THREE ESSAYS ON MONETARY ECONOMICS

by

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Abstract

The thesis consists of three studies on money, banking and monetary policy with modern monetary economic theory based on explicit micro-foundations.

As an introduction to the approach adopted by micro-founded monetary theory, the introductory chapter demonstrates the roles of money and capital in a quasi-linear environment with explicit informational frictions. When capital serves as the only record-keeping device, there could be two possible stationary equilibria: one is first-best and the other is not. In a suboptimal equilibrium, consumers are constrained by their capital rental income. Introducing fiat money, a better record-keeping technology with higher rate of return, can improve welfare by relaxing the liquidity constraint.

Chapter 2 studies the role of banking in financing investment. It is revealed that banking can mitigate underinvestment, raise capital-labour ratio, and improve welfare; and this effect is greatest under moderate inflation.

In Chapter 3, I introduce a record-keeping cost related to bank borrowing, and study the effects of such a banking cost on economic allocations and welfare, as well as its monetary policy implications. Main findings are: Costly banking emerges endogenously only with relatively high inflation and/or relatively low banking cost; the existence of costly banking may improve or reduce welfare relative to the case without banking; with higher inflation rate or banking cost, more people would choose not to deal with banks, which means larger welfare loss; inflation is less harmful with banking than without banking.

In Chapter 4, I investigate the trade-off between distribution effect and production effect of monetary policy with presence of idiosyncratic liquidity shocks. When liquidity shocks are observable, a type-contingent money transfer policy can desirably redistribute purchasing power among consumers. When the shocks are unobservable, an illiquid bond policy restores credit transactions on money through bond-money exchanges. Both policies have positive distribution effect, but the resulting inflation hampers production efficiency. I derive a sufficient condition under which the overall welfare can be improved by an inflationary monetary policy: if consumers are relative-risk-averse enough, the trade-off between distribution efficiency gain and production efficiency loss would result in net welfare enhancement.

Keywords: money; capital; banking; bonds; inflation; monetary policy.

Dedication

To my parents, XIANG Changquan and LIU Shangzhen,

and my wife, ZHANG Qiong.

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Chapter 1

Introduction: Money and Capital in a Quasi-linear Environment

1.1 Introduction

As an introduction to the approach adopted by micro-founded monetary economic theory, which is dubbed *New Monetarist Economics* in two papers by Williamson and Wright (2010a,b), this introductory chapter demonstrates the roles of money and capital in a quasi-linear environment with explicit informational frictions. All following three chapters are based on variants of this framework.

Lagos and Wright (2005) propose a unified framework for monetary economic analysis. Each time period is divided into a frictionless centralized market subperiod and a decentralized market subperiod characterized by lack of double coincidence of wants, limited commitment and anonymity. With presence of these frictions, money is endogenously demanded as a substitute for the missing record-keeping technology (Kocherlakota (1998)). Assuming quasi-linear preferences over two subperiods makes the distribution of money degenerate, so as to keep the problem tractable.

Aruoba, Waller and Wright (2006) introduce capital into this pure monetary economy. Capital is accumulated in the centralized market subperiod and can be used in decentralized market production as a complement to labor input. In another approach by Shi (1999), capital is introduced into a search model, and he studies extensive margin effect of inflation on capital accumulation due to change in search effort.

A variation of Lagos-Wright framework (see Rocheteau and Wright (2005)) modifies the environment to make it closer to standard real business cycle models. In this type of models, both subperiods are associated with centralized competitive markets. The micro-foundation of money is built in by explicitly modeling frictions (i.e., lack of double coincidence of wants and anonymity) in one of the subperiods. And the quasi-linear preferences remain to make distribution of money tractable. In particular, it gets rid of search and bargaining, while still endogenizing demand for money by explicitly assuming presence of frictions.

In this introductory chapter, I study the roles of money and capital in such a quasi-linear environment with competitive markets. The existence of capital renders autarky viable in spite of the frictions. There could be two possible stationary equilibria: one is first-best and the other is not. When parameterization leads to a suboptimal stationary equilibrium such that the capital rental rate is lower than the inverse of time discounting rate, $1/\beta$, consumers are constrained by their capital rental income. Introduction of money can improve welfare by raising rates of return of both money and capital. In this context Friedman rule is optimal, i.e., setting money growth rate equal to β can achieve first-best allocation.

When capital is incorporated and producers are allowed to hold it, capital actually serves as another substitute for the missing record-keeping technology, besides its role as a productive input. When its rate of return is too low in equilibrium, capital is not a good substitute. Then introduction of another substitute, fiat money, potentially can improve the welfare by raising rates of return of both.

1.2 A Quasi-linear Environment

Time is discrete and the horizon is infinite. Each period t is divided into two subperiods: day and night. There is measure one of infinitely-lived agents who are *ex ante* identical.

Preferences of a representative agent i are defined over stochastic sequences

$$\{e_t(i), c_t(i), n_t(i)\},$$

where $e_t(i)$ denotes consumption (if positive) or production (if negative) in the day, $c_t(i)$ consumption in the night, and $n_t(i)$ labor input during the night. The expected lifetime utility function is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[e_t(i) + \beta U_t \left(c_t(i), n_t(i) \right) \right],$$

where $\beta \in (0,1)$ is the time discounting rate between two consecutive periods. Note the utility function is quasi-linear in daytime consumption/production.

There are preference/technology shocks which are realized at the beginning of each night. The shocks are i.i.d. across agents and over time. With probability 1/2, an agent wants to consume but cannot produce; with probability 1/2, an agent can produce but does not want to consume. Therefore, the night utility function can be written as

$$U_t \left(c_t(i), n_t(i) \right) = \begin{cases} u\left(c_t(i)\right) & \text{w.p. } 1/2 \quad \text{(consumer)} \\ -g\left(n_t(i)\right) & \text{w.p. } 1/2 \quad \text{(producer)} \end{cases}.$$

The instantaneous utility function $u'(c)$ has properties: $u'' < 0 < u'$, $\lim_{c \to 0} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) = 0$. And the disutility function $g(n)$ satisfies $g' > 0$ and $g'' \geq 0$.

Capital is indispensable to production only in the night. The aggregate production possibilities can be described with a constant-returns-to-scale production function of the form $Y_t = F(K_t, N_t)$, where $K_t \equiv \int_0^1 k_t(j)dj$ denotes the aggregate capital stock available at night and $N_t \equiv \int_0^1 n_t(j)dj$ the aggregate labor input in the night. Capital is augmented in the usual way, with

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

denoting the capital available for production the next night. For simplicity I assume that $\delta = 1$. During each day, the existing capital held by agents can be traded on a competitive market, but cannot be consumed. Initially all agents are endowed with identical capital holdings $k_0 > 0$.

1.3 Pareto Optimal Allocation

As a benchmark, I consider the first-best allocation that a planner can implement in absence of all kinds of frictions. Since every agent is *ex ante* identical, it is natural to assume the planner treats everyone equally. Due to the quasi-linearity in day consumption, any lottery scheme in $\{e_t\}$ satisfying $E_0 e_t(i) = 0$ and day resource constraint $\int_0^1 e_t(j)\,dj = 0$ can be a solution. Therefore, the day subperiods are irrelevant and a trivial solution is $e_t(i) = 0\ \forall t$ for all agents. A planner only needs to consider capital accumulation across nights and consumption-production decision in nights.

Considering the preference/technology shocks in night, a planner asks each agent who turns out to be a producer to produce n_t. Given initial aggregate capital stock k_t, the total output will be $Y_t = F\left(k_t, \frac{1}{2}n_t\right)$. Then a consumption-investment decision is made: each consumer gets c_t for consumption (so aggregate consumption is $\frac{1}{2}c_t$), and the rest of output will be invested as capital stock, k_{t+1}, for future production.

The planner's problem is to maximize a representative agent's lifetime utility

$$\max \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2}u(c_t) - \frac{1}{2}g(n_t)\right]$$

subject to the night resource constraint

$$\frac{1}{2}c_t + k_{t+1} = F\left(k_t, \frac{1}{2}n_t\right), \tag{1.1}$$

given initial capital stock k_0.

In Appendix it is shown that the equilibrium path of $\{c_t, n_t, k_{t+1}\}$ can be characterized by

$$u_1(c_t)F_2\left(k_t, \frac{1}{2}n_t\right) = g_1(n_t),$$

$$u_1(c_t) = \beta u_1(c_{t+1})F_1\left(k_{t+1}, \frac{1}{2}n_{t+1}\right),$$

and the resource constraint (4.3).[1]

The focus is restricted to a stationary equilibrium where all real variables remain constant. Therefore, the equilibrium allocation $\{c^*, n^*, k^*\}$ is characterized by three equations:

$$u_1(c^*) = \frac{g_1(n^*)}{F_2\left(k^*, \frac{1}{2}n^*\right)}, \tag{1.2a}$$

$$F_1\left(k^*, \frac{1}{2}n^*\right) = \frac{1}{\beta}, \tag{1.2b}$$

$$\frac{1}{2}c^* + k^* = F\left(k^*, \frac{1}{2}n^*\right). \tag{1.2c}$$

[1] Note the subscript 1(2) stands for a partial derivative with respect to the first (second) argument.

1.4 A Nonmonetary Model

Now I assume that agents cannot commit, and it is impossible to monitor and record agents' transaction histories due to anonymity. Combined with the lack of double coincidence of wants in night introduced by preference/technology shocks, these frictions imply that any exchange must be based on *quid pro quo*.

In a model without capital, the only equilibrium for night market is autarky: no production and no consumption at all. As is well-known, money is essential in this scenario. It serves as a medium of exchange to make trade possible. Producers are willing to produce since their output can be sold for money which can be used to buy consumption goods in the future.

If one adds capital into the model and allows agents to rent out capital, then it turns out there will be production/consumption even without money. In this case, producers are willing to produce since they can accumulate capital which can be sold during the next day. Autarky with capital is viable, but autarky may or may not achieve the first-best allocation, depending on the parameterization.

1.4.1 Day Decision-making

To simplify notations, I drop time subscript t for all current variables, and add a superscript "+" to denote time period $t+1$ variables.

During the day, an agent's value with capital holding k_1 is

$$\begin{aligned} W(k_1) &= \max_{e,k_2} \{e + V(k_2)\} \\ \text{s.t. } e &= p(k_1 - k_2), \\ k_2 &\geq 0, \end{aligned}$$

where p is the price of capital in terms of day consumption and k_2 is the capital carried into night. Note the nonnegativity restriction on capital holding is implied by the presence of frictions, i.e., borrowing is ruled out due to anonymity and lack of commitment. Utilizing the budget constraint, one can rewrite the value function as

$$W(k_1) = \max_{k_2 \geq 0} \{p(k_1 - k_2) + V(k_2)\}.$$

Assuming $k_2 > 0$, the first order condition (FOC) is

$$p = V_1(k_2). \tag{1.3}$$

And the envelope condition is

$$W_1(k_1) = p. \tag{1.4}$$

1.4.2 Night Decision-making

At the beginning of a night, the preference/technology shocks are realized. As a result, half agents are consumers and the rest are producers. Assume there are competitive rental markets for capital and labor at night, with factor prices r and w, respectively.

Consumer: For an agent who turns out to be a consumer, she can rent out her capital carried to the night; and the rental income can be either consumed or invested for the future. Her value with capital holding k_2 is

$$\begin{aligned} V^c(k_2) &= \max_{c, k_{1c}^+} \left\{ u(c) + \beta W(k_{1c}^+) \right\} \\ \text{s.t. } c + k_{1c}^+ &= rk_2, \\ k_{1c}^+ &\geq 0. \end{aligned}$$

It is equivalent to

$$V^c(k_2) = \max_c \left\{ u(c) + \beta W\left(rk_2 - c\right) + \lambda_k \left(rk_2 - c\right) \right\},$$

where λ_k is the Lagrange multiplier associated with the nonnegativity constraint $k_{1c}^+ \geq 0$.

Utilizing condition (3.10), the FOC can be written as:

$$u_1(c) - \beta p^+ - \lambda_k = 0. \tag{1.5}$$

Envelope condition:

$$V_1^c(k_2) = \beta p^+ r + \lambda_k r = u_1(c) r, \tag{1.6}$$

where the second equality comes from (1.5). A consumer can choose to consume all rental income, or to invest some of it. (1.5) indicates that $\lambda_k = u_1(c) - \beta p^+ > 0$ if $k_{1c}^+ = 0$, and $\lambda_k = u_1(c) - \beta p^+ = 0$ if $k_{1c}^+ > 0$. Intuitively, when the current marginal utility $u_1(c)$ exceeds discounted future marginal utility βp^+, a consumer would choose to consume all rental income and does not put aside anything as capital investment.

Producer: For an agent who turns out to be a producer, he will rent out all capital and decide how much labor to input. Both capital and labor rental income will be held in form of capital into next day. His value with capital holding k_2 is

$$\begin{aligned} V^p(k_2) &= \max_{n, k_{1p}^+} \left\{ -g(n) + \beta W(k_{1p}^+) \right\} \\ \text{s.t. } k_{1p}^+ &= r k_2 + wn, \end{aligned}$$

which is equivalent to

$$V^p(k_2) = \max_n \left\{ -g(n) + \beta W\left(r k_2 + wn\right) \right\}.$$

FOC:

$$\frac{g_1(n)}{w} = \beta p^+. \tag{1.7}$$

Envelope condition:

$$V_1^p(k_2) = \beta p^+ r = \frac{g_1(n)}{w} r. \tag{1.8}$$

1.4.3 Equilibrium

Plugging (1.3), (1.6) and (1.8) into equation

$$V_1(k_2) = \frac{1}{2} V_1^c(k_2) + \frac{1}{2} V_1^p(k_2)$$

yields

$$p = \frac{1}{2} r \left[u_1(c) + \frac{g_1(n)}{w} \right].$$

This equation also holds for next period, i.e.,

$$p^+ = \frac{1}{2} r^+ \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \right].$$

Combining it with (1.7) results in

$$\frac{g_1(n)}{w} = \frac{\beta}{2} r^+ \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \right]. \tag{1.9}$$

Equation (1.3) implies that the capital brought into night is independent of capital holdings at the beginning of a day. In equilibrium, all agents rebalance their capital holdings during the day and bring same amount of capital into night. This degenerate distribution of capital balance is attributed to the quasi-linear preference assumption. And it follows that

$$k_2^+ = \frac{1}{2}k_{1c}^+ + \frac{1}{2}k_{1p}^+.$$

The market-clearing conditions at night for two consecutive periods can be written as

$$\frac{1}{2}c + k_2^+ = F\left(k_2, \frac{1}{2}n\right). \tag{1.10}$$

$$\frac{1}{2}c^+ + k_2^{++} = F\left(k_2^+, \frac{1}{2}n^+\right). \tag{1.11}$$

Now there are two cases to consider: either $\lambda_k > 0$ or $\lambda_k = 0$.

Case 1: $\lambda_k > 0$. Then $k_{1c}^+ = 0$, and the consumer budget constraint determines

$$c = rk_2, \tag{1.12}$$

$$c^+ = r^+ k_2^+. \tag{1.13}$$

Given an initial capital stock k_2, the equilibrium path of $\{n, c, k_2^+, n^+, c^+, k_2^{++}\}$ can be characterized by equations (1.9) through (1.13), as well as the transversality condition, where $r = F_1\left(k_2, \frac{1}{2}n\right)$ and $w = F_2\left(k_2, \frac{1}{2}n\right)$. And $k_{1c}^+ = 0, k_{1p}^+ = rk_2 + wn$.

Here the focus is on a steady state allocation. Then the equilibrium $\{n, c, k_2\}$ is determined by

$$\begin{aligned} \frac{g_1(n)}{w} &= \frac{\beta}{2}r\left[u_1(c) + \frac{g_1(n)}{w}\right], & \text{(1.14a)} \\ r &= \frac{c}{k_2}, & \text{(1.14b)} \\ \frac{1}{2}c + k_2 &= F\left(k_2, \frac{1}{2}n\right), & \text{(1.14c)} \end{aligned}$$

with $r = F_1\left(k_2, \frac{1}{2}n\right)$ and $w = F_2\left(k_2, \frac{1}{2}n\right)$. Consumers always consume all rental income and make no investment, i.e., $k_{1c}^+ = 0$. Producers save all rental income in form of capital, i.e., $k_{1p}^+ = rk_2 + wn$.

How does this equilibrium compare with the first-best allocation? Since $\lambda_k =$

$u_1(c) - \beta p^+ > 0$, or $u_1(c) > \beta p^+$, (1.7) implies

$$u_1(c) > \frac{g_1(n)}{w},$$

and hence (1.14a) means

$$F_1\left(k_2, \frac{1}{2}n\right) = r < \frac{1}{\beta}.$$

Therefore, the equilibrium is not first-best.

Case 2: $\lambda_k = 0$. Then $k_{1c}^+ > 0$, and it follows from (1.5) that $u_1(c) = \beta p^+$. Immediately (1.7) leads to

$$u_1(c) = \frac{g_1(n)}{w} = \frac{g_1(n)}{F_2\left(k_2, \frac{1}{2}n\right)}, \tag{1.15}$$

and (1.9) in stationarity results in

$$F_1\left(k_2, \frac{1}{2}n\right) = r = \frac{1}{\beta}. \tag{1.16}$$

Therefore, the stationary equilibrium $\{c, n, k_2\}$ can be determined by (1.15), (1.16) and (1.14c) which are exactly the three equilibrium conditions for first-best allocation. In this equilibrium, consumers are not constrained in consumption. Both consumers and producers make positive investment, with $k_{1c}^+ = rk_2 - c$ and $k_{1p}^+ = rk_2 + wn$.

Without money, no trade occurs in the night. Agents just rent out capital taken into the night. Out of the rental income they can consume or invest. This economy is viable without trade in night. As shown above, when

$$r = F_1\left(k_2, \frac{1}{2}n\right) < \frac{1}{\beta}, \tag{1.17}$$

the equilibrium is not first-best. Consumers are constrained in consumption. If there is a credit market, they would be willing to borrow to augment consumption, but due to anonymity and limited commitment such a credit market cannot exist. However, if there exists money, potentially consumers can use money to buy goods from producers. Producers would be willing to accept money if they believe money will be valued in the future. As a result, gains from trade can be exploited with aid of money.

1.5 A Monetary Model

Now fiat money is introduced to make night trade possible. It is intrinsically useless unbacked token which is portable, divisible and uncounterfeitable. Suppose initially nominal money stock M is evenly distributed to all agents. Then consumers can supplement their rental income with real money balances carried into the night. And producers can sell output in the night to earn money which could be exchanged for consumption in the future. The day transaction simply serve to rebalance each agent's money holdings and capital holdings such that the distributions of money and capital are degenerate. Suppose there is a money injection T by the government at the beginning of each day, resulting in a money growth rate $\mu \equiv M^+/M$.

1.5.1 Day Decision-making

During a day, the after-transfer money balances can be used to buy consumption goods and capital. All unspent money will be carried into the night. The value of an agent entering a day with capital holding k_1 and nominal money balance m_1 is

$$\begin{aligned}
W(k_1, m_1) &= \max_{e,k_2,m_2} \{e + V(k_2, m_2)\} \\
\text{s.t. } m_1 + T &= \phi e + p_1 (k_2 - k_1) + m_2, \\
m_2 &\geq 0, \\
k_2 &\geq 0,
\end{aligned}$$

where ϕ and p_1 are the prices of consumption and capital, respectively, in day transactions, and k_2, m_2 are nonnegative capital and money balances carried into night. Again, the nonnegativity restriction on money holdings is a direct result from the existence of frictions. Eliminating day consumption e by the budget constraint, one can rewrite the value function as

$$W(k_1, m_1) = \max_{k_2 \geq 0, m_2 \geq 0} \left\{ \frac{1}{\phi} (m_1 + p_1 k_1 + T - m_2 - p_1 k_2) + V(k_2, m_2) \right\}.$$

The first order conditions for an interior solution are:

$$\frac{p_1}{\phi} = V_1(k_2, m_2), \tag{1.18}$$

$$\frac{1}{\phi} = V_2(k_2, m_2). \tag{1.19}$$

Envelope conditions:

$$W_1(k_1, m_1) = \frac{p_1}{\phi}, \tag{1.20}$$

$$W_2(k_1, m_1) = \frac{1}{\phi}. \tag{1.21}$$

1.5.2 Night Decision-making

Consumer: For an agent who turns out to be a consumer, her value associated with entering the night with capital holding k_2 and nominal money balance m_2 is

$$\begin{aligned} V^c(k_2, m_2) &= \max_{c, k_{1c}^+, m_{1c}^+} \left\{ u(c) + \beta W(k_{1c}^+, m_{1c}^+) \right\} \\ \text{s.t. } p_2 c + p_2 k_{1c}^+ + m_{1c}^+ &= p_2 r k_2 + m_2, \\ k_{1c}^+ &\geq 0, \\ m_{1c}^+ &\geq 0, \end{aligned}$$

where p_2 is the price of night output which can be used for either consumption or investment. Using the budget constraint to eliminate m_{1c}^+ in the value function yields

$$V^c(k_2, m_2) = \max_{c, k_{1c}^+ \geq 0} \left\{ \begin{array}{c} u(c) + \beta W \left[k_{1c}^+, m_2 + p_2 \left(r k_2 - c - k_{1c}^+ \right) \right] \\ + \lambda_m \left[m_2 + p_2 \left(r k_2 - c - k_{1c}^+ \right) \right] \end{array} \right\},$$

where λ_m is the Lagrange multiplier associated with the nonnegative-money-holding constraint $m_{1c}^+ \geq 0$.

Utilizing envelope conditions (1.20) and (1.21), two FOC's can be derived as follows:

$$u_1(c) - \beta \frac{1}{\phi^+} p_2 - \lambda_m p_2 = 0, \tag{1.22}$$

$$\beta \frac{p_1^+}{\phi^+} - \beta \frac{1}{\phi^+} p_2 - \lambda_m p_2 \leq 0.$$

Equation (2.8) implies that $\lambda_m = \frac{u_1(c)}{p_2} - \beta \frac{1}{\phi^+} > 0$ if $m_{1c}^+ = 0$ and $\lambda_m = 0$ if $m_{1c}^+ > 0$. The marginal value of one dollar is $\frac{u_1(c)}{p_2}$ when it is spent at night. If the same one dollar is held into next day, the (discounted) marginal value will be $\beta \frac{1}{\phi^+}$. Apparently, if the former is larger, agents would not take money into next day; all money would be spent in night instead. Only when $\frac{u_1(c)}{p_2} = \beta\frac{1}{\phi^+}$, consumers are indifferent and would carry money into next day.

Combining the two FOC's leads to

$$\beta\frac{p_1^+}{\phi^+} - u_1(c) \leq 0,$$

which means $u_1(c) > \beta\frac{p_1^+}{\phi^+}$ if $k_{1c}^+ = 0$ and $u_1(c) = \beta\frac{p_1^+}{\phi^+}$ if $k_{1c}^+ > 0$. For a consumer, the cost of making investment is sacrifice of consumption at night, and the benefit is increase in next day's consumption. Therefore, the consumer will not make investment if the marginal cost $u_1(c)$ is larger than the (discounted) marginal benefit $\beta\frac{p_1^+}{\phi^+}$. If they are equal, the consumer is indifferent between consumption and investment at the margin.

Envelope conditions are:

$$V_1^c(k_2, m_2) = \beta\frac{1}{\phi^+}p_2 r + \lambda_m p_2 r = r u_1(c), \tag{1.23}$$

$$V_2^c(k_2, m_2) = \beta\frac{1}{\phi^+} + \lambda_m = \frac{1}{p_2}u_1(c), \tag{1.24}$$

where the second equality in each equation is derived from (2.8).

Producer: The value of a producer entering the night with capital holding k_2 and nominal money balance m_2 is

$$\begin{aligned} V^p(k_2, m_2) &= \max_{n, k_{1p}^+, m_{1p}^+} \left\{-g(n) + \beta W(k_{1p}^+, m_{1p}^+)\right\} \\ \text{s.t. } p_2 k_{1p}^+ + m_{1p}^+ &= p_2\left(r k_2 + w n\right) + m_2, \\ k_{1p}^+ &\geq 0, \end{aligned}$$

which can be rewritten in the form of

$$V^p(k_2, m_2) = \max_{n, k_{1p}^+ \geq 0} \left\{-g(n) + \beta W\left[k_{1p}^+, m_2 + p_2\left(r k_2 + w n - k_{1p}^+\right)\right]\right\}.$$

FOC's:

$$\frac{g_1(n)}{w} = \beta\frac{1}{\phi^+}p_2, \tag{1.25}$$

$$\beta\frac{p_1^+}{\phi^+} - \beta\frac{1}{\phi^+}p_2 \leq 0.$$

It follows that $p_2 = p_1^+$ if $k_{1p}^+ > 0$ and $p_2 > p_1^+$ if $k_{1p}^+ = 0$. When producers make investment decision, they are deciding whether to hold money or hold capital into next day. If holding money, the rate of return on money across night and day is p_2/p_1^+ while the rate of return on capital is 1. Therefore, if $p_2 > p_1^+$, they will not hold capital; and the coexistence of money and capital requires rate-of-return equality, i.e., $p_2 = p_1^+$.

Envelope conditions:

$$V_1^p(k_2, m_2) = \beta\frac{1}{\phi^+}p_2 r = r\frac{g_1(n)}{w}, \tag{1.26}$$

$$V_2^p(k_2, m_2) = \beta\frac{1}{\phi^+} = \frac{1}{p_2}\frac{g_1(n)}{w}, \tag{1.27}$$

where (2.13) is utilized when deriving the second equality in both equations.

1.5.3 Equilibrium

Note that

$$V_1(k_2, m_2) = \frac{1}{2}V_1^c(k_2, m_2) + \frac{1}{2}V_1^p(k_2, m_2).$$

Plugging (3.4), (2.11) and (2.14) into both sides of it leads to

$$\frac{p_1}{\phi} = \frac{1}{2}r\left[u_1(c) + \frac{g_1(n)}{w}\right]. \tag{1.28}$$

Similarly, plugging (2.6), (2.12) and (2.15) into both sides of

$$V_2(k_2, m_2) = \frac{1}{2}V_2^c(k_2, m_2) + \frac{1}{2}V_2^p(k_2, m_2)$$

gives rise to

$$\frac{1}{\phi} = \frac{1}{2}\frac{1}{p_2}\left[u_1(c) + \frac{g_1(n)}{w}\right]. \tag{1.29}$$

Combining (2.17) and (2.19) yields

$$r = \frac{p_1}{p_2}. \tag{1.30}$$

This is another rate-of-return equality condition. Across day and night, the rate of return on capital is r, while the rate of return on money is p_1/p_2. For all agents to hold both money and capital at the beginning of night, this condition must be satisfied.

To proceed, two cases need to be considered: either $\lambda_m > 0$ or $\lambda_m = 0$.

Case 1: $\lambda_m > 0$. Consumers are cash-constrained and they spend all money during the night. Also one has to consider four cases for the capital holdings: (i) $k_{1c}^+ > 0, k_{1p}^+ > 0$; (ii) $k_{1c}^+ > 0, k_{1p}^+ = 0$; (iii) $k_{1c}^+ = 0, k_{1p}^+ > 0$; (iv) $k_{1c}^+ = 0, k_{1p}^+ = 0$. In the appendix, I show that the only possible equilibrium is $k_{1c}^+ = 0, k_{1p}^+ > 0$, i.e., cash-constrained consumers would not hold any capital and producers hold both capital and money.

$k_{1p}^+ > 0$ implies $p_2 = p_1^+$, hence (2.13) gives

$$\frac{g_1(n)}{w} = \beta \frac{p_1^+}{\phi^+}. \tag{1.31}$$

Updating (2.17) by one period yields

$$\frac{p_1^+}{\phi^+} = \frac{1}{2} r^+ \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \right]. \tag{1.32}$$

Eliminating the left-hand side of (2.22) by (1.31), one obtains

$$\frac{g_1(n)}{w} = \frac{\beta}{2} r^+ \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \right]. \tag{1.33}$$

Since all capital investment is made by producers, the night output market clearing condition is

$$\frac{1}{2}c + \frac{1}{2}k_{1p}^+ = F\left(k_2, \frac{1}{2}n\right).$$

(2.14) and (2.15) indicate that when all agents rebalance capital and money holdings during each day, their assets carried into night are history independent. In particular, this results in equal capital holding $k_2^+ = \frac{1}{2}k_{1p}^+$. Therefore, the market clearing condition becomes

$$\frac{1}{2}c + k_2^+ = F\left(k_2, \frac{1}{2}n\right). \tag{1.34}$$

Since $m_{1c}^+ = 0$ and $k_{1c}^+ = 0$, the consumer budget constraint implies

$$p_2 = \frac{m_2}{c - rk_2}.$$

Considering the money market clearing condition $m_2 = M$, it becomes

$$p_2 = \frac{M}{c - rk_2}. \tag{1.35}$$

Given $p_2 = p_1^+$, (2.20) implies

$$r^+ = \frac{p_2}{p_2^+}.$$

Combining it with (2.21) gives rise to

$$r^+ = \frac{M}{c - rk_2} \cdot \frac{c^+ - r^+ k_2^+}{M^+} = \frac{1}{\mu} \cdot \frac{c^+ - r^+ k_2^+}{c - rk_2}. \tag{1.36}$$

The stationary equilibrium allocation $\{c, n, k_2\}$ can be characterized by the stationary version of (2.23), (1.36) and (2.24):

$$\frac{g_1(n)}{w} = \frac{\beta}{2} r \left[u_1(c) + \frac{g_1(n)}{w} \right], \tag{1.37a}$$

$$r = \frac{1}{\mu}, \tag{1.37b}$$

$$\frac{1}{2}c + k_2 = F\left(k_2, \frac{1}{2}n\right). \tag{1.37c}$$

with $w = F_2\left(k_2, \frac{1}{2}n\right)$ and $r = F_1\left(k_2, \frac{1}{2}n\right)$.

Case 2: $\lambda_m = 0$. Consumers are not cash-constrained and carry some money into next day. Again there are four cases for the capital holdings to consider. In the appendix, I show that the only possible equilibrium is $k_{1c}^+ > 0, k_{1p}^+ > 0$, i.e., consumers would hold both capital and money, and so do producers.

Given $k_{1c}^+ > 0$, $u_1(c) = \beta \frac{p_1^+}{\phi^+}$. And $k_{1p}^+ > 0$ implies $p_2 = p_1^+$. These two conditions combined with (2.13) result in

$$u_1(c) = \frac{g_1(n)}{w}.$$

Combining these conditions with (2.22), one gets

$$r^+ = \frac{1}{\beta} \frac{u_1(c)}{u_1(c^+)}.$$

Therefore, in stationary equilibrium it must be true that

$$r = \frac{1}{\beta}.$$

Another equilibrium condition is the night goods market clearing condition

$$\frac{1}{2}c + k_2 = F\left(k_2, \frac{1}{2}n\right).$$

It is clear that this is the first-best allocation.

1.5.4 Optimal Monetary Policy

Now a question is in order: what is the optimal money growth rate?

Apparently, when $\lambda_m > 0$, money is neutral but not super-neutral, i.e., the money growth rate affects equilibrium allocation. When $\mu > \beta$, one can deduce from (1.37a) and (1.37b) that

$$u_1(c) = \frac{2\mu - \beta}{\beta}\frac{g_1(n)}{w} > \frac{g_1(n)}{w}$$

and

$$r < \frac{1}{\beta}.$$

Therefore, the monetary equilibrium is not first-best when $\mu > \beta$.

When $\mu \to \beta$, $\frac{2\mu-\beta}{\beta} \to 1$ hence $u_1(c) \to \frac{g_1(n)}{w}$, and $r \to \frac{1}{\beta}$. At the same time, $\lambda_m \to 0$. Therefore, by setting $\mu = \beta$, the first-best allocation can be achieved. In other words, the Friedman rule is optimal in this environment.

1.6 Discussion

1.6.1 When Is Money Essential?

With only capital in the model, there are two possible equilibria: (1) When the capital rental rate $r < \frac{1}{\beta}$, the stationary allocation is not optimal; (2) When $r = \frac{1}{\beta}$, the allocation is first-best. If the parameterization leads to the first equilibrium, the suboptimality is due to presence of frictions. During each night, consumers' desire for consumption is constrained by their capital rental income; they are willing to borrow if they could. But borrowing cannot exist in this environment due to lack of commitment and anonymity. Without these frictions, the first-best allocation can always be achieved.

When the capital rental rate $r < \frac{1}{\beta}$, introduction of money can circumvent the frictions and improve welfare. In particular, setting money growth rate $\mu = \beta$ results in the optimal allocation. Comparing nonmonetary equilibrium conditions (1.14) and monetary equilibrium conditions (1.37), one can see that the only difference lies in capital rental rate r. If money is introduced as another asset, with its return kept high enough (by controlling money growth rate μ), then the capital rental rate will adjust upward accordingly, due to rate-of-return equality between money and capital. And resulting higher rental rate must be associated with welfare improvement. Intuitively, setting $\mu \in \left(\beta, \frac{k_2}{c}\right)$ should improve welfare away from a nonmonetary equilibrium characterized by (1.14).

All these findings are consistent with Lagos and Rocheteau (2008). They prove that when the socially efficient stock of capital is too low to provide liquidity as needed, agents overaccumulate productive assets to use as media of exchange, which is reflected by a low rental rate of capital in this model. And they also show that there exists a monetary equilibrium that dominates the nonmonetary one in terms of welfare.

In a simple overlapping generations (OLG) model with capital (see, for example, Chapter 6 in Champ and Freeman (2001)), it is known that when $r < r_n$ (the population growth rate) money can be welfare-improving and the optimal money growth rate is $\mu = 1$. In both OLG and quasi-linear models, introduction of money makes transactions possible, and the monetary exchange improves welfare, although different reasons preclude trade without money in the first place. In an OLG model, it is the special demographic structure that makes inter-generational trade impossible, while in a quasi-linear framework it is frictions of limited commitment and lack of record-keeping. In both models, money can improve welfare away from autarky only when money offers higher rate of return than capital alone does, i.e., $\frac{r_n}{\mu} > r$ in OLG model and $\frac{1}{\mu} > r$ in quasi-linear model. However, due to rate-of-return equality, the return of capital will be raised to the same level in monetary equilibrium.

1.6.2 Capital Is Another "Money"

Next one can compare a quasi-linear model without capital with one with capital. With presence of frictions mentioned above, the autarky in an economy without capital is painful - no output is produced/consumed. Introduction of fiat money can induce producers to make production. By producing and exchanging for money in the night, they can be rewarded in the next day when money can be used to buy consumption goods. Here money is a substitute to the missing record-keeping technology. With capital, production/consumption is viable even without money, as capital serves as a record-keeping device. Producers are willing to input labor in night production because they can accumulate capital by producing. Then their capital obtained in night can be sold in next day for consumption. Therefore, the existence of capital makes feasible a reward/punishment mechanism conditioned on past production/consumption history, which is fully revealed by capital holdings at the beginning of a day. Hence capital is essentially a substitute to record-keeping technology, besides its role as an input in production.

When $r < \frac{1}{\beta}$, capital is not a perfect substitute hence first-best cannot be achieved. Then introducing another (better) substitute, fiat money, can help improve welfare. In this sense, one can think of capital and fiat money as two "monies". When one "money", capital, alone cannot lead to optimality, introduction of another "money", fiat money, potentially can improve the welfare by raising rates of return on both "monies".

1.6.3 Capital Is Simply "Capital"

Now it is natural to think of a variant of the present model. Following Shi (1999), suppose that only consumers can hold capital. Then consumers make consumption-investment decisions as before, while producers' problem is just how much labor to input in the production. With this restriction imposed, evidently that there will be no production/consumption in autarky, even with presence of capital. This is because producers are not allowed to hold capital, so they do not have incentive to produce since their effort cannot be rewarded due to lack of record-keeping. In this case, money is always essential in improving welfare as a record-keeping device, as in models without capital. Here capital's only role is simply a productive input. In next chapter I will show that any monetary equilibrium is featured by *underaccumulation* of capital. In such a context consumers are investment-constrained, which is different from the result displayed here where consumers are consumption-constrained. Hence this setup is naturally suitable for studying the role of banking in financing investment as an extension to Berentsen, Camera and Waller (2007).

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1.8 Appendix

1.8.1 Solution to Planner's Problem

The planner's problem can be rewritten as a dynamic programming problem with value function

$$V\left(k_t\right) = \max_{k_{t+1}, n_t} \left\{ \frac{1}{2} u \left[2 \left(F\left(k_t, \frac{1}{2} n_t \right) - k_{t+1} \right) \right] - \frac{1}{2} g\left(n_t\right) + \beta V\left(k_{t+1}\right) \right\}.$$

FOC:

$$u_1(c_t) = \beta V_1\left(k_{t+1}\right), \tag{1.38}$$

$$u_1(c_t) F_2\left(k_t, \frac{1}{2} n_t\right) = g_1(n_t). \tag{1.39}$$

Envelope condition:

$$V_1\left(k_t\right) = u_1(c_t) F_1\left(k_t, \frac{1}{2} n_t\right). \tag{1.40}$$

Replacing t in (1.40) by $t+1$ and plugging into (1.38) yield the consumption Euler equation:

$$u_1(c_t) = \beta u_1(c_{t+1}) F_1\left(k_{t+1}, \frac{1}{2} n_{t+1}\right). \tag{1.41}$$

Then given initial capital stock k_0, the equilibrium paths of $\{c_t, n_t, k_{t+1}, c_{t+1,} n_{t+1}, k_{t+2}\}$ can be characterized by five equations: (1.39), time-t and time-$t+1$ versions of Euler equation (1.41) and resource constraint (4.3), as well as the transversality condition.

1.8.2 Four Cases When $\lambda_m > 0$

1. $k_{1c}^+ > 0, k_{1p}^+ = 0$. Given $k_{1c}^+ > 0$, $u_1(c) = \beta \frac{p_1^+}{\phi^+}$, then (2.8) gives $\lambda_m = \frac{u_1(c)}{p_2} - \beta \frac{1}{\phi^+} = \beta \frac{p_1^+}{\phi^+ p_2} - \beta \frac{1}{\phi^+} > 0$, which implies $p_2 < p_1^+$. But $k_{1p}^+ = 0$ determines that $p_2 > p_1^+$. A contradiction.

2. $k_{1c}^+ > 0, k_{1p}^+ > 0$. Given $k_{1c}^+ > 0$, $u_1(c) = \beta \frac{p_1^+}{\phi^+}$, then (2.8) gives $\lambda_m = \frac{u_1(c)}{p_2} - \beta \frac{1}{\phi^+} = \beta \frac{p_1^+}{\phi^+ p_2} - \beta \frac{1}{\phi^+} > 0$, which implies $p_2 < p_1^+$. But $k_{1p}^+ > 0$ determines that $p_2 = p_1^+$. A contradiction.

3. $k_{1c}^+ = 0, k_{1p}^+ = 0$. Then no one holds capital. It cannot be an equilibrium.

4. $k_{1c}^+ = 0, k_{1p}^+ > 0$. This is the only possible equilibrium.

1.8.3 Four Cases When $\lambda_m = 0$

1. $k_{1c}^+ > 0, k_{1p}^+ = 0$. Given $k_{1c}^+ > 0$, $u_1(c) = \beta \frac{p_1^+}{\phi^+}$, then (2.8) gives $\lambda_m = \frac{u_1(c)}{p_2} - \beta \frac{1}{\phi^+} = \beta \frac{p_1^+}{\phi^+ p_2} - \beta \frac{1}{\phi^+} = 0$, which implies $p_2 = p_1^+$. But $k_{1p}^+ = 0$ determines that $p_2 > p_1^+$. A contradiction.

2. $k_{1c}^+ = 0, k_{1p}^+ > 0$. Given $k_{1p}^+ > 0$, $p_2 = p_1^+$. And $k_{1c}^+ = 0$ means $u_1(c) > \beta \frac{p_1^+}{\phi^+}$, then (2.8) gives $\lambda_m = \frac{u_1(c)}{p_2} - \beta \frac{1}{\phi^+} > \beta \frac{p_1^+}{\phi^+ p_2} - \beta \frac{1}{\phi^+} = 0$. A contradiction with $\lambda_m = 0$.

3. $k_{1c}^+ = 0, k_{1p}^+ = 0$. Then no one holds capital. It cannot be an equilibrium.

4. $k_{1c}^+ > 0, k_{1p}^+ > 0$. This is the only possible equilibrium.

Chapter 2

Money, Capital and Banking[1]

2.1 Introduction

This chapter studies the role of banking in financing investment, as an extension to Berentsen, Camera and Waller (2007).

Berentsen, Camera and Waller (2007) incorporate credit to a Lagos-Wright (2005) framework by introducing banks. The banks accept nominal deposits and make nominal loans. In this way, banks can provide liquidity to cash-constrained consumers. Potentially this could improve welfare. Their main results are: Banking indeed improves welfare; the welfare gains come from paying interest on deposits, not from relaxing liquidity constraints of borrowers; the welfare improvement is greatest under moderate inflation.

Their paper focuses on consumer credit and studies the role of banking in financing consumption. I introduce capital to the model, so as to study the role of banking in financing investment. Regarding the welfare-improving role of banking, I obtain similar results comparable to theirs. In addition, the effect of banking on investment and capital accumulation is as follows: Banking can mitigate under-investment and raise capital-labor ratio; again this effect is greatest under moderate inflation.

[1] I would like to thank participants at the Brown Bag Seminar at Simon Fraser University and Canadian Economics Association 2009 Conference in Toronto for their helpful comments and suggestions.

2.2 Environment

This is the same quasi-linear environment as in the previous introductory chapter.

Time is discrete and the horizon is infinite. Each period t is divided into two subperiods: day and night. There is measure one of infinitely-lived agents who are *ex ante* identical.

Preferences of a representative agent i are defined over stochastic sequences

$$\{e_t(i), c_t(i), n_t(i)\},$$

where $e_t(i)$ denotes consumption (if positive) or production (if negative) in the day, $c_t(i)$ consumption in the night, and $n_t(i)$ labor input during the night. The expected lifetime utility function is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[e_t(i) + \beta U_t\left(c_t(i), n_t(i)\right)\right],$$

where $\beta \in (0,1)$ is the time discounting rate between two consecutive periods. Note the utility function is quasi-linear in daytime consumption/production.

There are preference/technology shocks which are realized at the beginning of each night. The shocks are i.i.d. across agents and over time. With probability 1/2, an agent wants to consume but cannot produce; with probability 1/2, an agent can produce but does not want to consume. Therefore, the night utility function can be written as

$$U_t\left(c_t(i), n_t(i)\right) = \begin{cases} u\left(c_t(i)\right) & \text{w.p. } 1/2 \quad \text{(consumer)} \\ -g\left(n_t(i)\right) & \text{w.p. } 1/2 \quad \text{(producer)} \end{cases}.$$

The instantaneous utility function $u'(c)$ has properties: $u'' < 0 < u'$, $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. And the disutility function $g(n)$ satisfies $g' > 0$ and $g'' \geq 0$.

Capital is required for production only in the night. The aggregate production possibilities can be described with a constant-returns-to-scale (CRS) production function of the form $Y_t = F(K_t, N_t)$, where $K_t \equiv \int_0^1 k_t(j)dj$ denotes the aggregate capital stock available at night and $N_t \equiv \int_0^1 n_t(j)dj$ the aggregate labor input in the night. Capital is augmented in the usual way, with

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

denoting the capital available for production the next night. For simplicity I assume that $\delta = 1$. During each day, the existing capital held by agents can be traded on a competitive market, but cannot be consumed. Initially all agents are endowed with identical capital holdings $k_0 > 0$.

2.3 Pareto Optimal Allocation

As a benchmark, I consider the first-best allocation that a planner can implement in absence of all kinds of frictions. Since every agent is *ex ante* identical, it is natural to assume that the planner treats everyone equally. Due to the quasi-linearity in day consumption, any lottery scheme in $\{e_t\}$ satisfying $E_0 e_t(j) = 0$ and day resource constraint $\int_0^1 e_t(j)\, dj = 0$ can be a solution. Therefore, the day subperiods are irrelevant and a trivial solution is $e_t(j) = 0\ \forall t$ for all agents. A planner only needs to consider capital accumulation across nights and consumption-production decision in nights.

Considering the preference/technology shocks in night, a planner asks each agent who turns out to be a producer to produce n_t. Given initial aggregate capital stock k_t, the total output will be $Y_t = F\left(k_t, \frac{1}{2}n_t\right)$. Then a consumption-investment decision is made: each consumer gets c_t for consumption (so aggregate consumption is $\frac{1}{2}c_t$), and the rest of output will be invested as capital stock, k_{t+1}, for future production.

The planner's problem is to maximize a representative agent's lifetime utility

$$\max \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2}u(c_t) - \frac{1}{2}g(n_t)\right]$$

subject to the night resource constraint

$$\frac{1}{2}c_t + k_{t+1} = F\left(k_t, \frac{1}{2}n_t\right), \tag{2.1}$$

given initial capital stock k_0.

It is straightforward to show that the allocation of $\{c_t, n_t, k_{t+1}\}$ can be characterized by

$$u_1(c_t)F_2\left(k_t, \frac{1}{2}n_t\right) = g_1(n_t),$$

$$u_1(c_t) = \beta u_1(c_{t+1})F_1\left(k_{t+1}, \frac{1}{2}n_{t+1}\right),$$

and the resource constraint (4.3).[2]

The focus is restricted to a stationary allocation where all real variables remain constant. And thus the first-best allocation $\{c^*, n^*, k^*\}$ is characterized by three equations:

$$u_1(c^*) = \frac{g_1(n^*)}{F_2\left(k^*, \frac{1}{2}n^*\right)}, \tag{2.2}$$

$$F_1\left(k^*, \frac{1}{2}n^*\right) = \frac{1}{\beta}, \tag{2.3}$$

$$\frac{1}{2}c^* + k^* = F\left(k^*, \frac{1}{2}n^*\right). \tag{2.4}$$

2.4 A Model without Banking

Now I assume that agents cannot commit, and it is impossible to monitor and record agents' transaction histories due to anonymity. Combined with the lack of double coincidence of wants introduced by preference/technology shocks, these frictions imply that any exchange must be based on *quid pro quo*.

Following Shi (1999), I assume that in the night only consumers can invest; producers can produce but cannot hold capital. For instance, one can think of night goods as perishable, and only consumers have access to a storage technology; therefore, only consumers can make investment. As demonstrated in the introductory chapter, if this restriction is relaxed, i.e., suppose that both consumers and producers can invest in the night, then capital effectively serves as a record-keeping device, besides its role as a productive input. There will be two possible cases of equilibrium: (1) With a low equilibrium capital return (i.e., $r < 1/\beta$), consumers are constrained by their capital rental income, and thus would not invest. Only producers hold capital as investment. In this case, introducing money could improve welfare. (2) With a high capital return (i.e., $r = 1/\beta$), consumers are not constrained, and capital is held by both consumers and producers as investment. Moreover, this equilibrium is Pareto efficient, hence money is inessential. If only consumers are allowed to hold capital, without aid of money there will be no exchange to take place, and there will be no production/consumption in equilibrium. Specifically, producers are not allowed to accumulate capital by producing; therefore, they are not willing to produce since they cannot be rewarded for their effort due to the lack of record-keeping. This

[2] For proof refer to Appendix of Chapter 1.

gives rise to the essentiality of fiat money, which serves as a substitute to missing record-keeping technology, while capital is simply a productive input.

Next I introduce fiat money which makes night trade possible. It is intrinsically useless unbacked token which is portable, divisible and uncounterfeitable. Now the production and exchange are viable in this economy. Suppose there are competitive rental markets for capital and labor at night, with factor prices r and w, respectively. Initially each agent is endowed with M units of fiat money. During the night, consumers can buy output from producers for consumption and investment using their real money balances, in addition to capital rental income. Producers are willing to input labor to produce since their output can be sold for money, which could be exchanged for consumption in the future when they want to consume. In this way, night production, hence consumption and investment are made feasible by circulation of fiat money. Note the day is simply used to rebalance each agent's money holdings and capital holdings such that the distributions of money and capital are degenerate. This tractability is the merit of quasi-linear preferences.

To simplify notations, I drop time subscript t for all current variables, and add a superscript "$+$" to denote time period $t+1$ variables. And suppose there is a money injection T by the government at the beginning of each day, with constant money growth rate $\mu \equiv M^{+}/M$.

2.4.1 Day Decision-making

During the day, all agents can work and consume. Capital is not needed in day production and the day goods cannot be transformed to capital. Therefore, aggregate capital accumulated in last night remains unchanged in this stage. However, capital trade among agents is allowed so that they can rebalance capital holdings before facing preference/technology shocks in the night. Also they can adjust money holdings by trading day consumption goods besides receiving money transfers.

The value of an agent entering a day with capital holding k_1 and nominal money balance m_1 is

$$\begin{aligned} W(k_1, m_1) &= \max_{e,k_2,m_2} \{e + V(k_2, m_2)\} \\ \text{s.t. } m_1 + T &= \phi e + p_1 (k_2 - k_1) + m_2, \\ m_2 &\geq 0, \\ k_2 &\geq 0, \end{aligned}$$

where ϕ is the price of consumption and p_1 is the price of capital, in day transactions, and $V(k_2, m_2)$ is the value associated with entering the night with capital holding k_2 and nominal money balance m_2. Note the nonnegativity restriction on asset holdings is implied by the presence of frictions, i.e., borrowing is ruled out due to anonymity and lack of commitment. After eliminating day consumption e by the budget constraint, one can rewrite the value function as

$$W(k_1, m_1) = \max_{k_2 \geq 0, m_2 \geq 0} \left\{ \frac{1}{\phi} (m_1 + p_1 k_1 + T - m_2 - p_1 k_2) + V(k_2, m_2) \right\}$$

The first order conditions (FOC's) for an interior solution are:

$$\frac{p_1}{\phi} = V_1(k_2, m_2), \tag{2.5}$$

$$\frac{1}{\phi} = V_2(k_2, m_2). \tag{2.6}$$

It is clear that the marginal values of capital and money holdings are constant, which means that agents' capital and money holding decisions are independent of previous balances. In symmetric equilibrium, everyone carries same amount of capital and money into night. This gives rise to degenerate distributions of both assets at the beginning of every night. This makes the model tractable without resorting to computational method (for example, see Molico (2006)).

Two envelope conditions are

$$W_1(k_1, m_1) = \frac{p_1}{\phi}, \tag{2.7a}$$

$$W_2(k_1, m_1) = \frac{1}{\phi}. \tag{2.7b}$$

2.4.2 Night Decision-making

Consumer: If an agent turns out to be a consumer, she can consume and invest out of her capital rental income and money balance carried from the day. Her value associated with entering the night with capital holding k_2 and nominal money balance m_2 is

$$\begin{aligned} V^c(k_2, m_2) &= \max_{c, k_{1c}^+, m_{1c}^+} \left\{ u(c) + \beta W(k_{1c}^+, m_{1c}^+) \right\} \\ \text{s.t. } p_2 c + p_2 k_{1c}^+ + m_{1c}^+ &= p_2 r k_2 + m_2, \\ m_{1c}^+ &\geq 0, \end{aligned}$$

where p_2 is the price of night output which can be used for either consumption or investment; k_{1c}^+ and m_{1c}^+ are capital and money holdings, respectively, taken to the next day. Again the nonnegative cash constraint is implied by the existence of frictions. In equilibrium it must be true that all consumers invest in the night, so the nonnegative-capital-holding constraint is always unbinding, hence it is safe to depress it to simplify notations. Using the budget constraint to eliminate m_{1c}^+ in the value function yields

$$V^c(k_2, m_2) = \max_{c, k_{1c}^+} \left\{ \begin{array}{c} u(c) + \beta W\left[k_{1c}^+, m_2 + p_2\left(rk_2 - c - k_{1c}^+\right)\right] \\ +\lambda\left[m_2 + p_2\left(rk_2 - c - k_{1c}^+\right)\right] \end{array} \right\},$$

where λ is the Lagrange multiplier associated with the nonnegative-cash constraint $m_{1c}^+ \geq 0$.

FOC's:

$$u_1(c) - \beta \frac{1}{\phi^+} p_2 - \lambda p_2 = 0, \tag{2.8}$$

$$\beta \frac{p_1^+}{\phi^+} - \beta \frac{1}{\phi^+} p_2 - \lambda p_2 = 0, \tag{2.9}$$

where envelope conditions (3.5) have been used.

It is clear that $\lambda = \frac{u_1(c)}{p_2} - \beta \frac{1}{\phi^+} > 0$ if $m_{1c}^+ = 0$ and $\lambda = 0$ if $m_{1c}^+ > 0$. The marginal value of one dollar is $\frac{u_1(c)}{p_2}$ when it is spent at night. If the same one dollar is held into next day, the (discounted) marginal value will be $\beta \frac{1}{\phi^+}$. Apparently, if the former is larger, agents would not save money and hold it into next day. Instead, all money would be spent in night when $\frac{u_1(c)}{p_2} > \beta \frac{1}{\phi^+}$. Only when $\frac{u_1(c)}{p_2} = \beta \frac{1}{\phi^+}$, consumers are indifferent and would carry money into next day.

Equation (2.9) implies that $\lambda > 0$ when $p_1^+ > p_2$. If capital is cheaper in night than in next day, naturally consumers would choose to spend all money to buy capital in night, rather than hold money and buy capital in next day. Also note that, in terms of capital, the rate of return of holding money across night and day is p_2/p_1^+, and the return on holding capital is 1. When consumers are cash-constrained, the rate-of-return equality between two assets does not hold.[3] Since credit transactions are precluded by the existence of frictions, capital must be purchased with cash. As a result, liquidity-constrained investors (consumers) cannot exploit the arbitrage opportunity, although capital offers better return.

Combining two FOC's leads to

$$u_1(c) = \beta \frac{p_1^+}{\phi^+}. \tag{2.10}$$

The cost of making investment is sacrifice of consumption at night, and the benefit is higher consumption in next day. Then for a consumer to both consume and invest, the marginal cost $u_1(c)$ must be equal to the (discounted) marginal benefit $\beta \frac{p_1^+}{\phi^+}$. Since night utility function is concave but day utility function is linear, there exists a satiation point $\widehat{c}$ for night consumption such that $u_1(\widehat{c}) = \beta \frac{p_1^+}{\phi^+}$. When night consumption reaches the satiation point, all the rest of income will be used to buy capital. Therefore, cash-constrained consumers would only invest more, not consume more, if given more cash. In this sense, I say consumers are cash-constrained for investment only, or investment-constrained.

Two envelope conditions are

$$V_1^c(k_2, m_2) = \beta \frac{1}{\phi^+} p_2 r + \lambda p_2 r = r u_1(c), \tag{2.11}$$

$$V_2^c(k_2, m_2) = \beta \frac{1}{\phi^+} + \lambda = \frac{1}{p_2} u_1(c), \tag{2.12}$$

where the second equality in both equations is derived using (2.8).

Producer: A producer just makes decision on how much labor to input. Since he can neither consume or hold output as capital, he must sell all his capital rental and labor wage income in terms of output in the market, and then carry all money into next day. The value of a producer entering the night with capital holding k_2 and

[3]Note in Section 1.5 such a rate-of-return equality holds, because the investors (producers) are *not* liquidity-constrained.

nominal money balance m_2 is

$$\begin{aligned} V^p(k_2, m_2) &= \max_{n, m_{1p}^+} \left\{ -g(n) + \beta W(0, m_{1p}^+) \right\} \\ \text{s.t. } m_{1p}^+ &= m_2 + p_2 \left(r k_2 + w n \right). \end{aligned}$$

where n is the labor input and m_{1p}^+ is the cash balance taken into next day. Using the budget constraint to eliminate m_{1p}^+ yields

$$V^p(k_2, m_2) = \max_n \left\{ -g(n) + \beta W \left[0, m_2 + p_2 \left(r k_2 + w n \right) \right] \right\}.$$

FOC:

$$\frac{g_1(n)}{w} = \beta \frac{1}{\phi^+} p_2, \tag{2.13}$$

where envelope conditions (3.5) have been used. It is clear that producers are equating marginal cost and (discounted) marginal benefit of output in utility terms.

Envelope conditions:

$$V_1^p(k_2, m_2) = \beta \frac{1}{\phi^+} p_2 r = r \frac{g_1(n)}{w}, \tag{2.14}$$

$$V_2^p(k_2, m_2) = \beta \frac{1}{\phi^+} = \frac{1}{p_2} \frac{g_1(n)}{w}, \tag{2.15}$$

where the second equality in both equations is derived utilizing (2.13).

2.4.3 Equilibrium

Note that

$$V_1(k_2, m_2) = \frac{1}{2} V_1^c(k_2, m_2) + \frac{1}{2} V_1^p(k_2, m_2). \tag{2.16}$$

Plugging (3.4), (2.11) and (2.14) into both sides of it leads to

$$\frac{p_1}{\phi} = \frac{1}{2} r \left[u_1(c) + \frac{g_1(n)}{w} \right]. \tag{2.17}$$

Similarly, plugging (2.6), (2.12) and (2.15) into both sides of

$$V_2(k_2, m_2) = \frac{1}{2} V_2^c(k_2, m_2) + \frac{1}{2} V_2^p(k_2, m_2) \tag{2.18}$$

gives rise to

$$\frac{1}{\phi} = \frac{1}{2}\frac{1}{p_2}\left[u_1(c) + \frac{g_1(n)}{w}\right]. \tag{2.19}$$

Combining (2.17) and (2.19) yields

$$r = \frac{p_1}{p_2}. \tag{2.20}$$

This is a rate-of-return equality condition. Across day and night, the rate of return on capital is r, while the rate of return on money is p_1/p_2. For all agents to hold both money and capital at the beginning of night, this condition must be satisfied.

To proceed, two cases need to be considered: either $\lambda > 0$ or $\lambda = 0$.

Case 1: $\lambda > 0$. Then $m_{1c}^+ = 0$, i.e., consumers are cash-constrained and they spend all money. The consumer budget constraint implies

$$p_2 = \frac{m_2}{c + k_{1c}^+ - rk_2}.$$

Since all agents will rebalance capital holdings in next day so that $k_2^+ = \frac{1}{2}k_{1c}^+$. Considering the money market clearing condition $m_2 = M$, one gets

$$p_2 = \frac{M}{c + 2k_2^+ - rk_2}. \tag{2.21}$$

Updating (2.17) by one period yields

$$\frac{p_1^+}{\phi^+} = \frac{1}{2}r^+\left[u_1(c^+) + \frac{g_1(n^+)}{w^+}\right]. \tag{2.22}$$

Eliminating the left-hand side by (2.10), one obtains

$$u_1(c) = \frac{\beta}{2}r^+\left[u_1(c^+) + \frac{g_1(n^+)}{w^+}\right]. \tag{2.23}$$

The night output market clearing condition is

$$\frac{1}{2}c + \frac{1}{2}k_{1c}^+ = F\left(k_2, \frac{1}{2}n\right).$$

Since $k_2^+ = \frac{1}{2}k_{1c}^+$, it becomes

$$\frac{1}{2}c + k_2^+ = F\left(k_2, \frac{1}{2}n\right). \tag{2.24}$$

Updating (2.20) for one period yields

$$r^+ = \frac{p_1^+}{p_2^+}. \tag{2.25}$$

Combining (2.10) and (2.13) gives rise to

$$p_1^+ \frac{g_1(n)}{w} = p_2 u_1(c).$$

Plugging it into (2.25) to eliminate p_1^+, and considering (2.21), one obtains

$$r^+ = \frac{1}{\mu} \cdot \frac{c^+ + 2k_2^{++} - r^+ k_2^+}{c + 2k_2^+ - rk_2} \cdot \frac{u_1(c)}{\frac{g_1(n)}{w}}. \tag{2.26}$$

Here the focus is again on a stationary equilibrium where all real variables are constant. The equilibrium allocation $\{c, n, k_2\}$ can be characterized by the stationary version of (2.23), (2.26) and (2.24):

$$u_1(c) = \frac{\beta}{2} r \left[u_1(c) + \frac{g_1(n)}{w}\right], \tag{2.27}$$

$$r = \frac{1}{\mu} \cdot \frac{u_1(c)}{\frac{g_1(n)}{w}}, \tag{2.28}$$

$$\frac{1}{2}c + k_2 = F\left(k_2, \frac{1}{2}n\right). \tag{2.29}$$

with $w = F_2\left(k_2, \frac{1}{2}n\right)$ and $r = F_1\left(k_2, \frac{1}{2}n\right)$.

Case 2: $\lambda = 0$. Consumers carry positive money balances into next day. Then (2.8) becomes

$$\frac{u_1(c)}{p_2} - \beta \frac{1}{\phi^+} = 0. \tag{2.30}$$

Combining (2.30) with (2.13) leads to one equilibrium condition:

$$u_1(c) = \frac{g_1(n)}{w}.$$

Combining (2.30) with (2.10) and (2.17) gives rise to

$$r^{+} = \frac{1}{\beta}\frac{u_1(c)}{u_1(c^{+})}.$$

In stationary equilibrium, it becomes

$$r = \frac{1}{\beta}.$$

Another equilibrium condition is the night goods market clearing condition (2.29). It is clear that this is the first-best allocation.

2.4.4 Optimal Monetary Policy

Apparently, when $\lambda > 0$, money is neutral but not super-neutral, i.e., the money growth rate affects equilibrium allocation. When $\mu > \beta$, one can deduce from (2.27) and (2.28) that

$$u_1(c) = \frac{2\mu - \beta}{\beta}\frac{g_1(n)}{w} > \frac{g_1(n)}{w}$$

and

$$r = \frac{1}{\mu} \cdot \frac{2\mu - \beta}{\beta} > \frac{1}{\beta}.$$

Therefore, the monetary equilibrium is not first-best when $\mu > \beta$. In particular, it is characterized with under-consumption, under-production and under-accumulation of capital.

If producers are allowed to hold capital, it has dual roles: a record-keeping device and a productive input. Any monetary equilibrium is featured by *over-accumulation* of capital.[4] As a stark contrast, if producers are not allowed to hold capital, it simply serves as a productive input, and then the monetary equilibrium is featured by *under-accumulation* of capital, reflecting that investors (consumers) are liquidity constrained in acquiring capital.

Also note that

$$r = \frac{1}{\mu} \cdot \frac{2\mu - \beta}{\beta} > \frac{1}{\mu},$$

which means that capital dominates money in inter-period rate of return. Again, the existence of such a rate-of-return dominance implies that investors (consumers) are

[4] Recall $r = \frac{1}{\mu} < \frac{1}{\beta}$ in Section 1.5. Also a similar result is highlighted in Lagos and Rocheteau (2008).

liquidity constrained in acquiring capital.

When $\mu \rightarrow \beta$, $\frac{2\mu-\beta}{\beta} \rightarrow 1$ and $\frac{1}{\mu} \cdot \frac{2\mu-\beta}{\beta} \rightarrow \frac{1}{\beta}$, hence $u_1(c) \rightarrow \frac{g_1(n)}{w}$, and $r \rightarrow \frac{1}{\beta}$. At the same time, $\lambda \rightarrow 0$. Therefore, by setting $\mu = \beta$, the first-best allocation can be achieved. In other words, the Friedman rule is optimal in this environment.

2.5 A Model with Banking

In a monetary equilibrium with $\mu > \beta$, $\lambda > 0$ and consumers are cash-constrained, as usual in most monetary models. However, in this setup, consumers are short of cash for investment, not for consumption. As can be seen from (2.10), "desired" consumption by consumers is uniquely determined given that night utility function is strictly concave. If a consumer holds positive capital, it must be the case that he has already consumed $\widehat{c}$ which is determined by (2.10). All the rest income would be used to buy capital. Since capital can be also purchased in next day, consumers would compare prices of capital in night and next day. When $\lambda > 0$, (2.9) implies $p_2 < p_1^+$. Therefore, it is cheaper to buy in the night than in the next day. Noticing this, consumers would like to buy infinite amount of capital after consumption need is satisfied. However, consumers are cash-constrained in the night. The most they can do is to spend all money they take into the night. The result is under-investment reflected by a too-high capital rental rate as $r > \frac{1}{\beta}$ in equilibrium. At the same time, producers are holding idle cash during the night. If they could lend their money to consumers, potentially this would reduce the inefficiency. However, credit is impossible in this environment due to anonymity of agents.

Berentsen, Camera and Waller (2007) incorporate banking into a monetary model without capital. The banks can record financial histories of agents, hence they can accept nominal deposits from producers and make nominal loans to consumers. However, the ability of record-keeping is restricted to financial transactions. Money is still needed to facilitate exchange between agents in output transactions. In this way, credit and money can coexist. In their model, the role of banking service is to finance consumption. Now I can consider an extension to their model. As is made clear now, consumers are cash-constrained for investment in a monetary model with capital, so it is naturally suitable for studying the role of banking in financing investment.

The financial intermediation is the same as in Berentsen, Camera and Waller

(2007). The service is provided by perfectly competitive banks at no cost.[5] Banks attract deposits from producers at the beginning of each night right after preference shocks are realized, promising (net) interest rate i_d which will be paid in the next day. The deposited money is then loaned to consumers in the same night at interest rate i, and the loan is repaid in next day. For simplicity, assume full enforcement by banks so that there is no default and no borrowing constraint. Note due to the quasi-linear environment one only needs to consider one-period credit, since there is no point in spreading repayment or redemption of debt over periods in order to smooth utility.

2.5.1 Day Decision-making

During the day, an agent has to settle all debt payment and redeem deposits, as well as rebalance money and capital holdings that she will take into the night. The value of an agent entering a day with capital holding k_1, nominal money balance m_1, loan balance l and deposit balance d is

$$\begin{aligned} W(k_1, m_1, l, d) &= \max_{e,k_2,m_2} \{e + V(k_2, m_2)\} \\ \text{s.t. } m_1 + p_1 k_1 + T + (1 + i_d)d &= \phi e + p_1 k_2 + (1 + i)l + m_2, \\ m_2 &\geq 0, \\ k_2 &\geq 0. \end{aligned}$$

Eliminating day consumption e by the budget constraint, one can rewrite the value function as

$$W(k_1, m_1, l, d) = \max_{k_2 \geq 0, m_2 \geq 0} \left\{ \begin{array}{c} \frac{1}{\phi}\left[(m_1 + p_1 k_1) + T + (1 + i_d)d - (1 + i)l - (m_2 + p_1 k_2)\right] \\ +V(k_2, m_2) \end{array} \right\}.$$

FOC's for an interior solution are:

$$\frac{p_1}{\phi} = V_1(k_2, m_2), \tag{2.31}$$

$$\frac{1}{\phi} = V_2(k_2, m_2). \tag{2.32}$$

[5] The implications of banking cost will be studied in Chapter 3.

Envelope conditions:

$$W_1(k_1, m_1, l, d) = \frac{p_1}{\phi}, \tag{2.33a}$$

$$W_2(k_1, m_1, l, d) = \frac{1}{\phi}, \tag{2.33b}$$

$$W_3(k_1, m_1, l, d) = -\frac{1+i}{\phi}, \tag{2.33c}$$

$$W_4(k_1, m_1, l, d) = \frac{1+i_d}{\phi}. \tag{2.33d}$$

2.5.2 Night Decision-making

Consumer: For an agent who turns out to be a consumer, she would not make any deposit since she is cash-constrained in equilibrium. Instead, she would like to borrow money from the bank. Therefore, her value associated with entering the night with capital holding k_2 and nominal money balance m_2 is

$$\begin{aligned} V^c(k_2, m_2) &= \max_{c, k_{1c}^+, m_{1c}^+, l^+} \left\{ u(c) + \beta W(k_{1c}^+, m_{1c}^+, l^+, 0) \right\} \\ \text{s.t. } p_2 c + p_2 k_{1c}^+ + m_{1c}^+ &= p_2 r k_2 + m_2 + l^+, \\ m_{1c}^+ &\geq 0. \end{aligned}$$

Note the consumer's available resources for consumption and investment are now augmented by the purchasing power of nominal loan balance l^+. Again the nonnegative-cash constraint remains, because no borrowing can be made after financial market closes at the beginning of each night. Suppose there is no default problem, for instance, by assuming banks have an enforcement technology to ensure borrowers to repay their loans. Therefore, consumers are not debt-constrained and they can borrow as much as they desire. Using the budget constraint to eliminate m_{1c}^+ in the value function yields

$$V^c(k_2, m_2) = \max_{c, k_{1c}^+, l^+} \left\{ \begin{array}{c} u(c) + \beta W\left[k_{1c}^+, m_2 + p_2\left(rk_2 - c - k_{1c}^+\right) + l^+, l^+, 0\right] \\ +\lambda\left[m_2 + p_2\left(rk_2 - c - k_{1c}^+\right) + l^+\right] \end{array} \right\},$$

where λ is the Lagrange multiplier associated with the constraint $m_{1c}^+ \geq 0$.

FOC's:

$$u_1(c) - \beta \frac{1}{\phi^+} p_2 - \lambda p_2 = 0, \tag{2.34}$$

$$\beta\frac{p_1^+}{\phi^+}-\beta\frac{1}{\phi^+}p_2-\lambda p_2=0, \tag{2.35}$$

$$\beta\frac{1}{\phi^+}-\beta\frac{1+i^+}{\phi^+}+\lambda=0, \tag{2.36}$$

where use has been made of envelope conditions (2.33).

Equations (2.34) and (2.35) imply that

$$u_1(c)=\beta\frac{p_1^+}{\phi^+}. \tag{2.37}$$

And (2.35) and (2.36) give rise to

$$\lambda=\beta\frac{1}{\phi^+}\left(\frac{p_1^+}{p_2}-1\right)=\beta\frac{1}{\phi^+}i^+. \tag{2.38}$$

Clearly $\lambda>0$ if $p_1^+>p_2$ and $\lambda=0$ if $p_1^+=p_2$; in addition, $\lambda>0$ if $i^+>0$ and $\lambda=0$ if $i^+=0$. Also it must hold that

$$1+i^+=\frac{p_1^+}{p_2}. \tag{2.39}$$

Envelope conditions:

$$V_1^c(k_2,m_2)=\beta\frac{1}{\phi^+}p_2r+\lambda p_2r=ru_1(c), \tag{2.40}$$

$$V_2^c(k_2,m_2)=\beta\frac{1}{\phi^+}+\lambda=\frac{1}{p_2}u_1(c), \tag{2.41}$$

where (2.34) is utilized when deriving the second equality in both equations.

Producer: It is straightforward that a producer would not borrow money in the night. And he would deposit all money in the bank, since he does not need money in the night while by depositing he can get interest payment in next day. His value of entering the night with capital holding k_2 and nominal money balance m_2 is

$$\begin{aligned}
V^p(k_2,m_2) &= \max_{n,m_{1p}^+,d^+}\left\{-g(n)+\beta W(0,m_{1p}^+,0,d^+)\right\}\\
\text{s.t. } m_{1p}^+ &= p_2\left(rk_2+wn\right)+m_2-d^+,\\
d^+ &\le m_2,
\end{aligned}$$

where d^+ is the deposit amount which cannot exceed money balance taken into the night. Note all financial transactions must occur at the beginning of a night. Therefore, producers' income from subsequent output sale cannot be deposited.

The producer's problem can be rewritten as

$$V^p(k_2, m_2) = \max_{n,d^+} \left\{ \begin{array}{c} -g(n) + \beta W\left[0, p_2\left(rk_2 + wn\right) + m_2 - d^+, 0, d^+\right] \\ +\lambda_d\left(m_2 - d^+\right) \end{array} \right\},$$

where λ_d is the Lagrange multiplier associated with deposit constraint $d^+ \leq m_2$.

FOC's:

$$\frac{g_1(n)}{w} = \beta\frac{1}{\phi^+}p_2, \tag{2.42}$$

$$-\beta\frac{1}{\phi^+} + \beta\frac{1+i_d^+}{\phi^+} - \lambda_d = 0, \tag{2.43}$$

where use has been made of envelope conditions (2.33).

(2.43) means

$$\lambda_d = \beta\frac{1}{\phi^+}i_d^+.$$

It is clear that given any positive deposit interest rate i_d^+ producers would deposit all money balance, i.e., $\lambda_d > 0$ and $d^+ = m_2$.

Envelope conditions:

$$V_1^p(k_2, m_2) = \beta\frac{1}{\phi^+}p_2 r = r\frac{g_1(n)}{w}, \tag{2.44}$$

$$V_2^p(k_2, m_2) = \beta\frac{1}{\phi^+} + \lambda_d = \frac{1+i_d^+}{p_2}\cdot\frac{g_1(n)}{w}, \tag{2.45}$$

where the second equality in (2.44) is derived from (2.42), and the second equality in (2.45) is derived from (2.42) and (2.43).

2.5.3 Equilibrium

Combining (2.37), (2.39) and (2.42) yields

$$1 + i^+ = \frac{u_1(c)}{\frac{g_1(n)}{w}}. \tag{2.46}$$

Plugging (2.31), (2.40) and (2.44) into both sides of (2.16) yields

$$\frac{p_1}{\phi} = \frac{1}{2} r \left[u_1(c) + \frac{g_1(n)}{w} \right]. \tag{2.47}$$

Similarly, plugging (2.32), (2.41) and (2.45) into both sides of (2.18) leads to

$$\frac{1}{\phi} = \frac{1}{2} \frac{1}{p_2} \left[u_1(c) + \frac{g_1(n)}{w} \left(1 + i_d^+\right) \right]. \tag{2.48}$$

To proceed, there are two cases to consider: either $\lambda > 0$ or $\lambda = 0$.

Case 1: $\lambda > 0$. Consumers are cash-constrained for investment. Then the consumer budget constraint gives rise to

$$p_2 = \frac{m_2 + l^+}{c + k_{1c}^+ - r k_2}.$$

Also it is known from (2.38) that $i^+ > 0$. Due to perfect competition in banking industry, the zero profit condition predicts $i_d^+ = i^+ > 0$. Then it must be true that $d^+ = m_2$. The credit market clearing condition imposes $l^+ = d^+$. Therefore, it follows that

$$l^+ = d^+ = m_2 = M,$$

where the last equality is the money market clearing condition. Since all agents will rebalance capital holdings in next day so that $k_2^+ = \frac{1}{2} k_{1c}^+$, it must be true that

$$p_2 = \frac{2M}{c + 2k_2^+ - r k_2}. \tag{2.49}$$

Rewriting (2.48) for next period and multiplying both sides by βp_2 result in

$$\frac{\beta p_2}{\phi^+} = \frac{\beta p_2}{2 p_2^+} \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \left(1 + i^{++}\right) \right].$$

Eliminating the left-hand side by (2.42) and using (2.49) lead to

$$\frac{g_1(n)}{w} = \frac{\beta}{2} \frac{1}{\mu} \frac{c^+ + 2k_2^{++} - r^+ k_2^+}{c + 2k_2^+ - r k_2} \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \left(1 + i^{++}\right) \right].$$

Imposing stationarity, this becomes an equilibrium condition

$$u_1(c) = \frac{\mu}{\beta} \frac{g_1(n)}{w}. \tag{2.50}$$

Hence (2.46) becomes

$$1 + i = \frac{\mu}{\beta}. \tag{2.51}$$

Updating (2.47) for one period yields

$$\frac{p_1^+}{\phi^+} = \frac{1}{2} r^+ \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \right].$$

Eliminating the left-hand side by (2.37), one obtains

$$u_1(c) = \frac{\beta}{2} r^+ \left[u_1(c^+) + \frac{g_1(n^+)}{w^+} \right].$$

Considering (2.50), in stationary equilibrium it becomes

$$r = \frac{2\mu}{\beta\,(\mu + \beta)}. \tag{2.52}$$

The night output market clearing condition is

$$\frac{1}{2} c + \frac{1}{2} k_{1c}^+ = F\left(k_2, \frac{1}{2} n \right).$$

Since $k_{1c}^+ = 2k_2^+ = 2k_2$, it becomes

$$\frac{1}{2} c + k_2 = F\left(k_2, \frac{1}{2} n \right). \tag{2.53}$$

In summary, the stationary equilibrium allocation $\{c, n, k_2\}$ can be characterized by (2.50), (2.52) and (2.53), with $w = F_2\left(k_2, \frac{1}{2}n\right)$ and $r = F_1\left(k_2, \frac{1}{2}n\right)$, while the equilibrium interest rate is determined by (2.51).

Case 2: $\lambda = 0$. Consumers carry positive money balances into next day. And (2.34) becomes (2.30). Combining (2.30) with (2.13) gives rise to one equilibrium condition:

$$u_1(c) = \frac{g_1(n)}{w}.$$

Combining (2.30) with (2.10) and (2.17) leads to

$$r^+ = \frac{1}{\beta} \frac{u_1(c)}{u_1(c^+)}.$$

In stationary equilibrium, it becomes

$$r = \frac{1}{\beta}.$$

Another equilibrium condition is the night goods market clearing condition (2.53). It is clear that this is the first-best allocation.

2.5.4 Optimal Monetary Policy

When $\mu > \beta$, (2.50) and (2.52) imply that

$$u_1(c) = \frac{\mu}{\beta}\frac{g_1(n)}{w} > \frac{g_1(n)}{w}$$

and

$$r = \frac{2\mu}{\beta(\mu+\beta)} > \frac{1}{\beta}.$$

Therefore, the monetary equilibrium is not first-best when $\mu > \beta$. Again the equilibrium is characterized by under-consumption, under-production, and under-accumulation of capital, as in the no-banking case.

When $\mu \to \beta$, $\frac{\mu}{\beta} \to 1$ and $\frac{2\mu}{\beta(\mu+\beta)} \to \frac{1}{\beta}$, hence $u_1(c) \to \frac{g_1(n)}{w}$, and $r \to \frac{1}{\beta}$. At the same time, $i \to 0$ and $\lambda \to 0$. Therefore, by setting $\mu = \beta$, the first-best allocation can be achieved. In other words, the Friedman rule is still optimal in the presence of banking.

2.6 Role of Banking

Now I compare the two equilibria when $\mu > \beta$, so as to study the effects of banking on consumption, investment, output and welfare. And the findings are presented as follows.

First, I can show that banking mitigates under-consumption. The wedge between marginal benefit of consumption and marginal cost of production represents inefficiency of under-production and hence under-consumption, when the ratio $u_1(c)/\frac{g_1(n)}{w}$ is greater than one. Therefore, I compare the values of this ratio in three cases: first-best (FB), monetary equilibrium without banking (NB) and monetary equilibrium with banking (B) and find that

$$\frac{u_1(c)}{\frac{g_1(n)}{w}} : \left(\frac{2\mu - \beta}{\beta}\right)_{NB} > \left(\frac{\mu}{\beta}\right)_B > (1)_{FB}.$$

This shows that the presence of banking can decrease the inefficiency caused by under-production and mitigate under-consumption.

In addition, banking mitigates under-investment. When checking the capital rental rate, one has the comparison as follows:

$$r : \left(\frac{2\mu - \beta}{\mu\beta}\right)_{NB} > \left(\frac{2\mu}{\beta(\mu + \beta)}\right)_B > \left(\frac{1}{\beta}\right)_{FB}.$$

Given a CRS production function, it is known that the capital-labor ratio is monotonically decreasing in rental rate. As a result, the ranking of capital-labor ratio in the three cases must be

$$\left(\frac{k_2}{n}\right)_{NB} < \left(\frac{k_2}{n}\right)_B < \left(\frac{k_2}{n}\right)_{FB}.$$

Clearly banking helps capital accumulation and mitigates under-investment. Moreover, another property of CRS production function is that the marginal product of labor, hence competitive wage rate, decreases in rental rate, so it follows that

$$w_{NB} < w_B < w_{FB},$$

i.e., the existence of banking sector helps raise the wage rate.

Furthermore, the effect of banking is greatest under moderate inflation. A decrease in rental rate indicates improvement of under-investment inefficiency. Therefore, the ratio of two capital rental rates, with banking and without banking,

$$\frac{r_B}{r_{NB}} = \frac{2\mu}{\beta(\mu + \beta)} / \frac{2\mu - \beta}{\mu\beta},$$

can be taken as a proxy to the degree of inefficiency improvement. As can be easily verified, this ratio reaches minimum at $\mu = 2\beta$, and it approaches 1 when $\mu \to \beta$ and $\mu \to \infty$. Figure 2.1 depicts such a relationship between the ratio and inflation rate for $\beta = 0.9$.

Berentsen, Camera and Waller (2007) have a similar result. They show the welfare improvement due to banking is highest under moderate inflation, and approaches zero when $\mu \to \beta$ and $\mu \to \infty$. Their explanation for the reason also applies here. When

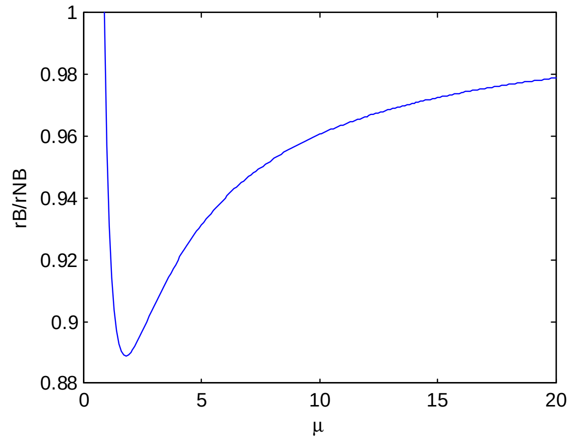

Figure 2.1: Ratio of Capital Rental Rates - r_B/r_{NB}

$\mu \to \beta$, the opportunity cost of holding money is low, and agents can insure themselves by holding money to a large extent, even without banks. When inflation rate is very high, money is not valued anyway, so that the presence of banking cannot improve the situation too much. Only when inflation is at intermediate level, banking has the greatest effect.

Lastly, it turns out the welfare gains of banking come from interest payments on deposits, not from relaxing borrowers' liquidity constraints. To see this, one can consider a deviation from equilibrium path as follows. If one agent decides not to borrow from the bank when he turns out to be a consumer in night, he must work more in the day to accumulate his own cash balance, in order to afford the same amount of consumption/investment in night without borrowing. However, he can still deposit all his idle money in night when he turns out to be a producer.

In stationary equilibrium, each consumer borrows l^+. During the day, if an agent decides not to borrow in night if he becomes a consumer, he must work more to earn the l^+ units of money now and carry it into night, so as to keep his night consumption/investment unaffected. By doing so, his day utility decreases by $\frac{1}{\phi}l^+$. In the night, if he is a consumer, he can afford the same consumption/investment without borrowing; if he is a producer, he works the same as others but has more money to deposit into the bank. In the next day, if previously a consumer, he can save the interest payment of $(1+i)l^+$; if previously a producer, he can get more interest payment by $(1+i)l^+$. In either case, his utility increases by $\frac{1}{\phi^+}(1+i)l^+$. The net welfare change due to this deviation across periods is $-\frac{1}{\phi}l^+ + \beta\frac{1}{\phi^+}(1+i)l^+$. Since in stationary equilibrium, $\phi^+ = \mu\phi$ and $1 + i = \frac{\mu}{\beta}$, the net welfare change turns out to be zero. In this quasi-linear environment, the dynamic problem can be decomposed into a sequence of identical static problems. Therefore, if this deviation is repeated over time, the lifetime welfare change would also be zero.

In this deviation strategy, the consumer's liquidity constraint is not relaxed by banking service (because his does not borrow), but he can receive interest payment for his deposit (when he is a producer). I have shown that this strategy does not change his welfare. Therefore, it proves that the welfare gains from banking, compared to the no-banking case, is not because of borrowers' liquidity constraints being relaxed, but because of banks paying interest on deposits.

The interest-paying banking sector has a similar function as the interest-paying government in Andolfatto (2010). In both cases, the interest payment on money balances raises rate of return on money, which is an incentive for producers to input more labor, since they are more willing to earn money by producing. Consequently, this yields higher output, consumption, investment, and welfare.

2.7 Conclusion

This chapter introduces capital to a quasi-linear monetary model, in order to study the role of banking in financing investment. As an extension to Berentsen, Camera and Waller (2007), same results regarding the welfare-improving role of banking have been derived. In addition, I obtain specific effects of banking on investment and capital accumulation, i.e., banking can mitigate under-investment and raise capital-labor ratio; this effect is greatest under moderate inflation. The benefit of banking does not come from providing liquidity to cash-constrained agents, but from paying interest on money deposits. Intuitively, raising rate of return on money closer to the "right" rate determined by the Friedman rule (i.e., $1/\beta$) makes money a better record-keeping device, hence welfare can be improved, in the sense that the friction (i.e., lack of record-keeping) can be overcome to a larger degree.

2.8 References

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Chapter 3

Costly Banking

3.1 Introduction

As is made well-known by Kocherlakota (1998), money is a substitute to the missing social record-keeping device. But why is such a record-keeping device missing in the first place? One answer to this may be that it is too costly to keep record of trade histories. Thus money emerges as a cost-efficient but imperfect substitute. This means record-keeping cost plays an important role in understanding monetary economy.

Berentsen, Camera and Waller (2007) introduce banking into a Lagos-Wright (2005) framework. They demonstrate that the existence of credit can improve welfare; this is not because the banking credit relaxes liquidity constraint, but because it enhances the return of money. In addition, they study the implications of default in credit market on monetary policy. In doing these, they assume costless banking, i.e., there is no cost related with recording financial histories. However, an important part of banking business is information process, which is resource-consuming. In particular, the cost related to loan making, such as screening, monitoring and auditing, must be borne by borrowers (usually) in the form of interest rate spread between loans and deposits. Naturally it helps to explicitly study such a banking cost, or financial record-keeping cost, in order to better understand the role of banking in a monetary economy.

I introduce heterogeneity in consumer preferences and consider a particular friction - a fixed cost related to borrowing. To be specific, when borrowing from a bank, the borrower has to pay a fixed amount of consumption goods, which will be used by the bank in keeping account and ensuring repayment. Otherwise, the environment is the same as in Berentsen, Camera and Waller (2007). The new ingredients make banking decisions more complex, and the existence of banking becomes an endogenous equilibrium outcome. In Berentsen, Camera and Waller (2007), costless banking is always used by consumers, and they take loans of equal amount from banks. In this study, depending on individual preferences and banking cost, a consumer makes decisions on: whether to use banking or not; if use banking, whether to borrow or deposit, and how much to borrow/deposit. Consumers' banking decisions will affect market interest rate, which eventually determines whether banking can exist or not. By studying such a model, I hope to look into the effects of intermediation cost on economic allocations and welfare, as well as its monetary policy implications.

This study is comparable to Chiu and Meh (2008). They develop a search-theoretic model to study the same interaction between costly banking and monetary policy. When doing so, they build the model on the setup in Silviera and Wright (2007) which characterizes a market for production projects ("ideas") that are used as an input for production. A more straightforward method is adopted in this chapter. It turns out such a simplification yields similar results. My main findings are as follows.

Emergence of banking is an endogenous equilibrium result. Depending on monetary policy and record-keeping cost, banking may or may not be used. It exists only with relatively high inflation and relatively low cost. Under low inflation, agents are less likely to be liquidity constrained; and high banking cost discourages borrowing. In both cases, interest rate would be driven down to zero, which makes depositing not worthwhile. As a result, costly banking cannot exist.

When costly banking does exist, it may improve or reduce welfare relative to the case without banking. Low but positive interest rate encourages borrowing, but the resulting increase in social deadweight loss due to banking cost actually lowers *ex ante* welfare. With higher interest rate and inflation rate, the benefit of banking becomes dominant and welfare is improved upon no-banking case.

With inflation rate or banking cost going up, more consumers would choose not to deal with banks, neither depositing nor borrowing. This enlarging proportion of agents outside banking services means more people are liquidity constrained, hence larger welfare loss.

Higher inflation rate depresses consumption level of all consumers, while the reaction to banking cost is quite different and depends on monetary policy. With low inflation, higher banking cost would raise consumption of all consumers, whether a bank user or not; with high inflation, bank users would consume more while those who stay away from banking would consume less.

Inflation rate and banking cost have opposite effects on nominal interest rate. It falls with higher banking cost since borrowing is discouraged. While higher inflation rate causes interest rate to rise, the usual Fisher relationship no longer holds. Due to the fixed borrowing cost, there is always a possibility of being liquidity constrained while banking cannot help. Such a possibility breaks the tie between inflation rate, nominal interest rate and real interest rate represented by a standard Fisher equation.

Inflation is less harmful with banking than without banking, since the liquidity shortage problem caused by inflation can be alleviated with aid of banking.

3.2 Environment

3.2.1 Basic Environment

Time is discrete and the horizon is infinite. Each period t is divided into two subperiods: day and night. There is measure one of infinitely-lived agents who are *ex ante* identical and indexed by $j \in [0, 1]$.

For each agent, preferences are defined over stochastic sequences $\{q_t, x_t\}$, where q_t denotes consumption in the day, and x_t consumption in the night. The expected utility function is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U_t(q_t) + x_t \right]$$

with discount rate between two days $0 < \beta < 1$.

There are preference/technology shocks which are realized at the beginning of each day. The shocks are i.i.d. across agents and over time. With probability 1/2, an agent wants to consume but cannot produce; with probability 1/2, an agent can produce but does not want to consume. Among consumers, there is another preference shock representing different degree of consumption demand. Therefore, the day utility function can be written as

$$U_t(q_t) = \begin{cases} \varepsilon_t u(q_t) & \text{w.p. } 1/2 \quad \text{(consumer)} \\ -c(q_t) & \text{w.p. } 1/2 \quad \text{(producer)} \end{cases},$$

where $\varepsilon_t \in [\varepsilon_{L,}\varepsilon_H]$ and is i.i.d. across consumers and time, with c.d.f. $F(\varepsilon)$ and p.d.f. $f(\varepsilon)$. The utility function $u(q)$ has properties: $u'' < 0 < u'$, $\lim_{q\to 0}u'(q) = \infty$ and $\lim_{q\to\infty}u'(q) = 0$. And the cost function $c(q)$ satisfies $c' > 0$ and $c'' \geq 0$. Each agent can both consume and produce in the night: consume if $x_t > 0$; produce if $x_t < 0$.

Agents are anonymous and lack commitment in days. Therefore, money is essential in the sense that it is necessary to facilitate exchange in day goods market. Now assume each agent is initially endowed with M units of fiat money. During the day, consumers can buy output from producers for consumption with their real money balances. And producers are willing to incur cost to produce since their output can be sold for money, which could be exchanged for consumption goods in the future when they are able to consume. In this way, day production and consumption are made feasible with the aid of fiat money. Note the night subperiod is simply used to rebalance each agent's money holdings such that the distribution of money is degenerate. This tractability is the merit of quasi-linear preferences. Suppose there is a (per capita) money injection T by the government at the beginning of each night, such that the money growth rate is $\mu \equiv M^+/M$. (To simplify notations, I drop time subscript t for all current variables, and add a superscript "+" to denote time period $t+1$ variables.) And I restrict $\mu \geq \beta$.

3.2.2 Banking

Berentsen, Camera and Waller (2007) incorporate banking into a monetary model. The banks can record financial histories of agents, hence they can accept nominal deposits from producers and make nominal loans to consumers. However, the ability of record-keeping is restricted to financial transactions, from which trading histories of agents cannot be inferred. Therefore, money is still needed to facilitate exchange between agents in output transactions. In this way, credit and money can coexist.

The financial intermediation is mostly the same in the current model. To be specific, the banking service is provided by perfectly competitive banks. Banks attract deposits from agents in the day right after preference shocks are realized, promising (net) interest rate i_d which will be paid in the night. The deposited money is then loaned to borrowers in the day at interest rate i_l, and the loan will be repaid in the night. Credit market closes before the production/transaction takes place. As a result, any sales receipt or unspent money has to be held into the night, with no interest earned. It is emphasized that banking is costly because of a fixed cost associated with borrowing. Whenever a loan is made, the borrower has to pay a fixed cost of k units of day goods, and the bank will use it up for record-keeping purposes. The banking cost observed in reality is in form of loan-deposit interest rate spread. Here a fixed cost simplifies the analysis while keeping the same essence of the friction of banking cost. Note due to the quasi-linear environment one only needs to consider one-period credit, since there is no point in spreading repayment or redemption of debt over periods in order to smooth periodic utility.

3.3 First-best Allocation

As a benchmark, I first consider the first-best allocation that a planner can implement in absence of all kinds of frictions. Since every agent is *ex ante* identical, it is natural to assume that the planner treats everyone equally. And I consider a stationary allocation. Due to the quasi-linearity in night consumption, any lottery scheme in $\{x\}$ satisfying $E_0 x(j) = 0$ and night resource constraint $\int_0^1 x(j)\, dj = 0$ can be a solution. Therefore, the night subperiods are irrelevant and a trivial solution is $x(j) = 0$ for all agents. A planner only needs to consider consumption/production decision in days.

After the preference/technology shocks are realized, a planner asks each agent who turns out to be a producer to produce q_p. Conditional on the realization of preference shock ε, each consumer is instructed to consume q_ε. For a planner, to maximize a representative agent's lifetime utility is equivalent to solve such a static problem:

$$\max_{q_p, \{q_\varepsilon\}} \left\{ \frac{1}{2} \int_{\varepsilon_L}^{\varepsilon_H} \varepsilon u(q_\varepsilon)\, dF(\varepsilon) - \frac{1}{2} c(q_p) \right\},$$

subject to the day resource constraint

$$\int_{\varepsilon_L}^{\varepsilon_H} q_\varepsilon dF(\varepsilon) = q_p.$$

It is easy to see that the first-best allocation $\left(q_p^*, \{q_\varepsilon^*\}\right)$ is characterized by

$$\varepsilon u'(q_\varepsilon^*) = c'\left(q_p^*\right), \tag{3.1}$$

$$\int_{\varepsilon_L}^{\varepsilon_H} q_\varepsilon^* dF(\varepsilon) = q_p^*. \tag{3.2}$$

3.4 Model

3.4.1 Decision-making of Banks

A representative bank accepts nominal deposits d, paying the nominal interest rate i_d, and makes nominal loans l at nominal rate i_l. The banking sector is perfectly competitive with free entry, so banks take these rates as given. Here banks are not allowed to issue banknotes. Only fiat money is accepted in day transactions. Therefore, The amount of loans is limited to not exceed deposit amount. For every loan, the bank charges k units of day goods which will be used to fully compensate for record-keeping cost. Because of symmetry, this charge is the same for all banks. But there will be no charge related to deposits. Each bank receives revenue from loan interest and the cost is interest repayment on deposits. Given deposits d, a bank solves the following problem to maximize profit:

$$\max_{l} \left\{ i_l l - i_d d \right\},$$

subject to

$$l \leq d.$$

In equilibrium, it cannot be the case that $i_l > i_d$, otherwise each bank would choose $l = d$ to make a positive profit, which cannot be true with free entry. If $i_l < i_d$, the best a bank can get is a negative profit by choosing $l = d$. Again this cannot be an equilibrium result. Therefore, it must be true that $i_l = i_d = i$. Given this, the bank's solution is

$$\begin{aligned} l &= d \text{ if } i > 0, \\ l &\leq d \text{ if } i = 0. \end{aligned} \tag{3.3}$$

In each case banks make zero profit.

3.4.2 Decision-making of Agents

Night Decision-making

During the night, an agent receives lump-sum transfer T of money from government, and chooses how much to consume/produce. At the same time, she has to settle debt payment or redeem deposit, as well as rebalance money holdings that she will take into next day. Since $i_l = i_d = i$ in equilibrium, the financial position can be fully revealed by banking account balance b. When $b > 0$, the agent has made a deposit; when $b < 0$, she has made a loan; when $b = 0$, she neither deposited nor borrowed during the day. The value of an agent entering a night with nominal money balance $\widehat{m}$ and banking account balance b is

$$W(\widehat{m}, b) = \max_{x, m^+} \left\{ x + \beta V(m^+) \right\},$$

$$\text{s.t. } \widehat{m} + (1+i)b + T = \phi x + m^+,$$

where ϕ is the price of consumption goods in night transactions, and $V(m^+)$ is the value of entering next day with nominal money balance m^+. After eliminating day consumption x by the budget constraint, one can rewrite the value function as

$$W(\widehat{m}, b) = \max_{m^+} \left\{ \frac{1}{\phi} \left[\widehat{m} + T + (1+i)b - m^+ \right] + \beta V(m^+) \right\}.$$

The first order condition (FOC) is

$$\frac{1}{\phi} = \beta V'(m^+). \tag{3.4}$$

A constant marginal value of money holdings means that money holding decision is independent of previous balances. Assuming a concave function $V(\cdot)$ (which turns out to be true in equilibrium), there is a unique m^+ to solve this equation. As a result, the distribution of money is degenerate at the beginning of next day.

The envelope conditions are

$$\frac{\partial W(\widehat{m}, b)}{\partial \widehat{m}} = \frac{1}{\phi}, \tag{3.5}$$

$$\frac{\partial W(\widehat{m}, b)}{\partial b} = \frac{1+i}{\phi}. \tag{3.6}$$

Day Decision-making

Producer: If an agent turns out to be a producer, he does not need money in the day. He would make decisions on how much to deposit and how much to produce. His continuation value of entering the day with nominal money balance m is:

$$V_p(m) = \max_{q_p, b} \left\{ -c(q_p) + W(m - b + pq_p, b) \right\},$$
$$\text{s.t. } b \leq m.$$

The FOC's are:

$$\begin{aligned} c'(q_p) &= \frac{p}{\phi}, \\ \lambda_p &= \frac{i}{\phi}, \end{aligned} \tag{3.7}$$

where λ_p is the Lagrange multiplier associated with the deposit constraint, and uses have been made of envelope conditions (3.5) and (3.6). If $i > 0$, then $\lambda_p > 0$ and $b = m$; the producer deposits all money that he has taken into the day.

The envelope condition is

$$V_p'(m) = \frac{1}{\phi} + \lambda_p = \frac{1+i}{\phi}. \tag{3.8}$$

Consumer: After the preference shock is realized, a consumer will make decisions on whether and how much to deposit or borrow, and how much goods to buy.

If a consumer with preference parameter ε and money balance m chooses not to borrow, then she can only use her own money to buy goods. Apparently she would not hold money unspent in hand for any given $i > 0$. Instead, she would make deposit to earn interest. Her problem is:

$$\max_q \left\{ \varepsilon u(q) + \frac{1+i}{\phi}(m - pq) \right\},$$
$$\text{s.t. } pq \leq m.$$

The FOC is

$$\varepsilon u'(q) = \frac{1+i}{\phi} p + \lambda_c,$$

where λ_c is the Lagrange multiplier associated with the liquidity constraint, and $\lambda_c \geq 0$ with $\lambda_c = 0$ if $pq < m$. If the liquidity constraint does not bind, a consumer buys $\widetilde{q}(\varepsilon)$ which is implicitly determined by

$$\varepsilon u'(\widetilde{q}) = \frac{1+i}{\phi} p, \tag{3.9}$$

and deposits $b = m - p\widetilde{q} > 0$. If she is liquidity constrained, she would consume $q = m/p$ with $b = m - pq = 0$.

If the same consumer decides to borrow, then the liquidity constraint is relaxed at the expense of a fixed cost. Her problem becomes:

$$\max_q \left\{ \varepsilon u(q) + \frac{1+i}{\phi}(m - pq) - \frac{1+i}{\phi} pk \right\}.$$

It is straightforward to show that she would consume $\widetilde{q}(\varepsilon)$ such that (3.9) is satisfied. As a result of borrowing, her banking account balance would be $b = m - p\widetilde{q} - pk < 0$.

Differentiating (3.9) with respect to ε results in

$$\widetilde{q}'(\varepsilon) = -\frac{u'(\widetilde{q})}{\varepsilon u''(\widetilde{q})} > 0.$$

Therefore, the "desired" consumption $\widetilde{q}$ is increasing in ε. When ε is small, $\widetilde{q}(\varepsilon)$ is also small so that $p\widetilde{q}(\varepsilon) \le m$; i.e., the consumer is not liquidity constrained. She buys $\widetilde{q}(\varepsilon)$ units of goods and deposits $b = m - p\widetilde{q}(\varepsilon) \ge 0$. When ε becomes larger, implied $\widetilde{q}(\varepsilon)$ is also larger. For big enough values of ε, $p\widetilde{q}(\varepsilon) > m$ and the consumer becomes constrained. She would decide whether to borrow to relax the liquidity constraint. If not borrow, she would just spend all money; if borrow, she would incur fixed cost and consume $\widetilde{q}(\varepsilon)$. Therefore, the value function of a consumer in the day is

$$V_c(m;\varepsilon) = \begin{cases} \varepsilon u(\widetilde{q}) + \frac{1+i}{\phi}(m - p\widetilde{q}), & \text{if } p\widetilde{q}(\varepsilon) \le m; \\ \max\left\{ \varepsilon u(\frac{m}{p}), \varepsilon u(\widetilde{q}) + \frac{1+i}{\phi}(m - p\widetilde{q}) - \frac{1+i}{\phi}pk \right\}, & \text{if } p\widetilde{q}(\varepsilon) > m. \end{cases}$$

It is clear that the value of ε determines whether a consumer is liquidity constrained, and whether to borrow if constrained. Moreover, since $\widetilde{q}(\varepsilon)$ is increasing in ε, the "tightness" of liquidity constraint is also increasing in ε. A consumer's solution must be based on her realization of ε value and take this form:

Lemma 3.1 *Given i, p and ϕ, for a consumer with real money balance m, there exist two threshold values ε_1 and ε_2, such that (1) If $\varepsilon_L \le \varepsilon \le \varepsilon_1$, the consumer consumes $q_\varepsilon = \widetilde{q}(\varepsilon)$ and deposits $b = m - p\widetilde{q}(\varepsilon) > 0$; (2) If $\varepsilon_1 < \varepsilon \le \varepsilon_2$, the consumer consumes $q_\varepsilon = m/p$ and $b = 0$; (3) If $\varepsilon_2 < \varepsilon \le \varepsilon_H$, the consumer incurs fixed cost to borrow; she consumes $q_\varepsilon = \widetilde{q}(\varepsilon)$ and her banking account balance is $b = m - p\widetilde{q}(\varepsilon) - pk < 0$.*

Next the task is to solve for ε_1 and ε_2. First I consider the case with $k=0$. Define

$$D_1(\varepsilon) \equiv \varepsilon u(\frac{m}{p}) - \left[\varepsilon u(\widetilde{q}(\varepsilon)) + \frac{1+i}{\phi}(m - p\widetilde{q}(\varepsilon))\right],$$

which is the value difference between two strategies: spending all money or consuming $\widetilde{q}(\varepsilon)$ with costless borrowing/saving. Taking first derivative of $D_1(\varepsilon)$ yields

$$\begin{aligned}
D_1'(\varepsilon) &= u(\frac{m}{p}) - u(\widetilde{q}(\varepsilon)) - \varepsilon u_1(\widetilde{q}(\varepsilon))\widetilde{q}_1(\varepsilon) + \frac{1+i}{\phi}p\widetilde{q}_1(\varepsilon) \\
&= u(\frac{m}{p}) - u(\widetilde{q}(\varepsilon)) - \left[\varepsilon u_1(\widetilde{q}(\varepsilon)) - \frac{1+i}{\phi}p\right]\widetilde{q}_1(\varepsilon) \\
&= u(\frac{m}{p}) - u(\widetilde{q}(\varepsilon)),
\end{aligned}$$

where the third equality is because of (3.9). It is clear that $D_1(\varepsilon)$ increases in ε if $\widetilde{q}(\varepsilon) < \frac{m}{p}$ and decreases if $\widetilde{q}(\varepsilon) > \frac{m}{p}$. Also the second derivative is

$$D_1''(\varepsilon) = -u'(\widetilde{q}(\varepsilon))\widetilde{q}'(\varepsilon) < 0.$$

Therefore, $D_1(\varepsilon)$ is strictly concave. Note that when $\widetilde{q}(\varepsilon) = \frac{m}{p}$, $D_1(\varepsilon) = 0$. In conclusion, there is a unique solution to $D_1(\varepsilon_1) = 0$ which is characterized by

$$\widetilde{q}(\varepsilon_1) = \frac{m}{p}. \tag{3.10}$$

With costless banking, it is always optimal to consume $\widetilde{q}(\varepsilon)$ rather than spend all money. When $\varepsilon < \varepsilon_1$, $\widetilde{q}(\varepsilon) < \widetilde{q}(\varepsilon_1) = \frac{m}{p}$. The consumer is not cash constrained. She would choose to consume $\widetilde{q}(\varepsilon)$ while making positive deposits. When $\varepsilon > \varepsilon_1$, $\widetilde{q}(\varepsilon) > \widetilde{q}(\varepsilon_1) = \frac{m}{p}$. The consumer is cash constrained. When banking is costless, she can borrow to relax the liquidity constraint so that she can still consume the "desirable" amount $\widetilde{q}(\varepsilon)$. Only when $\varepsilon = \varepsilon_1$, spending all money is equivalent to consuming $\widetilde{q}(\varepsilon)$, and the value difference $D_1(\varepsilon)$ achieves its maximum, 0.

The positive banking cost only affects cash constrained consumers with $\varepsilon > \varepsilon_1$. They have to decide whether to borrow to relax the constraint. Define

$$D_2(\varepsilon) \equiv \varepsilon u(\frac{m}{p}) - \left[\varepsilon u(\widetilde{q}(\varepsilon)) + \frac{1+i}{\phi}(m - p\widetilde{q}(\varepsilon)) - \frac{1+i}{\phi}pk\right],$$

which is the value difference between two strategies when a consumer is liquidity constrained: not borrowing and borrowing. When $D_2(\varepsilon) > 0$, the consumer is better

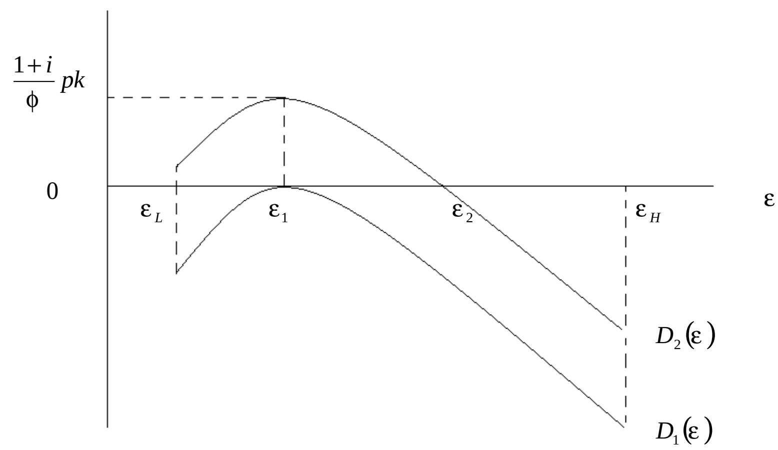

Figure 3.1: Threshold Values ε_1 and ε_2

off not borrowing and spending all her own money; when $D_2(\varepsilon) < 0$, she would choose to borrow and consume $\widetilde{q}(\varepsilon)$. Therefore, $D_2(\varepsilon) = 0$ defines the threshold value ε_2.

Note that

$$D_2(\varepsilon) = D_1(\varepsilon) + \frac{1+i}{\phi} pk,$$

which means it simply takes an upward shift of $D_1(\varepsilon)$ in a parallel manner to get $D_2(\varepsilon)$. Both curves are presented in Figure 3.1.

When observing the graph, it is tempting to say that the banking cost k is positively related to threshold value ε_2: with higher k, less constrained consumers would find it worthwhile to borrow, and only consumers with higher ε would borrow. However, the variation of k also has an impact on interest rate and price levels, which in turn affect consumer decisions. Realizing this, the general equilibrium effect of k is not easy to see at this stage.

Now the expected consumer value (before the ε shock is realized) can be written as

$$\begin{aligned} V_c(m) &= \int_{\varepsilon_L}^{\varepsilon_1} \left[\varepsilon u(\widetilde{q}(\varepsilon)) + \frac{1+i}{\phi} (m - p\widetilde{q}(\varepsilon)) \right] dF(\varepsilon) \\ &+ \int_{\varepsilon_1}^{\varepsilon_2} \varepsilon u(\frac{m}{p}) dF(\varepsilon) \\ &+ \int_{\varepsilon_2}^{\varepsilon_H} \left[\varepsilon u(\widetilde{q}(\varepsilon)) + \frac{1+i}{\phi} (m - p\widetilde{q}(\varepsilon)) - \frac{1+i}{\phi} pk \right] dF(\varepsilon). \end{aligned} \tag{3.11}$$

By envelope theorem, one can derive the expected marginal value of money if it is needed for consumption:

$$V_c'(m) = \int_{\varepsilon_L}^{\varepsilon_1} \frac{1+i}{\phi} dF(\varepsilon) + \int_{\varepsilon_1}^{\varepsilon_2} \varepsilon u'(\frac{m}{p})\frac{1}{p} dF(\varepsilon) + \int_{\varepsilon_2}^{\varepsilon_H} \frac{1+i}{\phi} dF(\varepsilon). \tag{3.12}$$

The intuition behind this expression is clear. If $\varepsilon_L \leq \varepsilon \leq \varepsilon_1$, the consumer deposits and every dollar of deposit yields utility of $\frac{1+i}{\phi}$. If $\varepsilon_1 < \varepsilon \leq \varepsilon_2$, the consumer spends all money to buy goods. One more dollar can buy $\frac{1}{p}$ units of goods which yield utility of $\varepsilon u'(\frac{m}{p})\frac{1}{p}$. If $\varepsilon_2 < \varepsilon \leq \varepsilon_H$, the consumer borrows. Taking one more dollar into the day can reduce loans by one dollar, which saves borrowing cost (including interest payment) by $1+i$ dollars; in terms of utility it is $\frac{1+i}{\phi}$.

3.4.3 Market Clearing

In the night, money market clearing condition is

$$m = M. \tag{3.13}$$

And the goods market clearing condition is

$$\int_0^1 x(j)\, dj = \frac{1}{2}\int_{\varepsilon_L}^{\varepsilon_H} x_\varepsilon dF(\varepsilon) + \frac{1}{2} x_p = 0. \tag{3.14}$$

During the day, goods market clearing condition is

$$\int_{\varepsilon_L}^{\varepsilon_1} \widetilde{q}(\varepsilon)\, dF(\varepsilon) + \int_{\varepsilon_1}^{\varepsilon_2} \frac{m}{p} dF(\varepsilon) + \int_{\varepsilon_2}^{\varepsilon_H} [\widetilde{q}(\varepsilon) + k]\, dF(\varepsilon) = q_p. \tag{3.15}$$

Banks make loans to borrowers with high ε, and deposits are from both producers and unconstrained consumers. Therefore, the credit market clearing condition $l = d$ becomes

$$\int_{\varepsilon_2}^{\varepsilon_H} [p\widetilde{q}(\varepsilon) + pk - m]\, dF(\varepsilon) = m + \int_{\varepsilon_L}^{\varepsilon_1} [m - p\widetilde{q}(\varepsilon)]\, dF(\varepsilon). \tag{3.16}$$

Multiplying both sides of (3.15) by p and adding (3.16) result in

$$2m = pq_p, \tag{3.17}$$

which is also the money market clearing condition.

3.4.4 Equilibrium

As is shown in Appendix, $V'(m)$ can be written as

$$\begin{aligned} V'(m) &= \frac{1}{2}V_p'(m) + \frac{1}{2}V_c'(m) \\ &= \frac{1+i}{\phi}\left[1 + \frac{1}{2}\int_{\varepsilon_1}^{\varepsilon_2}\left(\frac{\varepsilon}{\varepsilon_1} - 1\right)dF(\varepsilon)\right]. \end{aligned}$$

Combining it with lagged (3.4) leads to

$$\frac{1}{\phi^-} = \beta\frac{1+i}{\phi}\left[1 + \frac{1}{2}\int_{\varepsilon_1}^{\varepsilon_2}\left(\frac{\varepsilon}{\varepsilon_1} - 1\right)dF(\varepsilon)\right].$$

I focus on a stationary equilibrium such that all real variables, interest rate i, and threshold values ε_1 and ε_2 are constant. Then inflation rate $\mu = \frac{\phi}{\phi^-}$, which implies

$$\frac{\mu}{\beta} = (1+i)\left[1 + \frac{1}{2}\int_{\varepsilon_1}^{\varepsilon_2}\left(\frac{\varepsilon}{\varepsilon_1} - 1\right)dF(\varepsilon)\right]. \tag{3.18}$$

Now the usual Fisher equation $\frac{\mu}{\beta} = 1 + i$ no longer holds. Due to the fixed borrowing cost, there is always a possibility of being liquidity constrained while banking cannot help, i.e., when $\varepsilon \in (\varepsilon_1, \varepsilon_2)$. The presence of such a credit market friction breaks the tie between inflation rate, nominal interest rate and real interest rate represented by a standard Fisher equation.

Rearranging (3.18) yields

$$\frac{\frac{\mu}{\beta} - (1+i)}{1+i} = \frac{1}{2}\int_{\varepsilon_1}^{\varepsilon_2}\left(\frac{\varepsilon}{\varepsilon_1} - 1\right)dF(\varepsilon),$$

which equates the marginal cost and marginal benefit of liquidity. To understand this, consider an agent in night making decision on how much money to take into next day. The "subject" (gross) nominal interest rate is $\frac{\mu}{\beta}$ such that the inflation and time discounting can completely offset. But the market interest rate that can be earned in next day (if not liquidity constrained) is only $1 + i$. Therefore, the interest rate spread is equivalent to marginal cost of liquidity. Money is highly valued (i.e., more than $1+i$) by a consumer when she is trapped in liquidity constraint but finds it too costly to borrow (i.e., $\varepsilon_1 < \varepsilon \leq \varepsilon_2$) . In this sense, the right-hand side represents the marginal benefit of liquidity, or liquidity premium.

Combining (3.9) and (3.7) gives rise to

$$\varepsilon u'(\widetilde{q}(\varepsilon)) = (1+i)c'(q_p), \tag{3.19}$$

which defines an implicit function $\widetilde{q}(\varepsilon)$.

Combining (3.17) with (3.10) yields

$$\widetilde{q}(\varepsilon_1) = \frac{q_p}{2}. \tag{3.20}$$

Substituting (3.10) and (3.7) into $D_2(\varepsilon_2) = 0$ results in

$$\varepsilon_2 u(\widetilde{q}(\varepsilon_1)) = \varepsilon_2 u(\widetilde{q}(\varepsilon_2)) + (1+i)c'(q_p)\left[\widetilde{q}(\varepsilon_1) - \widetilde{q}(\varepsilon_2) - k\right]. \tag{3.21}$$

Considering (3.10), the goods market clearing condition (3.15) becomes

$$\int_{\varepsilon_L}^{\varepsilon_1} \widetilde{q}(\varepsilon)\, dF(\varepsilon) + \int_{\varepsilon_1}^{\varepsilon_2} \widetilde{q}(\varepsilon_1)\, dF(\varepsilon) + \int_{\varepsilon_2}^{\varepsilon_H} \left[\widetilde{q}(\varepsilon) + k\right] dF(\varepsilon) = q_p. \tag{3.22}$$

A stationary equilibrium is a collection of $(i, \varepsilon_1, \varepsilon_2, q_p, \{q_\varepsilon\})$ characterized by equations (3.18) through (3.22), while $\{q_\varepsilon\}$ follows the consumption rule specified by Lemma 3.1, i.e.,

$$q_\varepsilon = \begin{cases} \widetilde{q}(\varepsilon) & \varepsilon_L \leq \varepsilon \leq \varepsilon_1 \\ \widetilde{q}(\varepsilon_1) & \varepsilon_1 < \varepsilon \leq \varepsilon_2 \\ \widetilde{q}(\varepsilon) & \varepsilon_2 < \varepsilon \leq \varepsilon_H \end{cases}. \tag{3.23}$$

Once this equilibrium is obtained, other endogenous variables can be solved as shown in Appendix.

3.5 Economy without Banking

As a comparison, now I study properties of equilibrium in an economy with banking unavailable. Then I can say more about why and how presence of banking can affect economic allocations and welfare.

Without banking, unconstrained consumers consume quantity $\widetilde{q}(\varepsilon)$ and hold extra cash in hand into the night, while constrained consumers just spend all money they have taken into the day. It is straightforward to see that such an equilibrium, a collection of $(\varepsilon_1, q_p, \{q_\varepsilon\})$, is characterized by

$$\frac{\mu}{\beta} = 1 + \frac{1}{2}\int_{\varepsilon_1}^{\varepsilon_H} \left(\frac{\varepsilon}{\varepsilon_1} - 1\right) dF(\varepsilon), \tag{3.24}$$

$$\int_{\varepsilon_L}^{\varepsilon_1} \widetilde{q}\,(\varepsilon)\, dF(\varepsilon) + \int_{\varepsilon_1}^{\varepsilon_H} \widetilde{q}\,(\varepsilon_1)\, dF(\varepsilon) = q_p. \tag{3.25}$$

$$q_\varepsilon = \begin{cases} \widetilde{q}\,(\varepsilon) & \varepsilon_L \leq \varepsilon \leq \varepsilon_1 \\ \widetilde{q}\,(\varepsilon_1) & \varepsilon_1 < \varepsilon \leq \varepsilon_H \end{cases}, \tag{3.26}$$

where $\widetilde{q}\,(\varepsilon)$ is defined by

$$\varepsilon u'(\widetilde{q}\,(\varepsilon)) = c'(q_p). \tag{3.27}$$

Now I derive the welfare function of a representative agent in such a quasi-linear environment, with or without banking:

$$\begin{aligned} \mathcal{W} &= \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2}\int_{\varepsilon_L}^{\varepsilon_H} [\varepsilon u\,(q_\varepsilon) + x_\varepsilon]\, dF(\varepsilon) + \frac{1}{2}\left[-c(q_p) + x_p\right] \right\} \\ &= \frac{1}{1-\beta}\left[\frac{1}{2}\int_{\varepsilon_L}^{\varepsilon_H} \varepsilon u\,(q_\varepsilon)\, dF(\varepsilon) - \frac{1}{2}c(q_p) + \frac{1}{2}\int_{\varepsilon_L}^{\varepsilon_H} x_\varepsilon dF(\varepsilon) + \frac{1}{2}x_p\right] \\ &= \frac{1}{2\,(1-\beta)}\left[\int_{\varepsilon_L}^{\varepsilon_H} \varepsilon u\,(q_\varepsilon)\, dF(\varepsilon) - c(q_p)\right], \end{aligned} \tag{3.28}$$

where the last equality is because of night goods market clearing condition (3.14). Denote welfare with costly banking by $\mathcal{W}_{CB}$ and without banking by $\mathcal{W}_{NB}$.

Proposition 3.1 *In an equilibrium without banking,* $\frac{d\varepsilon_1}{d\mu} < 0$, $\frac{dq_p}{d\mu} < 0$, $\frac{d\mathcal{W}_{NB}}{d\mu} < 0$.

Proof. See Appendix. ■

When $\mu = \beta$, (3.24) yields $\varepsilon_1 = 1$, (3.25) and (3.27) coincide with first-best conditions. Holding money is a perfect way to insure against liquidity needs, and no one is liquidity constrained. With higher inflation rate, production is discouraged since producers are less willing to work to earn money given a lower rate of return. Meanwhile, the value of money $\left(\frac{1}{p} = \frac{q_p}{M}\right)$ is lowered, and more consumers become liquidity constrained, which leads to welfare loss. With banking unavailable, producers and unconstrained consumers (with $\varepsilon < \varepsilon_1$) have to hold idle cash in hands, but constrained consumers (with $\varepsilon > \varepsilon_1$) who highly value liquidity cannot get enough.

A credit market of money could potentially bring in mutual gains to both parties and hence enhance welfare. Due to anonymity and lack of commitment, such a nominal credit market cannot exist. Banking, a financial intermediation with record-keeping technology, can circumvent such frictions and make credit possible, as described in Berentsen, Camera and Waller (2007). However, there also exist frictions on credit market. Berentsen, Camera and Waller study default and credit rationing from aspects of banks. In their model the divisions of depositors and borrowers are fixed. In this chapter I look into implications of another credit market friction - financial record-keeping cost, focusing on consumers's banking decisions: whether to deal with banks, how much to deposit and how much to borrow.

3.6 Economy with Banking

In this section I examine the role of financial record-keeping cost in an economy where existence of banking is an *endogenous* result. For comparison, first I consider costless banking, a special case when $k = 0$. Next I investigate the case when $k > 0$. When the borrowing cost is too high, few consumers would find it worthwhile to borrow and the banking breaks down. If banking is not used, which emerges would be a pure monetary equilibrium without banking. Only when k takes intermediate values, costly banking exists.

3.6.1 Costless Banking

A special case is $k = 0$, i.e., banks can keep record of financial histories at no cost. Then (3.21) implies $\varepsilon_2 = \varepsilon_1$, hence the equilibrium reduces to a collection of $(i, \varepsilon_1, q_p, \{q_\varepsilon\})$ determined by

$$\frac{\mu}{\beta} = 1 + i, \tag{3.29}$$

$$\widetilde{q}(\varepsilon_1) = \frac{q_p}{2}, \tag{3.30}$$

$$\int_{\varepsilon_L}^{\varepsilon_H} \widetilde{q}(\varepsilon)\, dF(\varepsilon) = q_p, \tag{3.31}$$

$$q_\varepsilon = \widetilde{q}(\varepsilon), \tag{3.32}$$

where the function $\widetilde{q}(\varepsilon)$ is defined by

$$\varepsilon u'(\widetilde{q}(\varepsilon)) = \frac{\mu}{\beta} c'(q_p). \tag{3.33}$$

With the aid of costless banking, there are no liquidity constrained consumers. All consumers choose to consume their "desired" quantity $\widetilde{q}(\varepsilon)$. Meanwhile, those with $\varepsilon < \varepsilon_1$ deposit and those with $\varepsilon > \varepsilon_1$ borrow. With perfect credit market, liquidity premium of money does not exist, and the Fisher equation holds.

3.6.2 Costly Banking Not Used

When there exists banking cost, consumer decisions depend on the value of k. When k becomes larger, less consumers would choose to borrow, which drives down the interest rate in the credit market. When the cost is too high, to entice consumers to take loans, the interest rate would be driven down to zero. Then agents with extra cash to deposit would be indifferent between holding money in hand and making deposit into banks. When k exceeds a critical value, interest rate goes negative. Producers and unconstrained consumers would avoid paying interest on their deposits just by holding cash in hand, and the credit market breaks down. As an equilibrium result, the banking service is driven out of use due to the high cost. Moreover, a low enough inflation rate μ can bring interest rate down to zero, ensuing a pure monetary equilibrium without banking, even if such a financial record-keeping technology is available.

Proposition 3.2 *For any $\mu > \beta$, there exists a critical value $\overline{k}$ such that banking is not used when $k > \overline{k}$. For any $k > 0$, there exists a critical value $\underline{\mu}$ such that banking is not used when $\mu < \underline{\mu}$.*

Proof. For a given $\mu > \beta$, when k goes up, less consumers would find it worthwhile to borrow. For credit market to clear, interest rate will be driven down to induce consumers to borrow. Imposing $i = 0$ to the equilibrium conditions (3.18) through (3.22), one can solve for a collection of $\left(\overline{k}, \varepsilon_1, \varepsilon_2, q_p, \{q_\varepsilon\}\right)$. Due to continuity, $k > \overline{k}$ would result in negative interest rate.

For a given $k > 0$, when μ is lowered, value of money is raised and hence consumers are less likely to be liquidity constrained. More consumers making deposits and less consumers taking loans would drive down interest rate. Imposing $i = 0$ to the equilibrium conditions (3.18) through (3.22), one can solve for a collection of $\left(\underline{\mu}, \varepsilon_1, \varepsilon_2, q_p, \{q_\varepsilon\}\right)$. Due to continuity, $\mu < \underline{\mu}$ would result in negative interest rate. ■

3.6.3 Costly Banking

Next I focus on the case $0 < k \leq \overline{k}$ and investigate properties of the equilibrium with costly banking. Especially I look into the effects of k and μ, so as to understand the interaction of inflation and banking. To be specific, I want to answer the following questions:

1. How does the existence of costly banking depend on k and μ?
2. When costly banking exists, how does banking cost k affect output q_p, interest rate i, and banking/consumption patterns $\{q_\varepsilon\}$? What are the effects of μ?
3. How does availability of costly banking affect the welfare? Or how does $\mathcal{W}_{CB}$ compare with $\mathcal{W}_{NB}$ when k varies? when μ varies?

It seems difficult, if not impossible, to do comparative statics analytically. Thus I have to rely on numerical analysis. Specifically I choose functional forms generally used in the literature: $u(q) = 2q^{1/2}$ and $c(q) = q$, a uniform distribution for ε with $\varepsilon_L = 0, \varepsilon_H = 1$, and $\beta = 0.96$.

Existence of Costly Banking

The numerical analysis shows that $\overline{k}(\mu)$ is monotonically increasing in μ. With higher inflation, value of money is lowered. More consumers become constrained. Decrease in deposit provision and increase in loan demand would drive up the interest rate. Only a higher cost k can discourage borrowing and take interest rate down to hit lower bound.

It is also shown that $\underline{\mu}(k)$ is strictly increasing in k. Less consumers borrow with higher k, which would drive down the interest rate. Then higher inflation rate is needed as an offset force to encourage consumers to borrow, so as to keep interest rate from falling under zero.

It turns out $\overline{k}(\mu)$ and $\underline{\mu}(k)$ are a pair of inverse functions. Imposing $i = 0$ to the equilibrium conditions (3.18) through (3.22), I can solve for a locus of $\left(\underline{\mu}, \overline{k}\right)$, which

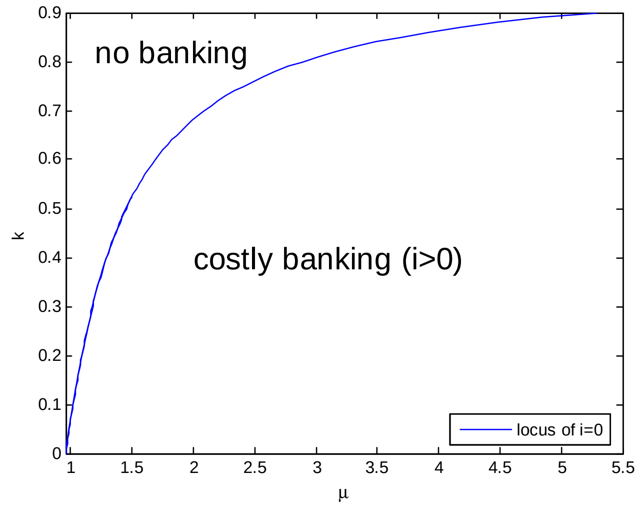

Figure 3.2: Existence of Costly Banking

is depicted in Figure 3.2. It separates the (μ, k) plane into two parts. Costly banking exists with a combination of relatively high inflation rate and relatively low banking cost. Banking is not used when inflation is too low and/or borrowing cost is too high.

Allocation Effects

First I investigate the effects of inflation rate change. Figure 3.3 plots the effects of μ variation for $k = 0.3$.

When inflation rate rises, real money balance falls; more consumers become liquidity constrained (ε_1 decreases), both output and welfare decrease. These effects are the same as in the economy when banking is unavailable. When costly banking exists, inflation also affects interest rate and consumers' consumption/banking decisions.

The response of interest rate is easy to understand. Considering the decrease in money value, real cost of interest is lowered, which encourages borrowing. As a result, higher demand for loans drives up the interest rate.

As can be seen from (3.19), higher interest rate means higher opportunity cost of consumption. Therefore, the "desired" consumption level $\widetilde{q}(\varepsilon)$ would drop. Since all bank customers, no matter a depositor (with $\varepsilon < \varepsilon_1$) or a borrower (with $\varepsilon > \varepsilon_2$), consume $\widetilde{q}(\varepsilon)$ in equilibrium, they all would decrease consumption when inflation rate goes up. Those constrained consumers who spend all money also consume less, since

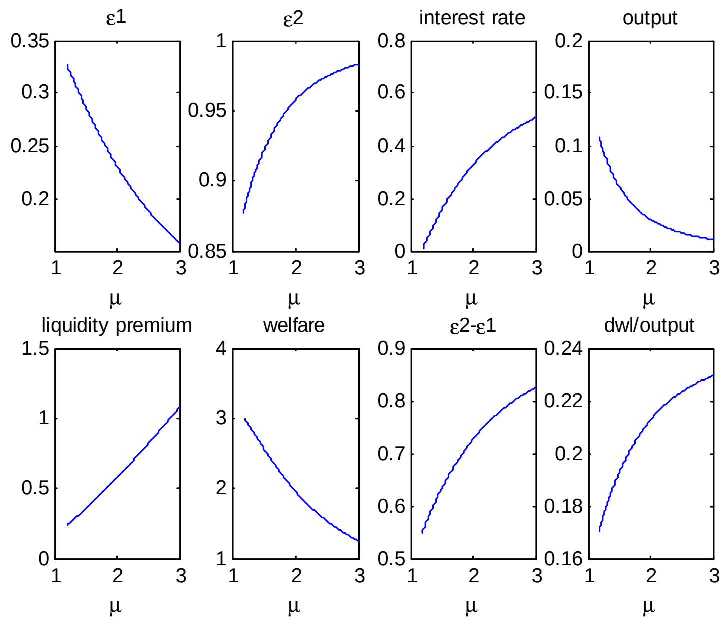

Figure 3.3: Comparative Statics - μ

their real money balances fall due to higher inflation. In summary, all consumers consume less when inflation rate rises.

For those constrained consumers (with $\varepsilon > \varepsilon_1$), the gross borrowing cost $(1+i)\,k$ increases with interest rate, while the benefit from borrowing drops because of decrease in "desired" level $\widetilde{q}\,(\varepsilon)$. Both factors contribute to explain the rise in ε_2 - less consumers find it worthwhile to borrow. Meanwhile more people choose not to use banking, as represented by an increase in $\varepsilon_2 - \varepsilon_1$. In summary, with higher inflation, less people - both depositors and borrowers - use banking services.

With a higher probability of being constrained, the liquidity premium of money is higher. Although less people borrowing results in a decrease in social deadweight loss on total record-keeping cost, due to a sharper decline in output the ratio of deadweight loss to output is increasing, which adds to the welfare loss.

As it turns out, banking cost has different effects under different inflation rates. Figure 3.4 plots the effects of k variation given a low inflation rate ($\mu = 1.02$):

It is natural that higher banking cost discourages borrowing - an increase in ε_2. With less borrowers, interest rate would be driven down. Such a decrease in interest rate has general effects on all agents. As nominal interest rate stands for marginal benefit of savings, it affects every consumer's consumption decision. With a lower

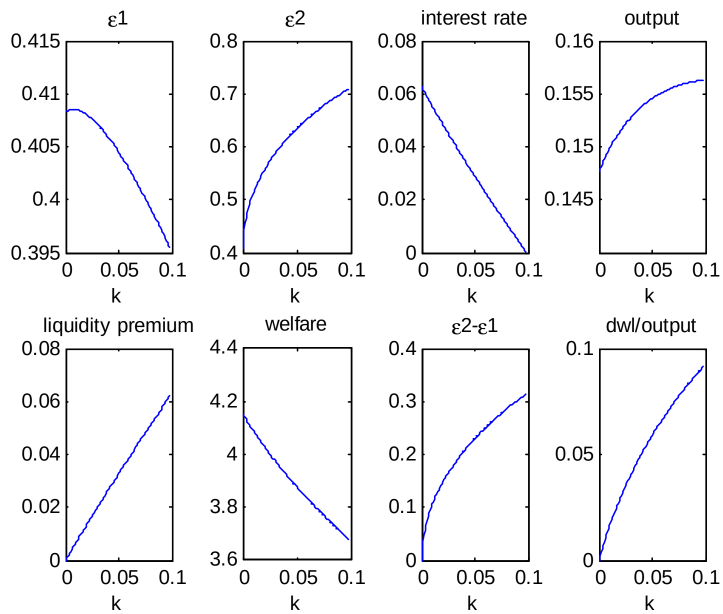

Figure 3.4: Comparative Statics - k ($\mu = 1.02$)

interest rate, the unrestricted optimal consumption level $\widetilde{q}(\varepsilon)$ rises, causing higher output as a general equilibrium result, hence the real money balance ($\frac{M}{p} = \frac{q_p}{2}$) also rises. Therefore, it is hard to tell how the threshold value ε_1 would change. In this numerical example, ε_1 slightly increases only with very small k, and drops sharply with larger k. It seems that the interest rate effect dominates the real money balance effect such that liquidity constraint is more likely to be tightened. Always there are more people who choose not to use banking, as represented by an increase in $\varepsilon_2 - \varepsilon_1$. In general, with higher banking cost, less people - both depositors and borrowers - use banking services.

Opposite to the case of μ increase, higher k would raise consumption level of all consumers. For banking users, they all consume $\widetilde{q}(\varepsilon)$ which is higher due to the interest rate effect. Consumption of those who just spend their own money is also higher due to the real money balance effect.

With a higher probability of being constrained, the liquidity premium of money is higher. The social deadweight loss on record-keeping takes a larger portion of total output. In a low inflation environment, it explains the welfare loss to a large extent. (In this example, while deadweight loss ratio increases to almost 10%, the welfare decreases by about 10%.)

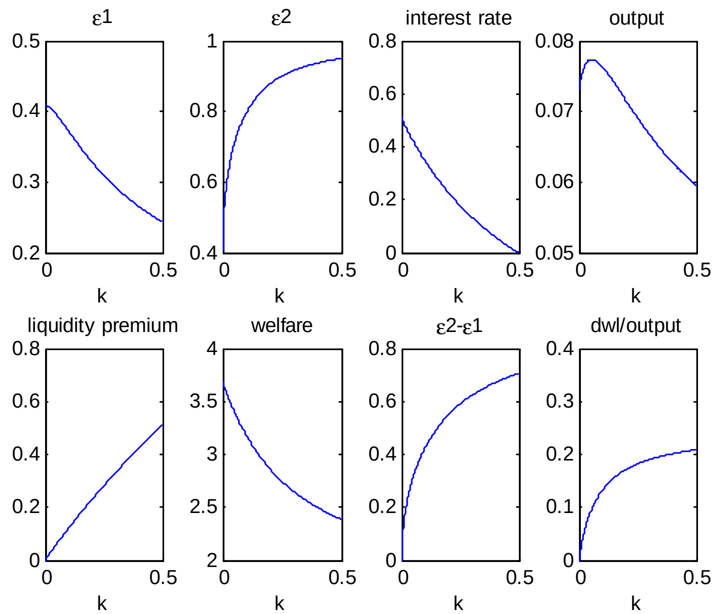

Figure 3.5: Comparative Statics - k $(\mu = 1.45)$

Figure 3.5 plots the effects of k variation for high inflation $(\mu = 1.45)$. The most striking difference from Figure 3.4 is that output falls when k becomes large. In this numerical analysis, I assume linear $c(q)$ function, hence (per worker) output q_p is pinned down by output market clearing condition. The interest rate effect is the same as in low inflation case: $\widetilde{q}(\varepsilon)$ increases because opportunity cost of consumption is lowered. All bank users would consume more. However, the problem of liquidity constraint is worse with high inflation, represented by a wider range of $(\varepsilon_1, \varepsilon_2)$. When k increases, more and more consumers become constrained and choose not to borrow. When the ratio of underconsumption population is high enough, the increase in consumption from unconstrained consumers is more than offset, and aggregate consumption demand is lowered, hence output falls and real money balance falls, exacerbating the liquidity shortage problem. Banking users would enhance consumption due to interest rate effect, but those who go away from banks would consume less due to the fall in real money balances.

Welfare Effects

Figure 3.6 shows the welfare effects of inflation for four cases: no banking, costless banking $(k = 0)$, costly banking with low k $(k = 0.05)$ and high k $(k = 0.3)$.

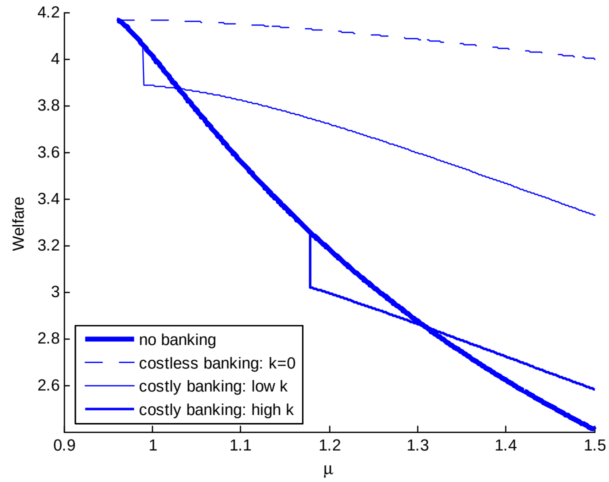

Figure 3.6: Welfare Effects - μ

It can be seen that the presence of banking does not necessarily improve welfare. For a given $k > 0$, when inflation rate is relatively low, the liquidity shortage problem is not too serious. Although emergence of banking can alleviate this problem, the resulting benefit is too small to offset the deadweight loss due to banking cost. Therefore, welfare actually declines when people begin using banking, which is accompanied by a low interest rate. When inflation rate is high, the benefit of banking becomes dominant and welfare is improved upon no banking case. In addition, the comparison of slopes shows that $\left|\frac{d\mathcal{W}_{CB}}{d\mu}\right| < \left|\frac{d\mathcal{W}_{NB}}{d\mu}\right|$, i.e., inflation is less harmful with banking than without banking. This is because the liquidity shortage problem caused by inflation can be alleviated by borrowing, hence less welfare loss.

Although banking can always improve welfare under high inflation, the magnitude of welfare improvement is not always increasing with μ. Figure 3.7 depicts the welfare difference between two economies: one with banking as an endogenous result ($\mathcal{W}_B$) and the other with banking prohibited ($\mathcal{W}_{NB}$), for three values of k.

When $k = 0$, the welfare improvement due to costless banking first rises and then falls, reaching a maximum at an "intermediate" value of inflation rate. This is exactly the same as Figure 2 in Berentsen, Camera and Waller (2007) and is because of the same reason: When μ is low, relying on money only is a good way to insure against liquidity needs, and liquidity shortage problem is not severe; so the welfare improvement of banking is not so significant. When μ is too high, money is not valued anyway, hence presence of nominal credit market cannot help much. Therefore,

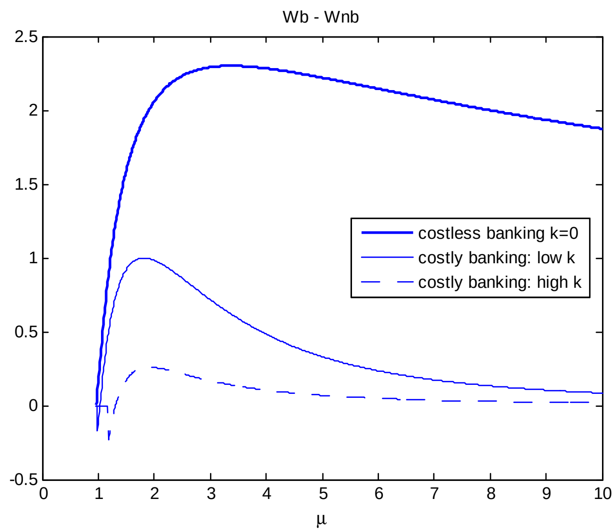

Figure 3.7: Welfare Effects of Banking

banking improves welfare the most only when μ takes an intermediate value. The existence of banking cost does not change this relationship, except for the welfare decrease related to low values of μ. This complexity is due to the welfare loss caused by fixed banking cost, as illustrated by Figure 3.6.

Figure 3.8 shows the welfare effects of banking cost for three inflation rates: $\mu = \beta, \mu = 1.02$, and $\mu = 1.25$. For a given inflation rate $\mu > \beta$, welfare falls with higher k. The direct effect is more social deadweight loss on each loan-making. And the indirect welfare loss is because less consumers choose to borrow and remain constrained. When interest rate is close to zero, welfare is actually even lower than if banking is not present. With high enough k, deadweight loss due to banking cost outweighs the benefit brought by banking services, hence banking causes net welfare loss, relative to the case when no one chooses to use banking.

Figure 3.9 plots the distribution of three types of equilibrium. The two loci are $i = 0$ and $\mathcal{W}_{CB} = \mathcal{W}_{NB}$ respectively. They divide the (μ, k) plane into three areas: no banking, "bad banking" ($\mathcal{W}_{CB} < \mathcal{W}_{NB}$) and "good banking" ($\mathcal{W}_{CB} > \mathcal{W}_{NB}$).

Intuitively, costly banking is welfare-improving with relatively high inflation (liquidity problem is severe) and relatively low borrowing cost (less intermediation cost is wasted). When inflation is relatively low and banking cost is relatively high, from the perspective of the society, it is not worthwhile to provide liquidity in form of costly banking.

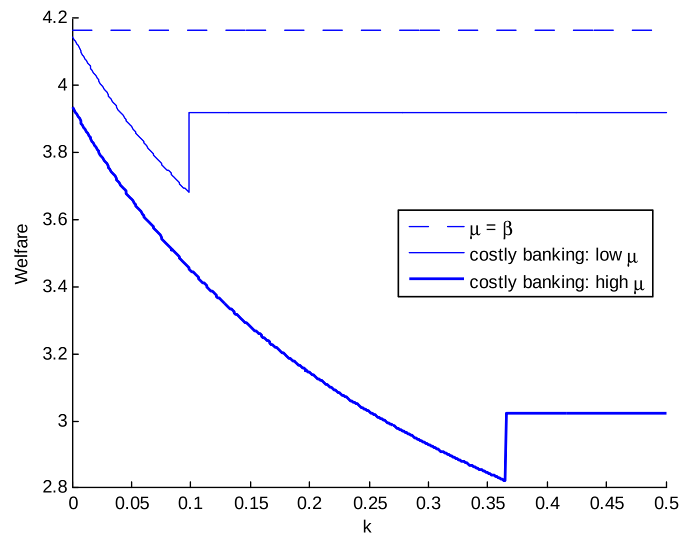

Figure 3.8: Welfare Effects - k

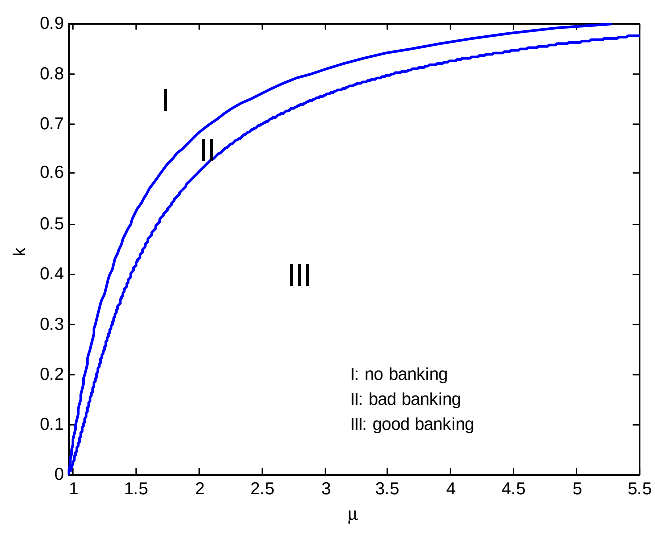

Figure 3.9: Distribution of Three Types of Equilibrium

3.7 Conclusion

This chapter studies the role of a banking cost related to its record-keeping function, and investigates the interaction between banking and monetary policy. It reveals that banking emerges endogenously only with relatively high inflation and relatively low cost. Costly banking may exist in equilibrium even when it is welfare-reducing. Inflation is less harmful with banking than without banking.

3.8 References

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3.9 Appendix

3.9.1 Derivation of $V'(m)$

Equation (3.12) can be rewritten as

$$\begin{aligned} V_c'(m) &= \frac{1+i}{\phi} + \frac{1}{p}\int_{\varepsilon_1}^{\varepsilon_2}\left[\varepsilon u'(\frac{m}{p}) - \frac{1+i}{\phi}p\right] dF(\varepsilon) \\ &= \frac{1+i}{\phi} + \frac{1}{p}\int_{\varepsilon_1}^{\varepsilon_2}\left[\varepsilon u'(\widetilde{q}\,(\varepsilon_1)) - \varepsilon_1 u'(\widetilde{q}\,(\varepsilon_1))\right] dF(\varepsilon) \\ &= \frac{1+i}{\phi} + \frac{1}{p}u'(\widetilde{q}\,(\varepsilon_1))\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon-\varepsilon_1)\, dF(\varepsilon) \\ &= \frac{1+i}{\phi} + \frac{1+i}{\phi}\frac{1}{\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon-\varepsilon_1)\, dF(\varepsilon), \end{aligned}$$

where (3.10) and (3.9) are used to get the second equality, while (3.9) is used again to derive the fourth equality. Recalling (3.8), the marginal value of money in day is

$$\begin{aligned} V'(m) &= \frac{1}{2}V_p'(m) + \frac{1}{2}V_c'(m) \\ &= \frac{1}{2}\frac{1+i}{\phi} + \frac{1}{2}\left[\frac{1+i}{\phi} + \frac{1+i}{\phi}\frac{1}{\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}(\varepsilon-\varepsilon_1)\, dF(\varepsilon)\right] \\ &= \frac{1+i}{\phi}\left[1 + \frac{1}{2}\int_{\varepsilon_1}^{\varepsilon_2}\left(\frac{\varepsilon}{\varepsilon_1} - 1\right) dF(\varepsilon)\right]. \end{aligned}$$

3.9.2 Solutions to Other Endogenous Variables in Equilibrium

Once the equilibrium $(i, \varepsilon_1, \varepsilon_2, q_p, \{q_\varepsilon\})$ is solved, the day price level p can be derived from (3.10) and (3.13):

$$p = \frac{M}{\widetilde{q}\,(\varepsilon_1)},$$

and then night price level ϕ by (3.7):

$$\phi = \frac{M}{\widetilde{q}\,(\varepsilon_1)\, c'(q_p)}.$$

The night asset position and allocation depend on the type of an agent in the preceding day. Money balance taken into the night is given by

$$\begin{aligned}\widehat{m}_p &= 2M,\\ \widehat{m}_\varepsilon &= 0.\end{aligned}$$

For a producer, his banking account balance at the beginning of night is

$$b_p = M.$$

For a consumer, Lemma 3.1 determines that

$$b_\varepsilon = \begin{cases} M - p\widetilde{q}(\varepsilon) & \varepsilon_L \leq \varepsilon \leq \varepsilon_1 \\ 0 & \varepsilon_1 < \varepsilon \leq \varepsilon_2 \\ M - p\widetilde{q}(\varepsilon) - pk & \varepsilon_2 < \varepsilon \leq \varepsilon_H \end{cases}.$$

The government budget constraint is

$$T = M^+ - M = (\mu - 1)\, M.$$

Plugging it into agents's night budget constraint, one obtains the consumption levels for producers:

$$x_p = \frac{2+i}{\phi} M = (2+i)\, c'(q_p)\widetilde{q}(\varepsilon_1),$$

and for consumers:

$$\begin{aligned} x_\varepsilon &= \begin{cases} \frac{1}{\phi}\left[iM - (1+i)p\widetilde{q}(\varepsilon)\right] & \varepsilon_L \leq \varepsilon \leq \varepsilon_1 \\ -\frac{1}{\phi}M & \varepsilon_1 < \varepsilon \leq \varepsilon_2 \\ \frac{1}{\phi}\left[iM - (1+i)\left(p\widetilde{q}(\varepsilon) + pk\right)\right] & \varepsilon_2 < \varepsilon \leq \varepsilon_H \end{cases} \\ &= \begin{cases} ic'(q_p)\widetilde{q}(\varepsilon_1) - (1+i)\, c'(q_p)\widetilde{q}(\varepsilon) & \varepsilon_L \leq \varepsilon \leq \varepsilon_1 \\ -c'(q_p)\widetilde{q}(\varepsilon_1) & \varepsilon_1 < \varepsilon \leq \varepsilon_2 \\ ic'(q_p)\widetilde{q}(\varepsilon_1) - (1+i)\, c'(q_p)\left(\widetilde{q}(\varepsilon) + k\right) & \varepsilon_2 < \varepsilon \leq \varepsilon_H \end{cases}. \end{aligned}$$

3.9.3 Proof of Proposition 3.1

For a given μ, a unique ε_1 can be solved from (3.24). Taking derivative w.r.t. μ leads to

$$\frac{d\varepsilon_1}{d\mu} = -\frac{2\varepsilon_1^2}{\beta \int_{\varepsilon_1}^{\varepsilon_H} \varepsilon dF(\varepsilon)} < 0.$$

Differentiating (3.25) w.r.t. ε_1 yields

$$\frac{dq_p}{d\varepsilon_1} = \frac{d\widetilde{q}\,(\varepsilon_1)}{d\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_H} dF(\varepsilon). \tag{3.34}$$

Since (3.27) holds for all $\varepsilon \in [\varepsilon_{L,}\varepsilon_H]$,

$$\varepsilon_1 u'(\widetilde{q}\,(\varepsilon_1)) = c'(q_p). \tag{3.35}$$

Differentiating w.r.t. ε_1 yields

$$u'(\widetilde{q}\,(\varepsilon_1)) + \varepsilon_1 u''(\widetilde{q}\,(\varepsilon_1))\frac{d\widetilde{q}\,(\varepsilon_1)}{d\varepsilon_1} = c''(q_p)\frac{dq_p}{d\varepsilon_1}. \tag{3.36}$$

Solving equations (3.34) and (3.36) results in

$$\frac{d\widetilde{q}\,(\varepsilon_1)}{d\varepsilon_1} = -\frac{u'(\widetilde{q}\,(\varepsilon_1))}{\varepsilon_1 u''(\widetilde{q}\,(\varepsilon_1)) - c''(q_p)\int_{\varepsilon_1}^{\varepsilon_H} dF(\varepsilon)} > 0,$$

$$\frac{dq_p}{d\varepsilon_1} = -\frac{u'(\widetilde{q}\,(\varepsilon_1))\int_{\varepsilon_1}^{\varepsilon_H} dF(\varepsilon)}{\varepsilon_1 u''(\widetilde{q}\,(\varepsilon_1)) - c''(q_p)\int_{\varepsilon_1}^{\varepsilon_H} dF(\varepsilon)} > 0.$$

Therefore,

$$\frac{dq_p}{d\mu} = \frac{dq_p}{d\varepsilon_1}\cdot\frac{d\varepsilon_1}{d\mu} < 0.$$

Considering (3.26), (3.28) becomes

$$\mathcal{W}_{NB} = \frac{1}{2\,(1-\beta)}\left[\int_{\varepsilon_L}^{\varepsilon_1} \varepsilon u\,(\widetilde{q}\,(\varepsilon))\,dF(\varepsilon) + \int_{\varepsilon_1}^{\varepsilon_H} \varepsilon u\,(\widetilde{q}\,(\varepsilon_1))\,dF(\varepsilon) - c(q_p)\right].$$

Differentiating w.r.t. ε_1 gives rise to

$$\begin{aligned}
\frac{d\mathcal{W}_{NB}}{d\varepsilon_1} &= \frac{1}{2\,(1-\beta)}\left[u'(\widetilde{q}\,(\varepsilon_1))\frac{d\widetilde{q}\,(\varepsilon_1)}{d\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_H} \varepsilon dF\,(\varepsilon) - c'(q_p)\frac{dq_p}{d\varepsilon_1}\right] \\
&= \frac{1}{2\,(1-\beta)}\frac{d\widetilde{q}\,(\varepsilon_1)}{d\varepsilon_1}\left[u'(\widetilde{q}\,(\varepsilon_1))\int_{\varepsilon_1}^{\varepsilon_H} \varepsilon dF\,(\varepsilon) - c'(q_p)\int_{\varepsilon_1}^{\varepsilon_H} dF\,(\varepsilon)\right] \\
&= \frac{1}{2\,(1-\beta)}\frac{d\widetilde{q}\,(\varepsilon_1)}{d\varepsilon_1}\left[u'(\widetilde{q}\,(\varepsilon_1))\int_{\varepsilon_1}^{\varepsilon_H} \varepsilon dF\,(\varepsilon) - \varepsilon_1 u'(\widetilde{q}\,(\varepsilon_1))\int_{\varepsilon_1}^{\varepsilon_H} dF\,(\varepsilon)\right] \\
&= \frac{1}{2\,(1-\beta)}\frac{d\widetilde{q}\,(\varepsilon_1)}{d\varepsilon_1}u'(\widetilde{q}\,(\varepsilon_1))\int_{\varepsilon_1}^{\varepsilon_H} (\varepsilon - \varepsilon_1)\,dF\,(\varepsilon) \\
&> 0,
\end{aligned}$$

where (3.34) is used to get the second equality, while the third equality is because of (3.35). As a result,

$$\frac{d\mathcal{W}_{NB}}{d\mu} = \frac{d\mathcal{W}_{NB}}{d\varepsilon_1} \cdot \frac{d\varepsilon_1}{d\mu} < 0.$$

Chapter 4

Optimal Monetary Policy: Distribution Efficiency versus Production Efficiency[1]

4.1 Introduction

What is the optimal monetary policy? The celebrated Friedman rule is a robust answer in extensive literature. In monetary exchanges, agents endure a cost today to receive a future benefit. The Friedman rule eliminates the inefficiency created by this lag, by keeping a rate of return on money high enough to offset the time discounting. However, the implied requirement of the government to run a deflation or pay interest on money is undesirable or infeasible. Realizing this limitation of Friedman rule, an optimal policy would be a *passive* one, i.e., constant money growth rate, since it implements highest possible rate of return on money and hence induces lowest possible intertemporal distortion. However, in reality many countries desire a low but positive inflation rate; for example, currently the Bank of Canada is targeting an inflation rate of 2%.

Usually the monetary policy is conducted through open market operations. The interest rate is determined in a bond market, leading to changes in the aggregate

[1] I am grateful to David Andolfatto for his advice. I would also like to thank Mei Dong, Janet Hua Jiang, Kenneth Kasa, Fernando Martin, Guillaume Rocheteau, Enchuan Shao, and participants at the Brown Bag Seminar at Simon Fraser University, Canadian Economics Association 2010 Conference in Quebec City, and Federal Reserve Bank of Chicago 2010 Workshop on Money, Banking, Payments and Finance for their helpful comments and suggestions.

money supply. According to the Friedman rule, nominal interest rate should be optimally set at zero. However, zero interest rate turns out not to be a popular policy in practice. In normal times more often we see a low but positive nominal interest rate.

This chapter provides one explanation for these discrepancies. I investigate the optimal monetary policy when there exist different liquidity needs among agents. Inflationary monetary policies can be devised to redirect purchasing power among consumers in a socially desirable manner, at the cost of inflation which discourages production. It is demonstrated that an *activist* monetary policy, i.e., positive inflation rate, can be welfare-improving relative to a passive one.

Consumer preference heterogeneity is introduced as a result of transient shocks affecting individual marginal utility of consumption. Agents lack commitment and are anonymous so that fiat money is essential as a record-keeping device, see Kocherlakota (1998).[2] Potentially a credit market could provide *ex post* insurance to idiosyncratic risks, but the credit market is ruled out by the very frictions which render money essential. As shown by Berentsen, Camera and Waller (2007), credit can be made possible by introducing financial intermediation with limited record-keeping technology. This chapter studies the role of monetary policy in providing the same insurance in absence of any other risk-sharing arrangement. Depending on whether preference shocks are observable, two monetary policies are feasible. When the shocks are observable, lump-sum transfers of money can be made contingent on consumer types, so as to redirect liquidity from agents with low desire to consume to those with high desire. When the shocks are unobservable, Kocherlakota (2003) reveals that, by issuing illiquid interest-bearing bonds, the government can restore credit transactions on money to some extent, in the sense that after-shock bond-money exchanges serve to channel liquidity among agents. Both policies have positive distribution effect, which is absent in a model with homogeneous consumers. Meanwhile, the resulting inflation associated with either policy hampers production efficiency, as is true in any monetary economy. This chapter studies the interaction of efficiency in these two dimensions.

It is demonstrated that the distribution effect and the production effect are always intertwined under a money transfer policy. I derive a sufficient condition under which the overall welfare can be improved by an activist monetary policy: if consumers

[2] By "essential" I mean the existence of money expands the set of implementable allocations, see Wallace (2001).

are relative-risk-averse enough, the trade-off between distribution efficiency gain and production efficiency loss would result in a welfare enhancement. Intuitively, high risk aversion implies that risk-sharing is highly valued, hence distribution efficiency becomes the primary concern. As a result, the optimal monetary policy necessitates a positive inflation rate. The same result applies when illiquid bonds are issued to facilitate credit transactions on money, where a positive nominal interest rate is optimal when consumers are risk averse enough.

Due to difficulty in tracking distribution of money holdings among agents, the distribution effect of monetary policy regarding its ability to provide insurance to idiosyncratic liquidity risk is rarely studied in vast micro-founded monetary literature, where only the production effect is emphasized. One exception is Molico (2006). Using numerical methods, he can capture the nondegenerate distribution of money holdings resulting from idiosyncratic uncertainty regarding consumption and production opportunities due to the random nature of search. On this basis, both distribution and production effects (which he calls "redistributive effect" and "real balance effect", respectively) of monetary expansion can be analyzed. In particular, he shows that there exists a trade-off: for low inflation the former effect dominates and leads to higher welfare, but under high inflation the opposite occurs.

Andolfatto (2009) embeds the spatial structure used by Kocherlakota (2003) in a quasi-linear environment with competitive markets. By this means he can analytically study the distribution effect of monetary policy with presence of consumer heterogeneity, which is caused by idiosyncratic risk concerning marginal utility of consumption. When doing so, he abstracts from the production effect of inflation by considering an endowment economy. Such a simplification yields a strong implication: the first-best allocation can be implemented by activist monetary policies simply due to their beneficial distribution effect.

By adopting Andolfatto (2009) framework, I study the interaction of distribution and production effects of monetary policy in an analytical manner. In particular, once both distribution and production issues are taken into account, it turns out the first-best can never be achieved by a feasible monetary policy.

Before emergence of micro-founded monetary economics, the implication of consumer heterogeneity to monetary policy was studied by, for instance, Grossman and Weiss (1983) and Rotemberg (1984). In their cash-in-advance models, transactions cost consideration leads to periodic money withdrawals from the bank. Agents differ in the timing they go to the bank to withdraw cash. Only those agents who are at the bank get the money injection. Such an asymmetry of monetary injection causes redistribution of purchasing power among agents and hence affects real activities, which is similar to the distribution effect studied in this chapter.

Kocherlakota (2005) points out two crucial defects in some micro-founded monetary literature: elimination of tax instruments other than inflation tax, and ignorance of the existence of assets other than money. He argues that "these two omissions are likely to matter greatly when understanding the nature of optimal monetary policy". In some sense, this study is an attempt to overcome these defects, since the type-contingent money transfer embodies a distribution effect underlying many non-inflation taxes, while the illiquid bonds exactly serve as the second asset other than money.

This study has the same spirit as Bhattacharya, Haslag and Martin (2005), which emphasizes that the basic intertemporal inefficiency argument underlying the Friedman rule ignores distribution effect of monetary policy. In Kocherlakota (2003), the societal benefit of illiquid bonds lasts for only one period, while it is persistent in the current model. Moreover, by construction he guarantees that the production is always at full capacity, hence only the distribution effect of monetary policy is emphasized. Boel and Camera (2006) construct a monetary economy with heterogeneity in discounting and consumption risk, where the illiquidity of bonds is an endogenized result. Shi (2008) studies the efficiency-improving role of illiquid bonds in an economy with search friction which is absent in this chapter.

In next section I describe the environment of the model and characterize the first-best allocation as a benchmark. Since the trade-off between distribution efficiency and production efficiency is an important theme in this study, I then introduce two measures of them which will be investigated later in models with different policies. A money transfer policy and a bond policy are studied in sequence in the following two sections. Section 6 provides a summary discussion of these two policies and Section 7 concludes. Most of the proofs are presented in Appendix.

4.2 Environment

Time is discrete and the horizon is infinite. Each period t is divided into two sub-periods: day and night. There is measure one of infinitely-lived households, each composed of a consumer and a producer. Each household belongs to one of two permanent groups, named group 1 and group 2, with equal measures.

During the day, all households live at a centralized location named location 0. They are all able to produce a perishable day output by exerting effort. Let $x_t \in \mathbb{R}$ denote household consumption (or production if negative) in the day at date t. Following Lagos and Wright (2005), I assume each household derives utility from day consumption/production which is linear in x_t.

Consumer heterogeneity is present only in the night. It is the result of an idiosyncratic shock on marginal utility. Specifically, a shock on consumer type denoted by ω_t is realized at the beginning of each night, with $\omega_t \in \{\omega_l = 1, \omega_h = \eta\}$ and $1 < \eta < \infty$. Such a shock is *i.i.d.* across consumers within each group and over time, and it is equally probable to be either type. A consumer derives utility $\omega_t u(c_t)$ from consuming $c_t \in \mathbb{R}_+$ units of night goods, where I assume a utility function $u(c)$ with a constant relative risk averse coefficient $\rho \equiv -cu''(c)/u'(c) > 0$, which satisfies $u'' < 0 < u'$, $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. For producers, the disutility from producing $y_t \in \mathbb{R}_+$ units of unstorable night goods is captured by a cost function $g(y_t)$ with $g' > 0$ and $g'' \geq 0$.

For a household $i \in [0,1]$, the expected lifetime utility function has a quasi-linear form as

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[x_t(i) + \omega_t(i)\, u(c_t(i)) - g(y_t(i))\right], \tag{4.1}$$

with discount rate between two days $0 < \beta < 1$.

A spatial structure for night transactions is incorporated as in Kocherlakota (2003). In particular, there are two other locations, location 1 and location 2. After the consumer type is realized, all producers of group 1(2) households go to location 2(1) to produce output, while all consumers of group 1(2) go to location 1(2) to consume night products. Such a special spatial structure implies that a household cannot consume its own output. The timeline is displayed in Figure 4.1.

Two assumptions are made about information and enforcement. First, agents are anonymous, and so it is impossible to monitor individual transaction histories. In addition, all trades are voluntary and the society cannot impose any penalties on agents; in particular, this means a producer cannot be forced to produce for others. Given these two assumptions and the spatial structure of this economy, the market outcome would be autarky in the night. Now I introduce fiat money into the economy. As is made clear by Kocherlakota (1998), money is essential since it serves as a substitute for the missing record-keeping technology to facilitate exchanges.

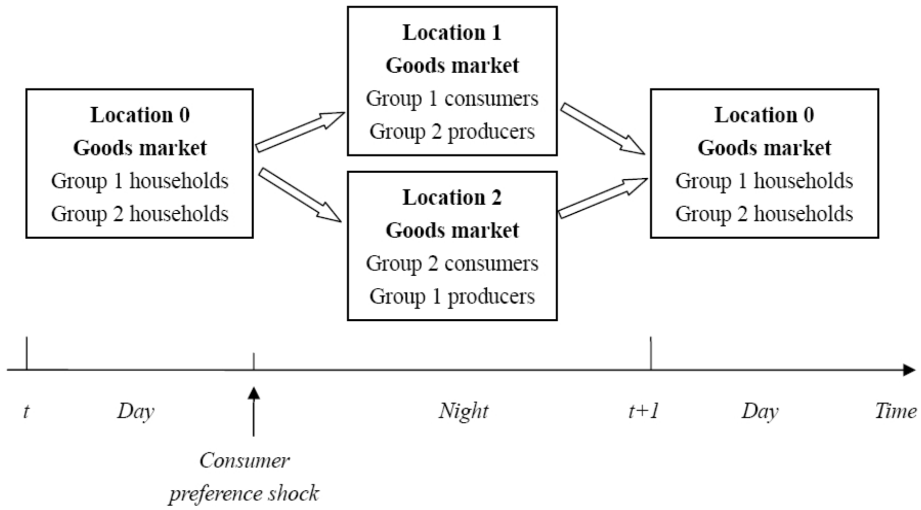

Figure 4.1: Timeline

Furthermore, it is assumed that all trades occur in a sequence of competitive markets. As shown in Rocheteau and Wright (2005), such a treatment preserves essentiality of money even without search friction. Zhu (2008) shows that the optimal inflation rate is positive in an overlapping generations model with search, where the exchange pattern of pairwise meetings is necessary for his result. In this study, the optimality of positive inflation can be supported in competitive markets without search.

As a benchmark, I now consider the first-best allocation that a planner can implement in absence of frictions. And my consideration is restricted to a stationary allocation.

Due to the quasi-linearity in day consumption, any lottery scheme in $\{x\}$ satisfying $E_0 x(i) = 0$ and day resource constraint $\int_0^1 x\,(i)\,di = 0$ can be a solution. Therefore, the day subperiods are irrelevant and a trivial solution is $x\,(i) = 0$ for all households. A planner only needs to consider consumption/production decision in the night.

Note that the two locations are symmetric in the night, i.e., each is resided by measure $1/4$ of type h consumers, measure $1/4$ of type l consumers, and measure $1/2$ of producers. Given this symmetry, I can focus on a representative household, regardless of its group. During the night, a planner can ask each producer to produce y. Conditional on type realization $j \in \{l, h\}$, each consumer is instructed to consume $c_j \in \{c_l, c_h\}$. Throughout this chapter, the welfare function is defined to be a representative household's expected lifetime utility:

$$\mathcal{W} \equiv \frac{1}{1-\beta}\left[\frac{1}{2}u\left(c_l\right)+\frac{1}{2}\eta u\left(c_h\right)-g\left(y\right)\right]. \tag{4.2}$$

The planner's problem is to maximize (4.2) subject to the night resource constraint:

$$\frac{1}{4}c_l+\frac{1}{4}c_h=\frac{1}{2}y. \tag{4.3}$$

It is easy to solve for the first-best allocation (c_l^*, c_h^*, y^*), which is characterized by

$$u'(c_l^*) = \eta u'\left(c_h^*\right), \tag{4.4a}$$

$$g'\left(y^*\right) = u'(c_l^*), \tag{4.4b}$$

$$c_l^*+c_h^* = 2y^*. \tag{4.4c}$$

4.3 Distribution Efficiency and Production Efficiency

In this economy with preference heterogeneity in consumers, there are two kinds of efficiency to consider: distribution efficiency and production efficiency. The distribution efficiency can be measured by

$$\mathcal{D} \equiv \frac{u'\left(c_l\right)}{\eta u'\left(c_h\right)}. \tag{4.5}$$

When $\mathcal{D}<1$, one has $-\Delta c u'\left(c_l\right)+\Delta c\eta u'\left(c_h\right)>0$; i.e., taking total consumption as given, a shift of a marginal unit of consumption Δc from a type l consumer to a type h one can increase total utility level. Therefore, it is distributively inefficient: c_l is relatively too high and c_h relatively too low. Furthermore, it is easy to understand that distribution efficiency increases (decreases) with $\mathcal{D}$ when $\mathcal{D}<1$ ($\mathcal{D}>1$). In optimum, it must be true that $\mathcal{D}=1$, i.e., full distribution efficiency necessitates equal marginal utility between two consumer types.

The production efficiency can be measured by

$$\mathcal{P} \equiv \frac{g'\left(y\right)}{\max\left\{u'\left(c_l\right), \eta u'\left(c_h\right)\right\}}. \tag{4.6}$$

When $\mathcal{P} < 1$, $-\Delta y g'(y) + \Delta y \max\{u'(c_l), \eta u'(c_h)\} > 0$. Under a given consumption distribution, by producing one more unit Δy at the margin, the utility gain from consumption by whom values it the most outweighs the disutility from production cost, hence a social welfare improvement. This implies inefficiency due to underproduction. Conversely, $\mathcal{P} > 1$ means inefficiency from overproduction. It is optimal to produce until marginal cost equates the marginal benefit. Moreover, the production efficiency increases when $\mathcal{P}$ goes up to 1 and decreases when $\mathcal{P}$ exceeds 1. Importantly, note that the welfare is closely linked to these two measures:

Lemma 4.1 *(i) Taking production as given, welfare is determined by the degree of distribution efficiency: it increases (decreases) with $\mathcal{D}$ when $\mathcal{D} < 1$ ($\mathcal{D} > 1$), and reaches maximum when $\mathcal{D} = 1$. (ii) Taking distribution as given, welfare is determined by the degree of production efficiency: it increases (decreases) with $\mathcal{P}$ when $\mathcal{P} < 1$ ($\mathcal{P} > 1$), and reaches maximum when $\mathcal{P} = 1$.*

Proof. See Appendix. ■

As a benchmark, the first-best conditions (4.4) prescribe both full distribution efficiency and full production efficiency. In general, under an environment where distribution efficiency and production efficiency are intertwined, the welfare effect of a monetary policy depends upon its influence on the direction and magnitude of changes in both dimensions.

4.4 Money Transfer

Since the society cannot impose penalties on agents, lump-sum taxation is not feasible. However, this does not rule out lump-sum transfer of money, which is the monetary policy considered now.

Assume the consumer type at night is public information. Then a type-contingent money transfer is feasible. Let M denote the nominal balances of aggregate money supply during the day. After the type shock is realized at the beginning of each night, a money transfer $T_j \geq 0$ is delivered to a type j consumer, $j \in \{l, h\}$. The transfer is financed by printing money such that the money supply is expanded at a constant growth rate $\mu \equiv M^+/M$ (Note: a superscript "+" stands for next period). Define type-contingent money growth rate $\mu_j \equiv 1 + T_j/M$. It follows that

$$0.5\mu_l + 0.5\mu_h = \mu, \tag{4.7}$$

with $\mu_j \geq 1$ and hence $\mu \geq 1$.

Households enter the day with $z \geq 0$ units of fiat money and the night with $m \geq 0$. Denote the value of money in the day by v_1 and in the night by v_2, and define $\phi \equiv v_1/v_2$. Denote real money balances $a \equiv v_1 z$ and $q \equiv v_2 m$, real money transfer $\tau_j \equiv v_2 T_j$, and real money stock $Q \equiv v_2 M$.

I focus on a stationary equilibrium such that all real variables are constant over time. Therefore, the (gross) inflation rate is constant: $v_1/v_1^+ = v_2/v_2^+ = \mu$.

4.4.1 Decision-making of Households

During the day, a household decides how much to consume and how much money to take to the night. The value of a household entering a day with real money balances a is

$$W(a) \equiv \max_{q \geq 0} \{a - \phi q + V(q)\},$$

where $V(q)$ is the value of entering the night with real money balances q.

Assuming $V(\cdot)$ is strictly increasing and concave (which turns out to be true in equilibrium), one can get the demand for real money balances q from the first-order condition:

$$\phi = V'(q). \tag{4.8}$$

It means all households enter the night with the same amount of money $0 < q < \infty$, regardless of how much money they have brought to the day. Such a history independence of money distribution results from the quasi-linearity of preferences (4.1). Applying the envelope theorem yields

$$W'(a) = 1. \tag{4.9}$$

At the beginning of a night, the type shock is realized, and consumers receive type-contingent transfers. Then the consumer and the producer in a household separate and go to different locations. Suppose the decisions on consumption c and production y are made by households before the departure; consumers and producers simply carry out the plan in their destinations. Since only consumers need money for purchase of goods in night, it is natural that all (after-transfer) money balances are held by consumers.

For a household with realized consumer type j, there is a cash constraint for consumption:

$$q + \tau_j - c_j \geq 0. \tag{4.10}$$

The real money balances taken into the next day are

$$a_j^+ \equiv v_1^+ z^+ = v_1^+ \left(q + \tau_j - c_j + y_j\right)/v_2,$$

which can be rewritten as

$$a_j^+ = \frac{\phi}{\mu}\left(q + \tau_j - c_j + y_j\right).$$

For a type j household, the value of entering the night with real money balances q is

$$V_j\left(q\right) \equiv \max_{c_j, y_j}\left\{\omega_j u\left(c_j\right) - g(y_j) + \beta W\left(\frac{\phi}{\mu}\left(q + \tau_j - c_j + y_j\right)\right)\right\},$$

subject to (4.10).

Using (4.9), one gets a first-order condition $g'(y_j) = \beta\phi/\mu$ which determines the desired production y_j. Clearly all producers, irrelevant of household types, produce identical output y satisfying

$$g'(y) = \beta\frac{\phi}{\mu}. \tag{4.11}$$

Depending on whether the cash constraint binds, there are two possible solutions to the desired consumption c_j. If the cash constraint does not bind, then c_j is characterized by

$$\omega_j u'\left(c_j\right) = \beta\frac{\phi}{\mu}. \tag{4.12}$$

If it binds, then $c_j = q + \tau_j$. In either case, it must be true that

$$V_j'\left(q\right) = \omega_j u'\left(c_j\right). \tag{4.13}$$

4.4.2 Market Clearing

The clearing condition for night goods market is (4.3), and that for money market is

$$q = Q. \tag{4.14}$$

4.4.3 Equilibrium

First note that since $V'(q) = 0.5V_l'(q) + 0.5V_h'(q)$, combining (4.8) and (4.13) yields

$$\phi = 0.5u'(c_l) + 0.5\eta u'(c_h). \tag{4.15}$$

Utilizing $\mu_j = 1 + \tau_j/Q$ and (4.14), one can rewrite the cash constraint (4.10) as

$$c_j \leq \mu_j q. \tag{4.16}$$

The next step is to pin down c_l and c_h. It can be shown that type h consumers must be cash constrained for any $\mu > \beta$, so that $c_h = \mu_h q$. For type l consumers, there are two cases to consider.

Case 1 *Type l consumers are not cash constrained.*

Then type l consumption is determined by (4.12): $u'(c_l) = \beta\phi/\mu$. Combining it with (4.15) leads to

$$\frac{2\mu - \beta}{\beta} u'(c_l) = \eta u'(c_h). \tag{4.17}$$

Considering (4.11), one obtains

$$g'(y) = u'(c_l). \tag{4.18}$$

The equilibrium allocation (c_l, c_h, y) is then characterized by (4.17), (4.18) and (4.3).

Note that the equilibrium allocation only depends on μ. In other words, a type-contingent money transfer policy (μ_l, μ_h) matters only to the extent that it affects the overall money growth rate μ. A variation in transfer (μ_l, μ_h) satisfying (4.7) and preserving a constant μ would not affect production side - naturally producers only care about aggregate inflation rate. Meanwhile, consumption patterns would also remain unchanged. To see this, note that as long as type l consumers are not cash constrained they would hold extra money in hands. Responding to a change in μ_l, they just save more or less money accordingly while keeping the same consumption. Consequently, type h consumers have to consume the same, given total output unaffected.

Case 2 *Type l consumers are cash constrained.*

Both type l and type h consumption are determined by (4.16) with equality. Without loss I assume money transfer policy takes the form of $(\mu_l, \mu_h) = (1, 2\mu - 1)$. As a result, $c_l = q$ and $c_h = (2\mu - 1)\, q$. Considering the market clearing condition (4.3), one obtains $q = y/\mu$, thus

$$c_l = \frac{y}{\mu} \text{ and } c_h = \frac{2\mu - 1}{\mu} y. \tag{4.19}$$

Combination of (4.11) and (4.15) gives rise to

$$g'(y) = \frac{\beta}{2\mu} \left[u'(c_l) + \eta u'(c_h) \right]. \tag{4.20}$$

The equilibrium allocation (c_l, c_h, y) is fully characterized by equations (4.19) and (4.20).

It is helpful to see how the division of two equilibrium cases depends on parameters. When type l consumers' cash constraint is weakly binding, i.e., $c_l(\mu) = q(\mu)$, at the threshold allocations must satisfy both Case 1 and Case 2 equilibrium conditions. Substituting (c_l, c_h) in (4.17) by (4.19) and utilizing the CRRA property of the utility function, one obtains

$$\eta = \frac{2\mu - \beta}{\beta} (2\mu - 1)^{\rho},$$

which defines an implicit function $\mu_1(\eta)$: strictly increasing and concave, and $\mu_1 = 1$ if $\eta = \eta_0 \equiv (2 - \beta)/\beta$. In any Case 1 equilibrium, unconstrained type l consumers do not spend all cash, but their decision on money savings depends on rate of return on money. In Appendix, it is proved that type l saving $s_l(\mu) \equiv q(\mu) - c_l(\mu)$ falls when inflation rate rises. Consequently, type l consumers become cash constrained (i.e., $s_l(\mu) = 0$) when $\mu \geq \mu_1(\eta)$. Figure 4.2 shows the division of two cases of equilibrium in (η, μ) plane.

For a given $\eta > \eta_0$, unconstrained type l consumers would become constrained when μ goes up, because the value of their money holdings is diluted by inflation. For a given μ, constrained type l consumers would become unconstrained when η rises. In this monetary economy, all consumption in night must be purchased using cash, hence the preference shock can be also explained as a liquidity shock. η measures the

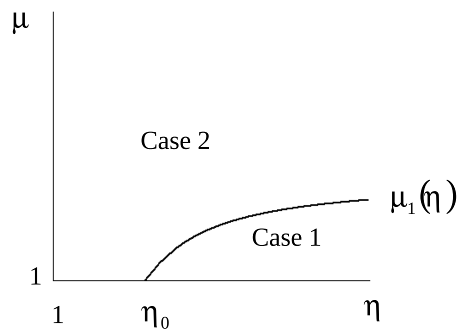

Figure 4.2: Division of Case 1 and Case 2 Equilibria

size of this liquidity shock. With a small shock, both types of consumers are cash constrained. With a large liquidity shock, the high demand from type h consumers elicits more output, hence raises the purchasing power of money. Type l consumers benefit from higher value of their money balances and become unconstrained when η is large enough.

4.4.4 Policy Implications

First let us examine how the two kinds of efficiency are related to a money transfer policy.

Lemma 4.2 *When type h consumers are cash constrained while type l not, neither distribution efficiency nor production efficiency can be achieved; both kinds of efficiency decrease with μ. In particular, $\frac{dc_l}{d\mu} \geq 0$, $\frac{dc_h}{d\mu} < 0$, $\frac{dy}{d\mu} < 0$ and $\frac{d\mathcal{W}}{d\mu} < 0$.*

Proof. Case 1 equilibrium is characterized by (4.17), (4.18) and (4.3). For $\mu \geq 1$,

$$\begin{aligned}
\mathcal{D} &= \frac{u'(c_l)}{\eta u'(c_h)} = \frac{\beta}{2\mu - \beta} < 1, \\
\mathcal{P} &= \frac{g'(y)}{\eta u'(c_h)} = \frac{g'(y)}{\frac{2\mu-\beta}{\beta} u'(c_l)} = \frac{\beta}{2\mu - \beta} = \mathcal{D} < 1.
\end{aligned}$$

Clearly both $\mathcal{D}$ and $\mathcal{P}$ decrease in μ. Since both measures are less than 1, smaller values mean greater inefficiency, hence both distribution efficiency and production efficiency decrease with μ. The allocation and welfare effects of inflation are revealed by a comparative static analysis, which is shown in Appendix. ■

It is expected that inflation harms production efficiency. This is the standard channel that monetary policy affects welfare: the rate of return on money falls with inflation, so that producers are less willing to produce to earn money, leading to greater inefficiency due to underproduction. However, it is surprising to notice that type-contingent money transfers aimed to raise distribution efficiency actually make it *worse*. Intuitively, lower rate of return on money related to higher inflation induces unconstrained type l consumers to save less and hence consume more. Meanwhile, higher inflation discourages production and leads to lower output. As a result, less consumption goods are left for type h consumers. For those receiving money transfers, there is always a trade-off between the fall in value of money and the rise in nominal money balances. In this case, the former effect dominates. With higher inflation, the deterioration in distribution efficiency can be found from the fact that c_l and c_h get closer, which means lower degree of risk-sharing. And the harm of inflation on production efficiency can be seen from the negative relationship between y and μ. Falls in both kinds of efficiency jointly explain the unambiguous fall in welfare.

Lemma 4.3 *When both types of consumers are cash constrained, given a type-contingent money transfer policy $(\mu_l, \mu_h) = (1, 2\mu - 1)$, (i) full distribution efficiency can be achieved by $\mu = \mu_2(\eta) \equiv \frac{1+\eta^{1/\rho}}{2}$; distribution efficiency increases with μ when $1 \leq \mu < \mu_2(\eta)$ and decreases when $\mu \geq \mu_2(\eta)$; (ii) production efficiency may increase or decrease with μ when $1 \leq \mu < \mu_2(\eta)$, but unambiguously decreases with μ when $\mu \geq \mu_2(\eta)$.*

Proof. (i) Given a CRRA utility function, Case 2 equilibrium conditions (4.19) imply that

$$\mathcal{D} = \frac{u'(c_l)}{\eta u'[(2\mu - 1)c_l]} = \frac{c_l^{-\rho}}{\eta[(2\mu - 1)c_l]^{-\rho}} = \frac{(2\mu - 1)^\rho}{\eta},$$

hence $\frac{d\mathcal{D}}{d\mu} = \frac{2\rho}{\eta}(2\mu - 1)^{\rho - 1} > 0$ for all $\mu \geq 1$ and $\rho > 0$. Define

$$\mu_2(\eta) \equiv \frac{1 + \eta^{1/\rho}}{2}.$$

For any given η, when $\mu = \mu_2(\eta)$, $\mathcal{D} = 1$. When $1 \leq \mu < \mu_2(\eta)$, $\mathcal{D} < 1$ and thus distribution efficiency rises with $\mathcal{D}$; since $\mathcal{D}$ is increasing in μ, distribution efficiency increases with μ. When $\mu \geq \mu_2(\eta)$, $\mathcal{D} > 1$; distribution efficiency falls with $\mathcal{D}$ and thus decreases with μ.

(ii) When $\mathcal{D} < 1$,

$$\mathcal{P} = \frac{g'(y)}{\eta u'(c_h)} = \frac{\frac{\beta}{2\mu}\left[u'(c_l) + \eta u'(c_h)\right]}{\eta u'(c_h)} = \frac{\beta}{\mu}\frac{1+\mathcal{D}}{2} < 1.$$

An increase in μ causes a fall in the first term but a rise in the second term. Such a trade-off leaves production effect of an inflation indeterminate. When $\mathcal{D} \geq 1$,

$$\mathcal{P} = \frac{g'(y)}{u'(c_l)} = \frac{\frac{\beta}{2\mu}\left[u'(c_l) + \eta u'(c_h)\right]}{u'(c_l)} = \frac{\beta}{\mu}\frac{1+\frac{1}{\mathcal{D}}}{2} < 1.$$

Now higher μ causes a fall in both terms, leading to a decrease in $\mathcal{P}$. Since production efficiency rises with $\mathcal{P}$ when $\mathcal{P} < 1$, it unambiguously decreases with μ. ■

Note here distribution efficiency and production efficiency are intertwined. With heterogeneous consumers, distribution efficiency not only matters on its own right, but also affects production efficiency. Therefore, the once-ignored distribution effect of monetary policy must be taken seriously.

The $\mu_2(\eta)$ curve is depicted in Figure 4.3 along with $\mu_1(\eta)$.[3] Area I is where both distribution and production efficiency fall with inflation, since type l consumers are unconstrained. In area IV, too high inflation (i.e., $\mu > \mu_2(\eta)$) associated with a transfer policy harms both kinds of efficiency, hence welfare falls with μ with certainty. Areas II and III are where moderate inflation (i.e., $\mu < \mu_2(\eta)$) could improve distribution efficiency but with ambiguous production effect.

Since $\mathcal{P} < 1$ in both cases of equilibrium, inefficiency due to underproduction always exists, and thus first-best allocation is not implementable. This result is in contrast with Andolfatto (2009). Isolated from consideration of production side, the optimal type-contingent transfer policy in an endowment economy (analogous to $\mu = \mu_2(\eta)$ in this model) can always achieve full distribution efficiency, hence the first-best. However, when production issue is taken into account, the production efficiency cannot be attained except when $\mu = \beta$. Since Friedman rule is not a feasible policy, the first-best allocation is not implementable.

Next, I investigate the optimal transfer policy. Specifically, the question to be answered is whether a positive inflation rate is optimal policy choice.

[3] $\mu_2(\eta)$ is concave when $\rho > 1$, and convex when $0 < \rho < 1$. It can be verified that $\mu_2(\eta) > \mu_1(\eta)$ for all $\eta \geq \eta_0$.

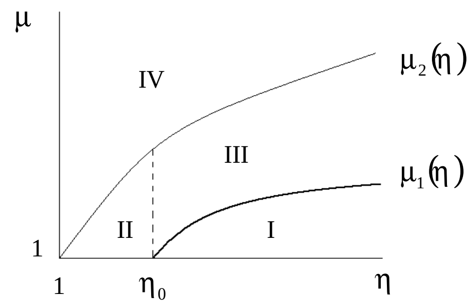

Figure 4.3: A Division of (η, μ) plane

Proposition 4.1 *When $\eta \leq \eta_0$, a type-contingent money transfer policy $(\mu_l, \mu_h) = (1, 2\mu - 1)$ is optimal at $\mu = \overline{\mu}$ with $1 < \overline{\mu} \leq \mu_2(\eta)$, if consumers are relative-risk-averse enough.*

Proof. The government chooses policy variable μ to maximize the welfare function (4.2) subject to equilibrium conditions (4.19) and (4.20). Such a Ramsey problem can be reformulated as

$$\max_{\mu} \mathcal{W} = \frac{1}{2(1-\beta)} \left[u\left(\frac{y}{\mu}\right) + \eta u \left(\frac{2\mu - 1}{\mu} y \right) - 2g(y) \right], \tag{4.21}$$

$$\text{s.t.} \quad g'(y) = \frac{\beta}{2\mu} \left[u'\left(\frac{y}{\mu}\right) + \eta u' \left(\frac{2\mu - 1}{\mu} y \right) \right]. \tag{4.22}$$

Differentiating (4.22) w.r.t. μ at $\mu = 1$ leads to

$$\frac{dy}{d\mu}|_{\mu=1} = \frac{(1+\eta) u'(y_1) + (1-\eta) y_1 u''(y_1)}{(1+\eta) u''(y_1) - \frac{2}{\beta} g''(y_1)},$$

where $y_1 \equiv y(\mu = 1)$. Therefore,

$$\begin{aligned} \frac{d\mathcal{W}}{d\mu}|_{\mu=1} &= \frac{u'(y_1)}{2(1-\beta)} \left[(1-\beta)(1+\eta) \frac{dy}{d\mu}|_{\mu=1} + (\eta - 1) y_1 \right] \\ &= \frac{u'(y_1) \left[(1-\beta)(1+\eta)^2 u'(y_1) + \beta(\eta^2 - 1) y_1 u''(y_1) - \frac{2}{\beta}(\eta - 1) y_1 g''(y_1) \right]}{2(1-\beta) \left[(1+\eta) u''(y_1) - \frac{2}{\beta} g''(y_1) \right]}. \end{aligned}$$

Note $\frac{d\mathcal{W}}{d\mu}|_{\mu=1} > 0$ if $(1-\beta)(1+\eta)^2 u'(y_1) + \beta(\eta^2-1) y_1 u''(y_1) < 0$, which is equivalent to[4]

$$\rho = -\frac{y_1 u''(y_1)}{u'(y_1)} > \frac{(1-\beta)(\eta+1)}{\beta(\eta-1)}.$$

Lemma 4.3 implies that $\frac{d\mathcal{W}}{d\mu} < 0$ for all $\mu > \mu_2(\eta)$, since welfare strictly decreases resulting from deterioration in both distribution and production efficiency. Because of continuity, there must exist a solution $\overline{\mu} \in (1, \mu_2(\eta)]$ to the maximization problem such that $\frac{d\mathcal{W}}{d\mu}|_{\mu=\overline{\mu}} = 0$. ■

When both consumer types are cash constrained, a type-contingent transfer policy definitely improves distribution efficiency relative to a passive ($\mu = 1$) policy, in the sense that it desirably switches the purchasing power away from type l consumers and toward type h consumers. On the other hand, any transfer policy must induce inflation which potentially harms production efficiency. Therefore, the distribution effect and production effect of a transfer may work in opposite directions, and thus the overall welfare effect of a transfer policy is ambiguous. High risk aversion means risk-sharing is highly valued when facing preference shocks, thus the distribution efficiency gain could more than offset the production efficiency loss (if exists), resulting in a welfare enhancement.

This welfare improvement would not persist when inflation goes up further, since rising production efficiency loss would finally dominate distribution efficiency gain. Furthermore, when $\mu > \mu_2(\eta)$, the overdone transfer worsens distribution efficiency and, in combination with its adverse production effect, results in lower welfare. Therefore, an optimal transfer policy necessitates a moderate inflation.

Some degree of transfer-induced inflation may be welfare-improving, only if both consumer types are cash constrained. This is the case when liquidity shock is small, i.e., $\eta \leq \eta_0$ (area II in Figure 4.3). With a large shock, i.e., $\eta > \eta_0$, only one cash constraint binds under low inflation (i.e., $1 \leq \mu < \mu_1(\eta)$). Then welfare unambiguously decreases with μ, as both distribution and production efficiency deteriorate with inflation. However, it might be optimal to raise inflation rate such that type l consumers become cash constrained (i.e., a point in area III in Figure 4.3).

Proposition 4.2 *When $\eta > \eta_0$, there are two possible cases for the optimal type-contingent money transfer policy $(\mu_l, \mu_h) = (1, 2\mu - 1)$: either $\mu = 1$ or $\mu = \widehat{\mu}$ where $\widehat{\mu} \in (\mu_1(\eta), \mu_2(\eta)]$.*

[4] This is a sufficient condition, not necessary if $g'' > 0$. It is sufficient *and* necessary if $g'' = 0$.

Proof. When $\mu \geq \mu_1(\eta)$, both cash constraints bind, and thus the equilibrium will be characterized by equations (4.19) and (4.20). Due to continuity at the threshold $\mu = \mu_1(\eta)$, it must be true that $\mathcal{D}(\mu = \mu_1(\eta)) < 1$ (a property of Case 1 equilibrium) and $\frac{d\mathcal{D}}{d\mu}|_{\mu=\mu_1(\eta)} > 0$ (a property of Case 2 equilibrium). Therefore, further transfer-induced inflation unambiguously improves distribution efficiency. This is potentially welfare-improving if the distortionary production effect is not too big. Let $\widehat{\mu}$ denote the solution to a problem of maximizing (4.21) subject to (4.22) and $\mu_1(\eta) < \mu \leq \mu_2(\eta)$. Then the optimal policy depends on how the welfare achieved with $\widehat{\mu}$ compares with that achieved with a passive policy: if $\mathcal{W}(\mu = 1) > \mathcal{W}(\mu = \widehat{\mu})$, $\mu = 1$ is optimal; if $\mathcal{W}(\mu = 1) < \mathcal{W}(\mu = \widehat{\mu})$, $\mu = \widehat{\mu}$ is optimal. ■

In conclusion, the presence of consumer heterogeneity, combined with high risk aversion, makes a moderate inflation socially desirable, as long as the monetary expansion is conducted through type-contingent transfers. Usually monetary policy is taken as equivalent to a control of the overall inflation level, but the way of how new money is injected also matters. In particular, a type-contingent transfer and associated inflation may be optimal. However, such a policy is no longer feasible when consumer type is private information. As proposed by Kocherlakota (2003), illiquid bonds can improve welfare in this situation.

4.5 Illiquid Bonds

The government issues two intrinsically valueless tokens, money M and bonds B. New bonds are sold during each day at a preset discount price $0 < \delta \leq 1$. All bonds mature in one period - they will be redeemed at par for money in the next day, and represent risk-free claims to future money. Bonds are illiquid in the sense that they cannot be used to purchase goods. However, in each night there exists a secondary market for outstanding bonds, where bonds can be exchanged for money at a competitive price δ_2. Such a market opens right after the consumer type shocks are realized, and closes before households leave for their trips.

The money supply evolves according to $M^+ = M + B - \delta B^+$, with a constant bond-money ratio $\theta \equiv B/M > 0$. Hence stationarity implies

$$\mu = \frac{1+\theta}{1+\delta\theta}. \tag{4.23}$$

Note $0 < \delta \leq 1$ implies $\mu \geq 1$ and nominal interest rate $i = 1/\delta - 1 \geq 0$.

4.5.1 Decision-making of Households

Since all old bonds will be redeemed into money at par, the composition of money-bond portfolio taken into the day is irrelevant; what matters is the total real balances a. Let b denote real holdings of newly-issued bonds purchased by a household in the day. The value of entering a day with real balances a is

$$W(a) \equiv \max_{q \geq 0, b \geq 0} \{a - \phi(q + \delta b) + V(q, b)\},$$

where $V(q, b)$ is the value of entering the night with real money q and real bonds b.

The real money demand q and real bond demand b are characterized by:

$$\phi = \frac{\partial V}{\partial q}(q, b), \tag{4.24}$$

$$\phi\delta = \frac{\partial V}{\partial b}(q, b). \tag{4.25}$$

And the same envelope condition (4.9) applies.

Households take portfolio (q, b) into a night. First the consumer type shocks are realized. Then the asset market for bonds opens. Let b_j denote the quantity of real bonds sold (which is negative for a purchase) by a type j household in the asset market. Due to limited commitment and lack of record-keeping, there is a trading restriction on short sales, i.e.,

$$b_j \leq b. \tag{4.26}$$

Next consumers and producers in the same households separate and go to their own destinations.

Naturally all cash is held by consumers who have liquidity needs in the night. Therefore, each consumer now faces a cash constraint

$$c_j \leq q + \delta_2 b_j. \tag{4.27}$$

The real portfolio balances taken into next day is

$$a_j^+ = \frac{\phi}{\mu}\left[(b - b_j) + (q + \delta_2 b_j - c_j) + y_j\right].$$

Note there are three sources of real balances for a household: bond holdings, unspent cash from consumer, and sales receipt from producer.

For a household with realized consumer type $j \in \{l, h\}$, the value of entering the night with portfolio (q, b) is

$$V_j(q,b) \equiv \max_{b_j, c_j, y_j} \left\{ \begin{array}{c} \omega_j u(c_j) - g(y_j) + \beta W \left(\frac{\phi}{\mu} \left[(b - b_j) + (q + \delta_2 b_j - c_j) + y_j \right] \right) \\ + \xi_j (b - b_j) + \lambda_j (q + \delta_2 b_j - c_j) \end{array} \right\},$$

where $\xi_j \geq 0$ and $\lambda_j \geq 0$ are Lagrange multipliers associated with constraints (4.26) and (4.27), respectively.

Again (4.11) determines the desired production y. The other two first-order conditions are

$$\xi_j = \delta_2 \omega_j u'(c_j) - \beta \frac{\phi}{\mu}, \tag{4.28}$$

$$\lambda_j = \omega_j u'(c_j) - \beta \frac{\phi}{\mu}. \tag{4.29}$$

Envelope theorem leads to $\frac{\partial V_j}{\partial q}(q,b) = \omega_j u'(c_j)$ and $\frac{\partial V_j}{\partial b}(q,b) = \delta_2 \omega_j u'(c_j)$. Then

$$\frac{\partial V}{\partial q}(q,b) = 0.5 \omega_l u'(c_l) + 0.5 \omega_h u'(c_h), \tag{4.30}$$

$$\frac{\partial V}{\partial b}(q,b) = 0.5 \delta_2 \omega_l u'(c_l) + 0.5 \delta_2 \omega_h u'(c_h). \tag{4.31}$$

Referring to (4.24) and (4.25), one obtains

$$\delta_2 = \delta.$$

4.5.2 Market Clearing

Still two market clearing conditions apply: (4.14) and (4.3) for money market and night output market, respectively. In addition, the bond market clearing conditions in the day and night are

$$b = \theta q, \tag{4.32}$$

$$b_l + b_h = 0. \tag{4.33}$$

4.5.3 Equilibrium

First note that combining (4.24) and (4.30) yields (4.15) again. In Appendix, it is proved that type l households buy bonds while type h households sell bonds in the asset market, i.e., $b_l < 0, b_h > 0$. Since $b_l < 0 < b$, a slack constraint (4.26) for type l consumers means $\xi_l = 0$ and thus (4.28) gives

$$u'(c_l) = \frac{1}{\delta}\beta\frac{\phi}{\mu}. \tag{4.34}$$

Using it to substitute ϕ in (4.15), one obtains

$$\frac{2\delta\mu - \beta}{\beta}u'(c_l) = \eta u'(c_h). \tag{4.35}$$

And a combination of (4.34) and (4.11) yields

$$g'(y) = \delta u'(c_l). \tag{4.36}$$

It seems that equations (4.35), (4.36) and market clearing condition (4.3) pin down the equilibrium allocation (c_l, c_h, y), given any monetary policy (δ, μ, θ). However, a stationary equilibrium exists only for a subset of policy variables. First, any feasible policy must satisfy (4.23). Second, there is another policy restriction which turns out to be part of the equilibrium.

Note comparing (4.28) and (4.29) leads to $\lambda_j \geq \xi_j$ for $\delta \leq 1$. Since Lagrange multipliers $\xi_j \geq 0$, $\lambda_j = 0$ if and only if $\delta = 1$ and $\xi_j = 0$; otherwise, $\lambda_j > 0$. When the first case is true, (4.23) implies $\mu = 1$, hence $\delta\mu = 1$. If $\xi_h = 0$, (4.28) and (4.34) yield $u'(c_l) = \eta u'(c_h)$, and thus (4.35) implies $\delta\mu = \beta$, a contradiction. Therefore, it must be true that $\lambda_h = \xi_h > 0$, i.e., type h consumers' cash constraint is strictly binding while type l consumers' cash constraint is weakly binding, which is a limit case of $\lambda_j > 0$ when $\delta \to 1$.[5] Generically what features an equilibrium is the second case: $\lambda_j > 0$, hence a binding cash constraint (4.27) for both types of consumers:

$$c_h = q + \delta b_h, c_l = q + \delta b_l.$$

It is clear that type h consumers must be selling bonds. But by how much? There are two possible cases: sell all or part of their bond holdings taken into the night. First

[5] Since $\xi_l = 0$, (4.28) and (4.29) imply that $\lambda_l \searrow 0$ when $\delta \nearrow 1$.

consider the case $b_h = b$. Utilizing (4.32) and (4.33) yields $c_h = q + \delta b = (1 + \delta\theta) q$ and $c_l = q - \delta b = (1 - \delta\theta) q$. And it follows from (4.3) that $q = y$. Therefore, another restriction on policy variables δ and θ is in place:

$$c_h = (1 + \delta\theta) y. \tag{4.37}$$

Note the binding constraint (4.26) for type h consumers implies $\xi_h > 0$. Since $\xi_h > \xi_l = 0$, (4.28) reveals that $u'(c_l) < \eta u'(c_h)$. As a result, (4.35) yields $\delta\mu > \beta$.

Next I consider the case $b_h < b$. Then the slack constraint (4.26) for type h consumers means $\xi_h = 0$. Due to $\xi_h = \xi_l = 0$, (4.28) yields $u'(c_l) = \eta u'(c_h)$, and thus (4.35) implies $\delta\mu = \beta$. Note this is true only as a limit case, since when $\delta\mu \searrow \beta$, (4.35) leads to $u'(c_l) \nearrow \eta u'(c_h)$, and then combining (4.28) and (4.34) implies $\xi_h \searrow 0$. Generically cash constrained type h consumers are also constrained by their bond holdings when trying to acquire more liquidity by selling bonds.

Given the two restrictions, (4.23) and (4.37), on feasible monetary policy, the government has only one degree of freedom in setting the trio of policy variables (δ, μ, θ). For instance, if the bond-money ratio θ is chosen as a policy instrument in open market operations, then both bond discount price δ (hence nominal interest rate i) and inflation rate μ will be endogenized as part of the equilibrium. In summary, the government can peg any one of the three policy variables; the equilibrium allocation (c_l, c_h, y) and the other two policy variables are then characterized by

$$\begin{aligned}
\frac{2\delta\mu - \beta}{\beta} u'(c_l) &= \eta u'(c_h), && \text{(4.38a)} \\
g'(y) &= \delta u'(c_l), && \text{(4.38b)} \\
c_l + c_h &= 2y, && \text{(4.38c)} \\
\mu &= \frac{1+\theta}{1+\delta\theta}, && \text{(4.38d)} \\
c_h &= (1+\delta\theta) y. && \text{(4.38e)}
\end{aligned}$$

4.5.4 Policy Implications

In a pure monetary economy, credit, either nominal or real, cannot exist due to lack of commitment and enforcement. Because the government has the ability to commit to repay its nominal debt (simply by printing money), bonds are generally accepted. Agents with different money demand in night can then engage in bond-money exchange and adjust their asset portfolios, through which liquidity is channeled from bond buyers (type l consumers) to sellers (type h consumers). Therefore, the bond-money trade amounts to a nominal credit transaction, with interest paid upfront in form of price discounting. In this sense, the introduction of illiquid bonds makes credit transactions on money viable. Moreover, the sufficiency of liquidity provision in this market is determined by the real interest rate, which is

$$R \equiv \frac{1+i}{\mu} = \frac{1}{\delta\mu},$$

by Fisher equation. Intuitively, high real interest rate will encourage agents to buy bonds, which in fact provides liquidity to bond sellers. Realizing this, it is helpful to derive the bounds of real interest rate prevailing in stationary equilibria.

Lemma 4.4 *For any stationary equilibrium with illiquid bonds, a feasible monetary policy is such that real interest rate* $1 \leq R < \frac{1}{\beta}$ *when* $\eta > \eta_0$ *and* $R_0(\eta) < R < \frac{1}{\beta}$ *when* $\eta \leq \eta_0$*, where* $R_0(\eta) \equiv \frac{2}{\beta(1+\eta)}$.

Proof. When $\eta > \eta_0$, it simply follows from restrictions $\theta > 0$, $0 < \delta \leq 1$ and (4.38d) that $R = \frac{1}{\delta\mu} \geq 1$.

When $\eta \leq \eta_0$, (4.37) implies $c_l < c_h$ for any $\theta > 0$, hence (4.38a) results in $\frac{1}{\delta\mu} > \frac{2}{\beta(1+\eta)} \geq \frac{2}{\beta(1+\eta_0)} \equiv 1$, a more stringent restriction on R. Therefore, the lower bound becomes $R > R_0(\eta) \equiv \frac{2}{\beta(1+\eta)}$.

In any equilibrium it must be true that $\delta\mu > \beta$. Therefore, an upper bound is $R = \frac{1}{\delta\mu} < \frac{1}{\beta}$. ■

It is known from Figure 4.2 that when $\eta \leq \eta_0$ both types of consumers are cash constrained with a constant money supply. If the real interest rate is set too low (i.e., $R < R_0(\eta)$), then even type l consumers would want to sell bonds for cash and thus consume more, since the opportunity cost of not holding bonds is low. Therefore, in this case the minimum real interest rate should be set higher, so as to encourage type l consumers to buy bonds.

Borrowing from Williamson (2009), I call $R < 1/\beta$ the case of "insufficient liquidity" and $R = 1/\beta$ "sufficient liquidity". $R < 1/\beta$ is related to the case $b_h = b$. It follows from (4.28) that $\delta v_2 \eta u'(c_h) > \beta v_1^+$, i.e., at the margin it is better to sell bonds and consume now rather than hold bonds until maturity. But due to the restriction of their bond holdings, type h consumers cannot get enough liquidity from bond sales as they desire. This resembles the case of a binding borrowing constraint, where the liquidity demand cannot be fully satisfied by borrowing, i.e., insufficient liquidity. The limit case $R = 1/\beta$ is associated with $b_h < b$. Unconstrained borrowing in a credit market would result in an efficient real interest rate $R = 1/\beta$. Similarly, with the same rate here, type h consumers could get sufficient liquidity from bond sales. (Otherwise they would just sell more.) As it turns out, in any stationary equilibrium, the sufficient liquidity cannot be implemented by a feasible bond policy, and type h consumers are always constrained by their bond holdings.

Next I need to investigate properties of the equilibrium. In any equilibrium, the policy making procedure is equivalent to a real interest rate targeting. To see this, note that $\delta = \frac{1}{\mu R}$ and so (4.38d) leads to $\delta\theta = \frac{\mu-1}{\mu(R-1)}$. Then the equation system (4.38) can be reduced to

$$(2-\beta R)\, u'\left[\left(1-\frac{\mu-1}{\mu\,(R-1)}\right)y\right] = \beta R\eta u'\left[\left(1+\frac{\mu-1}{\mu\,(R-1)}\right)y\right], \quad (4.39)$$

$$g'(y) = \frac{1}{\mu R}u'\left[\left(1-\frac{\mu-1}{\mu\,(R-1)}\right)y\right]. \quad (4.40)$$

Now it is clear to see that, given any policy target R, y and μ, hence all other allocation and policy variables, will be derived endogenously from these two equations.

Lemma 4.5 *For all feasible policies (δ, μ, θ) with implicit real interest rate*

$$\max\left\{R_0\left(\eta\right), 1\right\} < R < 1/\beta,$$

$\frac{d\mu}{dR} > 0$, $\frac{d\delta}{dR} < 0$, $\frac{d\delta}{d\mu} < 0$, $\frac{d(1+i)}{d\mu} > 0$, *and* $\frac{d\theta}{dR} > 0$.

Proof. See Appendix. ■

This implies a clear-cut relationship among inflation rate, nominal interest rate and real interest rate: they are all positively related as an equilibrium result. With the information available now, I can sketch the locus of feasible policy variables (δ, μ) with corresponding R in Figures 4.4 and 4.5.

In (δ, μ) plane, the two dashed hyperbolas $\delta\mu = 1$ and $\delta\mu = \beta$ represent the bounds $R = 1$ and $R = 1/\beta$, respectively. Given strict monotonicity of derivatives shown in Lemma 4.5, any equilibrium R is only compatible with a unique policy (δ, μ, θ), and evidently $R \to 1/\beta$ when $\mu \to \infty$. The lower bound of R depends on

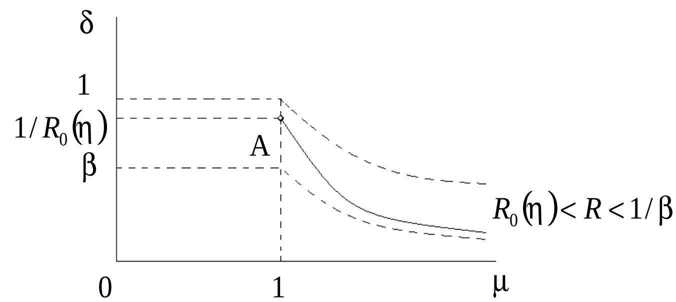

Figure 4.4: Feasible Policy (δ, μ) and R when $\eta \leq \eta_0$

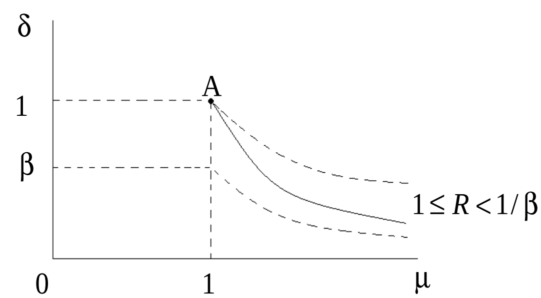

Figure 4.5: Feasible Policy (δ, μ) and R when $\eta > \eta_0$

η. When $\eta > \eta_0$, $R \geq 1$. In particular, $\delta = \mu = 1$ when $R = 1$.[6] When $\eta \leq \eta_0$, it must be true that $R > R_0(\eta)$. In the limit of $R = R_0(\eta)$, (4.39) indicates $c_l = c_h$ and $\mu = 1$, implying $\delta = 1/R_0(\eta) \in (\beta, 1)$ and $\theta = 0$.

Now let us look at how distribution efficiency and production efficiency can be affected by an illiquid bond policy.

Lemma 4.6 *With illiquid bonds, (i) neither full distribution efficiency nor full production efficiency can be achieved. (ii) distribution efficiency increases with real interest rate; production efficiency is positively related with real interest rate but negatively related with nominal interest rate.*

Proof. (i) Equilibrium condition (4.38a) implies

$$\mathcal{D} = \frac{u'(c_l)}{\eta u'(c_h)} = \frac{\beta}{2\delta\mu - \beta}. \tag{4.41}$$

It follows that $\mathcal{D} < 1$ since $\delta\mu > \beta$. As a result, (4.38b) yields

$$\mathcal{P} = \frac{g'(y)}{\eta u'(c_h)} = \frac{\delta u'(c_l)}{\frac{2\delta\mu-\beta}{\beta} u'(c_l)} = \frac{\delta\beta}{2\delta\mu - \beta} = \delta\mathcal{D}. \tag{4.42}$$

$\delta \leq 1$ and $\mathcal{D} < 1$ lead to $\mathcal{P} < 1$.

(ii) Because $\mathcal{D} < 1$ and $\mathcal{P} < 1$, efficiency improvement is positively associated with increase in value of both measures. Clearly $\mathcal{D}$, hence distribution efficiency, increases with $R = \frac{1}{\delta\mu}$. Since $\mathcal{P}$ rises with both $\mathcal{D}$ and δ, production efficiency increases with R and decreases with $(1+i) = \frac{1}{\delta}$. ■

Again the first-best allocation is not implementable, since any feasible policy would fail to realize full efficiency in either dimensions.

Along the policy locus in Figures 4.4 and 4.5, a fall in δ and associated rise in R will lead to improvement in distribution efficiency, but the effect on production efficiency is indeterminate. As a result, the overall welfare effect is uncertain. However, if a qualification on preferences is satisfied, a positive inflation may unambiguously

[6] In this case θ, together with y, is determined by an equation system:

$$\begin{aligned} (2-\beta)\, u'[(1-\theta)\, y] &= \beta\eta u'[(1+\theta)\, y], \\ g'(y) &= u'[(1-\theta)\, y]. \end{aligned}$$

It is straightforward to see that $\theta > 0$ given $\eta > \eta_0$.

improve the welfare relative to zero inflation. Indeed this is also a demonstration of the societal benefit of interest-bearing illiquid bonds, echoing Kocherlakota (2003).

When consumer types are unobservable, type-contingent money transfer is infeasible. If there are no bonds, the only possible monetary policy is a uniform lump-sum transfer resulting in $\mu_l = \mu_h = \mu \geq 1$. When $\eta > \eta_0$, the equilibrium for a "no-bonds" economy is again characterized by (4.17), (4.18) and (4.3), and hence $\frac{d\mathcal{W}}{d\mu} < 0$. When $\eta \leq \eta_0$, the equilibrium conditions become

$$\begin{aligned} c_l &= c_h = y, && (4.43) \\ g'(y) &= \frac{\beta}{2\mu}(1+\eta)\, u'(y). && (4.44) \end{aligned}$$

It is easy to verify that $\frac{d\mathcal{W}}{d\mu} < 0$. Therefore, the optimal policy is $\mu = 1$ in both cases. Next I show that the introduction of illiquid bonds potentially improves welfare.

Proposition 4.3 *When consumer types are unobservable, if consumers are relative-risk-averse enough, illiquid bonds are essential and resulting inflation is welfare-improving.*

Proof. Equilibrium conditions (4.38) can be reduced to

$$\begin{aligned} \left(2\delta\frac{1+\theta}{1+\delta\theta} - \beta\right) u'\left[(1-\delta\theta)\,y\right] &= \beta\eta u'\left[(1+\delta\theta)\,y\right], && (4.45) \\ g'(y) &= \delta u'\left[(1-\delta\theta)\,y\right]. && (4.46) \end{aligned}$$

with $c_l = (1-\delta\theta)\,y$, $c_h = (1+\delta\theta)\,y$, and $\mu = \frac{1+\theta}{1+\delta\theta}$.

If $\eta > \eta_0$, the allocation (c_{l1}, c_{h1}, y_1) corresponding to the optimal "no-bonds" policy $\mu = 1$ can be implemented by a "with-bonds" policy $(\delta = 1, \mu = 1, \theta = 1 - c_{l1}/y_1)$. To see this, plug the three policy variables into (4.45) and (4.46), and then one can see that the resulting allocation solution must also solve equations (4.17), (4.18) and (4.3) with $\mu = 1$. In Appendix I prove that $\frac{d\mathcal{W}}{d\delta}|_{\delta=1} < 0$ if

$$\rho > \frac{\ln(2-\beta) - \ln\beta - \ln\eta}{\ln(1-\beta)}.$$

Therefore, if consumers are risk averse enough, introducing interest-bearing bonds (i.e., $\delta < 1$), hence $\mu > 1$, improves welfare.

If $\eta \leq \eta_0$, the allocation (c_{l0}, c_{h0}, y_0) corresponding to the optimal "no-bonds" policy $\mu = 1$ can be implemented by a "with-bonds" policy $(\delta = \beta (1+\eta)/2, \mu = 1, \theta = 0)$, which is a limit case. To see this, just plug these policy variables into (4.45) and (4.46), and it can be verified that the solution must also solve equations (4.43) and (4.44) with $\mu = 1$. In Appendix I prove that $\frac{d\mathcal{W}}{d\theta}|_{\theta=0} > 0$ if

$$\rho > \frac{2(1-\beta)}{\beta^2(\eta-1)}.$$

Again, if consumers are risk averse enough, introducing illiquid bonds (i.e., $\theta > 0$), hence $\mu > 1$, improves welfare.[7] ■

The optimal "no-bonds" policy $\mu = 1$ corresponds to point A in Figures 4.4 and 4.5. Associated with the introduction of bonds is a trade-off between distribution efficiency gain and production efficiency loss. Moving down the policy locus, nominal interest rate becomes higher, which is detrimental to production efficiency, since the resulting high inflation discourages production. However, it is desired from the perspective of distribution efficiency. Lower bond price induces type l consumers to buy bonds. Through such bond transactions, (insufficient) liquidity is channeled from type l consumers to type h consumers, which helps improve distribution efficiency. When consumers are risk averse enough, the need for risk-sharing is high; improvement in distribution efficiency is valued so much that it outweighs the loss in production efficiency. As a result, a movement away from point A along the policy locus, i.e., implementing an inflation rate $\mu > 1$, leads to higher welfare.

When bonds are essential, an optimal policy involves $\delta < 1$ and hence a positive nominal interest rate, which is in contrast with the Friedman rule. Usually a zero nominal interest rate is optimal in an environment where only production efficiency is under consideration. When distribution efficiency also matters, and two kinds of efficiency clashes with each other, the Friedman rule is likely to cease to be optimal, especially when distribution efficiency matters much.

4.6 Discussion

In a monetary model with homogeneous consumers, for example in Berentsen, Camera and Waller (2007), the idiosyncratic risk is caused by a preference/productivity shock, and money is essential as a means of self-insurance. When consumers are het-

[7] The two restrictions on ρ are sufficient conditions. They are necessary if $g'' = 0$ but not if $g'' > 0$.

erogeneous in taste, there exists another risk caused by consumer preference shocks, which leads to a demand for additional insurance *among* consumers. Because of the absence of commitment, such an insurance cannot be provided by a credit market. Given that a benevolent government has the ability to commit, it has the potential to serve as the insurance provider.

When consumer types are observable, such an insurance can be realized by a type-contingent money transfer. As long as both types of consumers are cash constrained, type-contingent transfers can redirect purchasing power from consumers with low desire to consume to those with high desire. However, the improvement in distribution efficiency comes with a cost. The transfer-induced inflation is detrimental to production efficiency, for producers are discouraged to produce given their effort can only be exchanged for an asset which falls in value. Such a trade-off between distribution effect and production effect makes the total welfare impact uncertain.

When consumer types are unobservable, the type-contingent transfer is no longer feasible. By creating illiquid bonds, a risk-free claim to money, the government provides an instrument which makes a credit market viable. With the money-bond exchange, consumers can credibly commit to welfare-improving credit transactions. However, again the trade-off exists. Since real interest rate is relevant to the credit transaction decision, the gains from intertemporal trade is better exploited with a higher real interest rate. On the other hand, the high real interest rate necessarily leads to a high inflation rate, which hampers production efficiency.

As a matter of fact, under either transfer policy and bond policy, (4.11) and (4.15) hold and lead to the same relationship:

$$\mathcal{P} = \frac{\beta}{\mu}\frac{1+\mathcal{D}}{2}, \tag{4.47}$$

if $\mathcal{D} \leq 1$. In a production economy, production efficiency and distribution efficiency are always intertwined. In order to improve distribution efficiency, inflation is always necessary, either to finance money transfer or to pay interest on bonds. (4.47) reveals that inflation would result in a trade-off between changes in two kinds of efficiency.

If consumers are relative-risk-averse enough, i.e., the demand for risk-sharing is high enough, the gain from distribution efficiency improvement could more than compensate the loss in production efficiency, hence an inflation caused by either type-contingent money transfers or issuing interest-bearing bonds could be welfare-improving.[8]

Because a bond policy does not depend on the observability of consumer types, it is interesting to compare it with a transfer policy when consumer type is public information. In this study, it is difficult to compare the maximum welfare achievable under different policies, but at least one can see the difference of them in affecting distribution efficiency. A type-contingent transfer policy $(\mu_l, \mu_h) = (1, 2\mu_2(\eta) - 1)$ guarantees full distribution efficiency with a resulting inflation rate $\mu = \mu_2(\eta)$. A bond policy, however, cannot implement full distribution efficiency unless at the cost of infinite inflation. Therefore, it seems that transfer policy is a better choice when distribution efficiency is a major concern.

4.7 Conclusion

In this chapter, I investigate the optimal monetary policy in presence of individual liquidity shocks. Potentially a credit market could provide *ex post* insurance to smooth consumption over time. But the credit market is ruled out by lack of commitment and enforcement, the very frictions which render money essential. However, such an insurance can be provided by the government, which distinguishes from agents in its ability to commit.

Depending on the observability of consumer types, two monetary policies are feasible. A type-contingent money transfer can redistribute purchasing power among consumers in a socially desirable manner, while introducing illiquid interest-bearing bonds makes the credit transactions on money viable and hence channels liquidity among agents. Both have positive distribution effect. However, the resulting inflation hinders production efficiency. If consumers are relative-risk-averse enough, the trade-off between distribution efficiency gain and production efficiency loss associated with an activist monetary policy would result in net welfare enhancement.

[8] Given $g'' > 0$, high enough ρ is *sufficient* for optimality of inflationary policy. In general this condition can be relaxed with more convex disutility function, which implies the output reduction economizes cost to a larger degree, hence welfare loss due to underproduction is not so severe.

4.8 References

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4.9 Appendix

4.9.1 Proof of Lemma 4.1

(i) Given a constant output y, imposing the resource constraint yields $\mathcal{D} = \frac{u'(c_l)}{\eta u'(2y-c_l)}$. Differentiating with respect to $\mathcal{D}$ leads to

$$\frac{\partial c_l}{\partial \mathcal{D}} = \frac{\left[\eta u'\left(2y-c_l\right)\right]^2}{u''\left(c_l\right)\eta u'\left(2y-c_l\right) + u'\left(c_l\right)\eta u''\left(2y-c_l\right)} < 0.$$

Note that

$$\frac{\partial \mathcal{W}}{\partial \mathcal{D}} = \frac{\partial \mathcal{W}}{\partial c_l} \cdot \frac{\partial c_l}{\partial \mathcal{D}} = \frac{1}{2\left(1-\beta\right)}\left[u'\left(c_l\right) - \eta u'\left(2y-c_l\right)\right]\frac{\partial c_l}{\partial \mathcal{D}}.$$

Therefore, $\frac{\partial \mathcal{W}}{\partial \mathcal{D}} \gtreqless 0$ if and only if $u'\left(c_l\right) - \eta u'\left(2y-c_l\right) \lesseqgtr 0$, or equivalently, $\mathcal{D} \lesseqgtr 1$.

(ii) If $\mathcal{D} < 1$, (4.6) becomes $\mathcal{P} = \frac{g'(y)}{\eta u'(2y-c_l)}$. Taking c_l as constant and differentiating w.r.t. $\mathcal{P}$ lead to

$$\frac{\partial y}{\partial \mathcal{P}} = \frac{\left[\eta u'\left(2y-c_l\right)\right]^2}{g''\left(y\right)\eta u'\left(2y-c_l\right) - 2g'\left(y\right)\eta u''\left(2y-c_l\right)} > 0.$$

Since

$$\frac{\partial \mathcal{W}}{\partial \mathcal{P}} = \frac{\partial \mathcal{W}}{\partial y} \cdot \frac{\partial y}{\partial \mathcal{P}} = \frac{1}{1-\beta}\left[\eta u'\left(2y-c_l\right) - g'\left(y\right)\right]\frac{\partial y}{\partial \mathcal{P}},$$

it follows that $\frac{\partial \mathcal{W}}{\partial \mathcal{P}} \gtreqless 0$ if and only if $\eta u'\left(2y-c_l\right) - g'\left(y\right) \gtreqless 0$, or equivalently, $\mathcal{P} \lesseqgtr 1$.

If $\mathcal{D} \geq 1$, (4.6) becomes $\mathcal{P} = \frac{g'(y)}{u'(2y-c_h)}$. Taking c_h as constant, the same procedure as above would result in $\frac{\partial \mathcal{W}}{\partial \mathcal{P}} \gtreqless 0$ if and only if $u'\left(2y-c_h\right) - g'\left(y\right) \gtreqless 0$, or equivalently, $\mathcal{P} \lesseqgtr 1$.

4.9.2 Proof of $\frac{ds_l(\mu)}{d\mu} < 0$

Since $c_h\left(\mu\right) = \mu_h q\left(\mu\right) = \left(2\mu-1\right)q\left(\mu\right)$ and $c_l\left(\mu\right) + c_h\left(\mu\right) = 2y\left(\mu\right)$, it can be solved that

$$q\left(\mu\right) = \frac{2y\left(\mu\right) - c_l\left(\mu\right)}{2\mu-1},$$

then

$$s_l\left(\mu\right) \equiv q\left(\mu\right) - c_l\left(\mu\right) = \frac{2}{2\mu-1}\left[y\left(\mu\right) - \mu c_l\left(\mu\right)\right].$$

Differentiating w.r.t. μ results in

$$\frac{ds_l(\mu)}{d\mu} = -\frac{2}{2\mu-1}\left[\frac{2y(\mu)-c_l(\mu)}{2\mu-1} + \mu\frac{dc_l(\mu)}{d\mu} - \frac{dy(\mu)}{d\mu}\right].$$

Differentiating (4.18) w.r.t. μ yields

$$g''(y(\mu))\frac{dy(\mu)}{d\mu} = u''(c_l(\mu))\frac{dc_l(\mu)}{d\mu}.$$

Solving for $\frac{dc_l(\mu)}{d\mu}$ and then plugging into the previous equation, one obtains

$$\frac{ds_l(\mu)}{d\mu} = -\frac{2}{2\mu-1}\left[\frac{c_h(\mu)}{2\mu-1} + \mu\left(\frac{g''(y(\mu))}{u''(c_l(\mu))} - 1\right)\frac{dy(\mu)}{d\mu}\right] < 0,$$

after considering $\frac{dy(\mu)}{d\mu} < 0$ for all $\mu \geq 1$.

4.9.3 Proof of Lemma 4.2

Case 1 equilibrium conditions (4.17), (4.18) and (4.3) can be reduced to a two-equation system:

$$\begin{aligned}(2\mu-\beta)u'(c_l) &= \beta\eta u'(2y-c_l),\\ g'(y) &= u'(c_l).\end{aligned}$$

Taking derivative w.r.t. μ and solving for $\frac{dy}{d\mu}$ and $\frac{dc_l}{d\mu}$ yields

$$\begin{aligned}\frac{dy}{d\mu} &= \frac{2u'(c_l)}{2\beta\eta u''(2y-c_l) - \beta\eta\frac{u''(2y-c_l)}{u''(c_l)}g''(y) - (2\mu-\beta)g''(y)},\\ \frac{dc_l}{d\mu} &= \frac{g''(y)}{u''(c_l)}\frac{dy}{d\mu}.\end{aligned}$$

Clearly $\frac{dy}{d\mu} < 0$ and hence $\frac{dc_l}{d\mu} \geq 0$. Therefore,

$$\frac{dc_h}{d\mu} = 2\frac{dy}{d\mu} - \frac{dc_l}{d\mu} < 0.$$

Differentiating the welfare function (4.2) w.r.t. μ and using the equilibrium conditions leads to

$$\frac{d\mathcal{W}}{d\mu} = \frac{\mu-\beta}{\beta(1-\beta)}u'(c_l)\left[2 - \frac{g''(y)}{u''(c_l)}\right]\frac{dy}{d\mu} < 0.$$

4.9.4 Proof of $b_l < 0$ and $b_h > 0$

Suppose $b_l = 0, b_h = 0$. Then slack constraint (4.26) means $\xi_j = 0$ for $j \in \{l, h\}$, hence (4.28) implies

$$u'(c_l) = \eta u'(c_h) = \frac{1}{\delta}\beta\frac{\phi}{\mu}. \tag{4.48}$$

If $\delta < 1$, (4.29) yields $\lambda_j = \frac{1}{\delta}\beta\frac{\phi}{\mu} - \beta\frac{\phi}{\mu} > 0$, therefore, (4.27) is binding: $c_l = c_h = q$, which implies $u'(c_l) < \eta u'(c_h)$, a contradiction with (4.48). If $\delta = 1$, (4.23) implies $\mu = 1$ for any $\theta > 0$, hence $\delta\mu = 1$; however, combination of (4.15) and (4.48) yields $\delta\mu = \beta$, a contradiction.

Suppose $b_l > 0, b_h < 0$. Then constraint (4.26) means $\xi_h = 0$ and thus (4.28) gives

$$\eta u'(c_h) = \frac{1}{\delta}\beta\frac{\phi}{\mu}. \tag{4.49}$$

Consequently, for all $\delta \leq 1$, (4.29) yields $\lambda_h = \frac{1}{\delta}\beta\frac{\phi}{\mu} - \beta\frac{\phi}{\mu} \geq 0$, which means a cash constraint (4.27) for type h consumers:

$$c_h \leq q + \delta b_h. \tag{4.50}$$

On the other hand, there are two possible cases for type l consumers: (1) If $b_l = b$, then $\xi_l > 0$. (4.28) gives

$$u'(c_l) > \frac{1}{\delta}\beta\frac{\phi}{\mu}. \tag{4.51}$$

Thus (4.29) yields $\lambda_l > \frac{1}{\delta}\beta\frac{\phi}{\mu} - \beta\frac{\phi}{\mu} \geq 0$ for all $\delta \leq 1$. Therefore, (4.27) is binding for type l consumers:

$$c_l = q + \delta b_l. \tag{4.52}$$

Combining (4.50) and (4.52) yields $c_l > c_h$, hence $u'(c_l) < u'(c_h) < \eta u'(c_h)$. However, combination of (4.49) and (4.51) leads to $u'(c_l) > \eta u'(c_h)$, a contradiction. (2) If $b_l < b$, then $\xi_l = 0$. (4.28) gives

$$u'(c_l) = \frac{1}{\delta}\beta\frac{\phi}{\mu}. \tag{4.53}$$

If $\delta < 1$, (4.29) yields $\lambda_l = \frac{1}{\delta}\beta\frac{\phi}{\mu} - \beta\frac{\phi}{\mu} > 0$. This means (4.52) again, hence $c_l > c_h$ and $u'(c_l) < u'(c_h) < \eta u'(c_h)$. But (4.49) and (4.53) jointly give rise to $u'(c_l) = \eta u'(c_h)$, a contradiction. If $\delta = 1$, it must be true that $\delta\mu = 1$; but a combination of (4.49), (4.53) and (4.15) results in $\delta\mu = \beta$, a contradiction.

In conclusion, for (4.33) to hold it must be true that $b_l < 0, b_h > 0$.

4.9.5 Proof of Lemma 4.5

Differentiating (4.39) and (4.40) w.r.t. R and solving for $\frac{d\mu}{dR}$ result in $\frac{d\mu}{dR} = A/B$, where

$$\begin{aligned}
A &\equiv \mu\beta R u'(c_l)\left[\frac{c_l}{y}u''(c_l) + \frac{c_h}{y}\eta u''(c_h)\right] - 2\frac{\mu-1}{(R-1)^2}y\beta R^2\eta u''(c_h)u''(c_l) \\
&\quad -\mu^2R^2g''(y)\left\{\beta\left[u'(c_l)+\eta u'(c_h)\right] - \frac{\mu-1}{\mu(R-1)^2}y\left[(2-\beta R)u''(c_l)+\beta R\eta u''(c_h)\right]\right\}, \\
B &\equiv \frac{R^2}{R-1}yg''(y)\left[(2-\beta R)u''(c_l)+\beta R\eta u''(c_h)\right] - \frac{2}{\mu(R-1)}y\beta R^2\eta u''(c_h)u''(c_l) \\
&\quad +Ru'(c_l)\left[(2-\beta R)u''(c_l)\frac{c_l}{y} - \beta R\eta u''(c_h)\frac{c_h}{y}\right].
\end{aligned}$$

For all $\max\{R_0, 1\} < R < 1/\beta$, $\mu - 1 > 0$ and $2 - \beta R > 0$, hence $A < 0$. As to B, the first two terms are negative; the third term turns out to disappear since

$$\begin{aligned}
(2-\beta R)u''(c_l)\frac{c_l}{y} - \beta R\eta u''(c_h)\frac{c_h}{y} &= \frac{1}{y}\left[(2-\beta R)(-\rho u'(c_l)) - \beta R\eta(-\rho u'(c_h))\right] \\
&= -\rho\frac{1}{y}\left[(2-\beta R)u'(c_l) - \beta R\eta u'(c_h)\right] \\
&= 0.
\end{aligned}$$

Therefore, $B < 0$ and so that $\frac{d\mu}{dR} > 0$. Since $\delta = \frac{1}{\mu R}$, it is easy to verify that $\frac{d\delta}{dR} < 0$. Consequently, $\frac{d\delta}{d\mu} = \frac{d\delta}{dR}/\frac{d\mu}{dR} < 0$. Since $1 + i \equiv \frac{1}{\delta}$, $\frac{d(1+i)}{d\mu} = -\frac{1}{\delta^2}\frac{d\delta}{d\mu} > 0$.

$$\frac{d\theta}{dR} = \frac{d}{dR}\left(\frac{R(\mu-1)}{\mu(R-1)}\right) = \frac{R(R-1)\frac{d\mu}{dR} - (\mu-1)}{(R-1)^2} = \frac{R(R-1)A - (\mu-1)B}{B(R-1)^2}.$$

Substituting A and B into the numerator yields

$$\begin{aligned}
&R(R-1)\mu\beta R u'(c_l)\left[\frac{c_l}{y}u''(c_l)+\frac{c_h}{y}\eta u''(c_h)\right]\\
&-R(R-1)\mu^2R^2g''(y)\beta\left[u'(c_l)+\eta u'(c_h)\right]\\
&+(\mu R-1)\frac{\mu-1}{R-1}R^2g''(y)y\left[(2-\beta R)u''(c_l)+\beta R\eta u''(c_h)\right]\\
&+2(1-\mu R)\frac{\mu-1}{\mu(R-1)}y\beta R^2\eta u''(c_h)u''(c_l).
\end{aligned}$$

Since $\mu R-1=1/\delta-1\geq 0$, the numerator turns out to be negative. Meanwhile, the denominator is also negative due to $B<0$. In combination, it follows that $\frac{d\theta}{dR}>0$.

4.9.6 Proof of Proposition 4.3

Let $\chi\equiv\delta\theta$, equations (4.45) and (4.46) become

$$\left(2\frac{\delta+\chi}{1+\chi}-\beta\right)u'\left[(1-\chi)y\right] = \beta\eta u'\left[(1+\chi)y\right], \tag{4.54}$$

$$g'(y) = \delta u'\left[(1-\chi)y\right]; \tag{4.55}$$

and the welfare function becomes

$$\mathcal{W}=\frac{u\left[(1-\chi)y\right]+\eta u\left[(1+\chi)y\right]-2g(y)}{2(1-\beta)}.$$

Consequently,

$$\begin{aligned}
\frac{d\mathcal{W}}{d\delta} &= \frac{1}{2(1-\beta)}\left\{u'\left[(1-\chi)y\right](1-\chi)+\eta u'\left[(1+\chi)y\right](1+\chi)-2g'(y)\right\}\frac{dy}{d\delta}\\
&\quad+\frac{1}{2(1-\beta)}\left\{-u'\left[(1-\chi)y\right]+\eta u'\left[(1+\chi)y\right]\right\}y\frac{d\chi}{d\delta}.
\end{aligned} \tag{4.56}$$

Differentiating (4.54) and (4.55) w.r.t. δ yields

$$\begin{aligned}
& -\frac{2(1-\chi)}{(1+\chi)^2}u'[(1-\chi)y]\frac{d\chi}{d\delta} \\
& +\left\{\left(2\frac{\delta+\chi}{1+\chi}-\beta\right)u''[(1-\chi)y]+\beta\eta u''[(1+\chi)y]\right\}y\frac{d\chi}{d\delta} \\
& +\left\{\beta\eta u''[(1+\chi)y](1+\chi)-\left(2\frac{\delta+\chi}{1+\chi}-\beta\right)u''[(1-\chi)y](1-\chi)\right\}\frac{dy}{d\delta} \\
= & \frac{2}{1+\chi}u'[(1-\chi)y], \qquad (4.57)
\end{aligned}$$

$$\delta u''[(1-\chi)y]y\frac{d\chi}{d\delta}+\{g''(y)-\delta u''[(1-\chi)y](1-\chi)\}\frac{dy}{d\delta}=u'[(1-\chi)y]. \quad (4.58)$$

(1) If $\eta > \eta_0$. With a policy $(\delta=1,\mu=1,\theta=1-c_{l1}/y_1\equiv\theta_1)$, $\chi=\theta_1$, and equations (4.57) and (4.58) become

$$\begin{aligned}
\left\{\begin{array}{c}[(2-\beta)u''(c_{l1})+\beta\eta u''(c_{h1})]y_1\frac{d\chi_1}{d\delta} \\ +[\beta\eta u''(c_{h1})(1+\theta_1)-(2-\beta)u''(c_{l1})(1-\theta_1)]\frac{dy_1}{d\delta}\end{array}\right\} &= \frac{2}{1+\theta_1}u'(c_{l1}), \\
u''(c_{l1})y_1\frac{d\chi_1}{d\delta}+[g''(y_1)-u''(c_{l1})(1-\theta_1)]\frac{dy_1}{d\delta} &= u'(c_{l1}).
\end{aligned}$$

where $\frac{d\chi_1}{d\delta}\equiv\frac{d\chi}{d\delta}|_{\delta=1}$ and $\frac{dy_1}{d\delta}\equiv\frac{dy}{d\delta}|_{\delta=1}$. Solving $\frac{d\chi_1}{d\delta}$ and $\frac{dy_1}{d\delta}$ and then plugging into (4.56) yields

$$\begin{aligned}
\frac{d\mathcal{W}}{d\delta}|_{\delta=1} &= \frac{u'(c_{l1})[\eta u'(c_{h1})-u'(c_{l1})]}{2(1-\beta)\{2\beta\eta u''(c_{h1})u''(c_{l1})-g''(y_1)[(2-\beta)u''(c_{l1})+\beta\eta u''(c_{h1})]\}} \\
&\quad \cdot\left[-\frac{2}{1+\theta_1}g''(y_1)+2\left(\frac{1-\theta_1}{1+\theta_1}+\beta-1\right)u''(c_{l1})\right].
\end{aligned}$$

The first term is positive since $\mathcal{D}<1$ at $\delta=1$, but the sign of the second term is indeterminate.

Note (c_{l1},c_{h1}) solves $(2-\beta)u'(c_{l1})=\beta\eta u'(c_{h1})$, which leads to

$$\frac{1-\theta_1}{1+\theta_1}=\frac{c_{l1}}{c_{h1}}=\left(\frac{\beta\eta}{2-\beta}\right)^{-1/\rho}.$$

Therefore, the sufficient condition for $\frac{d\mathcal{W}}{d\delta}|_{\delta=1}<0$ turns out to be a restriction on parameters:

$$\left(\frac{\beta\eta}{2-\beta}\right)^{-1/\rho}+\beta-1>0.$$

Note that $\frac{\beta\eta}{2-\beta} > 1$ since $\eta > \eta_0$. Then the restriction boils down to

$$\rho > \frac{\ln(2-\beta) - \ln\beta - \ln\eta}{\ln(1-\beta)}.$$

(2) If $\eta \leq \eta_0$. In the limit case of $\theta = 0$, $\delta = \delta_0 \equiv \beta(1+\eta)/2$, $\mu = 1$, and $\chi = 0$. Denote $\frac{d\chi_0}{d\delta} \equiv \frac{d\chi}{d\delta}|_{\delta=\delta_0}$ and $\frac{dy_0}{d\delta} \equiv \frac{dy}{d\delta}|_{\delta=\delta_0}$. Solving (4.57) and (4.58) yields

$$\begin{aligned}
\frac{d\chi_0}{d\delta} &= \frac{u'(y_0)}{\beta\eta u''(y_0)\,y_0 - u'(y_0)}, \\
\frac{dy_0}{d\delta} &= \frac{u'(y_0)\left[(\beta\eta-\delta_0)\,u''(y_0)\,y_0 - u'(y_0)\right]}{\left[g''(y_0) - \delta_0 u''(y_0)\right]\left[\beta\eta u''(y_0)\,y_0 - u'(y_0)\right]}.
\end{aligned}$$

Now (4.56) becomes

$$\begin{aligned}
\frac{d\mathcal{W}}{d\delta}|_{\delta=\delta_0} &= \frac{1}{2(1-\beta)}\left\{(\eta-1)\,u'(y_0)\,y_0\frac{d\chi_0}{d\delta} + (\eta+1-2\delta_0)\,u'(y_0)\frac{dy_0}{d\delta}\right\} \\
&= \frac{u'(y_0)^2\left\{(\eta-1)\,y_0 g''(y_0) + \frac{\eta+1}{2}\left[\beta^2(1-\eta)\,y_0 u''(y_0) - 2(1-\beta)\,u'(y_0)\right]\right\}}{2(1-\beta)\left[g''(y_0)-\delta_0 u''(y_0)\right]\left[\beta\eta u''(y_0)\,y_0 - u'(y_0)\right]} \\
&= \frac{u'(y_0)^2\left\{(\eta-1)\,y_0 g''(y_0) + \frac{\eta+1}{2}u'(y_0)\left[\rho\beta^2(\eta-1) - 2(1-\beta)\right]\right\}}{2(1-\beta)\left[g''(y_0)-\delta_0 u''(y_0)\right]\left[\beta\eta u''(y_0)\,y_0 - u'(y_0)\right]}.
\end{aligned}$$

Then it is straightforward to see that a sufficient condition for $\frac{d\mathcal{W}}{d\delta}|_{\delta=\delta_0} < 0$ is

$$\rho > \frac{2(1-\beta)}{\beta^2(\eta-1)}.$$

Lastly note that $\frac{d\mathcal{W}}{d\delta}|_{\delta=\delta_0} < 0$ is equivalent to $\frac{d\mathcal{W}}{d\theta}|_{\theta=0} > 0$ since $\frac{d\theta}{d\delta} = \frac{d\theta}{dR}/\frac{d\delta}{dR} < 0$ by Lemma 4.5.