ALGEBRAIC DIFFICULTIES AS AN OBSTACLE FOR HIGH SCHOOL CALCULUS

by

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ABSTRACT

The mistakes in algebraic manipulations often hinder students' performance on calculus tasks. This observation is supported by the literature review, revealing that students, who are all supposedly strong in mathematics, are experiencing significant difficulties in their university level calculus course. Several researchers suggest that the biggest hurdle in first year calculus is the significant lack of proficiency with high school algebra. However, most studies do not itemize what such proficiency (or lack of proficiency) entails. As such, the purpose of this study is to identify several of the common algebraic mistakes made by students in a high school calculus course. This research focuses on prerequisite concepts for calculus, which are all covered in the mathematics curriculum for grades 10 – 12. The study identifies a variety of tasks in which algebraic errors hindered a correct solution, even though the calculus part of the problem was completed accurately. As an attempt to rectify the problem, a teaching approach is introduced, which is referred to as "Re-teaching in Context". This approach is aimed at improving student proficiency with the identified algebraic skills.

DEDICATION

I dedicate this thesis to my family, my friends, my fellow teachers, and my students (past and future). Many colleagues, companions and co-travellers, provided me with support and motivation on my journey in completing this thesis

To my wife, Donna, and three sons, you have all offered me never ending support and love. To my family and friends, you always gave uplifting encouragement to keep me going. To my students it is for you that I have endeavoured to make education better. To the students in my Masters cohort, you have helped my thoughts evolve and encouraged me on the overwhelming task of writing a thesis. To my senior supervisor, professor Rina Zazkis, you have helped me in so many ways, through one of the hardest tasks of my life and I deeply thank you.

I hope the thoughts and ideas of this thesis are used by other teachers to help students who are struggling with calculus and are used as a stepping-stone from which students will be able to fully appreciate the beauty and power of calculus.

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INTRODUCTION

It is a continuous source of concern and consternation to mathematics educators when seemingly strong high school mathematics students are unable to transition to calculus with the same level of ease and proficiency they have demonstrated in the past. As a calculus teacher, I share this concern, which in part motivated the study presented in this thesis.

In Chapter 1, I present my pathway as a teacher, focusing on the past several years of teaching calculus. In Chapter 2, I review research literature related to learning difficulties in calculus, as well as general problems in learning mathematics, focusing on errors in algebra.

While there appears to be a general agreement that weak algebraic skills hinder students' success in calculus, most studies do not specify the algebraic skills involved. As such, identifying particular algebraic errors that students make while performing calculus tasks became the objective of my study, the methodology for which is detailed in Chapter 3. To gather the data, students were asked to provide full solutions to mathematical problems, which would demonstrate their understanding of both algebra and calculus. Analysis of student solutions, presented in Chapter 4, shows examples in which students were able to complete the calculus portion of a task correctly, but did not arrive at a correct solution due to particular algebraic errors in their work. Furthermore, the data collected over the course of a school year show that algebraic skills are not improving while students are studying calculus and the students continue to struggle with the non-calculus components of their work. Having completed the analysis of common mistakes, in Chapter 5 I suggest a teaching approach which may enable students to transition more effectively into post-secondary calculus courses. I provide my observations from the initial implementation of this approach.

As mentioned, the focus of this thesis is to look at lacking competencies student's have in both algebra and calculus, highlighting particular errors that hinder their performance. The last chapter, Chapter 6, provides a summary of my work, bringing attention to particular contributions, as well as acknowledging the limitations of this study

1: PERSONAL HISTORY

This section gives a short history covering the fourteen years of my teaching career, mainly focusing on the last seven years, in which I taught calculus. During these last seven years, I have endeavoured to improve my teaching techniques in order to overcome student difficulties in the learning of calculus. I highlight my attempts to improve both the teaching and the learning of calculus, and conclude with the events that ultimately brought me to seek a Master's degree in secondary mathematics education and this thesis.

My teaching career began in 1996 at a secondary school in British Columbia, with a course load that included three blocks of Mathematics 8 and two blocks of Technology Education 8. After one year, I changed schools within the same district. I stayed at this second school for six years and taught Physics 11, Technology Education 8, Electronics 9/10, Computer Studies 9/10, Information Technology 11 and 12 and Advanced Placement Computer Science. The number of blocks varied from year to year, but physics and computer-based courses always remained a large part of my teaching load. Although I did not specifically teach mathematics, I gained valuable information about student learning in a variety of classroom settings.

In 2003, I moved back to the first school in which I had taught and began teaching mathematics exclusively. Since then, I have taught Mathematics 9 (adapted), Principles of Mathematics 10, Advanced Placement Statistics and Calculus 12. Although I am familiar with the content and curriculum of Principles of Mathematics (POM) 11 and 12, having never taught these courses I am not well versed with their specific method of instruction.

The remainder of this personal history focuses on my experiences teaching Calculus 12. In 2003, when I began teaching this course, I followed the mandated Ministry of Education curriculum. I used the district recommended textbook, Calculus 5e, written by James Stewart, and taught the course much like a first year university calculus course. I thought that the purpose of the course was to teach calculus as students would experience it in a post-secondary institution. I rationalized that by teaching “university style” the students would be more successful when they took the course for the second time at university or college. Throughout my first two years, I attributed the students’ greatest difficulties to their weaknesses with the pre-calculus skills learned in high school mathematics. Issues such as function notation, exponent laws, algebraic manipulation and trigonometry were not well understood and not used properly in calculus. I was confused and frustrated as to the reason for their weakness with algebra. The students in my calculus class were supposedly strong mathematics students having received an A or a B in POM 11 and POM 12. I decided to add review of essential prerequisite mathematics to the Calculus 12 course in order to address student weaknesses in pre-calculus mathematics. This in-depth review of all essential mathematical concepts needed for calculus would occur in the first four to six weeks of the full year course.

I began my third year of teaching calculus as planned with review in the first several weeks of the course. I was shocked at how many students could not get through these review topics without struggling. After the review, I anticipated that students with their NOW complete knowledge of prerequisite mathematics concepts would only struggle with the calculus and no longer have difficulty with the high school algebra. I was wrong. Students continued to have difficulty applying the concepts from the review and it was unclear to me why they were struggling. One explanation was they were not retaining the algebraic concepts. Another was that while students could use each discrete concept when prompted, they did not truly understand and were merely applying a memorized procedure often in an incorrect situation. The students were unable to determine when to apply a mathematical concept to a given problem appropriately. Upon reflection, I concluded that this attempt at review had little impact on student understanding.

Over the next few years, I tried to find a way to make the review session more effective. Initially, I felt the review session was too early in relation to when the mathematical concepts would be used in the calculus class. For example, the concept of graphing polynomials was reviewed in the third week of calculus, but then not used until the function sketching section of the derivative chapter, which was three months later. I felt that if the review of each mathematical concept were moved to a time closer to when it would be used in the class, the students would be more likely to remember the reviewed material and apply the concepts so I decided to give my classes two review sessions. One overall review at the start of the year would cover the basic topics for the year and a second shorter review would be given before each calculus section in which it was needed. The review immediately prior to each topic would contain the pre-calculus mathematics needed for that section. For example, just before the section on removable discontinuities, I would review factoring and simplification of algebraic fractions (rational expressions). In this way, the students would practice relevant skills closer to the time that they would need to apply them. At the end of the year, I concluded that this new structure was helping students more than the previous attempt, but there were still some problems. One problem was that there was too much material to cover in class-time. To properly review and adequately teach a single calculus concept required too much time for one class. In addition, many students continued to have difficulty applying the pre-calculus concepts and procedures to calculus.

In 1996, Dr. Veselin Jungic from Simon Fraser University asked me, to co-write an article on the indicators of success for the transition of students from high school to university calculus. The paper was later published in *Vector*, a local mathematics journal. Dr. Jungic encouraged me to start my M. Sc. in mathematics education and this experience planted the desire to look more broadly and more formally at the details that can help a high school student succeed in learning calculus. The experience also made me aware that numerous universities have done research on this or related topics. In 2008, I began a Master of Science in Secondary Mathematics Education at Simon Fraser University with the goal of writing a thesis on identifying algebraic difficulties commonly experienced by high school calculus students, and looking into possible solutions.

2: OVERVIEW OF PRIOR RESEARCH

This literature review looks at various studies that have investigated the aspects of mathematics that govern the success and/or lack of success in calculus. In essence, this literature review addresses the following question: **"What are the problems students have with learning calculus and why?"**

2.1 Overview of Literature Review

The first section (2.2) of the literature review verifies that the high school mathematics curriculum is meant to prepare students planning to study calculus at university. The prerequisite skills for first year university calculus were identified using lists provided by the BC Ministry of Education and Simon Fraser University. These lists were compared for similarities. The concepts from these lists were then compared against the BC high school principles of mathematics curriculum with the intent of showing that the students leaving high school with the aim to continue their post-secondary education with calculus, are being given the opportunity to learn the skills and concepts identified as essential for success in university calculus.

The second section (2.3) presents the studies that have looked at the main obstacles that limited student success with university calculus. Many of these articles clearly identify students' inability to use algebraic prerequisite calculus concepts as the largest problem students must overcome to succeed at calculus.

The third section (2.4) of the literature review reinforces the previous section by looking for the predictors of success with calculus. These studies show that the best predictor of a student's success at calculus is proficiency with prerequisite calculus concepts. These studies show that students with a strong ability with algebra do better in calculus than students with poor algebraic knowledge.

The first three sections collectively show that high school mathematics should prepare students for university calculus. The high school mathematics curriculum does indeed meet the list of calculus prerequisite skills provided by the BC Ministry of Education and Simon Fraser University yet numerous studies from around the world show that when students get to university calculus, many struggle due to their inability to use the mathematics learned in high school.

The fourth and fifth sections (2.5 and 2.6) of this literature review look at the specific details of the mistakes made in algebra learned in elementary and secondary mathematics. Some studies propose explanations as to why these difficulties occur. The research at the elementary level shows that some concepts are difficult for students to master and the incomplete knowledge of these concepts then continues to be problematic for students in high school and post secondary mathematics. The research at the middle school and high school levels looks at details of several of the more common algebraic errors, problems with the structure of the mathematics curriculum, and analyses the mathematics textbooks.

2.2 Prerequisite Skills for Calculus

This section compares the prerequisite skills for calculus as determined by two educational organizations, the B.C. Government's Ministry of Education and SFU's mathematics department for similarities. These lists of prerequisite skills are then compared to the high school mathematics curriculum confirming that the high school mathematics curriculum is teaching the concepts needed to properly preparing students for calculus.

2.2.1 Calculus Prerequisites of the B.C. Government's Ministry of Education

The B.C Government's Ministry of Education uses the Neufeld report (Neufeld, 1999), to identify the prerequisite skills essential for students studying Calculus 12. Calculus 12 is a secondary mathematics course, intended to be on par with first year calculus from SFU or the University of British Columbia(UBC).

Table 1 – BC Ministry of Education Prerequisites for Calculus 12 below, lists the concepts and skills (listed in descending order of importance) identified by the ministry as "essential" to "marginally important" for students to possess in order to attempt Calculus 12.

Table 1 – BC Ministry of Education Prerequisites for Calculus 12

#	List of skills and concepts (In descending order from most essential to least)
1.	The function concept
2.	Polynomial expressions
3.	Exponential expressions
4.	Straight line and linear functions
5.	Solving equations and inequalities
6.	Circular trigonometric functions
7.	Rational expressions
8.	Triangle trigonometry
9.	The quadratic function
10.	Logarithmic functions
11.	Radical expressions
12.	The geometry of lines and points
13.	Polynomial functions
14.	Quadratic relations
15.	Sequences and series
16.	The geometry of circles
17.	Basic knowledge of Calculus

It is significant to notice that concepts of highest priority are basic algebraic concepts.

2.2.2 Calculus Prerequisites of the Simon Fraser University

The Quantitative Assessment Topics is a list made by the Simon Fraser University that identifies the prerequisite skills for calculus. The skills from this list were used in the development of the Calculus Readiness Test. The purpose of the test is to determine whether incoming students have the necessary skills for calculus. The list in Table 2 can be found at the SFU department of mathematics site http://math.sfu.ca/ugrad/calctest/calctopics.shtml

Table 2 - SFU Prerequisites for University Calculus

#	**List of Skills**
1.	Use fractions, decimals, and percents to solve problems;
2.	Display skills such as simplifying a complex fraction and finding a percent equivalent to two or more sequentially applied percents;
3.	Compare and order fractions, decimals, and percents and find their appropriate locations on a number line;
4.	Interpret percent values greater than 100%;
5.	Use ratios and proportions to represent quantitative relationships;
6.	Solve problems involving proportions, such as scaling and finding equivalent ratios;
7.	Use factors, multiples, prime factorization, and relatively prime numbers to solve problems;
8.	Use arithmetic operations with fractions, decimals, and integers appropriately, as required in a given context;
9.	Recognize connections among the operations of arithmetic; for example, that certain operations are inverses of each other;
10.	Apply rules of arithmetic operations (commutative, associative and distributive properties) correctly;
11.	Use the correct order of operations in situations where more than one operation is performed;
12.	Interpret exponential notation and use laws of exponents for variables with integer exponents;
13.	Estimate the results of numerical computations and judge the reasonableness of the results;
14.	Verify the reasonableness of numerical computations and their results in a given context using an appropriate number of significant digits;

15.	Recognize and generalize numerical, geometrical, and other patterns;
16.	Plot linear and non-linear data, using appropriate scales;
17.	Given a verbal, graphical, or algebraic representation of a relationship, express it in a different form as required;
18.	Determine whether a relationship is a function;
19.	Distinguish among linear, exponential, and power functions;
20.	Compare linear and non-linear functions with respect to their rates of change;
21.	Interpret the meaning of intercept and slope of a linear function in a given context;
22.	Interpret the meaning of the intercepts and vertex of the graph of a quadratic function in a given context;
23.	Translate a verbal statement into algebraic language;
24.	Translate an algebraic statement into words;
25.	Evaluate algebraic expressions for specified values of the variables, including cases where a variable may take on negative or fractional values, and recognize that -x does not have to be negative and 1/x might be greater than 1.
26.	Generate and/or recognize equivalent forms of expressions, equations, inequalities, and relations;
27.	Determine any non-permissible values for the variable in an algebraic expression;
28.	Solve linear equations and inequalities, and systems of linear equations;
29.	Solve quadratic and rational equations;

30.	Solve linear equations and inequalities involving absolute value;
31.	Solve a literal equation for a specified variable, including cases when one or more variable(s) is (are) negative;
32.	Model and solve contextualized problems using various representations, such as graphs, tables, and equations; determine whether or not the results obtained fit the original context;
33.	Recognize and apply properties of parallel and perpendicular lines;
34.	Determine whether a given two-dimensional figure is a polygon; classify polygons according to the number of sides and how their sides and angles are related;
35.	Recognize and apply properties of isosceles and equilateral triangles;
36.	Recognize and apply the property that the sum of the angle measures in a triangle is 180 degrees;
37.	Classify three-dimensional objects with respect to the nature and number of their faces and angles;
38.	Apply properties of three-dimensional objects such as prisms, pyramids, spheres, cylinders or cones in problem solving;
39.	Recognize that if geometric objects are similar, then angle measures are preserved and side lengths are proportional;
40.	Given two similar geometric figures in a specified ratio, determine the ratios of their perimeters, areas, or volumes;
41.	Solve problems involving ratio and proportion in similar triangles;
42.	Use congruence and similarity to solve problems involving classes of two- and three-dimensional geometric objects;
43.	Use the Pythagorean Theorem to solve problems involving right triangles in various contexts;
44.	Describe sizes, positions, and orientations of shapes under

	transformations such as flips, turns, slides, and scaling;
45.	Use units of appropriate size and type to measure angles, perimeter, area, surface area, and volume in problem solving;
46.	Determine the circumference/perimeters and the area of triangles, parallelograms, trapezoids, and circles;
47.	Determine the surface area and volume of selected prisms, pyramids, spheres and cylinders;
48.	Decompose complex shapes into simpler ones to find areas or volumes;
49.	Solve problems involving perimeter and area of two-dimensional geometric figures;
50.	Solve problems involving surface area, and volume of three-dimensional geometric figures;
51.	Solve simple problems involving rates and derived measurements for such attributes as velocity and density;
52.	Use graphical representations of data to solve problems;
53.	Find, use, and interpret mean, weighted mean, or median as appropriate in the context of a given problem;
54.	Use principles of probability to make and test conjectures about the results of experiments and simulations;
55.	Compute probabilities for compound events;
56.	Construct sample spaces and distributions in simple cases;
57.	Use the concepts of conditional probability and independent events in problem solving;
58.	Differentiate between inductive and deductive reasoning;
59.	Interpret and correctly use connecting words, such as “and”, “or”, and “not”;

60.	Use examples and counterexamples to analyze conjectures;
61.	Distinguish between “if-then” and “if and only if” statements;
62.	Determine whether two statements are logically equivalent;
63.	State and interpret correctly the negation of a given statement;
64.	Analyze the validity of an argument;
65.	Given a relationship defined by a table, graph, or formula, tell whether or not it is a function;
66.	Given a representation of a function in any of the following forms: a table of values, a graph, an equation or formula, or a verbal description for a function; generate a different form of representation, as required, using function notation if appropriate;
67.	For any of the representations named in #2, distinguish between input values and output values, and/or provide an input value associated with a specified output value. Evaluate a function given in function notation for a specific value of the variable;
68.	Use the graph of a function to tell whether or not it is increasing (decreasing) on a specified interval;
69.	For linear functions: transform from the form ax + by = c to slope-intercept form and vice versa; given two points, or the slope and y-intercept, find the equation;
70.	Given two lines that are graphed on the same set of axes, tell which line has the greater (lesser) slope. Interpret the slope of a line as a rate of change. Find the point of intersection for two given lines by graphing or by solving the appropriate system of equations;
71.	Determine from their equations whether two lines are parallel, perpendicular, or neither. Write an equation of a line parallel or perpendicular to a given line and which passes through a given point.

	Write the equation of a vertical or horizontal line, given one of its points;
72.	Given a word problem or a description of a real-world event, model the situation by constructing an appropriate function or equation. Use this to find the information required, and interpret your solution in terms of the original situation. Recognize when real-world circumstances are best modelled using piecewise-defined functions;
73.	Determine the domain and range of a function by examining the graph, the formula, or the constraints of the situation being modelled;

74.	Given the graph of a function y = g(x), sketch any of the following graphs by applying suitable horizontal or vertical shifts, or by a suitable reflection:
	y = g(x) + k;
	y = g (x + k);
	y = -g(x);
	y = g(-x);

75.	Given the graph of a function y = f(x), sketch the graph of y = k• f(x) or y = f (kx). Identify a graph as a horizontal or vertical stretching or compression of another graph;
76.	Identify a graph, which can be seen as a sequence of translations, or as a sequence of a translation and a stretching or compression. Given a formula for f(x), write a formula for the function after such a sequence of transformations;
77.	Given formulas for the functions f(x) and g(x), find a formula for f(g(x)) or for g(f(x)). Given formulas or tables of values for f(x) and g(x), evaluate f(g(x)) or g(f(x)) for a specified value of x. Given the domains of f(x) and g(x), determine the domains of f(g(x)) and g(f(x));
78.	Express a complicated function as a composite of simpler functions;
79.	Given graphs of two functions f and g, sketch a graph of f+g or f–g. Given formulas for f and g, find the formulas for f + g and f – g;

80.	Use the definition of $\|x\|$ for any real number x to rewrite a function involving absolute values without using absolute value bars;
81.	Determine if a given function is one-to-one, and if it is, (a) given a formula for f(x), find a formula for $f^{-1}(x)$; (b) given a graph of f(x), sketch the graph of the inverse function; (c) evaluate $f^{-1}(b)$ for selected values of b from a given graph or table of values of f(x), or by using the formula;
82.	Find the zeros of a quadratic function by factoring, completing the square or the quadratic formula;
83.	Transform a quadratic function from standard form to vertex form by completing the square. Use the vertex form to find the maximum or minimum value of a quadratic function, or to sketch the graph without the use of calculator or computer;
84.	Given any of the following information, find a formula for a parabola: a) the x-intercepts and one other point; b) the vertex and one other point; c) the y-intercept and two other points;
85.	Match power functions of the form $f(x)=x^n$ for n = 1,2,3,4, or 5 with their graphs, without the use of a graphing calculator or computer. Describe the behaviour of the graph of any of these functions when the independent variable is close to zero or very large positively or negatively. Use algebra to find a formula for a power function if you know two data points;
86.	Given a formula, determine whether or not it defines a polynomial function, and if it is a polynomial, state the degree;
87.	Use the Rational Root Theorem to find the zeros of a given polynomial function;
88.	Given a polynomial function, identify the x- and y-intercepts. Determine the behaviour of the graph for large positive and negative values of the independent variable. Use this information to draw a rough sketch of the graph;

89.	Given a graph of a polynomial function, find an algebraic expression for the function that might produce the graph, and justify your choices;
90.	Given a rational function, identify any zeros or vertical asymptotes, and use the coefficients of the leading terms in the numerator and denominator to predict the behaviour of the graph for large positive and negative values of the independent variable;
91.	Given the graph of a rational function, find a plausible algebraic expression for a function that could produce this graph, and justify your choice;
92.	Write an exponential function to model a quantity that is growing (or decaying) by a fixed percentage in a given time period. Determine the percentage growth rate from the formula for an exponential function;
93.	Given an expression for an exponential function, identify the domain and range, the y-intercept, and the horizontal asymptote, and sketch the graph;
94.	Given expressions for two or more exponential functions, determine which function will have the steeper graph;
95.	Recognize that exponential and logarithmic functions are inverses of each other. Given one such function, write the appropriate expression for the inverse. Given a logarithmic function, sketch its graph;
96.	Relate the properties of logarithms to the corresponding properties of exponents. Apply properties of logarithms to solve logarithmic or exponential equations;
97.	Identify an angle given in radians as a real number related to the directed rotation of a ray about its endpoint, such that if a ray with its endpoint at the centre of a circle of radius r has rotated through an angle of θ radians in a counter-clockwise direction and the point of intersection of the ray with the circle has traversed an arc length of s, then $\theta = s/r$;
98.	Convert between radian measure and degree measure of an angle;

99.	Associate a point on the unit circle with a given angle θ, and define the sine and cosine of θ in terms of the coordinates of that point. Conversely, use trigonometric functions to find the coordinates of a point P on the unit circle associated with a given angle θ, or on a circle of any radius;
100.	Sketch the graphs of $y = \sin x$ and $y = \cos x$. Label intercepts and x-coordinates of turning points. State the domain, range, and period of the sine and cosine functions;
101.	Determine the amplitude, period, midline, and horizontal shift of any function of the form $y = A \sin (t – h) + k$ or $y = A \cos (t – h) + k$, and sketch the graph without using a calculator or computer;
102.	Given a sinusoidal graph, fit a suitable function to it. Identify sinusoidal behaviour in real-world situations;
103.	Define tan θ, cot θ, sec θ and csc θ in terms of sin θ and cos θ. Sketch the graphs of these functions, and determine the domain, range, and asymptotes for each one. Use the definitions to evaluate these functions for specific values of θ or to solve problems as needed;
104.	Recognize and use the Pythagorean relationships among the trigonometric functions to establish identities or to solve applied problems;
105.	Solve simple trigonometric equations and use reference angles to get all the solutions;
106.	Given a table of values, determine the type of function which best fits the given data: linear, quadratic, exponential, sinusoidal, etc.;
107.	Compare power, exponential, and logarithmic functions with respect to the values assumed by the function for large positive or negative values of the independent variable;
108.	Recognize an arithmetic sequence, and identify the common difference. Calculate an arbitrary term in a given arithmetic sequence;
109.	Recognize a geometric sequence, and identify the common ratio.

	Calculate an arbitrary term in a given geometric sequence;
110.	Recognize an arithmetic or geometric series and calculate S_n if required. Use the sigma notation to write S_n. Recognize an infinite geometric series, and find its sum.

2.2.3 Comparison of two Prerequisite Skills lists for Calculus

At first, one could conclude that the list of prerequisite skills for calculus from SFU, being significantly longer, includes skills and concepts that are missing from the BC Ministry Education Prerequisites for Calculus 12. However, upon inspection, the 110 points from the SFU list can be placed into two categories with respect to the BC Ministry Education Prerequisites for Calculus 12:

- Can be considered an explicit subset of the 17 concepts identified by BC Ministry Education Prerequisites for Calculus 12.
- Can be considered implied as a subset of the 17 concepts identified by BC Ministry Education Prerequisites for Calculus 12

For example, the following three points can be considered an explicit subset of point 15 from the BC Ministry Education Prerequisites for Calculus 12: 15: sequences and series.

The SFU Quantitative Assessment Topics 108 – 110 are as follows:

108. Recognize an arithmetic sequence, and identify the common difference. Calculate an arbitrary term in a given arithmetic sequence.

109. Recognize a geometric sequence, and identify the common ratio.

Calculate an arbitrary term in a given geometric sequence.

110. Recognize an arithmetic or geometric series and calculate S_n if required. Use the sigma notation to write S_n. Recognize an infinite geometric series, and find its sum.

An example of the SFU Quantitative Assessment Topics not directly identified, but implied by the BC Ministry Education Prerequisites for Calculus 12 are as follows:

The SFU Quantitative Assessment Topics 1 – 6:

1. Use fractions, decimals, and percents to solve problems;
2. Display skills such as simplifying a complex fraction and finding a percent equivalent to two or more sequentially applied percents;
3. Compare and order fractions, decimals, and percents and find their appropriate locations on a number line;
4. Interpret percent values greater than 100%;
5. Use ratios and proportions to represent quantitative relationships;
6. Solve problems involving proportions, such as scaling and finding equivalent ratios;

These six points can be interpreted as being implied in point 5 identified by BC Ministry Education Prerequisites for Calculus 12,

5: solving equations and inequalities.

When comparing the prerequisite skills, the analysis reveals that the more broadly defined points described by the BC Ministry of Education are covered by the more granular SFU list.

2.2.4 Comparison of Mathematics 10 – 12 to Prerequisite Calculus

To make a full analysis of the preparedness of high school students continuing on to university calculus, one must cross-reference the calculus prerequisite skills with the high school mathematics curriculum. The BC high school curriculum for Mathematics is defined in the “Principles of Mathematics” document. Table 3 compares the lists of pre-requisite skills as described by SFU and the BC Ministry of Education with the high school curriculum for Principles of Mathematics 10 (POM 10), Principles of Mathematics 11 (POM 11) and Principles of Mathematics 12 (POM 12). The prerequisite skills are referred to using the number given in the Tables 1 and 2. The Principles of Mathematics curriculum is referred to by Mathematics course and section.

Table 3 - Comparison of Mathematics Curriculum, BC Ed. Min. and SFU Calculus Prerequisites

Mathematics course	Curriculum section	Prerequisite skill number from BC Ministry of Education	Prerequisite skill number from Simon Fraser University
POM 10	A - Numbers	5	1
POM 10	A – Numbers	7	2
POM 10	A – Numbers	7	3
POM 10	A - Numbers	7	4
POM 10	A – Numbers	7	5
POM 10	A – Numbers	5, 7	6
POM 10	A – Numbers	5, 7	7
POM 10	A – Numbers	7	8
POM 10	A – Numbers	5	9
POM 10	A - Numbers	5	10
POM 10	A – Numbers	5	11
POM 10	A – Numbers	3, 10	12
POM 10	A – Numbers	5	13

POM 10	A – Numbers	5	14
POM 10 POM 12	B – Patterns A – Patterns	15	15
POM 10 POM 11	B – Functions/C – Space A – Relations/ B – Space	2, 4, 13	16
POM 10 POM 11	B – Functions/C – Space A – Relations/ B – Space	1, 2	17
POM 10	B - Functions	1, 2	18
POM 10 POM 11 POM 12	B – Functions A – Functions A - Functions	2, 3, 4	19
POM 10 POM 11	C – 2D Shapes A – Functions	2, 4	20
POM 10	C – 2D Shapes	4	21
POM 10	C – 2D Shapes	9	22
POM 10	B – Relations	1, 2	23
POM 10	B - Relations	1, 2	24

POM 11	A – Relations and Functions	7	25
POM 11	A – Relations and Functions	7	26
POM 11	A – Relations and Functions	7	27
POM 10 POM 11	B – Patterns and Relations A – Relations and Functions	4	28
POM 10 POM 11	B – Patterns and Relations A – Relations and Functions	7, 9	29
POM 10 POM 11	B – Patterns and Relations A – Relations and Functions	4	30
POM 11	A – Relations and Functions	7	31
POM 10	B - Relations	1, 2, 7	32
POM 10	B - Relations	12	33
POM 10 POM 11	C – Shape and Space B – Shape and Space	12	34
POM 10	C – Shape and Space	12, 16	35
POM 10	C – Shape and Space	12, 16	36

POM 10	C – Shape and Space	1, 12,16	37
POM 10	C – Shape and Space	1, 12	38
POM 10	C – Shape and Space	12, 16	39
POM 10	C – Shape and Space	1, 12	40
POM 10	C – Shape and Space	1, 12	41
POM 10	C – Shape and Space	1, 12	42
POM 11	C – Shape and Space	12, 16	43
POM 12	B – Shape and Space	12, 16	44
POM 10	C – Shape and Space	1, 12	45
POM 10	C – Shape and Space	1, 12	46
POM 10	C – Shape and Space	1, 12	47
POM 10	C – Shape and Space	1, 12	48
POM 10	C – Shape and Space	1, 12	49
POM 10	C – Shape and Space	1, 12	50
POM 10	C – Shape and Space	1, 5, 13	51
POM 10	B – Patterns and Relations	1, 4, 5, 6	52

POM 11	A – Relations and Functions	9,10, 13	
POM 12	C – Statistics and Probability	1, 5	53
POM 12	C – Statistics and Probability	1, 5	54
POM 12	C – Statistics and Probability	1, 5	55
POM 12	C – Statistics and Probability	1, 5	56
POM 12	C – Statistics and Probability	1, 5	57
POM 12	C – Statistics and Probability	5	58
POM 12	C – Statistics and Probability	5	59
POM 12	C – Statistics and Probability	5	60
POM 12	C – Statistics and Probability	5	61
POM 12	C – Statistics and Probability	5	62
POM 12	C – Statistics and Probability	5	63
POM 12	C – Statistics and Probability	5	64
POM 10	B – Patterns and Relations	1	65
POM 10	B – Patterns and Relations	1	66
POM 10	B – Patterns and Relations	1	67

POM 10	B – Patterns and Relations	1	68
POM 10	B – Patterns and Relations	1, 4	69
POM 10	B – Patterns and Relations	1, 4	70
POM 10	B – Patterns and Relations C – Shape and Space	1, 4	71
POM 10	B – Patterns and Relations	1	72
POM 10	B – Patterns and Relations	1	73
POM 12	B – Shape and Space	1, 12	74
POM 12	B – Shape and Space	1, 12	75
POM 12	B – Shape and Space	1, 12	76
POM 10 POM 12	B – Patterns and Relations B – Shape and Space	1, 12	77
POM 12	B – Shape and Space	1, 12	78
POM 12	B – Shape and Space	1, 12	79
POM 10 POM 12	B – Patterns and Relations B – Shape and Space	1, 4 1, 4	80 80

POM 12	B – Shape and Space	1	81
POM 11	A – Patterns and Relations	1, 5, 9	82
POM 11	A – Patterns and Relations	1, 5, 9	83
POM 11	A – Patterns and Relations	1, 5, 9	84
POM 12	A – Patterns and Relations	1, 2, 13	85
POM 11	A – Patterns and Relations	1, 2, 13	86
POM 10	B – Patterns and Relations	2, 5, 13	87
POM 11	A – Patterns and Relations	2, 5, 13	88
POM 11	A – Patterns and Relations	1, 13	89
POM 11	A – Patterns and Relations	1, 5, 7	90
POM 11	A – Patterns and Relations	1, 5, 7	91
POM 12	A – Patterns and Relations	1, 3, 10	92
POM 12	A – Patterns and Relations	1, 3,10	93
POM 12	A – Patterns and Relations	1, 3, 10	94
POM 12	A – Patterns and Relations	1, 3, 10	95
POM 12	A – Patterns and Relations	1, 3, 10	96

POM 12	A – Patterns and Relations	6, 16	97
POM 12	A – Patterns and Relations	6, 16	98
POM 12	A – Patterns and Relations	6, 8, 16	99
POM 12	A – Patterns and Relations	1, 6, 8	100
POM 12	A – Patterns and Relations	1, 6, 8	101
POM 12	A – Patterns and Relations	1, 6, 8	102
POM 12	A – Patterns and Relations	1, 6, 8	103
POM 10	C – Shape and Shape	1, 6, 8	104
POM 10	C – Shape and Shape	1, 6, 8	105
POM 12	A – Patterns and Relations	1, 4, 9, 10, 13, 14	106
POM 12	A – Patterns and Relations	15	107
POM 10	B- Patterns and Relations	15	108
POM 12	A – Patterns and Relations	15	109
POM 12	A – Patterns and Relations	15	110

2.2.5 Conclusions on Prerequisite Skills Needed for Calculus

The BC Ministry Education Prerequisites for Calculus 12 and the SFU Quantitative Assessment Topics have a common set of mathematical concepts needed for success at calculus. This list is compared with the high school curriculum for the “Principles of Mathematics” pathway and all the concepts identified as essential for success at calculus are covered. The mapping of the BC high school curriculum skills to the BC Ministry of Education and SFU prerequisite skills are detailed in Table 4. It appears that a student obtaining good marks in high school mathematics should be mastering the concepts needed for use in calculus. Therefore, students leaving high school with strong mathematics grades should, presumably, succeed in calculus.

However, this is not always the case. Some of the reasons for this are explored in this study.

2.3 Struggles/problems with University Calculus

In this section studies are presented which identify the problems of highest priority regarding students lacking success in post secondary calculus. These studies present the results from surveys done with students of first year calculus and interviews with calculus instructors. The results revealed that the biggest problems hindering student success in calculus are attributed to students' weak algebraic skills. When referring to "algebraic skills", or "algebra" in general, the studies leave it to the reader to infer what particular meaning they have assigned to these terms. My personal interpretation of these terms is explained in Chapter 3, following my research question.

A study in Brazil, at the Universidade Catolica de Brasilia, by Dias (2000) looked at identifying some of the problems for students learning first year calculus and offered instructional strategies to address these difficulties. Dias found the weak algebra skills of calculus students to be one of the biggest problems hindering the teaching and learning of calculus. Instructors were frustrated when students were not familiar with simple concepts such as function notation. Dias stated, "How does one manage to teach 200 freshmen first-year calculus, if they have difficulty with basic algebra and graphs?" (2000, p. 193). Some of the examples Dias identified were problems with conceptual understandings of graphs, domains, range and continuity. From Dias, one can conclude that students who lack algebraic skills will likely struggle in first year calculus.

Edge and Friedberg (1984) investigated students and the factors affecting success in first year calculus at the Illinois State University. Edge & Friedberg, concluded “that the computational material as well as the theory of calculus is frequently obscured by the lack of algebraic skills, which is assumed the students learned in high school” (1984, p.137).

Barry and Davis (2006) from the University of New South Wales studied ways to identify students who lacked the appropriate prerequisite skills. Once identified, these students were given supplemental instruction on prerequisite algebra to alleviate the some of the associated problems with learning calculus. Their study identified the frustration calculus instructors have with weak algebra skills, “… endless hours have been spent discussing the perceived decline in our student’s basic skills. In particular, the mathematical preparedness of our 1st-year students” (2006, p. 499).

While many researchers refer to lack of algebraic competence in general, only a few provide specific illustrations. For example, Kajander and Lovric (2005) searched for some of the specific mistakes made by students in calculus. One of their findings was the common algebraic misconception in treating functions as linear. Their research involved student surveys, which measured each student’s ability with specific prerequisite skills. In analyzing responses from student surveys on prerequisite skills, Kajander and Lovric found that many students made mistakes with algebraic manipulation. Kajander and Lovric noted the following algebraic misconception, “a common assumption that our students make, namely that all (or most) functions are linear: ‘ formulas’ such as

$$1/(a+b) = 1/a + 1/b$$

or

$$(a+b)^2 = a^2 + b^2$$

(or equivalent expressions for logarithm[ic], exponential or root functions) are found more and more often in students' university mathematics tests and on assignments" (2004, p. 154).

The common conclusion from the studies presented is that the most prevalent problem for students studying calculus is weak prerequisite skills (algebra). Thus students struggle when learning calculus if they do not have complete mastery of the mathematics learned in Mathematics 10 – 12. This is a surprising result given that students, who take post-secondary calculus, typically obtained good marks in high school mathematics.

2.4 The best predictor of success with university calculus

Many universities have looked for the best predictors of success at calculus. The universities use predictors to advise their students to take courses in which they are most likely to succeed. Using predictors, students at high risk of failing a first year calculus would be redirected to take a pre-calculus course. The results from the following studies show that the best predictor of a students' success in calculus is their algebraic proficiency.

Edge and Friedberg (1984) conducted a study out of Illinois State University that used three groups of students (235, 157, and 397 students). Their goal was to find the best predictors of success in a student's first course in calculus. They began by gathering data on several variables from each student.

These variables were analysed with the student's final calculus mark in search for a linear relationship and high correlation. The results found the best predictor of success in first year university calculus were the student's performance on an algebra skills test (given at the beginning of the year) and the student's high school ranking (high schools were ranked according to a national survey). The algebra skills were considered essential for working within the realm of calculus. The high school ranking was considered a measure of the student's social group and the group's positive attitude toward academic success, including long-term perseverance and competitiveness. Overall, their research of the best predictors show that the better a student leaves high school knowing his/her basic algebra, the better he/she will perform in university calculus.

In Jungic and Kraemer (2006), we looked at possible ways of predicting students' success with calculus. We gathered data on students' letter grades in Principles of Mathematics 11 and Principles of Mathematics 12 and the students' mark in university calculus from Simon Fraser University. A student that received an 'A' in Mathematics 11 did better in calculus than a student who received a 'B' in Mathematics 11. Similarly, a student who received an A in mathematics 12 did better in calculus than a student who received a B in mathematics 12. This study suggested that the reason students perform better in university calculus is that they have better mastery of algebra.

Lee (1998) performed a study that looked at the positive correlation between mathematical performance in high school and performance in university calculus. His study analyzed statistics of several BC high school students who finished their first year of university calculus. Several variables from high school were compared with their result in calculus to identify which variable makes the best predictor of a student's final mark in calculus. He found that the best predictor of success in calculus was the grade from the Mathematics 12 provincial exam. Lee concluded that a good predictor of a student's success at calculus was the student's overall mathematical ability, as measured by the grade on the Mathematics 12 exam.

One explanation as to why the high school algebra mark is a good predictor of success in calculus is found in the research from Kajander and Lovric (2005). Their study states, "Students' performance is strongly correlated to the time they spend doing mathematics in high school" (2005, p. 152). They concluded that the more mathematics studied in high school, the more successful a student was in first year calculus.

Pence (1995) looked at the correlation between performance on a short pre-test and their final grade in calculus. The pre-test was of interest as it only looked at a small number of algebra concepts: multiplication, squaring, exponents, approximation and square rooting. The students' performance on the pre-test was compared with their final grade in calculus. The results showed that students that performed better on the pre-test also performed better in calculus. This study reinforces the issue that students entering calculus, presumed strong in mathematics, were having problems with basic mathematical skills. This point is stated by Pence (1995), "… work from students who either dropped out or failed first semester calculus showed patterns of incomplete understanding of the operation of multiplication”(p. 7). This paper concluded that the students with better mathematics skills perform better at calculus.

Overall, the research on the best predictors shows that the better a student leaves high school knowing their algebra, the better he/she will perform in university calculus. The investigators explain this need for strong algebraic skills as follows; algebra is the working environment in which calculus is taught. If the students do not have a good working knowledge of algebra, they will not be able to manipulate and properly explore and ultimately understand the lessons of calculus.

A measure of a students’ knowledge in algebra is a good overall measure of their mathematical understanding and potential. A student with good algebra skills has good mathematical understanding and will have a greater potential to learn calculus.

These explanations give reason to ask the tasks, “What are the specific mistakes made with algebra and why?” These tasks are looked at in the next two sections of the literature review.

2.5 Identification of the specific errors made in algebra

The presented studies focus on the mistakes with pre-calculus algebra. The studies show a variety of errors and a variety of misunderstandings dealing with algebraic manipulation. The purpose of this section is to give insight into the broad range encompassed in the phrase “algebraic errors made in calculus”.

A series of articles by Barnard attempted to cover the entire spectrum of errors in mathematics. Barnard (2002a, 2002b, 2002c), categorized all of the errors identified into the following headings:

- Meanings attached to algebraic letters.

  Errors made due to a misunderstanding of the use of letters symbolically as a variable. Numerous studies reinforced this important point that many students do not truly understand the symbolic use of letters as a physical measurable characteristic. "… what is clear is that if letters don't have meaning for the pupil, very little algebra is possible other than fragmentary success instrumental rote learning of rules." Barnard (2002, p.10)

- Fractions

  Errors made due to a misunderstanding of a fraction, in its' parts or as a whole. The misunderstanding that a fraction has only one form limits what students can do with algebra and fractions. For successful algebraic manipulation, a student must be able to see a fraction in a multitude of forms.

  For example, 4/5 can be seen as; 4 ÷ 5, 8/10, 4 x 1/5,…

- Computations verses relationships

  This largely due to the changing role of the equal symbol in arithmetic and algebra

- Errors made because of implicit urge of students to find a final answer.

This is better explained by the situation in which students struggle with an answer such as $y = 3x$. Students prefer and sometimes forcibly change a task to have a numerical answer, such as $y= 1$ and $x= 3$.

- Expressions as single items

  Errors made because an expression is NOT manipulated on collectively as one object. For example, $(3x)^3$ is evaluated as $3x^3$.

The specific errors identified by Barnard (2002) in his studies are:

- Operating on one piece of a compound term
- False linearity
- The need for closure
- Confusion between operations
- Subtraction/minus sign
- Misapplied rules
- Inappropriate cancelling
- Rearranging Formulae
- Equations

From Barnard, one can see that the errors are occurring with concepts that should have been learned in elementary and junior mathematics courses

(Mathematics 8 – 10). These concepts are considered an implicit part of senior (Mathematics 11 and 12) and university calculus.

Another researcher who looks to classify error types is Matz (1982). In her investigation, Matz identified some specific errors and suggests a framework of explanations for these mistakes. In her article in the book, "An Intellectual Tutoring System", Matz concluded that many errors could be classified as one of three types of errors:

1. Errors generated by an incorrect choice of an extrapolation technique
2. Errors reflecting an impoverished (but correct) base knowledge
3. Errors arising during the execution of a procedure

One of the mistakes Matz analysed is the type of errors with the squaring of binomials. In her investigation, she found some students were treating the squaring operation as linear, as in the following example: $(a+x)^2 = a^2+x^2$. Matz attempted to explain the reason for the mistake as an error in recognition of a string. She explained string recognition to be when someone looks to understand a set of characters as a complete whole. From this perspective $(a+x)^2$ is seen to be similar to (a+x)2. These students do not look at each symbol having independent and specific meaning. Algebra requires that each symbol be recognized as having specific meaning. Every symbol must be identified and used correctly to understand the meaning of an algebraic expression.

In "Concepts of School Algebra", Zalman Usiskin (1999), described some of the problems faced when teaching and learning algebra. Usiskin identified some of the issues as problems concerning variables and problems with the transition from arithmetic to algebra. Many students have a vague understanding that a letter represents a variable. Without understanding that the letters in algebra represent an object, students will have difficulties. Usiskin explained that many students have difficulty moving from arithmetic to algebra.

Understanding the reasons for student mistakes, leads one to look at the studies that focus on how students learn. Jarre (2008), looked at the overall characteristics of high school students that lead to success at post secondary studies. Jarre summarized that students in high school focus too much on obtaining high grades through procedural imitation and not enough attention is given to thinking and understanding mathematically. Students succeed when they were able to understand concepts, and work with tasks and in conditions that were both old and new. Students fail to succeed when they learned how to answer a set of prescribed tasks. These two study methods present different outcomes when related to post-secondary education. A student, who studies to imitate the teacher's solution for a set of previously described tasks that will be on future tests, will generally score extremely well and have a greater chance of acceptance into post secondary school. The problems with these students, is that they often do not have the mathematical understanding needed to succeed in future mathematical studies. On the other hand, the student that studies to understand would do better in post-secondary studies, but will not obtain as high grades as the imitator leading to lower chances at university entry. Jarre (2008) stated in his conclusion, "(properly prepared) students who when faced with problems in a new context, can self-correct their own thinking, adapt and succeed... …students trained to imitate and to perform in exchange for grades, become confused when the context and conditions are not identical to the secondary school" (2008, p. 34).

2.6 The problems with learning mathematics

This section considers studies that try to explain the reasons for the difficulty in learning mathematics. It looks at both the general reasons for weak mathematics ability and the reasons for the weakness in specific areas of mathematics.

Many studies have looked at difficulties students have with fractions. These studies have searched for the source of students' inability to show competent performance with fractions. The problem with fractions is presented starting with the studies that have identified the size and extent of the problem. Following this are the studies that have tried to explain the problems with learning fractions. This section concludes with the studies that focused on identifying the reasons for the problems with the learning of algebra.

2.6.1 General Problems associated with Algebra

Sarah McKibben (2009) investigated the topic, "Should all 8th graders take Algebra?". Her investigation into mathematics in America showed that many students were taking algebra without having mastered some of the needed basic skills, such as fractions. She found that the curriculum is not set up for students to be taking algebra at such an early stage.

Two studies investigated mathematics textbooks that reflect the local curriculum. These studies focused on some of the shortcomings of these teaching resources. In “Were Our Textbooks a Mile Wide and an Inch Deep?” Gu (2010) looked at three textbooks at the middle school level. This study found that the textbooks contained too many topics to be reasonably covered in one year. In one textbook over thirty topics were covered. Upon analyzing the quality of presentation of three concurrent textbooks, Gu found that the topics were not given enough time to reasonably build a strong understanding of concepts. Overall topics were repeated year after year, with little depth and focused on teaching how to answer specific types of tasks. He stated, “in these three incoherent courses, topics were highly repetitive and unfocused” (2010, p. 42).

Raman (1998) looked at the concept of continuity and the differences in teaching methodology between textbooks for high school pre-calculus and university calculus. Her analysis brings to light some of the reasons students have difficulties with the transition from high school mathematics to university mathematics. Raman found the high school textbook to be too informal, and too often gave mathematics explanations too specific to answer a certain type of task.

One of the concepts studied by Raman (1998) was the difference in the ways definitions were presented. High school textbooks often gave an incomplete definition of a concept in order to prepare students to answer a single type of task. This teaching to a type of task limits the scope and breadth in which students connect and learn to understand definitions. In calculus textbooks, the student is expected to fully understand definitions and to be able to use definitions in a diverse number of situations.

Raman stated, “In Pre-calculus the graph of the function was used to determine the continuity of the function. In calculus… the graph is related to, but not an essential part of the reasoning” (p. 12). Raman revealed the gap between the methodology of high school mathematics and university calculus. High schools should look more to building mathematic understanding for future use. One of Raman’s concluding comments was, “we should think about how to build on students’ understanding to help them acquire an appropriate orientation to mathematics”(p. 18).

2.6.2 The problems with fractions

Fractions are one of the foundational concepts of rational expressions. Working with rational expressions is a common aspect of a calculus course. The essential need for a full understanding of fractions gives reason for presenting several studies that have investigated the problems with learning fractions.

The NCES report from 1999, *Long Term Trend Mathematics Summary Data Tables for age 17 Students*, indicated that the students of age 17 demonstrate a lack of proficiency with fraction concepts. A similar study by Mullis, Dossey, Owen and Philips (1991) showed that only 46 percent of high school seniors demonstrated success with fractions and simple algebra.

One reason for the problems with working competently with fractions is explained by the opinion that fractions are being taught to students at too early an age. Freudenthal (1981) and Kieran (1980) suggested that fractions and rational number concepts not be taught until students are older and more mature. This point is reinforced in conclusions from Brown and Quinn (2007), "a student must experience and master the diverse interpretations of fractional numbers. Therefore, they (*Brown and Quinn*) recommend that an in-depth consideration of rational numbers be postponed until such time as the student studies algebra" (p.23)

Ahia and Fredua-Kwarteng (2006) found that the concept of division of fractions is placed too early into the mathematics curriculum, as previous instruction does not align itself well. They found that division of integers formed strong real life connections while some division models were problematic when applying to division with fractions.

The problems with fractions are of great importance considering future mathematical learning. Students in later grades learn concepts that are the algebraic extension of fractions, such as rational expressions. As such, when concepts of fraction are improperly learned, students do not have the foundational skills needed for learning concepts such as rational expressions. These holes in a students' knowledge will manifest in the teaching and learning of future mathematical concepts. The future problems experienced with fraction - based concepts often become frustrating when the source of the problem is not understood. The difficulty could be an aspect of the new concept or it could be a lack of understanding from a foundational skill such as fractions.

Overall, the reasons for the misconceptions with algebra have great diversity. Some of the student difficulties are with basic multiplication while other problems come from misapplying correct rules in the wrong setting. The list of problems is vast and cannot be fully understood without full knowledge of all the details of learning mathematics for all students from grade 1 – 12, which is beyond the scope of this work. However, the explanations from the previous studies help understand some of the problems students have with learning arithmetic and algebra. These explanations help form strategies and alternate teaching methodologies aiming to improve the teaching of mathematics.

While the general mistakes have been identified and categorized, my focus is on the mistakes that surface in the context of calculus. As such, I intend to identify the specific mistakes with algebraic concepts within calculus tasks. Specifically, I look at the calculus tasks in which the calculus component has been completed correctly, yet the final answer is incorrect solely due to a mistake in algebra.

3: DESCRIPTION OF THE STUDY

3.1 Research Question

As a calculus teacher I am concerned with students' success, in particular with their ability to complete a calculus task correctly. While there may be a variety of reasons that interfere with correct performance, my specific interest is in prerequisite concepts and procedures. As such, the main purpose of this study is to address the following research question:

What lack of competency with algebraic concepts interferes with students' performance on calculus tasks?

Specifically this study looks at the algebraic concepts that students struggle with in calculus tasks solely preventing them from obtaining a correct solution. To reiterate, when students correctly complete the calculus components of a task, what are some of the specific algebraic errors that hinder students from obtaining a correct solution.

Note: It is recognized that in mathematics education the term "concept" can be interpreted as connected to conceptual understanding of a specific content or idea. In this thesis, however, the term "concept" is used in a colloquial manner: it refers to a particular curricular content that implies algebraic manipulation.

As shown in the previous chapter, several studies pointed out that algebra is the main hurdle for students in post secondary calculus. In particular, studies by Diaz (2000), Edge and Friedberg (1984), Kajander and Lovric (2005), Barry and Davis (2006), Jungic and Kraemer (2006), Lee (1998), and Pence (1995) identified that the main obstacle for students in first year calculus was their lack of competency with high school algebra. However these studies did not define explicitly what they mean by saying “algebra” or “algebraic concepts”.

Acknowledging the variety of possible interpretations of “algebra” (Mason, 1996, Kaput,2008), in my study “algebra” refers to algebraic concepts and procedures studied in high school. The specific subset of algebraic concepts that in my inspection appear most frequently in calculus tasks.

3.2 Overview of the study

This chapter describes the details of the study and is divided into three sections.

- Section 3.3 describes the setting in which this study was conducted. This includes detailed descriptions of the school, the Calculus 12 course and the students.
- Section 3.4 describes the procedure for task design. This includes all preparatory work and planning of the task. Much of the design work focused on the tasks for tests, which were modified such that a correct solution would have two independent parts: one using algebra and the other using calculus.
- Section 3.5 describes the procedure for data collection. This part explains how students’ work was selected to be presented as examples, as well as describes the purpose and format for interviews with students.

3.3 The Setting

3.3.1 The School

This study was conducted in a school from the lower mainland of British Columbia. The high school is typical of many high schools from the lower mainland in that it has student population from many ethnic backgrounds. Among this diversity, over 80% are of East-Asian descent. The high school has a reputation of high academic performance and many of the graduating class receive scholarships from well-known universities. Acceptance rates into top ranked American colleges and Universities are high compared with other schools across Canada. The Advanced Placement program is among the largest in Canada with over 600 Advanced Placement exams being written each year (data from the past 4 years). Approximately 75-80% of the graduating students go on to post secondary education. This is extremely high compared to a provincial average around 25 – 30%.

In the year of this study, there were three AP Calculus classes and four Calculus 12 classes. All the calculus classes are approximately 27-30 students in size. Almost half of the graduating class (approximately 450 students) took calculus.

In summary, the school has a large portion of students that are academically driven. There is a healthy competitive environment pushing and encouraging students to strive for high academic standards and not be embarrassed about scholastic work habits and “being smart”.

3.3.2 The Course: Calculus 12

The Calculus 12 course is described in the BC ministry education curriculum guide, (http://www.bced.gov.bc.ca/irp/pdfs/mathematics/2000math1012.pdf)*. Many of the topics from university first year calculus are covered. Below is a copy of the Calculus 12 course outline. Upon completion, students can decide to write the challenge exam offered by UBC and SFU to obtain a mark for first year calculus. The course begins in September and runs until the middle of June. The class is approximately 40 weeks long with 192 minutes per week, with 88 classes in total. In addition, Calculus 12 has no final exam. All test marks are from quizzes and/or chapter tests. This feature of having no final exam is a characteristic that appeals to many students.

NOTE: The curriculum for Calculus 12 has not been updated since 2000. New mathematics curriculum is to be implemented in mathematics grades 8 – 12, with the final phase of implementation occurring with the mathematics in grade 12 in 2012. These changes may affect student performance in calculus, but the main focus of these curriculum changes in mathematics 8 - 12 applies to students not taking calculus in university.

<u>Calculus 12 Outline</u>

<u>Term 1</u>

- Overview of course
    - Limits

    - Derivatives
    - Integration

- Limits (limits, continuity, at infinity, in finite space)
- The Derivative and Differentiation techniques

  (derivative as a limit, techniques: product, quotient, chain, rule, implicit differentiation, and logarithmic differentiation)

<u>Term 2</u>

- Applications of the Derivative

  (related rates, extrema, optimization, linear approximations)

- Curve Sketching

  (higher derivatives, asymptotes, concavity)

- L'hopital's rule

<u>Term 3</u>

- Integration

  (Riemann sums, fundamental theorem, substitution)

- Integration by parts

  (L.A.T.E – guidelines – Log, Algebra, Trig, Exponents Substitution)

- Applications of Integration

    (Area, Volume , Distance, Work)

- Analytic Geometry
- (Polar differentiation, polar coordinates)

3.3.3 The Students

The students are typically strong in mathematics and grades from mathematics 10, 11 and 12 were mostly A's and B's. The majority of the students are not the top mathematics students at the school as most of the top mathematics students take Advanced Placement Calculus (all future references to Advanced Placement are called AP). All of the students plan on going on to a post-secondary institute and most state that they plan on taking university calculus the following year. Many have heard that calculus is a difficult course at university and that taking calculus in high school will be a benefit to their academic performance.

The types of students in the two calculus classes used in this study had the typical variety and distribution of most high school classes. The students vary in their expected grade for calculus. Some of the students worked every hard and still struggle to get a grade of a C+ or a B. Other students put in little effort, did the bare minimum of homework and achieve B's or higher. All of the students intend to obtain a university degree, but many are unsure of the major topic of their bachelor's degree. Some students plan to go to highly reputable universities and others to local community colleges. The attrition rate in each class is approximately 1 – 3 students per class. Students that do drop Calculus 12 usually do so after the first major test on which they scored very low.

Calculus 12 is generally not needed for credit towards high school graduation and some universities do not consider the calculus course mark for university admission. For these two reasons, Calculus 12 is generally given a lower priority with the student's study time.

Overall the students in the calculus classes are considered students with strong mathematic ability, who have reasonably good study habits and, when motivated, work hard to obtain good grades. The students that do not drop the class typically obtain a reasonable degree of success with calculus.

3.4 The Tasks

3.4.1 The algebraic operations investigated

The following section describes the algebraic procedures and rules considered, namely, exponents, squaring of binomials and rational expressions. As mentioned earlier, these are the algebraic concepts that I have observed students to show lacking proficiency with the most frequently in calculus tasks.

Exponents

Multiplication of like terms with exponents

$$\left(x^{a}\right)\left(x^{b}\right)=\left(x^{a+b}\right)$$

Division of like terms with exponents

$$\frac{\left(x^{a}\right)}{\left(x^{b}\right)}=\left(x^{a-b}\right)$$

Terms with exponents raised to another exponent

$$\left(x^{a}\right)^{b}\left(x^{a\times b}\right)$$

Power form of radicals

$$\sqrt[b]{x^{a}}=\left(x^{\frac{a}{b}}\right)$$

Squaring of monomials

Monomials integer coefficients

$$(ax)^{2}=a^{2}x^{2}$$

Monomials with fractional coefficients

$$\left(\frac{a}{b}x\right)^{2}=\frac{a^{2}}{b^{2}}x^{2}$$

Monomials of degree 2 with fractional coefficients

$$\left(\frac{x^{2}}{b}\right)^{2}=\frac{x^{4}}{b^{2}}$$

Rational Expressions

Division of rational expressions

$$\frac{x^a + b}{x^c} = x^{a-c} + bx^{-c}$$

Factoring and simplification of rational expressions

$$\frac{x^2 + (b+c)x + bc}{x+b} = \frac{(x+b)(x+c)}{x+b}$$

3.4.2 Task design and selection

Each task selected or designed for this study has a separate and independent calculus and algebra component. A typical task presents an algebraic expression to which a calculus procedure could not be properly applied. In this situation, the expression must be simplified or manipulated. An essential aspect of the tasks was that mistakes in algebraic manipulation would still allow a student to continue and show their knowledge/ability with calculus. This feature was important as part of the scope of this study is to identify that some of the difficulties for students in calculus are not the calculus itself, but rather the prerequisite mathematics.

The following example shows a task that has a clear algebraic component and a clear calculus component.

Find the derivative of f(x): $f(x) = \left(x^3\right)^4$

In this task, the function must first be simplified to remove the brackets. The correct simplification results in f(x) = x^{12}. A typical mistake is to incorrectly simplify f(x) to x^7. In the cases where this happens, students can continue with $f(x)=x^7$ and show their ability to find the derivative using the power rule.

Power Rule

$$f(x) = x^n \qquad\qquad f'(x) = nx^{n-1}$$

An example of a correct solution for the above derivative task is provided below.

$f(x) = \left(x^3\right)^4$ is correctly simplified to $f(x) = x^{12}$,

which with power rule gives $f'(x) = 12x^{11}$

An example of a solution with a mistake in algebra, but the correct use of calculus is as follows,

$f(x) = \left(x^3\right)^4$ is incorrectly simplified to $f(x) = x^7$,

which with power rule gives $f'(x) = 7x^6$

In the previous example of a correct solution and incorrect solution, both solutions have the correct use of the power rule and thus calculus. The incorrect solution has an algebraic mistake. The mistake in the algebra is the sole reason that the answer is incorrect. (Note: An expert may consider that simplification of exponents introduced above as unnecessary as the task can be approached by using the chain rule. However, in the context of the calculus course here, the chain rule appears much later in the sequence of topic than the power rule for derivatives.)

Seven tasks were designed to be used in tests during the year of the study. Table 4 presents the list of specific tasks used in this study.

Table 4 List of tasks used in data collection

Example Number	Calculus Topic	Task	Algebraic Concept
1	Limits	Evaluate the limit $\lim_{x\to\infty}\frac{x^5\cdot x^7\cdot x^{1/2}}{\left(\left(x^2\right)^3\right)^2}$	Exponents
2	Derivatives	$Find\quad f'(x)$ $f(x)=(2x)^3(2x^5-\frac{1}{4x})$	Exponents
3	Integration	$Evaluate\ the\ following$ $\int\frac{\left(x^6\right)^{1/2}}{\left(x^3\right)\left(x^9\right)}dx$	Exponents
4	Integration	$Evaluate\ the\ following$ $\int\pi\left((12)^2-\left(\frac{-4}{3}x\right)^2\right)dx$	Squaring of monomials
5	Integration	$Evaluate\ the\ following$ $\int\pi\left(\left(\sqrt{x}\right)^2-\left(\frac{x^2}{27}\right)^2\right)dx$	Squaring of monomials
6	Derivatives	$Find\quad f'(x)$ $f(x)=\frac{x^4-81}{x+3}$	Rational Expressions
7	Integration	$Evaluate\ the\ following$ $\int\frac{x^3-1}{x^2}dx$	Rational Expressions

3.5 Data collection

The main data in this study were collected from students' work on the particular tasks during their regular tests. Additional data come from interviews with several students regarding a few specific errors.

3.5.1 Students' work on tasks

During the year of the study I gathered the work of 53 students from the two calculus classes that I taught on the seven tasks from five tests. (The specific tasks used as part of these test are presented in Figure 5.) This presents about 350 solutions that were examined (as some students missed a test or did not respond to a particular task).

In what follows I describe the general procedure used for selecting the actual examples of students' work presented in chapter 4. All the students' solutions for each particular task were examined at one time and placed into one of four categories:

1. A clear and correct solution. The work clearly shows all the parts to a correct solution.
2. A correct solution that lacks clarity. The solution is either skipping steps or not showing sufficient work, or is too messy to clearly identify the steps to their solution.
3. A clear and incorrect solution. The work is clear and shows that the calculus was done correctly and the algebra was done incorrectly.

4. An incorrect solution that lacks clarity. The solutions in this category may have a mistake in the algebra or a mistake in the calculus or both a mistake in calculus and algebra. Little can be identified from the work shown as to what the student was thinking in solving this task.

The students' works that were classified into Categories 1 and 3 were scanned into a jpeg format and saved for future analysis.

3.5.2 Selecting the Examples

After all the tasks were analyzed, the examples of students' work from Categories 1 and 3 were revisited. The examples of a correct solution were analysed and compared to identify the solution that best exemplified Category 1.

Further, the scanned examples of algebraic mistakes were analysed. Examples from the same task were first grouped into similar mistakes. With each task typically there were two or three ways of making an algebraic mistake. Within each group of similar mistakes the solution that best demonstrated characteristics of Category 3 were identified.

3.5.3 Interviews with students

Three students that made an algebraic mistake were asked to reflect on why they had made the mistake. The students were asked to remember what they were thinking when they had made a mistake. It was hoped that students' answers would provide some insight into the nature of presented errors.

Four students were asked to analyze and comment on several mistakes that other students had made. The mistake was rewritten in my handwriting so there could be no identification of the origin of the work.

I explained to students that the purpose of interviews was to better understand their (or theirs classmates) thinking. I listened to the students' comments and repeated them for clarity. I took comments during the interview and summarised several particular expressions that students used. Immediately after each interview I reconstructed what students said, attempting to keep my summary as close as possible to their actual phrases.

4: DATA ANALYSIS

4.1 Overview of the data analysis

The following section presents and analyzes the data for this study. The data is a set of examples of various students' work from Calculus 12 tests. The examples show that students are making mistakes with pre-requisite algebra yet are completing the calculus component of the task correctly. The examples are grouped into subsets showing a variety of students' work on the same test task. The first example of every group shows a correct solution. Subsequent examples from the same group show a mistake in algebra followed with a correct use of calculus. From this structure, each set of examples shows that students are using calculus correctly, but are not getting the correct solution due to a mistake in algebra.

The data are sectioned according to the following algebraic topics:

- Exponents
- Squaring of monomials and binomials
- Rational Expressions

As shown in Table 5, the examples are labelled with both a number and a letter, such as Example #1A. These examples are the same as presented in Chapter 3 – Table 4. All examples on the same task have the same number. The examples labelled with an A are all the examples of a correct solution. The examples labelled with B, C, D or E are examples of different incorrect solutions.

To keep the identity of the students anonymous, their names have been replaced with a label, such as S5.

Table 5 Table of Contents for Examples

Algebraic Concept	**Example Numbers**	**Calculus Topic**	**Page**
Exponents	1A, 1B, 1C	Limits	p. 62 - 64
	2A, 2B, 2C, 2D	Derivatives	p. 65 - 68
	3A, 3B, 3C, 3D	Integration	p. 69 – 71
Squaring Operation	4A, 4B	Integration	p. 72 - 75
	5A, 5B, 5C	Integration	p. 76 - 81
Rational Expressions	6A, 6B, 6C, 6D, 6E	Derivative	p. 82 - 86
	7A, 7B, 7C	Integration	p. 87 - 90

4.2 Errors with Exponents

Example 1A Exponents within limits (Correct solution)

Step A

Evaluate the following

4) $\lim\limits_{x\to\infty} \frac{x^5 \bullet x^7 \bullet x^{\frac{1}{2}}}{\left(\left(x^2\right)^3\right)^2}$

$$= \lim_{x\to\infty} \frac{x^{5+7+\frac{1}{2}}}{x^{12}}$$

$$= \lim_{x\to\infty} \frac{x^{12\frac{1}{2}}}{x^{12}} \quad \text{bigger}$$

4. ∞

Step B

In the following solution, S1 has correctly combined the three terms in the numerator and the multipart denominator.

In step A, S1 has decided to simplify the numerator and denominator. When like terms are multiplied together, the powers are added so the numerator being simplified to $x^{12.5}$ is correct. In the denominator there is one term, which is cubed and then squared. In this example, the exponent 2 is multiplied by 3 and then multiplied by 2 resulting in 12. S1 has come to the correct simplified version x^{12}.

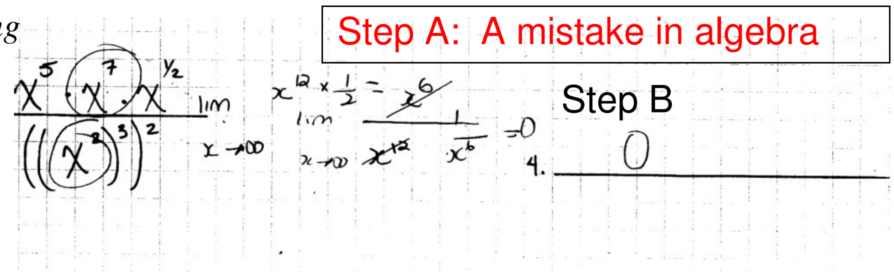
In step B, the calculus step, her comment of the exponent of the numerator being “bigger” shows she has identified the largest power and most significant part of the function. This limits task has been correctly completed with a final answer of positive infinity. S1 has correctly manipulated the exponents and correctly applied calculus showing a correct final answer.

Example 1B **Exponents within limits (Incorrect solution)**

In the following solution, S2 has incorrectly combined the three terms in the numerator. He has correctly applied the concepts of calculus, but because of a mistake in algebra has arrived at an incorrect solution.

Evaluate the following

4) $\lim\limits_{x \to \infty} \dfrac{x^5 \bullet x^7 \bullet x^{\frac{1}{2}}}{\left(\left(x^2\right)^3\right)^2}$



In step A, S2 has decided to simplify the numerator and denominator. Working with the numerator, he has started correctly by adding the exponents 5 and 7 to 12, but has incorrectly multiplied ½ with 12. Instead of adding the exponents to 12.5, he has multiplied the powers together resulting in 6. The interesting point of this error is that he has used the two different algebraic methods to combine terms needing the same method.

The calculus step is not explicitly seen, but the class has been taught to compare the power of the simplified numerator and denominator. When the numerator is greater than the denominator the solution is (+ or -) infinity. When the denominator is greater than the numerator (as in this case) the solution is zero.

S2 has applied the concepts of calculus correctly, but due to a mistake in algebra has come to an incorrect solution.

Example 1C **Exponents within limits (Incorrect solution)**

In the following solution, S3 has incorrectly combined the three terms in the numerator. He has correctly applied the concepts of calculus, but it is only by chance that he has come to a correct solution.

Evaluate the following

$$4)\ \lim_{x\to\infty}\frac{x^{5}\bullet x^{7}\bullet x^{\frac{1}{2}}}{\left(\left(x^{2}\right)^{3}\right)^{2}}$$

Step A: A mistake in algebra

$$4)\ \lim_{x\to\infty}\frac{\overset{35 \quad \frac{35}{2}}{x^{5}\cdot x^{7}\cdot x^{1/2}}}{\left(\left(x^{2}\right)^{3}\right)^{2} \; 12}$$

Step B

4. ∞

In Step A, S3 has identified that the like terms are to be simplified, but has incorrectly multiplied the exponents of the numerator. Instead of adding the exponents to 12.5, he has multiplied the exponents resulting with 35/2. The interesting point of this error is that he has used the same method of simplification of exponents for the denominator and numerator, even thought the two are obviously different. The numerator involves multiplying three terms. The denominator involves one term and three powers.

The calculus, step B, is not explicitly seen, but the simple comparison of power of the numerator with the denominator leads one to the correct solution. When the numerator is greater than the denominator the solution is infinity.

Example 2A Exponents in Derivatives (correct solution)

Find the derivative of f(x)

3. $f(x) = (2x)^3\left(2x^5 - \dfrac{1}{4x}\right)$

$f(x) = 8x^3\left(2x^5 - \dfrac{1}{4x}\right)$ STEP A

$f(x) = 16x^8 - \overset{2}{\cancel{8}}x^{\overset{2}{\cancel{3}}}\left(\dfrac{1}{\cancel{4x}}\right)$ STEP B

$f(x) = 16x^8 - 2x^2$

$f'(x) = 128x^7 - 4x$ STEP C

S4 has completed this derivative task completely correct. She has identified the task to be easier to work on if the function is first simplified through algebraic manipulation. In step A, she has expanded the first bracket $(2x)^3 = 8x^3$. In Step B, she correctly distributed $8x^3$ to both terms of the binomial and for clarity has rewritten the simplified function.

In step C, S4 has correctly applied the power rule to the function arriving at a correct answer.

Example 2B Exponents with Derivatives (Incorrect solution)

Find the derivative of f(x).

3) $f(x) = (2x)^3\left(2x^5 - \dfrac{1}{4x}\right)$

3) $f(x) = (\cancel{2x})^{\cancel{3}\,2}\left(2x^5 - \cancel{\dfrac{1}{4x}}\right)$

$\overset{1x^3}{\cancel{2x^3}}\left(2x^5 - \dfrac{1}{\underset{2}{\cancel{4x}}}\right) = 4x^8 - \dfrac{x^2}{2} = 32x^7 - x$

STEP A: A mistake in algebra

STEP B

STEP C

3. $32x^7 - x$

In part A, S5 first expanded $(2x)^3$. The student incorrectly expanded $(2x)^3$ to equal $2x^3$. The exponent 3 was incorrectly applied to only the x and not the 2.

The expansion of the binomial in step B is correct. The application of the power rule in step C is correct. This solution was incorrect because the coefficient for each term was incorrect. It is important to note that the step requiring calculus, applying the power rule, was completed correctly.

Example 2C **Exponents with Derivatives (Incorrect solution)**

Find the derivative of f(x).

3) $f(x) = \overset{8x}{(2x)^3}\left(2x^5 - \frac{1}{4x}\right)$

STEP A: A mistake in algebra

$f(x) = 16x^5 - \frac{8x}{4x}\ 2$

$f(x) = 16x^5 - 0$

STEP B

$16 \times 5 = 80$

$f'(x) = 80x^4$

STEP C

3. $80x^4$

In part A, S6 made a mistake expanding $(2x)^3$. S6 incorrectly expanded "$(2x)^3 = 2^3x = 8x$". He applied the exponent 3 to the 2 and not the x.

In part B, he has incorrectly distributed $8x(2x^5 - \frac{1}{4x})$ to be $16x^5 - 2$.

In part C, the calculus step (the power rule) has been correctly applied to the results from part B. The solution to this task is incorrect due to two mistakes in algebra, yet the calculus part needed in this task is done correctly.

Example 2D Exponents in Derivatives (Incorrect solution)

Find the derivative of f(x).

3) $f(x) = (2x)^3\left(2x^5 - \dfrac{1}{4x}\right)$

3) $f(x) = (2x)^3\left(2x^5 - \dfrac{1}{4x}\right)$

$8x^3\left(2x^5 - 4x^{-1}\right)$ STEP A: A mistake in algebra

$16x^8 - 32x^2$

$f'(x) = 128x^7 - 64x^1$ STEP B

3. $f'(x) = 128x^7 - 64x$

In step A, S7 expanded $(2x)^3 = 8x^3$ and in the same step wrote the fraction $\dfrac{1}{4x} = 4x^{-1}$. The student correctly expanded $(2x)^3$, but made a mistake when rewriting a fraction with negative exponents. The fraction should have been written as $\dfrac{1}{4x} = \dfrac{1}{4}x^{-1}$. The 4 from the denominator was incorrectly brought unchanged up to the numerator.

The expansion of the binomial in step B is correct. The application of the power rule in step C is correct. This solution, received part marks for the first term being correct, but did not receive marks for the incorrect second term. The student has correctly applied the rules of calculus, but a mistake in algebra has caused the answer to be incorrect.

Example 3A **Exponents with Integration (Correct solution)**

Evaluate the following

STEP A

4. $\int \frac{\left(x^6\right)^{\frac{1}{2}}}{\left(x^3\right)\left(x^9\right)} dx$ $\int \frac{x^3}{x^{12}} dx$

STEP B

$= \int \frac{1}{x^9} dx = \int x^{-9} dx$

$= -\frac{1}{8} x^{-8} + C$ STEP C

In this solution, S8 has done all parts correctly and has presented a correct solution. In step A, she has correctly simplified the numerator and denominator. In doing so, she has correctly multiplied 6 by ½ simplifying the numerator to x^3. In the denominator she has correctly added the exponents 3 and 9 resulting in x^{12}.

In step B, she has simplified the numerator and denominator resulting in x^{-9}. At this point the algebraic expression has been simplified to a point where integration can be easily applied. The final step, and only step requiring calculus S8 has found the antiderivative and arrived at a correct answer.

Example 3B Exponents with Integration (Incorrect solution)

Evaluate the following

4. $$\int \frac{\left(x^6\right)^{\frac{1}{2}}}{\left(x^3\right)\left(x^9\right)}\,dx$$

$$= \int \frac{X^{\frac{6}{2}}}{X^{12}}$$ STEP A

$$= \int \frac{X^3}{X^{12}}$$

$$= \int \frac{1}{X^4}$$ STEP B: A mistake in algebra

$$= \int X^{-4}$$ STEP C

$$= \frac{1}{-3} X^{-3} + C$$

In step A, S9 correctly simplified the numerator and denominator. She made a mistake in step B, when she combined the numerator and denominator, as follows," $\frac{x^3}{x^{12}} = x^{3-12} = x^{-9}$ ". The result from step B was correctly integrated (calculus) in step C. S9 arrived at an incorrect solution to a calculus task having completed the calculus parts correctly. A mistake in algebra is the sole reason for the mistake.

Example 3C **Exponents with Integration (Incorrect solution)**

Evaluate the following

4. $\displaystyle\int \frac{\left(x^6\right)^{\frac{1}{2}}}{\left(x^3\right)\left(x^9\right)}\,dx$

$= \displaystyle\int \frac{x^{\frac{12}{2}+\frac{1}{2}}}{x^{3\times 9}}$ STEP A: A mistake in algebra

$= \displaystyle\int \frac{x^{\frac{13}{2}}}{x^{27}}$

$= \displaystyle\int x^{\frac{13}{2}} \cdot x^{-27}$ STEP B

$= \displaystyle\int x^{\frac{13}{2}+\left(-\frac{54}{2}\right)}$

$= \displaystyle\int x^{-\frac{41}{2}}$ STEP C

$= -\dfrac{2x^{-\frac{39}{2}}}{39} + C$

In step A, S10 incorrectly simplified the numerator and denominator. She made a mistake when she simplified $\left(x^6\right)^{\frac{1}{2}}$ to $x^{\frac{13}{2}}$, and then $x^3 \bullet x^9$ to x^{27}. The following algebraic and calculus steps B and C were completed correctly. Her final answer is incorrect due to a mistake in algebra.

4.3 Errors with Squaring Operation

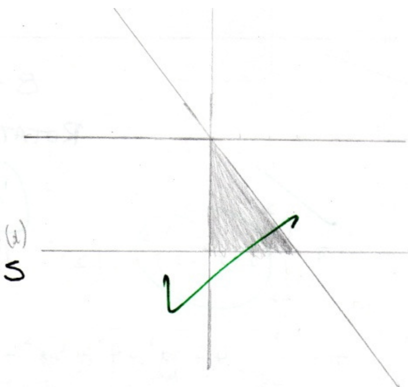
Example 4A **Squaring Operation with Integration (Correct solution)**

1. Find the volume formed when the bounded area is rotated about an axis.

$$f(x) = \frac{-4}{3}x$$
$$g(x) = -12$$
$$x = 0$$

Rotated about the x – axis

Step A



Step B

$-\frac{4}{3}x = -12$

$x = \frac{-36}{-4}$

$x = 9$

$f(x) = \frac{-4}{3}x$

$g(x) = -12$

$x = 0$

ROTATED ABOUT X-AXIS

$\pi\int_0^9 (12)^2 - (-\frac{4}{3}x)^2 \, dx$

Step C

$\pi\int_0^9 144 - \left(\frac{16}{9}x^2\right) dx$

Step D

$\pi\int_0^9 144 - \frac{16}{9}x^2 \, dx$

$144x\pi - \frac{16}{9}\cdot\frac{x^3}{3}\pi + c \Big|_0^9$

Step E

6/6

$144x\pi - \frac{16x^3}{27}\pi + c \Big|_0^9$

$1296\pi - 432\pi$

Step F

864π

In step A the S11 has found the bounded area (A and B). This involves graphing the equations provided in the task. The graph was used to find both the outer and inner radii. The graph was also used to find the left and right x values of the bounded area. The graph shows that the left boundary corresponds to x = 0, but the right x value must be calculated. Student S11 has found the correct x value by setting f(x) and g(x) to be equal and then solving. The mathematics involved in these initial steps is algebraic.

At this point (C) the student used the information of the bounded area to set up the integral. The general form of the integral is as follows,

$$Volume = \int \pi\left(r_{outer}\right)^2 - \int \pi\left(r_{inner}\right)^2$$

S11 has identified the inner and outer radii as well as the left and right x values of the bounded area. With the correct integral, the student simplified the two squared terms (D) and then uses calculus to get the indefinite integral with left and right x values (E). At this point S11 substituted the left and right x values. After simplification (F) by gathering the like terms S11 has the final answer 864π. This value 864π represents the volume formed when the bounded area is rotated about an axis.

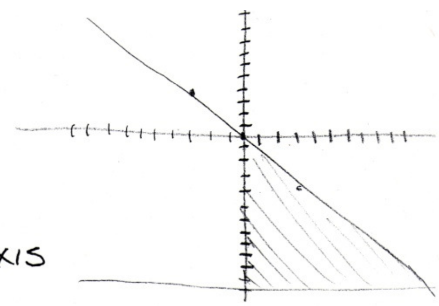
A complete solution is a blend of algebraic and calculus concepts. S11 has used algebra and calculus together to get a complete solution. It is important note that the calculus involved in the solution is in less than half the shown work.

Example 4B **Squaring operation with Integration (Incorrect solution)**

1. Find the volume formed when the bounded area is rotated about an axis.

$$f(x) = \frac{-4}{3}x$$
$$g(x) = -12$$
$$x = 0$$
$$\textit{Rotated about the } x - \textit{axis}$$

Step A



Step B

$$\frac{-4}{3}x = -12$$
$$x = \frac{-12}{\frac{-4}{3}}$$
$$x = -12\left(\frac{-3}{4}\right)$$

$$f(x) = \frac{-4}{3}x$$
$$g(x) = -12$$
$$x = 0$$

ROTATED ABOUT X-AXIS

$$\pi\int_0^9 (-12)^2 - (-4/3x)^2\,dx$$

$$-4/3 \cdot -4/3 = \frac{16}{9}$$

Step D A mistake in algebra

$$\pi\int_0^9 144 - \frac{16}{9}x\,dx$$

$$\pi\left(144x - \frac{16}{18}x^2\right)\Big|_0^9$$

Step E

$$\pi\left(144(9) - 16/18(9)^2\right)$$
$$\pi(1296 - 72)$$
$$= 1224\pi$$

Step F

This solution begins much like the example 4A. Part A shows the correct graph of the equations f(x), g(x) and x = 0 and produces the correct graph of the bounded area. In part B, the student has found the exact x value where f(x) and g(x) intercept(B). It is of some interest that the solution, x = 9 is not written here. The student has written 12*(3/4) and not evaluated this expression. Later, the value of x = 9 is correctly used when S12 set up the correct integral. In part C, the student displayed his knowledge of calculus and set up the integral correctly. The integral is again of the $Volume = \int \pi \left(r_{outer}\right)^2 - \int \pi \left(r_{inner}\right)^2$ form,

S12 set up an integral identical to the one in example 4A. Up to this point this solution was correct. In part D, S12, made a mistake applying the squaring operation. She wrote, "$\left(\frac{-4}{3}x\right)^2 = \frac{16}{9}x$".

S12 showed his work, but for some reason only applied the squaring operation to the $\frac{-4}{3}$ and not the x.

S12 used the incorrect result from step D in the two final steps. In step E, S12 correctly integrated the result from step D. S12 gathered like terms correctly in step F.

The final answer is incorrect. Student, S12, has applied his understanding of calculus correctly, but a mistake in algebra has caused his final answer to be incorrect.

Example 5A Squaring operation with Integration (Correct solution)

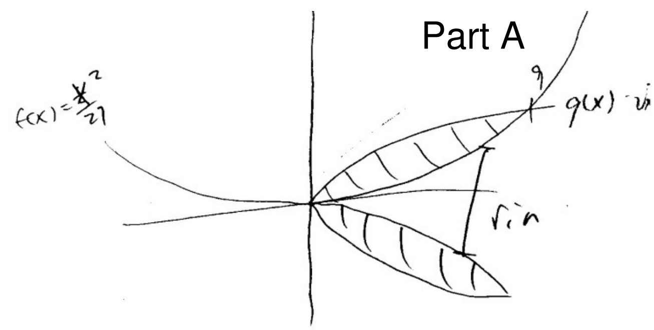
In these tasks, the students were given the instructions to stop at the indefinite integral.

Find the volume formed when the bounded area is rotated about an axis

$$f(x) = \frac{1}{3}\left(\frac{x}{3}\right)^2$$

$$g(x) = \sqrt{x}$$ Rotated about x axis

Part A



$r_{in} = \frac{x^2}{27}$

$r_{out} = \sqrt{x}$

Part B

$$\pi \int_0^9 (\sqrt{x})^2 - \left(\frac{x^2}{27}\right)^2 \, dx$$

Part C

$$\pi \int_0^9 \left(x - \frac{x^4}{729}\right) dx$$

Part D

$$\pi \left(\frac{x^2}{2} - \frac{x^5}{3645}\right)\Big|_0^9$$

$\frac{1}{27}x^2 = x^{\frac{1}{2}}$

$x^2 = 27x^{\frac{1}{2}}$

$x^{\frac{3}{2}} = 27$

$x = 9$

4/4

In part A, S13 has begun by graphing the two functions f(x) and g(x). She shaded the bounded region and also identified the points that mark the intersection of the two functions. This intersection marking the right boundary of the bounded area is seen to be x = 9, and the graph clearly shows the origin to be the left boundary (x = 0).

In part B the correct calculus is shown as the integral was correctly set up. The solution shows the integral specific to this task. This integral follows the general form but must be simplified before the indefinite integral can be calculated.

In part C, the squaring of the two terms is completed. It should be noted that S13 keeps function g(x) in radical form. The solution shows that square of the square root of x is x. The solution shows that f(x) has been simplified to $x^2/27$ and then squared correctly to $x^4/729$.

In part D, the solution shows the correct application of calculus concluding with the indefinite integral.

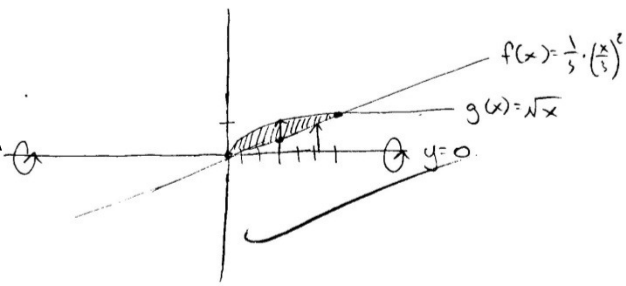
Example 5B **Squaring operation with Integration (Incorrect solution)**

Find the volume formed when the bounded area is rotated about an axis

$$f(x) = \frac{1}{3}\left(\frac{x}{3}\right)^2$$

$$g(x) = \sqrt{x}$$ Rotated about x axis

Step A



$$\sqrt[2]{\frac{x^2}{27}} = \sqrt[2]{x}$$

$$\boxed{x = 9}$$

$$\left(\frac{x^2}{27}\right)\left(\frac{x^2}{27}\right) = \frac{x^4}{729}$$

$r_{out} = \sqrt{x}$

$r_{in} = \frac{1}{3}\cdot\left(\frac{x}{3}\right)^2$

$$\int_{left}^{right} \pi r_{out}^2 - \pi r_{in}^2 \, dx$$

Step B

$$\int_0^9 \pi(\sqrt{x})^2 - \left(\frac{1}{3}\cdot\left(\frac{x}{3}\right)^2\right)^2$$

4/4

$$\pi\int_0^9 x^{\frac{1}{4}} - \frac{x^4}{729}\, dx$$

Step C: A mistake in algebra

$$= \pi\left(\frac{4x^{\frac{5}{4}}}{5} - \frac{x^5}{3645}\right)\Bigg|_0^9$$

Step D

In part A S14 has begun by graphing the two functions f(x) and g(x). He shaded the bounded region and has marked the left and right intersections of the two functions. The right boundary value has been calculated to the right of the graph. The left boundary can be identified from the graphed to be the origin.

In part B the correct calculus is shown as the integral was correctly set up. Both the left and right boundaries were placed in integral notation correctly (the same as in example 2A).

In part C, S14 needed to square two terms. The second term is squared successfully by squaring $x^2/27$ correctly to $x^4/729$. The student has made an algebraic mistake when he stated that $(x^{1/2})^2 = x^{1/4}$.

$$\left(\sqrt{x}\right)^2 = x^{1/4}$$

In part D, the student correctly found the indefinite integral of the solution from step C. Their final answer was incorrect because of an algebraic mistake. S14 showed that every step needing calculus was done correctly. Three steps of this solution needed calculus and all three steps were done correctly. The mistake in his final answer was due to a mistake with the squaring operation (an algebraic operation).

Example 5C Squaring operation with Integration (Incorrect solution)

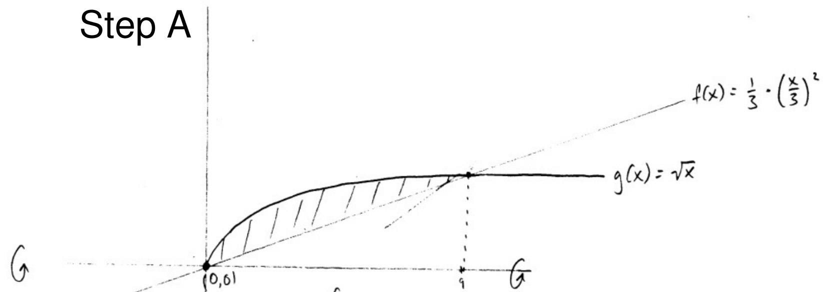
Find the volume formed when the bounded area is rotated about an axis

$$f(x) = \frac{1}{3}\left(\frac{x}{3}\right)^2$$

$$g(x) = \sqrt{x}$$ Rotated about x axis

.

Step A



2/4

$$\int \pi r^2 dx$$

Step B

$$\int \pi r_{out}^2 dx - \pi r_{in}^2 dx$$

$$\left(\frac{1}{3}\cdot\left(\frac{x}{3}\right)^2\right)\left(\frac{1}{3}\cdot\left(\frac{x}{3}\right)^2\right)$$

$$\frac{1}{9}\cdot\frac{1}{3}\left(\frac{x}{3}\right)^2\cdot\frac{1}{3}\left(\frac{x}{3}\right)^2\cdot\left(\frac{x}{3}\right)^4$$

$$\int_0^9 \pi\left(\sqrt{x}\right)^2 - \pi\left(\frac{1}{3}\cdot\left(\frac{x}{3}\right)^2\right)\Big)^2$$

$$\pi\int_0^9 \left(\sqrt{x}\right)^2 - \left(\frac{1}{3}\cdot\left(\frac{x}{3}\right)^2\right)\Big)^2$$

$$\pi\int_0^9 (x) - \left(\frac{1}{9}\cdot\frac{1}{3}\left(\frac{x}{3}\right)^2\cdot\frac{1}{3}\left(\frac{x}{3}\right)^2\cdot\left(\frac{x}{3}\right)^4\right)$$

Step C: A mistake in algebra

$$\pi\left(\frac{x^2}{2} - \left(\frac{1}{9}x\cdot\frac{1}{3}\left(\frac{\frac{x^2}{2}}{3}\right)^2\cdot\frac{1}{3}\left(\frac{\frac{x^2}{2}}{3}\right)^2\cdot\left(\frac{\frac{x^2}{2}}{3}\right)^4\right)\right)\Bigg|_0^9$$

$$\pi\left[\frac{x^2}{2} - \left(\frac{1}{9}x\cdot\frac{1}{3}\left(\frac{x^2}{6}\right)^2\cdot\frac{1}{3}\left(\frac{x^2}{6}\right)^2\cdot\left(\frac{x^2}{6}\right)^4\right)\right]\Bigg|_0^9$$ Step D

In part A S15 begun by graphing the two functions f(x) and g(x). She shaded the bounded region and identified the left value correctly as the origin, but made a mistake in identifying the right boundary. She found the right boundary to equal one and the correct value is nine. There is a table of values that showed she had incorrectly evaluated the f(x) function at x = 1 to be y = 1. The results from step A will be used to evaluate future steps in this example.

In part B the correct calculus is shown as the integral was correctly set up. The left and right boundaries were placed in the calculus-integral notation correctly.

In part C, S15 must square two terms. The first term was squared successfully $\left(\sqrt{x}\right)^2 = x$. S15 made an algebraic mistake when she stated incorrectly that $\left(\frac{1}{3}\bullet\left(\frac{x}{3}\right)^2\right)^2$ is expanded to $\frac{1}{9}\bullet\frac{1}{3}\left(\frac{x}{3}\right)^2\bullet\frac{1}{3}\left(\frac{x}{3}\right)^2\bullet\frac{x^4}{9}$. Analysis of the work suggests that the student has expanded the squared monomial as if it was a squared binomial.

In part D, only one of the terms is correctly integrated. The first term is correctly integrated while the remaining three terms are incorrectly integrated. The student has made no attempt to simplify the polynomial into a form similar to the first term in which she integrated correctly.

Her final answer is incorrect because of algebraic and calculus mistakes. This example is presented to show how extreme some of the algebraic mistakes can be.

4.4 Errors made with Rational Expressions

Example 6A Rational Expressions with Derivatives (Correct solution)

Find the derivative of f(x)

STEP A STEP B

$$7)\ f(x) = \frac{x^4-81}{x+3} = \frac{(x^2-9)(x^2+9)}{x+3} = \frac{(x^2+9)\ (x-3)\cancel{(x+3)}}{\cancel{x+3}}$$

$$= (x^2+9)(x-3) = x^3-3x^2+9x-27$$

STEP C

$$f'(x) = 3x^2-6x+9$$

7. $3x^2-6x+9$

In the above solution S16 has provided a correct solution. In step A, she has correctly expanded the binomial in the numerator.

$$\left(x^4-81\right)=\left(x^2+9\right)\left(x-3\right)\left(x+3\right)$$

In step B, she has correctly simplified the function, cancelling (x-3) in both the numerator and the denominator, resulting in a four-term polynomial.

In step C, she has correctly expanded the two binomials.

$$(x+3)\left(x^2+9\right)=x^3+3x^2+9x+27$$

In step D, she has correctly used the power rule and found the derivative. Her final answer is correct, due to successfully completing a combination of algebraic and calculus steps.

Example 6B Rational Expressions with Derivatives (Incorrect solution)

Find the derivative of f(x)

7) $f(x) = \dfrac{x^4 - 81}{x + 3}$

STEP A: A mistake in algebra

$f(x) = (x^4 - 81)(x^{-1} + \frac{1}{3})$

$f(x) = x^3 + \frac{x^4}{3} - 81x^{-1} - 27$ STEP B

$f'(x) = 3x^2 + \frac{4}{3}x^3 + 81x^{-2}$ STEP C

$f'(x) =$

7. $3x^2 + \frac{4}{3}x^3 + 81x^{-}$

In Step A, S17 has incorrectly manipulated the binomial expression in the denominator. He has rewritten (x+3) from the denominator as x^{-1} + 1/3. The mistake can be rewritten as: $\frac{1}{x+3} = x^{-1} + \frac{1}{3}$

In step B, S17 correctly expanded the two binomials. Using the result from Step B, S17 correctly used the power rule and found the derivative in step C. The algebraic mistake in step A caused S17 to arrive at an incorrect answer, yet the step involving calculus was done correctly.

Example 6C Rational Expressions with Derivatives (Incorrect solution)

Find the derivative of f(x)

7) $f(x) = \dfrac{x^4 - 81}{x + 3}$

$\boxed{\text{STEP A: A mistake in algebra}}$

$= x^4 - 81x^{-1} + 3$

$f(x) = 4x^3 + 81x^{-2}$

STEP B

7. $4x^3 + 81x^{-2}$

In Step A, S18 has incorrectly manipulated the binomial expression in the denominator. She seems to have made two mistakes. Initially, she moved (x+3) from the denominator and to the numerator as (x^{-1}+3). When she did this she incorrectly failed to multiply (x^{-1}+3) with the original numerator (x^4-81).

In Step B, S18 has correctly used the power rule on the result from step A to find the derivative. The algebraic mistake made in step A caused S18 to have an incorrect solution, even though the calculus was done correctly.

Example 6D Rational Expressions with Derivatives (Incorrect solution)

Find the derivative of f(x)

STEP A: A mistake in algebra

7) $f(x) = \dfrac{x^4-81}{x+3}$ $\quad \dfrac{x^4}{x} \; \dfrac{-81}{+3} = -27$

$\dfrac{x^4}{x} - 27$ STEP B

$f'(x) = 3x^2$ $\quad x^3 - 27$ STEP C

7. $f'(x) = 3x^2$

In step A, S19 has incorrectly separated the rational expression into two terms. She has written, “$\dfrac{(x^4-81)}{x+3} = \dfrac{x^4}{x} - \dfrac{81}{3}$”.

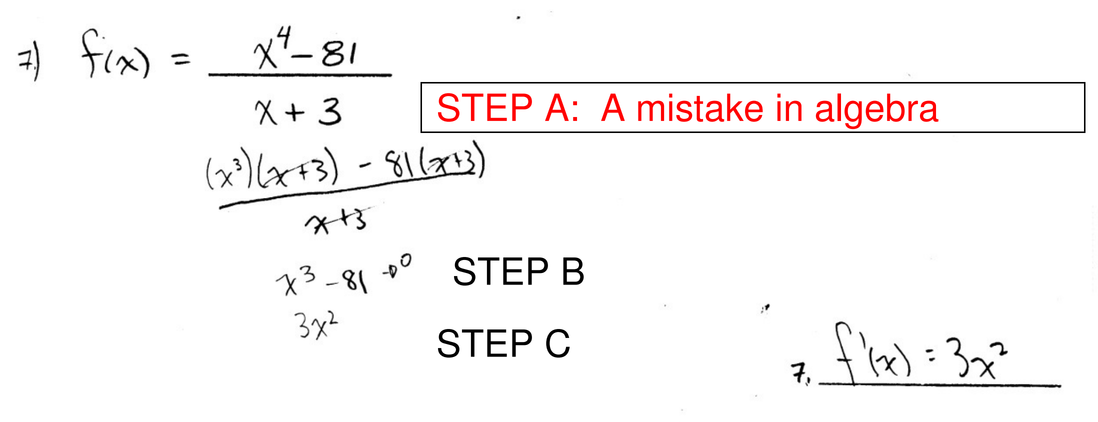
In step B, she has correctly simplified the two terms resulting with

$$= \frac{x^4}{x} - \frac{81}{3}$$
$$= x^3 - 27$$

In step C, she has applied the power rule correctly. This step requiring calculus was done correctly. The final answer in this solution is incorrect due to a mistake in algebra.

Example 6E Rational Expressions with Derivatives (Incorrect solution)

Find the derivative of f(x)



In step A, S20 has made a mistake with factoring. She has attempted to simplify the function by factoring (x+3) out from x^4 -81. She has incorrectly factored as follows, x^4 -81 = x^3 (x-3) - 81(x-3)

In step B, she has correctly worked with the result from step A and correctly simplified the function to x^3 -81. It is worth noting that she apparently thinks that her work at this point is correct. She has written,

$$\text{“}\frac{x^4 - 81}{x-3} = x^3 - 81\text{”}$$

In the final step, step C, S20 has correctly found the derivative of x^3 -81. Her work is lacking complete notation, but her final answer on the allocated blank line, correctly shows that she has derived the function in step B. In conclusion, S20 has made a mistake in factoring, but has done all of the calculus correctly. The final answer in this solution is incorrect due to a mistake in algebra.

Example 7A Rational Expressions with Integration (Correct solution)

20) $\int_1^2 \frac{x^3-1}{x^2}\,dx$

$= \int_1^2 (x - x^{-2})\,dx$ Step A

$= \frac{1}{2}x^2 + x^{-1}\Big|_1^2$ Step B

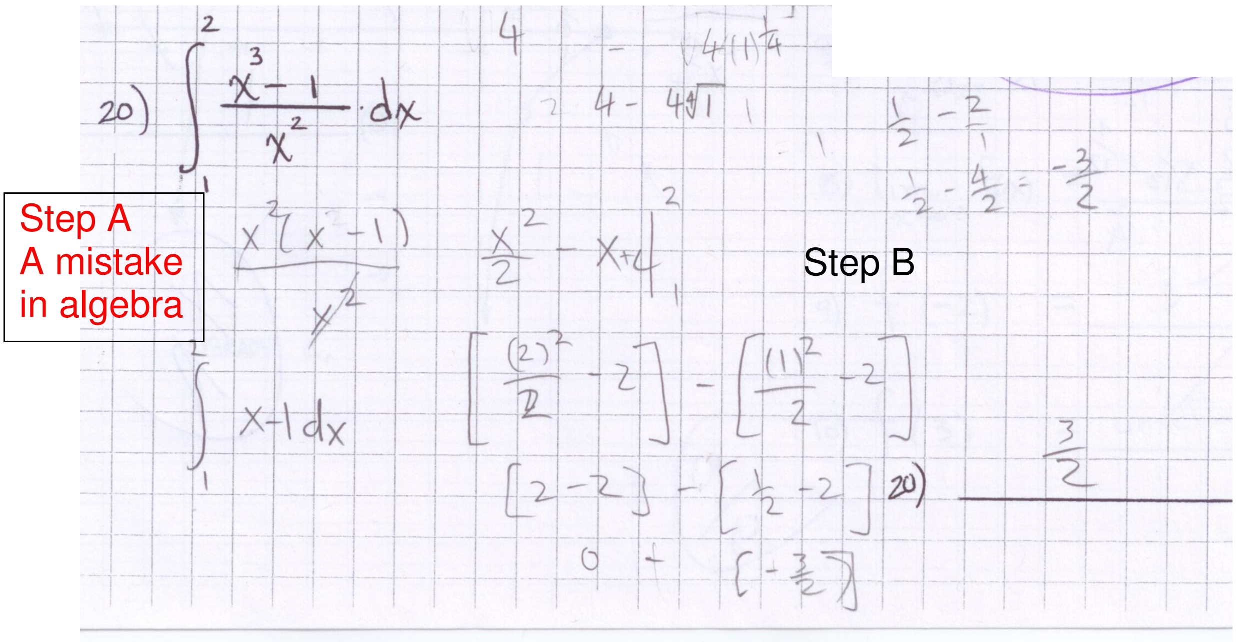
$= \left(\frac{1}{2}\times 4 + \cancel{\frac{1}{2}}\right) - \left(\cancel{\frac{1}{2}} + 1\right) = 2 - 1$ 20) 1

$= 1$

S21 has correctly completed this integration task. In part A, she correctly manipulated the rational expression into two terms, simplified each term and then re-written each term without a denominator. The result from part A allows her to easily integrate with calculus. In part B, she correctly found the indefinite integral followed with the definite integral. The final answer is based on correct algebraic manipulation and correct application of an understanding of calculus.

Example 7B Rational Expressions with Integration (Incorrect solution)

Evaluate the following integral

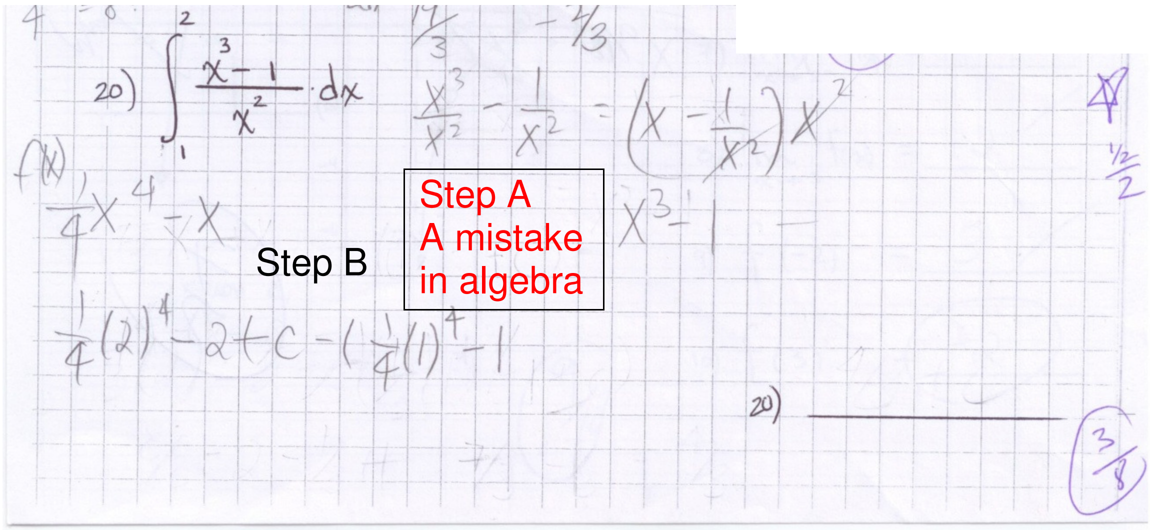


In step A, S22 incorrectly algebraically manipulated the expression, $x^3 - 1$ into $x^2(x-1)$. This result was then correctly manipulated and the expression $\frac{x^2(x-1)}{x^2}$ and correctly simplified to $(x-1)$. The results from step A were correctly operated on in step B, using calculus at first to find the indefinite integral. This was used to find the definite integral.

The solution was incorrect due an early mistake in algebra, which carried through and caused the final answer to be incorrect. The calculus parts of the solution were done correctly and a mistake in algebra was the sole reason for the answer to be incorrect.

Example 7C Rational Expressions with Integration (Incorrect solution)

Evaluate the following integral

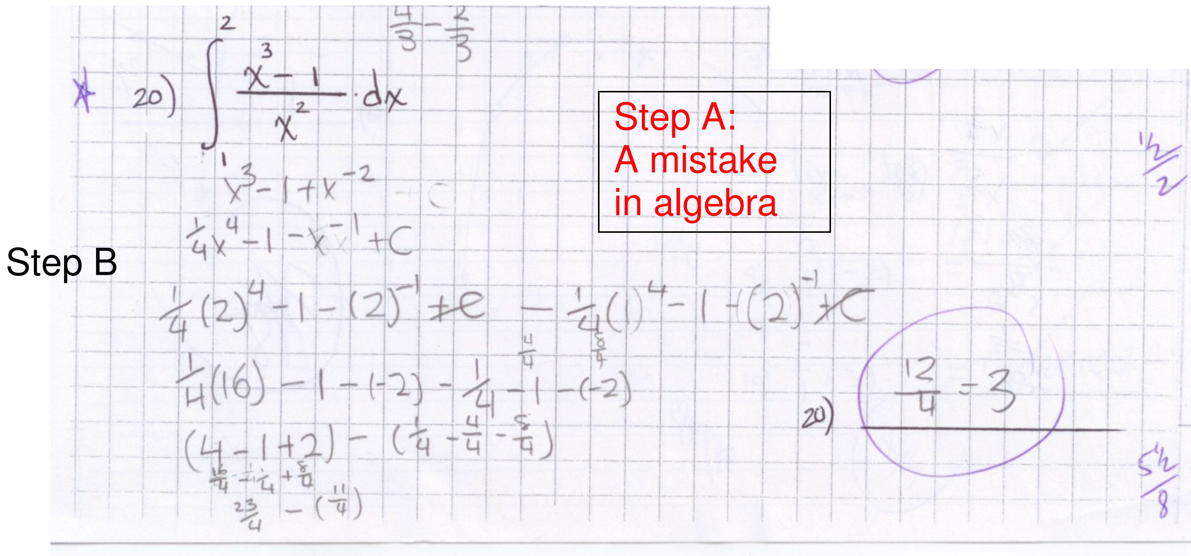


In step A, S23 correctly manipulated the expression into two separate terms. The work here is identical to the algebraic manipulation done in example 7A. The second part of step A has a mistake in algebra. The manipulation of $\frac{x^3}{x^2}-\frac{1}{x^2}$ into $\left(x-\frac{1}{x^2}\right)x^2$ is incorrect. He incorrectly factored x^2 out of the expression. The following distribution of x^2 back into the binomial is correct. The error in part A is confusing to understand as the act of factoring out and distributing into are inverse operations and the resulting function should be identical to the starting function.

Part B was correctly worked on using integration techniques using the results from step A. The calculus parts of the solution were done correctly and a mistake in algebra was the sole reason for the answer to be incorrect.

Example 7D Rational Expressions with Integration (Incorrect solution)

Evaluate the following integral



In part A, S24 incorrectly algebraically manipulated the expression $\dfrac{x^3-1}{x^2}$ into x^3-1+x^{-2}. In part C, he has correctly manipulated the result from part A, using correct calculus-integration techniques. The calculus parts of the solution were done correctly and a mistake in algebra was the sole reason for the answer to be incorrect.

4.5 Summary and analysis of errors

Table 6 lists the errors in algebra that impeded students' correct performance on calculus tasks.

Table 6 Summary of algebraic errors from student solutions

Example number	Student Error
1B	“ $x^{12} \square x^{\frac{1}{2}} = x^{12 \square \frac{1}{2}} = x^{6}$ ”
1C	“ $x^{5} \square x^{7} \square x^{\frac{1}{2}} = x^{5\times7\times\frac{1}{2}} = x^{\frac{35}{2}}$ ”
2B	“ $(2x)^{3} = 2x^{3}$ ”
2C	“ $(2x)^{3} = 8x$ ”
2D	“ $\frac{1}{4x} = 4x^{-1}$ ”
3B	“ $\frac{x^{3}}{x^{12}} = \frac{1}{x^{4}}$ ”
3C	“ [illegible] ” “ $(x^{3})(x^{9}) = (x^{3\times9}) = x^{27}$ ”
4B	“ $\left(\frac{-4}{3}x\right)^{2} = \frac{16}{9}x$ ”
5B	“ $(\sqrt{x})^{2} = x^{1/4}$ ”
5C	“ $\left(\frac{1}{3}\left(\frac{x}{3}\right)^{2}\right)^{2} = \frac{1}{9} \bullet \frac{1}{3}\left(\frac{x}{3}\right)^{2} \bullet \frac{1}{3}\left(\frac{x}{3}\right)^{2} \bullet \left(\frac{x}{3}\right)^{4}$ ”

6B	"$\frac{x^4-81}{x+3}=(x^4-81)(x^{-1}+3)$"
6C	"$\frac{x^4-81}{x+3}=x^4-81x^{-1}+3$"
6D	"$\frac{x^4-81}{x+3}=\frac{x^4}{x}-\frac{81}{3}=x^3-27$"
6E	"$\frac{x^4-81}{x+3}=\frac{x^3(x+3)-81(x+3)}{(x+3)}=(x^3-81)$"
7B	"$\frac{x^3-1}{x^2}=\frac{x^2(x-1)}{x^2}=x-1$"
7C	"$\frac{x^3-1}{x^2}=\frac{x^3}{x^2}-\frac{1}{x^2}=\left(x-\frac{1}{x^2}\right)x^2=x^3-1$"
7D	"$\frac{x^3-1}{x^2}=x^3-1+x^{-2}$"

The seventeen errors presented in the examples 1 – 7, show a variety of algebraic mistakes made by my students as part of their solutions of calculus tasks. The fact that students make mistakes in manipulating algebraic expressions is well known and has been acknowledged in a variety of studies. Further, several studies (e.g. Dias, 2000) identified weak proficiency with prerequisite algebraic skills as a major problem in Calculus. However, most studies did not itemize what essential algebraic skills students were lacking. A notable exception is in the study of Kajander and Lovric, who highlighted misapplication of linearity as a particular error made by calculus students. Such error was described previously by Matz (1982) working with younger students and further highlighted by Zazkis and Campbell (1996) working with prospective elementary school teachers.

The errors in examples 2B, 2C, 3C and 4B can be classified as the overgeneralization of linearity. This category is also referred to as “false linearity” in the general classification of algebraic errors provided by Barnard (2002). Other errors can also be seen as examples of particular categories identified by Barnard, such as operating on one piece of compound term (2B, 2C, 2D), confusion between operations (1B, 5C, 6C), inappropriate cancelling (6D) and misapplied rules (7D). Furthermore, this classification is not disjoint as many errors fit into more than one category identified by Barnard. For example, 2B can be seen either as operating on one piece of compound term as well as confusion between operations or misapplied rules. In supplement, 6B can be seen as an error due to a misunderstanding of fraction, operating on one piece of a compound term, rearranging formulae or misapplied rules. However, my goal was to demonstrate the existence of algebraic errors in students’ work, rather than to classify them.

4.6 Interviews with students

The following section presents the highlights from two separate sets of interviews with students. In the first set of interviews, students were asked to comment on their own mistakes made in a calculus test. In the second set of interviews, students were shown a mistake made by a different student and asked to explain why the mistake was made. The focus of these interviews was to gain insight into why students were making mistakes in algebra. The interviews were conducted in the calculus room after class. The students and I were gathered around a small desk and the mood was relaxed and informal.

4.6.1 Interviews regarding a students' interpretation of their own mistake

In three interviews, I asked students to reflect and analyse their own mistake made in a calculus test. Students were presented with a copy of their own work on a task from a recent test. In all three situations, the student had made a mistake with the algebraic component of the task, but had correctly completed the calculus part of the task. When asked to explain why they think they had made the mistake, all three students responded saying that they had made a stupid mistake.

I felt that the response from these three students was lacking objectivity and wanted to pursue the analysis of student mistakes. I thought that an objective analysis of a student's mistake could be better made through other students. The following interviews were conducted with students analysing another student's work.

4.6.2 Interviews with students analyzing another student's work

In both interviews, I have copied the work from the task of a recent test done by another student. In both cases, the student's work shows a mistake in algebra. In both interviews the students are asked to suggest an explanation as to why the mistake was made.

Interview #1: S25

Error example #1

$$\left(x^2+1\right)^2 \quad = x^4+1$$

Their dialogue from the interview with S25 follows,

Mr. K "What do you think the student who did this was thinking?"

S25 "The reason for the mistake was that the student was squaring the first and the last."

Mr. K "Does it make sense to apply the exponent to each term individually?"

S25 "Yes. It makes sense. The mistake is quite logical and makes sense."

Error example #2

$$\left(\frac{-4}{3}x\right)^2 = \frac{16}{9}x$$

In this example, S25 said the following,

S25 "The x is a part of the $\frac{16}{9}x$.

When the $\frac{-4}{3}$ is squared to $\frac{16}{9}$ the x is included."

Mr. K "Can you give me an example where this math procedure is correct?"

S25 "The problem is like 2(3x) = 6x.

The 2 multiplies the 3 to 6 and the x is included.

S25 seemed happy with this explanation.

Interview #2: S26

S26 was presented with the following example of a mistake in algebra and asked to explain why the mistake was made.

Error example #3

$$\left(\frac{1}{3}\left(\frac{x}{3}\right)^2\right)^2 = \frac{1}{9} \bullet \frac{1}{3}\left(\frac{x}{3}\right)^2 \bullet \frac{1}{3}\left(\frac{x}{3}\right)^2 \bullet \frac{x^4}{9}$$

In this interview S26, took some time thinking about what the student had done in making the above mistake. She was excited when she felt that she knew what the person was thinking when they made the mistake.

Mr. K "What do you think the student who did this was thinking?"

S26 "The student is expanding using FOIL *(first, outer, inner, last)*. You can check"

S26 then proceeded to check her explanation. She expanded the term $\left(\frac{1}{3}\left(\frac{x}{3}\right)^2\right)^2$ as if it was a squared binomial. Her answer was the same. I asked her if the answer with terms separated by the multiplication symbol "•" made sense.

Mr. K "Does it make sense to separate each term with the multiplication symbol?"

S26 "It makes complete sense when you think of $\left(\frac{1}{3}\left(\frac{x}{3}\right)^2\right)^2$ as if it was $\left(\frac{1}{3}+\left(\frac{x}{3}\right)^2\right)$ and that this is close enough"

4.6.3 Summary of student interviews

The student analysis of other students' work revealed that the algebraic mistakes made could be due to an incorrect application of an algebraic concept.

example #1: the mistake could have been made thinking that $(a+b)^2$ can be expanded with the same method as if it was (a+b)2

example #2: the mistake could have been made thinking that $(ax)^2$ can be expanded with the same method as if it was (ax)2

example #3: the mistake could have been made thinking that $(a \bullet b)^2$ can be

expanded with the same method as if it was $(a+b)^2$

The two interviews and analysis with calculus students show that the algebraic errors could be explained as the application of an incorrectly identified rule (Barnard, 2002). Further, the first two examples are special cases of misgeneralized linearity (Kajander & Lovric, 2005).

When I asked the entire class why they think these errors occur, the following statement from S27 offers some insight into the understanding the problem,

> "The problem is I have to know it all at the same time".

From this statement, one can deduce that part of the problem experienced by students in calculus is due to the first occurrence and large task of having to know most of the mathematics 10 though to 12 curriculum. Having come to this point in identifying the problems students experience in high school calculus, the next chapter presents ideas regarding a possible solution.

5: RE-TEACHING OF ALGEBRAIC CONCEPTS

The findings of the study presented in the previous chapter are rather disturbing for me as a teacher of calculus. On one hand, it appears that students can acquire basic concepts in calculus, such as limit or derivative, without any reliance on their algebraic skills. On the other hand, for many students in the calculus course, their insufficient competence in algebra hinders their correct performance on calculus tasks and as such interferes with their success in a course. I wondered how the problem could be addressed.

In the year following the study presented in Chapters 3 and 4, I created and used a pedagogical approach in an attempt to improve students' knowledge and ability to use pre-calculus algebra in Calculus 12 courses. I refer to this method of teaching both calculus and pre-requisite algebra as, "Re-teaching in Context" and describe my initial implementation. Much of the following are my opinions and ideas, but some of these are reinforced with quotes from students. To situate my ideas, I first focus on traditional review, as well as the criticisms of traditional review, to lay the foundational understanding on which Re-teaching in Context is based.

5.1 Traditional Review

Traditional review is typically done in the first few weeks of the school year. It is composed of the previously learned concepts that will be used at some time during the present mathematics course. These review concepts come from the previous mathematics courses and as such when each concept was learned can span many years into the past.

Throughout the traditional review period, the depth at which each concept is re-taught varies along a spectrum from a quick summarization to a complete re-teaching of a concept. It can be assumed that the teacher doing the review has made a decision on how much time to spend on each concept. The reviewing teacher understands how well and in what contexts the students will need to have mastered the previously learned mathematical concepts for success in the present course. The teacher also knows the general level of understanding (of the review concepts) the students bring into their class. With this perspective, the review done at the beginning of the year does three important things:

1. Covers the pre-requisite concepts needed for their course.
2. Reviews the concepts at a level and in a method that enables all students to reacquaint themselves to the review concepts.
3. Gives students the opportunity to identify the review concepts that they are having difficulty with, and to spend more time working on learning these concepts.

At the end of the review session students should know the review concepts and they often do well on tests covering the review concepts.

5.2 Problems with Traditional Review

Traditional review does not give the majority of students, mastery of essential concepts needed for use throughout the year in their new mathematics course. This is not because the students finish the review section not knowing the material. Students generally do know the review concepts at the end of the review session, but the problem is that many students cannot show proper use of these concepts later in the year.

There seems to be a contradiction concerning some students' ability with pre-calculus algebra. One asks, why do students finish the review session with marks that indicate they have a good working knowledge of pre-calculus, yet later in the school year, struggle with the same algebra concepts they apparently knew? This contradiction is due to two main reasons:

1. The length of time between when the review is done and when the review concepts will be used.

2. The absence of context in which the review concepts will be used.

Traditional review lacks this context and in doing so neglects the critical aspect of students needing to see how and when they will be using these review concepts.

5.2.1 The time between reviewing a concept and when the concept is later used

The review concepts taught in the beginning of the year are often not used until a much later date. With some algebra concepts, this length of time can be well over a month. Many students are not able to retain the review material for long lengths of time.

One reason for the weak proficiency with concept from previous mathematics courses lies with the mathematics curriculum. The mathematics curriculum is set up such that students rarely have to use significant knowledge from a previous chapter. Students learn a chapter's set of information and then do not use it again. This neglect of previously taught concepts thus reinforces the mindset "*learn for the moment and then forget it*". This point is reinforced with comments from students. The following is from an informal interview after a typical calculus class.

Mr. K: "Why doesn't reviewing in the first few weeks of a course work?"

S33: "I only remember how to work with the stuff I'm working on now. I have a friend who reviewed in the beginning of the year. She did all this stuff and now she's not sure where to use it and she's forgotten a lot of it"

This apparent inability to transfer concepts from short-term memory to long-term memory is seen as natural for many students. In a later interview students were asked to comment on the time between reviewing a concept and using the concept. On this day, review was conducted at the beginning of class and the calculus lesson used these algebraic concepts approximately forty-five minutes later.

Mr. K: "If I did review at the beginning of the class in which the review was used, would it be useful?"

S34: "No it wouldn't be because I might forget it."

S35: "After doing the review I want to use what I was taught. If I have to wait 45 minutes to use it, I'll not be sure how to use it."

Mr. K: "If the review in the beginning of the class is only 30 minutes from using it. Would that make it useful?"

S34: "No, it wouldn't be useful. I'd still forget the review."

S35: "Yup! Thirty minutes is too far away for me to remember"

These student's comments are examples of what some students think and feel about review. There are some students that, half way through a class, are not able to show proficient use of a concept that was reviewed from the beginning of class. The reasons behind this are veiled. Is it due to the lack of retention? Is it from a lack of true deep understanding? Is it due to a poor nights sleep? This study does not look into these reasons. Rather it suggests a possible solution.

5.2.2 Lack of Context

Students learn better when they see how they are going to use what is learned. When students learn concepts with traditional review they often do not see how they are going to apply them. Due to this absence of context, students build weaker connections with the review lesson and are not as likely to remember.

This point is reiterated by a student's comment regarding review

Mr. K: "What is the problem with review?"

S28: "After doing the review I want to use what I was taught. It's more confusing when you don't see how it's used right away."

S29: "When you do it (*review*) in the beginning of the year, you do it without knowing the need or the purpose"

The inability of some students to use pre-requisite concepts may be related to their lack of experience in using these concepts as part of a multistep problem. The majority of mainstream mathematics courses do not have significant use of concepts from earlier in the year. Students are often learning mathematics concepts in isolated mathematical environments.

In Principles of Mathematics 10, 11 and 12, students learn about factoring, expanding, logarithms, trigonometry and many more, but seldom see these concepts together. Students in a calculus class are introduced to tasks that involve multiple mathematical concepts. Students are expected to identify which concepts to use and then use their previous knowledge of these concepts as part of a multistep solution. In a calculus class, students' vast prerequisite knowledge is used like a toolkit to assist with the manipulation of a task. Students get little experience in high school mathematics with multistep problems using mathematical concepts like tools in building complex solutions.

To reiterate, when students review at the beginning of the year, they are not likely to be able to use these concepts as needed because of:

- The lack of context. Students do not see where and how they will use concepts reviewed at the beginning of the year, and…
- The lack of practice/experience that students have with using mathematical concepts as part of a bigger multistep problem.

5.3 Re-teaching in Context

It was clear from my findings presented in Chapter 4 that I was dissatisfied with the algebraic proficiency of my calculus students. I sought solutions that fostered their ability to use algebraic concepts and procedures as needed. From my experiences I formulated a teaching method, which would address the need for effective review of these same algebraic concepts. “Re-teaching in Context” was the result of these experiences and thoughts. This technique encapsulated the re-teaching of prerequisite concepts within a calculus lesson. Following is a description of “Re-teaching in Context”, the comments from several students regarding the effectiveness of this teaching approach and a particular example of a lesson that illustrates this strategy.

5.3.1 Description of Re-teaching in Context

“Re-teaching in Context” is the teaching approach in which the review is placed directly into the lesson. The calculus lesson is interrupted whenever the teacher comes across review of a prerequisite topic. An observer of this class would see a complete lesson in mathematics in which all prerequisite concepts needed for the lesson would be seamlessly embedded within the calculus concepts.

In this manner, two things happen,

- Firstly, the time between re-teaching and use of the prerequisite concept is effectively zero. The re-teaching of a concept is implemented immediately before it is used. In this way, there is no time to forget the review concept.
- Secondly, the mathematical concept is used in context. The students see where and how they are going to use this prerequisite knowledge within calculus. Students will experience and build on their ability to use previous knowledge in new multistep situations.

Several student opinions provide insight into the effectiveness of re-teaching in context. Several students have said they feel they are learning better within a method that allows them to see where it will be used. This method teaches that all mathematical concepts can be used in future mathematical situations.

The following student comments reinforce the effectiveness of this method:

Mr. K “What do you like about reviewing in context?”

S30 “ I like the reviewing in the middle of the lesson because if we’d done it earlier I probably would have forgotten it by now”

Mr. K “What do you like about reviewing in context?”

S31 “I think reviewing during the lesson helps because you can see how it is applied in the task”

Mr. K “What do you like about reviewing in context?”

S32 "Review in the middle of the lesson is better because then you know you have to apply it in the future"

5.3.2 A Sample Lesson with Re-teaching in Context

Following is a complete calculus lesson that illustrates how "Re-teaching in Context" may look like in a calculus lesson.

The following lesson has the following sections,

- An introduction to the concept of the derivative. In this first section the students practice calculating the slope of a function at several values of X.
- An explanation of the power rule. This is followed with examples and practice with various functions (NOTE: all of these functions are in a simplified form which allow direct application of the power rule.)
- The power rule applied to functions needing simplification. This section emphasizes that the power rule cannot be applied in all situations. Often there is the need for simplification through algebraic manipulation, in this case specifically with exponents.
- RE-TEACHING algebraic manipulation of exponents. The calculus lesson seamlessly changes over to a complete re-teaching of algebraic manipulation of exponents. This section ends with four practice tasks on simplification of exponents.
- Practice tasks with the power rule using functions needing algebraic manipulation of exponents.

Calculus

Derivative #3

Definition of the derivative

The derivative of a function f(x) = slope of the function = slope of f(x)

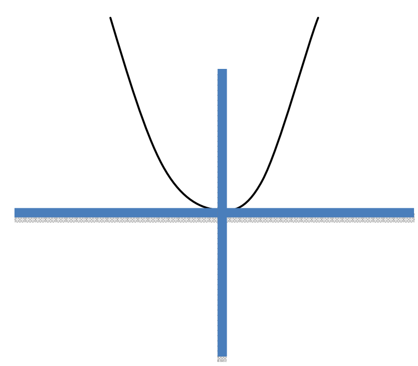
= rate of changes of f(x) with respect to x

= f '(x)

Getting familiar with f(x) and f'(x)

Example #1

$f(x) = x^2 - 2$



$f'(x) = 2x$

Look at different values of x and calculate the value of f(x) and the slope at these points

For example,

at x = 1 the functions value is $f(1) = 1^2 - 2 = -1$

the slope at x = 1, $f'(1) = 2(1) = 2$

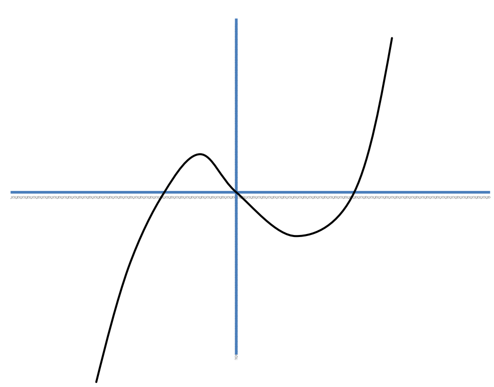
Compare these answers with the graph to confirm they are correct!

<u>Practice:</u> Calculate the value of f(x) and the slope at the following values,

at x = 2	f(2) =	f '(2) =
at x = -6	f(-6) =	f '(-6) =
at x = 0	f(0) =	f '(0) =

Check your answers with the graph

Example #2



$f(x) = x^3 - x^2 - 6x$

$f'(x) = 3x^2 - 2x - 6$

Practice: Calculate the value of f(x) and the slope at the following values,

at x = 2	f (2) =	f '(2) =
at x = 1	f (1) =	f '(1) =
at x = 0	f (0) =	f '(0) =
at x = -2	f (-2) =	f '(-2) =
at x = -5	f (-5) =	f '(-5) =

Check your answers with the graph

The derivative allows the calculation of the slope of f(x) at all values of x.

Finding the derivative. The derivative of simple functions is found using the POWER rule.

POWER RULE: starting with f (x), finds f '(x)

$f(x) = x^n \qquad f'(x) = n \bullet x^{n-1}$

EXAMPLES / PRACTICE

1) $f(x) = x^6$ $f'(x) =$ ___________	2) $f(x) = x^{99}$ $f'(x) =$ ___________
3) $f(x) = x^{-4}$ $f'(x) =$ ___________	4) $f(x) = 9^3$ $f'(x) =$ ___________
5) $f(x) = 5x^7$ $f'(x) =$ ___________	6) $f(x)=\frac{x^5}{3}$ $f'(x) =$ ___________
7) $f(x) = 8x^4 - 2x^{-1}$ $f'(x) =$ ___________	8) $f(x) = \frac{x^3}{6} + 5x^{0.4} - e^2$ $f'(x) =$ ___________

List some important points regarding the power rule.

THE FUNCTION MUST BE IN EXPANDED FORM TO USE POWER RULE

Find the derivative

9) $f(x) = 3(5x^4)$ f '(x) = ___________	10) $f(x) = \frac{1}{4x^3}(8x^5 - 2x^{-2})$ f '(x) = ___________
11) $f(x) = \frac{1}{4x^3}(8x^5 - 2x^{-2})$ f '(x) = ___________	12) $f(x) = x^{-7}(3x^2 + 9x^7)$ f '(x) = ___________

RE-TEACHING EXPONENTS

Examples,

a)
$$f(x) = (x^2)(x^5) = (x \bullet x)(x \bullet x \bullet x \bullet x \bullet x) = x^7$$
$$= x^{2+5} = x^7$$

b)
$$f(x) = \frac{(x^7)}{(x^3)} = \frac{(x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x)}{(x\cdot x\cdot x)} = \frac{(x\cdot x\cdot x\cdot x)}{1} = x^4$$
$$= x^{7-3} = x^4$$

c)
$$f(x) = (x^2)^4 = (x \bullet x)(x \bullet x)(x \bullet x)(x \bullet x) = x^{2+2+2+2} = x^8$$
$$= x^{2\times 4} = x^8$$

Summary of Exponent Rules

$a) \quad x^m \bullet x^n = x^{m+n}$

$b) \quad \dfrac{x^m}{x^n} = x^{m-n}$

$c) \quad \left(x^m\right)^n = x^{m+n}$

Practice with exponents. Simplify the following.

$a) \quad \dfrac{x^{12}}{x^3} =$ __________	$b) \quad \left(x^5\right)^8 =$ __________
$c) \quad \dfrac{x^7 \cdot x^8}{x^5} =$ ______	$d) \quad \dfrac{\left(x^5\right)^{-2} \bullet \left(x^4\right)}{x^{-9}} =$ ____________

Find the derivative

13) $f(x) = x^4 \bullet x^{-3} \bullet x^{-3}$ $f'(x) =$ ____________	14) $f(x) = \dfrac{x^4+3x^8}{x^3}$ $f'(x) =$ ________
15) $f(x) = 5x^2\left(3x^6 - 7x^{-2}\right)$ $f'(x) =$ ____________	16) $f(x) = \pi^2 + \left(x^4\right)^7$ $f'(x) =$ __________
17) $f(x) = \dfrac{1}{2x^2}\left(4x^8 - x^{-3}\right)$ $f'(x) =$ __________	18) $f(x) = \dfrac{11x^5 - 6x}{3x^4}$ $f'(x) =$ ___________

5.3.3 Summary of Re-teaching in Context

Re-teaching in Context can be summarized as embedding the teaching of prerequisite material into the calculus lesson immediately before it is used. In this manner, the prerequisite material is relearned and fresh in the students' mind as it's applied to a calculus task. Students' comments support the effectiveness of this teaching method. To summarize this chapter in a statement, it seems that

> **Mathematical ability and understanding are improved when using re-teaching in context because students get review of prerequisite concepts and then immediately see where and how they are going to use this relearned concept.**

It is not the intent of this study to suggest this is the only solution. The intent of this study is to better understand the mathematical difficulties of high school students and bring forth information that could lead to a solution.

6: SUMMARY AND CONCLUSION

In this chapter I summarize the findings of my study, “Algebraic difficulties as an obstacle for high school calculus”, and my efforts, as a calculus teacher, to deal with the identified issues. I further discuss particular contributions and limitations of this study and present several ideas for future investigations.

6.1 Summary of the study

As mentioned earlier, the study was motivated by my personal dissatisfaction with the algebraic skills that seemingly good mathematics students bring to their high school calculus class. This observation was confirmed by the research literature. In fact, researchers saw the lack of algebraic competency as one of the biggest problems students experience in their first calculus course at a postsecondary institute. Furthermore, algebraic proficiency was listed as the best predictor for success in a calculus course.

However, most research studies that I reviewed have not detailed what particular algebraic skills hinder student performance with calculus. As such, the goal of my study became to address the following research question: “**What lack of competency with algebraic concepts interfere with students’ performance on calculus tasks?”**

The school at which the study took place has an environment that encourages academic behaviour and a healthy competitive atmosphere. The school has a high academic reputation and a high percentage of the graduation class goes on to post secondary education. The study looked at students from two Calculus 12 classes. The students involved have reasonably strong study habits, have done well in previous mathematical classes and have identified that they plan on taking calculus in post secondary education.

This study identified that some of the students taking high school calculus do not show competent ability with prerequisite algebraic concepts taught in Principles of Mathematics 10 – 12. In particular, the study shows that some students do not have mastery of the following concepts:

1. Exponents

    Multiplication of like terms with exponents

    Division of like terms with exponents

    Distribution of exponents

    Terms with exponents raised with another exponent

2. Squaring of binomials and monomials

    Distribution of squaring exponent

    Squaring of a monomial with an exponent

3. Rational Expressions

    Manipulation of the denominator within rational expressions

    Simplification of rational expressions

    Incorrect factoring of perfect squares

    Failure to identify the need to factor a perfect square

The examples presented in Chapter 4 demonstrated a variety of mistakes that hindered students' performance on calculus task, while the calculus part of the task was completed correctly. While this may lead to a swift conclusion that algebraic proficiency does not interfere with the understanding of calculus concepts, I contend that it does interfere with correct performance and, as such, with students' success in the course.

6.2 Re-teaching In Context

To address these identified deficiencies in students' ability to remember and use prerequisite mathematical knowledge, I created and used a teaching method called **Re-teaching in Context.** Re-teaching in Context does away with the reviewing of typical prerequisite concepts in the beginning of the year. Instead, prerequisite concepts are only re-taught as part of the lesson in which they will be used. This lesson structure allows immediate use and practice of prerequisite concepts within calculus tasks.

The feedback and results from using Re-teaching in Context has shown some signs of success. For example, calculus student S29 was asked to describe how he felt re-teaching in context was affecting his learning. S29's comment, embodying the effectiveness of Re-teaching in Context, follows,

> "When you review *on-site,*
>
> You connect with it
>
> You know when to use it in a test
>
> You know how to do the review better

You connect with the memories of when you originally learned it

You know how to use it"

6.3 Contributions of this study

This study adds details to the findings from previous studies. In summary, the past studies reinforce each other in their conclusion that the largest problem holding students back from succeeding with calculus is a lack of proficiency with high school algebra. These weak skills in algebra are somewhat surprising as the literature shows that the high school mathematics includes the prerequisite skills for calculus. However, as mentioned above, most studies do not identify the specific algebraic concepts that cause students difficulty.

In response to this weakness in algebraic skills, this study focused on the most common skills with which calculus students struggle; those are manipulation of exponents, the squaring operation and rational expressions. It offers a detailed account of students' mistakes in the algebraic skills and procedures needed for completing calculus tasks.

Further, I suggest a pedagogical approach, which addresses the need for effective review of prerequisite skills. A teaching method that I have called "Re-teaching in Context", gives students the needed review in the context of a calculus lesson. The initial evidence for implementing Re-teaching in Context is positive and suggests it may be part of the solution for calculus students with weak algebraic skills.

6.4 Limitations of this study

The findings of this study are limited to its particular setting. This study was conducted with two Calculus 12 classes, containing approximately 55 students at a single school. The researcher, who was also the teacher of the calculus course, conducted this study. It is important when interpreting the data to take into account that teachers have different teaching methods and styles to which every student responds differently. Also, being the researcher and teacher may have biased the data.

Furthermore, this study only looked at three algebraic concepts. While it itemized the most frequent algebraic errors, other errors may be found in a more extended study. The new mathematics curriculum has made changes to mathematics 8 to 12. It is unclear how these changes will affect algebraic skills of future students taking calculus.

This study provides numerous examples showing the variety of ways students are making algebraic mistakes in calculus. However, it does not provide statistical evidence or numerical data, which would give weight to the magnitude of the identified issues.

6.5 Future Studies

Studies planned for the future should address the described limitations. I plan to conduct future studies in which I am not the teacher, thus removing some of the bias created in this study. In addition, future studies should look for advice and feedback from colleges and universities as to the effectiveness of Re-teaching in Context. Is Re-teaching in Context making the students more able to use the prerequisite skills needed for success in calculus?

As an educator and a researcher, I hope to continue studying methods, which aim to improve student learning. This study directly looks at high school Calculus 12. I hope to take the ideas and knowledge from this study and apply it to the teaching and learning of mathematics at all grade levels.

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