FINDING SHORTEST S-T PATHS WITH A RESTRICTED NUMBER OF ORIENTATIONS

by

Jillian N. Dicker
B.Sc., University of British Columbia, 2009

THESIS SUBMITTED IN PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the
School of Computing Science
Faculty of Applied Sciences

SIMON FRASER UNIVERSITY
Spring 2012

Approval

Name: **Jillian N. Dicker**

Degree: **Master of Science (Computing Science)**

Title of Thesis: **Finding Shortest s-t Paths with a Restricted Number of Orientations**

Examining Committee:

Chair: **Tamara Smyth**
Assistant Professor

Dr. Joseph Peters
Senior Supervisor
Professor

Dr. Thomas Shermer
Supervisor
Professor

Dr. Binay Bhattacharya
Internal Examiner
Professor

Date Approved: April 3, 2012

ABSTRACT

Finding the path between two points in a polygon which minimizes the Euclidean distance of the path has been studied extensively. In this thesis this problem is modified so that the path contains only a fixed number of orientations, and we wish to find the orientations which minimize the Euclidean length of the path between the two points. A method of finding such a set of orientations is given, and for the case where only two orientation are allowed an algorithm is presented which runs in $O(n^2 logn)$ time where n is the number of vertices in the polygon. Finally, previous results concerning the existence of *smallest paths* - paths which are minimum in both Euclidean distance and link distance - are generalized and it is shown that when the path between two points in a polygon is restricted to only include two orientations, such a path which is smallest always exists.

Contents

1 Introduction

Finding a path connecting two points s and t which minimizes the Euclidean distance (which will be referred to simply as *length*) is a well-studied problem. For the case where s, t and the path connecting them are contained in a simple polygon, the path can be found in linear time with an algorithm given by Chazelle [2]. In 1986 the problem of finding a path connecting s and t which minimizes the number of segments in the path (which will be referred to as *link-distance*) was solved by Suri [19] who gave a linear time algorithm for finding such a path which again relied on the triangulation of the polygon.

In this thesis we restrict the problem so that neither s nor t lie on the boundary of P, and the line segments making up the path between s and t cannot be oriented at any angle except those contained in $\mathcal{O}$. For example, if we set $\mathcal{O} = \{\frac{\pi}{2}, \pi\}$ then the path connecting s and t must contain only horizontal or vertical line segments. A path containing only segments with orientations contained in $\mathcal{O}$ is said to be an $\mathcal{O}$-path. In this thesis we explore the problem of finding the set $\mathcal{O}$ of m orientations for which the length of the $\mathcal{O}$-path connecting s to t is minimized.

The concept of such a set $\mathcal{O}$ was introduced by Guting [7], and was expanded upon in 1987 in a doctoral dissertation by Rawlins [17]. This area of Restricted Orientation Geometry is an attempt to bridge the gap between arbitrarily oriented geometry and rectilinear geometry. While this area still remains relatively unstudied, there have been some developments concerning shortest paths where the segment orientations of the path are restricted to some set $\mathcal{O}$. Finding a minimum-length $\mathcal{O}$-path that avoids a set of non-intersecting polygonal obstacles has been solved by Reich [18] using Steiner vertices to solve the problem for two allowable orientations, and then generalizing the result to m orientations. Nilsson et al. [16] give a good summary of past results in the area of shortest paths with restricted orientations, and give an algorithm which for the case of $\mathcal{O}$ being constrained to three allowable orientations finds a minimum-length $\mathcal{O}$-path which avoids a set of $\mathcal{O}$-polygonal obstacles. However, Nilsson et al. put heavy constraints on the problem to accommodate a plane-sweep approach.

In a paper more closely related the work of this thesis, Hershberger and Snoeyink [8] looked at finding the minimum length $\mathcal{O}$-path between two points in a simple $\mathcal{O}$-polygon and proved that a *smallest path* - that is, a path which simultaneously achieves minimum

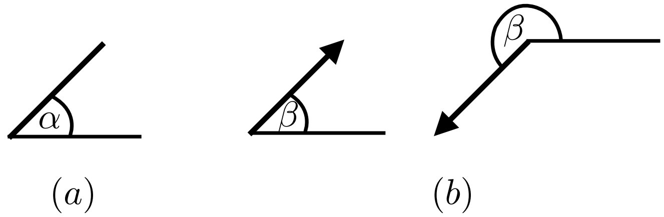


Figure 1: The line shown in (a) has orientation α, while the lines in (b) have direction β.

length and minimum link-distance - always exists when there are no more than three directions allowed. However we must be careful to note the difference between the terms *direction* and *orientation* - each orientation corresponds with two directions (see Figure 1). For example, the rectilinear case has two orientations - horizontal and vertical, but four directions - up, down, left and right. Therefore the proof given by Hershberger and Snoeyink does not correspond to a proof that there always exists a smallest path for 2 orientations. More recently, Mitchell et al. [15] revisited the problem of finding minimum-link paths and provided an algorithm which finds the minimum-link $\mathcal{O}$-path between two points in a simple polygon.

Before Hershberger and Snoeyink made a first attempt at generalizing the proof of the existence of a smallest path, there were a number of proofs published showing that a smallest path always exists if the path is constrained to be rectilinear. In 1989 McDonald [12] showed that a smallest rectilinear path always exists between any two points in a simple polygon, and provided an algorithm to find such a path. McDonald's method of finding the smallest path consisted of altering the given polygon to remove what he termed *unnecessary regions*, which are regions of the polygon that cannot contain any part of a shortest path. After removing the unnecessary regions of the polygon he shows the resulting shape is made up of lines segments and subpolygons, and this allows him to easily find smallest paths in each of the subpolygons and connect each of the smallest paths together to create the desired path. In 1991 De Berg [4] independently discovered a similar result for the less general problem of finding a rectilinear path connecting two points in a simple rectilinear polygon (as opposed to a simple arbitrarily oriented polygon, as considered by McDonald). The algorithm given by De Berg for finding a smallest path between two points s and t is quite different from McDonald's. First a divide-and-conquer algorithm for finding a minimum-link path is presented. This algorithm is based on a modified proof of Chazelle's polygon cutting theorem which allows the rectilinear polygon to be divided - by a single horizontal or vertical line - into two polygons with no more than $\frac{2n}{3}$ of the vertices contained in either polygon. Once the algorithm for finding a minimum-link path was found, it was shown that it could be modified to ensure the path produced has minimum length as well. A paper published in 1992 by McDonald and Peters [13] also proved the existence of a smallest rectilinear path connecting two points in a simple rectilinear polygon. The method used by McDonald and Peters differed from the algorithm presented in McDonald's thesis and first found a path which was shortest in Euclidean distance, then applied a series of path modifications to ensure the path had minimum link-distance as well. In 1999 Maheshwari and Sack [11] independently discovered yet another algorithm which finds the smallest rectilinear path between two points in a rectilinear polygon. The algorithm they provided was similar to the one described in McDonald's thesis and involved removing unnecessary regions of the polygon to simplify the process of finding the smallest path.

This thesis further generalizes the results discussed above and proves that for any set $\mathcal{O}$ of two allowable orientations there always exists a smallest $\mathcal{O}$-path connecting any two points in a simple polygon. Furthermore, it is shown by counterexample that for a set $\mathcal{O}$ containing more than two orientations a smallest path does not always exist.

An extension of the smallest path problem was investigated in 1997 by Yang et al. [21], who proved that given two pairs of points in a rectilinear polygon there does not always exist two non-crossing rectilinear paths connecting each pair of points in which both paths are smallest. Therefore the algorithm they present finds a pair of smallest paths connecting the two pairs of points only if such a pair of smallest paths exists, and otherwise finds a pair of paths which minimizes one of length or link-distance. Their paper used the concept of *extreme vertices* - vertices that must be contained on any smallest path - which is a key concept which this thesis elaborates on and uses throughout.

A number of papers have used the idea of finding *extreme vertices* - vertices that any minimum-length rectilinear path must contain - as a way to find minimum-length rectilinear paths [3, 12, 21]. Yang et al. [21] explicitly define what they call an *extreme sequence* which is a list of extreme vertices in the order in which they must be traversed between s and t. In this thesis we find the extreme sequence and find shortest paths between each pair of extreme vertices that is consecutive in the extreme sequence.

It is worth mentioning that if the smallest path does not exist or is not desired, the

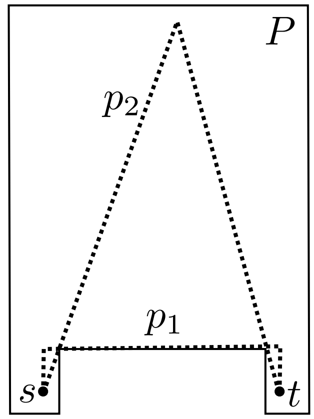


Figure 2: Path p_1 is an $s - t$ path with minimum length, while p_2 is a path with minimum link-distance. Clearly there is no path that has the length of path p_1 but with only two links.

shortest path problem can be varied so that we are looking for a path between two points in a polygon with length no greater than k_1 and link-distance no greater than k_2. This problem has been shown to be NP-Complete by Arkin et al. [1], and thus approximation algorithms have been developed to solve such problems in the plane [1], in arbitrary polygons [14], and avoiding rectilinear obstacles [5, 10].

The problems of finding minimum-length paths and minimum-length $\mathcal{O}$-paths between two points in a polygon have both been previously studied, and in this thesis I present a third type of restriction for which a minimum-length path can be found. Given a number m of allowable distinct orientations and two points in a simple polygon, find the set of orientations $\mathcal{O}$ for which the $\mathcal{O}$-path connecting the two points is as short as possible. In the context of this problem, an overall ”smallest path” would exist if and only if there was some set of orientations $\mathcal{O}$ for which the $\mathcal{O}$-path connecting the two points is both as short as is possible for any set of m orientations and has as few links as is possible for any set of m orientations. Thus it makes sense to refer to such a set as a *smallest set*. In this thesis I show that a smallest set does not always exist (see Figure 2 for an example of such a case). Since a smallest set is not guaranteed to exist, this thesis focuses on finding a set $\mathcal{O}$ which minimizes the Euclidean length of the path only.

The application of smallest paths can be easily seen in VLSI design where it is desirable to minimize the number of vias in a path (i.e. the link-distance of the path) along with the overall length of the path. In the past, VLSI design was restricted to orthogonal orientations, but now often allows any number of finite orientations to be used. Thus it is now practical to define a set $\mathcal{O}$ of orientations for which we aim to find smallest $\mathcal{O}$-paths between two points. Furthermore, if the designer were so inclined they could conceivably choose the set $\mathcal{O}$ to minimize the length of wire needed. Similarly, in the area of motion planning a robot with limited directions of travel could select these directions to minimize path length.

In this thesis I will prove that for any given pair of orientations $\mathcal{O} = \{\theta_1, \theta_2\}$, there must exist a smallest path between two points in any arbitrarily oriented polygon (where neither of the two points are contained on the boundary of P). Additionally, I give a counterexample showing that for any number of orientations greater than two, that is $|\mathcal{O}| \geq 3$, there does not always exists a smallest path. The main result here is a method of finding a set $\mathcal{O}$ of m orientations for which the $\mathcal{O}$-path between two points s and t in a simple polygon (where neither s nor t lies on the boundary of P) has minimum length over all possible sets $\mathcal{O}$. An explicit algorithm is given for the case where m is 2. The problem is broken up into subproblems where the length of the $s-t$ path is given by a continuous, differentiable function and therefore multi-variable calculus may be employed to find the minimum of such a function. It is proved that when m is two, each subproblem contains at most one minimum and therefore such a minimum can be found in constant time. Once the minimum is found for each subproblem, the solution to the problem will be the overall minimum of all the subproblem minimums found. The algorithm given runs in $O(n^2 logn)$ time where n is the number of vertices in the polygon.

2 Definitions

Definition 1. *Let a polygon P be defined by a set of n vertices $V = \{v_1, v_2, ..., v_n\}$ and a set of edges $E = \{e_1, e_2, ..., e_n\}$ where*

$$e_j = \begin{cases} (v_j, v_{j+1}) & \textit{if } j \neq n \\ (v_n, v_1) & \textit{otherwise} \end{cases}$$

Let the vertices be labelled in a counterclockwise order around the boundary of P, and P is closed. Let $bd(P)$ denote the boundary of P.

Definition 2. *A* simple polygon *$P = (V, E)$ is a polygon where E is not self-intersecting and no three consecutive vertices are co-linear.*

For the sake of limiting repetition, assume that we are always given a simple polygon $P = (V, E)$ and that $\mathcal{O}$ denotes a set of m distinct orientations.

Definition 3. *For a point z in the plane, let z_x denote the x-coordinate of z, and z_y denote the y-coordinate of z. Let the* length of a line segment*, denoted $\ell(\overline{xy})$ be the Euclidean length of the line $\overline{xy}$. Let the* length of a path*, denoted $\ell(S)$, be the sum of the lengths of all line segments in S.*

Definition 4. *A line segment $l = \overline{uv}$ where $u_y < v_y$ is* α-oriented *if $\alpha \in (0, \pi]$ is the counterclockwise angle l makes with the horizontal ray beginning at u travelling in the positive direction, when measuring in a counterclockwise manner. A line segment l is* γ-directed *if $\gamma \in (0, 2\pi]$ is the counterclockwise angle $\overline{uv}$ makes with a horizontal ray beginning at u travelling in the positive direction. See Figure 3. A line segment is said to be* $\mathcal{O}$-oriented *if its orientation is contained in $\mathcal{O}$.*
For some angle $\alpha \in (0, 2\pi]$, the complementary angle *of α is the angle $\phi = \alpha \pm \pi$ such that $\phi \in (0, 2\pi]$.*
Let $\theta(u, v)$ denote the orientation of the line $\overline{uv}$ and let $\theta_{dir}(u, v)$ denote the direction of the line $\overline{uv}$ from u to v.

Definition 5. *An* (α, β)-path *is a path consisting of a finite number of line segments where each line segment is either α-oriented or β-oriented. An* $\mathcal{O}$-path *is a path consisting of a*

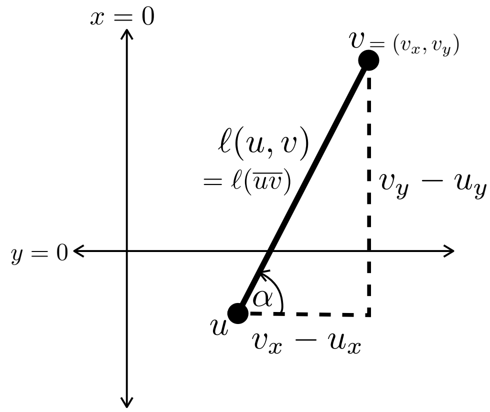


Figure 3: The orientation of $\overline{uv}$ is α, that is $\theta(u, v) = \alpha$. The length of $\overline{uv}$ is also shown.

finite number of line segments where each line segment is $\mathcal{O}$-oriented.

Let $\ell_{\alpha,\beta}(u, v)$ denote the length of the shortest (α, β)-path from u to v. Let $\ell_{\mathcal{O}}(u, v)$ denote the length of the shortest $\mathcal{O}$-path between u and v.

Definition 6. *For any $v \in V$ and $\phi \in (0, 2\pi]$, let the* projection point *from v onto polygon P, $Pr_\phi(v, P)$, be $x \in bd(P)$ such that $\overline{vx}$ is contained entirely in P, is ϕ-directed, and is as long as possible. If $Pr_\phi(v, P) = v$ the projection point is said to be* degenerate*. See Figure 4.*

Unless stated otherwise, the projection point from v, $Pr_\phi(v)$, means the projection point from v onto P.

Definition 7. *For simple polygon P, a* chord *is a line segment $\overline{pq}$ that is contained entirely in P where $p, q \in bd(P)$. A* maximum-chord *is a chord that is as long as possible. For any $v \in V$, let $C(v, \phi)$ for $\phi \in (0, 2\pi]$ denote the chord in P defined by $\overline{vp}$ where $p = Pr_\phi(v)$. See Figure 4.*

Definition 8. *For a set of line segments $S = \{s_1, s_2, ..., s_n\}$ which define a path or cycle, and two points $p, q \in S$, let the path along S from p to q be denoted $S(p \to q)$.*

If p and q both lie on the same segment, that is $\exists s_i \in S$, $p, q \in s_i$, then $S(p \to q) = \overline{pq}$. Otherwise p and q are on distinct edges $s_i, s_j \in S$ and $S(p \to q) = \{(p, v_{i+1})\} \cup \{s_{i+1}, ..., s_{j-1}\} \cup \{(v_j, q)\}$ where v_{i+1}, v_j are the endpoints of s_i and s_j connecting to s_{i+1} and s_{j-1}, respectively. See $E(p \to q)$ and $E(q \to p)$ in Figure 4.

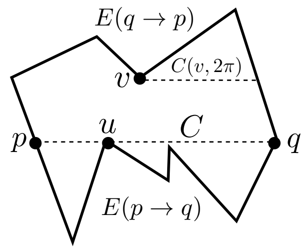


Figure 4: In the above figure, $C(v, 2\pi)$ is a chord but is not a maximum-chord, while C is the π-oriented maximum-chord through u. The points p and q are the projection points $Pr_{\pi}(u)$ and $Pr_{2\pi}(u)$, respectively. Note the projection point $Pr_{\frac{3\pi}{2}}(u)$ would be degenerate. The boundary of the polygon is made up of two subpaths $E(p \to q)$ and $E(q \to p)$ as shown above.

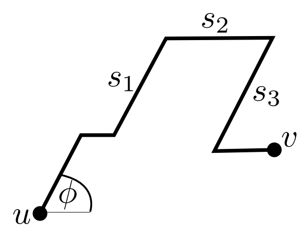


Figure 5: The segments s_1, s_2, s_3 form a u-turn on the (ϕ, π)-path from u to v.

Definition 9. *A* u-turn *is a path consisting of three line segments s_1, s_2, and s_3 such that s_1 and s_3 lie on the same side of the line containing s_2. A* staircase path *is a path containing no u-turns. See Figure 5.*

3 Smallest Paths

McDonald and Peters [13] showed that there always exists a smallest rectilinear path in a rectilinear polygon, and McDonald [12] showed that there always exists a smallest rectilinear path between two points in an arbitrary polygon, assuming neither of the two points is contained on the boundary of P. In this section I will generalize these results and show that for any two given orientations, that is $\mathcal{O} = \{\theta_1, \theta_2\}$, there always exists a smallest path between $s, t \in P - bd(P)$ where P is a simple polygon. The proof technique used is a modification of the one used by McDonald [12]. McDonald proved that for any simple polygon there exists a smallest rectilinear $s-t$ path. Since he didn't place any restrictions on the orientation of the polygon boundary, he actually proved that for any simple polygon there always exists a smallest orthogonal $s-t$ path (this can be shown by simply rotating the problem and applying his proof). In this section I generalize this proof further by assuming we are given any two orientations $\mathcal{O} = \{\theta_1, \theta_2\}$ and show that a smallest (θ_1, θ_2)-path between s and t exists. Most of the lemmas and theorems in this section are direct modifications of those given by McDonald, changed to accommodate non-orthogonal pairs of orientations.

Given a set of two orientations $\mathcal{O} = \{\theta_1, \theta_2\}$, the problem can be transformed so that McDonald's original proof can be applied directly to show that a smallest (θ_1, θ_2)-path exists. In this section we will discuss this transformation as well as give the modified proofs which directly prove the existence of a smallest (θ_1, θ_2)-path. It will also be shown via counterexample that for 3 or more given orientations, there does not always exists a smallest $s-t$ path.

Note that throughout the rest of this thesis when referring to a shortest $s-t$ path, we are referring to a shortest $s-t$ path that is contained entirely within P.

3.1 Proof Via Transformation

Here we will use the transformation matrix

$$T = \begin{bmatrix} 1 & 0 \\ -cot(\alpha) & csc(\alpha) \end{bmatrix}$$

to transform the simple polygon P, points s, t, and smallest (π, α)-path S from s to t into a new simple polygon P', new points s', t' and smallest path S' from s' to t'. Furthermore we will show that S' is a $(\pi, \frac{\pi}{2})$-path and has length equal to S. Thus if there doesn't exist a smallest (π, α)-path in P, then there does not exist a smallest $(\pi, \frac{\pi}{2})$-path in P'. Since McDonald proved there always exists a smallest $(\pi, \frac{\pi}{2})$-path in P', it must be that there always exists a smallest (π, α)-path in P.

In this section it will be shown that the transformation matrix transforms S into a $(\pi, \frac{\pi}{2})$-path while maintaining the length of S.

Lemma 3.1. *Using the transformation matrix T, an α-oriented line segment $\overline{uv}$ will be transformed into a vertical line segment $\overline{u'v'}$ where the length of $\overline{u'v'}$ is equal to the length of the original line.*

Proof.

$$\begin{bmatrix} x, & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -cot(\alpha) & csc(\alpha) \end{bmatrix} = \begin{bmatrix} x - ycot(\alpha), & ycsc(\alpha) \end{bmatrix}$$

Thus $u' = [u_x - u_y cot(\alpha), u_y csc(\alpha)]$ and $v' = [v_x - v_y cot(\alpha), v_y csc(\alpha)]$. First I will show that $u'_x = v'_x$, and thus $\overline{u'v'}$ is a vertical line.

$$\begin{aligned} u'_x - v'_x &= u_x - u_y cot(\alpha) - v_x + v_y cot(\alpha) \\ &= cot(\alpha)(u_x tan(\alpha) - u_y - v_x tan(\alpha) + v_y) \\ &= cot(\alpha)(tan(\alpha)(u_x - v_x) + v_y - u_y) \\ &= cot(\alpha)\left(\frac{u_y - v_y}{u_x - v_x}(u_x - v_x) - (u_y - v_y)\right) \\ &= 0 \end{aligned}$$

Now I will show that the length of the new segment $\overline{u'v'}$ is equal to the length of the original line.

$$sin(\alpha) = \frac{|u_y - v_y|}{\sqrt{(u_y - v_y)^2 + (u_x - v_x)^2}}$$
$$\Rightarrow csc(\alpha) = \frac{\sqrt{(u_y - v_y)^2 + (u_x - v_x)^2}}{|u_y - v_y|}$$

$$\begin{aligned}
\ell(\overline{u'v'}) &= |u'_y - v'_y| \\
&= |u_y csc(\alpha) - v_y csc(\alpha)| \\
&= csc(\alpha)|u_y - v_y| \\
&= \frac{\sqrt{(u_y - v_y)^2 + (u_x - v_x)^2}}{|u_y - v_y|}|u_y - v_y| \\
&= \sqrt{(u_y - v_y)^2 + (u_x - v_x)^2} \\
&= \ell(\overline{uv})
\end{aligned}$$

□

Lemma 3.2. *Using the transformation matrix T, an horizontal line segment $\overline{uv}$ will be transformed into a horizontal line segment $\overline{u'v'}$ where the length of $\overline{u'v'}$ is equal to the length of the original line.*

Proof.

$$\begin{bmatrix} x, & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -cot(\alpha) & csc(\alpha) \end{bmatrix} = \begin{bmatrix} x - ycot(\alpha), & ycsc(\alpha) \end{bmatrix}$$

Thus $u' = [u_x - u_y cot(\alpha), u_y csc(\alpha)]$ and $v' = [v_x - v_y cot(\alpha), v_y csc(\alpha)]$. Since $\overline{uv}$ is a horizontal line, $u_y = v_y$. Now I will show that $u'_y = v'_y$.

$$\begin{aligned}
u'_y &= u_y csc(\alpha) \\
&= v_y csc(\alpha) \\
&= v'_y
\end{aligned}$$

Now I will show that the length of $\overline{u'v'}$ is equal to the length of the original line.

$$\begin{aligned}
\ell(\overline{u'v'}) &= |u'_x - v'_x| \\
&= |u_x - u_y cot(\alpha) - v_x + v_y cot(\alpha)| \\
&= |u_x - v_x - u_y cot(\alpha) + v_y cot(\alpha)| \\
&= |u_x - v_x| \\
&= \ell(\overline{uv})
\end{aligned}$$

□

Now using the transformation matrix T we can transform the problem of finding a smallest (α, β)-path into the problem of finding a smallest rectilinear path, and thus use previous results to show such a path always exists.

Theorem 3.3. *For two orientations α, β, there always exists a smallest (α, β)-path from s to t.*

Proof. (direct)
First, w.l.o.g. rotate the problem so that $\beta = \pi$. Now transform the polygon P and points s and t by the matrix T, and McDonald proved that there exists a smallest rectilinear path S' from s to t. Now transform P, s, t and the path S' by the inverted matrix T^{-1} to get the original problem and a (π, α)-path S that has length equal to S' and consists of the same number of links as S (by Lemmas 3.1 and 3.2). Thus S has minimum length and a minimum number of links, and so is a smallest (π, α)-path. □

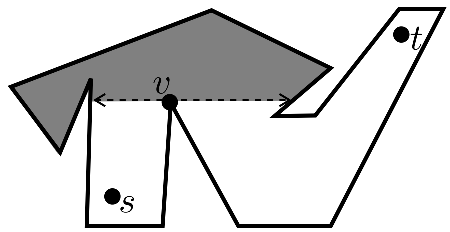


Figure 6: The shaded region is the middle region induced by $C(v, \pi)$ and $C(v, 2\pi)$, and the unshaded regions are the end regions induced by the chords. The vertex v is extreme with respect to π since s and t are in different end regions.

3.2 Direct Proof

Here we will prove directly that there always exists a smallest (α, β)-path between two points $s, t \in P - bd(P)$. The proof given in this section is a modified version of the proof given by McDonald [12].

The concept of an *extreme vertex* will be defined and it will be shown that all extreme vertices must be contained on any shortest $s-t$ path. This is a key insight that will be used throughout this thesis.

Definition 10. *For $\alpha, \beta \in (0, 2\pi]$ where $\alpha \neq \beta$, and $v \in V$, P is divided into three regions by the chords $C(v, \alpha)$ and $C(v, \beta)$ if and only if both $Pr_\alpha(v)$ and $Pr_\beta(v)$ are non-degenerate. A region is called the* middle region *if it is bounded by both chords. The two regions that are not the middle regions are called the* end regions *(see Figure 6). The chords themselves are considered part of the end regions.*
For a polygon which is divided into three regions by the chords $C(v, \alpha)$ and $C(v, \beta)$, let the end region *induced by $C(v, \alpha)$ ($C(v, \beta)$) be the end region bounded by $C(v, \alpha)$ ($C(v, \beta)$).*

Definition 11. *A vertex $v \in V$ is* extreme *with respect to orientation $\alpha \in (0, \pi]$ if s and t are in different end regions induced by $C(v, \alpha)$, $C(v, \alpha + \pi)$. See Figure 6.*
A vertex $v \in V$ is extreme with respect to $\mathcal{O}$ if v is extreme with respect to some $\theta_i \in \mathcal{O}$.

Definition 12. *Given $\mathcal{O}$, consider the sequence $\theta_1, \theta_2, \ldots, \theta_n$ of all orientations in $\mathcal{O}$ ordered counterclockwise. Let θ_i and θ_j be the orientations in $\mathcal{O}$ that are neighbours of ϕ with respect to the counterclockwise cyclic order. Then θ_i and θ_j are the* neighbouring orientations *of ϕ.*

Lemma 3.4. *For any two points x, y and the two neighbouring orientations of $\theta(x, y)$, $\theta_i, \theta_j \in \mathcal{O}$ - where if $\theta(x, y) \in \mathcal{O}$ then $\theta_i = \theta_j = \theta(x, y)$ - the shortest path (not necessarily in P) between x and y is a staircase path consisting of segments with orientations θ_i and θ_j.*

See Widmayer et al.[20] for a proof.

Lemma 3.5. *If S is a shortest path between two points s and t, then for any two points $x, y \in S$ the subpath $S(x \to y)$ is a shortest path between x and y.*

Proof. (by contradiction)
Recall by Definition 8 the path $S(x \to y)$ is the portion of the path S between points x and y. Say $S(x \to y)$ is not a shortest path between x and y, which means there is some path Q that is the shortest path between x and y. Therefore the length of the path $S(s \to x) \cup Q \cup S(y \to t)$ must be less than the length of S, which means S was not the shortest path between s and t. $\square$

Theorem 3.6. Extreme Point Theorem
Given points $s, t \in P$, if $v \in V$ is an extreme vertex with respect to $\mathcal{O}$ then the shortest $\mathcal{O}$-path from s to t must contain v.

Proof. (direct)
Let S be a shortest $\mathcal{O}$-path from s to t. Since v is extreme, s and t are in different end regions induced by two chords in P. This means that any path from s to t must pass through both chords. Let x be the point where S first intersects the first chord, and x' be the last point where S intersects the second chord. Note that by definition of an extreme point, the two chords are π apart and both intersect v, so will actually form one large chord across P. This means the line $\overline{xx'}$ will lie entirely in P.
Now S is made up of three subpaths: $S(s \to x)$, $S(x \to x')$ and $S(x' \to t)$. By Lemma 3.5 a shortest path is made up of shortest subpaths, so $S(x \to x') = \overline{xx'}$ which clearly contains v. $\square$

Lemma 3.7. *For the set of all vertices which are extreme with respect to $\mathcal{O}$, there is exactly one order in which the vertices are traversed on any shortest $\mathcal{O}$-path from s to t.*

Proof. (by contradiction)

Since all extreme vertices must be included on any shortest path from s to t, there is clearly at least one order. Let X be the set of extreme vertices in that order where $s, t \in X$ are the first and last points in X, and let S be a shortest path between s and t.

Take any two consecutive extreme vertices in X, say $u, v \in X$. Since S is a shortest path, the subpaths $S(s \to u)$, $S(u \to v)$ and $S(v \to t)$ are all shortest subpaths.

Say there exists a shortest path S' that traverses u and v in the opposite order. Then $S'(s \to v)$, $S'(v \to u)$ and $S'(u \to t)$ are all shortest subpaths. Since $S'(s \to v)$ is a shortest subpath, it is just as short as $S(s \to u) \cup S(u \to v)$ and we can construct a shortest path from s to t that consists of $S'(s \to v) \cup S(v \to t)$ that doesn't contain u. □

To prove that there always exists a smallest (θ_1, θ_2)-path from s to t, we first prove that there always exists a smallest (θ_1, θ_2)-path from s to any θ_1-oriented or θ_2-oriented chord. Then we will be able to construct a smallest (θ_1, θ_2)-path from s to t using the smallest path to a chord containing t. This is the same technique as used in McDonald's thesis.

Lemma 3.8. *For any horizontal maximum-chord C in P, the shortest (π, α)-path from s to C ends at a unique point.*

Proof. (by contradiction)

Assume there are two paths Q, R that are both shortest (π, α)-paths from s to C. Let $q, r \in C$ be the points at which Q and R intersect C, and let $m \in C$ be a point between q and r. Let P' be the polygon (not necessarily simple) as defined by the boundaries $Q \cup R \cup \overline{qr}$ and let m' be the path projection point $Pr_\alpha(m, P')$ or $Pr_{\alpha+\pi}(m, P')$, whichever is non-degenerate. We know m' cannot be contained on the line $\overline{qr}$ since the line $\overline{qr}$ is not α-oriented, and therefore $m' \in Q \cup R$. So w.l.o.g. say $m' \in Q$ (see Figure 7).

By Lemma 3.4, the length of the shortest (π, α)-path from m' to q is equal to $\ell(\overline{m'm}) + \ell(\overline{qm})$. Let $Q_s = Q(s \to m')$, $Q_q = Q(m' \to q)$ and Q_{new} denote the path $Q_s \cup \overline{m'm}$, which is a

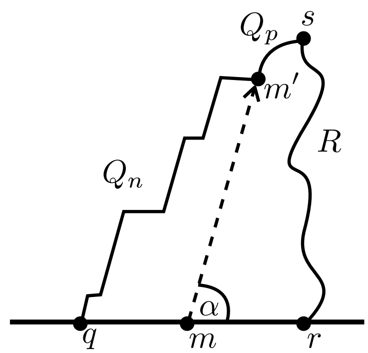


Figure 7: The configuration described in Lemma 3.8.

(π, α)-path from s to C contained entirely in P. Then the length of Q_{new} is:

$$\begin{aligned}
\ell_{\pi,\alpha}(Q_{new}) &= \ell_{\pi,\alpha}(Q_s) + \ell_{\pi,\alpha}(m', m) \\
&< \ell_{\pi,\alpha}(Q_s) + \ell_{\pi,\alpha}(m', m) + \ell_{\pi,\alpha}(m, q) \\
&\leq \ell_{\pi,\alpha}(Q_s) + \ell_{\pi,\alpha}(Q_q) \\
&= \ell_{\pi,\alpha}(Q) \\
&= \ell_{\pi,\alpha}(R)
\end{aligned}$$

and therefore Q_{new} is shorter than both Q and R, so neither were shortest paths from s to C. □

Definition 13. *In a shortest $\mathcal{O}$-path S between s and t, two points $u, v \in S$ are* consecutive extreme vertices *if $u, v \in V$ are extreme and there is no extreme vertex v_{mid} such that $v_{mid} \in S(u \to v)$.*

Lemma 3.9. *A shortest $\mathcal{O}$-path between s and t intsersects an $\mathcal{O}$-chord in one continuous piece.*

Proof. (by contradiction)
Let S be a shortest $\mathcal{O}$-path between s and t, and let $e_i = \overline{pq}$ be a segment of S contained entirely on C. Say S intersects C at some point $x \notin e_i$. S consists of the subpaths $S(s \to x) \cup S(x \to p) \cup S(p \to t)$ $(S(s \to q) \cup S(q \to x) \cup S(x \to t))$. By Lemma 3.5 the subpath $S(x \to p)$ $(S(q \to x))$ must be a shortest path from x to p (q to x), and

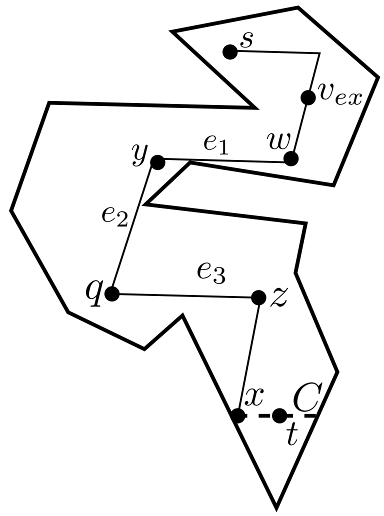


Figure 8: Configuration described in Lemma 3.10.

since $\theta(x,p) \in \mathcal{O}$ ($\theta(q,x) \in \mathcal{O}$) the shortest path is the straight line $\overline{xp}$ ($\overline{qx}$). However, we assumed x was not part of e_i. $\square$

Lemma 3.10. *For points $s, t \in P$ and $\alpha \in (0, \pi]$, let C be the horizontal maximum-chord containing t and let S be the shortest (π, α)-path from s to C ending at some point $x \in C$ plus the segment $\overline{xt}$. All u-turns in S must contain an extreme vertex.*

Proof. Let $e_1 = \overline{wy}$, $e_2 = \overline{yq}$, and $e_3 = qz$ form a u-turn in S and w.l.o.g. rotate and flip the problem so that $q_y < y_y$ and $q_x < z_x$ (see Figure 8). Let β denote the orientation of e_2, which is one of α or π. First I will show that there must be a vertex in e_2.

Say there exists some $x' \in e_3$ where $x' \neq q$ s.t. $Pr_\beta(x') \in e_1$. Let $x'' = Pr_\beta(x')$, $S_a = S(s \to x'')$, $S_b = S(x'' \to q)$, and $S_{new} = \overline{x''x'}$. Since $\overline{x''x}$ is an β-oriented line connecting e_1 and e_3, it must have length exactly equal to e_2.

1. $e_3 \nsubseteq C$

   In this case the length of the path $S(s \to x'') \cup \overline{x''x} \cup S(x' \to x)$ is less than the length of $S(s \to x)$, so $S(s \to x)$ was not a shortest path from s to C.

2. $e_3 \nsubseteq C$

   In this case the length of the path $S(s \to x'') \cup \overline{x''x}$ is less than the length of $S(s \to x)$, so $S(s \to x)$ was not a shortest path from s to C.

Thus $\forall x' \in \overline{xt}$, $Pr_\beta(x')$ hits an edge of P, so there must exist some vertex on e_2. Now it will be shown that for a vertex $v \in V$ that is on the middle segment of a u-turn in S, v is extreme.

Consider the β-oriented chords $C_1 = C(v, \beta)$ and $C_2 = C(v, \beta + \pi)$. $\overline{vy} \in C_1$ and $\overline{vq} \in C_2$, so both chords are of non-zero length and divide P into three regions: two end regions and a middle region. By Lemma 3.9, since $e_2 \in C_1 \cup C_2$, no other part of S can intersect $C_1 \cup C_2$. Since e_1 lies in the end region of P induced by C_1, $S(s \to y)$ lies entirely in that end region. Since e_3 lies in the end region of P induced by C_2, $S(q \to t)$ lies entirely in that end region. Therefore v is an extreme vertex. $\square$

Lemma 3.11. *For points $s, t \in P$ and $\alpha \in (0, \pi]$, the shortest (π, α)-path from s to the horizontal maximum-chord containing t plus the straight line to t is a shortest (π, α)-path from s to t.*

Proof. (direct)

Let C be the horizontal maximum-chord containing t and let S be the shortest (π, α)-path from s to C ending at some point $x \in C$ plus the segment $\overline{xt}$. Consider R, an arbitrary shortest (α, π)-path from s to t. If there are no u-turns in S then by Lemma 3.4 S is a shortest path from s to t. Otherwise S contains at least one u-turn, so let v' be the vertex contained on the u-turn where $S(v' \to t)$ is a staircase path. By Lemma 3.10 we know v' is extreme w.r.t. one of α or π, so must be contained in R. Since both $S(s \to x)$ and R are shortest paths, they are made up of shortest subpaths (as shown in Lemma 3.5). Thus both $S(s \to v')$ and $R(s \to v')$ must be shortest paths from s to v' and so are both of equal length. Furthermore, since $S(v' \to t)$ contains no u-turns, it is the shortest (π, α)-path from v' to t and therefore $R(v' \to t)$ is the same length as $S(v' \to t)$. Therefore S is a shortest path from s to t. $\square$

Lemma 3.12. *For a horizontal maximum-chord C in P, $s \in P$ and $\alpha \in (0, \pi]$, there exists a smallest (π, α)-path from s to C.*

Proof. (by induction on the number of bends in the minimum-link path)

Basis:

When the number of links in the minimum-link path from s to C is two or fewer, the theorem is obviously true.

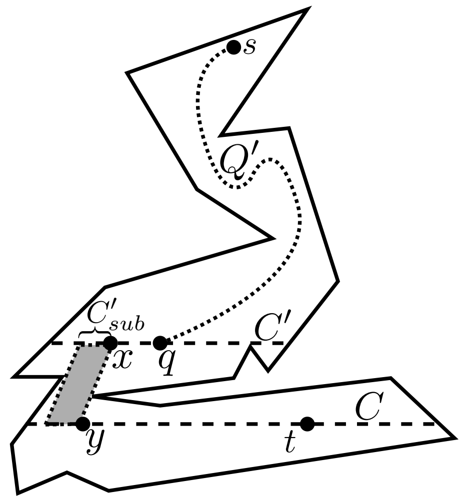


Figure 9: The configuration described in Theorem 3.13.

Inductive Hypothesis:

Assume that for all minimum-link paths from s to C with $\leq k$ links, we can find a smallest path from s to C.

Inductive Step:

Let Q be the minimum-link path from s to C, where Q has $(k+1)$ links. The $(k+1)^{th}$ segment of Q must be α-oriented, otherwise we would be able to shorten Q be removing the final link. Therefore the k^{th} segment of Q is π-oriented, and we can draw a maximum-chord C' that contains the k^{th} segment of Q. By the inductive hypothesis, a smallest path Q' exists from s to C' with $(k-1)$ segments. Let $q \in C'$ be the point at which Q' ends. See Figure 9.

Let C'_{sub} be the set of points on C' such that one α-oriented segment contained entirely in P can connect C' and C. Note that there is at least one point in C'_{sub}, which is the point at which the $(k+1)^{th}$ segment of Q intersected C'. Furthermore, $q \notin C'_{sub}$ since otherwise Q would not be a minimum-link path. Let $x \in C'_{sub}$ be the closest point to q, and let y be the point at which the α-oriented segment from x intersects C.

Now consider the path $S = (Q' \cup \overline{qx} \cup \overline{xy})$. From the Lemma 3.8, q is the unique point that is shortest from s to C', and by Lemma 3.11 we know that the shortest path from s to x has length equal to the length of Q' plus the length of the segment $\overline{qx}$. Additionally, the distance from C' to C via a single link is constant. Therefore S is a shortest path as well as a minimum-link path. □

Theorem 3.13. Smallest Path Theorem
For points $s, t \in P$ and $\theta_1, \theta_2 \in (0, \pi]$, there exists a smallest (θ_1, θ_2)-path between s and t.

Proof. (by construction)
First, find a minimum-link path from s to t and call this path Q. Let k be the number of links in Q and let the orientation of the final segment of Q be θ_1. Now w.l.o.g. we can rotate the problem so that $\theta_1 = \pi$.
Let C be a horizontal maximum-chord in P such that the last segment in Q is contained in C. By Lemma 3.12, there is a smallest path from s to C, say S. The path S has $(k-1)$ links and ends at some point $x \in C$. By Lemma 3.11, $S \cup \overline{xt}$ is a shortest path from s to t. Furthermore, $S \cup \overline{xt}$ has k links, which is minimum. □

For a set of more than 2 orientations, a smallest path does not always exist. To prove this, I will give a lemma concerning the shortest $\mathcal{O}$-path between two points then provide examples for all i greater than 3 allowable orientations where a smallest path is not possible.

Lemma 3.14. *For $s, t \in P$ and two consecutive extreme vertices $u, v \in V$, if $\overline{uv}$ is not contained entirely in P then there exists a vertex z that is extreme w.r.t. $\theta(u, v)$ and z lies between u and v on a shortest path from s to t.*

Lemma 3.15. *For $s, t \in P$ and two consecutive extreme vertices $u, v \in V$, if $\theta(u, v) \in \mathcal{O}$ then $\overline{uv}$ must be contained in P.*

Proof. (by contradiction)
To simplify the proof, w.l.o.g. we can rotate the problem so that $\theta_{dir}(u, v) = 2\pi$. Let Q be the shortest arbitrary path from u to v contained in P. If Q is a straight line from u to v then $\overline{uv}$ is contained entirely in P. Otherwise Q consists of at least two segments and contains at least one vertex that has y-value lower or higher than u_y (which is equal to v_y). Let z be the point in Q that maximizes $|z_y - u_y|$.
Consider the π-oriented maximum-chord through z, say C. Since $Q(u \to z)$ is contained entirely in one end region induced by C and $Q(z \to v)$ is contained entirely in the other end region, s must be contained in one end region while t is contained in the other. Thus

z is extreme w.r.t. π, which is in $\mathcal{O}$. Furthermore, since z lies on the path $Q(u \to v)$, u and v are not consecutive extreme vertices. $\square$

Lemma 3.16. *For any two consecutive extreme vertices u, v, the shortest path contained in P from u to v is either the line $\overline{uv}$, if $\theta(u, v) \in \mathcal{O}$, or a staircase path with two or more segments consisting of the two orientations in $\mathcal{O}$ that are the neighbouring orientations of $\theta(u, v)$.*

Proof. (by contradiction)
Let S be the shortest $\mathcal{O}$-path from u to v, which by Lemma 3.4 consists of at most two orientations which are the neighbouring orientations of $\theta(u, v)$, say $\alpha, \beta \in \mathcal{O}$. Note that if $\theta(u, v) \in \mathcal{O}$ then by Lemma 3.15 the line $\overline{uv}$ is contained in P and thus the shortest path in P between u and v is $\overline{uv}$. Otherwise $\alpha \neq \beta$ and we are looking for a staircase path in P from u to v.
By Lemma 3.11, a shortest path R exists that is a shortest path from s to a chord C plus a line segment on C, and by 3.10 all u-turns in R contain an extreme vertex. Since R and S are both shortest paths, u and v must be contained on both paths, and since they are consecutive extreme vertices there is no extreme vertex in $R(u \to v)$. Thus there are no u-turns in $R(u \to v)$ and $R(u \to v)$ is a staircase (α, β)-path.
Finally by Lemma 3.5, the length of $S(u \to v)$ must be equal to the length of $R(u \to v)$. $\square$

Theorem 3.17. *For three or more allowable orientations, there does not always exist a smallest path from s to t.*

Proof. (direct)
For $m = 3$, see Figure 10 for an example where there is no smallest path from s to t. By Lemma 3.16, the shortest path from s to v is a staircase (θ_1, θ_3)-path and the shortest path from v to t is a staircase (θ_2, θ_3)-path. Therefore a shortest path from s to t must consist of at least three segments since there are three orientations. However, as seen in Figure 10, there is a (θ_1, θ_2)-path from s to t that consists of two links. Therefore there is no $\mathcal{O}$-path connecting s and t which minimizes length and link-distance simultaneously.
For $m > 3$, imagine Figure 10 with $\theta_4 \ldots \theta_m \in \Upsilon$. Since the neighbouring orientations of $\theta(s, v)$ and $\theta(v, t)$ don't change, the shortest $\mathcal{O}$-path connecting s and t doesn't change.

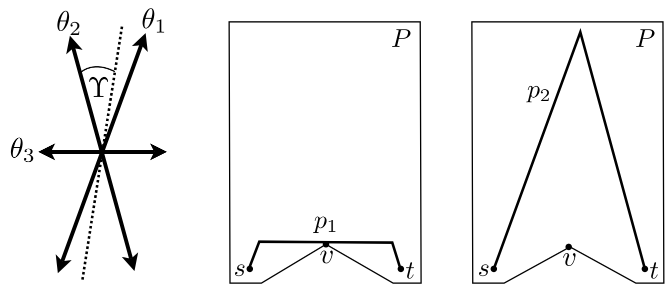


Figure 10: There is no smallest $(\theta_1, \theta_2, \theta_3)$-path between s and t. Any path of minimum length has at least three links.

Similarly, there still exists a 2-link path from s to t. Therefore there is no smallest $\mathcal{O}$-path connecting s and t for $\mathcal{O} = \{\theta_1, \ldots, \theta_m$ s.t. $\forall i > 3, \theta_i \in \Upsilon\}$. $\square$

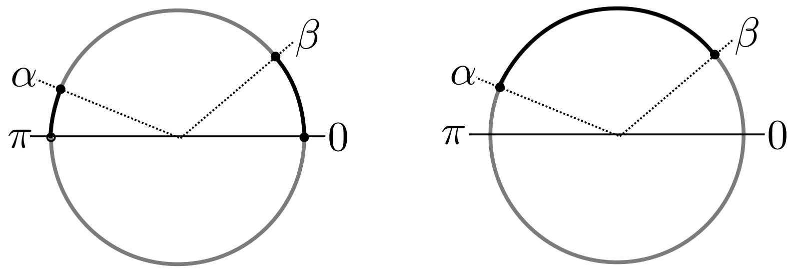


Figure 11: The figure on the left shows an interval of orientations where $\alpha > \beta$, and the figure on the right shows an interval of orientations where $\alpha < \beta$. Both are considered single, continuous intervals.

4 Constructing All Extreme Sequences

In order to compute the length of the shortest $\mathcal{O}$-path between s and t, it is useful to know the *extreme sequence* of $\mathcal{O}$, which is the set $X_{\mathcal{O}}$ of all vertices which are extreme with respect to $\mathcal{O}$, in order of traversal from s to t. To solve the problem of finding the set $\mathcal{O}^*$ out of all possible sets $\mathcal{O}$ that minimizes the length of the shortest $s-t$ path, it is not possible to check all possible sets $\mathcal{O}$ since the number of such sets is infinite. Instead, we will divide the problem into a finite set of subproblems which will later be shown to be solvable.

We adopt the usual notation that a square bracket ("[" or "]") bracketing a range on one endpoint implies that the endpoint is included in the range and a parenthesis ("(" or ")") implies that the endpoint is excluded. We also adopt that notation used by Rawlins [17] that an angle bracket ("$\langle$" or "$\rangle$") is used when we wish to make the statements that apply to either of the cases (i.e. "$\langle$" implies the statement applies to either "(" or "["). Furthermore. an interval of orientations that contains π - that is, an interval $\langle\alpha, \beta\rangle$ where $\alpha > \beta$ - is considered a single continuous interval (see Figure 11). Similarly, an interval of directions that contains 2π is considered a single continuous interval.

In this section we create an *extreme interval* for each vertex in P, which is the set of orientations for which v is extreme. It is proved that all extreme intervals that are non-empty are single continuous intervals, say $\langle\alpha, \beta\rangle$. Using these extreme intervals, we create a set of *Extreme Vertex Sets*. An Extreme Vertex Set is a set that contains all extreme vertices with respect to some orientation $\alpha \in (0, \pi]$. Thus for some orientation α, the set of vertices whose extreme intervals contain α will be the Extreme Vertex Set at α. Since all extreme

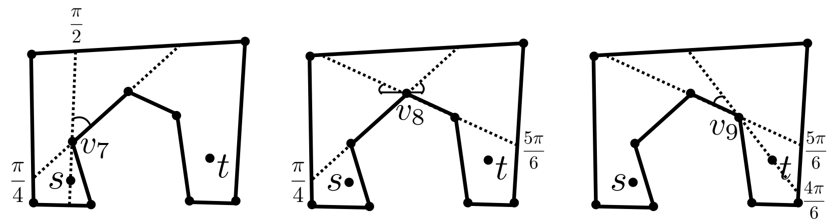


Figure 12: The vertex v_7 is extreme on the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$, v_8 is extreme on the interval $[\frac{5\pi}{6}, \frac{\pi}{4}]$, and v_9 is extreme on the interval $[\frac{4\pi}{6}, \frac{5\pi}{6}]$.

intervals are continuous, the number of distinct Extreme Vertex Sets is proportional to the number of extreme intervals. We will discuss this in more detail later in this section.

Note that we are eventually going to want to combine these Extreme Vertex Sets into extreme sequences, each of which will contain m Extreme Vertex Sets.

Figure 12 shows a polygon P and points $s, t \in P$ that will be used as an example throughout this section.

The method given in this section of finding all possible Extreme Vertex Sets is as follows:

1. For each vertex $v_i \in V$ find the interval in which v_i is extreme. The intervals for which v_7, v_8 and v_9 are extreme are shown in Figure 12.

2. Let Φ be the set of all orientations which bound the interval for which some vertex is extreme. In the example shown in Figure 12, Φ would consist of $\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{6}, \frac{5\pi}{6}\}$.

3. Arrange the orientations in Φ from $0 \rightarrow \pi$, and for each pair of consecutive orientations $\sigma_i, \sigma_{i+1} \in \Phi$ let $X_{(\sigma_i, \sigma_{i+1})}$ be the set of all vertices which are extreme for the interval (σ_i, σ_{i+1}). In our example,

$$\begin{aligned} X_{(\frac{\pi}{4}, \frac{\pi}{2})} &= \{v_7\} \\ X_{(\frac{\pi}{2}, \frac{4\pi}{6})} &= \emptyset \\ X_{(\frac{4\pi}{6}, \frac{5\pi}{6})} &= \{v_9\} \\ X_{(\frac{5\pi}{6}, \frac{\pi}{4})} &= \{v_8\} \end{aligned}$$

Note that for two consecutive orientations $\sigma_i, \sigma_{i+1} \in \Phi$, no vertex changes extremity at any orientation contained in the interval (σ_i, σ_{i+1}), which implies that the set of extreme vertices remains constant within (σ_i, σ_{i+1}).

Let $X_{(\alpha,\beta)}$ be the set of all extreme vertices with respect to the interval (α, β) and let X_ϕ be the set of all extreme vertices with respect to ϕ. Referring back to the start of this discussion, we are interested in finding the extreme sequence for $\mathcal{O}^*$. Finding such a sequence is reduced to finding the union of all sets $X_{\theta_j^*}$ where $\theta_j^* \in \mathcal{O}^*$. Let $I(\alpha)$ denote the interval (σ_k, σ_{k+1}) containing α. Then the set $X_{\theta_j^*}$ is equal to the set $X_{I(\theta_j^*)}$. This implies that to construct $X_{\mathcal{O}^*}$ we can find all intervals $I(\theta_j^*)$ where $\theta_j^* \in \mathcal{O}^*$, and add $X_{I(\theta_j^*)}$ to $X_{\mathcal{O}^*}$. A more formal definition of how to construct $X_{\mathcal{O}^*}$ is given in Eq. 1.

$$X_{\mathcal{O}^*} = \bigcup_{\theta_i^* \in \mathcal{O}^*} X_{I(\theta_j^*)} \tag{1}$$

Since the set $\mathcal{O}^*$ is unknown, the extreme sequence $X_{\mathcal{O}^*}$ is unknown as well. Therefore we create all possible extreme sequences by considering all possible combinations of m intervals in Φ, and combining their associated extreme vertex sets.

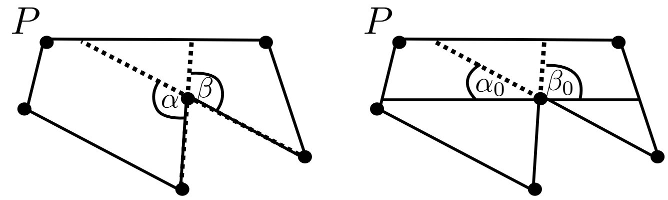
In our example if we were looking for a set $\mathcal{O}$ containing 2 orientations this would mean constructing the sets

$$\begin{array}{ccc}
X_{(\frac{\pi}{4},\frac{\pi}{2})} \cup X_{(\frac{\pi}{4},\frac{\pi}{2})} = \{v_7\} & X_{(\frac{\pi}{2},\frac{4\pi}{6})} \cup X_{(\frac{\pi}{2},\frac{4\pi}{6})} = \emptyset & X_{(\frac{4\pi}{6},\frac{5\pi}{6})} \cup X_{(\frac{4\pi}{6},\frac{5\pi}{6})} = \{v_9\} \\
X_{(\frac{\pi}{4},\frac{\pi}{2})} \cup X_{(\frac{\pi}{2},\frac{4\pi}{6})} = \{v_7\} & X_{(\frac{\pi}{2},\frac{4\pi}{6})} \cup X_{(\frac{4\pi}{6},\frac{5\pi}{6})} = \{v_9\} & X_{(\frac{4\pi}{6},\frac{5\pi}{6})} \cup X_{(\frac{5\pi}{6},\frac{\pi}{4})} = \{v_8, v_9\} \\
X_{(\frac{\pi}{4},\frac{\pi}{2})} \cup X_{(\frac{4\pi}{6},\frac{5\pi}{6})} = \{v_7, v_9\} & X_{(\frac{\pi}{2},\frac{4\pi}{6})} \cup X_{(\frac{5\pi}{6},\frac{\pi}{4})} = \{v_8\} & \\
X_{(\frac{\pi}{4},\frac{\pi}{2})} \cup X_{(\frac{5\pi}{6},\frac{\pi}{4})} = \{v_7, v_8\} & & X_{(\frac{5\pi}{6},\frac{\pi}{4})} \cup X_{(\frac{5\pi}{6},\frac{\pi}{4})} = \{v_8\}
\end{array}$$

Each of the extreme sequences constructed above represents a subproblem which we will later prove is solvable. The subproblem corresponding to the extreme sequence $X_{(\frac{\pi}{4},\frac{\pi}{2})} \cup X_{(\frac{\pi}{2},\frac{4\pi}{6})}$ seen above is of the form: Given $\theta_1 \in (\frac{\pi}{4}, \frac{\pi}{2})$ and $\theta_2 \in (\frac{\pi}{2}, \frac{4\pi}{6})$, minimize the length of the (θ_1, θ_2)-path between s and t.

In this section we will prove that each vertex is extreme for a single interval (or not at all), and show that there are $O(n)$ distinct Extreme Vertex Sets. Then for the case of two allowable orientations, that is $m = 2$, we prove there are at most $2n^2 + n$ possible extreme sequences which we must consider.

Definition 14. *For any $v \in V$, $\phi \in (0, 2\pi]$ is an* interior angle *of v if $Pr_\phi(v)$ is non-degenerate. An angle that is not an interior angle is called an* exterior angle.



(a) (b)

Figure 13: (a) The two closed intervals, α and β, for which the projections from v are non-degenerate, as described in Lemma 4.1. (b) The two closed intervals α_0 and β_0 containing all undirected orientations for which the projections from v are non-degnerate.

For any $v \in V$, the interior angle interval *of v is the set of all interior angles of v. The* exterior angle interval *of v is the set of all exterior angles of v.*

Definition 15. *For a vertex $v \in V$, the* incident edges *of v are the two edges which have v as an endpoint.*

Lemma 4.1. Double Projection Interval Lemma
For all $v_i \in V$ the set of directions $\Phi \subset (0, 2\pi]$ for which for all $\alpha \in \Phi$ both $Pr_\alpha(v_i)$ and $Pr_{\alpha+\pi}(v_i)$ are non-degenerate is either a pair of intervals or an empty set. Furthermore, the set of orientations for which both projection points are non-degenerate is a single interval.

Proof. (by construction)
Let $\Theta \subset (0, 2\pi]$ be the interior angle interval of v, and by Definition 14, for every direction $\varphi \in \Theta$, $Pr_\varphi(v_i)$ is non-degenerate. Let $\alpha = \theta_{dir}(v_i, v_{i+1})$ and $\beta = \theta_{dir}(v_{i-1}, v_i)$. Since the bounding directions of such an interval will be the directions of the incident edges of v_i, Θ is a closed interval defined by $[\alpha, \beta]$. If the interval $[\alpha, \beta]$ spans less than π then there is no orientation for which both $Pr_\alpha(v)$ and $Pr_{\alpha+\pi}(v)$ are non-degenerate, so let $\Phi = \emptyset$. Otherwise let α' be the direction opposite of α and let β' be the direction opposite of β. Then Φ is the pair of intervals $[\alpha, \beta']$ and $[\alpha', \beta]$. We now have Φ which is a pair of intervals or an empty set (see Figure 13(a)).
If one of the intervals in Φ is contained entirely in $(0, \pi]$ then that interval is the set within $(0, \pi]$ for which both projection points are non-degenerate. Otherwise, both intervals are contained partially in $(0, \pi]$ (see Figure 13(b)). Let $(\Theta \cap (0, \pi])$ be the set for which both projection points are non-degenerate, which is a single interval. $\square$

Definition 16. *For any $v \in V$, a point $x \in P$ is said to* switch regions *of $C(v, \phi)$ at $\phi' \in (0, 2\pi]$ if for any chord $C = C(v, \alpha)$ where $\alpha \neq \phi'$, the region induced by C that contains $C(v, \phi')$ also contains the point x.*

Lemma 4.2. Region Switch Lemma

For any $v \in V$ and $x \in P$ where $x \neq v$, there is at most one direction $\phi \in (0, 2\pi]$ at which x switches regions of $C(v, \phi)$.

Proof. (by contradiction)

Assume there are at least two distinct angles $\alpha, \beta \in (0, 2\pi]$ at which x switches regions. Let C_1 and C_2 be chords $C(v, \phi')$ and $C(v, \phi'')$ where $\phi' \neq \phi''$ such that both lie in the middle region induced by $C(v, \alpha)$ and $C(v, \beta)$. Since α and β are distinct, the chords C_1 and C_2 must exist. Clearly $C(v, \alpha)$ lies in one end region induced by $C_1 \cup C_2$ and $C(v, \beta)$ lies in the other. However, this means x lies in both end regions induced by $C_1 \cup C_2$, which is not possible. $\square$

Theorem 4.3. Extreme Interval Theorem

For $v \in V$ and $s, t \in P$ where $s \neq v$ and $t \neq v$, v is either never extreme or is extreme for a single interval in $(0, \pi]$.

Proof. (direct)

In order for v to have end regions induced at an orientation $\phi \in (0, \pi]$, both projection points $Pr_\phi(v)$ and $Pr_{\phi+\pi}(v)$ must be non-degenerate. Let $\Phi \subset (0, \pi]$ be the set of all such orientations. By Lemma 4.1, Φ is a single interval $\langle \alpha, \beta \rangle$.

By Lemma 4.2, there is at most one orientation contained in Φ, say $\phi_s \in \Phi$ ($\phi_t \in \Phi$), where s (t) switches regions of $C(v, \phi)$. If there is no orientation in Φ at which s (t) switches regions, then s (t) is either in an end region for all of Φ or is never in an end region. We will now deal with the case where ϕ_s (ϕ_t) exists.

The orientations in the set $\{\phi_s(\phi_t), \alpha, \beta\}$ are the only orientations at which s (t) can switch into or out of an end region. If ϕ_s (ϕ_t) is equal to either α or β, then it cannot switch regions for any orientation in the interval (α, β), which means either s (t) is in an end region for the whole interval Φ or s (t) is never in an end region.

If ϕ_s (ϕ_t) is not equal to either α or β, then there are two cases:

1. p is in an end region at α

   Then p will switch to the middle region at ϕ_s (ϕ_t), and stay in that region until β.

2. p is in the middle region at α

   Then p will switch to an end region at ϕ_s (ϕ_t) and stay in that region until β.

Therefore s (t) is in an end region for a single interval or not at all. If either s or t is never in an end region, then there can be no orientation at which s and t are both in end regions and therefore v is never extreme.

Say s is in an end region for $\langle \phi, \phi' \rangle$, and t is in an end region for $\langle \varphi, \varphi' \rangle$. If s and t are in the same end region, v is never extreme. Otherwise, v is extreme only in $\langle \phi, \phi' \rangle \cap \langle \varphi, \varphi' \rangle$. $\square$

Theorem 4.4. Extreme Set Theorem

Given $s, t \in P$ there are less than or equal to $2n^2 + n$ possible extreme vertex sequences with respect to 2 orientations.

Proof. (direct)

Every $v \in V$ has a single interval of orientations for which it is extreme, if it is extreme at all (by Theorem 4.3). Let Υ be the ordered set of all orientations within $(0, \pi]$ at which some vertex changes extremity. There are at most $2n$ orientations in Υ, so there are at most $2n$ possible sets of extreme vertices with respect to one orientation. Each extreme sequence contains all vertices extreme with respect to $\mathcal{O} = \{\theta_1, \theta_2\}$, so contains all vertices extreme w.r.t. θ_1 and all vertices extreme w.r.t. θ_2. Thus each extreme sequence is made up of the vertices from (at most) 2 extreme vertex sets. The number of possible combinations of Extreme Vertex Sets is $N \leq \binom{2n}{1} + \binom{2n}{2}$, which is equal to $2n^2 + n$. $\square$

5 Calculating the Length of a Shortest Path

In the last section we showed how to create a finite number of extreme sequences where each sequence corresponds to a subproblem of the form: Given a multiset of m open intervals $(\rho_1, \rho'_1), \ldots, (\rho_m, \rho'_m)$ and the set $X = \bigcup_{(\rho_i, \rho'_i)} X_{(\rho_i, \rho'_i)}$ of extreme vertices, find the set $\mathcal{O} = \{\theta_1, \ldots, \theta_m\}$ where $\theta_1 \in (\rho_1, \rho'_1), \ldots, \theta_m \in (\rho_m, \rho'_m)$ which minimizes the length of the $\mathcal{O}$-path between s and t. In this section we refer to these as $s - t$ *subproblems*, and focus on solving an instance of an $s - t$ subproblem.

The main goal of this section is to find a distance function $D(\mathcal{O})$ that gives the length of the shortest $\mathcal{O}$-path connecting s and t. We will find this function and prove it is continuous and differentiable, meaning we will be able to use calculus to find the minimum (if it exists).

We will now define some properties and notation that will be used throughout the section. Let Υ refer to a multiset of m non-overlapping, but possibly duplicate, ranges

$$\Upsilon = \{(\rho_1, \rho'_1), \ldots, (\rho_m, \rho'_m)\}$$

where for all intervals (ρ_i, ρ'_i) there does not exist any $\phi \in (\rho_i, \rho'_i)$ such that a vertex changes extremity at ϕ. Furthermore, let the ranges in Υ be ordered from $0 \to \pi$. We let X refer to the set of $\{s, t\}$ union with all vertices which are extreme with respect to any orientation contained in an interval in Υ. More formally, X is the set of all extreme vertices with respect to some $\phi \in (\rho_i, \rho'_i)$ where $(\rho_i, \rho'_i) \in \Upsilon$. Since it will be convenient to do so, we will also create a set X_c of pairs from X where $(u, v) \in X_c$ if and only if u and v are consecutive in X. Note that it makes sense to create X_c since by Lemma 3.7 there is exactly one order in which the elements in X can be traversed from s to t.

A function $D(\mathcal{O})$ will be defined which gives the distance of the shortest $\mathcal{O}$-path traversing the extreme sequence, which is equivalent to the distance of the shortest $\mathcal{O}$-path from s to t. We aim to find the set of orientations $\mathcal{O}^* = \{\theta_1, \theta_2, \ldots, \theta_m\}$ that minimizes the length of the $s - t$ path where $\theta_i \in (\rho_i, \rho'_i)$.

We label the orientations in $\mathcal{O}$ so that $\theta_1 < \ldots < \theta_m$. If two orientations $\theta_i, \theta_j \in \mathcal{O}$ are in the same interval (ρ_k, ρ'_k), then we insist that the θ_i or θ_j where the lower index is always less than the other.

Lemma 5.1. *For $s, t \in P$ and two consecutive extreme vertices $u, v \in V$, if $\theta(u, v) \in$*

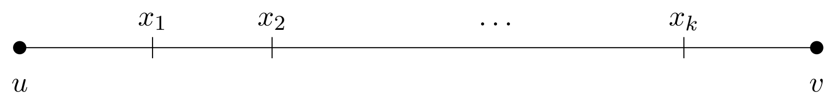


Figure 14: An example of the set $X_E = \{x_1, x_2, \ldots, x_k\}$ ordered from u to v.

$(\rho_1, \rho_1') \cup \ldots \cup (\rho_m, \rho_m')$ *then* $\overline{uv}$ *must be contained entirely in* P*.*

Proof. (by contradiction)
To simplify the proof, w.l.o.g. we can rotate the problem so that $\theta_{dir}(u, v) = 2\pi$. Let Q be the shortest arbitrary path from u to v contained in P. If Q is a straight line from u to v then $\overline{uv}$ is contained entirely in P. Otherwise Q consists of at least two segments and contains at least one vertex that has y-value lower or higher than u_y (which is equal to v_y). Let z be the point in Q that maximizes $|z_y - u_y|$.
Consider the π-oriented maximum-chord through z, say C. Since $Q(u \to z)$ is contained entirely in one end region induced by C and $Q(z \to v)$ is contained entirely in the other end region, s must be contained in one end region while t is contained in the other. Thus z is extreme w.r.t. π. Furthermore, since z lies on the path $Q(u \to v)$, u and v are not consecutive extreme vertices. □

Lemma 5.2. *For an extreme vertex* $v \in V$*, the direction of the final segment containing* v *(directed away from* v*) on the shortest arbitrary path from* s *to* v *will be the direction where* s *switches regions.*

Proof. Let $e \in S$ be the segment of S containing v, and let $\alpha \in (0, 2\pi]$ be the direction of the segment from v to u, the other endpoint of e. Consider the chord $C(v, \alpha)$. Since S is a shortest path, it cannot cross this chord except at e, so $S(s \to u)$ is contained entirely in one region bounded by $C(v, \alpha)$. Let the region bounded by $C(v, \alpha)$ that contains $S(s \to u)$ be A. Now let $\beta = \alpha \pm \delta$ for some arbitrary $\delta > 0$.
If there exists a point $p \in C(v, \beta)$ that lies on S, then the path $S(v \to p)$ can be replaced with $\overline{vp}$ and the new path $\overline{vp} \cup S(p \to s)$ is shorter than S. This is not possible since S is a shortest path, and thus the chord $C(v, \beta)$ does not intersect S.
Let B be the region bounded by $C(v, \beta)$ that does not contain $C(v, \alpha)$. Since e is contained on $C(v, \alpha)$ and therefore not contained in B, if any of S were contained in B then at some point S would have to cross $C(v, \beta)$ to enter the region B. However this cannot happen

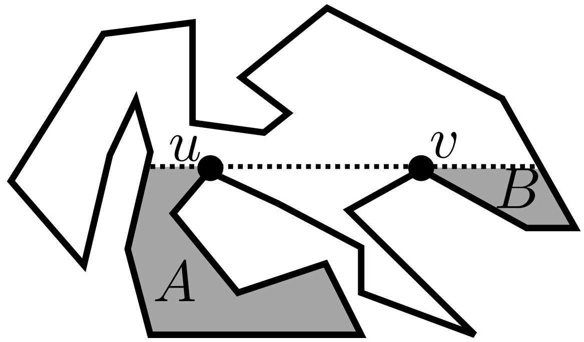


Figure 15: A representation of the proof described in Theorem 5.3.

since $C(v,\beta)$ does not intersect S. Therefore no portion of S is contained in B. Since B contains neither S nor $C(v,\alpha)$, both the point s and the chord $C(v,\alpha)$ lie in the same region induced by $C(v,\beta)$. Finally, since $\beta = \alpha \pm \delta$, s and $C(v,\alpha)$ are always contained in the same region induced by any chord and therefore s switches regions at α (by Definition 16). $\square$

Theorem 5.3. *For every pair of consecutive extreme vertices* $u, v \in V$

$$\theta(u,v) \notin ((\rho_1, \rho_1') \cup \ldots \cup (\rho_m, \rho_m'))$$

Proof. (by contradiction)

Assume that there exists some pair of consecutive extreme vertices $u, v \in V$ where $\theta(u,v) \in (\rho_1, \rho_1') \cup \ldots \cup (\rho_m, \rho_m')$. Let u be extreme w.r.t. $\alpha \in \mathcal{O}$ and v be extreme w.r.t. $\beta \in \mathcal{O}$. First, w.l.o.g. rotate the problem so that $\theta_{dir}(u,v) = 2\pi$. See Figure 15 for a diagram of the configuration described in this proof.

Let S be the shortest $\mathcal{O}$-path from s to t. By Lemma 5.1, $\overline{uv}$ must be contained entirely in P, which implies that $\overline{uv}$ must be contained in an end region induced by $C(u,\alpha)$ and $C(u,\alpha+\pi)$. Let $C = C(u,\alpha) \cup C(u,\alpha+\pi)$.

I will start by proving that neither $Pr_\pi(u)$ nor $Pr_{2\pi}(v)$ is degenerate.

If π is an exterior angle of u, then a line $\overline{up}$ where $\theta_{dir}(u,p) = 2\pi$ must lie in the middle region induced by any chord that induces two end regions. But that would mean that v lies in the middle region induced by $C(u,\alpha) \cup C(u,\alpha+\pi)$, which means v cannot be on S. Similarly, it can be shown that $Pr_{2\pi}(v)$ is non-degenerate.

Since both $u, v \in S$, v (u) must be in an end region induced by the α-chord (β-chord) through u (v). Let A be the end region induced by $C(u, \pi)$ and B be the end region induced by $C(v, 2\pi)$ (see Figure 15).

To ensure that v is in an end region induced by the α-chord through u, the line $\overline{uv}$ must be contained in an end region. This means that the entire chord $C(u, 2\pi)$ (and therefore the entire region B) will be contained in an end region induced by α. The more important consequence is that the other end region induced by α is a subset of A. Therefore A contains an end region induced by the α-chord through u. Similarly, B contains an end region induced by the β-chord through v. Since one of s, t lies in the end region induced by the α-chord through u, one of s, t lies in A. Similarly, since one of s, t lies in the end region induced by the β-chord through v, one of s, t lies in B. Since A and B are disjoint, it must be that one lies in A and the other in B. All that is left is to show that u changes extremity at $\theta(u, v)$. Since one of s or t lies in A, w.l.o.g. say s lies in A.

Consider R, the shortest arbitrarily oriented path from s to v, and let $\overline{xv}$ be the final segment of R that ends at v. By Lemma 5.2, s will switch regions at direction $\theta_{dir}(v, x)$. Since s lies in A, which is bounded by $C(u, \pi)$, the path R must intersect $C(u, \pi)$ to get out of A and reach v. Thus R must intersect $C(u, \pi)$ at some point x and $\overline{xv}$ will be the final segment on R. Therefore s switches regions at $\theta_{dir}(v, x)$ which is equal to π.

Since s switches regions at π, v will change extremity at π and therefore π must be one of the boundary orientations ρ_i or ρ'_i. Finally, this means that $\theta(u, v) \notin (\rho_1, \rho'_1) \cup \ldots \cup (\rho_m, \rho'_m)$. $\square$

The next lemma will show how to calculate the length of the shortest $\mathcal{O}$-path between two points assuming there are no obstacles/edges to avoid. This lemma will then be used to show how to calculate the length of the shortest $\mathcal{O}$-path between s and t.

Lemma 5.4.

$$\ell_{\mathcal{O}}(x, y) = \frac{\ell(\overline{xy})(|sin(\alpha - \theta(x, y))| + |sin(\beta - \theta(x, y))|)}{|sin(\beta - \alpha)|} \tag{2}$$

for $\alpha, \beta \in \mathcal{O}$ *which are the two neighbouring orientations of* $\theta(x, y)$*.*

Proof. (by cases)

First by Lemma 3.4, $\ell_{\mathcal{O}}(x, y) = \ell_{\alpha,\beta}(x, y)$ for the neighbouring orientations of $\theta(x, y)$. Let

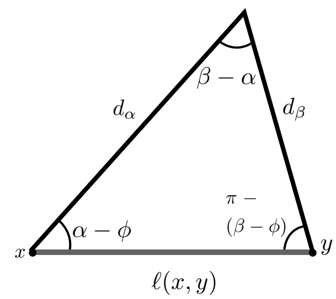


Figure 16: The configuration described in Lemma 5.4.

$\phi = \theta(x, y)$ and w.l.o.g. assume $\alpha < \beta$. By Figure 16 and the sin law,

$$\frac{d_\alpha}{sin(\pi - (\beta - \phi))} = \frac{d_\alpha}{sin(\beta - \phi)} = \frac{d_\beta}{sin(\alpha - \phi)} = \frac{\ell(\overline{xy})}{sin(\beta - \alpha)}$$

$$d_\alpha = \frac{\ell(\overline{xy}) sin(\beta - \phi)}{sin(\beta - \alpha)}, \quad d_\beta = \frac{\ell(\overline{xy}) sin(\alpha - \phi)}{sin(\beta - \alpha)}$$

And obviously the length of the lines d_α and d_β cannot be negative, so the total length of both line segments is

$$\begin{aligned} \ell_{\alpha,\beta}(x, y) &= d_\alpha + d_\beta \\ &= \left| \frac{\ell(\overline{xy}) sin(\beta - \phi)}{sin(\beta - \alpha)} \right| + \left| \frac{\ell(\overline{xy}) sin(\alpha - \phi)}{sin(\beta - \alpha)} \right| \\ &= \frac{\ell(\overline{xy})(|sin(\alpha - \phi)| + |sin(\beta - \phi)|)}{|sin(\beta - \alpha)|} \end{aligned}$$

□

Lemma 5.5. *Given an $s - t$ subproblem, the total length of the shortest $\mathcal{O}$-path between s and t is given by*

$$D(\mathcal{O}) = \sum_{(u,v) \in X_c} \left(\ell(\overline{uv}) \frac{|sin(\theta_i - \theta(u, v))| + |sin(\theta_j - \theta(u, v))|}{|sin(\theta_i - \theta_j)|} \right)$$

where for each $(u, v) \in X_c$, the orientations θ_i, θ_j are $\theta(u, v)$'s neighbouring orientations. Furthermore, $D(\mathcal{O})$ is a continuous, differentiable function.

Proof. (direct)

By Theorem 3.7 there is only one order in which the vertices in X are traversed, and by Lemma 3.16 the path between each $(u,v) \in X_c$ is a staircase path, so $\sum_{(u,v)\in X_c} \ell_{\mathcal{O}}(u,v)$ is equal to the length of the shortest $\mathcal{O}$-path from s to t. Now I will show that even though $D(\mathcal{O})$ includes absolute values, it is still a differentiable function.

By Lemma 5.3, $\theta(u,v) \notin (\rho_i, \rho'_i) \cup \ldots \cup (\rho_m, \rho'_m)$ and therefore $sin(\theta_i - \theta(u,v))$ and $sin(\theta_j - \theta(u,v))$ cannot change sign. Furthermore $\theta_i < \theta_j \Rightarrow sin(\theta_i - \theta_j) < 0$.

Each $(u,v) \in X_c$ falls into one of three cases:

1. $\theta_i < \theta(u,v) < \theta_j$

$$sin(\theta_i - \theta(u,v)) < 0 \qquad\qquad sin(\theta_j - \theta(u,v)) > 0$$

$$\Rightarrow \ell_{\mathcal{O}}(u,v) = \ell(\overline{uv}) \frac{-sin(\theta_i - \theta(u,v)) + sin(\theta_j - \theta(u,v))}{-sin(\theta_i - \theta_j)}$$

2. $\theta(u,v) < \theta_i$

$$sin(\theta_i - \theta(u,v)) > 0 \qquad\qquad sin(\theta_j - \theta(u,v)) > 0$$

$$\Rightarrow \ell_{\mathcal{O}}(u,v) = \ell(\overline{uv}) \frac{sin(\theta_i - \theta(u,v)) + sin(\theta_j - \theta(u,v))}{-sin(\theta_i - \theta_j)}$$

3. $\theta(u,v) > \theta_j$

$$sin(\theta_i - \theta(u,v)) < 0 \qquad\qquad sin(\theta_j - \theta(u,v)) < 0$$

$$\Rightarrow \ell_{\mathcal{O}}(u,v) = \ell(\overline{uv}) \frac{sin(\theta_i - \theta(u,v)) + sin(\theta_j - \theta(u,v))}{sin(\theta_i - \theta_j)}$$

Therefore we can write the distance equation as one of the following

$$\ell_{\mathcal{O}}(u,v) = \begin{cases} \ell(\overline{uv}) \frac{sin(\theta_i - \theta(u,v)) + sin(\theta_j - \theta(u,v))}{sin(\theta_j - \theta_i)} & \text{if } \theta(u,v) < \theta_i < \theta_j \\ \ell(\overline{uv}) \frac{-sin(\theta_i - \theta(u,v)) + sin(\theta_j - \theta(u,v))}{sin(\theta_j - \theta_i)} & \text{if } \theta_i < \theta(u,v) < \theta_j \\ -\ell(\overline{uv}) \frac{sin(\theta_i - \theta(u,v)) + sin(\theta_j - \theta(u,v))}{sin(\theta_j - \theta_i)} & \text{if } \theta_i < \theta_j < \theta(u,v) \end{cases}$$

Thus the distance $D(\mathcal{O})$ is a sum of continuous, differentiable functions and so must be continuous and differentiable as well. $\square$

6 Minimizing Length for Each Extreme Sequence

In the previous two sections we have shown how to divide our problem into subproblems, and for each subproblem we have a continuous, differentiable function $D(\mathcal{O})$ that gives the distance of the shortest $\mathcal{O}$-path between s and t. This section focuses on the problem allowing two orientations. Just as in the previous section, we are working within the confines of an $s-t$ subproblem, meaning we have a set X of all extreme vertices and two orientations θ_1, θ_2 where $\theta_1 \in (\rho_1, \rho_1')$ and $\theta_2 \in (\rho_2, \rho_2')$.

This section aims to prove that within an $s-t$ subproblem the function $D(\mathcal{O})$ has at most one minimum for $\theta_1 \in (\rho_1, \rho_1')$ and $\theta_2 \in (\rho_2, \rho_2')$. This means we can find that minimum in a finite amount of time, then check the boundary cases where θ_1 is equal to ρ_1 or ρ_1', or θ_2 is equal to ρ_2 or ρ_2'. When discussing an $s-t$ subproblem, *solving* the subproblem refers to finding the set $\mathcal{O}^* = \{\theta_1^*, \theta_2^*\}$ where $\theta_1^* \in (\rho_1, \rho_1')$ and $\theta_2^* \in (\rho_2, \rho_2')$ for which $D(\mathcal{O}^*)$ is as small as possible.

First, it is shown that given the set of extreme vertices in the subproblem, the problem can be represented by two vectors $\vec{L_1}$ and $\vec{L_2}$ so that the length of the shortest (θ_1, θ_2)-path from s to t is equal to the sum of the θ_1 and θ_2 components of $\vec{L_1}$ and $\vec{L_2}$. After the two vectors are found, it is shown the distance function - that is, the sum described above - contains at most one minimum. Since there is a single minimum, it is clear that the pair θ_1^*, θ_2^* which minimizes the distance function can be found to any degree of desired accuracy.

Let X_c represent the set of pairs of consecutive extreme vertices (including s and t), and to simplify the proofs, the problem is rotated so that $\rho_2' = \pi$.

6.1 General Mathematical Lemmas

The following section will give a few mathematical lemmas that will later be used to prove there is at most one minimum in an $s-t$ subproblem. The full proofs of these lemmas are given in the appendix.

Lemma 6.1. *For edges* $e_1 = (u, v)$ *and* $e_2 = (w, z)$ *where* $0 < \theta_{dir}(u, v) \leq \theta_{dir}(w, z) \leq \pi$*,*

$$\ell(\overline{uv})sin(x - \theta(u, v)) + \ell(\overline{wz})sin(x - \theta(w, z)) = \sqrt{S_v^2 + S_h^2}sin\left(x - tan^{-1}\left(\frac{S_v}{S_h}\right)\right)$$

where

$$S_v = \ell(\overline{uv})sin(\theta(u, v)) + \ell(\overline{wz})sin(\theta(w, z))$$

$$S_h = \ell(\overline{uv})cos(\theta(u, v)) + \ell(\overline{wz})cos(\theta(w, z))$$

and $tan^{-1}\left(\frac{S_v}{S_h}\right)$ *is contained in the interval* $[\theta_{dir}(u, v), \theta_{dir}(w, z)]$.

Lemma 6.2.

$$\begin{aligned}&\frac{d}{d\theta_1}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right)\\ &\quad = \frac{l_1 sin(\theta_2 - \phi_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-l_2 sin(\theta_2 - \phi_2)}{1 + cos(\theta_2 - \theta_1)}\end{aligned}$$

Lemma 6.3.

$$\begin{aligned}&\frac{d}{d\theta_2}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right)\\ &\quad = \frac{-l_1 sin(\theta_1 - \phi_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-l_2 sin(\theta_1 - \phi_2)}{1 + cos(\theta_2 - \theta_1)}\end{aligned}$$

Lemma 6.4.

$$\frac{d}{d\varphi}\left(\frac{l_1 sin(\varphi - \phi_1) \pm l_2 sin(\varphi - \phi_2)}{l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2)}\right) = \frac{\pm 2 l_1 l_2 sin(\phi_2 - \phi_1)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2}$$

6.2 Simplifying the Subproblem and Minimizing Distance

The idea of this section is the following: take all pairs of consecutive extreme vertices and sort them into two sets based on the orientation between the pair of vertices. One set will be all orientations in $(0, \rho_1)$ while the other will be all orientations in (ρ_1', ρ_2), and these sets will be referred to as X_a and X_b, respectively.

$$X_a = \{(u, v) \in X_c \text{ s.t. } \theta(u, v) \in (0, \rho_1]\}$$
$$X_b = \{(u, v) \in X_c \text{ s.t. } \theta(u, v) \in [\rho_1', \rho_2]\}$$

We treat each pair of vertices as a vector and for each set we calculate the vector sum of the pairs. Thus we create two representative vectors $\vec{L_1}$ and $\vec{L_2}$. See Figure 17 for a visualization of what will be done.

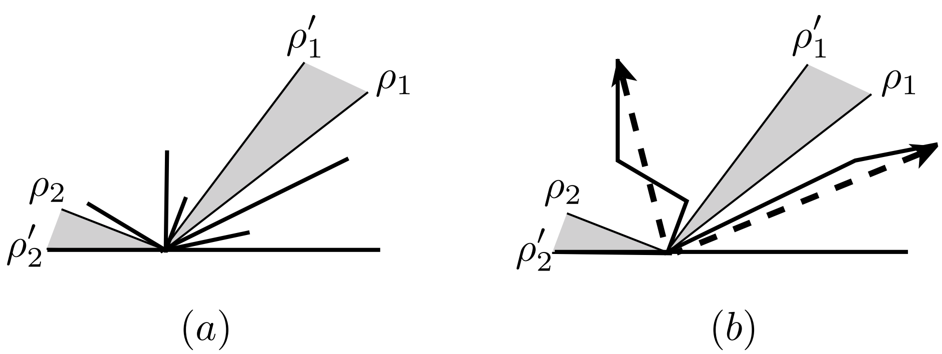


Figure 17: The solid lines shown in (a) represent the lines $\overline{uv}$ where u and v are consecutive extreme vertices. The dotted lines in (b) are the vectors that are created in this section, which are the vector sums of each set.

Lemma 6.5. *If we are given a set of consecutive extreme vertices Y where all pairs of vertices $(u, v) \in Y$ have orientations in the interval $(\alpha, \beta) \subset (0, \pi]$, then*

$$\sum_{(u,v) \in Y} \ell_{\mathcal{O}}(u, v) sin(x - \theta(u, v)) = l sin(x - \phi)$$

for some length l and orientation $\phi \in (\alpha, \beta)$.

Proof. (by construction)

First, for any $(u, v) \in Y$ where $\theta_{dir}(u, v) > \pi$, reverse the order of the pair so that (v, u)

is contained in Y and (u, v) is not. Note that this transformation will not change either $\ell_{\mathcal{O}}(u, v)$ or $\theta(u, v)$, so will not change the sum $\sum_{(u,v)\in Y} \ell_{\mathcal{O}}(u, v)sin(x - \theta(u, v))$. Now by Lemma 6.1 we can calculate the sum by adding two terms at a time, and still maintain that the resultant orientation ϕ is within (α, β). $\square$

To simplify the proof of Theorem 6.6, for some set of pairs of points Y, let $V(Y)$ denote the vertical distance spanned by all the pairs in Y and let $H(Y)$ denote the horizontal distance spanned by all the pairs in Y.

$$V(Y) = \sum_{(u,v)\in Y} \ell(\overline{uv})sin(\theta(u, v)) \qquad H(Y) = \sum_{(u,v)\in Y} \ell(\overline{uv})cos(\theta(u, v))$$

Theorem 6.6. *The distance of the shortest* (θ_1, θ_2)*-path from* s *to* t*,* $D(\theta_1, \theta_2)$*, is given by*

$$D(\theta_1, \theta_2) = \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)}$$

for constants $l_1, l_2 \geq 0$*,* $\phi_1 \in (0, \rho_1)$ *and* $\phi_2 \in (\rho_1', \rho_2)$ *where*

$$l_1 = \sqrt{V(X_a)^2 + H(X_a)^2} \qquad \phi_1 = tan^{-1}\left(\frac{V(X_a)}{H(X_a)}\right)$$

$$l_2 = \sqrt{V(X_b)^2 + H(X_b)^2} \qquad \phi_2 = tan^{-1}\left(\frac{V(X_b)}{H(X_b)}\right)$$

Proof. By Theorem 5.3, $\forall (u, v) \in X_c$, $\theta(u, v) \notin ((\rho_1, \rho_1') \cup (\rho_2, \rho_2'))$ and therefore $X_a \cup X_b = X_c$. As shown in Lemma 5.5, the distance of the shortest (θ_1, θ_2)-path in P from s to t is given by

$$\begin{aligned}
&\ell_{\theta_1,\theta_2}(X_c) \\
&\quad = \sum_{(u,v)\in X_c} \left(\ell(\overline{uv})\frac{|sin(\theta_1 - \theta(u, v))| + |sin(\theta_2 - \theta(u, v))|}{sin(\theta_2 - \theta_1)}\right) \\
&\quad = \sum_{(u,v)\in X_a} \left(\ell(\overline{uv})\frac{|sin(\theta_1 - \theta(u, v))| + |sin(\theta_2 - \theta(u, v))|}{sin(\theta_2 - \theta_1)}\right) \\
&\qquad + \sum_{(u,v)\in X_b} \left(\ell(\overline{uv})\frac{|sin(\theta_1 - \theta(u, v))| + |sin(\theta_2 - \theta(u, v))|}{sin(\theta_2 - \theta_1)}\right) \\
&\quad = \sum_{(u,v)\in X_a} \left(\ell(\overline{uv})\frac{sin(\theta_1 - \theta(u, v)) + sin(\theta_2 - \theta(u, v))}{sin(\theta_2 - \theta_1)}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(u,v)\in X_b} \left(\ell(\overline{uv}) \frac{-sin(\theta_1 - \theta(u,v)) + sin(\theta_2 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) \\
&= \sum_{(u,v)\in X_a} \left(\frac{\ell(\overline{uv}) sin(\theta_1 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) + \sum_{(u,v)\in X_a} \left(\frac{\ell(\overline{uv}) sin(\theta_2 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) \\
&- \sum_{(u,v)\in X_b} \left(\frac{\ell(\overline{uv}) sin(\theta_1 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) + \sum_{(u,v)\in X_b} \left(\frac{\ell(\overline{uv}) sin(\theta_2 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) \\
&= \frac{1}{sin(\theta_2 - \theta_1)} \left(\sum_{(u,v)\in X_a} \ell(\overline{uv}) sin(\theta_1 - \theta(u,v)) + \sum_{(u,v)\in X_a} \ell(\overline{uv}) sin(\theta_2 - \theta(u,v)) \right. \\
&\left. - \sum_{(u,v)\in X_b} \ell(\overline{uv}) sin(\theta_1 - \theta(u,v)) + \sum_{(u,v)\in X_b} \ell(\overline{uv}) sin(\theta_2 - \theta(u,v)) \right) \\
&= \left(\frac{1}{sin(\theta_2 - \theta_1)} \right) (l_1 sin(\theta_1 - \phi_1) + l_1 sin(\theta_2 - \phi_1) - l_2 sin(\theta_1 - \phi_2) + l_2 sin(\theta_2 - \phi_2)) \\
&\text{(by Lemma 6.5)} \\
&= \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)}
\end{aligned}
$$

where by Lemma 6.5, $\phi_1 \in (0, \rho_1]$, $\phi_2 \in [\rho_1', \rho_2]$ and

$$
\begin{aligned}
l_1 &= \sqrt{V(X_a)^2 + H(X_a)^2} & \phi_1 &= tan^{-1}\left(\frac{V(X_a)}{H(X_a)}\right) \\
l_2 &= \sqrt{V(X_b)^2 + H(X_b)^2} & \phi_2 &= tan^{-1}\left(\frac{V(X_b)}{H(X_b)}\right)
\end{aligned}
$$

□

Now that the distance function is simplified, the aim is to find the two orientations θ_1^*, θ_2^* where $\theta_1^* \in (\rho_1, \rho_1')$ and $\theta_2^* \in (\rho_2, \rho_2')$ which minimize $D(\theta_1, \theta_2)$. It will be shown that there exists at most one pair θ_1^*, θ_2^* which minimizes this distance function, meaning a hill-climbing algorithm can find such a minimum (if it exists) to any desired degree of accuracy. To prove there is at most one minimum, it will be shown that there is at most one place where both the θ_1-derivative and the θ_2-derivative of $D(\theta_1, \theta_2)$ are zero. Note that a place where both derivatives are zero is a necessary but not sufficient condition for a minimum.

Let us define the following functions which will be used in the remainder of this section

$$h_1(\theta_1) = \frac{l_1 sin(\theta_1 - \phi_1) + l_2 sin(\theta_1 - \phi_2)}{-l_1 sin(\theta_1 - \phi_1) + l_2 sin(\theta_1 - \phi_2)}$$
$$h_2(\theta_2) = \frac{-l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}$$

Lemma 6.7. *The distance $D(\theta_1, \theta_2)$ can be minimized only if the following equations are both satisfied (i.e. if they intersect)*

$$\theta_2 = f(\theta_1) = \theta_1 + cos^{-1}\left(\frac{l_1 sin(\theta_1 - \phi_1) + l_2 sin(\theta_1 - \phi_2)}{-l_1 sin(\theta_1 - \phi_1) + l_2 sin(\theta_1 - \phi_2)}\right)$$
$$\theta_1 = g(\theta_2) = \theta_2 - cos^{-1}\left(\frac{-l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}\right)$$

Proof. (direct)

By Theorem 6.6, the distance of the two lines is given by

$$D(\theta_1, \theta_2) = \frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}$$

If $D(\theta_1, \theta_2)$ is minimized then it must be true that $\frac{d}{d\theta_1} D(\theta_1, \theta_2) = \frac{d}{d\theta_2} D(\theta_1, \theta_2) = 0$. For ease of notation let $L(\theta) = l_1 sin(\theta - \phi_1)$ and $M(\theta) = l_2 sin(\theta - \phi_2)$. By Lemmas 6.2 and 6.3

$$\frac{d}{d\theta_1} D(\theta_1, \theta_2) = \frac{L(\theta_2)}{1 - cos(\theta_2 - \theta_1)} + \frac{-M(\theta_2)}{1 + cos(\theta_2 - \theta_1)}$$
$$\frac{d}{d\theta_2} D(\theta_1, \theta_2) = \frac{-L(\theta_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-M(\theta_1)}{1 + cos(\theta_2 - \theta_1)}$$

By setting the derivatives to zero and rearranging,

$$\frac{L(\theta_2)}{1 - cos(\theta_2 - \theta_1)} = \frac{M(\theta_2)}{1 + cos(\theta_2 - \theta_1)}$$
$$cos(\theta_2 - \theta_1) = \frac{-L(\theta_2) + M(\theta_2)}{L(\theta_2) + M(\theta_2)}$$

$$\frac{L(\theta_1)}{1 - cos(\theta_2 - \theta_1)} = \frac{-M(\theta_1)}{1 + cos(\theta_2 - \theta_1)}$$
$$cos(\theta_2 - \theta_1) = \frac{L(\theta_1) + M(\theta_1)}{-L(\theta_1) + M(\theta_1)}$$

Finally we rearrange again to get

$$\theta_2 - \theta_1 = cos^{-1}\left(\frac{-L(\theta_2) + M(\theta_2)}{L(\theta_2) + M(\theta_2)}\right)$$
$$\theta_1 = \theta_2 - cos^{-1}\left(\frac{-L(\theta_2) + M(\theta_2)}{L(\theta_2) + M(\theta_2)}\right)$$

$$\theta_1 = \theta_2 + cos^{-1}\left(\frac{-l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2)}{l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2)}\right)$$

$$\begin{aligned}
\theta_2-\theta_1 &= cos^{-1}\left(\frac{L(\theta_1)+M(\theta_1)}{-L(\theta_1)+M(\theta_1)}\right)\\
\theta_2 &= \theta_1 + cos^{-1}\left(\frac{L(\theta_1)+M(\theta_1)}{-L(\theta_1)+M(\theta_1)}\right)\\
\theta_2 &= \theta_1 + cos^{-1}\left(\frac{l_1 sin(\theta_1-\phi_1)+l_2 sin(\theta_1-\phi_2)}{-l_1 sin(\theta_1-\phi_1)+l_2 sin(\theta_1-\phi_2)}\right)
\end{aligned}$$

□

Lemma 6.8. *The function $\theta_2 = f(\theta_1) = \theta_1 + arccos(h_1(\theta_1))$ has θ_1-derivative > 1, and the function $\theta_1 = g(\theta_2) = \theta_2 - arccos(h_2(\theta_2))$ has θ_1-derivative < 1.*

Proof.

$$\begin{aligned}
\frac{d}{d\theta_1} f(\theta_1) &= \frac{d}{d\theta_1}\left(\theta_1 + arccos(h_1(\theta_1))\right)\\
&= 1 + \frac{-1}{\sqrt{1-h_1(\theta_1)^2}} * \frac{d}{d\theta_2} h_1(\theta_1)\\
&= 1 + \underbrace{\left(\frac{-1}{\sqrt{1-h_1(\theta_1)^2}}\right)}_{<0} * \underbrace{\left(\frac{-2l_1 l_2 sin(\phi_2-\phi_1)}{(l_1 sin(\theta_1-\phi_1)-l_2 sin(\theta_1-\phi_2))^2}\right)}_{<0}\\
&> 1
\end{aligned}$$

To find the θ_1-derivative of $g(\theta_2)$, first we will find its θ_2-derivative.

$$\begin{aligned}
\frac{d}{d\theta_2} g(\theta_2) &= \frac{d}{d\theta_2}\left(\theta_2 - arccos(h_2(\theta_2))\right)\\
&= 1 - \frac{-1}{\sqrt{1-h_2(\theta_2)^2}} * \frac{d}{d\theta_2}(h_2(\theta_2))\\
&= 1 - \underbrace{\left(\frac{-1}{\sqrt{1-h_2(\theta_2)^2}}\right)}_{<0} * \underbrace{\left(\frac{2l_1 l_2 sin(\phi_2-\phi_1)}{(l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2))^2}\right)}_{>0}\\
&> 1
\end{aligned}$$

Now define the following inverse function

$$\theta_1 = g(\theta_2) \Rightarrow \theta_2 = g^{-1}(\theta_1)$$

The θ_1-derivative of the line, $\frac{d}{d\theta_1}g^{-1}(\theta_1)$, is equal to the inverse of $\frac{d}{d\theta_2}g(\theta_2)$, and $\frac{d}{d\theta_2}g(\theta_2) > 1 \Rightarrow \frac{d}{d\theta_1}g^{-1}(\theta_1) < 1$. □

Definition 17. *A function $f(x)$ over domain X is said to be* 1-to-1 *if no horizontal or vertical line intersects $f(x)$ more than once.*

Lemma 6.9. *The functions*

$$\theta_2 = f(\theta_1) = \theta_1 + arccos(h_1(\theta_1))$$
$$\theta_1 = g(\theta_2) = \theta_2 - arccos(h_2(\theta_2))$$

intersect at most once.

Proof. (direct)
First, I will show that for $\theta_1 \in (\rho_1, \rho_1')$ and $\theta_2 \in (\rho_2, \rho_2')$ both $h_1(\theta_1)$ and $h_2(\theta_2)$ are continuous, and have ranges contained in $[-1, 1]$.
The functions $h_1(\theta_1)$ and $h_2(\theta_2)$ are made up of continuous functions, so are both continuous themselves except at the series of points where the denominator is zero. The denominator of $h_1(\theta_1)$ is

$$\underbrace{-l_1 sin(\theta_1 - \phi_1)}_{<0} + \underbrace{l_2 sin(\theta_1 - \phi_2)}_{<0}$$

and the denominator of $h_2(\theta_2)$ is

$$\underbrace{l_1 sin(\theta_2 - \phi_1)}_{>0} + \underbrace{l_2 sin(\theta_2 - \phi_2)}_{>0}$$

Therefore neither denominator is ever 0, so $h_1(\theta_1)$ and $h_2(\theta_2)$ both are continuous.
By lemma 6.4,

$$\frac{d}{d\theta_1}h_1(\theta_1) = \frac{-2l_1 l_2 sin(\phi_2 - \phi_1)}{(l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2))^2} < 0$$
$$\frac{d}{d\theta_2}h_2(\theta_2) = \frac{2l_1 l_2 sin(\phi_2 - \phi_1)}{(l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2))^2} > 0$$

meaning both derivatives are of constant sign.

The derivative of $h_1(\theta_1)$ is negative so its range is $(h_1(\rho_1'), h_1(\rho_1))$, and since the interval (ρ_1, ρ_1') is contained within (ϕ_1, ϕ_2), the range of $h_1(\theta_1)$ is contained within $(h_1(\phi_2), h_1(\phi_1))$. The derivative of $h_2(\theta_2)$ is positive so its range will be $(h_2(\phi_2), h_2(\pi))$, and since the interval (ρ_2, ρ_2') is contained within (ϕ_2, ϕ_1), the range of $h_2(\theta_2)$ is contained within $(h_2(\phi_2), h_2(\phi_1))$. By evaluating the functions at ϕ_1 and ϕ_2 we get

$$h_1(\phi_2) = -1 \qquad\qquad h_2(\phi_2) = -1$$
$$h_1(\phi_1) = 1 \qquad\qquad h_2(\phi_1) = 1$$

so both ranges are contained within $(-1, 1)$. Since $arccos(x)$ is continuous for $x \in (-1, 1)$, $arccos(h_1(\theta_1))$ and $arccos(h_2(\theta_2))$ are both continuous.

The value of $\theta_2 - \theta_1$ is always in the interval $(0, \pi)$, so

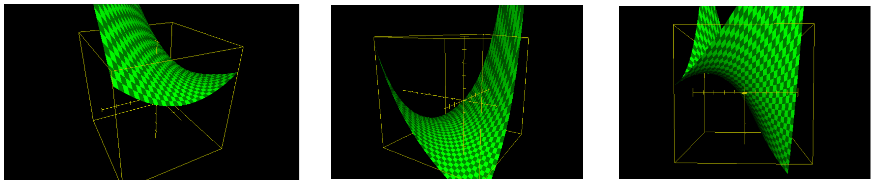
$$\theta_2 = \theta_1 + arccos(h_1(\theta_1)) \qquad\qquad \theta_1 = \theta_2 - arccos(h_2(\theta_2))$$
$$\underbrace{\theta_2 - \theta_1}_{\in(0,\pi)} = arccos(h_1(\theta_1)) \qquad\qquad \theta_1 - \theta_2 = -arccos(h_2(\theta_2))$$
$$\underbrace{\theta_2 - \theta_1}_{\in(0,\pi)} = arccos(h_2(\theta_2))$$

which means the values of $arccos(h_1(\theta_1))$ and $arccos(h_2(\theta_2))$ are always in the interval $(0, \pi)$. The function $cos^{-1}(x)$ is 1-to-1 for domain $(-1, 1)$ and range $(0, \pi)$, so both $arccos(h_1(\theta_1))$ and $arccos(h_2(\theta_2))$ are 1-to-1.

Finally, given that $arccos(h_1(\theta_1))$ and $arccos(h_2(\theta_2))$ are continuous and 1-to-1, it is clear that $f(\theta_1) = \theta_1 + arccos(h_1(\theta_1))$ and $g(\theta_2) = \theta_2 - arccos(h_2(\theta_2))$ are continuous and 1-to-1. As shown by Lemma 6.8, the θ_1-derivative of $f(\theta_1)$ is always greater than 1 and the θ_1-derivative of $g(\theta_2)$ is always less than 1. Consider a point at which $f(\theta_1)$ and $g(\theta_2)$ intersect. Since the θ_1-derivative of $f(\theta_1)$ is always greater than the θ_1-derivative of $g(\theta_2)$, the functions cannot intersect again. □

Theorem 6.10. *Given an $s - t$ subproblem for two allowable orientations, there is at most one pair of orientations $\theta_1 \in (\rho_1, \rho_1')$ and $\theta_2 \in (\rho_2, \rho_2')$ that minimizes $D(\theta_1, \theta_2)$.*

Proof. (direct)



$l_1 = 7$ $l_2 = 5$ $\phi_1 = 57°$ $\phi_2 = 103°$

$l_1 = 3$ $l_2 = 5$ $\phi_1 = 34°$ $\phi_2 = 120°$

$l_1 = 0.1$ $l_2 = 20$ $\phi_1 = 74.5°$ $\phi_2 = 163°$

Figure 18: The figures above show the graphs of three distance functions and the "minimum" point as found by setting both derivatives to 0.

An $s - t$ subproblem where $m = 2$ contains a multiset of two ranges $\theta_1 \in (\rho_1, \rho'_1)$ and $\theta_2 \in (\rho_2, \rho'_2)$. Lemmas 6.7 and 6.9 prove the theorem. □

While it has just been shown that there exists at most one pair of orientations where $D(\theta_1, \theta_2)$ is minimized, but we have not eliminated the possibility that there is no minimum. The following conjecture predicts such an outcome.

Conjecture 1. *FIX! Given an $s - t$ subproblem for two allowable orientations, there is no pair of orientations $\theta_1 \in (\rho_1, \rho'_1)$ and $\theta_2 \in (\rho_2, \rho'_2)$ that minimizes $D(\theta_1, \theta_2)$.*

In Theorem 6.10 I was able to prove there is at most one minimum by showing that there is at most one place where both partial derivatives of the distance function are 0. However, both partial derivatives being 0 is a necessary but not sufficient condition of a minimum and thus there may not be a minimum at the point found in this section.

When graphing the distance function, the point at which both partial derivatives are 0 looks to be a saddle point - that is, the point where both partial derivatives are 0 is not a minimum. Figure 18 shows three such examples, and is zoomed in to highlight the saddle point.

6.3 More Than Two Allowable Orientations

For three or more allowable orientations, it will be shown that there are m equations of three variables each that must intersect if there is a minimum at such a point. If these equations can be shown to intersect in some predictable way an algorithm can be developed to find the shortest $\mathcal{O}$-path for three or more orientations. This is an open problem in need of further research.

In this subsection there are m orientations $\theta_1, \ldots, \theta_m$ where $\theta_i \in [\rho_i, \rho_i']$ and $\theta_1 < \ldots < \theta_m$.

Lemma 6.11.

$$\frac{d}{d\theta_i} D(\mathcal{O}) = \frac{l_i sin(\theta_{i-1} - \phi_i)}{1 + cos(\theta_i - \theta_{i-1})} + \frac{l_{i+1} sin(\theta_{i+1} - \phi_{i+1})}{1 + cos(\theta_{i+1} - \theta_i)}$$

Proof. Let

$$\begin{aligned}
X_1 &= \{(u,v) \text{ s.t. } \theta(u,v) \in (0, \rho_1)\} \\
X_2 &= \{(u,v) \text{ s.t. } \theta(u,v) \in (\rho_1', \rho_2)\} \\
&\ldots \\
X_m &= \{(u,v) \text{ s.t. } \theta(u,v) \in (\rho_{m-1}', \rho_m)\}
\end{aligned}$$

By Theorem 5.3, for all consecutive extreme vertices $(u,v) \in X_c$, $\theta(u,v) \notin ((\rho_1, \rho_1') \cup \ldots \cup (\rho_m, \rho_m'))$ and therefore $X_1 \cup \ldots \cup X_m = X_c$. Thus the distance of the shortest $\mathcal{O}$-path from s to t is given by

$$\begin{aligned}
D(\mathcal{O}) &= \sum_{(u,v)\in X_c} \left(\ell(\overline{uv}) \frac{|sin(\theta_i - \theta(u,v))| + |sin(\theta_{i+1} - \theta(u,v))|}{sin(\theta_{i+1} - \theta_i)} \right) \quad \text{(by Lemma 5.5)} \\
&= \sum_{(u,v)\in X_1} \left(\ell(\overline{uv}) \frac{|sin(\theta_1 - \theta(u,v))| + |sin(\theta_m - \theta(u,v))|}{sin(\theta_1 - \theta_m)} \right) \\
&\quad + \sum_{(u,v)\in X_2} \left(\ell(\overline{uv}) \frac{|sin(\theta_2 - \theta(u,v))| + |sin(\theta_1 - \theta(u,v))|}{sin(\theta_2 - \theta_1)} \right) \\
&\quad + \ldots + \sum_{(u,v)\in X_m} \left(\ell(\overline{uv}) \frac{|sin(\theta_m - \theta(u,v))| + |sin(\theta_{m-1} - \theta(u,v))|}{sin(\theta_m - \theta_{m-1})} \right) \\
&= \sum_{(u,v)\in X_1} \left(\ell(\overline{uv}) \frac{-sin(\theta_1 - \theta(u,v)) - sin(\theta_m - \theta(u,v))}{sin(\theta_1 - \theta_m)} \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(u,v)\in X_2} \left(\ell(\overline{uv}) \frac{sin(\theta_2 - \theta(u,v)) - sin(\theta_1 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) \\
&+ \ldots + \sum_{(u,v)\in X_m} \left(\ell(\overline{uv}) \frac{sin(\theta_m - \theta(u,v)) - sin(\theta_{m-1} - \theta(u,v))}{sin(\theta_m - \theta_{m-1})} \right) \\
= &- \sum_{(u,v)\in X_1} \left(\frac{\ell(\overline{uv}) sin(\theta_1 - \theta(u,v))}{sin(\theta_1 - \theta_m)} \right) - \sum_{(u,v)\in X_1} \left(\frac{\ell(\overline{uv}) sin(\theta_m - \theta(u,v))}{sin(\theta_1 - \theta_m)} \right) \\
&+ \sum_{(u,v)\in X_2} \left(\frac{\ell(\overline{uv}) sin(\theta_2 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) - \sum_{(u,v)\in X_2} \left(\frac{\ell(\overline{uv}) sin(\theta_1 - \theta(u,v))}{sin(\theta_2 - \theta_1)} \right) + \ldots \\
&+ \sum_{(u,v)\in X_m} \left(\frac{\ell(\overline{uv}) sin(\theta_m - \theta(u,v))}{sin(\theta_m - \theta_{m-1})} \right) - \sum_{(u,v)\in X_m} \left(\frac{\ell(\overline{uv}) sin(\theta_{m-1} - \theta(u,v))}{sin(\theta_m - \theta_{m-1})} \right) \\
= &- \frac{l_1 sin(\theta_1 - \phi_1)}{sin(\theta_1 - \theta_m)} - \frac{l_1 sin(\theta_m - \phi_1)}{sin(\theta_1 - \theta_m)} + \frac{l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} - \frac{l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)} \\
&+ \ldots + \frac{l_m sin(\theta_m - \phi_m)}{sin(\theta_m - \theta_{m-1})} - \frac{l_m sin(\theta_{m-1} - \phi_m)}{sin(\theta_m - \theta_{m-1})} \quad \text{(by Lemma 6.5)} \\
= &\frac{-l_1 sin(\theta_1 - \phi_1) - l_1 sin(\theta_m - \phi_1)}{sin(\theta_1 - \theta_m)} + \frac{l_2 sin(\theta_2 - \phi_2) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)} \\
&+ \ldots + \frac{l_m sin(\theta_m - \phi_m) - l_m sin(\theta_{m-1} - \phi_m)}{sin(\theta_m - \theta_{m-1})}
\end{aligned}
$$

where by Lemma 6.5 $\phi_1 \in (0, \rho_1), \phi_2 \in (\rho'_1, \rho_2), \ldots, \phi_m \in (\rho'_{m-1}, \rho_m)$.

$$
\begin{aligned}
\frac{d}{d\theta_i} D(\mathcal{O}) = &\frac{d}{d\theta_i} \left(\frac{-l_1 sin(\theta_1 - \phi_1) - l_1 sin(\theta_m - \phi_1)}{sin(\theta_1 - \theta_m)} + \ldots \right. \\
&+ \frac{l_i sin(\theta_i - \phi_i) - l_i sin(\theta_{i-1} - \phi_i)}{sin(\theta_i - \theta_{i-1})} + \frac{l_{i+1} sin(\theta_{i+1} - \phi_{i+1}) - l_{i+1} sin(\theta_i - \phi_{i+1})}{sin(\theta_{i+1} - \theta_i)} \\
&\left. + \ldots + \frac{l_m sin(\theta_m - \phi_m) - l_m sin(\theta_{m-1} - \phi_m)}{sin(\theta_m - \theta_{m-1})} \right) \\
= &\frac{d}{d\theta_i} \left(\frac{l_i sin(\theta_i - \phi_i) - l_i sin(\theta_{i-1} - \phi_i)}{sin(\theta_i - \theta_{i-1})} \right. \\
&\left. + \frac{l_{i+1} sin(\theta_{i+1} - \phi_{i+1}) - l_{i+1} sin(\theta_i - \phi_{i+1})}{sin(\theta_{i+1} - \theta_i)} \right) \\
= &\frac{cos(\theta_i - \theta_{i-1})(l_i sin(\theta_i - \phi_i) - l_i sin(\theta_{i-1} - \phi_i))}{sin^2(\theta_i - \theta_{i-1})} + \frac{-l_i cos(\theta_i - \phi_i)}{sin(\theta_i - \theta_{i-1})} \\
&+ \frac{-cos(\theta_{i+1} - \theta_i)(l_{i+1} sin(\theta_{i+1} - \phi_{i+1}) - l_{i+1} sin(\theta_i - \phi_{i+1}))}{sin^2(\theta_{i+1} - \theta_i)} \\
&+ \frac{-l_{i+1}(-cos(\theta_i - \phi_{i+1}))}{sin(\theta_{i+1} - \theta_i)} \\
= &\frac{-l_i cos(\theta_i - \phi_i) sin(\theta_i - \theta_{i-1})) + l_i cos(\theta_i - \theta_{i-1}) sin(\theta_i - \phi_i)}{sin^2(\theta_i - \theta_{i-1})} \\
&- \frac{l_i cos(\theta_i - \theta_{i-1}) sin(\theta_{i-1} - \phi_i))}{sin^2(\theta_i - \theta_{i-1})}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{l_{i+1}sin(\theta_{i+1}-\theta_i)cos(\theta_i-\phi_{i+1})-l_{i+1}cos(\theta_{i+1}-\theta_i)sin(\theta_{i+1}-\phi_{i+1})}{sin^2(\theta_{i+1}-\theta_i)}\\
&+\frac{l_{i+1}sin(\theta_i-\phi_{i+1})cos(\theta_{i+1}-\theta_i)}{sin^2(\theta_{i+1}-\theta_i)}\\
=&\frac{l_isin(\theta_i-\phi_i-\theta_i+\theta_{i-1})-l_icos(\theta_i-\theta_{i-1})sin(\theta_{i-1}-\phi_i)}{sin^2(\theta_i-\theta_{i-1})}\\
&+\frac{l_{i+1}sin(\theta_{i+1}-\theta_i+\theta_i-\phi_{i+1})-l_{i+1}cos(\theta_{i+1}-\theta_i)sin(\theta_{i+1}-\phi_{i+1})}{sin^2(\theta_{i+1}-\theta_i)}\\
=&\frac{l_isin(\theta_{i-1}-\phi_i)-l_icos(\theta_i-\theta_{i-1})sin(\theta_{i-1}-\phi_i)}{sin^2(\theta_i-\theta_{i-1})}\\
&+\frac{l_{i+1}sin(\theta_{i+1}-\phi_{i+1})-l_{i+1}cos(\theta_{i+1}-\theta_i)sin(\theta_{i+1}-\phi_{i+1})}{sin^2(\theta_{i+1}-\theta_i)}\\
=&\frac{l_isin(\theta_{i-1}-\phi_i)(1-cos(\theta_i-\theta_{i-1}))}{sin^2(\theta_i-\theta_{i-1})}\\
&+\frac{l_{i+1}sin(\theta_{i+1}-\phi_{i+1})(1-cos(\theta_{i+1}-\theta_i))}{sin^2(\theta_{i+1}-\theta_i)}\\
=&\frac{l_isin(\theta_{i-1}-\phi_i)(1-cos(\theta_i-\theta_{i-1}))}{(1-cos(\theta_i-\theta_{i-1}))(1+cos(\theta_i-\theta_{i-1}))}\\
&+\frac{l_{i+1}sin(\theta_{i+1}-\phi_{i+1})(1-cos(\theta_{i+1}-\theta_i))}{(1-cos(\theta_{i+1}-\theta_i))(1+cos(\theta_{i+1}-\theta_i))}\\
=&\frac{l_isin(\theta_{i-1}-\phi_i)}{1+cos(\theta_i-\theta_{i-1})}+\frac{l_{i+1}sin(\theta_{i+1}-\phi_{i+1})}{1+cos(\theta_{i+1}-\theta_i)}
\end{aligned}
$$

□

By Lemma 6.11, finding where the distance function is minimized can be done by finding the point(s) where all derivatives are 0. It may be possible to use calculus to show that it is never possible for all derivatives to be 0 at the same time, and possibly it is the case that if they are all 0 there is a 3-D equivalent of a saddle point. Further research is needed to determine if there can be a point at which all derivatives are 0, and if so how many such points can exist. Without having a bound on how many local minimums exist, there is no guarantee that any algorithm can find the minimum set $\mathcal{O}^*$.

7 Algorithm

In this section an algorithm will be given that finds the two orientations θ_1^*, θ_2^* for which the (θ_1^*, θ_2^*)-path between s and t is of minimum length. The general idea is as follows: create all possible sets of extreme vertices with two allowable orientations, then find the overall minimum out of all subproblems.

Overall Algorithm Idea:

1. For each vertex $v \in V$ find the associated interval (α, β) for which v is extreme.
2. Arrange all extreme interval boundaries on a line and for each interval between consecutive boundaries create a list of associated extreme vertices in the order they are traversed on the $s - t$ path.
3. Create a set of Subproblems that consists of two sets of ordered extreme vertices and their associated interval boundaries.
4. For each Subproblem:
   (a) find θ_1, θ_2 that minimizes the total length of the path
   (b) find the length of the path at the interval boundaries
5. Find the minimum of the lengths reported above.

First I will prove a few lemmas that will allow the algorithm to be run more efficiently.

Lemma 7.1. *The projection point $Pr_\phi(v)$ can be determined in $O(logn)$, and a point p can be determined to be visible from v in $O(logn)$ time.*

Proof. (direct)
Guibas et al. [6] proved that the projection point can be calculated in $O(logn)$ time given $O(n)$ preprocessing and that P is triangulated (which takes $O(nlogn)$ time). If p is contained on the line between v and the projection point from v then p is visible from v. Otherwise p is not visible from v. □

In the next lemma we will show how to find a point on the boundary of the polygon which - for all orientations which induce a middle region - is always in the middle region. See Figure 19 for an example of such a point.

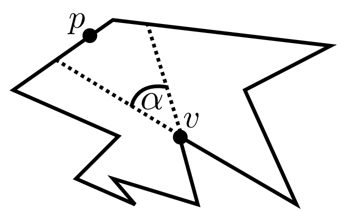


Figure 19: Any projection point from v in a direction contained in the interval α will result in a point that is always in the middle region - for all orientation which induce a middle region. The point p is an example of a point that is always in the middle region.

Lemma 7.2. *For any $v \in V$ there is always some point on the boundary of the polygon which - for all orientations which induce a middle region - is always in the middle region. Furthermore, this point can be found in $O(logn)$ time.*

Proof. (direct)

All orientations which induce a middle region are orientations $\phi \in \Phi$ for which both $Pr_\phi(v)$ and $Pr_{\phi+\pi}(v)$ are non-degenerate. By Lemma 4.1 Φ is a pair of intervals Φ_1 and Φ_2. Since P is a simple polygon it is clear that v must have at least one direction for which the projection point from v is degenerate, and let us call this direction φ. Let φ' be the complementary angle of φ. Since φ is degenerate it cannot be in Φ_1 or Φ_2 and thus φ' cannot be in Φ_1 or Φ_2. There is a single interval within $(0, 2\pi]$ for which the projection point from v is non-degenerate, namely the interior angle interval. Since $\Phi_1 \cup \Phi_2$ contain only non-degenerate directions and φ is a degenerate direction, $Pr_{\varphi'}(v)$ must be non-degenerate or the interior angle interval would not be a single interval. Let $p = Pr_{\varphi'}(v)$.

Consider the boundaries $E(p \to v)$ and $E(v \to p)$, one of which contains all projection points from v at directions $[\varphi, \varphi']$ and the other at directions $[\varphi', \varphi]$. Since Φ_1 and Φ_2 are intervals which are π apart and neither contains φ or φ', Φ_1 is contained in one of (φ, φ') or (φ', φ) and Φ_2 is contained in the other. Since φ' cannot induce two end regions, consider any orientation ϕ_q which induces two end regions and let $q = Pr_{\phi_q}(v)$ and $r = Pr_{\phi_q+\pi}(v)$. Since ϕ_q and $\phi_q + \pi$ are π apart, the points q and r cannot both be in the same interval $[\varphi, \varphi']$ or $[\varphi', \varphi]$. Thus one is in (φ, φ') and the other is in (φ', φ), which means p is not in an end region.

To find the point p, we can first find a direction φ for which $Pr_\varphi(v)$ is degenerate. Since the interior angle interval is $[\theta_{dir}(v_i, v_{i+1}), \theta_{dir}(v_i, v_{i-1})]$, we can find an exterior angle in constant time with the following equation:

$$\varphi = \begin{cases} \frac{\theta_{dir}(v_i, v_{i+1}) + \theta_{dir}(v_i, v_{i-1})}{2} & \text{if } \theta_{dir}(v_i, v_{i+1}) < \theta_{dir}(v_i, v_{i-1}) \\ \frac{\theta_{dir}(v_i, v_{i+1}) + \theta_{dir}(v_i, v_{i-1})}{2} + \pi & \text{otherwise} \end{cases}$$

Then we can find φ' in constant time and find the projection point $p = Pr_{\varphi'}(v)$ in $O(logn)$ time by Lemma 7.1. □

Lemma 7.3. *For a vertex $v \in V$ which is extreme at some orientation, there are two sets of disjoint vertices V_a and V_b where vertices in V_a are always traversed before v and vertices in V_a are always traversed after v. For a point p which is always contained in the middle region, one of the sets of vertices in $E(p \to v)$ or $E(v \to p)$ contains the set V_a and the other contains the set V_b.*

Proof. (direct)

By Lemma 7.2 let p be the point on E which is always in the middle region. Now consider $E(p \to v)$ and $E(v \to p)$. The end regions must be bounded by some subset of the boundary, say $E_s \subset E$ and $E_t \subset E$. Since p is never in an end region p is not in either E_s or E_t, and v is contained on both E_s and E_t as an endpoint where $E_s = E(q \to v)$ and $E_t = E(v \to r)$ for some points $q, r \in bd(P)$. Thus $E(q \to v) \subset E(p \to v)$ and $E(v \to r) \subset E(v \to p)$.

Finally, all vertices contained in the end region containing s are traversed before v and all vertices contained in the end region containing t are traversed after v. □

Lemma 7.4. *When θ_1 (θ_2) is fixed, θ_2 (θ_1) that minimizes the distance function can be found in $O(k)$ time where k is the number of decimal places θ_2 (θ_1) is accurate to.*

Proof. (direct)

In the proof of Lemma 6.7 it is shown that

$$\frac{d}{d\theta_1}D = \frac{l_1 sin(\theta_2 - \phi_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-l_2 sin(\theta_2 - \phi_2)}{1 + cos(\theta_2 - \theta_1)}$$
$$\frac{d}{d\theta_2}D = \frac{-l_1 sin(\theta_1 - \phi_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-l_2 sin(\theta_1 - \phi_2)}{1 + cos(\theta_2 - \theta_1)}$$

Now I will show that the second derivatives of both functions are always positive, and so the first derivatives can equal 0 at most once.

$$\begin{aligned}
\frac{d_2}{d^2\theta_1}D &= \frac{d}{d\theta_1}\left(\frac{l_1 sin(\theta_2-\phi_1)}{1-cos(\theta_2-\theta_1)}\right) + \frac{d}{d\theta_1}\left(\frac{-l_2 sin(\theta_2-\phi_2)}{1+cos(\theta_2-\theta_1)}\right) \\
&= \frac{-l_1 sin(\theta_2-\phi_1) * sin(\theta_2-\theta_1) * -1}{(1-cos(\theta_2-\theta_1))^2} + \frac{-(-l_2 sin(\theta_2-\phi_2)) * -sin(\theta_2-\theta_1) * -1}{(1+cos(\theta_2-\theta_1))^2} \\
&= \frac{l_1 sin(\theta_2-\phi_1) sin(\theta_2-\theta_1)}{(1-cos(\theta_2-\theta_1))^2} + \frac{l_2 sin(\theta_2-\phi_2) sin(\theta_2-\theta_1)}{(1+cos(\theta_2-\theta_1))^2} \\
&> 0 \quad \text{(since } \phi_1 < \theta_1 < \phi_2 < \theta_2\text{)}
\end{aligned}$$

$$\begin{aligned}
\frac{d_2}{d^2\theta_2}D &= \frac{d}{d\theta_2}\left(\frac{-l_1 sin(\theta_1-\phi_1)}{1-cos(\theta_2-\theta_1)}\right) + \frac{d}{d\theta_2}\left(\frac{-l_2 sin(\theta_1-\phi_2)}{1+cos(\theta_2-\theta_1)}\right) \\
&= \frac{-(-l_1 sin(\theta_1-\phi_1)) sin(\theta_2-\theta_1)}{(1-cos(\theta_2-\theta_1))^2} + \frac{-1(-l_2 sin(\theta_1-\phi_2)) * -sin(\theta_2-\theta_1)}{(1+cos(\theta_2-\theta_1))^2} \\
&= \frac{l_1 sin(\theta_1-\phi_1) sin(\theta_2-\theta_1)}{(1-cos(\theta_2-\theta_1))^2} + \frac{-l_2 sin(\theta_1-\phi_2) sin(\theta_2-\theta_1)}{(1+cos(\theta_2-\theta_1))^2} \\
&> 0 \quad \text{(since } \phi_1 < \theta_1 < \phi_2 < \theta_2\text{)}
\end{aligned}$$

Thus if we are given θ_1 we can do a binary search on $\frac{l_1 sin(\theta_2-\phi_1)}{1-cos(\theta_2-\theta_1)} + \frac{-l_2 sin(\theta_2-\phi_2)}{1+cos(\theta_2-\theta_1)}$ to find where it equals 0 (if it does at all). SInce the answer must be accurate to k decimals, we need to divide the interval into 10^k sub-intervals on which to perform the binary search. Then the search takes $O(log(10^k)) = O(k)$ probes, and on every probe the function $\frac{l_1 sin(\theta_2-\phi_1)}{1-cos(\theta_2-\theta_1)} + \frac{-l_2 sin(\theta_2-\phi_2)}{1+cos(\theta_2-\theta_1)}$ can be computed in constant time. $\square$

Lemma 7.5. *For a vertex v which is extreme for some orientation, two vertices $q \in V$ and $r \in V$ can be found in linear time so that $V(q \to v)$ contains all vertices which can be traversed before v and $V(v \to r)$ contains all vertices traversed after v (or vice versa).*

Proof. From Lemmas 7.2 and 7.3, we can find the point p such that $E(p \to v)$ contains all vertices traversed before (after) v and $E(v \to p)$ contains all vertices traversed after (before) v. Then in linear time we can find the edge $e \in E$ on which p is contained. Then $q, r \in V$ are the two vertices bounding e. $\square$

By Lemma 7.5, we can find vertices p and q such that $V(p \to v)$ contains all vertices in one end region and $V(v \to q)$ contains all vertices in the other end region. Since all vertices are indexed from $1 \to n$ in a counterclockwise order around P, a vertex v_i can be found to be in or out of the set $V(p \to v)$ in constant time by checking to see if the index is between p and v. We can find which set bounds the end region containing s and then store the pair of vertices v_a, v_b where $V(v_a \to v_b)$ is equal to one of either $V(p \to v)$ or $V(v \to q)$, whichever bounds the region containing s. Similarly we find and store the pair of vertices v_c, v_d where $V(v_c \to v_d)$ is the other of $V(p \to v)$ or $V(v \to q)$.

For the remainder of this section, a sorted list of vertices refers to a list of vertices sorted in order of traversal from $s \to t$, and any reference to finding, inserting, or removing a vertex from a sorted list is assumed to be performed in $O(logn)$ time since a binary search can be used along with a constant time query that determines if a vertex is before or after another vertex.

For each vertex $v_i \in V$, a set $\Theta(v_i)$ will be created which is guaranteed to contain the two orientations $\alpha, \beta \in (0, \pi]$ which bound the extreme interval of v_i. The set contains only four orientations and it will be shown later these orientations can be calculated in $O(nlogn)$ time. $\Theta(v_i)$ contains the two orientations ϕ_s and ϕ_t, which are the orientations where s and t change regions, and the orientations $\theta(v_i, v_{i-1})$ and $\theta(v_i, v_{i+1})$.

The list Φ will be created to include all orientations at which a vertex changes extremity. Two sets of vertices will be associated with each orientation $\phi \in \Phi$: a set of vertices that become extreme at ϕ, and a set of vertices that stop being extreme at ϕ. These lists will be denoted $V(\phi, EX)$ and $V(\phi, NEX)$, respectively.

To summarize, for each vertex $v_i \in V$ two lists are created which contain the following information

$$v_i(s) \leftarrow (v_a, v_b) \text{ s.t. } V(v_a \to v_b) \text{ contains all vertices traversed before } v_i$$

$$v_i(t) \leftarrow (v_c, v_d) \text{ s.t. } V(v_c \to v_d) \text{ contains all vertices traversed after } v_i$$

There will also be $O(n)$ vertex sets created of the following form

$$V(\phi, EX) \leftarrow \text{vertices that become extreme at } \phi$$

$$V(\phi, NEX) \leftarrow \text{vertices that stop being extreme at } \phi$$

and a set $\Theta(v_i)$ associated with each vertex which contains the orientations

$$\Theta(v_i) = \{\theta(v_i, v_{i-1}), \theta(v_i, v_{i+1}), \phi_s, \phi_t\}$$

Algorithm 7.1: PREPROCESS(P, s, t)

Triangulate P

$V_T \leftarrow V \cup \{s, t\}$

$\Phi \leftarrow \emptyset$ **comment:** set of orientations where some vertex changes extremity

for each $v_i \in V$

do
- Find $\Theta(v_i)$
- Find $\alpha, \beta \in \Theta(v_i)$ s.t. all orientations in (α, β) are extreme
- Add α and β to Φ
- Add v_i to $V(\alpha, EX)$ and to $V(\beta, NEX)$
- Find $v_i(s)$ and $v_i(t)$ for the vertex v_i

Sort Φ in ascending order

Create a list of extreme vertices, L_1, for the first Interval

for each (non-first) consecutive pair of orientations (σ_i, σ_{i+1}) in Φ

do
- $L_i \leftarrow L_{i-1}$
- **for each** $v_j \in V(\sigma_i, EX)$
  - **do** Add v_j to L_i
- **for each** $v_j \in V(\sigma_i, NEX)$
  - **do** Remove v_j from L_i

Algorithm 7.2: MINIMUMLENGTHPATH(P, s, t)

$Preprocess(P, s, t)$

comment: Φ is ordered from 0 up to π

$\Phi = \{\omega_1, \omega_2 \ldots, \omega_c\}$

for each $i \leftarrow c$ down to 1

do
- Rotate problem so that $\omega_{i+1} = \pi$
- $X_c \leftarrow \{(u, v)$ s.t. u and v are consecutive vertices in $L_i\}$
- $X_b \leftarrow \emptyset$
- $X_a \leftarrow \{(u, v) \in X_c$ s.t. $\theta(u, v) \in (0, \omega_i)\}$
- $V(X_b) \leftarrow 0, V(X_a) \leftarrow \sum_{(u,v)\in X_a} \ell(\overline{uv})sin(\theta(u, v))$
- $H(X_b) \leftarrow 0, H(X_a) \leftarrow \sum_{(u,v)\in X_a} \ell(\overline{uv})cos(\theta(u, v))$
- **for each** $j \leftarrow (i - 1)$ down to 1
- **do**
  - **for each** $v \in V(\omega_j, EX)$
  - **do**
    - Find $(v_a, v_b) \in X_c$ where v comes between v_a and v_b
    - $removeFromSum((v_a, v_b), i, j)$
    - $addToSum((v_a, v), i, j)$
    - $addToSum((v, v_b), i, j)$
  - **for each** $v \in V(\omega_j, NEX)$
  - **do**
    - Find the two pairs $(v_a, v), (v, v_b) \in X_c$
    - $removeFromSum((v_a, v), i, j)$
    - $removeFromSum((v, v_b), i, j)$
    - $addToSum((v_a, v_b), i, j)$
  - Recalculate l_1, l_2, ϕ_1 and ϕ_2
  - Check distance function for a minimum value
  - Check boundaries for a minimum value

return (overall minimum)

Algorithm 7.3: ADDTOSUM($(u, v), i, j$)

Add (u, v) to X_c

if $\theta(u, v) \in (0, \omega_j)$

then

do $\begin{cases} V(X_a) \leftarrow V(X_a) + \ell(\overline{uv}) sin(\theta(u, v)) \\ H(X_a) \leftarrow H(X_a) + \ell(\overline{uv}) cos(\theta(u, v)) \\ \text{Add } (u, v) \text{ to } X_a \end{cases}$

else if $\theta(u, v) \in (\omega_{j+1}, \omega_i)$

then

do $\begin{cases} V(X_b) \leftarrow V(X_b) + \ell(\overline{uv}) sin(\theta(u, v)) \\ H(X_b) \leftarrow H(X_b) + \ell(\overline{uv}) cos(\theta(u, v)) \\ \text{Add } (u, v) \text{ to } X_b \end{cases}$

Algorithm 7.4: REMOVEFROMSUM($(u, v), i, j$)

Remove (u, v) from X_c

if $\theta(u, v) \in (0, \omega_j)$

then

do $\begin{cases} V(X_a) \leftarrow V(X_a) - \ell(\overline{uv}) sin(\theta(u, v)) \\ H(X_a) \leftarrow H(X_a) - \ell(\overline{uv}) cos(\theta(u, v)) \\ \text{Remove } (u, v) \text{ from } X_a \end{cases}$

else if $\theta(u, v) \in (\omega_{j+1}, \omega_i)$

then

do $\begin{cases} V(X_b) \leftarrow V(X_b) - \ell(\overline{uv}) sin(\theta(u, v)) \\ H(X_b) \leftarrow H(X_b) - \ell(\overline{uv}) cos(\theta(u, v)) \\ \text{Remove } (u, v) \text{ from } X_b \end{cases}$

7.1 Proof of Correctness

In the following section it will be shown that the algorithm given in this thesis correctly determines the set $\{\theta_1, \theta_2\}$ which minimizes the length of the (θ_1, θ_2)-path connecting s and t.

Theorem 7.6. *Algorithm 7.1 correctly constructs $O(n)$ lists of extreme vertices in order of traversal from s to t, and two sets of vertices associated with every orientation in Φ - one containing all vertices which become extreme at that orientation and one containing all vertices which become non-extreme at that orientation. The algorithm also finds two lists $v_i(s) = \{v_s, v'_s\}$ and $v_i(t) = \{v_t, v'_t\}$ where $E(v_s \to v'_s)$ contains all vertices which are traversed before v_i and $E(v_t \to v'_t)$ contains all vertices traversed after v_i.*

Proof. (direct)

The first for loop correctly constructs the two sets of vertices that become extreme or non-extreme at each angle in Φ, and also finds the lists $v_i(s)$ and $v_i(t)$. The details are left out of the algorithm but can be directly inferred from Lemma 7.5.

Let (α, β) be the single interval in $(0, \pi]$ for which v_i is extreme (by Theorem 4.3 there is only one interval for which v is extreme). See the proof of Lemma 4.1 to verify that the orientations bounding the interval where both $Pr_{\varphi}(v_i)$ and $Pr_{\varphi+\pi}(v_i)$ are non-degenerate are contained in $\Theta(v_i)$. Now we will show by contradiction that the orientations bounding the extreme region must be contained in $\Theta(v_i)$.

Let $\alpha_1, \beta_1 \in (0, \pi]$ be two orientations such that for all $\phi \in [\alpha_1, \beta_1]$ both $Pr_{\phi}(v)$ and $Pr_{\phi\pm\pi}(v)$ are non-degenerate, and for all $\phi \notin ([\alpha_1, \beta_1]$ at least one of the projections is degenerate. Within the intervals $[\alpha_1, \beta_1]$, the only orientations at which s or t could enter or exit an end region are ϕ_s or ϕ_t.

A list L_i is then constructed for each consecutive pair of orientations which contains all vertices that are extreme in the interval. Since we begin with L_1 as an ordered list and all vertices are inserted in order, every list is correctly ordered.

□

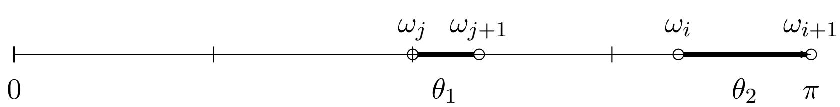


Figure 20: An example of the orientations θ_1 and θ_2 is shown above.

Lemma 7.7. *Algorithms 7.2, 7.3, and 7.4 correctly find the two orientations θ_1, θ_2 which minimize the length of the (θ_1, θ_2)-path connecting s and t.*

Proof. (direct)

The correctness of this algorithm hinges on correctly calculating l_1, l_2, ϕ_1 and ϕ_2 for each combination of i and j. First, all combinations of i and j are found by the two for-loops that are run. For each value of i, we rotate the problem so that $\omega_{i+1} = \pi$ and thus we have a set of orientations as shown in Figure 20.

First, we create the sets X_a and X_b with respect to the case where $j = i - 1$. Therefore all the pairs of consecutive extreme vertices in X_c will be in the set X_a and none will be in X_b (remember that $X_b = \{(u, v) \in X_c$s.t. $\theta(u, v) \in (\omega_{j+1}, \omega_i)\}$). Then we calculate $V(X_a), V(X_b), H(X_a)$, and $H(X_b)$ according to Theorem 6.6.

No pair of extreme vertices $(u', v') \in X_c$ can change sets from X_a to X_b, since if that were the case then at $j = k - 1 \Rightarrow \theta(u', v') \in (0, \omega_j)$ and at $j = k \Rightarrow \theta(u', v') \in (\omega_{j+1}, \omega_i)$. Thus $\theta(u', v') \in (0, \omega_{k-1})$ and $\theta(u', v') \in (\omega_{k+1}, \omega_i)$, which is not possible since $\omega_{k-1} \leq \omega_{k+1}$. Thus we only need to concern ourselves with modifying the sums to reflect any vertices which change extremity.

For each value of j from $(i - 1)$ down to 1 the algorithm goes through all vertices which change extremity at ω_j and modifies the sets X_a, X_b, and X_c, then modifies the summations $V(X_a), V(X_b), H(X_a)$, and $H(X_b)$ to reflect the new sets X_a and X_b. There are two for loops which I will now prove modify the summations and sets correctly.

1. v was not extreme in (ω_j, ω_{j+1}) but is extreme in (ω_{j-1}, ω_j)

   Then v becomes non-extreme at ω_j, so $v \in V(\omega_j, NEX)$. Since v is extreme in the interval we are now considering, we must add v to the list of extreme vertices and therefore must add v to X_c. So we find the two vertices which come before and after v on the path traversal from s to t, which are v_a and v_b in the algorithm. Then we remove (v_a, v_b) from X_c since they are no longer consecutive extreme vertices. Furthermore, we add (v_a, v) and (v, v_b) to X_c since they are now consecutive extreme vertices.

2. v was extreme in (ω_j, ω_{j+1}) but is not extreme in (ω_{j-1}, ω_j)

   Then v becomes extreme at ω_j, so $v \in V(\omega_j, EX)$. Since v is not extreme in the interval we are now considering, we must remove v from the list of extreme vertices and therefore must remove any pairs containing v from X_c. Once we remove (v_a, v) and (v, v_b) from X_c, v_a and v_b are now consecutive extreme vertices so we must add

(v_a, v_b) to X_c.

The implementation of Algorithms 7.3 and 7.4 which add/remove pairs of consecutive extreme vertices from X_c also ensures that the sets X_a and X_b are updated appropriately and that the summations $V(X_a), V(X_b), H(X_a)$, and $H(X_b)$ reflect the changes to X_a and X_b.

By Lemma 6.5 and Theorem 6.6, the following equations will give us the two lines we use to find the minimum distance

$$l_1 = \sqrt{(V(X_a))^2 + (H(X_a))^2}$$
$$l_2 = \sqrt{(V(X_b))^2 + (H(X_b))^2}$$
$$\phi_1 = tan^{-1}\left(\frac{V(X_a)}{H(X_a)}\right)$$
$$\phi_2 = tan^{-1}\left(\frac{V(X_b)}{H(X_b)}\right)$$

Finally the boundary of the interval must be checked as well, in case there is no minimum within the interval. □

7.2 Time Complexity

The time complexity of this algorithm will be shown to be $O(n^2 log(n) + kn^2)$.

Lemma 7.8. *For a vertex $v_i \in V$ the orientations ϕ_s and ϕ_t can be found in constant time given linear preprocessing (along with the triangulation of P).*

Proof. (direct)

For any fixed point x in a polygon, Guibas et al. [6] give a linear time algorithm that finds the shortest paths from x to all vertices of P. A modification of this algorithm can be made to store the orientation of the final segment of the shortest path (directed towards the vertex the shortest path ends at). Thus after preprocessing for both s and t, we can find the orientations of the final segments in constant time, which by Lemma 5.2 are the orientations at which s and t switch regions. □

Lemma 7.9. *Algorithm 7.1 runs in $O(n^2)$ time.*

Proof. (direct)

Triangulation takes $O(nlogn)$ time and the for loop iterates $O(n)$ times. Constructing Θ

takes constant time given linear time preprocessing (by Lemma 7.8). By Lemma 7.5 finding the lists $v_i(s)$ and $v_i(t)$ takes linear time. Thus the first for loop takes $O(n^2)$ time.
Sorting Φ takes $O(nlogn)$ time since Φ contains $O(n)$ orientations. Constructing the first list L_1 takes $O(nlogn)$ time since we need it sorted.
For all other intervals, each vertex that changes extremity is added or removed from L_i. Adding or removing v_j to or from the list L_i in takes $O(logn)$ time since ordering is maintained. Since each vertex changes extremity at most twice, addition/removal is performed $O(n)$ times in total and thus adding/removing vertices takes $O(nlogn)$ time in total. Duplicating the list L_{i-1} to put into L_i takes $O(n)$ time and happens $O(n)$ times and thus takes $O(n^2)$ time. Therefore the second for loop takes $O(n^2)$ time. □

Lemma 7.10. *Algorithm 7.2 takes* $O(n^2logn + kn^2)$ *time.*

Proof. (direct)
Finding all extreme sets takes $O(n^2)$ time by Lemma 7.9. The number of intervals is in $O(n)$, so the outer for-loop iterates $O(n)$ times. The inner-most for loops are visited $O(n^2)$ times since for each value of i, the inner for-loops are visited only if a vertex is changing extremity with respect to the current j value. Each vertex changes extremity twice, so for each value of i the inner for-loops will be visited $O(n)$ times. Algorithms 7.3 and 7.4 take $O(logn)$ time each since they do insertions and removals from a sorted list and constant time calculations. Thus the inner for loops take $O(n^2logn)$ time during the entire algorithm. Recalculating l_1, l_2, ϕ_1 and ϕ_2 takes constant time using the summations. Finding the minimum point (if it exists) can be reduced to finding if two functions intersect. These functions intersect at most once (by Lemma 6.9), and the θ_1-derivative of one function is always greater than the θ_1-derivative of the other (by Lemma 6.8), so before they intersect, one function has highest value and after they intersect, the other function has highest value. Thus a binary search of θ_1 on the interval $(0, \pi]$ can be performed to find where the fuctions intersect. The binary search can be performed in $O(log(10^k)) = O(k)$ time where k is the number of decimals the answer is accurate to. Calculating the boundary minimums can be done in $O(k)$ time by Lemma 7.4. Thus the j for loop takes $O(kn)$ time.
Rotating the problem and constructing X_a, X_b and X_c all take linear time. Calculating $V(X_a), H(X_a), V(X_b)$ and $H(X_b)$ can be done in linear time as well. Therefore the i loop takes $O(kn^2)$ time. □

8 Conclusion

Previous research in the area of restricted orientation geometry has focused on allowing a predetermined set of orientations and constructing geometric objects allowing only those orientations. In this thesis a new area of restricted orientation geometry was investigated in which the set of orientations is not given and instead only the number of orientations is limited. Previous results concerning smallest - that is, shortest in both Euclidean and link distance - paths were generalized and it was shown that for any two orientations θ_1, θ_2 there always exists a smallest (θ_1, θ_2)-path between s and t in any simple polygon P. This unifies previous research ([4], [13], [8]) which focused on finding smallest paths given any two rectilinear orientations. For three or more orientations, it was shown that there does not always exists such a smallest path.

It was also proved that when allowing two orientations of travel, an $s-t$ path can be found which minimizes the length of the shortest path. Furthermore, an algorithm to find such a path was given that runs in $O(n^2 logn + n^2 k)$ time where n is the number of vertices in P and k is the number of decimals θ_1, θ_2 (which produce a minimum) are accurate to. A conjecture was given that the single "minimum" point found is actually a saddle point and therefore not a minimum. If the conjecture can be shown to be true, it would imply that one of the orientations for which the shortest path is minimized must be an orientation between two vertices in P. Thus only one of θ_1 or θ_2 would need to be varied at a time, resulting in a simpler algorithm.

The problem of finding the set of orientations which minimizes the shortest path for more than two allowable orientations was defined, and a series of m equations (where m is the number of allowable orientations) is given which must all intersect for at any point which is a minimum. Further research may show these equations intersect either once, not at all, or in some predictable way that can be analyzed. Currently, any algorithm given to find a minimum for $m \geq 3$ cannot be guaranteed to run in any finite amount of time since there is no bound on the number of minimums possible. Furthermore, a good upper bound is needed for the number of combinations of m extreme vertex sets (with duplicates allowed).

This thesis was a first attempt at defining minimum distance paths for a restricted number of orientations, and there are many unsolved variations of the problems looked at. More specifically, there are three common version of the $s-t$ path problem [16]: the *one-shot problem* where s and t are given and the task is to find a path between the two points; the *semi-query problem* where only t is given and the task is to preprocess the environment so that for any s the path between s and t can be found as fast as possible; and finally the *full-query problem* where neither point is given and the task is to preprocess the environment so that finding a path between any s and t can be done as fast as possible. Only the one-shot problem was addressed in this thesis, while the other two are left as open problems. Further open problems include varying the distance metric used to define the shortest path, for example finding the orientations which minimize the link-distance of the $s-t$ path. This problem may prove to be difficult (or even impossible) due to the fact that link distance is a discreet value while solving the problem in the method described in this thesis required taking derivatives over continuous functions. An entirely different approach may be needed to solve the problem with respect to link distance.

It was shown via counterexample that there does not always exists a smallest path for $m \geq 3$, but no conditions were given upon which a smallest path does or does not exist. It is possible that placing restrictions on the problem could result in a smallest path always existing. For example, restricting the polygon to be an $\mathcal{O}$-polygon may be a restriction worth investigating. Similarly, a counterexample was given to show that it is not always possible to choose a set of orientations that simultaneously minimizes both the Euclidean and link distance of the $s-t$ path, and putting restrictions on the problem may remove the possibility of such a counterexample. Lastly, the algorithm developed in this thesis has a running time that may not be optimal, so proving the optimality of this algorithm or improving its running time are both problems of interest.

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A

Lemma A.1.

$$S_v = \sqrt{S_v^2 + S_h^2} sin\left(tan^{-1}\left(\frac{S_v}{S_h}\right)\right)$$
$$S_h = \sqrt{S_v^2 + S_h^2} cos\left(tan^{-1}\left(\frac{S_v}{S_h}\right)\right)$$

Proof. By Figure 21 we can see that

$$tan^{-1}\left(\frac{S_v}{S_h}\right) = \alpha \quad sin(\alpha) = \frac{S_v}{\sqrt{S_v^2 + S_h^2}} \quad cos(\alpha) = \frac{S_h}{\sqrt{S_v^2 + S_h^2}}$$
$$\Rightarrow sin\left(tan^{-1}\left(\frac{S_v}{S_h}\right)\right) = \frac{S_v}{\sqrt{S_v^2 + S_h^2}}$$
$$\Rightarrow cos\left(tan^{-1}\left(\frac{S_v}{S_h}\right)\right) = \frac{S_h}{\sqrt{S_v^2 + S_h^2}}$$

$\square$

Lemma 6.1. *For edges* $e_1 = (u, v)$ *and* $e_2 = (w, z)$ *where* $0 < \theta_{dir}(u, v) \leq \theta_{dir}(w, z) \leq \pi$*,*

$$\ell(\overline{uv}) sin(x - \theta(u, v)) + \ell(\overline{wz}) sin(x - \theta(w, z)) = \sqrt{S_v^2 + S_h^2} sin\left(x - tan^{-1}\left(\frac{S_v}{S_h}\right)\right)$$

where

$$S_v = \ell(\overline{uv}) sin(\theta(u, v)) + \ell(\overline{wz}) sin(\theta(w, z))$$
$$S_h = \ell(\overline{uv}) cos(\theta(u, v)) + \ell(\overline{wz}) cos(\theta(w, z))$$

and $tan^{-1}\left(\frac{S_v}{S_h}\right)$ *is contained in the interval* $[\theta_{dir}(u, v), \theta_{dir}(w, z)]$.

Proof.

$$\begin{aligned}
&\ell(\overline{uv}) sin(x - \theta(u, v)) + \ell(\overline{wz}) sin(x - \theta(w, z)) \\
&\quad = \ell(\overline{uv})(sin(x) cos(\theta(u, v)) - sin(\theta(u, v)) cos(x)) \\
&\qquad + \ell(\overline{wz})(sin(x) cos(\theta(w, z)) - sin(\theta(w, z)) cos(x)) \\
&\quad = sin(x)(\ell(\overline{uv}) cos(\theta(u, v)) + \ell(\overline{wz}) cos(\theta(w, z)))
\end{aligned}$$

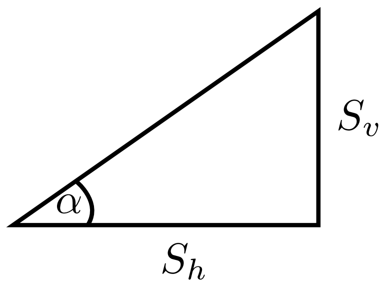

Figure 21: As used in Lemma A.1.

$$
\begin{aligned}
&\quad - cos(x)(\ell(\overline{uv})sin(\theta(u,v)) + \ell(\overline{wz})sin(\theta(w,z))) \\
&= sin(x)S_h - cos(x)S_v \\
&= \sqrt{S_v^2 + S_h^2}\left(sin(x)cos\left(tan^{-1}\left(\frac{S_v}{S_h}\right)\right) - cos(x)sin\left(tan^{-1}\left(\frac{S_v}{S_h}\right)\right)\right) \quad \text{(by Lem A.1)} \\
&= \sqrt{S_v^2 + S_h^2} sin\left(x - tan^{-1}\left(\frac{S_v}{S_h}\right)\right)
\end{aligned}
$$

Therefore the two edges can be considered vectors and the line found with length $\sqrt{S_v^2 + S_h^2}$ and orientation $tan^{-1}\left(\frac{S_v}{S_h}\right)$ is the vector sum of the two edges.

Using the parallelogram law of vector addition, if e_1 and e_2 form two adjacent edges of a parallelogram, then the sum vector must begin where the two edges meet and end at the opposite corner of the parallelogram. By translating $e_2 = (w, z)$ so that $w = u$, e_1 and e_2 form two adjacent edges of a parallelogram, and the sum vector lies in that parallelogram. Thus the resultant vector has orientation between e_1 and e_2, meaning between $\theta_{dir}(u, v)$ and $\theta_{dir}(w, z)$. $\square$

Lemma 6.2.

$$
\begin{aligned}
&\frac{d}{d\theta_1}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right) \\
&\quad = \frac{l_1 sin(\theta_2 - \phi_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-l_2 sin(\theta_2 - \phi_2)}{1 + cos(\theta_2 - \theta_1)}
\end{aligned}
$$

Proof.

$$
\frac{d}{d\theta_1}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right)
$$

$$
\begin{aligned}
&= \frac{d}{d\theta_1}\left(-\frac{l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2)}{sin(\theta_1-\theta_2)} - \frac{l_1 sin(\theta_1-\phi_1)-l_2 sin(\theta_1-\phi_2)}{sin(\theta_1-\theta_2)}\right)\\
&= -\frac{d}{d\theta_1}\left(\frac{l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2)}{sin(\theta_1-\theta_2)}\right) - \frac{d}{d\theta_1}\left(\frac{l_1 sin(\theta_1-\phi_1)-l_2 sin(\theta_1-\phi_2)}{sin(\theta_1-\theta_2)}\right)\\
&= -\left(\frac{d}{d\theta_1}(sin(\theta_1-\theta_2))^{-1}\right) * (l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2))\\
&\quad - \frac{1}{sin(\theta_1-\theta_2)} * \left(\frac{d}{d\theta_1} l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2)\right)\\
&\quad - \left(\frac{d}{d\theta_1}(sin(\theta_1-\theta_2))^{-1}\right) * (l_1 sin(\theta_1-\phi_1)-l_2 sin(\theta_1-\phi_2))\\
&\quad - \frac{1}{sin(\theta_1-\theta_2)}\left(\frac{d}{d\theta_1}(l_1 sin(\theta_1-\phi_1)-l_2 sin(\theta_1-\phi_2))\right)\\
&= -\left(\frac{-1 * cos(\theta_1-\theta_2)}{sin^2(\theta_1-\theta_2)}\right) * (l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2))\\
&\quad - 0\\
&\quad - \left(\frac{-1 * cos(\theta_1-\theta_2)}{sin^2(\theta_1-\theta_2)}\right) * (l_1 sin(\theta_1-\phi_1)-l_2 sin(\theta_1-\phi_2))\\
&\quad - \frac{1}{sin(\theta_1-\theta_2)}(l_1 cos(\theta_1-\phi_1)-l_2 cos(\theta_1-\phi_2))\\
&= \frac{cos(\theta_1-\theta_2)(l_1 sin(\theta_2-\phi_1)+l_2 sin(\theta_2-\phi_2))}{sin^2(\theta_1-\theta_2)}\\
&\quad + \frac{cos(\theta_1-\theta_2)(l_1 sin(\theta_1-\phi_1)-l_2 sin(\theta_1-\phi_2))}{sin^2(\theta_1-\theta_2)}\\
&\quad - \frac{sin(\theta_1-\theta_2)(l_1 cos(\theta_1-\phi_1)-l_2 cos(\theta_1-\phi_2))}{sin^2(\theta_1-\theta_2)}\\
&= \frac{l_1 cos(\theta_1-\theta_2)sin(\theta_1-\phi_1)-l_1 sin(\theta_1-\theta_2)cos(\theta_1-\phi_1)+l_1 cos(\theta_1-\theta_2)sin(\theta_2-\phi_1)}{sin^2(\theta_1-\theta_2)}\\
&\quad + \frac{-l_2 cos(\theta_1-\theta_2)sin(\theta_1-\phi_2)+l_2 sin(\theta_1-\theta_2)cos(\theta_1-\phi_2)+l_2 cos(\theta_1-\theta_2)sin(\theta_2-\phi_2)}{sin^2(\theta_1-\theta_2)}\\
&= \frac{l_1(sin(\theta_1-\phi_1)cos(\theta_1-\theta_2)-sin(\theta_1-\theta_2)cos(\theta_1-\phi_1))+l_1 cos(\theta_1-\theta_2)sin(\theta_2-\phi_1)}{sin^2(\theta_1-\theta_2)}\\
&\quad + \frac{-l_2(sin(\theta_1-\phi_2)cos(\theta_1-\theta_2)-sin(\theta_1-\theta_2)cos(\theta_1-\phi_2))+l_2 cos(\theta_1-\theta_2)sin(\theta_2-\phi_2)}{sin^2(\theta_1-\theta_2)}\\
&= \frac{l_1 sin(\theta_1-\phi_1-\theta_1+\theta_2)+l_1 cos(\theta_1-\theta_2)sin(\theta_2-\phi_1)}{sin^2(\theta_1-\theta_2)}\\
&\quad + \frac{-l_2 sin(\theta_1-\phi_2-\theta_1+\theta_2)+l_2 cos(\theta_1-\theta_2)sin(\theta_2-\phi_2)}{sin^2(\theta_1-\theta_2)}\\
&= \frac{l_1 sin(\theta_2-\phi_1)+l_1 cos(\theta_1-\theta_2)sin(\theta_2-\phi_1)}{sin^2(\theta_1-\theta_2)} + \frac{-l_2 sin(\theta_2-\phi_2)+l_2 cos(\theta_1-\theta_2)sin(\theta_2-\phi_2)}{sin^2(\theta_1-\theta_2)}\\
&= \frac{l_1 sin(\theta_2-\phi_1)(1+cos(\theta_2-\theta_1))}{sin^2(\theta_2-\theta_1)} + \frac{-l_2 sin(\theta_2-\phi_2)(1-cos(\theta_2-\theta_1))}{sin^2(\theta_2-\theta_1)}\\
&= \frac{l_1 sin(\theta_2-\phi_1)(1+cos(\theta_2-\theta_1))}{(1+cos(\theta_2-\theta_1))(1-cos(\theta_2-\theta_1))} + \frac{-l_2 sin(\theta_2-\phi_2)(1-cos(\theta_2-\theta_1))}{(1+cos(\theta_2-\theta_1))(1-cos(\theta_2-\theta_1))}
\end{aligned}
$$

$$= \frac{l_1 sin(\theta_2 - \phi_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-l_2 sin(\theta_2 - \phi_2)}{1 + cos(\theta_2 - \theta_1)}$$

□

Lemma 6.3.

$$\frac{d}{d\theta_2}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right)$$
$$= \frac{-l_1 sin(\theta_1 - \phi_1)}{1 - cos(\theta_2 - \theta_1)} + \frac{-l_2 sin(\theta_1 - \phi_2)}{1 + cos(\theta_2 - \theta_1)}$$

Proof.

$$\frac{d}{d\theta_2}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right)$$
$$= \frac{d}{d\theta_2}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)} + \frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right)$$
$$= \frac{d}{d\theta_2}\left(\frac{l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)}{sin(\theta_2 - \theta_1)}\right) + \frac{d}{d\theta_2}\left(\frac{l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2)}{sin(\theta_2 - \theta_1)}\right)$$
$$= \left(\frac{d}{d\theta_2}(sin(\theta_2 - \theta_1))^{-1}\right) * (l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2))$$
$$+ \frac{1}{sin(\theta_2 - \theta_1)} * \left(\frac{d}{d\theta_2} l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2)\right)$$
$$+ \left(\frac{d}{d\theta_2}(sin(\theta_2 - \theta_1))^{-1}\right) * (l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2))$$
$$+ \frac{1}{sin(\theta_2 - \theta_1)} \left(\frac{d}{d\theta_2}(l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2))\right)$$
$$= \left(\frac{-1 * cos(\theta_2 - \theta_1)}{sin^2(\theta_2 - \theta_1)}\right) * (l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2))$$
$$+ \frac{1}{sin(\theta_2 - \theta_1)} * (l_1 cos(\theta_2 - \phi_1) + l_2 cos(\theta_2 - \phi_2))$$
$$+ \left(\frac{-1 * cos(\theta_2 - \theta_1)}{sin^2(\theta_2 - \theta_1)}\right) * (l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2))$$
$$+ \frac{1}{sin(\theta_2 - \theta_1)} * 0$$
$$= \frac{-cos(\theta_2 - \theta_1)(l_1 sin(\theta_2 - \phi_1) + l_2 sin(\theta_2 - \phi_2))}{sin^2(\theta_2 - \theta_1)}$$
$$+ \frac{sin(\theta_2 - \theta_1)(l_1 cos(\theta_2 - \phi_1) + l_2 cos(\theta_2 - \phi_2))}{sin^2(\theta_2 - \theta_1)}$$
$$- \frac{cos(\theta_2 - \theta_1)(l_1 sin(\theta_1 - \phi_1) - l_2 sin(\theta_1 - \phi_2))}{sin^2(\theta_2 - \theta_1)}$$
$$= \frac{-l_1 cos(\theta_2 - \theta_1) sin(\theta_2 - \phi_1) + l_1 sin(\theta_2 - \theta_1) cos(\theta_2 - \phi_1) - l_1 cos(\theta_2 - \theta_1) sin(\theta_1 - \phi_1)}{sin^2(\theta_2 - \theta_1)}$$

$$
\begin{aligned}
&+ \frac{-l_2cos(\theta_2-\theta_1)sin(\theta_2-\phi_2)+l_2sin(\theta_2-\theta_1)cos(\theta_2-\phi_2)+l_2cos(\theta_2-\theta_1)sin(\theta_1-\phi_2)}{sin^2(\theta_2-\theta_1)}\\
=& \frac{-l_1(sin(\theta_2-\phi_1)cos(\theta_2-\theta_1)-sin(\theta_2-\theta_1)cos(\theta_2-\phi_1))-l_1cos(\theta_2-\theta_1)sin(\theta_1-\phi_1)}{sin^2(\theta_2-\theta_1)}\\
&+ \frac{-l_2(sin(\theta_2-\phi_2)cos(\theta_2-\theta_1)-sin(\theta_2-\theta_1)cos(\theta_2-\phi_2))+l_2cos(\theta_2-\theta_1)sin(\theta_1-\phi_2)}{sin^2(\theta_2-\theta_1)}\\
=& \frac{-l_1sin(\theta_2-\phi_1-\theta_2+\theta_1)-l_1cos(\theta_2-\theta_1)sin(\theta_1-\phi_1)}{sin^2(\theta_2-\theta_1)}\\
&+ \frac{-l_2sin(\theta_2-\phi_2-\theta_2+\theta_1)+l_2cos(\theta_2-\theta_1)sin(\theta_1-\phi_2)}{sin^2(\theta_2-\theta_1)}\\
=& \frac{-l_1sin(\theta_1-\phi_1)-l_1cos(\theta_2-\theta_1)sin(\theta_1-\phi_1)}{sin^2(\theta_2-\theta_1)}\\
&+ \frac{-l_2sin(\theta_1-\phi_2)+l_2cos(\theta_2-\theta_1)sin(\theta_1-\phi_2)}{sin^2(\theta_2-\theta_1)}\\
=& \frac{-l_1sin(\theta_1-\phi_1)(1+cos(\theta_2-\theta_1))}{sin^2(\theta_2-\theta_1)} + \frac{-l_2sin(\theta_1-\phi_2)(1-cos(\theta_2-\theta_1))}{sin^2(\theta_2-\theta_1)}\\
=& \frac{-l_1sin(\theta_1-\phi_1)(1+cos(\theta_2-\theta_1))}{(1+cos(\theta_2-\theta_1))(1-cos(\theta_2-\theta_1))} + \frac{-l_2sin(\theta_1-\phi_2)(1-cos(\theta_2-\theta_1))}{(1+cos(\theta_2-\theta_1))(1-cos(\theta_2-\theta_1))}\\
=& \frac{-l_1sin(\theta_1-\phi_1)}{1-cos(\theta_2-\theta_1)} + \frac{-l_2sin(\theta_1-\phi_2)}{1+cos(\theta_2-\theta_1)}
\end{aligned}
$$

□

Lemma 6.4.

$$
\frac{d}{d\varphi}\left(\frac{l_1sin(\varphi-\phi_1)\pm l_2sin(\varphi-\phi_2)}{l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2)}\right) = \frac{\pm 2l_1l_2sin(\phi_2-\phi_1)}{(l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2))^2}
$$

Proof.

$$
\begin{aligned}
&\frac{d}{d\varphi}\left(\frac{l_1sin(\varphi-\phi_1)\pm l_2sin(\varphi-\phi_2)}{l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2)}\right)\\
&\quad= \frac{l_1cos(\varphi-\phi_1)\pm l_2cos(\varphi-\phi_2)}{l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2)}\\
&\quad\quad+(-1)*\frac{(l_1sin(\varphi-\phi_1)\pm l_2sin(\varphi-\phi_2))(l_1cos(\varphi-\phi_1)\mp l_2cos(\varphi-\phi_2))}{(l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2))^2}\\
&\quad= \frac{(l_1cos(\varphi-\phi_1)\pm l_2cos(\varphi-\phi_2))(l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2))}{(l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2))^2}\\
&\quad\quad-\frac{(l_1sin(\varphi-\phi_1)\pm l_2sin(\varphi-\phi_2))(l_1cos(\varphi-\phi_1)\mp l_2cos(\varphi-\phi_2))}{(l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2))^2}\\
&\quad= \frac{l^2sin(\varphi-\phi_1)cos(\varphi-\phi_1)\mp l_1l_2sin(\varphi-\phi_2)cos(\varphi-\phi_1)}{(l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2))^2}\\
&\quad\quad+\frac{\pm l_1l_2sin(\varphi-\phi_1)cos(\varphi-\phi_2)-l_2^2sin(\varphi-\phi_2)cos(\varphi-\phi_2)}{(l_1sin(\varphi-\phi_1)\mp l_2sin(\varphi-\phi_2))^2}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{l^2 sin(\varphi - \phi_1)cos(\varphi - \phi_1) \mp l_1 l_2 sin(\varphi - \phi_1)cos(\varphi - \phi_2)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} \\
&- \frac{\pm l_1 l_2 sin(\varphi - \phi_2)cos(\varphi - \phi_1) - l_2^2 sin(\varphi - \phi_2)cos(\varphi - \phi_2)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} \\
= &\frac{\mp l_1 l_2 sin(\varphi - \phi_2)cos(\varphi - \phi_1)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} + \frac{\pm l_1 l_2 sin(\varphi - \phi_1)cos(\varphi - \phi_2)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} \\
&- \frac{\mp l_1 l_2 sin(\varphi - \phi_1)cos(\varphi - \phi_2)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} - \frac{\pm l_1 l_2 sin(\varphi - \phi_2)cos(\varphi - \phi_1)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} \\
= &\frac{\mp 2 l_1 l_2 sin(\varphi - \phi_2)cos(\varphi - \phi_1)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} + \frac{\pm 2 l_1 l_2 sin(\varphi - \phi_1)cos(\varphi - \phi_2)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} \\
= &\frac{\mp 2 l_1 l_2 (sin(\varphi - \phi_2)cos(\varphi - \phi_1) - sin(\varphi - \phi_1)cos(\varphi - \phi_2))}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} \\
= &\frac{\mp 2 l_1 l_2 sin(\phi_1 - \phi_2)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2} \\
= &\frac{\pm 2 l_1 l_2 sin(\phi_2 - \phi_1)}{(l_1 sin(\varphi - \phi_1) \mp l_2 sin(\varphi - \phi_2))^2}
\end{aligned}
$$

$\square$