

**RIEMANN ZERO SPACINGS AND MONTGOMERY'S
PAIR CORRELATION CONJECTURE**

by

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Abstract

This paper presents an overview of mathematical work surrounding Montgomery's pair correlation conjecture.

The first chapter introduces the Riemann zeta function and Riemann's method of computation of the first several zeros on the vertical line $\frac{1}{2} + it$.

Chapter 2 presents Montgomery's pair correlation conjecture following his original paper from 1971.

Chapter 3 concerns the Gaussian Unitary Ensemble of random matrices, used to model particle physics and having eigenvalue distribution paralleling the distribution of nontrivial zeros of the Riemann zeta function, as well as touching on similar matrix ensembles.

Chapter 4 presents empirical results of the distribution of nontrivial zeros, obtained computationally by Odlyzko, and the methods used to obtain them.

The final chapter presents brief highlights of recent results which contribute to the growing body of Riemann zeta-to-physics and Riemann zeta-to-random matrix theory correspondence.

Dedicated to my parents

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Chapter 1

The Riemann Zeta Function

1.1 Introduction

In the middle of the eighteenth century, Euler formally rearranged the series¹

$$\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

into what has become known as the Euler product formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(\sum_{k=0}^{\infty} \frac{1}{p^{ks}} \right) = \prod_{p \text{ prime}} \left(\frac{1}{1 - \frac{1}{p^s}} \right). \quad (1.1)$$

Formal equality between the left- and right-hand sides of (1.1) follows from the fundamental theorem of arithmetic, as for every integer $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, the term $\frac{1}{p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}} = \frac{1}{n}$ appears in the expansion on the right exactly once. While Euler was mainly interested in this series for integer values of s (see [11]), Dirichlet in the nineteenth century took an interest in letting s take real values:

$$\sum_{n=1}^{\infty} \frac{1}{n^s}, s \in \mathbb{R}, \quad (1.2)$$

where setting $s = 1$ gives the (divergent) harmonic series. Bernhard Riemann, who studied under Dirichlet, took a major step forward by extending the definition of this function to complex numbers $s = \sigma + it$, $\sigma > 1$, and naming the function $\zeta(s)$. The series converges

¹In fact Euler further made some astute observations connecting the growth of $\sum \frac{1}{n}$ to the density of primes (see [11]).

in the domain $\sigma > 1$ as shown by the integral test:

$$\begin{aligned} \left| \sum_{n=1}^{\infty} \frac{1}{n^s} \right| &\leq \sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| \\ &= \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \\ &\leq \int_1^{\infty} \frac{1}{x^\sigma} dx + 1 \\ &= \frac{1}{\sigma - 1} + 1 < \infty. \end{aligned}$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $\sigma > 1$ and uniformly for $\sigma \geq 1 + \epsilon$.

We now have a function which is analytic in the region $\sigma > 1$ and the Euler product formula (1.1) is valid in this region.

In order to analytically continue the function $\zeta(s)$ to the right half-plane $\Re(s) > 0$ we introduce the closely related Dirichlet eta function $\eta(s)$,

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s},$$

also known as the alternating zeta function. The function $\eta(s)$ allows us to write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} + \sum_{n=1}^{\infty} \frac{1}{n^s} &= 2 \sum_{n=2,4,\dots}^{\infty} \frac{1}{n^s} \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ &= 2^{1-s} \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= 2^{1-s} \zeta(s), \end{aligned}$$

so that

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}.$$

Since the last series converges for $\sigma > 0$, this gives the analytic continuation of $\zeta(s)$ to the region $\Re(s) > 0$.

The next step is to analytically continue $\zeta(s)$ to all of \mathbb{C} . This can be shown using the functional equation for $\zeta(s)$, for which we require the Gamma function:

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^{s-1} dt.$$

We recall that Γ satisfies $\Gamma(s+1) = s\Gamma(s)$ and interpolates the factorial function²: $\Gamma(n) = (n-1)!$ for positive integers n . Substituting $n^2\pi x$ for t gives

$$\Gamma\left(\frac{s}{2}\right) = (n^2\pi)^{\frac{s}{2}} \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1} dx.$$

Dividing both sides by $(n^2\pi)^{\frac{s}{2}}$ and summing over positive integers n then gives

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-n^2\pi x} x^{\frac{s}{2}-1} dx. \quad (1.3)$$

We also require the following function. Define

$$\varpi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x}.$$

This function is related to Jacobi's θ function, $\theta(x) := \sum_{-\infty}^{\infty} e^{-n^2\pi x}$, by the simple relation

$$2\varpi(x) = \theta(x) - 1.$$

Then since the series given by $\varpi(x)$ is uniformly convergent on $[0, \infty)$, we may interchange the order of summation and integration in (1.3) to obtain

$$\begin{aligned} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^{\infty} \left(\sum_{n=1}^{\infty} e^{-n^2\pi x} \right) x^{\frac{s}{2}-1} dx \\ &= \int_0^{\infty} \varpi(x) x^{\frac{s}{2}-1} dx \\ &= \int_0^1 \varpi(x) x^{\frac{s}{2}-1} dx + \int_1^{\infty} \varpi(x) x^{\frac{s}{2}-1} dx. \end{aligned} \quad (1.4)$$

²For historical reasons the function $\Gamma(n)$ is shifted by 1 from the factorial function, whereas the now-antiquated Π function satisfies $\Pi(n) = n!$

The Jacobi θ -function also satisfies a simple functional equation which allows us to write

$$2\varpi(x) + 1 = x^{-\frac{1}{2}} \left(2\varpi\left(\frac{1}{x}\right) + 1 \right). \quad (1.5)$$

Then (1.4) can be written as

$$\begin{aligned} & \int_1^\infty \varpi(x) x^{\frac{s}{2}-1} dx + \int_1^\infty \varpi\left(\frac{1}{x}\right) x^{\frac{s-3}{2}} dx + \frac{1}{2} \int_0^1 \left(x^{\frac{s-3}{2}} - x^{\frac{s}{2}-1} \right) dx \\ &= \frac{1}{s(s-1)} + \int_1^\infty \varpi(x) \left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}} \right) dx. \end{aligned}$$

At this point Riemann defines

$$\xi(t) = \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-\frac{s}{2}} \zeta(s),$$

however it is conventional today to define ξ as a function of s (Riemann also mentions this function in his paper without naming it). This function has a beautiful symmetry about the line $\sigma = \frac{1}{2}$ (and as we will see below shares all non-trivial zeros with $\zeta(s)$):

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) \quad (1.6)$$

so that we have

$$\xi(s) = \xi(1-s). \quad (1.7)$$

The function $\xi(s)$ is entire on the complex plane \mathbb{C} .

1.2 Location of Nontrivial Zeros

So-called trivial zeros of the ζ -function occur at negative even integers; this is clear from (1.6) since the function $\Gamma\left(\frac{s}{2}\right)$ has simple poles at these locations.

The complex, or nontrivial zeros, are precisely the zeros of the function $\xi(s)$. The statement that $\zeta(1+it) \neq 0$ for all real t is equivalent to the Prime Number Theorem. The conjecture that $\sigma = \frac{1}{2}$ for all nontrivial zeros, is the Riemann Hypothesis (RH), and has been verified for the first ten trillion zeros as of 2012 (see Chap. 4).

That the zeros occur for values of s having real part $0 \leq \sigma \leq 1$ follows from the symmetry of $\xi(s)$ about $\sigma = \frac{1}{2}$ and the fact that if $\sigma > 1$ then the logarithm $\log \zeta(s) = -\sum_p \log(1-p^{-s})$ remains finite.

The following argument will demonstrate that, in fact $0 < \sigma < 1$. Taking $\sigma > 1$, the Euler product formula for $\zeta(s)$ is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Taking the logarithm of each side of the above expression then gives

$$\log \zeta(s) = - \sum_p \log \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The right hand side can then be expanded via the Taylor series for $\log(1 - x)$ at $x = 0$:

$$\begin{aligned} \log \zeta(s) &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-sm} \\ &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} p^{-imt} \\ &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} e^{-imt \log p}. \end{aligned}$$

The real part of $\log \zeta(s)$ is thus

$$\Re \log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} \cos(mt \log p).$$

The inequality

$$3 + 4 \cos \theta + \cos(2\theta) \geq 0,$$

which is derived simply by expanding $2(\cos \theta + 1)^2$, can then be applied to the expression for $\Re(\log \zeta(s))$ as follows:

$$\begin{aligned} &3\Re(\log \zeta(\sigma)) + 4\Re(\log \zeta(\sigma + it)) + \Re(\log \zeta(\sigma + 2it)) \\ &= 3 \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} + 4 \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} \cos(mt \log p) + \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} \cos(2mt \log p) \\ &= \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-\sigma m} (3 + 4 \cos(mt \log p) + \cos(2mt \log p)). \end{aligned}$$

Since $\Re(\log(w)) = \log|w|$ for all complex w , this becomes

$$3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)| \geq 0,$$

which is equivalent to

$$|\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$$

The function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. Hence the Laurent series of $\zeta(\sigma)$ at $\sigma = 1$ is

$$\zeta(\sigma) = \frac{1}{1-\sigma} + a_0 + a_1(\sigma - 1) + a_2(\sigma - 1)^2 + \dots = \frac{1}{1-\sigma} + g(\sigma)$$

where $g(\sigma)$ is analytic at $\sigma = 1$. If $1 < \sigma \leq 2$ then $g(\sigma) = O(1)$ and so

$$\zeta(\sigma) = \frac{1}{1-\sigma} + O(1).$$

The mean value theorem can now be used to show that $\zeta(\sigma + it) = 0$ for no t . Suppose to the contrary that $\zeta(\sigma + it) = 0$ for $t \neq 0$. Then for any $\sigma > 1$,

$$\begin{aligned} |\zeta(\sigma + it)| &= |\zeta(\sigma + it) - \zeta(1 + it)| \\ &= |\sigma - 1| |\zeta'(\sigma_0 + it)| \\ &\leq A(\sigma - 1), \end{aligned}$$

where $1 < \sigma_0 < \sigma$ and A is dependent on t . It is also clear that $|\zeta(\sigma + 2it)|$ is bounded by some B dependent on t , within any neighbourhood not containing $s = 1$. Now by observing the degrees of the terms containing $\sigma - 1$, one sees that

$$\begin{aligned} \lim_{\sigma \rightarrow 1^+} |\zeta(\sigma)|^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| &\leq \lim_{\sigma \rightarrow 1^+} \left(\frac{1}{\sigma - 1} + O(1) \right)^3 A^4 (\sigma - 1)^4 B \\ &= 0. \end{aligned}$$

This contradicts (1.2), establishing the result that $\zeta(1 + it) \neq 0$ for any t .

The result is equivalent to the Prime Number Theorem, as proved independently by Hadamard and de la Vallée Poussin [33].

The Prime Number Theorem can be stated as follows:

$$\Psi(x) \sim x \quad \text{as } x \rightarrow \infty \tag{1.8}$$

where

$$\Psi(x) := \sum_{n \leq x} \Lambda(n)$$

and

$$\Lambda(n) := \begin{cases} \log p & \text{for } n = p^k \\ 0 & \text{otherwise.} \end{cases}$$

Selberg and Erdős proved the Prime Number Theorem independently by elementary methods (though there is some controversy in the history, see [6]). Here we present a brief outline of the analytic proof, which uses the fact $\zeta(1 + it) \neq 0$, following Apostol [1].

The first step of the proof is to define the integral

$$\Psi_1(x) = \int_1^x \Psi(t) dt,$$

which is a continuous piecewise linear function and is more convenient to deal with than $\Psi(x)$.

Next, after showing that

$$\Psi_1(x) \sim \frac{1}{2}x^2 \text{ as } x \rightarrow \infty$$

implies (1.8), the bulk of the proof consists of proving that $\Psi_1(x) \sim \frac{1}{2}x^2$. The following equation is helpful for this:

$$\frac{\Psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds, \quad \text{where } c > 1. \quad (1.9)$$

The quotient $-\zeta'(s)/\zeta(s)$ has a pole of order 1 at $s = 1$ with residue 1. By subtracting this pole a new formula is obtained:

$$\frac{\Psi_1(x)}{x^2} - \frac{1}{2} \left(1 - \frac{1}{x} \right)^2 = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) ds, \quad \text{for } c > 1. \quad (1.10)$$

Let

$$h(s) = \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right).$$

Then (1.10) can be rewritten as

$$\begin{aligned} \frac{\Psi_1(x)}{x^2} - \frac{1}{2} \left(1 - \frac{1}{x} \right)^2 &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s-1} h(s) ds \\ &= \frac{x^{c-1}}{2\pi} \int_{-\infty}^{\infty} h(c+it) e^{it \log x} dt. \end{aligned}$$

It is now required to show that

$$\lim_{x \rightarrow \infty} \frac{x^{c-1}}{2\pi} \int_{-\infty}^{\infty} h(c+it)e^{it \log x} dt = 0.$$

When $c > 1$, the Riemann-Lebesgue lemma shows that the integral converges. However, due to the factor x^{c-1} there is an indeterminate form here. Moving the line of integration to $c = 1$ causes this factor disappear. Since the integrand $x^{s-1}h(s)$ involves the quotient $\zeta'(s)/\zeta(s)$, it is necessary to take a few more steps before the proof is complete.

Lemma 1.1. *Suppose a function $f(s)$ has a pole of order k at $s = \alpha$. Then the quotient $f'(s)/f(s)$ has a first order pole at $s = \alpha$ with residue k .*

Proof. The function $f(s)$ equals $g(s)/(s - \alpha)^k$, where g is analytic and nonzero at α . Thus in some neighbourhood of α ,

$$f'(s) = \frac{g'(s)}{(s - \alpha)^k} - \frac{k g(s)}{(s - \alpha)^{k+1}} = \frac{g(s)}{(s - \alpha)^k} \left[\frac{-k}{s - \alpha} + \frac{g'(s)}{g(s)} \right]$$

and so

$$\frac{f'(s)}{f(s)} = \frac{-k}{s - \alpha} + \frac{g'(s)}{g(s)}.$$

□

This lemma implies that the function

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

is analytic at $s = 1$, since both terms have a first order pole at $s = 1$ with residue 1.

1.3 How Riemann Computed Zeros

Riemann's insight in making his famous conjecture is certainly praiseworthy. It later became clear, however, that Riemann had developed sophisticated methods to compute the first few zeros off the real line. The 'rediscovery' of this method is credited to Carl Ludwig Siegel (see [11]); the method is known as the Riemann-Siegel formula. We introduce the formula after some preliminary steps.

Recalling that the zeros we are looking for are those of the ξ function, and making use of the symmetry given by $\xi(s) = \xi(1-s)$, we have

$$\xi\left(\frac{1}{2} + it\right) = \xi\left(\frac{1}{2} - it\right) = \xi\left(\overline{\frac{1}{2} + it}\right) = \overline{\xi\left(\frac{1}{2} + it\right)},$$

indicating that $\xi\left(\frac{1}{2} + it\right)$ is a real-valued function of t . By expanding

$$\begin{aligned} \xi\left(\frac{1}{2} + it\right) &= \frac{1}{2} \left(\frac{1}{2} + it\right) \left(-\frac{1}{2} + it\right) \pi^{-\frac{1}{2}(\frac{1}{2}+it)} \Gamma\left(\frac{\frac{1}{2} + it}{2}\right) \zeta\left(\frac{1}{2} + it\right) \\ &= -\frac{1}{2} \left(\frac{1}{4} + t^2\right) \pi^{-\frac{1}{4}} \pi^{-\frac{it}{2}} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \zeta\left(\frac{1}{2} + it\right), \end{aligned}$$

we see that the argument of the complex part of the expression, namely

$$\pi^{-\frac{it}{2}} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \zeta\left(\frac{1}{2} + it\right),$$

is necessarily an integer multiple of 2π . Defining

$$\theta(t) := \arg\left(\pi^{-\frac{it}{2}} \Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right),$$

it follows that $\arg(\zeta(\frac{1}{2} + it)) = -\theta(t)$. We then define

$$Z(t) := e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right),$$

and show that $Z(t)$ is a real-valued function

$$\begin{aligned} Z(t) &= e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \\ &= e^{i\theta(t)} e^{-i\theta(t)} \left| \zeta\left(\frac{1}{2} + it\right) \right| \\ &= \left| \zeta\left(\frac{1}{2} + it\right) \right|. \end{aligned}$$

The function $Z(t)$ must be continuous since $\zeta(\frac{1}{2} + it)$ is analytic. We may thus find zeros on the critical strip $\sigma = \frac{1}{2} + it$ by finding sign changes in $Z(t)$.

Definition 1.2 (Riemann-Siegel Formula). *The Riemann-Siegel Formula for $Z(T)$ is given by*

$$Z(t) = 2 \sum_{k=1}^{\nu(t)} \frac{1}{\sqrt{k}} \cos[\theta(t) - t \log k] + R(t)$$

with

$$\begin{aligned}\nu(t) &= \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor, \\ p &= \sqrt{\frac{t}{2\pi}} - \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor = \sqrt{\frac{t}{2\pi}} - \nu(t), \\ R(t) &= (-1)^{\nu(t)-1} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} \sum_{k=0}^{\infty} c_k \left(\sqrt{\frac{t}{2\pi}} - \nu(t)\right) \left(\frac{t}{2\pi}\right)^{-\frac{k}{2}} \\ &= (-1)^{\nu(t)-1} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} \sum_{k=0}^{\infty} c_k(p) \left(\frac{t}{2\pi}\right)^{-\frac{k}{2}},\end{aligned}$$

and where the c_k are trigonometric functions which quickly become complicated (see (4.3) for more details).

1.4 Deriving the Formula

In his seminal paper "On the Number of Primes Less Than a Given Magnitude," [25] in order to prove the Prime Number Theorem Riemann establishes the following analytic continuation of $\zeta(s)$ to the complex plane:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s dx}{e^x - 1} \frac{1}{x}, \quad (1.11)$$

where the contour of integration begins at $+\infty$, moves in the negative direction along the real axis, circles the origin counterclockwise in the positive direction, and moves in the positive direction back along the real axis to $+\infty$. Following Edwards [11], two methods of splitting finite sums from (1.11) are recombined to reach the desired result. The first is to replace $(e^x - 1) = \sum e^{-nx}$ with

$$\frac{e^{-Nx}}{e^x - 1} = \sum_{n=N+1}^{\infty} e^{-nx}$$

which yields

$$\zeta(s) = \sum_{n=1}^N n^{-s} + \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{e^{-Nx} (-x)^s dx}{e^x - 1} \frac{1}{x}. \quad (1.12)$$

The second is to change the contour of integration in (1.11) so that the circle around the origin expands to include the poles $\pm 2\pi i, \pm 4\pi i, \dots, \pm 2M\pi i$, so that using the residue theorem gives

$$\zeta(s) = 2\Gamma(1-s)(2\pi)^{(s-1)} \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^M n^{s-1} + \frac{\Gamma(1-s)}{2\pi i} \int_{C_M} \frac{(-x)^s dx}{e^x - 1} \frac{1}{x}. \quad (1.13)$$

The combination of the two techniques gives

$$\zeta(s) = \sum_{n=1}^N n^{-s} + 2\Gamma(1-s)(2\pi)^{(s-1)} \sin\left(\frac{\pi s}{2}\right) \sum_{n=1}^M n^{s-1} + \frac{\Gamma(1-s)}{2\pi i} \int_{C_M} \frac{e^{-Nx}(-x)^s dx}{e^x - 1} \frac{1}{x}. \quad (1.14)$$

From this we obtain a formula for $\xi(s)$ valid for all N, M, s , by multiplying by the factor $\frac{1}{2}s(s-1)\Gamma(s/2)\pi^{-s/2}$ which allows for functional symmetry:

$$\begin{aligned} \xi(s) &= (s-1)\Gamma\left(\frac{s+1}{2}\right) \pi^{-s/2} \sum_{n=1}^N n^{-s} \\ &\quad - s\Gamma\left(\frac{3-s}{2}\right) \pi^{(s-1)/2} \sum_{n=1}^M n^{s-1} \\ &\quad - \frac{s\Gamma(2-s/2)\pi^{(s-1)/2}}{(2\pi)^{s-1} \sin(\pi s/2) 4\pi i} \int_{C_M} \frac{(-x)^s e^{-Nx} dx}{e^x - 1} \frac{1}{x}. \end{aligned} \quad (1.15)$$

Chapter 2

Montgomery's Pair Correlation conjecture

Hugh Montgomery's conjecture on the distribution of pairwise differences $\gamma - \gamma'$ for non-trivial zeros $\frac{1}{2} + i\gamma$ of the Riemann Zeta Function is a seminal example the fruitful overlap between number theory and physics. The realization of this possibility first took place when Montgomery reluctantly allowed himself to be introduced to physicist Freeman Dyson. When Montgomery told Dyson of his predictions, Dyson surprisingly was already familiar with distribution, as it is the distribution of the pairwise distances between eigenvalues of random Hermitian matrices.

Montgomery's conjecture is that, for fixed real numbers $\alpha < \beta$,

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ 2\pi\alpha / \log T \leq \gamma - \gamma' \leq 2\pi\beta / \log T}} 1 \sim \left(\int_{\alpha}^{\beta} 1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 du + \delta(\alpha, \beta) \right) \frac{T}{2\pi} \log T$$

as T tends to infinity, where the double summation is over all pairs of nontrivial zeros of $\frac{1}{2} + i\gamma$ of $\zeta(s)$ and where δ is the Dirac delta distribution, that is $\delta(\alpha, \beta) = 1$ if $0 \in [\alpha, \beta]$, $\delta(\alpha, \beta) = 0$ otherwise. We may rephrase the conjecture by saying that the pair-correlation (i.e. the 2-point correlation) function of the non-trivial zeros of the Riemann zeta function is

$$R_2(u) := 1 - \left(\frac{\sin \pi u}{\pi u} \right)^2,$$

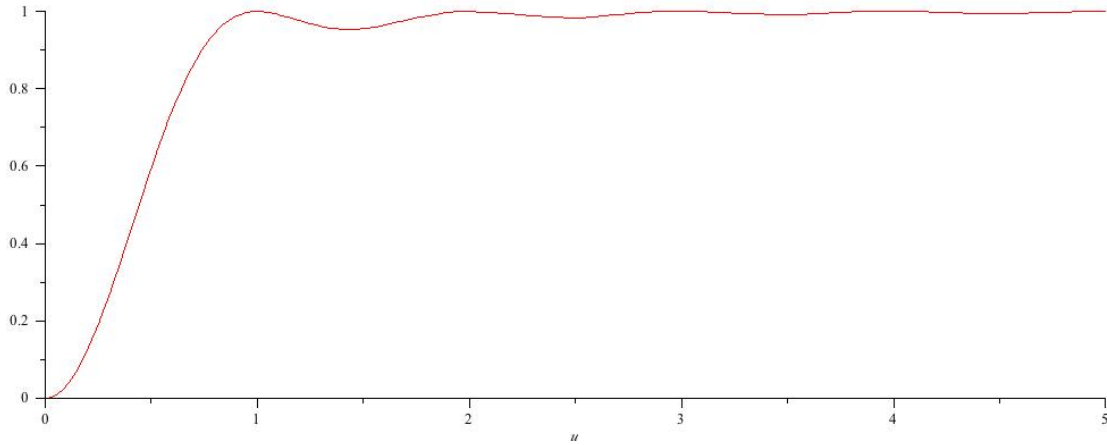


Figure 2.1: The function $1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$

where the notation R_2 will become clear in Chapter 3.

In his 1971 manuscript [18] Montgomery leads to this conjecture by first proving a theorem and three corollaries which will be outlined here.

Define

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'),$$

where α and $T \geq 2$ are real. Montgomery uses $w(u) = 4/(4 + u^2)$ here as a suitable weighting function.

Theorem 2.1. (Assume RH) For real $\alpha, T \geq 2$, and $F(\alpha)$ defined as above, $F(\alpha)$ is real with $F(\alpha) = F(-\alpha)$. If $T > T_0(\epsilon)$ then $F(\alpha) \geq -\epsilon$ for all α . For fixed α such that $0 \leq \alpha < 1$,

$$F(\alpha) = (1 + o(1))T^{-2\alpha} \log T + \alpha + o(1)$$

as T tends to infinity. This holds uniformly for $0 \leq \alpha \leq 1 - \epsilon$.

It is necessary here to find a kernel \hat{r} with which to convolve $F(\alpha)$. Defining \hat{r} to be the Fourier transform of r ,

$$\hat{r}(\alpha) = \int_{-\infty}^{\infty} r(u)e^{-2\pi i\alpha u} du$$

gives the formula

$$\sum_{0 < \gamma, \gamma' \leq T} r \left((\gamma - \gamma') \frac{\log T}{2\pi} \right) w(\gamma - \gamma') = \left(\frac{T}{2\pi} \log T \right) \int_{-\infty}^{\infty} F(\alpha) \hat{r}(\alpha) d\alpha. \quad (2.1)$$

We have

$$\begin{aligned} \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(\alpha) \hat{r}(\alpha) d\alpha &= \int_{-\infty}^{\infty} \sum_{\gamma, \gamma' \in [0, T]} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma') \hat{r}(\alpha) d\alpha \\ &= \sum_{\gamma, \gamma' \in [0, T]} w(\gamma - \gamma') \int_{-\infty}^{\infty} T^{i\alpha(\gamma - \gamma')} \hat{r}(\alpha) d\alpha \\ &= \sum_{\gamma, \gamma' \in [0, T]} w(\gamma - \gamma') \int_{-\infty}^{\infty} \hat{r}(\alpha) e^{2\pi i \alpha (\gamma - \gamma') \log T / 2\pi} d\alpha \\ &= \sum_{\gamma, \gamma' \in [0, T]} r \left((\gamma - \gamma') \frac{\log T}{2\pi} \right) w(\gamma - \gamma'). \end{aligned}$$

Since this theorem gives little information for the case $\alpha \geq 1$, attention here is restricted to kernels \hat{r} which vanish outside $[-1 + \delta, 1 - \delta]$.

Corollary 2.2. (Assume RH) If $0 < \alpha < 1$ is fixed then

$$\sum_{0 < \gamma, \gamma' \leq T} \left(\frac{\sin \alpha (\gamma - \gamma') \log T}{\alpha (\gamma - \gamma') \log T} \right) w(\gamma - \gamma') \sim \left(\frac{1}{2\alpha} + \frac{\alpha}{2} \right) \frac{T}{2\pi} \log T,$$

and

$$\sum_{0 < \gamma, \gamma' \leq T} \left(\frac{\sin(\frac{\alpha}{2})(\gamma - \gamma') \log T}{\frac{\alpha}{2}(\gamma - \gamma') \log T} \right)^2 w(\gamma - \gamma') \sim \left(\frac{1}{\alpha} + \frac{\alpha}{3} \right) \frac{T}{2\pi} \log T.$$

The proof of the corollary requires the following two kernels obtained by Fourier transform.

Lemma 2.3. For $0 < a < 1$,

$$r_1(u) = \frac{\sin(2\pi au)}{2\pi au}$$

has Fourier transform

$$\hat{r}_1(\xi) = \frac{1}{2a} \chi_a(\xi)$$

where χ_a is the characteristic function for the interval $[-a, a]$.

Proof. This is verified by Fourier inversion:

$$\begin{aligned}
 r_1(u) &= \int_{-\infty}^{\infty} \hat{r}_1(\xi) e^{2\pi i u \xi} d\xi \\
 &= \frac{1}{2a} \int_{-a}^a e^{2\pi i u \xi} d\xi \\
 &= \frac{e^{2\pi i a u} - e^{-2\pi i a u}}{2a(2\pi i u)} \\
 &= \frac{\sin(2\pi a u)}{2\pi a u}.
 \end{aligned}$$

□

Lemma 2.4. *The function*

$$s(u) = \left(\frac{\sin \pi u}{\pi u} \right)^2$$

has Fourier transform

$$\hat{s}(\xi) = (1 - |\xi|)\chi_1(\xi).$$

Proof. Applying Fourier inversion gives

$$\begin{aligned}
 s(u) &= \int_{-\infty}^{\infty} \hat{s}(\xi) e^{2\pi i u \xi} d\xi \\
 &= \int_{-1}^1 (1 - |\xi|) e^{2\pi i u \xi} d\xi \\
 &= \int_{-1}^0 (1 + \xi) e^{2\pi i u \xi} d\xi + \int_0^1 (1 - \xi) e^{2\pi i u \xi} d\xi \\
 &= \frac{1}{(2\pi i u)^2} (e^{2\pi i u} + e^{-2\pi i u} - 2) \\
 &= \frac{2}{(2\pi i u)^2} (\cos 2\pi u - 1) \\
 &= \left(\frac{\sin \pi u}{\pi u} \right)^2,
 \end{aligned}$$

where the last equality is due to trigonometric identities. □

Lemma 2.5. *For $0 < a < 1$,*

$$r_2(u) = \left(\frac{\sin \pi a u}{\pi a u} \right)^2$$

has Fourier transform

$$\hat{r}_2(\xi) = \frac{1}{a^2}(a - |\xi|)\chi_a(\xi).$$

Proof. Using the function $\hat{s}(\xi)$ from the previous lemma gives

$$\begin{aligned} \hat{s}\left(\frac{\xi}{a}\right) &= \left(1 - \frac{|\xi|}{a}\right)\chi_a(\xi) \\ &= \frac{1}{a}(a - |\xi|)\chi_a(\xi) \\ &= a \cdot \hat{r}_2(\xi). \end{aligned}$$

Thus,

$$\begin{aligned} r_2(u) &= \int_{-\infty}^{\infty} \hat{r}_2(\xi) e^{2\pi i u \xi} d\xi \\ &= \frac{1}{a} \int_{-\infty}^{\infty} \hat{s}\left(\frac{\xi}{a}\right) e^{2\pi i u \xi} d\xi \\ &= \int_{-\infty}^{\infty} \hat{s}(\xi) e^{2\pi i u \xi} d\xi \\ &= s(au) \\ &= \left(\frac{\sin \pi au}{\pi au}\right)^2. \end{aligned}$$

□

The first of Montgomery's corollaries can now be proved.

Proof. By the convolution formula (2.1) and for \hat{r}_1 as previously defined,

$$\sum_{\gamma, \gamma' \in [0, T]} \left(\frac{\sin(2\pi a(\gamma - \gamma') \frac{\log T}{2\pi})}{2\pi a(\gamma - \gamma') \frac{\log T}{2\pi}} \right) w(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u) \hat{r}_1(u) du.$$

The kernel \hat{r}_1 has support in $(-1, 1)$ and so the integral on the right hand side can be

computed as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} F(u)\hat{r}_1(u)du &= \frac{1}{2a} \int_{-a}^a F(u)du \\
&= \frac{1}{a} \int_0^a F(u)du \\
&= \frac{1}{a} \int_0^a [(1+o(1))T^{-2u} \log T + u + o(1)]du \\
&= \frac{1}{a} \int_0^a [(1+o(1))e^{-2u \log T} \log T + u + o(1)]du \\
&= \frac{1}{2a} + \frac{a}{2}
\end{aligned}$$

as $T \rightarrow \infty$. Using this expression for the integral in the convolution formula gives the desired result,

$$\sum_{\gamma, \gamma' \in [0, T]} \left(\frac{\sin(a(\gamma - \gamma') \log T)}{a(\gamma - \gamma') \log T} \right) w(\gamma - \gamma') \sim \left(\frac{1}{2a} + \frac{a}{2} \right) \frac{T}{2\pi} \log T.$$

The second formula is proved with the use of \hat{r}_2 . Using the convolution formula gives

$$\sum_{\gamma, \gamma' \in [0, T]} \left(\frac{\sin((a/2)(\gamma - \gamma') \log T)}{(a/2)(\gamma - \gamma') \log T} \right)^2 w(\gamma - \gamma') = \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(u)\hat{r}_2(u)du.$$

The symmetry of $F(u)$ and the support of \hat{r}_2 allows the integral to be computed as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} F(u)\hat{r}_2(u)du &= \frac{1}{a^2} \int_{-a}^a F(u)(a - |u|)du \\
&= \frac{2}{a} \int_0^a F(u)du - \frac{2}{a^2} \int_0^a uF(u)du.
\end{aligned}$$

The first of these two integrals is

$$\begin{aligned}
\frac{2}{a} \int_0^a F(u)du &= \frac{2}{a} \int_0^a [(1+o(1))T^{-2u} \log T + u + o(1)]du \\
&= \frac{2}{a} \int_0^a [(1+o(1))e^{-2u \log T} \log T + u + o(1)]du \\
&= \frac{1}{a} + a
\end{aligned}$$

as $T \rightarrow \infty$. The second integral is

$$\begin{aligned} \frac{2}{a^2} \int_0^a uF(u)du &= \frac{2}{a^2} \int_0^a u[(1+o(1))T^{-2u} \log T + u + o(1)]du \\ &= \frac{2a}{3} \end{aligned}$$

as $T \rightarrow \infty$. Combining the two gives

$$\begin{aligned} \int_{-\infty}^{\infty} F(u)\hat{r}_2(u)du &= \frac{1}{a} + a - \frac{2a}{3} \\ &= \frac{1}{a} + \frac{a}{3}, \end{aligned}$$

which implies

$$\sum_{\gamma, \gamma' \in [0, T]} \left(\frac{\sin((a/2)(\gamma - \gamma') \log T)}{(a/2)(\gamma - \gamma') \log T} \right)^2 w(\gamma - \gamma') = \left(\frac{1}{a} + \frac{a}{3} \right) \frac{T}{2\pi} \log T$$

as required. □

The next corollary follows from the second half of Corollary 2.2:

Corollary 2.6. (Assume RH) As T tends to infinity,

$$\sum_{\substack{0 < \gamma \leq T \\ \rho \text{ simple}}} 1 \geq \left(\frac{2}{3} + o(1) \right) \frac{T}{2\pi} \log T,$$

where ρ are the nontrivial zeros of the zeta function.

Proof. Let m_ρ denote the multiplicity of a nontrivial zero ρ . Then

$$\sum_{\substack{\gamma, \gamma' \in [0, T] \\ \gamma = \gamma'}} 1 = \sum_{\gamma \in [0, T]} m_\rho,$$

since each zero ρ is counted m_ρ^2 times on each side. Now since $w(0) = 1$ and we have $\sin 0/0$ whenever $\gamma = \gamma'$, the following inequality must hold:

$$\sum_{\gamma \in [0, T]} m_\rho \leq \sum_{\gamma, \gamma' \in [0, T]} \left(\frac{\sin((a/2)(\gamma - \gamma') \log T)}{(a/2)(\gamma - \gamma') \log T} \right)^2 w(\gamma - \gamma').$$

By letting $a = 1 - \delta$ and applying Corollary 2.2, one obtains, as $T \rightarrow \infty$,

$$\sum_{\gamma \in [0, T]} m_\rho \leq \left(\frac{1}{1 - \delta} + \frac{1 - \delta}{3} \right) \frac{T}{2\pi} \log T = \left(\frac{4}{3} + o(1) \right) \frac{T}{2\pi} \log T.$$

Note also that

$$\sum_{\substack{\gamma \in [0, T] \\ \rho \text{ simple}}} \geq \sum_{\gamma \in [0, T]} (2 - m_\rho) = 2N(T) - \sum_{\gamma \in [0, T]} m_\rho,$$

where $N(T)$ is the number of zeros in the critical strip with height less than or equal to T . Since, as estimated by Riemann and proved by von Mangoldt [11],

$$N(T) \sim \frac{T}{2\pi} \log T,$$

one may conclude that as $T \rightarrow \infty$

$$\sum_{\substack{\gamma \in [0, T] \\ \rho \text{ simple}}} \geq 2 \left(\frac{T}{2\pi} \log T \right) - \left(\frac{4}{3} + o(1) \right) \frac{T}{2\pi} \log T = \left(\frac{2}{3} + o(1) \right) \frac{T}{2\pi} \log T.$$

□

Corollary 2.7. (Assume RH) One can find a constant λ so that

$$\liminf_n (\gamma_{n+1} - \gamma_n) (\log \gamma_n / 2\pi) \leq \lambda < 1.$$

One may interpret this as saying that the difference between consecutive zeros dips below the average (that is, below $\log \gamma / 2\pi$), by a proportion greater than $1 - \lambda$, infinitely often. This fact, together with the fact that a positive density of zeros satisfy $\gamma_{n+1} - \gamma_n \leq 2\pi\lambda / \log \gamma_n$ ([18]) implies the similar assertion

$$\limsup_n (\gamma_{n+1} - \gamma_n) (\log \gamma_n / 2\pi) \geq \lambda' > 1$$

for some constant λ' .

Proof. To prove the corollary take $r(u) = \max(1 - (|u|/\lambda), 0)$, leaving the choice of λ for later. Since $\hat{r}(\alpha)$ is nonnegative and $\int_0^\infty \hat{r}(\alpha) d\alpha < \infty$, it can be seen that

$$\left(\frac{T}{2\pi} \log T \right) \int_{-\infty}^\infty F(\alpha) \hat{r}(\alpha) d\alpha \geq (1 + o(1)) \left(\lambda + 2\lambda \int_0^1 \alpha \left(\frac{\sin \pi \lambda \alpha}{\pi \lambda \alpha} \right) d\alpha \right) \frac{T}{2\pi} \log T.$$

One may assume all but finitely many nontrivial zeros are simple, for otherwise $\gamma = 0$. It follows that the terms with $\gamma = \gamma'$ contribute an amount which is asymptotic to $(T/2\pi) \log T$. Thus

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ 0 < \gamma - \gamma' < 2\pi\lambda / \log T}} 1 \geq \left(\frac{1}{2} + o(1) \right) C(\lambda) \frac{T}{2\pi} \log T$$

where

$$C(\lambda) = \lambda + \frac{1}{\pi^2 \lambda} \int_0^{2\pi\lambda} \frac{1 - \cos u}{u} du - 1.$$

□

Chapter 3

The Gaussian Unitary Ensemble

3.1 Introduction

Wigner's proposal for the probability density function of a sequence of energy levels with identical spin and parity (called simple sequence) was as follows:

$$p_W(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi}{4}s^2\right),$$

where $s = S/D$, while for cases with mixed spin and parity the probability density function is obtained by randomly superimposing the simple sequences making up the mix ([17]).

Definition 3.1. *A square matrix A is Hermitian if it is equal to its own conjugate transpose A^\dagger ; that is, if $A = (a_{ij})$ then $a_{ij} = \overline{a_{ji}}$ for all i, j .*

In quantum systems, eigenvalues E_n and wavefunctions $\psi_n(\mathbf{r})$ arise as solutions to the Schrodinger equation

$$H\psi_n(\mathbf{r}) = E_n\psi_n(\mathbf{r}),$$

where H is the Hamiltonian operator. Because H is Hermitian the eigenvalues are real. In bound systems, the eigenvalues are discrete and we have the density

$$d(E) = \sum_{n=1}^{\infty} \delta(E - E_n)$$

where δ is the Dirac delta distribution.

As the complexity of a system increases, for example in the case of increasing energy in an atomic nucleus, the calculations become intractable. Wigner had the bright idea of modeling the statistics with a certain kind of "random" ensemble which would preserve those symmetries which were guaranteed by the system in question, while being independent of the particular values of the inputs of the system. In this way general information on the level structures of similar nuclei is obtained, such that comparing data obtained on the energy levels particular of a particular isotope to the ensemble average can yield insight into unusual properties of the isotope under study. Thus was born the use of random matrix theory in nuclear physics.

The Gaussian Unitary Ensemble is one of a class of three matrix ensembles developed in the study of energy levels of physical systems as pioneered by E. Wigner (see [17]) to study the spectra of heavy atoms. Unlike the other two in this class, the Gaussian Symplectic Ensemble (GSE) and Gaussian Orthogonal Ensemble (GOE), the systems modeled by the GUE do not possess time-reversal invariance. In a classic 1963 article Dyson [10] shows that for a given symmetry G of a system Z represented by matrices, the matrices must be Hermitian for certain invariance properties to hold. Furthermore, he shows that under these invariance conditions the set Z is a direct product of three irreducible components: one corresponding to the algebra of \mathbb{R} , one to the algebra of \mathbb{C} and one to the algebra of the quaternions.

A system with integral total spin (or angular momentum) and which is symmetrical under time-reversal has a Hamiltonian matrix H which is symmetric (and therefore real symmetric). The same can be said of a system with time-reversal symmetry and rotational symmetry. The corresponding ensemble E_{1G} in the space T_{1G} of symmetrical real matrices is defined by two requirements (see [17]):

1. The probability $P(H)dH$ that a system of E_{1G} will belong to the volume element $dH = \prod_{j \leq k} dH_{jk}$ is invariant under real orthogonal transformations:

$$P(H')dH' = P(H)dH$$

where

$$H' = W^T H W$$

and

$$W^T W = W W^T = 1.$$

2. The probability density function $P(H)$ is a product of independent functions of a single variable:

$$P(H) = \prod_{j \leq k} f_{jk}(H_{jk}).$$

More specifically, Mehta shows further that the form is

$$P(H) = \text{const} \times \prod_j \exp[bH_{jj}] \prod_{j \leq k} \exp[-a(H_{jk})^2],$$

with a, b real and $a > 0$.

The system just described is known as the Gaussian Orthogonal Ensemble or GOE. To see why, note that condition 1 guarantees that probability density function is independent of the choice of orthogonal basis for the system. Generally, if the basis is changed by a unitary transformation $\psi \rightarrow U\psi$ then the Hamiltonian changes as $H \rightarrow U H U^{-1}$. Since U in the case must be real and symmetric as well as unitary, it must be an orthogonal matrix (see [29]).

In the case of physical systems with time-reversal symmetry and half-integer total spin (or angular momentum), the appropriate Hamiltonians take the form of self-dual Hermitian matrices. By definition a self-dual matrix is equal to its time reverse ($H = H^R$). Rotational symmetry is allowed but not required in this ensemble, called the Gaussian Symplectic Ensemble or GSE. The space T_{4G} of self-dual matrices thus contains T_{1G} as a small subset [29]. Matrices in the symplectic ensemble can either be written as $N \times N$ matrices with quaternion entries, or as $2N \times 2N$ matrices where the quaternion units are in 2×2 matrix form:

$$e_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$e_1^2 = e_2^2 = e_3^2 = -1$$

and

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2.$$

The conditions defining this ensemble are:

1. The probability $P(H)dH$ that a system of E_{4G} will belong to the volume element $dH = \prod_{j \leq k} dH_{jk}^{(0)} \prod_{\lambda=1}^3 \prod_{j < k} dH_{jk}^{(\lambda)}$ is invariant under symplectic transformations :

$$P(H')dH' = P(H)dH$$

where

$$H' = W^R H W.$$

2. The probability density function $P(H)$ is a product of independent functions of a single variable:

$$P(H) = \prod_{j \leq k} f_{jk}^{(0)}(H_{jk}^{(0)}) \prod_{\lambda=1}^3 \prod_{j < k} f_{jk}^{(\lambda)}(H_{jk}^{(\lambda)}),$$

where the index λ is used to denote the quaternion units.

Our third ensemble is mathematically simplest and corresponds to physical systems without time reversal symmetry. A Hamiltonian representing such a system may be represented by an arbitrary Hermitian matrix. Interestingly, though, this ensemble represents a situation which would be unrealistic in nuclear physics, as the atomic nucleus under observation would have to be placed into an environment completely mixing its entire "natural" (i.e. zero field) level structure [17]. This ensemble is the Gaussian Unitary Ensemble and will be defined after some preliminaries. In addition to these three ensembles, one can also consider two other cases which will not be covered here. The first is a mix of ensembles to represent a system where time reversal invariance is weakly violated. The second is the Gaussian ensemble of antisymmetric (anti-self-dual quaternion) Hermitian matrices, which Mehta describes as elegant though not physically relevant [17].

3.2 Determining the joint probability density function from the constraints

Following Mehta [17], we describe how the above pairs of requirements on each ensemble, that is invariance under symmetry and independence of matrix elements (in order to achieve

"randomness"), determine the form of the joint probability density function of matrix elements. First, due to a result of Weyl ([34]), the invariance requirement restricts $P(H)$ to depend on the traces of powers of H in the following way (see [17]).

Lemma 3.2. *The invariants of an $N \times N$ matrix H under nonsingular transformations given by*

$$H \rightarrow H' = AHA^{-1}$$

can be expressed in terms of the traces of the first N powers of H . Furthermore, the trace of the j th power of H is the sum of j th powers of its eigenvalues $\lambda_1, \dots, \lambda_N$,

$$\text{Tr } H^j = \sum_{k=1}^N \lambda_k^j,$$

and any symmetric function of the λ_k can be expressed in terms of the first N of these traces.

In the Gaussian Orthogonal Ensemble, we have

$$\text{Tr } H^2 = \sum_{j=1}^N \theta_j^2, \quad \text{Tr } H = \sum_{j=1}^N \theta_j,$$

where the θ_j are the eigenvalues of H . A real symmetric matrix H can be diagonalized by a real orthogonal matrix:

$$H = U\Theta U^{-1} = U\Theta U^T$$

where Θ is a diagonal matrix with elements $\theta_1 \leq \theta_2 \leq \dots \leq \theta_N$ and the columns of U are the normalized eigenvectors of H .

In the Symplectic Ensemble, we have the following [17]:

Theorem 3.3. *Given a quaternion-real, self-dual matrix H , there is a symplectic matrix U such that*

$$H = U\Theta U^{-1} = U\Theta U^R,$$

where Θ is diagonal, real and scalar.

That Θ is scalar means that it consists of N blocks of the form $\begin{bmatrix} \theta_j & 0 \\ 0 & \theta_j \end{bmatrix}$, along the main diagonal, so that there are N pairs of equal eigenvalues of H .

The matrix H corresponds to the Hamiltonian of a physical system that is invariant under time reversal, has odd spin, and does not have rotational symmetry. Due to the eigenvalues being paired, the energy levels in such a system are doubly degenerate. Here we have

$$\text{Tr } H^2 = 2 \sum_{j=1}^N \theta_j^2, \quad \text{Tr } H = 2 \sum_{j=1}^N \theta_j.$$

Lemma 3.4. *If three continuous, differentiable functions satisfy*

$$f(xy) = g(x) + h(y)$$

then each of the functions must take the form $a \log x + b$.

Proof. The proof is given in [17] and uses elementary calculus. □

Now consider a particular transformation $H = U^{-1}H'U$, where

$$U = \begin{bmatrix} \cos \theta & \sin \theta & 0 & \dots & 0 \\ -\sin \theta & \cos \theta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

is a matrix which is at once orthogonal, symplectic and unitary. Then differentiating $H = U^{-1}H'U$ with respect to θ gives

$$\frac{\partial H}{\partial \theta} = \frac{\partial U^T}{\partial \theta} H'U + U^T H' \frac{\partial U}{\partial \theta} = \frac{\partial U^T}{\partial \theta} UH + HU^T \frac{\partial U}{\partial \theta},$$

so that we have

$$\frac{\partial H}{\partial \theta} = AH + HA^T,$$

where

$$A = \frac{\partial U^T}{\partial \theta} U = \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since the probability density function

$$P(H) = \prod_{(\lambda)} \prod_{j \leq k} f_{jk}^{(\lambda)} \left(H_{jk}^{(\lambda)} \right)$$

is invariant under transformation by the matrix U , its derivative with respect to θ must be zero:

$$\sum \frac{\partial f_{jk}^{(\lambda)}}{\partial H_{jk}^{(\lambda)}} \frac{\partial H_{jk}^{(\lambda)}}{\partial \theta} = 0.$$

The form of A means this can be written in terms of a few entries of f and H :

$$\begin{aligned} & \left[\left(-\frac{1}{f_{11}^{(0)}} \frac{\partial f_{11}^{(0)}}{\partial H_{11}^{(0)}} + \frac{1}{f_{22}^{(0)}} \frac{\partial f_{22}^{(0)}}{\partial H_{22}^{(0)}} \right) \left(2H_{12}^{(0)} \right) + \frac{1}{f_{12}^{(0)}} \frac{\partial f_{12}^{(0)}}{\partial H_{12}^{(0)}} \left(H_{11}^{(0)} - H_{22}^{(0)} \right) \right] \\ & + \sum_{k=3}^N \left(-\frac{1}{f_{1k}^{(0)}} \frac{\partial f_{1k}^{(0)}}{\partial H_{1k}^{(0)}} H_{2k}^{(0)} + \frac{1}{f_{2k}^{(0)}} \frac{\partial f_{2k}^{(0)}}{\partial H_{2k}^{(0)}} H_{1k}^{(0)} \right) \\ & + \sum_{k=3}^N \left(-\frac{1}{f_{1k}^{(1)}} \frac{\partial f_{1k}^{(1)}}{\partial H_{1k}^{(1)}} H_{2k}^{(1)} + \frac{1}{f_{2k}^{(1)}} \frac{\partial f_{2k}^{(1)}}{\partial H_{2k}^{(1)}} H_{1k}^{(1)} \right) = 0. \end{aligned}$$

All of the two-term sums within braces must be constant, since they depend on mutually exclusive sets of variables and sum to zero. Thus we may write, for example,

$$-\frac{1}{f_{1k}^{(0)}} \frac{\partial f_{1k}^{(0)}}{\partial H_{1k}^{(0)}} H_{2k}^{(0)} + \frac{1}{f_{2k}^{(0)}} \frac{\partial f_{2k}^{(0)}}{\partial H_{2k}^{(0)}} H_{1k}^{(0)} = C_k^{(0)}.$$

Dividing both sides of this equation by $H_{1k}^{(0)} H_{2k}^{(0)}$ and applying Lemma 3.4 implies that $C_k^{(0)}$ must be zero. Thus

$$\frac{1}{H_{1k}^{(0)} f_{1k}^{(0)}} \frac{\partial f_{1k}^{(0)}}{\partial H_{1k}^{(0)}} H_{2k}^{(0)} = \frac{1}{H_{2k}^{(0)} f_{2k}^{(0)}} \frac{\partial f_{2k}^{(0)}}{\partial H_{2k}^{(0)}} H_{1k}^{(0)} = -2a$$

for some constant a , and on integration we have

$$f_{1k}^{(0)} \left(H_{1k}^{(0)} \right) = \exp \left[-a \left(H_{1k}^{(0)} \right)^2 \right].$$

Extending the argument to other terms gives the general formula

$$\begin{aligned} P(H) &= \exp \left(-a \operatorname{Tr} H^2 + b \operatorname{Tr} H + c \right) \\ &= e^c \prod_j \exp \left(b H_{jj}^{(0)} \right) \prod_{j \leq k} \exp \left[-a \left(H_{jk}^{(0)} \right)^2 \right] \\ &\times \prod_{\lambda} \prod_{j < k} \exp \left[-a \left(H_{jk}^{(\lambda)} \right)^2 \right]. \end{aligned}$$

Following Mehta, the Gaussian Unitary Ensemble (GUE) of matrices may be defined within the space of Hermitian matrices T_{2G} in the following way. Let $P(H)dH$ be the probability that an element $H = (H_{jk})$ of T_{2G} belongs to the volume element

$$dH = \prod_{j \leq k} d\Re(H_{jk}) \prod_{j < k} d\Im(H_{jk}).$$

Then the definition of the GUE is as follows.

Definition 3.5. *The Gaussian Unitary Ensemble is defined to be the set of matrices H in T_{2G} such that*

- (i) $P(H)dH = P(H')dH'$ whenever $H' = U^{-1}HU$ for any unitary matrix U
- (ii) *The real and imaginary parts of H_{jk} , $j < k$, are statistically independent, as are the diagonal elements H_{jj} .*

$P(H)$ may thus be expressed as the product of N^2 functions, each of a single real variable:

$$P(H) = \prod_{j \leq k} f_{jk}(\Re(H_{jk})) \prod_{j < k} g_{jk}(\Im(H_{jk})).$$

Furthermore, each independent variable has Gaussian distribution with mean 0 and variance 1. Thus Odlyzko [21], defines the GUE as the family of $N \times N$ Hermitian matrices $A = (a_{ij})$ where

$$\begin{aligned} a_{jj} &= \sqrt{2}\sigma_{jj} \\ a_{jk} &= \sigma_{jk} + i\eta_{jk} \quad \text{for } j < k \\ a_{kj} &= \sigma_{kj} - i\eta_{kj} \quad \text{for } k < j. \end{aligned}$$

The necessity to scale the diagonal elements by the factor $\sqrt{2}$ is one of the reasons which led to Dyson defining the circular ensembles (see Chapter 5) which do not share this minor defect.

Theorem 3.6 (Pair correlation function for GUE). *The pair correlation function for the eigenvalues of the Gaussian Unitary Ensemble is*

$$1 - \left(\frac{\sin \pi u}{\pi u} \right)^2.$$

Definition 3.7. *The joint probability density function of the eigenvalues of an $N \times N$ Hermitian matrix is*

$$P_N(x_1, \dots, x_N) = C_N \cdot \exp\left(-\sum_{j=1}^N x_j^2\right) \prod_{1 \leq j < k \leq N} |x_j - x_k|^2$$

where $-\infty < x_j, x_k < \infty$ for $1 \leq j < k \leq N$.

In order to normalize P_N to unity, i.e.,

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(x_1, \dots, x_N) dx_1 \cdots dx_N = 1,$$

the constant C_N may be chosen to satisfy $C_N^{-1} = 2^{-N(N-1)/2} \pi^{N/2} \prod_{j=1}^N j!$.

Definition 3.8. *The n -point correlation function for an $N \times N$ matrix is defined*

$$R_n(x_1, \dots, x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(x_1, \dots, x_N) dx_1 \cdots dx_N.$$

$R_2(x_1, x_2)$ is the 2-point or pair correlation function which will be calculated here.

Definition 3.9. *The n -level cluster function for an $N \times N$ matrix is defined*

$$T_n(x_1, \dots, x_n) = \sum_G (-1)^{n-m} (m-1)! \prod_{j=1}^m R_{h_j}(x_{n_1}, \dots, x_{n_{h_j}})$$

where G is any grouping of the indices $(1, 2, \dots, n)$ into m ordered subsets G_j with h_j elements each, and with $\sum_{j=1}^m h_j = n$.

These definitions yield a useful formula for the pair correlation function R_2 :

$$R_2(x_1, x_2) = R_1(x_1)R_1(x_2) - T_2(x_1, x_2).$$

Energy levels can be normalized to have mean spacing 1 by letting

$$\bar{x}_j = \frac{x_j}{\alpha}.$$

Then the limiting cluster function is

$$\bar{T}_n(\bar{x}_1, \dots, \bar{x}_n) = \lim_{n \rightarrow \infty} \alpha^n T_n(x_1, \dots, x_n).$$

Denoting by $r_2(GUE)$ the limiting 2-point correlation function for the GUE,

$$r_2(GUE) = \lim_{N \rightarrow \infty} \alpha^2 R_2(x_1, x_2),$$

we obtain, in the limit,

$$r_2(GUE) = 1 - \bar{T}_2(\bar{x}_1, \bar{x}_2).$$

Thus the theorem to be proved is that

$$\bar{T}_2(\bar{x}_1, \bar{x}_2) = \left(\frac{\sin \pi u}{\pi u} \right)^2$$

where $u = |\bar{x}_1 - \bar{x}_2|$, $\bar{x}_1 = x_1/\alpha_1$, $\bar{x}_2 = x_2/\alpha_2$, and α_j is the mean local spacing of the eigenvalues at x_j , $j = 1, 2$. A detailed proof is given by Pierce in [23]; a brief outline is given here. We start with Wigner's famous result on mean spacings of consecutive eigenvalues as N tends to infinity:

Proposition 3.10 (Semi-Circle law). *The density $R_1(x)$ of eigenvalues for the Gaussian Unitary Ensemble has the following behaviour as $N \rightarrow \infty$:*

$$R_1(x) \rightarrow \alpha(x) = \begin{cases} \frac{1}{\pi} \sqrt{2N - x^2} & \text{for } |x| < \sqrt{2N}, \\ 0 & \text{for } |x| \geq \sqrt{2N}. \end{cases}$$

The proof of this proposition is given in [17].

To compute \bar{T}_2 we will use oscillator wave functions $\phi_n(x)$, related to quantum harmonic oscillators (see e.g. [14]). This requires several definitions.

Definition 3.11 (Hermite polynomials). *Define for each $n \geq 0$,*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The first few Hermite polynomials are (see [31]):

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

⋮

$$H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240$$

⋮

These polynomials have generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \omega^n = e^{2x\omega - \omega^2}$$

which can be obtained by writing the Taylor series expansion for the function e^{-x^2} . This leads to the following useful relations satisfied by the Hermite polynomials. For every $n \geq 1$,

$$\begin{aligned} H'_n(x) &= 2nH_{n-1}(x) \\ H_{n+1}(x) &= 2xH_n(x) - H'_n(x). \end{aligned}$$

Following a useful method in [23], a scaled version of the Hermite polynomials $H_n(x)$ is introduced:

Definition 3.12 (*h-functions*). Define for each $n \geq 0$,

$$h_n(x) = H_n(x)e^{-\frac{1}{2}x^2}.$$

The above relations for the Hermite polynomials lead to recurrence relations for the h -functions: for every $n \geq 1$,

$$\begin{aligned} \left(\frac{d}{dx} + x\right) h_n(x) &= 2nh_{n-1}(x) \\ \left(-\frac{d}{dx} + x\right) h_n(x) &= h_{n+1}(x), \end{aligned}$$

where the expressions in parentheses on the left are "creation" and "annihilation" operators associated with quantum harmonic oscillators.

Definition 3.13 (Oscillator wave functions). *Define for each $n \geq 0$,*

$$\phi_n(x) = (-1)^n (\sqrt{\pi} 2^n n!)^{-\frac{1}{2}} e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The Oscillator wave functions are also known as Hermite functions and can be shown to form an orthonormal basis for $L^2(\mathbb{R})$. Thus

$$\int_{-\infty}^{\infty} \phi_n(x) \phi_m(x) dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases} \quad (3.1)$$

Lemma 3.14.

$$(i) \quad \lim_{m \rightarrow \infty} (-1)^m m^{1/4} \phi_{2m}(x) = \frac{1}{\sqrt{\pi}} \cos(\pi y)$$

$$(ii) \quad \lim_{m \rightarrow \infty} (-1)^m m^{1/4} \phi_{2m+1}(x) = \frac{1}{\sqrt{\pi}} \sin(\pi y)$$

where $\pi y = 2\sqrt{m}x$.

Proof. This follows from the asymptotic behaviour of the Hermite polynomials, detailed by Bateman [12]:

$$\lim_{m \rightarrow \infty} \frac{(-1)^m \sqrt{m}}{2^{2m} m!} H_{2m} \left(\frac{x}{2\sqrt{m}} \right) = \frac{1}{\sqrt{\pi}} \cos x$$

and

$$\lim_{m \rightarrow \infty} \frac{(-1)^m}{2^{2m} m!} H_{2m+1} \left(\frac{x}{2\sqrt{m}} \right) = \frac{1}{\sqrt{\pi}} \sin x.$$

□

The main result can now be proved:

Theorem 3.15.

$$\bar{T}_2(\bar{x}_1, \bar{x}_2) = \left(\frac{\sin \pi u}{\pi u} \right)^2$$

where $u = |\bar{x}_1 - \bar{x}_2|$, $\bar{x}_1 = x_1/\alpha_1$, $\bar{x}_2 = x_2/\alpha_2$, and α_1, α_2 are the respective mean local spacings of the eigenvalues at x_1, x_2 .

Proof. By the semi-circle law, as $N \rightarrow \infty$ the mean spacing at the origin is

$$\alpha = \frac{1}{\sigma(0)} = \frac{\pi}{(2N)^{1/2}}.$$

□

Since

$$T_2(x_1, x_2) = \left(\sum_{n=0}^{N-1} \phi_n(x_1) \phi_n(x_2) \right)^2,$$

it follows that

$$\begin{aligned} \overline{T}_2(\bar{x}_1, \bar{x}_2) &= \lim_{N \rightarrow \infty} \alpha^2 T_2(x_1, x_2) \\ &= \lim_{N \rightarrow \infty} \left(\frac{\pi}{(2N)^{1/2}} \right)^2 \left(\sum_{n=0}^{N-1} \phi_n(x_1) \phi_n(x_2) \right)^2. \end{aligned}$$

Here the Christoffel-Darboux identity comes in handy (see [30]):

Definition 3.16 (Christoffel-Darboux identity). *For a sequence of orthogonal polynomials the following relationship holds:*

$$\sum_{j=0}^{n-1} \frac{f_j(x) f_j(y)}{m_j} = \frac{k_{n-1}}{m_{n-1} k_n} \frac{f_n(x) f_{n-1}(y) - f_n(y) f_{n-1}(x)}{x - y}$$

where k_j is the leading coefficient of $f_j(x)$ and m_j is the norm $\int f_j^2(x) \omega(x) dx$ with respect to a suitable weighting function ω .

Applying the Christoffel-Darboux identity to the oscillator wave functions ϕ_j , with (3.1) in mind, we have

$$\sum_{n=0}^{N-1} \phi_n(x_1) \phi_n(x_2) = \left(\frac{N}{2} \right)^{1/2} \frac{\phi_N(x_1) \phi_{N-1}(x_2) - \phi_{N-1}(x_1) \phi_N(x_2)}{x_1 - x_2}.$$

Without loss of generality, it may be assumed that $N = 2m$ is even, so that

$$\sum_{n=0}^{2m-1} \phi_n(x_1) \phi_n(x_2) = \sqrt{m} \frac{\phi_{2m}(x_1) \phi_{2m-1}(x_2) - \phi_{2m-1}(x_1) \phi_{2m}(x_2)}{x_1 - x_2}.$$

Recalling $\bar{x}_1 = x_1/\alpha_1$ and $\bar{x}_2 = x_2/\alpha_2$, we have $\pi\bar{x}_1 = 2\sqrt{m}x_1$ and $\pi\bar{x}_2 = 2\sqrt{m}x_2$. By letting $x_1, x_2 \rightarrow 0$ as $\sqrt{m} \rightarrow \infty$, and applying Lemma 3.14, gives

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{n=0}^{2m-1} \phi_n(x_1)\phi_n(x_2) &= \lim_{m \rightarrow \infty} \left(\frac{2\sqrt{m}}{\pi} \right) \frac{\cos \pi\bar{x}_1 \sin \pi\bar{x}_2 - \sin \pi\bar{x}_1 \cos \pi\bar{x}_2}{\pi(\bar{x}_1 - \bar{x}_2)} \\ &= \lim_{m \rightarrow \infty} \left(\frac{2\sqrt{m}}{\pi} \right) \frac{\sin(\pi(\bar{x}_2 - \bar{x}_1))}{\pi(\bar{x}_1 - \bar{x}_2)}. \end{aligned}$$

We can then square both sides of the above:

$$\lim_{m \rightarrow \infty} \left(\sum_{n=0}^{2m-1} \phi_n(x_1)\phi_n(x_2) \right)^2 = \lim_{m \rightarrow \infty} \left(\frac{2(2m)}{\pi^2} \right) \left(\frac{\sin(\pi(\bar{x}_2 - \bar{x}_1))}{\pi(\bar{x}_1 - \bar{x}_2)} \right)^2.$$

By letting $u = |\bar{x}_1 - \bar{x}_2|$, this expression can now be used to obtain $\bar{T}_2(\bar{x}_1, \bar{x}_2)$:

$$\begin{aligned} \bar{T}_2(\bar{x}_1, \bar{x}_2) &= \lim_{N \rightarrow \infty} \alpha^2 T_2(x_1, x_2) \\ &= \lim_{N \rightarrow \infty} \left(\frac{\pi}{\sqrt{2N}} \right)^2 \left(\sum_{n=0}^{N-1} \phi_n(x_1)\phi_n(x_2) \right)^2 \\ &= \lim_{N \rightarrow \infty} \left(\frac{\pi}{\sqrt{2N}} \right)^2 \left(\frac{2N}{\pi^2} \right) \left(\frac{\sin(\pi(\bar{x}_2 - \bar{x}_1))}{\pi(\bar{x}_1 - \bar{x}_2)} \right)^2 \\ &= \left(\frac{\sin \pi u}{\pi u} \right)^2. \end{aligned}$$

Applying Proposition 3.10, the local mean spacings α_1, α_2 are

$$\begin{aligned} \alpha_1 &= \frac{1}{\sigma(x_1)} = \frac{\pi}{(2N - x_1^2)^{1/2}} \\ \alpha_2 &= \frac{1}{\sigma(x_2)} = \frac{\pi}{(2N - x_2^2)^{1/2}} \end{aligned}$$

and it then follows that

$$\bar{T}_2(\bar{x}_1, \bar{x}_2) = \lim_{N \rightarrow \infty} \frac{\pi^2}{(2N - x_1^2)^{1/2}(2N - x_2^2)^{1/2}} \left(\sum_{n=0}^{N-1} \phi_n(x_1)\phi_n(x_2) \right)^{1/2}.$$

Setting $\pi\bar{x}_1 = \sqrt{(2N - x_1^2)}x_1, \pi\bar{x}_2 = \sqrt{(2N - x_2^2)}x_2$ and again using Proposition 3.10, we then obtain

$$\bar{T}_2(\bar{x}_1, \bar{x}_2) = \left(\frac{\sin \pi u}{\pi u} \right)^2$$

as required.

Chapter 4

Empirical Results

Computation of Riemann zeta zeros has a century-and-a-half long history, beginning of course with Riemann's work introduced in Chapter 1. Unpublished notes deciphered by Siegel (see [11]) revealed Riemann's technique (Riemann-Siegel formula) as well as close estimates of several of the lowest zeros up the critical line. By 1925 Hutchinson had verified the Riemann Hypothesis for the first 138 nontrivial zeros using the Euler-Maclaurin summation formula. Electromechanical tabulating machinery was used to establish RH for the first 1041 zeros by Titchmarsh and Comrie; Turing improved this to 1104 zeros in 1953 (see below). The number of zeros on the real line grew well into the millions by the 1960s and passed one billion in the 1980s. However, these computations aimed at establishing t as real and did not produce the precise values for t which would be required to examine closely the distribution of the zeros. Furthermore, zeros very high up the critical line would be needed to study the asymptotic behaviour of the zeros, which the zeros are quite slow to reach.

Monumental achievements in computing large numbers of zeros high up the critical line, as well as extensive statistical evaluation of the distribution of the zeros, were made by Andrew Odlyzko. Due to this work, the empirical equivalence of the distribution of large Riemann zeros (and sometimes large zeros of other functions as well) to the GUE distribution (or other random matrix ensemble) is often referred to as the Montgomery-Odlyzko law.

In 1987 Odlyzko numerically computed sets of 10^5 zeros of the zeta function starting at zeros with indices 10^{10} , 10^{11} , 2×10^{11} and 10^{12} , with accuracy 10^{-8} . These were some of

the first results showing strong numerical support for the GUE hypothesis. However, even with the availability of supercomputers to provide thousands of hours of high powered precise computing, faster computational methods were needed for results much higher up the critical line. In 1988, with A. Schönhage, Odlyzko published a new algorithm for computing $\zeta(\frac{1}{2} + it)$ using the fast Fourier transform. This algorithm allowed the computation of 70 million zeros at height 10^{20} in 1989 followed by 175 million zeros at the same height in 1992. In 1998 Odlyzko computed 10000 zeros at height 10^{21} ; in 2001 ten billion zeros at height 10^{22} , and in 2002 twenty billion zeros at height 10^{23} .

4.1 Turing's Method

Turing's attempted construction of a mechanical computer designed to find Riemann zeros was cut short by the onset of World War II (see [5]). After the war he developed a new computational strategy to take advantage of digital computers which by now far exceeded the power mechanical devices. Recalling that the number $N(T)$ of zeros of $\zeta(\sigma + it)$ for $0 < \sigma < 1$ and $0 < t < T$ satisfies

$$N(T) = 1 + \frac{\theta(T)}{\pi} + S(T),$$

where $S(t) = \pi^{-1} \arg \zeta(1/2 + it)$. It is useful here to introduce Gram's law and Rosser's rule. Recall the definitions from Chapter 1 of $\theta(t)$ and $Z(t)$:

$$\theta(t) := \arg \left(\pi^{-\frac{it}{2}} \Gamma \left(\frac{1}{4} + \frac{it}{2} \right) \right),$$

and

$$Z(t) := e^{i\theta(t)} \zeta \left(\frac{1}{2} + it \right).$$

Definition 4.1 (Gram point, Gram interval). *Let $n \geq 1$. The n -th Gram point is defined (see e.g. [11]) as the unique positive real number g_n such that $\theta(g_n) = \pi n$. The interval*

$$I_n := [g_n, g_{n+1})$$

is then the n -th Gram interval.

Gram's law is that the function $Z(t)$ has a single root (i.e. changes sign once) in each Gram interval. The law is known to fail infinitely often; the first failure occurs in the 126th Gram interval. Rosser *et al.* [11] observed that 'missing' roots tend to occur in neighbouring intervals: if the n -th Gram interval has no roots then two roots will occur in the interval numbered $n - 2, n - 1, n + 1$ or $n + 2$. Specifically, define a Gram point g_n for which $(-1)^n Z(g_n) > 0$ as *good* and otherwise *bad*. Then define a Gram block of length $k \geq 1$ as an interval beginning and ending with good Gram points and having all its interior Gram points bad. Rosser's rule then specifically states that each Gram block $[g_n, g_{n+k})$ has k roots of $Z(t)$. Like Gram's law, Rosser's rule fails infinitely often (as proved by Lehman[16]), with the first failure at the Gram block beginning with $g_{13999525}$ [13].

A result by Littlewood (1924, see [11]), that $\int_0^T S(t)dt = O(\log T)$, implies that

$$\lim_{T \rightarrow \infty} 1/T \int_0^T S(t)dt = 0.$$

Thus it is reasonable to expect $S(t)$ to hover around the value 0, and with this in mind, following Gourdon in [26], we may call a Gram point g_n *regular* if $S(g_n) = 0$, that is if $N(g_n) = n + 1$. Turing's method involves finding a sequence (h_n) such that $(-1)^n Z(g_n + h_n) > 0$, the sequence $(g_n + h_n)$ increases, and h_n is small and, if possible, zero. Turing shows that if $h_m = 0$ and the nearby h_n (for n close to m) are relatively small then $S(g_m)$ must be an even integer. This is based on the even parity of zeros off the real line for a given height (symmetry of the functional equation (1.7)) and the fact that $S(g_m) = N(g_m) - m - 1$. Turing then shows that g_m must be regular, i.e. $S(g_m) = 0$ by showing that $-2 < S(g_m) < 2$. To do this he obtains a quantitative version of Littlewood's estimate,

$$\left| \int_{t_1}^{t_2} S(t)dt \right| \leq 2.3 + 0.128 \log \frac{t_2}{2\pi},$$

valid when $t_2 > t_1 > 168\pi$, and uses this to bound $S(g_m)$ as follows. For all $k > 0$,

$$\begin{aligned} & -1 - \frac{2.3 + 0.128 \log(g_m/2\pi) + \sum_{j=1}^{k-1} h_{m-j}}{g_m - g_{m-k}} \\ & \leq S(g_m) \leq 1 + \frac{2.3 + 0.128 \log(g_{m+k}/2\pi) + \sum_{j=1}^{k-1} h_{m+j}}{g_{m+k} - g_m}. \end{aligned}$$

4.2 Odlyzko's Methods

The Odlyzko-Schönhage algorithm improved calculation speed by a factor of 10^5 versus a straightforward implementation of the Riemann-Siegel formula. The implementation of the algorithm has several key components.

Odlyzko's methods prior to the Schönhage collaboration involved two steps. The first was to locate Gram blocks and search for the expected number of roots of $Z(t)$ in each block, establishing sign changes of $Z(t)$, and refining the vertical location of the roots with repeated computations of increasing accuracy. The second step was to evaluate $Z(t)$ at $t \pm 8 \times 10^{-9}$ for each possible root t , establishing that the sign change of $Z(t)$ was indeed at t or, in the case when this check failed, leading to the recalculation of the zero with an even higher precision algorithm.

Recall that on the line $\sigma = 1/2$, we have

$$\zeta(1/2 + it) = e^{-i\theta(t)} Z(t).$$

As t goes to infinity, $\theta(t)$ satisfies the asymptotic formula

$$\theta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + O(t^{-5})$$

where the terms in the expansion are given by a formula involving the Bernoulli numbers (see e.g. [11]). The Riemann-Siegel Z-function is a real-valued function of the real variable t (see (1.2); the formula used here is slightly different) and satisfies the Riemann Siegel expansion

$$Z(t) = 2 \sum_{n=1}^{\nu(t)} \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}} + R(t) \quad (4.1)$$

$$R(t) = (-1)^{\nu(t)-1} \left(\frac{t}{2\pi}\right)^{-\frac{1}{4}} \sum_{j=0}^M (-1)^j \left(\frac{t}{2\pi}\right)^{-\frac{k}{2}} \Phi_j(z) + R_M(t), \quad (4.2)$$

with $R_M(t) = O(t^{-(2M+3)/4})$, and where $z = 2(t - \nu(t)) - 1$ and the other notations are as in (1.2).

The first few $\Phi_j(z)$ are given by

$$\begin{aligned}\Phi_0(z) &= \frac{\cos(\frac{1}{2}\pi z^2 + \frac{3}{8}\pi)}{\cos(\pi z)} \\ \Phi_1(z) &= \frac{1}{12\pi^2}\Phi_0^{(3)}(z) \\ \Phi_2(z) &= \frac{1}{16\pi^2}\Phi_0^{(2)}(z) + \frac{1}{288\pi^4}\Phi_0^{(6)}(z).\end{aligned}\tag{4.3}$$

For $j > 2$ the expression for $\Phi_j(z)$ becomes complicated. However, explicit bounds for $R_M(t)$ have been found and we have

$$|R_0(t)| \leq 0.127t^{-3/4}, |R_1(t)| \leq 0.053t^{-5/4}, |R_2(t)| \leq 0.011t^{-7/4}.$$

Thus the choice $M = 1$ allows an absolute precision of $Z(t)$ smaller than 2×10^{-14} for zeros above the 10^{10} th zero, which is sufficient to locate the zeros.

4.3 Odlyzko-Schönhage Algorithm

The Odlyzko-Schönhage algorithm, still based on the Riemann-Siegel formula, has three main improvements to the computational methods just described. First is the application of the Fast Fourier Transform to greatly speed up computations. Second is a new method of rational function evaluation that allows the rapid evaluation at many points simultaneously. The method is based on Taylor series expansions around a small set of points. Third, the authors show that evaluation of $\zeta(\sigma + it)$ for t in a short interval can be reduced to evaluating simple exponential sums at an evenly spaced grid of points [20]. The Odlyzko-Schönhage algorithm allows $Z(t)$ to be evaluated efficiently in a range $T \leq t \leq T + \Delta$, where $\Delta = O(\sqrt{T})$. Then we have

$$Z(t) = \sum_{n=1}^{k_0-1} \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}} + \Re(e^{-i\theta(t)} F(t)) + \sum_{n=k_1+1}^m \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}} + R(t)$$

where $R(t)$ is as above and $F(t)$ is a complex function given by

$$F(t) = F(k_0 - 1, k_1; t) := \sum_{k=k_0}^{k_1} \exp(it \log k)\tag{4.4}$$

where $k_1 = \lfloor \sqrt{T/(2\pi)} \rfloor$ and k_0 is a fixed integer. For a fixed range $[T, T + \Delta]$, the values k_0 and k_1 are fixed and the majority of computing time is taken to compute $F(k_0 - 1, k_1; t)$. The algorithm evaluates $F(t)$ at evenly spaced values

$$t = T_0, T_0 + \delta, \dots, T_0 + (R - 1)\delta$$

by computing the discrete Fourier transforms

$$u_k = \sum_{j=0}^{R-1} F(T_0 + j\delta) \omega^{-jk}, \omega = \exp(2i\pi/R),$$

for $0 \leq k < R$. Thus by inverse Fourier transform,

$$F(T_0 + j\delta) = \frac{1}{R} \sum_{k=0}^{R-1} u_k \omega^{jk}.$$

The values $F(T_0 + j\delta)$ are obtained via Fast Fourier Transform (FFT) from the values u_k ; values of R are taken to be powers of 2 to make this more efficient. By rearranging (4.4), we can write $u_k = \omega^k f(\omega^k)$, where $f(z)$ is given by

$$f(z) = \sum_{k=k_0}^{k_1} \frac{a_k}{z - b_k}, \quad b_k = e^{\delta \log k}, \quad a_k = \frac{e^{iT_0 \log k}}{k^{1/2}} (1 - e^{iR\delta \log k}). \quad (4.5)$$

Here Odlyzko uses a Taylor series expansion on the variable z found in $a_k/(z - b_k)$ (see [20] for a detailed description).

4.4 Gourdon's Optimization

In work described in an unpublished 2004 article [13], Xavier Gourdon verifies the Riemann Hypothesis for the first 10^{13} zeros¹ and also computes two billion zeros starting with index 10^{24} .

The principal difference in Gourdon's approach is to use Chebychev interpolation in place of Taylor series at the step outlined at (4.5) above. Specifically, where Odlyzko uses a Taylor series of the complex variable z of $a_k/(z - b_k)$, Gourdon's approach is to use

Chebyshev interpolation of the function $a_k/(e^{i\theta} - b_k)$ for real θ . Taking as abscissas the points

$$\alpha_j = \theta_0 + L\gamma_j, \quad \gamma_j = \cos \frac{(2j+1)\pi}{2N}$$

for the interval $[\theta_0 - L, \theta_0 + L]$, the resulting interpolating polynomial of a function $G(\theta)$ on this interval is

$$P_{N,\theta_0,L}(\theta) = \sum_{j=0}^{N-1} \frac{G(\alpha_j)R_N(\theta)}{R'_N(\alpha_j)(\theta - \alpha_j)},$$

where $R_N(\theta) = \prod_{j=0}^{N-1}(\theta - \alpha_j)$.

4.5 Long-Range Correlations

Odlyzko chose not to investigate $\delta_n + \delta_{n+1} + \dots + \delta_{n+k}$ for $k > 1$. Already the behaviour of $\delta_n + \delta_{n+1}$ varied from the GUE prediction compared to δ_n . He found that higher order spacings were related to the distribution of primes rather than with the GUE.

Specifically, define the autocovariances of a set of δ_n be given by

$$c_k(H, M) = \frac{1}{M} \sum_{n=H+1}^{H+M} (\delta_n - 1)(\delta_{n+k} - 1).$$

An unpublished conjecture by Dyson, namely that c_k would be approximately $-1/2\pi^2 k^2$ for positive k was found to be true only for smaller and intermediate k . Correlations at larger k values were better explained through a formula by Landau relating to the primes:

$$\sum_{n=1}^N e^{i\gamma_n y} = \begin{cases} -\frac{\gamma N}{2\pi} e^{-y/2} \log p + O(e^{y/2} \log N) & \text{if } y = \log p^m, \\ O(e^{-y/2} \log N) & \text{if } y \neq \log p^m, \end{cases}$$

¹Empirical results such as these lead to improved bounds on number theoretic functions. A line of investigation beginning with Rosser in 1941 uses the verification of the Riemann Hypothesis on the first n critical zeta zeros to derive explicit estimates on certain number theoretic functions without assuming RH. Rosser and Schoenfeld used verification of RH for zeros up to index 3,502,500 to prove that for $x > 1.04 \times 10^7$, $|\psi(x) - x| < 0.0077629 \frac{x}{\log x}$, and $|\theta(x) - x| < 0.0077629 \frac{x}{\log x}$. More recently Ramaré and Saouter ([24]) used verification of RH up to height 3.3×10^9 to prove the following: for every real number $x \geq 10, 726, 905, 041$, there exists at least one prime number p such that

$$x \left(1 - \frac{1}{28,314,000}\right) < p \leq x.$$

for fixed $y > 0$ and as $N \rightarrow \infty$ (see [21]). The formula forces the spectrum of the δ_n to consist of point masses at frequencies which correspond to prime powers, which then leads to the behaviour of the c_k observed in Odlyzko's calculations.

4.6 Conclusions and Summary

The agreement between the known GUE distribution and the statistics calculated from Odlyzko's computed zeros is "generally good, and improves at larger heights" according to Odlyzko [19]. This modest statement leaves room for doubt on results that are in fact strong enough to become known as the Montgomery-Odlyzko law [32]; in part this is due to the fact that even as high as computationally possible up the critical we go, we are far from having a limiting distribution.

Chapter 5

Further Connections

In this chapter a very nontechnical outline of several recent approaches relating random matrices to the Riemann zeta function is presented. The results in Chapters 3 and 4 on the distribution of zeta zeros up the critical line demonstrate that the normalized GUE spacing provides a very accurate match to the nontrivial zero spacings. The circular matrix ensembles are invaluable in physical modeling and interesting in their own right. The spectral interpretation of Riemann zeros (the Hilbert-Polya approach) will be discussed, as will the log-normality of $|\zeta(s)|$ and a result concerning the apparent repulsion of nontrivial zeta zero differences by the heights of the zeros.

5.1 Circular Ensembles

Dyson introduced the circular ensembles to address some deficiencies in the Gaussian ensembles, such as the unequal weighting of matrix elements in the GUE.

To define the Circular Unitary Ensemble (CUE), characterize a system by a unitary matrix S giving the transition probabilities between states. The eigenvalues of unitary matrices take the form $e^{i\theta_j}$ for angles $0 \leq \theta < 2\pi$. The relation between S and the Hamiltonian H can be imagined as

$$S = e^{i\tau H} \text{ or } S = \frac{1 - i\tau H}{1 + i\tau H}$$

but only for a short range of energy levels.

A CUE matrix S is invariant under the transformation $S \rightarrow USV$ where U, V are any

unitary matrices satisfying $S = UV$.

Mehta demonstrates (see [17]) that in the limit $N \rightarrow \infty$, the n -level correlation and cluster functions for finite n are identical to those of the GUE. The same holds when comparing GOE to COE, and GSE to CSE.

In fact, the relationship between Dyson's CUE, COE, and CSE and the GUE, GSE and GOE is a fascinating one involving Cartan's classical compact irreducible symmetric spaces (see Katz and Sarnak, [15] for an excellent account). The CUE corresponds to $U(N)$, the COE to $U(N)/O(N)$, and the CSE to $U(2N)/USp(2N)$. Interestingly, the GUE is defined on the non-compact dual space of $U(N)$, which is $GL_N(\mathbb{C})/U(N)$; and similarly the GOE is defined on $GL_N(\mathbb{R})/O(N)$, the non-compact dual of $U(N)/O(N)$, and the GSE is defined on $U^*(2N)/USp(2N)$, the non-compact dual space of the $U(2N)/USp(2N)$.

The pair correlation of eigenvalues on random unitary matrices may be given with respect to Haar measure on $U(N)$ as follows:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \int_{U(N)} \sum_{\substack{1 \leq j, k \leq N \\ j \neq k}} f\left(\frac{N}{2\pi}(\theta_j - \theta_k)\right) dA_{Haar} = \int_{-\infty}^{\infty} f(u) \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du,$$

where $e^{i\theta_1}, \dots, e^{i\theta_N}$ are the eigenvalues of $A \in U(N)$.

Theorem 5.1. *In the ensemble $E_{\beta c}$ the probability of finding an eigenvalue of S in each of the intervals $(\theta_j, \theta_j + d\theta_j)$, $j = 1, \dots, N$ is given by*

$$P_{N\beta}(\theta_1, \dots, \theta_N) d\theta_1, \dots, d\theta_N,$$

where

$$P_{N\beta}(\theta_1, \dots, \theta_N) = C'_{N\beta} \prod_{1 \leq l < j \leq N} |e^{i\theta_l} - e^{i\theta_j}|^\beta.$$

Where $\beta = 2$ in the unitary ensemble case and $\beta = 1, 4$ in the orthogonal and symplectic ensemble cases, respectively, and $C'_{N\beta}$ is a constant which will disappear upon normalization

This is to show the match between the circular and gaussian ensembles. Mehta further proves that the n -level correlation function is

$$R_n(\theta_1, \dots, \theta_n) = \det[S_n(\theta_j - \theta_k)]_{j,k=1,\dots,n},$$

and that level density is thus

$$R_1(\theta) = S_N(0) = \frac{N}{2\pi}.$$

Similar results hold for the n -level cluster function of the circular ensembles, and in particular Mehta shows that for the CUE,

$$T_2(\theta_1, \theta_2) = \left(\frac{\sin \pi r}{\pi r} \right)^2,$$

so that the limiting distribution again matches when comparing the circular ensemble as compared to the gaussian ensemble.

5.2 Riemann Zeros and the Spectral Interpretation

The analogy between Riemann Zeros and physics has a history going back at least to the 1910s (see [27]) and the Hilbert-Polya conjecture, which states the possibility that a Hermitian operator exists with Riemann zeros as its eigenvalues. Much work since then has expanded on this approach in several ways, where different physical systems are taken as bases for the analogy to the mathematical ζ -function. The Hilbert-Polya conjecture posits that the nontrivial zeros of the Riemann zeta function may correspond to the eigenvalues of a self-adjoint linear operator (a Hamiltonian, H). While attempts to find such an operator explicitly have failed, much progress has been made in this direction (see for example the work of Connes, [7]).

Using a similarity between fluctuations of the counting function for the Riemann zeros (see e.g. [2]),

$$\mathcal{N}(t) = \sum_{n=1}^{\infty} \Theta(t - t_n),$$

where Θ denotes the unit step function, and vibration frequencies for a chaotic dynamical system, the function $\mathcal{N}(t)$ may be decomposed into a smooth part and a fluctuating part:

$$\mathcal{N}(t) = \langle \mathcal{N}(t) \rangle + \mathcal{N}_{fl}(t),$$

where the smooth part is

$$\langle \mathcal{N}(t) \rangle = \frac{\theta(t)}{\pi} + 1 = \frac{1}{\pi} \left[\arg \Gamma \left(\frac{1}{4} + \frac{1}{2}it \right) - \frac{1}{2}t \log \pi \right] + 1 = \frac{t}{2\pi} \log \left(\frac{t}{2\pi e} \right) + \frac{7}{8} + O \left(\frac{1}{t} \right)$$

and the fluctuating part is

$$\mathcal{N}_{fl}(t) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \Im \log \zeta \left(\frac{1}{2} + it + \epsilon \right).$$

Formally substituting the Euler product (1.1) for $\zeta(s)$ into the above expression gives

$$\mathcal{N}_{fl}(t) = -\frac{1}{\pi} \Im \sum_p \log \left\{ 1 - \frac{\exp(-it \log p)}{\sqrt{p}} \right\} \quad (5.1)$$

$$= -\frac{1}{\pi} \sum_p \sum_{m=1}^{\infty} \frac{\exp(-\frac{1}{2}m \log p)}{m} \sin(tm \log p). \quad (5.2)$$

In this way the fluctuations are seen as oscillatory contributions from the prime powers p^m , exponentially decreasing for fixed p as m increases. A prime p has a t -period

$$\tau_p = \frac{2\pi}{\log p}$$

equivalent to a wavelength in a physical system, which we compare to the following (see [8]). Let $N(E)$ be the number of eigenvalues λ of the Hamiltonian H , a quantum mechanical system obtained by quantizing a certain classical system. Then here we may also decompose $N(E)$ into a smooth and a fluctuating part:

$$N(E) = \langle N(E) \rangle + N_{fl}(E),$$

and the fluctuating part may be expanded asymptotically as

$$N_{fl}(E) \sim \frac{1}{\pi} \sum_{\gamma} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2 \sinh \left(\frac{m\lambda_{\gamma}}{2} \right)} \sin(T_{\gamma}^{\#} m E) \quad (5.3)$$

where the first sum is over periodic orbits γ and an orbit γ has period $T_{\gamma}^{\#}$ and instability exponent λ_{γ} . The similarity between (5.2) and (5.3) is striking and has been both enticing and frustrating for researchers such as Connes, as the sign difference is not easily accounted for in the spectral interpretation ([8]).

5.3 A central limit theorem and a promising conjecture

A theorem of Selberg (see [21]) is as follows

Theorem 5.2. For any rectangle $B \subset \mathbb{C}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \# \left\{ t \in [T, 2T] : \frac{\log \zeta(1/2 + it)}{\sqrt{(\log \log T)/2}} \in B \right\} = \frac{1}{2\pi} \iint_B \exp(-(x^2 + y^2)/2) dx dy.$$

This gives a Gaussian distribution of the real and imaginary parts of

$$\log \zeta(1/2 + iT) / \sqrt{(1/2) \log \log T},$$

independent from one another and with unit variance and zero mean, in the limit as T goes to infinity. Thus we can compare this to the distribution given by random matrix theory.

Let $Z(U, \theta)$ be the characteristic polynomial for the CUE. Keating and Snaith show that $\Re \log Z / \sqrt{1/2 \log N}$ and $\Im \log Z / \sqrt{1/2 \log N}$ independently tend to Gaussian distributions with mean 0 and variance 1 in the limit as N goes to infinity, providing a sort of central limit theorem as with Selberg's above for the Riemann zeta function. Thus to make a comparison between this ensemble's eigenvalue distribution and computed statistics of the zeta zero distribution, they associate N with $\log(T/2\pi)$ and choose the appropriate N . Using Odlyzko's computations [21] around height 10^{20} , with $t \approx 1.5 \times 10^{19}$, the appropriate value for N is 42. Interestingly, while both the computed distribution of $\Re \log \Lambda_A$ for matrices in $U(42)$ and the distribution of $\Re \log \zeta(1/2 + it)$ near the 10^{20} th zero deviate slightly from their asymptotic normal distribution (when scaled to have variance 1), the two curves match one another perfectly, and indication that the two limiting distributions (as $N \rightarrow \infty$ and as $\log(T/2\pi) \rightarrow \infty$ respectively) approach the asymptotic limit at the same rate. The formula by Keating and Snaith obtain a similar formula for $U(N)$ [28]:

Theorem 5.3. For any rectangle $B \subset \mathbb{C}$,

$$\lim_{N \rightarrow \infty} \text{meas} \left\{ A \in U(N) : \frac{\log \Lambda_A(e^{i\theta})}{\sqrt{\frac{1}{2} \log N}} \in B \right\} = \frac{1}{2\pi} \iint_B \exp(-(x^2 + y^2)/2) dx dy,$$

where Λ_A is the characteristic polynomial of the unitary matrix A :

$$\Lambda_A(s) = \det(I - A^* s) = \prod_{n=1}^N (1 - s e^{-i\theta_n}).$$

Some conjectures involving very complicated integrals over ratios of $\zeta(s)$ such as the following example by Conrey, Farmer, and Zirnbauer (given in [28]) provide evidence for very deep connections between $\zeta(s)$ and random matrices:

Conjecture 5.4. *Let $-1/4\Re(\alpha), \Re(\beta) < 1/4, 1/\log T \ll \Re(\gamma), \Re(\delta) < 1/4,$
 $\Im(\alpha), \Im(\beta), \Im(\gamma), \Im(\delta) \ll_\epsilon T^{1-\epsilon}$ for any $\epsilon > 0,$ and $s = 1/2 + it.$ Then*

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\alpha)\zeta(1-s+\delta)} dt \quad (5.4)$$

$$= \int_0^T \left(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A_\zeta(\alpha, \beta; \gamma, \delta) \quad (5.5)$$

$$+ e^{-\log \frac{t}{2\pi}(\alpha+\beta)} \frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)} A_\zeta(-\beta, -\alpha; \gamma, \delta) \right) dt + O(T^{1/2+\epsilon}), \quad (5.6)$$

where

$$A_\zeta = (\alpha, \beta; \gamma, \delta) = \prod_p \frac{(1 - 1/p^{1+\alpha+\beta})(1 - 1/p^{1+\gamma+\delta})}{(1 - 1/p^{1+\alpha+\delta})(1 - 1/p^{1+\beta+\gamma})} \int_0^1 \frac{\left(1 - \frac{e(\theta)}{p^{1/2+\gamma}}\right) \left(1 - \frac{e(-\theta)}{p^{1/2+\delta}}\right)}{\left(1 - \frac{e(\theta)}{p^{1/2+\alpha}}\right) \left(1 - \frac{e(-\theta)}{p^{1/2+\beta}}\right)} d\theta$$

and $e(\theta) = e^{2\pi i\theta}.$

The conjecture has a strong analogy with a theorem which the same authors proved (see [9]):

Theorem 5.5. *For $\Re(\gamma), \Re(\delta) > 0,$*

$$\int_{U(N)} \frac{\Lambda_A(e^{-\alpha})\Lambda_{A^*}(e^{-\beta})}{\Lambda_A(e^{-\gamma})\Lambda_{A^*}(e^{-\delta})} dA_{Haar} = \frac{z(\alpha+\beta)z(\gamma+\delta)}{z(\alpha+\delta)z(\beta+\delta)} + e^{-N(\alpha+\beta)} \frac{z(-\alpha-\beta)z(\gamma+\delta)}{z(-\beta+\delta)z(-\alpha+\gamma)},$$

where $z(x) = 1/(1 - e^{-x}).$

Some of the strength of this analogy comes from the consideration that both $z(x)$ and $\zeta(1+x)$ have poles at $x = 0$ with residue 1. Thus with N corresponding to $\log(t/2\pi),$ the symmetry of the two formulae is apparent, lending further strength to the random matrix theory-Riemann zeta function correspondence.

Snaith [28] gives a theorem, assuming the above conjecture, which sheds light on the question of zeta zero differences "repelling" zeta zeros.

5.4 Zero-Repulsion Phenomenon

A statistical analysis of low-lying zeros (see for example [22]) seems to indicate that the gaps between non-normalized, non-trivial Riemann zeros $1/2 + i\gamma$ are repelled by the

values taken by the imaginary parts γ . That is to say it that plots of values $\gamma_j - \gamma_k$, for all pairs j, k less than some fixed n , show distinctive local minima around the values 14.13, 21.02, 25.01, 30.42, etc. which the reader might recognize as the first few non-trivial Riemann zeros. However, examination of zeros higher up the critical line shows that this relationship is not as simple as for lower on the critical line. At sufficiently high ranges it becomes apparent that while Riemann zeros occur frequently within a short distance from local minima in the distribution of pair spacings, they do not exhibit the close match that occurs at short heights. Rather, what is responsible for this phenomenon seems to come from two facts. First, the lower Riemann zeros occur when local minima of $|\zeta(1 + it)|$ occur. Second, if we accept Conjecture 5.4 then following theorem of Snaith (see [28]) holds:

Theorem 5.6. *For f a suitable, even test function and assuming Conjecture 5.4, we have*

$$\sum_{0 \leq \gamma_j, \gamma_k \leq T} f(\gamma_j - \gamma_k) = \frac{1}{(2\pi)^2} \Re \left[2T \left(\frac{\zeta'}{\zeta} \right)' (1 + iX) - 2T \cdot B(iX) + T \left(\log \frac{T}{2\pi} \right)^2 - 2T \log \frac{T}{2\pi} + 2T + \frac{2\zeta(1 - iX)\zeta(1 + iX)A(iX)}{(2\pi)^{iX}} \left(\frac{T^{1-iX} - 1}{1 - iX} \right) \right],$$

where

$$A(s) = \prod_p \left(1 - \frac{1}{p^{1+s}} \right) \left(1 - \frac{2}{p} + \frac{1}{p^{1+s}} \right) \left(1 - \frac{1}{p} \right)^{-2},$$

$$B(s) = \sum_p \left(\frac{\log p}{p^{1+s} - 1} \right),$$

While the expression is very complicated, an examination of the formula reveals, assuming the theory is correct, the influence of $\zeta(1 + it)$ on the pairwise distribution of zeta zeros.

5.5 3,4, and n -point correlations

In [3], Bogomolny and Keating extend Montgomery's approach to 3- and 4-point correlations of the non-trivial Riemann zeta function and show that these, too, are identical to the equivalent correlation functions of the Gaussian Unitary Ensemble. In [4] the same authors extend the same results to all n .

5.6 Conclusions

The last two decades have seen an explosion in work connecting random matrix ensembles to L -functions and in particular the Riemann ζ -function. While these approaches as of yet hold no promises of a proof (or disproof) of the Riemann Hypothesis, the spectral interpretation of the Riemann zeta function has been strengthened in many ways. Meanwhile the rich and growing body of correspondence between random matrix theory and number theory has immense theoretical value in its own right, and the theory on both the mathematical and physical side continues to grow deeper and broader.

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