor if and only if H does. If such an H does not exist, G does not have a rooted $K_{2,4}$ minor.

2.2.2 1-separations

Suppose $\{A, B\}$ is a 1-separation of (G, Y) with cut vertex v such that $Y \subseteq A$. If G has no rooted $K_{2,4}$ minor, then clearly A also has none. If G has a rooted $K_{2,4}$ minor, then if, in a model of the minor, any subgraph S_i or T_j uses a vertex in B, it must also use the vertex v. Thus, at most one subgraph uses vertices in B, so we can find the same minor in A.

Suppose $B \cap Y = \{t_1\}$. If v was a terminal, then we have terminals in different components of $G - v$, meaning there is no rooted $K_{2,3}$ minor in $G - \{v\}$ and thus no rooted $K_{2,4}$ minor in G. When v is not a terminal, we obtain G' by contracting B onto the vertex v and use this vertex in place of t_1. Since G' is a minor of G, if G has no rooted $K_{2,4}$ minor, then G' has no rooted $K_{2,4}$ minor. If G has a rooted $K_{2,4}$ minor, then the vertex v must be used in the subgraph T_1 and all other subgraphs must be disjoint from B, otherwise we can not connect every S_i to every T_j. Thus, G' would also have a rooted $K_{2,4}$ minor.

Suppose $B \backslash A$ contains two terminals t_1 and t_2 and $A \backslash B$ contains the other two terminals. We may assume that $v \notin T_1$ and $v \notin T_3$. Since every rooted $K_{2,4}$ minor contains two internally disjoint (t_1, t_3)-paths, and any such path must pass through v, we conclude that G has no rooted $K_{2,4}$ minor in this case.

2.2.3 2-separations

By the discussion in Section 2.2.2 , we may assume that G is 2-connected. Suppose $\{C_1, C_2\}$ is a 2-separation of (G, Y) with $C_1 \cap C_2 = \{v, w\}$. We may assume that $|C_1 \cap Y| \leq |C_2 \cap Y|$ and we consider cases based on the value of $|C_1 \cap Y|$ (i.e. the number of terminals in C_1). For each case, we either find a minor, show one does not exist, or reduce the problem to G', a minor of G, (or two minors) with fewer 2-cuts such that G has a rooted $K_{2,4}$ minor if and only G' does.

No terminals in C_1

Suppose C_1 contains no terminals. We construct G' from G by contracting C_1 to an edge between v and w; any resulting loops or parallel edges are deleted. If G had no rooted $K_{2,4}$ minor then clearly G' also has none. If G had a rooted $K_{2,4}$ minor, then any subgraph S_i or T_j that used C_1 must also use either v or w. Thus, in G', we use the subgraphs as in G, restricted to G'. If some subgraph used both v and w, then we include the edge vw in that subgraph to ensure it remains connected. This clearly gives a rooted $K_{2,4}$ minor in G'.

Two terminals in C_1

Suppose C_1 contains two terminals, t_1 and t_2. If $t_1 = v, t_2 = w$, then clearly G has a rooted $K_{2,4}$ minor if and only if the subgraph induced by C_2 does. If $t_1 = v, t_2 \neq w$, then it is easy to see that in any model of a $K_{2,4}$ minor in G, $w \in T_2$. Let H be the subgraph induced on C_2 letting $w = t_2$. Then $H \leq_M G$ and G has a rooted $K_{2,4}$ minor if and only if H does. Thus, we may assume that $v, w \notin Y$.

When $v, w \notin Y$, we let H_2 be the subgraph induced on C_2 with $v = t_1$ and $w = t_2$ and H_1 be defined symmetrically on C_1 with t_3 and t_4. We show that G has a rooted $K_{2,4}$ minor if and only if H_1 has one, H_2 has one, or H_1 and H_2 both have $K_{2,2}$ minors between $\{t_1, t_2\}$ and $\{t_3, t_4\}$. If H_1 or H_2 has a rooted $K_{2,4}$ minor, then clearly G does, since both graphs are minors of G. If the $K_{2,2}$ minors both exist, they can be composed to get a $K_{2,4}$ minor of G.

Suppose G has a rooted $K_{2,4}$ minor and let $T_1, T_2, T_3, T_4, S_1, S_2$ be a model of the minor. If $v \in S_1$, then $w \in S_2$ since it must connect to vertices in both C_1 and C_2. This gives rooted $K_{2,2}$ minors in C_1 and C_2 between $\{v, w\}$ and $\{t_1, t_2\}$, or $\{t_3, t_4\}$, respectively. If instead $v \in T_1$ then $w \notin S_1 \cup S_2 \cup T_1 \cup T_3 \cup T_4$ since T_2 would not be able to connect to both S_1 and S_2, so $w \in T_2$. We see that H_2 has a rooted $K_{2,4}$ minor. Similarly we have that if $v \in T_3$ then H_1 has a rooted $K_{2,4}$ minor.

In the proceeding lemma we will show that finding rooted $K_{2,2}$ minors is equivalent to finding disjoint rooted paths. This is an important result that will be used throughout the thesis.

Lemma 2.2.2. *Let t_1, t_2, s_1, s_2 be distinct vertices in a graph H. Then H contains a rooted $K_{2,2}$ minor between $\{s_1, s_2\}$ and $\{t_1, t_2\}$ if and only if there exists paths $P_{i,j}$ (where $P_{i,j}$ connects s_i to t_j) such that $P_{1,1} \cap P_{2,2} = \emptyset$ and $P_{1,2} \cap P_{2,1} = \emptyset$.*

Proof. Clearly having a rooted $K_{2,2}$ minor gives us the desired paths. Suppose we have paths $P_{1,1}$, $P_{1,2}$, $P_{2,1}$, and $P_{2,2}$ that satisfy the stated properties. The proof that the subgraph $K = P_{1,1} \cup P_{1,2} \cup P_{2,1} \cup P_{2,2}$ of H contains a rooted $K_{2,2}$ minor is by induction on $|E(K)|$. If any non-end vertex has degree 2, we can contract an edge incident with it and win. We can also contract any edge that is in two paths. If there is an edge of K between two non-adjacent vertices on a path, then we can reroute that path to use that edge, reducing the number of edges in K. Otherwise, the graph K has the property that each vertex is of degree 4, except the endpoints of the paths which are of degree 2.

Since any end vertex has degree 2 and is not adjacent to two vertices on the same path, there is an edge s_1v_1 of $P_{1,2}$ where $s_1 \in P_{1,1}$ and $v_1 \in P_{2,2}$ and there is an edge s_2v_2 of $P_{2,1}$ with $s_2 \in P_{2,2}$ and $v_2 \in P_{1,1}$. We can contract the portion of $P_{1,1}$ between t_1 and v_2 onto t_1 and similarly contract a portion of $P_{2,2}$ onto v_1. We contract the remainder of $P_{1,1}$ and $P_{2,2}$ to be edges s_1v_2 and s_2v_1 respectively, giving us the desired $K_{2,2}$ minor. □

One terminal in C_1

Suppose that t_1 is the only terminal in C_1. If $t_1 = v$, then based on arguments similar to the above cases we see that G has a rooted $K_{2,4}$ minor if and only if the graph formed by contracting $C_1 \setminus C_2$ onto w does. Thus, we may assume that $Y \cap (C_1 \cap C_2) = \emptyset$. If the subgraph induced on $C_2 \setminus C_1$ is disconnected, then some component C_3 of this contains zero or one terminal, then $\{C_1 \cup C_3, C_2 \setminus C_3\}$ or $\{C_3 \cup \{v, w\}, C_1 \cup C_2 \setminus C_3\}$ would be 2-separations where one side has zero or two terminals. Since we have considered these cases previously, we may assume that $C_2 \setminus C_1$ is connected.

Let G' be formed by contracting $C_1 \setminus C_2$ to the terminal t_1. We show that G' has a rooted $K_{2,4}$ minor if and only if G does. Clearly if G' has a rooted $K_{2,4}$ minor then so too does G, since G' is a minor of G. If G has a rooted $K_{2,4}$ minor then in any model either S_1 uses v and S_2 uses w, or T_1 uses at least one of v or w. If the former occurs, then we clearly still have the minor in G'. If the latter occurs, we may assume that T_1 uses w. If v was also used by T_1 then the minor still exists in G'. If v was in some other T_i, then it was not connecting to anything in C_1 so the minor still exists in G'. If v was in S_1 or S_2 (or not part of the minor), then it is still connected to t_1 and so the minor still exists. Thus, G' has a rooted $K_{2,4}$ minor if and only if G does.

By Lemma 2.2.1 we may assume the edge vw is not present in G. If G has another 2-separation $\{A, B\}$, we may assume that the subgraph induced on A consists of a terminal joined by edges to two non-terminals. We know by the previous section that we cannot have $A \cap B = C_1 \cap C_2$. Therefore G is a subdivision of a 3-connected graph, so G has a unique planar embedding. If a degree 2 terminal is cofacial with another terminal in this embedding, we can add an edge between them without changing the existence of a rooted $K_{2,4}$ minor. Thus, we may assume that t_1 (and any other terminal of degree 2) is not cofacial with any other terminals.

Lemma 2.2.3. *If G is as above then either G has a vertex with exactly one degree 2 neighbour or G has structure as shown in Figure 2.3.*

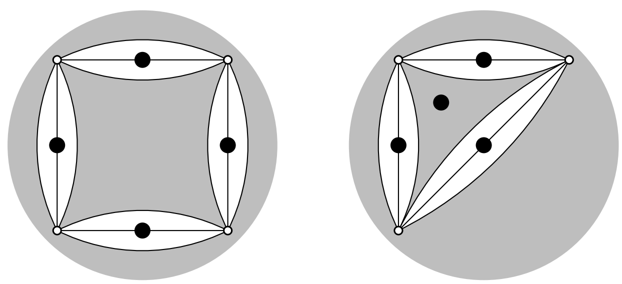


Figure 2.3: Graphs where all vertices adjacent to a terminal of degree 2 have at least two neighbours of degree 2.

Proof. Suppose G has no vertex with exactly one neighbour of degree 2 and consider the graph H with vertex set the vertices of G with a degree 2 neighbour, and an edge between them if they have a common degree 2 neighbour. The graph H has min degree ≥ 2, at most 4 edges and no parallel edges, so it must have a 3-cycle or a 4-cycle. This gives one of the obstructions depicted in Figure 2.3. □

Suppose G has the first structure in Figure 2.3. If we can find disjoint paths between opposite pairs of the non-terminal "white" vertices shown in the figure, then we would have a rooted $K_{2,4}$ minor. If such paths exist, one would have to be in the interior region and the other in the exterior.

Let v_1 be one of the "white" non-terminal vertices in the figure, with v_2 and v_3 being the non-terminal vertices cofacial with v_1. The facial neighbourhood of v_2 gives two paths between the v_2 and v_3. One such path will be in the interior region (and possibly contain the opposite non-terminal v_4) and the other path will be in the exterior region (and possibly contain v_4). So, if there is no path between v_2 and v_3 in one of the regions, then the facial path described above must use v_4, and so v_1 and v_4 are cofacial in that region, so there is no $K_{2,4}$ minor. If we consider paths from v_1 to v_4 in the facial neighbourhood of v_3, we can arrive at a similar conclusion. If we have a path from v_1 to v_4 in one region and a path from v_2 to v_3 in the other, we have our $K_{2,4}$ minor. If we do not have such paths, then either both pairs (v_1, v_4) and (v_2, v_3) are cofacial in the same region, or one pair (say (v_1, v_4)) is cofacial in both regions. If both pairs are cofacial in one region then all four vertices must be on a common face in that region and so we can embed the graph so that all 4 terminals are also in that face. If one pair is cofacial in both regions, then that pair gives a 2-cut with two terminals on each side. We have already assumed that we do not have such a cut, so we either have the desired paths that give the minor, or we have the 4 terminals on a common face.

Suppose G has the second structure in Figure 2.3. We claim that in this case G does not have a rooted $K_{2,4}$ minor. Reductions $(R1)$ – $(R3)$ give $G' = K_4$, which does not have a rooted $K_{2,4}$ minor. By Lemma 2.2.1, G does not have a rooted $K_{2,4}$ minor.

Let t_1 be a degree 2 terminal (with neighbours v and w) in a graph G such that G does not have the structures as shown in Figure 2.3, and let $G_v = G/t_1v$ and $G_w = G/t_1w$. Clearly, if G_v or G_w has a rooted $K_{2,4}$ minor then so does G. However, the converse does not hold. It may happen that G has a rooted $K_{2,4}$ minor but G_v and G_w do not. But this can happen only in very special situations as shown by the next lemma.

Lemma 2.2.4. *Let G be a graph as above and let t_1 be a terminal of degree 2 with neighbours v and w such that v has no other neighbours of degree 2 and such that there are no (R1), (R2) and (R3) reductions which can be performed on G. Then either t_1 is cofacial with another terminal t_2, in which case G has a rooted $K_{2,4}$ minor if and only if the graph formed by adding the edge t_1t_2 does, or t_1 is not cofacial with any other terminal, and G has a rooted $K_{2,4}$ minor if and only if at least one of G_v or G_w does.*

Proof. It is clear that if t_1, t_2 are cofacial that adding the edge t_1t_2 maintains planarity and does not affect the existence of a rooted $K_{2,4}$ minor. Thus, we may assume that t_1

is not cofacial with other terminals. It is also clear that if G has no rooted $K_{2,4}$ minor that neither G_v nor G_w will. Thus, we may assume that G has a rooted $K_{2,4}$ minor. Let $T_1, T_2, T_3, T_4, S_1, S_2$ form a model of such a minor. If $v \in T_1$ or $w \in T_1$ then G_v or G_w will have a rooted $K_{2,4}$ minor using the same model with the respective edge contracted. Thus, we may assume that in any such model $T_1 = \{t_1\}$. Since T_1 connects to S_1 and S_2, we may also assume that $v \in S_1$ and $w \in S_2$.

We can always find a model where each T_i ($i \in \{2, 3, 4\}$) consists of a path from t_i to some vertex u_i (possibly $u_i = t_i$), where S_1 and S_2 are each joined by an edge to u_i and to no other vertices on the path. To see this, let us consider a model where T_2, T_3, T_4 have the minimum number of edges. Each T_i is then clearly a tree. If any leaf vertex of this tree is adjacent to exactly one of S_1 or S_2, then that vertex is t_i since otherwise that vertex could be added to the subgraph it is adjacent to. Thus, either $T_i = t_i$ or T_i has a leaf vertex which is adjacent to both S_1 and S_2 and so T_i must be the path between these 2 vertices. If S_1 was joined to another vertex on the path, we could use a sub-path for t_i and add the rest to S_2, so it is not.

Subject to the above conditions on T_2, T_3, T_4, we can also assume that S_1 and S_2 are trees. Over all possible models, we will choose one that has the sizes of T_2, T_3, T_4 minimum and subject to this, he sizes of S_1 and S_2 minimum. Since G_v and G_w have no rooted $K_{2,4}$ minor when G does, the vertex v must be a cut vertex of S_1 and the vertex w must be a cut vertex of S_2. The embedding of the subgraph $S_1 + vt_1$ induces a natural clockwise ordering of the vertices t_1, u_2, u_3, u_4 with respect to v (which we may assume appear in that order). The embedding of S_2 also induces a clockwise ordering with respect to w, which must be the order u_4, u_3, u_2, t_1 (of Figure 2.4).

We can also see that u_2 and u_4 are not cofacial with t_1. Suppose $u_2 \in P(w, v)$. We know that if this occurs, $u_2 \neq t_2$, so there is an edge from u_2 to some vertex $y \in T_2$. Looking at the facial neighbourhood of u_2, there is a vertex $g_2 \in S_2$, x, and $g_1 \in S_1$ which occur clockwise in that order. As we proceed clockwise starting from g_2, we will eventually arrive at a vertex z which is in $T_2 \cup T_3 \cup T_4$ (it will be x unless we arrive at some other vertex first). If we proceed counterclockwise from z around u_2, we will eventually get to a vertex in $S_1 \cup S_2$. Since this facial path connects z to either S_1 or S_2, by minimality we must have $z \in \{u_3, u_4\}$. We can repeat this argument going counterclockwise from g_1 to x. From this we conclude that u_3 and u_4 are both cofacial with u_2. We may assume that u_3 occurs first in the clockwise order. We observe that the paths in S_1 to u_3 and u_2 will disconnect S_2

from u_4, meaning we would not have a $K_{2,4}$ minor. This is a contradiction, and so u_2 and u_4 cannot be cofacial with t_1.

Consider the subgraphs of S_1 and S_2 (together with edges to u_2 and u_4) consisting of the paths from v and w to u_2 and u_4. Let v_2 be the last vertex in the facial neighbourhood of v on the path in S_1 to u_2 and define v_4, w_2, w_4 in the obvious respective manners. This gives a partial representation of the model as shown in Figure 2.4 (where possibly $v_2 = u_2$, etc.).

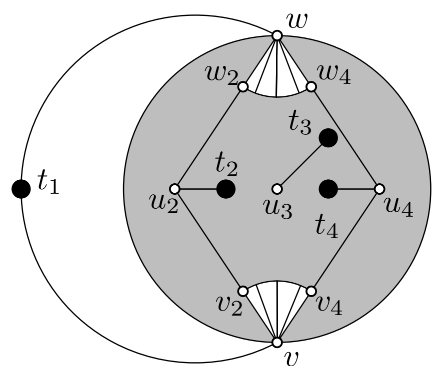


Figure 2.4: Structure of G when $v \in S_1$, $w \in S_2$

If the path P_v on the facial neighbourhood of v, clockwise from v_2 to v_4, is disjoint from all subgraphs of the $K_{2,4}$ model aside from S_1, then we could use P_v and would not need the vertex v in S_1, implyingthat G_v has a rooted $K_{2,4}$ minor. Thus, we may assume that some vertex in $S_2 \cup T_2 \cup T_3 \cup T_4$ is on P_v. A similar argument holds for the path P_w from w_4 to w_2 in the facial neighbourhood of w. Consider the vertex v_2' on P_v that is closest to v_2 and is also in $S_2 \cup T_2 \cup T_3 \cup T_4$. If this vertex is in T_i and is different from u_i, then we could let the subpath of P_v from v_2 to v_2' be in S_1 and find a model where T_i is smaller, contradicting minimality. We obtain similar conclusions for vertices v_4', w_2', w_4' whose definitions are similar to the definition of v_2'. Thus, each of $v_2', v_4', (w_2', w_4')$ is either u_2, u_3, u_4 or a vertex in $S_2(S_1)$.

Since we assumed that no (R1), (R2) and (R3) reductions could be performed, it is clear that the facial neighbourhoods of v and w are disjoint aside from vertices which are also cofacial with t_1. Thus none of u_2, u_3, u_4 can be in $P_v \cap P_w$. We observe that if $w_4' = u_2$ then we can eliminate w from S_2 by connecting to u_2 along the path from w_4 to w_4' since u_3 could not have connected to S_2 along the path from w_2 to u_2, as this would mean u_3 could not connect to S_1. Similar conditions about w_2', v_2', and v_4' follow. Also notice that if u_2 and u_4 both occur in P_w then they are not consecutive in the facial neighbourhood, since we would not be able to connect S_2 to u_3. Similarly for the facial neighbourhood of v. Moreover, there must be an edge from w to a vertex $w_3 \in P_w$ and an edge from v to a vertex $v_3 \in P_3$.

We observe that if $t_2 \neq u_2$, then t_2 is in the portion of graph bounded by the paths from v to w through u_2 and u_4, since otherwise we could find a path from t_2 to S_1 or S_2 that did not use u_1, contradicting the minimality of the size of T_2. Suppose that the path from v to u_2 intersects P_w at a vertex x as in Figure 2.5. Then either we can reroute the path to use the facial neighbourhood of x (and henceforth be disjoint from P_w), or there is a vertex y in the path in S_2 from w to u_2 that is also in the facial neighbourhood of x and the corresponding face containing x and y is not contained in the disk that is shaded darker in Figure 2.5. If such a y exists, then it cannot be a facial neighbour of w, since then w, x, y would be a 3-cut which isolates t_2 and so by minimality, t_2 is adjacent to w, and so y would not be in S_2. Since y is not a facial neighbour of w, we may assume it is on the path from w_2 to u_2. In this case, we can reroute S_2 to use the path from w_2 to x and the path from x to u_2 instead of the path from w_2 to u_2 and reroute S_1 along the facial neighbourhood of x to the vertex y and the path from y to u_2. This allows us to assume that the path in S_1 from v to u_2 does not intersect P_w.

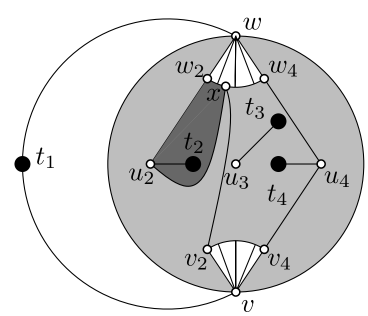


Figure 2.5: Structure of G when S_1 intersects P_w.

Suppose that none of u_2, u_3, u_4 are on $P_v \cup P_w$. If we consider the path in S_2 from u_3 to w, this path must intersect P_w; otherwise we could reroute S_1 along P_v and G_v would thus have a rooted $K_{2,4}$ minor. Similarly, the path from in S_1 from u_3 to v must intersect the P_w. Now, consider the first time the paths from u_3 to v and w hit a vertex on $P_v \cup P_w$. If one first hits P_v and the other P_w, then the subpaths to these segments along with P_v and P_w will give us a new model which shows that both G_v and G_w have rooted $K_{2,4}$ minors.

If both paths hit P_w first, we can follow P_w from u_3 to S_1 until it hits a vertex in S_1. This path cannot hit P_w again on the side opposite the intersection of the path in S_2 from u_3 to w. Thus, we can change S_2 by adding a segment of P_w so that the path in the new S_2 will not intersect P_v (and hence see that G_v has a rooted $K_{2,4}$ minor.

We can easily extend the above arguments to work if $u_2 \in P_w$ and $u_4 \notin P_w$ and $u_3 \notin P_u \cup P_w$. The only time this makes a difference is when both paths in $S_1 \cup S_2$ from u_3 to v and w hit P_w first. Here, we choose to continue the path which hits P_w closer to u_2 until it hits P_v and then include this into S_1; next we change S_2 by adding the second path from u_3 to P_w together with a segment of P_w from this path to w_4. This gives rise to a $K_{w,4}$ model in G_v.

We can also extend to the case where u_2 and u_4 are both P_w (or both in P_v). Again, the only difference is how we remake our model when the paths from u_3 to v and w both intersect P_w first. We cannot have both paths, each P_w. Thus we choose to extend to v the path which only intersects P_w on one side of w_3 (if one intersects w_3, we consider this as intersecting on both sides of w_3). This will allow us to find a model where S_1 does not use v and so G_v has a rooted $K_{2,4}$ minor.

We are left to consider cases where $u_3 \in P_v \cup P_w$. We may assume that $u_3 \in P_v$. As long as $u_4 \notin P_v$, we can find a path from u_3 to P_w. If we consider this path to be in T_3, then we can connect S_1 to T_3 along P_v, and we can let P_w be in S_2, replacing w. Thus, G_w would have a rooted $K_{2,4}$ minor. We can do similarly if $u_3 \in P_v$ and $u_2 \notin P_v$.

Finally, if u_2, u_3, u_4 are all in P_v, consider the vertex v_3 defined earlier. As above, we can simply find a path in S_2 between u_3 and P_w. Letting this path be in T_3 gives a model of a rooted $K_{2,4}$ minor where S_2 does not use w, and so G_w has a rooted $K_{2,4}$ minor. □

2.3 Three-Connected Graphs and the Main Theorem

For a 3-connected 4-terminal graph G, we define G^* as the graph obtained from G by performing the (R1), (R2), and (R3) reductions as described in Section 2.2. An (R2) reduction is only performed when there are no (R1) reductions to perform and an (R3) reduction is only performed when there are neither (R1) nor (R2) reductions to perform. This choice of ordering is important in reducing the number of cases we must consider in the sequel. Before stating the main theorem, we prove some preliminary results about 3-connected planar graphs, the relationship between G and G^*, and the behaviour of the reductions.

We prove our main theorem by considering a minimal counterexample to the claim that a graph has a rooted $K_{2,4}$ minor or it has one of the listed structures. Such a graph will have neither a structure nor the minor. Through a sequence of lemmas (2.3.7 – 2.3.10, we will continually strengthen the criteria for which graphs could be a minimal counterexample. After this, we consider several cases (Lemmas 2.3.14 – 2.3.18) which complete the proof of the theorem.

2.3.1 Important results

Our first three results deal with the connectivity and cut sets in 3-connected graphs.

Lemma 2.3.1. *Let G be a 3-connected planar graph, and $W \subset V(G)$ a vertex set such that all vertices in W are on a common face. Then $G - W$ is connected. If W forms a path P in G, then contracting P to a single vertex gives a 2-connected graph.*

Proof. Let $v, w \in V(G)\backslash W$ and let P_1, P_2, P_3 be three internally disjoint paths in G between them. If P_1, P_2, P_3 all intersected W, then we could embed $K_{3,3}$ in the plane by adding a vertex to the middle of the face containing W joined to a vertex of each path. Thus, at most two paths intersect W, so if we delete W, a path from v to w still exists and so $G - W$ is connected. If we contract P to a single vertex, we are using vertices from at most two of the given paths, so two paths must remain and G/W is 2-connected. □

Lemma 2.3.2. *Let G be a 3-connected planar graph, $W \subset V(G)$ and $x \in V(G)$ such that all vertices of W are on a common face . Then $H = G - (W \cup x)$ is connected unless there exists a 3-cut $\{x, w_1, w_2\}$ with $w_1, w_2 \in W$, separating G into two components, each of which contains a vertex not in W.*

Proof. If H is not connected, then by Observation 1.3.4 we have in G a tight cut set $\{v_1, v_2, \ldots v_k\} \subseteq (W \cup x)$ such that consecutive pairs of vertices are cofacial and other pairs are not. By Lemma 2.3.1 this cutset must contain x. Since x is cofacial with all vertices $k \leq 3$, and since G is 3-connected $k \geq 3$ and the result holds. □

Lemma 2.3.3. *Let G be a 3-connected planar graph, $W \subset V(G)$ and $x \in V(G)$ such that each vertex of W is cofacial with x (though not necessarily all on the same face) then $H = G - (W \cup x)$ is connected unless there exists a 3-cut $\{x, w_1, w_2\}$ with $w_1, w_2 \in W$.*

Proof. Suppose H is not connected. Then G has a tight cut set $S = \{v_1, v_2, \ldots, v_k\} \subset (W \cup x)$ such that there exists a vertex not in $W \cup x$ on each side of the cut, consecutive pairs of cut vertices are cofacial, and other pairs of cut vertices are not cofacial. This cutset will satisfy the requirement of the Lemma unless $x \notin S$. or $k \geq 4$. If $x \notin S$, then the component of $G - S$ containing x must contain at least one other vertex $y \notin (W \cup x$, and the other component contains a vertex $z \notin (W \cup x)$. If we consider the sets $\{v_1, v_2, x\}, \{v_2, v_3, x\}, \ldots, \{v_k, v_1, x\}$ then one of these must be a cutset separating y and z. Thus we may assume that $x \in S$. We now need only consider the case when $k \geq 4$ and $x \in S$. Let $S = \{x, v_2, \ldots, v_k\}$. Each of the sets $\{x, v_2, v_3\}, \ldots, \{x, v_k, v_2\}$ will be a cutset unless the three vertices are on a common face. At least one such set must be a cutset, since otherwise S was not a cutset. Thus we can find a cutset of the form $\{x, w_1, w_2\}$. □

We next turn our attention to the relationship between G and G^*. We showing that moving from G to G^* maintains 3-connectivity and does not change whether the graph has a rooted $K_{2,4}$ minor or not.

Lemma 2.3.4. *If G is a 3-connected 4-terminal planar graph, so too is G^*.*

Proof. Clearly all reductions (R1) – (R3) preserve the proper number of terminals and planarity. Thus, we only need to argue that 3-connectivity is also maintained.

Suppose we perform an (R1) reduction on a 3-connected graph G resulting in the graph G'. Any pair of vertices that is cofacial in G' was also cofacial in G. The only new facial adjacencies that are formed are between the vertices of the 3-cut used in the (R1) reduction, which may now be cofacial on an additional face (of size 3). Thus, if we have a 2-cut in G', it must use some pair of these vertices. However, when the vertices of a 2-cut are joined by an edge, they must be cofacial on at least three faces. This would imply they were cofacial on two faces in G, and not adjacent by an edge, contradicting the 3-connectivity of G. Clearly the graph has no 1-cut, since no vertex is put onto a face with itself by the reduction.

Suppose we perform an (R2) reduction and the resulting graph G' contains a 2-cut $\{v, w\}$. The vertices v and w appear on at least two common faces in G'. They must be on a common face in G' that they were not on in G since G is 3-connected. For each face of the 3-cut used in the reduction, if the terminal t was on that face then we did not add the corresponding edge in G', and if t was not on that face, then it is now on a face of size 3 with the corresponding vertices of the 3-cut. Thus, the only new facial adjacencies we created are between t and some of the vertices v_1, v_2, v_3 or between a pair of $\{v_1, v_2, v_3\}$.

A 2-cut is not formed between a pair of $\{v_1, v_2, v_3\}$ by the same proof as in the (R1) case. Thus, if we have a 2-cut, one of the vertices is t, and we may assume the other one is v_1.

Clearly the vertex t is on at most 3 faces and v_1 is on at least two of these faces. If v_1 was on the other face, then v_1, v_2, v_3 would have been on a common face in G and so would not be a 3-cut to use for an (R2) reduction. Thus v_1 and t are on exactly two common faces, and they are joined by an edge. This cannot give a tight 2-cut, since if we have a tight 2-cut where the vertices are joined by an edge, they are on at least 3 common faces.

If G' has a 1-cut, then there is some vertex that is on the same face twice. The only new facial adjacencies we create are on the faces of size three that contain t, so any such vertex would have also been a cut vertex in G.

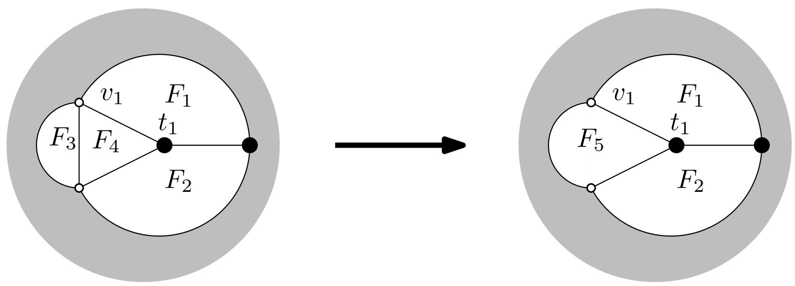


Figure 2.6: Performing an (R3) Reduction

Suppose we perform an (R3) reduction on G where neither (R1) nor (R2) reductions could be done. Let G' be the resulting graph. It is clear from Menger's Theorem that deleting this single edge cannot create a cut vertex in the graph.

If G' has a 2-cut, then it must use a pair of vertices that have a new facial adjacency. The only new facial adjacencies that are created are between t_1 and vertices on the face F_3 of G as seen in Figure 2.6. Thus, in G' one of the vertices in the 2-cut is t_1 and the other one is $v \in V(F_5)$. Since it is a 2-cut, v is on a second face with t_1, which we may assume is F_1. Clearly $v \neq v_1$. Since in G the faces F_1 and F_3 have two vertices v_1 and v in common, those vertices must be adjacent. However this would mean v_1 is a degree-3 non-terminal vertex in G, so we could have performed an (R1) reduction contradicting our assumption. This completes the proof. □

If we apply Lemma 2.2.1 to G, it is clear that G has a rooted $K_{2,4}$ minor if and only if G^* does.

2.3.2 The main theorem

We are now ready to state the main theorem of this section.

Theorem 2.3.5. *Let G be a 3-connected, 4-terminal planar graph. Then either G has a rooted $K_{2,4}$ minor, or the reduced graph G^* has one of the following five structures (for some ordering of its terminals $t_1, \ldots, t_4$):*

1. *(Three-Face – 3F) A face F such that $t_1, t_2, t_3 \in V(F)$.*
2. *(One With Others – OWO) Three faces F_1, F_2, F_3 such that $t_1, t_2 \in V(F_1)$, $t_1, t_3 \in V(F_2)$, and $t_1, t_4 \in V(F_3)$.*
3. *(Double Face – DF) Three faces F_1, F_2, F_3 and three vertices v_1, v_2, v_3 such that v_1, t_1, t_2, v_2 appear clockwise in that order on F_1, the vertices v_2, v_3, t_4, t_3 appear clockwise in that order on F_2, and $v_1, v_3 \in V(F_3)$.*
4. *(Happy Face – HF) Three faces F_1, F_2, F_3 and vertices $v_1, \ldots v_5$ such that v_1, t_1, t_2, v_2, v_4 appear clockwise in that order around F_1, vertices v_2, v_3, t_3, v_5 appear clockwise in that order around F_2, vertices $v_1, v_3 \in V(F_3)$ and v_2, v_4, v_5 form a 3-cut separating t_4 from all other terminals.*
5. *(Double Cut Join – DCJ) Two faces F_1, F_2 and five vertices $v_1, \ldots, v_5$ such that t_2, t_1, v_1, v_2 appear clockwise in that order around F_1, vertices t_1, t_2, v_4, v_3 appear clockwise in that order around F_2, vertices v_1, v_2, v_5 form a 3-cut separating t_3 from all other terminals and v_3, v_4, v_5 form a 3-cut separating t_4 from all other terminals. (The faces F_1 and F_2 may be the same face, when the edge t_1t_2 is not present).*

Some of these vertices shown may be equal (unless that produces a 2-cut), e.g. it is allowed that $v_1 = t_1$ in DF (but $v_2 = t_2$ would give $3F$) or $v_1 = v_4$ in HF or even $v_5 = t_3 = v_3$ in HF. Figure 2.7 illustrates the structures. When showing the existence of one of the structures, we will use the notation present beneath the diagram to indicate the key vertices appearing in the structures.

2.3.3 Proof of the Theorem 2.3.5

We begin the proof of Theorem 2.3.5 by showing that if a graph has one of the listed structures then it does not have a rooted $K_{2,4}$ minor.

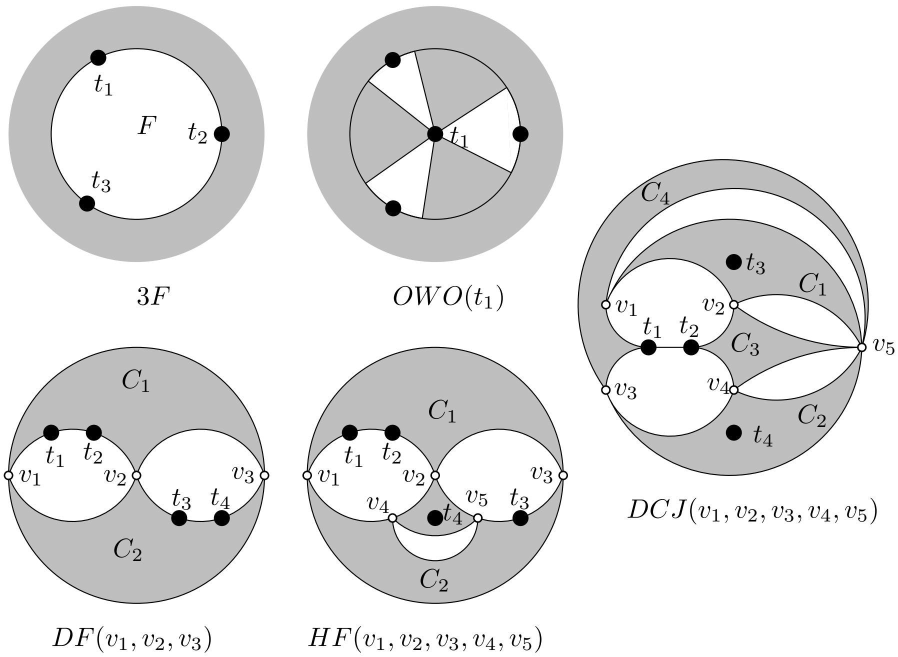


Figure 2.7: The five obstructions for the existence of a $K_{2,4}$ minor

Lemma 2.3.6. *Suppose G has one of the structures in Theorem 2.3.5. Then G has no rooted $K_{2,4}$ minor.*

Proof. If G has 3F structure, then there is no $K_{2,3}$ minor between some three of the terminals (if there was, we could add a vertex on the common face and get a planar embedding of $K_{3,3}$). So, there is also no $K_{2,4}$ minor. If G has $OWO(t_1)$ then $G - t_1$ has no rooted $K_{2,3}$ minor and so G has no rooted $K_{2,4}$ minor.

Suppose G has $DF(v_1, v_2, v_3)$ and a rooted $K_{2,4}$ minor. If v_2 was in no subgraph of the model, $G - v_2$ would have the minor as well, but $G - v_2$ has 3F. If $v_2 \in T_i$, then we can contract the path from t_i to v_2 and the resulting graph would have the minor, but, again it has 3F. Thus, we may assume that $v_2 \in S_1$. There is a path in $T_1 \cup S_1$ from v_2 to t_1. Deleting this path must leave the other terminals in the same component; otherwise S_2 cannot connect to all of them. It is easy to see that the only way for this to happen is that $v_1 \in S_1 \cup T_1$. A similar argument shows that $v_3 \in S_1 \cup T_4$ and so S_2 cannot connect to both T_3 and T_2.

Suppose G has $HF(v_1, v_2, v_3, v_4, v_5)$ and a rooted $K_{2,4}$ minor. If any of v_2, v_4, v_5 were in T_4, then we could contract T_4 and get a graph with $3F$ or DF which has no rooted $K_{2,4}$

minor, so we may assume that $T_4 \cap \{v_2, v_4, v_5\} = \emptyset$. Similarly to the DF case, we may assume $v_2 \in S_1$, otherwise we have an obstruction we have already discussed; similarly, $v_1 \in T_1 \cup S_1$. Further to this, we must also have $v_4 \in T_1 \cup S_1$, since otherwise $T_1 \cup S_1$ would separate T_4 from T_2. For S_2 to connect to T_2, we must have $v_3 \in T_2 \cup S_2$. To connect S_2 to T_4, we must have $v_5 \in S_2$. However, then $S_2 \cup T_2$ separates T_3 from S_1 and so G does not have a rooted $K_{2,4}$ minor.

Suppose G has $DCJ(v_1, v_2, v_3, v_4, v_5)$. Any rooted $K_{2,4}$ minor must induce a rooted $K_{2,2}$ minor on the terminals t_1 and t_2. This minor contains two paths between t_1 and t_2 passing through S_1 and S_2 respectively. The only way the paths can be routed so that none of their deletions separates the remaining terminals is to route one through v_1 and v_2 and to route the other through v_3 and v_4. Thus, we may assume that $\{v_1, v_2\} \subseteq S_1 \cup T_1 \cup T_2$ and $\{v_3, v_4\} \subseteq S_2 \cup T_1 \cup T_2$. However, this cannot be extended to a $K_{2,4}$ minor since v_5 is needed to connect S_1 to T_4 and to connect S_2 to T_3. □

To show that graphs not having a structure from Theorem 2.3.5 have a rooted $K_{2,4}$ minor, we will consider a minimal counterexample to this claim. This will be a graph G such that G^* has no rooted $K_{2,4}$ minor nor a structure from the theorem. We may assume that $G = G^*$, since if $G \neq G^*$ then G^* is a smaller counterexample. We say that G is *reduced* when $G = G^*$. The following series of lemmas will put restrictions on what such a minimal counterexample would look like.

Lemma 2.3.7. *Given a reduced 3-connected planar graph G such that one terminal (say t_1) is cofacial with two other terminals (say t_2 and t_3) possibly on different faces, then either G has $3F$ structure, or $OWO(t_1)$ or G has a rooted $K_{2,4}$ minor.*

Proof. If t_4 is cofacial with t_1 or if t_2 and t_3 are cofacial, then G has $3F$ structure or $OWO(t_1)$. If not, we consider the two paths P, Q from t_2 to t_3 in the facial neighbourhood of t_1. Since t_2 and t_3 are not cofacial, there is a vertex $p \in V(P)\backslash\{t_1, t_3\}$ that has a neighbour outside the facial neighbourhood of t_1. Similarly, there is a vertex $q \in V(Q)\backslash\{t_2, t_3\}$ that has a neighbour outside the facial neighbourhood of t_1. By Lemma 2.3.3 we can find paths in $G - (V(P \cup Q)\backslash\{p, q\})$ from t_4 to p and q, completing the $K_{2,4}$ minor, unless there is a 3-cut which uses t_1 and separates p or q from t_4. The existence of such a cut would either contradict that G is reduced or imply that t_1 and t_4 are cofacial, which we already said cannot occur. □

From now on we shall frequently use the notation $P(a, b)$ introduced in Section 1.1, where a and b are cofacial vertices and $P(a, b)$ is the clockwise traversal of a facial walk containing a and b.

Lemma 2.3.8. *Let G be a reduced, 3-connected planar graph such that two terminals t_1 and t_2 are cofacial. Then either G has one of the structures of Theorem 2.3.5 or G has a rooted $K_{2,4}$ minor.*

Proof. Let $S_1 = P(t_1, t_2)$, $S_2 = P(t_2, t_1)$ and let $G' = G - \{t_1, t_2\}$. We will consider two cases, first the case where G' is 3-connected and then the case where G' is not 3-connected.

1: We consider first the case where G' is 3-connected. Let $G'' = G/(E(S_1) \cup E(S_2))$ and let s_1, s_2 be the contracted vertices. To exhibit a $K_{2,4}$ minor in G, it is sufficient to exhibit a $K_{2,2}$ minor in G'' between $\{s_1, s_2\}$ and $\{t_3, t_4\}$. By Lemma 2.2.2, this minor exists if and only if we can find the pairs of disjoint paths mentioned in the lemma. We consider two possible cases, either G'' is 3-connected or it is not.

1.1: Suppose that G'' is 3-connected. Then we know that the required paths exist unless there is a face F containing s_1, s_2, t_3, t_4 in a bad order around the face (see Theorem 1.3.9). All vertices of S_1 and S_2 are on a common face in G', namely the face which used to contain t_1, t_2, so this must be the same face corresponding to F, since G'' is 3-connected. All vertices on this face are cofacial in G with at least one of t_1, t_2, so t_3, t_4 are cofacial with t_1 or t_2. By Lemma 2.3.7 this graph either has a structure or the minor. This completes case 1.1, and we may now assume that G'' is not 3-connected.

1.2: We now consider the case that G'' is not 3-connected. If G'' has a 1-cut $\{v\}$, then $v \in \{s_1, s_2\}$ and in G' this would give a 2-separation. Since G' is 3-connected, this is not possible and hence G'' is 2-connected. There are three types of 2-cuts which can exist in G''':

(1) The set $\{s_1, s_2\}$ could be a 2-cut.

(2) The set $\{s_1, v\}$ (or $\{s_2, w\}$) could be a 2-cut that isolates t_3 or t_4.

(3) The set $\{s_1, v\}$ (or $\{s_2, w\}$) could be a 2-cut that has t_3, t_4 on one side and s_2 on the other (or t_3, t_4 on one side, s_1 on the other).

Note that cuts of type (1) cannot exist, since if such a cut existed, then in G', for some $v_1 \in S_1$ and $v_2 \in S_2$, the set $\{v_1, v_2\}$ would be a 2-cut (since contraction of part of a face boundary does not introduce new cofacial pairs of vertices), but G' is 3-connected. We consider three possibilities, either we have only cuts of type (2), we have only cuts of type (3), or we have cuts of both types.

1.2.1: Suppose first that only cuts of type (2) exist. Since G is reduced, there are only three possibilities to consider: There is a single such cut; there are two such cuts, which both have s_1 as an endpoint; or there are two such cuts, one using s_1, the other using s_2. We consider these three cases separately.

1.2.1.1: Suppose there is a single cut, $\{s_1, v\}$ that isolates t_3. In G', there is a corresponding 3-cut of the form $\{v_1, w_1, v\}$, where $v_1, w_1 \in S_1$, which also isolates t_3. This is also a 3-cut in G, so it is reduced, and so the only vertex in the component containing t_3 is the terminal t_3. In G'', if we contract t_3v (creating a new vertex t_3) then the resulting graph H will be 3-connected. We see that H will have the two disjoint paths unless t_3, t_4, s_1, s_2 are on a common face of H. As in case 1.1, we see that this corresponds to t_4 being cofacial with t_1 or t_2 in G, so by Lemma 2.3.7, G has either a rooted $K_{2,4}$ minor or a structure. Thus, there is not a single cut of type (2), completing case 1.2.1.1.

1.2.1.2: Suppose there are two cuts, $\{s_1, v\}, \{s_1, w\}$. As in case 1.2.1.1, the first cut corresponds to a cut $\{v_1, w_1, v\}$ in G' and in G, and the second corresponds to a cut $\{v_2, w_2, w\}$ in G' and G. Since G is reduced, t_3 and t_4 are the only vertices on the smaller side of their respective cuts. We have single-edge paths from s_1 to t_3 and t_4. If $v = w$, then we can find a path from s_2 to v in $G'' - \{s_1\}$ completing the $K_{2,4}$ minor. If $v \neq w$, we can find a path from s_2 to t_4 in $G'' - \{v, s_1\}$ and we can find a path from s_2 to t_3 in $G'' - \{w, s_1\}$. These paths give the desired $K_{2,4}$ minor, so we are done the case with two cuts using the same vertex, completing case 1.2.1.2.

1.2.1.3: Suppose there are two cuts $\{s_1, v\}, \{s_2, w\}$. As in case 1.2.1.2, the first cut corresponds to a cut $\{v_1, w_1, v\}$ in G' and in G, and the second corresponds to a cut $\{v_2, w_2, w\}$ in G' and G. Since G is reduced, t_3 and t_4 are the only vertices on the smaller side of their respective cuts. If $v = w$, then G has $DCJ(v_1, w_1, v_2, w_2, v\}$. When $v \neq w$, we contract the edge t_3v and t_4w to create vertices v', w'. This new graph is 3-connected, so we can find the desired paths unless $\{v, w, s_1, s_2\}$ are on a common face. We see that this must be the face which contained t_1, t_2 in G, so we may assume that w is cofacial with t_2. Then the set $\{t_2, v_2, w\}$, must be a 3-cut in G which isolates t_4, and since G is reduced, we can apply Lemma 2.3.7. This complete all cases where we have only cuts of type (2).

1.2.2: We next suppose that only cuts of type (3) exist. There could be multiple such cuts which all use s_1 and are laminar. We will consider the cut $\{s_1, v\}$ for which the component containing t_3 and t_4 is smallest possible. We create the graph H by contracting the component not containing t_3, t_4 onto v. This will make an edge between s_1, v, and the resulting graph will be 3-connected. If we can find a $K_{2,2}$ minor from $\{s_1, v\}$ to $\{t_3, t_4\}$ in H, then G has a rooted $K_{2,4}$ minor. If we cannot, s_1, w, t_3, t_4 must be appear on a common face of H in a bad order. Since H is 3-connected, this must be one of the faces of H containing the edge s_1v. In G' and G, this means we have a 3-cut $\{v_1, w_1, v\}$ with $v_1, w_1 \in S_1$ such that v_1, v, t_3, t_4 appear on a common face in that order (up to swapping t_3, t_4). Then G has $DF(v, v_1, w_1)$ structure. Thus, we may assume that G does not have only cuts of type (3).

1.2.3: We now consider the case where cuts of both types (2) and (3) exist. As mentioned above, there may be multiple cuts of type (3), however we will take the smallest one, $\{s_1, v\}$. Any cuts of type (2) must be of the form $\{s_1, w\}$, since a cut of the form $\{s_2, x\}$ cannot isolate a terminal. We contract the component of the 2-cut $\{s_1, v\}$ containing s_2 onto v to create the graph H. The only 2-cuts in H are the cuts of type (2) which were also present in G''. If two such cuts exist, then as in cases 1.2.1.2 and 1,2,1,3, we can find the desired paths and complete the $K_{2,4}$ minor. If there is only a single cut $\{s_1, w\}$, isolating t_4, then we consider $H' = H/t_4w$, and let w' denote the vertex formed after contracting t_4w. Note that $w' \neq v$. The graph H' will be 3-connected, and so the desired paths exist unless $\{s_1, v, t_3, w'\}$ are cofacial in that order. Note that here we cannot swap t_3, w' in the ordering, since this would give a 2-cut which would isolate t_3.

In G' and G, the vertex s_1 in the cut $\{s_1, v\}$ corresponds to $v_1, w_1 \in V(S_1)$. Similarly, the cut $\{s_1, w\}$ corresponds to $v_2, w_2 \in S_1$. The vertices occur in the order v_1, v_2, w_2, w_1 (say) in S_1. Recall that w is cofacial with v and v_1 or with v and w_1; we may assume the former. Since G is reduced, we see that $v_1 = v_2$. This corresponds to G having $HF(w_1, v_1, v, v_2, w)$ structure.

2: We lastly consider the case where G' is not 3-connected. If there was a 1-cut, $\{v\}$ in G', then $\{t_1, t_2, v\}$ would be a 3-cut in G. Since G is reduced, this would mean that each side of the 3-cut consists of only one other terminal, and so G would be a graph on 5 vertices, with $3F$ structure. Thus we may assume that G' is 2-connected. Let $\{v, w\}$ be a 2-cut in G'. If this 2-cut corresponds to a 3-cut $\{v, w, t_1\}$ (or similarly $\{v, w, t_2\}$) in G, then since G is reduced, t_1 would be cofacial with another terminal t_j of degree 3. Since $\{v, w\}$ is a 2-cut in G', it is easy to see that $j \neq 2$. If $j \in \{3, 4\}$, then we are done by Lemma 2.3.7. Therefore, we have $\{t_1, t_2, v, w\}$ is a 4-cut in G, with the following pairs of vertices cofacial (up to relabelling of v, w): $\{t_1, t_2\}, \{t_2, v\}, \{v, w\}, \{w, t_1\}$. We consider two possibilities, either t_3 and t_4 are in the same component of the associated separation or they are in different components.

2.1: Suppose first that we have such a 4-cut where t_3 and $P(t_1, t_2)$ are on one side of the cut while t_4 and $P(t_2, t_1)$ are on the other. We let $S_1 = P(t_1, t_2) \cup P(t_2, v]$ and $S_2 = P(t_2, t_1) \cup P(t_1, w]$. We can join the components of S_1 in the facial neighbourhood of t_2 and the components of S_2 in the neighbourhood of t_1. If there is no path from t_3 to w that does not intersect S_1, then applying Lemma 2.3.3, to $S_1 \cup t_2$ we have a 3-cut using the vertex t_2 that isolates t_3. Since G is reduced, this would mean that t_2 and t_3 are cofacial, which would mean t_2 was cofacial with two terminals and we could apply Lemma 2.3.7. By Lemma 2.3.3, we can find a path from t_3 to S_1 which is disjoint from $\{w, t_1, t_2\}$ unless those three vertices form a 3-cut. If this occured, since G is reduced, t_3 would actually be on a common face with t_1 and t_2 and G would have $3F$ structure. Thus, we can always connect S_1 and S_2 to t_3. We can construct similar paths in the other component of the cut $\{v, w, t_1, t_2\}$ to complete the $K_{2,4}$ minor.

2.2: It remains to consider the case where every 2-cut $\{v, w\}$ in G' has both terminals on the same side. Again, we consider G and have the vertices $\{t_1, t_2, v, w\}$ being pairwise cofacial in the manner as stated above. There may be more than one such 2-cut, and we distinguish between two classes of cuts. Each cut separates $Q_1 = P(t_1, t_2)$ from $Q_2 = P(t_2, t_1)$ and we classify cuts based on which of Q_1 and Q_2 is in the same component as t_3 and t_4. We choose a cut $\{v_1, w_1\}$ such that Q_1 is in the same component as t_3 and t_4 and this component is minimal. We choose a cut $\{v_2, v_2\}$ such that Q_2 is in the same component as t_3 and t_4 and this component is minimal. At least one of the cuts $\{v_1, w_1\}$ and $\{v_2, w_2\}$ must exist and we may have $v_1 = v_2$ or $w_1 = w_2$ but not both pairs being equal.

2.2.1: We first assume that only one cut $\{v, w\}$ exists. Let H be the graph obtained from G by contracting the component with no terminals to be an edge between v and w. In $H' = H - \{t_1, t_2\}$ there are no 2-cuts, since ours was chosen to be minimal. Notice that in H, the path $P(t_2, t_1)$ is replaced by the edge joining v and w. We construct H'' by contracting $P(t_1, t_2)$ to a vertex s_1 and contracting the edge vw into a vertex s_2. Let us remark that our goal is to find a $K_{2,4}$ minor in G is thus reduced to finding a $K_{2,2}$ minor in H'' joining $\{t_3, t_4\}$ and $\{s_1, s_2\}$.

We now proceed as in case 1.2, looking to find two pairs of disjoint paths, one pair joining s_1, t_3 and s_2, t_4 respectively, and the other joining s_1, t_4 and s_2, t_3 respectively. If $\{s_1, s_2\}$

would be a cut of type (1) in H'', then in G, some $v' \in P(t_1, t_2)$ would be cofacial with v or w and therefore $\{v', w, t_1\}$ (or $\{v', v, t_2\}$) would be a 3-cut. Since G is reduced, this 3-cut would either separate one terminal (thus making it cofacial with t_1 or t_2), or would contradict our choice of v and w.

If one side contained no terminals, then it would not be a 2-cut in H''. If one side contained a single terminal, then that terminal would have to be t_2 (or t_1)), since otherwise we could apply Lemma 2.3.7. However, since G is reduced, this would imply that the vertex w (or v) did not exist, and so our cut $\{v, w\}$ would not exist. Thus, there can be no cut of type (1).

If there are no 2-cuts in H', then the graph has a rooted $K_{2,4}$ minor, since if it did not, we would again have $\{v', w\}$ or $\{v', v\}$ cofacial for some $v' \in P(t_1, t_2)$. If we only have cuts of type (2) or (3) in H'' which use s_1, then this is the same as above, thus there must be a 2-cut of type (2) or (3) that uses s_2. When we have only cuts of type (2), if there is a single such cut $\{s_2, v\}$ which isolates t_3, then we have the minor unless there is a face containing s_1, s_2, t_3, x. This face must be the face containing t_1 and w in G (or the face containing t_2 and v in G), but then t_1 and t_3 or t_2 and t_3 are cofacial and we can apply Lemma 2.3.7. If we have two cuts, $\{s_1, x\}$ isolating t_3 and $\{s_2, y\}$ isolating t_4, then similarly to above we have the minor unless there is a face in G containing w, x, y, v' for some $v' \in s_1$. The only face this can be is the face in G with t_1 and w, in which case there will be a 3-cut of the form $\{t_1, v', v''\}$ which isolates t_3, meaning Lemma 2.3.7 applies.

If we have only a cut of type (3), then G must be as in diagram 1 of Figure 2.8 (or we can find the paths), and if we have a cut of type (3) and a cut of type (2) it must be as in diagram 2. If G is as in diagram 1, we let $P(t_1, t_2) \cup P(t_2, w] \cup P[w, x] \subseteq S_1$ and $P(t_2, t_1) \cup P(t_1, v] \subseteq S_2$. We can join the components of S_1 in the facial neighbourhood of t_2 and join the components of S_2 in the facial neighbourhood of t_1. We let $T_3 = P(x, t_3]$, $T_4 = P[t_4, w)$. We can join v to T_3 in the facial neighbourhood of x and join v to T_4 in the facial neighbourhood of w. We add these paths to S_2, completing the $K_{2,4}$ minor. Diagram 2 proceeds in a similar manner, except that $T_4 = t_4$.

2.2.2: Let us now assume that cuts $\{v_1, w_1\}$ and $\{v_2, w_2\}$ both exist. The cuts $\{v_1, w_1\}$ and $\{v_2, w_2\}$ split G' into three components. Let C_1 be the component bounded by $\{t_1, v_1, w_1, t_2\}$ which contains no terminals. If we contract the interior of this component to the vertex v_1, then we claim that v_1 will be adjacent to w_1, t_1, t_2. These edges must exist unless some three of the four vertices are pairwise cofacial. However, this cannot occur since G is reduced

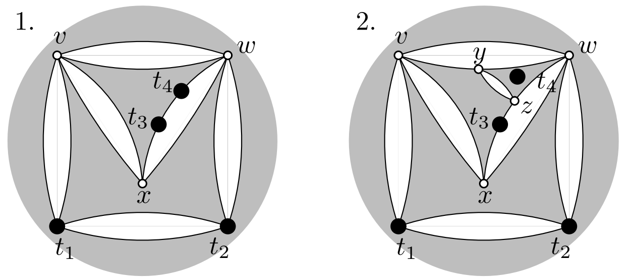


Figure 2.8: Possible structures for H''

and $\{v_1, w_1\}$ is a cut. Similarly, we could contract this component to the vertex w_1 and have w_1 adjacent to t_1, v_1, t_2. We define C_2 similarly with respect to v_2, w_2 and see that we could contract C_2 in a similar manner. We can obtain four different graphs $G_1 \ldots G_4$ by contracting the two components in each of two different ways. By the minimality of cuts $\{v_1, w_1\}$ and $\{v_2, w_2\}$, each of these graphs is 3-connected.

For each graph G_i, we must have a $K_{2,2}$ minor (which extends to a $K_{2,4}$ minor of G) unless t_3 and t_4 are on a common face with $x_1 \in \{v_1, w_1\}$ and $x_2 \in \{v_2, w_2\}$. Over all four graphs, we must have that t_3 and t_4 are on common faces with each possible pair $\{x_1, x_2\}$. This cannot occur since G is 3-connected, so we are done. Note that if $v_1 = v_2$ we instead have three graphs G_i to consider, but the same conclusion arises. This completes the proof of the lemma. $\square$

Lemma 2.3.9. *Let G be a reduced 3-connected graph with A, B a 3-separation, such that $t_1, t_2 \in V(A)$, $t_3, t_4 \in V(B)$. Then G has a rooted $K_{2,4}$ minor or one of the structures from Theorem 2.3.5.*

Proof. Let $A \cap B = \{v_1, v_2, v_3\}$. If one of these vertices is a terminal, say $v_1 = t_i$ then t_i and another terminal are cofacial since G is reduced. Thus we are done by Lemma 2.3.8, and we may henceforth assume that $\{v_1, v_2, v_3\} \cap \{t_1, \ldots, t_4\} = \emptyset$.

In G, there may be many 3-separations which have two terminals on each side. We choose one that has A minimal. Let $P_{i,j}$ be the facial path from v_i to v_j in A. Of the three such facial paths $P_{i,j}$, we may assume at least one of them does not contain a terminal, since if all three contained a terminal, then one terminal would be in two paths, and, since G is reduced, we would have t_1, t_2 cofacial, and Lemma 2.3.8 would apply. We may assume the path $P_{1,3}$ contains no terminals. Let A' be the graph formed from A by contracting $P_{1,3}$ to a single vertex v. Since $P_{1,3}$ is a facial path, A' is 2-connected by Lemma 2.3.1. If there is a 2-cut in A', then it must use the vertex v. No such 2-cut can isolate both terminals, since this 2-cut would give rise to a 3-cut in G that would contradict the minimality of A. Proceeding as in the proof of Lemma 2.3.8, we see that we can find a $K_{2,2}$ minor in A between $\{t_1, t_2\}$ and $\{P_{1,3}, v_2\}$ unless A has the structure as in Figure 2.9.

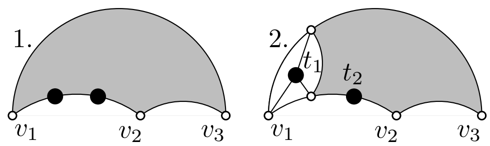


Figure 2.9: Graphs where A does not have the $K_{2,2}$ minor

We may assume A is not as in diagram 1 of Figure 2.9, since then Lemma 2.3.8 would be applicable. If A is as in diagram 2, we can complete a $K_{2,2}$ minor between $\{t_1, t_2\}$ and $\{v_1, P_{2,3}\}$ by using $P[v_1, t_2)$ and the rest of the outer face boundary as shown in the diagram. We may thus assume that in A we can always find a $K_{2,2}$ minor between $\{t_1, t_2\}$ and $P_{1,3}, v_2$.

We now choose a 3-separation such that B is minimal containing t_3, t_4. If the cut vertices are not the same as from the cut which made A minimal, we can find three disjoint paths between the vertices of the cuts by using Menger's theorem. Thus, we may assume they are the same vertices. Then it remains to show that in B, we can find a $K_{2,2}$ minor between $\{t_3, t_4\}$ and $\{v_2, \{v_1, v_3\}\}$.

Using B, we create the graph B' by identifying v_1 and v_3 to form the vertex w and need only find a $K_{2,2}$ minor in the planar graph B' between $\{t_3, t_4\}$ and $\{w, v_2\}$. If B' has a 2-cut, then there must be a 3-cut in B which uses v_1, v_3. Since B is minimal, such a cut would have to be as in diagrams 1 or 2 of Figure 2.10.

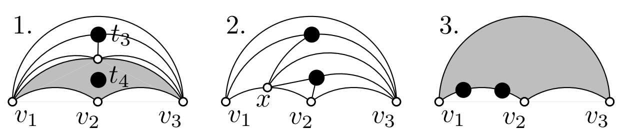


Figure 2.10: Interesting cases for B

If B is as in diagram 1, by 3-connectivity of G we can find disjoint paths from t_3 to v_1 and t_4 to v_2. We can find a path Q from v_2 to t_3 in the facial neighbourhood of v_1 unless v_1 and t_4 are cofacial. If we can not find a path from v_3 to t_4 which is disjoint from Q, then by Lemma 2.3.3, the vertex t_4 is of degree 3 and is adjacent to v_1. Thus, we can complete the $K_{2,2}$ minor in this manner unless t_4 is cofacial with v_1. If we swap v_1 and v_3 above, we can complete the minor unless t_4 and v_3 are cofacial. One of these must work, since t_4 being cofacial with v_1 and v_3 would mean t_4 was cofacial with t_1 and Lemma 2.3.8 would apply.

If B is as in diagram 2, then it only has the 6 indicated vertices. The edge v_2x must be present, by minimality of B. Thus, since v_3 is adjacent to both terminals and x is adjacent to both terminals and to v_2, we can complete the minor.

We last consider the case when B' is 3-connected. Then we can complete the minor unless v_1, v_2, t_1, t_2 are on a common face of B' in the interlaced order, as in diagram 3 of Figure 2.10. If B has this structure, then Lemma 2.3.8 applies. Thus, we can always complete the $K_{2,4}$ minor. □

Lemma 2.3.10. *Let G be a reduced 3-connected planar graph such that no pair of terminals is cofacial and there is no 3-separation which has exactly two terminals on one side. If G has an edge $e = v_1t_1$ and two vertices v_2, v_3 such that $\{v_1, t_1, v_2, v_3\}$ is a 4-cut in G that isolates the terminal t_2, then G has a rooted $K_{2,4}$ minor.*

Proof. Figure 2.11 shows what the structure of the 4-cut looks like for different scenarios in the proof. We attempt to construct a rooted $K_{2,2}$ minor in the component shown in diagram 1 by using $S_1 = P(t_1, v_2]$ and $S_2 = P[v_3, v_1]$. Since S_1 and S_2 are each on a single face, deleting either will not disconnect the graph, so by Lemma 2.3.1, we can get a $K_{2,2}$ minor between $\{S_1, S_2\}$ and $\{t_1, t_2\}$ unless $t_2 \in P(v_3, v_1)$ and the edge v_1v_3 is not present, or there exists a vertex $v \in P(t_1, v_2) \cap P(v_3, v_1)$ and $v_1v_3 \notin E(G)$. These possibilities are shown in diagrams 2 and 3 of Figure 2.11. In diagram 3, the edges v_3v and v_3v_2 may not be present.

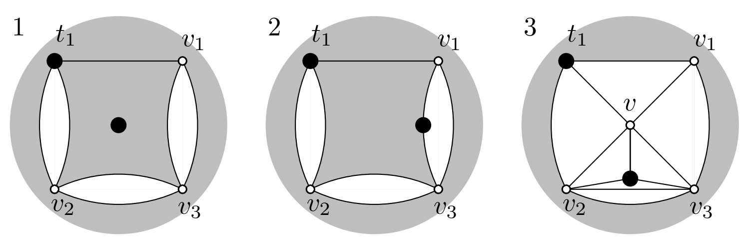


Figure 2.11: Structures of the 4-cut from an (R2) edge

We consider three cases: either S_1 and S_2 give a $K_{2,2}$ minor, or G is as in diagram 2, or G is as in diagram 3.

1: Let $H = G - \{t_1\}$. Then H is 3-connected, since any 2-cut in H would correspond to a 3-cut in G using t_1, and since G is reduced, one side either contains no vertices (and so this is not a cut) or contains a single terminal (which would be cofacial with t_1). When the above $K_{2,2}$ minor exists, then from the proof of Lemma 2.3.9, we see that a rooted $K_{2,2}$ minor between $\{S_1, S_2\}$ and $\{t_3, t_4\}$ exists in H. This gives the desired rooted $K_{2,4}$ minor in G.

2: If G is as in diagram 2 of Figure 2.11 we attempt to construct a $K_{2,2}$ minor by letting $S_1 = P(t_1, v_2] \cup P[v_2, v_3]$ and $S_2 = P(t_2, v_1]$. By Lemma 2.3.2 $G - S_1$ is connected unless $\{w, v_2, x\}$ is a 3-cut, where $w \in P(t_1, v_2)$ and $x \in P(v_2, v_3)$. Thus, we can ding a $K_{2,2}$ minor between (S_1, S_2) and (t_1, t_2) unless $t_2 \in P(v_2, v_3)$ and the edge v_2v_3 is not present (considered in case 2.1), or t_2 is of degree 3 with neighbours $\{w, v_2, x\}$ where $w \in P(t_1, v_2)$ and $x \in P(v_2, v_3)$. If t_2 has neighbours $\{w, v_2, x\}$ then $w \in P(t_2, v_1)$ and G is as in diagram 3 of Figure 2.11, except the edge vv_3 is not present (considered in case 3). If none of the exceptional cases occurs, we have the $K_{2,2}$ minor that can be extended to a rooted $K_{2,4}$ minor in G in the same way as explained in case 1.

2.1: If $t_2 \in P(v_2, v_3)$, then we see that t_2 is on both faces containing v_3 in the 4-cut. In this instance, we let $S_1 = P[v_1, v_3]$ and $S_2 = v_2$. From the proof of Lemma 2.3.9, the only obstructions for extending this to a $K_{2,2}$ minor between $\{S_1, S_2\}$ and $\{t_3, t_4\}$ require at least one of t_3 or t_4 to be on a face of the cut (see Figure 2.9. This cannot occur, since t_2 is on two of these faces and all vertices on the other face are in the facial neighbourhood of t_1 in G. This can be extended to a $K_{2,4}$ minor in G, since S_1 is connected to t_1 and t_2, and S_2 can be connected to both by 3-connectivity.

3: Suppose G is as in diagram 3 of Figure 2.11. If none of the edges vv_3 and v_2v_3 are present, then t_2 is on two faces of the 4-cut in G and the above argument holds, so we may assume at least one of the edges is present. From the proof of Lemma 2.3.9, we see that we can find a $K_{2,2}$ minor in $H = G - t_1$ between $\{t_3, t_4\}$ and $\{P[v_1, v_3], v_2\}$ unless one of the terminals (say t_3) is on $P[v_1, v_3]$ or we have the situation shown in Figure 2.9. Since no terminals are cofacial in G, the first case of Figure 2.9 does not occur. The second case of Figure 2.9 gives t_3 on a face F of the 3-cut $\{v_1, v_2, v_3\}$ of H and t_4 is of degree 3 with one neighbour in $V(F) \cap P[v_1, v_3]$, one additional neighbour in F and one additional neighbour in $P[v_1, v_3]$. We note that in this case, $F \supseteq P[v_3, v_2]$ since the face of the 3-cut consists of the facial neighbourhood of t_1 clockwise from v_2 to v_1, and so cannot contain a terminal. Also note that if the edge v_1v_3 is present, we can use this edge and neither obstruction can occur.

If the $K_{2,2}$ minor between $\{t_3, t_4\}$ and $\{P[v_1, v_3], v_2\}$ exists, this extends to a rooted

$K_{2,4}$ minor in G since $P[v_1, v_3]$ is adjacent to t_1 and t_2, while v_2 is adjacent to v which is adjacent to t_1 and t_2. Thus, we may assume that either $t_3 \in P(v_1, v_3)$ or $t_3 \in P(v_3, v_2)$ and t_4 is of degree 3. We consider these cases separately.

3.1: If $t_3 \in P(v_1, v_3)$, then the edge vv_3 must be present since t_2 and t_3 are not cofacial. We can try to construct the $K_{2,2}$ minor between $\{S_1 = P[v_3, v_2],\ S_2 = v_1\}$ and $\{t_3, t_4\}$. This will work unless $t_4 \in P(v_3, v_2)$ or t_4 is of degree 3 with neighbours $w \in P(v_1, v_3), x \in P(v_3, v_2)$ and v_3. If either of these occurs, we let S_1 be the facial neighbourhood of t_1 clockwise from v_2 to v_1 and $S_2 = v_3$. S_1 and S_2 both connect to (t_1, t_2) since v_3v is present. They also both connect to t_3 along $P(v_1, v_3)$ and to t_4 along $P(v_3, v_2)$ or

3.2: If $t_3 \in P(v_3, v_2)$ and t_4 is of degree 3 then the edge v_2v_3 is present. If the edge v_3v is present, then letting $S_1 = v_3v$ and letting S_2 be the facial neighbourhood of t_1 clockwise from v_2 to v_1 will work as in the above cases. If the edge v_3v is not present, we let $S_1 = P[v_3, t_1) \cup \{v_2v_3\}$ and $S_2 = P[v_1, v_3) \cup \{v_1v\}$. By construction, S_1 and S_2 are each connected to t_1, t_2, t_4. We can connect t_3 to S_1 using $P(t_3, v_2)$. We can connect t_3 to S_2 using the facial neighbourhood of t_4 unless some vertex $y \in P[v_2, t_1]$ is in the facial neighbourhood. However, if this occurs, then y, v, z (where z is the neighbour of t_4 on $P(v_1, v_3)$) is a 3-cut which isolates t_1. Since G is reduced, for this to occur, we must have $z = v_1$. This means that t_4 is on $P[v_1, v_3]$ and would be cofacial with t_2. Since we know this does not occur, we must be able to connect S_2 to t_3, completing the $K_{2,4}$ minor. □

By Lemmas 2.3.7, 2.3.8, and 2.3.9, we need only prove Theorem 2.3.5 for reduced internally 3-connected graphs G which have no pair of terminals cofacial. By Lemma 2.3.10 we may assume that in a minimal counterexample, G, there is no edge e incident with a terminal such that G/e admits an (R2) reduction.

Lemma 2.3.11. *If a reduced graph G is a minimal counterexample to Theorem 2.3.5 and has no pair of terminals cofacial, then for any edge e incident with a terminal, G/e is a 3-connected 4-terminal planar graph.*

Proof. Let $e = t_1v_1$. It's clear that G/e is planar. The vertex v_1 cannot be a terminal, since t_1 is not cofacial with any terminals. If G/e is not 3-connected, then G has a 3-cut of the form $\{t_1, v_1, v_2\}$ for some $v_2 \in V(G)$. However such a cut cannot exist, since if it has 0 terminals on one side, then since G is reduced it would not be a cut, and if it had a single terminal on one side, that terminal would be cofacial with t_1 since G is reduced. Thus G/e is 3-connected. □

Based on the above lemma, we see that for a minimal counterexample G, for any edge e incident with a terminal, $(G/e)^*$ must have one of the structures from Theorem 2.3.5. Before we proceed, we need the following lemmas about (R1) reductions.

Lemma 2.3.12. *Performing a reduction of type (R1) does not create any new reductions of type (R1) or (R2).*

Proof. To have a reduction of type (R1) or type (R2), it is necessary to for G to have 3 vertices which are pairwise cofacial. Performing an (R1) reduction, does not cause a pair of vertices that were not cofacial to become cofacial. So any 3 pairwise cofacial vertices after we perform an (R1) reduction were cofacial already before. Clearly we have not changed the location of any terminals with respect to the 3 cofacial vertices, so the number of terminals on each side of a 3-cut is the same before and after the (R1) reduction. □

It is not hard to see that the graph G' that we obtain after performing all reductions of type (R1) does not depend on the order of the reductions made. The proof is omitted as this result is not required.

Lemma 2.3.13. *Let G be a reduced 3-connected planar graph such that no pair of terminals is cofacial and there is no 3-separation which has exactly two terminals on one side and there is no edge t_1v_1 as in Lemma 2.3.10. Suppose that G has an edge $e = v_1t_1$ such that $\{v_1, t_1, v_2, v_3\}$ is a 4-cut in G and one side of the cut has no terminals. If G has no rooted $K_{2,4}$ minor then G/e has one of the structures from Theorem 2.3.5 if and only if the graph obtained by performing an (R1) reduction in G/e on $\{v_2, v_3, t_1\}$ has the structure (where t_1 denotes the vertex formed by contracting e).*

Proof. Let G and e be as described in the lemma. Suppose no pair of terminals is cofacial in G/e. Performing (R1) reductions in G/e will not make any pair be cofacial, and by Lemma 2.3.10 there are no (R2) reductions to perform. Thus neither G/e nor its reduction have a structure from Theorem 2.3.5. Thus, we may assume that some pair of terminals is cofacial in G/e. Since there were no cofacial terminals in G, one of the terminals is t_1 and we may assume it is cofacial with t_2. We assume that the vertices $\{v_1, t_1, v_2, v_3\}$ are pairwise cofacial in the cyclic order given. We consider two cases, either v_3 is a terminal or it is not.

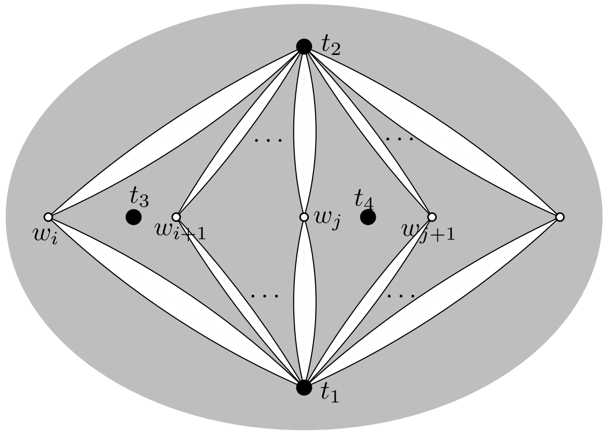
1: We first consider the case where v_3 is not a terminal. In this case, the terminal t_2 is cofacial with v_1 in G. If G/e has one of the structures from Theorem 2.3.5 then we are done, so we assume it does not. Performing the (R1) reduction on $\{v_2, v_3, t_1\}$ creates a triangular face T incident with t_1. For this to be relevant to making one of the structures in Theorem 2.3.5, this new face must be one of the faces indicated in the statement of the theorem. This clearly cannot be one of the faces from $3F$ or OWO structures, since they all have more than a single terminal. This also cannot create DF structure or DCJ structure, since the only faces which do not contain two terminals are faces where it only matters that a certain pair of vertices are cofacial, and the (R1) reduction did not make any new pairs of vertices cofacial. For HF structure the face on the left has two cofacial terminals, so t_1 is one of them. Observe that T is incident with t_1 (but not with another terminal), so T does not participate in the structure.

2: Thus, we may assume that $v_3 = t_2$. We consider two cases here, either t_1 and t_2 have v_1 and v_2 as their only common facial neighbours, or they have other common facial neighbours.

2.1: If t_1 and t_2 have no facial neighbours in common aside from v_1 and v_2, then we can construct a rooted $K_{2,4}$ minor in G letting $S_1 = P(t_2, v_1]$ and $S_2 = P(t_1, v_2] \cup P[v_2, t_2)$. Since t_1 and t_2 have no other facial neighbours in common, if S_1 and S_2 intersect, it can only be at the vertex $w \in P(t_2, v_1) \cap P[v_2, t_2)$. If this happens, we let $T_2 = t_2 w$ and exclude w from S_1 and S_2. Since w is adjacent to t_2, this will leave S_1 and S_2 connected. By construction, S_1 and S_2 are connected to t_1 and t_2. Let $H = G - \{t_1, t_2\}$ and H' be formed from H by contracting the side of the 2-cut $\{v_1, v_2\}$ containing no terminals to a single edge. If G' had a 1-cut or a 2-cut, then G would have a 3-cut involving t_1 and t_2 or a 4-cut involving t_1, t_2 and vertices in S. Since t_1 and t_2 are not cofacial and their only common facial neighbours are v_1 and v_2, this cannot happen. Thus H' is 3-connected. This means we can find a rooted $K_{2,2}$ minor between $\{t_3, t_4\}$ and $\{v_1, v_2\}$ in H' unless the four vertices are on a common face in the interlaced order (by Lemma 2.2.2). However, this cannot happen since then t_3 and t_4 would be cofacial with either t_1 or t_2.

2.2: We now consider the case where t_1, t_2 have common facial neighbours $\{w_1, w_2, \dots w_n\}$ ($n \geq 3$) which occur in that order clockwise around t_1. In $H = G - \{t_1, t_2\}$, the set $\{w_i, w_{i+1}\}$ is a 2-cut or $w_i w_{i+1} \in E(H)$ for $1 \leq i \leq n$. Let C_i be the bridge of H with cut set $\{w_i, w_{i+1}\}$. We consider two cases, either t_3, t_4 are in different components $C_i, C_j, i < j$ or both are in the same component C_k, as shown in Figure 2.12.

2.2.1: First, we suppose t_3 and t_4 are in different components, C_i and C_j, respectively where $1 \leq i < j \leq n$. We let S_1 be a path from w_{i+1} to w_j clockwise in the facial



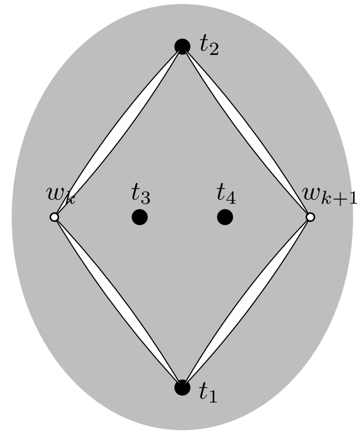


Figure 2.12: Structures of the 4-cut from an (R1) edge

neighbourhood of t_1 combined with $P[w_j, t_1) \cup P[w_{i+1}, t_2)$, and let S_2 be the path from w_{j+1} to w_i clockwise in the facial neighbourhood of t_1 combined with $P[w_{j+1}, t_2) \cup P[w_i, t_1)$. By construction, S_1 and S_2 are connected to t_1 and t_2. To see that t_3 can be connected to S_1, we observe that by Lemma 2.3.2, either $G - (\{t_2 \cup P[w_i, t_1])$ is connected or t_2 and t_3 are cofacial. Since the latter outcome does not happen, t_3 can be connected to A_1 without intersecting S_2, t_1, t_2. By symmetry, we can make the other necessary connections to complete the $K_{2,4}$ minor.

2.2.2: We now suppose that in G, the terminals t_3 are t_4 are both in the component C_k as in the right side drawing of Figure 2.12. We attempt to form a rooted $K_{2,4}$ minor using $S_1 = P[w_k, t_2) \cup P(t_1, w_k]$ and $S_2 = P(t_2, w_{k+1}] \cup P[w_{k+1}, t_1)$. We consider cases depending on if these two subgraphs are disjoint or if they intersect in different ways.

2.2.2.1: If these two subgraphs are disjoint, then they are each connected to t_1 and t_2. Letting $H = G - \{t_2, t_2\}$ and then contracting the component separated by $\{w_k, w_{k+1}\}$ not containing any terminals to be a single edge to obtain the graph H'. We see that H' is 3-connected in the same way as above. We can find a rooted $K_{2,2}$ minor between $\{t_3, t_4\}$ and $\{w_k, w_{k+1}\}$ unless the four vertices are on a common face. It is easy to argue that this cannot happen since no terminals are cofacial. this completes the proof when S_1 and S_2 are disjoint.

2.2.2.2: We next consider the possible ways for the subgraphs S_1 and S_2, as defined in case 2.2.2, to intersect. We first suppose that there exists $v \in P[w_k, t_2) \cap P[w_{k+1}, t_1)$ (the case when there exists $v \in P(t_1, w_k] \cap P(t_2, w_{k+1}]$ proceeds similarly). Such a v means that $\{v, t_2, w_{k+1}\}$ and $\{v, t_1, w_k\}$ are 3-cuts in G, and, since G is reduced, v is the only vertex of G not in C_k. Since G is 3-connected and reduced, v must have neighbour set $\{t_1, t_2, w_k, w_{k+1}\}$. Moreover, at least one of the edges t_1w_k and t_1w_{k+1} must be present, and is the edge e from the statement of the lemma. If both edges are present, we have a $K_{2,2}$ minor between $\{w_k, w_{k+1}\}$ and $\{t_1, t_2v\}$ which can be completed to a $K_{2,4}$ minor in G as before. If only the edge t_1w_k is present, we consider two constructions of a $K_{2,2}$ minor minus an edge between $\{S_1, S_2\}$ and $\{T_1, T_2\}$. First, we take $S_1 = w_k, S_2 = w_{k+1}, T_1 = t_1, T_2 = t_2v$. This gives a $K_{2,2}$ minor minus the edge between S_2 and T_1. We can also take $S_1 = w_k, S_2 = w_{k+1}v, T_1 = t_1, T_2 = t_2$. This gives a $K_{2,2}$ minor minus the edge between S_1 and T_2.

We would like to extend one of these to a $K_{2,4}$ minor in G, by finding a $K_{2,2}$ minor between $\{w_k, w_{k+1}\}$ and $\{t_3, t_4\}$ in C_k, while also making the connection for the edge missing from the first $K_{2,2}$. To get a $K_{2,2}$ minor plus the edge between S_2 and T_1, we will take $S_2 = P(t_1, w_{k+1}]$ and $S_1 = w_k$. We know that H', the graph obtained from G by deleting t_1 and t_2, and contracting the portion of the graph outside C_k to a single edge is 3-connected. By Lemma 2.3.1, we know that if in H' we contract S_2 to a single vertex to form H'', that this graph is 2-connected, and any 2-cut uses the vertex s_2 formed by the contraction. If H'' is 3-connected, then we can complete the minor, since the only obstruction would be a face in G containing w_k, t_3, t_4, and some vertex in S_2, which cannot occur.

Any 2-cut in H'' gives a 3-cut in G using two vertices in S_2, so it must isolate either t_3 or t_4 since G is reduced and has no 3-cut which isolates both t_3 and t_4. If cuts exist for both t_3 and t_4, then S_2 is connected to all terminals in H' and by Lemma 2.3.1 we can connect $S_1 = w_k$ to both terminals in $H' - S_2$. If only one terminal, say t_3 is isolated by a 2-cut, say $\{s_2, z_1\}$ then t_3 is of degree 3 in G and is adjacent to z_1 and two vertices in $S + 2$. From the proof of Lemma 2.3.8, we see that the only obstruction to finding the rooted $K_{2,2}$ minor is if, for some $u_1 \in P(t_1, w_{k+1})$ there is a face F_1 in G containing w_k, t_4, z_1, u_1 (in the stated order). If this occurs, we attempt to extend the other possible $K_{2,2}$ minus an edge to a $K_{2,4}$ minor and see that this can be done unless t_4 is of degree 3 with two neighbours in $P(t_2, w_k]$ and a third neighbour z_2, and for some $u_2 \in P(t_2, w_k]$, we have a face F_2 in G containing w_{k+1}, t_3, x_2, u_2 (in this order).

We observe that the vertices in each of sets $\{w_k, t_1, u_1\}$ and $\{w_{k+1}, t_2, u_2\}$ are pairwise cofacial in G. If either of these sets gave rise to a 3-cut, it would have no terminals on one side and so each set is on a common face. Since $\deg(t_3) = 3$ we see that z_2 is also a vertex on the face F_1 and similarly, z_1 is a vertex on the face F_2. Therefore $\{w_k, u_2, z_1\}$ and $\{w_{k+1}, u_1, z_2\}$ are 3-cuts. Combining these facts, we see that $z_1 = z_2$ and that G has structure as in Figure 2.13. Here we can let $S_1 = \{u_2z_1, z_1u_1\}$ and $S_2 = \{w_kv, w_{k+1}v\}$ which will connect each to all terminals, completing the $K_{2,4}$ minor.

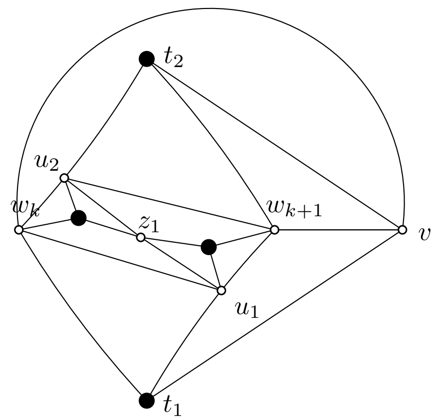


Figure 2.13: Special case of the 4-cut from an (R1) edge

2.2.2.3: We now need only consider the case where there exists $w \in P(w_k, t_2) \cap P(t_2, w_{k+1})$ or $x \in P(w_{k+1}, t_1) \cap P(t_1, w_k)$. In each case, the $K_{2,2}$ minor is easily found by letting $T_2 = t_2w$ or $T_1 = t_1x$, respectively, and so this can be completed to a $K_{2,4}$ minor in G as above. □

Based on the above two lemmas, we see that for any minimal counterexample G and any edge $e \in E(G)$ incident with a terminal then either G/e has a structure from Theorem 2.3.5 or we can perform an (R3) reduction in G/e, possibly after performing some (R1) reductions. We let $\mathcal{J}_1$ be the set of minimal counterexamples G where for some edge e incident with a terminal G/e has a structure from Theorem 2.3.5 and let $\mathcal{J}_2$ be the set of minimal counterexamples G where for every edge e incident with a terminal, to arrive at $(G/e)^*$ we must perform an (R3) reduction.

Graphs in $\mathcal{J}_1$

For graphs G in $\mathcal{J}_1$, when we contract an edge e incident with a terminal we must immediately see one of the structures from Theorem 2.3.5. Through the following lemmas, we show that G has a rooted $K_{2,4}$ minor.

Lemma 2.3.14. *If $G \in \mathcal{J}_1$, contracting an edge incident with a terminal does not create $3F$ or DF structure*

Proof. The only new facial adjacencies created by contracting the edge are between the terminal vertex t formed by the contraction of the edge e and other vertices. For any terminal, both $3F$ and DF structures have a pair of terminals cofacial that are not that terminal and so we do not create either structure. □

Lemma 2.3.15. *If $G \in \mathcal{J}_1$ has an edge e such that G/e gives OWO structure then G has a rooted $K_{2,4}$ minor.*

Proof. We first consider the possibility that G/e gives an instance of OWO structure. Since we may assume no pair of terminals are cofacial before contraction, after contraction our terminal is cofacial with all other terminals and so the only possible structure is that as in Figure 2.14.

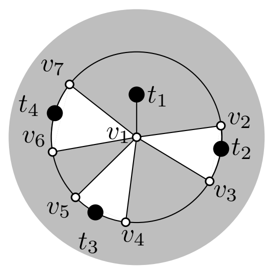


Figure 2.14: Graphs in $\mathcal{J}_1$ where contracting an edge gives OWO

Suppose G has structure as in Figure 2.14. Let $T_i = t_i$ for $1 \leq i \leq 4$. Observe that no vertex in $P[t_4, v_7]$ is cofacial with a vertex in $P[v_2, t_2]$ since this would give a v_1 and t_1 cofacial. Also, no vertex in $P[t_4, v_7]$ is cofacial with a vertex in $P[t_2, v_3]$ since this would give a 3-cut with two terminals on each side. Similarly, no vertex in $P[v_6, t_4]$ is cofacial with a vertex in $P[v_2, t_2]$. Let Q be the path from v_7 to v_3 using the facial neighbourhood of v_1t_1 and the facial neighbourhood of $P[v_2, t_2]$. The part of the path cofacial with v_1t_1 cannot intersect $P[v_4, v_5]$ and the part cofacial with $P[v_2, t_2]$ cannot intersect $P[t_3, v_5]$. We let S_1 be Q combined with the path in the facial neighbourhood of v_1 form v_3 to t_3. We let $S_2 = P[v_1, t_4) \cup P[v_1, t_2) \cup P(t_3, v_1]$. By construction, all the desired connections have been made, completing the $K_{2,4}$ minor. □

Lemma 2.3.16. *If $G \in \mathcal{J}_1$ has an edge e such that G/e gives HF structure then G has a rooted $K_{2,4}$ minor.*

Proof. The terminal in G incident with e must be either t_1 or t_2 and must not be on the face F_1 before contraction, since otherwise G has two terminals cofacial and we can apply Lemma 2.3.8. The set $\{v_1, v_2, v_3\}$ cannot be a 3-cut separating $\{t_1, t_2\}$ from $\{t_3, t_4\}$ before contraction, since then Lemma 2.3.9 would apply. Similarly, the terminal which is incident with the contracted edge cannot be in C_3. If the terminal incident with e is t_1, then in G, t_1 can be in C_3, or the edge e can split C_1 and C_2 or split C_1 and C_3. Note that if e splits C_1 and C_3 then t_1 would be cofacial with t_3, so we need not consider that case and by Lemma 2.3.10 we also need not consider the case where e splits C_1 and C_2 . The terminal incident with e cannot be t_2, since in G it cannot be in C_1 or C_3 and the edge e cannot split C_1 and C_3, since then t_2 would be cofacial with t_3. This gives the only possibility as seen in Figure 2.15.

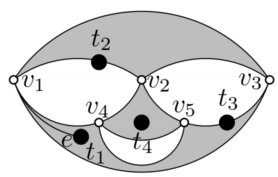


Figure 2.15: Graphs in $\mathcal{J}_1$ where contracting an edge gives HF

Suppose that G is as in Figure 2.15. Notice that $\{v_1, v_2, v_3\}$ is a 3-cut, so t_2 is of degree 3 with neighbours $\{v_1, v_2, v_3\}$. The edges v_2v_3, v_2v_5, v_2v_4 must be present, since otherwise G has an edge which can be contracted to give $3F$. Let $T_i = t_i$ for $q \leq i \leq 4$. Let $P[v_4, v_1] \cup P(t_3, v_5] \subseteq S_1$ and $P[v_2, t_3) \subseteq S_2$. Note that $t_1 \notin P[v_4, v_1]$ since otherwise contracting t_4v_2 would give 3F structure. By construction, S_2 is connected to all T_2, T_3, T_4 and S_1 is connected to each T_i, though S_1 has two components. To join the components of S_1, we use the edge v_4v_5 if present, or the path $P[v_5, v_4]$ if the edge is not present. In the latter case, we may assume $t_1 \notin P[v_5, v_4]$ since G/t_4v_2 would have OWO structure. We may also assume $t_3 \notin P[v_5, v_4]$ since $\{t_3, v_4, v_2\}$ would be a 3-cut isolating t_4 but not minimal.

To connect S_2 to T_1, we attempt to use the facial neighbourhood of v_1, clockwise from the edge e to the vertex v_3. This will connect S_2 to T_1 unless v_1 had a facial neighbour $x \in P(v_5, v_4)$ and the edge v_4v_5 is not present, or v_1 has a facial neighbour $y \in P[t_3, v_5]$. Suppose such a y exists. Then $\{y, v_1, v_3\}$ is a 3-cut which isolates t_3. The edge v_1v_3 must be present, since otherwise G/e has 3F structure. In this case, we let $S_1 = P[v_4, v_1] \cup \{v_5\}$ and $S_2 = P[y, v_2]$, unless $y = t_3$, in which case we use $P(y, v_2]$. By construction, we have all desired connections except T_1 to S_2. When $y \neq t_3$, there must be a path from t_1 to S_2 that is disjoint from S_1 since $G - P[v_4, v_1]$ is connected. If $t_3 = y$, then either such a path exists, or by Lemma 2.3.2 $\{t_3, v_1, z\}$ is a 3-cut which isolates t_1 for some $z \in P[v_4, v_1)$. Such a 3-cut cannot exist, since by minimality we would have t_1, t_2 cofacial, and G/t_4v_4 would have 3F structure.

We may now assume that v_1 has a facial neighbour $x \in P(v_5, v_4)$ and the edge v_4v_5 is not present. We consider two possibilities, either there exists $w \in P[v_3, t_3)$ such that $\{w, x, v_5\}$ isolates t_3 or no such w exists. If no such w exists, then we let $S_1 = P[v_2, t_3) \cup v_2v_4$ and $S_2 = P[v_5, x] \cup P[x, v_1]$. These subgraphs as constructed must be disjoint, since if S_1 and S_2 intersected a vertex of intersection would be the w described above. Moreover, they connect S_1 and S_2 to each T_1, giving a $K_{2,4}$ minor. Next, we consider the case where a w as described exists. Let $S_1 = v_2v_5 \cup v_5x$ and $v_1v_4 \subseteq S_2$. (Note that edge v_5x exists since otherwise contracting t_1v_4 gives 3F structure.) This has all the desired connections except for S_2 to T_3. To connect S_2 to T_3, we can connect v_1 to w in the facial neighbourhood of x. This will work unless t_2 is in the facial neighbourhood of x. Moreover, t_2 is not in this facial neighbourhood, since $\{x, v_4, t_2\}$ would be a 3-cut isolating t_1 and contradicting minimality of G. $\square$

Lemma 2.3.17. *If $G \in \mathcal{J}_1$ has an edge e such that G/e gives DCJ structure then G has a rooted $K_{2,4}$ minor.*

Proof. If in G, the set v_1, v_2, v_5 isolates t_3 and t_1, t_2, v_1, v_2 are on a common face, then G/t_3v_1 would give $3F$. Thus, either one end of e is one of $\{t_1, t_2\}$, or the contracted edge becomes v_5. Moreover, if t_3 is contracted to become v_5 then G/e has $HF(v_3, v_4, t_3, v_3, t_3)$ structure and so was already considered previously. By symmetry, we may assume that t_2 is contracted rather than t_1. Notice that in G, $\{v_1, v_2, v_5\}$ cannot separate $\{t_2, t_3\}$ from $\{t_1, t_2\}$, so the only possibility for G is as in Figure 2.16.

Suppose that G is as in Figure 2.16. We first observe that for $w \in P[v_2, v_6)$, $x \in P(v_6, v_4]$, if $\{w, x, v_6\}$ was a 3-cut that isolated t_2, then $\{t_1, w, x\}$ would also be such a 3-cut, and by minimality the edge e would not be present. Since the edge e is present, if we delete $P[v_2, v_6] \cup P[v_6, v_4]$ then by Lemma 2.3.3 the graph must remain connected, and so there is a path from t_2 to v_5 in C_4 that is disjoint from F_1, F_2. Let $T_i = t_i$ for all i.

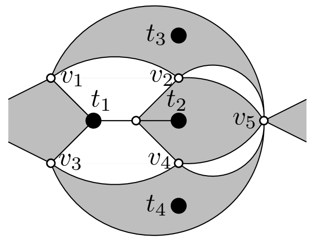


Figure 2.16: Graphs in $\mathcal{J}_1$ where contracting an edge gives DCJ

$S_1 = P[v_6, v_4] \cup P[v_2, v_6]$, $v_5 \in S_2$. By construction, S_1 is connected to each T_i, and S_2 is connected to T_2, T_3, T_4. To connect to S_2 to T_1 it is sufficient to connect S_2 to either v_1, v_3 or t_1. We can make this connection using either of the edges v_5v_1 or v_5v_3 or one of the paths $P[v_1, v_5]$ or $P[v_5, v_3]$. If none of these was usable, then t_3 and t_4 would be cofacial in the infinite face, which we know does not occur, so we can complete the $K_{2,4}$ minor. □

Combining Lemmas 2.3.14, 2.3.15, 2.3.16, 2.3.17 we see that for $G \in \mathcal{J}_1$, G has a rooted $K_{2,4}$ minor and is not a minimal counterexample. Thus, any minimal counterexample must be in $\mathcal{J}_2$.

Graphs in $\mathcal{J}_2$

Suppose $G \in \mathcal{J}_2$. To create an (R3) reduction, after performing the edge contraction t_1v_1 and possibly some (R1) reductions, there must be a terminal vertex t with the following three properties: t has degree 3, t has exactly one terminal neighbour, and w_1, w_2, the two non-terminal neighbours of t are joined by an edge.

When performing an (R1) reduction, the only vertices that experience a change in their neighbours are those that are involved in the 3-cut associated with the reduction. If we have a terminal t' that has degree 3 after preforming an (R1) reduction, then t' has neighbours t'', v, w and the 3-cut for the (R1) reduction consisted of vertices $\{t', t'', v\}$. However, the set t'', v, w would have given a 3-cut which produces an (R2) reduction. Such a 3-cut cannot exist, by Lemma 2.3.10. Thus, performing an (R1) reduction does not affect the neighbour set of t.

The only way performing an (R1) reduction can change the adjacency of any pair of terminals, is if both terminals are vertices in the 3-cut of the reduction. However, since an (R1) reduction does not change the facial adjacency of any pair of vertices, any such pair of terminals was cofacial after contracting the edge t_1v_1, so the only way to add an edge between them is if there is

Performing an (R1) reduction can change whether there is an edge between a pair of non-terminals if those non-terminals are two vertices in the 3-cut associated with the (R1) reduction. However, this cannot occur in this instance, since the (R1) reduction would have to be performed on the set $\{w_1, w_2, t_1\}$, where t_1 was the vertex formed after contraction. We cannot have $t = t_1$ since then G would not be reduced, and we cannot have $\{w_1, w_2, t_1\}$ being the neighbours of t and giving an (R1) reduction. Thus, performing an (R1) reduciton does not create any possible (R3) reductions, so the only way we can get an (R3) reduction is as in Figure 2.17.

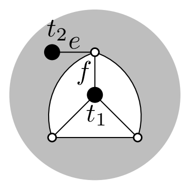


Figure 2.17: $\mathcal{J}_2$ Edge

We observe that when we contract the edge f in Figure 2.17, we arrive at an (R3) reduction in a similar manner. Thus, we see that every vertex adjacent to a terminal (even if the terminal may have degree more than 3) is adjacent to at least two terminals of degree 3. Let v_1 be a vertex in $V(G)$ that is adjacent to the largest number of terminals of degree 3. Suppose v_1 is adjacent to four terminals. Then no other vertex is adjacent to more than two terminals, as this would give a 3-cut which split the graph into more than two components. Thus, G is as in the first diagram in Figure 2.18. If v_1 is adjacent to three terminals t_1, t_2, t_3 of degree 3, then no other vertex, v_2, can be adjacent to the same three terminals, as this would result in $\{v_1, v_2\}$ being a 2-cut. Each terminal incident with v_1 has two other neighbours. The remaining terminal t_4 gives some required incidences with neighbours of the first three only when it is of degree 3. In that case it can only be adjacent to at most three neighbours of t_1, t_2, t_3. We see that at least two pairs of these neighbours must coincide, thus G is as in the second diagram of Figure 2.18 (where possibly $v_2 = v_5$).

The last possibility is that each non-terminal adjacent to a terminal of degree 3 is adjacent to exactly two such terminals. We first consider the possibility that there is a pair of terminals that have two common neighbours v_1, v_2 and have third neighbours v_3 and

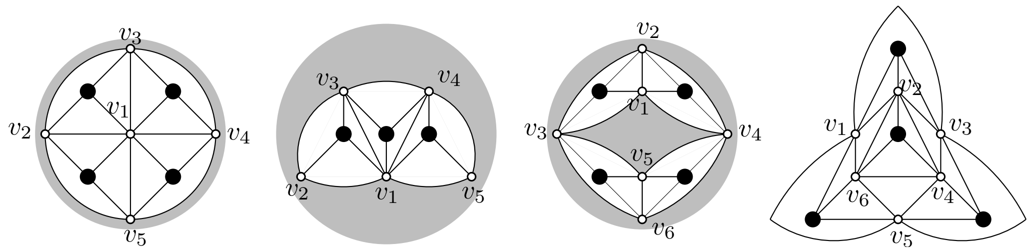


Figure 2.18: Cases for a $\mathcal{J}_2$ edge

v_4, respectively. If v_3 and v_4 were adjacent to the same second terminal of degree 3, then v_3 and v_4 would be cofacial, and $\{v_3, v_4, v_1\}$ or $\{v_3, v_4, v_2\}$ would be a 3-cut that would isolate two terminals. Since this does not occur, we must have G as in the third diagram of Figure 2.18. If each pair of terminals has at most one common neighbour then each pair must have exactly one common neighbour and G is the ten-vertex graph in the last diagram of Figure 2.18.

Lemma 2.3.18. *Graphs in $\mathcal{J}_2$ have rooted $K_{2,4}$ minors.*

Proof. We have proved above that graphs in $\mathcal{J}_2$ have one of the structures from Figure 2.18. If G is as in the first diagram, we can construct a $K_{2,4}$ minor letting $S_1 = v_1$ and $S_2 = P[v_2, v_3] \cup P[v_3, v_4]$.

If G is as in the second diagram of Figure 2.18, we consider two possibilities, either $v_2 = v_5$ or $v_2 \neq v_5$. If $v_2 = v_5$ then $\{v_2, v_3, v_4\}$ isolates the remaining terminal and so the graph has eight vertices and between each pair of non-terminals an edge is present. Thus, we can set $S_1 = \{v_1v_4\}$ and $S_2 = \{v_2v_3\}$. If $v_2 \neq v_5$, then the remaining terminal is adjacent to v_2, v_5 and not to v_1. By symmetry we may assume that it is adjacent to $\{v_2, v_4, v_5\}$. However, this gives $\{v_1, v_2, v_4\}$ as a 3-cut with two terminals on each side, so we may assume this does not occur.

If G is as in the third diagram, we let S_1 be the path in the facial neighbourhood of v_3, clockwise from v_6 to v_2 and let S_2 be the path in the facial neighbourhood of v_3 clockwise from v_1 to v_5. Since G is 3-connected, these paths cannot intersect and so this gives a rooted $K_{2,4}$ minor.

If G is as in the final diagram, then all indicated edges between non-terminals must be present, since each one is the unique edge deleted by an (R3) reduction created by the contraction of an edge. Thus, we let $S_1 = \{v_1v_2, v_2v_4\}$ and $S_2 = \{v_3v_5, v_5v_6\}$ giving us a rooted $K_{2,4}$ minor. $\square$

This completes the proof of Theorem 2.3.5, as Lemmas 2.3.7 – 2.3.10, and 2.3.13 show that any counterexample must be in $\mathcal{J}_1$ or $\mathcal{J}_2$ and Lemmas 2.3.14 – 2.3.18 show that there is no counterexample in either set.

2.4 Algorithm For Finding a Rooted $K_{2,4}$ Minor

We define the problem ROOTED $K_{2,4}$ MINOR as follows:

Input: A graph G and a set of four terminal vertices $T \subseteq V(G)$.

Task: Find a $K_{2,4}$ minor in G with the four terminal vertices forming the larger side of the bipartition, or give a certificate that such a minor does not exist.

Theorem 2.4.1. *There is a polynomial time algorithm that solves ROOTED $K_{2,4}$ MINOR on planar graphs.*

The algorithm for Theorem 2.4.1 follows the proof contained in Sections 2.2 and 2.3. It can be summarized as follows.

Algorithm for ROOTED $K_{2,4}$ MINOR problem when G is planar:

1. If G is disconnected:
   - 1.1 If G has a connected component C containing all four terminals take $G \leftarrow C$.
   - 1.2 Otherwise, G does not have a rooted $K_{2,4}$ minor.
2. If G has a 1-separation $\{A, B\}$ with $A \cap B = \{v\}$, let A be the component of the separation containing the least number of terminals.
   - 2.1 If A contains no terminals, take $G \leftarrow B$.
   - 2.2 If A contains one terminal t, take $G \leftarrow B$, letting v be a new terminal in B replacing t.
   - 2.3 If A contains two terminals, G has no rooted $K_{2,4}$ minor.
   - 2.4 Repeat step 2 until G is 2-connected.
3. If G has a 2-separation $\{A, B\}$ with $A \cap B = \{v, w\}$, let A be the component of the separation containing the least number of terminals.

3.1 If A contains no terminals, take $G \leftarrow B + vw$.

3.2 If A contains two terminals:

3.2.1 If v is a terminal, we take $G \leftarrow B \cup vw$ with w being the other terminal from A. Similarly if w is a terminal. For the remaining subcases we assume v, w are not terminals.

3.2.2 If A contains a rooted $K_{2,2}$ minor between $\{v, w\}$ and the two terminals in A, and B contains a rooted $K_{2,2}$ minor between $\{v, w\}$ and the two terminals in B, then the composition of these gives a rooted $K_{2,4}$ minor in G. [See Remark 1]

3.2.3 If only A contains the rooted $K_{2,4}$ minor, let $G \leftarrow A + vw$ with v and w becoming the terminals from B.

3.2.4 If only B contains the rooted $K_{2,4}$ minor, let $G \leftarrow B + vw$ with v and w becoming the terminals from A.

3.3 If A contains one terminal:

3.3.1 If the terminal in A is cofacial with another terminal, add an edge between them.

3.3.2 If the four terminals are each of degree 2 and have a total of 4 distinct neighbours between them, the terminals can either all be made cofacial, in which case G has no rooted $K_{2,4}$ minor, or we can exhibit a $K_{2,4}$ minor as in the discussion in Section 2.2.2.

3.3.3 Otherwise Let $G_v = B + vw$, taking v to be the terminal from A and $G_w = B + vw$, taking w to be the terminal from A. Take $G \leftarrow G_v$ and $G \leftarrow G_v$. If either has a rooted $K_{2,4}$ minor then G does, if neither does then G does not. [See Remark 2]

3.3.4 Repeat step 3 until G is 3-connected.

4. Perform any (R1), (R2), and (R3) reductions on G. [See Remark 3]

5. If G has one of the five structures from Theorem 2.3.5 then it has no rooted $K_{2,4}$ minor. [See Remark 4]

6. If G has one terminal cofacial with two other terminals, exhibit a rooted $K_{2,4}$ minor by considering the construction from the proof of Lemma 2.3.7.

7. If G has a pair of cofacial terminals, exhibit a rooted $K_{2,4}$ minor by considering the construction from the proof of Lemma 2.3.8.

8. If G has a 3-separation {A,B} with two terminals in each of A, B, then exhibit a rooted $K_{2,4}$ minor by considering the construction from Lemma 2.3.9.

9. If G has an edge $e = v_1t_1$ and two vertices v_2, v_3 such that $\{v_1, t_1, v_2, v_3\}$ is a 4-cut that isolates the terminal t_2, then exhibit a rooted $K_{2,4}$ minor as in the proof of Lemma 2.3.10.

10. Let L be the set of all edges in G. While there exists an edge $e \in L$ adjacent to a terminal, repeat the following steps:

    10.1 Form $G' = G/e$ and perform any (R1), (R2), and (R3) reductions on G' to get G''. [See Remark 5]

    10.1.1 If G'' has none of the five structure, let $G \leftarrow G''$. Add any new edges of G'' to L and remove any edges from L that are no longer present in G. [See Remark 6]

    10.1.2 If G'' has one of the five structures, let $L \leftarrow L - \{e\}$.

11. For any edge e in $E(G)$ contracting e and performing reductions results in a graph with one of the five structures.

    11.1 If G has one of the structures from case 6, 7, 8, or 9, use the appropriate construction to obtain a $K_{2,4}$ minor.

    11.2 If there is an edge that gives OWO after contraction, exhibit a rooted $K_{2,4}$ minor as in the proof of Lemma 2.3.15.

    11.3 If there is an edge that gives HF after contraction, exhibit a rooted $K_{2,4}$ minor as in the proof of Lemma 2.3.16.

    11.4 If there is an edge that gives DCJ after contraction, exhibit a rooted $K_{2,4}$ minor as in the proof of Lemma 2.3.17.

    11.5 If no edge contraction gives the structure, then each contraction gives an (R3) reduction, and G has one of the structures from Figure 2.18. We can exhibit a rooted $K_{2,4}$ minor by considering the constructions from Lemma 2.3.18

Remark 1. *By Theorem 2.2.2, checking for a rooted $K_{2,2}$ minor is equivalent to checking twice for disjoint rooted paths. This can be done in polynomial time. See, for example, [9].*

Remark 2. *To avoid an exponential number of subcases, we always choose a cut that is maximal subject to isolating a single terminal. This means we will have at most one cut for each terminal, so at most 16 subcases to check.*

Remark 3. *Note that G remains 3-connected after performing these reductions.*

Remark 4. *For each structure, we can look at every possible set of vertices in separations that define the specific structure. (at most $O(n^5)$ cases) and see if the required terminals are on the correct faces. We only need to do this if there is at least one pair of cofacial terminals.*

Remark 5. *Note that by Lemma 2.3.11, $G' = G/e$ remains 3-connected. If we choose our reductions carefully, we will be required to perform at most two (R1) reductions. If we were able to perform an (R2) reduction, we could instead use step 9 and exhibit the minor.*

Remark 6. *Each reduction performed adds at most 3 edges to L. Each of these 3 edges is added to a face of G. The edges removed from that face by the (R1) reduction will be removed from L if they were still in it (in which case L does not grow in size). If none of those edges remained in L, then they were removed because contracting them would give a structure after reductions. Contracting this new edge will also give a structure, and so does not need to be added to L.*

Chapter 3

Delta-Wye Transformations

3.1 Introduction

A connected graph G is said to be *delta-wye reducible* if G can be reduced to a single vertex by repeatedly applying the following four reductions and two transformations:

- **Loop reduction** - a loop can be deleted.
- **Degree-one reduction** - an edge incident with a vertex of degree one can be deleted along with the degree one vertex.
- **Series reduction** - if a vertex v has degree two and neighbours u, w, the edges uv and vw are replaced by an edge uw and the vertex v is deleted.
- **Parallel reduction** - two edges having the same endpoints may be replaced by a single edge having the same endpoints.
- **Delta-wye transformation** - If edges uv, vw, uw exist, they can be deleted and replaced by a new vertex x adjacent to u, v, w.
- **Wye-delta transformation** - If a vertex x has degree three with neighbours u, v, w, then v can be deleted and edges uv, vw, uw can be added.

If G has a distinguished set of terminals T then we add the restriction that no operation can delete a terminal vertex and consider a graph *reducible* if we can arrive at a graph on the vertices of T.

We also allow the following additional operation:

- **FP-assignment** A terminal vertex of degree one whose neighbour is not a terminal can be reduced using a degree-one reduction, making the neighbour into a terminal.

As noted by Gitler [5], transformations of this type may affect the embedding but not the structure. For our purposes, there are times when we wish to not allow FP-assignments to be performed on certain terminal vertices. This allows us to ignore those restrictions and perform an inverse FP-assignment at the end of the reductions. In particular, when we are dealing with terminal vertices that arise from splitting a graph along a cut-set, we do not want to allow an FP-assignment on those vertices. When we are dealing with terminal vertices which are not placeholders, then we freely allow FP-assignments on those vertices.

The first important result in this area is due to Epifanov [3] who showed in 1966 that all (non-terminal) planar graphs are reducible. Gitler [5] later extended that result to graphs with no K_5 minor and graphs with no $K_{3,3}$ minor. Truemper [16] showed the class of reducible graphs is minor closed. This result was later extended by Archdeacon et al. [2] to show that terminal reducibility is minor-closed.

Theorem 3.1.1 ([2])**.** *Suppose that H is a terminal-minor of G. If G is terminal delta-wye reducible to a graph G' then H is reducible to a minor of G'. In particular, when G is reducible to only terminals, then so too is H.*

There has been much work on finding the list of excluded minors for the class of graphs which are reducible. A known result from the literature gives the first known obstructions.

Lemma 3.1.2. *The graphs of the* Petersen Family*, consisting of the seven graphs which are equivalent to the Petersen graph under delta-wye and wye-delta transformations are minor-minimal irreducible.*

There has been additional progress in finding excluded minors by Yu [22], [23] showing (constructively) that there at least 68 billion minor-minimal obstructions. These obstructions fall into 20 delta-wye equivalent families.

Our main approach when it comes to trying to reduce large graphs will be to split them apart into smaller graphs and reduce each part individually. We must ensure that when doing this we remain faithful to the structure of the original graph so that the operations can be carried back to the larger graph. For example, we cannot perform a series reduction if the middle vertex has neighbours in the portion of the graph we are not currently considering, since we could not carry this operation back to the graph.

Lemma 3.1.3. *Let $\{A, B\}$ be a separation of the graph G with terminal set T such that $A \cap B \cap T = \emptyset$. Let $A \cap B = W$. Define A' to be the subgraph of G induced on A with terminal set $W \cup (T \cap A)$ and define B' similarly. If A' is delta-wye reducible to A'' and B' is reducible to B'' without performing FP-assignments on vertices of W in either graph, then G is reducible to a minor of the graph G'' formed by identifying A'' and B'' on vertices of W.*

Proof. Suppose we have a sequence of operations $\{O_1, O_2, \ldots O_a\}$ which were performed on A' to arrive at A'' and a sequence of operations $\{N_1, N_2, \ldots, N_b\}$ which were performed on B' to arrive at B''. Since A' is an induced subgraph of G and none of the operations performed depended on the degree of a vertex in W, we can perform $\{O_1, \ldots, O_a\}$ on G to arrive at the graph G'. In G' the subgraph B looks as it did in G except for the possibility of adding or removing edges between vertices in W. Any edges added to W will not prevent us from performing $\{N_1, \ldots, N_b\}$ on G' since none depend on the degree of vertices in W and if any edges are removed, then we have a minor of B in G' and so we can reduce to a minor of B. $\square$

If we modify how we rejoin the graphs we can allow for FP-assignments on vertices in W. For any vertex in W which was reduced by an FP-assignment, we join the vertices in A'' and B'' by an edge instead of identifying them. This is easily seen to be equivalent.

3.2 Four Terminal Planar Graphs

We now consider the case of planar graphs G with 4 terminals, $T = \{t_1, t_2, t_3, t_4\}$. Our main goal is to classify when such a graph is reducible and when it is not.

Theorem 3.2.1. *Let G be a 3-connected 4-terminal planar graph. Then G is delta-wye reducible if and only if it has no rooted $K_{2,4}$ minor on the same set of terminals.*

We first show the easy direction.

Lemma 3.2.2. *If a 4-terminal graph has a rooted $K_{2,4}$ minor then it is not delta-wye reducible.*

Proof. We observe that the terminal graph $K_{2,4}$ with T consisting of the larger side of the bipartition is not delta-wye reducible. Combined with Theorem 3.1.1 we see that a graph with such a minor is not reducible. $\square$

To prove the other direction, we will use the structural characterization of when a graph has a rooted $K_{2,4}$ minor from Theorem 2.3.5. To help with this, we will make use of some prior results:

Theorem 3.2.3 (Gitler [5]). *A 2-connected 3-terminal plane graph is reducible to a copy of K_3 on the terminal vertices.*

Moreover, it can be done in a planar manner. That is, a delta-wye transformation is only performed on a facial triangle, so that the resulting graph remains a plane graph. All future results discussing plane graphs also maintain planarity throughout the process.

Theorem 3.2.4 (Gitler [5]). *A k-terminal planar graph with all terminals on a common face can be reduced to the $k \times k$ half grid (or a minor of it), that is shown in Figure 3.1.*

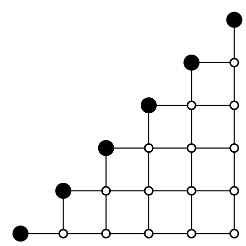


Figure 3.1: $k \times k$ half grid for $k = 6$

It is in fact possible to do this without using FP-assignments. Without using FP-assignments, we are able to reduce to the graph in Figure 3.1 with the terminals being joined to the diagonal vertices by edgs. We can perform a series reduction to bring the outermost terminals back to the desired positions. We can perform Wye-Delta operations on the vertices indcident with the other terminals to get the figure shown plus diagonal edges. These edges can be eliminated by moving them to the bottom or right side of the grid and then performing series reductions.

Theorem 3.2.5 (Gitler [5]). *A $k \times k$ half grid with terminals along the diagonal can be reduced to the $k \times k$ quarter grid (or a minor of it), that is shown in Figure 3.2.*

Theorem 3.2.6 (Archdeacon et al. [2]). *Let G be a 4-terminal planar graph with at least three terminals on a common face. Then G is reducible to a subgraph of K_4.*

We also require the following lemmas which extend the above results.

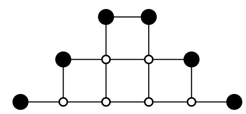


Figure 3.2: $k \times k$ quarter grid for $k = 6$

Lemma 3.2.7. *Let G be a 4-terminal planar graph with three terminals on a common face. Then G can be reduced to a subgraph of K_4 by performing FP-assignments on at most one of the terminal vertices.*

Proof. Combining Theorem 3.2.6 and Gitler's observation about FP-assignments means that any such graph is reducible to a minor of the 8-vertex graph in diagram 1 of Figure 3.3. We perform an FP-assignment on the middle terminal and then a delta-wye transformation on one of the triangles to arrive at the graph in diagram 2. We perform a wye-delta transformation on the indicated vertex. This gives two new triangles. We perform a delta-wye transformation on the one which does not reverse the first transformation and then an FP-assignment on the middle terminal to arrive at the graph in diagram 3. Performing a wye-delta on the indicated vertex brings us to a K_4 with two dangling terminals. We can repeat this process starting with the other triangles to reduce the remaining dangling terminals, arriving at K_4. Notice that by symmetry, we can choose any of the terminals we wish and perform FP-assignments only on that terminal.

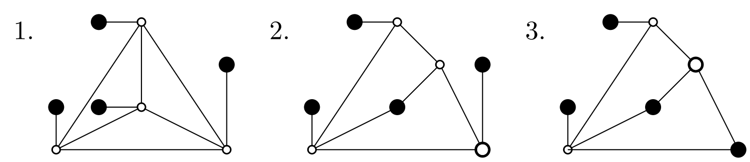


Figure 3.3: 4 terminal reduction to K_4

If G reduces to a minor of the 8-vertex graph in Figure 3.3, then by Theorem 3.1.1 the above result still holds and we get a minor of K_4 on the terminals. □

The second lemma is similar, but the proof is simpler and so it is omitted.

Lemma 3.2.8. *Let G be a 3-terminal planar graph. Then G can be reduced to a subgraph of K_3 without performing FP-assignments.*

Lemma 3.2.9. *Let G be a 4-terminal planar graph. Adding or removing an edge between a pair of terminals on a common face does not affect the reducibility of G.*

Proof. Let G^+ be the graph formed by adding an edge between a pair of terminals on a common faces of G. Suppose G is not reducible. Then G^+ is not reducible, since it contains G as a minor. Suppose G is reducible. We look at how we can preform our operations in G^+. It is easy to see that the ability to perform any of the four reductions is not affected. Any triangle in G will still be present after adding an edge as will any degree-3 non-terminal vertices. Thus, any delta-wye or wye-delta transformations that could be performed in G can also be preformed in G^+. We see that adding an edge may prevent us from performing an FP-assignment, but this is the only possibility. If this occurs, we recall that without FP-assignments we are able to reduce G to a subgraph of K_4 with each terminal being one of the vertices of the K_4 or connected to it by an edge. This means we can reduce G^+ to the first graph shown in Figure 3.4. Performing a delta-wye transformation on the highlighted triangle gives the second graph in the figure. Performing wye-delta transformations on the highlighted terminals gives the third diagram. From this graph, we perform a delta-wye transformation on the highlighted triangle to arrive at the fourth diagram. From this graph, a wye-delta transformation on the indicated vertex and an FP-assignment on the degree 1 terminal give K_4. Therefore, G is reducible if and only if G^+ is. □

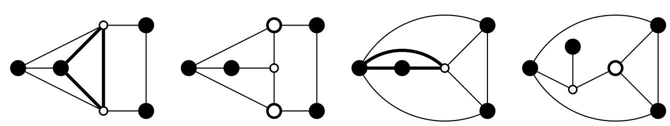


Figure 3.4: Reduction of an extended K_4

We will also use the following observations (see e.g. Gitler [5]):

- If two neighbours of a degree-3 nonterminal vertex are adjacent that edge can be deleted by a delta-wye transformation followed by a series reduction.
- If two consecutive neighbours of a degree-4 nonterminal vertex are adjacent the edge between them can be moved to be between the other pair of neighbours by a delta-wye transformation followed by a wye-delta transformation.

Lemma 3.2.10. *A 3-connected 4-terminal planar graph which has one of the structures of Theorem 2.3.5 is delta-wye reducible.*

Proof. We consider each of the five structures separately.

3F Structure: This is the result of Theorem 3.2.6.

OWO Structure: See Appendix A.1.1.

DF Structure: See Appendix A.1.2.

HF Structure: See Appendix A.1.3.

DCJ Structure: See Appendix A.1.4. □

We recall from Theorem 2.3.5 that a graph has rooted $K_{2,4}$ minor unless *possibly after preforming (R1), (R2), and (R3) reductions* it has one of the five structures. Thus, we need also show that each reduction can be accomplished as a series of delta-wye operations.

Lemma 3.2.11. *An (R1) reduction can be accomplished as a series of delta-wye operations.*

Proof. We apply Lemma 3.1.3 along the cut vertices of the (R1) reduction. By Lemma 3.2.8 we can reduce the side with no terminals to a K_3 on the cut vertices without using any FP-assignments. □

Lemma 3.2.12. *An (R2) reduction can be accomplished as a series of delta-wye operations.*

Proof. For an (R2) reduction, we consider cases based on the number of cut faces the interior vertex is incident with. If the internal terminal is not incident with any of the cut faces, we apply Lemma 3.1.3 and 3.2.7 and get a minor of K_4 on the vertices while only performing FP-assignments on the internal terminal vertex. If the terminal is incident with two of the faces, we have a 3-cut using the terminal that gives an (R1) reduction. Applying the proceeding lemma gives the desired result.

If the internal terminal is incident with a single face of the cut, then when we split that component along the cut (via 3.1.3), we have 4 terminals on a common face. By Lemma 3.2.4 we can reduce this to a 4×4 half grid without using FP-assignments. It is easy to check that this can be reduced to the graph in Figure 3.5. By a sequence of delta-wye operations, we can let the dangling terminal be any of the four. Thus, we can let it be the terminal on the interior of the component. After preforming an FP-assignment on this terminal, we have the desired minor of K_4.

□

Lemma 3.2.13. *An (R3) reduction can be accomplished as a series of delta-wye operations.*

Proof. By Lemma 3.2.9 we can delete the edge between the pair of terminals without affecting the reducibility. We then perform a delta-wye on the triangle and then an FP-assignment

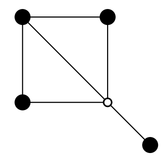


Figure 3.5: 4-terminals on a common face

on the terminal before adding the deleted edge back, arriving at the same graph as we obtained from performing the (R3) reduction. □

Combining Lemmas 3.2.10 – 3.2.13 yields the other direction of Theorem 3.2.1, completing the proof.

3.2.1 Irreducible graphs

We have seen already that $K_{2,4}$ with terminals on the larger side of the bipartition is irreducible since it has all non-terminal vertices of degree 4 and the graph has no triangles. Not only is it minor-minimal irreducible, but no delta-wye or wye-delta transformations can be performed on it. This is different from say the Petersen family for non-terminal graphs, where we have the whole family being minor-minimal irreducible and delta-wye equivalent to each other. Let us examine what happens if we ignore the minor-minimality condition. In particular, what is the structure of a four-terminal planar graph which cannot have any delta-wye operations performed on it?

Lemma 3.2.14. *Let G be a connected four-terminal planar graph such that no delta-wye operations, series-parallel operations, or FP-assignments can be performed on G. Then G is either a minor of K_4 on the terminal vertices or all of the following properties would hold:*

(I1) all terminal vertices of G are of degree 2,

(I2) all non-terminal vertices of G are of degree 4,

(I3) all faces of any planar embedding of G are of size 4.

Proof. We let $\mathcal{F}, \mathcal{E}, \mathcal{V}$ denote the set of faces, edges, and vertices of G, respectively. If G has only terminal vertices, then it is clear that G is a minor of K_4, so we may assume that G has at least one non-terminal vertex. Let t_1 be a terminal vertex in G. If t_1 had degree 1, then its neighbour must be another terminal, since otherwise we could perform an FP-assignment in G. However, we can apply Lemma 3.1.3 and see that $G - t_1$ is reducible to a subgraph of K_3 and so the graph is not irreducible. Thus, any terminal vertex must have degree at least 2. For $v \in V(G)$ is a non-terminal vertex, $deg(v) \geq 4$ since otherwise we could perform a degree-1 reduction, a series reduction, or a wye-delta transformation. Let F be a face in an embedding of G. If $|F| \leq 3$ we could perform a loop reduction, a parallel reduction, or a delta-wye transformation, so we must have $|F| \geq 4$.

Combining the first two conditions we have that

$$|\mathcal{V}| \leq \frac{|\mathcal{E}|}{2} + 2$$

and the last condition tells us that for any embedding

$$|\mathcal{F}| \leq \frac{|\mathcal{E}|}{2}.$$

Adding the two conditions yields

$$|\mathcal{V}| + |\mathcal{F}| \leq |\mathcal{E}| + 2$$

which by Euler's formula must hold with equality. Thus, all inequalities must hold with equality and so we have the desired result. □

We will describe a simple construction for all graphs satisfying (I1) – (I3). Given such a graph, we can construct a polygonal surface by identifying with each face of the graph a unit square and joining the squares at the corresponding edges. The theory of the intrinsic metric of polygonal surfaces due to Aleksandrov [1] tells us that there are two necessary and sufficient conditions for when such a construction will give us a convex polyhedron, so long as we consider a doubly-covered convex polygon to be a convex polyhedron.

(1) "The positive curvature condition": for each vertex, the sum of the angles glued together at this vertex must be at most 2π.

(2) "The Euler condition": if f, e, and v denote the number of faces, edges, and vertices respectively, then $f - e + v = 2$ must hold.

Moreover, the vertices of the polyhedron will be at the vertices where the sum of the angles is less than 2π. For our construction, the first criteria holds since we are gluing either 2 or 4 squares together at each vertex, so the sum of the angles is π or 2π at each vertex.

Since there are four vertices where the sum of the angles is less than 2π, the polyhedron will have 4 vertices, and so is a (possibly degenerate) tetrahedron. Moreover, at these four points, the sum of the angles is equal to π, so if we cut the polyhedra along the three edges incident with a vertex, it will unfold into a triangle. Thus, the tetrahedron is an isosceles tetrahedron. The second criteria holds since our construction is from a connected planar graph.

Consider the infinite regular square tiling of the plane. If we choose any 3 non-collinear vertices of this tiling, we obtain a triangle. We can extend this triangle to a similar triangle with four times the area as in Figure 3.6 and fold it into a polyhedron. When one of the angles of the triangle is a right angle, this will give a doulbe sided polygon. We can let the vertices of the polyhedron be vertices of a (planar) graph with edges present if there is an edge of the grid between the vertices. This will give a graph with the desired structure. Thus, we see that there is a correspondence between graphs of the desired structure and graphs constructed from this, and so this construction gives all desired graphs. This can be seen as an extension of the results of Thurston [15] on triangulations of the sphere.

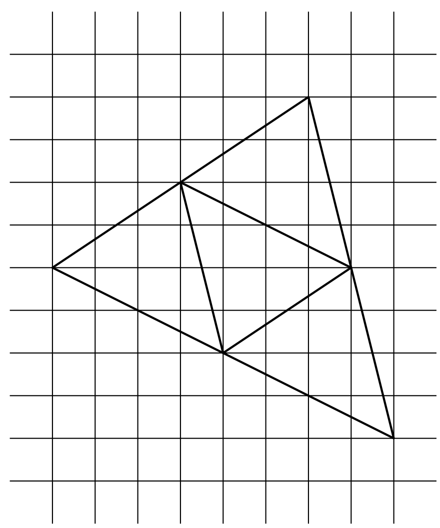


Figure 3.6: Construction of a regular tetrahedron

3.3 Cubic Graphs

A graph G is *apex* if it is non-planar and has a vertex $v \in V(G)$ such that $G - v$ is planar. A graph G is *doublecross* if any drawing of G in the plane has at least two pairs of crossing edges and there is a drawing of G in the plane with exactly two pairs of crossing edges, which cross in the same face. The graph *Starfish* is the 20-vertex graph in Figure 3.7.

For $k \geq 1$, a graph is *cyclically $k-$connected* if G has girth $\geq k$, and $|\delta_G(X)| \geq k$ for every $X \subset V(G)$ such that both X and $V(G) - X$ include the vertex set of a cycle of G.

We say a graph G is *theta-connected* if G is cubic, cyclically 5-connected, and $|\delta_G(X)| \geq 6$ for all $X \subset V(G)$ with $|X|, |V(G) - X| \geq 7$. A cubic graph cannot be k-connected for k greater than three. However, if a cubic graph is theta-connected, it is in a sense highly connected, since one side of any small cutset has few vertices and no cycles.

The following characterization of theta-connected graphs is due to Robertson, Seymour, and Thomas.

Theorem 3.3.1 ([12]). *Let G be a theta-connected cubic graph. Then G does not have the Petersen graph as a minor if and only if either G is apex, or G is doublecross, or G is isomorphic to Starfish.*

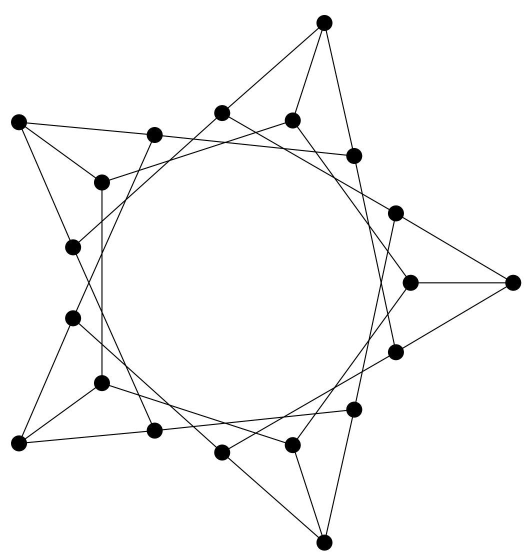


Figure 3.7: Starfish

Theorem 3.3.2. *Let G be a theta-connected cubic graph. Then either G is delta-wye reducible or G has a the Petersen graph as a minor, in which case G is not reducible.*

Proof. By Theorem 3.3.1 we know that G either has the Petersen graph as a minor or G is apex, or G is double cross, or G is isomorphic to starfish. If G has the Petersen graph as a minor, then by contracting the paths $\nu(e), e \in E(G)$ to single edges gives a Petersen minor and hence G is not delta-wye reducible by Theorem 3.1.2. Suppose G is apex. By Theorem 3.2.3, we have that an apex graph where the apex vertex has degree 3 is delta-wye reducible. Since G is cubic, the apex vertex is of degree 3 and so G is reducible.

Suppose G is double cross. Then G has either two pairs of crossing edges in the infinite face or one edge e that crosses two other edges in the infinite face. If such an e exists then then G is apex, since deleting either end of e gives a planar graph. Thus, we may assume that G has two pairs of crossing edges. We can replace the eight endpoints of the four crossing edges with terminals and by Theorems 3.2.4 and 3.2.5 this is reducible to a minor of an 8×8 quarter grid. See Appendix A.2.1 for how this graph can be reduced.

Suppose G is isomorphic to starfish. Then it can be checked that G is delta-wye reducible. □

3.4 Planar Duality

When we restrict ourself to planar graphs, a natural question to look at is what happens when we take planar duals. By Epifanov's result mentioned above we know that all planar graphs are delta-wye reducible, so taking the dual of a planar graph gives another reducible graph. However, a list of operations that reduce a plane graph give us a sequence of operations that reduce the dual graph, as described in the proceeding lemma.

Lemma 3.4.1. *Suppose G is a plane graph with dual graph H. Performing a delta-wye transformation on a facial triangle of G (resp. H) corresponds to performing a wye-delta transformation on H (G). Performing a series reduction on G (H) corresponds to performing a parallel reduction on H (G). Performing a loop deletion in G (H) corresponds to performing a degree-one reduction in H (G).*

What happens if we take the planar dual of a terminal graph? Our graph now has special faces that are not allowed to be eliminated. We can also carry over our terminal operations and their restrictions to this new framework. We are not allowed to perform a delta-wye operation on a terminal face and we cannot perform a parallel reduction if the face of size two is a terminal face. An FP-assignment corresponds to deleting a loop where one of the incident faces is a terminal face and making the other face incident with the loop into the a terminal face. This cannot be performed if the new face was already a terminal face. We call this new loop deletion operation an LD-assignment. If we have a loop edge e such that exactly one face incident with e is a terminal face, we perform an *LD-assignment* on e by deleting e and letting the new face formed be a terminal face.

Studying graphs with terminal faces is clearly equivalent to studying graphs with terminal vertices, so we would like to look at the interesting question of graphs with both terminal vertices and terminal faces. In this case, we will call a graph reducible if it can be reduced to a graph where all vertices are terminal vertices and all faces are terminal faces. We call these graphs *doubly terminal*. When we have a graph with certain faces labelled as terminals, an embedding of the graph in the plane is implicitly assumed. We begin by looking at the simplest case of graphs with one terminal vertex and one terminal face.

Theorem 3.4.2. *Let G be a plane graph with one terminal vertex and one terminal face. Then G is reducible to an isolated terminal vertex surrounded by a terminal face.*

Proof. We first observe that we are able to apply Lemma 3.1.3 to split the graph along the vertices of the terminal face and that any reductions that can be performed on the remainder of the graph can be performed on the entire graph.

If we ignore the restriction on the terminal face, then G is reducible to a single terminal by a series of operations $\{O_i\}_{i=1}^{k}$. If we now perform the O_i on G respecting the restrictions on the terminal face, we will either arrive at the desired single terminal graph or try to perform either a parallel reduction or delta-wye reduction on the terminal face. In either case, we can apply Lemma 3.1.3 arriving at either a three-terminal graph which is reducible to a subgraph of K_3 or a four-terminal graph with all terminals on a common face which is reducible to a subgraph of K_4. If we now add the terminal face back to these graphs, it is routine to verify that each can be reduced to the single terminal. $\square$

As we increase the number of terminal vertices and faces, the problem becomes more difficult. If we allow two terminal vertices and one terminal face (which, by duality is the same as allowing two faces and one vertex) we have a partial result.

Theorem 3.4.3. *Let G be a plane graph with two terminal vertices and one terminal face such that at least one terminal vertex is incident with the terminal face. Then G is reducible to a subgraph of K_2 on the terminal vertices with a terminal face.*

Proof. We first observe that once a terminal vertex is incident with a terminal face they can never become non-incident by performing any of the operations. Knowing that, we can simply repeat the proof of Theorem 3.4.2 and obtain the desired result. □

Lemma 3.4.4. *Let G be a connected plane graph with two terminal vertices and one terminal face. Then G is K_2 with a terminal face or there is some operation (series-parallel reduction, FP-assignment, etc.) that can be performed on G.*

Proof. Let V, E, and F be the number of vertices, edges, and faces respectively in the embedding of G. All faces have size at least 4, aside for possible the terminal face. If the terminal face had size one, we could perform an LD-assignment, so we assume it has size at least 2. So

$$E \geq \frac{4(F-1)+2}{2} = 2F - 1$$

Each terminal vertex must have degree at least 2, since otherwise we could perform an FP-assignment, or the terminals would be adjacent, and the graph would be reducible to K_2. Each other vertex has degree at least 4. So

$$E \geq \frac{4(V-2)+4}{2} = 2V - 2.$$

Rearranging the above inequalities, we get $F \leq E + \frac{1}{2}$ and $V \leq \frac{E}{2} + 1$. Combining these we get $F - E + V \leq \frac{3}{2}$, which violates Euler's Formula. Thus, there must be a non-terminal face of size at most 3, or a non-terminal vertex of degree at most 3. □

We can also extend the results when we have terminal vertices incident with terminal faces. More generally, if we have one terminal face and k terminals, all incident with the terminal face, we can use Lemma 3.1.3 to obtain the following analog of Theorem 3.2.4.

Theorem 3.4.5. *Let G be a plane graph with k terminal vertices and one terminal face such that all terminal vertices are incident with the terminal face. Then G is reducible to a subgraph of a $k \times k$ half grid with the terminal vertices on the diagonal and the infinite face as the terminal face.*

In the proof of Theorem 3.2.4, the face with all the terminals is never eliminated, so the result follows directly. The last case we discuss here is when G has two terminal vertices and two terminal faces. We present the following minor-minimal irreducible graph.

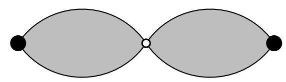


Figure 3.8: Irreducible graph with 2 terminal vertices and 2 terminal faces

If G and H are planar duals of each other then an edge contraction in G corresponds to an edge deletion in H. This means that the idea of minors for graphs with terminal faces is well-defined. For doubly terminal graphs G and H, we say G is a *doubly-terminal minor* of H if we can obtain G from H be a series of edge deletions and contractions.

This notion of minors and the irreducible graph from Figure 3.8 tells us that when G has at least two terminal vertices and and two terminal faces it will in general not be reducible.

Chapter 4

Conclusion and Future Work

In this chapter, we outline possible directions for future research.

4.1 Rooted $K_{2,4}$ Minors

In the thesis, we found a characterization for when a four-terminal planar graph has a rooted $K_{2,4}$ minor. A natural question is what happens when we remove the restriction of planarity. If we take any of the planar obstructions and add edges between any pairs of terminals we wish, the resulting graph still has no rooted $K_{2,4}$ minor. We also note that the OWO and $3F$ obstructions have no rooted $K_{2,4}$ minor because there is no rooted $K_{2,3}$ minor for some three of the terminals. So, if we take a three-terminal planar graph with three terminals on a common face and add a terminal connected arbitrarily to the planar graph, this will not have a $K_{2,4}$ minor.

Notice also that if we start with any planar obstruction and join any non-planar graph to a vertex, edge or triangle in the graph that this will still be an obstruction. This tells us that we must again consider some low-connectivity reductions for these graphs. The reductions from the planar case for 1- and 2-cuts will behave in a similar manner for non-planar graphs, as will the (R1), (R2), and (R3) reductions.

Conjecture 4.1.1. *Let G be a four-terminal graph. Let H be formed from G by performing all low-connectivity and (R1), (R2) and (R3) reductions. Then either H is planar and G has a rooted $K_{2,4}$ minor if and only if G does, or H has a terminal t such that H/t has no rooted $K_{2,3}$ minor, or G has a rooted $K_{2,4}$ minor.*

There are many other natural extensions to this problem. One is to ask what would happen if we were looking for $K_{2,n}$ minors in n-terminal graphs. There are some asymptotic results due to Kawarabayashi [7], but a more general structural result would also be interesting. We can also look at the problem of specifying all 6 vertices of the $K_{2,4}$ minor. This problem can be modelled as finding a rooted $K_{2,6}$ minor where we specify the six vertices of the largesnt bipartite class. To see this, we take our graph where we wish to find a $K_{2,4}$ minor with all vertices specified and add two terminals adjacent to each of the two vertices required to be on the small side of the bipartition. It is easy to verify that any $K_{2,6}$ minor in this graph must use the two original vertices in distinct big vertices or that the original vertices are in the subgraphs for the two new terminals. It can be shown that if a minor of the second type exists that it can be reduced to a minor of the first type.

4.2 Delta Wye Transformations

The problem of determining the complete list of excluded minors is still open and is the most well known problem in the area. Terminal reducibility has been a useful tool for helping with general reducibility, since a terminal planar graph can be used to model an apex graph. However, studying terminal graphs does not give the complete picture for general reducibility. Consider the result from this thesis where we characterized that a suitably connected four-terminal planar graph is reducible if and only if it does not have a rooted $K_{2,4}$ minor. If we have an apex graph with the apex vertex having degree 4, we would model this using a four-terminal graph. If this terminal graph is reducible to $K_{2,4}$, then the apex graph is reducible. So while we know that 3-apex graphs are reducible, the characterization of 4-terminal planar graphs does not tell us the complete story for 4-apex graphs. When is a 4-apex graph reducible and when is it not? What about for apex vertices of larger degree? Is there some number k such that if G is k-apex and not i-apex for any $i < k$ then G is not reducible?

Because terminal graphs were being used to study apex reducibility the focus has been on planar terminal graphs. If we consider terminal non-planar graphs there are some interesting questions we can ask. If we consider non-planar graphs which are reducible (e.g. K_5), we can add terminals to these graphs. When is such a graph with added terminals reducible? For a general graph G, it is reducible to some equivalent family of graphs $\mathcal{F}$. For what placement of terminals on G is the resulting graph reducible to a subfamily of $\mathcal{F}$ with terminals on some of the vertices?

Appendix A

Graph Reductions

Here we show that various graphs from the thesis are delta-wye reducible. We use the following three operations D_i ($i = 2, 3, 4$, where i referes to the degree of the vertex involved) during our reductions. We show how each can be realized as a series of delta-wye operations.

D2: If a degree-2 terminal has neighbours v_1, v_2, the edge v_1v_2 can be deleted, if present. Figure A.1 shows how this is done using a delta-wye transformation followed by an FP-assignment.

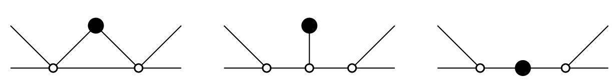


Figure A.1: D2 Reduction

D3: If a degree-3 non-terminal has neighbours v_1, v_2, the edge v_1v_2 can be deleted, if present. Figure A.2 shows how this is done using a delta-wye transformation followed by a series reduction.

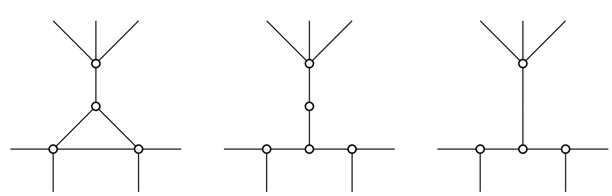


Figure A.2: D3 Reduction

D4: If a degree-4 non-terminal has neighbours v_1, v_2, v_3, v_4, then if the edge v_1v_2 is present it can be replaced by the edge v_3v_4. Figure A.3 shows how this is done using a delta-wye transformation followed by a wye-delta transformation.

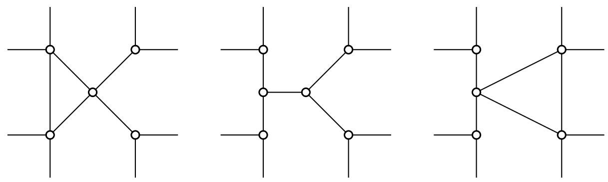


Figure A.3: D4 Swap

A.1 Main Theorem Cases

Here we show how to reduce the various structures from Theorem 2.3.5.

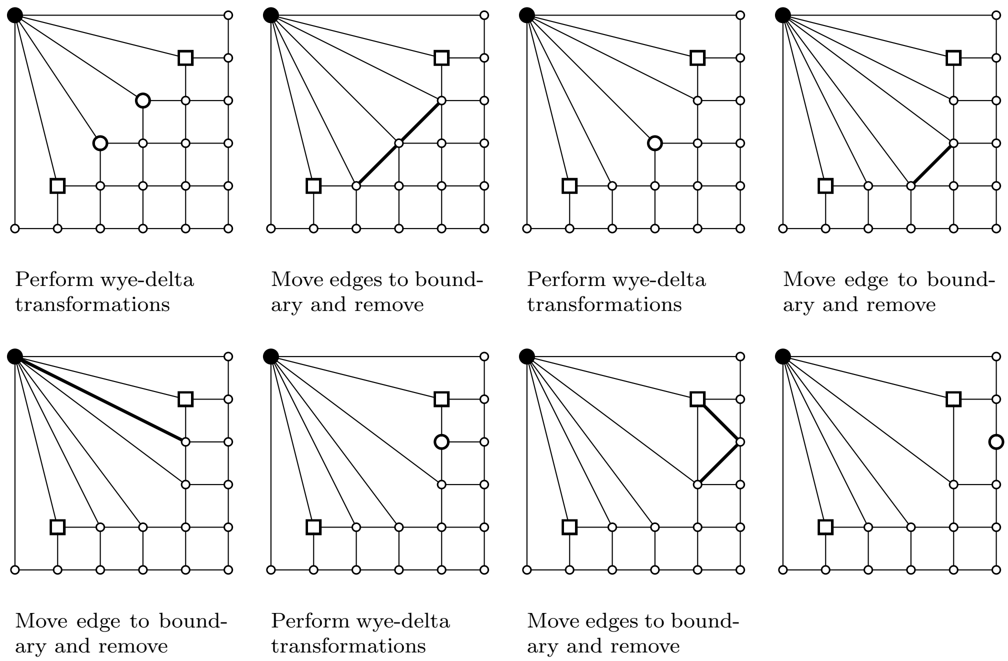
A.1.1 Graphs with OWO Structure

Let G be a graph with OWO Structure. We apply Lemma 3.1.3 to the partition $\{H = G - t_1, N(t_1)\}$. We assume H has k terminals, which can be seen to lie on a common face of an embedding of H. By Theorem 3.2.4, we see that H can be reduced to a graph J which is a minor of a half $k \times k$ grid with the terminals along the diagonal. We may assume that J is the whole $k \times k$ grid, since it suffices to show the reducibility of this. By Lemma 3.1.3 this means that G can be reduced to a graph G' consisting of a $k \times k$ grid with three terminals on the diagonal and the fourth terminal adjacent to all vertices on the diagonal.

If two consecutive diagonal vertices of G' are non-terminal, we can reduce G' to consist of t_1 joined to the diagonal of a $(k-1) \times (k-1)$ grid by removing one of the consecutive pair via the diagonal fixing algorithm shown in Figure A.4. We represent the the diagonal vertices before and after the ones we are working with as square vertices. We notice that in the algorithm, the square vertices remain unchanged.

Thus, we may assume that G can be reduced to a minor of the first graph in Figure A.5. We apply Lemma 3.1.3 to this graph using the separation formed by cutting along the diagonal. By Lemma 3.2.5 this is reducible to the second graph in Figure A.5.

In Figures A.6 and A.7, we show how to reduce this to a graph with three terminals on a common face. By Theorem 3.2.6, this is reducible, so all graphs with OWO structure are reducible.



Continue repeating the last two operations on successive layers of the grid until a series reduction is preformed on the boundary. This removes a row from the grid. Repeat on the other side to remove a column.

Figure A.4: Diagonal fixing algorithm

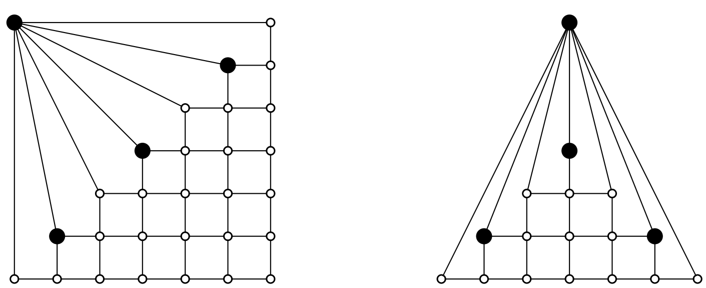


Figure A.5: OWO after reducing to half and quarter grid

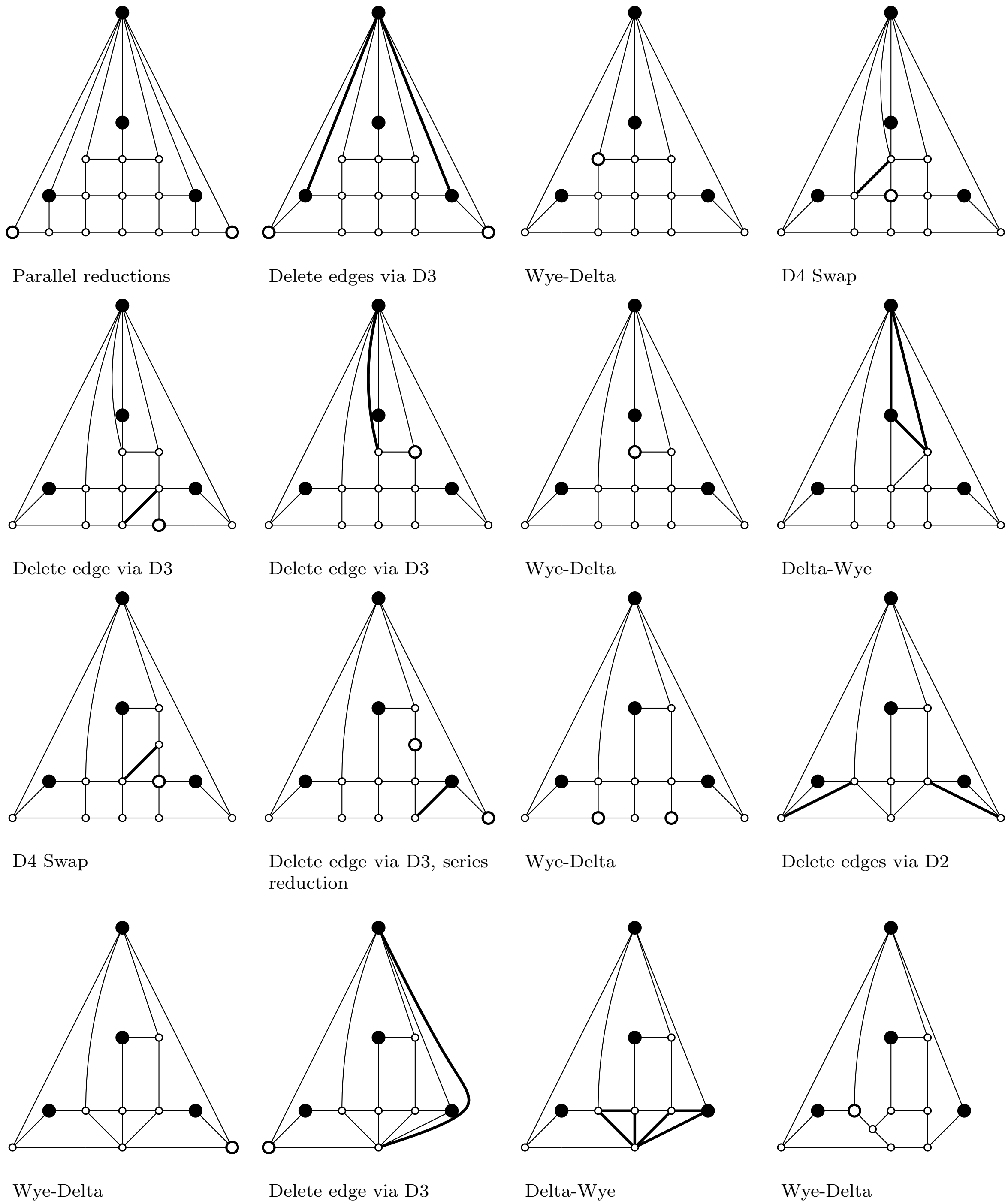


Figure A.6: OWO Reductions 1

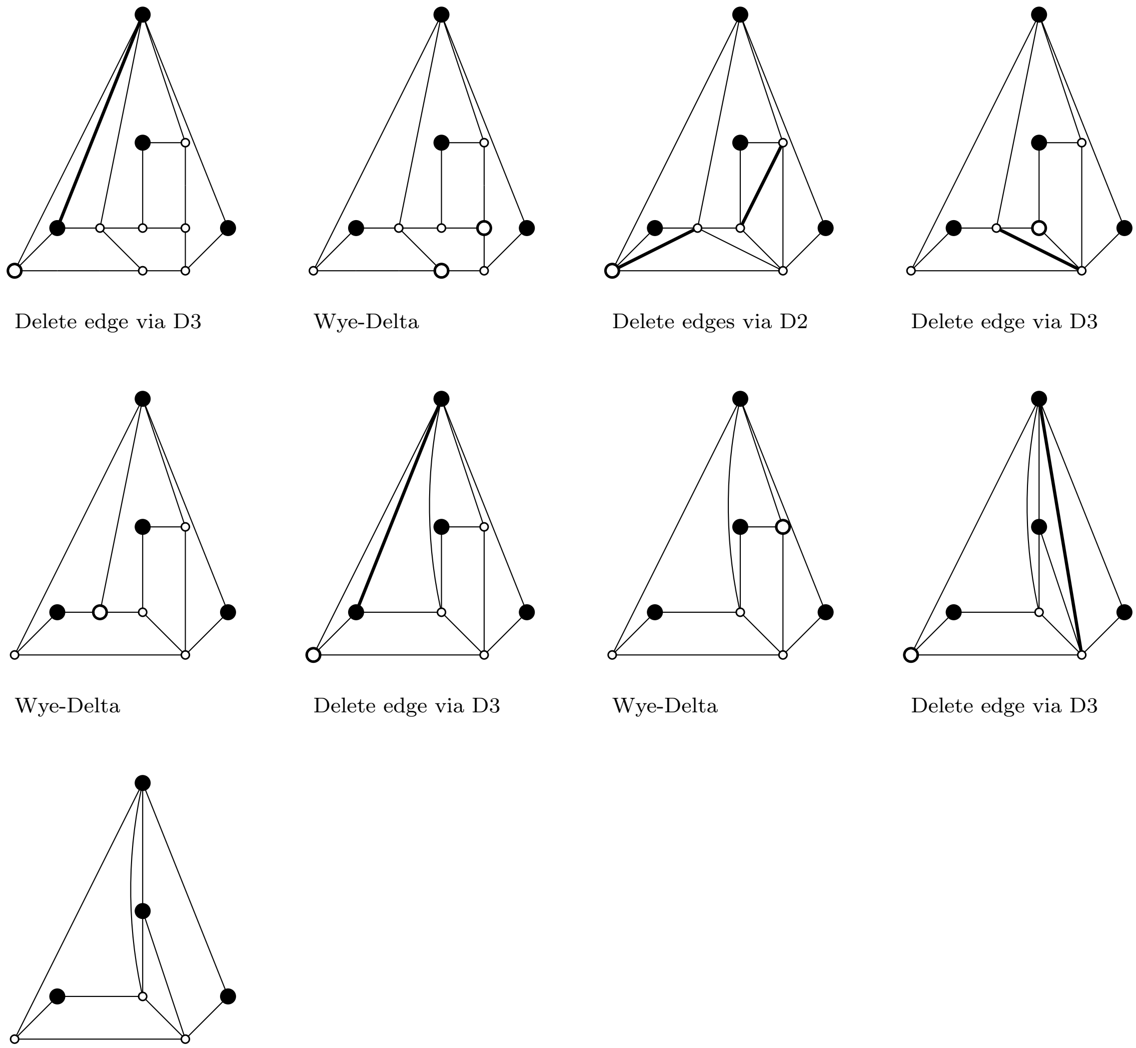


Figure A.7: OWO Reductions 2

A.1.2 Graphs with DF Structure

Let G be a graph with DF Structure. We split the graph into two subgraphs along the cut vertices v_1, v_2, v_3 and make them terminals in each component. We can reduce each graph to a 5×5 quarter grid and then identify corresponding vertices. Figure A.8 shows the graph before and after the two components have been joined together.

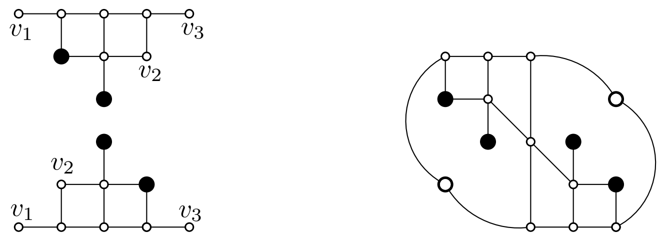


Figure A.8: DF afer reducing to quarter grids

Figure A.9 shows how to reduce this to a graph with three terminals on a common face. By Theorem 3.2.6, this is reducible, so all graphs with DF structure are reducible.

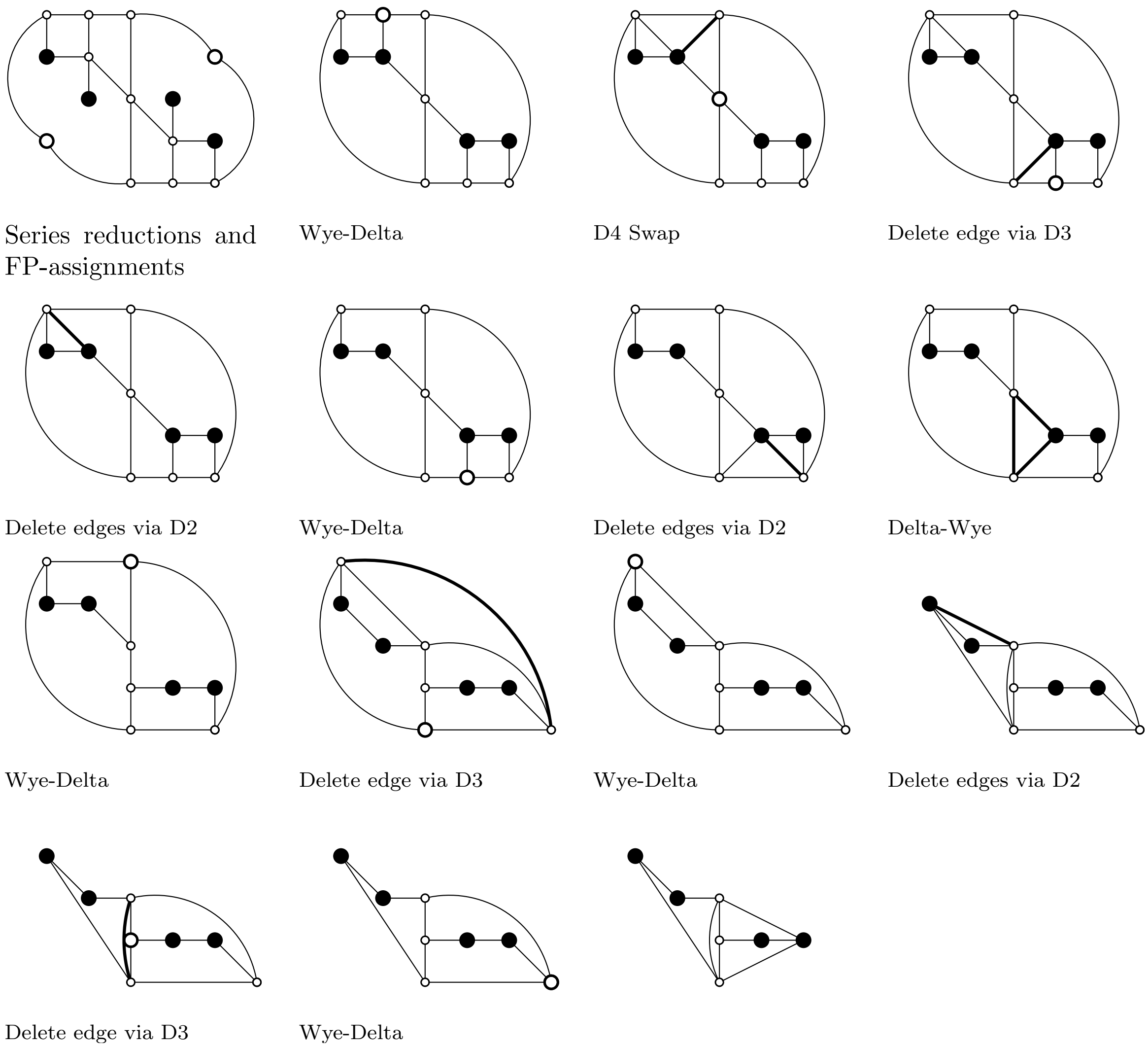


Figure A.9: DF Reductions

A.1.3 HF Structure

Let G be a graph with HF Structure. We split the graph into three subgraphs by cutting on v_1, v_2, v_3, v_4, v_5. Two of these reduce to 5×5 quarter grids and the other reduces to a minor of K_4. We identify corresponding vertices. Figure A.10 shows the graph before and after the three components have been joined together.

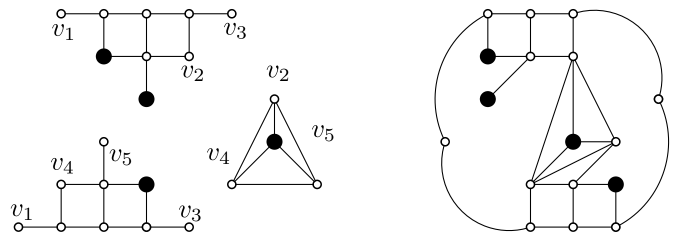
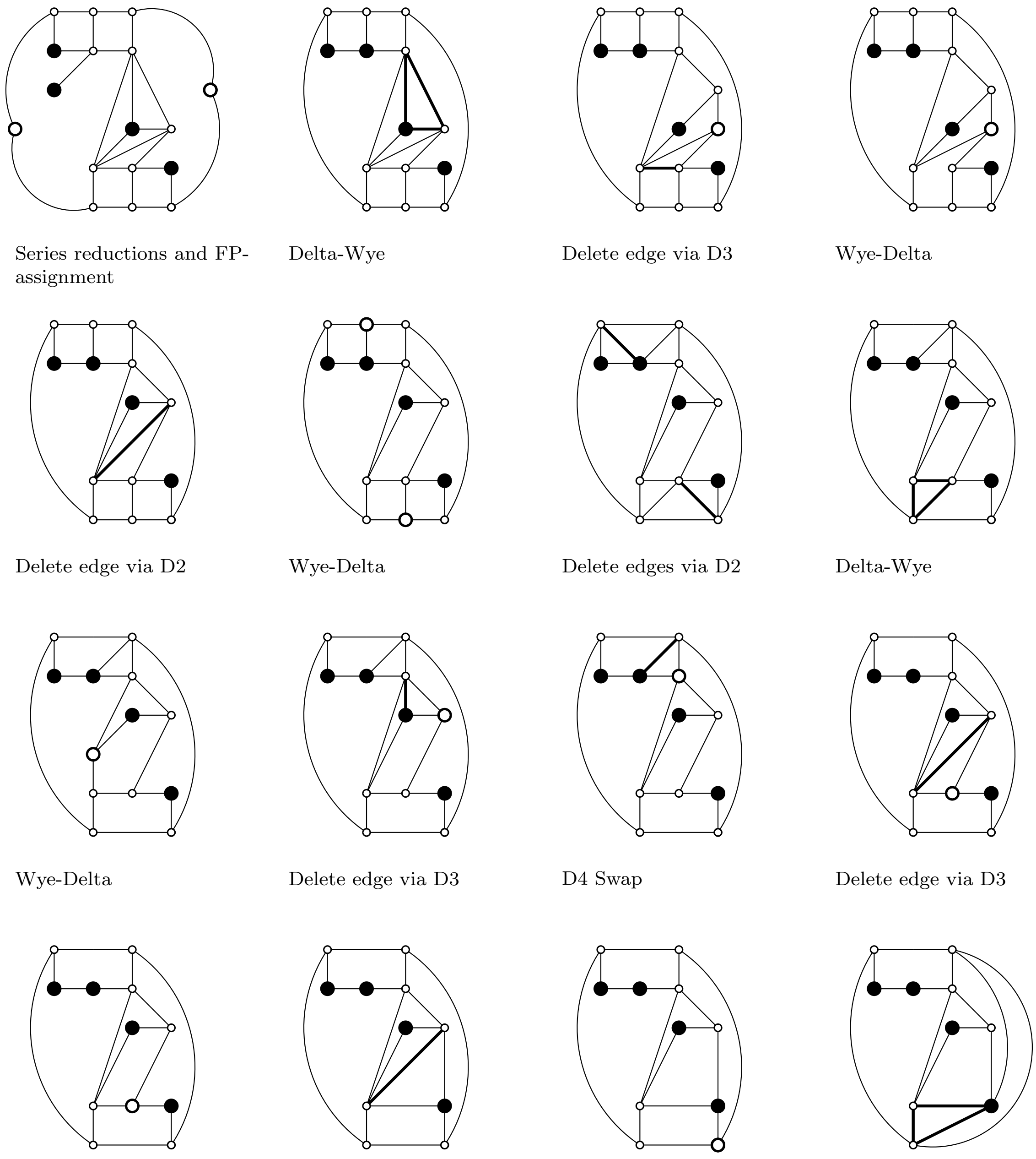


Figure A.10: HF afer reducing to quarter grids and K_4

In Figures A.11 and A.12, we show how to reduce this to a graph with three terminals on a common face. By Theorem 3.2.6, this is reducible, so all graphs with HF structure are reducible.



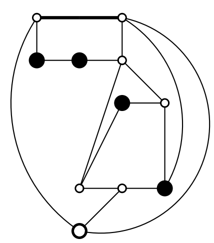
Series reductions and FP-assignment | Delta-Wye | Delete edge via D3 | Wye-Delta

Delete edge via D2 | Wye-Delta | Delete edges via D2 | Delta-Wye

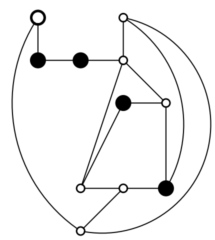
Wye-Delta | Delete edge via D3 | D4 Swap | Delete edge via D3

Wye-Delta | Delete edge via D3 | Wye-Delta | Delta-Wye

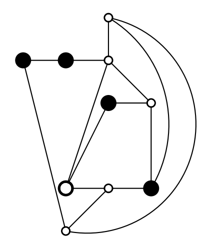
Figure A.11: HF Reductions 1



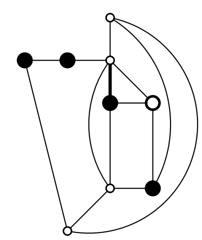
Delete edge via D3



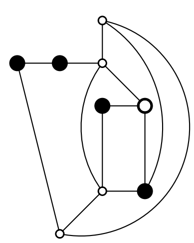
Series reduction



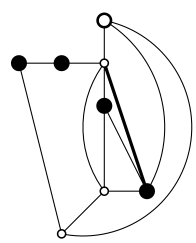
Wye-Delta



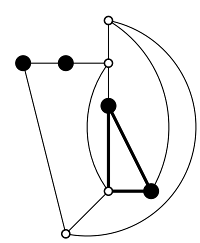
Delete edge via D3



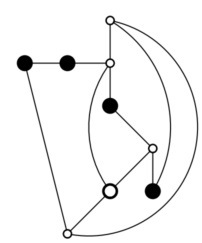
Wye-Delta



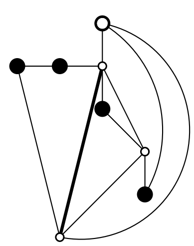
Delete edge via D3



Delta-Wye



Wye-Delta



Delete edge via D3

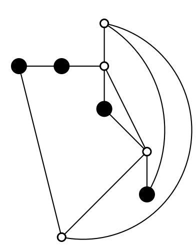


Figure A.12: HF Reductions 2

A.1.4 Graphs with DCJ Structure

Let G be a graph with DCJ structure. We split the graph into five pieces by cutting on $v_1, v_2, v_3, v_4, v_5, t_1, t_2$ (one piece is the single edge t_1t_2). Each of these is reducible to a graph on at most four vertices. Figure A.13 shows the graph before and after the five components have been joined together.

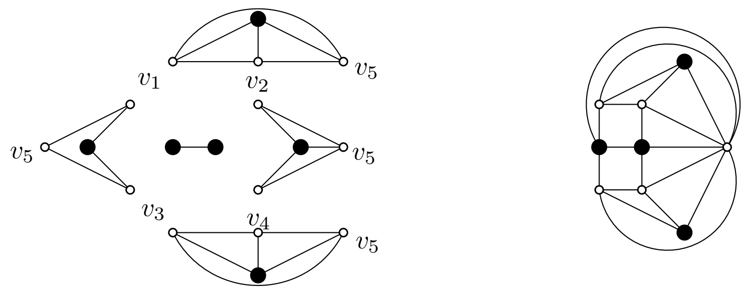
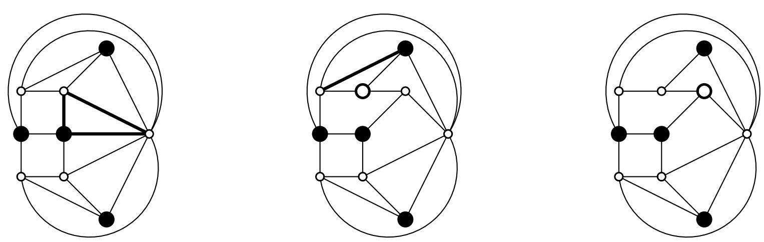
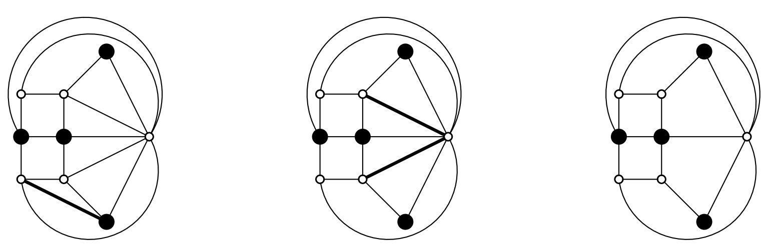


Figure A.13: DCJ afer reducing

In Figure A.14, we show how to reduce this to a graph with OWO structure. By Appendix A.1.1 this graph is reducible, so all graphs with DCJ structure are reducible.



Delta-Wye | Delete edge via D3 | Wye-Delta



Repeat above steps symmetrically | Delete edges via D2

Figure A.14: DCJ Reductions

A.2 Doublecross Graphs and Starfish

In this section, we give reductions for two cases from Theorem 3.3.2. We consider graphs which have doublecross structure and the graph Starfish.

A.2.1 Graphs with Doublecross Structure

In the proof of Theorem 3.3.2, we show that graphs with doublecross structure are reducible to the graph in Figure A.15

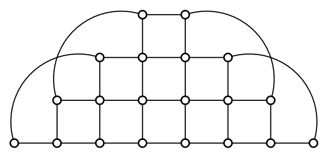


Figure A.15: Doublecross reduced to a quarter grid

In Figures A.16 and A.17, we show how to reduce this to an apex graph, where the apex vertex has degree of 3. First, we implement the reductions of Fig A.16. These reductions have been contained to the right side of the graph, aside for needing that one vertex on the left was degree 3. We can mirror this set of reductions on the left of the graph and then implementing the reductions in Figure A.17. The resulting graph is apex, with an apex vertex of degree 3, so by Theorem 3.2.3, this graph is reducible, and so all doublecross graphs are reducible.

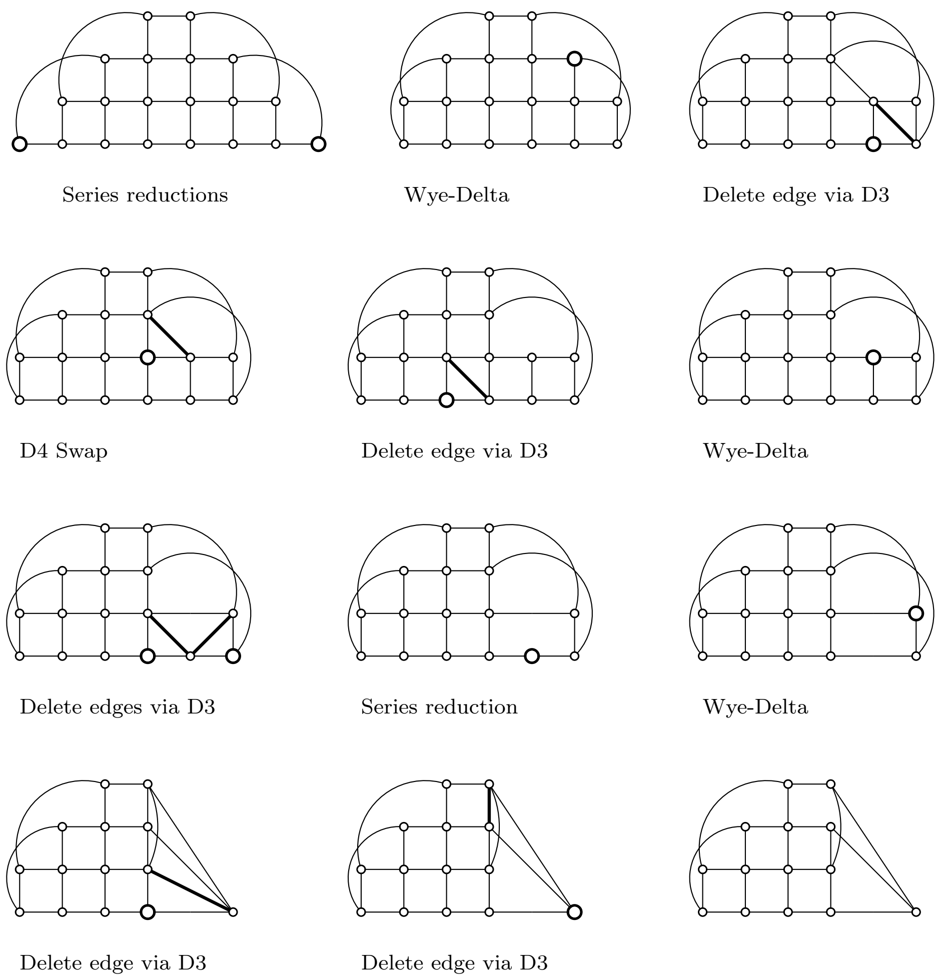


Figure A.16: Doublecross Reductions 1

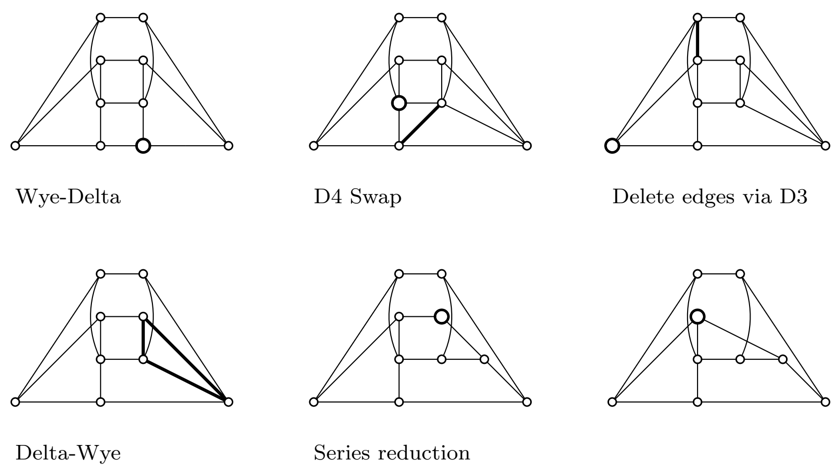


Figure A.17: Doublecross Reductions 2

A.2.2 Starfish

Starfish is the graph from Figure 3.7. Because it is highly non-planar, to increase readability, we encode the graph as three embedded 5-cycles and five vertices $\{1, 2, 3, 4, 5\}$. We label each embedded vertex in with which (if any) of the five vertices are adjacent to it. If we decompose the graph of starfish, we see that to start, each 5-cycle will be labelled $\{1, 2, 3, 4, 5\}$ in the same cyclic order. In Figures A.18, A.19, and A.20, we show a series of steps that reduces starfish to an apex graph, where the apex vertex has degree 3. By theorem 3.2.3 this is reducible, so starfish is reducible.

Throughout the steps of the reduction, we update the labels on the vertices as the adjacencies change. We mention three operations that occur frequently and how the labels are update. When we perform a Wye-Delta operation on an embedded vertex with two embedded neighbours and one label, the neighbouring vertices will are joined together and the label is added to both vertices. We also can perform the inverse of this. If there are two adjacent embedded vertices v, w with a common label t, we can delete t from the labels of v, w, delete the edge vw, and add a new vertex x adjacent to v and w with the label t. When an embedded vertex v has two labels and the corresponding vertices are adjacent, we add a vertex w of degree 1 adjacent to v and move the labels from v to w.

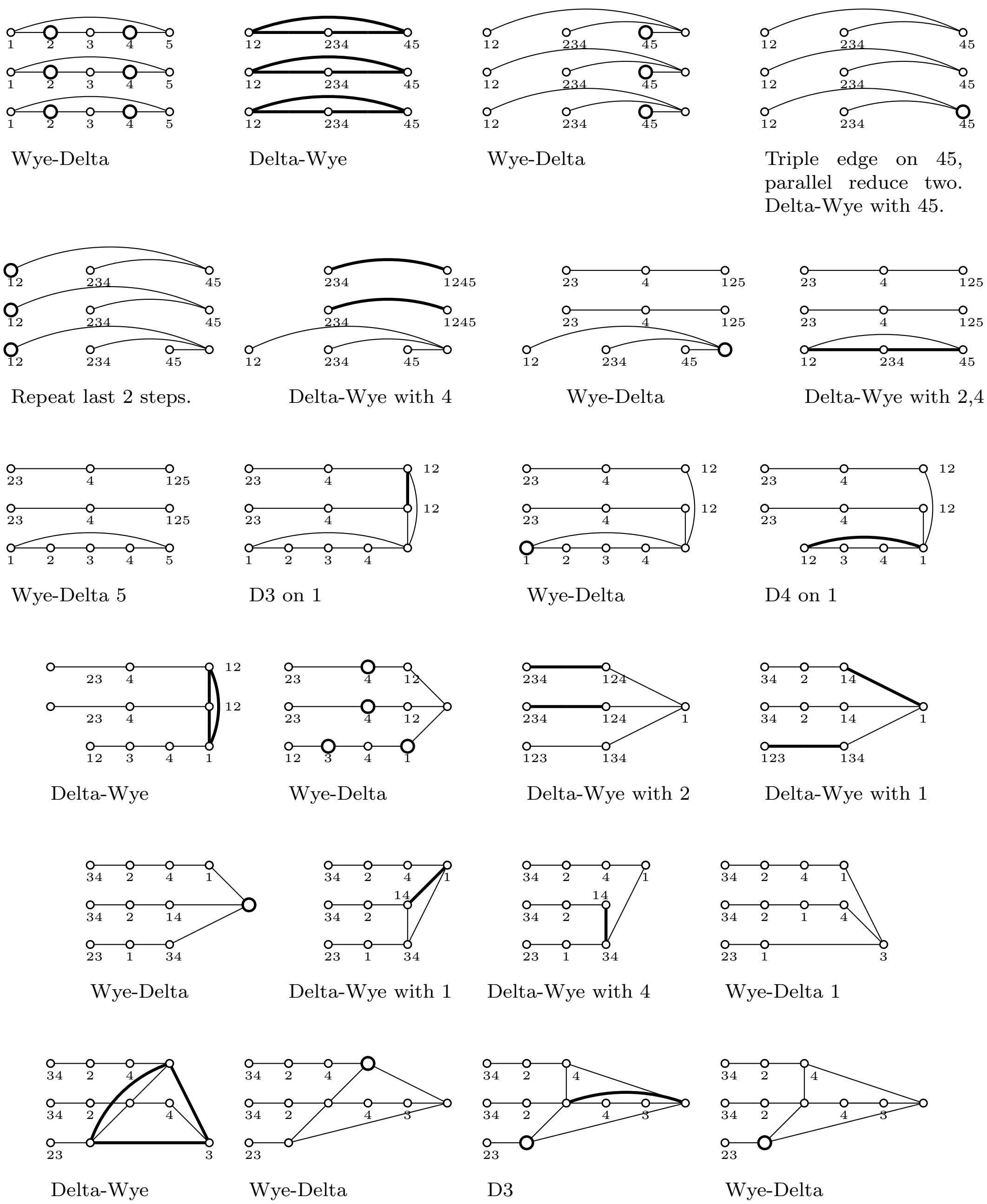


Figure A.18: Starfish Reductions 1

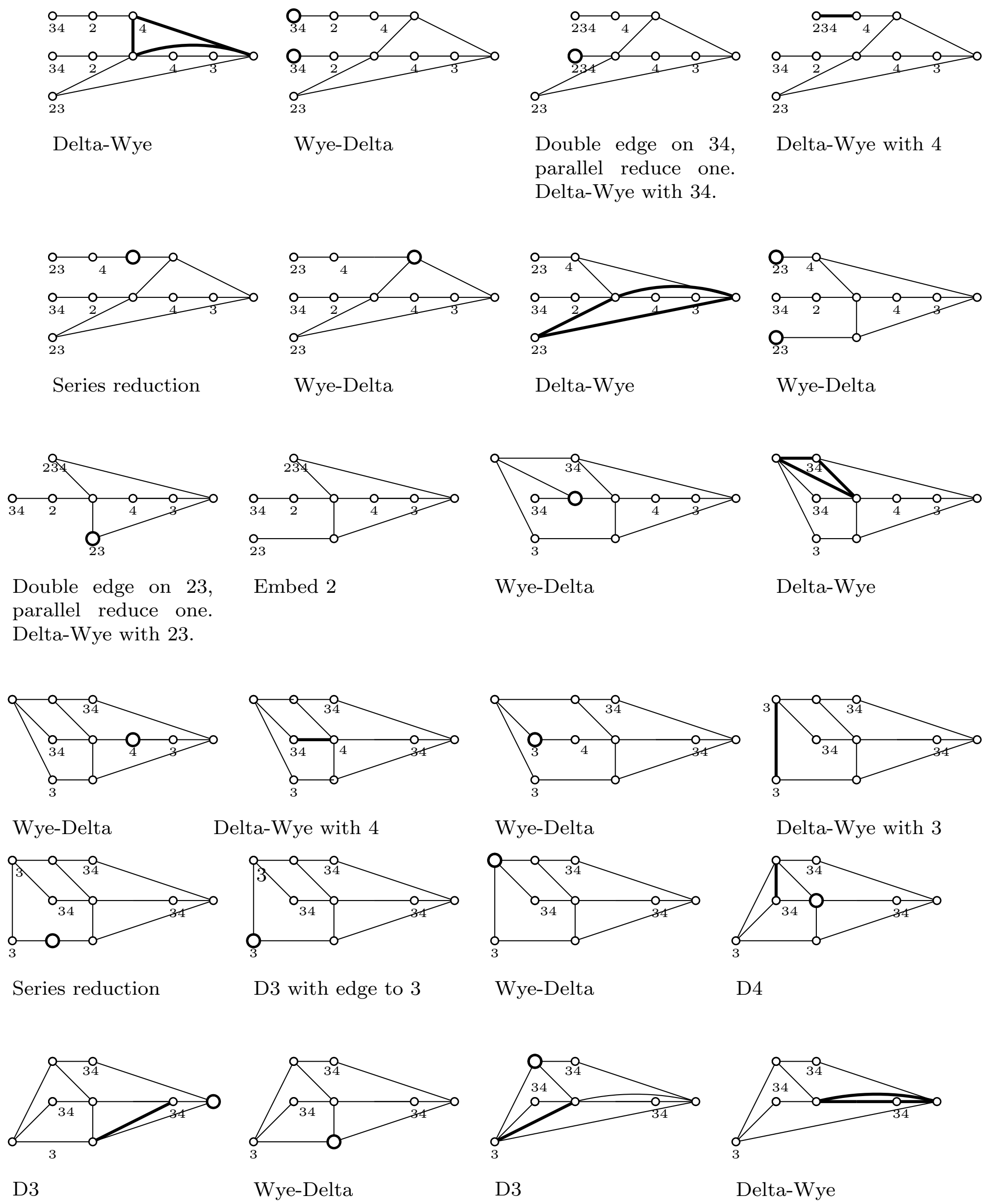


Figure A.19: Starfish Reductions 2

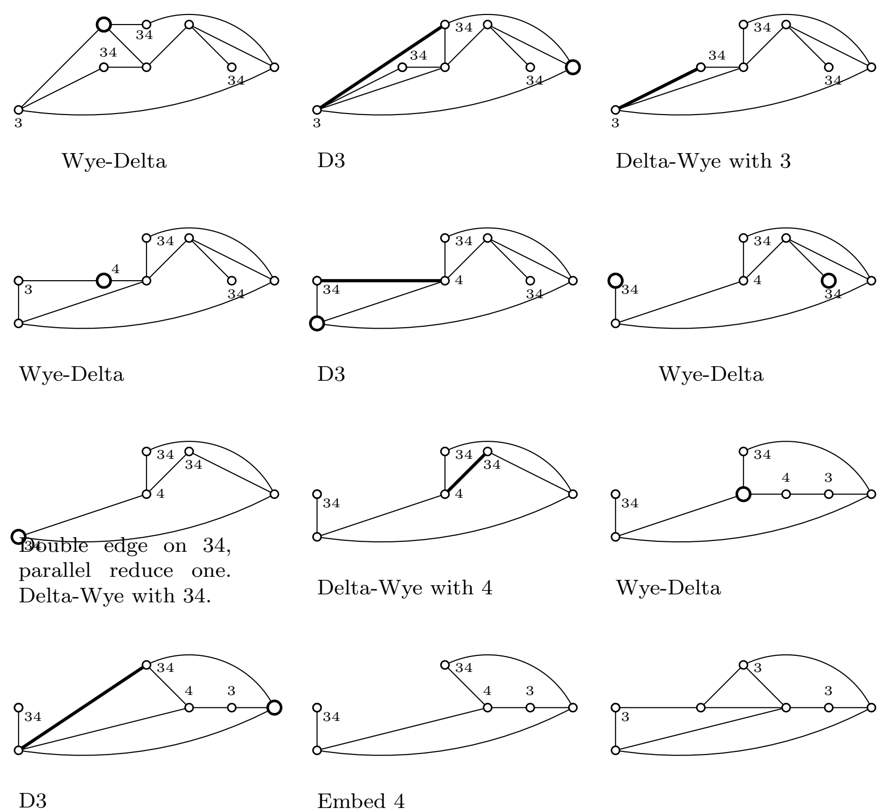


Figure A.20: Starfish Reductions 3

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