COMPRESSION OF SOCIAL NETWORKS

by

Hossein Maserrat
B.Sc., Sharif University of Technology, 1999
M.Sc., Sharif University of Technology, 2002

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APPROVAL

Name: Hossein Maserrat

Degree: Doctor of Philosophy

Title of Thesis: Compression of Social Networks

Examining Committee: Dr. Hao (Richard) Zhang
Chair

Dr. Jian Pei, Professor
Senior Supervisor

Dr. Tom Shermer, Professor
Supervisor

Dr. Binay Bhattacharya, Professor
SFU Examiner

Dr. Shen, Heng Tao, External Examiner Professor,
Information Technology & Electrical Engineering,
University of Queensland

Date Approved: July 18, 2012

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Abstract

Compressing social networks can substantially facilitate mining and advanced analysis of large social networks. Preferably, social networks should be compressed in a way that they still can be queried efficiently without decompression. Arguably, neighbor queries, which search for all neighbors of a query vertex, are the most essential operations on social networks. In this thesis we develop a social network compression framework, which allows efficient in-neighbor and out-neighbor query answering without replicating the data.

As a simple setting of the framework, we introduce the novel Eulerian data structure and prove an upper bound on its bits-per-edge rate. Using the notion of k-linearization, a lossless compression scheme is introduced as a practical compression scheme. The experiments on a dozen real world social networks confirm the effectiveness of our design. Finally, we use our framework to develop a lossy compresson scheme, which aims at capturing the community structure of a given network using a predefined amount of storage. We affirmatively evaluate the lossy compression scheme on both synthesis and real world social networks.

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Chapter 1

Introduction

Real life social networks, thanks to the popularity of World Wide Web, are getting larger and larger, to the extent that fitting them in the main memory is a challenge. The web graph has billions of nodes and tens of billions of edges. Facebook, as a friendship network, has more than 900 million nodes and 125 billion edges[1].

From a practical point of view, most of the social network analysis methods and algorithms implicitly assume the network fits into main memory. Therefore, social network compression indirectly addresses the scalability issues for a wide variety of algorithms.

Roughly speaking, data compression is the process of representing the data using fewer number of bits than trivial representation. Compression is useful in practice because it reduces the consumption of critical resources such as main memory, communication capacity, or cache memory. A compression scheme usually exploits the statistical redundancy of the data to achieve memory efficiency.

For example, in a text corpus, some letters statistically are more frequent than others. Therefore, by assigning shorter codes to the frequent and longer codes to the infrequent letters, one can obtain a better bits-per-letter cost on average. However, the statistical redundancy often significantly depends on the characteristics of the domain of data. As a consequence, the compression scheme has to be designed accordingly.

[1] `http://newsroom.fb.com/content/default.aspx?NewsAreaId=22`

On the other hand, the objective of data compression could raise a variety of requirements. For example, as for data archival purposes, probably the most critical measure is the compression rate. In this case, arguably, for any future process of the archived data a decompression phase is affordable.

As a more realistic scenario, let us assume we want to apply a pattern mining algorithm on a dataset that does not fit into main memory. Although, a compression scheme here could be effective for fitting the data, it has to support a number of efficient basic operations directly on the compressed version of the data depending on the requirements of the algorithm.

Having the advances in distributed computing in mind, one might argue, by using more computers it is always possible to fit a large network in the main memory. Although the argument is somewhat reasonable, it raises another issue. Since the capacity of communication, in a distributed computing platform, is often the main bottleneck, an extremely challenging problem is to partition the network so that the amount of communication between different computing units is minimum.

From this point of view, a compact representation of the network could be beneficial not only because it reduces the number of computing units, but also it implicitly increases the utility of the communication capacity. A compact representation often encodes the correlated nodes (edges) together. Therefore, a better compression method potentially could lead to a better utilization of the locality of the data.

To answer queries regarding the correlation of the nodes and edges in a social network, an expensive link analysis usually is required. For example, given a few nodes of a network, to find out if they belong to a dense subgraph, one has to apply an extensive link analysis method, not only on the given vertices but also on the neighbors of them. However, potentially, a compression method can provide implicit information to facilitate answering such queries.

In order to use the existing social network analysis algorithms on a compressed network, the compression framework has to often support both efficient in-neighbor [2] and out-neighbor [3] queries. Depending on the application, we might also want to perform efficient incremental updates directly on the compressed network. The focus of this dissertation is to develop a compression framework for social networks that addresses these additional requirements. We refer to such a compression framework as a General Purpose Social Network (GPSN) compression framework. To the best of our knowledge there is no compression method for social network in the literature that qualifies as a GPSN compression framework.

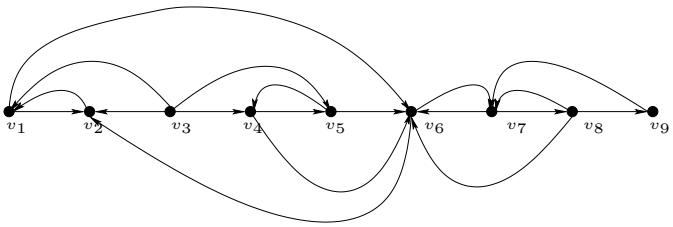
For the rest of this chapter, we present a high level overview of the major results of the dissertation. In Section 1.1, we illustrate the core idea for the General Purpose Social Network (GPSN) compression framework. Section 1.2 discusses an upper-bound on bits-per-edge rate for a simple setting of the framework, which we refer to it as Eulerain data structure. In Sections 1.3 and 1.4 we review two lossless and lossy compression schemes, respectively.

1.1 GPSN Compression Framework

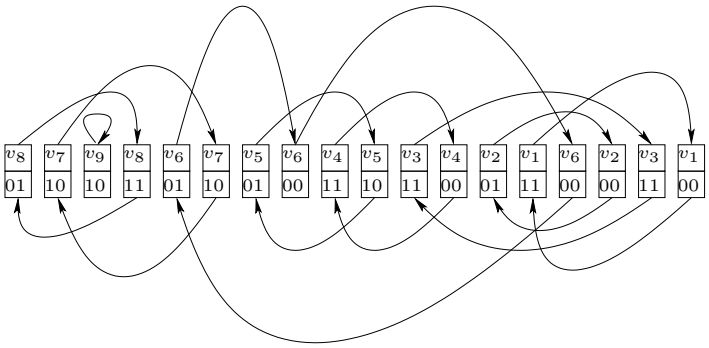
For the purpose of discussion, let us assume for a graph G, we can find an ordering of the nodes in which all the edges have length of at most k (that is all the edges connecting two nodes that are at most k positions away in the ordering, k is a small integer). Then, we can encode the edges that connect the position i to the positions $i+1, i+2, \cdots, i+k$ using k bits. Another k bits would be necessary to represent the edges from i to $i-1, \cdots, i-k$. In this scenario, we can save memory assuming k is small enough relative to the average degree. Note that finding such an ordering may not be possible for a real world social network. However, we can relax the requirements and instead of an ordering, ask for a sequence of the nodes (with possible replication) in which all the edges connect two nodes that are at

[2] An in-neighbor query, given a node u, asks for all nodes v such that there is an edge from v to u.

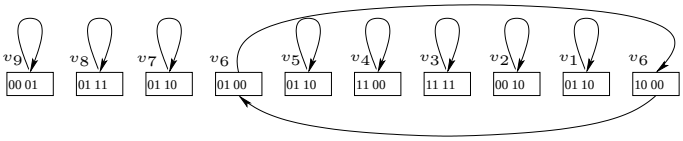
[3] An out-neighbor query, given a node u, asks for all nodes v such that there is an edge from u to v.



(a) Graph G



(b) A representation of G using a sequence of nodes in which all the edges connect two nodes next to each other.



(c) A representation of G using a sequence of nodes in which all the edges connect two nodes at most two positions away.

Figure 1.1: The illustration of GPSN Framework. Refer to the text for more details.

most k positions away from one another. Each copy of the node u in this new sequence is responsible for representing a subset of edges incident to u. Thus, for each copy of u in the sequence we need to keep $2k$ bits to encode the edges associated to that copy and a pointer to the next copy of u.

Figure 1.1(a) displays a graph G, and Figure 1.1(b) illustrates a sequence of the nodes where for any edge in G, there are two nodes next to each other in the sequence. A 2-bits label represents the edges for each position. For example, the label 01 indicates the respected position does not have any edge going to its left position, and it does have an edge

going to its right node. The arcs are pointers to the next copy of the same node. Figure 1.1(c) illustrates a sequence in which the maximum length of all edges is 2. The 4-bits labels inside the box encode the edges and the arcs are pointers to the next copy of the same node. If there is only one copy of the node in the sequence, the pointer points to itself.

Let us assume we can find a sequence S of length l, in which all the edges are connecting two nodes at most k positions away from one another. In such a case, the network can be represented using exactly $l \times (2k + \lceil \log_2 l \rceil)$ bits; we need $\lceil \log_2 l \rceil$ to encode the next pointer and $2k$ bits to encode the associated edges for each position. This framework has several benefits. First, similar to the order-based methods it reduces the social network compression to an elegant combinatorial problem. Second, it allows efficient incremental updates, in-neighbor and out-neighbor queries. Third, intuitively in the sequence S, we expect the nodes that belong to a dense subgraph appear close to each other, and if a node is part of several (overlapping) dense subgraphs, the node would appear several times in the sequence.

To have a solid compression scheme within the GPSN framework, we need to develop two modules: (1) a linearization algorithm/heuristic that given a network, outputs a sequence of the nodes, and (2) an encoding protocol to encode the edges associated to each position of the sequence.

As an example, we will formulate a trivial representation schema using the principles of GPSN framework. Let us assume that the linearization algorithm enumerates all the edges in arbitrary order, and represents each edge by its source and destination. This would produce a sequence of length $2|E|$ of the nodes. Then, precisely there is one edge associated to each position; each edge connects a position with an even index to its next position. Therefore, the parity of the index of the positions is enough to recover the edges of the network. The representation scheme merely stores a pointer to the next appearance of the same node for each position. In total, the scheme uses exactly $2|E|(\lceil \log_2(|E|) \rceil + 1)$ bits. Note that, this is comparable to the adjacency list representation where each node stores both lists of incoming and outgoing edges.

Note that to be able to compare the results with previous work in the literature, similar to [8, 9, 14], we assume: (1) It is possible to assign new identifiers to the vertices. (2) The data structure is not responsible for maintaining the mapping between the old and new identifiers. (3) Finally, there are no labels for edges. Please note that we can straightforwardly extend the data structure to remove the above assumptions.

Contribution. We introduce the novel GPSN compression framework. Our method for compressing social networks, comparing to the state-of-the-art methods, has the advantage of supporting efficient in-neighbor[4], out-neighbor queries[5], and incremental updates[6]. The GPSN framework exploits a sequence S, in which the nodes that belong to a dense subgraph of a given network, appears in proximate positions to achieve compression.

1.2 Eulerian Data Structure

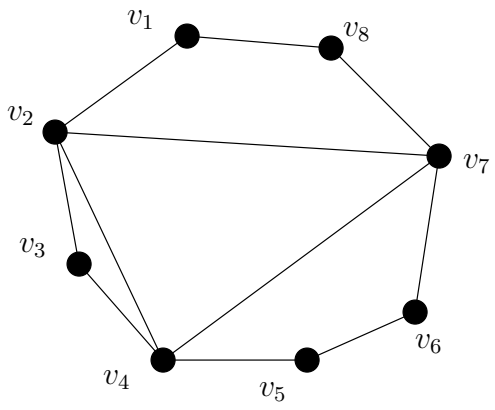
A k-linearization of a given graph G is a sequence of nodes in which all the edges have a length at most k. In Chapter 3, we show that finding the optimal 1-linearization is possible in linear time. Note that any graph would have an optimal 1-linearization regardless of the existence of the Eulerian path. Constructing such a linearization is, in fact, equivalent to decomposing the graph into the minimum number of edge-disjoint paths such that each edge appears exactly once in one of the paths. However, the problem of finding an optimal k-linearization, when k is given as part of the input, is NP-Hard.

The **Eulerian data structure** for a graph G stores an optimal 1-linearization L of G using an array of the same length as L. Let $v(i)$ be the vertex in G that appears at the position i of L. For the cell i of the array, we keep the following two pieces of information: (1) Edge Information: two bits specifying if edges $(v(i-1), v(i))$ and $(v(i), v(i-1))$ belong to $e(G)$, respectively. (2) Pointer: a pointer to the next appearance of $v(i)$, if this is the

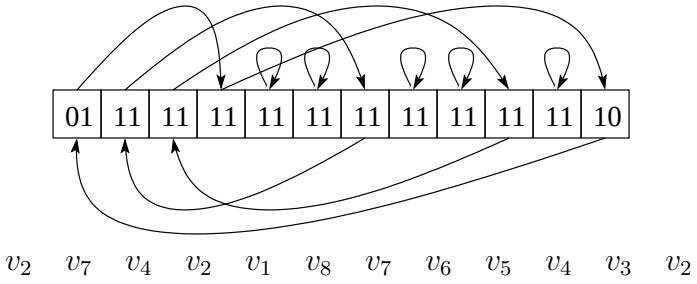
[4]Given the node v in a graph G, in-neighbor query asks for all the nodes u such that (u, v) be an edge G

[5]Given the node v in a graph G, out-neighbor query asks for all the nodes u such that (v, u) be an edge G

[6]Incremental update is the operation of applying the changes directly on the compressed version of the graph



(a) Graph G



(b) Illustration of the 1-linearization of G and the Eulerian data structure (pointers are depicted by arcs).

01, 3	11, 6	11, 9	11,11	11, 4	11, 5	11, 1	11, 7	11, 8	11, 2	11,10	01, 0
0	1	2	3	4	5	6	7	8	9	10	11
v_2	v_7	v_4		v_1	v_8		v_6	v_5		v_3	

(c) The Eulerian data structure. Each cell stores 2-bits label and a pointer to next appearance of the same node. The index of the first appearance of a node is the new identifier of that node.

Figure 1.2: The Eulerian data structure. The new identifiers of the nodes are illustrated.

last appearance of $v(i)$, then the pointer points to the first appearance of the vertex.

Example 1 (Eulerian Data Structure)**.** *In figure 1.2, the Eulerian Data Structure for graph G is depicted. The graph is undirected; therefore, each edge is actually equivalent to two directed edges. The number of directed edges for G is* 22*. The length of the optimal* 1*-linearization is* 12*, therefore, a pointer takes* 4 *bits. For each position we need* $4+2$ *bits. In total,* $12 \times (4+2) = 82$ *bits is necessary to encode G. The compression rate is* $82/22 \approx 3.72$ *bits per edge.*

Theorem 1, in Chapter 3, states an upper-bound on the compression rate of the Eulerian data structure: an Eulerian data structure, to encode a graph G, uses up to

$$(1 - \frac{fre(G)}{2} + \frac{1}{\bar{d}})\big(\lceil \log_2(|V(G)|) + \log_2(\bar{d}+1)\rceil + 1\big),$$

bits per edge on average, where $\bar{d}$ is the average degree of G, and $fre(G)$ is the fraction of reciprocal edges. The edge $(u, v) \in E(G)$ is a reciprocal edge if and only if $(v, u) \in E(G)$.

Contribution. As a simple setting of the GPSN framework, we introduce the novel Eulerian data structure, and a nontrivial upper bound on its bits-per-edge rate is provided.

1.3 Lossless Compression Scheme

The real world networks, in many cases, exhibit some kind of locality property which can be used to further improve our method. The notion of k-linearization introduced in Chapter 3 can be used for lossless compression.

As a lossless compression scheme, we can extend the Eulerian data structure by using k-linearization instead of 1-linearization. However, comparing to 1-linearization, finding an optimal, or even near optimal, k-linearization is challenging.

Example 2. *Figures 1.1(b) and (c) depict two possible encoding of graph G using* 1*-linearization and* 2*-linearization, respectively. Using the* 1*-linearization, we need to use* 2 *bits to store the local information for each position. Since in this case there are* 18 *positions, the pointer uses* 5 *bits. In total we need* $18 \times (5+2) = 126$ *bits, approximately* 6.63 *bits per edge on average.*

Using the 2*-linearization, we need to use* 4 *bits to store the local information for each position. Since the linearization has* 10 *positions, each pointer takes* 4 *bits. In total it uses* $10 \times (4 + 4) = 80$ *bits, approximately* 4.21 *bits per edge. The saving of using the* MP_2 *linearization is substantial.*

Can we save more by moving from 2*-linearization to* 3*-linearization? The length of an* k*-linearization for any* k *cannot be less that the number of vertices in the graph.* G *has* 9 *vertices. Therefore, in the best case of using an* 3*-linearization, the length of the sequence is* 9*. For each position we have to use* 6 *bits to store the local information and* 4 *bits to encode the pointer. Thus, using an* 3*-linearization takes at least* $9 \times (6 + 4) = 90$ *bits. Using* 3*-linearization instead of* 2*-linearization cannot improve the compression rate.*

To be more adaptive, we extent the notion of k-linearization to adapt the specific properties of social networks. Multi-level linearization uses several value of k to linearize a network. The formal definition can be found in Chapter 4. The heuristic to build the multi level linearization is as follows: we start with a random vertex and a starting value for k. At each step we append to the list, the vertex that has the largest number of edges with the last k nodes of the list. We remove these edges from the graph and iterate until no edge is left. If none of the last k vertices of the list have a neighbor, we pick a random node with non-zero degree and continue from there.

As we are removing the edges of the graph, it becomes sparser and sparser and eventually the rear part of the linearization may have very few new edges to encode. To be adaptive, we watch the average density for the recent positions in the list (the last 1000 positions), once it drops below a certain density threshold, we reduce k by multiplying it to a predefined reducing factor.

For experimental study, we used the data sets from the SNAP project (Stanford Network Analysis Package, `http://snap.stanford.edu/data/`). These data sets are from very different domains, such as friendship networks, web graphs, peer-to-peer networks, collaboration networks, citation networks, and co-purchasing networks.

Comparing to results reported in [14], our method has slightly better compression rate.

However, the real advantage of our lossless compression comes from the ability of answering both in-neighbor and out-neighbor queries without replicating the data. More experimental results are provided in Chapter 4.

Contribution. The lossless compression scheme is a practical setting of the GPSN framework, in which a notion of multi-level linearization is utilized. We introduce a greedy heuristic to build a "good" multi-level linearization for a given graph. An extensive set of experiments validate our design.

1.4 Lossy Compression Schema

Although, there are a few pioneering studies on social network compression, they only focus on lossless approaches. In Chapter 5, we tackle the novel problem of lossy compression of social networks. The trade-off between memory and "information-preserved" in a lossy compression presents an interesting angle for social network analysis, and at the same time makes the problem very challenging. We propose a lossy graph compression approach, based on our GPSN framework. We do so by designing an objective function to measure the quality of a given linearization with the respect of community structure of the network. We present an interesting and practically effective greedy algorithm to optimize such an objective function. Our experimental results on both real data sets and synthetic data sets demonstrate the promise of our method. However, there is no theoretical analysis to guarantee the performance of our heuristic.

From the practical point of view, the noise edges and vertices in large social networks do not help in analysis of the network. Instead, they may compromise the quality of social network analysis. An appropriate lossy compression of a social network can discard the noise edges and vertices in the network. Consequently, the lossy compression may be presented as a higher quality input for social network analysis. In other words, lossy compression of social networks can serve as a preprocessing step in social network analysis. This goes far beyond just saving space. The experimental study in Chapter 5 verifies our claim.

The idea of representing the edges of a network by a sequence of vertices naturally

inspires a lossy compression scheme. To explain such a scheme, a dual definition of k-linearization could prove to be useful. The definition of k-covered edges is presented in Section 5.3. The intuition is as follows: a subset $A \subseteq E(G)$ is k-covered by a sequence S of the nodes of a given graph G, if for all $e \in A$, the length of e according to S is less than or equal to k. $E_k(S)$ would be the largest set that is k-covered by S.

The setting for the lossy compression problem is as follows: given two parameters k and l, find a sequence S of nodes with length l such that $E_k(S)$ captures the community structure of the network. Since communities are the building blocks of a social network, such a scheme would cover the majority of the edges of the network assuming reasonable values for k and l.

We design an objective function to measure how well a given sequence S captures the community structure of a social network. A critical issue would be to design such an objective function that is sensitive to the loss of community structure. Instead of developing a utility function parameterized by k, we consider a utility function that optimizes for edges of short spans in the corresponding sequence S. For this purpose, we introduce a parameter α $(0 < \alpha < 1)$ that controls the preference for shorter spans. We build the connection between parameter α and parameter k in Section 5.4. The objective function is as follows:

$$\bar{f}(S, G) = \sum_{1 \leq i \leq length(S)} (\sum_{e \in E_i} \alpha^{span(e)})^2, \tag{1.1}$$

where E_i is the set of edges associated to the position i of the sequence S, and $span(e)$ is the stretch of edge e according to sequence S. Sections 5.3 and 5.4 discuss the objective function and the reasoning behind it in details.

A critical question is that what would the lost of information in lossy compression scheme do to the community structure of the network. A set of experiments introduced in Section 5.5 aims to evaluate the effect of lossy compression on the community structure of the network. Chapter 5 provides detailed results on real world and synthesis networks.

Contribution. Our lossy compression scheme is designed to capture the global community structure of social networks using a predefined amount of storage. We introduce an objective function which quantifies how well a sequence S captures the community structure

of a given network. Our objective function turns the problem of lossy compression to an optimization problem. For optimizing such an objective function we develop a local search heuristic. Finally, we evaluate our method on both synthesis and real world networks.

1.5 Structure of the Thesis

In this chapter, we provide a high level and intuitive introduction of the core idea of the GPSN compression framework. Chapter 2 reviews the related work. Then, we justify the design of our GPSN framework by mentioning three solid compression schemes, designed base on the principles of this framework. The first scheme, the Eulerian data structure, is one of the simplest settings for which the framework is not trivial. The simplicity in this case allows us to prove a theoretical upper-bound on the compression rate. The detailed proof of the upper bound can be found in Chapter 3. A lossless compression scheme is discussed in details in Chapter 4. For the third and last study, we present an overview of the design of the lossy compression scheme. Chapter 5 discusses the lossy compression scheme in details. We conclude the thesis in Chapter 6.

Chapter 2

Related Work

In the last decade, World Wide Web [3] has been the focus of much research, both from academic and industrial perspectives. Among other things, it has been a diverse and incredibly huge source of data. Particularly, it is rich with different types of social networks. In this thesis, a social network is a graph which models the binary relations among a certain set of objects. The nodes of the graph represent objects and the edges represent the binary relations. Often, relations come from a social phenomenon. A binary relation could be a one way or two ways relation.

For example, in a *friendship network,* a node is a person and an undirected edge is representing the friendship between two persons. The *web graph* is an abstraction for the web pages (URLs) and the hyper links among them. The nodes are representing web pages. There is a directed edge from web page x to web page y, if there is a hyper link from x to y. Likewise, a *collaboration network* typically models a set of individuals and the collaborative relationship among them.

Social networks have been exploited in many different applications, such as search engines, epidemic analysis for diseases, modeling information and behaviour cascades [45, 21]. However, the huge amount of information always comes with the scalability challenges, in both time and memory aspects. In this chapter, we focus on the literature related to compression of social networks.

2.1 Categorization of Related Work

Compressibility is highly related to the amount of "regularity" in a network. It is a well known fact that social networks are rich with dense subgraphs. Depending on the type of network, these dense subgraphs might have different meanings. For example, in a friendship network a dense subgraph is usually corresponding to a community of friends, while in a web graph, a dense subgraph could be corresponding to several pages making an online manual for a certain product. Likewise, a dense bipartite subgraph in a web graph could be corresponding to a set of "survey" web pages that link to the same set of "reference" web pages.

All the compression methods for social networks, at least implicitly, exploit the existence of such dense subgraphs to achieve storage efficiency. We organize the compression methods in the two categories of the aggregation-based and the order-based approaches. The idea of compression in the aggregation-based approach is to reduce the number of nodes or edges, by aggregating a set of edges (or nodes) into one super-edge (or super-node). In these methods, the compression rate is the ratio of the number of edges in the compressed network over the number of edges in the original network. In the order-based methods, the saving comes from using fewer number of bits to encode a particular edge. The compression rate for these methods is the average number of bits that is necessary to encode an edge. To compare the compression rate of the two approaches sometimes we need to have an assumption for the size of an identifier of a node. In those cases, we assume an identifier is encoded by a 32-bits integer.

Having the four criteria of compression rate, in-neighbor and out-neighbor queries, and incremental update efficiency in mind we review and compare compression methods in the literature. Unfortunately comparing the compression rate is not always trivial for the following reasons: first, different methods are using different criteria as the compression rate, and second, the performance of different methods are reported on different datasets, which may not be publicly available.

Often, the compression method reassigns new identifiers to the nodes of the network. In

the scenarios that the original identifiers are informative, storing a two-way mapping from the original identifiers to the new ones is necessary. Consistent with the literature [41, 8, 14], to evaluate the compression rate we ignore the cost of this mapping. This is a reasonable assumption because the size of such a mapping is proportional to the number of nodes, and the number of nodes is often much smaller compared to the number of edges. As the last remark, all the compression methods that we describe in this thesis assume the input is a simple directed graph (i.e., a directed graph without multiple edges).

We categorize the literature related to the compression methods in the following four categories:

- **The aggregation-based compression approach (Section 2.2):** The general idea of this approach is to find a set of nodes, that have similar neighbors and replace them by a single super-node. Likewise, a super-edge represents a set of edges [23, 17]. Raghavan and Garcia-Molina [50] recursively used this idea to decompose a web graph into a hierarchical structure. They introduced the notion of S-node to aggregate several nodes of a web graph into one super-node. Similar to this work, the method of Navlakha *et al.* [41] represents a social network by a summary graph S and a correction set C. We will explain this method as an example of aggregation-based approach in details. Buehrer and Chellapilla [11] used a different aggregating idea to compress web graphs. However, their approach does not replace any node in the web graph, rather they find semi-complete bipartite subgraphs[1] and aggregate the edges by introducing a virtual node. The challenge in their approach is how to mine the directed semi-complete bipartite subgraph. We explain the idea of their approach in Section 2.2, and leave the mining technique for Section 2.4.

- **The order-based compression approach (Section 2.3):** In the order-based approach, finding a "good" ordering of the nodes is essential. Intuitively, a "good" ordering is an ordering in which (1) the nodes with similar neighbors are close to each

[1] A semi-complete bipartite subgraph is a directed bipartite graph in which all the possible directed edges from one (source) partition to the other (destination) partition exist.

other, and (2) the nodes that belong to a dense subgraph, fall in proximate positions [8, 6, 7]. The existing order-based methods either use external information to establish the ordering (e.g. lexical ordering of URLs) or they rely on a heuristic to produce it.

- **Clustering and dense subgraph mining (Section 2.4):** Dense subgraph mining and clustering in the social networks are relevant to compression schemes. Often, the intuition and the design of the compression scheme determines what kind of mining or clustering technique should be used. We discuss a semi-complete bipartite mining technique as a part of an aggregation-based compression method [11]. As another example, we explain a clustering technique [55] that has been used for reordering the nodes of a network, in an order-based compression method [6]. Later, we present a structural clustering method [63] that partitions the nodes based on the similarity of their neighbors. This is particularly interesting, because both aggregation-based and order-based approaches exploit the existence of the nodes with similar neighbors to achieve compression. Finally, we discuss the spectral clustering method [44, 38, 65, 64], a major clustering technique that has the advantages of being scalable and having a solid theoretical foundation. The spectral clustering method as a preprocessing step embeds the nodes of the graph into a low-dimensional space. Likewise, the order-based approach also as a preprocessing step needs to embed the nodes into a one-dimensional space (an ordering of the nodes can be considered as an embedding of the nodes into a line).

- **Graph layout problems (Section 2.5):** Order-based methods rely on an ordering of the nodes with two specific properties. A possible approach to find such a "good" ordering is to design an objective function that encapsulate the desired properties of a "good" ordering. The challenge in this direction has two folds: the design of the objective function and the algorithm to optimize such a function. A family of combinatorial problems, known as graph layout problems have a long history in the graph theory discipline [19, 5, 53]. The goal in these problems is to find an ordering of the nodes that minimizes/maximizes a given objective function. In the classic variants

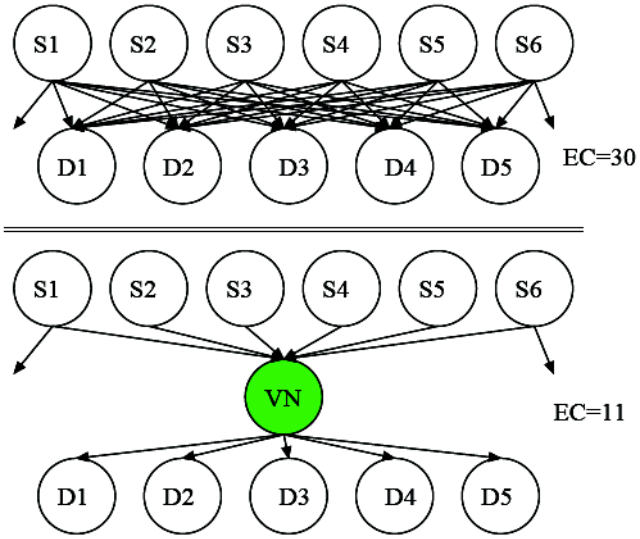


Figure 2.1: Reprinted from [11]. The top graph is an example of a semi-complete bipartite subgraph. The example illustrates the idea of aggregating a set of directed edges by introducing a virtual node.

of graph layout problem the objective functions are motivated by applications such as VLSI circuit design, numerical analysis and scheduling [19]. Even though these objective functions do not have an explicit connection to the compression problem, studying them would shed light on the nature and the challenges of this approach.

2.2 The Aggregation-Based Approach

In this section, we discuss two aggregation-based methods: (1) the Virtual Node Compression scheme [11], and (2) the Summary Graph Compression Scheme [41].

2.2.1 The Virtual Node Compression Scheme

The idea of using semi-complete bipartite subgraph for network compression is illustrated in Figure 2.1. A set of directed edges, from the source nodes S_i $(i = 1, \ldots, 6)$ to the destination nodes D_i $(i = 1, \ldots 5)$, are encoded by introducing a virtual node VN, using one level of

indirection. In the original graph, there are $5 \times 6 = 30$ edges. However, after introducing the virtual node the number of edges reduces to $5 + 6 = 11$. The decrease in the number of edges is $30 - 11$.

Generally, if we have a sources and b destinations, the decrease in the number of edges is $ab - a - b$. The effectiveness of the method highly depends on the existence of large semi-complete bipartite subgraphs. This is particularly an intuitive idea for Web graph. The set of source nodes in a semi-complete bipartite subgraph could be corresponding to a set of web pages on the same topic which all of them have references to a small set of "reference" pages. For example, considering the articles on the coming US presidential election, an statistically significant fraction of them would be sharing references to Wikipedia pages of Barack Obama, Mitt Romney, Republican party, Democratic Party, and more. Section 2.4.1 explains the method for mining such subgraphs.

2.2.2 The Summary Graph Compression Scheme

In Figure 2.2, graph G (left) and its compressed representation (right) are depicted. For simplicity, the method is illustrated using an undirected graph, however it is trivially extendible to directed case. The compressed representation consists of a summary graph $S = (V_S, E_S)$ and a set of edge corrections C. In the summary graph S, each node is corresponding to a set of nodes in the original graph G, e.g. d, e and f are replaced with z in the summary graph. Edge (z, y) in the summary graph represents six edges in the original graph, namely: $(h, d), (h, e), (h, f), (g, f), (g, e)$ and (g, d). Observe that according to summary graph, (g, d) is an edge in the original graph, which is not the case In order to be able to correct such mistakes, a correction set C is maintained. In this case C contains correction $-(g, d)$, which means after recovering the original graph, (g, d) has to be removed. Likewise $+(a, e)$ means, after recovery, edge (a, e) has to be added to the graph. In addition to the summary graph and the set of corrections, the compression scheme maintains a mapping from the nodes of the original graph to the nodes of super-nodes of the summary graph.

In the next two subsections, we explain two algorithms for computing a compressed

representation (S, C), where S is a summary graph and C is a set of corrections. The objective is to find a representation (S, C) such that the sum of $|E_S| + |C|$ is minimized. Before explaining the algorithms, notice that the edge set E_S and the correction set C can be optimally determined based on the super-nodes in V_S.

Assuming V_S is fixed, we show how to compute E_S and C. Let v and u be two super-nodes in V_S. We define Π_{uv} as the set of all pairs (a, b) such that $a \in A_u$ and $b \in A_v$, where A_u and A_v are the two set of nodes in the original graph, that are replaced by super-nodes u and v, respectively. Observe that some of the pairs in Π_{uv} might not be an edge in the original graph. Let $A_{uv} \subseteq \Pi_{uv}$ be the set of all those pairs that are also an edge of the original graph. Now, for encoding the edges in A_{uv} there are two possibilities: (1) add (u, v) to the summary graph E_S and $\Pi_{uv} - A_{uv}$ to the correction set C with the negative sign. (2) add A_{uv} to C with the positive sign. The cost would be $|\Pi_{uv}| - |A_{uv}| + 1$ for the former case, and $|A_{uv}|$ in the case of latter. Therefore, one must add the super-edge (u, v) if and only if $|A_{uv}| > (|\Pi_{uv}| + 1)/2$. Then, the corrections have to be chosen accordingly. We can conclude that the task here is to find a good partitioning of the nodes in the original graph into super-nodes.

The Greedy Algorithm

The greedy algorithm starts with having each node of the original graph as a super-node ($V_S = V_G$). Then, iteratively it tries to merge the super-nodes, based on a cost reduction function $s(u, v)$, until no improvement is possible. We define a neighborhood N_v for each super-node v as follows:

$$N_v = \{u \in V_S | \exists a \in A_v \wedge \exists b \in A_u \rightarrow (a, b) \in E_G\}$$

In other words, u is in neighborhood of v if and only if for some node $a \in A_v$ and $b \in A_u$, edge (a, b) belongs to E_G. Given two super-nodes u and v, let c_{vu} be the cost of encoding the edges in A_{uv}:

$$c_{vu} = \min\{|\Pi_{uv}| - |A_{uv}| + 1, |A_{uv}|\}$$

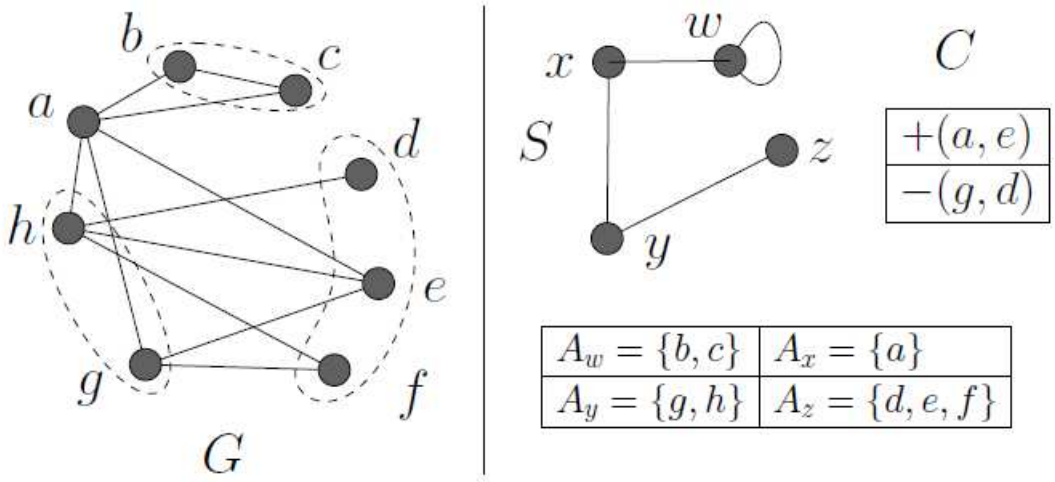


Figure 2.2: Reprinted from [41]. Graph G (left) is the original graph and its compressed representation (right). For simplicity the idea of the method is illustrated on an undirected graph.

Notice that c_{vu} for node u that does not belong to N_v is equal to zero (since $|A_{vu}|$ is equal to zero). Also we define:

$$c_v = \sum_{u \in N_v} c_{vu},$$

where, c_v is the cost of encoding all the edges incident to v. Now, the cost reduction of merging two super-nodes u and v into super-node w is defined as follows:

$$s(u, v) = \frac{c_u + c_v - c_w}{c_u + c_v}$$

The algorithm has three different phases: initialization, iterative merging and output. As the initialization, V_S is set to be V_G, and cost reduction $s(.)$ would be computed for all the pairs that are at most 2 hops away. Note that, if two nodes are more than 2 hops away, they do not share any neighbors in the original graph, and therefore merging them would not have any benefit. A standard max heap structure keeps all the pairs (u, v) with a positive $s(u, v)$.

During the merging phase, iteratively the pair (x, y) with maximum $s(.)$ is chosen. Then, the super-nodes x and y have to be removed from V_S, and merged into the super-node w. Finally, w has to be added to V_S. Notice that as a side effect of the merging operation the value of $s(.)$ for some pairs might change. Precisely, those pairs that contain x, y or any neighbor of them have to be considered for cost reduction revaluation. Algorithm 1 is the pseudo code for this heuristic.

The Randomized Algorithm

Since recomputing the cost reduction function for all the affected pairs of super-nodes in each step of greedy algorithm is expensive, the randomized algorithm is proposed to reduce the time complexity. Similar to greedy algorithm, the heuristic initializes V_S to V_G. Then, it iteratively merge nodes to form the final set of super-nodes. These super-nodes are divided into two categories, unfinished (U) and finished (F). The algorithm starts with all nodes in the U category, and ends when all the nodes are in the F category.

In each iteration a node u, uniformly at random, is chosen from the U category. The reduction cost of this node with respect to all nodes in its 2 hop neighborhood is computed. Let v be the node such that $s(v, u)$ is the largest. If merging u and v gives a positive reduction, they should be merged into super-node w, otherwise we add u to the F category. Algorithm 2 is the pseudo code for the randomized heuristic.

2.2.3 Discussion

The Virtual Node scheme uses an adjacency list table to store the compressed graph, therefore incremental updates and out-neighbor queries are efficient. However, the in-neighbor queries are not efficient, without replicating the data. Also, since the method uses a fast heuristic for mining the semi-complete bipartite subgraph, it is scalable. A disadvantage of the method is that the effectiveness, from the compression-rate point of view, depends on the existence of semi-complete bipartite subgraphs. As it is clear from the experiments in [11], web graphs tend to have relatively large semi-complete bipartite subgraphs. The

Algorithm 1 Reprinted from [41]. The pseudo code for three phases of greedy algorithm for computing the summarization graph G_S and correction set C. The algorithm consisted of three phases: Initialization, Iterative merging and Output.

```
/*Initialization phase*/
V_S = V_G; H = {};
for all u, v ∈ V_S that are 2 hops apart do
  if s(u, v) > 0 then
    insert (u, v, s(u, v)) into H;
  end if
end for
/*Iterative merging phase*/
while H ≠ {} do
  Choose pair (u, v) ∈ H with the largest s(u, v) value;
  w = u ∪ v; /* merge super-nodes u and v */
  V_S = V_S − {u, v} ∪ {w};
  for all x ∈ V_S that are within 2 hops of u or v do
    Delete (u, x) and (v, x) from H;
    if s(w, x) > 0 then
      insert (w, x, s(w, x)) into H;
    end if
  end for
  for all x, y ∈ V_S such that x or y is in N_w do
    Delete (x, y) from H;
    if s(x, y) > 0 then
      insert (x, y, s(x, y)) into H;
    end if
  end for
end while
/*Output phase*/
E_S = C = {};
for all u, v ∈ V_S do
  if |A_uv| > (|Π_uv| + 1)/2 then
    Add (u, v) to E_S;
    Add −(a, b) to C for all (a, b) ∈ Π_uv − A_uv;
  else
    Add +(a, b) to C for all (a, b) ∈ A_uv;
  end if
end for
return representation R = (S = (V_S, E_S), C);
```

Algorithm 2 Reprinted from [41]. Pseudo code for the randomized heuristic.

```
U = V_S = V_G; F = {};
while U ≠ {} do
  Pick a node u randomly from U;
  Find the node v with the largest value of s(u, v) within two hops of u;
  if s(u, v) > 0 then
    w = u ∪ v;
    U = U − {u, v} ∪ {w};
    V_S = V_S − {u, v} ∪ {w};
  else
    Remove u from U and put it in F;
  end if
end while
```

compression rate for the reported web graphs is between 0.15-0.35, that is the number of edges of the compressed network over the original graph is between 0.15-0.35. To best of our knowledge, there is no result available concerning other types of social networks.

In the case of the summary graph scheme [41], the summary graph S and the correction set C, both can be represented as an adjacency list table. For correction set C, a label from $\{+, -\}$ should be stored for each edge. Therefore, it can be shown that the average running time of the out-neighbor query is $O(d_{avg})$, where d_{avg} is the average degree. The incremental updates naively can be done in $O(1)$, however, it compromises the compression rate. As for the compression running time there is no precise analysis. We believe the merging operation in this method is quite expensive and it is likely that a large portion of the network has to be processed for a single merge operation. As an evidence for our claim, the largest network in the dataset collection of [11] has about 600 thousand edges. The compression rate for the only web graph in the dataset collection is about 0.20, while, in the case of friendship network, it is about 0.80.

An advantage of the aggregation-based approach is that as a side product, they produce a summarization of the network. The summary graph, as it is illustrated in [41], can be used for visualisation purposes, and/or to get a general idea of the global structure of the network. However, the disadvantage is that unless the network has really dense subgraphs, the method is not effective. Precisely speaking, we can show the existence of subgraphs

with a density strictly higher than 1/2 is necessary, and for substantial saving we need a density close to one. Comparing the performances of the two compression methods could be informative. First, the excellent compression rate of Virtual Node scheme suggests that the web graphs have relatively large semi-complete bipartite subgraphs. We believe, this is the reason that the summary graph scheme also works well on the web graphs. At the same time, the poor performance of the summary graph scheme on the friendship networks is an evidence for the lack of relatively large very dense subgraphs.

2.3 Order-Based Approach

Methods in this category rely on a particular ordering of the nodes in which similar nodes, i.e., nodes with the similar set of neighbors, fall in the proximate positions. The ordering might come as an external information [8], or is computed using the link analysis of the graph [14, 6]. In either case, the ordering should have the following two properties:

- Closeness Similarity: the proximate nodes tend to have similar neighbors.
- Edge Locality: the edges of the graph tend to connect a pair of "close by" nodes.

Next, we explain the major ideas of the WebGraph framework [8, 6, 7, 14] that exploits the lexicographical ordering of the URLs to achieve impressive compression rate on web graphs. It is also worth mentioning that WebGraph uses and improves ideas from the Connectivity Server [4] and the LINK database [52].

2.3.1 WebGraph Compression Framework

There are two critical components for the WebGraph framework:

- ζ *codes* [9]: this is a family of codes, particularly designed for storing integers that come from a power law distribution[2] in a certain exponent range. Unlike fixed length

[2] A power law distribution is a distribution whose density function (mass function in the discrete case) $p(x)$ is proportional to $Lx^{-\alpha}$, where L is a constant and α is the exponent.

code, the length of the code to represent a given number x depends on the value of x. A ζ code to represent a number x would use at most $2\lceil \log_2(x) \rceil$ bits. The unary encoding of a number x is $x-1$ zeros, followed by a one. The general idea for these codes can be explained as follows: for a given number x, first we write unary encoding of $\lceil log_2(x) \rceil$, followed by $\lceil log_2(x) \rceil$ bits encoding number x. However, we have to mention this is only to demonstrate the general idea, while the actual encoding is more complicated. The key observation is that most integers coming from a power law distribution are small numbers, therefore this approach on average would use fewer number of bits, comparing to the fixed length codes.

- The compression format [8]: the framework uses the adjacency list table (e.g. table 2.1) of a network as the starting point. The rows of the adjacency table is ordered lexicographically based on the URLs of the web pages. Then, by exploiting the closeness similarity of the nodes, and the application of ζ codes, an impressive storage efficiency is achievable.

The details of ζ codes are out of scope for this report, and can be found in [8]. However, we will discuss the major ideas of the compression format for the rest of this subsection.

Naive Representation [8]: The naive representation is an adjacency list table, in which the nodes are ordered according to the lexicographic order of the corresponding URLs. The adjacency list for each node is preceded with the out degree of the node. Also the list is sorted in the increasing order. Table 2.1 is an example of the naive representation.

Gap coding [8]: the advantage of the sorted out-neighbor list in the naive representation is that the gaps between consecutive out-links are going to follow a pattern. In fact, experimentally it is shown that they follow a power low distribution [8]. To exploit this observation, instead of storing the adjacency list, one can represent the out-neighbors by storing the gap list. For a given node x let $S(x) = (s_1, s_2, \cdots, s_k)$ be the adjacency list, then, instead one can store the gap list: $g(x) = (s_1 - x, s_2 - s_1 - 1, s_3 - s_2 - 1, \cdots, s_k - s_{k-1} - 1)$.

Node	Outdegree	Out-neighbors
...	...	...
15	11	13, 15, 16, 17, 18, 19, 23, 24, 203, 315, 1034
16	10	15, 16, 17, 22, 23, 24, 315, 316, 317, 3041
17	0	
18	5	13, 15, 16, 17, 50
...	...	...

Table 2.1: Reprinted from [8]. Naive representation using adjacency List and outdegree

Table 2.2: Reprinted from [8]. Gap coding representation

Node	Outdegree	Out-neighbors
...	...	...
15	11	3, 1, 0, 0, 0, 0, 3, 0, 178, 111, 718
16	10	1, 0, 0, 4, 0, 0, 290, 0, 0, 2723
17	0	
18	5	9, 1, 0, 0, 32
...	...	...

Note that, since the adjacency list $S(x)$ is sorted, except possibly for the first one, all these numbers would be non-negative. To eliminate this exception, one can handle the first number as a special case. The first number in the list would be encoded using the following bijective mapping:

$$\nu(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 2|x| - 1 & \text{if } x < 0 \end{cases}$$

Table 2.2 illustrates the gap coding technique.

Reference Coding [8]: the reference coding technique exploits the similarity between the adjacency lists of the proximate positions in the lexicographic order. The idea is that instead of representing $S(x)$ directly, one can represent it as a modified version of another list $S(y)$. Where, y is a node, or equivalently a row of the adjacency table, that appears before x. We call $S(y)$ the reference list, and $x - y$ the reference number. To encode $S(x)$ with respect to $S(y)$, both the reference number and a vector of bits of length $|S(y)|$ have

Table 2.3: Reprinted from [8]. Reference coding representation

Node	Outdegree	Ref.	Copy list	Extra Nodes
...	...	...	...	...
15	11	0		13, 15, 16, 17, 18, 19, 23, 24, 203, 315, 1034
16	10	1	01110011010	22, 316, 317, 3041
17	0			
18	5	3	11110000000	50
...	...	...	...	...

to be stored. The vector specifies those neighbors of y that are also among the neighbors of x. Finally, those neighbors of x that do not appear in the adjacency list of y have to be stored ($S(x) \backslash S(y)$). Table 2.3 illustrates the idea.

The nontrivial part of this technique is the choice of reference number for a particular node. For that, one can assume there is a window size $W > 0$. The reference number r ($0 \leq r \leq W$) for the row i of the table is chosen such that encoding the row i by referencing to the row $i - r$, obtains the best compression rate. Having a larger window size W would improve the compression rate by providing more choices in the reference coding technique, however, would make the compression algorithm slower and less memory efficient. To find such a reference number for each row, one need to scan all the W previous rows, therefore, the running time for reference coding technique is $O(|E|W)$.

2.3.2 The Shingle Ordering Heuristic

The nice properties of the lexicographical order of the nodes based on the corresponding URLs is the crucial reason for the excellent performance of the WebGraph framework. Unfortunately, the framework cannot be used for other types of social networks trivially. Recently, Chierichetti *et al.* [14] extended the WebGraph framework to general social networks. The central idea is to introduce an ordering based on Jaccard coefficient [10]:

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|},$$

where A and B are two sets. Jaccard coefficient is a similarity measure for two sets A and B. Shingle ordering is a heuristic based on this notion. Let $S(x)$ and $S(y)$ be the sets of neighbors for x and y, respectively. Let σ be a random permutation of the nodes of the network. Then, for node x:

$$M_\sigma(x) = \sigma^{-1}(min_{a \in S(x)}\{\sigma(a)\})$$

Intuitively, $M_\sigma(x)$ is the neighbor of x that appears first among all other neighbors according to σ. We call $M_\sigma(x)$ the shingle of x. It is easy to prove the following:

$$Pr[M_\sigma(x) = M_\sigma(y)] = \frac{|S(x) \cap S(y)|}{|S(x) \cup S(y)|} = J(S(x), S(y))$$

The shingle ordering heuristic sorts the nodes based on their shingle. The idea is that if the neighbors of two nodes highly overlap, there is a good chance that they both have the same shingle, therefore it is likely that they will be close to each other. For breaking the ties a second (and third ...) shingle can be used.

2.3.3 Layered Label Propagation

Very recently, Vigna *et al.* [6] introduced the *Layered Label Propagation* (LLP) algorithm for reordering the nodes of a graph, and showed that using the output of this algorithm the compression rate of WebGraph framework can be improved, for both web graphs and social networks. It is worth mentioning, the layered label propagation algorithm in essence is a clustering method built up on the work of [51, 56]. The LLP algorithm uses a variant of label propagation algorithms, known as Absolute Pott Model (APM), as a subroutine. APM has a parameter γ as an input and guarantees that the resulting clusters have the density at least $\gamma/(\gamma+1)$. The value of γ in this algorithm is related to the resolution level of the resulting clusters. For values close to zero the algorithm returns fewer, sparser clusters. As γ increases the clusters become smaller and denser. More details on this algorithm is provided in Section 2.4.2.

The intuitive description of the LLP algorithm is as follows: we run the APM clustering algorithm with k different parameters $\gamma_1, \gamma_2, \cdots, \gamma_k$. Let λ_i be the label of node v

corresponding to parameter γ_i. We assign the tuple $(\lambda_1, \cdots, \lambda_k)$ to v as a label. Then, we sort the nodes lexicographically based on this label. Although, there is no discussion in [6] regarding the value of k, presumably it is the smallest number that resolves (almost) all the ties in the lexicographic order. The parameter γ_i is chosen uniformly at random from $\{2^{-i}|i = 0, \cdots l\} \cup 0$, where l is a fixed constant. We should mention that the intuition behind this choice is unclear for us and there is no discussion provided by the authors [6].

2.3.4 Discussion

The average neighborhood query time for the WebGraph framework is $O(Cd_{avg})$, where C is the limit on the length of the reference chains and d_{avg} is the average degree. Since C is a predefined small constant number, the neighborhood query time on average is $O(d_{avg})$. The framework does not allow sub linear incremental updates and in-neighbors query answering.

Assuming the order of the nodes is provided, the compression time is $O(|E|W)$. The running time to build the lexicographic order is $O(|V|log|V|)$, however, the time to build the shingle order depends on the number of edges as well ($O(|V|log|V| + |E|)$). The running time of LLP algorithm is proportional to $|E|^{1.3}$ [54][3].

Similar to the aggregation-based approach the compression rate of the order-based approach on web graphs is extremely good, while for other types of social networks the performance is quite poor. The compression rate for order-based approach is measured by the average number of bits that is necessary to represent a single edge. The WebGraph framework uses 2-4 bits/edge to encode a typical web graph. However, when the framework is coupled with the shingle ordering heuristic, the compression rate for other type of social networks is in the range of 10-15 bits/edge [14]. Also the order that is produced by the LLP algorithm comparing to both the shingle heuristic ordering and lexicographic ordering in most cases can improve the compression rate by up to 25 percent [6].

[3]However, this running time is based on experimental analysis, rather than analytical analysis. That is why we do not use the big O notation for this case.

2.4 Clustering and Dense Subgraph Mining

Any compression method, at least implicitly, exploits the existence of the dense subgraphs of a given social network in order to achieve compression. This naturally brings the topic of dense subgraph mining and clustering into picture. Clustering in the social networks is a broad and diverse area.

The techniques and algorithms in this area very much depend on the problem formulation. An early study formulate the problem as a min-max problem. The min-max cut method [20] partitions the nodes of a graph into clusters $C_1, \cdots, C_k$ such that it minimizes the number of edges between different clusters, and maximizes the number of edges inside the clusters. Later, the notion of modularity is proposed to measure the quality of a clustering [43], and a fast greedy algorithm to optimize it was proposed in [16]. The SCAN algorithm [63] introduced as a extension of the popular DBSCAN algorithm [22] to the social networks. A comprehensive survey on the clustering method can be found here [27]. Dense subgraph mining also is studied extensively. A recent work [61] introduces a mapping technique that maps edges and nodes to a multi-dimensional space in which dense areas are corresponding to dense subgraphs. Furthermore, many papers studied the problem of dense subgraph mining specific to the web graph [30, 26, 34]. The goal here is not to cover all the aspects of clustering and dense subgraph mining, rather we explain a few methods that have clear connections to the existing compression schemes.

In this section, we explain a dense subgraph mining method that successfully has been used in a compression framework. A clustering method known as Absolute Pott Model, which has been used for reordering the nodes of a network in the WebGraph framework, is discussed. In addition, we explain a structural clustering algorithm (SCAN), which uses the similarity between the neighborhood of the nodes as a measure for partitioning the nodes. Since both order-based and aggregation-based approaches use the same notion of neighbors similarity to achieve compression, we believe structural clustering potentially can be beneficial. Finally, we consider the spectral clustering method. Spectral clustering particularly is interesting, because as a preprocessing step it uses an embedding of the nodes

to a low-dimensional space. Note that, the ordering of the nodes in the order based methods can be considered as an embedding to the one-dimensional space.

2.4.1 Mining Semi-Bipartite Subgraphs

In this section, we explain a mining technique for finding a set of source nodes that all share a set of destination nodes as out neighbors (the top graph in Figure 2.1 displays an example). As we described in Section 2.2.1, this particular subgraph exploits by the Virtual Node scheme to achieve compression. The mining of this type of bipartite subgraph can be seen as a frequent itemset mining problem, where the out-neighbors of each vertex is a transaction. Items are corresponding to the vertices of the graph. Then, the goal is to find those sets of items that appeared in at least m transactions. The parameter m is called the minimum support. The number of transactions and the number of items are both equal to the number of vertices in the graph. The full enumeration of those patterns is particularly challenging, because in our setting the minimum support is 2 (since an itemset with support of 2 could be benefical for purpose of compression). However, in our setting there is no need to recall all such frequent itemsets.

The mining method relies on a two phases heuristic: a clustering phase, in which the vertices are clustered based on their similarity, i.e., Jaccard coefficient of their neighbor set. Next, in a pattern mining phase, the frequent patterns are discovered using the fast frequent itemset mining method introduced in [12].

Phase 1: Clustering

The clustering method uses k independent shingles for each vertex (see section 2.3.1 for the definition of the shingle). These shingles can be considered as k independent samples from the out-neighbors of each vertex. Therefore, if the out-neighbors of two vertices u and v highly overlap, it is likely that the sets of shingles of u and v also highly overlap. Note that, k is a small fixed number.

The clustering algorithm is as follows: first, we obtain a table in which the rows are the

vertices and the columns are the shingles. Then, we sort the rows of the table based on the lexicographic ordering of the shingle vectors. We traverse across the table to cluster the rows with the same shingle together, starting from column one. If the size of a cluster is larger than a threshold we move to the next shingle and refine the cluster to smaller parts. We continue this until the size of all clusters are less than the threshold, or we reach to the last shingle. Note that, this method biases the left-most shingle.

Phase 2: Pattern Mining

The objective of the pattern mining phase is to find common subsets of neighbors of a given cluster of vertices. Pattern P is a subset of vertices of the network. The frequency of P is the number of those vertices $v \in C$, that have all the vertices in P as a neighbor. The compression performance of P is defined as follows (see section 2.2.1):

$$Compression(P) = freq(P)|P| - |P| - freq(P) \tag{2.1}$$

The idea of the pattern mining phase is to first reorder the out-neighbor list of the nodes such that the vertex Ids with higher frequency sort first. Then, one can treat the out-neighbor lists as string. The out-neighbor lists would be added to a prefix tree one by one[4]. Each node in the tree is associated with a vertex Id as the label. The *prefix* of the node t in the tree is the sequence of vertex Ids, starting from the root, and along the path to t.

Table 2.4 is a toy example of a typical cluster of vertices, passed to mining phase. The vertex Ids in the out-neighbor list are sorted according to their frequency. Figure 2.3 is illustrating the corresponding prefix tree. Note that, before adding the out-neighbor list to the tree, we remove the vertices with frequency one. Each node of the tree, in addition of its label, is associated with a list of transactions, i.e., out-neighbor lists, that share the prefix of the node. We refer to the vertices in this list as the *support vertices* of the node.

[4]A prefix tree (or a trie) is a data structure to store a set of strings. In prefix tree, the nodes are labeled with letters, then the string associated with each node is the sequence of labels, starting from the root, along the path to that node. All the descendants of a node have a common prefix of the string associated with that node, and the root is usually associated with the empty string. Values are normally not associated with every node, only with leaves and some inner nodes that correspond to keys of interest.

Vertex Id	Out-neighbor List
23	1,2,3,5,6,10,12,15
55	1,2,3,5
102	1,2,3,20
204	1,7,8,9
13	1,2,3,8
64	1,2,3,5,6,10,12,15
43	1,2,3,5,6,10,22,31
431	1,2,3,5,6,10,21,31,67

Table 2.4: An example of a cluster of vertices passed to mining phase

For example, there are two nodes in the prefix tree (Figure 2.3) with label 8. The prefix sequence of the first one is $1, 8$ and second one is $1, 2, 3, 8$. The out-neighbor sequence of the vertex 204 is the only sequence that supports $1, 8$ (after removing 7 which has frequency 1), while the out-neighbor sequence of vertex 13 supports $1, 2, 3, 8$.

The merit of using the prefix tree is that it suggests a heuristic approach to exploit the trade off between the size and the frequency of candidate patterns. The prefix associated with each node of the prefix tree, could be a potential pattern. For leaf t of the tree, the size of the prefix of t is large, however, the frequency of the prefix is small. As we move towards the root from t, the size of prefix decreases and the frequency increases. The algorithm in one scan of the prefix tree, generates those patterns with high compression performance. Then, it sorts the patterns based on their compression performance and replace the edges for each pattern with a virtual node (see section 2.2.1) starting from the top of the list. More details of the mining algorithm can be found here [12].

2.4.2 Label Propagation Clustering

The label propagation algorithms are a family of algorithms that share the following idea: at the beginning of the algorithm each node has a unique label. During each round, every node would update its label according to a predefined rule. The algorithm terminates as soon as no update is possible.

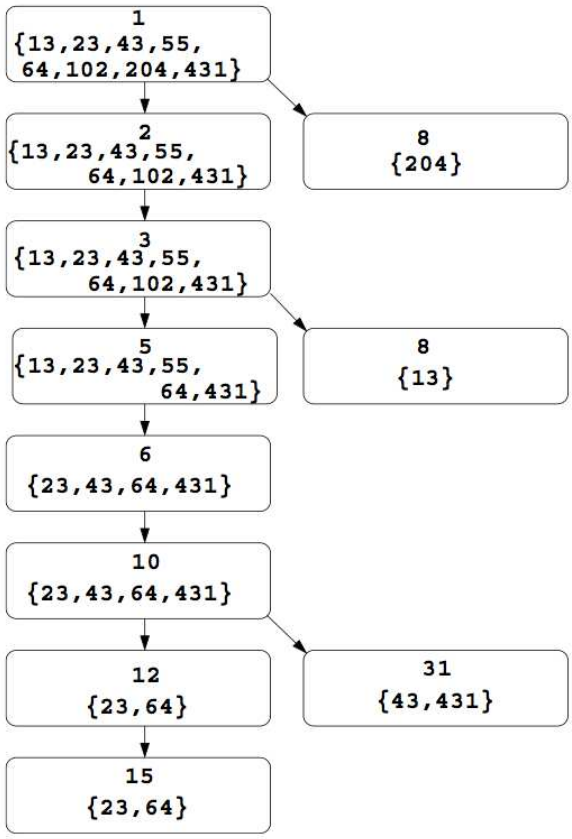


Figure 2.3: Reprinted from [11]. The prefix tree for the cluster of vertices in Table 2.4

The simplest rule for updating a node's label would be the majority vote among the neighbors of that node. However, using the majority vote as the updating rule, it has been observed that the label propagation algorithm tends to produce a very large cluster, containing most of the nodes. To resolve this problem a variety of methods have been proposed [42, 2].

The following variant of the label propagation family, called Absolute Pott Model (APM) [56], has been used by the WebGraph compression framework as a subroutine to reorder the vertices of a network: let $\lambda_1, \cdots, \lambda_k$ be those labels that have appeared on the neighborhood of node u. Let k_i $(1 \leq i \leq k)$ be the number of neighbors of u having label λ_i. Let v_i be the number of nodes in the whole graph with label λ_i. According to the APM algorithm, the label of u will be updated to λ_i such that it maximises the following:

$$k_i - \gamma(v_i - k_i)$$

The intuition is that for adding a particular node to cluster λ_i, the number of non-existing edges $(v_i - k_i)$ is also important. It is not hard to see that the density of resulting communities would be at least $\gamma/(\gamma + 1)$. Algorithm 3 presents the pseudo code for the APM algorithm.

Algorithm 3 The APM algorithm. λ is a function that will provide the cluster labels. $\lambda^{-1}(l)$ is the set of nodes having label l.

```
π ← a random permutation of G's nodes;
for all x: λ(x) ← x, v(x) ← 1;
while no update is possible do
  for all i = 0, 1, ··· , n − 1 do
    for every label l, k_l ← |λ^{-1}(l) ∩ N_G(π(i))|
    l̂ ← argmax_l[k_l − γ(v(l) − k_l)]
    decrement v(λ(π(i)))
    λ(π(i)) ← l̂
    increment v(λ(π(i)))
  end for
end while
```

2.4.3 Structural Clustering Method

We explain a clustering method in this section which we believe potentially can be used in both aggregation-based and order-based compression frameworks. The idea of this approach is to view the clusters as a set of nodes that have similar neighbors. A cluster is consisted of a "nucleus" and a "boundary". The vertices in the nucleus does not belong to any other cluster. However, vertices in the boundary of a cluster might belong to other clusters as well. First, we define the similarity measure for two nodes u and v [63]:

$$\sigma(u, v) = \frac{|N(v) \cap N(u)|}{\sqrt{|N(v)| \times |N(u)|}},$$

where $N(u)$ is the set of the neighbors of u. Having this notion of similarity in mind the ϵ-neighborhood of u is the set of those neighbors of u that their similarity to u is at least ϵ:

$$N_\epsilon(u) = \{v \in N(u) | \sigma(u, v) \geq \epsilon\}$$

The method uses two parameters, ϵ and μ. A node u is a core if the ϵ-neighborhood of it, has at least μ vertices. Then, if we only consider the core vertices and the edges among them, in this new network, a connected component of the core vertices is forming the nucleus of a cluster. The boundary of a cluster is consisted of those non-core vertices that are belonging to $N_\epsilon(u)$ for at least a vertex u in the nucleus.

In [63], the formalism to capture this idea has been introduced. Moreover, an algorithm to discover all the clusters for given ϵ and μ is developed. The algorithm works as follows: in one scan it identifies the core vertices. If a core vertex u is not assigned to a cluster yet, the algorithm generates a new cluster Id for it. Then, in a breadth first search style, it grows the cluster around u. The algorithm is linear in the number of edges of the network.

Note that, the reference coding technique in the order-based approach also uses the similarity of the neighborhood of the vertices. In this technique the out-neighbor list of one node is encoded by referring to the out-neighbor list of another node. To have an effective reference coding we need to find an ordering in which the nodes with similar neighbors are close to each other. Since the structural clustering method clusters the nodes based on the

similarity of their neighbors, potentially it can be exploited to generate an ordering in which reference coding technique is effective. However, the structural clustering method assumes the graph is undirected. An immediate challenge is to extend the idea to directed networks.

2.4.4 Spectral Clustering

Spectral clustering method uses the algebraic properties of the adjacency matrix of the network. The solid theoretical foundations and the running time efficiency are two critical advantages of the method. The spectral clustering method as a preprocessing step embeds the nodes of the graph into a low-dimensional space. Likewise, the order-based approach also as a preprocessing step embeds the nodes into a one-dimensional space (an ordering of the nodes can be considered as an embedding of the nodes into a line). However, spectral clustering assumes that the network is symmetric, i.e., undirected. A popular technique to apply the spectral clustering on directed networks is to simply ignore the direction of the edges. Here, we cover the basic concepts and algorithms of the spectral clustering. We also provide a theoretical analysis to reveal the intuition behind the method.

Definitions and Backgrounds

For a square matrix X, a non-zero vector $\mathbf{v}$ is an eigenvector of X if there is a real number λ such that:

$$X\mathbf{v} = \lambda\mathbf{v}$$

The real number λ is the eigenvalue of X corresponding to $\mathbf{v}$. A square matrix X is symmetric positive semidefinite if and only if it is symmetric and for any $\mathbf{v} \in \mathbb{R}^{\mathbf{n}}$, we have $\mathbf{v}^T X\mathbf{v} \geq 0$. It is a basic fact in linear algebra [15], that a $n \times n$ symmetric positive semidefinite matrix has n distinct, pairwise orthogonal eigenvectors, and the eigenvalues corresponding to them are non-negative.

The adjacency matrix A of an undirected graph G is a $|V(G)| \times |V(G)|$ square matrix, where $A_{i,j} = A_{j,i} = 1$, if i and j are connected and $A_{i,j} = 0$ otherwise[5]. Let d_i be the degree

[5]To be perfectly precise, we assume that $V(G) = \{1, \cdots, |V(G)|\}$

of vertex i, then, the degree matrix D is a diagonal $|V(G)| \times |V(G)|$ matrix, where $D_{i,i} = d_i$ and $D_{i,j} = 0$ if $i \neq j$. The unnormalized Laplacian matrix of G is defined as:

$$L = D - A$$

For vector $\mathbf{v}$, $\mathbf{v}_i$ is the entry corresponding to node i. The following proposition summarizes the basic properties of L [38]:

Proposition 1. *Let $n = |V(G)|$. The following properties holds for unnormalized Laplacian matrix L:*

1. *For every vector $\mathbf{v} \in \mathbb{R}^n$ we have:*

$$\mathbf{v}^T L \mathbf{v} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} (\mathbf{v}_i - \mathbf{v}_j)^2$$

2. *L is symmetric positive semidefinite.*

3. *L has n non-negative real-valued eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$*

Proof. Item 1

$$\mathbf{v}^T L \mathbf{v} = \mathbf{v}^T D \mathbf{v} - \mathbf{v}^T A \mathbf{v} = \sum_{i=1}^{n} d_i \mathbf{v}_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_i \mathbf{v}_j$$

Notice that $d_i = \sum_{j=1}^{n} A_{i,j}$. Therefore we have:

$$\mathbf{v}^T L \mathbf{v} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_i \mathbf{v}_j$$

Since $A_{i,j} = A_{j,i}$, we have:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_i^2 = \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_j^2 \right)$$

Therefore,

$$\begin{aligned} \mathbf{v}^T L \mathbf{v} &= \frac{1}{2} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_i^2 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_i \mathbf{v}_j + \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \mathbf{v}_j^2 \right) \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} (\mathbf{v}_i - \mathbf{v}_j)^2 \end{aligned}$$

Item 2

L is symmetric because both D and A are symmetric, and is positive semidefinite because according to part (1), for $\mathbf{v} \in \mathbb{R}^n$, we have:

$$\mathbf{v}^T L\mathbf{v} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}(\mathbf{v}_i - \mathbf{v}_j)^2 \geq 0$$

Item 3

L is symmetric positive semidefinite, thus, it has n non-negative eigenvalues. Observe that for the constant one vector $\mathbb{1}$, we have $L\mathbb{1} = 0 \times \mathbb{1}$. Therefore, the smallest eigenvalue of L is zero. □

As a convention, we sort the eigenvalues of a Laplacian matrix L, in a non-decreasing order: $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. λ_1 is the first eigenvalue of L, λ_2 is the second and so on. Likewise, the i-th eigenvector is the eigenvector corresponding to the i-th eigenvalue. There are many different variants of spectral clustering methods in the literature. The common ingredient, however, is that somehow they use the eigenvectors of the Laplacian for a preprocessing step. For example, let $\mathbf{v}$ be an eigenvector of the Laplacian L , with a non-zero eigenvalue. Then, the mapping $\varphi(i) = \mathbf{v}_i$ is an embedding of the vertices to a line. This is particularly interesting, because the vertices of a network can be considered as a set of points in a high dimensional Euclidean space. So, one can think of the spectral methods as a dimension reduction technique. We like to remind the reader that many clustering methods, e.g. K-means, perform much better on low dimensional data.

Algorithm 4 shows a simple variant of spectral clustering algorithms.

Algorithm 4 Pseudo code for the spectral clustering

1: Let A be the adjacency matrix and D be the diagonal degree matrix
2: Let $L = D - A$
3: Let $\mathbf{v^1}, \mathbf{v^2}, \cdots, \mathbf{v^k}$ be the first k eigenvectors of L
4: Let $U \in \mathbb{R}^{k \times n}$ be the matrix consisted of $\mathbf{v^1}, \cdots, \mathbf{v^k}$ as columns
5: Let $\mathbf{y}_i \in \mathbb{R}^k$ be the vector corresponding to the i-th row of U
6: Cluster $\{\mathbf{y}_i | i = 1, \cdots, n\}$ using K-mean algorithm

Justification

Unfortunately, there is no solid theorem that relates the eigenvalues and eigenvectors of the Laplacian of a network to the clusters of it. However in this section, we try to justify the relevance of the eigenvectors in an intuitive fashion. We show that the second eigenvector is related to the relaxation of a variant of min-cut problem, which is NP-hard otherwise. The material in this section comes from [38].

A natural formulation for node clustering is to find a partitioning such that it minimizes the number of edges with end points in different partitions. For a subset C of vertices $\bar{C}$ is the set of those vertices that does not belong to C. $Cut(C, \bar{C})$ is the number of edges that have one end in C and the other end in $\bar{C}$. For a partition $C_1, \cdots, C_k$ of vertices, we define:

$$Cut(C_1, \cdots, C_k) = \frac{1}{2} \sum_{i=1}^{k} Cut(C_i, \bar{C}_i)$$

In particular for $k = 2$, the problem of finding a minimum Cut can be solved in polynomial time. However, in many cases it separates a single vertex from the rest of the network. To resolve this problem the notion of $RatioCut$ tries to favor a sort of balanced partitioning:

$$RatioCut(C_1, \cdots, C_k) = \frac{1}{2} \sum_{i=1}^{k} \frac{Cut(C_i, \bar{C}_i)}{|C_i|}$$

Unfortunately, finding the partitioning that minimizes the $RatioCut$,even for $k = 2$, is NP-hard [60]. For $k = 2$, we show that how the spectral clustering can be seen as solving a relaxed version of the $RatioCut$ minimization problem.

The objective is to find a subset $C \subset V$ such that:

$$\min_{C \subset V} RatioCut(C, \bar{C})$$

First, we rewrite the problem using the Laplacian matrix L. For a subset $C \subset V$, $f = (f_1, \cdots, f_n)^T \in \mathbb{R}^n$ is defined as follows:

$$f_i = \begin{cases} \sqrt{|\bar{C}|/|C|} & \text{if } i \in C \\ -\sqrt{|C|/|\bar{C}|} & \text{if } i \in \bar{C} \end{cases} \tag{2.2}$$

Having this definition for f, we can write:

$$\begin{aligned}
f^T L f &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{i,j}(f_i - f_j)^2 \\
&= \frac{1}{2}\sum_{i\in C}\sum_{j\in \bar{C}} A_{i,j}\Big(\sqrt{|\bar{C}|/|C|} + \sqrt{|C|/|\bar{C}|}\Big)^2 \\
&+\frac{1}{2}\sum_{i\in \bar{C}}\sum_{j\in C} A_{i,j}\Big(-\sqrt{|\bar{C}/|C|} - \sqrt{|C|/|\bar{C}|}\Big)^2 \\
&= Cut(C,\bar{C})\Big(\frac{|\bar{C}|}{|C|} + \frac{|C|}{|\bar{C}|} + 2\Big) \\
&= Cut(C,\bar{C})\Big(\frac{|\bar{C}|+|C|}{|C|} + \frac{|C|+|\bar{C}|}{|\bar{C}|}\Big) \\
&= RatioCut(C,\bar{C}) \times |V|
\end{aligned}$$

Therefore, minimizing $f^T L f$ is equivalent to minimizing the $RatioCut$:

$$\min_{C\subset V} f^T L f, \text{ subject to:}$$

$$f_i \text{ defined as in Equation 2.2}$$

Moreover, the multiplication of f and the constant one vector is:

$$f\mathbb{1} = \sum_{i=1}^{n} f_i = \sum_{i\in C}\sqrt{\frac{|\bar{C}|}{|C|}} - \sum_{i\in \bar{C}}\sqrt{\frac{|C|}{|\bar{C}|}} = |C|\sqrt{\frac{|\bar{C}|}{|C|}} - |\bar{C}|\sqrt{\frac{|C|}{|\bar{C}|}} = 0$$

Also the square of norm 2 of vector f is:

$$\|f\|^2 = \sum_{i=1}^{n} f_i^2 = |C|\frac{|\bar{C}|}{|C|} - |\bar{C}|\frac{|C|}{|\bar{C}|} = n$$

Hence, the addition of these two new constraints would not change the solution of the minimization problem:

$$\min_{C\subset V} f^T L f, \text{ subject to:}$$

1) f_i defined as in equation 2.2
2) $f\mathbb{1} = 0$
3) $\|f\| = \sqrt{n}$

However, this is still a NP-hard problem [38]. The variables f_i can take two particular values, according to equation 2.2. The immediate relaxation of this optimization problem is to relax variables f_i to take any real value. This leads to the following optimization problem:

$$\min_{f \in \mathbb{R}^n} f^T L f, \text{ subject to:}$$

$$1)\ f\mathbb{1} = 0$$
$$2)\ \|f\| = \sqrt{n}$$

This problem can be solved in polynomial time. In fact, by the Rayleigh-Ritz theorem (e.g. see Section 5.5.2 of [39]) it can be shown that the eigenvector corresponding to the second smallest eigenvalue is the solution for this optimization problem. So, in some sense the second eigenvector of L is an estimation for minimizing the *RatioCut*. However, this solution is not a partitioning yet. In order to get the partitioning we need another step. A simple heuristic to obtain the partitions would be:

$$\begin{cases} v_i \in C & \text{if } f_i \geq 0 \\ v_i \in \bar{C} & \text{if } f_i < 0 \end{cases}$$

This simple heuristic is not working for $k > 2$. Instead, we can consider f_i as points in $\mathbb{R}$ and use a more general clustering algorithm like K-means to partition the points. This is exactly the spectral clustering method we presented in Algorithm 4. Notice that the first eigenvector, that is used in Algorithm 4, is the constant one vector, therefore, it does not have any influence in the output of K-means algorithm. Also we like to note that, using a similar argument one can show that the following standard trace minimization problem is a relaxation of *RatioCut* problem for general k:

$$\min_{H \in \mathbb{R}^{k \times n}} Tr(H^T L H), \text{ subject to:}$$

$$H^T H = I,$$

where, $Tr(X)$ for a square matrix X is the sum of the entries on the diagonal of X. The same Rayleigh-Ritz theorem [39] tells us that the solution of this optimization problem is the matrix H consisted of the first k eigenvectors of L as columns.

2.5 The Graph Layout Problem

In the classical graph theory, the problem of finding a "good" ordering of the nodes with respect to minimizing/maximizing a given objective function has a long history. The applications of this family of problem include VLSI circuit design, numerical analysis and more. These problems fall in the class of combinatorial optimization problems. Often for nontrivial objective functions, finding the ordering that has the optimal cost is NP-hard [29, 47]. In this section, we refer to these family of problems as *graph layout problems*.

Both graph layout problem and order-based compression approach are looking for a "good" ordering of the nodes. The notion of goodness in a layout problem is captured by an objective function, while in the order-based compression framework the goodness of an ordering is measured by the compression rate. A promising direction to pursue is to introduce an objective function for a layout such that it captures the requirements of the compression framework.

We review the results and heuristic algorithms for a few classical variants of the graph layout problems. Even though, typically the objective function in these classical variants are not explicitly connected to the compression framework, we believe studying them would help us to understand the nature and challenges of these family of problems.

2.5.1 Definitions and Theoretical Results

In this section, first we present a few formal definitions, and illustrate them using a toy example. Also, we give a short review of the theoretical algorithmic results. The material of this section mostly comes from the Díaz *et al.* survey [19]. A layout φ of an undirected graph $G = (V, E)$ is a bijective function $\varphi : V \to \{1, \cdots, n\}$, where n is the number of vertices of the graph. Given a graph G, a layout φ of G, and an integer i, we define the set of left vertices of i:

$$L(i, \varphi, G) = \{u \in V : \varphi(u) \leq i\},$$

likewise the set of right vertices of i is defined as follows:

$$R(i, \varphi, G) = \{u \in V : \varphi(u) > i\}$$

Now, having these definitions, we can define the *edge cut* at position i:

$$\theta(i, \varphi, G) = |\{uv \in E : u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|$$

The *vertex cut* at position i of φ is defined as:

$$\delta(i, \varphi, G) = |\{u \in L(i, \varphi, G) : \exists v \in R(i, \varphi, G) : uv \in E\}|$$

Finally, the *length* of $uv \in E$ in the layout φ is defined as:

$$\lambda(uv, \varphi, G) = |\varphi(u) - \varphi(v)|$$

Figure 2.4(a) displays a sample graph, and Figure 2.4(b) illustrates a layout φ of the same graph. A vertical line after position i and before position $i + 1$ separates two sets of vertices $L(i, \varphi, G)$ and $R(i, \varphi, G)$. Then, $\theta(i, \varphi, G)$ would be the number of edges that cross the vertical line. $\delta(i, \varphi, G)$ would be the number of vertices at the left of vertical line which have at least an edge crossing the vertical line. Finally, $\lambda(uv, \varphi, G)$ is the distance of the two ends of the edge uv, assuming the distance of any two consecutive vertices is one.

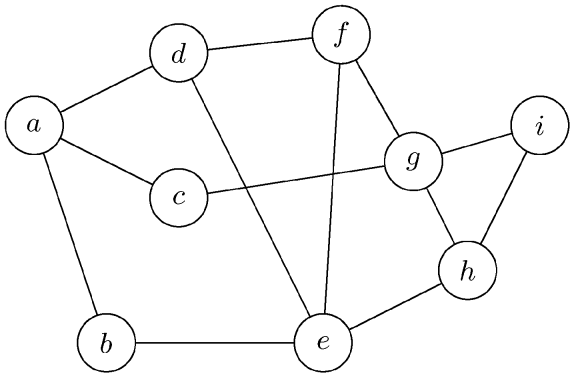
Having these measures, we define the following layout problems:

Definition 1. *Bandwidth problem (Bandwidth)*
Given a graph $G = (V, E)$, find a layout φ^* such that $BW(\varphi^*, G)$ is minimized, where:

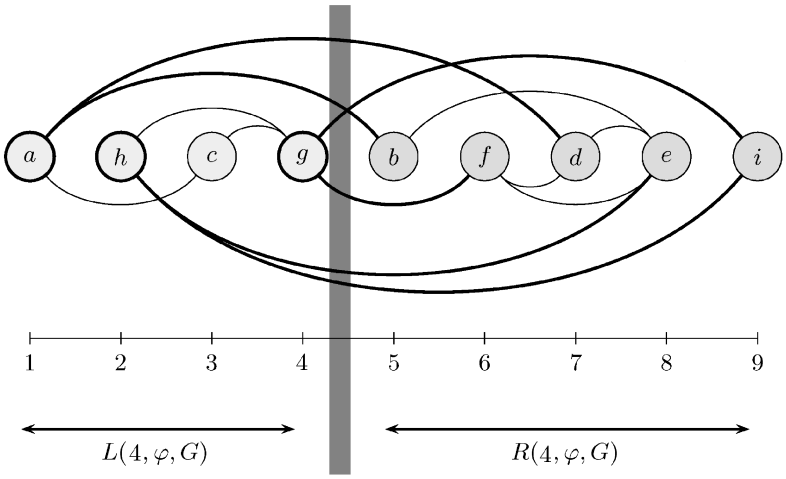
$$BW(\varphi, G) = \max_{uv \in E} \lambda(uv, \varphi, G)$$

Definition 2. *Linear Arrangement problem (MinLA)*
Given a graph $G = (V, E)$, find a layout φ^* such that $LA(\varphi^*, G)$ is minimized, where:

$$LA(\varphi, G) = \sum_{uv \in E} \lambda(uv, \varphi, G)$$



(a) Graph G



(b) Layout φ of graph G is illustrated. In this layout of G, $\lambda(gi, \varphi, G) = 5, \delta(4, \varphi, G) = 3$ and $\theta(4, \varphi, G) = 6$.

Figure 2.4: Reprinted from [19]. A graphical illustration of a few measures for a layout φ.

Definition 3. *Cut Width problem (Cutwidth)*

Given a graph $G = (V, E)$, find a layout φ^* such that $CW(\varphi^*, G)$ is minimized, where:

$$CW(\varphi, G) = \max_{1 \leq i \leq |V|} \theta(i, \varphi, G)$$

Definition 4. *Vertex Separation problem (VertSep)*

Given a graph $G = (V, E)$, find a layout φ^* such that $VS(\varphi^*, G)$ is minimized, where:

$$VS(\varphi, G) = \max_{1 \leq i \leq |V|} \delta(i, \varphi, G)$$

All of these problems have been proven to be NP-hard for general undirected graphs [19]. Except MinLA, the other three are NP-hard even when the input is restricted to graphs with maximum degree 3. Surprisingly, Bandwidth problem is NP-hard, when the input is restricted to trees with maximum degree 3.

An α-approximation algorithm guarantees that the output is at most α times worse than the optimal solution [1, 25], where α is the approximation factor. To the best of our knowledge, the Bandwidth problem has a $O(\log^3 n\sqrt{\log \log n})$-approximation algorithm. The best approximation factor known for MinLA is $O(\sqrt{\log n} \log \log n)$. There are $O(\log^2 n)$-approximation algorithms for both VertSep and CutWidth [24, 19].

Even thought, these approximation algorithms are interesting from a theoretical point of view, they are not practical. The major tool for establishing these upper bounds is a generalized linear programming technique, known as the Ellipsoid method. A linear program is an optimization problem in which the goal is to maximize/minimize a linear objective function, subject to a set of linear inequality constraints. The power of the Ellipsoid method is that, under some conditions, the running time does not depend on the number of constraints. Therefore, if the linear program has polynomial number of variables and exponential number of constraints, the Ellipsoid method could solve the problem in polynomial time. In practice, the method suffers from a very poor running time. This motivates us to discuss the heuristic algorithms.

2.5.2 Heuristics

In this subsection we briefly discuss the heuristics for the layout problems. However, we should mention that the layout problems are not initially motivated by the social network analysis applications. As a result, there are not much experimental results on the performance of these heuristics on social networks. The techniques and ideas that we describe here are general and can be used for any layout problem.

We discuss three different approaches, spectral sequencing, a multi-scale framework and local search technique. The first technique is merely based on the second eigenvector of the laplacian of the network, while the multi-scale framework is using a coarsening approach, to recursively reduce the size of the network.

Spectral Sequencing: The algorithm starts by computing the second eigenvector $\mathbf{v_2}$ of Laplacian matrix L (see section 2.4.4). Then, the layout φ is defined as follows:

$$\varphi(x) < \varphi(y) \iff \mathbf{v_2}(x) < \mathbf{v_2}(y),$$

where x and y are two nodes of the graph, and $\mathbf{v_2}(x)$ is the entry of vector $\mathbf{v_2}$ corresponding to node x. The ties would be broken arbitrarily. This is a very simple, yet effective heuristic. As it is mentioned in [49], this ordering is particularly effective for those networks that have a sort of geometric structure, e.g. VLSI networks.

Local Search: The local search technique starts with an initial layout φ and improves the layout by applying local changes. The critical design factor in a local search scheme is the definition of local changes. The following two operations are introduced in [49]:

1. Flip2: randomly chooses two vertices and switch their position in the layout φ.

2. Flip3: randomly chooses three vertices and rotates their positions in the layout φ.

The reason for choosing these operations is efficiency of applying the operation and computing the cost for the new layout. This method is somehow the complement of spectral sequencing. The performance of this local search scheme on the network with geometric

structure is worse than spectral sequencing, However, it outperforms the spectral sequencing method for the random networks.

A natural idea to overcome the weakness of the local search is to use the spectral sequencing method as the initialization phase [48]. This combination is quit effective and improves the result both for geometric and random graph. To the best of our knowledge, there is no reported results for the performance of these methods on the social networks.

Multi-Scale Framework: Figure 2.5 illustrates the basic ideas of multi-scale framework. The algorithm consisted of three phases: coarsening, initialization and uncoarsening. In the coarsening phase, recursively, a few nodes aggregate in a super-node, and the relations between these super-nodes are captured by weighted edges. The coarsening phase stops when the size of the graph is small enough, e.g. ten super-nodes. Then, for the coarsest aggregated network, using an exhaustive search, the best ordering is computed. In the uncoarsening phase, using the layout from each level, the algorithm initiate the layout for the next finer level. Then, a local search technique is deployed to improve the finer layout. The framework has been used successfully for solving different problems [33, 57, 13]. One can think of this framework as a recursive local search scheme. Arguably, coarsening phase is the most critical step of the framework. Here, we explain the coarsening phase in [57] that is based on an algebraic multi-grid (AMG) technique [58].

In the AMG technique, a matrix P projects the fine graph to the coarse graph. Let A^f be the adjacency matrix of the fine graph. The projection is as follows:

$$A^c \leftarrow P^T A^f P,$$

where A^c is the adjacency matrix of the coarse graph. Projection matrix P is a real-valued $n \times m$ matrix, in which n and m are the number of nodes in the fine and coarse graph, respectively. Notice that using the AMG projection, the coarse graph will be a weighted graph, even if the fine graph is unweighted. In the rest of this section, we explain how to construct the projection matrix P, and the meaning behind this algebraic projection.

Let G_f be the fine network. We start by choosing a set of seed nodes $S \subset V(G_f)$. One

can think of these seed nodes as the center of the future super-nodes of the coarse graph. We have to choose subset S so that for node $i \notin S$:

$$\sum_{j \in S} A^f_{i,j} \Big/ \sum_{j \in V(G_f)} A^f_{i,j} \geq \theta, \tag{2.3}$$

where, θ is a threshold parameter. Intuitively, this means that any node of the fine graph should be "well represented" by its neighbors in S. The set S can be constructed in one scan. If a node i is not "well represented" by S, i.e., does not satisfy the inequality 2.3, we greedily add it to S. Let $I : S \to \{1, \cdots, |S|\}$ be a bijective function from seed nodes S to integers $\{1, \cdots, |S|\}$. The $n \times |S|$ projection matrix P is defined as follows:

$$P_{i,I(j)} = \begin{cases} A^f_{ij} / \sum_{k \in N^S_i} A^f_{ik} & \text{if } i \in V(G_f) \backslash S \wedge j \in N^S_i \\ 1 & \text{if } i \in S \wedge j = i \\ 0 & \text{otherwise} \end{cases}$$

Where, N^S_i is the neighbors of i in S. One can think of $P_{i,I(j)}$ as the likelihood of i belonging to super-node $I(j)$. The merit of this approach is that a node i proportionally could aggregate to several super-nodes. This leads to a more accurate coarsening process.

In the coarse graph G^c, the weight of the edge that connecting two super-nodes p and q is:

$$A^c_{p,q} = \sum_k \sum_{l \neq k} P_{k,p} A^f_{k,l} P_{l,q}$$

This means that the contribution of every edge (k, l) in G^f to the edge (p, q) in G^c is proportional to the likelihood of k and l belonging to super-nodes p and q, respectively.

2.6 Summary

We discussed two completely different approaches towards compressing social networks. The first approach uses the idea of aggregating a set of nodes with a similar set of neighbors to a super-node, while the second approach exploits an ordering of the nodes in which the similar nodes tend to be close to each other.

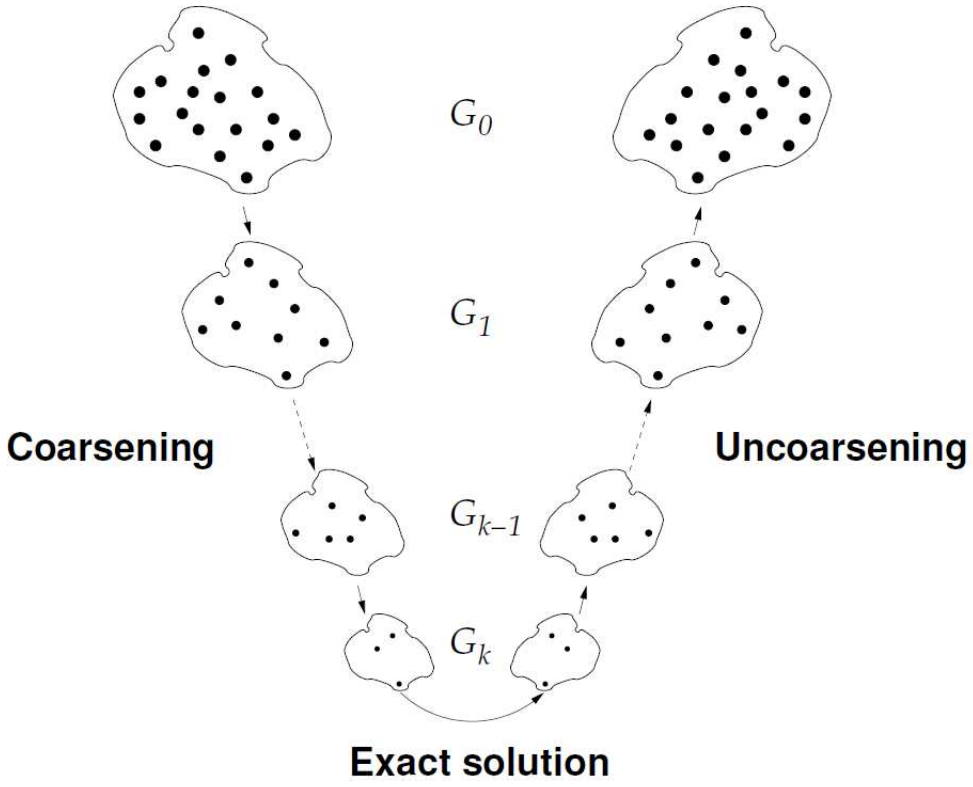


Figure 2.5: Reprinted from [57]. The illustration of multi-scale framework.

	Inc. Update	Out-Neigh. Query	In-Neigh. Query	Compression Time	Summarization
Virtual Node [11]	$O(1)$	$O(d_{avg})$	$O(\|V\|)$	$O(\|E\| \log \|E\|)$	no
Graph Summary [41]	$O(1)$	$O(d_{avg})$	$O(\|V\|)$	-	yes
WebGraph+Lex.[8]	$O(\|E\|)$	$O(d_{avg})$	$O(\|V\|)$	$O(\|E\| + \|V\| \log \|V\|)$	no
WebGraph+Shingle [14]	$O(\|E\|)$	$O(d_{avg})$	$O(\|V\|)$	$O(\|E\| + \|V\| \log \|V\|)$	no
WebGraph+LLP [6]	$O(\|E\|)$	$O(d_{avg})$	$O(\|V\|)$	proportional to $\|E\|^{1.3}$	no

Table 2.5: Comparison of the compression methods

We reviewed a few methods for the dense subgraph mining and clustering of social networks. A semi-complete bipartite subgraph mining technique and a clustering algorithm known as Absolute Pott Model had been explained. The semi-complete bipartite subgraph mining is a basic component of the Virtual Node compression scheme, and the Absolute Pott Model is used in the LLP algorithm. A structural clustering algorithm that partitions the nodes based on the similarity of their neighbors has been discussed. We remind that both the reference encoding technique in the order-based approach and the aggregation technique exploit the similarity of neighbors of a group of nodes to achieve compression. The spectral clustering is discussed for three different reasons: (1) it has solid theoretical foundation, (2) the method is scalable, and most importantly, (3) the spectral clustering is based on an embedding of the nodes to a low dimension space. Note that, the order-based compression methods also use an embedding of the nodes to a one-dimensional space.

Finally, we reviewed a family of combinatorial graph problems, known as graph layout problems. The goal is to find an ordering of the nodes that minimizes a given objective function. Different variants of these problems have been introduced and the hardness and approximation results along with a few major heuristic algorithms have been reviewed. This line of research can be inspiring if one tries to capture the notion of a "good" ordering for the order-based approach by introducing a combinatorial objective function.

Table 2.5 summarizes and compares different features of the compression algorithms. Generally speaking, the aggregation-based methods have the advantage of allowing efficient incremental updates and providing a summarization of the communities (dense subgraphs) of the network. On the other hand, the order based methods are more efficient regarding the compression time. They both can answer the out-neighbor query efficiently, however, the order-based methods have larger over-head due to bit-level operations. To answer the in-neighbor query efficiently, both approaches has to replicate the data by storing the transpose of the data.

Comparing the compression rate of the aggregation-based methods and order-based methods, is not trivial for two reasons: first, the two approaches use different measures

as the compression rate. Second, they use different datasets to report the experimental results. We can resolve the first issue by assuming that to store an identifier of a node, one uses 32 bits in the aggregation-base approach. With this assumption, a compression rate of 0.15, i.e., the number of edges in compressed graph over the number of edges in the original graph, is corresponding to $0.15 \times 32 = 3.36$ bits/edge. Using this criteria, roughly one can imply that the compression rate of the order-based approach is better comparing to the aggregation-based approach. Among the order-based methods, WebGraph+LLP, i.e., the WebGraph framework coupled with the LLP ordering, has the best compression rate.

We believe both aggregation and order-based approaches suffer from major disadvantages that disqualify them as a general purpose storage scheme for social networks. The order-based methods does not allow incremental updates, while the aggregation methods are only effective if the social network is rich with a certain type of very dense subgraphs (depending on the aggregation technique). Both approaches do not allow efficient in-neighbor query answering without replicating the data.

2.7 How this thesis is related and different from the existing work

In Chapter 3, we will introduce GPSN compression framework, a novel compression framework that can be regarded as a generalization of the order-based approach. This framework allows efficient in-neighbor and out-neighbor queries, and incremental updates. We provide an upper-bound on the compression rate for a simple setting of our framework. Similar to order-based methods the framework exploits a "lineup" of nodes. However, the order-based approach uses an ordering of nodes in which each node appear exactly once. As oppose to the GPSN framework in which, the "lineup" is a sequence of nodes, with possible replications. We refer to this sequence as a "linearization" of nodes.

On the other hand, similar to the aggregation-based approach, existing of dense subgraphs is essential to the effectiveness of the GPSN compression framework. The framework tries to position the nodes that are part of a dense subgraph, in the proximate positions

of the sequence. Chapter 4 discusses a lossless compression method designed based on the principles of the GPSN framework. We compare the compression rate of our lossless compression scheme with WebGraph+Shingle [14].

The lossy compression scheme is presented in Chapter 5. Motivated by graph layout problems, we turn the problem of finding a good "linearization" of nodes to an optimization problem. Although, our objective function is more complicated comparing to those that we have discussed in Section 2.5. We believe this complexity is due to the combinatorial nature of the network compression problem, and is unavoidable. The heuristic to optimize such an objective function is designed according to the complex combinatorial nature of the objective function, and is among the contributions of this dissertation.

Chapter 3

GPSN Compression Framework

In this chapter, we discuss our approach towards having a General Purpose Social Network (GPSN) compression framework. A GPSN compression framework allows network analysis algorithms directly operate on the compressed version of a network. This is particularly interesting because most of the algorithms implicitly assume that the network fits in the main memory. However, in the age of the online social media, applications have to analyze and manipulate networks with hundreds of millions of nodes and billions of edges.

For the rest of this section, we take a formal approach to provide a theoretical analysis of our framework. We introduce the notion of k-cover and k-linearization, and using these notions, provide an upper bound on the compression rate of a simple setting of our framework. The theoretical analysis provided in this section is interesting and could serve as an evidence of the effectiveness of our framework. However, in practice often a more realistic setting is desirable. We leave the discussion on the practical aspects of the framework for the next two chapters.

3.1 The GPSN Compression Framework

Note that, in Section 1.1, we already discussed the core idea of the GPSN compression framework in an intuitive fashion. We start the section by introducing two definitions that

will be useful for the future discussions:

Definition 5 (k-cover)**.** *We say an edge $e = (u, v) \in E(G)$ is k-covered by a sequence S if and only if there are copies of u and v in S that are at most k positions away. $E_k(S)$ would be the set of all edges that are k-covered by S. We might drop the subscript k if it is obvious from the context.*

Definition 6 (k-linearization)**.** *We say a sequence S is a k-linearization of a graph G if and only if $E_k(S) = E(G)$. The length of k-linearization S is the same as the length of S.*

In this section, we define the framework and briefly discuss how by making appropriate choices for each of its component, one can tackle different problems. The GPSN framework is consisted of three components:

Definition 7 (GPSN Framework)**.** *The GPSN framework is consisted of the following components:*

1. *A sequence S of the nodes with the parameter k. Each position S_i of the sequence S is associated with two pieces of information: a pointer to the next copy of the same node, and a vector of $2k$ bits to encode the local edges corresponding to position S_i.*

2. *A data structure to store sequence S, $2k$ bits labels and the next pointer associated to each position.*

3. *An algorithm to find sequence S that k-covers the edges of network.*

Note that the sequence S in the GPSN framework might be a k-linearization of a given graph, or it might only k-covers the majority of the edges. We call it a lossless compression in the former case or a lossy compression scheme for the latter one. The merit of the GPSN framework is coming from the flexibility of its components. In fact, the focus of this dissertation is to show that by making appropriate choices for each component, we can achieve a variety of objectives:

- **Upper Bound on Compression Rate**

The first natural question regarding our method is on the possibility of proving an upper bound on the compression rate. We know that for a k-linearization of length l, the GPSN framework uses $l \times (2k + \log_2(l))$ bits. Hence, to have an upper bound on the compression rate, it is sufficient to prove an upper bound on the length of the k-linearization. Having such an upper bound for arbitrary value of k is not easy to obtain. However, we can have such an analysis for the special case of $k = 1$. For the rest of this chapter we prove an upper bound for the compression rate of our framework when it is restricted to the simple setting of $k = 1$. For this special case, we use a simple array data structure to represent the input network. From an algorithmic point of view, we show that to find the shortest sequence S that 1-covers all the edges of a graph G, one has to cover the edges of G by the least number of paths, such that any edge appears exactly once in the exactly one path. This can be seen as a generalization of the Eulerian path problem where the objective is to cover the edges of the graph by exactly one path.

- **Lossless Compression**

  As the second major part of the thesis, we focus on evaluating our framework on real world social networks, as a lossless compression scheme. The setting is as follows: we fix the parameter k, and design a greedy heuristic to find a sequence S that k-linearize a given graph G. We use an array data structure to store the sequence S and a binary tree to store the meta-data (see Chapter 4). The focus of the experimental study is to evaluate the compression rate for different values of k. We also report the query processing time of our lossless compression scheme.

- **Lossy Compression**

  In Chapter 5, we change the settings of our problem substantially. Instead of looking for a sequence S that is a k-linearization of a given graph G, we fix k and l, and ask for a sequence S of length l that k-covers those "important" edges of G. For the purpose of our work, an edge is "important" if it contributes to the global community structure

of the network. In this new setting, since both k and l are fixed, the framework uses a predefined number of bits to represent the graph. Therefore, we have to use a criteria other than compression rate to measure the effectiveness of our lossy compression scheme. We use two criteria: first, we ask how well it preserves the community structure of the network. Second, we measure the utility of a single bit, that is the number of edges over the number of bits in the compressed network. We introduce an objective function $f_k(S)$ to measure the quality of the sequence S with respect to the community structure of the network. We develop a local search heuristic to optimize $f_k(S)$. Our heuristic starts with a random sequence S of length l, and improves it by iterative insertion and deletion of the nodes. Therefore, to store S we need a data structure that supports efficient insertion and deletion operations. We have designed such a data structure by combining the advantages of linked lists and arrays.

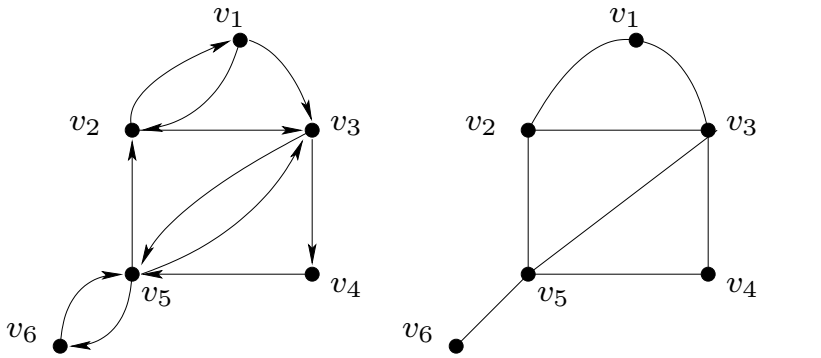
3.2 Theoretical Analysis

We start the formal discussion of this chapter by reviewing the background on graph theory, then, we proceed by relating the concept of 1-linearization to the well studied notion of Eulerian path. We conclude the chapter by an upper bound on the compression rate of our framework, when it incorporates a 1-linearization of the input graph.

3.2.1 Notions

In this dissertation, a social network is a *directed graph* $G = (V, E)$ where V is a set of vertices and $E \subseteq V \times V$ is a set of edges. We also refer to V by $V(G)$ and to E by $E(G)$. For an edge $e = (u, v)$, we refer to u as the *source* of e and v as the *destination* of e. Note that $(u, v) \neq (v, u)$.

A *simple directed graph* is a directed graph in which there does not exist a self-loop, i.e., no edge (u, u) for any vertex u, and there is at most one edge from a source u to a destination v. We consider simple directed graphs only. However, it is straightforward to generalize our results and algorithms to deal with directed graphs that contain self-loops



a) Directed graph G b) The underlying undirected graph $\bar{G}$ of G

$S_1 = (v_6, v_5, v_4, v_3, v_5, v_2, v_3, v_1, v_2)$
$S_2 = (v_6, v_5, v_4, v_3, v_1, v_2, v_5)$
$S_3 = (v_4, v_5, v_3, v_4, v_6, v_5, v_2, v_3, v_1, v_2)$

c) A set of sequences

Figure 3.1: A directed graph and its underlying undirected graph

and multiple edges between two vertices.

In an *undirected graph*, edges do not carry direction information, i.e., $\{u, v\} = \{v, u\}$. For a directed graph G, we can obtain the *underlying undirected graph* $\bar{G}$ of G such that $\{u, v\} \in E(\bar{G})$ if and only if $(u, v) \in E(G)$ or $(v, u) \in E(G)$. The *degree* of u is the number of edges that have u as an endpoint.

Figure 3.1 shows an example.

Hereafter, we call a simple directed graph simply a graph if there is no ambiguity. Occasionally, we use the notion of undirected graphs which will be mentioned explicitly.

For a graph G, the *transpose* of G, denoted by G^T, is a graph such that $V(G^T) = V(G)$ and $(u, v) \in E(G^T)$ if and only if $(v, u) \in E(G)$.

In a graph G, an edge $(u, v) \in E$ is called *reciprocal* if $(v, u) \in E$ as well. In such a case, u and v are immediately connected in both directions. Let $Fre(G)$ be the fraction of reciprocal edges in $E(G)$, i.e.,

$$Fre(G) = \frac{\text{number of reciprocal edges in } E(G)}{|E(G)|}.$$

Therefore, $G = G^T$ if and only if $Fre(G) = 1$.

3.2.2 Neighbor Queries in Directed Graphs

In a directed graph G, there are two types of neighbors. For a vertex $u \in V(G)$, $v_1 \in V(G)$ is an *out-neighbor* of u if $(u, v_1) \in E(G)$. Moreover, $v_2 \in V(G)$ is an *in-neighbor* of u if $(v_2, u) \in E(G)$. An *out-neighbor query* on u intents to find the set of out-neighbors of u. Similarly, an *in-neighbor query* on u would search for all in-neighbors of u.

Example 3 (Neighbor queries)**.** *In Figure 3.1(a), an out-neighbor query on v_5 in G returns $\{v_2, v_3, v_6\}$. An in-neighbor query on v_5 returns $\{v_3, v_4, v_6\}$. Please note that v_3 and v_6 are both out-neighbors and in-neighbors of v_5, since there are reciprocal edges between v_3 and v_5 as well as between v_5 and v_6.*

All the existing methods for compressing web graphs or social networks encode only outgoing edges. Consequently, those methods can only answer out-neighbor queries directly.

3.2.3 Eulerian Paths

A *path* P of length k in a graph G is a sequence of edges $(u_1, u_2), (u_2, u_3), \ldots, (u_k, u_{k+1})$, where $(u_i, u_{i+1}) \in E(G)$ $(1 \leq i \leq k)$. For the purpose of simplicity, we often write path P as $(u_1, u_2, \ldots, u_{k+1})$. P is a *simple path* if $u_1, \ldots, u_{k+1}$ are unique among one another.

Definition 8 (Eulerian path)**.** *An **Eulerian path** for an undirected graph is a path in the graph which visits each edge of the graph exactly once.*

Example 4 (Eulerian path)**.** *In Figure 3.1(b), path S_1 is an Eulerian path for $\bar{G}$, S_2 is not because it does not visit edge $\{v_5, v_3\}$. Likewise, S_3 is not an Eulerian path, because it is not a path – $\{v_4, v_6\}$ is not an edge in the graph $\bar{G}$.*

It is well known that a connected undirected graph G has an Eulerian path if and only if it has at most two vertices with odd degree.

A simple algorithm to construct the Eulerian path which dates back to 1883, known as Fleury's algorithm, is as follows: we start with a vertex of odd degree. If there is no such a

vertex, we start with any vertex. At each step we move across an edge whose deletion does not disconnect the graph, unless there is no other choice. We repeat this process until no edge is left.

3.2.4 1-Linearization

Here, we establish the connection between the notions of 1-linearization and Eulerian path for a given graph G. The following proposition is immediate:

Proposition 2 (Lower bound of 1-linearization)**.** *Given a directed graph G, the lower bound for the length of the 1-linearization of G is*

$$|E(\bar{G})| + 1 = (1 - \frac{Fre(G)}{2})|E(G)| + 1,$$

moreover the bound is tight if and only if $\bar{G}$ has an Eulerian path, or equivalently it has at most two vertices of odd degrees.

Example 5 (Lower bound of 1-linearization)**.** *The lower bound for the length of 1-linearization of G in Figure 3.1 is 9, since $|E(\bar{G})| = 8$. We can write $|E(\bar{G})|$ in terms of $|E(G)|$ using $Fre(G)$. Since $Fre(G) = 6/11$, $|E(\bar{G})| = (1 - \frac{6/11}{2})|E(G)| = 8$.*

Unlike the Eulerian path construction problem which is an existence problem, finding the optimal 1-linearization is an optimization problem. No matter what structure a graph has, always there exists an optimal (shortest) 1-linearization for the graph. The following lemma gives the length of an optimal 1-linearization of an arbitrary directed graph.

Lemma 1. *The minimum length for an 1-linearization of an arbitrary directed graph G is $|E(\bar{G})| + \max\{n_{odd}/2, 1\}$, where n_{odd} is the number of vertices with odd degrees in $\bar{G}$.*

Proof. In any undirected graph $\bar{G}$, the sum of degrees of all vertices is even. Thus, the number of vertices with odd degrees is also even.

We first prove by induction that for any graph G we can obtain an 1-linearization of length $|E(\bar{G})| + \max\{n_{odd}/2, 1\}$.

Basis: For $n_{odd} = 0$ and $n_{odd} = 2$, the claim follows from proposition 2.

Induction: Since n_{odd} must be even, we assume that the lemma holds for $n_{odd} = 2k$ ($k \geq 1$), and consider the case when $n_{odd} = 2(k+1)$. There are two subcases.

In the first subcase, there are two vertices u and v with odd degrees such that they are not connected in $\bar{G}$. We add an edge $\{u, v\}$ to $\bar{G}$ and call the new graph $\bar{G}_*$. Since $\bar{G}_*$ has only $2k$ vertices with odd degrees, applying the induction assumption, we can obtain an 1-linearization of $\bar{G}_*$ with length $|E(\bar{G}_*)| + k = |E(\bar{G})| + k + 1$. Since $E(\bar{G}) \subset E(\bar{G}_*)$ the 1-linearization of $\bar{G}_*$ is also an 1-linearization for $\bar{G}$.

In the second subcase, all vertices with odd degrees are connected. We arbitrarily take two vertices u and v with odd degrees and remove the edge $\{u, v\}$ from $\bar{G}$. Let us call the resulting graph $\bar{G}_*$. Again, $\bar{G}_*$ has $2k$ vertices with odd degrees. Therefore, there is an 1-linearization with length $|E(\bar{G}_*)| + k = |E(\bar{G})| - 1 + k$ for $\bar{G}_*$. Since the 1-linearization for $\bar{G}_*$ does not cover $\{u, v\}$, we have to add u and v to the end of the sequence. Therefore we just build an 1-linearization for $\bar{G}$ of length $|E(\bar{G})| + k + 1$, as desired.

We prove that $|E(\bar{G})| + \max\{n_{odd}/2, 1\}$ is also a lower bound for the 1-linearizations for G. For $n_{odd} = 0$ and $n_{odd} = 2$, simply applying Proposition 2 gives the bound.

Now, let us consider the case where $n_{odd} \geq 4$. Let $v(i)$ be the vertex that appears in the position i of an 1-linearization L. For a vertex u with an odd degree in $\bar{G}$ which appears in neither the first nor the last position of L, we denote by $P_u = \{i | v(i) = u\}$ the set of all appearances of u in L. There are in total at least $n_{odd} - 2$ such vertices.

For any interior position i in L, there are two edge slots in L: $(i-1, i)$ and $(i, i+1)$. Consider all the edge slots associated with the positions in P_u. At least one of these slots must be a "waste", that is, there is no edge appearing in the slot or the edge in the slot also appears in some other slot. Otherwise, the degree of u is $2|P_u|$, which is an even number.

In the best scenario, two vertices with odd degrees can share a wasted edge slot. Therefore, we have at least $(n_{odd} - 2)/2$ wasted edge slots. In addition, we need $|E(\bar{G})|$ edge slots to cover the edges of $\bar{G}$. Therefore, in total L has to have at least $|E(\bar{G})| + n_{odd}/2 - 1$ edge slots. Hence, the length of L cannot be smaller than $|E(\bar{G})| + \max\{n_{odd}/2, 1\}$. □

Please note that the induction in the proof of Lemma 1 also gives an linear algorithm

Algorithm 5 to find an optimal 1-linearization of G

Input: Graph G
Output: L

```
1: Let O be the set of all vertices in G with odd degree
2: EE = ∅; RE = ∅
3: // eliminate all the nodes with odd degree by adding or removing edges between pair
   of them
4: while there are u, v ∈ O do
5:    if (u, v) ∉ E(G) then
6:       add (u, v) to E(G)
7:       add (u, v) to EE
8:    else
9:       remove (u, v) from E(G)
10:      add (u, v) to RE
11:   end if
12:   remove u and v from O
13: end while
14: Let L be the Eulerian tour of G given by Hierholzer's algorithm
15: for all (u, v) ∈ RE do
16:   add u and v to the end of L
17: end for
```

to find an optimal 1-linearization of a graph G. Before presenting the algoirthm we explain Hierholzer's algorithm [32] that is a linear time (linear to the number of edges) algorithm for finding an Eulerian path in a given graph. The algorithm is as follows: choose any vertex v as the starting vertex, and follow a trail of edges from that vertex until returning to v. Note that if the degree of all edges are even it is not possible to stuck in any vertex other than v, because by entering any vertex other than v, the number of remaining edges for that vertex will be an odd number and therefore nonzero. Thus, we will endup at v. However, the tour might not cover all the edges. In case there is a node in the current tour that has unused edges, start another trail, using only unused edges, from that vertex, and join the new tour to the old one. By the use of the simple doubly linked list data structre to maintain the list of all nodes with unused edges and the list of unused edges for each node, this algorithm can be implemented in linear time.

Algorithm 5 returns an optimal 1-linearization L of a given graph G. The proof of

Lemma 1 is an induction and therefore constructive. In fact each iteration of the while loop in the algorithm is corresponding to one step of induction in the proof of the lemma. Since the complexity of finding an Eulerian path is linear in the number of edges, finding an optimal 1-linearization of a graph G is also of the same complexity.

Example 6. *In Figure 3.2(a), G has four vertices of odd degrees, namely v_9, v_{10}, v_{11} and v_{12}. Therefore, the lower bound on the length of 1-linearization is $15+4/2 = 17$. Therefore, the 1-linearization of Figure 3.2(b) is optimal.*

3.3 Upper Bound on Compression Rate

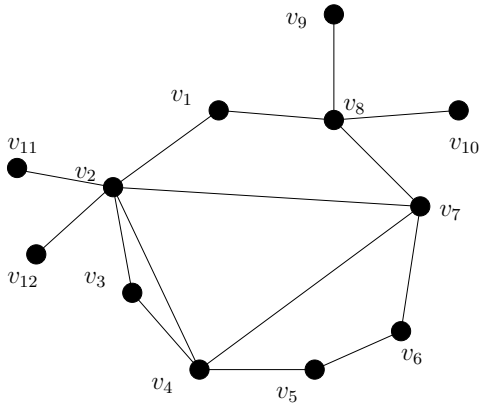
Based on the notion of 1-linearization, we present a novel data structure for encoding graphs. To keep our discussion simple, similar to [8, 9, 14], we assume (1) we are allowed to renumber the vertices; (2) for each vertex there is an identifier which can be used for referring to the vertex. However, our data structure does not maintain an index of the identifiers; and (3) the edges are not labeled. Please note that we can straightforwardly extend the data structure to remove the above assumptions.

Definition 9 (Eulerian data structure)**.** *The **Eulerian data structure** for a graph G stores an optimal 1-linearization L of G using an array of the same length as L. Let $v(i)$ be the vertex in G that appears at the position i of L. For cell i of the Eulerian data structure, we keep the following two pieces of information*

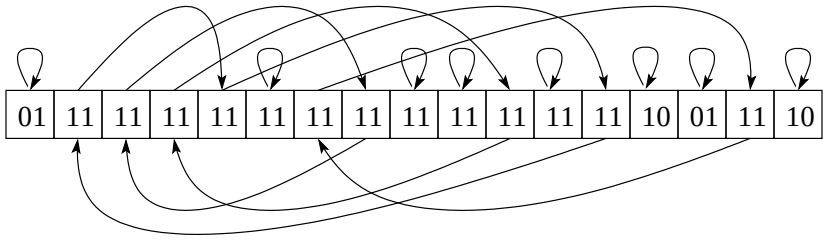
***Local information:** two bits specifying if edges $(v(i-1), v(i))$ and $(v(i), v(i-1))$ belong to $E(G)$, respectively.*

***Pointer:** a pointer to the next appearance of $v(i)$. If this is the last appearance of $v(i)$, then the pointer points to the first appearance of the vertex.*

Example 7 (Eulerian data structure)**.** *In Figure 3.2(b), the Eulerian data structure of G, in Figure 3.2(a), using an optimal 1-linearization is illustrated. Here we show the pointers by arcs. Since the length of the linearization is 17, we need $\lceil \log_2 17 \rceil = 5$ bits to encode each*



(a) Graph G. The edges that does not have direction are equivalent with two directed edges



(b) Illustration of the 1-linearization of G and the Eulerian data structure (pointers are depicted by the arcs).

01, 0	11, 4	11, 7	11,10	11,12	11, 5	11,15	11, 3	11, 8	11, 9	11, 3	11,11	11, 1	10,13	01,14	11, 6	10,16
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
v_{11}	v_2	v_7	v_4		v_1	v_8		v_6	v_5		v_3		v_{12}	v_9		v_{10}

(c) The Eulerian data structure. Each cell stores a 2-bits label and a pointer to the next appearance of the same node. The index of the first appearance of a node is the new identifier of that node.

Figure 3.2: The Eulerian data structure. The new identifiers of the nodes are illustrated.

pointer. Therefore, for each position we need $5+2$ *bits. In total we need* $17 \times (5+2) = 119$ *bits.*

We prove an upper bound on the size of the Eulerian data structure:

Lemma 2. *An Eulerian data structure to encode a graph* G *uses at most*

$$\frac{|V(\bar{G})|\Big(\bar{d}+1\Big)}{2}\Big(1+\log(|V(\bar{G})|)+\log(\bar{d}+1)\Big)$$

bits, where $\bar{d}$ *is the average degree of* $\bar{G}$*.*

Proof. Let L be an optimal 1-linearization of $\bar{G}$ (therefore for G as well). Since there are at most $|V|$ vertices of odd degrees in $\bar{G}$, the upper bound for the length of L is $|E(\bar{G})| + |V(\bar{G})|/2$. Using 2 bits to store the local information and $\lceil \log_2(|E(\bar{G})| + |V(\bar{G})|/2) \rceil$ bits for the pointer for each cell, in total the Eulerian data structure uses at most

$$\Big(|E(\bar{G})| + |V(\bar{G})|/2\Big)\Big(2 + \lceil \log_2(|E(\bar{G})| + |V(\bar{G})|/2) \rceil\Big)$$

bits. We can write $|E(\bar{G})| = |V(\bar{G})|\bar{d}/2$, where $\bar{d}$ is the average degree of $\bar{G}$. Therefore:

$$\frac{|V(\bar{G})|\Big(\bar{d}+1\Big)}{2}\Big(1+\log(|V(\bar{G})|)+\log(\bar{d}+1)\Big)$$

□

We have the following result on the compression efficiency of the Eulerian data structure.

Theorem 1. *An Eulerian data structure to encode a graph* G *uses at most*

$$(1 - \frac{Fre(G)}{2} + \frac{1}{\bar{d}})\Big(\lceil \log_2(|V(\bar{G})|) + \log_2(\bar{d}+1) \rceil + 1\Big)$$

bits per edge on average, where $\bar{d}$ *is the average degree of* $\bar{G}$*. Moreover, using this data structure, it is possible to answer the in-neighbor and out-neighbor queries for any vertex* v *in* $O(\sum_{u \in N_v} deg(u) \log |V(G)|)$ *time, where* $deg(u)$ *is the degree of vertex* u *in* $\bar{G}$ *and* N_v *is the set of out-neighbors/in-neighbors of* v *in out-neighbor/in-neighbor queries.*

Proof. To get the bits/edge rate, we divide the upper bound on the size of the data structure by the number of edges of G. We have

$$\Big(\frac{|E(\bar{G})| + |V(\bar{G})|/2}{|E(G)|}\Big)\Big(2 + \lceil \log_2(|E(\bar{G})| + |V(\bar{G})|/2)\rceil\Big).$$

Notice that we can write the ratio of $|E(\bar{G})|/|E(G)|$ in terms of $Fre(G)$ and also

$$\frac{|V(\bar{G})|}{2|E(G)|} \leq \frac{|V(\bar{G})|}{2|E(\bar{G})|}$$

which is precisely the inverse of the average degree of $\bar{G}$. Therefore, the bits/edge rate is at most:

$$\Big(1 - \frac{Fre(G)}{2} + \frac{1}{\bar{d}}\Big)\Big(2 + \lceil \log_2(|E(\bar{G})| + |V(\bar{G})|/2)\rceil\Big)$$

We use $|E(\bar{G})| = \frac{|V(\bar{G})|\bar{d}}{2}$ to further simplify the inside of the logarithm, and obtain

$$\begin{aligned}&\Big(1 - \tfrac{Fre(G)}{2} + \tfrac{1}{\bar{d}}\Big)\Big(2 + \lceil \log_2(|V(\bar{G})|\bar{d}/2 + |V(\bar{G})|/2)\rceil\Big)\\ &= \Big(1 - \tfrac{Fre(G)}{2} + \tfrac{1}{\bar{d}}\Big)\Big(2 + \lceil \log_2(|V(\bar{G})|) + \log_2(\bar{d}+1) - \log_2(2)\rceil\Big)\end{aligned}$$

The upper bound is proved.

For analysis of the query processing time, we assume that the data structure is a sequence of bits. Hence reading a pointer takes $O(\log(|V(G)|))$ time.

Since each vertex u of G appears in at least one position in the Eulerian data structure, we use the index of the first position of u in L as the identifier for the vertex. Therefore, for an out-neighbor/in-neighbor query on vertex u, we have to return the positions of the first appearances of all out-neighbors/in-neighbors of u. Fetching the local information (only two bits) for one position takes constant time. Reading the pointer takes $O(\log |V(G)|)$ time (the number of bits for each pointer). Since the number of copies for a vertex u is $\lceil deg(u)/2\rceil$ in the optimal 1-linearization of G, traversing over the linked list takes $O(deg(u) \log_2 |V(G)|)$ time. By traversing the linked list corresponding to vertex u we can retrieve the positions of all neighbors of u. However, for a neighbor v of u, the retrieved position may not be the first appearance of v. Therefore, for each retrieved neighbor v, we have to traverse the linked list for v to get the first appearance. So, answering an out-neighbor/in-neighbor query takes $O(\log(|V(G)|) \sum_{u \in N_v} deg(u))$ time in total. □

	Number of vertices			
Average degree	10^3	10^5	10^7	10^9
10	0.83	0.71	0.66	0.64
100	0.91	0.74	0.67	0.64
500	1.00	0.80	0.71	0.67

Table 3.1: Comparison of compression using the Eulerian data structure against the baseline schema. The baseline schema is the adjacency list. We presented the ratio of the number of bits that the Eulerian data structure uses over the number of bits that adjacency list representation uses.

As a baseline for representing a graph G with sublinear in-neighbor and out-neighbor queries, the adjacency list representation uses $2\lceil \log_2 |V(G)| \rceil$ bits to encode an edge. For social networks in practice, it is reasonable to assume that the average degree increases logarithmically with respect to the number of nodes in the graph. Therefore, asymptotically (that is, assuming the number of nodes approaches infinity) the Eulerian data structure uses half of the number of bits that the baseline schema uses due to the following equation:

$$\lim_{|V(G)|\to\infty} \frac{(1+\frac{1}{\log_2 |V(G)|})(\log_2(|V(G)|) + \log_2 \log_2(|V(G)|) + 1)}{2\log_2(|V(G)|)} = \frac{1}{2}$$

Readers familiar with the literature of the social network analysis might recall the densification law [36]. The densification law claims that the number of edges in a real world social network G grows proportional to $|V(G)|^{1+\alpha}$ where α is a real number between zero and one. In such a case we can redo the asymptotical analysis:

$$\lim_{|V(G)|\to\infty} \frac{(1+\frac{1}{|V(G)|^{\alpha}})((1+\alpha)\log_2(|V(G)|) + 1)}{2\log_2(|V(G)|)} = \frac{1+\alpha}{2}$$

In this equation, since the number of edges grows proportional to $|V(G)|^{1+\alpha}$ the average degree is proportional to $|V(G)|^{\alpha}$.

Table 3.1 compares the compression rate of our method and the baseline schema for a number of combinations of $\bar{d}$ and $|V(G)|$. Clearly, the larger and the sparser the graph,

the less bits the Eulerian data structure uses. Real life social networks are often large and sparse, therefore, the Eulerian data structure is capable of compressing social networks.

To the best of our knowledge, the Eulerian data structure is the first schema that allows answering both out-neighbor and in-neighbor queries in sublinear time, and provides a nontrivial upper bound on the number of bits per edge. Note that to prove the upper bound given by Theorem 1 we did not use any assumption on the type of the network, therefore the result is not limited to the social networks and is valid for arbitrary graphs.

3.4 Summary

In this chapter, we introduce a compression framework. Theoretically, we prove an upper bound on the compression rate for a simple setting of the framework. This theoretical analysis can serve as a good evidence for the storage efficiency of our framework. For the next two chapters of this dissertation, we are going to consider two more practical settings for our framework.

In Chapter 4, we use the principles of the GPSN framework to design a lossless compression scheme for social networks. The lossless compression scheme uses a relaxed version of k-linearization, which we call multi level linearization. We introduce this new notion to be adaptive to the "skewed" structure of the real world social network.

Chapter 4

Lossless Compression

The GPSN framework has three components, namely: (1) a linearization S of the given graph, (2) a data structure to store the sequence S, and (3) an algorithm to build the sequence S. In the previous chapter, we used the notion of k-linearization to prove an upper bound for the compression rate of the GPSN framework. However, in a more realistic setting having a fixed parameter k for linearizing a real world social network is not always practical. For the rest of this chapter, first, we introduce a relaxed notion for linearizing a social network. Then, we explain the data structure to store the linearization. We present a simple greedy heuristic to build such a linearization. Finally, we conclude the chapter with the experimental results.

4.1 Multi Level Linearization

We start our discussion based on the known fact that the edges in real world social networks are highly correlated. The building blocks of social networks are communities. The edges are often local in the sense that if a node belongs to a community, a statistically significant fraction of its neighbors tend to be in the same community. Therefore, a large fraction of the edges in a given social network are intra-community edges. Intuitively, the parameter k in the k-linearization is trying to capture the degree of locality in the network, that is the

size of the communities in a given social network.

The size and structure of such communities have been the focus of many recent works. While Leskovec *et al.* [37] suggested that the communities are typically no larger than a hundred nodes, it is unrealistic to assume all of them have the same size. Therefore, using multiple values of k to linearize a social network is a more effective approach. The notion of multi-level linearization is designed to capture such intuition. A multi-level linearization is a sequence S of nodes, such that S is partitioned to several segments and each segment uses its own value of k. The definition of multi-level linearization is as follows:

Definition 10 (Multi-Level Linearization)**.** *A multi-level linearization of a given graph G is a sequence S and a list of integer pairs $(i_1, k_1), \cdots, (i_m, k_m)$, such that $i_1 = 0 < i_2 < \cdots, < i_m < length(S)$ and $k_1 > k_2, \cdots, k_m > 0$. Then, assuming $S_t = S[i_t, i_{t+1} - 1]$ we have*[1]*:*

$$\cup_{t=1}^{m} E_{k_t}(S_t) = E(G),$$

where $E_{k_t}(S_t)$ is the set of all edges that are k-covered by S_t. We refer to this list as the neighborhood scheme of S.

A commonly used notion for measuring the locality in social networks is the average clustering coefficient [62], that is:

$$Acc(G) = Acc(\bar{G}) = \frac{1}{|V(\bar{G})|} \sum_{v \in V(\bar{G})} \frac{2|E_v|}{|N_v|(|N_v| - 1)},$$

where E_v is the set of edges that have both ends among the neighbours of v. In social networks, the number of vertices with a small degree is much more than the number of vertices with a large degree. Consequently, average clustering coefficient has a bias towards the vertices with small degrees. That is, since the low degree vertices, e.g. vertices with degree one, tend to have large clustering coefficient, the average clustering coefficient is misleadingly large. Therefore, to measure the locality of a network we use Global Clustering

[1] *The notation $S[i_t, i_{t+1} - 1]$ refers to the subsequence of S that starts from position i_t and ends at position $i_{t+1} - 1$*

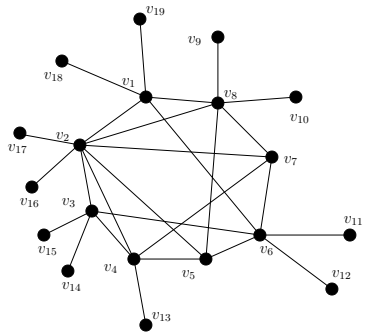
Coefficient [62], which is the ratio of three times the number of triangles over the number of paths of length two:

$$Gcc(G) = Gcc(\bar{G}) = \frac{2\sum_{v\in V(\bar{G})} |E_v|}{\sum_{v\in V(\bar{G})} |N_v|(|N_v|-1)},$$

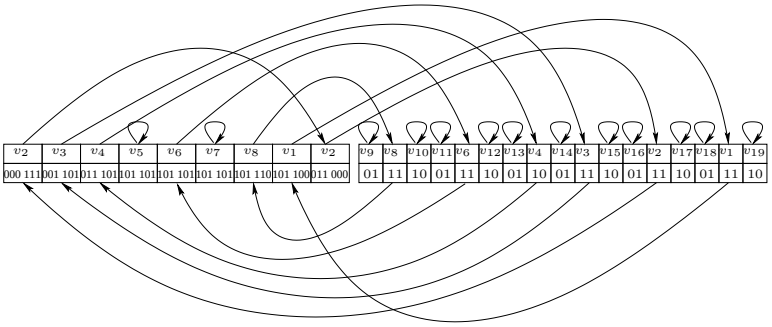
Roughly speaking, Gcc measures the likelihood that there is an edge between two vertices when they have a common neighbor. Consider a social network with a large Gcc value. Suppose vertices u, v, w and t are four consecutive vertices in a linearization, and there are edges between u and v, v and w, as well as w and t. There is a good chance that u and w are connected, since both have v as a neighbor. Moreover, if u and w are connected, using the same argument, there is a good chance that u and t are also connected. This argument can be used for farther vertices to u again and again. Depending on the value of the Gcc, it does make sense to keep more bits to encode the edges of u, specifying if $(u, v), (u, w)$ and (u, t) belong to $E(G)$ or not.

There is a tradeoff between the length of the linearization against the amount of local information. The tradeoff highly depends on the structure of the graph. Intuitively, for a large sparse graph where each vertex has the same out degree, and the destinations of the out-edges are picked randomly, increasing k would not influence the length of the linearization significantly. However, for a random dense graph G where the existence of an edge from every node to another is independently determined by a probability of 50%, increasing k up to $|V(G)|$, the number of vertices, is actually beneficial. Note that the extreme case of storing $|V(G)|$ bits for each vertex is roughly corresponding to the adjacency matrix representation of G.

Example 8. *Figures 4.1(a) is illustrating a graph G. G consists of a set of high degree vertices, which make the core of the network. A set of dangling low-degree vertices are loosely connected to the core. Note that this is consistent with the structure of the social networks that come from the real world. Figures 4.1(b) shows a multi-level linearization of G. The neighborhood scheme for this linearization is $\{(0, 3), (9, 1)\}$. The first segment of the sequence (positions 0 to 8) encode the edges between the vertices of the core of G, while the rear part (positions 9-25) encode the dangling edges.*



(a) Graph G



(b) Multi-level linearization of G.

Figure 4.1: Multi-Level Linearization. The neighborhood scheme is $\{(0,3),(9,1)\}$. Arcs point to the next appearance of the same vertex. The cells are indexed from left to right. Position 0 is the most left cell and position 25 is the most right cell.

Each position in the first segment of the sequence uses 6 *bits to encode the edges and* 5 *bits for the next pointer. The second segment uses* 2 *bits for the labels and* 5 *bits for the pointers. In total, it uses* $9 \times (6+5) + 17 \times (2+5)$*=218, which makes a compression rate of* $218/(28 \times 2) = 3.89$*. Note that each undirected edge is equivalent to two directed edges.*

4.2 Data Structure

We present the Eulerian data structure in this section:

Definition 11 (Multi-Level Data Structure)**.** *Let S be a multi-level linearization of a given graph G, and $(i_1, k_1), \cdots, (i_m, k_m)$ be the neighborhood scheme of S. The **Multi-Level Data Structure** stores the graph G using m arrays of the size $i_{a+1} - i_a$ for $1 \leq a \leq m$, where m is the length of the neighborhood scheme and i_{m+1} is equal to the length of S. The a-th array ($1 \leq a \leq m$) encodes the information corresponding to the positions i_a up to $i_{a+1} - 1$ of the linearization. In cell i of the a-th array, we keep the following two pieces of information:*

***Local information:** k_a bits to encode the edges and non-edges from $S(i_a + i)$ to $S(i_a + i + 1), \cdots, S(i_a + i + k(i))$, likewise, another k_a bits to encode edges and non-edges from $S(i)$ to $S(i - 1), \cdots, S(i - k(i))$.*

***Pointer:** a pointer to the next appearance of $S(i)$. If this is the last copy, the pointer points to the first appearance of the vertex.*

In this data structure, the index of the first appearance of a node is the identifier of that node.

4.3 Compression Heuristic

Unlike 1-linearization, finding the most memory-efficient multi-level linearization is highly challenging[2]. Instead, in this section, we focus on the designing of a heuristic that finds a "good" multi level linearization without much cost.

To ensure that we can handle large social networks, we use a straightforward greedy heuristic for linearizing a graph. The pseudo code is shown in Algorithm 6. Note that this algorithm assumes the network fits in the main memory. We start with a random vertex and a starting value of k. At each step we append to the list the vertex that has the largest number of edges with the last k nodes in the list. We remove these edges from the graph and iterate until no edge is left. If none of the last k vertices in the list have a neighbor,

[2]Finding the shortest multi-level linearization is a generalization of the problem of finding the shortest k-linearization. Also, the latter problem is the generalization of Min-Bandwidth problem [47] which is a NP-hard problem.

Algorithm 6 to find a multi level linearization of G

Input: K, reducing factor RF ($0 \leq RF \leq 1$), density threshold DT ($0 \leq DT \leq 1$) and Graph G
Output: Linearization L of G

```
1: initialize L to an empty list
2: while |E(G)| ≥ 1 do
3:    let u be a random node with nonzero degree
4:    append u to L
5:    {let X be the set of the last K vertices in L}
6:    while X has at least one neighbor in V(G) − X do
7:       let v be the node which has the most number of edges to and from X
8:       remove all edges between v and vertices in X
9:       edgecount+ = deg_old(v) − deg_new(v)
10:      append v to L
11:      if Length(L)%1000==0 then
12:         if edgecount/(2 ∗ K ∗ 1000) < DT then
13:            K = K ∗ RF
14:         end if
15:         edgecount = 0
16:      end if
17:   end while
18: end while
```

then we pick a random node with non-zero degree and continue from there.

As we are removing the edges of the graph, the graph becomes sparser and sparser and eventually the rear part of the linearization may have very few new edges to encode. To be adaptive, we watch the average local density for the recent positions in the list (the last 1000 positions[3] as shown in Algorithm 6). Once it drops below a certain density threshold DT, we reduce k by multiplying it to a predefined reducing factor RF. Note that, this algorithm guarantees that the length of the neighborhood scheme is short, i.e. $O(\log k)$, assuming $0 < RF < 1$.

We chose to use a simple heuristic to examine the feasibility of using the GPSN framework for compressing social networks. Our method leaves space for further improvement which could be the subject for future work.

[3]Experimentally we used the last 100 up to 10000 positions for computing the average local density. It turns out that the algorithm is not sensitive to this number as long as it is more than a few hundreds.

Name	$\|V\|$	$\|E\|$	Acc	Gcc	Fre
amazon0302	262111	1234877	0.424	0.236	0.542
amazon0312	400727	3200440	0.411	0.160	0.531
ca-CondMat	23133	186878	0.633	0.264	1
ca-HepPh	12006	236978	0.611	0.659	1
cit-HepPh	34546	421534	0.296	0.145	0.003
cit-Patents	3774768	16518947	0.091	0.067	0
email-Enron	36692	367662	0.497	0.085	1
email-EuAll	265009	418956	0.309	0.004	0.260
p2p-Gnutella08	6301	20777	0.015	0.020	0
p2p-Gnutella24	26518	65369	0.009	0.004	0
soc-Slashdot0902	82168	870161	0.061	0.024	0.841
soc-LiveJournal1	4846609	68475391	0.312	0.288	0.374
web-Google	875713	5105039	0.604	0.055	0.306
web-Stanford	281903	2312497	0.610	0.096	0.276

Table 4.1: The dataset stats. (Acc, Gcc, and Fre are defined in Section 3.2.1)

4.4 Experiments

To the best of our knowledge, there is no existing social network compression method that can answer out-neighbor and in-neighbor queries in sublinear time without replicating the network. For example, in case of adjacency list, to find the in-neighbors of a given node u one has to search all other vertices. However, the existing methods that answer out-neighbor queries can be made comparable to ours in functionality by encoding a given graph G and also its transpose G^T.

4.4.1 Experimental Setup

We used the data sets from the SNAP project (Stanford Network Analysis Package, `http://snap.stanford.edu/data/`). The data sets in the SNAP project are organized in different categories. From each category we chose the data sets with the smallest and the largest *Gcc* values in order to test the effect of our method with respect to social networks of different degrees of locality. Those data sets are from very different domains, such as social

networks, web graphs, peer-to-peer networks, collaborative networks, citation networks, and co-purchasing networks. Table 4.1 provides the statistics of these networks and a short descriptions. The description of the datasets according to `http://snap.stanford.edu/data`:

- *a*mazon0302: this network is collected by crawling Amazon website. It is based on *"Customers Who Bought This Item Also Bought"* feature of the website. If a product x is frequently co-purchased with product y, the graph contains a directed edge from x to y. The network was collected in March 02 2003.

- *c*a-CondMat: the Arxiv COND-MAT (Condense Matter Physics) is a collaboration network from the e-print arXiv and covers scientific collaborations among authors of papers submitted to Condense Matter category. If an author x co-authored a paper with author y, the graph contains an undirected edge from x to y. If the paper is co-authored by k authors this generates a completely connected subgraph on k nodes.

  The data cover papers in the period of January 1993 to April 2003. This period starts within a few months of the inception of the arXiv, and thus represents essentially the complete history of its COND-MAT section.

- *c*a-HepPh: Arxiv HEP-PH (High Energy Physics - Phenomenology) is a collaboration network from the e-print arXiv, and covers scientific collaborations among authors, based on papers submitted to the High Energy Physics Phenomenology category. If an author x co-authored a paper with author y, the graph contains an undirected edge from x to y. If the paper is co-authored by k authors this generates a completely connected (sub)graph on k nodes.

  The data cover papers in the period of January 1993 to April 2003. This period starts within a few months of the inception of the arXiv, and thus represents essentially the complete history of the HEP-PH section.

- *c*it-HepPh: the Arxiv HEP-PH (High Energy Physics Phenomenology) is a citation graph from the e-print arXiv, and covers all the citations within a dataset of 34,546

papers with 421,578 edges. If a paper x cites paper y, the graph contains a directed edge from x to y. If a paper cites a paper outside the dataset, the graph does not represent any information about this.

The data cover papers in the period from January 1993 to April 2003. This period starts within a few months of the inception of the arXiv, and therefore represents essentially the complete history of the HEP-PH section.

- *c*it-Patents: the U.S. patent dataset is maintained by the National Bureau of Economic Research. The data set cover 37 years (January 1, 1963 to December 30, 1999), and include all the utility patents granted during this period, including 3,923,922 patents. The citation graph includes all citations made by patents granted between 1975 and 1999, including 16,522,438 citations. For the patents dataset there are 1,803,511 nodes for which there is no information about their citations (only in-links are available).

- *e*mail-Enron: the Enron email communication data cover all the email communication within a dataset of about half million emails. This data was originally made public, and posted to the web, during a federal investigation. Nodes of the network are email addresses and if an address x sent at least one email to address y, the network contains an undirected edge from x to y. Note that for non-Enron email addresses, we only observe their communication with the Enron email addresses.

- *e*mail-EuAll: this network is generated using email data from a large European research institution. For a period from October 2003 to May 2005, we have anonymous information about all incoming and outgoing emails of the research institution. For each email message, we know the time, the sender and the recipient of that email. In total, we have 3,038,531 emails, sent or received by 287,755 different email addresses.

- *p*2p-Gnutella08: a snapshot of the Gnutella peer-to-peer file sharing network from August 2002. The nodes of the network represent hosts, and edges represent connections between the Gnutella hosts.

- *p*2p-Gnutella24: another snapshot of the Gnutella peer-to-peer file sharing network from August 2002. The nodes of the network represent hosts, and edges represent connections between the Gnutella hosts.

- *s*oc-Slashdot0902 slashdot.org is a technology-related news website which is known for its specific user community. The website features user-submitted and editor-evaluated technology oriented news. In 2002, the website introduced the Slashdot Zoo feature which allows users to tag each other as friends or foes. The network represents friend/foe links among the users of Slashdot. The network is collected in February 2009.

- *s*oc-LiveJournal1: LiveJournal is a free on-line social network with almost 10 million members; a significant fraction of these members are highly active. For example, roughly 300,000 users update their content in any given 24-hour period. LiveJournal allows members to maintain journals, individual and group blogs, and it allows people to declare which other members are their friends. The data was released on 2006

- *w*eb-Google: this network is the web-graph of google.com domain. The nodes of the network represent web pages in the domain, and directed edges represent hyper-links among them. The data set released in summer 2002 as part of Google Programming Contest.

- *w*eb-Stanford: this network is the web-graph of stanford.edu domain. The nodes of the network represent web pages in the domain, and directed edges represent hyper-links among them. The data was collected in 2002.

We implemented our algorithms using C++, on top of the SNAP[4] library which is publicly available at `http://snap.stanford.edu/`. We used a heterogeneous Linux based cluster to run most of the experiments. To report the running time, we selected a subsets of our experiments and ran them on a core(TM)2 Duo 2.66GHz Linux system with 2GB of main memory.

[4]Stanford Network Analysis Package

Our method has three parameters: Reducing Factor (RF), (Starting) neighborhood size (K) and Density Threshold (DT). The last two parameters are more important than the first one, since they have direct control on the generated linearization. Therefore, we conducted an extensive experimental study on different values of those two parameters for each network in our collection. Particularly, we are interested in the tradeoff between the length of the linearization and the neighborhood size. That is, whether starting from a larger neighborhood size would reduce the length of the linearization significantly.

We measured the compression performance using the bits/edge rate, as the previous studies did. In addition, we also report query processing time for the multi level data structure.

Another interesting tradeoff in our method is between the out-neighbor query processing time and in-neighbor query processing time. An implementation decision is how to store the local information for each position. There are two options: (1) for each position of the linearization, we use the first k bits to record the out-edges to the previous k vertices in the linearization sequence, and use the next k bits to record the out-edges to the next k vertices. (2) we use $2k$ bits to encode both the out-edges and in-edges between the current position and the next k positions.

The first option biases the compression scheme towards the out-neighbor queries. To answer an in-neighbor query on a vertex u, we have to scan the k positions preceding and following every occurrence of u in the linearization sequence. We implemented the second option in our experiments which is not biased on any specific types of neighbor queries.

4.4.2 Comparison of the Compression Rates

Table 4.2 summarizes the results on compression rate. While the performance of our method varies on different data sets, the interesting observation here is the strong negative correlation between the bits/edge rate and the value of locality measures, Fre, Gcc and the average degree of the network. In particular, Fre and Gcc are larger in Amazon0302 than in Amazon0312, however the performance of our method is better on Amazon0312. We believe that

(K, reducing factor)	(10, 1)	(10, 0.9)			(15, 0.9)			(20, 0.9)			(30, 0.9)		
Density threshold	0	0.15	0.25	0.30	0.15	0.25	0.30	0.15	0.25	0.30	0.15	0.25	0.30
amazon0302	15.38	14.61	13.99	14.43	15.08	13.97	14.16	15.09	13.98	14.49	15.39	14.07	14.49
amazon0312	14.35	13.32	12.70	12.79	13.57	12.74	12.84	13.92	12.73	12.90	14.08	12.79	12.86
ca-CondMat	7.89	7.69	6.96	6.69	8.35	7.16	6.77	8.94	7.33	6.93	9.55	7.56	7.26
ca-HepPh	5.24	5.09	4.76	4.63	5.00	4.59	4.57	5.20	4.65	4.53	5.51	4.79	4.69
cit-HepPh	17.07	15.65	14.59	14.23	15.99	14.69	14.29	16.47	14.85	14.31	16.97	15.02	14.48
cit-Patents	31.59	27.69	25.95	25.75	27.63	25.97	25.69	27.73	25.95	25.69	27.78	25.97	25.78
email-Enron	8.72	8.11	7.39	7.26	8.53	7.47	7.27	8.88	7.52	7.31	9.19	7.64	7.44
email-EuAll	30.73	25.31	22.96	22.55	25.63	22.97	22.55	25.56	22.97	22.61	25.81	23.11	22.72
p2p-Gnutella08	30.36	25.48	22.90	21.63	26.70	23.88	23.42	29.82	27.13	26.88	33.84	33.21	33.21
p2p-Gnutella24	35.76	29.51	25.59	24.33	28.67	25.69	24.93	29.41	26.90	26.02	31.25	28.94	28.10
soc-Slashdot0902	16.17	14.19	12.68	12.14	14.55	12.69	12.15	14.63	12.68	12.17	14.75	12.74	12.19
soc-LiveJournal1	16.13	14.48	13.96	13.97	14.50	13.92	13.93	14.49	13.95	13.93	14.56	13.91	13.95
web-Google	12.84	12.22	11.63	11.66	12.29	11.58	11.68	12.74	11.61	11.70	12.99	11.59	11.65
web-Stanford	10.79	10.27	10.17	10.76	10.19	10.23	10.41	10.14	10.05	10.22	10.19	9.88	9.92

Table 4.2: The average number of bits per edge. The worse cases happen on those data sets that have very poor locality measures (Gcc and Fre)

this is due to the higher average degree in Amazon0312 than Amazon0302.

It is interesting to look at the difference between email-Enron and email-EuAll data sets from the same category. the email-Enron data set has one of the best bits/edge rates and email-EuAll has one of the worse. We believe this is a footprint of the difference in communication patterns in industry and in academia.

The results clearly shows that our method can exploit the locality properties of social networks. Our best result for the LiveJournal data set is 13.91 bits/edge, while the best result of BV scheme for the same data set is 14.308 (reported in [14]). Please note that BV scheme supports only the out-neighborhood queries. To answer both out-neighbor and in-neighbor queries, the BV schema needs 2×14.308 bits per edge, assuming that encoding the transpose of the graph has approximately the same rate. Moreover, our method is flexible for incremental updates. We only need to encode the incremental subgraph. The BV schema does not allow sublinear updates.

4.4.3 Query Processing Time

We report the query processing time for two types of queries. An adjacency query reports whether a query edge (u, v) belongs to E. A neighbor query searches for all out-neighbors and in-neighbors of a query vertex u.

	adj queries(ns)		Neigh. queries(ns)	
dataset	comp.	SNAP	comp.	SNAP
amazon0302	800	750	951	72
amazon0312	1170	790	1753	46
ca-CondMat	390	420	777	30
ca-HepPh	520	400	1849	19
cit-HepPh	1300	480	2745	28
cit-Patents	1400	930	1842	91
email-Enron	620	500	5539	31
email-EuAll	530	670	21518	148
p2p-Gnutella08	640	320	1663	34
p2p-Gnutella24	600	320	1488	50
soc-LiveJournal1	3050	1130	9734	49
soc-Slashdot0902	1380	610	7884	35
web-Google	810	830	4110	66
web-Standford	890	810	39939	49

Table 4.3: The average access time per edge for processing adjacency queries and (in+out) neighbor queries.

We used $K = 20, RF = 0.9$ and $DT = 0.25$ as the default values for the parameters. Table 4.3 compares the average access time for the adjacency queries performed on the compressed graphs (comp.) and on the original graphs (SNAP), using the SNAP implementation of the graph data structure. We ran 1 million adjacency queries and 1 million neighborhood queries, and normalized the time by the number of edges that those queries returned. The time is in nano second.

Our method spends up to 3 times more time to answer an adjacency query than that on the original graph. In most cases, extra cost in our method is very minor. For neighbor queries, the query answering time depends on the efficiency of the linearization. We believe at least one order of magnitude in the reported neighbor query time is due to bit-level encoding of the information. That is because to read a pointer we have to read it one bit by one bit.

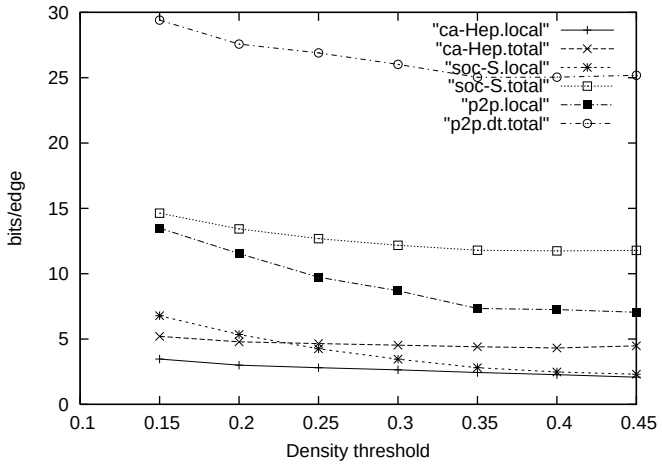


Figure 4.2: The trade off between the bits/edge rate of local information and that of pointers on three data sets: ca-HepPh, p2p-Gnutella24, and soc-Slashdot0902. ($K = 20$, $RF = 0.9$)

4.4.4 Tradeoff between Local Information and Pointers

We divide the compression rate in our method into two parts, the rate for encoding the local information, and that for encoding the pointers. The total compression rate is simply the sum of the two.

We studied the tradeoff between the local bits/edge rate and that of the pointers, while we varied the parameters of our method. We report here the tradeoff for two parameters: the density threshold (DT) and the starting window size (K). We chose three data sets: ca-HepPh which has the best compression rate, p2p-Gnutella24 which has the worse compression rate, and soc-Slashdot0902 which has about the average compression rate.

In the first experiment, we varied $DT = 0.15$ to 0.45 with step 0.05, and fixed the other two parameters $K = 20$ and $RF = 0.9$. Figure 4.2 shows the local information bits/edge rate and the total bits/edge rate. Clearly, the compression rate is insensitive to parameter DT.

In the second experiment, we fixed $DT = 0.25$ and $RF = 0.9$, and varied K from 1 to 30. Figure 4.3 shows the results. Increasing k leads to better compression rates on the

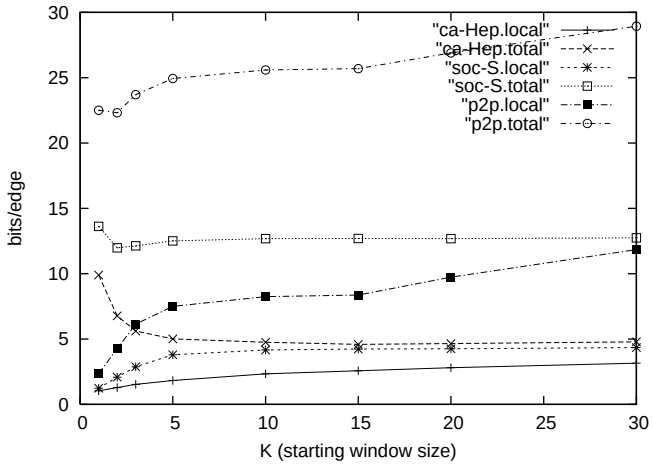


Figure 4.3: The trade off between the bits/edge rate of local information and that of pointers on three data sets: ca-HepPh, p2p-Gnutella24, and soc-Slashdot0902. ($DT = 0.25$ and $RF = 0.9$)

ca-HepPh and Soc-Slashdot0902 data sets. However, when k is 5 or larger, increasing k does not gain big advantage. Therefore, setting k to a value between 5 and 10 is a good experimental choice.

4.5 Summary

In this chapter, we tackled the problem of compressing social networks in a neighbor query friendly way. We developed an effective social network lossless compression approach achieved by a novel notion of multi level linearization of directed graphs. To the best of our knowledge, our approach is the first that can answer both out-neighbor and in-neighbor queries in sublinear time, without replicating the network. An extensive empirical study on more than a dozen benchmark real data sets justifies the effectiveness of our method.

Chapter 5

Lossy Compression

From both practical and theoretical points of views, compression plays an important role in social network analysis. Although there are a few pioneering studies on social network compression [9, 4, 52], they only focus on lossless approaches. In this chapter, we tackle the problem of lossy compression of social networks. The trade-off between memory and "information-preserved" in a lossy compression presents an interesting angle for social network analysis, and at the same time makes the problem very challenging. We propose a lossy graph compression approach, based on our GPSN framework. We do so by designing an objective function to measure the quality of a given linearization with respect to community structure of social networks. We present an interesting and practical greedy algorithm to optimize such an objective function. Our experimental results on real data sets and synthetic data sets demonstrate the promise of our method.

5.1 Overview

Partly motivated by the recent success of many online social networking sites such as Facebook and Twitter, managing and analyzing huge social networks have attracted dramatic interest from both industry and academia. An essential challenge is that many social networks are huge and ever growing. For example, from January to September 2008, the

number of active users of Facebook grew from 60 million to 140 million, and Facebook has over 900 million active users nowadays.

As illustrated by several recent studies [8, 14, 40, 6], compressing social networks can substantially facilitate mining and advanced analysis of huge social networks. Social network compression plays an important role in social network analysis from both practical and theoretical points of view.

Practically, many advanced social network analysis tools are sensitive to the input size. Those methods are highly efficient when the data can be held completely or largely into main memory, but are very costly if most of the data is out of memory. If social networks can be compressed effectively and efficiently, it may help such advanced analysis tools to handle much larger social networks.

Theoretically, compression is always achieved by utilizing some local "regularity" in data. Thus, the compressibility of a social network can provide valuable insights into the structure of the network. For example, if a subnetwork can be compressed well, it may indicate that the members in the subnetwork share some regularity, or the subnetwork follows some structural patterns that are shared by similar subnetworks.

As indicated by a literature review in Chapter 2, almost all the existing social network compression methods target at lossless compression. We argue that lossy compression in fact is very interesting and useful for social networks, because it is well-known that large social networks are often noisy.

From the practical point of view, noise edges and vertices in large social networks may hinder the quality of social network analysis. An appropriate lossy compression of a social network should discard the noise edges and vertices in the network. Consequently, the lossy compression may be present as the input of higher quality for social network analysis. In other words, lossy compression of social networks can serve as a preprocessing step in social network analysis. This goes far beyond just space saving.

From the theoretical point of view, lossy compression of social networks can help to discover the importance of edges and vertices in a social network, and identify noise edges and

vertices. Suppose ideally we have a lossy compression method that preserves the important information about a social network and filters out noise. If the method can assign to each element in a social network (e.g., each edge and vertex) a priority of being included into a lossy compression, then the priority can be regarded as a good indicator of the importance of the element. The lower the priority of an element, the more likely the element is noise.

Lossy compression of social networks is interesting and important for social network analysis. At the same time, it is a very challenging problem. To achieve good lossy compression of social networks, we have to develop good methodology that can detect and preserve important information in social networks.

In this chapter, we tackle the novel problem of community preserving lossy compression of social networks, and make several important contributions. First, we advocate community preserving lossy compression of social networks due to the importance of communities in social networks. To the best of our knowledge, we are the first to identify and tackle the problem. Second, we propose a sequence graph compression approach. We design a simple yet meaningful objective function that optimizes for community structure preservation. We are not aware of any other objective function for the same purpose in the literature. A heuristic algorithm is developed. Last, we report an empirical study on both synthetic and real data sets, which verifies the effectiveness of our method.

The rest of the chapter is organized as follows. In Section 5.2, the notations and the essential idea of our compression method using the GPSN framework is discussed. The design of the community preserving objective function is covered in Section 5.3. We present the compression algorithm in Section 5.4, and report the experimental results in Section 5.5. Section 5.6 summarizes the chapter.

5.2 Graph Linearization for Lossy Compression

For the sake of simplicity, in this chapter, we model a social network as an *undirected simple graph* $G = (V, E)$, where V is a set of vertices, $E \subset V \times V$ is a set of edges, and $(u, u) \notin E$ for any $u \in V$. We also refer to V by $V(G)$ and to E by $E(G)$. Our discussion can be

straightforwardly extended to directed and non-simple graphs.

We first formulate a notion of sequence graph.

Definition 12 (Sequence graph)**.** *A graph G_s is a (k,l)-**sequence graph**, if $|V(G_s)| = l$ and there is a bijection ϕ between $V(G_s)$ and the set of integers $\{1,\ldots,l\}$ such that for every edge $(x,y) \in E(G_s)$, $|\phi(x) - \phi(y)| \leq k$. We call k the **local range size**, l the **sequence length**, and $span(x,y) = |\phi(x) - \phi(y)|$ the **span** of edge (x,y).*

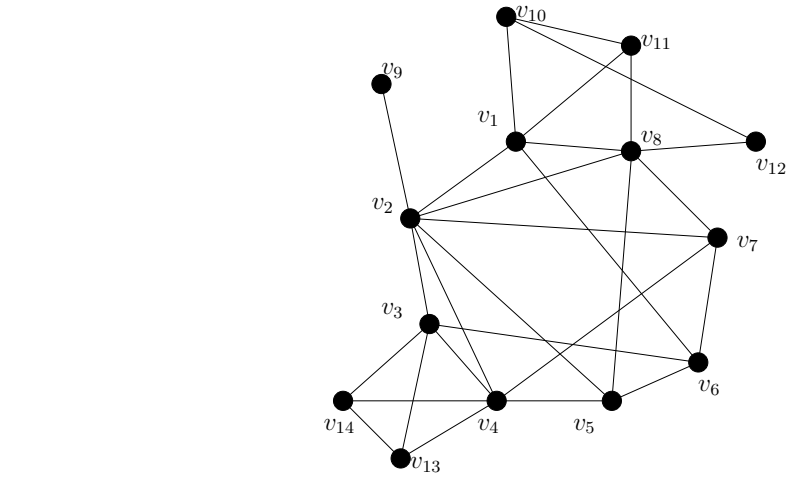
Intuitively, in a sequence graph, the vertices can be lined up into a sequence so that all edges are "local", that is, the two end points locate within a segment of at most k in the sequence. Since ϕ is a bijection between $V(G_s)$ and integers $\{1,\ldots,l\}$, hereafter, we may simply refer to the vertices in G_s by integers in $\{1,\ldots,l\}$, and may draw the vertices of a sequence graph in a sequence and omit the integers if they are clear from the context.

Example 9 (Sequence graph)**.** *Graph G_s in Figure 5.1(b) is a $(3,15)$-sequence graph. Please note that we simply line up the vertices in a sequence and omit the integers in the graph. The function $\psi(\cdot)$ in the figure is for Example 10 and should be ignored at this moment.*

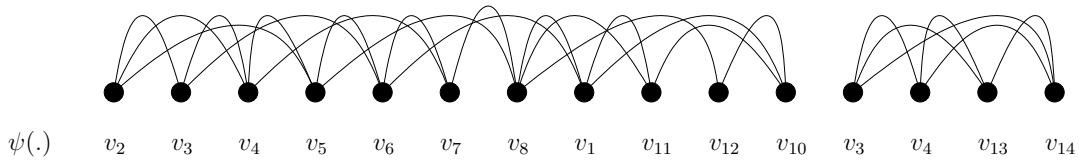
To store a (k,l)-sequence graph, for each vertex, we only need to allocate $2k$ bits to represent the edges involving the vertex. This representation can also enable efficient neighborhood queries — finding all neighbors of a vertex u takes only $O(k)$ time.

The general idea behind graph compression using linearization is that we try to "unfold" a graph into a sequence graph, so that many vertices have the associated edges in their local ranges. Then, storing the corresponding sequence graph can save space, because many edges are stored using only 2 bits each, one for each end point. We refer to this process as "unfolding" because a vertex in the original graph may be mapped to several vertices in the sequence graph.

Definition 13 (Graph linearization)**.** *A (k,l)-sequence graph G_s is a (k,l)-**linearization** of a graph G if there exists a function $\psi : V(G_s) \to V(G)$ such that (1) for every edge $(x,y) \in E(G_s)$, $(\psi(x),\psi(y)) \in E(G)$, and (2) there does not exist two edges $(x,y),(x',y') \in$*



(a) Graph G



(b) A $(3, 15)$-sequence graph G_s that is a lossy linearization of G.

Figure 5.1: A graph G and its lossy representation using a $(3, 15)$-sequence graph G_s.

$E(G_s)$, $(x, y) \neq (x', y')$ such that $(\psi(x), \psi(y)) = (\psi(x'), \psi(y'))$. To keep our notation simple we overload the symbol ψ by writing $\psi(x, y) = (\psi(x), \psi(y))$.

*G_s is a **lossless linearization** of G if for every edge $(u, v) \in E(G)$, there exists an edge $(x, y) \in E(G_s)$ such that $\psi(x, y) = (u, v)$. Otherwise, G_s is a **lossy linearization** of G.*

The second condition in the definition of (k, l)-linearization ensures that an edge in the original graph is encoded at most once in the linearization. This condition helps us to design a simple yet effective objective function for lossy compression in the next section.

Example 10 (Lossy linearization)**.** *The $(3, 15)$-sequence graph G_s in Figure 5.1(b) is a lossy linearization of graph G in Figure 5.1(a). The mapping $\psi(\cdot)$ from the nodes of G_s to*

Symbol	Meaning
G_s	a (k, l)-sequence graph
G	a graph to be compressed
k	the neighbor range size
l	the sequence length
ϕ	the mapping between $V(G_s)$ and the set of integers $\{1, \ldots, l\}$
$span(x, y)$	the span of edge $(x, y) \in E(G_s)$, $\lvert\phi(x) - \phi(y)\rvert$
$N_s(x)$	the neighbor set of x, $\{y \in V(G_s) \lvert\lvert \phi(x) - \phi(y)\rvert \leq k\}$
ψ	the mapping from $V(G_s)$ to $V(G)$ in a linearization
$\psi(x, y)$	$(\psi(x), \psi(y))$
p	a path
$span(p)$	the span of a path in G_s, $\max_{1 \leq i \leq m}\{\phi(u'_i)\} - \min_{1 \leq i \leq m}\{\phi(u'_i)\}$
$P_m(G_s)$	the set of length-m paths in G_s

Table 5.1: Some frequently used notions.

the nodes of G is depicted.

In general, a graph G may have multiple (k, l)-lossy linearizations. Finding the best (k, l)-lossy linearization for a graph G is a novel problem not touched by any previous work. To make the problem concrete, we need to explore how to quantify the "loss of information" and assess the degree of community preservation in lossy compression. We answer this question by designing an objective function in the next section.

The frequently used notions are summarized in Table 5.1.

5.3 Objective Function

Consider the following optimization problem. Given a graph G and parameters $l > 0$ and $k > 0$, find a (k, l)-lossy linearization G_s for G and the mapping $\psi : V(G_s) \to V(G)$ such that a utility objective function $f(G_s)$ is maximized, where $f(G_s)$ measures the degree of

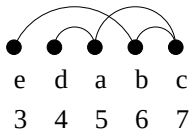


Figure 5.2: The span of a path.

information preservation in G_s.

Lossy compression trades off some edges in the original graph G for space saving. What information in G should be preserved in priority? Since communities are the essential building blocks of social networks, in this chapter, we focus on lossy compressions of social networks that preserve communities. We regard a dense area in a graph as a potential community. We intently avoid an exact definition of community, since different applications may have different definitions.

We want to obtain a utility function that optimizes for edges of short spans in the corresponding sequence graph. Instead of developing a utility function parameterized by k, we introduce a parameter α ($0 < \alpha < 1$) that controls the strength of preference on shorter spans. We will build the connection between parameters α and k in Section 5.4.1.

A *path* $p = (u_1, u_2, \ldots, u_m)$ in a graph G is a series of edges such that $(u_i, u_{i+1}) \in E(G)$, $1 \leq i < m$. The *length* of path p is $(m-1)$, the number of edges involved in the path. In a linearization G_s of G under mapping ψ, path $p' = (u'_1, u'_2, \ldots, u'_m)$ in G_s is the *embedding* of path p if $\psi(u_i, u_{i+1}) = (u'_i, u'_{i+1})$ for $1 \leq i < m$.

Definition 14 (Span of path)**.** *Let G_s be a linearization of graph G, $p = (u_1, u_2, \ldots, u_m)$ a path in G, and $p' = (u'_1, u'_2, \ldots, u'_m)$ the embedding of p in G_s. The* ***span*** *of p is*

$$span(p) = \max_{1 \leq i \leq m} \{\phi(u'_i)\} - \min_{1 \leq i \leq m} \{\phi(u'_i)\}$$

Example 11. *Figure 5.2 shows a segment of a $(3, l)$-sequence graph. For path $p = (d, a, c, b, e)$, the span is $7 - 3 = 4$.*

Let us start our design of the objective function by considering a simple function. Suppose G_s is a (k, l)-linearization of G, where $k = l = |V(G)|$. If we only consider individual

edges, and try to decrease the sum of spans of all edges, then we have the following utility function:

$$f_1(G_s) = \sum_{(x,y)\in E(G_s)} \alpha^{span(x,y)}$$

Utility function f_1 has the following two properties.

Property 1: the shorter the spans of edges in the sequence graph, the higher the utility. This property is consistent with our goal of preserving community information. A community typically has a high density, which means there exist many edges among the set of vertices belonging to the community. If the vertices of a community are placed in proximate positions in the sequence, the spans of the edges within the community tend to be short. The edges of long spans contribute little to the utility. The utility decreases exponentially with respect to the span. This property encourages the arrangement of vertices belonging to the same community in the close-by positions, and discourages the inclusion of edges crossing communities far away in the original graph G.

Property 2: the more edges included in the compression, the higher the utility. Consider two linearization graphs G_s and G'_s such that $V(G_s) = V(G'_s)$ and $E(G_s) \subset E(G'_s)$. Then, $f_1(G_s) < f_1(G'_s)$. This property encourages a linearization graph to include as many edges as possible in addition to optimizing for short span edges.

Utility function f_1 is sensitive to individual edges. We can extend it to incorporate community information better. Instead of edges, we can consider how paths of a certain length are represented in a sequence graph. Generally, a community as a dense subgraph has many short paths traversing members within the community. If a sequence graph preserves the community information, then the members of the community are lined up close to one another in the sequence graph and thus the paths in the community fall into short ranges of the sequence.

To incorporate the above idea, let $P_m(G_s)$ be the set of paths of length m in a sequence graph G_s. We can extend utility function f_1 to

$$f_m(G_s) = \sum_{p\in P_m(G_s)} \alpha^{span(p)}$$

Clearly, utility function f_m is a generalization of f_1. The longer the paths considered, the more community oriented the utility function becomes. At the same time, the optimization problem becomes more challenging when the value of m increases.

Observe that the function f_1 takes its maximum value when the span of each edge is one, and that is basically an adjacency representation of the graph. In this chapter, we focus on the simplest nontrivial setting $m = 2$ as the first step. Interestingly, several recent studies, such as [28], suggested that even considering random walks of short length can generate high quality results in network analysis. Note that for $m \geq 3$, the problem is computationally more expensive. Optimizing f_m for larger values of m is the subject of future studies.

For the sake of simplicity, we omit the subscript 2. In the rest of the chapter, we tackle the optimization of the following objective function:

$$f(G_s) = f_2(G_s) = \sum_{p \in P_2(G_s)} \alpha^{span(p)} \tag{5.1}$$

5.4 Linearization Method

In this section, we simplify the objective function, build the connection between parameters α and k, and develop a greedy heuristic linearization method.

How difficult is the problem of finding the optimal lossy linearization using utility function f in Equation 5.1, that is, finding a sequence graph maximizing the objective function? In literature, there is a family of *graph layout problems* [19], whose objective is to find an *ordering* of nodes to optimize a particular objective function. Many variants of these problems have been shown to be NP-hard [47, 14]. To the best of our knowledge, even no constant factor approximation algorithm for any variation of these problems is known [59, 19]. Note that our setting is even more complex, since one node can appear in several positions in a sequence graph. These evidences suggest that very likely the problem is not solvable in polynomial time, unless $P = NP$. Therefore, in this section, we design a greedy heuristic method.

In order to obtain effective greedy heuristics, we try to bound the objective function. We observe the following.

Theorem 2 (Bounds)**.** *Let G_s be a sequence graph. Then,*

$$f(G_s) \leq \sum_{p \in P_2(G_s)} (\alpha^{1/2})^{span(e_1)+span(e_2)} \tag{5.2}$$

$$f(G_s) \geq \sum_{p \in P_2(G_s)} \alpha^{span(e_1)+span(e_2)} \tag{5.3}$$

Proof. For any path $p = e_1e_2$ in G_s, we have $span(p) \geq \max\{span(e_1), span(e_2)\} \geq \frac{span(e_1)+span(e_2)}{2}$. Since $0 < \alpha < 1$, Equation 5.2 follows immediately.

Apparently, $span(p) \leq span(e_1) + span(e_2)$. Thus, Equation 5.3 holds.

Heuristically, if we can obtain a sequence graph optimizing the lower bound in Theorem 2, the sequence graph have a good chance to boost the objective function f. Note that, optimizing the lower bound in Theorem 2 is computationally less expensive.

Let E_i be the set of edges incident to vertex i in G_s and P_i be the set of those paths of length two that have vertex i as the middle vertex. Then,

$$(\sum_{e \in E_i} \alpha^{span(e)})^2 = \sum_{p=e_1e_2 \in P_i} \alpha^{span(e_1)+span(e_2)}$$

The equation holds, because after expanding the left side, for any possible pair of edges e_1 and e_2 in E_i, we have the term $\alpha^{span(e_1)+span(e_2)}$. Moreover, each possible pair of edges in E_i is corresponding to a path in P_i, and vice versa. Therefore, we optimize the lower bound in Theorem 2 if we optimize the following nice double summation:

$$\bar{f}(G_s) = \sum_{1 \leq i \leq |V(G_s)|} (\sum_{e \in E_i} \alpha^{span(e)})^2$$

5.4.1 Connection between Parameters α and k

We have the following interesting upper bound for the span of any given edge in the optimal sequence graph, assuming the value of α is given.

Theorem 3. *For a given α, the maximum span of all edges in the optimal sequence graph is at most* $\log_\alpha \frac{\alpha(1-\alpha)}{4}$.

Proof. Let $w_i = \sum_{e \in E_i} \alpha^{span(e)}$, where E_i is the set of edges associated with position i.

$$w_i = \sum_{e \in E_i} \alpha^{span(e)} < 2 \sum_{t=1}^{\infty} \alpha^t$$

Notice that, in E_i, there are at most two edges with span equal to t for any possible t. However, $\sum_{t=1}^{\infty} \alpha^t$ is the sum of a geometric sequence and is equal to $\alpha/(1-\alpha)$. Thus we can rewrite the inequality in the following form:

$$w_i < \frac{2\alpha}{1-\alpha}$$

The contribution of edge e is at most:

$$\begin{aligned} w_i^2 - (w_i - \alpha^{span(e)})^2 &= 2\alpha^{span(e)} w_i - \alpha^{2span(e)} \\ &< \frac{4\alpha^{span(e)+1}}{1-\alpha} - \alpha^{2span(e)} \end{aligned}$$

This value should be larger than the contribution of a single isolated edge, otherwise removing this edge and adding it to the end of the sequence graph would increase the objective function. Thus,

$$\alpha^2 < \frac{4\alpha^{span(e)+1}}{1-\alpha} - \alpha^{2span(e)}$$

$$\frac{\alpha(1-\alpha)}{4} < \frac{\alpha^2(1-\alpha)}{4\alpha - \alpha^{span(e)}(1-\alpha)} < \alpha^{span(e)}$$

Since $0 < \alpha < 1$ and $span(e)$ is an integer, we have

$$span(e) < \log_\alpha \frac{\alpha(1-\alpha)}{4}$$

Our problem formulation assumes a parameter k is given as the maximum local range size for the sequence graph. However, the objective function uses the parameter α. Theorem 3 builds the analytical ground to connect α with k. We use the equation $k = \log_\alpha \frac{\alpha(1-\alpha)}{4}$ to estimate α. Specifically, to estimate α given k, we do a binary search on the interval $[0, 1]$, and stop when the value of $\log_\alpha \frac{\alpha(1-\alpha)}{4}$ is between k and $k - 0.01$. Using this estimate of α, experimentally we observe that in the resulting sequence graphs the spans of an extremely small fraction of edges are more than $k/2$. This is consistent with Theorem 3. Therefore, we use the equation $2k = \log_\alpha (1-\alpha)\alpha/4$ to estimate α.

Algorithm 7 Compression Algorithm

Input: G: input network, k: local range,
l: length of compression ($l \geq |V(G)|$)
Output: $SeqG$: sequenced compression

```
1: Initialize SeqG with a random ordering of nodes
2: α ← EstimateAlpha(k)
3: repeat
4:    b = f(SeqG, α)
5:    for all u ∈ V(G) do
6:       IPos = NULL, DPos = NULL
7:       (IPos, Nbh) ← ReAllocate(u, G, SeqG, α)
8:       if (IPos ≠ NULL) and (Length(SeqG) = l) then
9:          DPos ← SeqG.LowestBenf(IPos − k, IPos + k)
10:      end if
11:      x ← UtilityIncrease(IPos, Nbh, SeqG)
12:      y ← UtilityDecrease(DPos, SeqG)
13:      if x − y > 0 then
14:         Insert(IPos, Nbh, SeqG)
15:         Delete(DPos, SeqG)
16:      end if
17:   end for
18:   a = f(SeqG, α)
19: until convergence condition
```

5.4.2 A Greedy Heuristic Method

In this section, we develop a greedy heuristic method for the community preserving lossy compression problem. We will use a local search heuristic.

Overview and General Ideas

The basic operation for local improvement in our heuristic is that, given a node, we find a position in the sequence to insert a new copy of the node, and find a position to delete such that the total change in the objective function is positive after the insertion and deletion. Similar to most local search heuristic methods, our method does not have any theoretical guarantee for the convergence time or the quality of the result. However, using an extensive set of experiments in the next section we verify the effectiveness of our design in practice.

Algorithm 7 shows our main algorithm. we initialize G_s with a random ordering of the vertices of G (Line 1). Later, we show that the random initialization would help us to improve the quality of our lossy compression by aggregating several independent outputs of this algorithm. There is no edge in G_s at this stage. Then, iteratively we consider all vertices for possible reallocation. The $ReAllocate(u, G, G_s, \alpha)$ procedure (Algorithm 8) returns a position in G_s for possible insertion of an extra copy of u and its associated edges. If the length of G_s is already l, the algorithm searches the local range of the insertion point for a possible deletion. We apply the changes if they improve the objective function.

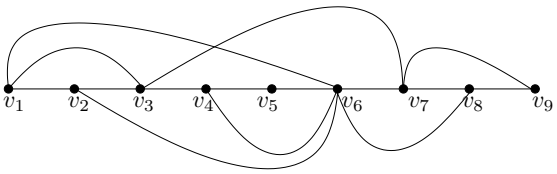
To implement this algorithm we need a data structure to store the sequence graph G_s, which allows fast insertion and deletion operations. We explain our data structure next.

SeqGraph Data Structure

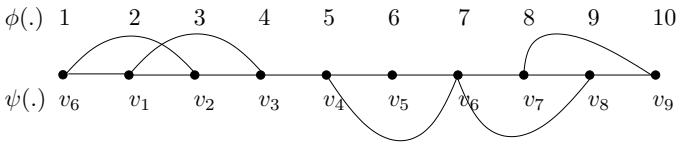
Similar to the Eulerian data structure we need to store a sequence of cells (Figures 5.3(a) and (b)), where each cell represents a position in G_s. Each cell contains two pieces of information: a *next-copy pointer* to the next copy of the same vertex, and a vector of $2k$ bits to represent the local edges. All copies of the same vertex form a cyclic linked list, which we refer to it as a *vertex cycle*. The Eulerian data structure [40] uses an array, in which the cost of inserting and deleting a cell is linear.

In our heuristic algorithm, we have to frequently insert and delete cells. The linear cost for these operations is too expensive. Thus, we need a better data structure to avoid the cost of shifting long segments in the sequence graph in insertions and deletions. Specifically, we divide the cells into segments. Each segment has up to M cells and is stored in an array. Then, the SeqGraph data structure is a double linked list of segments. In Figure 5.3(c), a, b and c are the segments. In each cell, a next-copy pointer is stored. For example, in the first cell of segment a, the next-copy pointer b:4 points to the fourth cell in segment b. To point to a cell, unlike the Eulerian data structure [40] where an integer index can be simply used, we need to use an index consisting of a pointer to a segment and an offset in that segment.

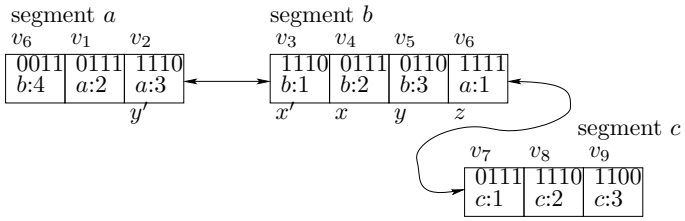
Fortunately, we only need to search within the range size k. That is, in Algorithm 8, we



(a) Graph G



(b) $(2, 10)$-Sequence Graph G_s and the mapping ψ to original vertices of G



(c) SeqGraph data structure (the cell ids in cells, such as $b : 4$, are the next-copy pointers to the next copy of the same vertices)

Figure 5.3: A graph G and its representation by SeqGraph data structure.

only need to compute the exact distance between positions i_1 and i_2 if the distance is up to k. This can be achieved efficiently by searching a small neighborhood.

Let M be the maximum number of cells in a segment. Without loss of generality, we assume M is an even number. If an insertion operation causes a segment to have M cells, we split the segment into two segments of equal size. Moreover, if a deletion operation results in the sum of the lengths of two consecutive segments equal to $M/2$, we merge them. This is to avoid having many tiny segments in the data structure.

For an insertion (deletion) operation, a shift in the affected segment is necessary. Moreover, the next pointers should be updated for those that point to the positions affected by

the insertion and deletion, that is, the position of insertion or deletion and the positions thereafter in the same segment. A nice property of our SeqGraph data structure is that all other segments are not affected. Finally, those edges that pass over the insertion (deletion) position should be updated, too.

Example 12. *In Figure 5.3(c), let us call the cells b:*2*, b:*3 *and b:*4*, x, y and z, respectively, as labeled in the figure. To insert a copy of v_7 at position b:*2*, a shift in the segment b is needed. Moreover, we have to change the next-copy pointers of all the cells that point to x, y and z to b:*3*, b:*4 *and b:*5*, respectively. Note that we can find these pointers by following the vertex cycles of x, y and z, respectively. The edges associated with those cells that are at most* 2 *positions away from the inserted cell should be updated, that is x, x', y and y'. Finally, the newly inserted cell should be added to the vertex cycle of v_7.*

The cost of the insertion and deletion operations depends on the sizes of segments, vertex cycles and maximum span of edges. The time complexity of an insertion or deletion operation is $O(M\Delta + k^2)$, where Δ is the maximum degree of the graph.

The Reallocation Procedure

For a node u, the reallocation procedure (Algorithm 8) partitions the edges incident to u into groups. Each group is associated with a copy of u in G_s. Let C_1 be the set of positions of copies of u that already exist in G_s, and C_2 be the set of positions of potential new copies. C_2 is generated as follows: let $E(G \setminus G_s)$ be the set of edges $(u, v) \in E(G)$ that are not represented in G_s. For $(u, v) \in E(G \setminus G_s)$, we add a potential new copy of u right behind a random copy of v in G_s (Lines 3-7).

We insert these new potential copies at the beginning, and delete them all at the end of the procedure (Lines 1-12). In the edge reassigning step (Lines 13-26), each edge is added to the group for which it has the best contribution (Line 19). The reassigning process stops when no improvement is possible. At the end, for the existing copies of u, that is, C_1, the associated edges will be updated if they have been changed (Lines 27-29). For those potential new copies, that is, C_2, we remove them and return the best in the set as the

position of a potential new copy of u in G_s. We also return the edges associated to this potential new copy (Lines 30-35).

Running Time Analysis

Let d_v be the degree of node v. We have at most d_v groups for each vertex. Inserting and deleting the groups in the SeqGraph data structure takes $O(d_v(M\Delta + k^2))$ time. In an iteration of the edge reassigning process, all edges have to be considered for reassigning. Each edge takes $O(d_v k)$ time. In total each iteration takes $O(d_v^2 k)$ time.

We notice that the number of iteration in practice is extremely low, typically 2 or 3. We do not have any theoretical bound on the number of iterations. However, since the contribution of any single edge is increasing, the convergence is guaranteed.

Likewise, the overall time complexity of Algorithm 7 depends on the convergence condition and the number of iterations. However, the complexity of a single iteration can be estimated as follows. Assuming the number of iterations for the reallocation process is at most I, the running time is proportional to $\sum_{v \in V(G)} d_v(d_v k I + M\Delta + k^2)$. Since k and M are constants, the sum is proportional to $\sum_{v \in V(G)} d_v(d_v I + \Delta)$. Notice that d_v is at most Δ, therefore, the time complexity of an iteration is $O(|E|\Delta I)$.

5.5 Empirical Evaluation

In this section, we report a systematic empirical evaluation.

5.5.1 Evaluation Methodology

Our compression algorithm assumes the parameters k, the local range size, and l, the length of linearization, are given. Having these parameters fixed, the size of compression can be computed precisely. Therefore, compression rate does not have a straightforward meaning here. Instead, we consider a measure to assess the utility of a single bit.

Definition 15 (Bit-utility rate)**.** *The* ***bit-utility rate*** *is the ratio of the number of edges encoded in the lossy compression over the total number of bits.*

To evaluate the quality of a lossy compression, one has to look at the quality of community preservation. This is a challenging task. It is hard to find a ground truth for the community structure of real world networks. Moreover, our method does not explicitly identify communities of networks.

For a sanity check of our method, we design the following methodology. Using the distance of the vertices in the sequence graph compression, we define a *proximity graph* (Definition 16), which is a weighted graph on the vertex set of the original network. The weight of edge (u, v) is an indicator for u and v belonging to the same community. Then, we run a simple community structure detection algorithm on both the original network and the proximity graph. A significant improvement in community detection should be interpreted as the effectiveness of our community preservation lossy compression method. To have a ground truth, we use synthesis networks with implanted communities.

The second experiment is concerned with the effect of missing edges in the lossy compression. We compare three different centrality measures on both the network with the original set of edges and the network with the set of those edges that are encoded in the compressed version. We use a collection of real world networks for this experiment.

5.5.2 Synthesis Networks

The LFR [35] benchmark is a random model to generate networks with implanted communities of variable size, while the degree of the nodes follows a power law distribution. We use the algorithm by Clauset *et al.* [16] for detecting the community structure. This is a simple and scalable greedy modularity-based algorithm. As reported by Fortunato [27] in his Figure 33, this algorithm has a poor performance on the LFR benchmark. A significant improvement of the performance of this algorithm over our lossy representation of the network supports our claim that our method is more than a compression framework and can serve as a preprocessing step for clustering analysis.

To generate synthesis networks, we use the same settings as those used by Fortunato [27]. We set the parameters as follows: average degree to 20, maximum degree to 50, exponent

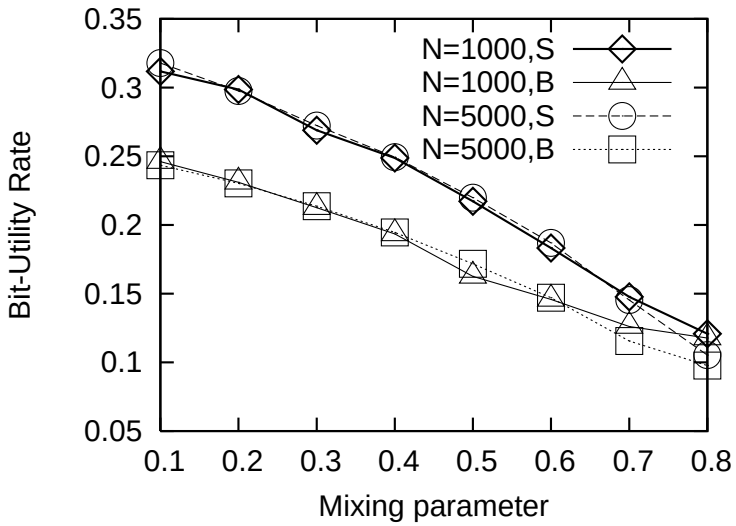


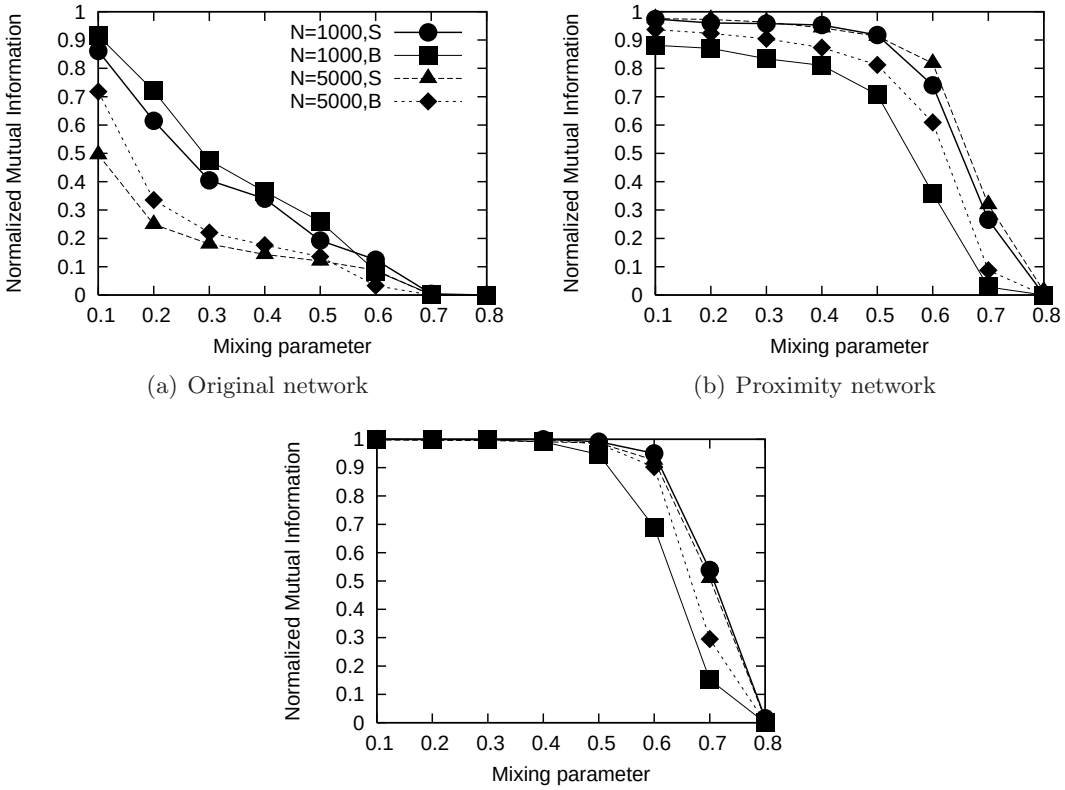
Figure 5.4: The bit-utility-rate of our lossy compression.

for degree distribution to 2, exponent for the community size distribution to 1.

In Figure 5.4, N is the number of nodes, S stands for the data sets where the community sizes are made between 10 and 50, while B stands for the data sets where the community sizes are set between 20 and 100. The X-axis is the mixing parameter, which is the fraction of edges going outside of the community for any particular vertex, used by the data generator [35]. The Y-axis is the bit-utility rate. A larger bit-utility rate means that the lossy compression scheme makes a better use of the available space. As expected, if the density of the communities decreases, the bit-utility rate decreases, too.

We define a proximity graph as follows.

Definition 16 (Proximity graph)**.** *Let G_s be a linearization of G, and $\psi : V(G_s) \to V(G)$ be the mapping. Note that $V(G_s) = \{1, \cdots, |V(G_s)|\}$. The **proximity graph** of G with respect to G_s is defined as follows. Consider $(u, v) \in E(G)$ and $(i, j) \in E(G_s)$ such that $|i - j| \leq k$ and $\psi(i) = u, \psi(j) = v$. Without losing the generality we assume $i < j$. The*



(a) Original network

(b) Proximity network

(c) Aggregated proximity network

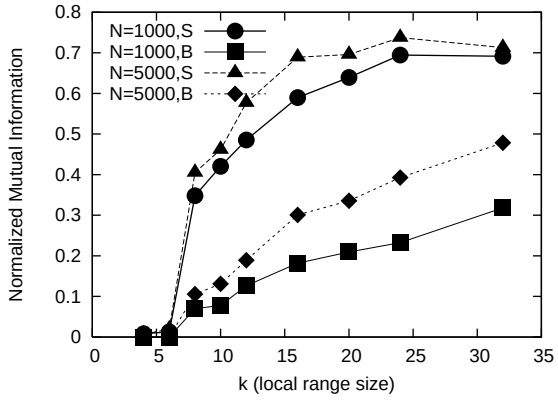
Figure 5.5: The performance of the algorithm by Clauset *et al.* [16] on the original network, the proximity network derived from one compression, and the aggregated proximity network derived from five independent compressions.

weight of undirected edge (u, v) *in the proximity graph is*

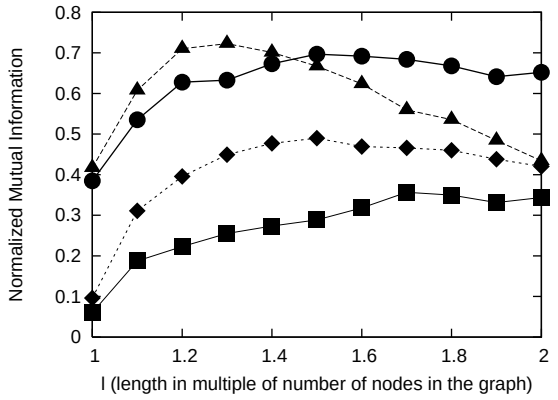
$$\sum_{(i,l)\in E(G_s), l\geq j} \alpha^{|i-l|} + \sum_{(j,l)\in E(G_s), l\leq i} \alpha^{|j-l|}$$

The first (second) term is over all the edges associated with position i (j) *that pass over position* j (i)*. If there is no such a pair of* (i, j)*, then the weight for* (u, v) *is zero. If there are more than one such pair, for each of those pairs, we compute the weight and take the sum over all of them.*

The merit of this weighting schema is that, when i and j belong to a dense part of the



(a) $\mu = 0.6$ and $L = 1.2 \times N$



(b) $\mu = 0.6$ and $K = 8$

Figure 5.6: The effect of (a) local range size (k) and (b) length of sequence (l).

sequence graph G_s, $(\psi(i), \psi(j))$ receives a relatively large value, even if there is no direct edge between i and j. In practice, there is no need to explicitly store the proximity graph, rather one can compute the weights from the linearization graph on the fly. Note that, since the unfolding algorithm uses a random order of the vertices as the starting seed, to exploit the power of randomization, one can use several independent linearizations to derive an aggregate proximity graph, i.e., the weight of (u, v) is the sum of its weights in the independent proximity graphs.

Figure 5.5 shows the results of the algorithm by Clauset *et al.* [16] on the original

graph, the proximity graph and the aggregate proximity graph on five independent lossy linearization. For these experiments $k = 16$ and $l = 1.2 \times N$, where N is the number of vertices in the original graph. The Y-axis is the Normalized Mutual Information (NMI) [18, 27], which measures the similarity between two clusterings, i.e. community structures, of the same set of nodes. We compute NMI using the code at `http://sites.google.com/site/andrealancichinetti/mutual`.

The results clearly show that the algorithm is not effective on the original graph, but performs significantly better on the proximity graph. Moreover, the results on the aggregate proximity graph are comparable to the state-of-the-art algorithms reported by Fortunato [27] in his Figure 33. This strongly suggests that our method captures the community structure of the network, and can serve as an effective preprocessing step.

Figure 5.6 shows that our framework can benefit from increasing local range size k. However, the benefit of increasing the length of lossy linearization is limited.

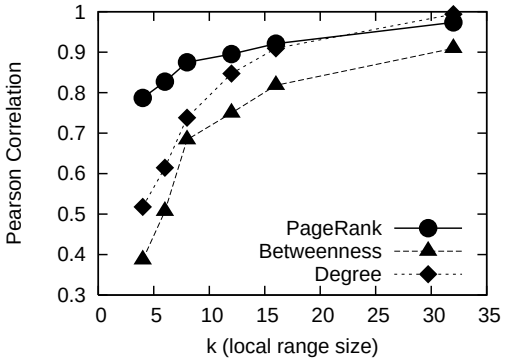
5.5.3 Centrality

Betweenness [31] is a centrality measure of vertices for a given graph. It gives higher values to those vertices that occur on many shortest paths between other pairs of vertices. PageRank [46] is another centrality measure that uses random walks in a network to evaluate the importance of nodes. Degree of a vertex also can be considered as a simple centrality measure. In this section, we evaluate the effect of information loss in lossy compression on those centrality measures.

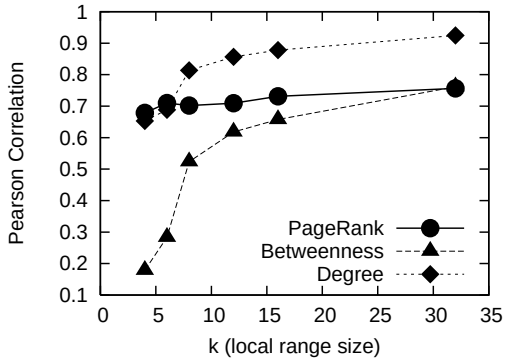
For a graph G and its lossy linearization G_s, $\psi(G_s)$ is a graph on vertex set $V(G)$. Edge $(u, v) \in E(G)$ is an edge in $\psi(G_s)$ if and only if there exists $(i, j) \in E(G_s)$ such that $(u, v) = (\psi(i), \psi(j))$. Intuitively, $\psi(G_s)$ consists of those edges in G that are encoded in G_s.

Each vertex has a centrality score. A centrality measure on a network G can be regarded as a *centrality vector* of size $|V(G)|$. To make comparison we use the Pearson correlation of the centrality vectors of G and $\psi(G_s)$.

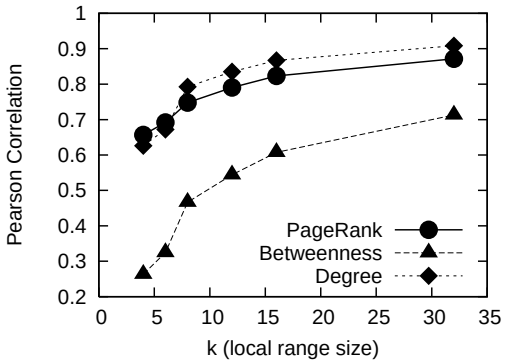
Figure 5.7 shows the results on three real world networks whose statistics are given in



(a) Collaboration network



(b) Wiki vote network



(c) Email exchange network

Figure 5.7: Pearson Correlation of centrality measures on original and the compressed version

Table 5.2. As the value of k increases the correlation between the two centrality vectors increases.

5.5.4 Running Time Efficiency

The running time of our method depends on the number of iterations in Algorithm 7. For our setting the number of iterations is typically between 15 to 30. Figure 5.8 depicts the running time of our method with respect to the graph size on the same LFR benchmarks [27]. The running time of a single iteration, when parameter k is fixed, is scaling almost linearly.

	Description	#nodes	#edges
ca-GrQc	Coll. net. of Arxiv Gen Relativity	5242	14990
em-ExCh	UVR Email Exchange	1133	5451
wiki-vote	Wikipedia who-votes-on-whom network	7115	100763

Table 5.2: Statistic of data-sets

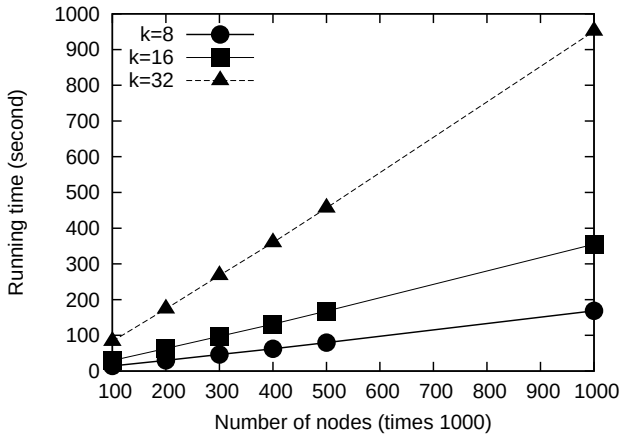


Figure 5.8: The running time of a single iteration of Algorithm 7.

5.6 Summary

In this chapter, we developed a lossy compression scheme for social networks. In our design, the size of compression is given, and the objective is to preserve the community structure of a given network as much as possible. We tackled this problem by introducing a notion of sequence graph. Moreover, we designed an objective function that measures the quality of a sequence graph with respect to the community structure of the network. Also, we designed a non-trivial heuristic to optimize our objective function. Finally, we validated our method using both synthesis and real world networks.

Algorithm 8 Reallocation Procedure

Input: G: original graph, $SeqG$: sequence graph, $u \in V(G)$, α: weighting parameter
Output: $IPos$: potential position to insert a new copy of u, Nbh: the edges associated to the new copy of u

$C_1 = \psi^{-1}(u)$; $C_2 = \emptyset$;
for all $\{v|(u,v) \in E(G \setminus SeqG)\}$ **do**
  Let p be a random member of $\psi^{-1}(v)$;
  $C_2 = C_2 \cup \{p\}$;
  $N(p) = \{+1\}$; /* neighbors of p in $SeqG$ */
  $U(p) = \alpha$; /* utility of p */
  $SeqG.Insert(u, p)$;
end for
for all $p \in C_1$ **do**
  $N(p) = SeqG.Nbh(p)$; $U(p) = \sum_{a \in N(p)} \alpha^{|a|}$;
end for
$C = C_1 \cup C_2$;
repeat
  for all $\{v|(u,v) \in E(G)\}$ **do**
    Let p_u and p_v be s.t. $(\psi(p_1), \psi(p_2)) = (u, v)$;
    /* $p_u \in C$ and $p_v \in \psi^{-1}(v)$ */
    $a = Dist(p_u, p_v)$; /* the distance in $SeqG$ */
    $U(p_u) = U(p_u) - \alpha^{|a|}$;
    Let $p_u^* \in C$ and $p_v^* \in \psi^{-1}(v)$ be s.t. maximize:

$$(\alpha^{|dist(p_u^*, p_v^*)|} + U(p_u^*))^2 - U(p_u^*)^2$$

    $U(p_u^*) = U(p_u^*) + \alpha^{dist(p_u^*, p_v^*)}$;
    if $p_u^* \neq p_u$ **then**
      $N(p_u^*) = N(p_u^*) + \{dist(p_u^*, p_v^*)\}$;
      $N(p_u) = N(p_u) - \{dist(p_u, p_v)\}$;
    end if
  end for
until Convergence
for all $p \in C_1$ **do**
  $SeqG.UpdateNeighbor(p, N(p))$;
end for
for all $p \in C_2$ **do**
  $SeqG.Delete(p)$;
end for
Let $p \in C_2$ s.t. $U(p)$ is maximum;
$IPos = p$; $Nbh = N(p))$;
return $(IPos, Nbh)$;

Chapter 6

Conclusion

The central idea of the thesis is to use a notion of "linearization of the nodes" in order to design a General Purpose Social Network (GPSN) compression framework. The study has three main components: the Eulerian data structure, the lossless GPSN compression scheme and the lossy GPSN compression scheme. All three components are different settings of the same framework. The Eulerian data structure is the simplest setting for which we can have an analytical analysis. The lossless compression scheme, comparing to the Eulerian data structure, is a more general setting of the framework, where the goal is to represent all the edges of the network using the least number of bits. However, for the lossy compression scheme, we assume that there is a constraint on the available storage, and the goal is to encode as much as "information" as possible.

6.1 Summary

We like to highlight the following main contributions of this dissertation:

- **The Eulerian Data Structure.** The novel Eulerian data structure is a non-trivial data structure which is memory efficient for any type of network, including real world social networks. To the best of our knowledge, this is the first data structure for network data that allows both efficient in-neighbor and out-neighbor queries without

replicating the data.

A solid process for evaluating a computational system is to provide a proof for efficiency and correctness of the system. The Eulerian data structure is a setting of the GPSN compression framework for which it is possible to prove an upper bound on the bits-per-edge rate. To the best of our knowledge this is the only nontrivial upper bound on the bits-per-edge rate of any representation scheme for social networks. We compare the upper bound with the trivial baseline data structure that can answer both in-neighbor and out-neighbor queries, which is the adjacency list representation of the network and the transpose of it. Chapter 3 provides an in-depth discussion on the Eulerian data structure.

- **Lossless Compression Scheme.** We exploit the notions of k-linearization, and multi-level linearization introduced in Section 3.2.4, in order to design a general purpose lossless compression scheme. Interestingly, this approach reduces the social network compression to a combinatorial optimization problem. However, we believe finding the optimal solution of this problem is NP-hard. We develop a heuristic algorithm to construct the compressed network. A comprehensive set of experiments has been conducted to study the compression rate of the lossless compression scheme. Chapter 4 presents the experimental study for the lossless GPSN compression scheme.

  The lossless compression scheme is a practical setting of the GPSN framework, and there is no upper bound on its compression rate. However, experimentally, we compare the bits-per-edge rate of lossless compression scheme with the results reported in [14] for those publicly available datasets. Note that the framework used in this study does not support sub-linear incremental updates, nor efficient in-neighbor queries. Therefore, to justify our lossless compression scheme, we do not require a significant improvement of the compression rate, rather we show that the compression rate is comparable.

- **Lossy Compression Scheme.** The challenge of developing a lossy network compression framework has two folds. First, it is the matter of designing an efficient representation scheme to represent the links, and second, having a mechanism in place to determine whether a given edge, or vertex, has to be included in the compression. For this part of the thesis, we show that the GPSN compression framework, in addition to being storage-efficient, offers a natural edge ranking mechanism. Chapter 5 discusses the lossy compression scheme in details.

  For the lossy compression, the goal is to encode as much as possible information using a given amount of memory. Therefore, the compression rate does not have a straightforward meaning here. However, a notion of utility can be useful. Bit-utility-rate measures the amount of information that a single bit represents, that is the number of edges over the number of bits. For example, the meaning of a bit-utility-rate of 0.5 is that for representing a certain number of edges, the lossy scheme uses twice as many bits. We evaluate the bit-utility-rate of our lossy compression scheme.

  One might argue that the edges in a real world social network are not of the same importance. The majority of them form a community structure for the network, while others considered being abnormal or even depending on the context, regarded as noise. From this perspective, a legitimate question is that what would the lossy compression scheme do to the community structure of the network. A set of experiments introduced in Section 5.5 evaluate the effect of lossy compression on the community structure of the network. We show that our framework is quit effective in capturing the global community structure of a given social network.

6.2 Future Work

There are several future directions for this dissertation, both from theoretical and practical perspectives. We distinguish five interesting directions for further investigation:

- **Hardness Result.** From a theoretical point of view, a hardness result for the problem of finding the optimal k-linearization, when k is fixed, would be interesting.

- **Approximation Algorithm.** Assuming the problem is NP-hard, the next natural question would concern the existence of a nontrivial approximation algorithm for finding the optimal k-linearization. The optimal 1-linearization is already a k approximation for the problem, and since finding the optimal 1-linearization can be done in linear time, therefore we have a k-approximation algorithm. Then, the accurate question is that can we have an approximation algorithm with a factor better than k?

- **Upper Bound.** Having an upper bound on the length of the optimal k-linearization would give us an upper bound on the compression rate. Proving an upper bound for general graphs might be hard. However, having some assumption on the structure of the network could make the problem easier. For example, proving an upper bound parametrized by the number of edges and the Global Clustering Coefficient might not be out of reach.

- **Better Linearization.** As for the practical side, exploring alternative methods for linearizing a given network, is of great interest. We believe the study of this dissertation should be regarded as a baseline both for lossless and lossy compression. As a possible future direction, developing linearization methods based on spectral analysis of the network could be quit fruitful both for lossless and lossy compression.

- **Distributed Computing.** Another interesting direction for exploring further is the possibility of extending the GPSN framework to a distributed network management system. We believe the notion of linearization suggests a natural partitioning paradigm that relies on partitioning of the edges rather than the nodes of a given network.

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