

TWO-SIDED IDEALS IN LEAVITT PATH ALGEBRAS

by

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Abstract

Leavitt path algebras are a natural generalization of the Leavitt algebras, which are a class of algebras introduced by Leavitt in 1962. For a directed graph E , the Leavitt path algebra $L_K(E)$ of E with coefficients in K has received much recent attention both from algebraists and analysts over the last decade, due to the fact that they have some immediate structural connections with graph C^* -algebras.

So far, some of the algebraic properties of Leavitt path algebras have been investigated, including primitivity, simplicity and being Noetherian. We explicitly describe two-sided ideals in Leavitt path algebras associated to an arbitrary graph. Our main result is that any two-sided ideal I of a Leavitt path algebra associated to an arbitrary directed graph is generated by elements of the form $(v + \sum_{i=1}^n \lambda_i g^i)(v - \sum_{e \in S} ee^*)$, where g is a cycle based at vertex v , and S is a finite subset of $s^{-1}(v)$. We first use this result to describe the necessary and sufficient conditions on the arbitrary-sized graph E , such that the Leavitt path algebra associated to E satisfies two-sided chain conditions. Then we show that this result can be used to unify and simplify many known results for Leavitt path algebras some of which have been proven by using established methodologies from C^* -algebras.

Keywords:

Leavitt path algebras; generators of two-sided ideals; Noetherian rings; Artinian rings

Subject Terms:

algebra; noncommutative algebras; Leavitt path algebras; two-sided ideals; two-sided chain conditions; graph algebras

To my family

“Believe me, the reward is not so great without the struggle. ”

— Wilma Rudolph

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Chapter 1

Introduction

Leavitt path algebras over a field K are the focus of this thesis. Let E be a directed graph, that is a collection of vertices and directed edges connecting them. The *Leavitt path algebra* associated to E is formed by taking the K -algebra generated by the set of vertices and edges of a graph E and then adding relations involving them. Leavitt path algebras are a natural generalization of Leavitt algebras, defined by Leavitt in 1962 [29]. The goal of Leavitt was to find rings which do not have Invariant Basis Number (IBN), that is, for which there exist isomorphic free modules of different ranks. This class of K -algebras are called Leavitt algebras and are denoted by $L_K(m, n)$.

Cuntz, almost 10 years later, constructed related C^* -algebras (also called Cuntz algebras) denoted by \mathcal{O}_n for $n \geq 1$ [20]. In his paper, Cuntz investigated these C^* -algebras and showed that they are simple besides having many other C^* -algebraic properties. Although the work of Cuntz was independent of that of Leavitt, the algebras share strong connections. For example $L_{\mathbb{C}}(1, n)$ is a dense subalgebra of \mathcal{O}_n . In other words, if K is the field of complex numbers, \mathcal{O}_n can be seen as a completion of $L_K(1, n)$ in an appropriate norm.

Cuntz and Krieger [21] generalized the definition of \mathcal{O}_n by constructing a C^* -algebra from a finite matrix A whose entries consists of 0s and 1s, and every row and every column of A is non-zero. The assumption that the entries consisting of 0s and 1s was only for convenience. They showed that all constructions and results also extend to matrices with entries in \mathbb{Z}_+ . With this definition, the algebras \mathcal{O}_n discussed above arise in this way from the 1×1 matrix $[n]$, or, equivalently, from the $n \times n$ matrix all of whose entries are 1s. These algebras are denoted by \mathcal{O}_A where A is a (finite) square matrix.

Kumjian et al. [27] noticed that relations given in the definition of \mathcal{O}_A also make sense for infinite matrices A whenever the rows of A contain only finitely many 1s. These matrices can also be seen as the adjacency matrices of row-finite directed graphs, in which there are

only finitely many edges emanating from each vertex. Hence they expanded the definition of \mathcal{O}_A to C^* -algebras associated to row-finite graphs, and these are denoted by $C^*(E)$ for a row-finite graph E . Later the definition was generalized to C^* -algebras over arbitrary graphs in [31], and have been the subject of much investigation since. For example, to better understand the algebraic properties of these algebras, Ara et al. [10] constructed algebraic analogues of \mathcal{O}_A , and they denoted these by $\mathcal{CK}_A(K)$. In the case that A is the $n \times n$ matrix all whose entries consist of 1s, $\mathcal{CK}_A(K)$ gives exactly $L_K(1, n)$.

In 2005, Abrams and Aranda-Pino [1] constructed Leavitt path algebras (please see 2.4.1 for the definition). When $K = \mathbb{C}$, $L_{\mathbb{C}}(E)$ is the algebra described in [30], where it is presented as

$$\text{span}\{S_{\mu}S_{\nu}^* \mid \mu, \nu \text{ are paths in } E, s(\mu) = s(\nu)\}.$$

This \mathbb{C} -algebra along with certain rules for forming products was used by Raeburn to investigate the C^* -algebra $C^*(E)$ by completing this algebra with respect to an appropriate norm. One major difference between $L_{\mathbb{C}}(E)$ and $C^*(E)$ is that the elements of $L_{\mathbb{C}}(E)$ can be seen as linear combinations of elements of the form pq^* , where p and q are paths in E , unlike the situation in $C^*(E)$ [17].

Other than this major difference, the two classes of algebras share amazing similarities. These two classes of algebras have some immediate structure-theoretic connections, and many theorems in one class have analogues in the other. For most of the known results, the graph-theoretical properties on the directed graph E that characterize $C^*(E)$ satisfies a C^* -algebraic property are exactly the same that are needed for the Leavitt path algebra $L_K(E)$ to satisfy the corresponding purely algebraic property. For example, the necessary and sufficient conditions on the underlying graph E such that $C^*(E)$ is simple (respectively, purely infinite simple, finite-dimensional) in the category of graph C^* -algebras are precisely the same with the conditions such that $L_K(E)$ is simple (respectively, purely infinite simple, finite-dimensional) in the category of K -algebras [9]. Moreover, the results for Leavitt path algebras are independent of base field K , and hence hold for \mathbb{C} in particular.

This intimate relationship between the two classes of algebras has been mutually beneficial: the results found in graph C^* -algebras help to determine which results may be true for Leavitt path algebras and to identify which direction should be taken to prove them, and Leavitt path algebras help to identify the sort of things one should expect to hold for graph C^* -algebras. In addition, both of the classes are associated to directed graphs, providing one with graph-theoretic tools that can be used to study both classes of algebras. Graphs, which are combinatorial objects, give visual representations of these algebras, and

make it easier to find examples and counter-examples. These are the reasons why Leavitt path algebras have been drawing attention both from algebraists and analysts since their introduction in 2005 [2, 11, 24].

Many properties of Leavitt path algebras have been investigated with respect to the underlying graph. These properties include, but are not limited to, being simple [1], being purely infinite simple [2], being finite-dimensional [4], being exchange [14], and being Noetherian (equivalently locally finite) [5]. In addition, the ideal structure has been investigated in terms of defining the lattice of ideals [35]. Our aim is to complete the algebraic picture by characterizing the generators of two-sided ideals in Leavitt path algebras. Our main result is the following.

Theorem 3.2.1. *Let I be any two-sided ideal of $L_K(E)$. Then there exists a generating set for I consisting of elements of I of the form*

$$(v + \sum_{k=2}^m \lambda_k g^{r_k})(v - \sum_{e \in S} ee^*)$$

where $v \in E^0$, $\lambda_2, \dots, \lambda_m \in K$, r_2, \dots, r_m are positive integers, S is a finite (possibly empty) subset of E^1 consisting of edges with source vertex v , and, whenever $\lambda_k \neq 0$ for some $2 \leq k \leq m$, g is the unique cycle based at v .

This result says that one can have an idea what the generators might be for any two-sided ideal in a Leavitt path algebra by observing the vertices of the graph. In addition, we see that we may omit the second factor, $v - \sum_{e \in S} ee^*$, in case the graph has no vertices emitting infinitely many edges. We make this precise.

Theorem 3.1.5. *Let E be a row-finite graph. Let I be any two-sided ideal of $L_K(E)$. Then I is generated by elements of the form $v + \sum_{k=1}^m \lambda_k g^k$, where $v \in E^0$, g is a cycle at v and $\lambda_1, \dots, \lambda_m \in K$.*

In addition, we use these two results to give necessary and sufficient conditions on the directed graph E so that the associated Leavitt path algebra satisfies two-sided chain conditions, namely being two-sided Noetherian and two-sided Artinian.

Noetherian rings, in which any ascending chain of left (or right) ideals terminate, lie in the core of ring theory, since they give an idea about the complexity of the ring, as we will see in Chapter 2. Artinian rings are analogues of Noetherian rings with the requirement any descending chain of left (or right) ideals must terminate. These two chain conditions together give a great deal of information about the structure of the ring. We see that, for Leavitt path algebras, the two-sided chain conditions depend on the hereditary saturated

subsets of the vertices (defined in Chapter 2, Section 2.2), and also upon the vertices emitting infinitely many edges.

We start Chapter 2 with some useful background information about chain conditions. In Section 2.1, we give some definitions and facts on the ring theoretical properties that we will use to prove our results. In Sections 2.3 and 2.4, we will define Leavitt algebras and Leavitt path algebras. We give two possible approaches for the definition of Leavitt path algebras. Even though they are equivalent, both uses have appeared in the literature, and one makes Leavitt path algebras a direct algebraic analogue of graph C^* -algebras, whereas the other approach sees a Leavitt path algebra as a quotient of a path algebra. To better understand Leavitt path algebras, we will state and prove some known results about them in Section 2.5. Most of these results are algebraic versions of results in graph C^* -algebras given in [30].

In Chapter 3, we prove Theorem 3.1.5 and Theorem 3.2.1, which characterize the generators of two-sided ideals in Leavitt path algebras. We first consider row-finite graphs in Section 3.1, and then arbitrary graphs in Section 3.2, by using the ideas of the former. In Section 3.3, we give some examples of graphs and ideals to demonstrate the tools we used to prove these results.

In Chapter 4, we give the necessary and sufficient conditions on the graph so that the corresponding Leavitt path algebra satisfies the two-sided chain conditions described above. First, in Section 4.1, we give some background. Then in Section 4.2, first we prove some results by using Theorem 3.2.1. Then we characterize two-sided Noetherian Leavitt path algebras. Once again, we will first consider row-finite graphs (Theorem 4.2.5), and later give the general version (Theorem 4.2.12), by using similar ideas. In Section 4.3, we give the necessary and sufficient conditions to be a two-sided Artinian Leavitt path algebra. Finally, in Section 4.4, we offer some explicit examples.

In Chapter 5, we show some of the well-known results in the theory of Leavitt path algebras that can be deduced by using Theorem 3.2.1. We show that many known results can be unified and simplified in terms of giving shorter and simpler proofs.

In Chapter 6, we explicitly define graph C^* -algebras, and give examples for the structure similarities and differences between the two classes of graph algebras. We conclude this section with giving Question 6.1.15, and we show that our main result, namely Theorem 3.2.1, cannot be translated into the class of graph C^* -algebras via an example.

Chapter 2

Preliminaries

2.1 Ring Theory

We give some background in ring theory. We follow the approaches of [25, 26, 32].

We begin with defining Noetherian rings, which are named after Emmy Noether. As we will see in Theorem 2.1.3, Noetherian rings play an important role in ring theory, as they give an idea about the complexity of the ring.

Definition 2.1.1. An algebra is said to be *left Noetherian* if it satisfies the ascending chain conditions (a.c.c.) on its left ideals; that is, given any chain of left ideal

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots$$

there exists a positive integer k such that

$$I_k = I_{k+1} = \cdots .$$

We make the remark that one can define right Noetherian rings analogously.

Definition 2.1.2. An algebra that is both left and right Noetherian is called *Noetherian*.

The next theorem gives a useful characterization for Noetherian algebras.

Theorem 2.1.3. *Let A be an algebra. The following are equivalent:*

- (i) *A is a left (right) Noetherian algebra;*
- (ii) *every non-empty set of left (right) ideals of A contains a maximal element under inclusion;*
- (iii) *every left (right) ideal of A is finitely generated.*

The proof is analogous to the proof for Noetherian modules given in [23, Theorem 1].

Proof. (i) \Rightarrow (ii): Assume A is a left Noetherian algebra. Let S be a non-empty set of ideals of A . Choose any ideal I_1 in S . If I_1 is maximal element of S , then we are done. So assume that I_1 is not maximal. Then there is some I_2 in S such that $I_1 \subset I_2$. If I_2 is maximal, then (ii) holds and we are done. Proceeding this way we can see that if (ii) fails we can create an infinite strictly increasing chain of elements of S , contradicting (i).

(ii) \Rightarrow (iii): Assume every non-empty set of ideals of A contains a maximal element under inclusion. Let I be any left ideal of A , and let S be the collection of all finitely generated left ideals of I . Note that $\{0\}$ is in S , hence S is non-empty. By assumption, S contains a maximal element, say J . If $J \neq I$, then there exists $x \in I \setminus J$. Since $J \in S$, it is finitely generated by assumption. Hence the left ideal generated by J and x is also finitely generated. However, this contradicts the maximality of J , implying $I = J$ and all left ideals are finitely generated.

(iii) \Rightarrow (i): Assume every left ideal of A is finitely generated. Let

$$I_1 \subseteq I_2 \subseteq \cdots$$

be a chain of left ideals of A . Let

$$J = \bigcup_{i=1}^{\infty} I_i$$

and note that J is a left ideal of A . By the assumption, J is finitely generated by, say, x_1, x_2, \dots, x_n . Since x_i is in J for all i , each x_i lies in some left ideal I_{j_i} . Let $m = \max\{j_1, j_2, \dots, j_n\}$, and note that $x_i \in I_m$ for every i . Hence the left ideal generated by x_1, \dots, x_n is contained in I_m , that is, $J \subseteq I_m$. Note that this implies $I_m = J = I_k$ for all $k \geq m$, which proves that A is a left Noetherian algebra. \square

Let us consider some examples first before proceeding with the definition of two-sided Noetherian algebras.

Example 2.1.4. *The set of integers, \mathbb{Z} , is a Noetherian ring, since it is a Principal Ideal Domain, that is, all of its ideals are generated by single element.*

Example 2.1.5. *Let*

$$R = \left\{ \left(\begin{array}{cc} a & b \\ 0 & c \end{array} \right) \mid a \in \mathbb{Z}, b, c \in \mathbb{Q} \right\}.$$

The ring R is right Noetherian, but not left Noetherian.

Proof. To show that it is right Noetherian, we will use Theorem 2.1.3 and show that any right ideal I of R is finitely generated.

First assume that $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in I$ for some $a \in \mathbb{Z} \setminus \{0\}$, $b, c \in \mathbb{Q}$. Then note that $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in I$. We see that the right ideal of R generated by elements in I of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ is finitely generated by Example 2.1.4.

Now consider elements in I of the form $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$. Let $x_1 = \begin{pmatrix} 0 & b_1 \\ 0 & c_2 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 & b_2 \\ 0 & c_2 \end{pmatrix}$ be two elements in I such that $x_1 \neq x_2 \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}$ for some $r \in \mathbb{Q}$, as otherwise the right ideal generated by x_2 is the same with the right ideal generated by x_1 .

We compute

$$\begin{pmatrix} 0 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{c_2}{c_1} \end{pmatrix} - \begin{pmatrix} 0 & b_2 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & b_1 \frac{c_2}{c_1} - b_2 \\ 0 & 0 \end{pmatrix}.$$

Note that $b_2 \neq b_1 \frac{c_2}{c_1}$, otherwise $x_1 = x_2 \begin{pmatrix} 0 & 0 \\ 0 & \frac{c_2}{c_1} \end{pmatrix}$.

Hence we get $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, where b is not zero, and this implies both $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are in I . Thus, the right ideal of R generated by elements of I of the form $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}$ is also finitely generated, implying that I is finitely generated.

To see that R is not left Noetherian, first let $x_i = \begin{pmatrix} 0 & \frac{1}{p_i} \\ 0 & 0 \end{pmatrix}$, where p_i is the i^{th} prime. Then note that the ascending chain

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \cdots \subset \langle x_1, x_2, \dots, x_n \rangle \subset \cdots$$

clearly does not terminate. □

By using the definition of Noetherian algebras on left or right ideals, we can define an analogous algebraic property by using two-sided ideals.

Definition 2.1.6. An algebra is said to be *two-sided Noetherian* if it satisfies the ascending chain condition (a.c.c.) on two-sided ideals.

We note that the two-sided Noetherian condition is weaker than the Noetherian condition: two-sided ideals can also be considered as left or right ideals, hence if the set of left ideals or right ideals satisfy the ascending chain condition, then two-sided ideals satisfy it as well, implying that a Noetherian algebra is also a two-sided Noetherian algebra. As it can be easily seen, these two concepts coincide for commutative algebras.

Now we analogously define Artinian rings, which are named after Emil Artin. While Noetherian rings deal with ascending chains of ideals, Artinian rings deal with descending chains of ideals.

Definition 2.1.7. An algebra is said to be *left (right) Artinian* if it satisfies the descending chain condition (d.c.c.) on left (right) ideals; that is, given any chain of left (right) ideal

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

there exists a positive integer k such that

$$I_k = I_{k+1} = \cdots .$$

Definition 2.1.8. An algebra that is both left and right Artinian is called *Artinian*.

Example 2.1.9. Any division ring is Artinian, as it has no nontrivial right or left ideals.

Example 2.1.10. The ring of $n \times n$ matrices over a division ring is Artinian.

The following result is analogous to Theorem 2.1.3 which is stated for Noetherian algebras.

Proposition 2.1.11. *The following are equivalent:*

- (i) *A is a left (right) Artinian algebra;*
- (ii) *every nonempty set of left (right) ideals of A contains a minimal element under inclusion.*

As in the case with Noetherian algebras, we can define two-sided Artinian algebras in a similar fashion.

Definition 2.1.12. An algebra is said to be *two-sided Artinian* if it satisfies the descending chain condition (d.c.c.) on two-sided ideals.

The ascending chain condition and the descending chain condition are also known together as *chain conditions*, and they are connected in the following way: A consequence of the Akizuki-Hopkins-Levitzki Theorem shows that a left (right) Artinian ring is also a left (right) Noetherian ring [28]. Next, we give examples of rings and check the chain conditions for them.

Example 2.1.13. *Consider the polynomial ring $K[x]$ where K is a field. Then the residue ring $K[x]/(x^n)$ is both Artinian and Noetherian for all positive integers n since it is a finite dimensional vector space of dimension n .*

Example 2.1.14. *The ring \mathbb{Z} is Noetherian, but not Artinian. All rings with a finite number of ideals, like $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{Z}$, and fields are Artinian and Noetherian.*

Example 2.1.15. *The polynomial ring $\mathbb{Z}[x_1, x_2, \dots]$ is not Noetherian since it contains the infinite chain*

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \dots$$

of ideals. It is not Artinian either since

$$\langle x_1 \rangle \supset \langle x_1^2 \rangle \supset \langle x_1^3 \rangle \supset \dots$$

is a chain that doesn't terminate.

We next define graded algebras, which arise when there is a natural notion of degree.

Definition 2.1.16. An algebra A is called a *graded algebra* if it is the direct sum of additive subgroups:

$$A = \bigoplus_{n \in \mathbb{N}} A_n = A_0 \oplus A_1 \oplus \dots$$

such that $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$. The elements of A_k are said to be *homogeneous of degree k* , and A_k is called the *homogeneous component of A of degree k* .

More generally, one can replace \mathbb{N} by a monoid or semigroup G . In which case, A is called G -graded algebra.

An ideal I of the graded algebra A is called a *graded ideal* if $I = \bigoplus_{k=0}^{\infty} (I \cap A_k)$.

Example 2.1.17. *The polynomial ring $K[x]$, where K is a field, is an \mathbb{N} -graded K -algebra.*

Example 2.1.18. *Let V be a n -dimensional vector space over a field K . The exterior algebra $\Lambda(V)$ over V is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $x \otimes x$ such that $x \in V$, i.e., $\Lambda(V) = T(V)/I$,*

where $T(V)$ is the tensor algebra of V . The product on this algebra is called the exterior product or the wedge product, denoted by \wedge , and defined as $x \wedge y = x \otimes y \pmod I$.

We let $\Lambda^k(V)$ be the subspace of $\Lambda(V)$ spanned by elements of the form $x_1 \wedge x_2 \wedge \cdots \wedge x_k$, where $x_i \in V$ for $i = 1, \dots, k$. Then it is known that

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V),$$

and this makes the exterior algebra a graded algebra, as $(\Lambda^i(V)) \wedge (\Lambda^j(V)) \subset \Lambda^{i+j}(V)$.

2.2 Graph Theory

In this section we give some graph-theoretic definitions and properties.

A directed graph $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and functions $r, s : E^1 \rightarrow E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. For each $e \in E^1$, $r(e)$ is the *range* of e and $s(e)$ is the *source* of e . If $s(e) = v$ and $r(e) = w$, then we say that v emits e and that w receives e . A vertex which emits no edges is called a *sink*. A graph is called *finite* if E^0 is a finite set. A graph is called *row-finite* if every vertex is the source of at most finitely many edges. A vertex that emits infinitely many edges is called an *infinite emitter*. If a vertex is either a sink or an infinite emitter, we call it a *singular vertex*. If a vertex is not singular, then we call it a *regular vertex*.

A path μ in a graph E is a sequence of edges $\mu = e_1 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. We define the source of μ by $s(\mu) := s(e_1)$ and the range of μ by $r(\mu) := r(e_n)$. An edge $e \in E^1$ is an *exit* to the path $\mu = \mu_1 \cdots \mu_n$ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$.

If we have $r(\mu) = s(\mu) = v$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle* based at v .

A *closed path based at v* is a path $\mu = e_1 \cdots e_n$, with $e_j \in E^1$, $n \geq 1$ and such that $s(\mu) = r(\mu) = v$. We denote the set of all such paths by $CP(v)$. A *closed simple path based at v* is a closed path based at v , $\mu = e_1 \cdots e_n$, such that $s(e_j) \neq v$ for $j > 1$. We denote the set of all such paths by $CSP(v)$.

Note that a cycle is a closed simple path based at any of its vertices. However the converse may not be true, as a closed simple path based at v may visit some of its vertices (but not v) more than once.

Example 2.2.1. Consider the graph given in Figure 2.1. Note that the path xyz is a closed path based at v , but not a closed simple path as $s(y) = v$. In addition, note that zxy is a closed simple path based at w , but not a cycle as v is visited twice. The only cycles in this graph are x and yz both based at v , and zy based at w .

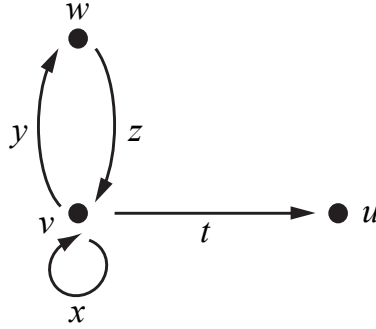


Figure 2.1: The graph defined in Example 2.2.1.

Let v be a vertex in E^0 . If there is no cycle based at v , then we let $g = v$ and call it a *trivial* cycle based at v . If g is a cycle based at v of length at least 1, then g is called a *non-trivial* cycle.

For a given graph E we define a preorder \geq on the vertex set E^0 by: $v \geq w$ if and only if $v = w$ or there is a path μ such that $s(\mu) = v$ and $r(\mu) = w$. We say that a subset $H \subseteq E^0$ is *hereditary* if $w \in H$ and $w \geq v$ imply $v \in H$. We say a set H is *saturated* if whenever $0 < |s^{-1}(v)| < \infty$ and $\{r(e) \mid s(e) = v\} \subseteq H$, then $v \in H$.

In words, H is saturated in case whenever v is a vertex having the property that v emits at least one but at most finitely many edges, and all of the vertices to which the edges emanating from v point are in H , then v is in H as well.

The *hereditary saturated closure* of a set $X \subseteq E^0$ is defined as the smallest hereditary and saturated subset of E^0 containing X . For the hereditary saturated closure of X we use the notation given in [3]: $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$$\Lambda_0(X) := \{v \in E^0 \mid v \geq x \text{ for some } x \in X\}, \text{ and for } n \geq 1,$$

$$\Lambda_n(X) := \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X).$$

Example 2.2.2. Consider the graph E given in the Figure 2.2. We find the hereditary saturated closure of u_2 , hence we let $X = \{u_2\}$. First we see that $\Lambda_0(X) = \{u_2, u_3, u_4, \dots\}$, as $u_2 \geq u_i$ for $i \geq 2$. Next we get $\Lambda_1(X) = \{u_1\} \cup \{u_2, u_3, \dots\}$, as u_1 emits only one edge and the range of that edge is u_2 which is in $\Lambda_0(X)$. Note that even though the ranges of the edges emanating from the v_i 's are in Λ_1 , the v_i 's are emitting infinitely many edges. Thus we are done, and we also have $\overline{\{u_2\}} = \{u_1, u_2, \dots\}$. We can also see that this is the smallest nontrivial hereditary saturated subset of E^0 .

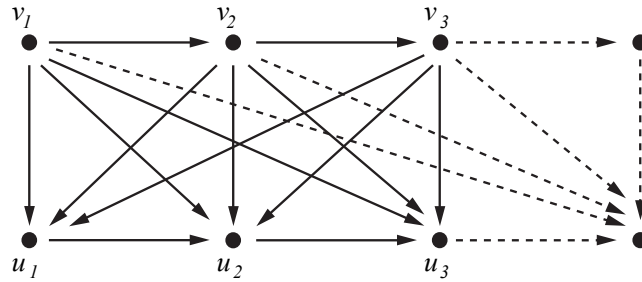


Figure 2.2: Graph defined in Example 2.2.2.

We say the graph E satisfies Condition (L) in case every cycle in E has an exit, while the graph E satisfies Condition (K) in case no vertex in E is the base of exactly one cycle. Note that Condition (K) implies Condition (L) [24]. To see this, let $g = e_1 \cdots e_n$ be a cycle based at some vertex v . Condition (K) implies that there is a cycle $h = f_1 \cdots f_m \neq g$ also based at v . We cannot have $e_i = f_i$ for all $i \leq \min\{m, n\}$, as that would imply that the longer of p or q would visit v twice, and hence cannot be a cycle. Thus, let j be the least index such that $e_j \neq f_j$. Then we get $s(e_j) = s(f_j)$, and f_j is an exit for p .

2.3 Leavitt Algebras

Leavitt path algebras can be considered as a natural generalization of Leavitt algebras.

Definition 2.3.1. Given a field K and a nonnegative integer n , the *Leavitt K -algebra $L(1, n)$* of type $(1, n)$ is the algebra with generators $x_i, y_j, 0 \leq i, j \leq n$, and defining relations which, in matrix form, can be written as

$$(x_0, \dots, x_n)(y_0, \dots, y_n)^T = 1, (y_0, \dots, y_n)^T(x_0, \dots, x_n) = I_{n+1},$$

where I_r denotes the identity matrix of size $r \times r$.

Leavitt algebras were constructed by Leavitt in 1960s to give examples of rings without invariant basis number [29].

Definition 2.3.2. A ring R is said to have *invariant basis number (IBN)* if whenever the free left R -module R^m is isomorphic to R^n with $m, n \in \mathbb{N}$, then $m = n$.

IBN can be seen as the analogue of the Dimension Theorem for vector spaces, as it implies that any two bases for a free module over an IBN ring have the same cardinality.

2.4 Leavitt Path Algebras

In this section we define Leavitt path algebras and give preliminary results about them.

Definition 2.4.1. Let $E = (E^0, E^1, r, s)$ be any directed graph, and let K be a field. We define the *Leavitt path K -algebra* $L_K(E)$ associated with E as the K -algebra generated by a set E^0 together with a set $\{e, e^* | e \in E^1\}$, which satisfy the following relations:

1. $vv' = \delta_{v,v'}v$ for all $v, v' \in E^0$.
2. $s(e)e = er(e) = e$ for all $e \in E^1$.
3. $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
4. (The “CK1 relations”) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E^1$.
5. (The “CK2 relations”) $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ for every regular $v \in E^0$.

The elements of E^1 are called *real edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* | e \in E^1\}$ is denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. We say that a path in $L_K(E)$ is a *real path* (resp., a *ghost path*) if it contains no terms of the form e^* (resp., e).

The conditions CK1 and CK2 are called the *Cuntz-Krieger relations*, and are inherited from graph C^* -algebras.

We note that this algebra can be seen as the free K -algebra $K[E^0 \cup E^1 \cup (E^1)^*]$ with the given relations, hence the multiplication is defined by concatenation of elements of $E^0 \cup E^1 \cup (E^1)^*$ with coefficients from K .

The *length* of a real path (resp., ghost path) μ , denoted by $|\mu|$, is the number of edges it contains. The length of $v \in E^0$ is 0.

Another way of looking at Leavitt path algebras is as a quotient of the path algebra over the *extended graph* of E . First we recall the definition of a path algebra over an arbitrary graph E .

Definition 2.4.2. Let $E = (E^0, E^1, r, s)$ be any directed graph, and let K be a field. The *path K -algebra over E* is defined as the free K -algebra $K[E^0 \cup E^1]$ with the relations:

- $vv' = \delta_{v,v'}v$ for every $v, v' \in E^0$.
- $e = er(e) = s(e)e$ for every $e \in E^1$.

This algebra is denoted by $A(E)$.

Next we define the extended graph over E .

Definition 2.4.3. Given a graph E we define the *extended graph of E* as the new graph $\hat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e^* \mid e \in E^1\}$ and the function r' and s' are defined as

$$r'|_{E^1} = r, \quad s'|_{E^1} = s, \quad r'(e^*) = s(e) \text{ and } s'(e^*) = r(e).$$

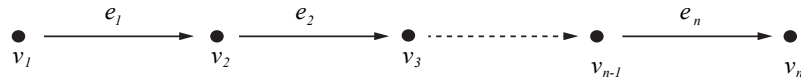
Now we can define the Leavitt path algebra of E with coefficients in K as the path algebra over the extended graph \hat{E} :

Definition 2.4.4. Let $E = (E^0, E^1, r, s)$ be any directed graph, and let K be a field. We define the *Leavitt path K -algebra $L_K(E)$* associated with E as the path algebra over the extended graph \hat{E} , with relations:

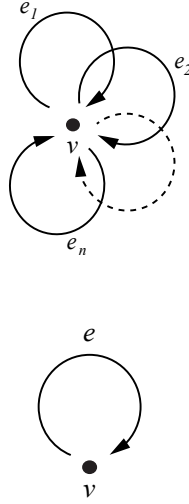
1. $e^*f = \delta_{e,f}r(e)$ for every $e \in E^1$ and $f^* \in (E^1)^*$.
2. $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$ for every regular $v \in E^0$.

Many well-known algebras are of the form $L_K(E)$ for some graph E . Here we give some examples to demonstrate this point.

1. The matrix algebra $M_n(K)$: Let E be the graph defined by $E^0 = \{v_1, \dots, v_n\}$, $E^1 = \{e_1, \dots, e_{n-1}\}$, $s(e_i) = v_i$, $r(e_i) = v_{i+1}$ for $i = 1, \dots, n-1$. The fact that $M_n(K) \cong L_K(E)$ can be seen by defining a map $\phi : L_K(E) \rightarrow M_n(K)$ such that $\phi(v_i) = e(i, i)$, $\phi(e_i) = e(i, i+1)$, and $\phi(e_i^*) = e(i+1, i)$, where $e(i, j)$ denotes the standard (i, j) -matrix unit in $M_n(K)$.



2. The Leavitt algebra $A = L(1, n)$ for $n \geq 2$: Let E be the graph defined by $E^0 = \{v\}$, $E^1 = \{e_1, \dots, e_n\}$ such that $s(e_i) = r(e_i) = v$ for every i . Then $L(1, n) \cong L_K(E)$.
3. Laurent polynomial algebras $K[x, x^{-1}]$: Consider the graph E defined by $E^0 = \{v\}$, $E^1 = \{e\}$ such that $s(e) = r(e) = v$. Then clearly $K[x, x^{-1}] \cong L(E)$.



2.5 Preliminary Results

The following are algebraic analogues of the results given in [30]. Most of the results given here are straightforward, but we state and prove them here for the sake of completeness.

Lemma 2.5.1. *Suppose that E is an arbitrary directed graph and $L_K(E)$ is the Leavitt path algebra of E . Then:*

1. *If $ef \neq 0$, then $r(e) = s(f)$;*
2. *If $ef^* \neq 0$, then $r(e) = r(f)$.*

Proof. We use the relations (1), (2) and (3) given in the definition of Leavitt path algebra.

1. $ef = (er(e))(s(f)f) = e(r(e)s(f))f = 0$ unless $r(e) = s(f)$.
2. $ef^* = (er(e)(s(f^*)f^*)) = e(r(e)r(f))f^* = 0$ unless $r(e) = r(f)$.

□

Definition 2.5.2. Suppose E is an arbitrary directed graph and $n \in \mathbb{N} \cup \{0\}$. Then E^n denotes the set of paths of length n . Moreover, we let $E^* := \bigcup_{n \geq 0} E^n$.

Notation. We extend the range and source maps to E^* by setting $r(v) = v = s(v)$ for $v \in E^0$.

Remark 2.5.3. Note that $\mu^*\mu = r(\mu)$ where $\mu = x_1 \dots x_n$ is a real path. To see this consider the following calculations.

$$\begin{aligned}
\mu^*\mu &= (x_1x_2 \dots x_n)^*x_1x_2 \dots x_n \\
&= x_n^* \dots x_2^*(x_1^*x_1)x_2 \dots x_n \\
&= x_n^* \dots x_2^*r(x_1)x_2 \dots x_n \\
&= x_n^* \dots x_2^*s(x_2)x_2 \dots x_n \\
&= x_n^* \dots x_3^*(x_2^*x_2)x_3 \dots x_n \\
&= \dots \\
&= r(x_n) = r(\mu).
\end{aligned}$$

The relation (4) in the definition of Leavitt path algebra and Lemma 2.5.1 extends to the paths as follows.

Corollary 2.5.4. Suppose E is an arbitrary directed graph and $L_K(E)$ be the Leavitt path algebra of E . Let $\mu, \nu \in E^*$. Then:

$$1. \mu^*\nu = \begin{cases} \mu^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^* \\ \nu' & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise;} \end{cases}$$

2. if $\mu\nu \neq 0$, then $\mu\nu$ is a path in E ;

3. if $\mu\nu^* \neq 0$, then $s(\mu) = s(\nu)$.

Proof. For (1), first assume that $n := |\mu| \leq |\nu|$, and write $\nu = \alpha\nu'$ where $|\alpha| = n$. Then

$$\mu^*\nu = \mu^*(\alpha\nu') = (\mu^*\alpha)\nu'.$$

If $\mu = \alpha$, then by Remark 2.5.3 we get

$$\mu^*\nu = r(\mu)\nu' = s(\nu')\nu' = \nu'.$$

If $\mu \neq \alpha$, then write $\mu = e_1 \dots e_n$, $\alpha = f_1 \dots f_n$, and let i be the smallest integer such

that $e_i \neq f_i$. Then using the idea in Remark 2.5.3 yields

$$\begin{aligned}
\mu^* \nu &= (\mu \alpha) \nu \\
&= (e_n^* \dots e_1^* f_1 f_2 \dots f_n) \nu' \\
&= [e_n^* \dots e_i^* (e_{i-1}^* \dots e_1^* f_1 \dots f_{i-1}) f_i \dots f_n] \nu' \\
&= [e_n^* \dots e_i^* r(e_{i-1}) f_i \dots f_n] \nu' \\
&= [e_n^* \dots e_i^* f_i \dots f_n] \nu' \\
&= 0.
\end{aligned}$$

This gives (1) when $|\mu| \leq |\nu|$. Note that if $|\mu| > |\nu|$, we can use the same idea by writing $\mu = \beta \mu'$.

Parts (2) and (3) follow from (1) and (2), respectively, of Lemma 2.5.1. \square

Corollary 2.5.5. *Suppose E is an arbitrary graph and $L_K(E)$ is the Leavitt path algebra of E . If $\mu, \nu, \alpha, \beta \in E^*$, then we have*

$$(\mu \nu^*)(\alpha \beta^*) = \begin{cases} \mu \alpha' \beta^* & \text{if } \alpha = \nu \alpha' \\ \mu (\beta \nu')^* & \text{if } \nu = \alpha \nu' \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it follows that every non-zero finite product of the real and ghost edges has the form $\mu \nu^$ for some $\mu, \nu \in E^*$ with $r(\mu) = r(\nu)$.*

Proof. The formula part follows from part (1) of Corollary 2.5.4. To see the last statement, let x be a non-zero monomial, that is, a product of finitely many e 's and f^* 's.

Any adjacent e 's can be combined into one single term μ . Since x is non-zero, we get that μ is a path. Similarly, we can combine any adjacent f^* 's into one single term ν^* . Hence we see that x is a product of terms of the form $\mu \nu^*$ where $\mu, \nu \in E^*$. The formula above implies that we can combine this product into one term of the same form, hence the result follows. \square

We see that Corollary 2.5.5 describes the monomials spanning $L_K(E)$. For convenience, we give a detailed list for these monomials the way it is stated in [1].

Corollary 2.5.6. *$L_K(E)$ is spanned as a K -vector space by monomials*

1. kv with $k \in K$ and $v \in E^0$, or
2. $ke_1 \dots e_a f_1^* \dots f_b^*$ where $k \in K$; $a, b \geq 0$, $a + b > 0$, $e_1, \dots, e_a, f_1, \dots, f_b \in E^1$.

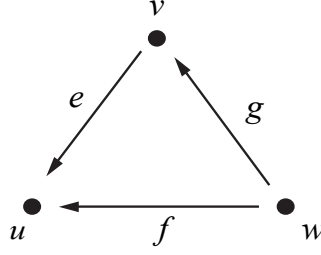


Figure 2.3: Graph of the Leavitt path algebra defined in Example 2.5.7.

Notation. We extend the range and source maps to the set of monomials of $L_K(E)$ by setting $r(\mu\nu^*) = r(\nu^*) = s(\nu)$ and $s(\mu\nu^*) = s(\mu)$ for μ and ν in E^* .

We give an example which demonstrates Corollary 2.5.6.

Example 2.5.7. Let E be the graph given in Figure 2.3. Note that $L_K(E)$ is spanned as a K -vector space by the monomials $\{u, v, w, e, f, g, e^*, f^*, g^*, ge, gef^*, ef^*, fe^*, fe^*g^*, ff^*, e^*g^*\}$.

Next, we present some results from [1] involving closed simple paths.

Lemma 2.5.8. Let $\mu, \nu \in \text{CSP}(v)$. Then $\mu^*\nu = \delta_{\mu,\nu}v$.

Proof. Let μ and ν be two arbitrary paths, and let $\mu = e_1 \dots e_n$ and $\nu = f_1 \dots f_m$.

Case I: Assume $n = m$, but $\mu \neq \nu$. Let i be the first index where μ and ν differ, that is, $e_j = f_j$ for $j < i$ and $e_i \neq f_i$. Then we get

$$\begin{aligned} \mu^*\nu &= e_n^* \dots e_1^* f_1 \dots f_m = \delta_{e_1, f_1} e_n^* \dots e_2^* r(e_1) f_2 \dots f_m \\ &= \delta_{e_2, f_2} e_n^* \dots e_2^* f_2 \dots f_m = \dots = \delta_{e_2, f_2} \dots \delta_{e_i, f_i} e_n^* \dots e_i^* f_i \dots f_m = 0. \end{aligned}$$

Case II: Let $\mu = \nu$. By following the same procedure above, we get

$$\mu^*\nu = \delta_{e_1, f_1} \delta_{e_2, f_2} \dots \delta_{e_n, f_n} r(e_n).$$

Case III: Now consider the case where $\mu, \nu \in \text{CSP}(v)$ with $n \neq m$. Without loss of generality we may assume that $n < m$. Let $\nu = \nu_1\nu_2$ where ν_1 and ν_2 are real paths and $|\nu_1| = |\mu|$. If $\mu = \nu_1$, then $r(\mu) = r(\nu_1) = s(\nu_2)$, but this contradicts with v being closed simple path based at v . So $\mu \neq \nu_1$, and by Case I, we obtain $\mu^*\nu = \mu^*\nu_1\nu_2 = 0$. \square

Lemma 2.5.9. For every $\mu \in \text{CP}(v)$ there exist unique $\mu_1, \dots, \mu_m \in \text{CSP}(v)$ such that $\mu = \mu_1 \dots \mu_m$.

Proof. Let $\mu = e_1 \dots e_n$ and $T = \{t \in \{1, \dots, n\} \mid r(e_t) = v\}$. Label all the elements of T so that $T = \{t_1, \dots, t_n\}$ so that $t_1 < t_2 < \dots < t_n = m$. Then $c_1 = e_1 \dots e_{t_1}$, and $c_i = e_{t_{i-1}} \dots e_{t_i}$ for $1 < i < n$ prove existence.

To prove uniqueness, let $\mu = c_1 \dots c_r = d_1 \dots d_s$ where $c_i, d_j \in \text{CSP}(v)$. Multiply all sides by c_1^* , and we get $0 \neq v c_2 \dots c_r = c_1^* d_1 \dots d_s$. By using Lemma 2.5.8, we get $c_1 = d_1$. Then we use induction to show that the statement holds. \square

For the rest of the section, we will give results concerning the algebraic properties of Leavitt path algebras. These results mostly depend on the underlying directed graph E .

The first result we give is regarding the units of Leavitt path algebras, but let us give some definitions first.

Definition 2.5.10. An algebra A is said to be an *unital algebra*, if A has an unit, that is, an element 1 with the property $1x = x1 = x$ for all $x \in A$.

Definition 2.5.11. An algebra A is called an *algebra with local units* if for every finite set S of elements in A , there exists $e \in A$ such that $ex = xe = x$ for every $x \in S$.

We note that $L_K(E)$ might be a unital algebra or an algebra with local units depending on the size of the graph E :

Lemma 2.5.12. *If E^0 is finite then $L_K(E)$ is a unital K -algebra. If E^0 is infinite, then $L(E)$ is an algebra with local units (specifically, the set generated by finite sums of distinct elements of E^0).*

Proof. First suppose that E^0 is finite. Hence let $E^0 = \{v_1, \dots, v_n\}$. We claim that $\sum_{i=1}^n v_i$ is the unit element of the algebra. Note that $(\sum_{i=1}^n v_i)v_j = \sum_{i=1}^n \delta_{i,j}v_i = v_j$. Next, let $e \in E^0$ and compute

$$\left(\sum_{i=1}^n v_i\right)e = \sum_{i=1}^n v_i s(e)e = s(e)e = e.$$

Similarly, $\sum_{i=1}^n v_i f^* = f^*$ for $f^* \in (E^1)^*$. Since $L_K(E)$ is generated by $E^0 \cup E^1 \cup (E^1)^*$, we conclude that $\sum_{i=1}^n v_i x = x$ for all $x \in L_K(E)$. Analogously, we obtain $x \sum_{i=1}^n v_i = x$ as well.

Now suppose that E^0 is infinite. Let $\{x_i\}_{i=1}^t$ be a finite collection of elements in $L_K(E)$. We use Lemma 2.5.6 to write

$$x_i = \sum_{j=1}^{n_i} \lambda_j^i v_j^i + \sum_{k=1}^{m_i} \kappa_k^i a_k^i$$

where $\lambda_j^i, \kappa_k^i \in K \setminus \{0\}$, and a_k^i are monomials of type (2). Let

$$V = \bigcup_{i=1}^t \{v_j^i, s(a_k^i), r(a_k^i) \mid i = 1, \dots, n_i; j = 1, \dots, m_i\},$$

and then by using the same idea above, one can easily show that $\sum_{v \in V} v$ is a finite sum of vertices such that $\sum_{v \in V} va_i = a_i \sum_{v \in V} v = a_i$ for every i . \square

Next we prove that $L_K(E)$ is a \mathbb{Z} -graded algebra for any graph E .

Lemma 2.5.13. *Let E be an arbitrary graph. Then $L_K(E)$ is a \mathbb{Z} -graded algebra, with grading induced by*

$$\deg(v) = 0 \text{ for all } v \in E^0; \deg(e) = 1 \text{ and } \deg(e^*) = -1 \text{ for all } e \in E^1.$$

That is, $L(E) = \bigoplus_{n \in \mathbb{Z}} L(E)_n$, where $L(E)_0 = KE^0 + A_0$, $L(E)_n = A_n$ for $n \neq 0$ where

$$A_n = \sum \{ke_1 \dots e_a f_1^* \dots f_b^* \mid a + b > 0, e_i, f_j \in E^1, k \in K, a - b = n\}.$$

Proof. The fact that $L_K(E) = \sum_{n \in \mathbb{Z}} L_K(E)_n$ follows from Lemma 2.5.6. The grading on $L_K(E)$ follows directly from the fact that $A(\hat{E})$ is \mathbb{Z} -graded, and that the relations CK1 and CK2 are homogeneous in this grading. \square

Definition 2.5.14. We call an ideal I of $L_K(E)$ *graded* in case, whenever $x = \sum_{j=-m}^n x_j \in I$ for homogeneous elements x_j of $L_K(E)$ of degree j , then $x_j \in I$ for all $-m \leq j \leq n$.

Remark 2.5.15. *If Y is a set of homogeneous elements in a \mathbb{Z} -graded ring, then the ideal $I = \langle Y \rangle$ generated by Y is a graded ideal.*

The following are from [3]. The first result does not depend on the underlying graph.

Proposition 2.5.16. *If I is an ideal of $L_K(E)$, then $I \cap E^0$ is a hereditary and saturated subset of E^0 .*

Proof. If $I \cap E^0$ is empty, then it is both hereditary and saturated, and we are done. So assume $I \cap E^0$ is not empty.

To show that $I \cap E^0$ is hereditary, let $v, w \in E^0$ be in I and $v \leq w$. Then by the definition of the preorder, we can find a path $\mu = e_1 \dots e_n$ such that $s(\mu) = s(e_1) = v$ and $r(\mu) = r(e_n) = w$. Note that $e_1^* v e_1 = e_1^* e_1 = r(e_1) = s(e_2)$ is in I . If we repeat the argument n times, we get $r(e_n) = w \in I$.

To show that $I \cap E^0$ is saturated, consider $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$ and $\{r(e) \mid s(e) = v\} \subseteq I$. Since v is not a sink or an infinite emitter, we get $v = \sum_{\{e \in E^1 \mid s(e)=v\}} ee^*$. Note that for $e \in E^1$ with $s(e) = v$, we have $r(e) \in I$. So $er(e)e^* = ee^* \in I$, and $v \in I$. \square

The next result gives a characterization of the underlying graph E so that the associated Leavitt path algebra is simple. We will omit the original proof now, and give a shorter proof by using Theorem 3.2.1 in the Applications Chapter.

Theorem 2.5.17. *Let E be an arbitrary graph. The Leavitt path algebra $L_K(E)$ is simple if and only if E satisfies the following conditions.*

- (i) *The only hereditary and saturated subsets of E^0 are \emptyset and E^0 .*
- (ii) *E satisfies Condition (L).*

Chapter 3

Two-Sided Ideals

In this Chapter we will give a description for the generators of two-sided ideals of Leavitt path algebras. We will give the proof for Leavitt path algebras associated to row-finite graphs first. Then we will prove the case for arbitrary graphs. This two-step approach is fairly common in this area: Theorems are usually first developed for the row-finite graphs, and then extended to the arbitrary case. We will also show how to get the first result as the corollary of the second one. In the final section, we will use some examples to demonstrate the tools used. The material from the first section, “Two-Sided Ideals for the Row-Finite Case”, appears in [18], and the material from the second section, “Two-Sided Ideals for Arbitrary Case”, appears in [6].

3.1 Two-Sided Ideals for the Row-Finite Case

With the introductory remarks now complete, we begin our discussion of the main result with the following important observation.

Remark 3.1.1. *Let I be an ideal of $L_K(E)$ and let $\mu = \lambda_1\mu_1 + \cdots + \lambda_n\mu_n$ be in I , where μ_1, \dots, μ_n are real paths in I and $\lambda_1, \dots, \lambda_n$ are in K . Note that $s(\mu_i)\mu r(\mu_i)$ is in I and every surviving real path has the same source and the same range. Also note that*

$$\mu = \sum_{w \in R} \sum_{v \in S} v\mu w,$$

where $R = \{r(\mu_1), \dots, r(\mu_n)\}$, $S = \{s(\mu_1), \dots, s(\mu_n)\} \subset E^0$. Hence μ can be written as $\nu_1 + \cdots + \nu_m$, where

1. $\nu_1, \dots, \nu_m \in I$,

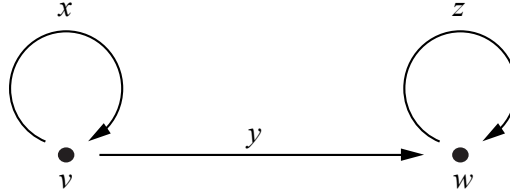


Figure 3.1: Graph of the Leavitt path algebra defined in Example 3.1.2.

2. for $1 \leq i \leq m$, ν_i is a sum of monomials whose sources are all the same and whose ranges are all the same.

Example 3.1.2. Consider the directed graph given in Figure 3.1. Consider an ideal I of $L_K(E)$ with $xyz + yz + xy + x^2 + z \in I$. Then by using Remark 3.1.1, we see that

$$\begin{aligned}\mu_1 &:= v(xyz + yz + xy + x^2 + z)v = x^2, \\ \mu_2 &:= v(xyz + yz + xy + x^2 + z)w = xyz + yz + xy \in I, \\ \mu_3 &:= w(xyz + yz + xy + x^2 + z)w = z \in I, \text{ and} \\ xyz + yz + xy + x^2 + z &= \mu_1 + \mu_2 + \mu_3.\end{aligned}$$

Notation. Let $L_K(E)_R$ (resp., $L_K(E)_G$) be the subring of elements in $L_K(E)$ whose terms involve only real edges (resp., only ghost edges).

Lemma 3.1.3. Let I be a two-sided ideal of $L_K(E)$ and $I_{\text{real}} = I \cap L_K(E)_R$. Then I_{real} is the two-sided ideal of $L_K(E)_R$ generated by elements of I_{real} having the form $v + \sum_{i=1}^n \lambda_i g^i$, where $v \in E^0$, g is a cycle based at v and $\lambda_i \in K$ for $1 \leq i \leq n$.

Proof. Let J be the ideal of $L_K(E)_R$ generated by elements in I_{real} of the indicated form. Our claim is $J = I_{\text{real}}$. Towards a contradiction, suppose $I_{\text{real}} \setminus J \neq \emptyset$; choose $\mu \in I_{\text{real}} \setminus J$ of minimal length. By Remark 3.1.1, we can write $\mu = \tau_1 + \cdots + \tau_m$ where each τ_i is in I_{real} and is the sum of those paths whose sources are all the same and whose ranges are all the same. Since $\mu \notin J$, one of the $\tau_i \notin J$. Replacing μ by τ_i , we may assume that $\mu = \lambda_1 \mu_1 + \cdots + \lambda_n \mu_n$ where all the μ_i have the same source and the same range. First we claim that one of the μ_i must have length 0, i.e., $\mu_i = v$ for some vertex $v \in E^0$. Suppose not. Then for each i we can write $\mu_i = e_i \nu_i$ where $e_i \in E^1$. So $\mu = \sum_{i=1}^n \lambda_i e_i \nu_i$. Now

$$e_i^* \mu = \sum_{\{j | e_j = e_i\}} \lambda_j \nu_j \in I_{\text{real}}$$

and has smaller length than μ . So $e_i^* \mu \in J$ and hence clearly $e_i e_i^* \mu \in J$. Then

$$\mu = \sum_{\text{distinct } e_i} e_i e_i^* \mu \in J,$$

a contradiction. So we can assume without loss of generality that $\mu_1 = v$, with v a vertex. Since all the terms in μ have the same source and the same range, each μ_i is a closed path based at v . Multiplying by a scalar if necessary we can write $\mu = v + \lambda_2 \mu_2 + \cdots + \lambda_n \mu_n$.

Case I: There exists no, or exactly one, closed simple path at v . If there are no closed simple paths at v then we get $\mu \in J$, a contradiction. If there is exactly one closed simple path g based at v then necessarily g must be a cycle. Furthermore, the only paths in E which have source and range equal to v are powers of g . Then $\mu = v + \sum_{i=2}^n \lambda_i g^{m_i} \in J$, a contradiction.

Case II: There exist at least two distinct closed simple paths g_1 and g_2 based at v . Without loss of generality, we may assume that g_1 is a cycle. As $g_1 \neq g_2$ and neither is a subpath of the other, $g_2^* g_1 = 0 = g_1^* g_2$. Without loss of generality assume $|\mu_2| \geq \cdots \geq |\mu_n| \geq 1$. Then for some $k \in \mathbb{N}$, $|g_1^k| > |\mu_2|$. Multiplying by $(g_1^*)^k$ on the left and g_1^k on the right, we get

$$\mu' = (g_1^*)^k \mu (g_1)^k = v + \sum_{i=2}^n \lambda_i (g_1^*)^k \mu_i (g_1)^k.$$

If $(g_1^*)^k \mu_i (g_1)^k = 0$ for every i , then we get $\mu' = (g_1^*)^k \mu (g_1)^k = v \in J$. Then $\mu = \mu v \in J$, a contradiction. Note that if $0 \neq (g_1^*)^k \mu_i (g_1)^k$, then $(g_1^*)^k \mu_i \neq 0$. Since $|g_1^k| > |\mu_i|$, we get $g_1^k = \mu_i \mu'_i$ for some path μ'_i . Since the μ_i are closed paths based at the vertex v , one gets from the equation $(g_1)^k = \mu_i \mu'_i$ that $\mu_i = (g_1)^r$ for some integer $r \leq k$. So μ_i commutes with $(g_1)^k$ and thus each non-zero term $(g_1^*)^k \mu_i (g_1)^k = \mu_i$.

Since $g_2^* g_1 = 0$, $g_2^* \mu_i = 0$ for every $i \in \{2, \dots, n\}$ such that $(g_1^*)^k \mu_i (g_1)^k \neq 0$ and so we get $g_2^* \mu' g_2 = g_2^* v g_2 = v \in I \cap L_K(E)_R = I_{\text{real}}$, which implies that v is in J . Then $\mu = \mu v \in J$, a contradiction. \square

It can be easily shown that the analogue of Lemma 3.1.3 is true for $I_{\text{ghost}} = I \cap L_K(E)_G$. We state this for the sake of completeness.

Lemma 3.1.4. *Let I be a two-sided ideal of $L_K(E)$. Then I_{ghost} is the two-sided ideal of $L_K(E)_G$ generated by elements of the form $v + \sum_{i=1}^n \lambda_i (g^*)^i$, where $v \in E^0$, g is a cycle at v and $\lambda_i \in K$ for $1 \leq i \leq n$.*

Now we are ready to prove Theorem about the generators of two-sided ideals in Leavitt path algebras associated to row-finite graphs.

Theorem 3.1.5. *Let E be a row-finite graph. Let I be any two-sided ideal of $L_K(E)$. Then I is generated by elements of the form $v + \sum_{k=1}^m \lambda_k g^k$, where $v \in E^0$, g is a cycle at v and $\lambda_1, \dots, \lambda_m \in K$.*

Proof. Let J be the two-sided ideal of $L_K(E)$ generated by I_{real} . By Lemma 3.1.3, it is enough if we show that $I = J$. Suppose not. Choose $x = \sum_{i=1}^d \lambda_i \mu_i \nu_i^*$ in $I \setminus J$, where d is minimal and $\mu_1, \dots, \mu_d, \nu_1, \dots, \nu_d$ are real paths in $L_K(E)_R$. By Remark 3.1.1, $x = \alpha_1 + \dots + \alpha_m$, where each $\alpha_j \in I$ and is a sum of those monomials all having the same source and same range. Since $x \notin J$, $\alpha_j \notin J$ for some j . By the minimality of d , we can replace x by α_j . Thus we can assume that $x = \sum_{i=1}^d \lambda_i \mu_i \nu_i^*$, where for all i, j , $s(\mu_i \nu_i^*) = s(\mu_j \nu_j^*)$ and $r(\mu_i \nu_i^*) = r(\mu_j \nu_j^*) = w \in E^0$. Among all such $x = \sum_{i=1}^d \lambda_i \mu_i \nu_i^* \in I \setminus J$ with minimal d , select one for which $(|\nu_1|, \dots, |\nu_d|)$ is the smallest in the lexicographic order of $(\mathbb{Z}^+)^d$. First note that we have $|\nu_i| > 0$ for some i , otherwise x is in $I_{\text{real}} \subseteq J$. Let e be in E^1 . Then note that

$$xe = \sum_{i=1}^d \lambda_i \mu_i \nu_i^* e = \sum_{i=1}^{d'} \lambda_i \mu'_i (\nu'_i)^*$$

either has fewer terms ($d' < d$), or $d = d'$ and $(|\nu'_1|, \dots, |\nu'_d|)$ is smaller than $(|\nu_1|, \dots, |\nu_d|)$. Then by minimality, we get xe is in J for every $e \in E^1$. Since $|\nu_i| > 0$ for some i , w is not a sink and emits finitely many edges. Hence we have

$$x = xw = x \sum_{\{e_j \in E^1 \mid s(e_j)=w\}} e_j e_j^* = \sum_{\{e_j \in E^1 \mid s(e_j)=w\}} (xe_j) e_j^* \in J.$$

Our assumption was that $x \in I \setminus J$, hence we get a contradiction, so the result follows. \square

Remark 3.1.6. *We note that the Theorem does not hold for arbitrary graphs. An example is the “infinite clock” (Figure 3.2: Let $E^0 = \{v, w_1, w_2, \dots\}$ and $E^1 = \{e_1, e_2, \dots\}$ with $r(e_i) = w_i$ and $s(e_i) = v$. Then the two-sided ideal generated by $v - e_1 e_1^*$ is not generated by the elements of the desired form.*

3.2 Two-Sided Ideals for the Arbitrary Case

We are already in position to present the main result.

Theorem 3.2.1. *Let I be any ideal of $L_K(E)$. Then there exists a generating set for I consisting of elements of I of the form*

$$\left(v + \sum_{k=2}^m \lambda_k g^{r^k} \right) \left(v - \sum_{e \in S} ee^* \right)$$

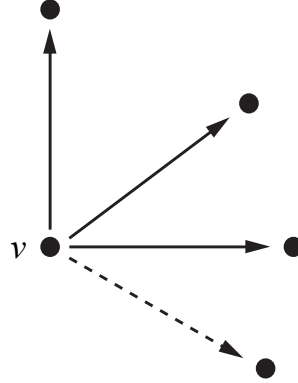


Figure 3.2: Graph of the Leavitt path algebra defined in Example 3.1.6.

where $v \in E_0$, $\lambda_2, \dots, \lambda_m \in K$, r_2, \dots, r_m are positive integers, S is a finite (possibly empty) subset of E^1 consisting of edges with source vertex v , and, whenever $\lambda_k \neq 0$ for some $2 \leq k \leq m$, g is the unique cycle based at v .

Proof. Let J be the ideal of $L_K(E)$ generated by all the elements of I which have the form described in the statement of the Theorem 3.2.1. We want to show that $I = J$. We note that $I \cap E_0 \subseteq J$ (by choosing $\lambda_k = 0$ for $2 \leq k \leq m$, and $S = \emptyset$).

First we prove a specific case of the general result: namely, that any element of I of the form

$$(\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k) \left(v - \sum_{e \in S} e e^* \right)$$

is in J , where S is a finite subset of $s^{-1}(v)$, each $\lambda_i \in K$, and each a_i is assumed to be a real path in E . Towards a contradiction, suppose not. That is, suppose that there are elements in $I \setminus J$. Over all possible vertices $w \in E_0$, all finite subsets T of $s^{-1}(w)$, and all possible $\kappa_i \in K$ ($1 \leq i \leq n$) find an element of $I \setminus J$ of the form $(\kappa_1 a_1 + \kappa_2 a_2 + \dots + \kappa_n a_n)(w - \sum_{e \in T} e e^*)$ for which n is minimal; let x denote this minimal value. So we have

$$x = (\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_k a_k) \left(v - \sum_{e \in S} e e^* \right) \in I \setminus J$$

for some $v \in E^0$, $\lambda_i \in K$ ($1 \leq i \leq k$), and S a finite (possibly empty) subset of $s^{-1}(v)$.

We argue by contradiction on the minimality of k that no such element exists.

Since for $w \in E^0$ we have $w(v - \sum_{e \in S} e e^*) = \delta_{v,w}(v - \sum_{e \in S} e e^*)$, we may assume that each a_i has $r(a_i) = v$.

Let S_x denote the set $\{s(a_i) \mid \lambda_i \neq 0\}$. For each $w \in S_x$ we have $w x \in I$. But $x = \sum_{w \in S_x} w x$, so $x \notin J$ gives $w x \notin J$ for some w . Since $w x$ has the correct form, we

conclude by the minimality of k that $wa_i \neq 0$ for all $1 \leq i \leq k$. Thus we may assume that each of the paths a_i , $1 \leq i \leq k$, has the common source vertex.

Rephrased, we may assume that $x = (\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k)(v - \sum_{e \in S} ee^*)$, where for all i, j , $s(a_i) = s(a_j)$, and $r(a_i) = r(a_j) (= v)$. Among all such x with minimal k , select one for which $(|a_1|, \dots, |a_k|)$ is smallest in the lexicographic order of $(\mathbb{Z}^+)^k$. Multiplying by λ_1^{-1} if necessary, we may assume that

$$x = (a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k) \left(v - \sum_{e \in S} ee^* \right).$$

We analyze the various possible cases for x , and show in each case we are led to a contradiction.

In the first case, suppose $|a_i| > 0$ for every $i \in \{1, \dots, k\}$. Let A denote the set

$$\{f \in E^1 \mid f^* a_i \neq 0 \text{ for some } 1 \leq i \leq k\}.$$

Note that A is finite. Furthermore, we see that $f^* x$ is in J for every $f \in A$, as $f^* x$ is of the correct form (since $|a_i| > 0$ for all i), and either $f^* x$ has fewer terms than x does, or $f^* x$ has the same number of terms as x , in which case $(|f^* a_1|, \dots, |f^* a_k|)$ is smaller than $(|a_1|, \dots, |a_k|)$. But then $ff^* x \in J$ for all $f \in A$, which yields that

$$\sum_{f \in A} ff^* x \in J.$$

But this last expression is precisely x (by again using $|a_i| > 0$ for all i), so we have $x \in J$, a contradiction.

In the other case, suppose $|a_i| = 0$ for some i . By the minimality assumed on $(|a_1|, \dots, |a_k|)$, this gives $|a_1| = 0$. Since the a_i are real paths, this means that a_1 is a vertex, necessarily $a_1 = v$. Since all of the a_i are assumed to start and end at the same vertex as each other, we get that each a_i is in fact a closed path starting and ending at v (Note that each a_i for $i \geq 2$ is a nontrivial closed path based at v , otherwise we would have combined v with such a_i to get a shorter expression.). There are three subcases to consider; we obtain a contradiction in each.

First, suppose there are no simple closed paths based at v . Then necessarily there are no closed paths at all based at v , so that the sum $a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k$ reduces to the expression $a_1 = v$, so that $x = (v + 0)(v - \sum_{e \in S} ee^*)$ is in J , contradicting the assumption that $x \in I \setminus J$.

Secondly, suppose there is exactly one simple closed path at v . Then necessarily this path g must be a cycle, and is the unique cycle based at v . But then any closed path based

at v must be a power of this cycle, i.e., for each $2 \leq i \leq k$ we have $a_i = g^{r_i}$ for some positive integer r_i , so that x in this case has the indicated form, so $x \in J$, again contradicting the assumption that $x \in I \setminus J$.

Finally, suppose there are at least two distinct simple closed paths based at v . Consider the set

$$F = \{f \in E^1 \mid f^*a_i \neq 0 \text{ for some } 2 \leq i \leq k\}.$$

There are two subcases here. Suppose first that $F \cap S \neq \emptyset$. Let $f \in F \cap S$. Now note that

$$\begin{aligned} ff^*x &= (ff^* + \lambda_2 ff^*a_2 + \cdots + \lambda_k ff^*a_k) \left(v - \sum_{e \in S} ee^* \right) \\ &= ff^* \left(v - \sum_{e \in S} ee^* \right) + \left(\sum_{i=2}^k \lambda_i ff^*a_i \right) \left(v - \sum_{e \in S} ee^* \right). \end{aligned}$$

But $f \in S$ yields that the first summand is zero; thus

$$ff^*x = \left(\sum_{i=2}^k \lambda_i ff^*a_i \right) \left(v - \sum_{e \in S} ee^* \right).$$

Note that ff^*a_i is either 0 or a_i (since $a_i \neq v$), so the displayed expression for ff^*x has the correct form, so that $ff^*x \in J$ by the minimality of k . Furthermore,

$$x - ff^*x = \left(v + \sum_{\{a_j \mid f^*a_j=0\}} \lambda_j a_j \right) \left(v - \sum_{e \in S} ee^* \right)$$

is also of the correct form, and the left hand factor has fewer than k nonzero terms (since $f \in F$ gives $f^*a_i = 0$ for some $2 \leq i \leq k$), so that $x - ff^*x \in J$ by the minimality of k . So we have

$$x = ff^*x + (x - ff^*x) = x \in J,$$

again a contradiction.

For the second of the two subcases, which will complete the proof, suppose $F \cap S = \emptyset$. Then in particular a_2 is a closed path based at v , for which $e^*a_2 = 0$ for all $e \in S$. Write $a_2 = fa'_2$ for some edge f and real path a'_2 (possibly of length zero). Among all closed paths based at v having initial edge f , choose one of minimal length, call it g_1 . Then g_1 is necessarily a cycle, and, since it has the same initial edge as does a_2 , we have $e^*g_1 = 0$ for all $e \in S$. In particular, this gives that

$$\left(v - \sum_{e \in S} ee^* \right) g_1 = g_1.$$

By the hypotheses of this subcase, there exists a second simple closed path g_2 based at v . In particular, $g_2^*g_1 = 0$. Pick an integer t for which $|g_1^t| > |a_k|$. Let y denote the element $(g_1^t)^*xg_1^t$ of $L_K(E)$. Since $x \in I$ we have $y \in I$. Using $(v - \sum_{e \in S} ee^*)g_1 = g_1$, we get

$$y = (g_1^t)^*xg_1^t = v + \lambda_2(g_1^t)^*a_2g_1^t + \cdots + \lambda_k(g_1^t)^*a_kg_1^t$$

is in I . We now argue exactly as in the proof of Case II of Theorem 3.1.5, as follows. If $(g_1^t)^*a_i g_1^t \neq 0$, then $(g_1^t)^*a_i \neq 0$. Since $|g_1^t| > |a_i|$, this gives that a_i is an initial segment of g_1^t , i.e., $g_1^t = a_i b_i$ for some real path b_i . Since the a_i are closed paths based at v , and g_1 is a cycle, we get from the equation $g_1^t = a_i b_i$ that $a_i = g_1^{r_i}$ for some integer r_i . In particular, each a_i commutes with g_1^t , which yields that whenever a term of the form $(g_1^t)^*a_i g_1^t$ is nonzero, then necessarily it equals a_i , so that $(g_1^t)^*a_i g_1^t \neq 0$ implies $(g_1^t)^*a_i g_1^t = g_1^{r_i}$ for some positive integer r_i .

Thus we may write the element y of I as

$$y = v + \delta_2 \lambda_2 g_1^{r_2} + \cdots + \delta_k \lambda_k g_1^{r_k},$$

where $\delta_i = 1$ if $\lambda_i (g_1^t)^*a_i g_1^t \neq 0$, and $\delta_i = 0$ otherwise. Since $g_2^*g_1 = 0$ this yields that

$$g_2^*y g_2 = g_2^*v g_2 = v,$$

so that $v \in I$. But $I \cap E^0 \subseteq J$, so $v \in J$, so that $x = vx \in J$, the final contradiction required to establish the “real part” part of the proof.

To summarize, we have shown that any element of I of the form

$$(\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_k a_k) \left(v - \sum_{e \in S} ee^* \right),$$

where S is a finite subset of E_1 , each a_i is a real path in E , and $\lambda_i \in K$, is in the ideal generated by elements of I of the indicated form.

Now we prove that any arbitrary element of I is in J . Again working towards a contradiction, suppose that $I \setminus J \neq \emptyset$, and let

$$x = (\lambda_1 a_1 b_1^* + \cdots + \lambda_n a_n b_n^*) \left(v - \sum_{e \in S} ee^* \right) \in I \setminus J,$$

where each a_i and b_i is a real path in E , and n is minimal. As above, we may choose $\lambda_1 = 1$, and we may assume that $s(a_i) = s(a_j)$ and $r(b_i^*) = r(b_j^*) (= v)$ for every i and j . Among all such x , select one for which $(|b_1|, \dots, |b_n|)$ is smallest in the lexicographic order of $(\mathbb{Z}^+)^n$.

Suppose $|b_i| > 0$ for some i . Write $b_i = e_i b'_i$ for some edge e_i and some real path b'_i (possibly of length 0). If $e_i \in S$, then

$$b_i^* \left(v - \sum_{e \in S} ee^* \right) = (b'_i)^* e_i^* \left(v - \sum_{e \in S} ee^* \right) = (b'_i)^* e_i^* - (b'_i)^* e_i^* = 0.$$

So we may assume that if $|b_i| > 0$ in the indicated expression for x , then $e_i \notin S$.

First, suppose $|b_i| > 0$ for every i . As above, let e_i denote the initial edge of b_i , and write $b_i = e_i b'_i$; then as shown in the previous paragraph, we may assume $e_i \notin S$. Note that for any edge $f \in s^{-1}(v) \setminus S$ we have $(v - \sum_{e \in S} ee^*)f = f$. So for any $f \in s^{-1}(v) \setminus S$ we have

$$xf = (a_1 b_1^* + \cdots + \lambda_n a_n b_n^*) \left(v - \sum_{e \in S} ee^* \right) f = \sum_{i=1}^n \lambda_i a_i b_i^* f = \sum_{\{i | e_i = f\}} \lambda_i a_i (b'_i)^* \in I.$$

We note that, since $fr(f) = f$, this expression is of the correct form. So if the number of monomial terms in xf is less than n , then $xf \in J$. If the number of monomial terms in xf is n , then since $(|b'_1|, \dots, |b'_n|) < (|b_1|, \dots, |b_n|)$, the minimality condition implies $xf \in J$. So either case gives $xf \in J$. In particular, for each e_j which appears as the initial edge of some b_j in the expression for x , we have $x e_j e_j^* \in J$. But this in turn yields

$$x = \sum_{\{\text{distinct } e_j \mid 1 \leq j \leq n\}} x e_j e_j^* \in J,$$

a contradiction.

On the other hand, suppose $|b_j| = 0$ for some $1 \leq j \leq n$. So one of the b_j , say b_1 , is of the form v for some $v \in E_0$. Without loss of generality, assume that $|b_1| = \cdots = |b_u| = 0$ for some $u \geq 1$, and that $|b_j| > 0$ for $j \geq u + 1$. Then we have

$$x = (a_1 + \lambda_2 a_2 + \cdots + \lambda_u a_u + \lambda_{u+1} a_{u+1} b_{u+1}^* + \cdots + \lambda_n a_n b_n^*) \left(v - \sum_{e \in S} ee^* \right).$$

Let

$$T = \{f \in E^1 \mid b_i^* f \neq 0 \text{ for some } u + 1 \leq i \leq n\};$$

so T is the set of edges which appear as the initial edge of some real path b_i , $u + 1 \leq i \leq n$. Note that T is finite. As indicated above, minimality implies that $S \cap T = \emptyset$. Again using minimality, an argument analogous to one used previously yields that xf is in J for all $f \in T$, hence

$$\sum_{f \in T} x f f^* \in J.$$

Write $b_i = f_i b'_i$ for each $u + 1 \leq i \leq n$. Then for $f \in T$ we have $b_i^* f f^* = 0$ unless $f = f_i$, in which case $b_i^* f f^* = b_i^*$. This yields that $b_i^*(v - \sum_{f \in T} f f^*) = 0$ for $u + 1 \leq i \leq n$, which in turn implies that

$$x - \sum_{f \in T} x f f^* = (a_1 + \lambda_2 a_2 + \cdots + \lambda_u a_u) \left(v - \sum_{f \in T} f f^* \right).$$

We are now in position to invoke the result established in the first part of the proof: since each a_i ($1 \leq i \leq u$) is a real path, the displayed expression is in J . Thus we have both $x - \sum_{f \in T} x f f^*$ and $\sum_{f \in T} x f f^*$ are in J , which gives $x \in J$, the final contradiction needed to establish our main result. \square

Not surprisingly, the description of the generating sets for ideals of Leavitt path algebras in the row-finite case will follow from the description of the generating sets in the general case. However, this conclusion is not completely immediate. Specifically, to establish the row-finite case from the general case we must show that in the row-finite case, any ideal generated by an element of the form $(v + \sum_{i=2}^m \lambda_i g^i)(v - \sum_{e \in S} e e^*)$ can in fact be generated by some collection of vertices, together with elements of the form $f(h) = w + \sum_{i=2}^{\ell} \kappa_i h^i$, where $\kappa_i \in K$ and h is a cycle based at w .

We first prove a lemma.

Lemma 3.2.2. *Let E be any graph. Let $v \in E^0$ be a finite emitter, and let S denote any subset of $s^{-1}(v)$. Then these two ideals of $L_K(E)$ are equal:*

$$\left\langle v - \sum_{e \in S} e e^* \right\rangle = \langle \{r(f) \mid f \in s^{-1}(v) \setminus S\} \rangle.$$

Proof. For convenience we let y denote $v - \sum_{e \in S} e e^*$, A denote $\langle y \rangle$, and B denote the second displayed ideal. Note that, since v is a finite emitter, then for $S = s^{-1}(v)$ we get the trivial statement that the ideal $\{0\}$ is generated by the empty set. So we consider the situation where $s^{-1}(v) \setminus S$ is nonempty.

Let $f \in s^{-1}(v) \setminus S$. Since $f \notin S$ we get $y f = v f = f$, so that $f^* y f = f^* f = r(f)$. Thus each $r(f)$ in the generating set for B is in A , so $B \subseteq A$.

Conversely, since v is a finite emitter, the CK2 relation at v gives

$$v = \sum_{e \in s^{-1}(v)} e e^* = \sum_{e \in S} e e^* + \sum_{f \in s^{-1}(v) \setminus S} f f^*,$$

so that

$$y = v - \sum_{e \in S} e e^* = \sum_{f \in s^{-1}(v) \setminus S} f f^* = \sum_{f \in s^{-1}(v) \setminus S} f r(f) f^* \in B.$$

So $y \in B$, so that $A \subseteq B$. \square

Proposition 3.2.3. *Let E be an arbitrary graph, let v be a finite emitter, let S be a subset of $s^{-1}(v)$, and let g be a cycle based at v having initial edge e_1 . Write $g = e_1 p$ for some real path p in E . Let w denote $r(e_1)$, and let g_w denote the cycle pe_1 based at w (so that g_w is the cycle g , shifted to be based at w rather than v). Let*

$$z = \left(v + \sum_{i=1}^m \lambda_i g^i \right) \left(v - \sum_{e \in S} ee^* \right).$$

1. If $e_1 \in S$, then $\langle z \rangle = \langle \{r(f) \mid f \in s^{-1}(v) \setminus S\} \rangle$.
2. If $e_1 \notin S$, then $\langle z \rangle = \langle w + \sum_{i=1}^m \lambda_i g_w^i \rangle$.

Proof. We use throughout that $e_1 e_1^* g = g$. For convenience we let $t(x) = \sum_{i=1}^m \lambda_i x^i \in K[x]$.

1. We compute

$$e_1 e_1^* z = e_1 e_1^* (v + t(g)) \left(v - \sum_{e \in S} ee^* \right) = (e_1 e_1^* + t(g)) \left(v - \sum_{e \in S} ee^* \right).$$

But $e_1 \in S$ gives $e_1 e_1^* (\sum_{e \in S} ee^*) = e_1 e_1^*$, so that expanding the last term in the display gives

$$= e_1 e_1^* - e_1 e_1^* + t(g) - t(g) \sum_{e \in S} ee^* = t(g) \left(v - \sum_{e \in S} ee^* \right).$$

So $e_1 e_1^* z = t(g)(v - \sum_{e \in S} ee^*) \in \langle z \rangle$, so that

$$\begin{aligned} z - e_1 e_1^* z &= z - t(g) \left(v - \sum_{e \in S} ee^* \right) \\ &= (v + t(g)) \left(v - \sum_{e \in S} ee^* \right) - t(g) \left(v - \sum_{e \in S} ee^* \right) \\ &= v \left(v - \sum_{e \in S} ee^* \right) \\ &= \left(v - \sum_{e \in S} ee^* \right) \in \langle z \rangle. \end{aligned}$$

Now using Lemma 3.2.2, we get that $\langle \{r(f) \mid f \in s^{-1}(v) \setminus S\} \rangle \subseteq \langle z \rangle$.

Conversely, since

$$v - \sum_{e \in S} ee^* = \sum_{f \in s^{-1}(v) \setminus S} ff^* = \sum_{f \in s^{-1}(v) \setminus S} fr(f)f^*,$$

we get $v - \sum_{e \in S} ee^*$ is in the ideal generated by the indicated vertices, hence so is z , so that

$$\langle z \rangle \subseteq \langle \{r(f) \mid f \in s^{-1}(v) \setminus S\} \rangle$$

as well.

2. Since $e_1 \notin S$ we have $(v - \sum_{e \in S} ee^*)e_1 = e_1$. Clearly $e_1^*ze_1 \in \langle z \rangle$. Now compute

$$e_1^*ze_1 = e_1^*(v + t(g)) \left(v - \sum_{e \in S} ee^* \right) e_1 = e_1^*(v + t(g))e_1 = e_1^*e_1 + e_1^*t(g)e_1.$$

But if w denotes $r(e_1)$, this last expression is precisely $w + t(g_w)$, so that

$$\left\langle w + \sum_{i=1}^m \lambda_i g_w^i \right\rangle = \langle w + t(g_w) \rangle \subseteq \langle z \rangle.$$

On the other hand, writing $g = e_1p$, we have that $v + t(g) = p^*(w + t(g_w))p$, so that $v + t(g)$, and therefore

$$z = (v + t(g)) \left(v - \sum_{e \in S} ee^* \right),$$

are in $\langle w + t(g_w) \rangle$ as desired. □

We now get as a consequence of Theorem 3.1.5 the analogous result for the row-finite case.

Alternative Proof of Theorem 3.1.5. By Theorem 3.2.1, I has a generating set consisting of elements of the type

$$\left(v - \sum_{i=1}^m \lambda_i g^i \right) \left(v - \sum_{e \in S} ee^* \right).$$

But since E is row-finite, Proposition 3.2.3 yields that the ideal generated by any element of this type can in fact be generated by elements of the desired form. □

3.3 Examples

In this section we will give a couple of examples of two-sided ideals for different types of graphs, and show that they are generated by elements of the desired form. Our goal is to break down the given polynomials into smaller components.

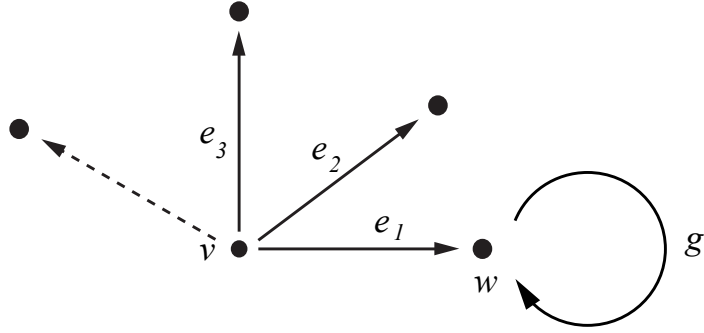


Figure 3.3: Graph of the Leavitt path algebra defined in Example 3.3.1 and 3.3.2.

Example 3.3.1. Let E be the directed graph given in Figure 3.3. Consider the two-sided ideal I in $L_K(E)$ generated by $\mu = v + e_1gg^*e_1^* - e_2e_2^*$. Then

$$e_1^*\mu e_1 = r(e_1) + gg^* = w + gg^* \in I.$$

Here we consider two cases with respect to the field K .

Case I: If K has characteristic 2, then note that $w + gg^* = w - gg^* \in I$ and

$$\begin{aligned} \mu &= v + e_1gg^*e_1^* - e_2e_2^* \\ &= v - e_1e_1^* + e_1e_1^* - e_1gg^*e_1^* - e_2e_2^* \\ &= (v - e_1e_1^* - e_2e_2^*) + (e_1e_1^* - e_1gg^*e_1^*) \\ &= (v - e_1e_1^* - e_2e_2^*) + e_1(w - gg^*)e_1^*. \end{aligned}$$

Hence $\langle \mu \rangle = \langle v - e_1e_1^* - e_2e_2^*, w - gg^* \rangle$.

Case II: If K is any other field, then $(w + gg^*)g = 2g \in I$ and hence $g \in I$. Then $g^*g = w \in I$, and $e_1gg^*e_1^* \in I$. Note that $\mu - e_1gg^*e_1^* = v - e_2e_2^*$. We get that $\langle \mu \rangle = \langle v - e_2e_2^*, w \rangle$

Example 3.3.2. Let E be the directed graph given in Figure 3.3. Consider the two-sided ideal I in $L_K(E)$ generated by $\mu = v - e_1gg^*e_1^*$. Then

$$e_1^*\mu e_1 = r(e_1) - gg^* = w - gg^* \in I.$$

Note that

$$e_1(w - gg^*)e_1^* = e_1e_1^* - e_1gg^*e_1^* \in I \text{ and}$$

$$e_1e_1^* - e_1gg^*e_1^* - \mu = v - e_1e_1^* \in I.$$

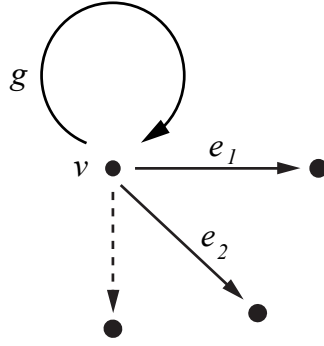


Figure 3.4: Graph of the Leavitt path algebra defined in Example 3.3.3.

Now we have

$$\mu = v - e_1 g g^* e_1^* = e_1 (w - g g^*) e_1^* + (v - e_1 e_1^*).$$

Hence $\langle \mu \rangle = \langle w - g g^*, v - e_1 e_1^* \rangle$.

Example 3.3.3. Let E be the directed graph given in Figure 3.4. Consider the two-sided ideal I in $L_K(E)$ generated by $\mu = v + e_1 e_1^* g^* - e_2 e_2^* - g$. Then $e_1 e_1^* \mu = e_1 e_1^* + e_1 e_1^* g^* \in I$. Note that

$$\mu - e_1 e_1^* \mu = v - e_1 e_1^* - e_2 e_2^* - g \in I.$$

We then have

$$g^* (\mu - e_1 e_1^* \mu) g = v - g \in I \text{ and}$$

$$x := e_1 e_1^* + e_2 e_2^* \in I.$$

We see that $e_1^* x e_1 = r(e_1) \in I$ and $e_2^* x e_2 = r(e_2) \in I$. As a result, we obtain $\langle \mu \rangle = \langle r(e_1), r(e_2), v - g \rangle$.

Chapter 4

Two-Sided Chain Conditions

In this chapter, we will give the necessary and sufficient conditions on the directed graph E so that the associated Leavitt path algebra satisfies two-sided chain conditions by using the main theorems from previous section. In the first section, we will state some preliminary concepts and results. In the next two sections, we will characterize two-sided Noetherian Leavitt path algebras and two-sided Artinian Leavitt path algebras with respect to the underlying directed graph. In the final section, we will present some examples to illustrate the results we have found. The results we demonstrate here appear in [18] and [6].

4.1 Preliminaries

In this section we classify those Leavitt path algebras which satisfy the ascending chain condition (resp., descending chain condition) on two-sided ideals. To do so, we first recall some notation.

Definition 4.1.1. If H is a hereditary subset of E^0 , then the *breaking vertices* of H is defined to be the set

$$B_H = \{v \in E^0 \setminus H \mid v \text{ is an infinite emitter, and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty\}$$

and, for any $v \in B_H$, we let

$$v^H = v - \sum_{s(e)=v, r(e) \notin H} ee^*.$$

Example 4.1.2. We consider the graph given in Figure 2.2. Let $H = \{u_1, u_2, \dots\}$, and by Example 2.2.2, we know that H is hereditary subset of E^0 . Let f_i be the edge with $s(f_i) = v_i$ and $r(f_i) = v_{i+1}$. We make the following observation. We see that $v_1 \in E^0 \setminus H$, and v_1

is an infinite emitter. In addition, $|s^{-1}(v) \cap r^{-1}(E^0 \setminus H)|$ consists of one edge; f_1 . Hence, $v_1 \in B_H$. Similarly, we see that v_i is in B_H for each $i \in \mathbb{N}$, hence B_H is infinite.

We also get that

$$v_i^H = v_i - \sum_{s(e)=v_i, r(e) \notin H} ee^* = v_i - f_i f_i^*$$

for each $i \in \mathbb{N}$.

Definition 4.1.3. A pair (H, S) is called an *admissible pair* if H is a hereditary saturated subset of E^0 and $S \subseteq B_H$.

We let \mathcal{L}_E denote the set of admissible pairs of E , and order these elements by setting

$$(H, S) \leq (H', S') \text{ in case } H \subseteq H' \text{ and } S \subseteq H' \cup S'.$$

Definition 4.1.4. If H is a hereditary saturated subset of E^0 and $S \subseteq B_H$, then $I(H, S)$ denotes the ideal of $L_K(E)$ generated by $\{v \mid v \in H\} \cup \{v^H \mid v \in S\}$.

Definition 4.1.5. If H is a saturated hereditary subset of E^0 and $S \subseteq B_H$, let $I_{(H, S)}$ denote the ideal in $L_K(E)$ generated by $\{v \mid v \in H\} \cup \{v_H \mid v \in S\}$.

For any admissible pair (H, S) we define the graph $E \setminus (H, S)$ by setting:

$$(E \setminus (H, S))^0 = (E^0 \setminus H) \cup \{v' \mid v \in B_H \setminus S\}, \text{ and}$$

$$(E \setminus (H, S))^1 = \{e \in E^1 \mid r(e) \notin H\} \cup \{e' \mid e \in E^1, r(e) \in B_H \setminus S\}.$$

Here the symbols v' and e' denote symbols not in the original graph E . The range and source functions r and s are extended to $E \setminus (H, S)$ by defining $s(e') = s(e)$ and $r(e') = r(e)'$.

Example 4.1.6. Consider the graph given in Figure 4.1, and let $H = \{w\}$ which is hereditary saturated subset of E^0 . There are no infinite emitters in this graph, hence $B_H = \emptyset$. Consider the admissible pair (H, \emptyset) . Then

$$(E \setminus (H, \emptyset))^0 = \{v, u\}, \text{ and}$$

$$(E \setminus (H, \emptyset))^1 = \{x, t\}.$$

The resulting graph is given in Figure 4.2.

Example 4.1.7. Consider the graph given in Figure 2.2, and let $H = \{u_1, u_2, \dots\}$. In Example 2.2.2, we showed that $B_H = \{v_1, v_2, \dots\}$. Let $S = \{v_1\} \subset B_H$, hence (H, S) is an admissible pair. Once again, let f_i be the edge with $s(f_i) = v_i$ and $r(f_i) = v_{i+1}$. Then

$$(E \setminus (H, S))^0 = \{v_1, v_2, \dots\} \cup \{v'_2, v'_3, \dots\}, \text{ and}$$

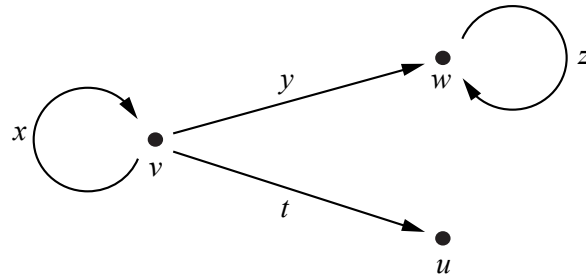


Figure 4.1: Graph for Example 4.1.6.

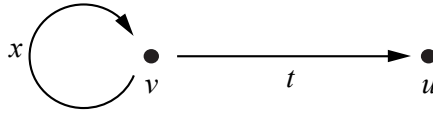


Figure 4.2: The graph of $E \setminus (H, \emptyset)$ in Example 4.1.6.

$$(E \setminus (H, S))^1 = \{f_1, f_2, \dots\} \cup \{f'_1, f'_2, \dots\}$$

where $s(f'_i) = s(f_i) = v_i$ and $r(f'_i) = r(f_i)' = v'_{i+1}$. The resulting graph is given in Figure 4.3.

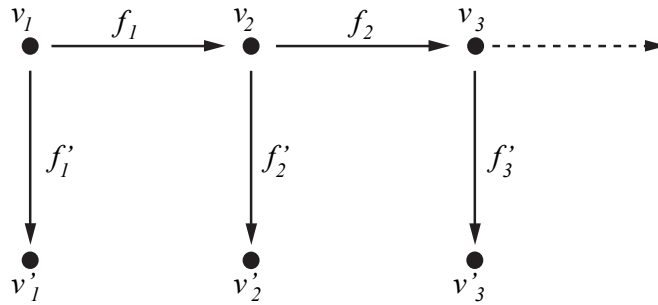


Figure 4.3: The graph of $E \setminus (H, S)$ in Example 4.1.7.

A theorem of Tomforde [35, Theorem 5.7], stated for countable graphs, plays a central role in the current discussion. The key result upon which the proof of [35, Theorem 5.7] relies is the so-called “Graded Uniqueness Theorem” [35, Theorem 4.8]. The Graded Uniqueness Theorem was extended from countable graphs to graphs of arbitrary size in both [24, Theorem 3.2] and [12, Theorem 3.5]. (Indeed, one can also show that all of the machinery which

supports the proof of [35, Theorem 4.8] holds verbatim for arbitrary graphs as well.) With the appropriate extension of the Graded Uniqueness Theorem in hand, a close examination of the remainder of the proof of [35, Theorem 5.7] yields that the following result holds for graphs of arbitrary size.

Theorem 4.1.8. *(Extension of [35, Parts (1) and (2) of Theorem 5.7] to graphs of arbitrary size.) Let E be a directed graph.*

(i) *The map $(H, S) \mapsto I_{(H,S)}$ is a lattice isomorphism from the lattice \mathcal{L}_E of admissible pairs to the lattice \mathcal{H}_E of graded ideals of $L_K(E)$.*

(ii) *For any admissible pair (H, S) there is an isomorphism of K -algebras*

$$L_K(E)/I_{(H,S)} \cong L_K(E \setminus (H, S)).$$

4.2 Two-Sided Noetherian Leavitt Path Algebras

Now the Theorems are in hand, we are going to put the pieces together to get the two-sided Noetherian results. First we will do this for row-finite graphs by using Theorem 3.1.5, and then for the arbitrary graphs by using 3.2.1

First we need some preliminary results that we get as consequences of Theorem 3.2.1.

Lemma 4.2.1. *Let I be a two-sided ideal of $L_K(E)$, where E is an arbitrary graph. Suppose g, h are two non-trivial cycles based at distinct vertices u, v respectively. Suppose $u + \sum a_r g^r = p(g)$ and $v + \sum b_s h^s = q(h)$ both belong to I , where $p(x)$ and $q(x)$ are polynomials of smallest positive degree in $K[x]$ with $p(0) = 1 = q(0)$ such that $p(g) \in I$ and $q(h) \in I$. If $u \geq v$, then $v \geq u$ and $\langle p(g) \rangle = \langle q(h) \rangle$.*

Proof. Let the non-trivial cycle g be given by the edge sequence $e_1 \cdots e_n$ with $r(e_i) = w_i$ for all i and that $w_n = u$. For any i , let $g_i = e_{i+1} \cdots e_i$ be the shifted non-trivial cycle based at w_i and $p(g_i) = w_i + \sum a_r g_i^r$. Clearly, $p(x)$ is a polynomial of smallest positive degree such that $p(g_i) \in I$.

Let μ be a path from u to v . We claim that v must lie on the cycle g . Because otherwise, $\mu^*g = 0$ and so $\mu^*p(g)\mu = \mu^*u\mu + \sum a_r \mu^*g^r\mu = v \in I$. This contradicts the fact that $\deg q(x) > 0$. So we can write $g = \mu\nu$ where ν is the part of g from v to u . Thus, in particular, $v \geq u$. We claim that the cycle $h = \nu\mu$. Indeed, if h contains an edge f with $s(f) = w_i$ for some i and $f \neq e_{i+1}$, then $f^*g_i = 0$ and we get $f^*p(g_i)f = f^*w_i f = w_{i+1} \in I$, contradicting the fact that $\deg p(x) > 0$. Hence $h = \nu\mu$. Since $\mu^*g\mu = h$, we get $\mu^*p(g)\mu = p(h) \in I$. By the minimality of $q(x)$, we have $q(x)$ is a divisor of $p(x)$ in $K[x]$. Similarly,

since $\nu^*q(h)\nu = q(g) \in I$, we conclude that $p(x)$ is a divisor of $q(x)$. Thus $q(x) = kp(x)$ for some $k \in K$. Since $p(0) = 1 = q(0)$, $q(x) = p(x)$. Hence $q(h) = \mu^*p(g)\mu \in \langle p(g) \rangle$. Likewise, $p(g) = \nu^*q(h)\nu \in \langle q(h) \rangle$. Hence $\langle p(g) \rangle = \langle q(h) \rangle$. \square

The next Lemma and its proof are implicit in the proof of Lemma 7 in [2].

Lemma 4.2.2. *Let E be an arbitrary graph and $S \subseteq E^0$. If $v \in \overline{S}$, and there is a non-trivial cycle based at v , then $u \geq v$ for some $u \in S$.*

Proof. We recall that $\overline{S} = \bigcup_{n \geq 0} \Lambda_n(S)$. Let k be the smallest non-negative integer such that $v \in \Lambda_k(S)$. We prove the statement by induction on k , the statement being true by definition when $k = 0$. Assume $k > 0$ and that the statement holds when $k = n - 1$. Let $k = n$. Since $v \in \Lambda_n(S) \setminus \Lambda_{n-1}(S)$, $0 < |s^{-1}(v)| < \infty$ and $\{w_1, \dots, w_m\} = r(s^{-1}(v)) \subseteq \Lambda_{n-1}(S)$. Since v is the base of a non-trivial cycle g , one of the vertices, say, w_j lies on the cycle g and so $w_j \geq v$. Since $w_j \in \Lambda_{n-1}(S)$ and is the base of a cycle, by induction there is a $u \in S$ such that $u \geq w_j$. Then $u \geq v$, as desired. \square

Remark 4.2.3. *If $p(x) \in K[x]$ is the polynomial of smallest degree > 0 such that $p(g) \in I$ and $p(0) = 1$, then for any polynomial $q(x) \in K[x]$ satisfying $q(g) \in I$ we must have $p(x)|q(x)$. First we use the division algorithm and get $q(x) = p(x)s(x) + r(x)$ where either $\deg r(x) < \deg p(x)$ or $r(x) = 0$. In either case, we see that $r(g) \in I$. We claim that $r(x) = 0$. Otherwise, then write $r(g) = \lambda_0v + \lambda_1g + \dots + \lambda_n g^n$, and let k be the smallest index such that $\lambda_k \neq 0$. Then note that*

$$r'(g) := (1/\lambda_k)(g^*)^k r(g) = v + \lambda'_{k+1}g + \dots + \lambda'_n g^{n-k} \in I.$$

Hence we get $\deg r'(x) < \deg p(x)$, $r'(g) \in I$ and $r'(0) = 1$, a contradiction.

We also need the following Lemma, whose proof is given in the first paragraph of the proof of Theorem 5.7 in [35].

Lemma 4.2.4. *Let E be an arbitrary graph and let H be a hereditary and saturated subset of vertices in E . If I is the two-sided ideal generated by H , then $I \cap E^0 = H$.*

Theorem 4.2.5. *Let E be a row-finite graph. Then the following are equivalent:*

- (i) $L_K(E)$ has a.c.c. on two-sided ideals;
- (ii) $L_K(E)$ has a.c.c. on two-sided graded ideals;
- (iii) The hereditary saturated closures of the subsets of the vertices in E^0 satisfy a.c.c.

Proof. (iii) \Rightarrow (i) Suppose the ascending chain condition holds on the hereditary saturated closures of the subsets of E^0 . Let I be a two-sided ideal of $L_K(E)$. By Theorem 3.1.5 and by Remark 4.2.3, I is generated by the set

$$T = \{v + \sum_r \lambda_r g^r = p(g) \in I \mid v \in E^0, g \text{ is a cycle (may be trivial) based at } v \text{ and } p(x) \in K[x] \text{ is a polynomial of smallest degree such that } p(g) \in I \text{ and } p(0) = 1\}.$$

It is well known that two-sided Noetherian is equivalent to every two-sided ideal being finitely generated, so we wish to show that I is generated by a finite subset of T .

Suppose, towards a contradiction, there are infinitely many $p_i(g_i) = v_i + \sum \lambda_r g_i^r \in T$ with $i \in H$, where H is an infinite set. Assume that for each i , g_i is a non-trivial cycle based at v_i and that $\deg p_i(x) > 0$. By Lemma 4.2.1, we may assume that for any two i, j with $i \neq j$, $v_i \not\leq v_j$. Well-order the set H and consider it as the set of all ordinals less than an infinite ordinal κ . Define $S_1 = v_1$ and for any $\alpha < \kappa$, define $S_\alpha = \cup_{\beta < \alpha} S_\beta$ if α is a limit ordinal, and define $S_\alpha = S_\beta \cup \{v_{\beta+1}\}$ if α is a non-limit ordinal of the form $\beta + 1$. By the hypothesis the ascending chain of hereditary saturated closures of subsets $\bar{S}_1 \subseteq \bar{S}_2 \subseteq \dots \subseteq \bar{S}_\alpha \subseteq \dots$ becomes stationary after a finite number of steps. So there is an integer n such that $\bar{S}_n = \bar{S}_{n+1} = \dots$. Now $v_{n+1} \in \bar{S}_{n+1} = \bar{S}_n$ and by Lemma 4.2.2, there is a $v_i \in S_n$ such that $v_i \geq v_{n+1}$. This is a contradiction. Hence the set $W = \{p_i(g_i) \in T \mid \deg p_i(x) > 0\}$ is finite.

So by the previous paragraph, if there are only finitely many $p_i(g_i)$ in T with $\deg p_i(x) = 0$, that is, only finitely many vertices in T , then we are done. We index the vertices v_α in T by ordinals $\alpha < \kappa$, where κ is an infinite ordinal. Then as before, we get a well-ordered ascending chain of hereditary saturated closure of subsets $\bar{S}_1 \subseteq \bar{S}_2 \subseteq \dots \subseteq \bar{S}_\alpha \subseteq \dots$ ($\alpha < \kappa$) where $S_1 = \{v_1\}$ and the S_α are inductively defined as before. Since, by hypothesis, this chain becomes stationary after a finite number of steps, there is an integer n such that $\bar{S}_\alpha = \bar{S}_n$ for all $\alpha > n$. Thus $\{v_\alpha \mid \alpha < \kappa\} \subseteq \bar{S}_n$. Since the ideal generated by the finite set $S_n = \{v_1, \dots, v_n\}$ contains \bar{S}_n , we conclude that the ideal I is generated by the finite set $W \cup S_n$. Thus the Leavitt path algebra is two-sided Noetherian.

(i) \Rightarrow (iii) Conversely, suppose $L_K(E)$ is two-sided Noetherian. Consider an ascending chain of hereditary saturated closures of subsets of vertices $\bar{S}_1 \subseteq \bar{S}_2 \subseteq \dots$ in E^0 . Consider the corresponding ascending chain of two-sided ideals $I_1 \subseteq I_2 \subseteq \dots$, where for each integer i , I_i is the two-sided ideal generated by \bar{S}_i . By hypothesis, there is an integer n such that $I_n = I_i$ for all $i > n$. We claim that $\bar{S}_i = \bar{S}_n$ for all $i > n$. Otherwise, we can find a vertex $w \in \bar{S}_i \setminus \bar{S}_n$ and since $w \in I_i = I_n$, $w \in I_n \cap E^0 = \bar{S}_n$ by Lemma 4.2.4 and this is a contradiction.

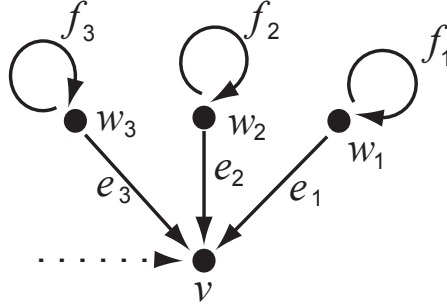


Figure 4.4: Graph of the Leavitt path algebra defined in Example 4.2.8.

(ii) \Leftrightarrow (iii) This follows directly from [11], Theorem 5.3. \square

Remark 4.2.6. We note that this result only shows that the a.c.c. on graded ideals is sufficient to get a.c.c. on all ideals, and that we are not proving that every ideal in a two-sided Noetherian Leavitt path algebra is graded. As an example we can consider $K[x, x^{-1}]$, which is the Leavitt path algebra of the graph with one vertex and one loop. Note that although this Leavitt path algebra has infinitely many ideals, it is nonetheless Noetherian, but has only the trivial graded ideals.

Now we easily obtain the following result.

Corollary 4.2.7. Every Leavitt path algebra with a finite graph is two-sided Noetherian.

We conclude by presenting another example of a non-Noetherian Leavitt path algebra.

Example 4.2.8. Let $E = (E^0, E^1, r, s)$ be the directed graph where $E^0 = \{v, w_1, w_2, w_3, \dots\}$ and $E^1 = \{e_1, e_2, \dots\} \cup \{f_1, f_2, \dots\}$ is such that $r(e_i) = v$ and $s(e_i) = r(f_i) = s(f_i) = w_i$. The graph of this Leavitt path algebra is given in Figure 4.4.

Note that if we let $S_i = \{w_1, \dots, w_i\}$, then $\overline{S}_1 \subsetneq \overline{S}_2 \subsetneq \dots$ is a non-terminating ascending chain of hereditary saturated closures of sets in E^0 . Hence by Theorem 4.2.5, $L_K(E)$ is not two-sided Noetherian. Indeed, $\langle w_1 \rangle \subsetneq \langle w_1, w_2 \rangle \subsetneq \dots$ is a non-terminating ascending chain of ideals in $L_K(E)$.

The current goal is to show how Theorem 3.2.1 allows us to identify the two-sided Noetherian Leavitt path algebras for arbitrary graphs. In our verification of the following useful result, we will use 2.5.16 which says that if I is an ideal of $L_K(E)$, then $E^0 \cap I$ is a hereditary saturated subset of E^0 .

Proposition 4.2.9. *Let E be an arbitrary graph and let I be an ideal of $L_K(E)$. Let H denote the hereditary saturated subset $I \cap E^0$ of E^0 , and let L denote the ideal of $L_K(E)$ generated by $H \cup C$, where C is the collection of elements of I of the form*

$$w + \sum_{i=1}^k \kappa_i h^i$$

where h is a cycle based at the vertex w , and $\kappa_i \in K$.

Suppose $x = (v + \sum_{i=1}^k \lambda_i g^i)(v - \sum_{e \in S} ee^*) \in I$.

(i) *If v is an infinite emitter, then either*

$v \in B_H$ (in case $s^{-1}(v) \cap r^{-1}(E^0 \setminus H) \neq \emptyset$), or

$\langle x, L \rangle = \langle v, L \rangle = L$ (in case $s^{-1}(v) \cap r^{-1}(E^0 \setminus H) = \emptyset$).

(ii) *If v is a finite emitter, then $\langle x, L \rangle = \langle v, L \rangle = L$.*

Proof. (i) First assume that g is not a trivial cycle, and let e_1 denote the initial edge of g .

If $f \in s^{-1}(v) \setminus (S \cup \{e_1\})$, then $f^*g = 0$, and $e^*f = 0$ for all $e \in S$. If g is trivial, then let $f \in s^{-1}(v) \setminus S$. In either case, we get $f^*xf = f^*vf = r(f) \in I \cap E^0 = H$. So if v is an infinite emitter, we have shown that the range vertices of all edges emitted by v , except perhaps for those in the finite set $S \cup \{e_1\}$, are in H . So $|s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty$. Thus if $0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)|$, then $v \in B_H$ by definition. If on the other hand we have $0 = |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)|$, then $r(e_1) \in H$ and $r(e) \in H$ for all $e \in S$. Since $x \in I$ this gives $v \in I$, so that $\langle v, L \rangle \subseteq \langle x, L \rangle$. Since the reverse containment is clear, we get the desired conclusion.

(ii) follows directly from Proposition 3.2.3. □

We now relate the chain conditions on the set of admissible pairs to the chain conditions on the hereditary saturated subsets and sets of breaking vertices.

Lemma 4.2.10. *Let E be an arbitrary graph. Then the following are equivalent.*

- (i) *The lattice \mathcal{L}_E of admissible pairs (H, S) of E satisfies the a.c.c. with respect to the partial order indicated above.*
- (ii) *The lattice \mathcal{H}_E of all hereditary saturated subsets of E satisfies the a.c.c. (under set inclusion), and, for each $H \in \mathcal{H}_E$, the corresponding set B_H of breaking vertices is finite.*

Proof. Suppose the a.c.c. holds in \mathcal{L}_E . Let

$$H_1 \subseteq H_2 \subseteq \cdots$$

be an ascending chain of hereditary saturated subsets of vertices in E . Then we get an ascending chain of admissible pairs

$$(H_1, \emptyset) \leq (H_2, \emptyset) \leq \cdots \text{ in } \mathcal{L}_E$$

(where \emptyset is the empty set). By hypothesis, there is an integer n such that

$$(H_n, \emptyset) = (H_{n+1}, \emptyset) = \cdots .$$

This implies that

$$H_n = H_{n+1} = \cdots ,$$

showing that a.c.c holds in \mathcal{H}_E . Let $H \in \mathcal{H}_E$. Then the corresponding set B_H of breaking vertices of H must be finite, since otherwise B_H would contain an infinite ascending chain of subsets indexed by positive integers

$$S_1 \subsetneq \cdots \subsetneq S_n \subsetneq \cdots ,$$

and this would then give rise to a proper ascending chain

$$(H, S_1) < \cdots < (H, S_n) < \cdots \text{ in } \mathcal{L}_E,$$

contradicting the fact that a.c.c. holds in \mathcal{L}_E .

Conversely, suppose the a.c.c. holds in \mathcal{H}_E , and that B_H is a finite set for each $H \in \mathcal{H}_E$. Consider an ascending chain of admissible pairs

$$(H_1, S_1) \leq (H_2, S_2) \leq \cdots \text{ in } \mathcal{L}_E.$$

This gives rise to an ascending chain

$$H_1 \subseteq H_2 \subseteq \cdots \text{ in } \mathcal{H}_E$$

and so there is an integer k such that $H_i = H_k$ for all $i \geq k$. So from the k^{th} term onwards, the given chain of admissible pairs is of the form

$$(H, S_k) \leq (H, S_{k+1}) \leq \cdots ,$$

where S_k, S_{k+1}, \dots are subsets of B_H . Observe that since $B_H \cap H = \emptyset$, it follows from the definition of \leq on \mathcal{L}_E that we have an ascending chain

$$S_k \subseteq S_{k+1} \subseteq \cdots .$$

Since B_H is a finite set, there is a positive integer m such that $S_{k+m} = S_{k+m+i}$ for all $i \geq 0$. This establishes the a.c.c. in \mathcal{L}_E . \square

By a completely analogous argument, we get the following result as well.

Lemma 4.2.11. *Let E be an arbitrary graph. Then the following are equivalent.*

- (i) *The lattice \mathcal{L}_E of admissible pairs (H, S) of E satisfies the d.c.c. with respect to the partial order indicated above.*
- (ii) *The lattice \mathcal{H}_E of all hereditary saturated subsets of E satisfies the d.c.c. (under set inclusion), and, for each $H \in \mathcal{H}_E$, the corresponding set B_H of breaking vertices is finite.*

Here now is our main consequence of Theorem 3.2.1.

Theorem 4.2.12. *Let E be an arbitrary graph and K any field. Then the following are equivalent:*

- (i) *$L_K(E)$ is two-sided noetherian;*
- (ii) *$L_K(E)$ is two-sided graded noetherian;*
- (iii) *The a.c.c. holds in the set \mathcal{H}_E of all hereditary saturated subsets of E (under set inclusion), and, for each $H \in \mathcal{H}_E$, the corresponding set B_H of breaking vertices is finite.*

Proof. (iii) \Rightarrow (i) Let I be an ideal of $L_K(E)$. We seek to show that I is finitely generated. To this end, let $H = I \cap E^0$. Then as noted previously, H is a hereditary saturated subset of E^0 . Let $J_1 \subseteq I$ be the ideal of $L_K(E)$ generated by H .

By considering the hereditary saturated closures of finite subsets of H , the a.c.c. condition in \mathcal{H}_E implies that $H = \overline{M}$, the hereditary saturated closure of a finite subset M . Thus J_1 is the ideal generated by the finite set M .

Let J_2 be the ideal generated by the set

$$C = \left\{ v + \sum_{i=1}^n \lambda_i g^i \in I \mid v \in E^0, \text{ and } g \text{ is a nontrivial cycle based at } v \right\}.$$

We will follow the ideas in the proof of Theorem 4.2.5 to show that J_2 is finitely generated. By Lemma 4.2.1, we can assume that J_2 is generated by a subset T of C with the property that for any two $v + \sum_{i=1}^n \lambda_i g^i, w + \sum_{i=1}^m \mu_i h^i$ in T , $v \not\leq w$ and $w \not\leq v$. We claim that T is a finite set. Suppose, by way of contradiction, T has infinitely many elements. Denote

a countably infinite number of elements of T by $v_k + \sum_{i=1}^{n_k} \lambda_i g_k^i$, $k \in \mathbb{N}$. For each positive integer n define $T_n = \{v_1, \dots, v_n\}$, and let \bar{T}_n be its hereditary saturated closure. By hypothesis, the ascending chain

$$\bar{T}_1 \subseteq \dots \subseteq \bar{T}_n \subseteq \dots$$

becomes stationary after a finite number of terms, say, $\bar{T}_m = \bar{T}_{m+1} = \dots$ for some integer m . Then $v_{m+1} \in \bar{T}_{m+1} = \bar{T}_m$. But then, by Lemma 4.2.2, there is a $v_j \in \{v_1, \dots, v_m\}$ such that $v_j \geq v_{m+1}$, a contradiction. Thus T is a finite set and J_2 is finitely generated (by T).

Let $L = J_1 + J_2$ and thus L is the ideal generated by the finite set $M \cup T$.

By Theorem 3.2.1, the ideal I has a set of generators for which each element in the generating set has the form

$$x = \left(v + \sum_{i=1}^k \lambda_i g^i \right) \left(v - \sum_{e \in S} ee^* \right),$$

where $S(x)$ is some finite subset of $s^{-1}(v)$. By the previous paragraph, to show that I is finitely generated it suffices to show that there exists a finite set x_1, x_2, \dots, x_n of elements of I for which, for each expression x in I having the displayed form, $\langle x, L \rangle \subseteq \langle x_1, x_2, \dots, x_n, L \rangle$.

So let $x = (v + \sum_{i=1}^k \lambda_i g^i)(v - \sum_{e \in S} ee^*) \in I$. If v is a finite emitter, then $\langle x, L \rangle = \langle v, L \rangle = L$ by Proposition 4.2.9(ii). So L itself already captures all of the expressions having v a finite emitter.

Now let v be an infinite emitter. By Proposition 4.2.9(i), either $\langle x, L \rangle = \langle v, L \rangle = \langle L \rangle$, or $v \in B_H$. Thus we need only consider those vertices v in B_H .

Since B_H is finite by hypothesis, this yields that there are only finitely many infinite emitters w in E for which there exists an expression of the form

$$x = \left(w + \sum_{i=1}^k \lambda_i g^i \right) \left(w - \sum_{e \in S(x)} ee^* \right)$$

in I . Call this finite set W .

Let $w \in W$, and suppose $(w + \sum_{i=1}^k \lambda_i g^i)(w - \sum_{e \in S} ee^*) \in I$. We claim that $\langle x, L \rangle = \langle x', L \rangle$, where $x' = (w + \sum_{i=1}^k \lambda_i g^i)(w - \sum_{e \in T(x)} ee^*) \in I$ and $T(x) \subseteq \{e \in S(x) \mid r(e) \notin H\}$.

But this is straightforward: for each $\ell \in s^{-1}(w)$ having $r(\ell) \in H$, we have $\ell\ell^* = \ell r(\ell)\ell^* \in L$, so that

$$\begin{aligned} x &= \left(w + \sum_{i=1}^k \lambda_i g^i \right) \left(w - \sum_{\ell \in S(x) \setminus \{\ell \in S(x) \mid r(\ell) \in H\}} \ell\ell^* \right) \\ &\quad - \left(w + \sum_{i=1}^k \lambda_i g^i \right) \left(\sum_{\{\ell \in S(x) \mid r(\ell) \in H\}} \ell\ell^* \right). \end{aligned}$$

Since the second summand is in L , we get the desired conclusion.

To complete the proof, we thus need only establish that for each of the finitely many $w \in B_H$ for which there exists an expression of the form $x = (w + \sum_{i=1}^k \lambda_i g^i)(w - \sum_{e \in S(x)} ee^*) \in I$, and each of the finitely many corresponding (finite) subsets $U(x) \subseteq \{e \in S(x) \mid r(e) \notin H\}$, that there exist finitely many elements x_1, \dots, x_n of I for which any element of the form $x = (w + \sum_{i=1}^k \lambda_i g^i)(w - \sum_{e \in U(x)} ee^*)$ is in the ideal $\langle x_1, \dots, x_n, L \rangle$. For a given choice of w and $U(x)$, let $p(t) \in K[t]$ be a polynomial of smallest degree with the properties that $p(0) = 1$ and $x = p(g)(w - \sum_{e \in U(x)} ee^*) \in I$, where $p(g)$ denotes $(w + \sum_{i=1}^k \lambda_i g^i)$. We note that $p(t)$ could possibly have degree 0, i.e., $p(g) = w$ is allowed.

We claim that the ideal $\langle x, L \rangle$ contains every other expression of the form $z = (w + \sum_{i=1}^m \mu_i g^i)(w - \sum_{e \in U(z)} ee^*) \in I$, where where $U(z)$ is a finite subset of $s^{-1}(w)$ for which $U(x) = U(z) \setminus \{e \in U(z) \mid r(e) \in H\}$, and $q(t) = \sum_{i=0}^m \mu_i t^i \in K[t]$ is some polynomial with $q(g) = w + \sum_{i=1}^m \mu_i g^i$. That we may eliminate the edges e in $U(z)$ for which $r(e) \in H$ is a result of the process described previously. This yields $z \in \langle L, z' \rangle$, where $z' = (w + \sum_{i=1}^m \mu_i g^i)(w - \sum_{e \in S(z')} ee^*) \in I$. By the minimality of the degree of $p(t)$, a standard division algorithm argument yields that $q(t) = p(t)p_1(t)$ for some $p_1(t) \in K[t]$, so that $z' = p_1(g)p(g) = p_1(g)x \in \langle x \rangle \subseteq \langle x, L \rangle$, as desired.

In conclusion, we have shown that if we denote by x_1, x_2, \dots, x_n the elements

$$\left\{ p(g) \left(w - \sum_{e \in S} ee^* \right) \mid w \in B_H, S \subseteq \{e \in s^{-1}(w) \mid r(e) \notin H\}, \text{ and } p(t) \text{ is the} \right. \\ \left. \text{monic polynomial of smallest degree for which } p(g) \left(w - \sum_{e \in S} ee^* \right) \in I \right\},$$

then $\langle x, L \rangle \subseteq \langle x_1, x_2, \dots, x_n, L \rangle$, thus establishing the result.

That (i) \Rightarrow (ii) is obvious.

Finally, (ii) \Rightarrow (iii) follows from the lattice isomorphism between the lattice \mathcal{L}_E of admissible pairs of E and the lattice \mathcal{H}_E of graded ideals of $L_K(E)$ established in Theorem 4.1.8(i), together with Lemma 4.2.10. \square

We note that the result established in Theorem 4.2.12 for arbitrary-sized graphs immediately yields the identical result established for row-finite graphs in Theorem 4.2.5.

4.3 Two-Sided Artinian Leavitt Path Algebras

With the two-sided Noetherian result now in hand, we begin to build the machinery which will allow us to achieve the two-sided Artinian result for Leavitt path algebras. Of use will be the following two results.

Proposition 4.3.1. [14, Proposition 1.17] *Let E be an arbitrary graph. Suppose E satisfies Condition (L) but does not satisfy Condition (K). Then there exists a hereditary saturated subset H of E^0 for which the graph $E \setminus (H, \emptyset)$ does not satisfy Condition (L).*

Proof. Since Condition (K) does not hold in E , there is a vertex v which is the base of exactly one simple closed path $c = e_1 \cdots e_n$ with $e_i \in E^1$. By Condition (L), c has exits. Let $A = \{f \in E^1 \mid f \text{ an exit of } c\}$, and let $B = \{r(f) \mid f \in A\}$. Let H denote the hereditary saturated closure of B in E , and let c^0 denote the vertices of c , i.e., $c^0 = \{r(e_i) \mid 1 \leq i \leq n\}$. We claim that $H \cap c^0 = \emptyset$. Indeed if there is a vertex $w \in H \cap c^0$, then by Lemma 4.2.1 there exists $u \in B$ such that $u \geq w$. This would then give rise to another simple closed path based at v , a contradiction. Hence $H \cap c^0 = \emptyset$. If we consider the graph $E \setminus (H, \emptyset)$, we get by definition that $c^0 \subset (E \setminus (H, \emptyset))^0$, $\{e_1, \dots, e_n\} \subset (E \setminus (H, \emptyset))^1$, and thus c is a cycle in $E \setminus (H, \emptyset)$ with no exits. Therefore Condition (L) does not hold in $E \setminus (H, \emptyset)$, as desired. \square

We immediately use Proposition 4.3.1 to get the following.

Proposition 4.3.2. *Let E be an arbitrary graph and K any field. Suppose \mathcal{P} is a ring-theoretic property such that:*

- (i) *if a ring R has \mathcal{P} , then any factor ring of R also has \mathcal{P} , and*
- (ii) *for any Leavitt path algebra $L_K(E)$, if $L_K(E)$ satisfies \mathcal{P} then E satisfies Condition (L).*

Then for any Leavitt path algebra $L_K(E)$ which satisfies \mathcal{P} , the graph E satisfies Condition (K).

Proof. Suppose $L_K(E)$ has property \mathcal{P} . By Proposition 4.3.1, if E does not satisfy Condition (K) then we may find a hereditary saturated subset H of E^0 for which $E \setminus (H, \emptyset)$ does not satisfy Condition (L). But $L_K(E \setminus (H, \emptyset)) \cong L_K(E)/I_{(H, \emptyset)}$ by Theorem 4.1.8(ii), so that by hypothesis (i) $L_K(E \setminus (H, \emptyset))$ satisfies \mathcal{P} , and thus by hypothesis (ii) yields Condition (L) on $E \setminus (H, \emptyset)$, a contradiction. \square

We are now ready to prove the two-sided Artinian result.

Theorem 4.3.3. *Let E be an arbitrary graph and K any field. Then $L_K(E)$ is two-sided Artinian if and only if the graph E satisfies Condition (K), the d.c.c. holds for hereditary saturated subsets of E^0 , and for each hereditary saturated subset H , the corresponding set B_H of breaking vertices is finite.*

Proof. Suppose $L_K(E)$ is two-sided Artinian. We first show that E satisfies Condition (L). Suppose, on the contrary, Condition (L) does not hold. Then there is a cycle c based at a vertex v for which c has no exits. Consider the following descending chain of ideals of $L_K(E)$:

$$\langle v - c \rangle \supseteq \langle v - c^2 \rangle \supseteq \cdots \supseteq \langle v - c^{2^n} \rangle \supseteq \cdots$$

By hypothesis, there is an integer k such that $\langle v - c^{2^k} \rangle = \langle v - c^{2^{k+1}} \rangle$. So in particular we can write

$$v - c^{2^k} = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^* (v - c^{2^{k+1}}) \gamma_i \delta_i^*,$$

where $\alpha_i, \beta_i, \gamma_i$ and δ_i are paths in E . Multiplying both expressions on the left and right by v , we conclude that if a term $v \alpha_i \beta_i^* (v - c^{2^{k+1}}) \gamma_i \delta_i^* v$ is nonzero, then $v = s(\alpha_i) = s(\beta_i) = s(\gamma_i) = s(\delta_i)$ and further $r(\alpha_i) = r(\beta_i)$ and $r(\gamma_i) = r(\delta_i)$. Since c has no exits, we argue as in the proof of Proposition 5.1.5 that $\alpha_i \beta_i^* = c^{t_i}$ and $\gamma_i \delta_i^* = c^{u_i}$ for some $t_i, u_i \in \mathbb{Z}$. In particular, all factors in the sum commute, and we may write

$$v - c^{2^k} = \sum_{i=1}^n \lambda_i v - c^{2^{k+1}} c^{w_i}.$$

But then arguing on both the smallest and largest degrees in the right hand expression we see that $w_i = 0$ for all $1 \leq i \leq n$. So we have $v - c^{2^k} = \lambda(v - c^{2^{k+1}})$ for some $\lambda \in K$, which is impossible, again by a comparison of degrees of homogeneous terms. Hence Condition (L) holds in E .

But any homomorphic image of a two-sided Artinian ring is again two-sided Artinian. Thus, using the previous paragraph, we may invoke Proposition 4.3.2 (where \mathcal{P} is “two-sided Artinian”) to conclude that E satisfies Condition (K).

Furthermore, since $L_K(E)$ satisfies the d.c.c. on all ideals it necessarily satisfies the d.c.c. on graded ideals. Then by Lemma 4.2.11 this yields the desired properties on the hereditary saturated subsets of E and the sets of breaking vertices in E .

Conversely, if Condition (K) holds then by Theorem 5.1.1 we have that every ideal of $L_K(E)$ is graded. In particular, the lattice of all ideals is the same as the lattice of graded ideals. But the indicated conditions on the hereditary saturated subsets of E and the sets of breaking vertices in E implies that the lattice of graded ideals has the d.c.c. by Lemma 4.2.11. Thus $L_K(E)$ is two-sided Artinian. \square

We note that in fact we have proven that the d.c.c. on principal ideals of $L_K(E)$ is equivalent to $L_K(E)$ being two-sided Artinian, since the d.c.c. on principal ideals was sufficient to establish that Condition (K) holds in E .

The following observation compares and contrasts the Artinian result Theorem 4.3.3 with its Noetherian counterpart Theorem 4.2.12. While two-sided Noetherian is equivalent to being graded two-sided Noetherian, not all ideals in a Noetherian Leavitt path algebra are necessarily graded. Indeed, $K[x, x^{-1}] = L_K(R_1)$, where R_1 is the directed graph consisting of one vertex v and one edge e such that $s(e) = r(e) = v$, is graded Noetherian (there are no nontrivial graded ideals), and therefore Noetherian, but all of the (infinitely many) nontrivial ideals of $K[x, x^{-1}]$ are nongraded. On the other hand, two-sided Artinian is not equivalent to being graded two-sided Artinian, as the same $K[x, x^{-1}]$ example demonstrates. However, in a two-sided Artinian Leavitt path algebra, every ideal is necessarily graded (as Condition (K) holds in such algebras). We note that in order to achieve the Noetherian result, we invoked the explicit description of the generating sets of arbitrary ideals afforded by Theorem 3.2.1. However, we did not need to directly use the result of Theorem 3.2.1 for the Artinian result. (We did utilize Theorem 5.1.1, a consequence of Theorem 3.2.1, in the proof of the Artinian result; however, we could have instead simply invoked [24, Theorem 3.8] in its place.)

4.4 Examples

We now offer some explicit examples which we hope will help the reader to clarify these ideas.

If $L_K(E)$ is two-sided Artinian, it need not be two-sided Noetherian, and, likewise, the two-sided Noetherian condition does not imply the two-sided Artinian condition.

Clearly $K[x, x^{-1}] = L_K(R_1)$ provides an example of the latter. For the former, consider the following example.

Example 4.4.1. Let $P_\omega = \bigcup_{n \in \mathbb{N}} P_n$ be the “pyramid” graph of length ω described in [13] and pictorially represented here (Figure 4.5). Specifically, for $n \geq 1$, P_n denotes the subgraph of P_ω consisting of vertices in the first n “rows”, together with all edges they emanate. This graph is acyclic, so Condition (K) is vacuously satisfied. The hereditary saturated subsets of P_ω correspond exactly to the subgraphs P_n ; these clearly do not satisfy a.c.c., but do satisfy d.c.c. Since there are no infinite emitters in P_ω , the sets of breaking vertices B_H are empty. Hence by Theorem 4.3.3 the Leavitt path algebra $L_K(P_\omega)$ is two-sided Artinian, but, by Theorem 4.2.12, not two-sided Noetherian.

Example 4.4.2. Consider the graph E in Figure 4.6.

So in particular for each $j \in \mathbb{N}$, there is an edge from v_j to every u_i , $i \in \mathbb{N}$.

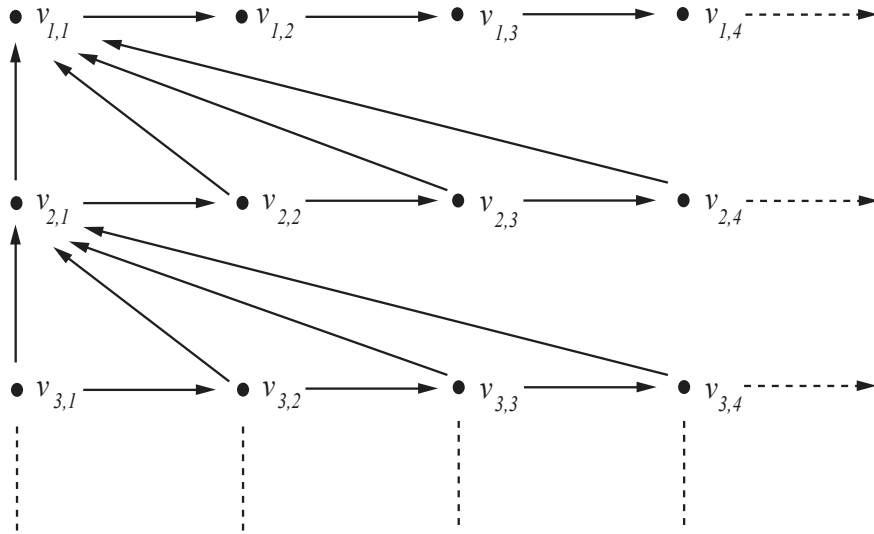


Figure 4.5: Graph of the Leavitt path algebra defined in Example 4.4.1.

Since there is only one edge which emanates from each u_i , if H is a hereditary saturated subset of E^0 which contains some u_j , then H contains u_i for all $i \in \mathbb{N}$ (For $i > j$ use the hereditary property, and for $i < j$ use the saturated property.). With this observation in mind, it then follows easily that E^0 contains precisely three hereditary saturated subsets: \emptyset , $\{u_i \mid i \in \mathbb{N}\}$, and E^0 . If we denote $\{u_i \mid i \in \mathbb{N}\}$ by H , then each of the infinitely many v_i are breaking vertices for H . Therefore, even though there are very few hereditary saturated subsets in E^0 , $L_K(E)$ does not satisfy either chain condition by Theorems 4.2.12 and 4.3.3.

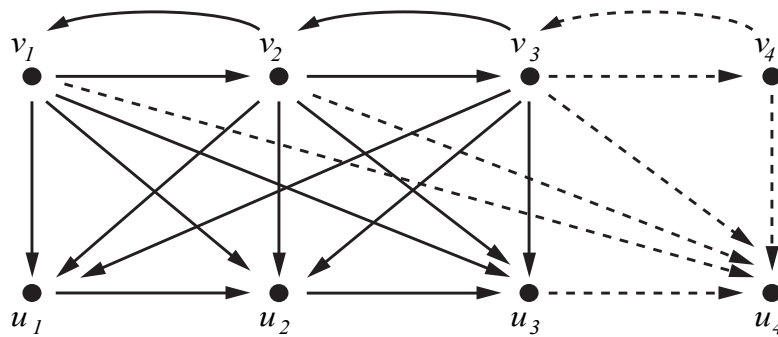


Figure 4.6: Graph of the Leavitt path algebra defined in Example 4.4.2.

Chapter 5

Applications

5.1 Some Applications of the Main Theorem

We now give some applications of Theorem 3.2.1. We will give shorter and simpler proofs for some of the known results in Leavitt path algebras. The results presented here also appear in [6].

We start with the following result, which was established for arbitrary-sized graphs in [24, Theorem 3.8], using techniques significantly different than we will use here.

Theorem 5.1.1. *Let E be an arbitrary graph, and K any field. Then E satisfies Condition (K) if and only if every ideal of $L_K(E)$ is graded.*

Before beginning the proof, we note that a similar result for countable graphs is presented as [35, Theorem 6.16]. Although the statement of [35, Theorem 6.16] indeed holds also for arbitrary graphs, the tools used in its proof relies on the desingularization process, a process which may be utilized only for countable graphs (see e.g. [7]).

Proof. Let I be an ideal of $L_K(E)$. If E satisfies Condition (K) then there is no vertex in E which is the base of a unique cycle. So, by Theorem 3.2.1, I has a generating set of the form

$$\left\{ v - \sum_{e \in S} ee^* \mid v \in V \subseteq E^0, S \subseteq s^{-1}(v) \right\}.$$

But any element of the form $v - \sum_{e \in S} ee^*$ is homogeneous of degree 0. So I is an ideal generated by homogeneous elements (of degree 0), hence is graded by Remark 2.5.15.

Conversely, if E does not satisfy Condition (K) then by [35, Proposition 6.12] there is a graph F which does not satisfy Condition (L), and an onto ring homomorphism $\phi : L_K(E) \rightarrow L_K(F)$ for which ϕ preserves the respective gradings. (We give the proof of this

result in Proposition 4.3.1) But for a graph F which does not satisfy Condition (L) one can build a non-graded ideal in $L_K(F)$ (specifically, the ideal $\langle v + c \rangle$ where c is a cycle without exits based at v), which implies the existence of a non-graded ideal in $L_K(E)$. \square

In addition, we may use Theorem 3.2.1 to obtain information about generating sets for the graded ideals of the Leavitt path algebra $L_K(E)$ for any graph E . In particular, this allows us (in the implication (1) \Rightarrow (3)) to give a more direct proof of the key piece of [35, Theorem 5.7(1)].

Theorem 5.1.2. *Let E be an arbitrary graph and K any field. Then the following are equivalent for an ideal I of $L_K(E)$:*

- (i) I is a graded ideal;
- (ii) I is generated by elements of the form $v - \sum_{e \in S} ee^* \in I$, where $v \in E^0$ and S is a finite (perhaps empty) subset of $s^{-1}(v)$;
- (iii) I is generated by the subset $H \cup Y$, where $H = I \cap E^0$ and $Y = \{v - \sum_{e \in s^{-1}(v), r(e) \notin H} ee^* \in I, \text{ with } v \in B_H\}$.

Proof. (i) \Rightarrow (ii) By Theorem 3.2.1, I is generated as an ideal by elements in I of the form

$$x = \left(v - \sum_{k=1}^n \lambda_k g^k \right) \left(v - \sum_{e \in S} ee^* \right) = \left(v - \sum_{e \in S} ee^* \right) - \sum_{k=1}^n \lambda_k g^k \left(v - \sum_{e \in S} ee^* \right),$$

where S is a finite subset of $s^{-1}(v)$. Let m denote the number of edges in the cycle g . Since I is graded, each of the graded components of x is in I . Since $\deg(v - \sum_{e \in S} ee^*) = 0$, we have that the degree 0 component of x is $v - \sum_{e \in S} ee^*$, while the degree mk component of x for $k \geq 1$ is $\lambda_k g^k (v - \sum_{e \in S} ee^*)$. This implies that x belongs to the ideal generated by elements in I of the form $v - \sum_{e \in S} ee^*$, as desired.

(ii) \Rightarrow (iii) Consider a generator $y = v - \sum_{e \in S} ee^* \in I$. If S is empty, then $y = v \in I \cap E^0 = H$. Suppose $y \notin \langle H \rangle$, the ideal generated by H . Then $v \notin H$. If $r(e) \in H$ for any $e \in S$, then $e = er(e) \in \langle H \rangle$ and so $ee^* \in \langle H \rangle$. Subtracting from y all those terms ee^* for which $r(e) \in H$ (and thus removing such e from S), we may assume that $r(e) \notin H$ for every $e \in S$. Observe that this process will not exhaust all of S , since otherwise, $\sum_{e \in S} ee^* \in \langle H \rangle$ and $v = y - \sum_{e \in S} ee^* \in I \cap E^0 = H$, a contradiction. If there is an $f \in E^1$ with $s(f) = v$ and $r(f) \notin H$, then f must belong to S , because, otherwise,

$$z = (v - ff^*)(v - \sum_{e \in S} ee^*) = v - ff^* - \sum_{e \in S} ee^* \in I,$$

which implies that $y - z = ff^* \in I$. From this we get $r(f) = f^*(ff^*)f \in I \cap E^0 = H$, a contradiction. Thus the finite set S is precisely the set $\{e \in s^{-1}(v) \mid r(e) \notin H\}$. If v is a finite emitter, then

$$y = v - \sum_{e \in S} ee^* = \sum_{f \in s^{-1}(v) \setminus S} ff^* = \sum_{f \in s^{-1}(v), r(f) \in H} ff^* \in \langle H \rangle.$$

If on the other hand v is an infinite emitter, then v is a breaking vertex of H and so $y = v - \sum_{e \in S} ee^* \in Y$. This proves (iii).

(iii) \Rightarrow (i) As noted in Remark 2.5.15, any ideal generated by homogeneous elements (here, of degree 0) is graded. \square

Corollary 5.1.3. *Let E be an arbitrary graph, and K any field. Then every nonzero graded ideal of $L_K(E)$ contains a vertex.*

Proof. Let I be a nonzero graded ideal of $L_K(E)$. By Theorem 5.1.2, I is generated by elements of the form $v - \sum_{e \in S} ee^* \in I$, where $v \in E^0$ and S is a finite (perhaps empty) subset of $s^{-1}(v)$. If $S = \emptyset$ for some such v then we are done. If v is a finite emitter, then $S \neq s^{-1}(v)$ (since otherwise the expression $v - \sum_{e \in S} ee^*$ is zero); if v is an infinite emitter, then $S \neq s^{-1}(v)$ as well, as S is finite. Thus in either case there exists $f \in s^{-1} \setminus S$, and we get

$$r(f) = f^*f = f^* \left(v - \sum_{e \in S} ee^* \right) f \in I.$$

\square

We make a final observation regarding the graded ideals of $L_K(E)$ for arbitrary E . Since any element of the form $v - \sum_{e \in S} ee^*$ for any finite subset S of $s^{-1}(v)$ is an idempotent, condition (2) of Theorem 5.1.2 yields the following

Corollary 5.1.4. *Let E be any graph, K any field, and I any graded ideal of $L_K(E)$. Then $I^2 = I$.*

With Theorem 5.1.1 and Corollary 5.1.4 in mind, the following result follows almost immediately.

Proposition 5.1.5. *Let E be any graph and K be any field. Then $I^2 = I$ for every ideal I of $L_K(E)$ if and only if E satisfies Condition (K).*

Proof. Suppose $I^2 = I$ for every ideal of $L_K(E)$. We first claim that E must satisfy Condition (L). Because, on the contrary, there would be a cycle c in E based at a vertex v for

which c has no exits; we show that this would yield that the ideal $I = \langle v - c \rangle$ has $I^2 \neq I$. Since $I^2 = I$, $v - c$ can be written as a K -linear combination of non-zero elements of the form

$$x = \alpha\beta^*(v - c)pq^*(v - c)\gamma\delta^*$$

for suitable paths in E . Moreover, as $v(v - c)v = v - c$, we may assume that $v xv = x$, and that $v = s(\alpha) = s(\beta) = s(p) = s(q) = s(\gamma) = s(\delta)$, $r(\alpha) = r(\beta)$, $r(p) = r(q)$ and $r(\gamma) = r(\delta)$. Since c has no exits, the expressions $\alpha\beta^*$, pq^* , and $\gamma\delta^*$ must be of the form c^j for some $j \in \mathbb{Z}$. In particular, each of these expressions commutes with $v - c$, and so we get $x = (v - c)^2 c^t$ for some integer t . This then yields $v - c = (v - c)^2 \sum_{i=1}^m k_i c^{t_i}$ for some $k_i \in K$, $t_i \in \mathbb{Z}$. But this is not possible by comparing degrees on both sides.

Thus we have shown that E satisfies Condition (L). Since the property that $I^2 = I$ for every ideal I is preserved under homomorphic images, we conclude from Proposition 4.3.2 that E satisfies Condition (K).

Conversely, suppose E satisfies the Condition (K). By Theorem 5.1.1, every ideal I of $L_K(E)$ is graded and so $I^2 = I$ by Corollary 5.1.4. \square

Remark 5.1.6. In [16, Theorem 3.15] it was shown that an arbitrary graph E satisfies Condition (K) if and only if $I^2 = I$ for every left (or right) ideal of $L_K(E)$. We thus obtain from Proposition 5.1.5 that in a Leavitt path algebra $L_K(E)$, $I^2 = I$ for every (two-sided) ideal I if and only if $I^2 = I$ for every one-sided ideal I .

Lastly, we conclude this Chapter by noting that many known properties of Leavitt path algebras derive almost immediately from Theorem 3.2.1. Indeed, with Theorem 3.2.1 in hand, we may re-establish a number of results in a manner different from the proofs originally provided in the literature. We offer the proof of one such result.

Proposition 5.1.7. *If the graph E satisfies Condition (L), then every nonzero ideal I of $L_K(E)$ contains a vertex.*

Proof. By Theorem 3.2.1, I is generated by elements of the form

$$x = \left(v + \sum_{r=1}^n k_r g^r \right) \left(v - \sum_{e \in S} ee^* \right).$$

belonging to I where $v \in E^0$, g is a cycle based at v , and S is a finite subset of $s^{-1}(v)$. Since some such x is necessarily nonzero, for this x we have that S must be a proper subset of $s^{-1}(v)$; let $f \in s^{-1}(v) \setminus S$. Let e_1 denote the initial edge of g , so that $g = e_1 p$ for a path p with $r(p) = v$.

If $f \neq e_1$, then $f^*g = 0$ and we get $f^*xf = f^*vf = r(f) \in I$ and we are done.

If $f = e_1$, then $f^*gf = pf = h$, a cycle based at $r(f) = w$. Since $(\sum_{e \in S} ee^*)f = 0$, we get $f^*xf = f^*vf + \sum_{r=1}^n k_r f^*g^r f = w + \sum_{r=1}^n k_r h^r \in I$. Now by Condition (L), h has an exit e' , which we can assume to be at w (Indeed, if the e' is an exit at a vertex u on h , and if μ is the path from w to u and ν is the path from u to w along h , then $\mu^*(w + \sum_{r=1}^n k_r h^r)\mu = \mu^*w\mu + \sum_{r=1}^n k_r \mu^*h^r\mu = u + \sum_{r=1}^n k_r c^r \in I$ where $c = \nu\mu$ is the cycle based at u). As $(e')^*h = 0$, the element $(e')^*(w + \sum_{r=1}^n k_r h^r)e' = (e')^*ve' + 0 = r(e')$ and belongs to I . Hence I contains a vertex in this case as well. \square

Additional results for arbitrary graphs which follow from Proposition 5.1.7 and Corollary 5.1.3 include the Graded Uniqueness Theorem [35, Theorem 4.6], the Cuntz-Krieger Uniqueness Theorem [35, Theorem 6.8], and the Simplicity Theorem [3, Theorem 3.1]. Please see [12, Sec. 3] for a complete description.

Chapter 6

Graph C^* -algebras

6.1 Extending the Results to Graph C^* -algebras

In this section we give the relationship between graph C^* -algebras and Leavitt path algebras. The definitions here can also be found in [17, 19, 33].

We first define graph C^* -algebras.

Definition 6.1.1. A $*$ -algebra is an associative algebra A over the complex numbers \mathbb{C} with an *involution*: a map $a \mapsto a^*$ from A to A such that $(\lambda a + \mu b)^* = \bar{\lambda}a^* + \bar{\mu}b^*$, $(a^*)^* = a$ and $(ab)^* = b^*a^*$.

Definition 6.1.2. A C^* -algebra is a $*$ -algebra A with norm $a \mapsto \|a\| : A \rightarrow [0, \infty)$ which satisfies the usual axioms for a norm on a vector space:

$$\|ab\| \leq \|a\|\|b\| \quad \text{and} \quad \|a\|^2 = \|a^*a\| \quad (\text{the } C^* \text{-identity}),$$

and for which the normed space $(A, \|\cdot\|)$ is complete in the sense that Cauchy sequences converge.

Definition 6.1.3. Let A be a C^* -algebra. An element a in A for which a^*a is a projection is called a *partial isometry*.

Definition 6.1.4. An element h of a C^* -algebra A is said to be *self-adjoint* if $h^* = h$. An element a of A is said to be *positive* if $a = h^2$ for some self-adjoint element $h \in A$.

We can introduce a partial order on the self-adjoint elements of a C^* -algebra by defining $S \leq T$ if and only if $T - S \leq 0$.

Definition 6.1.5. If E is a graph we define a *Cuntz-Krieger E -family* to be a set of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a set of partial isometries $\{s_e \mid e \in E^1\}$ with orthogonal ranges which satisfy the *Cuntz-Krieger relations*:

1. $s_e^*s_e = p_{r(e)}$ for every $e \in E^1$,
2. $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$,
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$.

The graph C^* -algebra $C^*(E)$ is defined to be the C^* -algebra generated by a universal Cuntz-Krieger E -family.

As it can be seen, the relations 4. and 5. in the definition of Leavitt path algebra are inherited from the definition of $C^*(E)$, and makes Leavitt path algebras a purely algebraic analog of the graph C^* -algebras.

We have already stated the necessary and sufficient conditions on the graph so that the associated Leavitt path algebra is simple in 2.5.17. To make the comparison, we state the graph C^* -algebra version as well. The following result appears in [17, Theorem 2.1.23].

Theorem 6.1.6. *Let E be a row-finite graph. Then $C^*(E)$ is simple if and only if E satisfies Condition (L) and E^0 has no saturated hereditary subsets other than \emptyset and E^0 .*

We note that the conditions on the graph E are precisely the same for both $C^*(E)$ and $L_K(E)$ to be simple.

In addition, we present results about purely infinite and simple (purely infinite simple) Leavitt path algebras and graph algebras as another example of the relationship between the two classes.

Before stating these results, let us define purely infinite simple rings and purely infinite simple C^* -algebras.

Definition 6.1.7. An idempotent e in a ring R is called *infinite* if eR is isomorphic as a right R -module to a proper direct summand of itself. R is called *purely infinite* in case every nonzero right ideal of R contains an infinite idempotent. R is *purely infinite simple*, if it is both purely infinite and simple.

Definition 6.1.8. A C^* -algebra A is *purely infinite simple* if every nonzero hereditary subalgebra of A contains an infinite projection.

Definition 6.1.9. Let E be a directed graph. If g is a cycle in E , and v is a vertex in E^0 , then we say that v *connects to cycle g* if $v \geq w$ for some vertex w in g .

The next result appears as [3, Theorem 4.3].

Theorem 6.1.10. *Let E be a directed graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

- (i) The only hereditary saturated subsets of E^0 are E^0 and \emptyset ;
- (ii) Every cycle in E has an exit;
- (iii) Every vertex connects to a cycle.

The next Theorem describes purely infinite simple graph C^* -algebras. We combine [22, Corollary 2.14], [22, Remark 2.16] and [17, Theorem 2.1.13] for an easier comparison.

Theorem 6.1.11. *Let E be a directed graph. Then $C^*(E)$ is purely infinite simple if and only if E satisfies the following properties.*

- (i) The only hereditary and saturated subsets of E^0 are E^0 and \emptyset ;
- (ii) Every vertex in E has an exit;
- (iii) Every vertex connects to a cycle.

It has been also noted that simple Leavitt path algebras and graph algebras share the same dichotomy [17]:

Theorem 6.1.12. *Let E be a row-finite directed graph. If $L_K(E)$ is simple, then either*

1. $L_K(E)$ is purely infinite simple, or
2. $L_K(E)$ is a limit of finite dimensional matrix rings.

For the $C^*(E)$ version, we need the definition of an approximately finite-dimensional C^* -algebra. The approximately finite-dimensional property for C^* -algebras corresponds to being a limit of finite dimensional matrix rings for rings.

Definition 6.1.13. A C^* -algebra is an approximately finite-dimensional (AF-algebra) if it can be written as the closure of the increasing union of finite-dimensional C^* -algebras; or, equivalently, if it is the direct limit of a sequence of finite-dimensional C^* -algebras.

The corresponding graph C^* -algebra result can be obtained by combining [22, Corollary 2.13], [22, Remark 2.16] and [17, Theorem 2.1.13].

Theorem 6.1.14. *Let E be a directed graph. If $C^*(E)$ is simple, then either*

- (i) $C^*(E)$ is an AF-algebra (if E contains no cycles); or
- (ii) $C^*(E)$ is purely infinite (if E contains a cycle).

As it can be seen, these two classes of algebras share a close relationship. Thus, it is natural to ask to following question.

Question 6.1.15. *Is it also possible to give characterization for the closed two-sided ideals in $C^*(E)$?*

By considering Theorem 3.2.1, one may naively think that these ideals could be generated by elements of the form

$$\left(p_v + \sum_{k=2}^m \lambda_i S_g^{r_k} \right) \left(p_v - \sum_{e \in X} S_e S_e^* \right),$$

where $v \in E^0$, $\lambda_i \in \mathbb{C}$, X is a finite (possibly empty) subset of $s^{-1}(v)$, g is the unique cycle based at v , and $r_k \in \mathbb{N}$. However, Pere Ara pointed out to us that this is not the case. To see this we consider the directed graph E consisting of one vertex $v \in E^0$ and one edge x such that $s(x) = r(x) = v$. Then $C^*(E)$ becomes the universal C^* -algebra generated by the unitary element S_x . Recall that this C^* -algebra is the algebra of complex-valued continuous functions on the unit circle, and it is well-known that this algebra is generated by trigonometric polynomials [34, Theorem 4.25]. This together with the fact that the graph is row-finite reduce the question to the following. Is any closed two-sided ideal of the continuous functions on the unit circle generated by elements of the form

$$1 + \sum_{k=1}^N \lambda_i e^{ikx},$$

where $\lambda_i \in \mathbb{C}$?

We let C be a proper closed interval in the unit circle, and let I_C be the closed two-sided ideal consisting of continuous functions vanishing on C . This ideal cannot be expressed as the closure of the ideal generated by an elements of the form $1 + \sum_{k=1}^N \lambda_i e^{ikx}$ as such an element has at most finitely many zeros in C .

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