Using Dynamic Geometry to Explore Linear Algebra Concepts: the Emergence of Mobile, Visual Thinking

by

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Abstract

This dissertation sheds lights on aspects of students' thinking as they interacted with a dynamic geometric diagram of the concepts of eigenvector and eigenvalue. Given that the phenomenon of thinking is not directly observable, I attend to their use of the dragging tool, shifts in their attention and emerging ways of communicating the concepts through gestures and speech. I present the transcripts of one-on-one videotaped interviews with five university students and analyze isolated episodes.

My analytic frame is informed by the theories of shifts of attention and instrumental genesis. The latter reveals evidence of the transformation of tool into an instrument of semiotic mediation by the process of internalization while the former highlights the significant role of attention and awareness in learning and understanding mathematics. The complementary use of the theories enables me to analyze the cognitive development of a student in a digital technology environment, because the student's use of different dragging modalities can provide easily-visible evidence of shifts in her structure of attention and consequently can reveal her understanding of the concepts. Moreover, the dynamic geometric diagram stimulated the formation of kinaesthetic and dynamic imagery, as evidenced by the students' ways of communicating. I thus incorporate aspects of embodied cognition into my analysis in order to account for the important role played by the body in students' exploring and communication.

My analysis suggests that the students mostly used a synthetic-geometric mode of thinking, but more importantly, their thinking involved facilities of *process and time* and *vision, spatial sense, kinesthetic (motion) sense*. These facilities enabled them to communicate dynamic and kinesthetic imagery using embodied expressions and gestures. I thus argue that dynamic geometric representations of eigenvectors enabled the students to develop *dynamic-synthetic-geometric thinking*. I also discuss the role of dynamic geometric diagram of the concepts in enabling students to experiment with the behaviour of eigenvectors. This is in opposition to static diagrams that can be found in textbooks. I conclude this dissertation with some pedagogical suggestions in terms of the use of dynamic geometric diagrams of the concepts of eigenvector and eigenvalue.

Keywords: Students' thinking, linear algebra concepts, dynamic geometric environments, instrumental genesis, shifts of attention, dynamic and kinesthetic imagery

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I also express my gratitude to the participants of my study who volunteered their time and opened their thoughts.

Finally, but most importantly, I would like to thank my family for their constant love and support.

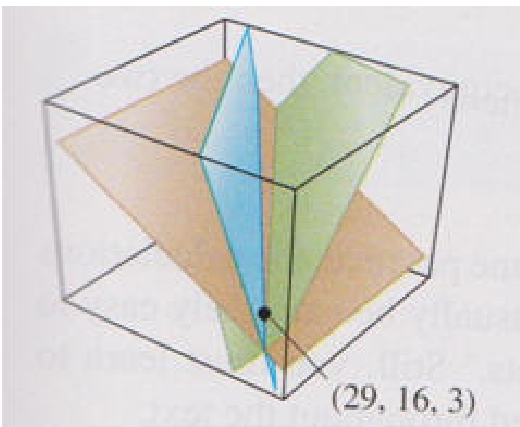
Table of Contents

List of Figures

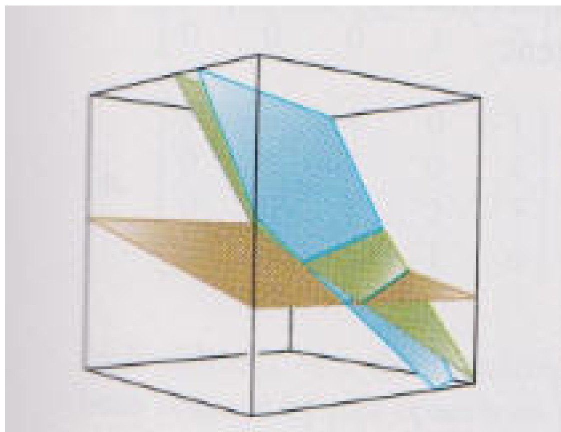
1. Introduction

In fall 2006 and spring 2007, I taught sections of an elementary linear algebra course at a medium-sized North-American university. The university offered multiple sections of the course each semester and the course outline, schedule, assignments and final exam were written by the course coordinator to structure all sections in a similar way. Each class contained about fifty students. Being a new instructor, in fall 2006, I was encouraged by the course coordinator to attend and learn from experienced instructors' lectures. I thus chose to attend an instructor's classes whose lectures seemed to me cohesive and clear. He employed a traditional, lecture-based approach with no integration of technology. Following his lead closely, I gained skill in making explicit connections among concepts and in selecting appropriate examples to introduce or exemplify ideas.

Given that the course was an introduction course to linear algebra, it was based on a matrix-vector approach, with a particular emphasis on Euclidean vector spaces to provide students with geometric representational models of concepts. In fact, the geometric representations were limited to the same static diagrams that one can find in a linear algebra textbook. The first topic of the course was studying systems of linear equations and finding the solution sets of systems using the Gauss-Jordan elimination method. I observed that many students seemed to struggle with understanding that the solution set of a system of linear equations is invariant under elementary row operations. A few were applying the elimination or substitution methods that they had learned before, even if a given system involved more than three variables. Drawing their attention to planar geometric representations of systems with two variables seemed to help them understand the possible outcomes of such systems (i.e. no solution/parallel lines, infinitely many solutions/the same line, or unique solution/two intersecting lines), but it was still difficult for them to visualize the possible outcomes of systems with three variables. Although a textbook may include diagrams of linear systems with three variables and three equations, students seemed to find it challenging to understand these diagrams. Consider, for example, the two diagrams in Figure 1. They depict the geometric representation of two different systems of linear equations each with three linear equations in three variables, and are meant to illustrate that one system has a single solution of (29, 16, 3) and the other is an inconsistent system (no solution). The diagrams use quadrilaterals bounded in cubes to illustrate geometric representations of equations with three variables. The two diagrams are drawn from different angles of view, but use the same color shade and size. A few students who were not familiar with the illustration of three-dimensional geometry on a piece of paper considered these diagrams as illustrations of the intersections of three rectangles captured from different angles of view.



(a) A system of three planes with the solution set of (29, 16, 3) (Lay, 2006, p. 7)



(b) An inconsistent system of three planes (Lay, 2006, p. 9)

Figure 1. Geometric representations of two linear systems

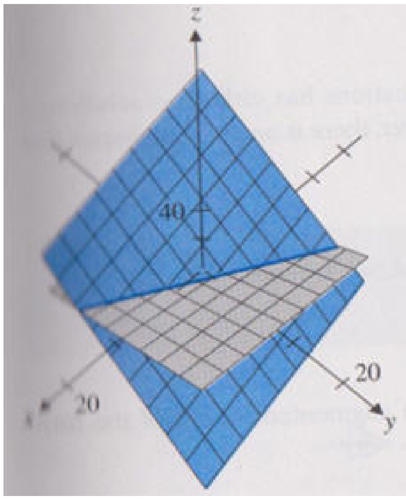
As mentioned above, the course included a particular emphasis on Euclidean vector spaces that helped most of students become acquainted with the geometry of linear systems in two- and three-dimensional spaces. Still, it seemed to me that the students' understanding was compromised by the overwhelming number of new definitions and theorems, their personal lack of geometric intuition, and the failure to coordinate algebraic representations with geometric ones. As a result, many of them developed only a limited procedural knowledge of matrices and vectors. It seemed that my attempt to provide them with clear and cohesive notes was not enough.

This experience caused me to wonder about other teaching approaches, particularly ones in which geometrical representations could be used more widely and successfully. It also made me reflect on my own ways of understanding linear algebra concepts as I was doing my teaching preparation. I realized my inclination was to make sense of geometrical representations of concepts (where they were provided) before focusing on the formal definitions. I found myself sketching diagrams for my own understanding of ideas and also introducing ideas to my students using these diagrams.

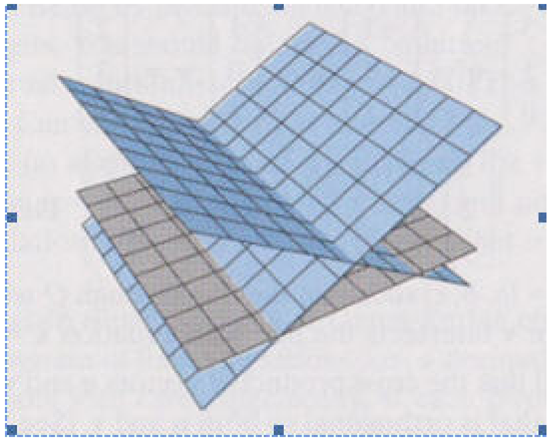
Pursuing my geometric intuition, I started reviewing textbooks written by different authors to find better resources where students would be given opportunities to construct geometrical representations and also to coordinate algebraic representations with geometric ones. The textbooks mostly used similar approaches to introducing concepts, all characterized by a limited number of diagrams and a few examples on linear algebra applications in other fields. Reviewing a sample[1] of linear algebra textbooks published from 1998 to 2009, I found that mostly textbooks do not include a diagram for systems of linear equations with three variables and Lay's book includes more diagrams in comparison with others. In Figure 2, I include another example of a pair of diagrams that are used to illustrate systems of linear equations.

My review made me conjecture that these linear algebra textbooks were partly to blame for students' lack of geometric intuition and their reliance on procedural knowledge. I became interested in the potential of technology as a tool to provide students with powerful representational models in linear algebra where they can build their own, more geometric, understanding of concepts. This was the beginning of my journey to explore the use of technology to represent linear algebra concepts geometrically. Before that, I never used calculators or computer technology as a computational or representational tool to learn linear algebra. I remember my undergraduate linear algebra courses were theory-based approaches that focused on the theory of vector spaces.

[1] This was an informal review of the existing linear algebra textbooks in a library of a medium-sized university.



a) A system of two planes with the line of intersection (infinitely many solutions) (Poole, 2003, p. 77)



b) A system of three planes with no solution (Poole, 2003, p. 83)

Figure 2. Geometric representations of two linear systems

Reviewing the literature, I learned about the Linear Algebra Curriculum Study Group's recommendation (1991) concerning the use of technology as a tool to run computations. This recommendation encouraged a few instructors to integrate Computer Algebra Systems (CASs) into linear algebra courses, particularly in North America. Although the plotting facilities of CASs, in particular *Maple* and *MATLAB*, provide visual representations of some basic linear algebra concepts, the use of these visual representations was not the main focus of this software. Instead, instructors, as well as researchers, valued CASs for their use as a computational tool.

Besides CASs though, I found other digital technologies being used, such as dynamic geometry environments (DGEs). In particular, it seemed to me that the integration of dynamic geometry software into teaching linear algebra facilitated students' articulation of the meaning of linear transformation, in problem-solving situations, using terms such as *operations, positioning and relationships* (Sierpinska, Dreyfus, & Hillel, 1999). These findings motivated me to investigate in more depth the effects of the use of dynamic geometric representations on students' understanding of the concepts of eigenvector and eigenvalue. I was intrigued by the geometric and dynamic affordances of this software, which contrasted with the algebraic, static, and symbolic representations that were prevalent in textbooks. Given that DGEs offer the possibility for geometric interactive representation of concepts (in two-dimensional vector spaces), I wondered whether they could facilitate students' understanding of the concepts of eigenvector and eigenvalue.

Elementary linear algebra students are mostly introduced to the concepts of eigenvector and eigenvalue after they learn the concept of linear transformation. But, they use the procedural algebraic method to find eigenvalues (i.e. the roots of the characteristic equation, $\det(A-\lambda I)=0$) and *then*, subsequently, to find a particular set of associate eigenvectors (i.e. the non-trivial solutions of $(A-\lambda I)x=0$ given a specific value for λ) of a particular square matrix. The algebraic method does not reveal the connections among the concepts of linear transformation, eigenvectors and eigenvectors. In fact, algebraic strategies (such as writing a characteristic equation, finding its roots and then finding the associated eigenvectors) lead students to develop procedural knowledge of how to find eigenvalues and their associated eigenvectors of square matrices without being explicitly aware that they are identifying a special vector that is transformed into its scalar multiple under a given matrix of transformation.

Similar to other textbook geometric representations, understanding the static geometric diagrams of eigenvectors and eigenvalues was difficult for some students. Consider, for example, the diagram in Figure 3. It depicts two examples of transformed vectors, Au and Av, and is meant to illustrate that v is an eigenvector, but *u* is not. Some of its deficiencies are: it is only limited to two examples of vectors; the vector u is visually almost at right angles to v; the vector Av looks like a free vector starting at v and ending at Av. A is introduced as a matrix rather than a linear transformation and the diagram is labelled "effect of multiplication by A". Also, the domain and range of the transformation, A, are both presented in one diagram.

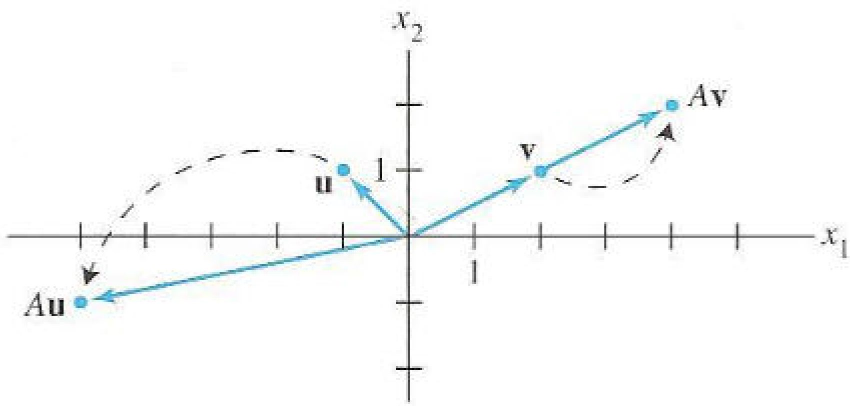


FIGURE 1 Effects of multiplication by A.

Figure 3. Geometric representation of eigenvectors (Lay, 2006, p. 303)

This diagram is different from a diagram that one might draw to visualize a problem-solving situation (such as graphing linear equations with two variables to visualize the solution set of the system). It illustrates the final result of two matrix-vector multiplications. To construct it, one needs *first* to find the eigenvalue of 2 (by finding the roots of the characteristic equation) and *then* to use the eigenvalue to find the associated eigenvector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. This led many students to ignore the diagram and just focus on learning the algebraic procedure for finding eigenvalues and eigenvectors. Also, it is unlikely to ask students to draw a diagram for eigenvectors but very common to ask them to find eigenvalues and eigenvectors of a given square matrix.

Having explored a few online Java sketches, I began seeing DGEs' affordances to generate infinitely many examples of concepts, to represent a concept in both arithmetic and geometric representational systems at the same time, and to promote geometric intuition of concepts. This motivated me to design several sketches using *The Geometer's Sketchpad* (Jackiw, 1989) to represent basic linear algebra concepts.

I made a dynamic geometric sketch that could be used to identify eigenvector(s) and then the associated eigenvalue(s) of any 2×2 matrix. This representation, unlike static ones, does not require finding eigenvectors and eigenvalues of a square matrix in order to be able to construct the geometric representation. It is not limited to a few examples of vectors. In fact, it enables one to experiment with linear transformations of infinitely many vectors, to identify specific vectors (eigenvectors) that are being transformed into their scalar multiples, and then to find the associated scaling factor (eigenvalue). Unlike the algebraic method, the dynamic geometric diagram privileges and foregrounds the concept of eigenvector over the concept of eigenvalue. My approach in this dissertation is captured in the phrase ‘eigenvectors and their associated eigenvalues’, whereas the norm (in courses and textbooks) is the algebraic privileging of ‘eigenvalues and their associated eigenvectors’. This is because the property of an eigenvector (i.e. invariant collinearity) of a 2×2 matrix becomes evident from its geometric representation and the span of it is always a line; but the associated eigenvalue comes with a generality within that property, namely the same dilation factor no matter which vector we choose along the line of the span.

My speculation about the potential use of DGEs to provide students with powerful representational models and awareness of the affordances of *Sketchpad* led me to focus on two main research questions. (1) Given that I had seen how students’ typical experiences led them to highly procedural, algebraic conceptions of eigenvalues and eigenvectors, I wondered what effect dynamic geometric representations of concepts would have on their conceptions. (2) Supposing they do develop more geometric understandings through their interactions with DGEs, I wondered how they would relate these representations to the more symbolic and static ones that are to be found in undergraduate textbooks.

This dissertation is organized into 8 chapters, including this Introduction presented as Chapter 1. Chapter 2 includes a review of the existing literature on students’ learning difficulties in linear algebra. It discusses students’ conceptual difficulties, the use of different representational modes in linear algebra, and the need for the development of different modes of thinking. I also include a brief examination of textbook representations of concepts of vector, linear transformation, eigenvector and eigenvalue, and the basis of a vector space. Chapter 3 focuses on literature related to the use of CASs and DGEs in the teaching and learning mathematics, with a particular focus on linear algebra. In this chapter, I explain why I opted for a DGE and discuss some of the ways in which I thought DGEs could help address some of the difficulties discussed in Chapter 2.

Chapter 4 includes a review of theories on attention and awareness, visualization and visual imagery, and tools and instruments. I include Mason's theory of shifts of attention since it provides a broad theoretical scope in order to highlight the significant role of attention and awareness in learning and understanding mathematics. Given that my participants interacted with the dynamic geometric sketches through a use of dragging tool (mouse), I also draw on the theory of instrumental genesis. Lastly, I include a review of visualization and different categories of visual imagery, given the visual and interactive nature of the sketches that affected the participants' ways of thinking about the concepts. The use of kinesthetic and dynamic imagery by participants additionally led me to include theories on embodied cognition and gesture. My review of these theories made me conjecture that triangulating a participant's dragging modalities, shifts in her attention, and her use of imagery could provide me with a richer understanding of her learning and thinking process.

In Chapter 5, I describe my research methodology. I include a description of the design of the sketches that the participants of this study interacted with. I also discuss my data collection technique and provide information about the academic background of the participants.

Analyses of data are presented in Chapter 6. It includes a detailed analysis of the participants' interactions with the *eigen* sketch in order to identify eigenvector(s) and associated eigenvalue(s) of the given four matrices. I first provide an analysis of each participant's interactions with the sketch applying each theoretical framework independently. Then, I provide a synthesized analysis by triangulating my analysis using the three theoretical frameworks jointly.

In Chapter 7, I provide an extension of the analysis of the data that I presented in Chapter 6. I discuss the results of the participants' interaction with the *eigen* sketch from three perspective: (1) a focus on the participants' understanding of eigenvectors and eigenvalues, (2) their linguistic expressions and (3) the gestures used by participants in describing the geometric representation of eigenvectors.

Chapter 8 includes my response to the research questions posed in Chapter 5. I also discuss my contribution to research and the limitations of my study. At the end, I provide pedagogical implications of this study.

This dissertation has three Appendices. Appendix A shows the worksheet given to the participants. Appendix B includes transcripts of the participants' discourse and a description of their interaction with the dynamic geometric representation of the concepts. Appendix C includes a list of questions that I used to prompt the participants during the interviews.

2. The teaching and learning of linear algebra

Learning linear algebra is a challenging experience for many university students and its teaching is also regarded as a frustrating experience for instructors (Dorier, 2000; Hillel, 2000). Although research on learning and teaching linear algebra has been going on since the late 1980s, teaching and learning difficulties still persist. In 1991, a group of linear algebra instructors in the U.S. formed the Linear Algebra Curriculum Study Group (LACSG) that aimed at improving the undergraduate linear algebra curriculum. They recommended a matrix-oriented linear algebra course, including a wide variety of application problems to address the needs of client disciplines and to attract students' interest. They also recommended the use of technology, especially Computer Algebra Systems (CASs) as a tool to carry out computations (Carlson, Johnson, Lay & Porter, 1993). In fact, Gilbert Strang, an expert professor in linear algebra, has represented some linear algebra concepts in his book using a Computer Algebra System (MATLAB) since 1988 (see Strang, 1988; Strang & Borre, 1997). A much smaller number of instructors have used Dynamic Geometry Environments (DGEs) such as *Cabri-géomètre II* or *The Geometer's Sketchpad* (see, for example, Sierpinska, Dreyfus & Hillel, 1999; Meel & Hern, 2005) to represent linear algebra concepts geometrically. A review of studies on the implementation of technology in the teaching and learning mathematics and, more specifically, in linear algebra is provided in Chapter 3. In this chapter, however, I focus on some of the possible sources of difficulties that students and instructors are faced with in learning and teaching linear algebra.

2.1. Linear Algebra Curriculum reforms

Until the 1960s, the study of linear equations comprised the main part of the linear algebra curriculum. But mathematics curriculum reform drastically changed the teaching of linear algebra. The axiomatic theory of vector spaces became the core part of undergraduate linear algebra courses. The theory provides a unified and generalized approach to and setting for modelling finite- and infinite-dimensional linear problems.

The use of an axiomatic theory of vector spaces seems natural and useful from a mathematician's perspective, but it does not take into account students' needs, abilities and their mathematical background. To address these, as I mentioned above, LACSG recommended a matrix-oriented course including a variety of application problems, whereas the theoretical approach focuses on introducing the axiomatic theory of vector spaces (Carlson, Johnson, Lay & Porter, 1993). A less apparent approach has also been used by Banchoff and Wermer (1991) that involves the use of vector geometry to provide intuitive meaning for basic linear algebra concepts.

2.2. Learning linear algebra and possible sources of students' difficulties

Despite the U.S. curriculum reforms that took place in 1991, research shows that students still have difficulties in learning linear algebra concepts. In his book, Jean-Luc Dorier (2000) provides a substantial overview of research work on the teaching and learning of linear algebra in both France and North America. Studies conducted in France were longitudinal, aimed at identifying the nature of students' difficulties, evaluating experimental teaching, and providing an epistemological analysis of linear algebra concepts. Dorier (2000) refers to a survey study (see Robert & Robinet, 1989) in which French students criticized the overwhelming number of new definitions and theorems, representational language and the use of formalism in first-year university linear algebra courses. Dorier also considers French instructors' observations about students' lack of basic logic and set theory to help them understand the formal representational language, and also students' lack of geometric intuition to allow them to visualize geometric representations of the basic concepts. Being involved in several studies, Dorier and his colleagues conclude that students had difficulties understanding the use of concepts, interpreting them in more intuitive contexts such as plane geometry, and making connections among and within concepts (Dorier, Robert, Robinet & Rogalski, 2000). Given that the difficulties are perceived as being due to formalism, they refer to them jointly as the *formalism obstacle*. A formalist approach to vectors, for example, involves seeing them simply as elements of any vector space. In contrast, a non-formalist approach to a vector is to consider it as an ordered array of numbers or as an arrow in two (or more) dimensions that has a certain magnitude and direction. The obstacle often makes students develop automatic behaviour in problem-solving situations. One such behaviour involves finding a reduced form of a given matrix whenever they can, regardless of the actual problem (Sierpinska, Dreyfus & Hillel, 1999). Studies suggest that anchoring the notions of vector space, linear transformation, and eigenvector in geometric intuitions can help students overcome the formalism obstacle (Sierpinska, Dreyfus & Hillel, 1999; Dorier, Robert, Robinet & Rogalski, 2000). As Dorier *et al*. (2000) suggest, the use of analytic geometry can promote visualization of linear equations, curves, skewed surfaces, and their solution sets in connection with geometric loci.

Another source of difficulty is associated with the representational language of linear algebra. A formal symbolic language is used to present definitions of concepts and the presentation mostly follows a systematic approach by means of reference to other definitions and previously proof-based theories. According to Hillel (2000), the formal symbolic language of linear algebra consists of three modes of description—abstract, algebraic and geometric—and their associated representational systems. The abstract mode involves the language and concepts of the general theory including vector spaces, dimension, linear transformations of vector spaces, general eigenvalue theory, and so on. The algebraic mode provides the language and concepts of R^n including *n*-tuples, matrices, solution of systems of linear equations, and so on. The geometric mode is the language of the two- and three-dimensional spaces, including directed line segments, points, lines, planes, and the transformation of geometric figures. The modes are not equivalent, but co-exist and are sometimes interchangeable. For example, a geometric representation of a vector (use of an arrow in a plane that has vertical shift of x and horizontal shift of y from any starting point) is not equivalent to its algebraic representation ($v = (x, y)$), but they co-exist and can be interchangeable when working on a problem involving the coordinate grid. In his study, Hillel (2000) shows that North American students had difficulties with the use of different modes of description of basic objects and processes, and with the transition from one mode to another, particularly from the algebraic to the abstract mode.

Sierpinska (2000) argues the need for the development of different modes of thinking to understand different modes of description and representation. Synthetic-geometric, analytic-arithmetic and analytic-structural are three different modes of thinking corresponding to different modes of representation. She points out that these three modes of thinking are “equally useful, each in its own context, and for specific purposes, and especially when they are in interaction” (p. 233). Analytic-arithmetic thinking involves describing a proper set-up to carry out the computations. For example, consider a system of three linear equations when the task is to find the solution set. The analytic-arithmetic mode of thinking invokes use of Gauss-Jordan elimination to find the solution set. Synthetic-geometric thinking involves use the geometric mode of description to visualize equations geometrically in two- and three- dimensional space. In the example above, it draws on the visual representation of the intersection set of three planes in a three-dimensional space. Analytic-structural thinking “synthesizes the algebraic elements of the representations into structural wholes” (p. 235): for example, finding a solution set of a system of linear equations through determining the singularity or non-singularity of the coefficient matrix of the system. In other words, analytic-structural thinking enables thinking of an object in terms of its properties, whereas analytic-arithmetic thinking specifies an object by a formula. The development of different modes of thinking is not exclusive to a linear algebra context. For example, Sfard (1991) argues for an operational-structural duality of mathematical conceptions more generally. According to her, mathematical concepts such as function or number can be conceived both structurally (as objects) and operationally (as processes). Although these two modes are ostensibly incompatible, in fact they are complementary. However, the transition from an operational mode to a structural one is a long and difficult process.

In the linear algebra context, Sierpinska makes a distinction between analytic-arithmetic and analytic-structural thinking, and notes that, historically, the dialectic between analytic-arithmetic and analytic-structural thinking has provoked the development of mathematical ideas in linear algebra and also in the calculus. She further argues that synthetic-geometric thinking supports the geometric intuition of concepts and allows the identification of invariants in reference to several representations of concepts. However, synthetic-geometric arguments do not belong to linear algebra, since linear algebra studies properties that do not have to be geometric. According to her, there are common features between synthetic-geometric and structural ways of thinking in linear algebra. One feature is *independence from the coordinate* *system* and another is that both ways of thinking are *based on properties of objects not calculations*. As she exemplifies:

> In the structural mode, the notion of eigenvalue cannot be reduced anymore to that of a root of a polynomial. It must be thought of as a scalar related to invariant one-dimensional subspaces of a linear operator. It is an object of reflection and a concept; not an outcome of a calculation. (p. 236)

Synthetic geometry concerns only geometric properties of figures or concepts, whereas analytic geometry goes further to include relations between figures. As I mentioned before, Dorier *et al*. (2000) suggest the use of analytic geometry to promote visualization and to enable the development of analytic thinking. But Hillel argues that the students' familiarity with analytic geometry can become an obstacle in the transition from a geometric mode to an abstract mode of description (e.g. working with standard coordinates being an obstacle to thinking about the notions of basis and change of basis). He also mentions that the language of R^n can become an obstacle to learning the axiomatic theory of vector spaces and conceptualizing mathematical objects such as functions, matrices or polynomials as vectors.

The aforementioned difficulties are due to the nature of linear algebra and can be categorized as conceptual difficulties. Considering the existence of such conceptual difficulties, learning and understanding linear algebra is a cognitively demanding process. The research of Alves Dias indicates that understanding linear algebra necessitates the development of cognitive flexibility between modes of thinking. She hypothesizes that cognitive flexibility is not a semiotic process and cannot be reduced to change of semiotic registers[2] (Alves Dias & Artigue, 1995). In this connection, other researchers acknowledge that understanding linear algebra requires a trans-object level of thinking (Hillel, 2000; Dorier & Sierpinska, 2001). According to Dorier and Sierpinska,

[2] Semiotic registers refers to registers of semiotic representation. For example, axes, points, segments and arrows are elements of a register for graphing representation. According to Duval (2006), changing register signals a threshold of mathematical comprehension for students. He suggests the coordination of several registers of semiotic representations is required in order for students understand mathematical concepts.

such a trans-object level of thinking consists of the building of conceptual structures out of what, at previous levels, were objects, actions on these objects and transformations of both objects and actions. In other words, a trans-object level of thinking calls for cognitive flexibility among modes of thinking, modes of description and representations. For example, a trans-object level of thinking about the eigenvectors of a square matrix requires thinking flexibly between perceiving eigenvectors as *n*-tuples (an algebraic mode of thinking and description) and eigenvectors as special vectors resulting from a specific transformation that can form a linear subspace called an eigenspace (abstract mode of thinking and description).

International studies show linear algebra students mostly develop procedural knowledge and their level of thinking does not advance to a trans-object level (Hillel & Sierpinska, 1994; Alves Dias and Artigue, 1995; Stewart, 2008). The persistence of conceptual difficulties in learning linear algebra has motivated several educators to look for other ways of representing concepts. Given that calculators and computer technology show the potential to reduce cognitive processing load in doing computations, several educators have integrated these technologies in to their teaching practices. Chapter 3 provides a more detailed literature review of the use of technology in the teaching and learning of linear algebra.

In the next section, I will evaluate that the way linear algebra textbooks represent concepts highly influences the development of procedural knowledge and perhaps affects the achievement of a trans-object level of thinking. In order to identify textbooks' contributions to students' ways of learning and thinking, in the following section, I include a brief evaluation of a popular textbook introducing the concepts of system of linear equations, linear transformation, basis, and eigenvector and eigenvalue.

2.3. Teaching approaches: textbooks' representations of concepts

At the undergraduate level, in North America, two approaches to teaching linear algebra —theoretical and practical—are prevalent. The theoretical one focuses on a systematic development of the axiomatic theory of vector spaces where proof plays an important role. The practical one is a matrix-vector approach, including a variety of applications as recommended by LACSG. Both approaches normally start by introducing systems of linear equations and finding the solution sets of systems using the Gauss-Jordan elimination method (see Lay, 2006; Anton & Rorres, 2004). The method relies on notions of elementary row operations and the relation of row equivalence of matrices. It can become tedious for students who do not see a practical reason for carrying out such operations. Moreover, students do not understand why the solution set of a system of equations is invariant under elementary row operations (Sierpinska, 2000). Conceptualizing the solution set of a system (n linear equations in n variables) as a set of n-tuples is a problematic concept for some students (de Vries & Arnon, 2004). Although researchers have identified that the sources of difficulties include a lack of prior knowledge in elementary set theory (Dorier *et al*., 2000) and equivalence classes (Sierpinska, 2000), linear algebra textbooks do not usually include a supplementary section on the concepts of equivalence relations and set theory. Furthermore, the connection between systems of linear equations and the concepts of vector and vector spaces are not emphasized in textbooks. For example, Lay (2006) introduces the concept of vector to connect "equations involving vectors" to "ordinary systems of equations" and further states that "we will use vectors to mean a list of numbers" (p. 28). This symbolic introduction to the concept of vector is disjoint from its embodied representation in physics and mechanics. The rich yet complex mathematical structure of a vector (as a force, a transformation, a velocity, a quantity having magnitude and direction, an element of a vector space) requires a flexible way of introducing the concept of vector (Watson, Panayotis & Tall, 2003). Watson *et al*.'s study shows that the concept of vector can be a representative of each of the embodied, symbolic and formal mathematical worlds. They believe that the development of these three mathematical worlds, in the particular development of the concept of vector, is sequential. However, my brief analysis of linear algebra textbooks shows that the connection among the three worlds there, especially between the embodied and symbolic worlds, is not explicitly developed.

The textbooks' representation of the concept of linear transformation may very well lead to certain difficulties for students. In his textbook, Lay (2006) first introduces a linear transformation in terms of matrix-vector multiplication; a transformation $T(x) = Ax$

where A is a $m \times n$ matrix and x is an $n \times 1$ vector. This representation may invite students to think of A, at best, as a coefficient matrix of a system of linear equations and, at worst, as a random array of numbers (Meel & Hern, 2005), rather than realizing that the column vectors of A are the coordinates of the images of the standard basis vectors under this transformation. The axiomatic definition of linear transformation, in textbooks, is mostly included after introducing the concept of linear transformation as a matrix-vector multiplication. But, students do not spontaneously refer to the axiomatic definition in problem-solving situations (Sierpinska, Dreyfus & Hillel, 1999). Sierpinska (2000) observes that students mostly think of the defining properties of linear transformation ($T(k_1v_1 + k_2v_2) = k_1T(v_1) + k_2T(v_2)$, for all vectors v_1 and v_2 and scalars k_1 and k_2) in terms of an equation. They try to perform typical actions on this apparent 'equation' for certain vectors and scalars. This is an example of thinking in terms of prototypical examples and also procedural thinking rather than a trans-object level of thinking. Although thinking in terms of prototypical examples brings some progress in understanding the notion of linear transformation, it is not sufficient to generalize the concept and to interrelate it with the matrix representation of a linear transformation (Sierpinska, 2000).

The use of the matrix-vector multiplication representation of a linear transformation is continued when introducing the concept of eigenvectors and eigenvalues (see Figure 4). The symbolic verbal definition of the concepts of eigenvector and eigenvalue in textbooks represents the concepts in terms of matrix-vector multiplication, emphasizing the equality $Ax = \lambda x$. The definition does not draw attention to the fact that eigenvectors are special vectors that are transformed into a scalar multiple of themselves. Also, it does not include the fact that eigenvalues are the dilation factors of the linear transformation represented by *A*.

This textbook representation leads to a procedural algebraic method for finding eigenvalues and associated eigenvectors of a square matrix. It also causes students to develop an automatic reaction to the term "eigenvector". As observed by Sierpinska *et al*.'s (1999), many students in their classes automatically set up the characteristic equation to find its roots regardless of whether this is necessary or useful in solving the given problem. In connection to this, another study draws attention to students' lack of geometric meaning for eigenvectors (Stewart & Thomas, 2007).

DEFINITION An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

Figure 4[3]. Formal definition of eigenvector and eigenvalue (Lay, 2006, p. 303)

In most linear algebra textbooks, the concepts of basis and change of basis are introduced after the theory of vector spaces. Sierpinska *et al*. (1999) notice that students have a tendency to visualize the representation of the basis in the horizontal/vertical position (as they turned their heads to view the basis in the horizontal/vertical position), although they accepted that a basis can be provided by any pair of non-collinear vectors in a two-dimensional space. Furthermore, Hillel's (2000) study shows that students have difficulties in finding the matrix representation of a linear transformation when the vector spaces, R^n , have nonstandard bases. This suggests that their understanding of the concept of basis is limited to the standard basis in finite-dimensional vector spaces, such as R^2 and R^3.

The limited number of diagrams in textbooks (see Lay, 2006; Kolman & Hill, 2000; Banchoff & Wermer, 1991) and the vague descriptions of diagrams are not sufficiently helpful in providing the geometrical meaning of concepts. Chartier's (2006) review of *Linear Algebra Through Geometry*, written by Banchoff and Wermer (1991), claims that 92% of the drawings represent situations in R , R^2 and R^3, 6% of them illustrate situations in R^4 , and only 2% of the drawings (2 out of 92 drawings) illustrate general situations in an arbitrary vector spaces. She points out that there are almost no drawings to accompany notions such as vector space, vector subspace, spanned subspace and basis. Furthermore, the text does not suggest referring to an associated

[3] The footnote 1 on the end of the definition states that an eigenvector must be a non-zero vector, but an eigenvalue could be zero.

drawing in R^2 and R^3 when it introduces abstract vector spaces. She further studied French mathematicians' use of drawings, by asking them to fill out questionnaires prompting them to produce diagrams they might use in their teaching. Her findings reveal that French mathematicians made limited use of figural models to indicate the geometric meaning of concepts in R^2 and R^3, nor did they extend these models to R^n. These findings indicate that students are assumed to comprehend and to conceptualize abstract modes of description without extensive preparation. However, studies show that students have serious difficulties in learning and understanding linear algebra (see Dorier *et al*., 2000; Sierpinska, 2000; Hillel, 2000). Hence, new approaches to teaching linear algebra are needed, particularly ones that provide students with powerful representational models where they can build their own, more geometric, understanding of concepts.

2.4. Representing linear algebra concepts: a geometric approach

Harel (2000) compares achievements of two groups of students in linear algebra. Group A was only presented with an abstract description of ideas, whereas group B was presented with abstract ideas and also references to geometric interpretations of these ideas. The comparison showed that the achievements were significantly different. Group B students used more geometric interpretations and provided more correct answers to a test on the vector-space concept than Group A students. These findings led to Harel proposing three pedagogical principles for designing and implementing linear algebra curricula. First, the *concreteness principle* is that students should be able conceptualize abstract ideas and structures from concrete contexts and structures. For example, geometric representations of vector spaces with dimension less than or equal to three can provide concrete contexts to conceptualize basic concepts of linear algebra. He recommends incorporating MATLAB in an instructional design to meet the conditions of the concreteness principle.

Second, the *necessity principle* involves offering problem-solving activities that are realistic to students, so that students can apply their learning to the solutions of problems and modify their strategies when they encounter cognitive conflicts. Third, the *generalizability principle* asserts that concrete contexts should allow and encourage the generalizability of concepts. For example, a spanning set of a vector space can be explored to lead students to grasp that a minimal spanning set (a basis) can still be used to generate the entire vector space.

In connection to Harel's concreteness principle, the implementation of the dynamic geometric diagram of concepts (in two-dimensional vector spaces) in linear algebra curricula seems to be a promising approach that could enable students' to conceptualize basic concepts. Dynamic geometry software offers the possibility for geometric representation of concepts such as vector, span of (a set of) vector(s) or linear transformation. It is also possible to design a sketch to represent a concept in both arithmetic and geometric representational systems. Such representations can provide concrete contexts for abstract notions and also can prompt synthetic-geometric thinking about concepts. Through the interaction with the sketch, a student can identify the invariants in reference to arithmetic and geometric representations of a concept and may come to conceptualize the concept. Understanding the invariance property of the concept can enable the student to generalize the concept in higher dimensions. But the sketch representation may not be seen as a realistic problem-solving activity to fulfill Harel's necessity principle.

2.5. Summary

In this chapter, I provided a literature review of studies on students' difficulties in learning linear algebra. According to Hillel (2000), the formal representational language of linear algebra consists of three different modes of description (abstract, algebraic and geometric) and their associated representational systems. The use of different modes of description of basic objects and processes, and the transition from one mode to another, both cause difficulties for students. To help students understand different modes of description and representation, Sierpinska (2000) argues the need for the development of three different modes of thinking (synthetic-geometric, analytic-arithmetic and analytic-structural). In connection to this, other researchers acknowledge that understanding linear algebra requires a trans-object level of thinking (Hillel, 2000; Dorier & Sierpinska, 2001). However, international studies show that linear algebra students mostly develop procedural knowledge and their level of thinking does not advance to a trans-object level (Hillel & Sierpinska, 1994; Alves Dias & Artigue, 1995; Stewart, 2008).

Students' lack of geometric intuition in order to visualize geometric representations of the basic concepts also causes difficulties in learning and understanding linear algebra. It seems that a lack of geometric intuition and the development of primarily procedural knowledge are both associated with the way linear algebra textbooks represent concepts. In section 2.3, I provided a brief evaluation of a popular textbook in use in North America in order to stress its representational approaches in introducing the concepts of system of linear equations, linear transformation, basis, and eigenvector and eigenvalue. My evaluation led me to argue for new approaches to teaching linear algebra, particularly ones that provide students with powerful, dynamic, representational models where they can build their own, more geometric, understanding of concepts.

A review of studies on the use of computer algebra systems (CASs) and dynamic geometry environment (DGE) in teaching and learning linear algebra is now provided in Chapter 3, in order to offer insights into the design of learning environments and possible difficulties that students may be faced with in such environments.

3. The teaching and learning of linear algebra in technology—enhanced environments

As discussed in Chapter 2, studies show that students develop primarily procedural knowledge of linear algebra concepts, and often fail to grasp connections between concepts (see Dorier *et al*., 2000; Sierpinska, 2000; Hillel, 2000; Harel, 2000). To promote students' understanding, instructors have studied the implementation of a variety of technological tools such as graphing calculators, computer algebra systems (Harel, 2000), and dynamic geometry software (Sierpinska, Dreyfus & Hillel, 1999; Meel & Hern, 2005). Graphing calculators and computer algebra systems illustrate static and algebraic representations of concepts, and also facilitate computations. On the other hand, dynamic geometry software has been shown to facilitate students' construction of their own mathematical objects and to avoid the obstacle of formalism. For example, the study by Sierpinska, Dreyfus and Hillel (1999) shows that students articulated the meaning of linear transformations, in problem-solving situations, using terms such as *operations, positioning* and *relationships*. Although it is possible to represent some concepts geometrically using CASs, the resulting representations are not inherently interactive, and thus are not as well suited to exploring, conjecturing and constructing in a geometric mode. The sections below include a brief review of literature on the use of CASs and DGEs in teaching and learning mathematics, with a particular focus on linear algebra.

3.1. Use of Computer Algebra Systems in learning and teaching mathematics

Recent digital technologies, such as CASs and symbolic calculators (TI-89 and TI-92), have changed the possibilities for improving student learning (see Trouche, 2005;

Lagrange, 2005; Artigue, 2005; Kieran, 2003). Lagrange (2005) suggests that computer algebra environments could assist students in problem-solving situations, provide visual representations of situations, and enable pattern perception and investigation. But such affordances of CASs do not happen so easily. Factors such as students' prior knowledge in algebra and about the system, the types of problem-solving situations, and the representations of concepts all influence the affordances of CASs and so their integration in teaching and learning. Moreover, the integration of symbolic calculators (such as the TI-89) in mathematics curricula challenges both the *pragmatic* and *epistemic values*[4] of traditional paper-and-pencil techniques. For example, using paper-and-pencil techniques to find the limit of a function retains little pragmatic value when students have access to the limit command on a symbolic calculator. However, using a symbolic calculator may make the epistemic value of limit more visible, since its routinization is no longer a necessity. Similarly, in the context of linear algebra, using paper-and-pencil techniques to find the determinant of a matrix retains little pragmatic value when students have access to the determinant command on a graphing calculator.

Although graphing calculators provide potentialities for visualization (such as the graphical representation of a function) and animation (zooming in and out), they do not necessarily help students to establish relationships between algebraic and graphical representations (Trouche, 2005). Trouche's study shows that a student was able to articulate the meaning of a function having an infinite limit by referring to the table of values in a graphic calculator environment (rather than referring to the graph of the function), since it was difficult for him to define an appropriate window for graphing the function on a large scale. Four months later, the student could not articulate the concept of limit in a symbolic calculator environment. This shows that the student's conception of limit changed from a process to an operation as the representational environment changed from graphical to symbolic. Each environment imposes potentialities and

[4] Lagrange (2005) uses the routines of pragmatic and epistemic value to evaluate techniques that one may use in a problem solving situation. The pragmatic value focuses on techniques' productive potential (such as efficiency, cost and field of validity), whereas, the epistemic value concerns the contribution of techniques to the understanding of the objects they involve.

constraints on the student's processes of adaptation to computer algebra environments and consequently influences the development of concepts.

In a linear algebra context, the use of technology as a tool to run computations has been recommended by the LACSG since 1991. This recommendation has directed some instructors' attention to the possible integration of CASs in linear algebra courses, particularly in North America. Although CASs facilitate computations and illustrate symbolic representations of concepts, they may not necessarily assist students in understanding the concepts. A recent study on students' ways of thinking in a technology-assisted environment (a *linear algebra with Maple* course) shows that the software did not have positive, significant effects on students' ways of thinking (Pruncut, 2008). Pruncut's study confirms the findings of other studies in regard to students' behaviour and difficulties in CAS environments. She observes that students' behaviour consisted of oscillating among several techniques and strategies (a phenomenon also observed by Defouad, 2000), use of trial-and-error procedures without attempting to validate the results, and memorizing strategies to apply in different situations (also observed by Trouche, 2005). These common observations led her to conjecture that such behaviour may occur in any CAS environment. She also identifies didactical obstacles caused by the CAS environment: for instance, learning concepts using *Maple* commands could result in thinking about these concepts in terms of commands. And students' reliance on *Maple* calculations could also lead them to complex situations that might be difficult to manage.

Although the plotting facilities of CASs, in particular *Maple* and *MATLAB*, provide visual representations of some basic linear algebra concepts, the use of these visual representations has not been a focus in studies. The literature only includes their use as a computational tool; the full evaluation of CASs visual tools is beyond the scope of my study.

3.2. Use of Dynamic Geometry Software in teaching and learning mathematics

The use of DGEs has been found to be effective in teaching a variety of school mathematics subject areas, including geometry and algebra, and, also, the teaching of calculus, both at the high school and undergraduate levels (Hollebrands, 2003; Falcade, Laborde & Mariotti, 2007; Habre & Abboud, 2006). Given that a DGE simulates a geometric environment, it is widely used and studied in the context of the geometry curriculum. However, the dynamic and interactive features of DGEs, enhanced by the dragging tool, enables the design of interactive sketches appropriate for representing algebraic concepts. For example, Falcade *et al*. (2007) designed a teaching experiment using *Cabri-géomètre* (Baulac, Bellemain & Laborde, 1988) to introduce students to the idea of function as covariation between dependent and independent variables. Their findings show that the combined use of the Dragging and the Trace tools enabled students to develop meaning for both variation and covariation. The DGEs also enabled students to perform multiple actions and generate a large number of examples effortlessly (Hollebrands, 2007; Laborde, 1992; Mariotti, 2000).

However, despite its appropriateness to other undergraduate subject areas, its potential in courses such as linear algebra has received little attention. Meel and Hern (2005) designed interactive web-based tools for various concepts of linear algebra using *The Geometer's Sketchpad*. They provide some technical details of the design of *GridMaster, Transformer 2D* and *Eigenizer*, and include samples of exploratory activities. Figure 5 shows a snapshot of the *Eigenizer* tool that enables students to explore eigenvalues and eigenvectors of matrices. Their description includes students' responses to several prompts about the tools. Although, they used prior research findings to design these tools carefully, they do not provide insight into the *effect* of these presentations on students' ways of thinking. Based on their classroom observations and on students' responses to prompts, Meel and Hern suggest that the use of these tools facilitates transition between algebraic and geometric modes of representations. Their work sparked my interest to study the effect of dynamic geometric representations on students' thinking. It seems that the existence of two modes of representations and the use of these tools could result in the development of different modes of thinking, as discussed in section 2.2.

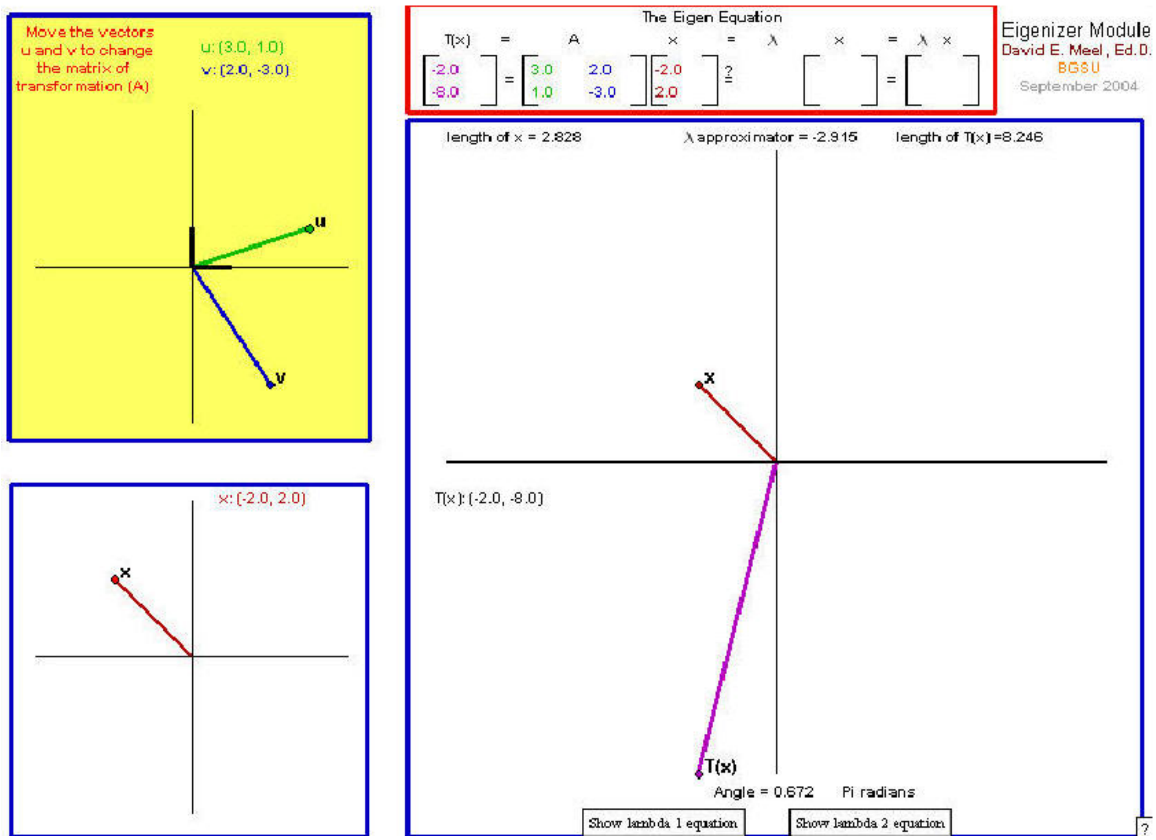


Figure 5. A snapshot of Eigenizer tool

Cabri-géomètre II software has also been used as a pedagogical aid to design a learning environment to help students to develop geometric intuitions about basic linear algebra concepts (see Sierpinska, Deryfus & Hillel, 1999). Sierpinska *et al*. were particularly concerned with students' difficulties in distinguishing vector from its coordinates with respect to a particular basis and a linear transformation from its matrix representation relative to a given basis. They believed that introducing vectors as arrays of numbers and reducing the notion of linear transformation to linear substitutions on the entries of vectors (e.g. $T(x, y) = (2x, -x + 2y)$) are sources of students' difficulty. Using *Cabri*, they designed a geometric model of a two-dimensional vector space, one where vectors represented position of points with respect to a fixed point (the origin). Using *Cabri*, they also provided sketches where students could verify the linearity of a given transformations by checking the conditions of the axiomatic definition (a transformation preserving the operations of vector addition and scalar multiplication). Furthermore, the representation consisted of arbitrary vectors and their image vectors under a transformation (not given by formulas) to prevent students from thinking in terms of prototypes or typical examples.

Sierpinska *et al*. (1999) hypothesized that the dynamic and interactive representation of concepts would enable students to construct mathematical objects by means of identifying the invariants with respect to several semiotic representations. For example, identifying the defining invariants of a linear transformation is to notice that it preserves the operations of vector addition and scalar multiplication. Although the geometric model enabled students to construct mathematical objects, there were discrepancies between students' interpretations of mathematical objects and the intended interpretations of the *Cabri* representations of linear transformations. For example, students called a dynamic pair of vectors (one free and one dependent) a linear transformation if the dilations of the free vector were accompanied by proportional dilations of the dependent vector. They also did not explicitly verify whether the given transformation preserves the operation of vector addition. This suggests that students could not see the link between the definitional properties of linear transformations ($T(v+w)=T(v)+T(w)$, $T(kv)=kT(v)$) and the geometric model of representation, because they tended to interpret $T(kv)=kT(v)$ in terms of proportions between lengths of vectors rather than a multiplication property of a transformation of a vector space. According to Sierpinska *et al*. (1991), "the dynamic *Cabri* representation of the dilation property of linear transformation led students to focus on a proportionally changing pair of vectors" (p. 35), when they were asked to verify linearity of a transformation.

Not all of the difficulties resulted from the *Cabri* representation. Students also had difficulties using the axiomatic definition of linear transformation in problem situations, and in understanding the notions of basis and the span of a plane by a pair of non-collinear vectors. Despite the aforementioned difficulties, Sierpinska *et al*. pointed out that the *Cabri* environment helped students articulate given problem-solving situations (in the context of linear transformation), using terms such as *operations*, *positioning* and *relationships*. For example, in the problem-solving situation: "put five vectors v_1, v_2, w_1, w_2 and v on the screen; assume that $w_1=T(v_1)$, $w_2=T(v_2)$ under a linear transformation T. From the information given, would you know where the vector $T(v)$ should be? Can you construct it?" (p. 31). A student, after interacting with the sketch and analyzing arithmetic representations of the images of the basis vectors (w_1 and w_2), states that “when we apply the same *operations* on *v* we should get $T(v)$ ” (p. 34). The student’s use of the term *operations* suggests that he was not just manipulating *Cabri* figures; instead, he was interacting with *Cabri* figures in a meaningful manner, although he seemed unaware that *operations* are not necessarily linear transformations.

3.3. Summary

In this chapter, I focused on the prior studies that integrated digital technologies (CASs and DGEs) in the teaching and learning of linear algebra. Although, the use of technology as a to date tool to run computations has been recommended by the LACSG since 1991, only a few studies have reported on the impact of the integration of digital technologies on students’ understanding.

In regard to the use of CASs, several researchers have observed that students’ behaviour consisted of oscillating among several techniques and strategies (Pruncut, 2008; Defouad, 2000), use of trial-and-error procedures without attempting to validate the results, and memorizing strategies to apply in different situations (Pruncut, 2008; Trouche, 2005). These common observations led Pruncut to conjecture that such behaviour may occur in any CAS environment.

On the other hand, Sierpinska *et al.* (1999) hypothesize that the dynamic and interactive representations of concepts would enable students to construct mathematical objects through identifying the invariants in reference to several semiotic representations. Their study led them to conclude that the *Cabri* environment helped students articulate given problem-solving situations (in the context of linear transformation), using terms such as *operations*, *positioning* and *relationships*.

Sierpinska *et al*.’s findings motivated me to investigate the effect of the use of dynamic geometric representations on students’ conceptualizations of eigenvectors and eigenvalues. Using *The Geometer’s Sketchpad*, I designed several sketches to represent the main concepts of linear algebra that undergraduate students are introduced to in their first linear algebra course. To justify the use of a DGE to represent mathematical ideas and concepts further, I now provide a review of theories on the role of tools, visualization, and attention in the development of mathematical thinking and understanding.

4. Theoretical influences

This chapter provides an overview of the multiple theoretical frameworks that I draw on in order to study the effect of the dynamic geometric representation on students' conceptualizations of eigenvectors and eigenvalues. I start with Mason's theory of shifts of attention, since it provides a broad theoretical account in order to highlight the significant role of attention and awareness in learning and understanding mathematics. Given that my study involves the use of dragging tool, I draw on the theory of instrumental genesis to understand the participants' use of this tool and its internalization into an instrument. I also include a review of research findings on the specific use of different dragging modalities since the participants used different dragging strategies and modalities while they interacted with the dynamic geometric representation of eigenvectors and eigenvalues. I then include a review of visualization and different categories of visual imagery given the visual and interactive nature of this representation. Moreover, I consider here the importance of kinaesthetic and dynamic imagery in mathematical thinking and draw on the theories of embodied cognition in the development of mathematical thinking. Theories of the role of embodied cognition and gestures in mathematics education have broadened my perspective in terms of noticing the role of time and motion—and not just visualization—in the development of mathematical thinking and learning.

4.1. Attention and awareness

Mason (2008) believes that attention and awareness are two aspects of the human psyche in the developmental process of mathematical being. Awareness refers to what enables us to act, calling upon our conscious and unconscious powers, and is closely connected to the sensitivity to detect changes and to choose proper actions in certain situations (Gattegno, 1987; Mason, 2008). To educate one's awareness involves drawing attention to actions which are being carried out with lesser or greater awareness. Attention can be drawn not only to mathematical objects, relationships and properties, but also to manifestations of mathematical themes, and to heuristic forms of mathematical thinking.

According to Mason, the structure of attention comprises both macro and micro levels; *how* something is being attended to is as important as *what* is being attended to. At the macro level, Mason describes the nature of attention as follows: “attention can vary in multiplicity, locus, focus and sharpness” (p. 5). At the micro level, he distinguishes five different states of attending: holding wholes, discerning details, recognizing relationships, perceiving properties, and reasoning on the basis of agreed properties. Holding wholes occurs when a student gazes at a definition, a collection of symbols and/or a diagram. The student may not focus on anything in particular, while “waiting for things to come to mind” (p. 36). Looking at the wholes, the student may discern and identify useful sub-wholes or details. Discerning details is a process that is involved in and contributes to subsequent attending. As the student discerns details, she may recognize relationships between symbolic and geometric representations of mathematical concepts. When she becomes aware of possible relationships in the particular situation, she may perceive these as instantiations of a property. As she continues attending, she can use the perceived properties as a basis for mathematical reasoning.

Molina and Mason (2009) note that these described states of attention are not levelled or ordered. They often last for a few micro-seconds and alternate among other states. Those that become stable and robust against alteration for varying periods of time may block further development of awareness.

Mason argues that different states of attention can be triggered more prominently than others by different cues. The flexibility of shifts among various forms of attention is a factor that influences one’s awareness. Also, a lack of accumulated necessary experience of the different forms of attention can cause difficulties for learners. For example, a learner’s attention may not advance to recognizing relationships or a learner may not attend to reasoning on the basis of properties. This suggests providing learners with opportunities where their attention can be drawn to identifying the invariants of a mathematical concept which would enable them to perceive properties of the concept.

Mason's theory of shifts of attention seems very relevant to use in analyzing a student's attention in a mathematical activity. But given the important role of the digital tool in DGE-based activity, a suitable theory for analyzing a student's structure of attention involved in such an activity needs to take into account the effect of a student's interactions with the tool. The following section includes a review of the theory of instrumental genesis to describe processes involved in the interactions between a learner and a digital tool. Given that the participants of my study mainly used the dragging tool, I also include a review of research findings on the specific use of different dragging modalities.

4.2. Tools and instruments

The theory of instrumental genesis (Verillon & Rabardel, 1995) draws on actions and procedures undertaken by a student in using a tool. The tool can be transformed into an internally oriented tool (an instrument of semiotic mediation) by the process of internalization (Vygotsky, 1978) that occurs through semiotic processes. For example, given a specific task in a dynamic geometry environment, the dragging tool can be transformed into an instrument referring to the idea of function as covariation between dependent and independent variables (Falcade *et al*., 2007). Similarly, with the *eigen* sketch, the dragging tool can be transformed into an instrument for detecting the presence of the geometric representation of an eigenvector as a non-zero special vector collinear with its transformation under a 2×2 matrix.

The development of instrumental genesis is a complex process that depends upon several factors, such as the potentialities and constraints of the tool, actions and procedures taken by the student, the student's knowledge of mathematical concept in the task, and also the student's awareness of the affordances of the tool. The two interconnected components of instrumental genesis—instrumentalization and instrumentation—are used to describe the processes involved in the interactions between the student and the tool. The *instrumentalization* process, directed toward the tool, involves development of the skill necessary to use the tool, as well as the personalization and the transformation of the tool. It is about what the student thinks the tool was designed for and how the student uses the tool. It requires attending to tool use.

The *instrumentation* process, directed by the tool, involves the constraints and potentialities of the tool that shapes the student's knowledge acquisition (Trouche, 2005). This involves a shift of attention from tool use to what the tool can do, so that the tool becomes not the object of attention, but something that focuses and directs attention in particular ways—a mediating tool.

These two components are concerned mostly with processes involved in transforming a tool into an instrument, not the role of the instrument in knowledge acquisition. As researchers point out, the role of the instrument in cognitive development is a delicate issue (Verillon and Rabardel, 1995), and the theory of instrumental genesis has shortfalls in putting forward the potentialities of the instrument in the development of mathematical thinking. As discussed above, Mason's theory of shifts of attention appears potentially fruitful in terms of revealing the developmental process of mathematical being. In my study, I attempted to show that in a digital technology environment, instrumentation and instrumentalization processes cause shifts of attention. To do so, I suggest combining the theory of instrumental genesis with the theory of shifts of attention to enable a deeper analysis of the cognitive development of a learner in a digital technology environment.

Moreover, prior study of the use of the dragging tool from a cognitive perspective suggests that dragging can mediate the relationships between perceptual and conceptual entities. Arzarello, Olivero, Paola, and Robutti (2002) point out that "dragging supports the production of conjectures: exploring drawings by moving them, looking at the ways after which their forms change (or do not change), [and] allows users to discover their invariant properties" (p. 66). They identify different dragging, modalities such as *wandering* dragging, *guided* dragging, *dummy-locus* dragging, and *line* dragging. *Wandering* dragging refers to moving a draggable object on the screen randomly, without a plan, in order to explore the relationships among the other parts of the object in the sketch. *Guided* dragging involves dragging an object in order to locate a particular configuration. *Dummy-locus* dragging refers to dragging an object in such a way that dragging preserves a discovered property. *Line* dragging is dragging along a line in order to preserve the regularity of the discovered configuration. For example consider an arbitrary triangle and its circumcircle: wandering dragging involves randomly dragging vertices or edges of the triangle in order to explore the relationship between triangles and the center of its circumcircle. Guided dragging, for example, could involve dragging vertices in a way to make the triangle look like a right-angle triangle. An example of dummy-locus dragging in this setting would be to drag the vertex (opposite to hypotenuse) in such a way as to make isosceles triangles, thus the vertex which is dragged follows a straight path (even though the locus of the vertex may not be visible). An example of a line dragging would be to drag a vertex of the triangle along the edge (that passes through it) toward the other vertex.

In connection with the use of tool, Artigue (2002) points out a theoretical perspective on the use of ostensive objects in mathematics education:

> Mathematical objects are not directly accessible to our senses: they are non-ostensive objects; we work with them through ostensive representations which can be of very diverse nature: discourse in natural language, schemas, drawings, symbolic representations, gestures, manipulatives. Work with ostensive objects both shapes the development of the associated non-ostensive objects, and is shaped by the state of development of these. (p. 270)

This suggests that dynamic interactive diagrams can also act as ostensive objects to shape the development and the state of development of mathematical objects. The developmental processes of a learner can be described through analyzing shifts in her attention. Nonetheless, these representations may stimulate the formation of visual imagery. Next, I focus on the role of visualization and dynamism in thinking in learning mathematics.

4.3. Visualization

The role of visualization, visual imagry and visual thinking has been long recognized in learning and generalizing mathematical concepts (see Krutetskii, 1976; Presmeg 1986; Zimmermann & Cunningham, 1991). A mental visual image is a mental representation of an object or a process. *Visualization* or *visual imagery* refers to the processes of forming and transforming visual images with or without the presence of the object or process. Visualization involves internal processes in the mind, and can be identified through external activities such as articulating mental images, drawing diagrams on paper or even manipulating diagrams on a computer screen. Arguably, Individuals vary in their ability to use visual imagery, since their external activities vary. Five categories of visual imagery of relevance to mathematics are: (1) concrete, pictorial; (2) pattern; (3) memory images of formulae; (4) kinaesthetic; (5) dynamic imagery (Presmeg, 1986). Concrete imagery involves a concrete image of an object. For example, a concrete image of a vector in two-dimensional space involves visualizing it as a fixed arrow in the plane. Although concrete imagery is identified as the most prevalent imagery used by high school students, studies show that use of it can promote thinking in terms of prototypical examples, and so hinder mathematical generalization (Presmeg, 1997; Sierpinska, Deryfus & Hillel, 1999).

Pattern imagery is imagery in which concrete details are disregarded and relationships are depicted. An example of pattern imagery is to recall the span of vectors as a pattern: the span of one arbitrary vector is a line and the span of two arbitrary vectors is a plane, without evoking any particular image of a line or plane. This recall of span disregards the linear relationship between two vectors in finding the span, and considers the span of two linearly dependent vectors to be a plane (the span of two linearly dependent vectors is a line). But one could argue that pattern imagery involves the quick recall of some mathematical facts. Considering imagery as a continuum of visual mental images may lead one not to include pattern imagery as a distinct category of visual imagery. However, given that pattern imagery does not derive from a prototypical example, Presmeg argues that its use could be effective in the development of mathematical thinking and generalization.

The memory image of a formula is another category of visual imagery. It refers to evoking a formula. For example, one may evoke the vector equation

$x_1v_1 + x_2v_2 + x_3v_3 = 0$ (where $\{v_1, v_2, v_3\}$ is a set of vectors in R^n and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a solution set) in verifying the relationship among a given set of vectors (i.e. linear dependence or independence). The use of memory images of formulae have been shown to be ineffective in certain circumstances the development of mathematical thinking (Presmeg, 1992).

The two last categories of imagery—kinesthetic and dynamic imagery—both involve motion. Dynamic imagery refers to a mental construction of a movement: it involves moving and transforming objects in the mind. It could lead to the kinesthetic sensation of making that movement (Kosslyn, Ganis & Thompson, 2009). Kinesthetic imagery involves bodily movements in evoking mathematical ideas or concepts. For example, a dynamic image of a vector in two-dimensional space involves evoking an arrow and moving it everywhere on the plane, whereas kinesthetic imagery of a vector involves use of fingers of the hand to draw, trace or position a vector in two- or three-dimensional space. I talk about dynamic and kinaesthetic imagery in Chapter 7.

Studies of mathematicians' ways of thinking about mathematical concepts and ideas have shown the importance of dynamic and kinesthetic imageries in their thinking (Núñez, 2006; Sinclair & Gol Tabaghi, 2010). Despite this, research on students' ways of thinking reveals the absence of the use of kinesthetic and dynamic imagery (Presmeg, 1992).

4.4. Dynamism in thinking

Recent research, drawing on neuroscientific theories, suggests the central importance of the body in thought (Seitz, 2000). This perspective confirms the importance of motor capacities and perception in the development of cognition. Cognitive scientists assume that, "the human propensity for categorization is structured by metaphoric, imagistic, and schematizing abilities that are themselves undergirded by perceptual and motor capacities" (Seitz, 2000, p. 25 referred to in Jackson, 1983; Johnson, 1987). Drawing on biological and cognitive studies, Seitz hypothesizes that "we think kinesthetically, too" (p. 24) and "movement and thinking do not exist in a biological and cognitive vacuum" (p. 28). He proposes that there exist three core cognitive abilities central to human action. The first is motor logic and the motor organization that comprises one's neuromuscular skill with regard to the articulation and ordering of movement. Motor organization refers to organizing movement and motor logic refers to the syntax of that movement. The second is kinesthetic memory that enables one to think in terms of movement by mentally reconstructing objects and imposing motion on and positioning them in space. This is to say that kinesthetic memory comprises kinesthetic and dynamic imageries. The last, kinesthetic awareness concerns having information about our bodies and objects coming in contact with them. These abilities reveal the relationship of thought to movement that are well evidenced from the use of sign language and aesthetic (dance) movement. In particular, kinesthetic sense and memory has a significant role in learning through the senses, hands and body. In summary, Seitz asserts:

> Thinking is an embodied activity. Although humans may be best characterized as symbol-using organisms, symbol use is structured by action and perceptual systems that occur in both natural environments and artifactual contexts. Indeed, human consciousness may arise not just from some novel feature of human brains, but way of the body's awareness of itself through its exteroceptive and proprioceptive senses. Indeed, the body structures thought as much as cognition shapes bodily experiences (p. 36).

His claims are interesting in light of Lakoff and Núñez's (2000) work on "where mathematics comes from". They propose that "mathematical objects are embodied concepts—that is, ideas that are ultimately grounded in human experience and put together via normal human conceptual mechanisms, … [such as] conceptual metaphors" (p. 366). Other evidence of the importance of kinesthetic sense in learning mathematics is offered by mathematicians such as William Thurston (1994), who includes "process and time" and "vision, spatial sense, kinesthetic (motion) sense" as two of the six major facilities that are important for mathematical thinking (pp. 4-5). More recent research in mathematics education that studies the way in which mathematicians think about mathematical concepts and ideas reveals the role of time, motion and gesture in mathematicians' description of mathematical concepts (see Sinclair and Gol Tabaghi, 2010). These theories and studies challenge inattention to (and sometimes ignorance of) the role of time, motion, and gesture in mathematical thinking.

In regard to the role of gesture, Núñez (2006) writes that gestures have become "a forgotten dimension of thought and language" (p. 174). Researchers claim that gestures provide complementary content to speech content (Kendon, 2000) and gestures are co-produced with abstract metaphorical thinking (McNeill, 1992). Within a psychological perspective, McNeill identifies different types of gesture: *deictic*, *iconic*, *metaphoric*, *beat*. A *deictic* gesture is a pointing gesture that can be used to indicate existing or virtual, objects or events. Pointing to an object on a computer screen using index finger can be an example of deictic gesture. An *iconic* gesture bears a close relationship to the semantic content of speech. An example of iconic gestures is to use hands and arms to illustrate a L-shaped desk as it is described with words. A *metaphoric* gesture is pictorial like an iconic gesture, but it illustrates an abstract idea rather than a concrete picture or event. Tracing a triangle in the space as describing it with words can be an example of a metaphoric gesture. Beat gestures are repeated gestures that have the same form regardless of the content, such as a poet making rhythm as she reads. In the context of mathematics, the distinction between iconic and metaphoric gestures is not clear given the debatable status of mathematical objects as being abstract or concrete.

McNeill's gesture classification has been used in analyzing students' use of gesture in learning mathematics (Cook, Mitchell & Goldin-Meadow, 2008), teachers' use of gesture (Alibali and Nathan, 2007) and also mathematician's use of gesture (Núñez, 2006) in teaching mathematics. Other studies have shown that gestures play an important role in cognition and can contribute to creating mathematical ideas (Arzarello *et al*., 2005; Edwards, Radford, & Arzarello, ESM special issue 2009).

A more recent perspective on gesture highlights the relationship between gesture and diagram seen as a technology that accompanies the development of mathematical ideas. This perspective draws on Châtelet's work whose interest was the implications of gesture on diagram rather than any sort of classification of gesture. According to de Freitas and Sinclair:

> Gestures, for Châtelet, are elastic and never exhausted; they cannot be reduced to a set of descriptive instructions. If a gesture functions in terms of reference or denotation or exemplification, it is already stale and domesticated. Châtelet is concerned with gesture as a kind of interference or intervention that has driven mathematics and the sciences forward, not as a semiotic divorced from the event, but as a dynamic process of excavation that conjures the sensible in sensible matter (2011, p.6).

Like gestures, diagrams are "the natural accomplice of thought experiment" and "reveal themselves capable of appropriating and conveying all this talking with the hands" (Châtelet, 2000, p. 11). This suggests that, for Châtelet, gestures give rise to sketches and diagramming, and diagrams give rise to new possibilities for gesturing.

4.5. Summary

As discussed in section 4.1, Mason's theory of shifts of attention appears potentially fruitful in terms of revealing the developmental process of a learner's mathematical understanding. Accumulated experience of the different forms of attention and the flexibility of shifts among various forms of attention are important factors in the mathematical development of the learner according to Mason. Moreover, biological and cognitive studies reveal that attention can influence mental processes and perception, and human ability to rapidly shift attention both within and between sensory modalities contributes to learning process (Seitz, 2000). These authors suggest that providing the learner with opportunities where her attention can be drawn to identifying the invariants of a mathematical concept would enable her to perceive properties of the concept. In my research, the use of dynamic geometric diagram of the concepts of eigenvector and eigenvalue provide an opportunity for the participants to study their mathematical developmental process.

Given that the study involves the use of a dragging tool, I draw on the theory of instrumental genesis to identify evidence of instrumentation and instrumentalization processes in the participants' use of dragging tool. As I mentioned in section 4.2, the theory of instrumental genesis provides a framework to analyse students' interactions with tools and transformation of tools into instruments. But it falls short in putting forward the potentialities of the instrument in the development of mathematical thinking. Therefore, I suggest the complementary use of the theory of instrumental genesis and the theory of shifts of attention to enable analysing cognitive development of a learner in a digital technology environment.

Beyond the role of the dragging tool and its effect on shifts of attention, I conjecture that the dynamic geometric representation may stimulate the formation of imagery. I then provided a review of the literature on visualization and different types of visual imagery in learning mathematics, in section 4.3. In section 4.4, drawing on recent theories on thinking, I discussed the importance of the role of time, motion and gesture in the development of mathematical thinking.

Drawing on these accounts, I decided to analyze each participant's interactions with the sketches using the theories of shifts of attention, instrumental genesis and the dragging tool, and embodied cognition and gestures independently. Then, I provide a synthesized analysis for each participant's interaction by triangulating my analysis using these three theoretical frameworks jointly. The analysis of data begins in Chapter 6. The next chapter describes the methodology of my study.

5. Methodology of the study

In this chapter, I describe the sketches that I designed to represent the concepts of vector, scalar multiple of a vector, and eigenvector and eigenvalue. I then introduce the participants of my study and describe their academic background and their familiarity with Sketchpad. In section 5.3, I include the task that I used to collect data using task-based interviews. Given that my research methodology involves clinical interviewing, in section 5.4, I briefly review the use of clinical interviewing by Piaget (Ginsburg 1981) and diSessa's (2007) recent argument on clinical interviewing as a scientific method of data collection.

In Chapter 1, I included the two main research questions that guided my research. My overview of the multiple theoretical frameworks and the methodology of my study enabled me to structure those questions and to develop another research question. The goal of my study is to respond to the research questions:

1. What is the effect of dynamic geometric representations of eigenvectors on a student's modes of thinking?

2. How do students relate these representations to the more symbolic and static ones that are found in undergraduate textbooks?

3. What can the complementary use of the theory of instrumental genesis and the theory of shifts of attention offer in regard to analyzing a participant's interaction with the *eigen* sketch?

5.1. Design of sketches

Using The Geometer's Sketchpad (Jackiw, 1989), I designed three sketches to represent the concepts of vector, scalar multiple of a vector, and eigenvector and eigenvalue. The first two sketches (one illustrating the concept of vector, arithmetically and geometrically, and the other depicting the concept of scalar multiples of a vector) were intended to familiarise anyone who had not used Sketchpad before with the aspects of the software. The sketches are fundamentally dynamic, representing relationships and behaviour over time.

As shown in Figure 6, the *vector* sketch includes a draggable vector v, its numerical representation, and a 'show axes' button. The user can select the tip of vector v and drag it on the sketch, so that she could notice numerical changes that occur as she drags v. She can also select the vector (the segment part of the vector v) and drag it on the sketch, so that she might notice invariant properties of the vector v. Although the construction of the *vector* sketch involved specifying a Cartesian coordinate system, I decided not to show that system (unless one clicks on the 'show axes' button) in order to support the geometric intuition of the concept of vector as suggested by Sierpinska *et al*. (1999). I was also inspired by the literature that I reviewed in section 2.2. Thus, I included the geometric and arithmetic representations of a vector to enable one to identify invariants of a vector with respect to two different representations. The use of two representations, arithmetic and geometric, can also bring forth the coordination between embodied and symbolic mathematical worlds that Watson *et al*. (2003) describe (see section 2.3).

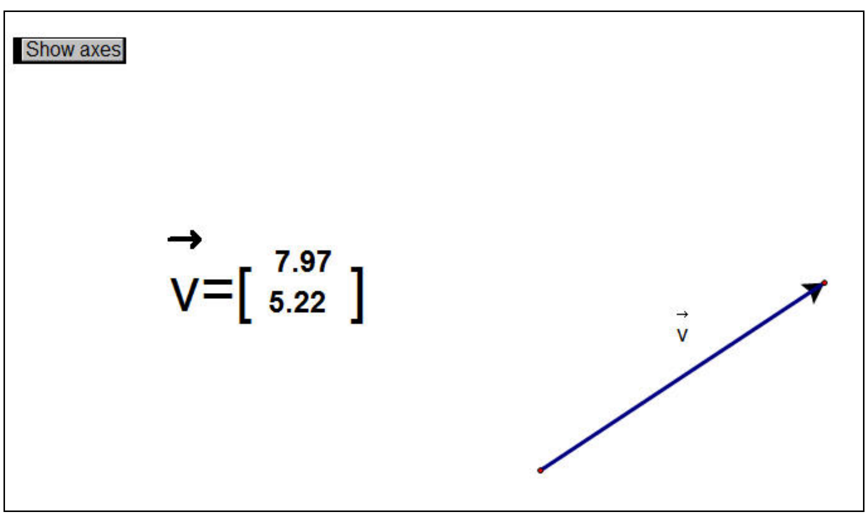


Figure 6. A snapshot of the vector sketch

The second sketch is one representing a scalar multiple of vector v. As shown in Figure 7, the sketch includes both arithmetic and geometric representations of v and av.

It also includes a slider for a to enable the user to change its value. Following Sierpinska *et al*.'s design of sketches, I included a coordinate-free representation of vectors v and av to enable participants to develop a geometric intuition of the concept of scalar multiple of a vector. The user can drag a and describe the behaviour of av. It is also possible to drag the tip of vector v or the segment part of it to generate arbitrary vectors.

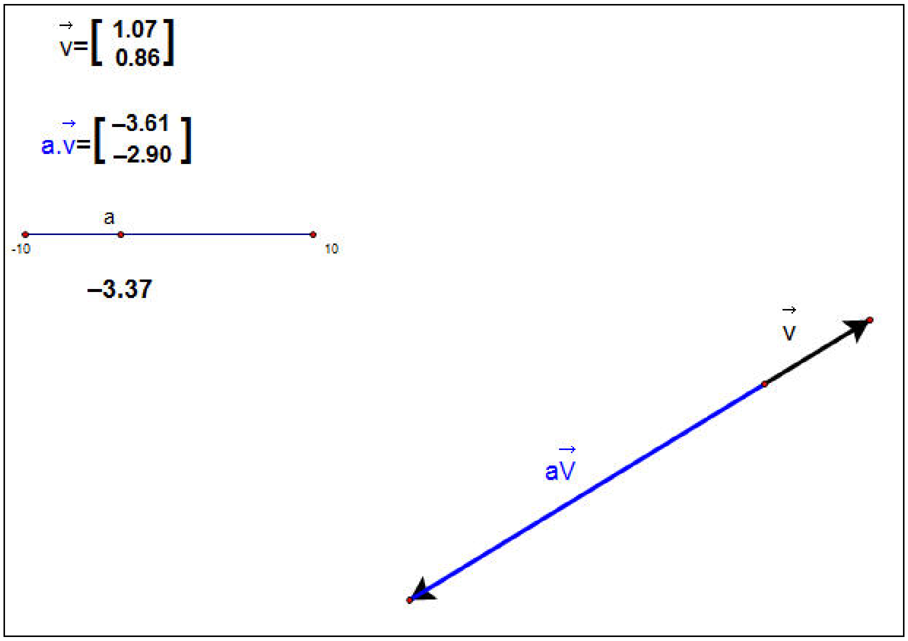


Figure 7. A snapshot of the sketch representing a scalar multiple of vector v

The last sketch, which I called *eigen*, was designed to enable exploration of the eigenvectors and eigenvalues of a 2×2 matrix. As shown in Figure 8, the sketch includes a draggable vector x and its image vector Ax. The sketch also includes an arithmetic representation of Ax (i.e. matrix-vector multiplication). The user can change the values of matrix A. It is also possible to represent the vectors on a Cartesian coordinate system by clicking on the 'Show axes' button. The sketch shown in Figure 8 is the improved version of the *eigen* sketch that I mostly used in collecting data. In improving the first version of *eigen* sketch design, I drew on Sierpinska *et al*.'s study design (1999), which suggests using a coordinate-free representation of vectors and transformations to enable students to develop synthetic-geometric mode of thinking.

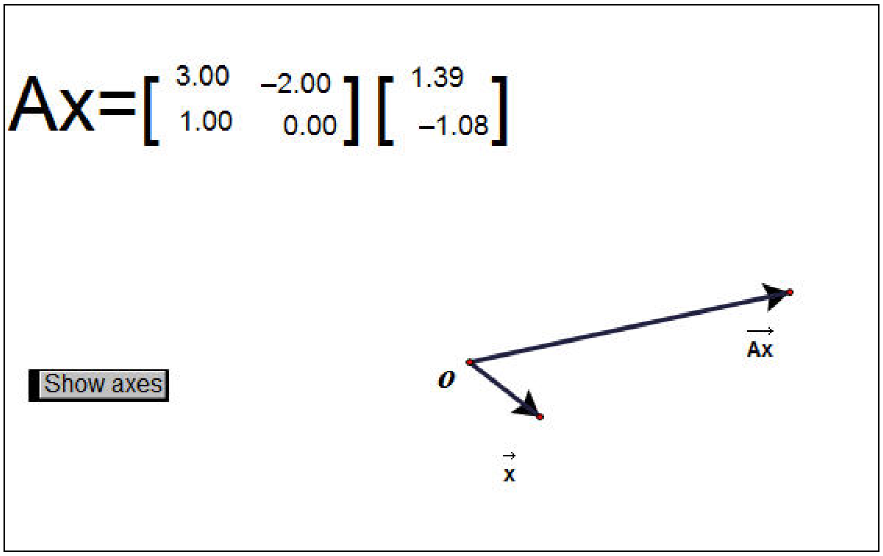


Figure 8. A snapshot of the improved version of the eigen sketch

The earlier design of the *eigen* sketch, as shown in Figure 9, includes a Cartesian coordinate system and also geometric representations of column vectors of the matrix A. I used this first sketch as the basis of an interview with two participants (Mike and Jack). I then noticed that the geometric representations of the column vectors of A are not really necessary in order to identify the eigenvectors and eigenvalues of the given matrices. Moreover, one of the two participants (Mike) who used the sketch became puzzled by the geometric representation of the column vectors.

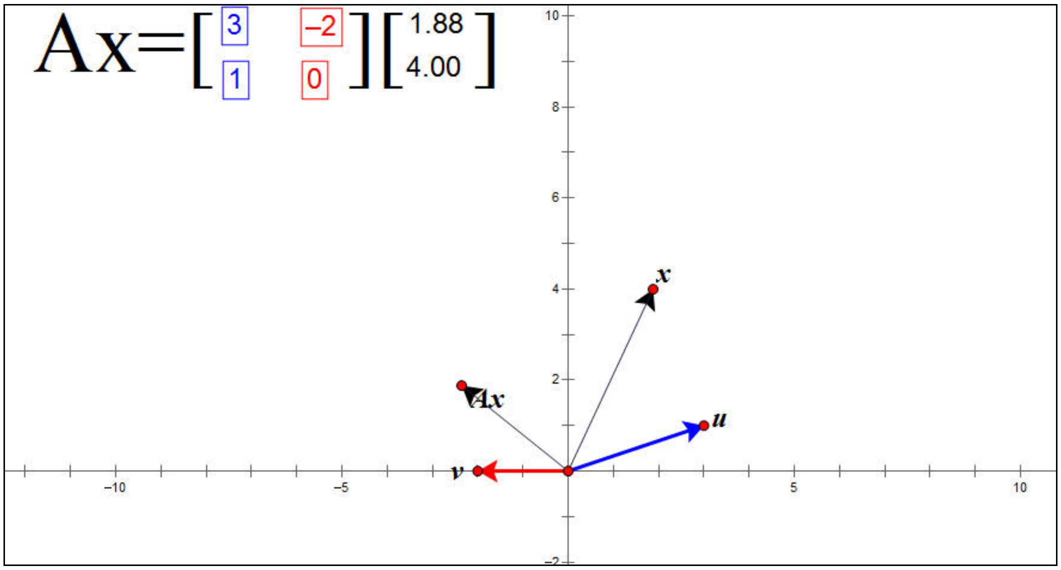


Figure 9. A snapshot of the first version of the eigen sketch

5.2. Participants

The participants in the study were four undergraduate students and one graduate student, three males and two females, from a large North American university. The participants were selected from among students who had either completed a linear algebra course (3 students) in the Mathematics Department or were enrolled in a linear algebra course at the time of the interview (2 students). The three students who had previously taken linear algebra were chosen because they had prior experience using *Sketchpad*. The two students, at the time of interview, enrolled in a linear algebra class were students who had volunteered to participate in the study after I had made a general request to the whole class. A total of eight students volunteered and I interviewed them all, but I used a slightly different methodology for the 6 that were not included in this study. In particular, these six students were not given the definition of the concepts of eigenvector and eigenvalue. For this dissertation, I decided to focus on the five interviews in which participants were given the definition so that I would be able to study the particular question of how students relate the symbolic definition of the concepts to the dynamic diagrams.

The participants all volunteered their time. Of the five, three of them were relatively familiar with *Sketchpad* because of its being part of their education course work. Appendix B includes transcripts of the interviews and a description of the participants' interaction with the sketches and their gestures. I used pseudonyms to identify the participants.

Mike and Jack

Mike was a graduate student pursuing a Master of Science degree in secondary mathematics education. He had completed a linear algebra course during his bachelor's degree, but said that he could not recall the concepts of eigenvector and eigenvalue at the beginning of the interview. The interview was conducted by Dr. Nathalie Sinclair. I mainly observed the conversation between her and Mike in order to gain some interviewing experience. Mike was relatively familiar with *Sketchpad*.

Jack was a third-year undergraduate student pursuing his Bachelor of Science degree. He had completed a linear algebra course during his second year of study. He was relatively familiar with *Sketchpad* because of being a part of his education course work.

The *eigen* sketch that Mike and Jack interacted with was my first design, a snapshot of which is shown in Figure 9. They neither interacted with the *vector* sketch nor with the sketch representing a scalar multiple of vector v because, they were both familiar with *Sketchpad*.

Kate, Tom and Rose

Kate, Tom and Rose interacted with the improved version of the eigen sketch (a snapshot of which is shown in Figure 8), the *vector* sketch and the sketch representing a scalar multiple of vector v. At the beginning, before they started their interaction with the sketches, they were prompted with the question "what is an eigenvector?"

Kate was pursuing a certificate program in education. She had completed a linear algebra course during her Bachelor of Science degree program and had used *Sketchpad* before in her spare time to expand her knowledge of geometry.

Tom was a second-year undergraduate student pursuing a bachelor degree in science. He successfully completed both calculus I and II courses, and was concurrently enrolled in a linear algebra course at the time of interview. He had not used *Sketchpad* before.

Rose was a first-year student who was pursuing her undergraduate degree, majoring in science. She had successfully completed a calculus course and, at the time of interview, she was concurrently enrolled in a linear algebra course. She did not recall studying matrices and vectors in high school. It was her first time using *Sketchpad*.

5.3. The interview task

The participants were given the sketches and a worksheet. The worksheet included a formal definition of eigenvectors and eigenvalues (see Appendix A) and a task as shown in Figure 10.

> Given a sketch that represents matrix A and an arbitrary vector x. Double click on the entries of A to change their values to the givens below, then drag x to find eigenvector(s) and associated eigenvalue(s), if they exist.
>
> (a) $A=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$, (b) $A=\begin{bmatrix}4 & -1\\ 1 & 2\end{bmatrix}$, (c) $A=\begin{bmatrix}1 & 6\\ 5 & 2\end{bmatrix}$, and (d) $A=\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}$

Figure 10. The interview task used to collect data

The task includes four different transformation matrices and invites the participants to find eigenvectors and associated eigenvalues of the given matrices using the *eigen* sketch. These four matrices were chosen to provide examples of four different cases that may occur in finding eigenvectors and eigenvalues of 2×2 matrices. The matrix in (a) has two sets of eigenvectors and associated eigenvalues: a set of $t\begin{bmatrix}1\\ 1\end{bmatrix}$ (where $t \in R-\{0\}$) with associated eigenvalue of $\lambda = 1$ and a set of eigenvectors of $t\begin{bmatrix}2\\ 1\end{bmatrix}$ (where $t \in R-\{0\}$) with associated eigenvalue of $\lambda = 2$. The matrix (b) has only one set of eigenvectors of $t\begin{bmatrix}1\\1\end{bmatrix}$ (where $t \in R - \{0\}$) with associate eigenvalue of $\lambda = 3$. The matrix (c) has two sets of eigenvectors; one associated with a positive eigenvalue and another with a negative one. The sets of eigenvectors of matrix (c) are $t\begin{bmatrix}1\\1\end{bmatrix}$ (where $t \in R - \{0\}$) with eigenvalue of $\lambda = 7$ and $t\begin{bmatrix}6\\-5\end{bmatrix}$ (where $t \in R - \{0\}$) with associated eigenvalue of $\lambda = -4$. The matrix (d) does not have any real eigenvectors or corresponding eigenvalues because the characteristic equation ($(1-\lambda)^2 + 6 = 0$) does not have any real roots.

5.4. Data collection

Data was collected using one-on-one, task-based semi-structured, clinical interviews. Each interview lasted about 30 minutes and was videotaped. The participants were given the sketches and the worksheet that included the definition of the concepts of eigenvector and eigenvalue taken from Lay's book (as shown in Figure 4, Chapter 2) and the task (as shown in Figure 10). By providing them with the definitions, the *vector* and scalar multiple of vector sketches, I tried to avoid putting participants in an unfamiliar situation. For example, the *vector* and scalar multiple of a vector sketches enabled participants who were not familiar with the dynamic diagrams of a vector to familiarise themselves with such representations and the dragging affordances of Sketchpad. Also, the provided definitions were useful since no one recalled the definition of eigenvectors and eigenvalues (as I predicted).

The videotapes captured the participants' interactions with the sketch, their speech and bodily movements. I watched the videotapes a number of times and captured single images using a stop-frame technique. I also transcribed everything that was said and what my participants wrote. I present the transcripts of the interviews, description of the participants' interaction with the sketches and snapshots of their gestures in Appendix B. To analyze the data, I watched the videotapes many times and isolated episodes of dragging by means of time codes. I attended to the participants' gaze directions and how they moved among different foci in the interview. Their speech also reflected what they were attending to and how the structure of their attention shifted as they interacted with the *eigen* sketch. I also considered their use of gestures. I made use of their written responses when it helped me to complete my analysis, but I did not focus just on the written production of the participants which is generally very sparse.

The use of semi-structured task-based clinical interviewing seemed to be a suitable method for data generation and collection in respect to my research questions which were: (1) what is the effect of dynamic geometric representations of eigenvectors on a student's modes of thinking? (2) how do students relate these representations to the more symbolic and static ones that are found in undergraduate textbooks? (3) what can the complementary use of the theory of instrumental genesis and the theory of shifts of attention offer in regard to analyzing a participant's interaction with the eigen sketch? I would not have been able to identify the effect of dynamic geometric representations on the students' ways of thinking if I had not used the clinical interview technique.

According to Ginsburg (1981), the roots of clinical interviewing go back to Piaget who used it to explore the richness of children's thought, to identify the structure of that thought, and to evaluate the child's cognitive competence. My purpose for the use of clinical interviewing is *identification* that is to identify and describe cognitive processes underlying interesting intellectual phenomena. To identify the structure of thought processes, the clinical interview may involve some degree of standardization and the interviewer's prompts are contingent on the participant's engagement with a task. In my study, the use of the task and sketches provided some degree of standardization. To semi-structure the interviews, I used a few common questions to prompt the participants as they interacted with the *eigen* sketch. A list of these questions is provided in Appendix C. However, the participants were prompted differently depending on their interaction with the sketches. I should acknowledge that in a few situations during the interviews I found myself acting more as an instructor rather than as an interviewer. For example, I directed them to drag x in a specific quadrant when they missed out a position in the quadrant where x and Ax were collinear. In a few situations, I needed to confirm their findings of eigenvectors and eigenvalues to assure them that they were doing it right.

Reflecting on my data-collection technique, I concur with diSessa (2007) that clinical interviewing provides rich data when it derives naturally from a form of mutual inquiry through engaging participants in a task. The technique enabled me to triangulate (a) the modes of each participant's thinking; (b) the range of strategies and dragging modalities that the five participants used; (c) something of the dynamics of the conceptualizations. The dynamics of conceptualizations addresses issues like how they develop, how much confidence and facility the participants exhibit, and how ideas may shift.

5.5. Summary

In chapter 5, I described the design of the sketches that I used in collecting data and also my rationale of the design by making connections to my findings in the literature. I discussed the importance of the use of clinical interviewing technique in collecting data in reference to my study questions. Qualitative analysis of data, which follows in Chapter 6, is based on the theoretical constructs described in the previous chapter. Appendix B includes data transcripts and description of the participants' interaction with the sketches.

6. Analysis of data

This chapter includes analysis of the the participants' interactions with the sketches drawing on multiple theoretical frameworks (as overviewed in Chapter 4). I used four main categories to conduct my data analysis. I first analyse the episodes in terms of these three theoretical perspectives: 1) instrumental genesis and dragging tool, 2) embodied cognition and 3) shifts of attention. The last category provides a reflection on the participant's interactions merging all the theories to give an insight of the participant's modes of thinking. At the end of this chapter, I include a comparative summary of the participants' interactions with the sketches.

I begin my analysis of the participants' interactions with the *eigen* sketch with attending to their use of the dragging tool. I incorporate aspects of embodied cognition into my analysis in order to account for the important role played by the body in participants' exploring and communication. I then use Mason's theory of shifts of attention to analyze the participants' structures of attention. In this way my analysis of their use of the dragging tool, and linguistic and gestural expressions can provide evidence of shifts in the structure of the participants' attention and consequently reveal their understanding of the concepts. In my overview of the underlying theoretical frameworks in Chapter 4, I began with Mason's theory of shifts of attention since it provides a broader theoretical scope.

Given that data is analyzed using different theoretical frameworks, I often need to include the same excerpts in more than one category. Also, I realize that I am making a number of interpretations of the intentions or purposes of the participants when I use the verb 'verify'. For instance, on page 61, I say "Mike also used line dragging to verify the collinearity of x and Ax [...]". In this way, I make explicit the understanding I have of their surface actions and words.

As I mentioned in Chapter 5, two participants of my study, Mike and Jack, interacted with the first design of the *eigen* sketch (a snapshot of the sketch is shown in

Figure 9, Chapter 5). The others, Kate, Tom and Rose, interacted with the improved version of the *eigen* sketch (a snapshot of it is shown in Figure 8, chapter 5). The order of presentation reflects the chronological order of my interviews.

6.1. Mike

6.1.1. Dragging modalities and strategies

6.1.1.1. Finding eigenvectors and eigenvalues of matrix (a)

Mike began his interaction with the sketch by using wandering dragging to drag the vector x slowly around its given position (fourth quadrant of the coordinate system). As he continued dragging, x overlapped with the x-axis and Ax overlapped with the vector u, as shown in Figure 11.

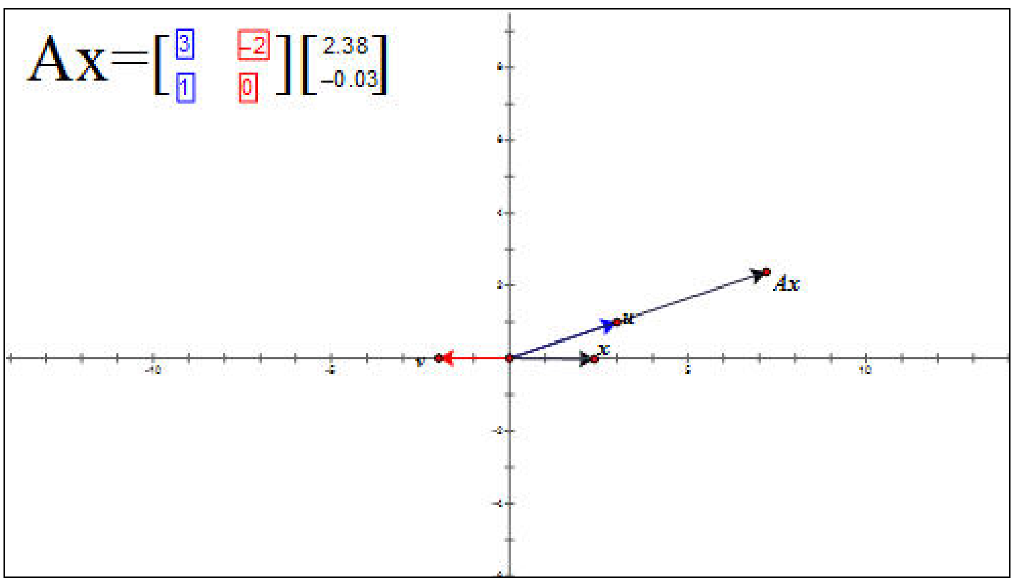


Figure 11[5]. A snapshot of the first eigen sketch shows vectors u and Ax are co-linear as x is collinear with x-axis

[5] The vector u and v on the sketch are the geometric representations of the column vectors of the transformation matrix A.

This representation enabled him to use line dragging to drag x toward the origin, along the line where x and the x-axis overlapped, until Ax became the same length as u. Mike's use of line dragging provides evidence of the instrumentalization process, in that it enabled him to explore the varying lengths of Ax and to make Ax the same length as u while keeping u and Ax overlapped. Although the relationship between Ax and u is not a mathematically interesting one to identify, his exploration of places where Ax overlaps with u is interesting. Seeing that Ax overlapped with the vector u may have enabled Mike to recall the symbolic representation $Ax = \lambda x$ from the given definition. This represents the instrumentation process, since through the use of dragging tool Mike recalled $Ax = \lambda x$. This means that the dragging tool enabled him to focus on and direct his attention to the equality or congruence of the two lengths.

Recalling the given definition, he noticed that he needed to make Ax equal to λx, so he used guided dragging and dragged the vector x in the first quadrant until he found a position where x and Ax overlapped. With respect to Arzarello et *al.*'s dragging modalities, Mike used guided dragging since he dragged x to make Ax equal to λx (instead of, say, dragging x increasingly away from the origin). Next, he used line dragging to explore further the relationship between x and Ax. As he dragged x along the line where x overlapped Ax for $\lambda = 1$, far away from the origin in the first quadrant, he realized that x and Ax stay overlapped and must have the same length. Having explored the collinearity between x and Ax using line dragging, he dragged the vector x into the third quadrant in an anti-clockwise direction. Upon finding a position where x overlapped with Ax in the third quadrant, he used line dragging and dragged x along the line away from the origin to verify whether the collinearity between x and Ax would hold and possibly also to see whether the ratio of lengths of x and Ax would remain invariant. In fact, Mike used line dragging to test the relationship between x and Ax.

The interviewer prompted him by asking "do you think there is another eigenvector?" in turn [24]. In response, Mike used a circular clockwise direction to drag x from the third quadrant into the first quadrant. He dragged x slowly (perhaps he was uncertain about the existence of another eigenvector), in the first quadrant, until he found a position where x and Ax overlapped for $\lambda = 2$. His dragging modality in finding

$\lambda = 2$ was in contrast to those that he used before, which were more exploratory, in which he happened to find an overlap position. Now, he *intentionally* used the dragging tool to find a position where x and Ax overlapped. This is because he has already found such position (i.e. he identified the first eigenvector and its associated eigenvalue, $\lambda = 1$), thus he knew he needed to drag x to find another position where x and Ax overlap. His intentional use of the dragging tool provides evidence of an instrumentalization process.

Mike then dragged x back and forth between the two positions where x and Ax overlapped in the first quadrant (for both $\lambda = 1$ and $\lambda = 2$) as he attended to the geometric and the arithmetic representation of eigenvectors to distinguish the eigenvector (2.7, 2.74) associated with $\lambda = 1$ from the eigenvector (5, 2.3) associated with $\lambda = 2$. This modality of dragging is a kind of guided dragging that Mike used to verify the two positions where x and Ax overlapped. It seems that Mike considered 2.7 and 2.74 to be equal values, and 2.3 to be equal to 2.5. He might have been aware that the sketch has precision issues, or he might have thought that a range of values would be accepted. However, in approximating the eigenvalue of matrix (b), he became precise in finding the exact value. I return to the question of approximation versus exactness in the final chapter.

His use of the different dragging modalities shows evidence of both the instrumentalization and instrumentation processes, in that the use of the dragging tool enabled him to find positions where x and Ax overlapped and also to verify the collinearity of x and Ax where they overlapped.

6.1.1.2. Finding eigenvectors and eigenvalues of matrix (b)

Based on the fact that Mike attended to the collinearity of the vectors, as well as to the ratio of their lengths, he could be seen as using a geometric understanding of an eigenvector's position in regard to its transformation under A. He used the guided dragging, carefully attending to the positions of the two vectors on the sketch, and found a position where x and Ax overlapped (for which $\lambda = 3$). He continued dragging x slowly around the overlapped position where he realized that the geometric representation of the two vectors was not very accurate since the two vectors, on the sketch, almost overlapped for an eigenvalue ranging from 2.7 to 3.2 for matrix (b). He then dragged x in a circular clockwise direction into the third quadrant and stopped dragging when he noticed that the two vectors overlapped. He used line dragging and dragged x along the overlapped line away from the origin as he attended to the geometric and arithmetic representations on the sketch to find the exact value of the associated eigenvalue. He then dragged x back into the first quadrant where x and Ax overlapped before. He stopped dragging and said "that [overlap position] is the only one there". His statement suggests that he visually verified the collinearity of the two overlapped positions in the first and the third quadrants since he concluded "that is the only one there".

6.1.1.3. Finding eigenvectors and eigenvalues of matrix (c)

Mike used guided dragging and found a position, in the first quadrant, where x and Ax overlapped for $\lambda = 7$. He then used line dragging to drag x along the overlapped line to verify the collinearity of the two vectors in the first quadrant. He also checked whether he could get a range of eigenvalues for matrix (c) as he said "it [eigenvalue] definitely looks like 7". He then dragged x in a clockwise direction into the third quadrant, where he noticed that the two vectors overlapped in the third quadrant, for $\lambda = 7$. This strategy is similar to the strategy that he used in finding eigenvectors and associated eigenvalue for matrix (b).

As he was explaining his strategy in turn [36] and dragging x in a circular path into the other quadrants, he found a position where x and Ax were collinear but had opposite directions (x was in the fourth quadrant whereas Ax was in the second quadrant). Noticing this geometric configuration of the two vectors, Mike immediately announced λ to be -4 (the sketch did not show the actual value of the eigenvalue, it only showed the absolute value of the ratio of length of Ax to the length of x). He then dragged x in a circular clockwise path into the second quadrant to verify further the collinearity of the two vectors and the value of the associated eigenvalue. He found a position where the two vectors were collinear (x being in the second quadrant and Ax being in the fourth quadrant) and said "yeah, having negative four there too". Mike's use of the dragging tool to identify eigenvectors associated with a negative eigenvalue shows evidence of the instrumentalization process in that he worked with a geometric manifestation of the two collinear vectors (having opposite directions). It also reveals the instrumentation process, since Mike found out that the two opposite but collinear vectors were also eigenvectors of matrix (c). This suggests that the dragging tool was transformed from a tool simply to move a point into an instrument detecting the presence of the geometric representation of an eigenvector associated with a negative eigenvalue.

6.1.1.4. Finding eigenvectors and eigenvalues of matrix (d)

Mike dragged x slowly in a circular anti-clockwise direction. He then dragged it in a circular clockwise direction, this time more speedily. He immediately realized that the two vectors do not overlap in this case and commented that "this [matrix (d)] one is not too promising." His dragging actions showed that he had developed strategies such as dragging in both circular clockwise and anti-clockwise directions to try to find positions where x and Ax overlapped. Mike also used line dragging to verify the collinearity of x and Ax, dragging x far away from the origin (in the quadrant where x and Ax overlapped). He also verified the collinearity of the two vectors in the opposite quadrant of the one in which he first identified their collinearity.

6.1.2. Embodied cognition: gesture and speech

6.1.2.1. Mike: Eigenvectors line up

Shortly after his first interaction with the *eigen* sketch, Mike was prompted to explain his thoughts. In doing so, he used the verb "to line up" (see turn [1]). His use of this verb could be related to his interpretation of $Ax = \lambda x$ in the definition. Thus, "to line up" in the same direction for Mike can be an embodied way of describing Ax is equal to λx. It is interesting to note that he persisted with "line up" even when the word "overlapping" was offered in turn [4]. The verb "to line up" was used five times during the process of completing the task. His use of the verb "line up" in turns [36] and [43] suggests that he developed dynamic imagery of the geometric representation of an eigenvector. In explaining his strategy of finding eigenvectors, in turn [36], he said "[...] if I go around 360 degrees I am interested in the spots like there [position where x overlapped Ax]" as he dragged x in a circular path. This shows that the use of the dragging tool enabled him to construct his kinaesthetic imagery of finding eigenvectors.

Through his interaction with the sketch, Mike started thinking about vector x as a material object that can be moved by him since he said"[...] I started moving this [...]" in turn [3]. After completing the task, in turn [43], he moved his index finger as a vector. He said "I would move x around [rotates his right index finger as tracing a circle as shown in Figure 15] 360 degrees to see if these two cases [puts his hands together as shown in Figure 13 and14] showed up". This suggests that the dynamic diagram gave rise to gesture.

6.1.2.2. Mike gestured at eigenvectors associated with positive or negative eigenvalues

At the beginning of the interview, Mike used his right index finger to trace the equality, $Ax = \lambda x$, on the given worksheet as he tried to recall the definition (as shown in Figure 12).

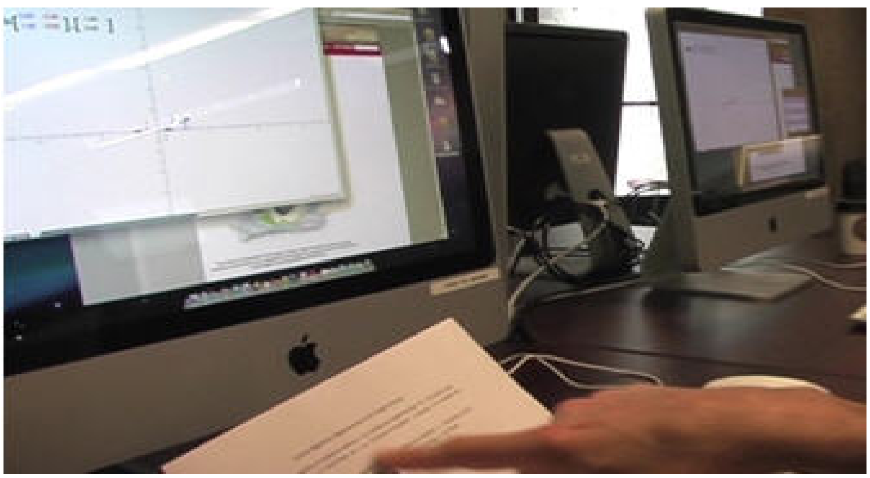


Figure 12. Mike's use of index finger to trace the equality $Ax = \lambda x$

He then used the mouse pointer several times to indicate the geometric representation of the vectors x and Ax on the sketch, as he tried to match the symbolic representations used in the definition with the ones used on the sketch. He also used the mouse pointer to indicate the arithmetic representation of the vector x and the transformation matrix on the *eigen* sketch. His use of the mouse pointer to indicate the symbols on the sketch is a kind of deictic gesture. After putting aside the worksheet, he used his right index finger to indicate the vectors x and Ax in the sketch as he tried to match the symbols on the sketch with the ones used on the definition. At this point, he moved his right index finger along the geometric representation of the vectors u and v as he was tracing the vectors from tail to tip. His use of his right index finger to move along the vectors can be seen as an iconic gesture which suggests evidence of dynamic imagery.

In explaining how he went about finding eigenvectors, Mike used his hands to illustrate eigenvectors. He put up his hands with fingers extended and placed his right-hand palm upward on top of his left-hand palm downward (both hands slightly slanted to the right such that his right-hand little finger overlapped with his left index finger) as he said "[...] x and Ax to line up in the same direction, [...]", shown in Figure 13. He then rotated his hands while keeping his wrists together such that his right- hand extended fingers pointing to the right and his left-hand pointing to the left , as shown in Figure 14, to illustrate collinear eigenvectors that have opposite directions. He also rotated his right index finger around (like tracing a circle in space), as shown in Figure 15, while he said "I would move x around 360 degrees to see if these two cases showed up". Referring to McNeill's classification of gestures, Mike used metaphoric gestures in describing his way of finding eigenvectors.

As mentioned above, Mike started with deictic and then iconic gestures, and at the end he used metaphoric ones. Using deictic gestures, he referred to the symbols and the objects, and matched symbols on the definition to their geometric referents on the sketch, but did not have much sense of what they are since he did not recall the definition. At the end, his hands and arms have become the vectors and he used gestures to express his dynamic imagery that was triggered from his interaction with the sketch. This confirms Châtelet's (2000) ideas on the diagram/gesture relationship in that the dynamic representation (diagram) gave rise to gesture, since the gestures came right out of his interaction with the dynamic representation of concepts. These gestures are new and, as such, imply the development of new understandings for Mike. Mike's gestures and speech suggest that he experienced an embodied geometric description of finding an eigenvector as he used his hands to gesture the geometric representation of x and Ax where x is an eigenvector. This also suggests that he developed synthetic-geometric modes of thinking through his interaction with the dynamic representations of the concepts.

Figure 13. *Mike's hands point to the same direction*

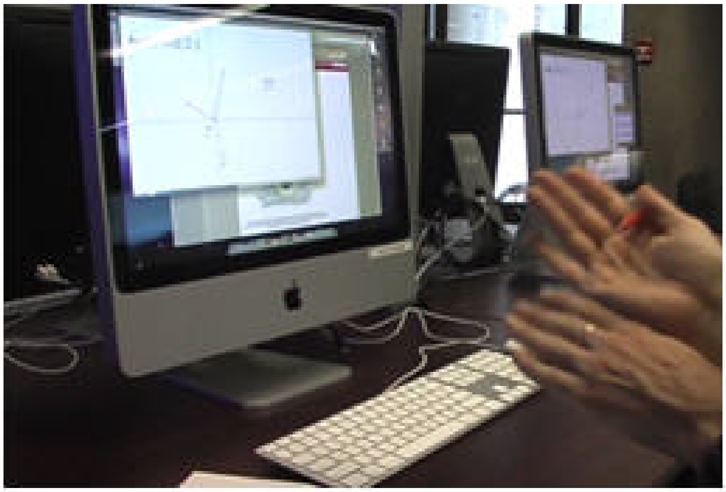


Figure 14. *Mike's hands point to the opposite directions*

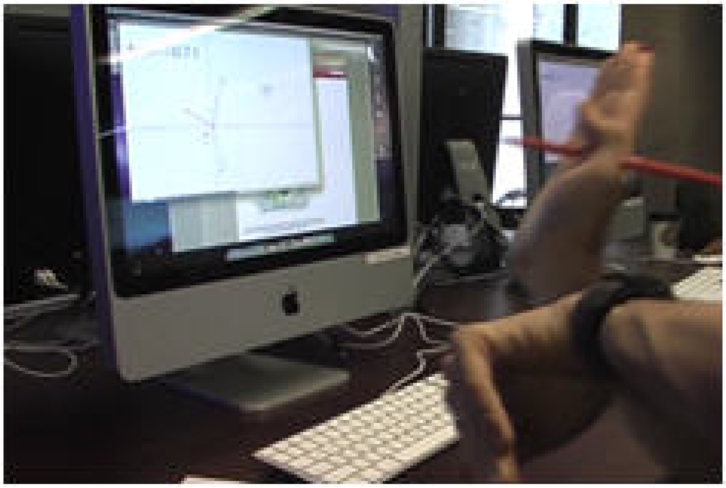
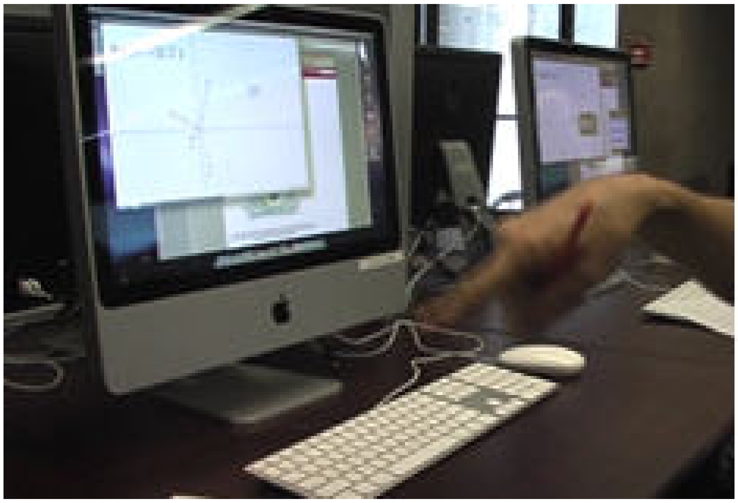


Figure 15. *Mikes' index finger rotates around a circle*



6.1.3. *Shifts of attention*

Mike first focused his attention on the definition since he gazed at the formal definition of eigenvector on the given worksheet. He then shifted his locus of attention to the sketch and dragged the vector x slowly around its given position. His initial interaction with the dragging tool suggests that he waited for visual feedback from the sketch and his attention was what Mason calls holding wholes. As he dragged the vector x, his attention shifted to the position of Ax on the sketch. He focused on a position where Ax overlapped with the vector u and when x overlapped with the x-axis. His focus on these positions resulted from his recall of the definition, as he said "I knew the two things need to be the same" (see turn [3]). His attention was blocked for a few seconds because he focused on the position where u and Ax overlapped as he thought that was what he was looking for.

He then re-read the definition and shifted his attention back and forth between the definition and the sketch matching the symbolic notations used on the definition with the symbolic or geometric representations on the sketch. This matching process enabled him to shift his attention to λ, and more particularly, to its absence in the sketch. Given that the sketch represents a geometric representation of an arbitrary vector x and its transformation under the matrix A, this allows a visible representation of eigenvectors (i.e. x and its transformation being collinear), whereas the associated eigenvalue is not visible.

In turn [17], he articulated "λ is what I am multiplying x by, so that it ends up being the same as the Ax." This suggests that Mike started recognizing the relationship between the symbolic representation (i.e. $Ax = \lambda x$) and the geometric representation on the sketch. Mike then found a position on the sketch and said "yeah, that would be it because now I have Ax here and a scalar multiplication of x by an amount which is λ, to make it equal to the same thing, so yeah that is right." (see turn [19]). This suggests that Mike instrumentalized the dragging tool and attended to the direction and the length of the vectors x and Ax as he identified a position where Ax overlapped x and both had the same length. He then used line dragging to explore further the relationship between x and Ax, and concluded that $\lambda = 1$. This shows that the use of line dragging enabled Mike to recognize the invariance property of the eigenvalue. At this point, his attention was drawn to recognizing properties of eigenvectors and eigenvalues, although he did not show any evidence of understanding that there are infinitely many eigenvectors associated with $\lambda = 1$.

Mike continued dragging x into the other quadrants as he attended to the geometric configuration of the two vectors on the sketch. Thus, he identified a position in the third quadrant where x and Ax overlapped for $\lambda = 1$. He then dragged x more and found another position in the first quadrant where x and Ax overlapped, for which $\lambda = 2$. Although he was first puzzled by distinguishing the two eigenvectors from each other, the use of the dragging tool and attending to the arithmetic representation of x on the sketch enabled him to differentiate them. This happened through dragging x back and forth between the two positions where x and Ax overlapped and attending to the geometric and numeric changes on the sketch. It seems that his attention is transitory; alternating between discerning details and recognizing geometric relationships, in this stage. His actions also reveal that his attention was completely drawn to the vector x and to Ax. He was not attending anymore to u and v.

In finding an eigenvector and its associated eigenvalue of matrix (b), Mike used guided dragging and immediately noticed a position in the first quadrant where the two vectors overlapped. After finding the position, he shifted his attention to finding the exact value of the eigenvalue. Although he used the Measure command and Calculator tool, he could not find the exact value because the geometric representation of the sketch suggested that x and Ax overlap for a range of eigenvalues from 2.7 to 3.2 (on a square grid scaled 2 centimetres per unit). Overall, Mike's strategy was to make x line up with Ax in the first quadrant, approximate the eigenvalue, drag x into the third quadrant to find a position where x and Ax overlap and verify the invariance property of eigenvalues in the third quadrant. He used the same strategy in finding the eigenvalue of 7 for matrix (c). He attended to the position of the two vectors and to the ratio of the two lengths, but he did not articulate that there were infinitely many eigenvectors associated with the eigenvalue of 7. This suggests that his attention was transitory, alternating between recognizing geometric relationships and perceiving properties of specific eigenvectors and eigenvalues. After finding the eigenvalue of 7, he dragged x in circular paths as he explained his strategy in turn [36]. Doing so, he noticed a position where the two vectors were collinear, but not in the same direction as shown in Figure 16.

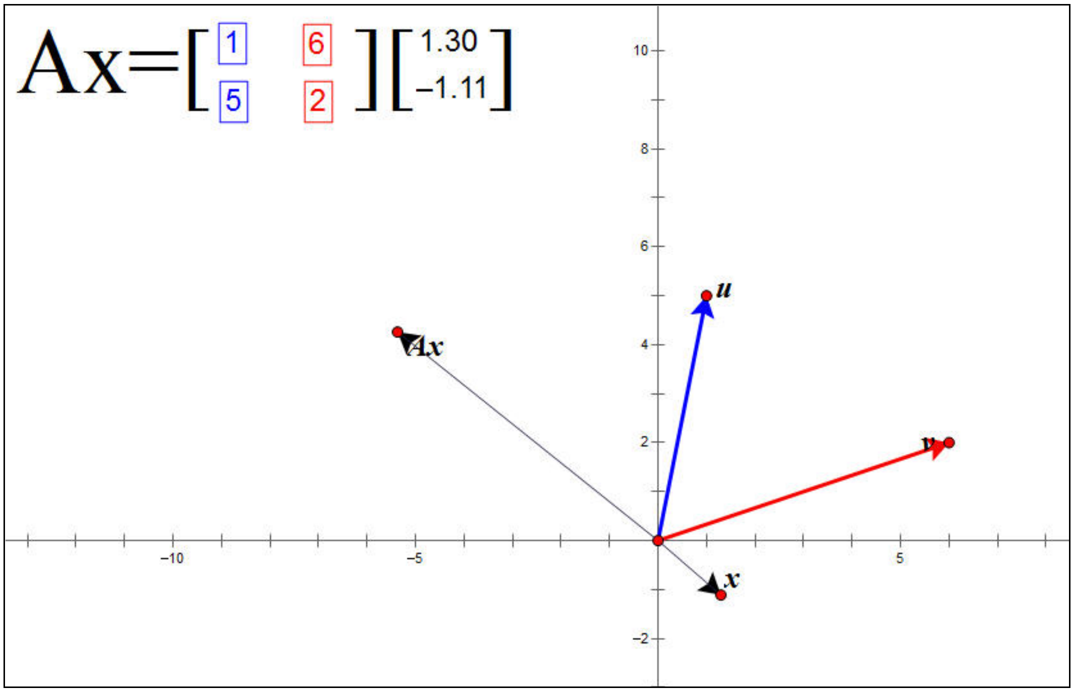


Figure 16. ***A snapshot of the eigen sketch shows that x and Ax are collinear vectors with opposite directions***

The representation shown in Figure 16 made him shift his attention to the definition. He gazed at the definition and said "now we want to Ax equal to um this would work. Because, in this case lambda would be negative four" (see turn [38]). This suggests that he became aware of the collinear property between x and Ax (where x was an eigenvector) as he was able to identify the position where x and Ax were collinear but had opposite directions. It seems that his attention was perceiving properties of eigenvectors as a special vector that lines up with its scalar multiples in the opposite or the same direction. This implies that the dynamic interactive representation of eigenvectors enabled him to shift the state of his attention. He had also become more aware of his ways of dragging, thus started to drag x in both clockwise and anti-clockwise directions several times. Mike's interactions suggest that the dragging tool caused shifts in his attention in which he identified another geometric interpretation of lining up of the two vectors.

In finding an eigenvalue and an associated set of eigenvectors of matrix (d), he dragged x in circular clockwise and anti-clockwise paths as tried to find a position where the two vectors were collinear.

6.1.4. Dragging, shifts of attention and gesture

The use of wandering dragging enabled Mike to notice that Ax changes as he drags x. He immediately attended to the length-wise and direction-wise changes that occurred to Ax as he dragged x. This observation made him shift his attention back and forth between the sketch and the definition, so that he matched the symbolic representations used in the definition with the ones used on the sketch. From the definition, he knew that $Ax = \lambda x$ thus he dragged x so as to make it equal to Ax. This modality of dragging can be classified as an *intentional* dragging modality because he *intentionally* dragged x to make it equal to Ax. Upon finding a position where x and Ax overlapped he used the line dragging modality to verify the invariance property of eigenvalues. Shifts in his attention at both the macro- and micro levels enabled him to use different dragging modalities and to develop dragging strategies such as dragging x in clockwise and anti-clockwise circular paths.

Moreover, the use of the dragging tool and the dynamism of the representation affected his modes of thinking as evidenced from his gestures and speech. At the very beginning of his interaction with the *eigen* sketch he mostly produced deictic gestures using his right index finger or the mouse pointer. After completing the task, when he explained his strategy of finding eigenvectors and eigenvalues (see turn [43]), he used metaphoric gestures to illustrate his mental imagery of the geometric representation of eigenvectors as shown in Figures 13 and 14. This suggests that the dragging tool contributed to his ways of communicating the concepts since he used his hands and arms as he explained his strategy of finding eigenvectors. In fact, he provided an embodied geometric description of finding an eigenvector and used his hands to gesture the geometric representation of x and Ax where x is an eigenvector. This suggests that his mode of thinking was synthetic-geometric. In contrast, when he was asked “what is an eigenvector?”, Mike described an eigenvector as a vector that resulted from a multiplication operation between a vector and a matrix (see turn [45]). In other words, at this point, he recognised a relationship that the eigenvector has to the product operation of the vector and matrix, but did not perceive any properties of the eigenvector. The result of this multiplication, according to Mike, was equal to the matrix multiplication by a scalar. He made an error by saying “matrix multiplied by scalar”, since that was not the case. He did not also pay attention to the fact that the commutative law of multiplication does not hold for matrix and vector multiplication. Surprisingly and strikingly, he did not integrate the ways that he went about finding eigenvectors (see turn [43]) in his final description of eigenvectors (see turn [45]). His description in turn [45] suggests that Mike used analytic-arithmetic mode of thinking. It seems that his mode of thinking suddenly shifted from synthetic-geometric to analytic-arithmetic in a few minutes. The shift might have happened because of the given prompt (that was “what is an eigenvector?”), which is static and depersonalised. In contrast, the prompt “Tell me how you are looking for the other one” in turn [35], which focuses more on human process, triggered the use of synthetic-geometric mode of thinking.

6.2. Jack

6.2.1. Dragging modalities and strategies

6.2.1.1. Finding eigenvectors and eigenvalues of matrix (a)

As with Mike, Jack began his interaction with wandering dragging to drag the vector x around its given position (fourth quadrant of the coordinate system) where he noticed that dragging x changes the position of Ax. His use of wandering dragging provides evidence of an instrumentalization process in that it enabled him to explore the relationship between x and Ax. In turn [57], Jack recognized relationships between x and Ax as he said “by dragging it, it is maintaining the eigenvectors”. He stopped dragging, re-read the definition and asked about the sketch representation of the associated eigenvalue. After being told that lambda does not appear on the sketch, Jack seemed to infer that he needed to make the two vectors collinear. He dragged x into the first quadrant until it overlapped with Ax and said “I guess I could have λ there” in turn [59]. In doing so, he *intentionally* dragged x in the first quadrant to find a position where x and Ax overlapped. He approximated the eigenvector by reading off the coordinate of x from the arithmetic representation on the sketch, and the associated lambda to 2 by attending to the ratio of the length of two vectors. While I invited him to seek other eigenvectors, Jack used line dragging to drag x along its path collinear with Ax (see Figure 17) to verify further the relationship between x and Ax. His use of line dragging shows instrumentalization process. By using line dragging, Jack noticed that eigenvalue is a fixed scalar as he said "λ still looks like two, x has changed" in turn [63]. This shows evidence of an instrumentation process because Jack noticed the invariance property of an eigenvalue.

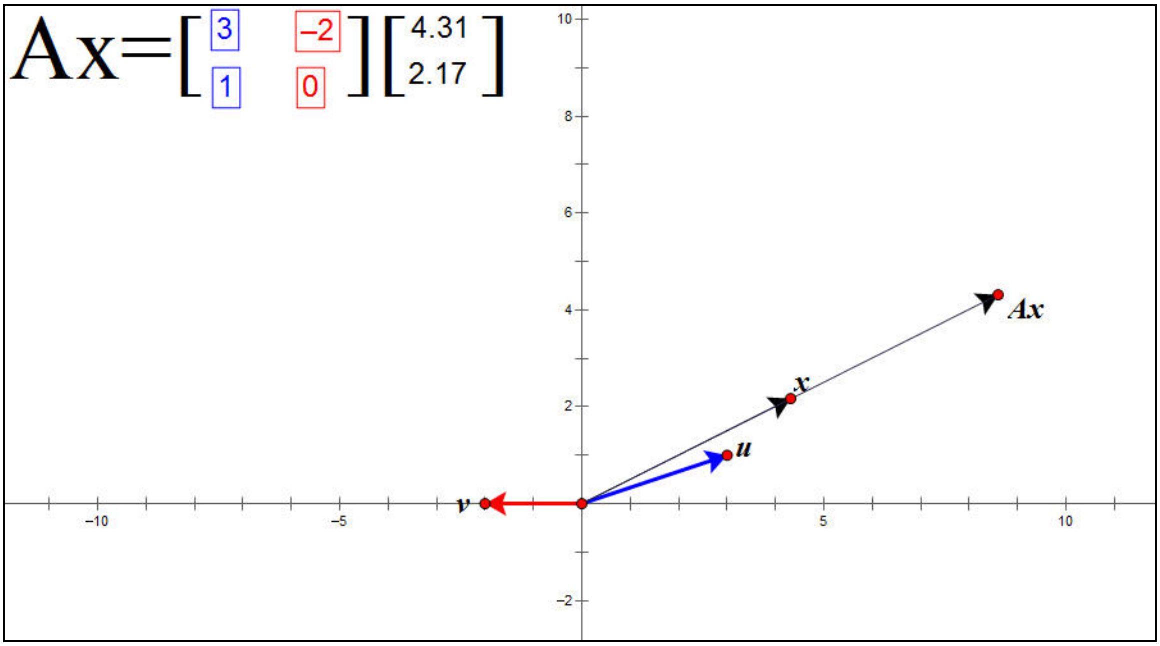


Figure 17. ***A snapshot of the eigen sketch shows vectors x and Ax overlapped in the same direction.***

In response to my prompt in turn [66], Jack used guided dragging to drag x in an anti-clockwise direction into the third quadrant. After finding a position where x and Ax overlapped, he used line dragging to drag x along its path collinear with Ax in the third quadrant. He then used line dragging to drag x back and forth along the straight line passing through the origin and said "the value of lambda wouldn't change but there are infinitely many eigenvectors", in turn [71]. His use of line dragging suggests evidence of instrumentalization process in that he dragged x along the straight line passing through the origin. His dragging strategy also suggests evidence of an instrumentation process because he articulated that "there are many eigenvectors" associated with $\lambda = 2$.

Jack was about to proceed to the next question when I prompted him to look for a different eigenvector from the set he gave before. After dragging x in an anti-clockwise circular fashion, Jack could not identify another set, until I asked him to drag x to (1, 1). At this point, Jack noticed that the two vectors overlapped for λ of 1. The use of line dragging enabled him to verify the collinearity of x and Ax where they overlapped for $\lambda = 1$.

6.2.1.2. Finding eigenvectors and eigenvalues of matrix (b)

In finding the eigenvectors and eigenvalues of matrix (b), Jack used guided dragging and dragged x in a circular path to find a position where x and Ax overlap. He found a position in the third quadrant where the two vectors overlapped. After approximating the lambda, he was prompted to find the associated eigenvectors. In response to the prompt, he used line dragging to drag x along the line (where the two vectors overlapped) away from the origin in the third quadrant. He then dragged x along the line passing through the origin into the first quadrant. As he dragged x, he mentioned eigenvectors "are the ones on this line". The use of line dragging to locate the set of eigenvectors shows evidence of an instrumentation process because Jack located infinitely many eigenvectors associated with a single eigenvalue.

6.2.1.3. Finding eigenvectors and eigenvalues of matrix (c)

Upon changing the entries of the matrix A to (c), Jack noticed that x and Ax both fall in the third quadrant. He used guided dragging and dragged x so that the two vectors overlapped in the third quadrant. He then dragged x along the line (of overlap) passing through the origin into the first quadrant. In finding another set of eigenvectors, he used guided dragging to drag x in an anti-clockwise circular path focusing on the positions where x and Ax overlap. Finding nothing, I asked him to drag x into the second quadrant. He dragged x from the first quadrant into the second quadrant as he carefully attended to the position of Ax on the sketch. He found a position where x and Ax were collinear but had opposite directions. He then dragged x along the collinear line passing through the origin. It seems that he used line dragging to verify the invariance property of the negative eigenvalue ($\lambda = -4$). This is evidence of instrumental genesis in that the dragging tool was transformed from a tool into an instrument for detecting another interpretation of 'lining up'.

6.2.1.4. Finding eigenvectors and eigenvalues of matrix (d)

He used guided dragging and dragged x in anti-clockwise circular paths with different radii several times. He then continued dragging x in a spiral fashion, starting far from the origin and ending at the origin. This suggests that Jack developed new dragging strategies (dragging in a circular or spiral path) to verify the existence of eigenvectors on the given sketch.

6.2.2. Embodied cognition: gesture and speech

6.2.2.1. Jack: Eigenvectors are scalar transformations that line up geometrically

In finding the first set of eigenvectors of matrix (a) and the associated eigenvalue, Jack said "I guess I line them up. I guess I could have lambda there" in turn [59]. He made such an inference based on the given equality as he said "I looked at this [$Ax = \lambda x$] and I realized that there was a scalar transformation" (see turn [61]). Jack's use of the verb "to line up" could be related to his interpretation of $Ax = \lambda x$ in terms of collinearity of Ax with λx. I discuss this in Chapter 7. Jack used the verb "to line up" three times during the process of completing the task. With his second use of the verb "to line up", he used his hands to represent the position of the two vectors as shown in Figures 18 and 19. This suggests that he developed dynamic imagery of the geometric representation of an eigenvector and an embodied way of describing collinearity of the two vectors.

From the very beginning of his engagement with the task, Jack used the term "linear transformation", although the term is not indicated on the given definition. It seems that Jack recalled the concept of linear transformation from his course work. His use of the term shows that he constantly got to coordinate the geometric representation of the concepts of vectors, linear transformation of vectors and eigenvectors with their symbolic representations as indicated in the given definition (see turns [61], [65] and [69]). Jack also used the term "linear transformation" in describing the set of eigenvectors associated with a negative eigenvalue of matrix (c).

Figure 18. Jack's hands are positioned parallel to each other

Figure 19. Jack's hands are exactly placed on each other

6.2.2.2. Jack gestured at eigenvectors associated with positive eigenvalues and diagrammed an eigenvector associated with a negative eigenvalue

Jack used his right index finger to point to the given symbols on the worksheet as he read the definition. He then used the mouse pointer several times to indicate the matrix A and the geometric representation of the vectors x and Ax on the sketch as he tried to match the symbolic representations used in the definition with the ones used on the sketch. His use of the mouse pointer and his right index finger to indicate the symbols on the sketch and on the worksheet is a kind of deictic gesture.

In finding the set of eigenvectors of matrix (a) associated with lambda of 1, Jack found a position where x and Ax overlapped and wrote down $x = Ax = \lambda x$ on the worksheet. Although, he knew $x = Ax$, he was slightly puzzled with coordinating the geometric representation of the vectors (when $\lambda = 1$) with the symbolic representation (i.e. $x = Ax$). In expressing his way of seeing the geometric representation of eigenvectors (when $\lambda = 1$), he used his hands to represent the two vectors as shown in Figure 18. He put up his hands, moved his right-hand and placed it exactly on his left-hand (as shown in Figure 19) to illustrate the geometric representation of vectors when lambda was one. This shows that the use of the dragging tool enabled him not only to coordinate the geometric representation with the symbolic one but also to evoke a kinaesthetic imagery of eigenvector associate with $\lambda = 1$. Referring to McNeill's classification, Jack used metaphoric gestures to communicate kinaesthetically his imagery of the eigenvector for $\lambda = 1$. And, according to Châtelet (2000), the dynamic diagram of the eigenvectors associated with $\lambda = 1$ gave rise to gestures that shown in Figures 18 and 19.

Jack also used his hands after seeing the geometric representation of eigenvectors associated with a negative eigenvalue on the sketch. He positioned his hands extended fingers in an angular shape but not attached from his wrists as shown in Figure 20. He then moved his right-hand toward his left-hand, placed it on the top of his left hand (as shown in Figure 21) and moved it away from his left-hand. He repeated this gesture a few times. Then he said "because of ninety degrees um I'm trying to recall" as he held his hands in an angular form attached from his wrists for a few seconds as shown in Figure 22. It is hard to say whether Jack used his hands to illustrate opposite vectors (a vector and its dilation by a negative factor) or to depict a vector and its quarter-turn rotational transformation (since he mentioned ninety degrees). Nevertheless, his gesture suggests that he tried to illustrate a geometric representation of an eigenvector. His gesture is a metaphoric gesture in McNeill's classification which enabled him to construct his kinaesthetic imagery of eigenvectors.

Figure 20. ***Jack positions his hands in an angular shape***

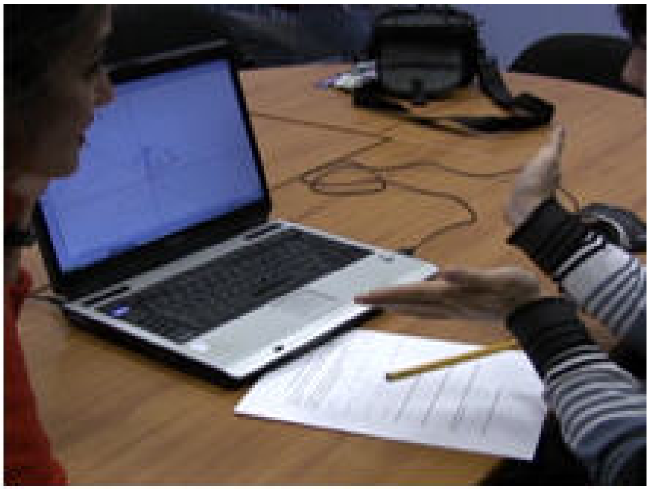


Figure 21. ***Jack brings his hands together.***

Figure 22. ***Jack holds his hands in an angular form for a few seconds.***

His use of the dragging tool to gesture his way of looking for eigenvectors is also interesting. In turn [92], he said "I tried to make x touch Ax" as he dragged x in a spiral fashion beginning far from the origin, turning in an anti-clockwise direction, and ending at the origin. His strategy of dragging was a new strategy that he developed through his interaction with the sketch. He then drew a diagram to illustrate eigenvectors for a negative eigenvalue as shown in Figure 23. In fact, his diagram conveys 'talking with hands' that occurred in describing the negative eigenvalue of matrix (c) (see Figure 20). This seems consonant with Châtelet's assertion that "the gesture envelopes before grasping and sketches its unfolding long before denoting or exemplifying" (2000, p.10).

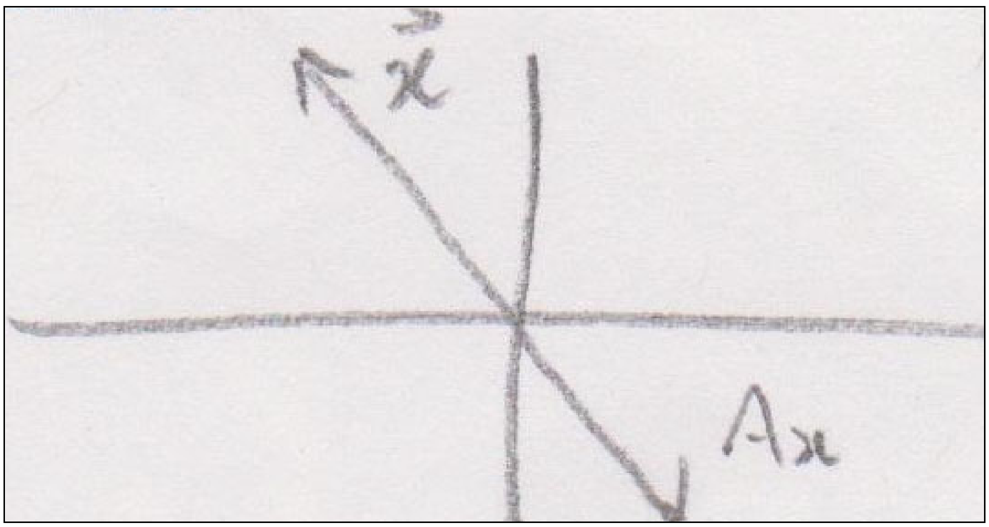


Figure 23. Jack's drawing of eigenvectors

As mentioned above, Jack started with deictic gestures, and at the end he used metaphoric gestures. Using deictic gestures, Jack referred to the symbols and the objects. At the end, his hands and arms have become the vectors expressing his dynamic imagery of eigenvectors. This implies that the gestures came right out of his interaction with the dynamic representation of the concepts.

Jack's speech and gesture show that he perceived matrix multiplication in terms of linear transformation. His interactions with the sketch suggest that he coordinated geometric representation with the given symbolic representation of the concept of eigenvectors and eigenvalues. Using his hands, dragging tool and drawing he provided an embodied geometric description of finding eigenvectors. This suggests that Jack drew on a synthetic-geometric mode of thinking though his interaction with the sketch.

He also coordinated synthetic-geometric with analytic-arithmetic thought, since he made explicit references to the definition and the concept of linear transformation.

6.2.3. *Shifts of attention*

Jack focused his attention on the definition, discerning details as he articulated every symbol one by one. He then shifted his locus of attention to the sketch and to the draggable vector x. His initial tentative dragging suggests he did not quite know what to expect and, indeed, he looked back at the definition again. As he engaged in wandering dragging, Jack's attention involved holding wholes as he watched and waited for feedback, not quite knowing what he was looking for. In turn [57], Jack began to recognise relationships as he said "by dragging it, it is maintaining the eigenvectors". This suggests that he came to notice the relationship between two vectors in that changing the position of x results in a changing of the vector Ax on the screen. He then shifted his attention to lambda, and, more particularly, to its absence from the sketch. After being told that λ does not appear on the sketch, Jack seemed to infer that he needed to make the two vectors collinear, thereby shifting his attention to perceiving properties. In turn [61], Jack responded to my prompt by focusing his attention back to the definition. He reasoned in terms of the properties of the definition, noticed that Ax is a vector that is a scalar multiple of x. In going back to the definition, Jack inferred the collinearity from the scalar transformation whereas, in his actions with the sketch, the collinearity preceded the identification of lambda. He approximated λ to 2 and the eigenvector by attending to the ratio of the length of two vectors and by reading off the coordinate of x from the arithmetic representation on the sketch. By using line dragging, Jack noticed that an eigenvalue is a fixed scalar as he said "lambda still looks like two, x has changed" in turn [63]. This suggests that Jack's attention shifted to perceiving the invariance property of eigenvalues.

In turn [65], Jack integrated the symbolic representation of the definition with the geometric representation on the sketch. He inferred from the definition that Ax is the linear transformation of x, he then referred to the sketch saying that "it [Ax] looks like it's a linear transformation of this [x]". The use of the term 'linear transformation' suggests that he shifted his attention to the relationship between x and Ax. Upon finding the position were the two vectors overlapped, he engaged in line dragging to drag x along the line of invariant collinearity. The use of line dragging enabled him to move from tentative statement about visual perception ("they look very") to one that seems more certain ("they are all on the same axis") in turn [71]. At this point, Jack's attention was involved in reasoning on the basis of the properties.

Given his haste to move to the next question, and his statement that there should only be one eigenvector, I infer that Jack's attention was blocked to the possibility of finding another. This is exacerbated by the difficulty he had in seeing the second eigenvector, probably because of the unit value of lambda—the two vectors coincide, which makes them difficult to see. He himself admitted that and clapped his hands (as shown in Figure 19) to show a geometric representation of the two vectors that lined up and had the same length.

In finding the eigenvalue and eigenvectors of the matrix (b), Jack used an explicit circular dragging strategy. His dragging strategy suggests a shift in attention that involved two components: first, an awareness that there could be more than one eigenvector, and second, use of line dragging that was intended not only to locate one eigenvector, but also to identify all possible eigenvectors. Using the same dragging strategy, he found a set of eigenvectors associated with a positive eigenvalue for matrix (c) and used a parametric notation to represent the set of eigenvectors (as he wrote $[1.28\lambda, 1.32\lambda]$). But, he did not identify another set of eigenvectors associated with a negative eigenvalue for matrix (c). It seems that his stable and robust state of attention (i.e. attending only to the positions where the two vectors overlap) blocked him from realising another interpretation of collinearity (i.e. positions where the two vectors were collinear but had opposite directions). My intervention (i.e. inviting him to drag x into the second quadrant) helped Jack re-direct his attention to the existence of another interpretation of 'lining up', one where the two vectors were collinear but had opposite directions. His use of parameters (λ and λ_2) in representing the sets of eigenvectors of matrix (c) is unclear in the sense that he did not identify whether λ and λ_2 refer to the eigenvalues or there are parameters to represent the set of vectors. In any case, he made explicit references to the concept of scalar transformation.

In seeking an eigenvector and its associate eigenvalue of matrix (d), he first dragged x in an anti-clockwise circular path as he drew his attention to find a position where the two vectors were collinear. He then dragged x in a spiral fashion, that varied both the angle and the distance from the origin of the vector, as he said "I cannot make x to touch Ax".

6.2.4. *Dragging, shifts of attention and gesture*

Focusing on the definition, Jack discerned details from the definition as he articulated every symbol one by one. He then interacted with the sketch. His attention was drawn to the changes that occurred to Ax as he dragged x. He shifted his attention back and forth between the sketch and the definition, matched the symbolic representations used in the definition with the ones used on the sketch, and immediately mentioned that "it doesn't output lambda". From the given equality on the definition, he inferred that "there was a scalar transformation, so the vectors have to be collinear" (see turn [61]). This made him to use an *intentional* dragging modality to drag x to make it collinear with Ax. The use of line dragging enabled him verify the invariance property of eigenvalues and also to realize that there are infinitely many eigenvectors associated with a single eigenvalue.

His constant coordination between the geometric representation of the concepts on the sketch and symbolic representations from the definition reveals evidence of shifts in his attention at both macro-level and micro-level. The shifts in his attention enabled him to use different dragging modalities and to develop dragging strategies such as dragging x in clockwise and anti-clockwise circular and spiral paths.

Moreover, similar to Mike, the use of the dragging tool and the dynamism of the representation affected Jack's modes of thinking as evidenced from his gestures and speech. At the very beginning of his interaction with the *eigen* sketch he mostly produced deictic gestures using his right index finger or the mouse pointer. After finding eigenvectors and associated eigenvalues of matrix (a), he used metaphoric gestures to illustrate his mental imagery of the geometric representation of eigenvectors (for eigenvalue of one) as shown in Figures 18 and 19. This suggests that the dragging tool affected his ways of communicating the concepts since he started using his hands and arms as he articulated his mental imagery. He also used his hands and arms (as shown in Figures 20, 21 and 22) after visualizing the sketch representation of eigenvectors associated with the negative eigenvalue for matrix (c). Although he did not make any gestures using his hands, in response to my prompt about finding eigenvector, he drew a diagram to represent the geometric position of two vectors (when eigenvalue is negative as shown in Figure 23). His words suggest evidence of the development of embodied description, as he said "I tried to make x touch Ax" in turn [92].

In his response to my prompt (what is an eigenvector?), he referred to the multiplication operation between a matrix and a vector, and between a matrix and a scalar. However, Jack used the term "linear transformation" to describe the relationship between x and Ax as shown in turns [61] and [65]. This suggests that Jack perceived matrix multiplication in terms of linear transformations even though his final response contained the word "multiplication" (see turn [94]). Given only his final response, one could argue that Jack used the analytic-arithmetic mode of thinking. But, considering the entire process of completing the task shows that his thought process integrated the synthetic-geometric mode of thinking with an analytic-arithmetic one. This is because he constantly coordinated geometric representation with the given symbolic representation of the concept of eigenvectors and eigenvalues. His final response, like Mike, might be because of the given prompt (that was "what is an eigenvector?"), which is static and depersonalised.

6.3. Kate

6.3.1. K.1 Dragging modalities and strategies

6.3.1.1. Finding eigenvectors and eigenvalues of matrix (a)

Kate started dragging x after carefully attending to the definition. She used wandering dragging to drag x slowly around its given position in the first quadrant. As she dragged x in a clockwise direction, she noticed that the position of x and Ax was changing from being perpendicular to being overlapped. She stopped dragging when x and Ax overlapped for $\lambda = 2$ and gazed at the screen. She then started dragging x in an anti-clockwise direction until the two vectors became about perpendicular to each other, and then dragged x in a clockwise direction until the vectors overlapped for $\lambda = 1$. She then used line dragging to explore further the relationship between the two vectors and said "these two [points to $Ax = \lambda x$] are the same so Ax is the same as x" in turn [105]. Although Kate found two positions, in the first quadrant, where x and Ax overlapped, she first attended to the position where x and Ax overlapped and had the same length. Kate's use of wandering dragging provides evidence of an instrumentalization process in that she dragged x to explore the relationship between x and Ax, and thus found positions where x and Ax overlapped. She identified λ by attending to the geometric representation of x and Ax. This suggests evidence of instrumentation process in that line dragging enabled her to coordinate the geometric relationship between the two vectors on the screen with the symbolic representation (as she wrote $Ax = x$ in turn [105]), finding $\lambda = 1$). As she dragged x along the overlapped line, she said "the top value is the same as the bottom value" in turn [109]. She generalized her findings as she wrote down $\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix}$ on a paper sheet to represent the set of eigenvectors associated with $\lambda = 1$ in turn [111]. Her use of line dragging in finding the set of eigenvectors is another evidence of instrumentation because she dragging enabled her to find out that there were infinitely many eigenvectors associated with $\lambda = 1$.

She then *intentionally* dragged x in a clockwise direction in the first quadrant to find another position where x and Ax overlapped. She said "for this situation Ax is not the same as x" in turn [112]. She wrote down $Ax = 2x$ and $\vec{x} = \begin{bmatrix} 2a \\ a \end{bmatrix}$ on the given paper sheet. After identifying the sets of eigenvectors and associated eigenvalues of matrix (a), Kate tried to locate another position where λ is 3 as she said "can we get a three?". This suggests that she did not recall that a 2×2 matrix can have at most two eigenvalues.

Kate continued dragging x in clockwise and anti-clockwise directions to explore further the relationship between x and Ax. In doing so, she found a position in the third quadrant where x and Ax overlapped and said "that's still positive two. It's possible to go to the opposite direction" as she dragged x along the line passing through the origin into the first quadrant. This suggests evidence of instrumentation in that through the use of dragging tool, she noticed x and Ax preserve collinearity in the first and third quadrants.

6.3.1.2. Finding eigenvectors and eigenvalues of matrix (b)

Kate used guided dragging to drag x in the third quadrant. She immediately found a position where x and Ax overlapped. She then dragged x in an anti-clockwise direction into the first quadrant and found a position where the vectors overlapped. She used line dragging to drag x back and forth along the line where x and Ax overlapped. She then conjectured that "actually the λ is the ratio of this length [points to Ax] to this one [points to x]" in turn [119]. Despite her conjecture, she confused the ratio of the lengths of x and Ax with the relationship (or the ratio) between the coordinates of the vector x. She immediately realized her error, approximated lambda to be 3 by attending to the ratio of the lengths and represented the set of eigenvectors by writing down $\vec{x} = \begin{bmatrix} 3.18 \\ 2.32 \end{bmatrix} = \begin{bmatrix} 1.5a \\ a \end{bmatrix}$ in turn [127]. As she dragged x along the line (where x and Ax overlapped) far away from the origin, her approximated ratio between the coordinates of the vector x (i.e. $x_1 / x_2 = 1.5$) did not hold. In turn [131], she noticed that the ratio between the coordinates of the vector x is 1.1 (the actual ratio is one). This difficulty happened because the sketch representation of the collinearity of the two vectors is not error-free. In this case, the two vectors are collinear when the ratio of x- and y-coordinates of vectors is equal to one (i.e. $x_1 / x_2 = 1$). But, the sketch, for the transformation matrix (b), suggests that the two vectors are collinear when x_1 / x_2 ranges from 0.7 to 1.3 (on a square grid scaled 2 centimetres per unit).

In turn [131], she described that "lambda is a fixed number" that can be found when the two vectors "are on the same line". This suggests that the use of line dragging and the dynamism of the representations enabled her to clarify her understanding of the relationships between the two vectors as she dragged vector x. Her interaction with the dragging tool suggests evidence of instrumentation process since she attended to the ratio of the lengths of the two vectors.

She then dragged x in circular clockwise and anti-clockwise paths into all of the four quadrants as she attended to the behaviour of x and Ax in the first and third quadrants. This dragging strategy enabled her to recognize the collinearity of the two positions, in the first and third quadrants, where the vectors overlapped.

6.3.1.3. Finding eigenvectors and eigenvalues of matrix (c)

She used guided dragging to find a position where x and Ax overlap in the first quadrant. After, finding the position, she used line dragging to drag x along the line (where x and Ax overlapped) away from the origin. She then dragged x into the third quadrant in a clockwise circular direction and found a position where the two vectors overlap in the third quadrant. Her dragging strategies suggest that she verified the collinearity of x and Ax in the first and third quadrants (for $\lambda = 8$). She found the set of eigenvectors (i.e. $\begin{bmatrix} a \\ a \end{bmatrix}$) by attending to the relationship between the x- and *y*-coordinates of the vector x from the arithmetic representation on the sketch. She also approximated the value of the associated eigenvalue by attending to the ratio of the length of two vectors.

To find a set of eigenvectors associated with the negative eigenvalue, I prompted her by asking her to drag x into the fourth quadrant. She dragged x in a clockwise path into the fourth quadrant and stopped dragging when x and Ax became collinear. This representation drew her attention to another interpretation of collinearity. It also shows evidence of instrumentation, in that the dragging tool was transformed into an instrument for detecting the presence of the geometric representation of an eigenvector associated with a negative eigenvalue. In turn [143] she used the Measure command and Calculator tool to calculate the ratio between the lengths of x and Ax. Although, she attended to the direction of the two vectors, she thought that the lambda is 4 until I prompted her in turns [144] and [146]. It seems that her attention was blocked by calculating the value of the ratio using the Measure command and Calculator tool. Her immediate attention to the direction of the vectors enabled her to notice that lambda was -4. This shows instrumentalization and instrumentation processes in that by the use of dragging tool she explored another interpretation of collinearity of two vectors.

6.3.1.4. Finding eigenvectors and eigenvalues of matrix (d)

Kate used guided dragging and dragged x in anti-clockwise and clock-wise circular paths several times. She concluded that "there is no lambda" as she said "because you are not able to put x vector, on the same line with vector Ax". Her dragging strategies suggest that Kate developed new dragging strategies (dragging in circular paths with different radii) to verify the existence of eigenvectors on the given sketch.

6.3.2. Embodied cognition: gesture and speech

6.3.2.1. Kate: Eigenvectors are parallel and on the same line

In her exploration of geometric representation of x and its transformation under the matrix (a), Kate said "they're parallel but how about non-parallel". This shows that she used the terms "parallel" and "non-parallel" in referring to the relationship between the two vectors. She also used the term "parallel" in turns [121] and [152]. Her use of the term "parallel" could be triggered by her initial interaction with the sketch. In her initial interaction, Kate dragged x in the first quadrant in a clockwise direction as she attended to the changes in the position of Ax. In doing so, she noticed that the position of x and Ax was changing from being perpendicular to being overlapped. This representation could have made her to articulate the existence of eigenvectors in reference to parallel property of vectors.

Upon recognition of the relationship among eigenvectors associated with an eigenvalue, Kate described the existence of eigenvectors using the term "on the same line" in turns [131] and [152]. This suggests that the use of line dragging enabled her to develop dynamic imagery of the geometric representation of infinitely many eigenvectors. It also enabled her to expand her visual perception of eigenvectors from "they're parallel" to "they're on the same line". In turn [152], in describing her ways of finding eigenvectors, she said "because they're equal that means they're parallel to each other or I can say they're on the same line". This shows that the dynamic representation of concepts enabled her to construct her geometric imagery of eigenvectors.

She also used the term "the same" to refer to the equality between two lengths in turn [105]. Although her use of the term "the same" could be related to the use of the equals sign on the definition, the term “the same” could also imply an embodied way of describing “equal”.

6.3.2.2. Kate gestured at a span of eigenvectors that it geometrically is a line

Kate used her right index finger and also the mouse pointer to indicate the geometric representation of the vectors x and Ax on the sketch as she interacted with the sketch. Her use of the mouse pointer and her right index finger is evidence of deictic gesture.

In approximating the eigenvalue of matrix (b), kate used her right and left index fingers to indicate the lengths of x and Ax. Shortly after, she used her right index finger and thumb to indicate (or to measure) the lengths of x and Ax. These can be classified as iconic gestures. In approximating the positive eigenvalue of matrix (c), she did not just rely on her visual perception. She used her fingers as a measurement unit. She first measured the length of x by using the distance between her index finger and thumb. She then used the length of x (the distance between her index finger and thumb) to approximate the length of Ax, thus she found the ratio of two lengths. This can be classified as an iconic or metaphoric gesture.

In her final explanation to my prompt “what is an eigenvector?”, Kate said “[…] if these two vectors can be on the same line so at this time this x is an eigenvector” as she gestured a line rising her left hand extend fingers in space as shown in Figure 24. Kate used metaphoric gesture to kinaesthetically communicate her imagery of a set of eigenvectors (or an eigenspace).

Kate’s metaphoric gesture, similar to Mike’s and Jack’s gestures, came right out of her interaction with the dynamic representation of eigenvectors. This suggests that she developed synthetic-geometric modes of thinking through her interaction with the dynamic representations of the concepts. It also confirms Châtelet’s ideas on the importance of the gesture/diagram relationships in the development of the concept of eigenvectors and eigenvalues.

Figure 24. Kate's gesture that describes eigenvectors

6.3.3. Shifts of attention

Kate focused her attention on the definition, read it twice word by word and said "it's too hard to me to understand the concept right now." She discerned details from the definition as she described lambda as a real number and Ax as a matrix times a vector. She then shifted her attention to Ax as she said "a matrix times a vector should be still a matrix, um, I cannot remember that one." She even used a numeric example to validate her conjecture that a matrix times a vector is a matrix as shown in Figure 25. But she did not recall how to do matrix-vector multiplication, so that she left it incomplete.

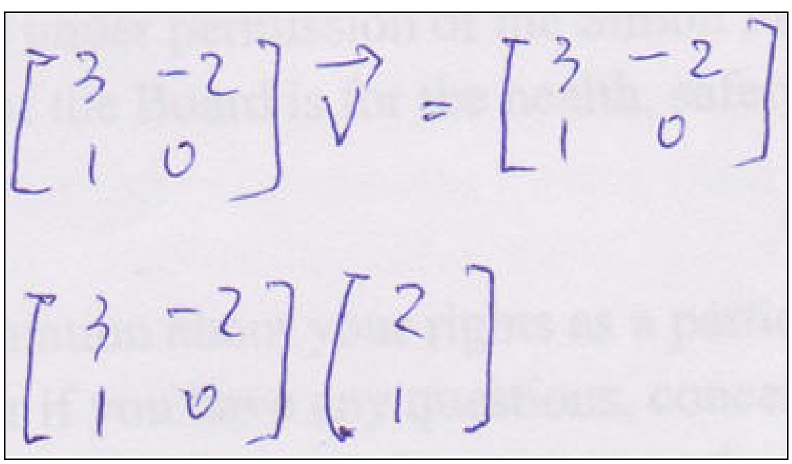


Figure 25. Kate's example of a matrix-vector multiplication

I drew her attention to the sketch by asking her to drag x. She then shifted her locus of attention to the sketch and to the draggable vector x. Her initial dragging suggests that her state of attention involved in holding wholes as she watched the changes in the position of Ax while dragging x. She began to recognise relationships between x and Ax by noticing that the position of two vectors was changing from being perpendicular to being overlapped as she dragged x in the first quadrant in a clockwise direction. Using wandering dragging, she found two positions, in the first quadrant, where the two vectors overlapped. She attended to the position where two vectors overlapped and had the same length. It seems that, in the beginning of her exploration, her interpretation of $Ax = \lambda x$ was to search for vectors that have the same direction and the same length. She estimated $\lambda = 1$ in turn [107], attended to the arithmetic representation of the eigenvector in turn [109] and generalized her finding by wring down $\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix}$ on a paper sheet. She then shifted her attention to the sketch to identify another set of eigenvectors. After identifying the two lambdas of matrix (a) and associated sets of eigenvectors, in her exploration of the *eigen* sketch, Kate tried to locate another position where lambda is 3 as she said “can we get a three?”. She even attended to a position where the two vectors did not overlap as she said “how about non-parallel”. This suggests that she started to perceive properties of eigenvectors. The use of dragging tool and dynamism of the representation enabled her to explore further the relationships between two vectors, thus she noticed that the two vectors preserve collinearity in the third quadrant.

In turn [119], Kate attended to the length of the vectors as she said “actually the lambda is the ratio of this length [points to Ax] to this one [points to x]”. Despite her statement, she confused lambda as being the ratio of the two lengths with being the ratio of the coordinates of vector x. She immediately realized her error by attending to the ratio of lengths of the two vectors. In turn [131], she described that “lambda is a fixed number” that can be found when the two vectors “are on the same line”. This suggests that her use of line dragging and the dynamism of the representations enabled her to distinguish an eigenvalue as a ratio of two lengths (when two vectors overlap) from the geometric representation of eigenvectors. In response to my prompt asking her about the number of eigenvectors, she said “many, infinitely many [...] on the same line” in turn [131]. This shows that she perceived properties of eigenvectors (invariant collinearity) and eigenvalues (as the ratio of lengths of two overlapped vectors).

Kate also experienced a difficulty reading off the values of eigenvectors from the sketch in turn [131]. She attended to the ratio between *x*- and *y*- coordinates of vector x to find a parametric representation of the set of eigenvectors. Using the numeric values, she calculated ratio equal to 1.1 (the actual ratio is one). This difficulty happened because the sketch representation of the collinearity of the two vectors is not error-free. As mentioned before, the two vectors are collinear when the ratio of the coordinates of vectors is equal to one (i.e. $x_1 / x_2 = 1$). But, the sketch, for the transformation matrix (b), shows that the two vectors are collinear when x_1 / x_2 ranges from 0.7 to 1.3 (on a square grid scaled 2 centimetres per unit).

Using guided dragging, Kate found a set of eigenvectors associated with the positive eigenvalue for matrix (c). After receiving a prompt in turn [138], she dragged x into the fourth quadrant and attended to their directions until she noticed a position where vectors had opposite directions, but they were still collinear. This representation drew her attention to another interpretation of collinearity. She then used the Measure command and Calculator tool to find the ratio of Ax and x since she thought that the ratio of the lengths was the lambda (as it was the case when the two vectors overlapped). Although, she shifted her attention to the direction of the two vectors as she said “they’re opposite to each other” in turn [139], she concluded that the lambda is 4. It seems that her attention was blocked by considering the ratio of the lengths as lambda. My prompts in turns [146] and [148] helped her to re-draw her attention to the direction of x and Ax, thus she concluded that lambda was -4. In turn [149], she reasoned on the basis of agreed properties of eigenvectors as she said “yeah they have opposite directions so the lambda should be negative”. This suggests that Kate perceived properties of eigenvectors and eigenvalues.

In finding an eigenvector and an associated eigenvalue of matrix (d), she dragged x in anti-clockwise and clockwise circular paths as she drew her attention to finding a position where the two vectors were collinear.

6.3.4. *Dragging, shifts of attention and gesture*

Reading the definition, Kate discerned details about Ax, λx and $Ax = \lambda x$. She articulated that λ is a real number acting as a scalar multiple. But she was surprised by finding out that a matrix-vector multiplication would result in a vector. She interacted with the sketch drawing her attention to the changes that occurred to Ax as she dragged x. Her strategy of dragging x enabled her to find positions in the first quadrant where the two vectors overlapped. She then coordinated the symbolic representations used in the definition ($Ax = \lambda x$) with the geometric representation of Ax and x on the sketch (Ax overlapped with x and had the same length), and found the lambda and the associated set of eigenvectors of matrix (a). In finding another set of eigenvectors, she dragged x *intentionally* to make x co-linear with Ax. The dynamic geometric representation of the concepts enabled her to articulate lambda as the ratio of the lengths of x and Ax in turn [121]. The use of line dragging enabled her to verify the invariance property of eigenvalues and also to realize that there are infinitely many eigenvectors associated with one eigenvalue (see turn [131]).

As mentioned above, she articulated λ as a real number (right after reading the definition), λ as a ratio of lengths (in turn [121]), and λ as a fixed number (in turn [131]). This reveals evidence of shifts in her attention both macro-level and micro-level. The shifts in her attention enabled her to use different dragging modalities and to develop dragging strategies such as dragging x in clockwise and anti-clockwise circular paths.

Moreover, the use of the dragging tool and the dynamism of the representation affected Kate's modes of thinking as evidenced from her gesture and speech. At the very beginning of her interaction with the *eigen* sketch she produced deictic and iconic gestures using her fingers or the mouse pointer. In her final explanation of an eigenvector, Kate produced a metaphoric gesture. She positioned her left hand extended fingers in space to illustrate a line (see Figure 24). This suggests that Kate developed a dynamic imagery of a set of eigenvectors (or an eigenspace) and used her hand to communicate kinaesthetically her imagery. Her gesture was resulted from her interaction with the dynamic representation of the concepts.

Kate's description of an eigenvector in turn [154] included the collinear property of eigenvectors as well as the multiplication operation. She used the multiplication operation in describing Ax and λx. She then emphasized that the results of both multiplication operations are vectors and said "if these two vectors [the results of products] can be on the same line so at this time this x is an eigenvector". Her explanation suggests that her thought processes integrated two modes: synthetic-geometric and analytic-arithmetic. It seems that her thought processes had a potentiality to achieve to analytic-structural mode of thinking since Kate articulated her thinking of an eigenvector of a 2×2 matrix in terms of its property (collinearity) in [152] and [154].

6.4. Tom

6.4.1. Dragging modalities and strategies

6.4.1.1. Finding eigenvectors and eigenvalues of matrix (a)

After reading the definition, Tom used wandering dragging to drag *x* everywhere on the sketch. In doing so, he noticed a position where x and Ax overlapped in the first quadrant. He then re-read the definition, looked back to the sketch and said “lambda seems to be 2”. This suggests that Tom instrumentalised the dragging tool, in that he used it to explore the relationships between the two vectors. It seems that he identified lambda by attending to the geometric representation of x and Ax. This also suggest that he coordinated the symbolic representation of eigenvectors (i.e. $Ax = \lambda x$) with the geometric representation (i.e. the position where the two vectors overlapped) on the sketch, thus he approximated the eigenvalue to be 2. Using line dragging, he dragged x along the line (where the two vectors overlapped) away from the origin in the first quadrant. He found two eigenvectors ($x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $x = \begin{bmatrix} 9 \\ 4.5 \end{bmatrix}$) associated with $\lambda = 2$. The use of line dragging suggests an instrumentalization process in that Tom developed a new dragging modality. It also provides evidence of an instrumentation process, in that the dragging tool was transformed into an instrument detecting the presence of the geometric representation of eigenvectors associated with $\lambda = 2$.

In response to my prompt asking him to find a different eigenvector from the other he gave before, he dragged x *intentionally* to make x collinear with Ax in the first quadrant. He found a position where x and Ax overlapped and had the same length. He then used line dragging to drag x along the line (where the two vectors overlapped), thus explored further the relationship between x and Ax. This suggests that the use of line dragging enabled him to verify the invariance property of eigenvalues.

6.4.1.2. Finding eigenvectors and eigenvalues of matrix (b)

Tom used guided dragging to drag x in a clockwise direction in the first quadrant. He immediately found a position where x and Ax overlapped. He approximated the value of lambda (by attending to the ratio of two lengths) and the eigenvector of $x = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$ (by attending to the arithmetic representation of x on the sketch). He then used line dragging to drag x further along the line as he said "it [overlap position]'s the same thing because of being multiple" in turn [161]. This provides evidence of an instrumentation process in that the use of dragging tool enabled him to perceive the relationships among a set of eigenvectors associated with $\lambda = 3$ (his approximation of eigenvalue was 4). In response to my prompt about the relationship between eigenvectors, he said "scalar [multiple] with the same basis." in turn [165]. His articulation suggests that Tom used the notion of basis in identifying a vector space (i.e. eigenspace associated with $\lambda = 3$).

Tom then used guided dragging to drag x in a circular anti-clockwise direction to explore further the relationship between x and Ax. In doing so, he noticed that x and Ax overlapped in the third quadrant. However, he immediately said "that [overlap position] is the same thing", and continued dragging. This suggests that he completely perceived properties of eigenvectors (invariant collinearity) and eigenvalues (as the ratio of lengths of two vectors when they overlap).

His interactions with the sketch suggest that he developed clockwise and anti-clockwise circular dragging strategies in verifying relationships between the two vectors. He also used different dragging modalities (wandering dragging, intentionally dragging, line dragging and guided dragging) in his interaction with the sketch. The strategies and modalities enabled him to perceive properties of eigenvectors and eigenvalues.

6.4.1.3. Finding eigenvectors and eigenvalues of matrix (c)

Tom used guided dragging to find a position where x and Ax overlap in the first quadrant. As he dragged x, he attended to the changes in the position of Ax as he said "first, it [Ax] goes to the opposite direction" in turn [168]. After dragging x a few times into all of the four quadrants, he found a position in the first quadrant where x and Ax overlapped. He approximated the eigenvalue and the eigenvector.

I then prompted him to drag x to find a different eigenvector from the one he gave before. He dragged x slowly in an anti-clockwise circular direction and said "I guess it looks like tracing each other". He stopped dragging when x and Ax lined up, stared at the screen for a few seconds and said "this one goes to the opposite direction". He then used line dragging to drag x along the line where they were collinear, and said "six times, it's six times more than x, I mean opposite direction". This shows evidence of an instrumentation process in that the dragging tool was transformed into an instrument detecting the presence of the geometric representation of an eigenvector associated with the negative eigenvalue. This geometric representation led Tom to attend to another interpretation of the collinearity of two vectors.

6.4.1.4. Finding eigenvectors and eigenvalues of matrix (d)

Tom used guided dragging and dragged x in clockwise circular paths with different radii a few times. He then dragged x in anti-clockwise circular paths with different radii and said "[...] the Ax is never on top of the x". His dragging strategies suggest that Tom developed new dragging strategies (dragging in circular paths with different radii) for finding eigenvectors using the *eigen* sketch.

6.4.2. Embodied cognition: gesture and speech

6.4.2.1. Tom: Eigenvectors are scalar multiples

Tom used the concepts of scalar multiple and basis in describing eigenvectors. In turns [161] and [165], he described the relationship among eigenvectors using the two clauses: "being multiple" and "scalar [multiple] with the same basis". His use of the term "multiple" in describing eigenvectors could be triggered by his use of line dragging on the sketch. This suggests that the use of line dragging enabled him to develop dynamic imagery of the geometric representation of infinitely many eigenvectors associated with one eigenvalue.

Tom also mentioned that eigenvectors are "linear transformations" and "eigenvector changes [...] if you have a multiple of the same vector" in turn [179]. He, like

Jack, used the term “linear transformation”, although the term is not indicated on the given definition. It seems that Tom recalled the concept of linear transformation from his course work. His articulation in turns [179] and [181] suggest that he became aware that an eigenvector is a special vector that becomes collinear with its scalar multiple as a result of linear transformation.

His articulation of the geometric relationship between x and Ax in turn [170] suggests that he started developing an embodied meaning for the notions of linear transformation and vector. He used the verbs “to rotate” and “to go” as he said “rotates, like when I go clockwise it [Ax] goes anti-clockwise”. In fact, the subjects of all these action verbs are vectors, but he used the clause “I go” since he imposed motion on vector x by dragging it around. The use of action verbs to describe mathematical objects was resulted from his interaction with the sketch. His use of action verbs conveys a sense of motion in his thinking which is in accordance with mathematicians' use of motion in describing the concept of eigenvectors (see Nathalie and Gol Tabghi, 2010).

6.4.2.2. Tom gestured and diagrammed even before his interaction with the eigen sketch

At the very beginning of his interview, before interacting with the *eigen* sketch, Tom used his right index finger and drew a vector on the desk as he said “I guess a vector is a line”. He immediately drew (on a sheet of paper) a ray starting from the origin and ending with an arrow in the first quadrant, and then extended it into the third quadrant to illustrate his mental imagery of a vector as shown in Figure 26.

His drawing and description of a vector suggest that Tom evoked geometric thinking of a vector since he used his right index finger to trace it on the desk and then on the paper. His gesture can be classified as an iconic or metaphoric gesture. His use of gesture and diagram in describing his thought process of a vector confirms Châtelet's ideas on the diagram/gesture relationships.

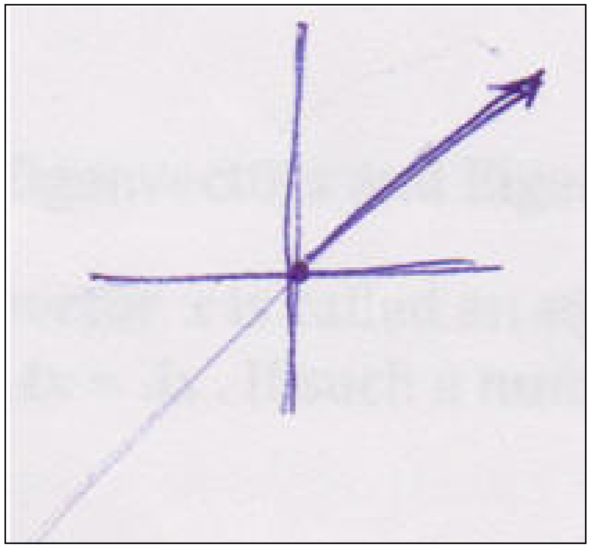


Figure 26. Tom's representation of a vector on a sheet of paper

In describing a scalar multiple of a vector, Tom evoked it as a straight line that goes to infinity. He used his hands and arms to communicate kinaesthetically his mental imagery of a scalar multiplication of a vector. He said "it [a scalar multiple of a vector] is just a straight line that goes to infinity" as he moved his right hands extend index finger toward up right corner and his left hands extended index finger and arms down toward the left corner of his body as shown in Figure 27. His gesture is a metaphoric gesture referring to McNeill's classification. His description of vector and scalar multiple of vector suggest that he employed synthetic-geometric mode of thinking that Sierpinska (2000) describes.

Figure 27. Tom gestures as describes a scalar multiple of a vector

While he interacted with the sketch, in describing the relationship between x and Ax, Tom used the verb “to rotate” in turn [170] as he rotated his right index finger around a circular path to represent the behaviour of Ax. He used the same gesture in describing eigenvectors in turn [179]. He first rotates his right index finger around a circular path and then he moved his right hand extended index finger back and forth along a path as he was tracing a straight path (as shown in Figure 28) to communicate kinesthetically his mental imagery of finding eigenvectors. It seems that his gestures in turn [170] and [179] resulted from his interaction with the dynamic geometric diagrams of the concepts. This confirms Châtelet’s idea on the diagram/gesture relationships in the development of mathematical thought processes.

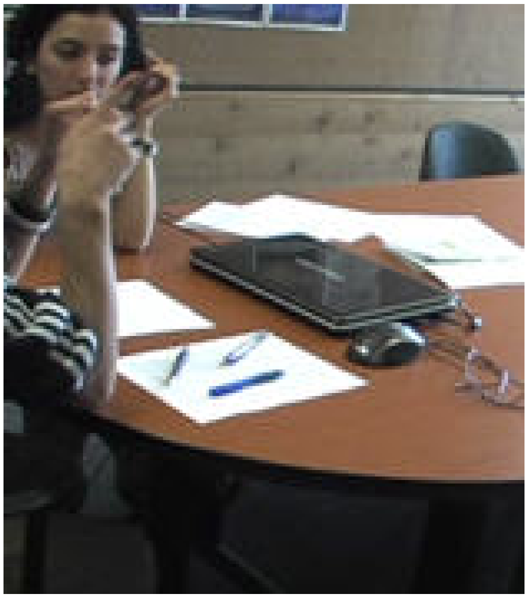


Figure 28. Tom rotates and moves his index finger as describes eigenvectors

Tom, who revealed synthetic-geometric mode of thinking at the beginning of the interview, had made connections among the concepts of linear transformation, eigenvectors, eigenvalues, scalar multiples and basis through interacting with the eigen sketch. His mode of thinking had not been changed, but became extended given that he made connections among concepts as he interacted with the sketch.

6.4.3. Shifts of attention

Tom focused his attention on the definition since he first read the formal definition of eigenvector on the given worksheet. He then shifted his locus of attention to the sketch and dragged the vector x randomly everywhere on the sketch. His initial interaction with the dragging tool suggests that he waited for visual feedback from the sketch and his attention was holding wholes. As he dragged x, he drew his attention to the position of Ax on the sketch. He stopped dragging when Ax overlapped with x in the first quadrant. He shifted his locus of attention to the definition and re-read the definition. He then shifted his locus of attention to the sketch and approximated the eigenvalue by attending to the lengths of x and Ax. It seems that he coordinated the symbolic notations used on the definition with the geometric representations of vectors on the sketch, thus he approximated the eigenvalue ($\lambda = 2$) and the eigenvector ($x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$) of matrix (a). He then used line dragging to explore further the relationship between x and Ax and found out that another eigenvector (i.e. $x = \begin{bmatrix} 9 \\ 4.5 \end{bmatrix}$) that associates with $\lambda = 2$. This shows that the use of line dragging enabled Tom to perceive the invariance property of the eigenvalue and also the existence of more than one eigenvector associated with $\lambda = 2$.

In response to my prompt asking him to find a different eigenvector from the others he gave before, Tom continued dragging x into the other quadrants as he attended to the geometric configuration of the two vectors on the sketch. In doing so, he attended to x and Ax directions, identified a position in the first quadrant where x and Ax overlapped and approximated the eigenvalue and the eigenvector (or the basis of eigenspace).

In finding an eigenvalue and an associated set of eigenvectors of matrix (b), Tom used guided dragging and immediately noticed a position in the first quadrant where the two vectors overlapped. After finding the position, he approximated the eigenvalue and the basis of the eigenspace. He then used line dragging to drag x along the line (where x and Ax overlapped) and mentioned that there exist infinitely many eigenvectors that are “scalar [multiple] with the same basis” in turn [165]. This suggests that Tom perceived the invariant collinearity of eigenvectors associated with the eigenvalue ($\lambda = 3$) and also made connections among the concepts of basis, scalar multiples, and eigenvectors.

Overall Tom's strategy was to drag x to find a position where x and Ax overlap and then to approximate the eigenvalue and the basis of eigenspace. He used the same strategy in finding the eigenvalue of 7 and $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for matrix (c).

In response to my prompt asking him to find a different eigenvector from the one he gave before, he dragged x in an anti-clockwise circular path and stopped dragging when he noticed a position where the two vectors were collinear, but not in the same direction. He said "this one goes to opposite direction", used line dragging to drag x along the line (where x and Ax lined up), and in turn [174] said "eigenvalue is probably negative six. Eigenvector is one and one". This suggests that he attended to the direction of vectors, their ratio, and position on the sketch, therefore he immediately approximated the eigenvalue to be -6. This suggests that his attention was involved in reasoning on the basis of the properties of eigenvectors (invariant collinearity) and eigenvalues (dilation factor).

In finding an eigenvector and an associated eigenvalue of matrix (d), he first dragged x in clockwise circular paths (with different radii) as he drew his attention to find a position where the two vectors were collinear. Finding nothing, he dragged x in an anti-clockwise circular path and said "[…] the Ax is never on top of the x".

6.4.4. Dragging, shifts of attention and gesture

Tom discerned details from the definition as he read it to himself. He then interacted with the sketch drawing his attention to the changes that occurred to Ax as he used wandering dragging to drag x everywhere on the sketch. He shifted his attention back to the definition when he found a position where x and Ax overlapped. He then drew his attention to the sketch and approximated the two specific eigenvectors associated with one eigenvalue for matrix (a). This suggests that he matched the symbolic representations used in the definition with the ones used on the sketch as he shifted his attention back and forth between the sketch and the definition.

From the given equality on the definition and his initial interaction with the sketch, he inferred that he needed to find positions where x and Ax become collinear. This made him to use intentionally dragging modality to drag x to make it collinear with Ax in finding the second eigenvalue of matrix (a). His use of line dragging enabled him to verify the invariance property of eigenvalues and also to realize that there were infinitely many eigenvectors associated with one eigenvalue.

As Tom interacted with the *eigen* sketch, he made connections among the concepts of linear transformation, eigenvectors, eigenvalues, scalar multiples and basis see turns [165] and [179]. Through the interaction, his attention shifted at both macro-level and micro-level. The shifts in his attention enabled him to use different dragging modalities and to develop dragging strategies such as dragging x in clockwise and anti-clockwise circular paths.

Despite revealing synthetic-geometric mode of thinking in describing the concepts of vector and scalar multiples of a vector (before interacting with the *eigen* sketch), Tom's way of communicating the concepts of eigenvectors was changed as evidenced from his gestures and speech. Before his interaction with the *eigen* sketch, he mostly produced iconic gestures in describing the notions of vector and scalar multiples of a vector. In finding the eigenvalue and associated set of eigenvectors of matrix (c), he gestured to communicate kinaesthetically his mental imagery of the changes in the position of vectors. He also used action verbs, "to rotate" and "to go", as he rotated his right index finger around. His use of action verbs conveys a sense of motion in his thinking. I should recall that Tom used the verb "to go" even before his interaction with the sketch in describing a scalar multiple of a vector he said "it[a scalar multiple of a vector] is just a straight line that goes to infinity".

In his response to my prompt (what is an eigenvector?), he said "they're linear transformations". Then, he referred to the concept of eigenvalue as he rotated his right index finger around, and to the concept of eigenvector as he moved his right hand index finger back and forth along a straight path (see turn [179]). His gestures illustrate an embodied way of finding an eigenvector using the eigen sketch. His statement, "the eigenvector is on top of Ax" in turn [181], is also evidence of the development of embodied description of eigenvectors. His responses in turn [179] and [181] suggest that Tom used synthetic-geometric mode of thinking. Although he used the term "linear transformation", it is hard to comment on interactions among different modes of thinking by only analysing turns [179] and [181]. His mode of thinking might not been changed, but it became extended given that he made connections among concepts as he interacted with the sketch.

6.5. Rose

6.5.1. Dragging modalities and strategies

6.5.1.1. Finding eigenvectors and eigenvalues of matrix (a)

Rose used wandering dragging, not appearing to expect anything in particular, to drag x randomly from its given position (at about (4, 1)). She stopped dragging when she observed that x and Ax were positioned perpendicular to each other (vector x being in a vertical position and vector Ax being in a horizontal position) and said "here it [position of x and Ax on the sketch] is ninety degrees" (see Figure 29). Her interaction with the dragging tool suggests that she was guided by the visual representation generated by the sketch. It also shows evidence of an instrumentalization process in that she was using the dragging tool in order to identify the relationship between the vectors.

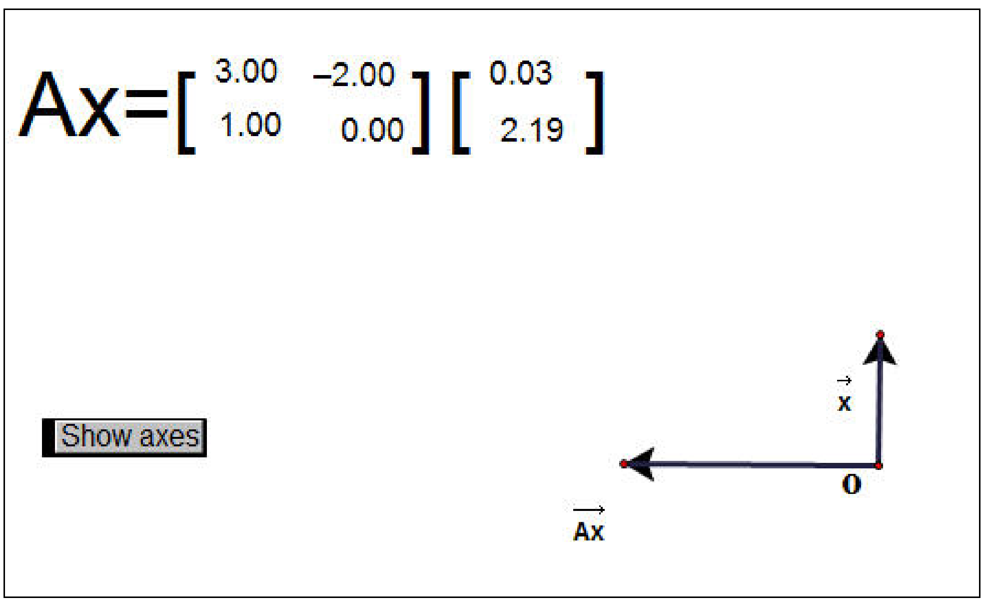


Figure 29. Snapshot of sketch where x and Ax are perpendicular to each other

In response to my prompt, “is that what you are asked to find?”, she switched her attention to the definition and then used wandering dragging to drag x in the first quadrant. She dragged x trying to keep its length fixed, thus using a kind of locus dragging. In doing so, she noticed that x and Ax overlapped. She continued dragging x slightly down from the overlapped position and then used an anti-clockwise direction to drag it back up. This made her notice that the length of vector Ax changes, as she dragged vector x in a clockwise direction in the first quadrant, from being smaller than to being twice as long as x, and then to being more than twice as long as x. Noticing these changes, she conjectured that "um it seems no matter [what] x is, Ax is twice [its] value". To verify her conjecture, she used line dragging to drag x along the line where x and Ax overlapped in the first quadrant. She then looked back to the definition, and said "that means Ax is going to be just two times x" and then wrote down $2x = \lambda x$ right below $Ax = \lambda x$ on the worksheet.

Her interaction suggests that line dragging and the dynamism of the representation enabled her to verify her conjecture about the ratio of the lengths of two vectors. It also enabled her to coordinate the geometric representation of the sketch with the symbolic one from the definition as she wrote $2x = \lambda x$. The use of line dragging suggests evidence of an instrumentalization process, in that she used line dragging to verify the collinearity of the vectors. It also reveals evidence of an instrumentation process, since she realized that the ratio of the lengths of two vectors was invariant despite dragging x along the line (where x and Ax overlapped) away from the origin in the first quadrant.

In response to my prompt asking her to find an eigenvector associated with $\lambda = 2$, Rose said "x is just two, isn't it?" in turn [194]. Her comment suggests that she had difficulties with identifying a vector. Indeed, she was attending to the lengths of x and Ax rather than to their components. The use of line dragging enabled her to identify the eigenvector, $x = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and its scalar multiple, $x = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$ in turns [196] and [198]. The use of line dragging enabled her to recognize that there were infinitely many eigenvectors associated with $\lambda = 2$ (dragging x only in the first quadrant). This suggests evidence of an instrumentation process in that Rose recognized infinitely many eigenvectors were associated with $\lambda = 2$.

In finding a different eigenvector from the one she gave before, she used guided dragging to drag x to find a position where x and Ax overlapped. She dragged x from the first quadrant into the fourth and then into the third. She found a position in the third quadrant and said “right here. But this time it is equal to it” (see turn [202]). She immediately mentioned that the eigenvalue was one (since x and Ax had the same length) and the eigenvector was “negative seven and negative seven” in turns [204] and [206]. This suggests evidence of instrumentation process in that Rose perceived eigenvalue as the ratio of lengths (where x and Ax overlapped).

6.5.1.2. Finding eigenvectors and eigenvalues of matrix (b)

Rose used guided dragging to drag x in the first quadrant. She immediately found a position where x and Ax overlapped, said “it [the ratio of lengths]’s three times”, and approximated the eigenvector of $x = \begin{bmatrix} 4.02 \\ 3.34 \end{bmatrix}$ by attending to the arithmetic representation of x on the sketch. She then used line dragging to drag x further along the line (in the first quadrant) where x and Ax overlapped and said “it [overlap position] goes to infinity that way” in turn [210]. To verify the collinearity of the two vectors in the third quadrant, she did not drag x along the line (where x and Ax overlapped) passing through the origin into the third quadrant. In fact, she avoided dragging very close to zero and also through zero because as x gets closer to zero (point O on the sketch), x and Ax both approach the zero vector. After receiving the prompt in turn [213], she recognized that she could continue dragging along the line through the origin, passing from the first quadrant to the third. In this case, the line (excluding (0,0)) was the geometric representation of the eigenspace associated with the eigenvalue of 3 since an eigenvector is a non-zero vector.

In response to my prompt asking her to look for a different eigenvector, she used guided dragging to drag x in a clockwise circular path into all of the four quadrants. She found a position where x and Ax overlapped in the third quadrant, but she immediately noticed that the position was not a new one, and said “it [overlapped position] looks the same to me”. This suggests that the dynamic geometric diagram enabled her to develop a visual imagery of the position of vectors on the sketch.

6.5.1.3. Finding eigenvectors and eigenvalues of matrix (c)

Rose used guided dragging to drag x in a clockwise circular direction in the first quadrant to find a position where x and Ax overlap. She immediately found such a position and approximated the eigenvalue to be 7. It seems that she read the value of eigenvalue by attending to the sketch representation of the ratio of the two lengths (as she had activated the Calculator tool and Measure command in verifying her approximation of eigenvalue of the matrix (b)). I then prompted her to drag x to find a different eigenvector from the one she gave before. She dragged x into the all four quadrants as she attended to the changes that occurred to Ax. She noticed that the geometric behaviour of x and Ax was different from the previous ones. And, she said "usually they [x and Ax] go in the same direction but this [points to Ax using her right index finger] goes opposite direction. Is it because there is no negative value on this one [matrix]?" in turn [224]. Even though she attended to the direction of two vectors, she related it to the positive values of the entries of the matrix (c). Receiving my comment in turn [225], she focused on the geometric representation of the two vectors and found a position where x and Ax were collinear (but not overlapping). It seems that she read the value of the eigenvalue by attending to the sketch measure of the ratio of two lengths as she thought it was 4. She then attended to the arithmetic representation of x on the sketch and used a numerical example of vector x and Ax, thus realized that the eigenvalue is -4. This shows the occurrence of instrumental genesis in that the dragging tool was transformed into an instrument detecting the presence of the geometric representation of an eigenvector associated with the negative eigenvalue. This geometric representation led Rose to attend to another interpretation of the collinearity of two vectors. In turn [230], she reflected on her findings from comparing the geometric representations of two collinear vectors having positive and negative eigenvalues.

6.5.1.4. Finding eigenvectors and eigenvalues of matrix (d)

Rose used guided dragging and dragged x in clockwise circular paths with different radii a few times. She then mentioned that x and Ax "do not meet at all". The use of different dragging modalities (such as wandering dragging, locus dragging and guided dragging) and strategies (dragging in circular paths with different radii) suggest that Rose developed new strategies in finding eigenvectors using the eigen sketch.

6.5.2. Embodied cognition: gesture and speech

6.5.2.1. Rose: Eigenvectors lie on top of each other

Before Rose interacted with the sketch, she described a vector in terms of its components as she said "you know that in the class we learned like x_1 and y_1". This suggests that she evoked a symbolic representation of a vector. According to Watson et al.'s account of the development of the concept of vector, Rose had not developed an embodied representation of a vector nor was she evidently aware of the connections between symbolic and embodied worlds. Her difficulties with identifying a vector were persisted when she was asked to give an eigenvector associated with $\lambda = 2$ for matrix (a). Her comment in turn [194] suggests that she was attending to the length of vectors rather than to the components of each vector. As she interacted with the sketch, she became acquainted with the geometric representation of vectors and the coordination between the geometric representation of concepts with the arithmetic representation on the sketch. After completing the task, in describing her way of finding eigenvectors she said "I tried to make the vectors lie on top of each other and then find the scaling value". This suggests that the dynamic geometric diagram enabled her to develop a mental imagery of the concept of vectors and eigenvectors in which eigenvectors "lie on top of each other".

In turns [210] and [212], she used the verb "to go" to articulate the behaviour of Ax. She said "it [overlapped position] goes to infinity that way" (in turn [210]) while using line dragging to drag x along the line where x and Ax were collinear. Her statements show that she developed a dynamic imagery of the geometric representation of an eigenspace of a 2×2 matrix (that is, a straight line), in that she used the verb "to go" to communicate her dynamic imagery of the representation.

Her use of the verb "to look" suggests evidence of the use of dynamic imagery as she said "it [overlapped position] looks the same to me" in exploring eigenvectors of matrix (b). She perhaps recalled her dynamic imagery of finding eigenvectors in the first quadrant and then imagined a line would pass through the overlapped vectors in the first and third quadrants.

6.5.2.2. Rose produced deictic gestures

Rose used her right index finger to point to the symbols on the definition as she read the definition. She also used it to point to the vectors on the sketch as indicated in turn [224]. Her gestures were deictic gestures.

6.5.3. Shifts of attention

Rose first focused her attention on the definition, discerning details as she articulated every symbol one by one. She then shifted her locus of attention to the sketch and to the draggable vector x. Using wandering dragging, she encountered a position (see Figure 29) where the two vectors were perpendicular to each other and so, her attention shifted to the relationship between x and Ax. Her attention was blocked by the right-angle relationship, which she seems to think is important (with good reason, since perpendicularity is frequently important in geometric configurations).

My prompt (is that what you are asked to find?) helped her to draw her locus of attention to the definition and then back to the sketch. She continued dragging x and tried not to change its length. As she dragged x in a clockwise direction in the first quadrant, she realized that the length of vector Ax changes from being smaller than to being twice as long as x, and then to being more than twice as long as x. Attending to those changes, she asked "is x double Ax?". Her interactions suggest that she attended to the lengths of x and Ax (where x and Ax overlapped) and estimated the ratio between the two lengths. She dragged x along the line where x and Ax overlapped, then looked back to the definition, said "that means Ax is going to be just two times x" and then wrote down $2x = \lambda x$ right below $Ax = \lambda x$ on the worksheet. This suggests that she attended to the relationships between the geometric representation of the sketch and the symbolic one from the definition since she wrote $2x = \lambda x$. She identified $\lambda = 2$, but had difficulties in identifying eigenvectors associated with $\lambda = 2$. Her statements in turns [192] and [194] suggest that she focused her attention on the positions of vectors (where the two vectors overlapped) trying to discern more details about the relationship between the two vectors rather than the position of x (such as (4, 2)). My prompt in turn [195] enabled her to direct her attention to the arithmetic representation of the vector x on the sketch, thus she found the associated eigenvectors with $\lambda = 2$. This suggests that Rose started to recognize the invariant collinearity of eigenvectors and also the existence of infinitely many eigenvectors associated with $\lambda = 2$ (see turn [200]).

In response to my prompt asking her to find a different eigenvector of matrix (a), Rose continued dragging x into other quadrants as she attended to the geometric configuration of the two vectors on the sketch. In doing so, she attended to x and Ax directions and the lengths of vectors as she said “right here. But this time it is equal to it. Isn’t it?” in turn [202]. This suggest that she attended to the position and the lengths of the two vectors where x and Ax overlapped. It also suggests that her state of attention was drawn to discerning details, such as the position of the vector, its length and its relationship to Ax.

In finding an eigenvector and associated eigenvector of matrix (b), Rose used guided dragging and immediately noticed a position in the first quadrant where the two vectors overlapped. She approximated the eigenvalue by attending to the ratio of lengths as she said “it’s three times”. She then approximated the eigenvector by attending to the arithmetic representation of x on the sketch. She then attended to the relationship between x and Ax as she dragged x along the line where x and Ax overlapped. Her statements in turns [210] and [212] suggest that she recognized the existence of infinitely many eigenvectors associated with $\lambda = 3$. In response to my prompt asking her to look for a different eigenvector from the one she found before, she dragged x in the third quadrant and found a position where x and Ax overlapped. She immediately noticed that the position was not a new one, and said “it looks the same to me”. This suggests her attention was drawn to comparing the visual representation of the two overlapped vector in the first quadrant (length-wise and position-wise) with the representation in the third quadrant.

Rose used a similar dragging strategy in finding the positive eigenvalue and an associated eigenvector of matrix (c). She could also read the ratio of the two lengths from the sketch (since she had activated the Calculator tool and Measure command in verifying her approximation of eigenvalue of the matrix (b)). In response to my prompt asking her to find a different eigenvector, she dragged x and said “oh it changes this time” in turn [222]. This suggests that she attended to the changes that occurred in the direction of the two vectors. She then said “usually they [x and Ax] go in the same direction, but this [Ax] goes opposite direction. Is it because there is no negative value on this one [matrix]?” in turn [224]. Her articulation suggest that she attended to the direction of x and Ax. But, her attention was blocked by the values of the entries of matrix (c) since they were all positive. My prompt in turn [225] helped her to shift her attention from the entries of the matrix to the geometric representation of the two vectors that represent collinearity of vectors (but not overlapping). She stopped dragging, gazed at the representation for a few seconds, and said “oh. It’s completely straight here”. This suggests that Rose’s attention was drawn to another interpretation of the collinearity of two vectors. It seems that she read the value of the eigenvalue by attending to the sketch measure of the ratio of two lengths as she thought it was 4. She then attended to the arithmetic representation of x on the sketch and used a numerical example of vector x and Ax, thus realized that the eigenvalue is -4. Her reflection of the geometric representation of eigenvectors associated with positive or negative eigenvalues in turn [230] suggest that her state of attention was involved in perceiving geometric properties of positive and negative eigenvalues.

In finding an eigenvector and associated eigenvalue of matrix (d), she dragged x in clockwise circular paths with different radii a few times as she drew her attention to find a position where x and Ax were collinear.

6.5.4. Dragging, shifts of attention and gesture

Rose drew her attention to the definition trying to make sense of the symbols and the equality $Ax = \lambda x$. She then interacted with the sketch using wandering dragging to drag x around its given position. She stopped dragging when she observed that the vector x and Ax were positioned perpendicular to each other. It seems that her attention was blocked for a few seconds by the right-angle relationship, which she seemed to think was important. My prompt helped her to draw her locus of attention to the definition and then back to the sketch. She then used wandering dragging to drag x as she attended to the lengths of x and Ax and to their position on the sketch. Attending to the changes, she conjectured that “um it seems no matter [what] x is, Ax is twice [its] value ". She then looked back to the definition, and said "that means Ax is going to be just two times x" and then wrote down $2x = \lambda x$ right below $Ax = \lambda x$ on the worksheet. This shows that she coordinated the symbolic representations used in the definition ($Ax = \lambda x$) with the geometric representation of Ax and x on the sketch and found the lambda and the eigenvectors of matrix (a). In interacting with the sketch, she developed some dragging strategies. These strategies caused shifts in the structure of her attention. For instance, the use of line dragging suggests that she shifted her attention to a position where x and Ax overlapped. It also shows that her state of attention was drawn to discerning details, such as the position of the vector, its length and its relationship to Ax .

Rose's use of dragging tool to explore further the relationship between x and Ax where they overlapped enabled her to develop a dynamic imagery of the geometric representation of an eigespace of a 2×2 matrix (that is a straight line), in that she used the verb "to go" to communicate her dynamic imagery of the geometric representation. Visualizing the constant overlap of the two vectors led her to become acquainted with the geometric representation of a set of eigenvectors. The *eigen* sketch representation enabled her to see how dynamic geometric interactive representation relate to the symbolic representations of eigenvectors and eigenvalues.

Despite revealing analytic-arithmetic mode of thinking in describing the concepts of vector (before interacting with the *eigen* sketch), Rose's way of communicating the concept of eigenvectors was changed as evidenced from her final statement. In that, she said "I tried to make the vectors lie on top of each other and then find the scaling value." In contrast to the order used in pencil-and-paper algebraic approaches, she described finding the eigenvector first, and then the "scaling factor," or the eigenvalue. Her embodied description of finding eigenvectors is evidence that Rose has started to develop synthetic-geometric mode of thinking.

6.6. Summary

In this chapter, I included a detailed analysis of the each participant's interactions with the *eigen* sketch drawing on multiple theoretical frameworks. An extension of this analysis is included in Chapter 7 to extend my understanding of the participants' development of mathematical thinking and learning. In this section, I include a comparative summary of the participants' interactions with the sketches.

Mike did not recall the concepts of eigenvector and eigenvalue at the beginning of the interview. Initially, he frequently shifted his locus of attention between the *eigen* sketch and the definition. He dragged x in non-circular and circular (clockwise and anti-clockwise) paths and used wandering dragging, intentional dragging, line dragging and guided dragging. His interactions with the *eigen* sketch suggest that his attention was structured by *perceiving properties* of eigenvectors since he understood eigenvector as a special vector that lines up with its scalar multiples in the opposite or the same direction. At the end of the interview, he used his hands and arms as vectors to gesture eigenvectors in a way that communicated dynamic, visual imagery that seemed triggered from his interaction with the *eigen* sketch. He employed a synthetic-geometric mode of thinking throughout his interactions with the *eigen* sketch.

Jack, unlike Mike, recalled the concepts of eigenvector and eigenvalue and made a connection to the notion of linear transformation. His dragging strategies and modalities were similar to Mike's, except that he also dragged x in a spiral path at the end of his interaction with the *eigen* sketch. His state of attention included *reasoning on the basis of the properties* of eigenvectors and eigenvalues, which is the most sophisticated according to Mason. During the interview, he used his hands as vectors, thus also communicating dynamic, visual imagery of eigenvectors that seemed triggered from his interaction with the *eigen* sketch. He integrated the synthetic-geometric mode of thinking with an analytic-arithmetic one since he constantly coordinated geometric representation with the given symbolic representation of the concepts of eigenvectors and eigenvalues. He explicitly articulated that there are infinitely many eigenvectors associated with one eigenvalue. He also sketched a diagram of an eigenvector associated with a negative eigenvalue.

Kate, like Mike, did not recall the concepts of eigenvector and eigenvalue. She was surprised to discover that a matrix-vector multiplication would result in a vector. Her dragging strategies and modalities were similar to those of Mike and Jack. She explicitly articulated that lambda is the ratio of the lengths of two overlapped vectors and there are infinitely many eigenvectors. Her state of attention, like Jack, included *reasoning on the basis of agreed properties* of eigenvectors. She integrated two modes of thinking—

synthetic-geometric and analytic-arithmetic. Kate had the potential of achieving an analytic-structural mode of thinking since she articulated her thinking of an eigenvector of a 2×2 matrix in terms of its property (collinearity). She used a gesture that was different from those of Jack and Mike: instead of using her arms or hands as eigenvectors she used her left hand to gesture a span of a set of eigenvectors (or an eigenspace) that is a line.

Tom evoked synthetic-geometric thinking of a vector and a multiple of a vector from the beginning of the interview. He recalled studying the concept of eigenvector, but did not know more about it. By interacting with the *eigen* sketch, Tom made connections among the concepts of linear transformation, eigenvector, eigenvalue, scalar multiple of a vector and basis. His dragging modalities were similar to ones used by Mike, Jack and Kate. He went beyond Jack's and Kate's articulation of the existence of infinitely many eigenvectors as he used the idea of "scalar [multiple] with the same basis" in describing a set of eigenvectors. His state of attention included *reasoning on the basis of the properties* of eigenvectors (invariant collinearity) and eigenvalues (dilation factor). Although, his mode of thinking did not change, it was extended given that he made connections among concepts. He also used his right index finger as a vector and rotated it around as described his mental imagery of eigenvectors.

Rose, like Mike and Kate, did not recall the concepts of eigenvector and eigenvalue. She employed an analytic-arithmetic mode of thinking in describing the concepts of vector (before interacting with the *eigen* sketch) and had difficulties with identifying a vector. Upon dragging x, she, like Kate, attended to the position where x and Ax were perpendicular to each other. But, unlike Kate, her attention was blocked for a few seconds since she thought perpendicularity of x and Ax was important to know in order to complete the task. She used wandering dragging, locus dragging guided dragging and line dragging. Her interactions with the *eigen* sketch suggest that her state of attention included *perceiving geometric properties* of positive and negative eigenvalues. She also articulated that there exist infinitely many eigenvectors associated with one eigenvalue. At the end of the interview, she described her way of finding eigenvectors as she said "I tried to make the vectors lie on top of each other and then find the scaling value". This suggests that Rose started to develop a synthetic-geometric mode of thinking.

7. An extension of the analysis

This chapter provides an extension of the analysis of data that I presented in Chapter 6. It starts with an overview of the interview task and the participants. Then, I discuss the analysis of the participants' interaction with the *eigen* sketch from three main perspectives: 1) a focus on the participants' mathematical understanding of eigenvectors and eigenvalues, 2) their linguistic expressions and 3) the gestures used by the participants in describing a geometric representation of eigenvectors. I have chosen to present the analysis through these perspectives since the participants developed their understanding of the properties of eigenvectors and eigenvalues through interaction with the *eigen* sketch. Also, their linguistic and gestural expression of the concepts reveal emergent ways of communicating about the concepts and often aspects of their thinking.

I draw on the differences between the two representations (symbolic verbal definition and the *eigen* sketch) of the concepts and their effects on the students' modes of thinking. The symbolic verbal definition of the concepts of eigenvector and eigenvalue (as shown in Figure 4) emphasizes the equality $Ax = \lambda x$. The equals sign can be interpreted in terms of the vector equality of Ax and λx (i.e. both length-wise and position-wise) or in terms of collinearity between the two vectors Ax and λx. I argue that the collinear interpretation of the equality sign in $Ax = \lambda x$ requires the use of a synthetic-geometric mode of thinking, whereas the equality interpretation can result from the use of analytic-arithmetic thinking.

7.1. An overview of the participants and the task

The participants were four undergraduate students (Jack, Tom, Kate and Rose) and one graduate student (Mike). Of the five, Mike, Jack, and Kate had completed a linear algebra course at the time of the interview. Jack recalled the concepts of eigenvector and eigenvalue as he read the given definition, but Mike and Kate seemed to have trouble making sense of the given definition. The other two, Rose and Tom, were enrolled in a linear algebra course at the time of the interviews and had already been introduced to the concepts of eigenvector and eigenvalue. Tom recalled the concepts of vector and scalar multiple of a vector and described them. He also recalled studying the concept of eigenvector, but did not seem to know more about it. Rose did not recall studying matrices and vectors in high school. She seemed to recall only a symbolic representation of the concept of vector and did not recall the notion of eigenvector.

The participants were given the versions of the *eigen* sketch (see Figures 8 and 9) and the worksheet (see Appendix A) that included a formal definition of eigenvector and eigenvalue and the task. The task was to find the eigenvectors and eigenvalues of the given four matrices using the *eigen* sketch.

7.2. Participants' mathematical understanding of eigenvectors and eigenvalues

The participants first focused on reading the definition, a symbolic verbal specification of the concepts of eigenvector and eigenvalue that presents them in terms of matrix-vector multiplication, emphasizing the equality $Ax = \lambda x$. Then they read the task and interacted with the *eigen* sketch, which was designed to enable exploration and discovery of the eigenvectors and associated eigenvalues of the first matrix, $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$. This matrix has two sets of eigenvectors; one set has a dilation factor of one ($\lambda = 1$), thus a geometric representation of the task involved finding vectors x such that they are collinear with their image vector (Ax), and x and Ax have the same length. Another set has a dilation factor of two ($\lambda = 2$) and a geometric representation involved finding vectors x such that they are collinear with their image and each image is twice as long as x.

As the participants interacted with the *eigen* sketch, they coordinated the geometric representation on the sketch with the algebraic one from the definition, and explored relationships between the two. Their interaction enabled them to perceive properties of eigenvectors and eigenvalues. I identified three aspects of their work that I would like to discuss in more detail here, both because they shed light on some issues that were raised in the literature review and because they involve particular (and unexpected) differences between the two representations:

1) equity versus collinearity: coordination between the symbolic and geometric representation of eigenvectors;

2) infinitely many eigenvectors associated with each particular eigenvalue;

3) the geometric representation of a set of eigenvectors associated with a negative eigenvalue.

7.2.1. *Equality versus collinearity: coordination between the algebraic and geometric representation of eigenvectors*

The participants tried to coordinate the symbolic representation, $Ax = \lambda x$ and the geometric representation of eigenvectors on the *eigen* sketch. Both representations use the symbol x to indicate a vector, Ax to indicate the image of x, A to indicate the matrix of transformation; but the *eigen* sketch does not show λ explicitly. In coordinating the two representations, only Mike and Jack verbally acknowledged that λ is not on the sketch. Others did not comment on the absence of λ from the *eigen* sketch.

In their exploration of the eigenvectors and eigenvalues of $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, Mike and Kate first identified the set of eigenvectors associated with $\lambda = 1$, whereas Jack, Tom and Rose first identified the set of eigenvectors associated with $\lambda = 2$. It seems that Mike interpreted the equality $Ax = \lambda x$ in terms of equity as he said he wanted to find the "same-length vectors that line up". Kate, like Mike, focused on identifying vectors with the same length and said "these two [points to $Ax = \lambda x$] are the same so Ax is the same as x". Kate also mentioned that "they're [x and Ax] parallel". Their interpretation of the equality $Ax = \lambda x$ in terms of equality (length-wise and position-wise) made them first identify eigenvectors associated with $\lambda = 1$ because the geometric representation of an eignvector associated with $\lambda = 1$ and its image are two equivalent collinear vectors. It is noteworthy that Mike and Kate found positions where x and Ax were collinear for both $\lambda = 1$ and $\lambda = 2$ as they dragged x, but they only articulated eigenvectors associated with $\lambda = 1$. Then, when prompted by the interviewers to find a different eigenvector from the ones they gave before, they found the eigenvectors associated with $\lambda = 2$.

In contrast, for Jack and Tom, the equality $Ax = \lambda x$ was more about the collinearity of x and Ax (rather than equality of the two vectors), because they first identified a set of eigenvectors associated with $\lambda = 2$. Jack and Tom referred to the notion of scalar transformation. Jack said that he needed to "line them [x and Ax] up" because he knew that "the vectors have to be collinear". Tom, like Jack, was aware of the shared geometric relationship between the vectors of a set of eigenvectors because he commented that eigenvectors are "scalar [multiple] with the same basis".

Rose, who initially attended to the angles between x and Ax, focused on the lengths of the two vectors when they became collinear for $\lambda = 2$. She relied on her algebraic strategies and wrote $2x = \lambda x$ right below $Ax = \lambda x$ on the worksheet. But it is hard to say whether she interpreted the equality as collinearity given that she had hard time distinguishing the concept of eigenvector from that of eigenvalue.

Mike's and Kate's first interpretation of the equality sign (from the $Ax = \lambda x$) were related to the general use of equality sign in mathematics, that is to consider the equals sign as equality. Mike and Kate both employed analytic-arithmetic thinking when they first interacted with the *eigen* sketch. Recall that Kate conjectured that "a matrix times a vector should be still a matrix" and Mike tried to make Ax equal to λx (length-wise and position-wise) at the beginning of their interaction. Unlike Mike and Kate, Jack and Tom interpreted the equals sign (from the $Ax = \lambda x$) as collinearity which is more relevant in a linear algebra context. Jack and Tom referred to the notion of linear transformation and employed synthetic-geometric thinking from the beginning of their interaction with the sketch.

Despite their different interpretations, the *eigen* sketch and the dragging tool enabled them to visualize the collinearity of Ax with x (for specific vectors x) and thus to understand the geometric meaning of the equality sign embedded in $Ax = \lambda x$. I discuss the effect of the *eigen* sketch on their modes of thinking in Chapter 8. In connection with the equals sign, Larson et *al*. (2009) showed that students were uncertain about the balance of the equation, $Ax = 2x$, because the left side included the matrix A while the right side included the scalar 2. The participants' interactions with the DGE sketch suggest that the use of the *eigen* sketch could help students to overcome their difficulties in interpreting the equality sign in $Ax = \lambda x$.

7.2.2. Infinitely many eigenvectors associated with each particular eigenvalue

Through their interactions with the *eigen* sketch, in particular with the use of line dragging, the participants became aware of the existence of infinitely many eigenvectors associated with one eigenvalue. All participants except Mike used the term 'infinity' to describe the number of eigenvectors associated with an eigenvalue, and they did not limit themselves to a single example; rather, they provided at least two eigenvectors associated with a particular eigenvalue. Jack, Kate and Rose communicated their exploration of infinitely many eigenvectors associated with one eigenvalue when they were engaged in finding eigenvectors and their associated eigenvalues of the first matrix. Tom talked about the existence of infinitely many eigenvectors when he identified a set of eigenvectors of the second matrix. Mike, like the others, used line dragging to explore the relationship between x and Ax when they were collinear, but he did not comment on the number of eigenvectors and he was not prompted to explain.

Jack provided two examples of eigenvectors associated with one eigenvalue. He used line dragging as he said, "I would deduce that the value of lambda wouldn't change but that there are infinitely many eigenvectors". Kate provided a symbolic notation, $\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix}$ of a set of eigenvectors, and shortly after said "many, infinitely many [...] so if they're on the same line we find lambda is a fixed number". Tom, in his first finding of eigenvectors, provided two examples of eigenvectors associated with one eigenvalue. After that, he provided a basis for each eigenspace and commented that there are "infinitely" many eigenvectors that are "scalar [multiple] with the same basis". Rose also used the term 'infinity', but she was uncertain about it as she said "would not be infinity!". As mentioned above, Mike never explicitly commented on the existence of infinitely many eigenvectors associated with an eigenvalue, but the use of the dragging tool enabled him to find more than one position where x and Ax were collinear and associated with the same eigenvalue. For instance, after finding a position in the first quadrant where x and Ax were collinear and associated with $\lambda = 1$, he dragged x into the third quadrant and found a position where x and Ax were collinear and still associated with $\lambda = 1$ (in working on the first given matrix).

The *eigen* sketch enabled the participants to change the specific numerical values of entries of matrix A to make it correspond to the second matrix, $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$. This matrix has only one set of eigenvectors. The participants' strategies were similar to the strategies they used in exploring eigenvectors of the first matrix. The exploration of eigenvectors of the first and second matrices enabled Jack, Kate and Tom to articulate the span of a set of eigenvectors. They communicated the geometric representation of the span of the set of eigenvectors when they interacted with the geometric representations of eigenvectors of the second matrix. Tom explicitly mentioned the concept of "basis". Jack and Kate both used the term "line" in describing a set of eigenvectors, although the *eigen* sketch does not display a line. Jack said eigenvectors are "all the ones on this line" as he dragged x along its collinear path with Ax far away from the origin and then in toward the origin. Kate, like Jack, stated that eigenvectors lie "on the same line".

As mentioned above, the participants were not explicitly asked to find multiple eigenvectors associated with each eigenvalue. Despite that, all but one of them identified at least two eigenvectors associated with an eigenvalue. In fact, the use of line dragging enabled them to explore the relationship between x and Ax where they were collinear, and thus to develop an awareness of the existence of infinitely many eigenvectors, a fact that is rather hidden in the algebraic procedure for finding eigenvectors of a square matrix based on finding the eigenvalues first.

7.2.3. *Geometric representation of a set of eigenvectors associated with a negative eigenvalue*

The third matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ that the participants were given has two sets of eigenvectors; one has a dilation factor of seven ($\lambda = 7$), and the geometric representation of its set of eigenvectors is similar to the other two matrices (i.e x and Ax are collinear and have the same direction). Another set has a dilation factor of negative four ($\lambda = -4$). The geometric representation of it can be shown by finding a vector x collinear with Ax where the two vectors have exactly opposite directions and Ax is four times longer than x.

The participants first explored the geometric representation of a set of eigenvectors and an associated positive eigenvalue. Their strategies were similar to the strategies that they used in identifying eigenvectors of the first two matrices. Then, they were *all* prompted to find another set of eigenvectors. They dragged x and found a position where x and Ax were collinear and had opposite directions. Mike, Jack and Tom approximated the value of the negative eigenvalue, but Kate and Rose initially thought that the eigenvalue was positive.

Mike found the position where x and Ax were collinear and had opposite directions and called it "an interesting point", re-read the definition and said "now we want to Ax equal to, um, this would work. Because, in this case lambda would be negative four". Jack, like Mike, said "it is still like the opposite eigenvector", re-read the definition and said "it [lambda] would be minus eight". Tom, who carefully attended to the direction of the two vectors, said "this one goes to the opposite direction", used line dragging, and said "eigenvalue is probably negative six".

Kate, like Tom, attended to the direction of the two vectors and stated that "they're opposite to each other". Despite her attention to the direction of the vectors, Kate first thought that the eigenvalue was positive, then she re-attended to the directions, of the two vectors and said "they have opposite directions so the lambda should be negative". Rose, like Kate, commented on the vectors' directions as she said "usually they [x and Ax] go in the same direction but this [Ax] goes [in the] opposite direction". She first thought that the eigenvalue was 4 by attending to the ratio of vectors. Then, she attended to the arithmetic representation of x on the sketch and used a numerical example of vector x and Ax, and thus realized that the eigenvalue was -4.

The last given matrix, $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$, does not have any eigenvectors in the plane or real-number eigenvalues. The participants interacted with the *eigen* sketch trying to find eigenvectors of A, and they realized that the behaviour of two vectors differed from the other three matrices in the sense that x and Ax never became collinear.

The geometric representation of an eigenvector associated with a negative eigenvalue re-directed the participants' attention to a broader interpretation of the collinearity of vectors; that is, two vectors can be collinear but still have opposite directions. Although this representation was new for everyone, they referred to the direction of collinear vectors to coordinate the geometric representation of opposite-directed vectors with the existence of a negative eigenvalue.

7.3. Linguistic expressions used in describing the concepts of eigenvector and eigenvalue

Given that all the participants (except Rose) used linguistic and gestural expressions to communicate their thinking of concepts, I provide a summary of the emerging ways in which the participants communicated their mathematical thinking through linguistic expressions and gestures. I focus on each mode, one at a time, in the sections below and also provide a comparison with the linguistic and gestural expressions that mathematicians used in describing eigenvectors, as described in Sinclair and Gol Tabaghi (2010). I provide this comparison because the students' ways of thinking (in particular Mike's and Jack's thinking) about the eigenvectors was similar to mathematicians' ways of thinking, including both temporal and kinetic aspects.

Mike used the verb "to line up" to describe the geometric representation of eigenvectors. He used the verb "to line up" five times in phrases such as "I am lining these two up" and "I want the x to line up with the Ax". In coordinating the two representations, he said "lambda is what I am multiplying x by, so that it ends up being the same as the Ax". After identifying the first eigenvector, he said "eigenvalue is the scalar multiple applier that made these two things [x and Ax] line up". Upon completing the task, Mike was prompted to explain his way of finding eigenvectors in which he gestured (his gesture is described in the next section) as he talked. He said "at first I was looking for where I can get x and Ax to line up in the same direction [gestures], but then the third one I realized could be in the same direction [gestures] or [gestures] opposite direction". It is interesting that Mike's exploration of the eigenvectors of the third matrix shifted his ways of thinking of negative eigenvalues. It seems that Mike's expressions were being shifted from a physical perception (i.e. use of the verb 'line up' in a sense that vectors are physically one on top of the other) to the mathematical notion of scalar multiple and then again to a physical perception.

Jack used the verb "to line up" and the notions of "collinear", "linear transformation", and "scalar transformation" during the process of completing the task. Jack used the verb "to line up" three times and in his second use of it his speech was accompanied by a gesture. Unlike Mike, Jack referred to the notion of scalar transformation to justify his action of lining up the vectors as he said "there was a scalar transformation, so the vectors have to be collinear". Jack used the term "linear transformation" three times, although the term is not mentioned in the given definition. After completing the task, he was prompted to describe his way of finding eigenvectors in which he said "I tried to make *x* touch *Ax*". Jack, unlike Mike, used mathematical notions at the beginning of his interaction with the *eigen* sketch, but after completing the task he used a more physical description by using the verb "to touch".

Kate used the term "the same" to describe the relationship between x and Ax, as she said "Ax is the same as x" upon finding the first set of eigenvectors of the first matrix ($\lambda = 1$). After finding the set of eigenvectors of the first matrix, she dragged x to explore further the relationship between x and Ax. Then, she stated that "they're [x and Ax] parallel but how about non-parallel?". In exploring the fourth matrix, she commented that it does not have any eigenvectors because she was not able "to put x vector on the same line with vector Ax". In her final explanation to my prompt "what is an eigenvector?", Kate said "[…] if these two vectors can be on the same line so at this time this x is an eigenvector" as she gestured. In her final expression, "vectors […] on the same line", Kate meant to point out the collinear property of eigenvectors. Kate's expression, like Mike and Jack, includes a trace of physicality because vectors can be on the same line.

Tom used the concepts of scalar "multiple", "basis" and "linear transformation" in describing eigenvectors. In exploring the eigenvectors of the first matrix, he described the relationship among eigenvectors using the two phrases: "being multiple" and "scalar [multiple] with the same basis". After completing the task, in response to "what is an eigenvector?", Tom said " they're linear transformations" and "you have a multiple of the same vector". Tom, like Jack, used the term "linear transformation", although it is not included in the given definition. Tom also described his way of finding eigenvectors as he said "when the eigenvector is on top of Ax then that's where they [eigenvectors] exist or the other one going on top of it that exists". Tom used the term "on top of" to describe physically the positions of vectors where eigenvectors exist.

Rose described a vector in terms of its components as she said "you know that in the class we learned like x_1 and y_1". Despite her symbolic description, she was unable to identify a vector numerically since she was attending to the length of the vector rather than to the components of it. As she interacted with the sketch, she became acquainted with the geometric representation. She, unlike the others, did not use any physical language to describe what she was doing while dragging the vectors. However, after completing the task, in describing her way of finding eigenvectors she said "I tried to make the vectors lie on top of each other and then find the scaling value".

At the beginning of the interview, Mike, Kate and Rose did not recall the concepts of eigenvector and eigenvalue; moreover, they had difficulties in making sense of the formal definition of the concepts. By the end of the interview, they had developed an understanding of the behaviour of eigenvectors as evidenced by their use of the verbs "to lie on" and "to line up" to describe eigenvectors. In fact, the expressions suggest ways of communicating the dynamic and kinesthetic imagery of eigenvectors that participants developed through their interaction with the *eigen* sketch. Jack and Tom, unlike the others, used the concept of linear transformation to describe eigenvectors. However, like others, they also evoked dynamic and kinesthetic imagery, as evidenced by their use of the terms “touch” and “line up” in describing the behaviour of eigenvectors.

The use of such expressions suggests the development of temporal and kinetic ways of thinking about eigenvectors which are similar to mathematicians’ ways of thinking. In their study, Sinclair and Gol Tabaghi (2010) identify diverse temporal and kinetic ways of thinking that a group of mathematicians used to describe eigenvectors. The mathematicians neither talked about the equality $Ax = \lambda x$ nor about matrix-vector multiplication. Instead, they used metaphors (such as arrows, clocks, ellipses, resonance, and plates) as well as gestures to describe eigenvectors in terms of special vectors that are mapped into their scalar multiples under a linear transformation. One of the mathematicians, similar to Mike and Jack, used the verb “to line up” and gestured eigenvectors using his index fingers as vectors. Their metaphors and gestures convey a sense of motion and time in their thinking and, in particular, the gestures underline the continuity of motion and time in describing the effect of a matrix on a vector.

7.4. Gestures used by the participants

In the beginning of the interview, all participants used their fingers to point to the representation of concepts on the *eigen* sketch and also to indicate symbols used in the definition. Mike, Jack and Kate even used the mouse pointer (in place of their fingers) several times to indicate the representations of the concepts on the sketch. As they interacted with the *eigen* sketch, all participants, except Rose, produced gestures that emerged from their interaction with the *eigen* sketch.

Mike used his right index finger to trace the equality, $Ax = \lambda x$, on the given worksheet (as shown in Figure 12). He also moved his right index finger along the geometric representation of the vectors u and v on the screen as he was tracing the vectors from tail to tip. In explaining how he went about finding eigenvectors, Mike used his hands to illustrate eigenvectors. He put up his hands extended fingers and placed his right-hand palm upward on his left-hand palm downward (slightly slanted to the right such that his right-hand little finger overlapped with his left index finger) as he said “ x and Ax to line up in the same direction”, shown in Figure 30. He then rotated his hands while keeping his wrists together such that his right- hand extended fingers pointing to the right and his left-hand pointing to the left, as shown in Figure 31, as he said "opposite direction". His hands metaphorically became vectors to gesture the geometric representation of eigenvectors.

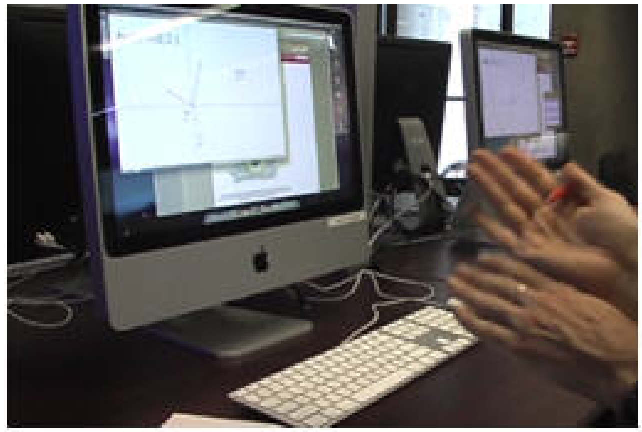


Figure 30. ***(same as Figure 13) Mike's hands point to the same direction.***

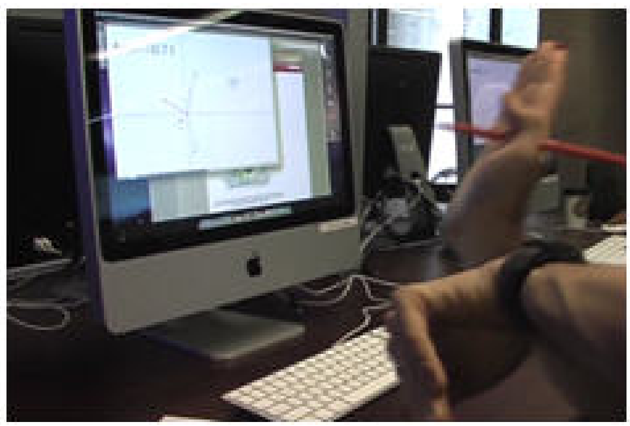


Figure 31. ***(same as Figure 14). Mike's hands point to the opposite directions.***

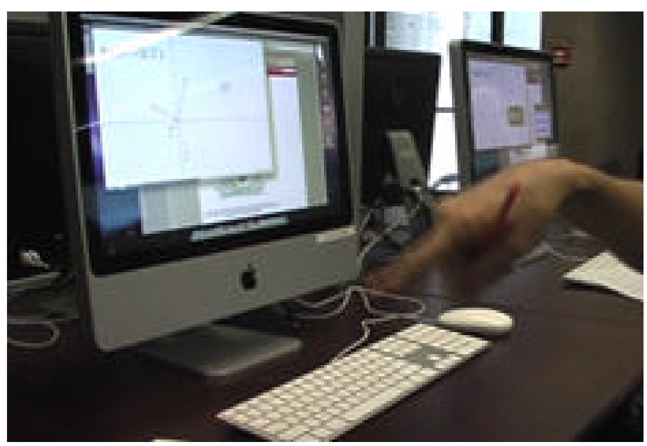


Figure 32. ***(same as Figure 15). Mikes' index finger rotates around a circle.***

He also rotated his right index finger around (like tracing a circle), as shown in Figure 32, while he said, “I would move x around 360 degrees to see if these two cases showed up”. In this case his index finger acted as a vector.

Jack, like Mike, used his hands as vectors. He found the set of eigenvectors of matrix (a) associated with $\lambda = 1$ and then used his hands to represent the two vectors as shown in Figure 33. He put up his hands, moved his right-hand and placed it exactly on his left-hand (as shown in Figure 34) to illustrate the geometric representation of vectors when lambda was one as he said “I am not very used to the Sketchpad and they kind of lined up”.

Figure 33. (same as Figure 18). Jack’s hands are positioned parallel to each other

Figure 34. (same as Figure 19). Jack’s hands are exactly placed on each other

Jack also used his hands after seeing the geometric representation of eigenvectors associated with a negative eigenvalue on the sketch. He said “Oh yeah, I guess it is still like the opposite eigenvector” as he positioned his hands extended fingers in an angular shape but not attached from the wrists as shown in Figure 35. He moved his right-hand toward his left-hand, placed it on the top of his left hand (as shown in Figure 36) and moved it away from his left-hand. He repeated this gesture a few times. Then he said “because of ninety degrees um I’m trying to recall” as he hold his hands in an angular form attached from the wrists for a few seconds as shown in Figure 37. It is hard to say whether Jack used his hands to illustrate opposite vectors (a vector and its dilation by a negative factor) or to depict a vector and its quarter turn rotational transformation (since he mentioned ninety degrees). However, he used his hands as vectors to communicate his mental imagery of eigenvectors that arose through his interaction with the *eigen* sketch.

In response to the prompt “how did you go to find eigenvectors?”, he said “I tried to make x touch Ax” as he dragged x in a spiral fashion beginning far from the origin, turning in an anti-clockwise direction, and ending at the origin. His strategy of dragging can be considered as a new gesture in that he dragged x to find a position where x “touches” Ax. He then said “I guess for the third one we tried to make that happen” as he drew a diagram to illustrate eigenvectors for a negative eigenvalue as shown in Figure 38. Jack not only used his hands as vectors to gesture a geometric representation of eigenvectors, but also sketched a diagram to represent eigenvectors associated with a negative eigenvalue.

***Figure 35.** (same as Figure 20). Jack positions his hands in an angular shape*

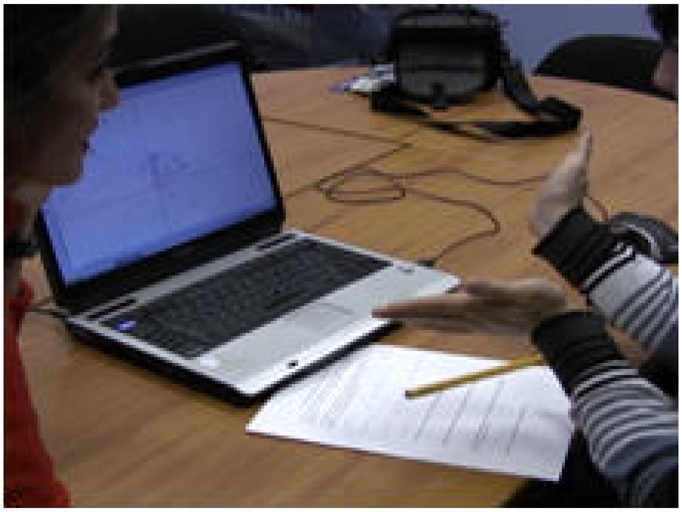


***Figure 36.** (same as Figure 21). Jack brings his hands together.*

***Figure 37.** (same as Figure 22). Jack holds his hands in an angular form for a few seconds.*

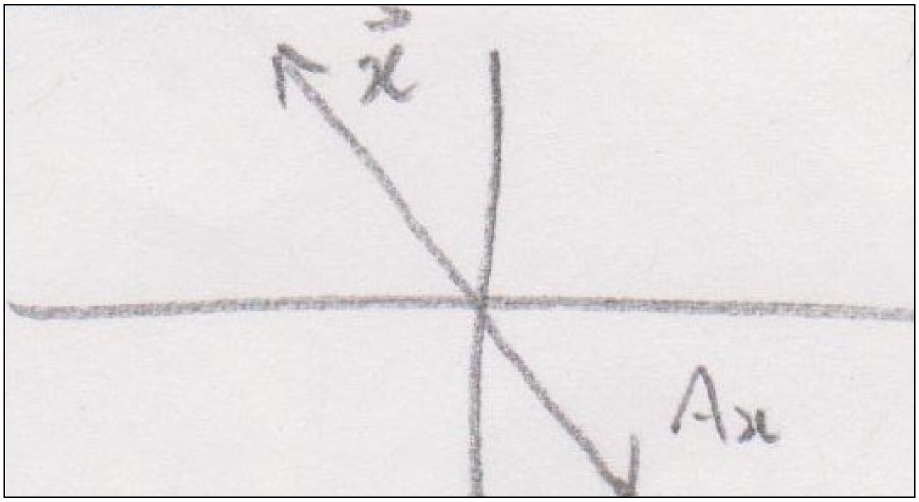


Figure 38. ***(same as Figure 23). Jack's drawing of eigenvectors***

Kate used her right and left index fingers to indicate the lengths of x and Ax in approximating the eigenvalue of the second matrix. Shortly after, she used her right index finger and thumb to indicate (or to measure) the lengths of x and Ax. Kate used her fingers as a measurement unit, in approximating the positive eigenvalue of the third matrix. She first measured the length of x by using the distance between her index finger and thumb. She then used the length of x (the distance between her index finger and thumb) to approximate the length of Ax, thus she found the ratio of two lengths. In her final explanation to my prompt "what is an eigenvector?", Kate said "[…] if these two vectors can be on the same line [gestures] so at this time this x is an eigenvector". She held her left hand extended fingers as shown in Figure 39 and moved it forward and backward as she said "on the same line". Her gesture depicts a span of a set of eigenvectors and does not include the associated eigenvalue.

Figure 39. ***(same as Figure 24) Kate's gesture that describes eigenvectors***

Tom used his right index finger and traced a vector on the desk as he said "I guess a vector is a line" before interacting with the *eigen* sketch. He immediately drew a ray starting from the origin and ending with an arrow in the first quadrant, and then extended it into the third quadrant to illustrate his mental imagery of a vector as shown in Figure 40.

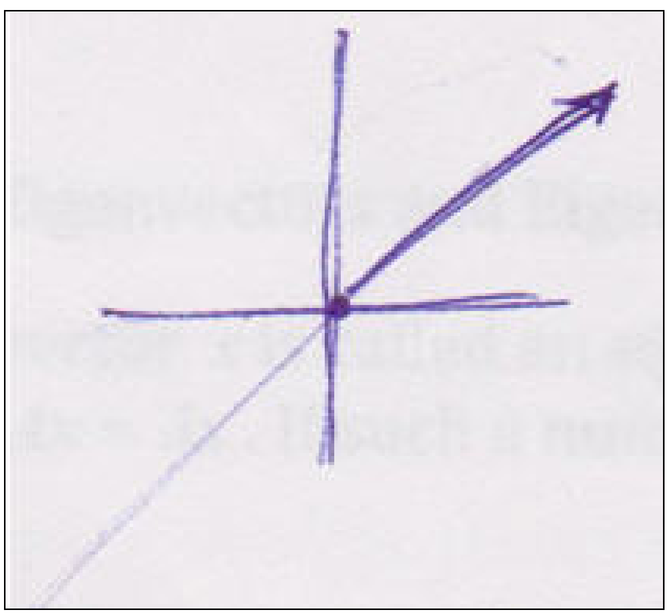


Figure 40. ***(same as Figure26). Tom's representation of a vector on a sheet of paper***

In describing a scalar multiple of a vector, Tom evoked it as a straight line that goes to infinity. He used his hands and arms to kinaesthetically communicate his mental imagery of a scalar multiplication of a vector. He said "it [a scalar multiple of a vector] is just a straight line that goes to infinity" as he moved his right hands extend index finger toward up right corner and his left hands extended index finger and arms down toward the left corner of his body as shown in Figure 41.

Figure 41. ***(same as Figure 27) Tom gestures as describes a scalar multiple of a vector***

During his interaction with the eigen sketch, in describing the relationship between x and Ax, Tom used the verb "to rotate" as he moved his right index finger around a circular path to represent the behaviour of Ax. He first rotated his right index finger around a circular path and then he moved his right hand extended index finger back and forth along a path as he was tracing a straight path (as shown in Figure 42) to communicate his mental imagery of finding a set of eigenvectors. This gesture was given rise from his interaction with the sketch.

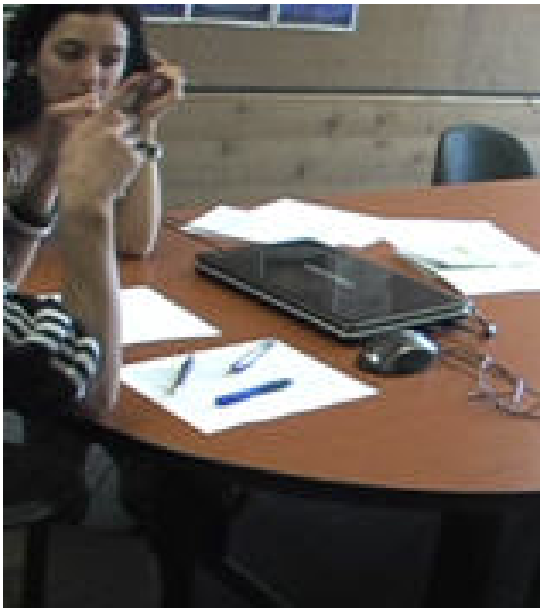


***Figure 42.** (same as Figure 28). Tom rotates and moves his index finger as describes eigenvectors*

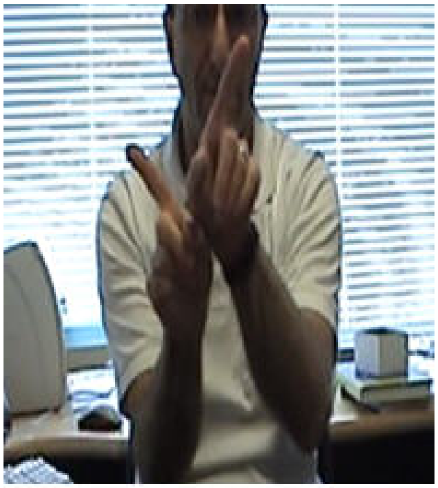
As mentioned above, Mike and Jack used their hands as vectors to gesture geometric representation of eigenvectors associated with a positive or a negative eigenvalue. In addition to using their hands, they also developed new dragging strategies that can be considered as gestures. Tom, unlike Jack and Mike, used his right index finger as a vector and Kate used her hand to gesture the span of a set of eigenvectors.

The participants' use of hands and fingers in describing eigenvectors that arose from their interaction with the *eigen* sketch offer evidence of a time- and motion-based conceptualization of the concept of eigenvectors, as was the case in mathematicians' use of gestures. Sinclair and Gol Tabaghi (2010) included a mathematician's gestures whose conceptualization of eigenvectors was affected by his use of *Eigenizer* tool (see Figure 2). The mathematician, JJ, used his index fingers to gesture a vector and its image under a linear transformation as he described eigenvectors. He gestured and said:

> [puts up his hands extended index fingers, see Figure 43a] you go up here and [rotates his hands, extends right index fingers, see Figure 43b] sort of move around as you play with this. [right and left index fingers coming towards each other, see Figure 43c] If you set the matrix up by some inputs they're gonna come inside and then obviously you say

[opens up his hands, see Figure 43d] what is the important direction [right and left index fingers are placed on each other, see Figure 43e] when the two line up.

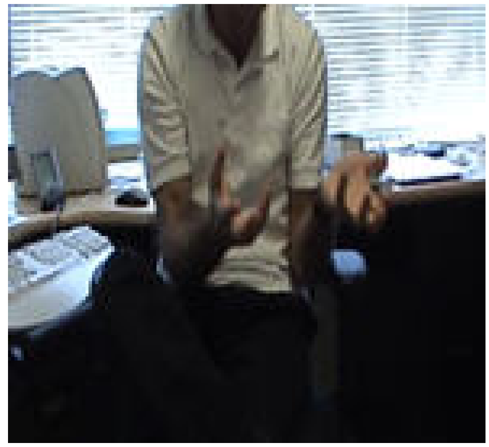
a) index fingers are vectors



b) moves vectors around

c) vectors are getting toward each other

d) opens up his hands



e) vectors line up

Figure 43. Gestures accompanying JJ's description of eigenvectors.

JJ, similar to Mike and Jack, used the verb "to line up" as he described the process of finding eigenvectors. JJ's used his index fingers as vectors, whereas Mike and Jack used their hands as vectors. Despite their distinct ways of gesturing a vector, their gesticulation is similar in the sense that they moved their hands and fingers to make them line up. This suggests that the dynamic interactive representation of eigenvectors has given them a way of moving their body (or dragging vector on the sketch) to gesture the process of finding eigenvectors and also the invariant collinear property of eigenvectors.

It is worth pointing out that Mike's and Jack's verbal utterances in describing the invariance property of eigenvectors (i.e. line up, same direction or opposite direction) and their gestures occurred together. Their speech described an action performed in relation to finding eigenvectors, and their gestures showed the collinear property of eigenvectors. Both gestures and speech conveyed the same semantic intent thus were coequally generated according to psycholinguistic theories (Quek et *al*., 2002). In contrary to Mike's and Jack's discourse, JJ's gestures occurred before his speech. He moved and rotated his index fingers before he said "sort of moving around". His index fingers lined up before he said "the two line up".

It is not easy to say why JJ's gestures occurred before his speech whereas Mike's and Jack's gestures happened at the same time as their verbal expressions in describing the invariance properties of eigenvectors. One possibility is that JJ gestured the process of finding eigenvectors by moving his fingers (as vectors) in many directions. Thus he drew his attention first to his fingers' movement rather than to his speech. But, Mike and Jack gestured the collinear property of eigenvectors and did not move their hands in any directions apart from making them collinear. Thus, their gestures and linguistic expressions were coequally generated. Another possibility is the importance of dynamic and kinaesthetic imagery in JJ's thinking about eigenvectors that enabled him to construct his imagery of eigenvectors before he communicated their properties linguistically.

The results suggest that analyzing gestural and linguistic expressions of the participants reveals the emerging ways of communicating the concepts of eigenvector and eigenvalue inspired through interaction with the *eigen* sketch. The gestural expressions also relate to the participants' understanding of the concepts. For examples, the participants' gesticulation of the collinear property of eigenvectors reveals their understanding of properties of the concept of eigenvectors. Prior research on the effect of gesture on learning suggests that gesture can play a causal role in learning (Cook, Mitchell and Goldin-Meadow, 2008). These authors suggest that gesture analysis, in addition to linguistic expressions, could inform educators better about students' understanding of mathematical concepts. The gesturing also reveals evidence of temporal and kinetic dimensions of thinking that are discussed in Chapter 8.

7.5. Summary

In this chapter, I described the participants' understanding of the concepts of eigenvector and eigenvalue. The *eigen* sketch representation enabled the participants to explore the collinearity of Ax with λx (when x is an eigenvector). As a result, they extended their interpretation of the equals sign embedded in $Ax = \lambda x$ from equality between Ax and λx to collinearity between the two. By exploring the collinear property of eigenvectors through the use of dragging tool, they developed an awareness of the existence of infinitely many eigenvectors associated with a fixed eigenvalue. Moreover, the geometric representation of an eigenvector associated with a negative eigenvalue re-directed the participants' attention to a broader interpretation of collinearity of vectors that is when vectors are collinear but have opposite directions.

Given the importance of linguistic and gestural expressions in communicating mental imagery, in sections 7.3 and 7.4, I provided a summary of the participants' expressions. In their linguistic description of the behaviour of eigenvectors they used verbs such as "to lie on", "to line up", "to be on", and "to touch" to communicate their dynamic and kinesthetic imageries that were stimulated through interacting with the *eigen* sketch. In some cases, their linguistic expressions were accompanied by gestural expressions. In particular, four (out of five) participants used their hands or fingers as vectors to gesture the geometric representation of eigenvectors. My comparison of the participants' linguistic and gestural expressions with the ones of mathematicians (as described by Sinclair and Gol Tabaghi, 2010) led me to conclude that my participants' ways of thinking became similar to mathematicians' ways of thinking about eigenvectors both temporally and kinetically. The participants' use of hands and fingers in describing eigenvectors (that were given rise from their interaction with the *eigen* sketch) offer evidence of a time- and motion-based conceptualization of the concept of eigenvectors. I discuss further the effect of the *eigen* sketch on their modes of thinking in Chapter 8.

8. Research contributions, limitations and implications

In this final chapter, I discuss my research contribution to the literature drawing both on the results of this study and on the theories discussed in Chapters 2 and 4. I then provide answers to the research questions posed in Chapter 5. At the end, I discuss the limitations of my study and some of its pedagogical implications.

8.1. Modes of students' thinking in understanding the concepts of eigenvectors and eigenvalues

In Chapter 2, I reviewed prior studies' findings on possible sources of students' difficulties in learning linear algebra. Given claims for the existence of three modes of description and their associated representational systems, Sierpinska (2000) argues for the development of different modes of mathematical thinking: synthetic-geometric, analytic-arithmetic and analytic-structural. My study has shed light on aspects of the students' thinking as they interacted with the dynamic geometric representation of the concepts of eigenvector and eigenvalue.

Similar to students' modes of thinking, a study of mathematicians' ways of thinking about mathematical concepts and ideas in general—not specified to linear algebra—suggests the existence of different modes of thinking (see Burton, 2004). Yet, evidence offered by mathematicians such as Thurston (1994) reveals more about the aspects of mathematicians' thinking as opposed to a classification of different modes of thinking. *Process and time* and *vision, spatial sense, kinesthetic (motion) sense* are two of the six major facilities of mind whose importance Thurston highlights in the mathematical thinking of a mathematician. More recent research that studies the way in which mathematicians think about mathematical concepts and ideas reveals the role of time and motion in mathematicians' description of mathematical concepts, and in particular the concept of eigenvector (see Sinclair and Gol Tabaghi, 2010). These findings challenge the lack of attention to (and sometimes ignorance of) the role of time and motion in mathematical thinking. In particular, it draws attention to the aspects of students' thinking about mathematical concepts.

Turning to my study, I used Sierpinska's classification of the modes of thinking in linear algebra to classify the participants' ways of thinking as they interacted with the *eigen* sketch. I found that the participants mostly integrated the synthetic-geometric mode of thinking with the analytic-arithmetic one. One reason for this is that they were given the *eigen* sketch and the symbolic verbal definition of the concepts. Thus, they needed to coordinate the geometric representation of vectors from the *eigen* sketch with the given symbolic verbal definition of the concepts. Another reason is the design of the *eigen* sketch itself, which includes an arithmetic representation of Ax (i.e. matrix-vector multiplication) and a geometric representation of the vectors, x and Ax. The sketch enabled the participants to notice numerical changes that occurred as they changed the vector x by dragging it on the screen. My design of the *eigen* sketch that was inspired by the literature enabled the participants to develop cognitive flexibility among modes of thinking, modes of description and representations. My findings suggest that the use of the dynamic geometric diagram (that includes both the geometric and arithmetic representations of the concepts) could enable the development of synthetic-geometric and analytic-arithmetic modes of thinking and the ability to move flexibly between them.

Nevertheless, given the powerful role of geometry in the development of mathematical ideas and the genesis of the concept of eigenvector in studying discrete mechanical systems[6], the participants mostly relied on a synthetic-geometric mode of thinking more than on other modes. In fact, they perceived an eigenvector as a special

[6] According to Hawkins (1975), the concept of eigenvector first arose in the 18th century describing the motion of a string fixed at one end and swing at the other end. In the latter part of the 18th century, D'Alembert solved the string problem by defining a system of differential equations with constant coefficients describing horizontal and vertical displacement of the string. He described the roots of these equations in terms of the stability of the mechanical system under consideration.

vector that was collinear with its scalar multiple and the associated eigenvalue as the ratio of the lengths of the special vector and its scalar multiple (where the vectors had the same direction). The use of the dragging tool and the dynamism of the representation affected their modes of thinking as evidenced by their gestures and speech. Drawing on my findings, below I respond to the first research question.

What is the effect of dynamic geometric representations of eigenvectors on a student's modes of thinking? My study reveals that interacting with the *eigen* sketch enabled the participants to develop dynamic and kinesthetic images of the concepts. They mostly used a synthetic-geometric mode of thinking, but more importantly, their thinking involved Thurston's facilities of *process and time* and *vision, spatial sense, kinesthetic (motion) sense.* These facilities enabled them to communicate dynamic and kinesthetic imagery using embodied expressions and gestures. Their use of gestures enabled them to make explicit (at least to me, the researcher) the "implicit dynamism of thinking" (Leung, 2008).

I thus argue that dynamic geometric representations of eigenvectors enabled the participant to develop *dynamic-synthetic-geometric* thinking. This mode of thinking comprises kinesthetic and dynamic imagery, thus enabling one to reconstruct mentally objects, to impose motion on them and to position them in space. This affirms my conjecture, from chapter 4, that the dynamic geometric representation of eigenvectors stimulated the formation of kinesthetic and dynamic imagery.

The development of dynamic-synthetic-geometric thinking might anticipate the development of structural ways of thinking in linear algebra since they share common features. One feature is the *independence from the coordinate system* and another is that both ways of thinking are *based on properties of objects not calculations*. These features are the same ones identified by Sierpinska (2000) in discussing the development of synthetic-geometric and structural ways of thinking. But, they have become more apparent when I take into account the participants' use of the facilities of *process and time* and *vision, spatial sense, kinesthetic (motion) sense.* In fact, the participants' embodied expressions in communicating the collinear property of eigenvectors—without providing a specific numerical example—might enable the development of analytic-structural thinking that is thinking of an object in terms of its properties.

8.2. Dynamic geometric representations as diagrammatic experiments: the role of time and motion

In Chapter 7, I described the participants' understanding of the behaviour of eigenvectors, focusing on their understanding of (1) the equality $Ax = \lambda x$ as vector equality or collinearity between Ax and λx, (2) the existence of infinitely many eigenvectors associated with a fixed eigenvalue, and (3) the geometric representation of a set of eigenvectors associated with a negative eigenvalue. The participants' use of verbs such as "to be on", "to lie on", "to line up" and "to touch" in describing the behaviour of eigenvectors evidence their ways of communicating their dynamic and kinesthetic imagery of eigenvectors. It is also evident from the data that dynamic geometric representations gave rise to gestures and other embodied expressions. The participants' use of hands and fingers as vectors in describing eigenvectors that arose from their interaction with the *eigen* sketch offers evidence of time- and motion-based conceptualization of the concept of eigenvectors. This suggests that their aspects of thinking have become mobile and temporal, similar to the mathematicians' ways of thinking about mathematical concepts and ideas (see Núñez, 2006; Sinclair & Gol Tabaghi, 2010).

In addition to the development of aspects of thinking, the *eigen* sketch exploration of eigenvectors and eigenvalues enabled the participants to understand the concepts of eigenvectors and eigenvalues by identifying the invariant properties of them. In contrast to this, as I discussed in Chapter 2, prior research shows that many students automatically set up the characteristic equation to find its roots regardless of whether this is necessary or useful in solving the given problem (Sierpinska, Dreyfus and Hillel, 1999). This is because students are mostly introduced to a procedural algebraic method for finding eigenvalues and, subsequently, their associated eigenvectors. The method requires them first to find eigenvalues (by finding the roots of the characteristic equation ($\det(A - \lambda I) = 0$)) and then to find the associated eigenvectors (by finding non-trivial solutions for $(A - \lambda I)x = 0$ given λ). This method leads students to the development of procedural knowledge for finding eigenvalues and eigenvectors. It does not reveal that they are identifying a special vector that is being transformed into its scalar multiple under a given linear transformation. It neither draws attention to the invariant collinearity of eigenvectors nor reveals a geometric interpretation of a set of eigenvectors.

In my study, I did not ask the participants to find eigenvectors algebraically nor make a comparison between a textbook representation and dynamic representation of a linear algebra concept. But, I can infer my response to the second question by looking specifically at how the participants managed to coordinate the formal definition of the concepts of eigenvector and eigenvalue with the *eigen* sketch representation of the concepts. Below is my response to the second question.

How do students relate these representations to the more symbolic and static ones that are found in undergraduate textbooks? My study shows ways in which the participants coordinated the *eigen* sketch representation of the objects and concepts with the given formal definition of eigenvector and eigenvalue. They matched the symbols on the sketch with the ones used in the definition. Then they dragged the vector x to find a position where the image of x under matrix A (i.e. the vector Ax) was collinear with a scalar transformation of the vector x (i.e. λx). As a result, they first identified an eigenvector and then they drew attention to the scaling factor of x that was the associated eigenvalue. More importantly, the *eigen* sketch enabled them to experiment with the behaviour of eigenvectors in order to identify the invariant properties of eigenvectors and eigenvalues. In fact, the dragging tool and the dynamism of the representation offered a diagrammatic experiment where time and motion are inherited characteristics of such diagrams.

In contrast to dynamic diagrams, the absence of time and motion in the static diagrammatic representation of eigenvectors that was shown in Figure 44 makes it difficult to understand the construction of the diagram. In fact, the diagram illustrates the final result of two different matrix-vector multiplications, Av and Au. A student must find the eigenvalue and associate eigenvector (v) (presumably using the procedural algebraic method) before sketching v and Av, whereas u is an arbitrary vector not an eigenvector of A. Further, the use of curved dashed arrows may seem unnecessary for the student. This static diagram (that was discussed in detail in Chapter 1) neither reveals the process of finding the eigenvector v nor does it draw attention either to the invariance property of v (i.e. its collinearity with Av) or to the existence of infinitely many eigenvectors collinear with v. It, in contrast to the *eigen* sketch, neither stimulates the formation of dynamic and kinaesthetic imagery nor enables students to communicate the mobile and temporal aspects of mathematics.

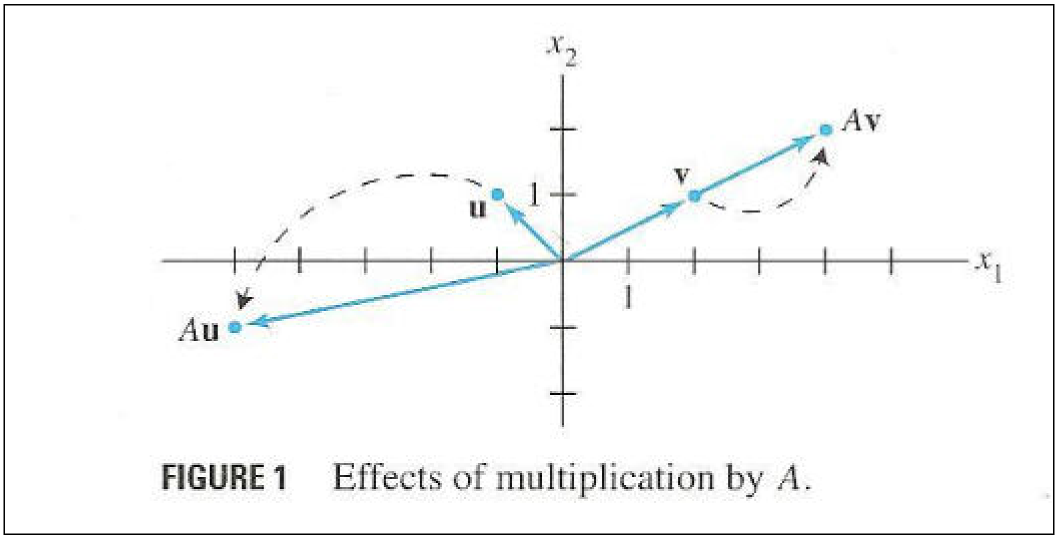


Figure 44. (same as Figure 3) Geometric representation of eigenvectors (Lay, 2006, p. 303)

In connection with the use of the *eigen* sketch as diagrammatic experiment, Châtelet's (2000) historical investigation of diagrams shows that they have played a central role in the development of new mathematical ideas. These diagrams do not simply illustrate or translate an already available content; they invent new spaces and new ways of conceptualizing that emerge from the mobile, material acts of experimenting on the page. For Châtelet, the diagram is a mid-station between the embodied gesture and the more formal mathematics; he writes that diagrams "can transfix a gesture, bring it to rest, long before it curls into a sign" (p. 10). For Châtelet, the gesture is an impulse in the sense that "one is infused with the gesture before knowing it" (p. 10).

It is helpful to think about how Châtelet might have seen the diagram in Figure 44 in light of his theory. The curved dashed arrows reflect the dynasties of cutting-out gestures. Those gestures are meant to indicate linear transformation (or mapping); one suggests dilation and the other a rotation although they both have a similar construct. The diagram captures or transfixes the cutting-out gestures, thus creating a new fold on the surface. The diagram is born with these cutting-out gestures. No longer are the tips of the curved dashed arrows cleaving to the points Av or Au; the arrows embody the effort of abstraction (that is, the linear transformation of v and u under matrix A) by participating in the concrete process of constituting a system of linear transformation.

Châtelet's philosophical view has given rise to a more recent hypothesis that working systematically with dynamic imagery could increase students' material interaction (see de Freitas and Sinclair, 2011). This is in contrast to the static and confining aspects of textbook diagrams that take away students' inventive acts by which mathematics can be grasped through using gestures and embodiment.

Drawing on Châtelet's (2000) thesis about diagrams as thought experiments and my analysis of the participants' interaction with the *eigen* sketch, I agree with de Freitas and Sinclair that interacting with dynamic geometric representations could increase one's material interaction and consequently could enable one to invent new spaces and new ways of conceptualizing that emerge from the mobile, material acts of experimenting on the page.

In discussing my response to the first and second research questions, I mostly drew on the theories of cognition, dynamism in thinking and embodied cognition. However, I could not understand the development of aspects of the participants' thinking without analysing their use of dragging tool and the effect of the use of different dragging modalities and strategies on shifts in the structure of their attention. Triangulating a participant's dragging modalities, shifts in her attention, and her use of imagery did provide me with a richer understanding of her learning and thinking process, as I conjectured in Chapter 1. To respond to my last research question, I drew on the complementary use of theories of instrumental genesis, dragging modalities and shifts of attention in framing the study.

8.3. Complementary use of the theory of instrumental genesis and the theory of shifts of attention

Based on the theory of instrumental genesis, I identified evidence of both instrumentation and instrumentalization processes in the participants' use of the dragging tool. This evidence was mostly concerned with processes involved in transforming a tool into an instrument, rather than the role of the instrument itself in knowledge acquisition. As researchers have pointed out, the role of the instrument in cognitive development is a delicate point (e.g.Verillon and Rabardel, 1995), and the theory of instrumental genesis has shortcomings in putting forward the potentialities of the instrument in the development of mathematical thinking. For example, consider the participants' initial use of line dragging. The use of line dragging is evidence of an instrumentalization process since all participants dragged x along its path collinear with Ax. It is also evidence of an instrumentation process because it enabled the participants to identify the existence of infinitely many eigenvectors.

But what can be said about the participants' knowledge acquisition? Mason's theory of shifts of attention provides some assistance in answering this question by revealing the developmental process of the participants' mathematical understanding as they interacted with the *eigen* sketch. Indeed, Figure 45 shows how the use of different dragging modalities and the different states of attending can be brought together in analyzing the participants' initial interaction with the *eigen* sketch. In particular, it shows shifts in the locus and the available state of attention as a participant interacts with the *eigen* sketch using the dragging tool. In general, as I hypothesized in Chapter 4, providing the participants with the *eigen* sketch caused shifts in the structure of their attention and, consequently, enabled them to identify the invariants of the concepts of eigenvector and eigenvalue. These findings enabled me to provide my response to the third research question here.

What can the complementary use of the theory of instrumental genesis and the theory of shifts of attention offer in regard to analyzing a participant's interaction with the eigen sketch? I argue that the complementary use of these theories provides an understanding of the participants' development of mathematical thinking in a DGE-based task. As discussed above, Figure 45 illustrates an instance of the complementary use of the theories. More specifically, observing how the participants used the dragging tool while they were engaged in the interview task, I noticed that some participants dragged the vector x with the intention of making it collinear with Ax. I called this dragging modality *intentional dragging*. It is to drag a point with the intention of producing a certain configuration. Intentional dragging differs from guided dragging as it is used to produce a certain configuration that has been identified beforehand, whereas guided dragging, dragging that is more exploratory, enables one to locate a particular configuration.

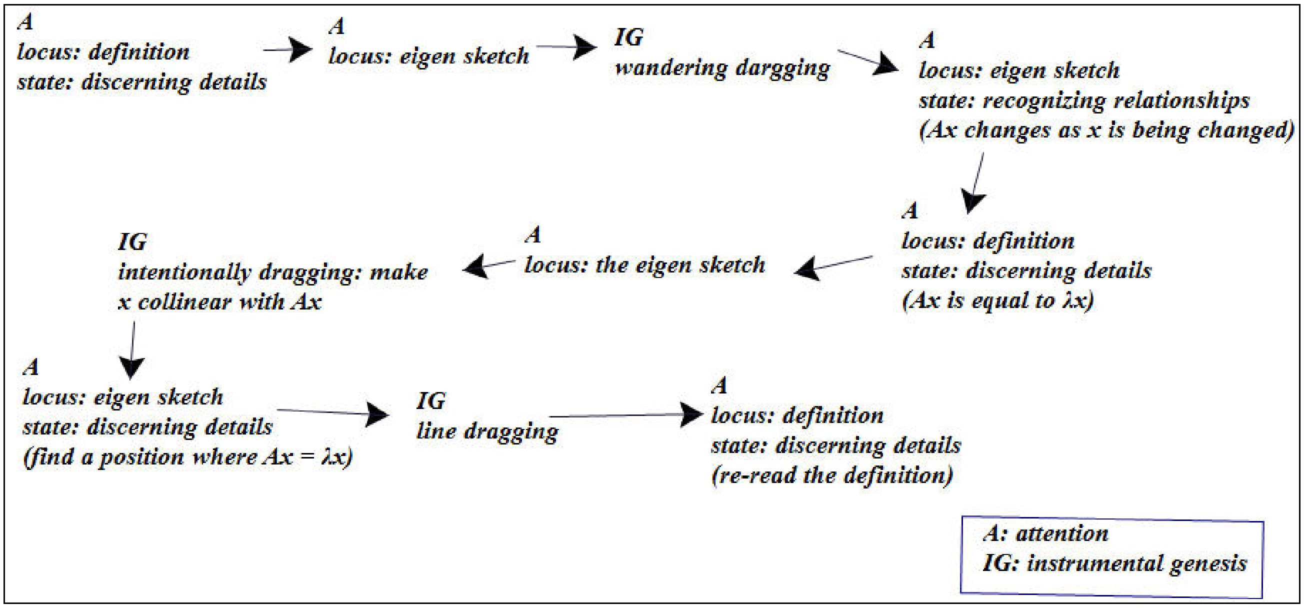


Figure 45. ***An illustration of the complementary use of the theories of instrumental genesis and shifts of attention in analysing an initial interaction of a participant with the*** **eigen** ***sketch to find a set of eigenvectors***

As evidenced by my analysis of the participants' interaction with the *eigen* sketch (in Chapter 6), I drew on the multiple theoretical frameworks in order to study the effect of the dynamic geometric representation on students' understanding of eigenvectors and eigenvalues. Given that the participants interacted with the dragging tool, I first analyzed the data using the theory of instrumental genesis. But my intention was to provide insight into the participants' understanding of the concepts. Mason's theory of shifts of attention enabled me to analyze the participants' structures of attention and the ways of attending as they used the dragging tool to interact with the *eigen* sketch. I found that a participant's use of different dragging modalities can provide easily-visible evidence of shifts in the structure of her attention and consequently can reveal her understanding of the concepts.

8.4. Research limitations

While my research enabled me to answer my research questions effectively, I identified a few limitations of my study in terms of the participants' first introduction to the concepts and their use of gestures during or after their interaction with the *eigen* sketch.

The design of the *eigen* sketch itself has a few limitations in terms of representing collinearity of two vectors and non-exact numerical values for components of eigenvectors.

8.4.1. The participants and their use of gestures

The participants of my study were all introduced to the concepts of eigenvector and eigenvalue through a static algebraic approach in their linear algebra course. This approach frames the concept of eigenvector in terms of matrix-vector multiplication, emphasizing the equality $Ax = \lambda x$. As I discussed in Chapter 2, it leads to a procedural algebraic method for finding eigenvalues and then the associated eigenvectors of a square matrix. None of the participants mentioned seeing a geometric representation of the concepts in their linear algebra course. The result of my study could have been different if the participants have seen a geometric representation of the concepts of eigenvectors and eigenvalue before.

The dynamic geometric approach affected their emerging ways of communicating the concepts of eigenvector and eigenvalue through gestures. As I described in Chapter 7, three participants used their hands or fingers as vectors to gesture a geometric representation of eigenvectors and one participant used her hand to gesture the span of a set of eigenvectors. These gestures that arose from their interaction with the *eigen* sketch offer evidence of a time- and motion-based conceptualization of the concept of eigenvector. But they are not perfect models depicting eigenvectors because eigenvalue either is equal to one or is not in gestures since gestures are limited by what hands, arms and fingers can do.

8.4.2. *The* eigen *sketch: approximation versus exactness*

The use of the *eigen* sketch in finding eigenvectors and eigenvalues of square matrices has limitations because the geometric representation of collinearity of the two vectors is not error-free and the arithmetic representation of eigenvectors is not exact.

The geometric representation of collinearity of the two vectors on the *eigen* sketch created some difficulties for the participants in identifying eigenvectors and their associated eigenvalues of matrix (b) from the interview task. First, the two vectors, on the sketch, almost overlapped for a scalar ranging from 2.7 to 3.2 (on a square grid scaled 2 centimetres per unit). This caused one participant to have difficulty in finding the exact value of the eigenvalue even though he activated the Measure command and the Calculator tool. Second, another participant found the eigenvector $\vec{x} = \begin{bmatrix} 3.18 \\ 2.32 \end{bmatrix}$ and then she generalized it into $\vec{x} = \begin{bmatrix} 3.18 \\ 2.32 \end{bmatrix} = \begin{bmatrix} 1.5a \\ a \end{bmatrix}$. As she dragged x along the line (where x and Ax overlapped) far away from the origin, her approximated ratio between the coordinates of the vector x (i.e. $x_1 / x_2 = 1.5$) did not hold. In this case, the two vectors are collinear when the ratio of the coordinates of vectors is equal to one (i.e. $x_1 / x_2 = 1$). But, the *eigen* sketch, for the transformation matrix (b), suggests that the two vectors are collinear when x_1 / x_2 ranges from 0.7 to 1.3 (on a square grid scaled 2 centimetres per unit).

The arithmetic representation of eigenvectors on the *eigen* sketch is not exact. The sketch enables one to read components of specific examples of eigenvectors to a certain precision that can be set up in advance. Through the use of line dragging, one could explore the existence of infinitely many eigenvectors associated with a fixed eigenvalue and then could generalize the findings by approximating the ratio of x_1- and x_2-coordinates of an explored eigenvector (i.e. $\vec{x} = \begin{bmatrix} 3.18 \\ 2.32 \end{bmatrix} = \begin{bmatrix} 1.5a \\ a \end{bmatrix}$). This would yield an approximation for a basis of an eigenspace, whereas the procedural algebraic method allows one to find exact basis of an eigenspace.

Approximation is a feature of a dynamic geometry environment as I identified evidence of its use in the participants' interaction with the *eigen* sketch. It is decidedly not a feature of the concepts themselves.

8.5. Pedagogical implications and suggestions

Drawing on my findings, I recommend the integration of dynamic geometric representations of concepts in teaching elementary linear algebra. These representations would enable students to build their own, more geometric, understanding of the concepts. Through interaction with dynamic representations, students could produce a greater coordination between symbolic and static representations that are mostly found in textbooks and dynamic ones. This would help students to develop cognitive flexibility among modes of description and representations, as well as among the different modes of thinking. As my study findings suggest, interaction with dynamic geometric representations of concepts could enable students to develop dynamic-synthetic-geometric mode of thinking.

The integration of dynamic geometric diagram of concepts could change teaching and learning approaches in a way to enable students to explore concepts and to build their own knowledge and to facilitate reasoning on the basis of properties, as well as perceiving properties as being instantiated in some situation. The participants of my study who were introduced into the concepts through a static algebraic approach but they did not recall the concepts. It seems to me that a balance between the two approaches —dynamic geometric and static algebraic— would enable students to overcome their learning difficulties. The question is which of the approaches could or should be used first to introduce students to linear algebra ideas and concepts. In this study, as I mentioned above, my participants were introduced to linear algebra concepts first through a static algebraic approach in a course and then they interacted with the *eigen* sketch. In finding eigenvectors and eigenvalues of matrix (a), they first attended to finding the value of an eigenvalue and then read off the components of an associated eigenvector from the *eigen* sketch. But, the *eigen* sketch suggests first finding eigenvectors and then one could approximate an associated eigenvalue with a set of eigenvectors. In other words, eigenvectors are visible but associated eigenvalues need to be approximated which might encourage using an algebraic method. The participants' strategy of finding eigenvalues before eigenvectors could be related to their introduction to the concepts through static algebraic approach in the sense that it mostly requires them to calculate or approximate numerical values.

I suggest the use of the *eigen* sketch (or similar) to introduce students first into the geometric representation of the concepts. This would require the design of suitable assessment items to draw students' attention to the properties of the concepts rather than asking them to find exact numerical values of eigenvectors and associated eigenavlues. Referring to my discussion of a linear algebra textbook's representations of concepts in Chapter 2, and to the participants' difficulties in recalling the concepts even though they learned them before, I affirm that linear algebra textbooks were partly to blame for students' lack of geometric intuition and their reliance on procedural knowledge. The integration of dynamic geometric representations of concepts could potentially fulfill Harel's (2000) concreteness principle, in the sense that these representations could provide concrete contexts for abstract notions. These representations could also fulfill Harel's (2000) generalizability principle, in that through the interaction with them, a student could identify the invariants in reference to arithmetic and geometric representations of a concept and may come to generalize the concept in higher dimensions.

The dynamic and interactive features of DGEs, enhanced by the dragging tool, enable a student to explore the relationships among objects presented in a sketch. The sketch represents the relationships and behaviour over time. In fact, the dragging tool and the dynamism of the representation offer a diagrammatic experiment (as it was the case in the *eigen* sketch) where time and motion are inherited characteristics of such diagrams. Recent versions of CAS software, such as *Maple* and *Mathematica*, are also enhanced by dynamic and interactive features, and may be able to provide students with opportunities to coordinate the geometric and symbolic aspects of eigenvalues and eigenvectors as they did in this study. However, the different modes of interaction afforded in these CAS programs may affect the instrumentation process and resulting shifts of attention that students experienced in my study.

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Appendices

Appendix A. The interview task

Let A be an $n \times n$ matrix. A nonzero vector x is called an *eigenvector* of A if and only if there exist a number λ such that $Ax = \lambda x$. If such a number λ exists, it is called an *eigenvalue* of A.

Given a sketch that represents matrix A and an arbitrary vector x. Double click on entries of A to change their values to givens below, then drag x to find eigenvector(s) and associated eigenvalues(s), if exist.

$a)\quad A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

$b)\quad A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

$c)\quad A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$d)\quad A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

Appendix B. Transcripts of the interviews

In this section, I included transcripts of the interviews and describe the participants' interaction with the sketches. I used the four main segments in describing each participant's interactions: (1) introduction to eigenvectors and eigenvalues, (2) recognition of the relationship among eigenvectors associated with an eigenvalue, (3) geometric representation of the negative eigenvalue of matrix (c), and (4) post-description of eigenvectors.

Mike

Mike was a second-year university student pursuing a Master of Science degree in secondary mathematics education. He had completed a linear algebra course during his bachelor's degree, but said that he could not recall the concepts of eigenvector and eigenvalue at the beginning of the interview. Mike was relatively familiar with *The Geometer's Sketchpad*. Mike interacted with the first design of the *eigen* sketch.

M.1 First Segment: introduction to eigenvectors and eigenvalues

Mike was not asked to describe the concept of vector or eigenvector at the beginning of the interview. Instead, he was given the worksheet and the *eigen* sketch. Mike gazed at the formal definition of eigenvector on the sheet, and he then started dragging the vector x slowly around its given position (fourth quadrant of the coordinate system). He noticed that dragging the vector x also changed the position of vector Ax on the screen. As he continued dragging, x overlapped with the x-axis and Ax overlapped with the vector *u*, as shown in Figure B1.

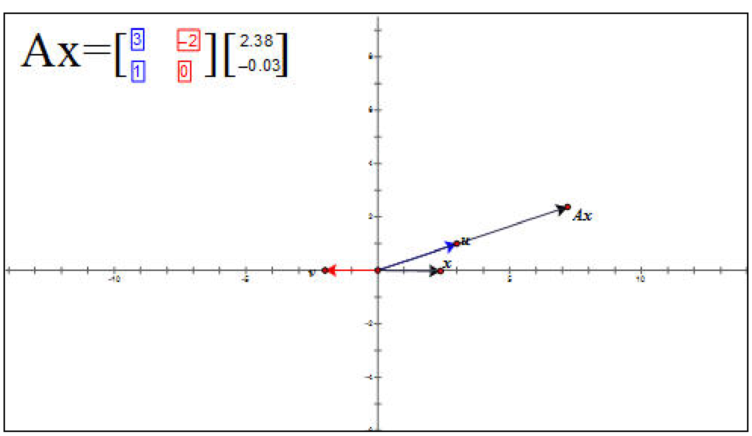

Figure B1. A snapshot of the first eigen sketch shows vectors u and Ax are co-linear as x is collinear with x-axis

Mike then used the line dragging strategy and dragged *x* toward the origin along the line segment where *x* and the *x*-axis overlapped, until *Ax* became the same length as *u.* He then said:

[1] Mike: Um, ok, I am lining these two [*Ax* and *u*] up and I am not totally convinced that is what I am supposed to be doing. So, I am gonna re-read the instruction to see if I have done the right things.

[2] Int: What made you think that was something that you wanted to do?

[3] Mike: Because from when I first read this, um, I remembered that I need to get *Ax* to equal to λx [writes down the equality on the worksheet using his right index finger as shown in Figure B2] and yeah [immediately points with the mouse] here is *Ax*. Yeah I knew the two things need to be the same [drags *x* along the line where *Ax* and *u* overlapped]. Just when I started moving this so I saw I could do that, but I do not like have the confidence that was right thing to do.

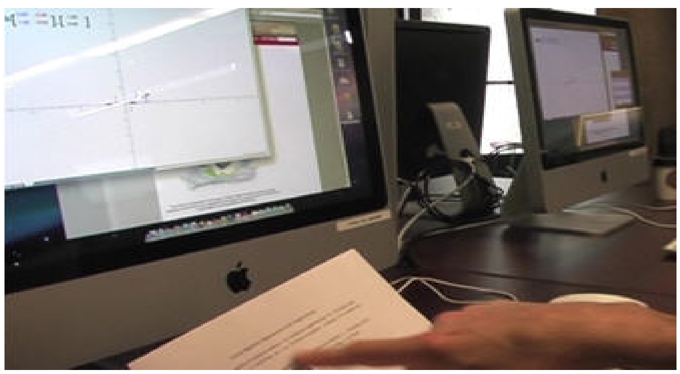

Figure B2. Mike's use of index finger to trace the equality $Ax = \lambda x$

[4] Int: So you have right there, *Ax* overlapping *u*?

[5] Mike: Yeah, that is what I did.

[6] Mike: [reads the definition aloud] So let *A* be a *n* by *n* matrix, there is *A* and non-zero vector *x* [points to the geometric representation of *x* using the mouse] this is my *x* right here, and it is an eigenvector of *A* if and only if there exists a number lambda such that *Ax* equals λx . If such a number lambda exists it is called an eigenvalue of *A*. Ok, given a sketch here it is [points to the sketch using the mouse] that represents matrix *A* and an arbitrary vector *x,* right here [points to the arithmetic representations of *A* and *x* using the mouse], double click on the entries of *A* to change values [points to the vectors using the mouse] I do not need to switch it because the first one is exactly this one, then drag *x* [points to *x* using the mouse] which I did, to find eigenvectors and associated eigenvalues

which appear up there [points to the arithmetic representation].

[7] Int: Yeah

[8] Mike: So what I am having trouble figuring out um [drags *x* slowly] is, it says that I want *Ax* equal to λx and that [points to the vector *Ax* using his right index finger] is obviously right there. So what I am having trouble figuring out now is what would lambda times *x* be? Um [drags *x* slowly]

[9] Mike: I am confused. I see the (3, 1) [points to the column vectors of *A* using his right index finger] matches up with the *u* here [moves his right index finger along vector *u*] and (-2, 0) [points to the column vectors of *A* using his right index finger] matches up with *v* here [moves his right index finger along vector *v*].What I am having trouble to see is how those two [points to *Ax* and *u* using his right index and middle fingers] go together with the *x* [points to *x* using his right index finger]. Um, I guess I am struggling to see [looks at the definition] what lambda is in all this [points to the sketch using his right hand].

[10] Int: Yeah, lambda is not there, right?

[11] Mike: Aha.

[12] Int: So you have described what the *u* corresponds to and what the *v* corresponds to. What is the *Ax*?

[13] Mike: *Ax* is this [points to the arithmetic representation of *A*] times this [points to the arithmetic representation of *x*]. If I multiply this [points to the matrix *A*] is a 2 by 2 and this [points to the arithmetic representation of *x*] is 2 by 1, so I could multiply though. I assume if I multiply those I would get this [points to the vector *Ax*]. Okay, um [drags x slowly into the fourth quadrant and then into the first quadrant until *Ax* overlaps *u*].

[14] Int: We want *Ax* to be equal to λx. What do we know? We have x.

[15] Mike: I know this is x.

[16] Int: What do you think λx looks like?

[17] Mike: λ is what I am multiplying x by [points to the vector x on the sketch using his right index finger], so that it ends up being [drags x slowly into the first quadrant] the same as the Ax.

[18] Int: Right, and what does multiplying x by λ mean?

[19] Mike: [drags vector x into the first quadrant] That would mean something like that [x overlaps Ax]. I think yeah [stops

dragging and gazes at the sketch] that would be it because now I have Ax here and a scalar multiplication of x by an amount which is λ, to make it equal to the same thing so yeah that is right.

Upon noticing that x and *Ax* overlapped in the first quadrant, Mike stopped dragging. He pointed to the geometric representations of x and Ax using the mouse at the same time as he said "that [vector *Ax*] would be my eigenvector". He then pointed to the arithmetic representation of *x* using the mouse at the same time as he said "these [components of vector *x*] would be my eigenvalues, I believe yeah". The interviewer prompted him to re-read the definition to see what the eigenvalue was. He re-read the definition aloud and then gazed back at the sketch. He then said that both the geometric and arithmetic representations of *x* represent eigenvectors, and "the eigenvalue is the scalar multiple applier that made these two things [x and Ax] line up". Next, he used line dragging to explore further the relationship between *x* and *Ax.* As he dragged *x* along the line where *x* and *Ax* overlapped, far away from the origin, he realized that *x* and *Ax* stay overlapped and have the same lengths. Thus, he concluded that lambda was one.

M.2 Second Segment: recognition of the relationship among eigenvectors associated with an eigenvalue

After Mike found the eigenvalue of 1 and perceived the invariant property of eigenvalue, the interviewer asked:

[24] Int: Do you think there is another eigenvector?

[25] Mike: [drags x in an anti-clockwise direction into the third quadrant] I do not think so [notices that x and Ax overlapped for $\lambda = 1$], but oh down here minus one. Is that negative one? [drags x along the line away from the origin in the third quadrant] oh no that is still one.

[26] Int: Can you drag a little bit more?

[27] Mike: [drags x into the first quadrant, finds a position where x and Ax overlapped for lambda equal to 2, and uses line dragging] Um I am confused. It looks like an eigenvector, a lambda two.

[28] Int: Is this the same eigenvector you had before?

Mike then dragged x back and forth between the two positions where x and Ax overlapped in the first quadrant and said, "it seems it is the same direction". The interviewer drew his attention to the arithmetic representation of eigenvectors, so that he noticed that $\lambda = 1$ associated with eigenvector (2.7, 2.74) and $\lambda = 2$ associated with eigenvector (5, 2.3). He then said "this is an eigenvector with lambda one this is an eigenvector with lambda two". He then dragged x into the third quadrant to explore the relationship between the two vectors, x and Ax, further.

Mike then changed the values of A to correspond to (b). He immediately found a position where x and Ax overlapped in the first quadrant and used line dragging. He said:

[31] Mike: So looks like 3, um, I imagine it is 3 because we probably set up for whole numbers, but is there somewhere that I can tell it is 3 here not 2.7 or something else.

At that point, the interviewer directed him to use the length measure command to measure vectors x and Ax, and the calculator tool to find the ratio of two lengths. Despite doing so, he found it difficult to identify the exact value of the eigenvalue, since the two vectors, on the sketch, almost overlapped for an eigenvalue ranging from 2.7 to 3.2 for matrix (b). Mike continued dragging x in a circular clockwise direction to find another set of eigenvectors and another eigenvalue. He stopped in the third quadrant when he noticed that the two vectors overlapped. He used line dragging to drag x along the overlapped line away from the origin as he attended to the ratio of the lengths. He then dragged x into the first quadrant and said "I can convince myself anywhere between 3 to 3.4 [...] that is the same thing. That is the only one there".

M.3 Third Segment: geometric representation of the negative eigenvalue of matrix (c)

Mike changed the values of A to correspond to matrix (c). He immediately found a position where they overlapped and approximated λ to be 7. He then used line dragging and said "it is interesting because this one it does not seem to be as much doubt about it. I could not. It definitely looks like 7". He then dragged x in a clockwise direction and noticed that x and Ax overlapped in the third quadrant for $\lambda = 7$. He was about to move to the next matrix when the following interaction took place:

[35] Int: Tell me how you are looking for the other one.

[36] Mike: Since I want the x to line up with the Ax [drags x in the first quadrant where the two vectors overlapped] I am gonna have to find some spot [drags x in a circular path toward second quadrant] that does not matter how long I extend x [uses line dragging to darg x along y-axis] there are not gonna meet. So the only possibility if I can get to the spot that they go to the same direction so if I go around 360 degrees I am interested in the spots like there [drags x in an anti-clockwise direction into the third quadrant where x overlaps with Ax and the uses line dragging].

[37] Int: Ok

[38] Mike: Actually very useful to have this [points to the ratio using his right index finger] and it's the same as the other side [drags x into the first quadrant and then back into the fourth quadrant].Although this is an interesting point [line dragging] um, [reads the definition] now we want to Ax equal to, um, this would work. Because, in this case lambda would be negative 4. We have this going opposite direction [points to the Ax] four times perfectly matching that, so lambda is -4 for this one. So I am wondering about the previous ones, I have to go back to check them. Um, I imagine that I have the same thing over here [drags x into the second quadrant in a circular clockwise direction] yeah having negative four there too okay so there is two possibilities where they line up perfectly or they are in the opposite direction.

M.4 Fourth Segment: post-interaction description of eigenvector

Mike changed the values of A to correspond to matrix (d). He then dragged x slowly in a circular anti-clockwise direction and said "this one is not too promising." He then dragged x in a clockwise direction quite speedily using a circular path and said "no". The interviewer then asked him "how did you go to find eigenvectors?" He responded:

[43] Mike: At first I was looking for [points to the sketch] where I can get x and Ax to line up in the same direction [puts two hands extended fingers in the same direction as shown in Figure B3], but then the third one I realized could be in the same direction or opposite direction [right hand extended fingers points to right direction and left hand extended fingers points to left direction as shown in Figure B4]. So I would move x around [rotates his right index finger as tracing a circle as shown in Figure B5] 360 degrees to see if these two cases showed up.

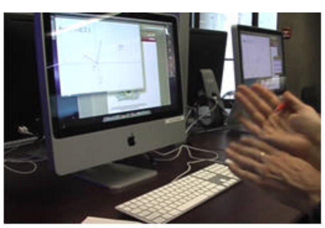	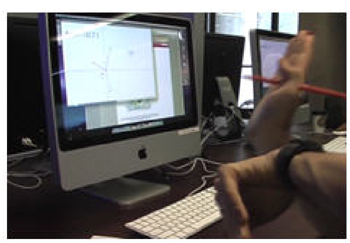	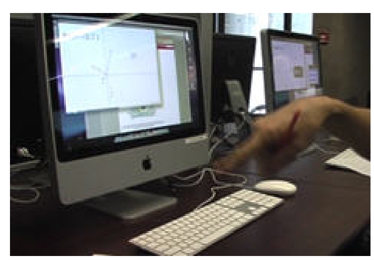
Figure B3. Mike's hands point to the same direction	**Figure B4. Mike's hands point to the opposite directions**	**Figure B5. Mikes' index finger rotates around a circle**

After completing the activity, the interviewer asked “what is an eigenvector?” He described:

[45] Mike: An eigenvector is a vector that when I multiply it by a matrix it has to also equal to the same matrix multiplied by a scalar.

Jack

Jack was a third-year university student pursuing his Bachelor’s of Science degree. He completed a linear algebra course during his second year of study. He was relatively familiar with The Geometer’s Sketchpad because of being a part of his course work. He, like Mike, interacted with the eigen sketch, a snapshot of which is shown in Figure 9.

J.1 First Segment: introduction to eigenvectors and eigenvalues

Jack began by reading aloud the formal definition of eigenvector from the given worksheet. He immediately pointed to each symbols given in the definition (using his right index finger) saying “this is a matrix, this is a vector, this is a scalar, and this is a vector and it’s saying if this lambda exists it is an eigenvalue. Okay so, now I read this and I understood but I am supposed to take that and modify it”. He looked at the sketch, then back to the definition and said “this one [points to matrix (a) using the mouse pointer] is already there”. He read the definition over again to himself, and dragged the vector x by a very small amount in the fourth quadrant. He stopped dragging and said “I see,” returned to the definition, said “now I’m confused,” and looked back to the definition again. I prompted him by asking to drag x. He started dragging x into the other quadrants somewhat randomly as he attended to the changes in the position of Ax while he dragged x. He then asked:

[55] Jack: Yes, but to what end? Is it finding eigenvectors?

[56] I: Yeah, you’re finding eigenvectors and eigenvalues.

[57] Jack: Oh. I see. I see. So by dragging it, it is maintaining the eigenvectors or but, um, it doesn’t output lambda. [stopes dragging, gazes at the definition] Should it be outputting lambda?

[58] I: Yes, it doesn’t show the lambda on the sketch, but you might be able to see it as you drag.

[59] Jack: I see, yeah so I guess I line them up [drags vector x directly into the first quadrant until it overlaps with Ax]. I guess I could have lambda there. And then should I change this value [points with mouse to the matrix]?

[60] I: Could you tell me how you got into that if you line them up it’s going to be what you looking for?

[61] Jack: Because I looked at this [puts his right index finger on $Ax = \lambda x$] and I realized that there was a scalar transformation, so the vectors have to be co-linear.

[62] I: So, Could you approximate the values? What is lambda and what is x?

[63] Jack: lambda looks like 2, I don't know my xs, but I guess it is right here [points to the vector x using the mouse, writes down $\lambda \approx 2$, $x_1 = \begin{bmatrix} 2.79 \\ 1.41 \end{bmatrix}$, and drag x along the line where Ax and x overlapped away from the origin in the first quadrant] lambda still looks like two, x has changed. Should I write down this as well? [writes down $x_2 = \begin{bmatrix} 8.58 \\ 4.25 \end{bmatrix}$].

J.2 Second segment: recognition of the relationship among eigenvectors associated with an eigenvalue

After Jack found the position where the two vectors where overlapped for $\lambda = 2$, I prompted him by asking:

[64] I: Okay. But how does this x [point to the sketch] relate to this one [point to the definition]?

[65] Jack: Um. I guess it's a linear transformation of this [points to his written vectors on worksheet], because of the definition. But assuming that I do not know that I guess it [points to the sketch using his index right finger] looks like it's a linear transformation of this [points to his written vectors on the worksheet].

[66] I: Can you continue dragging x?

[67] Jack: You mean to here [drags x in an anti-clockwise direction into the third quadrant stopes when he sees that the two vectors overlapped]?

[68] I: So what's the lambda here? And what is x?

[69] Jack: Um. [drags x along the line where x and Ax overlapped in the third quadrant, looks back at the definition, then to the sketch]. So [...] are you trying to hint that all values of x are linear transformations of each other?

[70] I: What do you mean by linear transformation? What type of linear transformation?

[71] Jack: Times by a scalar, so I guess I would at this point I would probably realise that they look very [drags x toward origin along the line where the two vectors overlap], that they are all on the same axis [drags x away from origin into the first quadrant] I guess and I would deduce [drags x more quickly back and forth along the straight line passing

through the origin] that the value of lambda wouldn't change but that there are infinitely many eigenvectors.

Jack was about to proceed to the next question when I prompted him to look for another set of eigenvectors. After dragging in an anti-clockwise circular fashion, Jack could not identify another eigenvector, and said, "it makes sense since there should only be one." I asked him to drag x to (1, 1). At this point, Jack noticed that the two vectors overlapped and wrote down $x = Ax = \lambda x$ on the worksheet. He then dragged x slowly in the first quadrant to find a position where x and Ax overlapped when λ=2, and then dragged x back to a position where x and Ax overlapped when λ=1. He then looked at the equality and said "does not make sense, oh yeah it does lambda is 1". I prompted him asking about the sketch representation of λ=1. He said "I guess I had trouble figuring out that the lambda was one, but yeah that is pretty apparent [gazes at the sketch]. It is just confused me, I am not very used to the sketchpad and they kind of lined up [holds his hands extended fingers and brings them together placing his right hand on the tops of his left hand as shown in Figure B6 and B7]".

<table>
<tr><td></td><td></td></tr>
<tr><td>Figure B6. Jack's hands are positioned parallel to each other</td><td>Figure B7. Jack's hands are exactly placed on each other</td></tr>
</table>

Having completed matrix (a), Jack turned his attention to each of the three others. He changed A to correspond to matrix (b), randomly dragged vector x using different radii into all the quadrants, and asked "is the eigenvector non-existent?" After some more circular dragging further from the origin, he hit upon a vector in the third quadrant, and said:

[78] Jack: [drags x into the third quadrant] So they're lined up, um so there's a lambda. I don't know why I said non-existent. Whatever this length is [drags the mouse pointer along the vector Ax] divided by that length [drags the mouse pointer along x].

[79] I: You can approximate the lambda.

[80] Jack: 1.5 oh sorry 2.5. I don't know what that is. Um, write down I guess [writes down $\lambda \approx 2.5$ on the worksheet].

[81] I: What are the eigenvectors?

[82] Jack: [drags x along the line where the two vectors overlapped far away from the origin] All the ones on this line and [drags x along the line into the first quadrant] all the ones on this line. Yeah it seems.

[83] I: Do you think that there will be another eigenvector?

[84] Jack: Just because it is a linear equation, it should only be one, I guess, but assuming that I do not know that if I drag that [vector x] around a circle I could find out.

J.3 Third Segment: geometric representation of the negative eigenvalue of matrix (c)

Jack changed the matrix to (c). He dragged x slowly in the third quadrant and found a position where the two vectors overlapped. He then dragged x along the line away from the origin in the third quadrant and then he dragged x into the first quadrant passing through the origin. He noticed that x and Ax stay overlapped in the first quadrant, approximated lambda to be 8 and read the components of the eigenvector (1.28, 1.32) from the arithmetic representation on the sketch. In the worksheet, however, he wrote down $[1.28\lambda, 1.32\lambda]$ to represent the set of eigenvectors associated with $\lambda = 8$. I then prompted him to find another set of eigenvectors. He dragged x in an anti-clockwise direction, in a speedy fashion, as he focused only on the position where vector x overlapped with vector Ax. Finding nothing, I invited him to drag vector x slowly into the second quadrant. Doing so, Jack found a position where x and Ax were collinear and said "it's a linear transformation um makes it I guess, oh yeah I guess it is still like the opposite eigenvector [positions his hand as shown in Figure B8 and brings his right hand to place it on the top of his left hand as shown in Figure B9] because of ninety degrees um I'm trying to recall" as he hold his hands in an angular form for a few seconds as shown in Figure B10. He then looked back at the definition. He then dragged x along its collinear path with Ax and said "oh yeah, yeah. I guess it would be -8". He then wrote down $\lambda \approx 8, -8$, $x_2 = [-2.09\lambda_2, 1.81\lambda_2]$ on the worksheet.

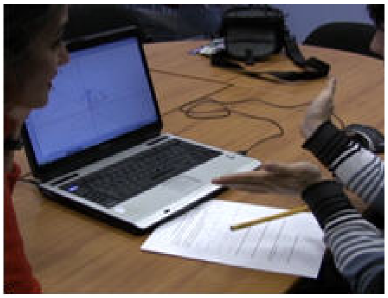		
Figure B8. Jack positions his hands in an angular shape	**Figure B9. Jack brings his hands together.**	**Figure B10. Jack holds his hands in an angular form for a few seconds.**

J.4 Fourth Segment: post-interaction description of eigenvector

Jack changed the matrix to (d). He dragged x in an anti-clockwise circular path and said "I guess this one has no eigenvalues or eigenvectors". He then continued dragging in a spiral fashion, that varied both the angle and the distance from the origin of the vector, as he said "I cannot make x to touch Ax". In response to my prompt, how he went about trying to find the eigenvectors, Jack said:

[92] Jack: I tried to make x touch Ax [dragging vector x in a spiral fashion beginning far from the origin, turning in an anti-clockwise direction, and ending at the origin] and I guess for the third one we tried to make that happen [drawing two vectors on the worksheet that are collinear but not overlapping as shown if Figure B11].

After completing the activity, I asked Jack what is an eigenvector? Jack described as wrote it down on the worksheet:

[94] Jack: a vector local to a specific matrix that whose multiplication by the matrix yields the same result as the multiplication of a specific scalar, that is the eigenvalue, by the same matrix.

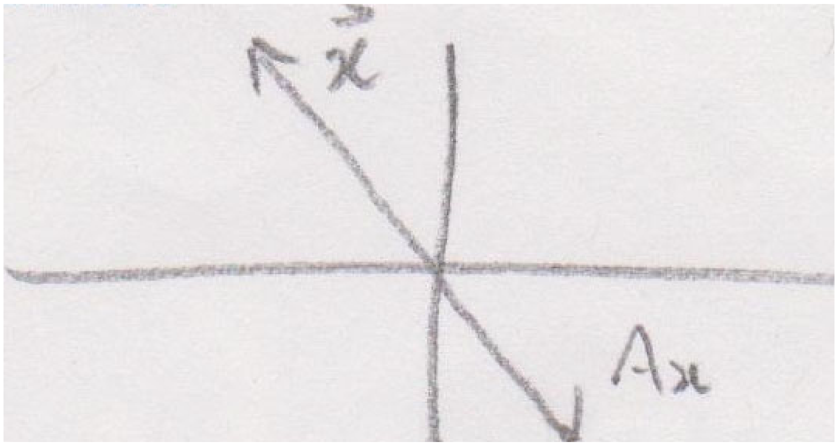

Figure B11. Jack's drawing of eigenvectors

Kate

Kate was pursuing a teaching certificate degree. She had completed a linear algebra course during her Bachelor of Science degree program and had used *The Geometer's Sketchpad* before in her spare time to expand her knowledge of geometry. She volunteered her time to participate in my study. In response to my first prompt, what is a vector, she said "a vector is a line with an arrow which includes the direction and the magnitude as well. That is a vector, so it is like this one [draws it on a paper as shown in Figure B12] the arrow show the direction and the length of the segment shows the magnitude. I think yeah".

Figure B12. Kate's drawing of a vector

K.1 First Segment: introduction to eigenvectors and eigenvalues

Kate first read the formal definition of eigenvector word by word from the given worksheet and stopped for ten seconds and re-read it and said "it's too hard to me to understand the concept right now". I prompted her by asking "what could you say about λx and Ax?". She responded that lambda is a real number acting as a scalar multiple and Ax is matrix times a vector. She further said "a matrix times a vector should be still a matrix, um, I cannot remember that one". She then immediately used an example of a matrix and an arbitrary vector to verify her conjecture that a matrix times a vector is a matrix, as shown in Figure B13. But she did not recall how to do matrix-vector multiplication, so that she left it incomplete.

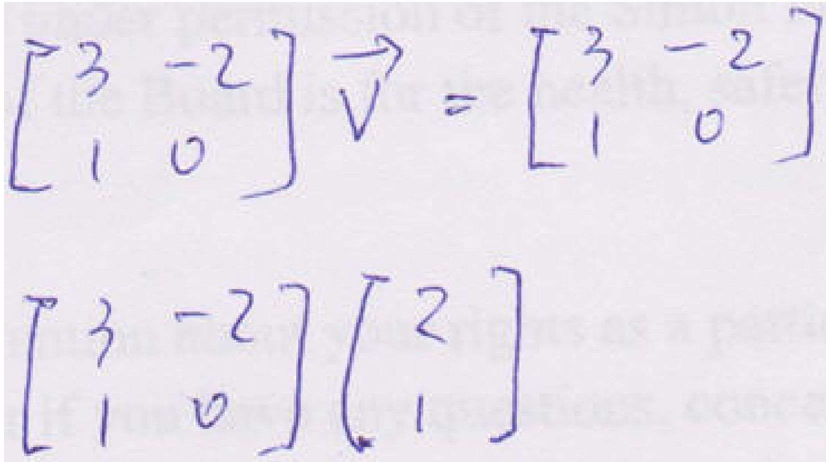

Figure B13. Kate's example of a matrix-vector multiplication

I drew her attention to the sketch asking her to drag x to find the eigenvectors and eigenvalues of A. Kate started dragging x slowly at the first quadrant around the given position (at about (1.5, 3.5)) and said "if you change x you change both, you change the direction as well as the magnitude". As she dragged x in the first quadrant in a clockwise direction, she noticed that the position of two vectors was changing from being perpendicular to being overlapped. She stopped dragging when the two vectors overlapped for $\lambda = 2$ and gazed at the screen. She then started dragging x in an anti-clockwise direction until the two vectors became about perpendicular to each other, and then dragged x in a clockwise direction until the vectors overlapped for $\lambda = 1$. She then used line dragging to further explore the relationship between the two vectors and said:

[105] Kate: This one has the same, right? so if these two [points to $Ax = \lambda x$] are the same so Ax is the same as x [writes down $Ax = x$].

[106] I: What is lambda?

[107] Kate: The lambda is one.

[108] I: For what values of x?

[109] Kate: [uses line dragging] They're the same, the top value is the same as the bottom value, right [points to the arithmetic representation of vector x on the sketch].

[110] I: This is a particular example which is 5.19 and 5.19.

[111] Kate: It's a special case. So x is like a and a [writes down $\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix}$ on a given paper sheet].

[112] Kate: At this time Ax equals λx, the lambda is one. This is case one. Case two, I still need Ax [drags x in a clockwise

direction until x and Ax overlap], so for this situation Ax is not the same as x so that means lambda, if um lambda is not one.

[113] I: Can you approximate lambda?

[114] Kate: Lambda is about two [writes down $Ax = 2x$ and $\vec{x} = \begin{bmatrix} 2a \\ a \end{bmatrix}$ on the given paper sheet].

I prompted her to drag x more. She first used line dragging, and then dragged x in a clockwise direction in the first quadrant and asked "can we get a three?". She then dragged x in an anti-clockwise direction in the first quadrant and said "they're parallel but how about non-parallel". She stopped dragging and said "they're not the same anymore." At this time, I prompted her to drag x into other quadrants. She dragged x in a clockwise direction into the third quadrant where she noticed that x and Ax overlapped. She said "that's still positive two. It's possible to go to the opposite direction", as she dragged x into the first quadrant in a clockwise direction and then back into the third quadrant.

K.2 Second Segment: recognition of the relationship among eigenvectors associated with an eigenvalue

Next, Kate changed the matrix to (b). She dragged x slowly in the third quadrant and she found a position where the two vectors overlapped. She then dragged x in a speedy fashion to the first quadrant, and used line dragging.

[119] Kate: Actually the lambda is the ratio of this length [uses her index fingers to indicate the length of Ax] to this one [uses her index fingers to indicate the length of x] and I want to find the ratio. I use my calculator [takes out her calculator]. I just want to find out the relationship between, so this one is 3.18 and 2.32.

[120] I: What did you calculate?

[121] Kate: The ratio, if there is a ratio because for this I could not see anything. If you change this matrix A, look at this x and Ax [uses the mouse pointer to point to x and Ax] they're parallel. right? That is almost 1.5 [divides 3.18 by 2.32].

[122] I: Yeah, you can give approximation. How about lambda?

[123] Kate: I think the lambda is 1.5.

[124]: I: You think lambda is 1.5?

[125] Kate: It's longer it should be 3.

[126] I: We are looking for the lambda and also for x.

[127] Kate: This is where I got confused. I think the lambda should be the ratio of the lengths [used her right index finger and thumb to indicate the lengths of x and Ax] so it is 3 [writes down λ=3 on the given paper sheet]. So we changed the matrix, this time the x is 3.18 and 2.32 okay. So we can see which is 1.5a [writes down $\vec{x}=\begin{bmatrix}3.18\\2.32\end{bmatrix}=\begin{bmatrix}1.5a\\a\end{bmatrix}$].

[128] I: Do you want to drag more to find other ones?

[129] Kate: [uses line dragging] This one is still 3.

[130] I: So how many xs can you find?

[131] Kate: Many [uses line dragging], infinitely many [...] so if they're on the same line we find lambda is a fixed number is a constant as long as A is determined. [uses clockwise circular dragging to drag x into the third quadrant] so for this one [drags x back into the first quadrant] it [the ratio] is still 3, right? [drags x away from the origin almost along the line where x and Ax overlapped] This one [ratio between the x- and y-coordinates of vector x] is not 1.5 anymore. The previous one, I found here so the number should be on the same line. This one [ratio between the x- and y-coordinates of vector x] is 1.1, it is a bit bigger.

K.3 Third Segment: geometric representation of the negative eigenvalue of matrix (c)

Upon changing the matrix to (c), x and Ax both fell into the first quadrant. Thus, she started dragging x in the first quadrant and immediately noticed a position where the two vectors overlapped. She used line dragging for a few seconds to drag x along the line (where x and Ax overlapped) away from the origin. She then dragged x into the third quadrant in a clockwise circular direction. She identified the position where the two vectors overlapped (it was on the same line as the first quadrant one) in the third quadrant. She stopped dragging and gestured in the sense that she used her index finger and thumb as a measurement unit to approximate the length of Ax and said:

[135] Kate: It [the ratio of two lengths or lambda] is about 6.

[136] I: How about x?

[137] Kate: The top and the bottom are the same [writes down $\lambda=6$]. if lambda is six, x is a and a. [drags x along the line into the first quadrant passing through the origin] Two values are the same.

[138] I: Okay. Do you want to drag x more? Drag it into the fourth quadrant.

[139] Kate: [drags x in a clockwise direction into the fourth quadrant, finds a position where the two vectors are collinear, stops dragging] If they're opposite to each other.

[140] I: What is x ? What is lambda?

[141] Kate: [uses the mouse pointer as a marker to find the ratio between the two lengths] The lambda is two, three, that is five. No, six.

[142] I: We can actually measure the ratio.

[143] Kate: [uses the Measure command and Calculator tool] So it is four, the lambda is four.

[144] I: Um… how about x?

[145] Kate: x this one is a , -a. Okay.

[146] I: I agree with x and the lambda's magnitude.

[147] Kate: You see [points to the sketch representation of the ratio] the ratio is always the same.

[148] I: Yes, but how this is different from the previous one?

[149] Kate: Oh I see the lambda should have be negative four…yeah they have opposite directions so the lambda should be negative.

K.4 Fourth Segment: post-description of eigenvector

Kate changed the matrix to (d) and dragged x in an anti-clockwise circular path until x traversed a full circle. She then stopped and dragged x in a clockwise direction and then again in an anti-clockwise circular path, and said "there's no lambda". I prompted her asking why there is not, in which she said "because you are not able to put x vector on the same line with vector Ax".

After completing the task, I asked her "how did you go to find eigenvectors?" She responded:

[152] Kate: Okay. So when I tried to find eigenvectors I just tried to find, because A times x is a vector and lambda times x is also a vector. Because they're equal that means they're parallel to each other or I can say they're on the same line if they have the same starting point so what I need to do is to put x vectors and Ax vectors on the same line.

After completing the activity, I asked her "what is an eigenvector?" She described:

[154] Kate: An eigenvector is such a vector when you times this vector by a matrix which is n times n matrix um you can find, if you can find um, a real number lambda and times this lambda by the vector as well and the lambda times x because lambda times x is a vector, right, and A times x is

also a vector if these two vectors can be on the same line [uses her left hand and arm to depict a line in space as shown in Figure B14] so at this time this x is an eigenvector.

Figure B14. Kate's gesture that describes eigenvectors

Tom

Tom was a second-year undergraduate student pursuing a bachelor degree in science. He successfully completed both calculus I and II courses, and was enrolled in a linear algebra course at the time of interview. He had not used *The Geometer's Sketchpad* before. In response to my first prompt, what is a vector, he said "um I guess a vector is a line [uses his right index finger to draw it on the desk]." He then drew a vector on a sheet of paper as shown in Figure B15 and mentioned that vectors are "used more differently in physics". I also asked him about a scalar multiple of a vector. He said "it is just a straight line that goes to infinity" and gestured moving his right hands extend index finger toward up right corner and his left hands extended index finger and arms down toward the left corner of his body as shown in Figure B16.

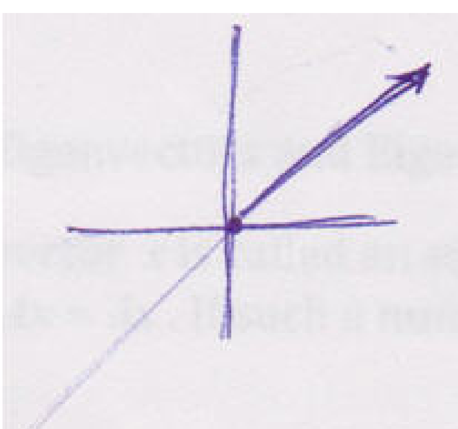

Figure B15. Tom's representation **Figure B16. Tom gestures as**

of a vector on a sheet of paper	**describes a scalar multiple of a vector**

T.1First Segment: introduction to eigenvectors and eigenvalues

Tom read the definition from the worksheet and then drew his attention to the sketch. He dragged *x* randomly everywhere on the sketch. He stopped dragging when the two vectors overlapped (in the first quadrant) and re-read the definition. He then looked back to the sketch and mentioned that "λ seems to be 2." I prompted him asking to find *x.* He then used line dragging to drags *x* along the line where *x* and *Ax* overlapped, and wrote down $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 2$ on the worksheet. He further dragged *x* along the line and wrote down a scalar multiple of *x*, $x = \begin{bmatrix} 9 \\ 4.5 \end{bmatrix}$.

I asked him to drag *x* more in the first quadrant to find another set of eigenvectors. He immediately noticed a position where the two vectors overlapped and also had the same length and direction. He then wrote down $\lambda = 1$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

A.4.2 Second Segment: recognition of the relationship among eigenvectors associated with an eigenvalue

Tom changed *A* to correspond to matrix (b). He dragged *x* in a clockwise direction in the first quadrant. He then stopped when they overlapped:

[160] Tom: this one seems to be four times [writes down $\lambda = 4$ on the worksheet].

[161] Tom: [looks back to the sketch] um 1.5 [writes down $x = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$, drags *x* along the line where the two vectors overlap] it's the same thing because of being multiple.

[162] I: how many eigenvectors did you find for this lambda?

[163] Tom: um infinity

[164] I: how they're connected to each other?

[165] Tom: um scalar with the same basis.

I prompted him to drag *x* more. He then dragged *x* in a circular anti-clockwise direction and noticed a position where the two vectors overlapped in the third quadrant. He immediately said "that is the same thing", and continued dragging. He then dragged *x* in a clockwise circular direction and concluded that he could not find any more for matrix (b).

T.3 Third Segment: geometric representation of the negative eigenvalue of matrix c

Tom changed *A* to correspond to matrix (c). He dragged *x* in a circular clockwise direction.

[168] Tom: first, it [*Ax*] goes to the opposite direction.

[169] I: opposite direction?

[170] Tom: rotates, like when I go clockwise it [*Ax*] goes anti-clockwise [moves his right hand extended index fingers in a circular path]

He then dragged x more and stopped where the two vectors overlapped in the first quadrant. He stared at the screen to approximate the value of lambda and x. He then wrote down $\lambda = 8$ (the actual eigenvalue was 7) and $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. I prompted him to drag x more slowly to find another eigenvector. He dragged x slowly in an anti-clockwise circular direction and said "I guess it looks like tracing each other". He then stopped dragging when x and Ax lined up. He stared at the screen for a few seconds and said "this one goes to the opposite direction". He then used line dragging and said "six times, it's six times more than x, I mean opposite direction".

[173] I: are you saying that you got an eigenvector there?

[174] Tom: that would be eigenvector. Eigenvalue is probably negative six. Eigenvector is one and one.

[175] I: no it is not.

[176] Tom: -1 and 1

T.4 Fourth Segment: post-description of eigenvector

Tom changed *A* to correspond to matrix (d). He dragged the vector *x* slowly in a circular clockwise direction as he observed the behaviour of *Ax.* He used this strategy to drag *x* in a circular path with different radii, he then dragged *x* in an anti-clockwise circular direction. He then said "this one it seems they don't. It seems always it is never the *x* value I mean the *Ax* is never on top of the *x*".

After completing the activity, the following interaction took place:

[178] I: what is an eigenvector?

[179] Tom: um it seems that eigenvectors, well, it seems that they're linear transformations. Um so the eigenvalue behaves according to the changes take around [rotates his right

index finger around], but eigenvector changes as you increase if you have a multiple of the same vector [moves his right hand extended index finger back and forth along a straight path as shown in Figure B17]. That is all I think.

[180] I: how did you go about finding eigenvectors?

[181] Tom: I guess when the eigenvector is on top of *Ax* then that's where they exist or the other one going on top of it that exists.

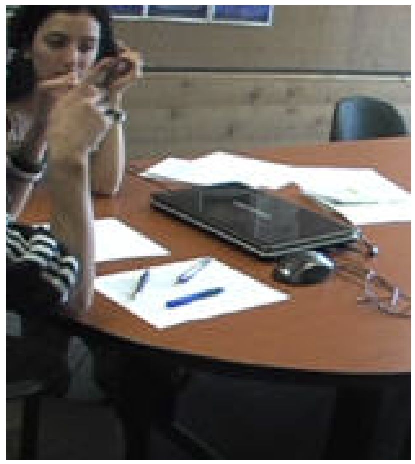

Figure B17. Tom rotates and moves his index finger as describes eigenvectors

Rose

Rose is a first-year university student who was pursuing her undergraduate degree, majoring in science. She had successfully completed a calculus course and, at the time of interview, she was enrolled in a linear algebra course. She did not recall studying matrices and vectors in high school. It was her first time using *The Geometer's Sketchpad*. Before she interacted with the sketches, we had the following exchange:

[185] I: What is a vector?

[186] Rose: A vector I know that it is just like when we are multiplying something by that; that would be the vector component [...] you know that in the class we learned like x_1 and y_1.

R.1 First Segment: introduction to eigenvectors and eigenvalues

Rose read the formal definition of eigenvector from the given worksheet and tried to make sense of the definition, saying that "ok, it is just *A* times this [*x*] is equal this [λ] times that [*x*], if such a number exist it is eigenvalue of that [*A*]. Is that what is trying to say?" She used her index finger to point to the symbols on the definition. Then, she began to explore the sketch using the default matrix (a). She dragged vector *x* randomly from its given position (at about (4, 1)). She stopped dragging when she observed that the vector *x* and Ax were positioned perpendicular to each other (vector *x* being in a vertical position and vector Ax being in a horizontal position) and said "here it is ninety degrees" (see Figure B18).

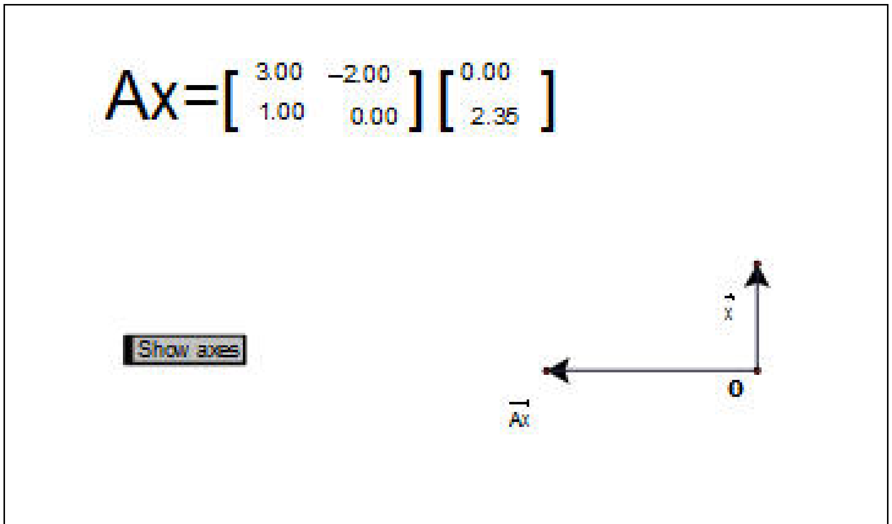

Figure B18. Snapshot of sketch where x and Ax are perpendicular to each other.

I prompted her by asking "is that what you are asked to find?" Then, she looked at the definition and then looked back to the sketch. She continued dragging the vector *x* randomly in the first quadrant and also tried not to change its length. As she dragged *x*, she attended to the length of vector *Ax* and to its position on the sketch, and noticed that *x* and *Ax* overlapped. She continued dragging *x* slightly down from the overlapped position and then used an anti-clockwise direction to drag it back up. This made her notice that the length of vector *Ax* changes, as she dragged vector *x* in a clockwise direction in the first quadrant, from being smaller than to being twice as long as *x*, and then to being more than twice as long as *x*. Noticing these changes, she asked "is *x* double *Ax*?" and conjectured that "um it seems no matter [what] *x* is, *Ax* is twice [its] value". To verify her conjecture, she dragged *x* away from the origin (point O), maintaining the position where *x* and *Ax* were overlapped in the first quadrant. She then looked back to the definition, and said "that means *Ax* is going to be just two times *x*" and then wrote down $2x = \lambda x$ right below $Ax = \lambda x$ on the worksheet.

R.2 Second Segment: recognition of the relationship among eigenvectors associated with an eigenvalue

After identifying that lambda is 2 for matrix (a), I prompted her to find eigenvectors. In response, she said:

[192] Rose: What do you mean? I do not know. It was two.

[193] I: The sketch can help you to find it.

[194] Rose: x is just two times, isn't it?

[195] I: How would you read the vector?

[196] Rose: x component and y component: 6 and 3 [drags x far away from the origin along the line where the two vectors overlapped]

[197] I: As you move along the line what else would you have?

[198] Rose: 12 and 6. This is two times that [writes down $x = \begin{bmatrix} 12.58 \\ 6.40 \end{bmatrix}$, $x = (12.58, 6.40)$, x is $2y$ on the worksheet].

[199] I: How many xs do you have?

[200] Rose: Would not be infinity!

[201] I: Yeah, drag x more to see if you can find another set of eigenvectors.

[202] Rose: [stops dragging] Right here. But this time it is equal to it. Isn't it?

[203] I: Yeah, what is the lambda?

[204] Rose: Just one.

[205] I: Good! What are the eigenvectors?

[206] Rose: Negative seven and negative seven. Um, there is no multiplying because it is the same length.

Having completed the first matrix, Rose turned her attention to each of the three others. She changed A to correspond to matrix (b). She randomly dragged x in the first quadrant, and immediately said "it's three times", and wrote down $x = \begin{bmatrix} 4.02 \\ 3.34 \end{bmatrix}$ and $x = (4.02, 3.34)$ on the worksheet. I then told her to use the Measure command to measure the lengths of x and Ax, and Calculator tool to find the ratio of two lengths. After calculating the ratio, she used line dragging to drag vector x along the line (in the first quadrant) where x and Ax overlapped.

[210] Rose: It goes to infinity that way [drags x along the line where x and Ax overlapped, away from the origin in the first quadrant].

[211] I: How about other way?

[212] Rose: [drags x into the fourth quadrant and then into the third quadrant. She finds a position where x and Ax overlapped in the third quadrant and then drags along the overlapped line away from the origin] Yeah it goes to negative infinity.

[213] I: Keep it in line and drag it down.

[214] Rose: [drags *x* along the line (where *x* and *Ax* overlapped) from the first quadrant into the third quadrant passing through the origin] Oh that looks better.

Next I prompted her to look for another set of eigenvectors. After dragging *x* in a clockwise circular path, she found a position where *x* and *Ax* overlapped in the third quadrant, but she immediately noticed that the position was not a new one, and said "it looks the same to me."

R.3 Third Segment: geometric representation of the negative eigenvalue of matrix c

Next, Rose changed the matrix to (c). She dragged *x,* immediately found an eigenvector and approximated the eigenvalue to be 7. She could also read the ratio of the two lengths from the sketch (since she had activated the Calculator tool and Measure command in verifying her approximation of eigenvalue of the matrix (b)). Next, I prompted her to find another set of eigenvectors.

[219] I: Do you want to drag more?

[220] Rose: Around?

[221] I: Yeah

[222] Rose: Oh it changes this time.

[223] I: What changes?

[224] Rose: Usually they [*x* and *Ax*] go in the same direction but this [points to *Ax* using her right index finger] goes opposite direction. Is it because there is no negative value on this one [matrix]?

[225] I: No, I do not think it is because of having no negative entries.

[226] R: Oh. It's completely straight here.

I prompted her to find the eigenvalue. She then said "it is 4 but other way is 7". It seems that she read the value of the eigenvalue by attending to the sketch measure of the ratio of two lengths rather than approximating it visually. Then I commented that "four is the ratio of the two lengths". She stopped and gazed at the screen. Using a numerical example of vector *x* and *Ax*, she realized that the eigenvalue is -4 and said:

[230] Rose: it's [*x*] multiplied by negative four to get it in the negative side. The other one [eigenvalue] is positive, this one is negative. Those are located in these quadrants [points to the second and fourth quadrants]. This one is like this [points to the collinearity of two opposite directed vectors]. They're [*x* and *Ax*] also in the same quadrant, this one is not. So you have to multiply by a negative so

the vector would be opposite direction of what it [*x*] would be. Is that right?

R.4 Fourth Segment: post-description of eigenvector

Having found eigenvectors and eigenvalues for the matrix (c), she turned her attention to the last matrix and changed A to correspond to matrix (d). Rose dragged x in a clockwise circular path to establish that there were no eigenvectors and said, "these ones [x and Ax] do not meet at all."

When asked (in the following question) how she went about trying to find the eigenvectors, Rose said "I tried to make the vectors lie on top of each other and then find the scaling value." In her final reflection, she commented that the eigenvector is the vector x, whatever it may be, that gives the value for Ax.

Appendix C. List of common questions used in the interviews

What is the lambda? And what is *x*?

What is *x* ? What is lambda?

Do you think there will be another eigenvector?

Can you drag *x* a little bit more?

What are the eigenvectors?

Can you approximate lambda?

Do you want to drag *x* more to find other eigenvectors?

How did you go to find eigenvectors?

What is an eigenvector?