STATISTICAL METHODS IN RELIABILITY TESTING

by

Alexey Antonitsin
Spct., Vologda State Dairy Academy, 2009

A PROJECT SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the
Department of Statistics and Actuarial Science
Faculty of Science

SIMON FRASER UNIVERSITY
Spring 2012

Approval

Name: Alexey Antonitsin

Degree: Master of Science

Title of Project: Statistical Methods in Reliability Testing

Examining Committee:

Chair: **Richard Lockhart**
Professor

Steven K. Thompson
Senior Supervisor
Professor

Carl Schwarz
Professor

Derek Bingham
Internal Examiner
Associate Professor

Date Approved: March 23, 2012

Abstract

In this project we examine different sampling plans as part of reliability control procedures in manufacturing. The systematic sampling conventionally used by many companies can be replaced by a skip-lot sampling plan which provides the same level of protection against defective products while requiring fewer resources. In this project we investigate skip-lot sampling procedure and its statistical properties. We will also compare the medium rank regression method of fitting Weibull distribution parameters with the maximum likelihood method. Finally, we conduct a simulation study to empirically investigate the performance of the proposed method and its properties. This project considers a wide range of reliability sampling procedures and makes extensive use of simulation methods.

Keywords: Reliability testing, Skip-lot sampling, Weibull analysis.

To my mother

Acknowledgments

More than two years that I have spent at the Department of Statistics and Actuarial Science at SFU stand particularly prominent in my life. I really enjoyed being a graduate student at a department with such a great faculty with strong passion for Statistics, wonderful teaching skills, and ability to create the right atmosphere for learning. I want to thank my great classmates for providing a lot of help in understanding the material and for just being great people to spend time with.

My special thanks go to Steven K. Thompson for being my supervisor and providing great guidance throughout these two years. I am very grateful for all the support, ideas, and wise advices Professor Thompson gave me during my Master's Degree.

Finally I want to thank my family and friends for their help and encouragement. I am especially grateful to Svetlana Smirnova for her assistance and inspiration.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

In recent years the popularity and usage of reliability tests and analysis have significantly grown. Many companies outsource the production process of product components or even the whole product to second party companies. In order to stay competitive on the market these companies require procedures to control the quality and reliability of outsourced products. Even if the manufacturing line at the outsourcing company is automated, there are still variations in the characteristics of outgoing items due to the material used, line operators, transportation and other controllable and non-controllable factors.

The terms "reliability" and "quality" are often used interchangeably. In fact, these terms have different meanings: reliability is concerned with the performance of a product through some desired period (product lifetime), whereas quality is concerned with the performance of a product at a particular point in time. Therefore, reliability tests are usually more complex procedures than quality tests. Nevertheless, quality testing and reliability testing both measure the "goodness" of a product and product reliability depends on the initial product quality. Most reliability tests are designed to answer one or both of the following questions:

1. Does the product comply with the reliability criteria set by the company?
2. Is one version of the product better than the other one?

If the product has different failure types and each tested item needs to be checked after every time unit of the test, the reliability testing could be very expensive. Therefore, companies often try to optimize the cost of reliability. As a result of this, it is common practice to test only a sample of the produced items and design an efficient test procedure.

This thesis is specially focused on the sampling procedure used to select items from a population of produced items. The theoretical and simulation study results show that the proposed sampling procedure is more efficient than typical procedure currently used in the industry. Besides the sampling scheme, we review methods to fit the Weibull distribution to the data. Therefore this thesis examines the complete procedure of reliability testing. The skip-lot sampling scheme that is used in the proposed method was developed by Dodge and studied by Perry (1971) in his PhD thesis. Stephens (2000) notes that although skip-lot sampling procedures were developed years ago, their potential for widespread application in industry have not been fully realized yet. Nevertheless a lot of standards have been formulated base on skip-lot sampling—for example, ANSI/ASQC Standard S1-1996.

Chapter 2

Reliability Testing

Many manufactures use reliability testing to meet and exceed customer demand for high quality and reliable products. As defined by the Advisory Group on Reliability of Electronic Equipment (AGREE), reliability is the "probability of performing without failure a specified function under given conditions for a specified period of time." Therefore reliability testing usually involves simulation of conditions under which the item will be used during its lifespan. Reliability does not compare the product to some predefined specifications, such as the case with quality assurance, but rather investigates the performance over a predefined period of time. For example, smartphone devices can undergo an accelerated life test. In this test, devices are exposed to events that simulate real life situations that happen to smartphones like drops, spills, or excessive heating. The goal of this test is to find out whether the produced items meet the specified minimum reliability requirement.

To reduce the cost of reliability testing, manufacturers employ sampling schemes to select items that represent all produced devices. As in any sampling, it is assumed that we can make appropriate inference about the true population characteristics based on appropriately selected samples.

Reliability sampling can be formulated in terms of testing a statistical hypothesis:

H_o: μ (mean life) is greater or equal to 20 hours

H_1: μ is less than 20 hours

The concept of producer's risks and consumer's risks are similar to Type I and Type II errors:

Producer's Risks (a): the probability of failing satisfactory items—the probability of rejecting H_o when it holds. It is associated with the level of reliability which has a high probability of acceptance, and, therefore, low fraction of non-conforming units.

Consumer's Risk (b): the probability of passing flawed items—the probability of accepting H_o when it is false.

It is possible to minimize those risks by taking a larger sample size, but in practice reliability engineers usually set tolerable $\boldsymbol{a}$ and $\boldsymbol{b}$ to reduce costs associated with testing large numbers of items.

When developing a sampling plan and a procedure for reliability sampling, the following questions should be answered:

1. Is the testing procedure representative of real life events?
2. Does the criteria to pass/fail comply with consumer and producer risks?
3. What sample size should be drawn?

Besides that, it is important to decide *a priori* what constitutes a failure, what units of measurement will be used, and when the test will be terminated.

Often, produced items are naturally combined in lots. Therefore, a reliability engineer can develop a sampling scheme that will first sample lots as primary units and then, in each chosen lot, sample devices as secondary units. Then the sampling scheme should consist of two sampling plans: one for primary units and one for secondary units. A sampling plan for secondary units is often referred to in the literature as a "reference sampling plan".

In Chapter 4 we examine a sampling scheme called Skip-lot Sampling Plan 2 with Double Sampling as a reference plan (SkSP-2DSP). Here primary units (lots) will be selected by skip-lot sampling procedure. As its name suggests, skip-lot procedure allows skipping some lots from inspection. It can be viewed as a type of adaptive sampling, where a sample size depends on the value of interest. SKSP-2DSP samples more from manufacturing lines with poor quality or reliability of produced devices and samples less from manufacturing lines that produce devices and lots that have historically passed quality and reliability inspections.

Secondary units are sampled by a Double Sampling Plan (DSP). A DSP provides better discrimination between acceptable and non-acceptable lots. Besides that, ANSI ASQC Z1.4-1993 specify that DSPs require less units than single sampling plans (SSPs) but more units than a multiple plan. Sometimes a SSP with a criteria for pass/fail set to zero non-conformative units favors the customer while a setting to one non-conformative unit favors the producer

(Vijayaraghavan 1998). This conflict of interest can be overcome with a DSP that evokes a second sample if one non-conformative unit was found in the first sample. Therefore a DSP that allows one non-conformative unit during first sampling and zero non-conformative units during second sampling is a compromise between the SSP settings that allow only one non-conformative unit or only zero non-conformity units.

Reliability test criteria can be specified in units of time or failures. In time-terminated tests the items are tested for a certain amount of time, whereas in failure-terminated tests the items are tested until a certain number of failures. It is not uncommon in time-terminated tests to have many survived units when the test is finished. Special methods are developed and used to take into account of censored failure times. For example, in fitting the Weibull distribution in Chapter 3, we have adjusted the ranks for failed items to account for the number of censored items.

There are two methods of measurement in reliability testing. ANSI ISO ASQC A3534 2 1993 defines them as follows:

Method of attributes: Noting the presence (or absence) of some characteristic or attribute in each of the items in the group under consideration and counting how many items do (or do not) possess the attribute, or how many such events occur in the item, group, or area.

Method of variables: Measuring and recording the numerical magnitude of a characteristic for each of the items in the group under consideration; this involves reference to a continuous scale of some kind.

When using the method of attributes, a reliability engineer can specify the maximum number of non-conforming units that are allowable in the sample. It should be noted that the manufacturer should not knowingly produce any number of defective products and it is always better to have zero non-conforming items.

Variables sampling plans are more often used for reliability sampling. Time to failure often can be described by Weibull or Exponential distribution. The test specification may be written in terms of:

1. Mean life μ
2. Hazard rate at t (number of items failing at time t)
3. Reliable life (some point of time beyond which predefined proportion of items will survive)

The following formulas allow the transition from one specification to another:

Weibull: $\boldsymbol{f(t) = \left(\frac{\beta}{\eta}\right)\left(\frac{t}{\eta}\right)^{\beta-1} \exp\left(\frac{t}{\eta}\right)^{\beta}}$		
Life Characteristic	**Notation**	**Formula**
Proportion failing before time t	$F(t)$	$F(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^{\beta}}$
Reliability	$\rho_r = 1 - F(t)$	$\rho_r = e^{-\left(\frac{t}{\eta}\right)^{\beta}}$
Mean life	μ	$\mu = \eta\Gamma\left(1 + \frac{1}{\beta}\right)$
Hazard rate	$h(t)$	$h(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}$
Cumulative hazard	$H(t)$	$H(t) = \left(\frac{t}{\eta}\right)^{\beta}$

Table 2.1: Life Characteristics for Weibull distribution

There are two misconceptions in reliability testing that are worth reviewing. First, it is common to assume that sample size should depend on a lot size or be a certain percentage of it. In most cases this is not true, because a sample size depends only on the variability of a given statistic from a random sample, which usually is not dependent on population size.

Second, many engineers believe that setting the number of non-conformative items for lot acceptance to zero will provide the best customer protection. We will see in the simulation chapter that customer risk depends on several factors. For example, plans with larger sample sizes have a lower probability of acceptance for a given number of non-conformative units.

Chapter 3

Parameter Estimation

3.3 Estimating parameters of Weibull distribution

The main challenge of fitting distributions to reliability data is finding the type of distribution and the values of the parameters that give the highest probability of producing the observed data. One of the most common probability density functions used in industry is the **Weibull Distribution**. It was invented by W. Weibull in 1937, who found it to be so flexible that it effectively worked on a very wide range of problems. In this chapter, we will discuss the family of Weibull distributions, various methods of estimating parameters, and the goodness-of-fit for Weibull distributions.

3.3.1 Weibull Distribution

There are numerous distributions that can model failure data, such as Normal, Exponential, Rayleigh, Weibull, Gamma, Lognormal, and others. The relationships between various distributions are shown in Figure 3.1, where the direction of each arrow represents a step from the general to a special case.

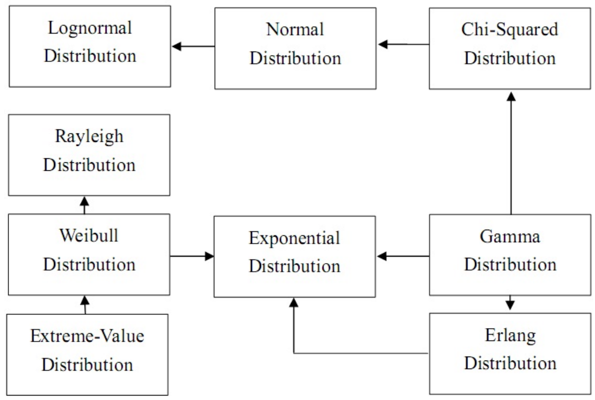

Figure 3.1: General and special case distributions

Weibull distribution can be applied to a large number of situations. The main advantage of using this distribution is its ability to handle small samples of failure data and its flexibility in fitting different failure modes. Small samples are common in reliability testing where tests are often *destructive* in nature and require costly resources.

Weibull can be fitted in two ways: (i) two-parameter or (ii) three-parameter distributions. The complete Weibull equation includes three parameters:

$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}, \tag{3.1}$$

where

β (beta) – slope or shape parameter,

γ (gamma) – location parameter,

η (eta) – scale parameter or characteristic life (time when 63.2% of the units will fail), and

$$f(t) \geq 0, T \geq max(0,\gamma), \beta > 0, \eta > 0, -\infty < \gamma < \infty. \tag{3.2}$$

Three-parameter fit is less common in reliability where majority of failures have non-zero probability from the beginning of the lifecycle. Also, support of three-parameter Weibull distribution depends on the location parameter which complicates maximum likelihood estimation. Most common workaround for this problem is using profile likelihood for search for γ (see [8]).

Two-parameter fit (assuming γ=0) is more common in reliability testing and more efficient with the same sample size. Furthermore, in some situations β—the slope parameter—can also be assumed to be known, reducing fit to one parameter. This method is referred to in some literature as Weibayes method. The method can be useful when 2 conditions are met:

1. Two-parameter Weibull fit produce unsatisfactory wide confidence intervals for estimators.
2. Prior knowledge in point estimation form of β is available.

Abernathy (1996) recommended Weibayes as best practice for all small samples, 20 failures or less, if a reasonable point estimate of β is available. Using Weibayes method, $\widetilde{\beta}$ can be assumed from some historical data or prior knowledge, leaving $\widetilde{\eta}$ as the single parameter, which can be estimated using maximum likelihood as:

$$\widetilde{\eta} = \left[\Sigma_{i=1}^{N} (\frac{{t_i}^{\beta}}{r})^{\frac{1}{\beta}} \right], \tag{3.3}$$

where

$t-$ time units,

$r-$ number of failed units +1,

$N-$ total number of failures plus suspensions,

$\beta-$ point estimate of slope.

On the one hand the assumption improves the method by shrinking the confidence interval for obtained estimates, while on the other hand it produces inaccurate results if the assumption about β was wrong.

The Weibull-Bayesian model (which is actually a true "WeiBayes") offers an alternative to one-parameter Weibull, by including the variation and uncertainty that might have been observed in the past on the shape parameter (β). Assuming prior distributions of β and η are independent, we obtain the following posterior pdf:

$$f(\beta, \eta | Data) = \frac{L(\beta, \eta) f(\beta) f(\eta)}{\int_0^\infty \int_0^\infty L(\beta, \eta) f(\beta) f(\eta) d\eta d\beta}, \tag{3.4}$$

where

$f(\eta)-$ prior distribution of η (Jeffrey's prior $f(\eta) = \frac{1}{\eta}$),

$f(\beta)-$ prior distribution of β (usually normal, lognormal, exponential or uniform).

The slope parameter β determines the shape of the Weibull curve (See Figure 3.2).

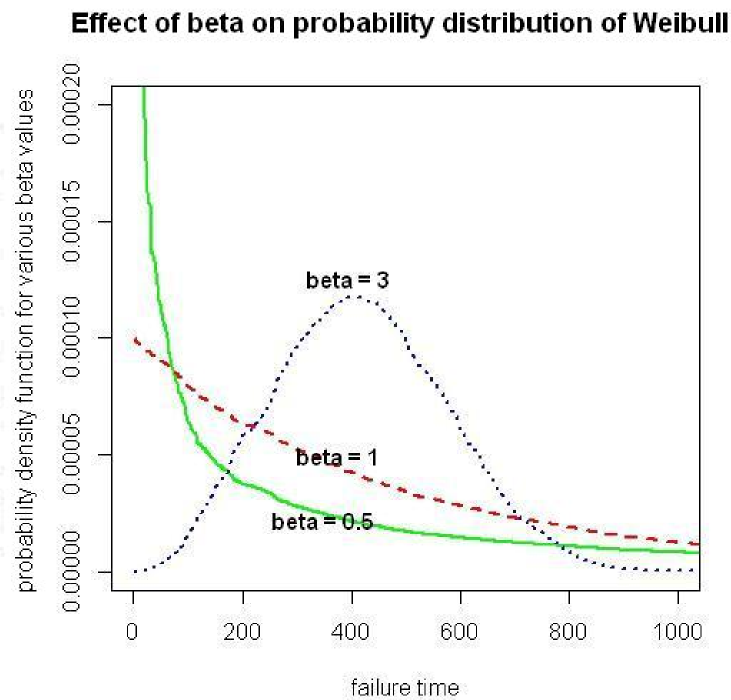

Figure 3.2: Weibull curves for different values of β

This effect of β can be translated into various modes of failures, as given in Table 3.1.

β **value**	**type of failure**	**meaning**
$\beta < 1$	infant mortality	high probability of failing at early stages
$\beta = 1$	random failures	failures are independent of time
$1 < \beta < 4$	early wear out	can be due to generic failure modes, such as corrosion
$\beta > 4$	rapid wear out	steep curve with fast wear out at some point

Table 3.1: Types of failures corresponding to β values

Figure 3.3 shows the width of distribution peaks for various values of η.

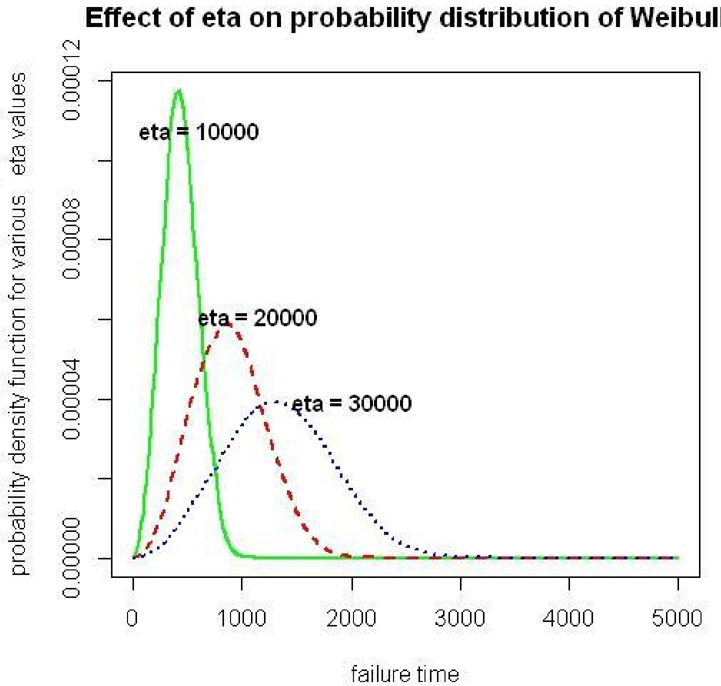

Figure 3.3: Weibull curves for different values of η

3.3.2 Weibull Parameter Estimation

There are several different methods of estimating Weibull parameters, such as: Maximum Likelihood, MLE with Reduced Bias Adjustment (RBA), and median rank regression. Olteanu and Freeman (2010) have investigated the performance of MLE and MRR methods and concluded that the median rank regression method is the best combination of accuracy and ease of interpretation when the sample size and number of suspensions are small. This method is popular in industry because fitting can be easily visualized.

First, we will examine the fitting of two-parameter Weibull using median rank regression method. Median rank regression determines the best-fit straight line by least squares regression curve fitting. This method proceeds as follows:

1. Obtain failure data

2. Consider the following equation representing Weibull CDF:

$$F(T) = 1 - e^{-\left(\frac{T}{\eta}\right)^{\beta}} \tag{3.5}$$

This equation can be linearized by taking logs:

$$\ln\left(1 - F(T)\right) = -\left(\frac{T}{\eta}\right)^{\beta} \tag{3.6}$$

$$\ln(T) = \ln(\eta) + \frac{1}{\beta}\ln\left(\ln\left(\frac{1}{1 - F(T)}\right)\right) \tag{3.7}$$

Putting it in form of

$$Y = mx + c \tag{3.8}$$

we get

$$Y = \ln\left(\ln\left(\frac{1}{1 - F(T)}\right)\right) \tag{3.9}$$

$$m = \beta \tag{3.10}$$

$$x = \ln(T) \tag{3.11}$$

$$c = \beta \ln(\eta) \tag{3.12}$$

3. Calculate median ranks: Rank failure times in ascending order. Mean ranks $y = \frac{R_i}{n+1}$ are less accurate for the skewed Weibull distribution, therefore median ranks are preferable [9]. Median ranks can be calculated from equation (3.13):

$$\sum_{k=i}^{N} \binom{N}{k} (MR)^k (1 - MR)^{N-k} = 0.50 = 50\% \tag{3.13}$$

Bernard used an approximation of it as follows:

$$MR = \frac{i - 0.3}{N + 0.4}, \tag{3.14}$$

where

i – failure order number,

N – total sample size.

Rank adjustments are used when failure times are censored for some items. Censored items cannot be excluded from the analysis. In reliability censored items are sometimes referred to as suspended or suspensions. The formula in equation (3.15) gives the ranks adjusted for the presence of suspensions. It is used for every failure and requires an additional calculation for reverse ranks (items ranked in descending order).

$$\text{Adjusted Rank} = \frac{(\text{Reverse Rank}) * (\text{Previous Adjusted Rank}) + (n+1)}{\text{Reverse Rank} + 1} \quad (3.15)$$

4. Fitting the line: The best fit line in MRR is defined here as the one that minimizes the sum of squared differences between the true and estimated values. There are other approaches to measure the accuracy of fit besides least squares. Narula et al. (1999) has described approaches that minimizes the sum of absolute errors and sum of relative errors (compared with actual value). The conclusion of this study was that least squares performs better when errors are distributed normally without outliers. Least squares can be minimized either in x-direction or y-direction. Experiments by Weibull found that life data errors in the x-direction are generally higher than in the y-direction. Therefore an estimate obtained by minimizing x-errors would be preferable.

   Suppose we have a set of points $(x_1, y_1), (x_2, y_2)\ldots$ obtained by linearization of life data. We can use the ordinary least squares to estimate the slope $(\hat{b})$ and the intercept $(\hat{a})$ of the straight line defined by the equation $y = \hat{b}x + \hat{a}$, as follows:

$$\hat{b} = \frac{\Sigma_{i=1}^{n} x_i y_i - \frac{\Sigma_{i=1}^{n} x_i \Sigma_{i=1}^{n} y_i}{N}}{\Sigma_{i=1}^{n} x_i^2 - \frac{(\Sigma_{i=1}^{n} x_i)^2}{N}} \quad (3.16)$$

$$\hat{a} = \frac{\Sigma_{i=1}^{n} y_i}{N} - \hat{b}\frac{\Sigma_{i=1}^{n} x_i}{N} = \bar{y} - \hat{b}\bar{x} \quad (3.17)$$

Another popular method for Weibull parameter estimation is Maximum Likelihood Estimation (MLE). It aims to find the combination of parameters β and η that maximize the probability of the observed data.

MLE method produces best estimates with large sample sizes. MLE estimates tend to be optimistic (predict longer life) with small samples [9]. Joint density function $\prod_{i=1}^{n} \left(\frac{\beta}{\eta}\right)\left(\frac{t_i}{\eta}\right)^{\beta-1} \exp\left(-\frac{t_i}{\eta}\right)^{\beta}$ would describe the likelihood function for Weibull parameters for n failed items. To account for the k suspended items, a factor $\big(1-F(T)\big)$ should be multiplied, where $F(T) = 1 - \exp{(-t/\eta)}^{\beta}$. So, the complete likelihood function (L) becomes:

$$L = \prod_{i=1}^{n} \left(\frac{\beta}{\eta}\right)\left(\frac{t_i}{\eta}\right)^{\beta-1} \exp\left(-\frac{t_i}{\eta}\right)^{\beta} \prod_{j=1}^{k} \exp\left(-\frac{t_i}{\eta}\right)^{\beta}. \quad (3.18)$$

To find the maximum of this function with respect to β and η, we should differentiate logarithm of the likelihhod with the corresponding parameter. Maximum likelihood estimators for β and η does not have short form and are usually found by software.

After estimating $\tilde{\beta}$ its value can be used to compute $\tilde{\eta}$ by the following expression:

$$\tilde{\eta} = \left(\frac{\sum_{i=1}^{n} t_i^{\tilde{\beta}}}{r} \right)^{\frac{1}{\tilde{\beta}}}, \tag{3.19}$$

where

r – number of failed items +1.

Similar to Bernard's rank correction in MRR, we use a 'Reduced Bias Adjustment' (RBA) for improvement of accuracy in the MLE method. This adjustment aims to improve $\tilde{\beta}$ by minimizing median bias. For small samples most of the uncertainty is in $\tilde{\beta}$. Abernathy used an adjustment factor C_4 defined as:

$$C_4 = \sqrt{\frac{2}{n-1}} \frac{\left(\frac{n-2}{2}\right)!}{\left(\frac{n-3}{2}\right)!}, \tag{3.20}$$

where

n – number of complete failures.

He suggested that the value of $\tilde{\beta}$ gets affected to $(7/2)^{th}$ power of C_4. So, the new $\tilde{\beta}$ would be defined as follows:

$$\tilde{\beta}_{RBA} = \tilde{\beta}(C_4)^{3.5} \tag{3.21}$$

Correction factors improve estimates of $\tilde{\beta}$. New $\tilde{\eta}$ can be calculated from new $\tilde{\beta}$. For alternative method of reducing bias of MLE estimator based on idea of Jacknife see Quenoulle [7].

Some comparisons between MLE and MRR estimates will be performed in Section 3.3.4.

3.3.3 Goodness of fit

There are different statistical measures of goodness-of-fit such as Chi-Square, Kolmogoroff-Smirnoff, likelihood-ratio test, Anderson-Darling, etc. MRR can produce a plot so that a reliability engineer can see how close the least squares line is to the data (Figure 3.4).

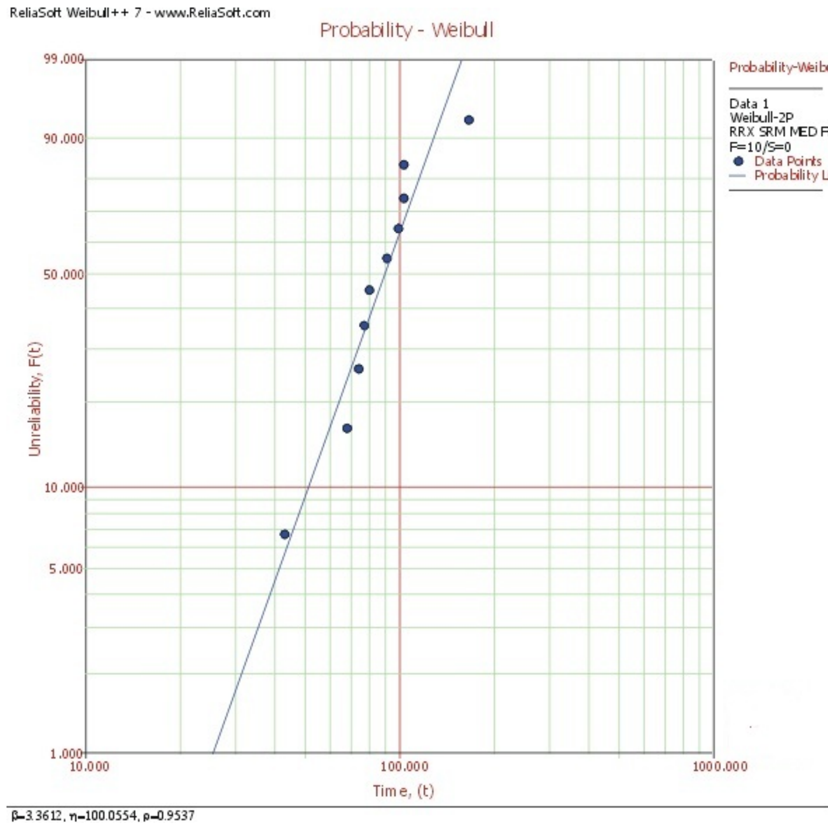

Figure 3.4: Median rank regression method on Weibull paper.

One of the simple measures of how close the model is to the true data is correlation coefficient between median ranks and the data. Although very popular in industry, it is not very indicative because it uses median ranks for plotting y on x, which artificially increases the observed correlation. Abernathy introduces a critical correlation coefficient (CCC) i.e., tenth percentile of simulated correlation coefficients as a threshold between satisfactory and unsatisfactory fits. CCC is estimated by the following procedure:

1. Simulate 1000 correlation coefficients by Monte Carlo trials using median ranks.

2. Choose the 100th value (10th percent of all values).

Therefore, fit is assumed acceptable if the obtained correlation coefficient is greater than that of the 10th percentile of simulated coefficients. Simulation is intended to approximate the distribution of the coefficient.

3.3.4 Case study

In this section we illustrate the theory provided previously in this chapter with a case study. We will fit the parameters of Weibull distribution and perform goodness of fit analysis.

1. Generate 20 values from Weibull ($\beta = 3$, $\eta = 20$).
2. Sort and rank them from lowest to highest.
3. Apply Bernard's approximation (without suspensions) to compute new ranks.
4. Linearize the equation using equation (3.13).
5. Fit least squares linear regression to get the slope (m) and intercept (c) estimates.
6. Simulate 1000 experiments fitting MRR to Weibull data. Then select the highest correlation value of the lowest 100 values of correlation coefficient. In our case it was CCC = 0.965.

Table 3.2 shows calculation of median rank regression method. Correlation coefficient of actual data is equal to 0.974. We conclude that it is higher than the CCC and, therefore, the fit is acceptable.

Rank	**Failure time**	**Bernard's rank**	**X**	**Y**	**Calculated Y**	**Residuals**	**Standard Residuals**
1	9.60	0.03	2.26	−3.51	−2.89	−0.62	−2.18
2	10.64	0.08	2.36	−2.50	−2.49	−0.01	−0.04
3	13.32	0.13	2.59	−1.99	−1.62	−0.37	−1.32
4	13.75	0.18	2.62	−1.64	−1.50	−0.14	−0.51
5	13.77	0.23	2.62	−1.36	−1.49	0.13	0.44
6	13.95	0.27	2.64	−1.14	−1.44	0.30	1.07
7	14.00	0.32	2.64	−0.94	−1.43	0.49	1.72
8	15.55	0.37	2.74	−0.76	−1.02	0.25	0.89
9	15.68	0.42	2.75	−0.60	−0.98	0.38	1.35
10	19.01	0.47	2.94	−0.45	−0.24	−0.22	−0.77
11	19.55	0.52	2.97	−0.31	−0.13	−0.18	−0.65
12	19.67	0.57	2.98	−0.17	−0.10	−0.07	−0.25
13	19.78	0.62	2.98	−0.04	−0.08	0.04	0.15
14	19.80	0.67	2.99	0.09	−0.08	0.17	0.60
15	20.39	0.72	3.02	0.23	0.04	0.19	0.68
16	21.45	0.76	3.07	0.37	0.23	0.13	0.47
17	23.73	0.81	3.17	0.52	0.63	−0.11	−0.38
18	23.82	0.86	3.17	0.69	0.64	0.04	0.15
19	24.83	0.91	3.21	0.89	0.80	0.08	0.29
20	30.96	0.96	3.43	1.18	1.66	−0.49	−1.72

Table 3.2: Intermediate values obtained from an experimental set

Now we can compare the maximum likelihood estimates against median regression estimates. Table 3.3 shows values for both methods along with true values.

	MRR	**MLE**	**TRUE**
beta	3.89	3.74	3
eta	20.19	20.1	20

Table 3.3: Comparison between MRR and MLE methods

Appendix A provides results from comparison of two methods by simulation study. MLE and MRR methods been used to fit Weibull parameters to 10000 series, each contained 20 randomly generated values from Weibull distribution ($\beta = 3$, $\eta = 20$). We can see that both estimators are biased but MLE produces lower MSE for both parameters.

	Bias		**Var**		**MSE**	
	beta	**eta**	**beta**	**eta**	**beta**	**eta**
MLE	0.22	−0.07	0.38	2.46	0.4284	2.4649
MRR	−0.12	0.19	0.42	2.63	0.4344	2.6661

Table 3.4: Simulation study results

A recent study by Genschel and Meeker [14] that aimed to compare the two methods of estimation concluded that for the majority of cases MLE method showed significantly better results. The reason MRR is still used in industry is its simple methodology and ability to visualize the fit. Also, a serious drawback for using MLE is its "optimism", overestimating lifecycle of the item, for small samples which is not desirable in many industry applications.

As will be shown below MRR method could be significantly inaccurate if:

(i) dataset contains a large number of suspensions,

(ii) the graph between failure time and number of failures is highly skewed, as shown in Table 3.5.

Failure Number	State (F/S)	Life (Case 1)	Life (Case 2)
1	F	10	10
2	S	11	97
3	S	12	98
4	S	13	99
5	F	100	100

Table 3.5: Input set for analysis of differences between MLE and MRR methods

MRR method does not differentiate between the two situations on the basis of failure time, as long as the number of failures and the total time are the same, and provides the same result for (β, η) = (0.81, 113.96). The corresponding values for MLE were different for the two cases—(1.33, 69.2) for Case 1, and (0.93,213.43) for Case 2. This illustrates that (i) MLE is more accurate than MRR for skewed failure times (ii) MRR underestimates β values in comparison to MLE ([14]).

In conclusion, it can be said that the number of Weibull parameters to fit and fitting method should be chosen based on the data and purpose of the analysis. For example, if there are small datasets with no suspensions, we can use two-parameter Weibull fitted by Median Rank Regression method. If sample size is large and there are suspensions, MLE method will provides parameter estimation with the smallest MSE.

Chapter 4

Sampling units for reliability analysis

The purpose of this chapter is to examine different sampling plans for monitoring reliability of the production process. In this chapter we will introduce comparison methodology for two sampling plans and show the current method used in industry. Also we will investigate properties of the proposed sampling plan and mathematical derivations for its operating characteristics.

4.1 Comparison

Different sampling schemes can be compared on the basis of their operating characteristics, which show the probability of acceptance as a function of a proportion defective. Operating characteristic can be plotted as an OC curve. Two plans will be considered identical if their OC curves are the same (see figure 4.1).

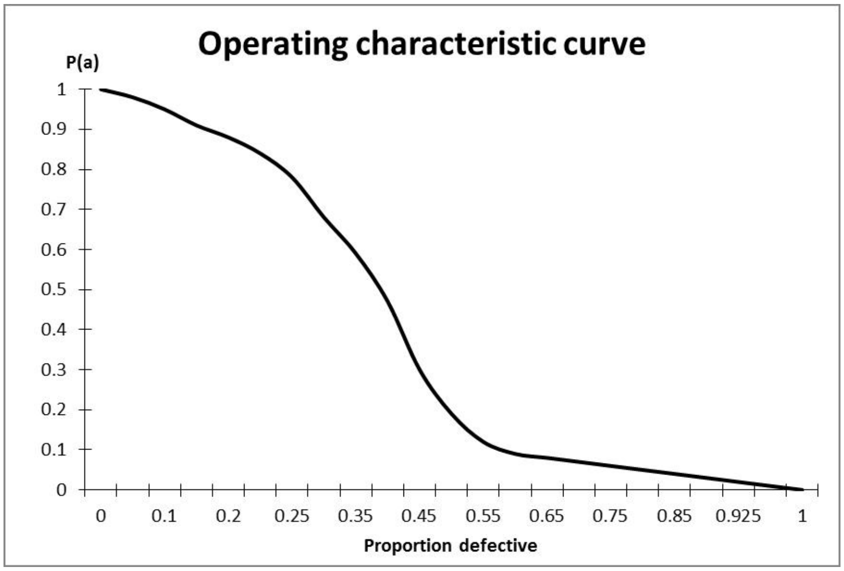

Figure 4.1: Example of Operating Characteristic Curve: OR=9

Operating ratio (OR) is another important measure for comparison. It is defined as the value of fraction defective (p) at probability of lot acceptance $P_a = 0.1$ divided by p at $P_a = 0.95$. As the OC curve become steeper, the operating ratio becomes smaller. Accordingly, small values of OR indicates ability of the plan to detect changes in product quality or reliability with high statistical power. An ideal operating characteristic function is one that accepts lots with probability one if the reliability is equal or greater than the specification and rejects lots with probability one if the reliability is less than a predetermined level (see figure 4.2).

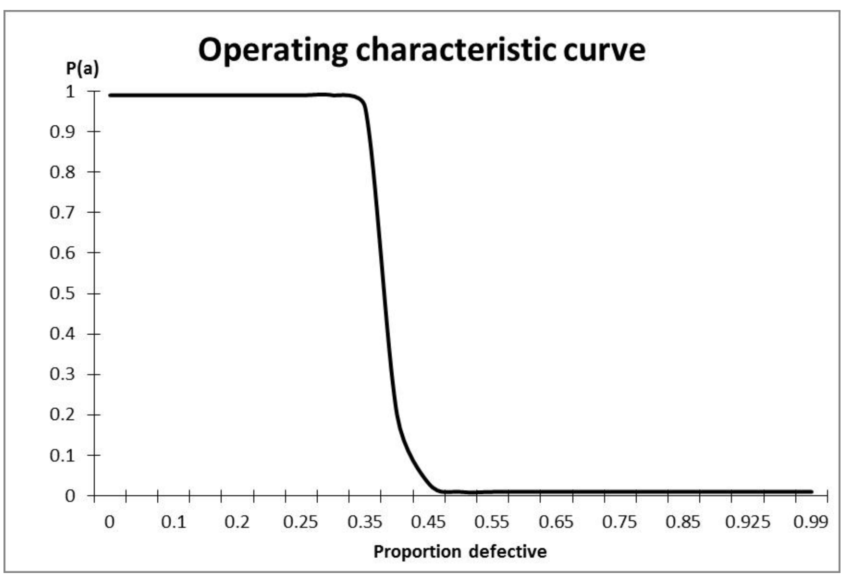

Figure 4.2: Ideal OC Curve (defective acceptance proportion=0.4): OR=1

In developing a sampling plan, a reliability team usually decides the probability of consumer and producer risks that is tolerable. After that, a sampling plan with a specific OC should be constructed to meet these goals. This process includes choosing a sampling design, setting the values of sample size, and choosing the number of nonconformitive items that are tolerable.

4.2 Current Method

One of the practices for assuring the quality or reliability of products in the smartphone industry is acceptance sampling. An acceptance sampling plan is a set of rules that specify sampling procedures and criteria for accepting or rejecting a lot based on observed information from the sample. Criteria can be based on a number of non-conforming items (attribute-based) or observed variables (variable-based). For example, acceptance of a lot with 3 or less defective items in the sample will be an attribute-based criterion, whereas acceptance of a lot with the mean life of sampled units greater than or equal to 20 days will be a variable-based criteria.

Currently one of the common methods to draw a sample for the smartphone reliability test is using a systematic sampling design for primary units (smartphone lots) and simple random sampling for secondary units (devices). The lot is rejected if any device in the sample fails before a certain number of time units. For example, a company produces 2000 devices per week, which represents one lot. Suppose that reliability specifications for device life during extreme heating is set to 20 hours. To monitor reliability of the produced devices from each production line, every fourth week 20 devices are sampled and exposed to extreme heating for 70 hours (see figure 4.3). Every 4 hours devices are inspected and all functionality is tested. If life of any sampled devices is less than 20 hours then the manufacturing line stops and a product investigation is launched. The main disadvantages of the sampling design used in this procedure is that it does not differentiate production lines base on their reliability history, and it is not the most powerful design for the given sample size. It is desirable to sample less from production lines with a good reliability history and test more devices from production lines where reliability of devices has historically been low.

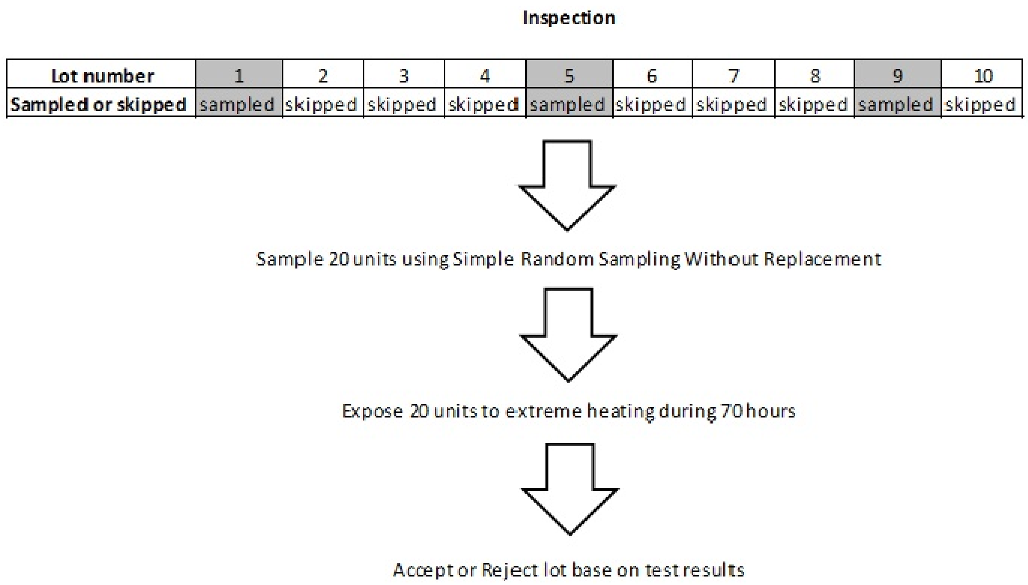

Figure 4.3: Algorithm of the current method

4.3 Proposed Method

The proposed method is similar to skip-lot sampling inspection with double-sampling as the reference plan (SkSP-2DSP) initially proposed by Perry and elaborated by Vijayaraghavan and Soundararajan (1998).

4.3.1 Skip-lot Sampling Plan with Double Sampling as the Reference Plan

SKSP-2DSP has several advantages compared to other sampling schemes. It is generally has more power when the sample size is small. Therefore it is especially beneficial for situations with destructive testing that are frequently encountered in reliability. This sampling plan consists of 2 stages: sampling for primary units using SKSP-2 methodology and sampling for secondary units using double sampling plan (DSP).

Sampling for primary units (lots) proceeds in several steps (see figure 4.4):

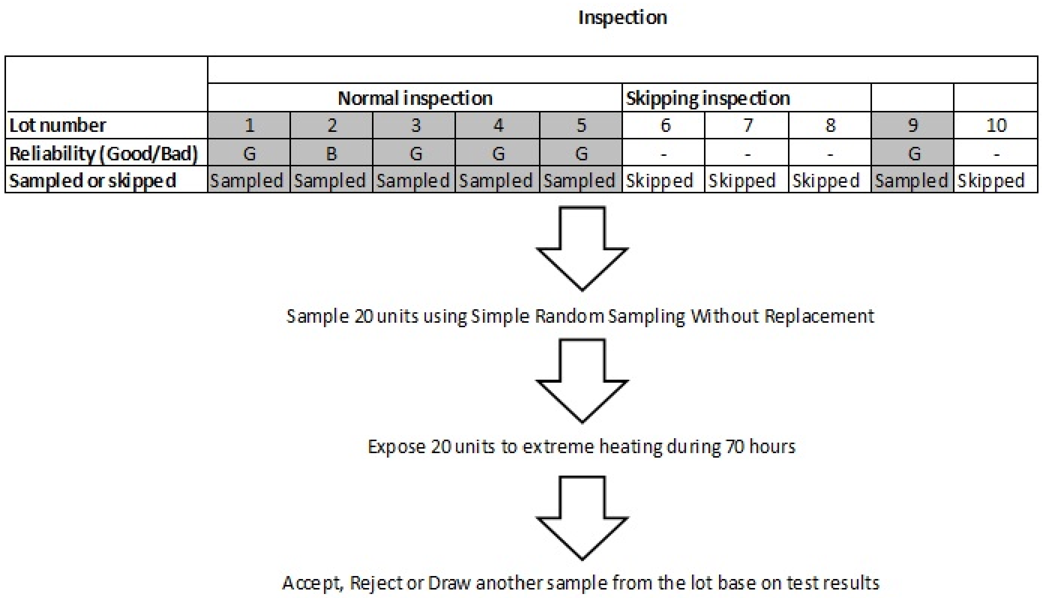

Figure 4.4: Algorithm of the proposed method

1. Perform sampling on every lot from the process.
2. When $\boldsymbol{i}$ consecutive lots are accepted, switch to skipping inspection, perform sampling for a fraction of lots.
3. If one of the lots has been rejected, switch to inspection of every lot.

Skipping inspection shold be performed in such a way that every lot has a probability of being selected. For example, if we want to test 30% of lots we can use the Bernoulli random number generator with $p = 0.3$. Therefore every lot will be a candidate for this test procedure. This procedure will assure that unacceptable lots are not submitted intentionally during a skipped step as can be done during systematic design.

Some properties of $P_a(f, i)$ where f – frequency of lot sampling during skipping inspection, i – number of consecutively accepted lots that triggers a skipping inspection which are readily proven, and P – probability of accepting a lot according to the [12] plan, are:

- For fixed i and given reference plan, and $f_1 > f_2$

$$P_a\left(f_1, i\right) \leq P_a\left(f_2, i\right)$$

- For fixed f and given reference plan and integers $i > j$

$$P_a(f, i) \leq P_a(f, j)$$

- For fixed f and i

$$P_a(f, i) \geq P$$

These properties will be further examined during simulation study in chapter 5.

The sampling for secondary units proceeds as follows:

1. Draw a sample of n_1 units from a lot using simple random sampling without replacement.
2. Count number of non-conforming items d_1 or calculate variable value:
   (a) If d_1 non-conforming units in the sample is less than or equal to c_1, then the lot is accepted without further inspection.
   (b) If d_1 is in the interval from c_1 to c_2, then a second sample n_2 is drawn from the lot using SRSWR.
   (c) If $d_1 + d_2$ exceeds c_2, then the lot is rejected. If the lot is rejected, manufacturing process should be stopped and qualitative analysis should be launched.

After this analysis all devices that are below specification should be replaced with items from accepted lots. The OC function $P_a(p)$ of the SkSP-2DSP plan at reliability level p can be derived using either Power Series or Markov Chain approaches.

4.3.2 Power Series Approach

This section shows how to calculate P_a for SkSP-2DSP. This derivation is analogous to the derivation by Perry (1970) but has been extended on several occasions ([10], [12]). As mentioned above, the plan has two levels of inspection:

1. Normal inspection, where each lot is sampled.
2. Skipping inspection, where only a fraction f of the lots are sampled.

Therefore, to calculate probability of acceptance (P_a), we need to obtain both P_a for normal inspection and P_a for skipping inspection.

	Inspection Level																		
	Normal Inspection										Skipping Inspection								
Lot Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Reliability	B	G	G	G	B	G	G	G	G	G	G	-	G	-	G	-	G	-	B
						interval value(i)													
	interval value(U)										interval value(V)								

Table 4.1: Example of inspection

where

$B-$ "bad lot"-sample with lots that have reliability below specifications,

$G-$ "good lot"- sample with lots that have reliability above specifications,

"-" $-$ skipped lot,

$U-$ expected number of lots during normal inspection,

$V-$ expected number of lots during skipping inspection,

$i-$ number of accepted lots that triggers skipping inspection,

$U+V-$ represents the expected number of lots in a complete cycle of normal and skipping.

Sequence	**Number of lots**	**Probability**
B	1	Q
GB	2	PQ
GGB	3	P^2Q
$GGGB$	4	P^3Q
etc.	etc.	etc.

Table 4.2: Sequence of lot samples

where

$P-$ probability of lot acceptance,

$Q-$ probability of lot been rejected.

Perry notes that table 4.2 represents a probability distribution since the probability terms sums to one, i.e., $Q(1+P+P_2+\dots)=1$. Therefore, the sum of first i probability terms from the distribution is:

$$\sum_{j=0}^{i-1} QP^j = 1 - P^i \tag{4.1}$$

Equation (4.1) represents the probability of a failure sequence with length i or less lots. It is the probability of not accepting all the next i lots. To derive U and V we note that U consist of i consecutive lots accepted just before the start of skipping inspection plus all lots from failure sequences. We will use the following notation:

$Z -$ expected number of failure sequences before having i consecutively accepted lots,

$h -$ expected number of lots in each of these failure sequences.

According to definition of i, h should be smaller than or equal to i. Therefore, U is equal to:

$$U = Zh + i \tag{4.2}$$

The value of h is the average of the distribution of the first i terms in table 4.2

$$h = \frac{1Q}{1-P^i} + \frac{2PQ}{1-P^i} + \cdots + \frac{iP^{i-1}Q}{1-P^i} \tag{4.3}$$

that simplifies to

$$h = \frac{1 - P^i(1+Q^i)}{Q(1-P^i)} \tag{4.4}$$

We can find Z, the expected number of failure sequences before the start of skipping inspection, from the distribution of all the possible number of failure sequences (see table 4.3).

Number of Failure Sequences Before i Lots Accepted	**Probability**
0	P^i
1	$P^i\left(1-P^i\right)$
2	$P^i\left(1-P^i\right)^2$
3	$P^i\left(1-P^i\right)^3$
etc.	etc.

Table 4.3: Distribution of Failure Sequences

Therefore:

$$\begin{aligned} Z &= 0P^i + 1(1-P^i)P^i + 2(1-P^i)^2P^i + \dots \\ &= P^i(1-P^i)[1 + 2(1-P^i) + 3(1-P^i)^2 + \dots] \end{aligned} \tag{4.5}$$

Using the formula for infinite series sum $\sum_{n=0}^{\infty} (n+1)\, z^n = \frac{1}{(1-z)^2}$ for all $z \leq 1$, we obtain:

$$Z = \frac{1-P^i}{P^i} \tag{4.6}$$

Now, putting all formulas together, we get:

$$\begin{aligned} U &= Zh + i \\ &= \frac{[1-P^i(1+Q^i)](1-P^i)}{Q(1-P^i)P^i} + i \end{aligned} \tag{4.7}$$

To derive V, the expected number of lots during the skipping period in terms of f, i, and P, Perry notes that V is equal to $\frac{1}{f}$ times the number of lots inspected in such periods.

We denote H as an expected number of lots inspected in the skipping period. Therefore, H is equal to the average number of lots in a terminal-reject sequence, that can be found from the distribution in table 4.1:

$$H = 1 \times Q + 2 \times P \times Q + 3 \times P^2 \times Q + \dots \tag{4.8}$$

Therefore, using the same formula as in (4.6), we obtain:

$$\begin{aligned} H &= Q(1 + 2P + 3P^2 + \dots) \\ &= \frac{Q}{(1-P)^2} \\ &= \frac{1}{1-P} \end{aligned} \tag{4.9}$$

so V can be shown as:

$$V = \frac{1}{f(1-P)} \tag{4.10}$$

Now we can obtain the probability of acceptance P_a for the skip-lot plan. First we will derive the probability of being rejected $P_r = 1 - P_a$, the proportion of lots which are rejected:

$$\begin{aligned}P_r &= \frac{\text{expected number of lots rejected}}{\text{expected number of lots submitted}} \\ &= \frac{Z+1}{U+V}\end{aligned}$$

The numerator consists of expected number of lots rejected during a normal inspection period plus 1 lot rejected during a skipping period.

Substituting all previous derivations we have:

$$\begin{aligned}P_r &= \frac{\frac{1-P^i}{P^i}+1}{\frac{1-P^i}{QP^i}+\frac{1}{fQ}} \\ &= \frac{f(1-P)}{(1-f)P^i+f}\end{aligned} \tag{4.11}$$

Now, $P_a = 1 - P_r$ or

$$\begin{aligned}P_a &= 1 - \frac{f(1-P)}{(1-f)P^i+f} \\ &= \frac{(1-f)P^i+fP}{(1-f)P^i+f}\end{aligned} \tag{4.12}$$

Now we can construct the OC curve as a function of the skipping parameters f, i, and also P the probability of accepting a lot under DSP, given by:

$$P = R(c_1, m_1) + \sum_{r=c_1+1}^{c_2} q(r; m_1)R(c_2 - r; m_2), \tag{4.13}$$

$$m_1 = n_1 p, \qquad m_2 = n_2 p, \qquad q(r, m) = \exp(-m)m^r/r! \tag{4.14}$$

and

$$R(c_1; m) = \sum_{r=0}^{c_1} q(r; m) \tag{4.15}$$

We have used proportion of defective (p) for the above calculations. In reliability testing variable-based criterias are more common then attribute-based criterias. Technical Report TR3 (1961) provide sampling plans and procedures for transition from variable plan to attribute plan. The lifetime of the product is considered to be Weibull-distributed random variable Y with unknown parameter η (eta) and β (beta). Since p is the proportion of product failing before time t, it can be used as "percent defective" in any attributes plan. The relationship is illustrated below:

$$P = F(t) = 1 - e^{-(t/\eta)^{\beta}} \tag{4.16}$$

This can be shown by the following example.

$n_1 = 20$

$n_2 = 20$

$c_1 = 0$

$c_2 = 1$

$i = 3$

$f = 0.25$

$P = 0.1$ (from fitting Weibull to the data and finding the cdf)

$$P_a(p) = \frac{[fP + (1-f)P^i]}{[f + (1-f)P^i]} = \frac{0.25P + 0.75P^3}{0.25 + 0.75P^3}$$

$$P = R(c_1, m_1) + \sum_{r=c_1+1}^{c_2} q(r; m_1)R(c_2 - r; m_2)$$

$$m_1 = n_1 p = 20 \times 0.1 = 2$$

$$m_2 = n_2 p = 20 \times 0.1 = 2$$

$$R(c_1, m_1) = \sum_{r=0}^{c_1} q(r; m_1) = \frac{\exp(-m_1)m_1^r}{r!} = \frac{\exp(-2)2^0}{0!} = \exp(-2)$$

$$\sum_{r=c_1+1}^{c_2} q(r; m_1)R(c_2 - r; m_2) = q(1; 2)R(0; 2) = \frac{\exp(-2)2^1}{1!}\frac{\exp(-2)2^0}{0!} = 2\exp(-4)$$

$$P = \exp(-2) + 2\exp(-4) = 0.172$$

$$P_a(p) = \frac{0.043}{0.542} = 0.169.$$

4.4 Markov Chain Approach to Determine P_a

In this section we will derive probability of acceptance using Markov Chain methods. The SKSP-2 sampling plan is a Markov Chain because the probability of accepting the present lot is dependent only upon the outcome of testing the preceding lot. We define states for the chain and determine a one-step transition probability matrix.

A trial for the sampling plan corresponds to sampling from a lot which is under consideration. Outcomes of these trials will be the states of the chain. Using the double sampling plan, all tested lots will be either accepted or rejected even if the immediate result of the sampling can be accepted, rejected, or unable to reach decision (in which case another sampling is performed from the same lot).

Therefore we can categorize all states of Markov Chain into two main groups: normal inspection states and skipping inspection states. We will use N to denote states of the chain where normal inspection is used and S to represent skipping inspection. For the normal inspection we add an integer that shows the number of preceding consecutive accepted lots, e.g., $N_0, N_1, N_2 \ldots, N_i$. For the skipping inspection we will use only three states to specify all outcomes:

SA–lot sampled and accepted during skipping inspection,

SR–lot rejected during skipping inspection,

SN–lot skipped during skipping inspection.

Therefore, there are $i + 4$ states in this Markov Chain. The transition probability matrix is shown in table 4.4.

		To this state							
		N_o	N_1	N_2	$\ldots$	N_i	**SA**	**SR**	**SN**
From this state	N_O	Q	P	-		-	-	-	-
	N_1	Q	-	P		-	-	-	-
	$\ldots$								
	N_{i-1}	Q	-	-		P	-	-	-
	N_i	-	-	-		-	fP	fQ	1-f
	SA	-	-	-		-	fP	fQ	1-f
	SR	Q	P	-		-	-	-	-
	SN	-	-	-		-	fP	fQ	1-f

Table 4.4: Transition Matrix

Now to derive the probability of acceptance we need to sum the limiting or long run probabilities of all states where a lot has been accepted, which are $N_1, N_2, \ldots, N_i, SA$, and SN. From the transition matrix in table 4.4 we can see that there are $i + 4$ states in this Markov Chain and therefore it is finite. Besides that, every state can be reached from another in a finite number of steps; therefore, the chain is recurrent and irreducible. Perry also noted that N_0, SA, and SN steps can recur in exactly one transition, therefore the chain is aperiodic.

Therefore, Markov Chain representing the sampling plan is finite, recurrent, irreducible, and aperiodic. Now we will derive the limiting distribution–long run probabilities of the chain occupying each of its states. Markov Chain ergodic theorem corollary states that when a chain possesses the aforementioned properties, the limiting distribution will be the same as the unique stationary distribution. This stationary distribution for each state π_i can be derived from following equations:

$$\pi_i = \sum_{\text{all } j} \pi_j P_{ji} \tag{4.17}$$

$$\sum_{\text{all } i} \pi_i = 1 \tag{4.18}$$

where

P_{ij} – is the one-step transition probability of going from state i to state j.

i, j – states of the chain.

As in the previous section we will first derive P_r:

$$\begin{aligned} P_r &= 1 - P_a \\ &= \pi_{N_0} + \pi_{SR} \end{aligned} \tag{4.19}$$

π_{N_0} and π_{SR} can be found from the following equations:

$$\pi_{N_0} = Q\left(\pi_{N_0} + \pi_{N_1} + \cdots + \pi_{N_{i-1}} + \pi_{SR}\right), \tag{4.20}$$

$$\pi_{N_1} = P\left(\pi_{N_0} + \pi_{SR}\right), \tag{4.21}$$

$$\pi_{N_j} = P^{j-1}\pi_{N_1} \quad \text{for } j = 2, 3, \ldots, i, \tag{4.22}$$

$$\pi_{SN} = (1-f)\left(\pi_{N_i} + \pi_{SN} + \pi_{SA}\right), \tag{4.23}$$

$$\pi_{SA} = fP\left(\pi_{N_i} + \pi_{SN} + \pi_{SA}\right), \tag{4.24}$$

$$\pi_{SR} = fQ\left(\pi_{N_i} + \pi_{SN} + \pi_{SA}\right). \tag{4.25}$$

and the fact that

$$\pi_{N_0} + \pi_{N_1} + \cdots + \pi_{N_i} + \pi_{SN} + \pi_{SA} + \pi_{SR} = 1 \tag{4.26}$$

Note that $\pi_{N_i} = P^{i-1}\pi_{N_1}$ and

$$\begin{aligned}\pi_{N_1} + \cdots + \pi_{N_i} &= \pi_{N_1} + \pi_{N_1}P + \cdots + \pi_{N_1}P^{i-1} \\ &= \pi_{N_1}\left(1 + P + \cdots + P^{i-1}\right) \\ &= \frac{(1-P^i)}{Q}\pi_{N_1} \qquad (4.27)\end{aligned}$$

From the same logic

$$\pi_{N_1} + \cdots + \pi_{N_{i-1}} = \frac{(1-P^{i-1})}{Q}\pi_{N_1} \qquad (4.28)$$

Now we can rewrite equation (4.26) as

$$\pi_{N_0} + \frac{(1-P^i)}{Q}\pi_{N_i} + \pi_{SN} + \pi_{SA} + \pi_{SR} = 1 \qquad (4.29)$$

then by substituting $\pi_{N_1} + \cdots + \pi_{N_{i-1}}$ by $\frac{(1-P^{i-1})}{Q}\pi_{N_1}$ in the equation (4.20)

$$P\pi_{N_0} = \left(1 - P^{i-1}\right)\pi_{N_1} + Q\pi_{SR} \qquad (4.30)$$

or using equation (4.21)

$$P\pi_{N_0} = \pi_{N_1} - P\pi_{SR} \qquad (4.31)$$

From equation (4.23) we have

$$f\pi_{SN} = (1-f)P^{i-1}\pi_{N_1} + (1-f)\pi_{SA} \qquad (4.32)$$

From equation (4.24) we have

$$(1-fP)\,\pi_{SA} = fP^i\pi_{N_1} + fP\pi_{SN}$$

Now if we multiply both parts of equation (4.32) by P and add to the above equation we obtain

$$\pi_{SA} = \frac{P^i}{Q}\pi_{N_1} \qquad (4.33)$$

Then Perry suggests multiplying equation (4.32) by $(1-fP)$ and equation for $(1-fP)\pi_{SA}$

by $(1-f)$ and then adding the resulting equations:

$$\pi_{SN} = \frac{(1-f)P^{i-1}}{fQ}\pi_{N_1} \tag{4.34}$$

Now we can put the above results in equation (4.29) to obtain:

$$\pi_{N_0} + \frac{(f + P^{i-1} - fP^{i-1})}{fQ}\pi_{N_1} + \pi_{SR} = 1 \tag{4.35}$$

If we subtract equation (4.31) from equation (4.30) the following will be evident:

$$\pi_{SR} = P^{i-1}\pi_{N_1} \tag{4.36}$$

Now by subtracting these into equation (4.35), multiplying by P and then subtracting equation (4.30) π_{N1} is equal to:

$$\pi_{N_1} = \frac{fQP}{f + (1-f)\,P^i} \tag{4.37}$$

Then we can rewrite equation (4.33) and (4.34) as:

$$\pi_{SN} = \frac{P^i\,(1-f)}{f + (1-f)\,P^i} \tag{4.38}$$

$$\pi_{SR} = \frac{fQP^i}{f + (1-f)\,P^i} \tag{4.39}$$

$$\pi_{SA} = \frac{fP^{i+1}}{f + (1-f)\,P^i} \tag{4.40}$$

Finally, we can rewrite equation (4.35) as:

$$\pi_{N_0} = \frac{fQ\,(1-P^i)}{f + (1-f)\,P^i} \tag{4.41}$$

Therefore,

$$\begin{aligned} P_r &= \pi_{N_0} + \pi_{SR} \\ &= \frac{fQ\,(1-P^i)}{f + (1-f)\,P^i} + \frac{fQP^i}{f + (1-f)\,P^i} \\ &= \frac{f\,(1-P)}{f + (1-f)\,P^i} \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} P_a &= 1 - P_r \\ &= \frac{f + P^i\,(1-f) - fQ}{f + (1-f)\,P^i} \\ &= \frac{fP + (1-f)P^i}{f + (1-f)P^i} \end{aligned} \tag{4.43}$$

which matches equation (4.12) obtained through the power series approach.

Chapter 5

Simulation

In this chapter the properties of the proposed sampling method will be examined through theory and simulation. In particular the relationships between $P(a)$ and its parameters: f, i, fraction defective and p will be investigated and compared to the OC curve for SkSP-2DSP and systematic sampling plans.

During simulation, failure times were generated for the 1000 lots with 100 devices in each lot. Failure times were generated from Weibull distributions with three different sets of parameters. The first type of parameter distribution values were supposed to produce failure times that would satisfy the reliability requirements so that a lot would be accepted. Distributions from the second type of parameter values were supposed to produce failure times that would generate “defective” lots. Distributions from the third type of parameter values were supposed to produce failure times that would generate “doubtful” lots that would require a second sampling as per the SKSP-2DSP procedure. After that the median rank regression method was used to fit Weibull parameters to all the devices in a lot (100) and all lots (1000) to obtain population values for the comparison.

5.1 Frequency (f)

To reveal the dependence between $P(a)$ and f Equation 4.12 will be transformed as follows:

$$\begin{aligned} P_a &= \frac{(1-f)P^i + fP}{(1-f)P^i + f}, \quad \text{assuming } f > 0 \\ &= \frac{P^i - fP^i + fP}{P^i - fP^i + f} \\ &= 1 - \frac{f - fP}{P^i - fP^i + f} \\ &= 1 - \frac{1-P}{\frac{P^i}{f} - P^i + 1} \end{aligned} \tag{5.1}$$

It can be seen from Equation (5.1) that as f increases the ratio P^i/f decreases and overall $P(a)$ decreases, assuming other parameters are fixed; in other words, the more primary units are sampled, the lower the probability that the lot will be accepted. Figure 5.1 shows the relationship between f and P^i obtained through the simulation.

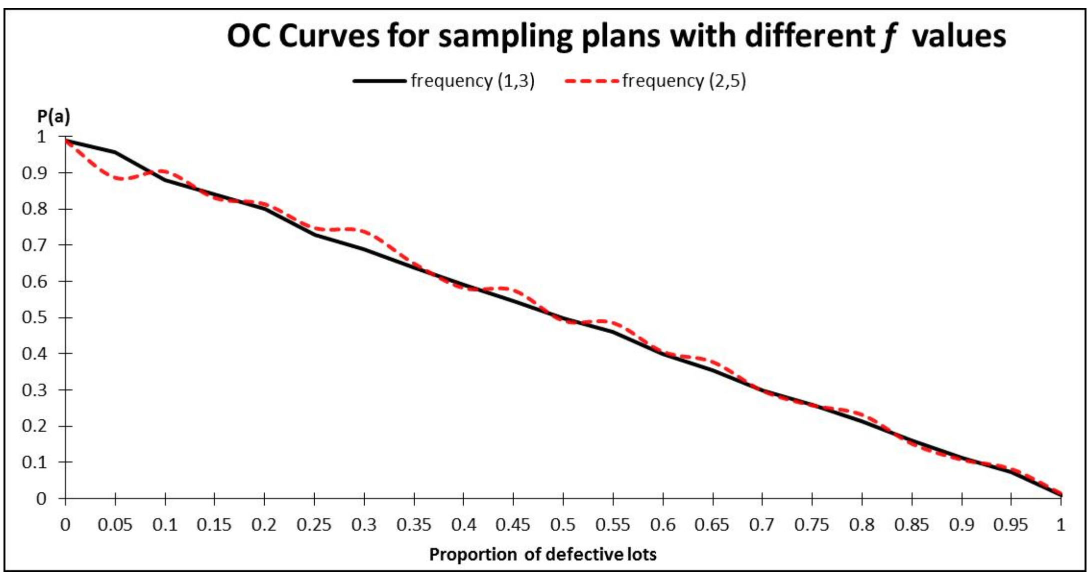

Figure 5.1: OC curve for sampling plan ($n_1 = 20$, $n_2 = 20$, $f_1 = 1$, $f_2 = 3$, $c_1 = 1$, $c_2 = 0$, $i = 5$) and ($n_1 = 20$, $n_2 = 20$, $f_1 = 2$, $f_2 = 5$, $c_1 = 1$, $c_2 = 0$, $i = 5$)

It can be seen from the OC curve for the sampling scheme that samples taken every second lot during normal inspection and every fifth lot when skipping inspections has a slightly bigger $P(a)$ value than the sampling scheme that samples every lot during normal inspection and every third when skipping inspections. It is also worth noting that when $f = 1$ the probability of acceptance will be equal to P—probability of acceptance in the reference sampling plan.

5.2 Number of accepted lots that triggers skipping inspection (i)

Now $P(a)$ will be considered as a function of i only. As with frequency, Equation 4.12 can be rewritten as:

$$P_a = \frac{(1-f)P^i + fP}{(1-f)P^i + f} = 1 - \frac{f - fP}{P^i(1-f) + f} \tag{5.2}$$

Now as i increases the denominator decreases (P is less than 1), and therefore $P(a)$ also decreases. It is logical that when the number that triggers skipping inspection is increased there will be a bigger probability for the lot to fail, because more lots will be subject to inspection. When lots are skipped it is assumed that they have been accepted. Figure 5.2 shows the result from simulation, assuming P and f are fixed.

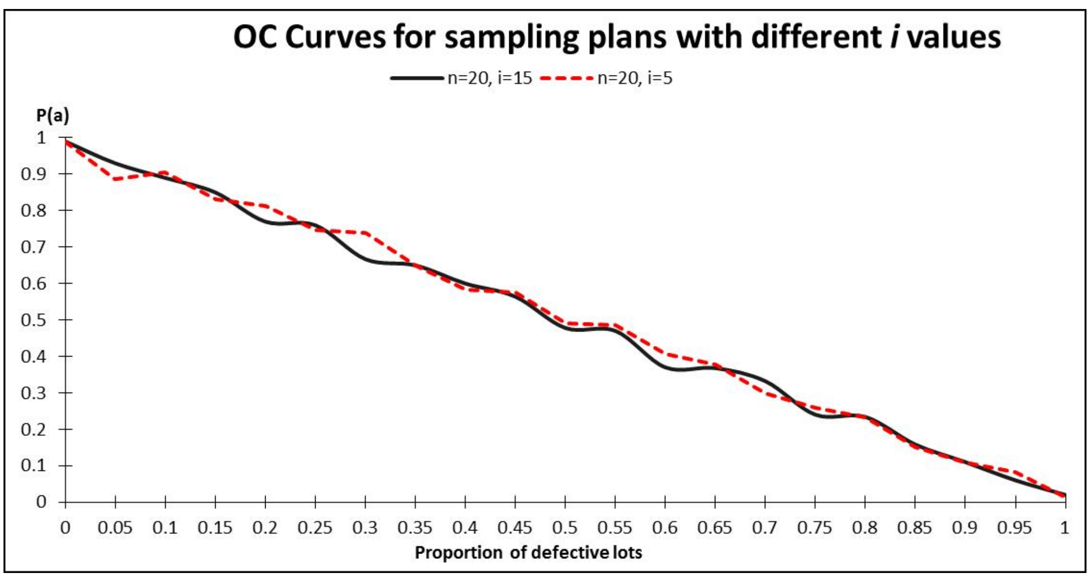

Figure 5.2: OC curve for sampling plan ($n_1 = 20$, $n_2 = 20$, $f_1 = 1$, $f_2 = 3$, $c_1 = 1$, $c_2 = 0$, $i = 5$) and ($n_1 = 20$, $n_2 = 20$, $f_1 = 1$, $f_2 = 3$, $c_1 = 1$, $c_2 = 0$, $i = 15$)

It is hard to see the difference between the two lines but when the proportion of defective lots is less than 30%, the sampling scheme with $i = 5$ has lower values than when using the sampling scheme with $i = 15$ for 90% of the time. When the proportion of defective lots is bigger than 70%, the values are very similar because skipping inspection can not be triggered in either case.

5.3 Probability of lot acceptance (P)

The dependency between $P(a)$ and probability of lot acceptance P will be examined. P itself is a function of the proportion of defective lots, sample size and acceptance threshold. Equation 4.12 can be rewritten as follows:

$$\begin{aligned} P_a &= \frac{(1-f)P^i + fP}{(1-f)P^i + f} \\ &= 1 - \frac{f(1-P)}{P^i(1-f) + f} \end{aligned} \tag{5.3}$$

It can be seen that as P increases, the numerator in Equation (5.3) decreases, the denominator increases and overall $P(a)$ increases. Figure 5.3 shows the result from the simulation study.

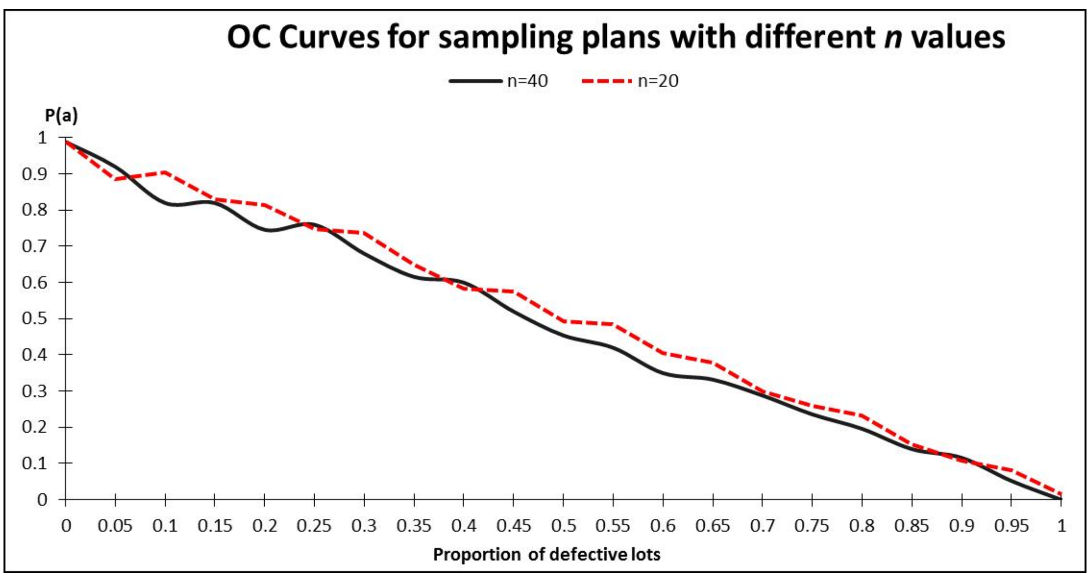

Figure 5.3: OC curve for sampling plan ($n_1 = 20$, $n_2 = 20$, $f_1 = 1$, $f_2 = 3$, $c_1 = 1$, $c_2 = 0$, $i = 5$) and ($n_1 = 40$, $n_2 = 20$, $f_1 = 1$, $f_2 = 3$, $c_1 = 1$, $c_2 = 0$, $i = 5$)

In this case two different sample sizes, 20 and 40, were tried. The differences between the two OC curves can be seen: the OC curve for the sampling scheme with $n = 20$ generally has a higher $P(a)$ than the OC curve for the sampling scheme with $n = 40$.

Now the skip-lot plan that has approximately the same OC curve as the plan with systematic sampling, will be shown.

Figure 5.4 presents the OC curves for the method with systematic sampling for primary units and simple random sampling for secondary units.

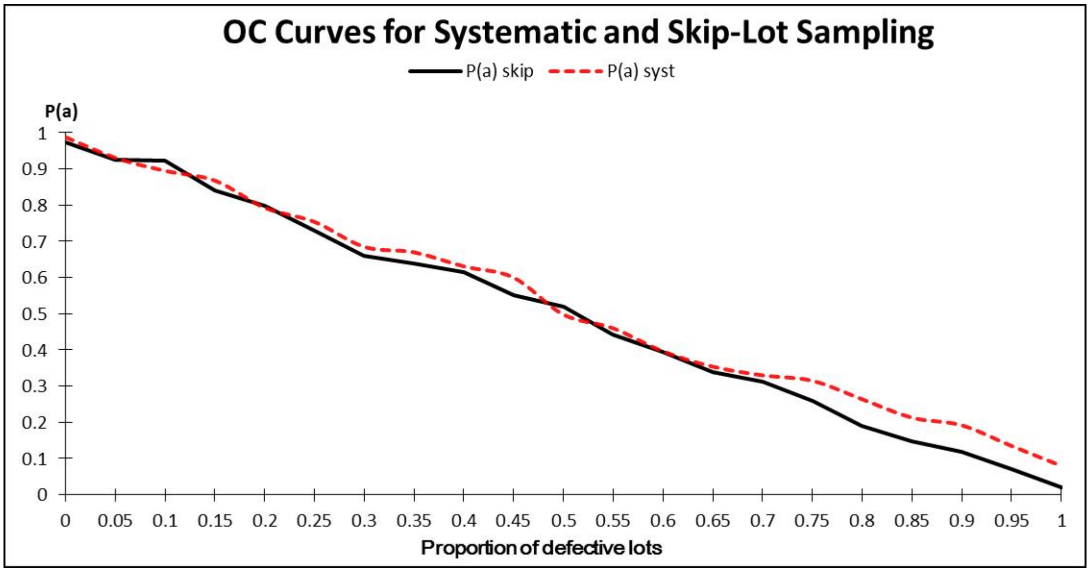

Figure 5.4: OC curve for SkSP-2DSP sampling plan ($n_1 = 20$, $n_2 = 20$, $f_1 = 2$, $f_2 = 4$, $c_1 = 1$, $c_2 = 0$, $i = 5$) and systematic sampling ($f = 3$, $n = 20$, $c = 0$)

It can be seen that the line for SkSP-2DSP is steeper than the systematic sampling curve; it also more efficient in terms of sample size. When the proportion of defective lots is less that 20% and skipping inspection was triggered, the sample size was on average 266 lots which is 20% lower than with systematic sampling (333 lots).

Chapter 6

Conclusions

There are numerous applications for control-based sampling in industry. Control of reliability based on population sampling often requires expensive resources and does not always produce better results than control based on sampling [10]. Therefore, a statistically powerful sampling scheme can be used to ensure that production meets predefined levels of reliability. When a control engineer tests products from several different manufacturing lines it is often the case that some lines, overtime, produce items that constantly pass reliability tests while other lines require more control. Therefore, it is a natural idea to develop a sampling scheme that will sample less from manufacturing lines with historically satisfactory reliability. Skip-lot sampling allows this differentiation and therefore often reduces the frequency (and cost) of control.RB1983

The skip-lot sampling scheme that is used in the proposed method was developed by Dodge and studied by Perry (1971) [11]. SkSP-2 compared with SkSP-1 has an additional reference sampling plan: a special plan for sampling units in the lot. Although the methodology was developed several decades ago, the SkSP-2 plan still has great potential to be more widespread in industry.

Weibull distribution is the most commonly used distribution to approximate life data in reliability testing. It is a very flexible distribution that can mimic the characteristics of many other distributions, based on the value of the shape parameter, β. Besides β, there are two other parameters that determine Weibull pdf. It is common in reliability testing to assume that the location parameter, γ, is to be fixed at 0, therefore reducing the fit to two parameters. In Weibayes method, the β parameter could also be assumed to be fixed at some historical value. In Bayesian setup for Weibull analysis it is common to use non-informative distribution for η and normal, exponential or uniform distribution *for* β.

In chapter 3 two methods for fitting parameters to the Weibull distribution were examined. The median rank regression method was seen to be less preferable than the maximum likelihood method for the majority of situations. The MLE-RBA method helps to reduce further bias in maximum likelihood estimators.

SkSP-2DSP plan can be defined fully by the following parameters:

f_1 and f_2—frequency of sampling during normal and skipping inspections,

n_1 and n_2—sample size during normal and skipping inspections,

c_1 and c_2—criteria for acceptance during normal and skipping inspections.

An advantage of using the SkSP-2DSP plan is that it has a lower frequency of sampling than systematic sampling when the tests are consistently being passed. This encourages the manufacturer to improve the production process. The reduction in sampling does not affect the ability of the scheme to differentiate between satisfactory and non-satisfactory lots. A double sampling plan used as a reference plan was chosen because its operating characteristics are superior to simple random sampling and it also benefits manufactures with reliable production.

Sampling schemes for SkSP-2DSP and systematic sampling have been compared on the basis of their operating characteristics,which show the probability of acceptance as a function of a proportion of defective items. Operating characteristic can be plotted as an OC curve. It was seen that the SkSP-2DSP plan provided a similar OC curve to systematic sampling, given a smaller sample size (266 lots versus 333 lots for systematic sampling when proportion of defective lots is 20%). Therefore, SkSP-2DSP plan can be recommended as an efficient sampling scheme for reliability testing.

Probability of acceptance can be derived either with Markov Chain's approach or a power series approach. In both cases the final result is:

$$P_a = \frac{(1-f)P^i + fP}{(1-f)P^i + f}$$

where

$f-$ frequency of sampling during skipping inspection,

$i-$ number of accepted lots that triggers skipping inspection,

$P-$ probability of lot acceptance.

During simulation the relationship between parameters of SkSP-2DSP and probability of acceptance were examined. Simulation and theoretical results showed that for increasing i – probability of acceptance decreasing, for increasing f – probability of acceptance decreasing, and for increasing P – probability of acceptance also increasing.

In future work the criteria for passing a lot can be improved. As seen in chapter 2 it is possible to set criteria for the Weibull mean. It can be advantageous to use a medium value for Weibull because reliability data is usually skewed. Also it will be interesting to test the SkSP-2DSP plan with several levels of sampling frequency or even make sampling frequency a function of reliability for previously tested lots.

Appendix A

Simulation Study for Estimating β and η

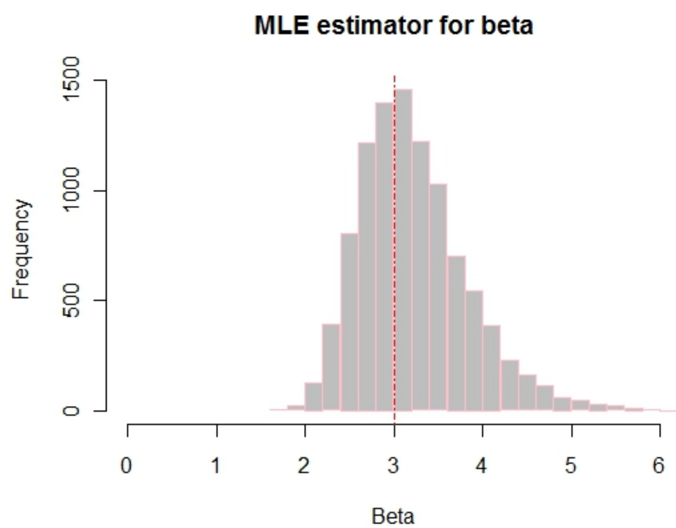

Figure A.1: Simulation study for estimating β using MLE

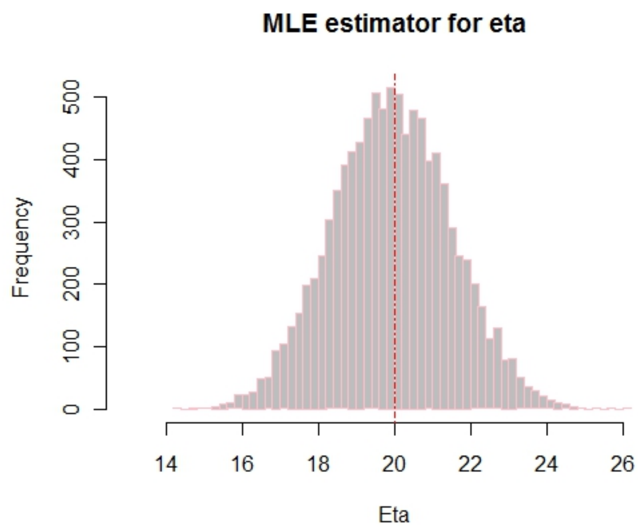

Figure A.2: Simulation study for estimating η using MLE

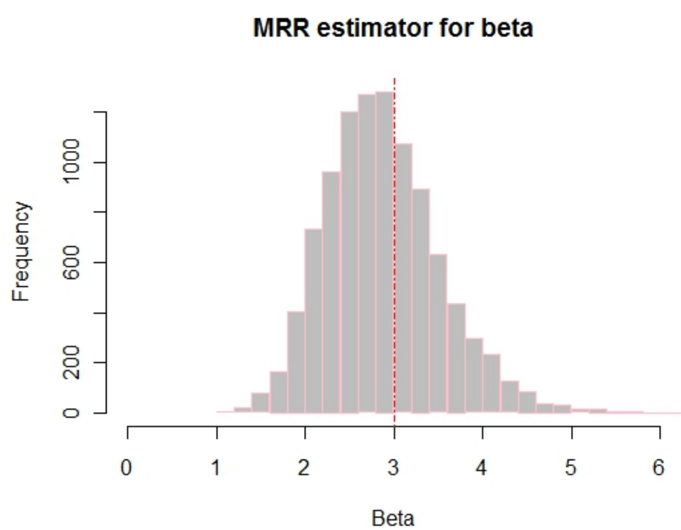

Figure A.3: Simulation study for estimating β using MRR

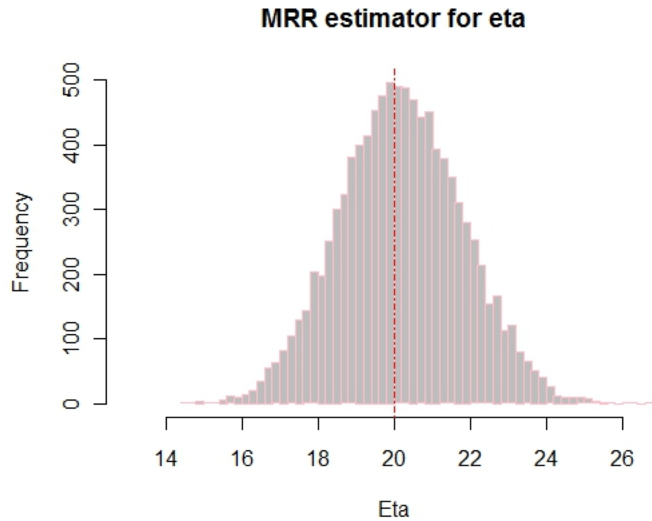

Figure A.4: Simulation study for estimating η using MRR

Appendix B

Partial R Code for the Simulation Study

B.1 Main Code

```
source ("label_matrix.R")
source ("cdf_combo.R")
# 3 classes of lots according to failure time
# (i) fully defective
# (ii) fully good
# (iii) partially defective
# User-defined conditions : frequency, sample size, criteria, i-number that triggers skipping
inspection
# (1) systematic, (2) SKSP first draw and (3) SKSP second draw
# total 9 variables (f1,n1,c1,f2,n2,c2,f3,n3,c3)
library(mixdist)
#input variables from csv file
lot=1000 #number of lots
item=100 #number of items in the lot
perdef=0.25 #share of defective lots
pergood=0.5 #share of acceptable lots
perdoub=1-(perdef+pergood) #share of doubtful
i<- read.csv(file="input.csv",header=FALSE,sep=",")
input = i$V1 # this matrix contains values f1,n1,c1,f2,n2,c2,f3,n3,c3 respectively f1 = input
```

```
[1]
n1 = input [2]
c1 = input [3]
f2 = input [4]
n2 = input [5]
c2 = input [6]
f3 = input [7]
n3 = input [8]
c3 = input [9] # creation of matrix
i<-matrix (runif(lot*item),nrow=lot)
items_matrix<- label_matrix (i)
write.csv(items_matrix,"data.csv",append=F)
# SHUFFLING AND SIMULTANEOUS ALLOCTION PROCESS
# lets shuffle the lots and take first as defective (dlot),
# next as good(glot), and last as doubtful (rlot)
# sls : shuffled lot sequence
sls<- sample.int(lot,lot) #sample 0-n, number
dlot<- matrix (1:(lot*perdef*item),nrow=lot*perdef)
for (i in 1:lot*perdef)
dlot [i,] = rweibull (item,2,60)
write.csv(dlot,"dlot.csv",append=F)
#allocation of values 'dlot' in items_matrix on the positions described by sls
for (i in 1:lot*perdef)
for (j in 1:item)
items_matrix[sls[i],j] = dlot[i,j];
# similarly, do the same for glot and rlot
glot<- matrix (1:(lot*pergood*item),nrow=lot*pergood)
for (i in 1:(lot*pergood))
glot [i,] = rweibull (item,2,300)
write.csv(glot,"glot.csv",append=F)
for (i in 1:(lot*pergood))
for (j in 1:item)
items_matrix[sls[i+lot*perdef],j] = glot[i,j];
rlot<- matrix (1:(lot*perdoub*item),nrow=lot*perdoub)
for (i in 1:(lot*perdoub))
rlot [i,] = rweibull (item,2,105.95)
```

```
write.csv(rlot,"rlot.csv",append=F)
for (i in 1:(lot*perdoub))
for (j in 1:item)
items_matrix[sls[i+(lot*pergood+lot*perdef)],j] = rlot[i,j];
write.csv(items_matrix,"items_matrix_shuffled.csv",append=F)
# since elemnts in the lot are randomly generated we could
#just take first 20 values which will be the same as SRS
# TRUE VALUE COMPUTATION FOR SYSTEMATIC AND SKIP-LOT
cdf_t<- NULL
count_systematic_t = 0
count_skiplot_t = 0
n4 = n2+n3
c4 = c2+c3
true_count<- NULL
for (i in 1:lot)
{
cdf_t [i] = cdf_combo (items_matrix[i,],20)
q = cdf_t [i]
if ((q*n1)<c1)
{
count_systematic_t = count_systematic_t + 1
} # cdfcombo is a function that has output as a weibull cdf at particular point-neither pweibull
or survreg worked for this if ((q*n4)<c4)
{
count_skiplot_t = count_skiplot_t + 1
} # end if - skiplot } # end for
count_systematic_t
count_skiplot_t
true_count = c(count_systematic_t,count_skiplot_t)
write.csv(cdf_t,"true_values.csv",append=F)
write.csv(true_count,"true_count.csv",append=F)
cat (paste("actual number of good units expected from systematic sampling = ",count_systematic_t,'\n'))
cat (paste("actual number of good units expected from skip-lot sampling = ",count_skiplot_t,'\n'))
# APPROACH 1 : systematic sampling
# S1 contains the values of the randomly selected cells.
S1_row = lot/f1
```

```
S1_row
S1_n = S1_row*n1
#out of x lots, only s1_row will get selected, and n1 items from each lot
S1_0 = items_matrix [,c(1:n1)]
S1 = S1_0 [c(1:S1_row),]
write.csv(S1,"S1.csv",append=F)
# matrix extraction completed
passed<-NULL
failed<-NULL
# cdf test for lots in S1 (rows)
for (i in 1:S1_row)
{
x = cdf_combo(S1[i,],20)
if ( x*n1 > c1)
{
failed<-c(failed,x)
cat (paste(i,x,"failed",'\n'))
} else
{
passed<-c(passed,x)
cat(paste(i,x,"passed",'\n'))
}
write.csv(passed,"passed.csv",append=F)
write.csv(failed,"failed.csv",append=F)
}
# APPROACH 2 : Skip-lot: In this approach, we will have 3 groups: "rejected", "selected",
"doubtful"
# c2<c3. For E(lot)<c2 -> select. for E(lot)>c3 - reject. Else add n3, check skip lot for c3
# finally, either "rejected" or "selected".
# S2 would be similar to S1
S2_row = lot/f2
S2_n = S2_row*n2
S2_0 = items_matrix [,c(1:n2)]
S2 = S2_0 [c(1:S2_row),]
write.csv(S2,"S2.csv",append=F)
# However, S3 will contain columns starting from n2+1 -> n2+n3
```

```
S3 = items_matrix [,c(n2+1:n3)]
write.csv(S3,"S3.csv",append=F)
selected<- NULL
rejected<- NULL
doubtful<- NULL
cs = 0 # count for skip
nlots_2 = 0 # count for selected samples
# cdf test for lots in S1 (rows)
i1 = 0
while (i1<S2_row)
{
i1 = i1+1
x = cdf_combo (S2[i1,],20)
if ( x*n2 < c2)
{
selected<-c(selected,x)
cs=cs+1
nlots_2 = nlots_2+1
cat (paste(i1,x,"selected",cs,'\n'))
} # end condition E(n) <c2
else
if ( x*n2 > c3)
{
rejected<- c(rejected,x)
cat (paste(i1,x,"rejected",'\n'))
cs = 0
} # end condition E(n)>c3
else
{
doubtful<-c(doubtful,x)
cat (paste(i1,x,"doubtful",'\n'))
# tie breaker
y = cdf_combo (S3[i1,],20)
if (y*n3 <c3)
{
selected<-c(selected,x)
```

```
cat (paste(i1,y,"selected",'\n'))
cs = cs+1
nlots_2 = nlots_2 + 1
}
else
{
rejected<- c(rejected,x)
cat (paste(i1,y,"rejected",'\n'))
cs = 0
}
} # end of tie
if (cs>5) # SKIPPING VALUE (i)
{
nlots_2 = min (nlots_2+f3, nlots_2 + (S2_row - i1))
i1 = i1 +f3
}
} #end for
# FINAL RESULTS:
# Quantity of lots
nlots_1 = length (passed) *f1
# Quality of lots
good_1 = length (passed)
bad_1 = length(failed)
good_2 = length (selected)
bad_2 = length (rejected)
p_a_bad_syst<-(1-(bad_1/(bad_1+good_1)))
cat (paste("p(a) systematic = ",p_a_bad_syst,'\n')) prop_bad_syst<-(lot-count_systematic_t)/lot
cat (paste("true proportion of bad lots systematic = ",prop_bad_syst,'\n'))
p_a_bad_skip<-(1-(bad_2/(bad_2+good_2)))
cat (paste("p(a) skip = ",p_a_bad_skip,'\n'))
prop_bad_skip<-(lot-count_skiplot_t)/lot
cat (paste("true proportion of bad lots skip = ",prop_bad_skip,'\n'))
```

B.2 CDF fitting

```
# This function generates best fit alpha, beta for a given dataset
# uses it to get cdf, test the cdf at any desired value
# returns point value for cdf (t)
# custom function for specific situation
cdf_combo<- function (x,t)
{
n = length (x) # number of points
# converting weibull to linear and applying least squares
y1 = c(1:n)
for (i in 1:n)
{
q = (i-0.3)/(n+0.4)
y1[i] = log(log(1/(1-q)))
}
x0 = sort (x)
x1 = log(x) # sorting before applying regression
z = lm (y1 ~ x1)
beta = z$coeff["x1"]
eta1 = z$coeff["(Intercept)"]
eta = exp (-(eta1/beta))
cdf = 1 - exp (-((20/eta)^beta)) # ficing x=20 for cdf
return (cdf );
} Simulation study:
be = matrix (runif(40000),nrow=10000)
be[,]=0
betaeta_matrix = label_matrix_method (be)
# Loop initiation
for (i in 1:10000)
{
random_num= rweibull (20,3,20)
sorted_num = sort (random_num)
p_mle = parameters_mle(sorted_num)
p_mrr = parameters_mrr(sorted_num)
# store values in original matrix
```

```
betaeta_matrix [i,] = c (p_mle,p_mrr)
}
hist(betaeta_matrix[,1], breaks=50, col="grey", border="pink", main =
"MLE estimator for beta",xlab="Beta",xlim=c(0,6) )
abline(v = 3, col = "red", lty="dotdash")
```

Bibliography

[1] Gelman A., John B. Carlin, Hal S. Stern, and Rubin Donald B. *Bayesian Data Analysis.* Chapman & Hall/CRC, 2nd edition, 2004.

[2] Liu C.-C. *A Comparison Between the Weibull and Lognormal Models Used to Analyze Reliability Data.* PhD thesis, University of Nottingham, 1997.

[3] Olteanu D.A. and Freeman L.J. The evaluation of median rank regression and maximum likelihood estimation techniques for a two-parameter weibull distribution. *Quality Engineering*, 2010.

[4] Hirosi H. Bias correction for maximum likelihood estimates in the two parameter weibull distribution. *IEEE Transactions on Dielectrics and Electrical Insulation*, 6:66–68, 1996.

[5] Martz H.F. and Waller R.A. *Bayesian Reliability Analysis.* John Wiley & Sons, Inc., New York, 1982.

[6] Dimitri Kececioglu. *Reliability Engineering Handbook*, volume 1. Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1991.

[7] Quenouille M.H. Notes on bias in estimation. *Biometrika*, 61:353–360, 1956.

[8] Lockhart R.A. and Stephens M.A. Estimation and tests of fit for the three-parameter weibull distribution. *Society*, 56(3):491–500, 2011.

[9] Abernethy R.B. *The New Weibull Handbook.* Robert B. Abernethy, 536 Oyster Road, North Palm Beach, 5th edition, 2006. FL 33408–4328.

[10] Abernethy R.B., Breneman J.E., Medlin C.H., and Reinman G.L. Weibull analysis handbook. Technical report, Air Force Wright Aeronautical Laboratories Technical Report AFWAL-TR-83-2079, 1983. Available at `http://handle.dtic.mil/100.2/ADA143100`.

[11] Perry R.L. *A System of Skip-Lot Sampling Plans for Lot Inspection.* PhD thesis, Rutgers University, New Brunswick, New Jersey, 1971. p. 95.

[12] Brugger R.M. A simplification of the Markov Chain approach to continuous sampling plan formulation. *QEM 21-230-12*, March 1972. Ammunition Procurementand Supply Agency.

[13] Thompson S.K. *Sampling.* Wiley, New York, 2nd edition, 2002.

[14] Genschel U. and Meeker W.Q. A comparison of maximum likelihood and median rank regression for weibull estimation. *Quality Engineering*, 22:236–255, 2010.

[15] Wallodi Weibull. A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18:293–297, 1951.

[16] Meeker W.Q. and Escobar L.A. *Statistical Methods for Reliability Data.* John Wiley & Sons, Inc., New York, 1998.