PRICING AND DYNAMIC HEDGING OF SEGREGATED FUND GUARANTEES

by

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Abstract

Guaranteed minimum maturity benefit and guaranteed minimum death benefit offered by a single premium segregated fund contract are priced. A dynamic hedging approach is used to determine the value of these guarantees. Cash flow projections are used to analyze the loss or profit to the insurance company. Optimally exercised reset options are priced by the Crank-Nicolson method. Reset options, assuming they are exercised only when the funds exceed a given threshold, are priced using simulations. Finally, we study the distribution of the loss or profit for segregated funds with reset option under a dynamic hedging strategy with an allowance for transaction costs.

Keywords: Guaranteed Minimum Maturity Benefit, Guaranteed Minimum Death Benefit, Charge Rate, Black-Scholes European Put Option, Dynamic Hedging, Hedge Error, Transaction Cost, Transaction Costs Adjusted Hedge Volatility, Reset Option, Crank-Nicolson

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Chapter 1

Introduction

A segregated fund is a type of equity-linked insurance contract commonly sold in Canada. Policyholders are usually offered a broad selection of investment choices and these funds are fully separated from the company's general investment funds (Wikipedia, http://en.wikipedia.org/wiki/Segregated_fund). The segregated fund normally provides a guaranteed minimum maturity benefit (GMMB) and a guaranteed minimum death benefit (GMDB). The benefit guarantee is 75% or higher percentage of the initial fund value (Moodys Looks At Guaranteed Segregated Funds In Canada & Their Risks, August 2001). In case of death within the term of the contract or at maturity, the benefit amount is the maximum of the accumulated fund value and the guarantee.

Segregated fund guarantees are financial guarantees. In most cases the payoffs of the benefits are zero, while due to some poor investment a considerable amount of additional money is needed to compensate the gap between the fund value and the guarantee level. The risk arisen by the highly skewed payoff distribution cannot be diversified by pooling segregated fund contracts with the same maturity date.

One approach commonly used in practice is called dynamic hedging which considers the segregated fund as a special type of European put option (Great-West Life Segregated Fund Policies Information Folder, May 2010). Many issues of this approach have been discussed in mathematical finance such as *Leland* (1985) and *Toft* (1996), and was applied to the segregated fund by *Boyle* and *Hardy* (1997) and *Hardy* (2000, 2001, 2002). A hedging portfolio consisting of some risk-free bond and risky asset is held at the beginning of the contract term and the segregated fund is several times before it matures. *Leland* (1985) introduced the transaction costs adjusted hedge volatility (*Leland*'s volatility) with which the hedge errors net of transaction costs tends to zero as the hedge interval decreases. *Toft* (1996) derived expressions for the mean and variance of the hedge error and transaction cost. In *Boyle* and *Hardy* (1997) and *Hardy* (2000, 2002) dynamic hedging was used to hedge a segregated fund contract. The results derived were based on simulation and *Leland*'s volatility was not adopted.

Segregated fund contract lasts for at least 10 years (Moodys Looks At Guaranteed Segregated Funds In Canada & Their Risks, August 2001). Within the contract term, policyholders are given the option to reset the contract several times within some period (usually one or two times per year). Upon reset, the GMMB and GMDB would be reset to the guarantee levels of the current fund value, and the contract lasts for another 10 years from the time of reset. This feature adds more complexity to the valuation of the segregated fund. In *Armstrong* (2001) some techniques for the optimal reset decisions to a simplified segregated fund contract were discussed. *Windcliff*, *Forsyth* and *Vetzal* (2001a, 2001b and 2002) discussed the valuation of segregated funds with reset options by employing finite difference methods; one of which, commonly used by financial engineers, is call the Crank-Nicolson method. This method requires some discretization techniques and optimality assumptions to approximately solve a collection of partial differential equations (PDE) backwards for the price of the contract. The uncertainties of the contract duration and the guarantee benefits add more volatility to the risks of the segregated fund.

The main focus of this project is to discuss some risk issues arisen by the features of the segregated fund and apply dynamic hedging approach to both no-reset-allowed (standard) and reset-allowed (extendable) segregated fund contracts. For simplicity, no lapse is assumed. Two types of the segregated fund contracts are considered. Type I offers 100% GMMB level and 75% GMDB level and Type II offers 75% GMMB level and 100% GMDB level. We use the traditional geometric Brownian motion to model the risky asset value that the premium of the segregated fund is invested into. We also assume that the capital set by dynamic hedging approach is put into a zero-coupon bond which offers a constant risk-free interest rate. Note that all the analysis in this project do not consider either model risk or parameter risk. In different time period or other situations, the model employed and the parameters estimated in this project might not be accurate. The idea is to provide a convenient way to focus on the main purpose of this project.

The remainder of the project is organized as follows: in Chapter 2, the model is introduced and the standard segregated fund is priced. Chapter 3 discusses dynamic hedging approach and the corresponding cash flows. In the following chapter, the reset option is priced and discussed, and the distribution of the loss or profit for segregated funds with reset option under a dynamic hedging strategy is studied.

Chapter 2

The model

2.1 The model for the asset

We assume that the market price of the asset in which the segregated fund is invested follows a geometric Brownian motion. That is, if S_t is the asset price at time t, then

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{2.1}$$

where μ is the drift rate, σ is the volatility and B_t stands for a standard Brownian motion. This implies that the return on asset over discrete time intervals follows an independent normal distribution. That is,

$$log\frac{S_{t_2}}{S_{t_1}} \sim N\left((\mu - \frac{1}{2}\sigma^2)(t_2 - t_1), \sigma^2(t_2 - t_1)\right),$$

where $t_2 > t_1 \geq 0$ and $N(a, b)$ stands for a normal distribution with mean a and variance b.

We also assume that the fund premium is invested in S&P/TSX Composite Index (the name was TSE 300 before May 1, 2002). Based on the monthly data from 2000 to 2009 (Data source: Yahoo! Finance, http://finance.yahoo.com), the estimated values of the parameters in Equation (2.1) are $\mu = 0.04$ and $\sigma = 0.16$.

Figure 2.1 shows four types of asset price sample paths for 10 years based on the estimated parameters. The dashed and dotted parallel lines represent the 100% and 75% levels of the initial asset value. Figure 2.1 (a) and (b) give two opposite price path trends, while Figure 2.1 (c) and (d) show the price paths that fluctuate around the 100% guarantee level, the difference being that (c) ends up at the safe zone (above both parallel lines) and (d) is subject to the 100% guarantee risk.

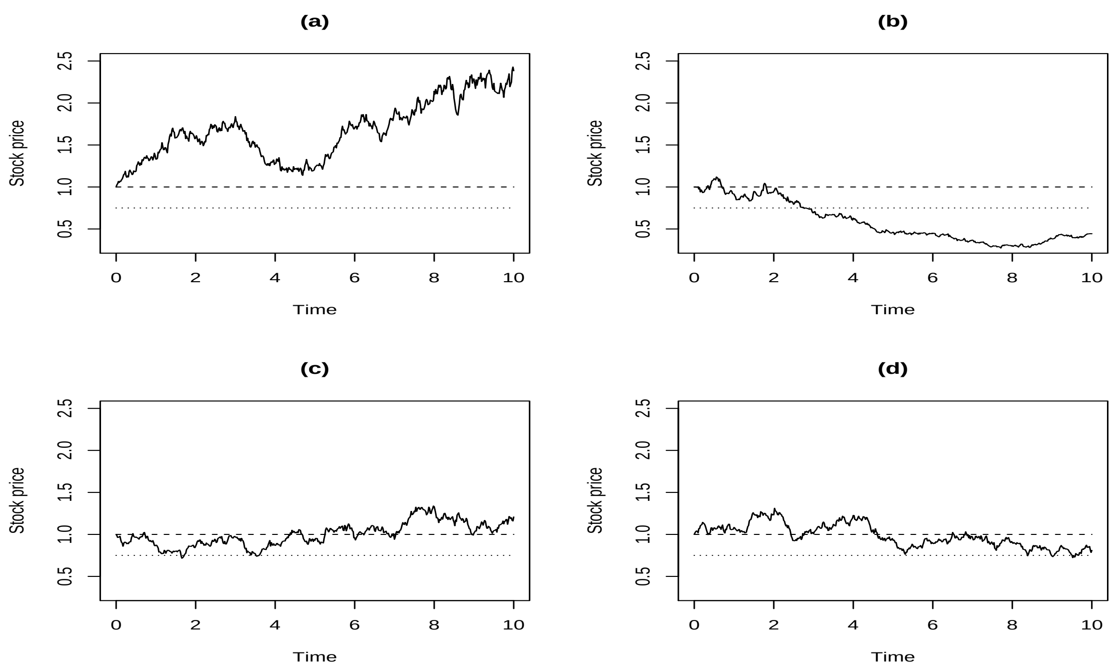

Figure 2.1: Examples of asset price processes

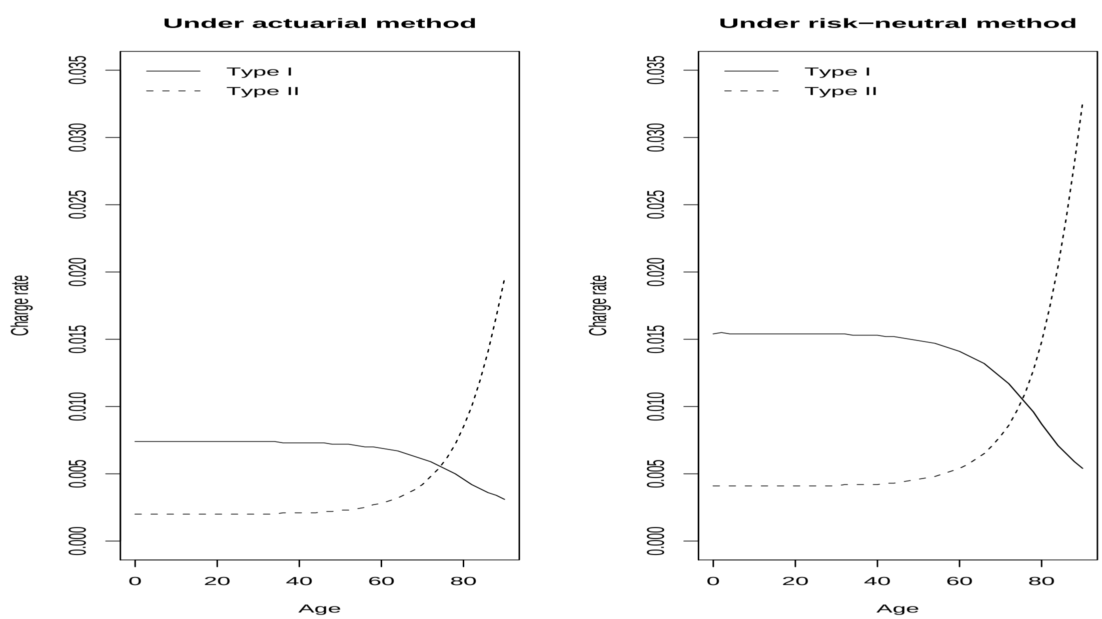

Figure 2.2: The instantaneous charge rates

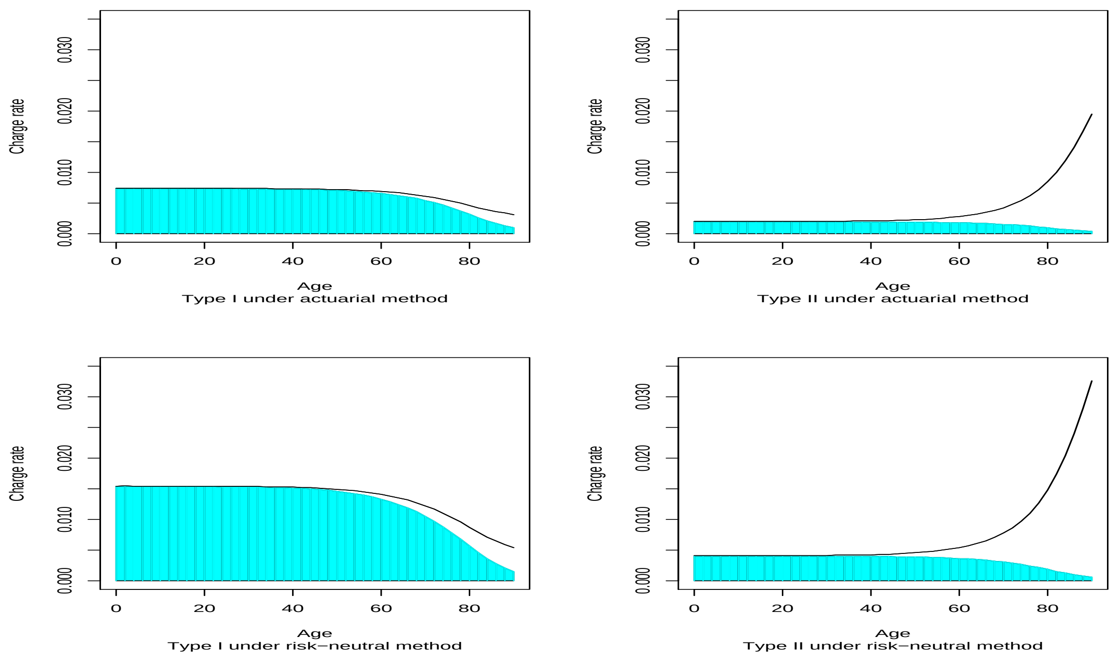

Figure 2.3: The composition of the instantaneous charge rates

2.2 Pricing the segregated fund guarantees

The fees are charged periodically from policyholders and made up of two components. The first includes surplus margins, management charge to cover operation cost of the fund, etc. The second is the insurance charge to cover the benefit protection in the guarantees. In this project, the first charge is assumed to be zero and we assume the fees are deducted from the fund annually. For further analysis, we define the loss function of the segregated fund at the payoff time t as

$$\mathcal{L}(G, t) = max(G - S_t(1-m)^t, 0)$$

or

$$\mathcal{L}(G, t) = max(G - S_t e^{-f_m t}, 0)$$

for a given guarantee G , the annual charge rate m (or the equivalent instantaneous charge rate f_m) and $t = 0, 1, \ldots, T$. This definition shows that the loss occurs when the performance of the asset is so poor that the asset amount after charge deductions is below the guarantee. In this project we follow the indifference principle which suggests that the expected total insurance charges should at least cover the expected future costs. Then in traditional actuarial notations we have

$$\int_0^T f_m \mathbb{E}[S_t] e^{-t(f_m+r)} {}_t p_x dt$$
$$= {}_T p_x \mathbb{E}[\mathcal{L}(G_m, T)] e^{-Tr} + \int_0^T \mathbb{E}[\mathcal{L}(G_d, t)] e^{-tr} \mu(x+t) {}_t p_x dt, \qquad (2.2)$$

where T is the term of the segregated fund contract (for the standard segregated fund contracts, T is 10 years), x is the age of the policyholder when the policy is written, G_d and G_m are the GMDB and GMMB guarantees respectively, and r, the risk-free force of interest, is assumed to be 2.25% (Data source: TD Canada Trust prime rate effective on April 22, 2009, http://www.tdcanadatrust.com) in this project. The two terms of the right hand side of Equation (2.2) can be viewed as the expected costs for GMMB and GMDB respectively. The left hand side can be considered as the total expected charges during T years. Assuming a constant force of mortality μ_x for the mortality decrements in fraction year x (*Bowers*, *Gerber* and etc 1997) and under the actuarial method which states that the asset price accumulates with drift μ, Equation (2.2) becomes

$$\sum_{k=0}^{T-1} \int_0^1 f_m S_0 e^{-(k+s)(f_m+r-\mu)} {}_k p_x e^{-s\mu_{x+k}} ds$$
$$= {}_T p_x \mathbb{E}[\mathcal{L}(G_m, T)] e^{-Tr} + \sum_{k=0}^{T-1} \int_0^1 \mathbb{E}[\mathcal{L}(G_d, k+s)] e^{-(k+s)r} {}_k p_x \mu_{x+k} e^{-s\mu_{x+k}} ds,$$

where

$$\begin{aligned} \mathbb{E}[L(G,t)] \quad &= \quad G\Phi\left(\frac{log(\frac{G}{S_0}) - (\mu - f_m - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) \\ &\quad -S_0 e^{(\mu - f_m)t} \Phi\left(\frac{log(\frac{G}{S_0}) - (\mu - f_m + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right). \end{aligned}$$

with a normal cumulative density function $\Phi(\cdot)$. Under the risk-neutral method which replaces drift μ with r, Equation (2.2) becomes

$$\sum_{k=0}^{T-1} \int_0^1 f_m S_0 e^{-(k+s)f_m} {}_k p_x e^{-s\mu_{x+k}} ds$$
$$= \quad {}_T p_x P(S_0 e^{-Tf_m}, G_m, 0, T) + \sum_{k=0}^{T-1} \int_0^1 P(S_0 e^{-(k+s)f_m}, G_d, 0, k+s) {}_k p_x \mu_{x+k} e^{-s\mu_{x+k}} ds,$$

where $P(\cdot,\cdot,\cdot,\cdot)$ represents the Black-Scholes European put option formula which is defined as

$$P(S_{t_0}, K, t_0, T) = Ke^{-r(T-t_0)}\Phi\left(-d_2(t_0)\right) - S_{t_0}\Phi\left(-d_1(t_0)\right), \tag{2.3}$$

where S_{t_0} is the asset price at time t_0, K is the strike, T is the maturity date,

$$d_1(t_0) = \frac{log\frac{S_{t_0}}{K} + (r + \frac{1}{2}\sigma^2)(T - t_0)}{\sigma\sqrt{T - t_0}},$$

and

$$d_2(t_0) = d_1(t_0) - \sigma\sqrt{T - t_0}.$$

Figure 2.2 shows the instantaneous charge rates with respect to different ages under both actuarial and risk-neutral approaches (The mortality data is from Complete Life Table, Canada, 2000 to 2002: males, Statistics Canada.). We can see that the charge rates are relatively insensitive to ages unless the policyholder gets relatively old (e.g. age 60). It shows that Type I contract requires higher charge rates than Type II contract when the policyholder is aged below 78 since GMMB dominates the cost. On the other hand, however, the Type II contract seems more sensitive to mortality risk when the policyholder gets older. It should be also noted that the charge rates under the actuarial method are in general lower than those under risk-neutral method. This is due to the higher asset drift than risk-free force of interest rate, which would in turn affect the expected values of asset price.

Figure 2.3 helps us understand the evolution of the charge rate with age. The shaded areas represent the parts to cover the GMMB. And the areas between upper line and the shaded areas represent the charges for the GMDB. As we can see, the charge for the GMMB dominates the total charge for the segregated fund when the policyholder is relatively young while the GMDB part of the cost would increase with age. On the other hand, the GMDB charge increment with age is greater for Type II contract than Type I contract, which explains sharper changes of the charge rate for Type II contract.

Chapter 3

Dynamic hedging approach

3.1 Discrete time hedging strategy

The spirit of the Black-Scholes model is the possibility to set up a portfolio consisting of risky asset and risk-free bond that replicates the payoff of the derivative. The market is free of friction and the portfolio is assumed to be continuously adjusted. *Leland* (1985) showed that the change of the derivative price and the change of the value of replicating portfolio during some time interval are different. The difference is called hedge error.

Dynamic hedging is a strategy that adjusts the hedging portfolio several times before maturity. Suppose we intend to hedge a European put option. The value of the hedging portfolio is given by Equation (2.3). The Black-Scholes formula indicates that the initial hedging portfolio should consist of $\Phi\left(-d_2(t_0)\right)$ units of risk-free bond and a short position of $\Phi\left(-d_1(t_0)\right)$ shares of risky asset.

Suppose at the first hedge time t_1, The required hedging portfolio, H^+, is given by

$$H^+ = Ke^{-r(T-t_1)}\Phi\left(-d_2(t_1)\right) - S_{t_1}\Phi\left(-d_1(t_1)\right).$$

On the other hand, the holding position of the hedging portfolio cannot be self-adjusted. The accumulated hedging portfolio from t_0 to t_1, H^-, is given by

$$H^- = Ke^{-r(T-t_1)}\Phi\left(-d_2(t_0)\right) - S_{t_1}\Phi\left(-d_1(t_0)\right).$$

The hedge error at time t_1, $N(t_1)$ is then shown as

$$\begin{aligned} H(t_1) &= H^- - H^+ \\ &= Ke^{-r(T-t_1)}\left[\Phi\left(-d_2(t_0)\right) - \Phi\left(-d_2(t_1)\right)\right] \\ &\quad -S_{t_1}\left[\Phi\left(-d_1(t_0)\right) - \Phi\left(-d_1(t_1)\right)\right]. \end{aligned} \tag{3.1}$$

The transaction cost is defined as the fees to trade the risky asset. Let $TC(t_1)$ denote the transaction cost at t_1, then

$$TC(t_1) = kS_{t_1}\left|\Phi\left(-d_1(t_1)\right) - \Phi\left(-d_1(t_0)\right)\right|, \tag{3.2}$$

where k is the one-way transaction cost rate. *Toft* (1996) derived the expressions for the means of the hedge errors and transaction costs if the price of the asset follows Equation (2.1). In general, let t_{j-1} and t_j denote the two consecutive hedge times before maturity, where $T \geq t_{j-1} > t_j \geq t_0$. The hedge interval Δt is equal to $t_j - t_{j-1}$. Following *Toft*'s derivation and assuming that the hedge errors and transaction costs are discounted at the risk-free rate r, the respective expected present values of the hedge error and transaction cost at time t_j, given S_{t_0}, are

$$\begin{aligned} \mathbb{E}[H(t_j,T)|S_{t_0}] &= Ke^{-r(T-t_0)}\left[\Phi\left(-d_2^*(t_0,t_{j-1})\right) - \Phi\left(-d_2^*(t_0,t_j)\right)\right] \\ &\quad -S_0e^{(\mu-r)(t_j-t_0)}\left[\Phi\left(-d_1^*(t_0,t_{j-1})\right) - \Phi\left(-d_1^*(t_0,t_j)\right)\right], \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \mathbb{E}[TC(t_j,T)|S_{t_0}] &= kS_0e^{(\mu-r)(t_j-t_0)}\left\{2\Phi\left(\Upsilon, -d_1^*(t_0,t_j), \kappa_1\right) - \Phi\left(-d_1^*(t_0,t_j)\right)\right. \\ &\quad \left. +\Phi\left(-d_1^*(t_0,t_{j-1})\right) - 2\Phi\left(-d_1^*(t_0,t_{j-1}), \Upsilon, \kappa_2\right)\right\}, \end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
d_1^*(t_{j-1}, t_j) &= \frac{log\frac{S_{t_{j-1}}}{K} + (r + \frac{1}{2}\sigma^2)(T - t_j) + (\mu + \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{T - t_{j-1}}}, \\
d_2^*(t_{j-1}, t_j) &= d_1^*(t_{j-1}, t_j) - \sigma\sqrt{T - t_{j-1}}, \\
\Upsilon &= d_1^*(t_0, t_{j-1})\sqrt{\frac{(T - t_0)(T - t_j)}{t^*}} - d_1^*(t_0, t_j)\sqrt{\frac{(T - t_0)(T - t_{j-1})}{t^*}}, \\
\kappa_1 &= \frac{(t_j - t_0)\sqrt{T - t_{j-1}} - (t_{j-1} - t_0)\sqrt{T - t_j}}{\sqrt{(T - t_0)t^*}}, \\
\kappa_2 &= \frac{(t_{j-1} - t_0)\sqrt{T - t_{j-1}} - (t_{j-1} - t_0)\sqrt{T - t_j}}{\sqrt{(T - t_0)t^*}}, \\
t^* &= (T - t_{j-1})(t_j - t_0) + (T - t_j)(t_{j-1} - t_0) \\
&\quad -2(t_{j-1} - t_0)\sqrt{(T - t_j)(T - t_{j-1})},
\end{aligned}
$$

and $\Phi(\cdot, \cdot, \rho)$ denotes the bivariate standard normal distribution with correlation coefficient ρ. The derivation of Equations (3.3) and (3.4) are shown in Appendix A. Further, the expected present value of total hedge errors net of transaction costs during the lifetime of the option is denoted by $\Psi(t_0, T)$, where

$$\Psi(t_0, T) = \sum_{j=1}^{\varpi} \{\mathbb{E}[H(t_j, T)|S_{t_0}] - \mathbb{E}[TC(t_j, T)|S_{t_0}]\} \tag{3.5}$$

and ϖ stands for the total hedge times during $T - t_0$. Note that in this project we only consider the time-based hedging strategy, i.e., Δt is constant during the term of the derivative.

3.2 Transaction costs adjusted hedge volatility

The value of the hedge errors net of transaction costs depends on the hedge interval. If the interval is too large, the transaction costs are small but the hedge errors are large. On the other hand, if the interval gets relatively small, the hedge errors decrease but the transaction costs tend to be high. *Leland* (1985) introduced a modified hedge volatility for which the modified hedge errors, net of transaction costs, are almost surely zero as the hedge interval width tends to zero. If we let $\hat{\sigma}^2$ denote *Leland*'s volatility, then with the one-way transaction cost rate k,

$$\hat{\sigma}^2 = \sigma^2 \left(1 + \frac{2k\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{\Delta t}}\right).$$

With *Leland*'s volatility we can get the expected present values of hedge error and transaction cost similar to Equations (3.3) and (3.4), that is

$$\begin{aligned}\mathbb{E}[\hat{H}(t_j,T)|S_{t_0}] &= Ke^{-r(T-t_0)}\left[\Phi\left(-\hat{d_2}^*(t_0,t_{j-1})\right)-\Phi\left(-\hat{d_2}^*(t_0,t_j)\right)\right]\\ &\quad -S_0e^{(\mu-r)(t_j-t_0)}\left[\Phi\left(-\hat{d_1}^*(t_0,t_{j-1})\right)-\Phi\left(-\hat{d_1}^*(t_0,t_j)\right)\right], \end{aligned}\quad (3.6)$$

$$\begin{aligned}\mathbb{E}[\widehat{TC}(t_j,T)|S_{t_0}] &= kS_0e^{(\mu-r)(t_j-t_0)}\left\{2\Phi\left(\hat{\Upsilon},-\hat{d_1}^*(t_0,t_j),\hat{\kappa}_1\right)-\Phi\left(-\hat{d_1}^*(t_0,t_j)\right)\right.\\ &\quad \left.+\Phi\left(-\hat{d_1}^*(t_0,t_{j-1})\right)-2\Phi\left(-\hat{d_1}^*(t_0,t_{j-1}),\hat{\Upsilon},\hat{\kappa}_2\right)\right\}, \end{aligned}\quad (3.7)$$

where

$$\begin{aligned}\hat{d_1}^*(t_{j-1},t_j) &= \frac{log\frac{S_{t_{j-1}}}{K}+(r+\frac{1}{2}\hat{\sigma}^2)(T-t_j)+(\mu+\frac{1}{2}\sigma^2)\Delta t}{\sqrt{\hat{\sigma}^2(T-t_j)+\sigma^2(t_j-t_{j-1})}},\\ \hat{d_2}^*(t_{j-1},t_j) &= \hat{d_1}^*(t_{j-1},t_j)-\sqrt{\hat{\sigma}^2(T-t_j)+\sigma^2(t_j-t_{j-1})},\\ \hat{\Upsilon} &= \hat{d_1}^*(t_0,t_{j-1})\frac{\sqrt{(T-t_j)(\hat{\sigma}^2(T-t_{j-1})+\sigma^2(t_{j-1}-t_0))}}{\sigma\sqrt{t^*}}\\ &\quad -\hat{d_1}^*(t_0,t_j)\frac{\sqrt{(T-t_{j-1})(\hat{\sigma}^2(T-t_j)+\sigma^2(t_j-t_0))}}{t^*},\\ \hat{\kappa}_1 &= \frac{\sigma((t_j-t_0)\sqrt{T-t_{j-1}}-(t_{j-1}-t_0)\sqrt{T-t_j})}{\sqrt{(\hat{\sigma}^2(T-t_j)+\sigma^2(t_j-t_0))t^*}},\\ \hat{\kappa}_2 &= \frac{\sigma((t_{j-1}-t_0)\sqrt{T-t_{j-1}}-(t_{j-1}-t_0)\sqrt{T-t_j})}{\sqrt{(\hat{\sigma}^2(T-t_{j-1})+\sigma^2(t_{j-1}-t_0))t^*}},\end{aligned}$$

and the expected present value of total hedge errors net of transaction costs can be written as

$$\hat{\Psi}(t_0,T)=\sum_{j=1}^{\varpi}\left\{E[\hat{H}(t_m,T)|S_{t_0}]-E[\widehat{TC}(t_m,T)|S_{t_0}]\right\}. \quad (3.8)$$

In the following part of the project k is assumed to be 0.005. Tables 3.1 and 3.2 show the hedge costs based on different hedge strategies. We assume a 10-year at-the-money European put option with $S_0 = \$100$. As expected, the initial costs with *Leland*'s volatility are higher than those in Table 3.1 as long as the positive transaction costs are charged. The hedge errors in Table 3.2 which turn out to be positive (negative values of the hedge errors (net of the transaction costs) mean extra asset should be bought, while positive ones mean the selling of the asset), along with the transaction costs, make the hedge errors net of the

Table 3.1: The hedge costs with σ

	Put price	Hedge errors	Transaction costs	Hedge errors net of transaction costs	Total costs
Annually	13.5872	-0.0803	0.4268	-0.5071	14.0943
Monthly	13.5872	-0.0066	1.3778	-1.3844	14.9716
Weekly	13.5872	-0.0015	2.8485	-2.8501	16.4373
Daily	13.5872	-0.0002	7.5312	-7.5314	21.1186

Table 3.2: The hedge costs with $\hat{\sigma}$

	Put price	Hedge errors	Transaction costs	Hedge errors net of transaction costs	Total costs
Annually	13.9850	0.3101	0.4242	-0.1141	14.0991
Monthly	14.9340	1.3180	1.3387	-0.0208	14.9548
Weekly	16.3015	2.6740	2.6842	-0.0102	16.3117
Daily	20.1631	6.5188	6.5240	-0.0052	20.1683

transaction costs relatively small. Specifically, under daily hedge strategy, the hedge errors net of the transaction costs are almost zero using *Leland*'s volatility, while the costs reach a relatively high level with σ.

Figure 3.1 shows the hedge errors net of the transaction costs at each hedge time based on annual, monthly, weekly and daily hedge strategies, respectively. The solid lines represent the costs based on σ and the dotted lines correspond with *Leland*'s volatility. The same put option as above is assumed. The costs are lower for *Leland*'s volatility and are almost zero for the monthly hedge. The sharp changes at the end of the term reflect the relatively large amount of the traded asset shares.

Note that for a segregated fund contract, annual charges may make the total hedge errors net of transaction costs not strictly decreasing as the hedge frequency increases. However,

Table 3.3: The hedge errors net of the transaction costs with σ. The annual charge is deducted

Annual charge rate	Annually	Monthly	Weekly	Daily
1%	-0.5726	-1.5372	-3.1628	-8.3571
2%	-0.6310	-1.7188	-3.5372	-9.3461
3%	-0.6708	-1.8519	-3.8120	-10.0721

Table 3.4: The hedge errors net of the transaction costs with $\hat{\sigma}$. The annual charge is deducted

Annual charge rate	Annually	Monthly	Weekly	Daily
1%	-0.1776	-0.1637	-0.2917	-0.6780
2%	-0.2300	-0.3237	-0.6081	-1.4257
3%	-0.2789	-0.4875	-0.9335	-2.1970

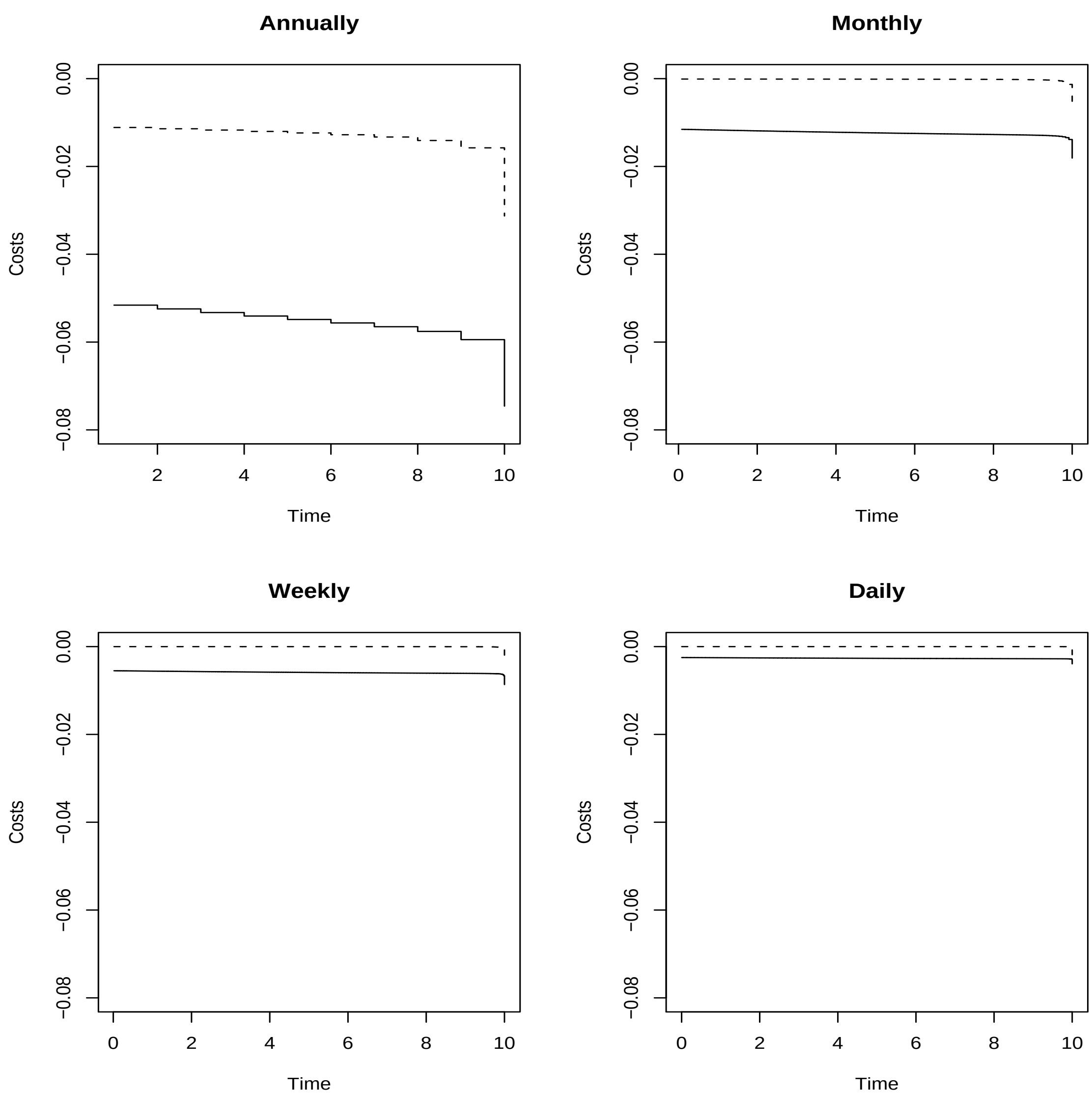

Figure 3.1: The hedge errors net of the transaction costs at each hedge time

Leland's volatility still has a positive effect on the hedge costs. As shown in Tables 3.3 and 3.4, from annual to daily hedge, the total hedge errors net of the transaction costs with σ increase up to about fifteen times, while they only increase by three times for 1% of the annual charge rate and seven times for 3% of the annual charge rate under the effect of *Leland*'s volatility.

3.3 Hedging for the segregated fund contract

3.3.1 The initial hedge cost

The hedge cost for the segregated fund contract includes two parts: the cost to set up the hedging portfolio (HP cost) and the hedge error net of the transaction cost (H&T) reserve. The HP cost is determined by the modified Black-Scholes European put option formula incorporating *Leland*'s volatility. Assuming deaths can only occur at the end of the year, the formula for the HP cost is

$$\begin{aligned} MP &= \sum_{j=0}^{T-1} \hat{P}(S_0(1-m)^T, G_d, 0, j+1) \cdot {}_{j|}q_x \\ &\quad + \hat{P}(S_0(1-m)^T, G_m, 0, T) \cdot {}_T p_x, \end{aligned} \tag{3.9}$$

where G_m and G_d represent the GMMB and GMDB guarantees, respectively, and MP denotes the initial hedge cost for the $\$G_m$ GMMB and $\$G_d$ GMDB contract for a policyholder aged x at time 0 with asset price S_0. Equation (3.9) states that the HP cost for the segregated fund contract consists of a collection of hedging portfolios whose initial values are defined in Equation (2.3) incorporating *Leland*'s volatility with different durations. Therefore, to fund the H&T reserve, we should calculate the expected present value of total hedge errors net of the transaction costs for each of those hedge portfolios. Let ${}_0V_T^{(H\&T)}$ denote the H&T reserve for a T-year contract at time 0, then

$${}_0V_T^{(H\&T)} = \sum_{j=0}^{T-1} \hat{\Psi}(0, j+1) \cdot {}_{j|}q_x + \hat{\Psi}(0,T) \cdot {}_T p_x. \tag{3.10}$$

When applying Equation (3.10) the charges should be taken into account. That is, the annual charge should be deducted from the segregated fund before the hedging portfolio is modified at each integer year. Figures 3.2, 3.3 and 3.4 show some numerical results based on annual, monthly and weekly hedge strategies, respectively, for a ten-year contract with a

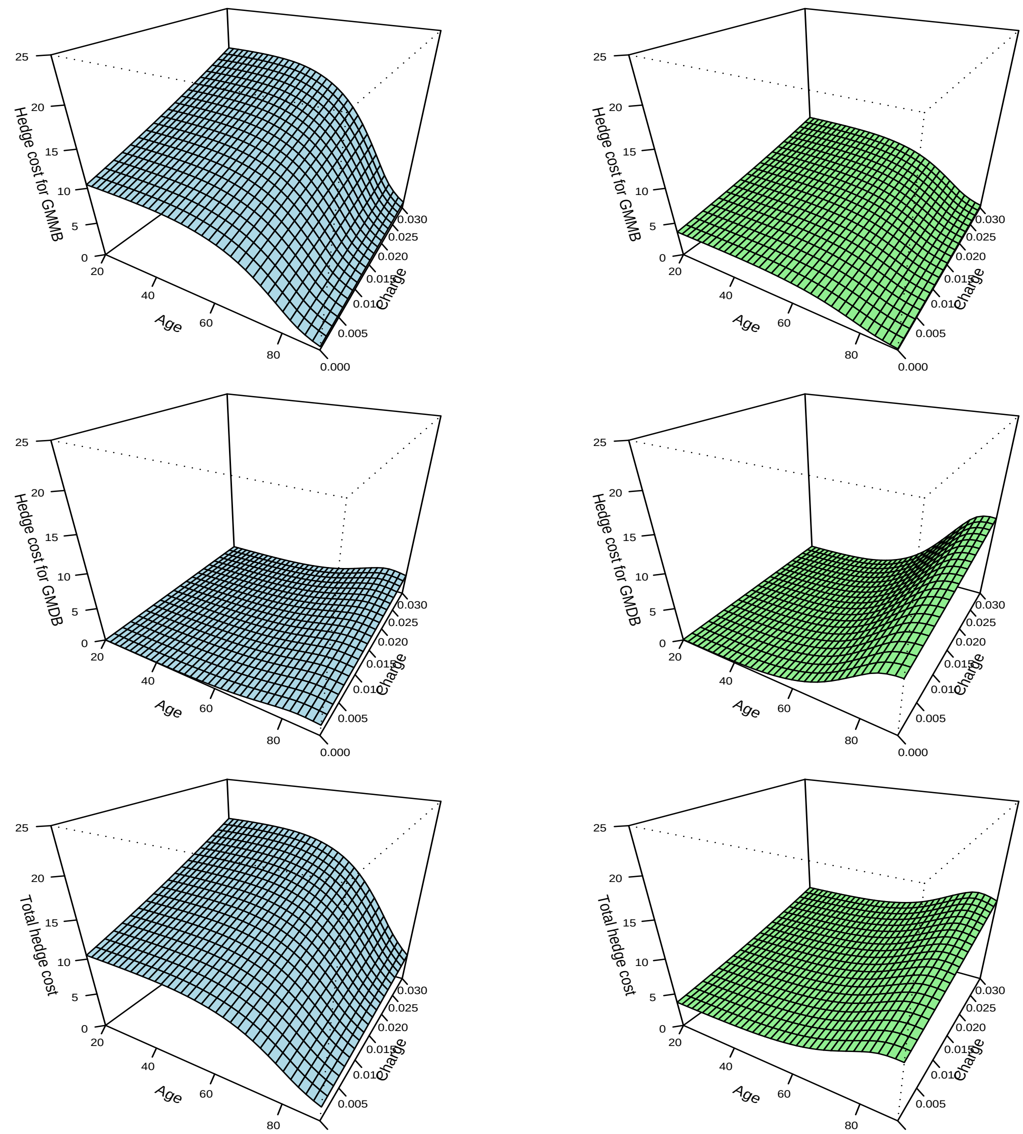

Figure 3.2: The initial hedge costs for the annually hedging strategy

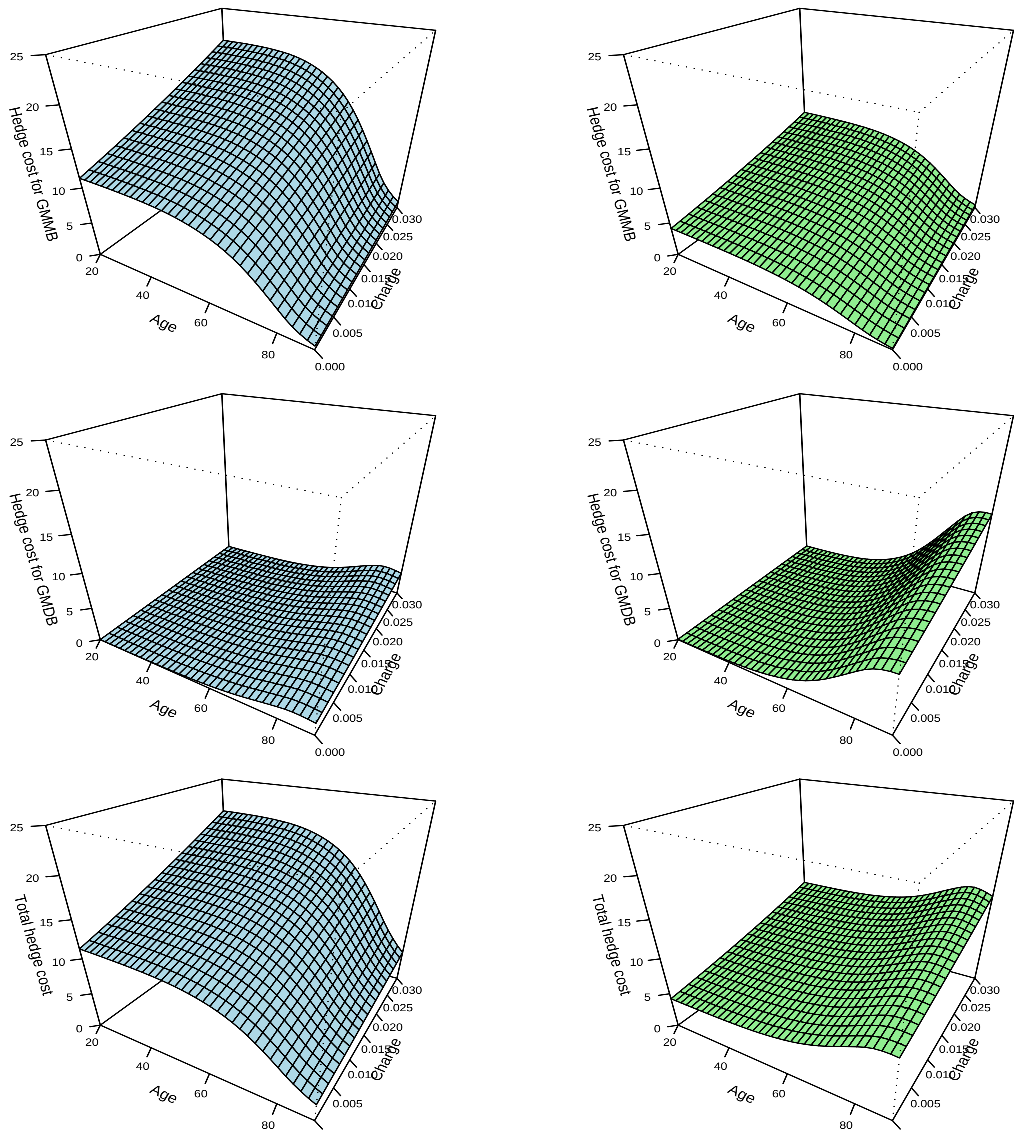

Figure 3.3: The initial hedge costs for the monthly hedging strategy

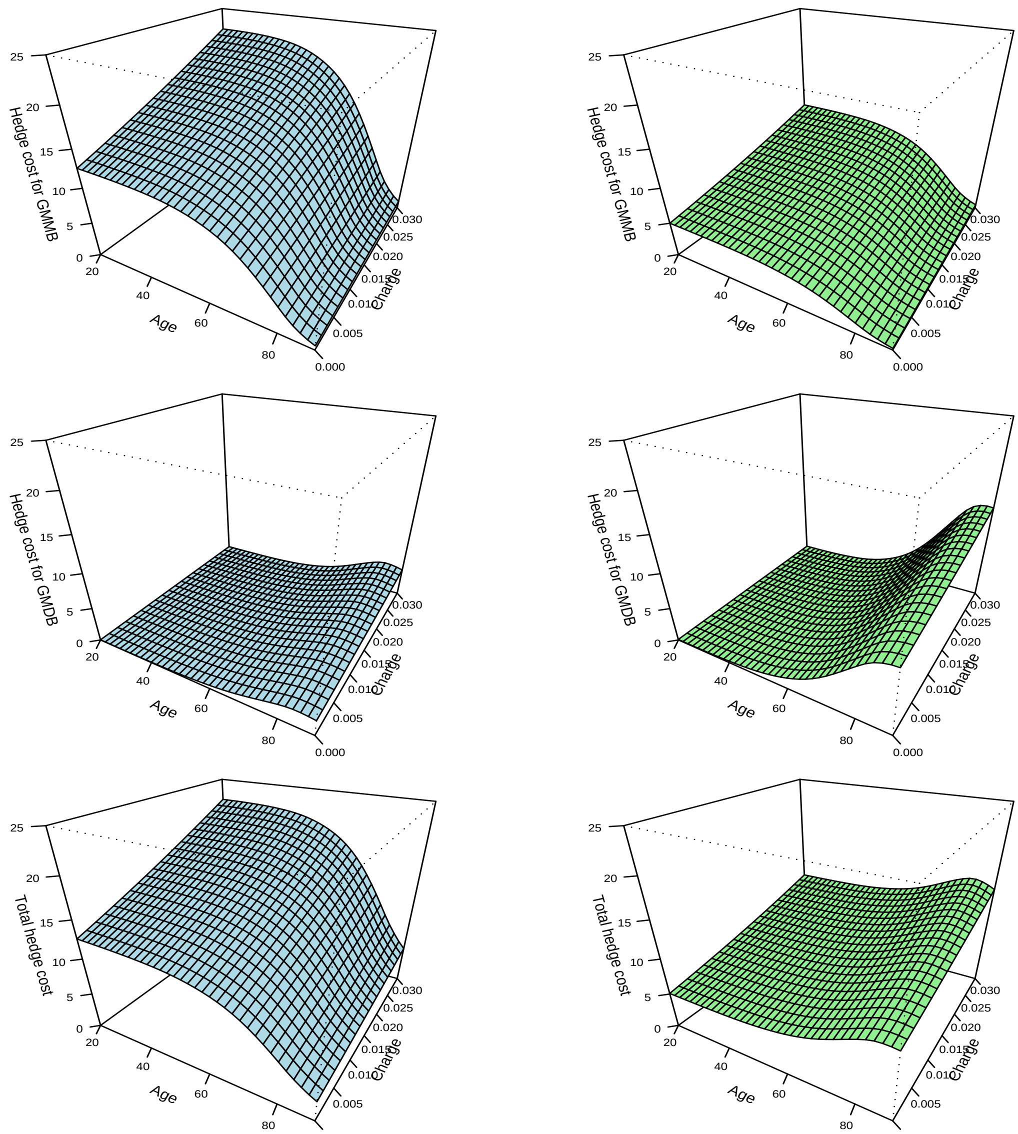

Figure 3.4: The initial hedge costs for the weekly hedging strategy

single premium \$100. The graphs on the left hand side are for Type I contracts and those on the right hand side for Type II contracts. We first note that for Type I contracts the initial hedge costs for the GMMB dominate the total hedge costs for relatively young policyholders. In the case of Type II contracts the initial hedge costs for the GMDB dominate the total hedge costs when the policyholder is relatively old, which leads to relatively high total hedge costs for the contracts. Second, for Type I contracts the change of the charges has much effect on the total hedge cost for relatively young policyholders. This is mainly because of the 100% GMMB offered by Type I contracts. The charge deduction could be considered as a negative return on the segregated fund. As the charge increases, it is more likely that the fund at maturity stays below the initial deposit. The increasing chance of the positive payoff calls for higher hedge costs from the insurance company.

3.3.2 Re-balancing the hedging portfolio

As we mentioned earlier, after the initial hedge cost is calculated, periodical re-balancing of the hedging portfolio is needed until the contract matures. The balancing frequency depends on the hedge strategy. Given a hedge time t_j, where $T \geq t_j > 0$, based on newly gathered information at time t_j, denoted by I_{t_j} (i.e. the asset price S_{t_j} and policyholders who are still alive l_{x+t_j}), the revised HP cost and H&T reserve for the revised portfolio at time t_1 are

$$\begin{aligned} MP|I_{t_j} &= \sum_{J=0}^{T-t_j-1} \hat{P}(S_{t_j}(1-m)^T, G_d, t_j, t_j+J+1) \cdot {}_{J|}q_{x+t_j} \cdot l_{x+t_j} \\ &\quad + \hat{P}(S_{t_j}(1-m)^T, G_m, t_j, T) \cdot {}_{T-t_j}p_{x+t_j} \cdot l_{x+t_j} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} {}_{t_j}V_T^{(H\&T)}|I_{t_j} &= \sum_{J=0}^{T-t_j-1} \hat{\Psi}(t_j, t_j+J+1) \cdot {}_{J|}q_{x+t_j} \cdot l_{x+t_1} \\ &\quad + \hat{\Psi}(t_j, T) \cdot {}_{T-t_j}p_{x+t_j} \cdot l_{x+t_j}. \end{aligned} \tag{3.12}$$

respectively.

Results of the hedge cost movement during the lifetime of the contracts from Equations (3.11) and (3.12) are shown From Figures 3.5 to 3.10. The simulated asset prices used are from four types of asset price paths in Figure 2.1. We assume that the annual charge rate $m = 1\%$ and a pool of 100 insureds whose ages are all 30, 60 and 80 respectively with \$1 put

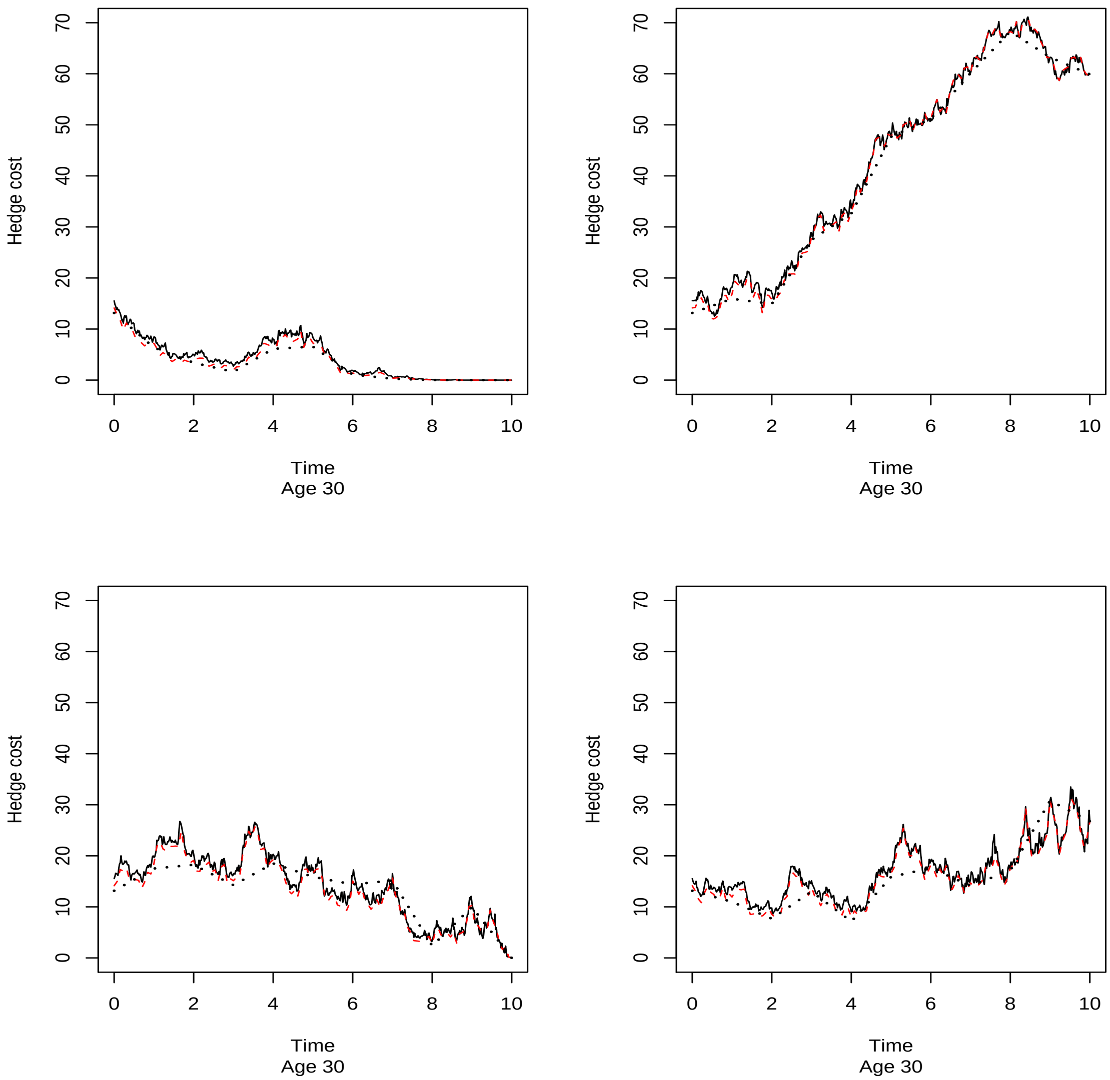

Figure 3.5: Type I, Age 30, $100 single premium

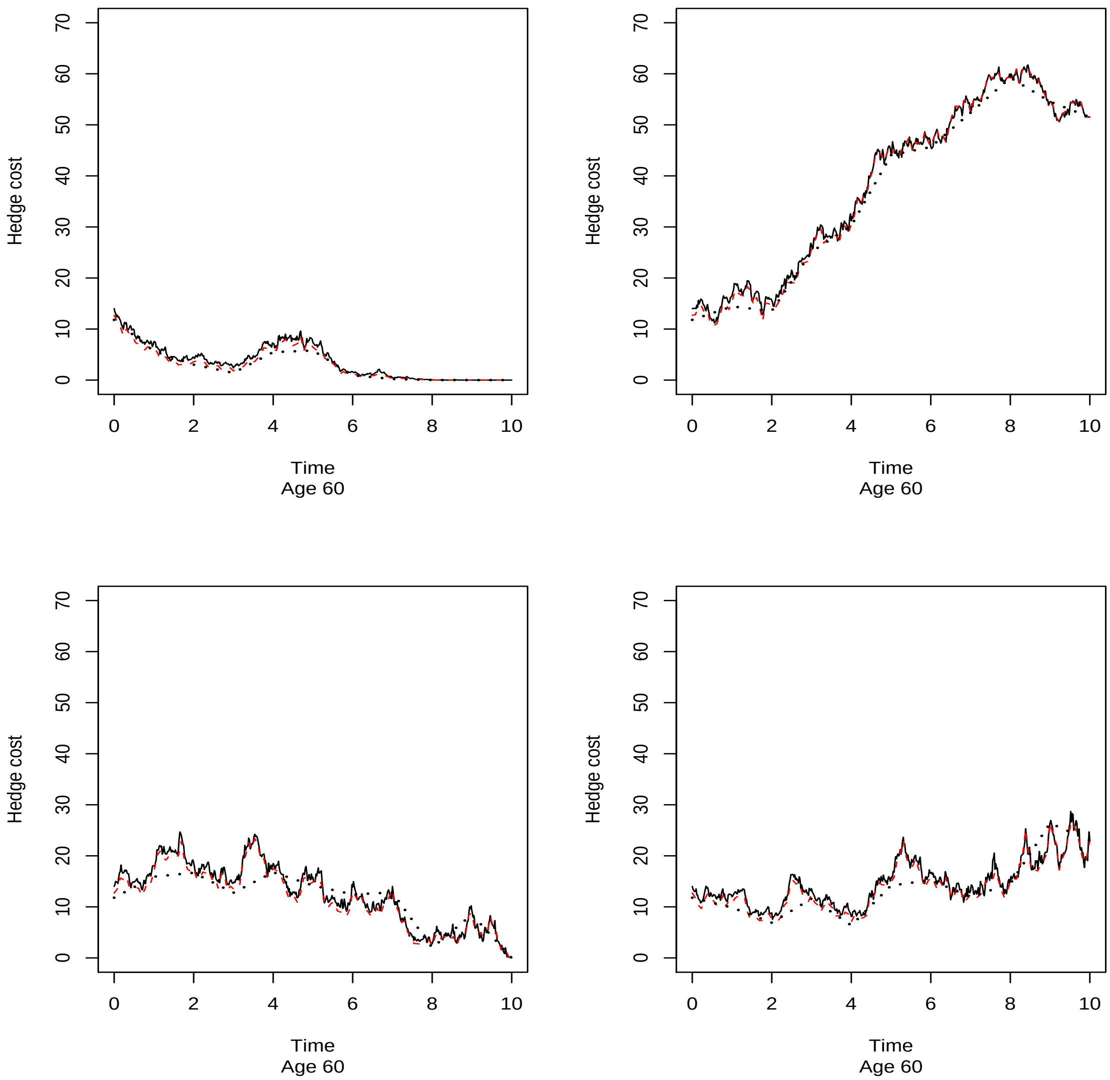

Figure 3.6: Type I, Age 60, $100 single premium

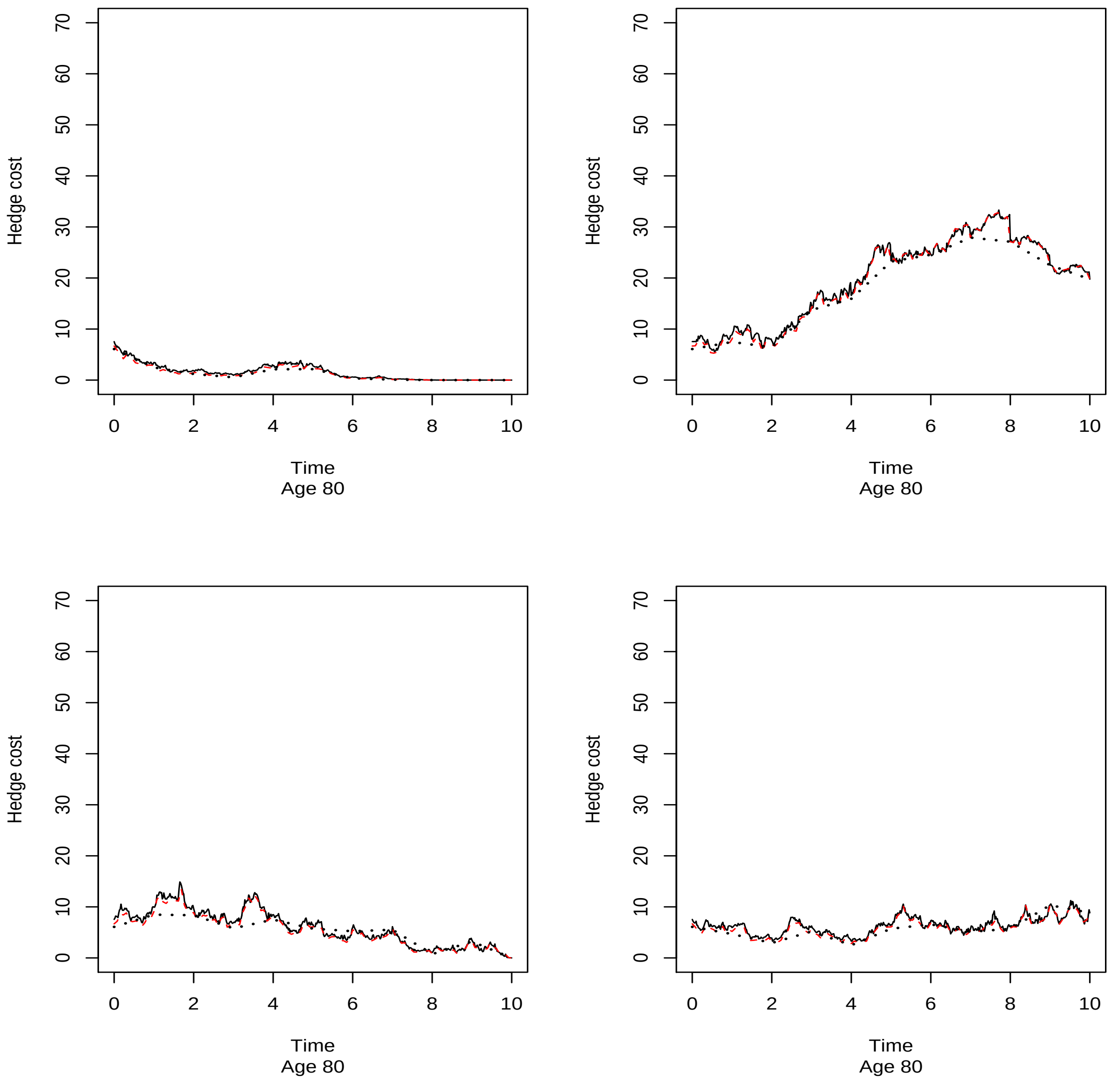

Figure 3.7: Type I, Age 80, $100 single premium

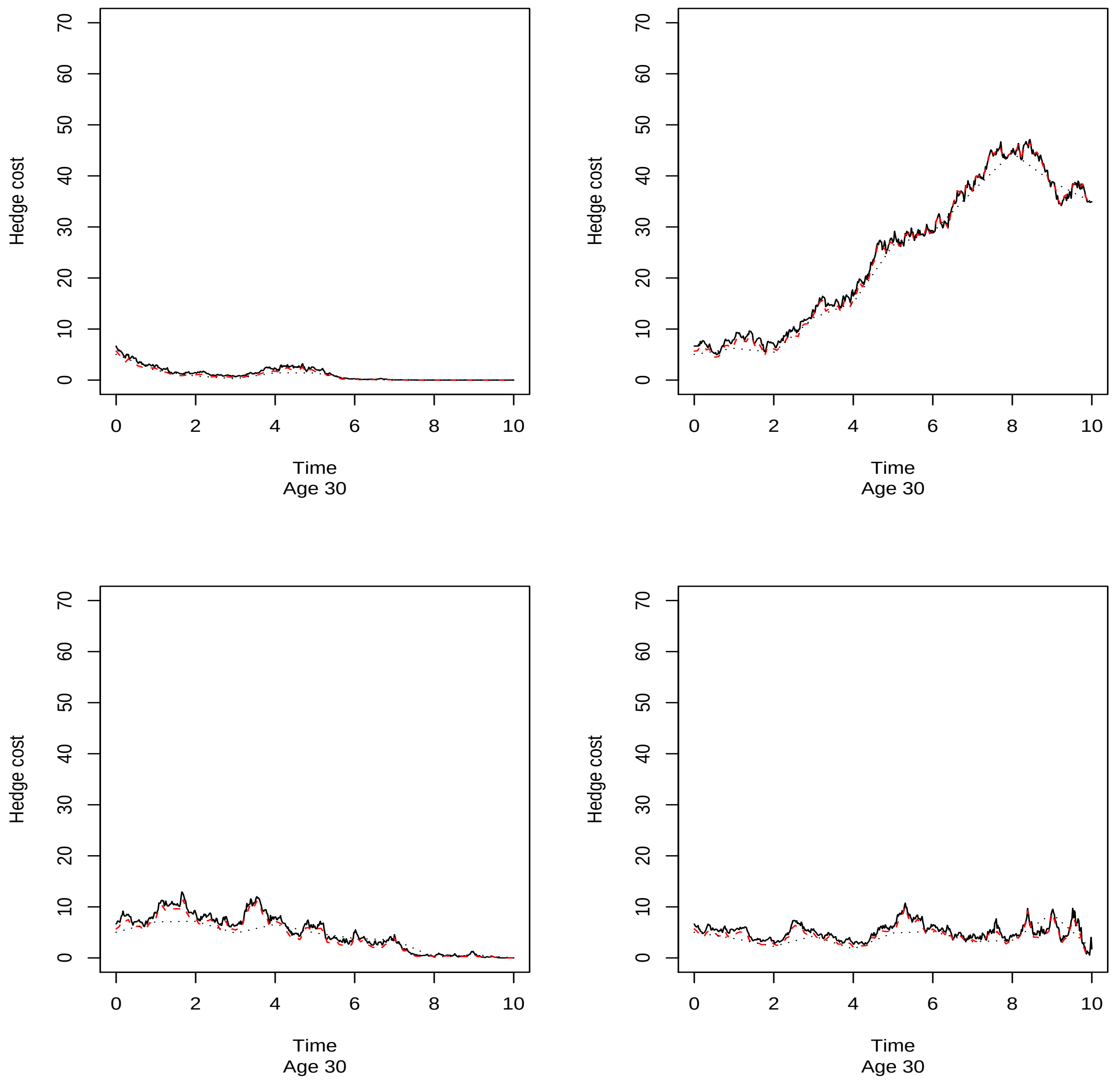

Figure 3.8: Type II, Age 30, \$100 single premium

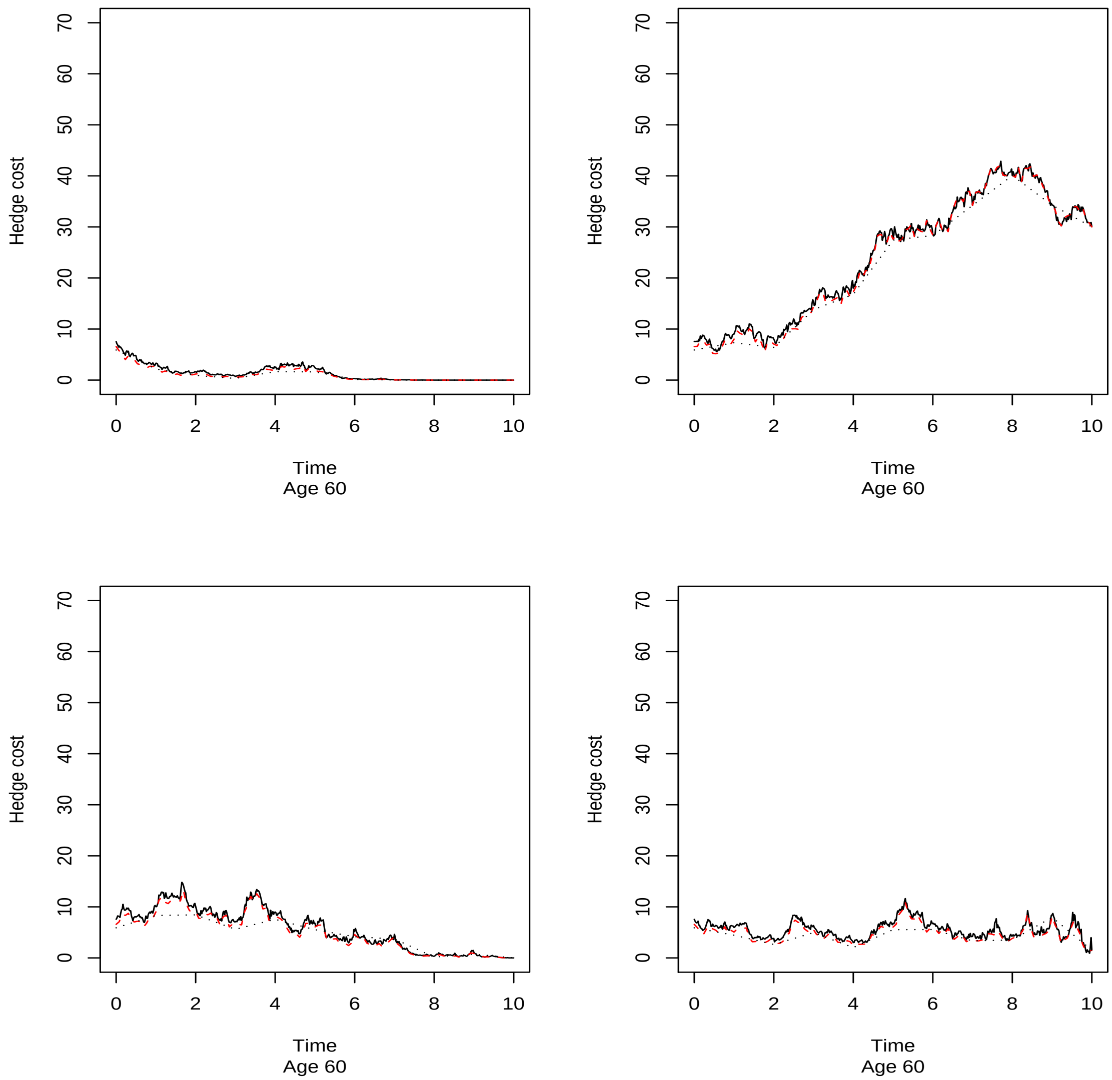

Figure 3.9: Type II, Age 60, \$100 single premium

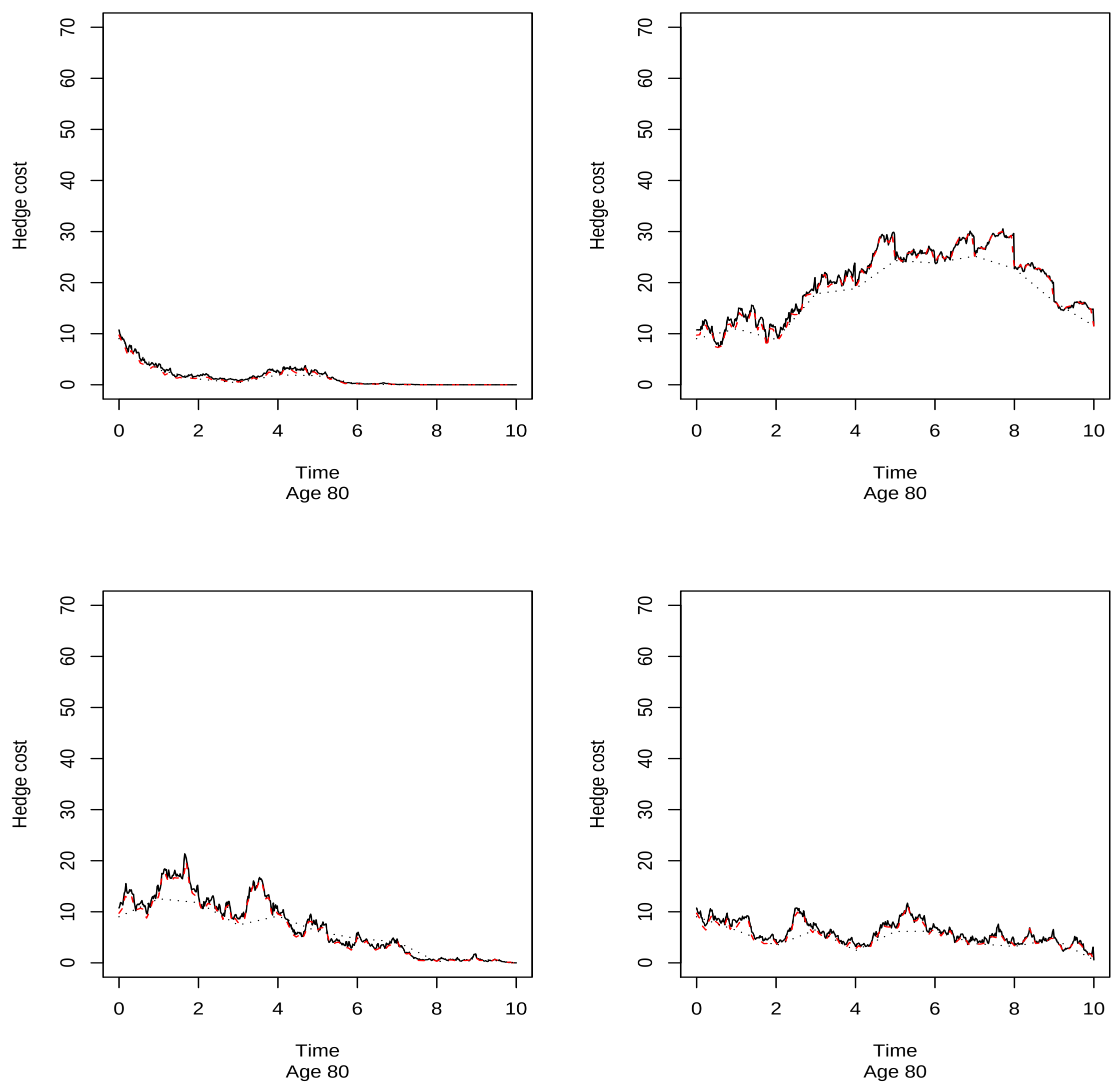

Figure 3.10: Type II, Age 80, $100 single premium

into Type I or Type II segregated fund contracts. The solid lines represent the results from monthly hedge strategy; the dashed lines represent weekly hedge results and the dotted lines are for the annual ones. In general, The hedge cost is reduce when the asset price raises and increased for the drop of the asset price. For example, in graphs at the top left, the asset price increases to a relatively high level after year 6. As a result, the hedge cost is gradually reduced to zero. For another example, the poor investment in the top right graphs makes the segregated fund deep in the money. In turn, a relatively high hedge cost is required. It should also be noted that under the annual hedge strategy the hedge portfolio is adjusted only at the end of each year. It ignores many details of the asset price fluctuation within each year. As a result, we would expect relatively large cash flow volatility.

3.4 The cash flows

To analyze the effect of the dynamic hedging approach on the segregated, the loss or profit of the insurance company needs to be studied. cash flows are projected based on the incomes and outgoes from the company. For the segregated fund, the cash flows include the charge incomes and benefit payments. Under the dynamic hedging approach, it also involves cost for the hedge errors and transaction costs. Assuming that each policyholder puts \$1 into the segregated fund, at the beginning of the 10-year contract, the net cash flow is the initial hedge cost net of the initial charge. Let CF_t denote the net cash flow at time t, then,

$$CF_0 = l_x \left(m - MP - {}_0V_{10}^{(H\&T)} \right).$$

During the lifetime of the contract, the net cash flows are a bit complicated. The composition of the cash flows for year $t = 1, 2, \ldots, 9$ can be seen in Table 3.5. For further analysis, we decompose the cash flows into four parts for year $t = 1, 2, \ldots, 9$. The first part is the annual charge applicable to policyholders who are still alive at the beginning of year t+1. The income of the second part is the projection of the initial hedge cost from year t-1 to t, denoted by

$$\begin{aligned} MP^{(p)}|I_{t-1} &= \sum_{j=0}^{10-t} \mathcal{H}^-(G_d, t-1, t+j) \cdot {}_{j|}q_{x+t-1} \cdot l_{x+t-1} \\ &\quad + \mathcal{H}^-(G_m, 11-t, T) \cdot {}_{11-t}p_{x+t-1} \cdot l_{x+t-1}, \end{aligned}$$

where

$$\mathcal{H}^-(G, \tau_1, \tau_2) = Ge^{-r(\tau_2 - \tau_1 - 1)} \Phi(-\hat{d}_2(G, \tau_1, \tau_2)) - S_{\tau_1+1} \Phi(-\hat{d}_1(G, \tau_1, \tau_2))$$

Table 3.5: The cash flows in the mid-years

Part	Income	Outgo	Net cash flow
A	$l_{x+t}S_t \cdot (1-m)^t \cdot m$		$l_{x+t}S_t \cdot (1-m)^t \cdot m$
B	$MP^{(p)}\|I_{t-1}$	$MP\|I_t$	
		$(l_{x+t-1} - l_{x+t})\mathcal{L}(G_d, t)$	$\mathcal{HE}_t^{(act)}$
C		$\mathcal{HT}_t^{(act)}$	$-\mathcal{HT}_t^{(act)}$
D	$\left({}_{t-1}V_{10}^{(H\&T)}\|I_{t-1}\right) e^r$	${}_tV_{10}^{(H\&T)}\|I_t$	$\mathcal{RE}_t$

$$\hat{d}_1(G, \tau_1, \tau_2) = \frac{log \frac{S_j(1-m)^{\tau_2}}{G} + (r + \frac{1}{2}\hat{\sigma}^2)(\tau_2 - \tau_1)}{\hat{\sigma}\sqrt{\tau_2 - \tau_1}},$$

and

$$\hat{d}_2(G, \tau_1, \tau_2) = \hat{d}_1(G, \tau_1, \tau_2) - \hat{\sigma}\sqrt{\tau_2 - \tau_1}$$

for $\tau_2 = \tau_1 + 1, \ldots, 10$. The outgoes of the second part consist of two components: one is the revised initial hedge cost based on the information at year t, $MP|I_t$; the other is the payoff of the GMDB from the actually terminated contracts at the end of year t. Then the net cash flow (defined as the actual hedge error) occurring at the end of year t is denoted by

$$\mathcal{HE}_t^{(act)} = MP^{(p)}|I_{t-1} - MP|I_t - (l_{x+t-1} - l_{x+t})\mathcal{L}(G_d, t).$$

For the third part, we recognize the hedging transaction costs for both terminated and valid contracts at the end of year t as

$$\begin{aligned} \mathcal{HT}_t^{(act)} &= 0.005 \times S_t \Big\{ |\Phi(-\hat{d}_1(G_d, t, t)) - \Phi(-\hat{d}_1(G_d, t-1, t))| \cdot (l_{x+t-1} - l_{x+t}) \\ &\quad + \sum_{j=0}^{9-t} |\Phi(-\hat{d}_1(G_d, t, t+j+1)) - \Phi(-\hat{d}_1(G_d, t-1, t+j+1))| \cdot {}_{j|}q_{x+t} \cdot l_{x+t} \\ &\quad + |\Phi(-\hat{d}_1(G_m, t, 10)) - \Phi(-\hat{d}_1(G_m, t-1, 10))| \cdot {}_{10-t}p_{x+t} \cdot l_{x+t} \Big\}. \end{aligned}$$

The last part is defined as H&T reserve error at year t, $\mathcal{RE}_t$. It is the difference between the projection of H&T reserve from year t-1 to t, $\left({}_{t-1}V_{10}^{(H\&T)}|I_{t-1}\right) e^r$, and the H&T reserve needed at year t, ${}_tV_{10}^{(H\&T)}|I_t$.

To sum up, we write the net cash flow at the end of year t ($t = 1, 2, \ldots, 9$) as

$$CF_t = l_{x+t}S_t \cdot (1-m)^t \cdot m + \mathcal{HE}_t^{(act)} - \mathcal{HT}_t^{(act)} + \mathcal{RE}_t.$$

Similarly, the cash flow at the end of the last year is

$$CF_{10} = MP^{(p)}|I_9 - (l_{x+t-1} - l_{x+t})\mathcal{L}(G_d, 10) - l_{x+t}\mathcal{L}(G_m, 10) - \mathcal{HT}_{10}^{(act)}.$$

It should be noted for the monthly and weekly hedging strategies, net cash flows at fraction year need to be considered. They are similar to those at integer year. The difference is: Part A in Table 3.5 is zero since charge is deducted at the beginning of each year; based on the assumption of GMDB is payable at the end of the year, there is no GMDB payoff or transaction costs for GMDB payoff.

Figures 3.11 to 3.16 show the net cash flows generated by the dynamic hedging approach, corresponding to the asset price paths in Figure 2.1. For monthly and weekly hedge strategies those outstanding positive cash flows at the integer years are partially or mostly attributed to the annual charges. Under the annual hedge strategy, the hedging portfolio is adjusted only at the end of each year, which general relatively large net cash flows at each hedge time. On the other hand, portfolio adjustments are also needed within each year. This leads to relatively smooth cash flows. Specifically, look at the third column in Figure 3.11 which corresponds to graph (c) in Figure 2.1 and the bottom left one in Figure 3.5. From year 7 to year 8, there is a relatively large return on the fund investment. In turn, the cash for the hedge cost is released at the end of year 7. However, the large hedging portfolio adjustment also causes relatively high transaction costs. As a result, a relatively large loss is generated at the end of year 7 under the annual hedge strategy. On the other hand, the segregated fund is gradually hedged from year 7 to year 8 under monthly or weekly strategy. The outstanding loss at the end of year 7 under annual hedge strategy is replaced with monthly or weekly cash flows within year 7.

The amount of the net cash flow in each year is an important issue. The speed at which the cash is being released back to the company is another critical issue. The net present value (NPV) of the cash flows provides a good basis for the analysis of the dynamic hedging approach. Under the dynamic hedging approach, the NPV of the cash flows can be expressed as

$$NPV = \sum_{j=0}^{T/\Delta t} CF_j (1+i_d)^{-j\Delta t},$$

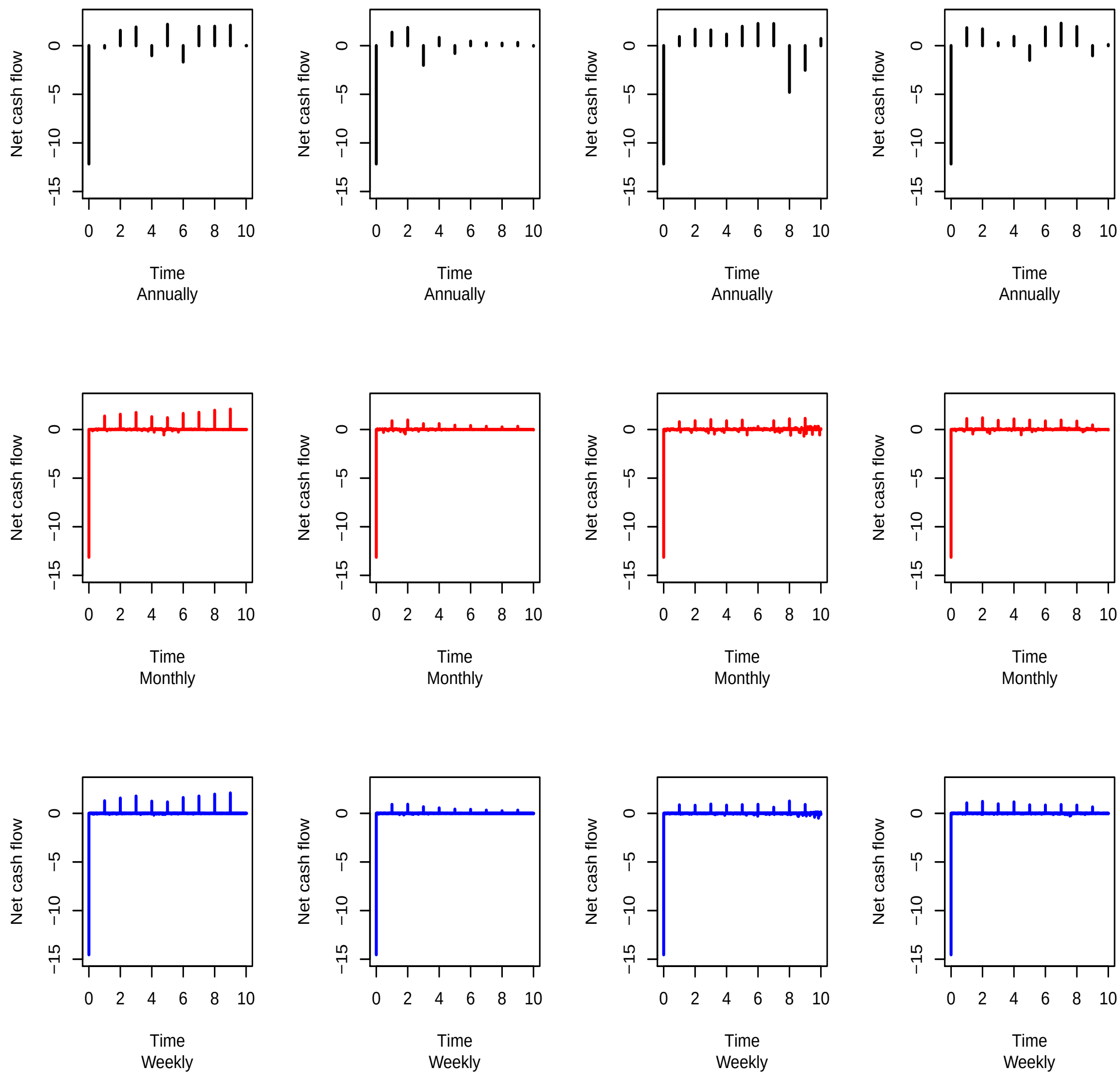

Figure 3.11: Net Cash flows of Type I contract, Age 30

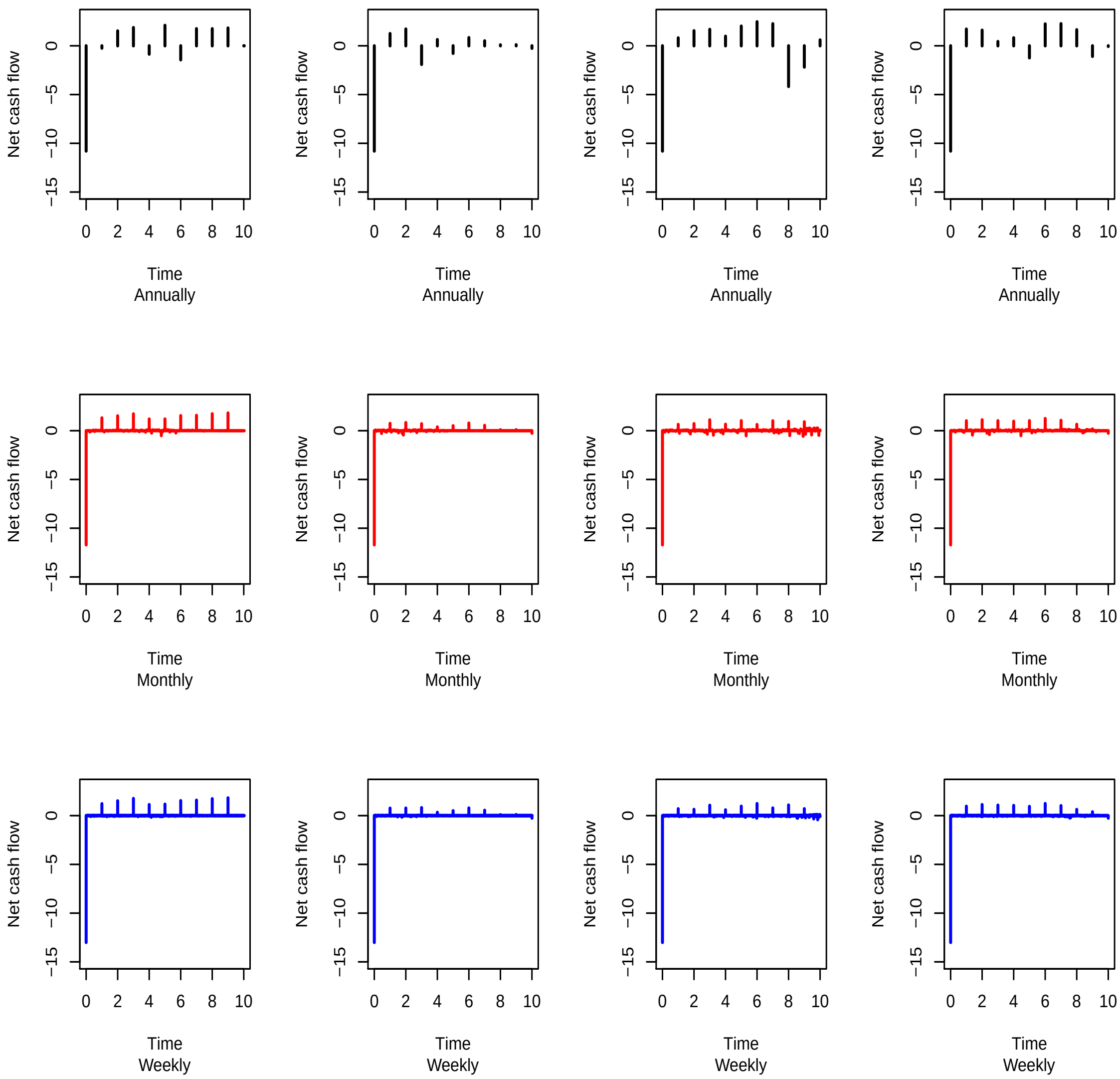

Figure 3.12: Net Cash flows of Type I contract, Age 60

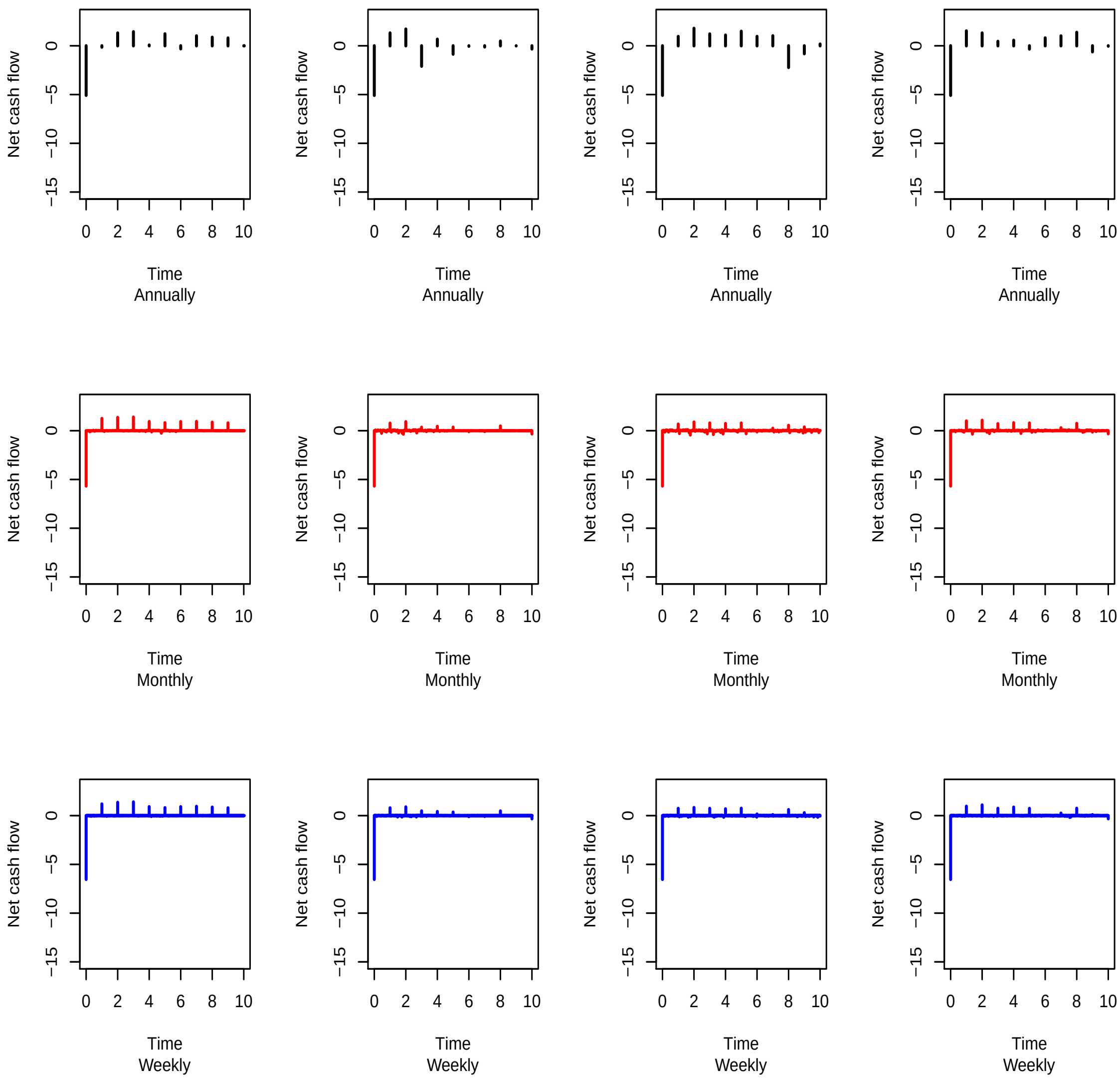

Figure 3.13: Net Cash flows of Type I contract, Age 80

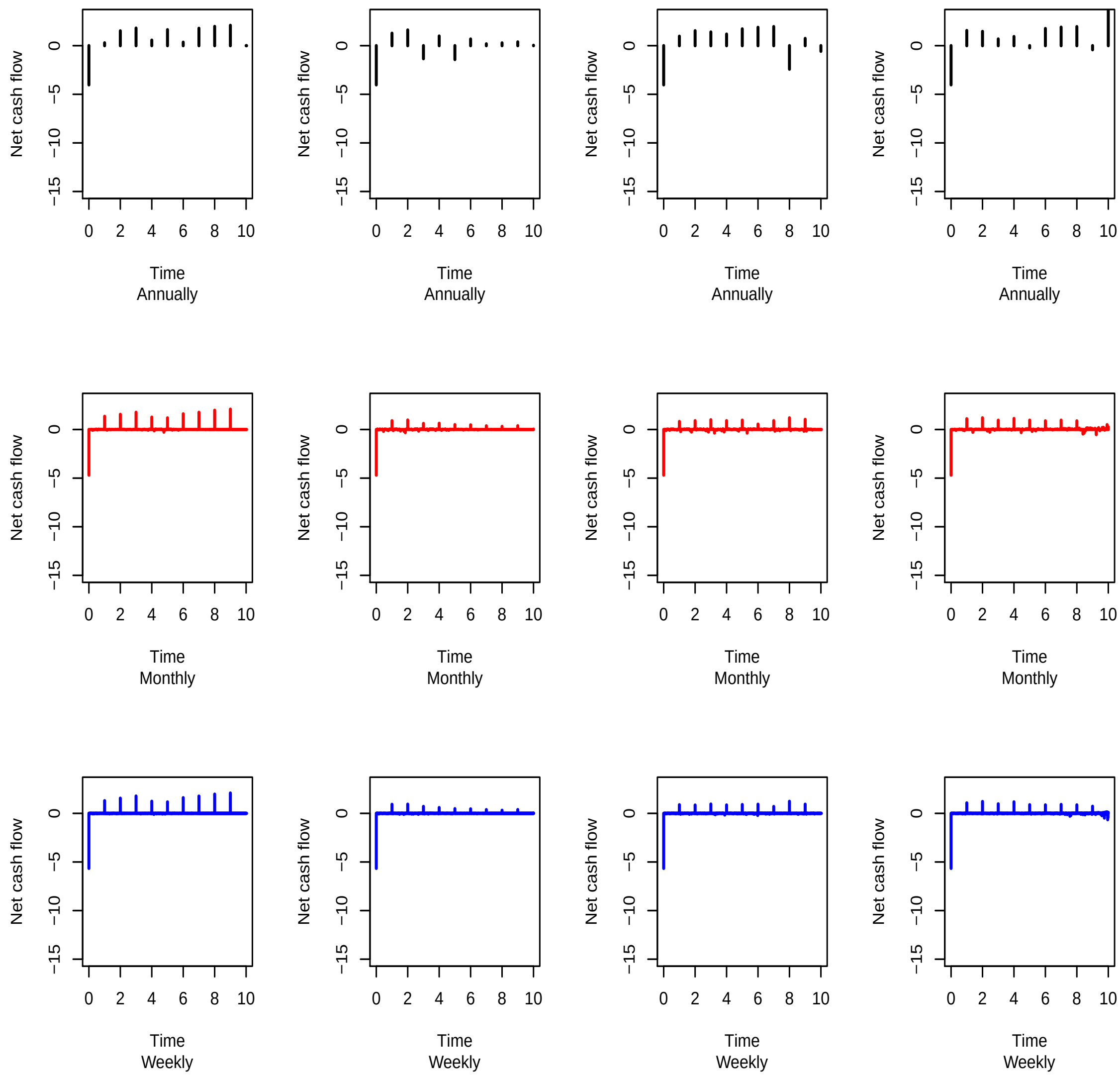

Figure 3.14: Net Cash flows of Type II contract, Age 30

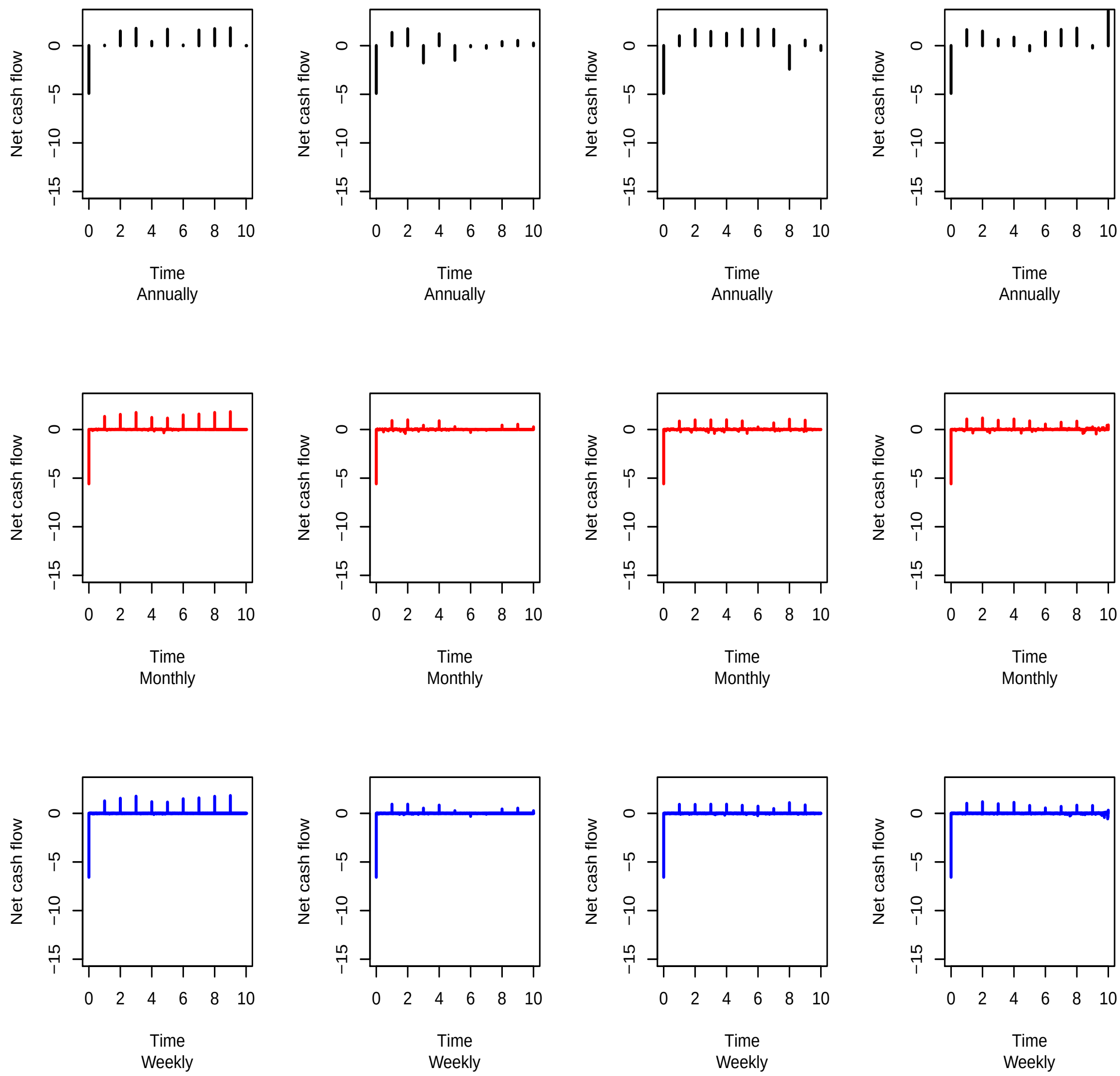

Figure 3.15: Net Cash flows of Type II contract, Age 60

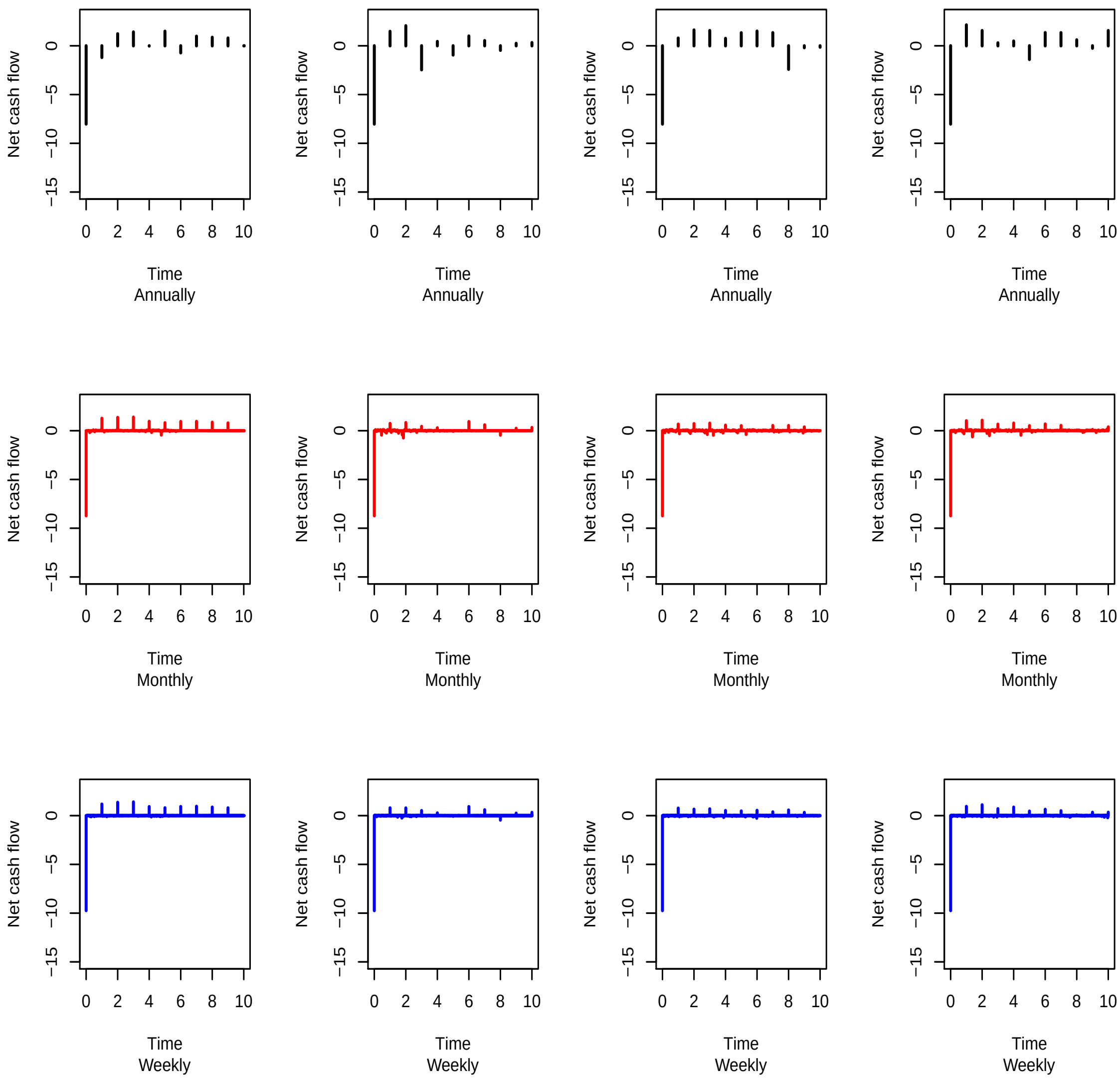

Figure 3.16: Net Cash flows of Type II contract, Age 80

where i_d is the annual interest rate to discount the cash flows.

The mean and standard deviation of the simulated NPV of the cash flows are shown in Figure 3.17. A Policyholder aged 30 with \$100 single premium Type I contract is chosen. Under the dynamic hedging approach, the NPV of the cash flows with higher annual charge and lower discount rate are subject to greater mean and standard deviation. On the other hand, however, with the help of the dynamic hedging, the NPV of the cash flows is relatively insensitive to both interest rate and charge rate. Specifically, monthly hedge strategy generates relatively small standard deviation, which indicates better effect on controlling the risk volatility as more asset price details are captured.

To help further analyze the dynamic hedging approach, the simulated distribution and the percentiles of the NPV of the cash flows are shown in Figures 3.18 and 3.19. Type I contract is chosen. Annual charge rate and interest rate for discount are assumed to be 1% and r respectively. The solid line represents the one without dynamic hedging which has a long left tail. The segregated fund has a positive mean (1.9922) and a standard deviation of 15.4465. As derived in Chapter Two, the actuarial solution of the annual charge is about 0.75%. The 0.25% surcharge in our example may be attributed to the average surplus of the segregated fund. The curve for the dynamic hedging approach, on the other hand, is more concentrated. The mean and standard deviation are -3.7828 and 3.2641 respectively. In Figure 3.19, we can see that without a dynamic hedging approach, there is a 20% chance of generating quite poor NPV of the cash flows, while for most of the time the NPV of the cash flows ends up within the range from 0 to -20 under monthly hedging strategy. For the dynamic hedge approach, both figures do not indicate a high chance of getting positive NPV since transaction costs are involved in the hedge strategy and 1% per year may be not sufficient enough to cover both guarantees and transaction costs. However, they reveal that dynamic hedging approach is in favor of reducing the left tail and volatility of the distribution.

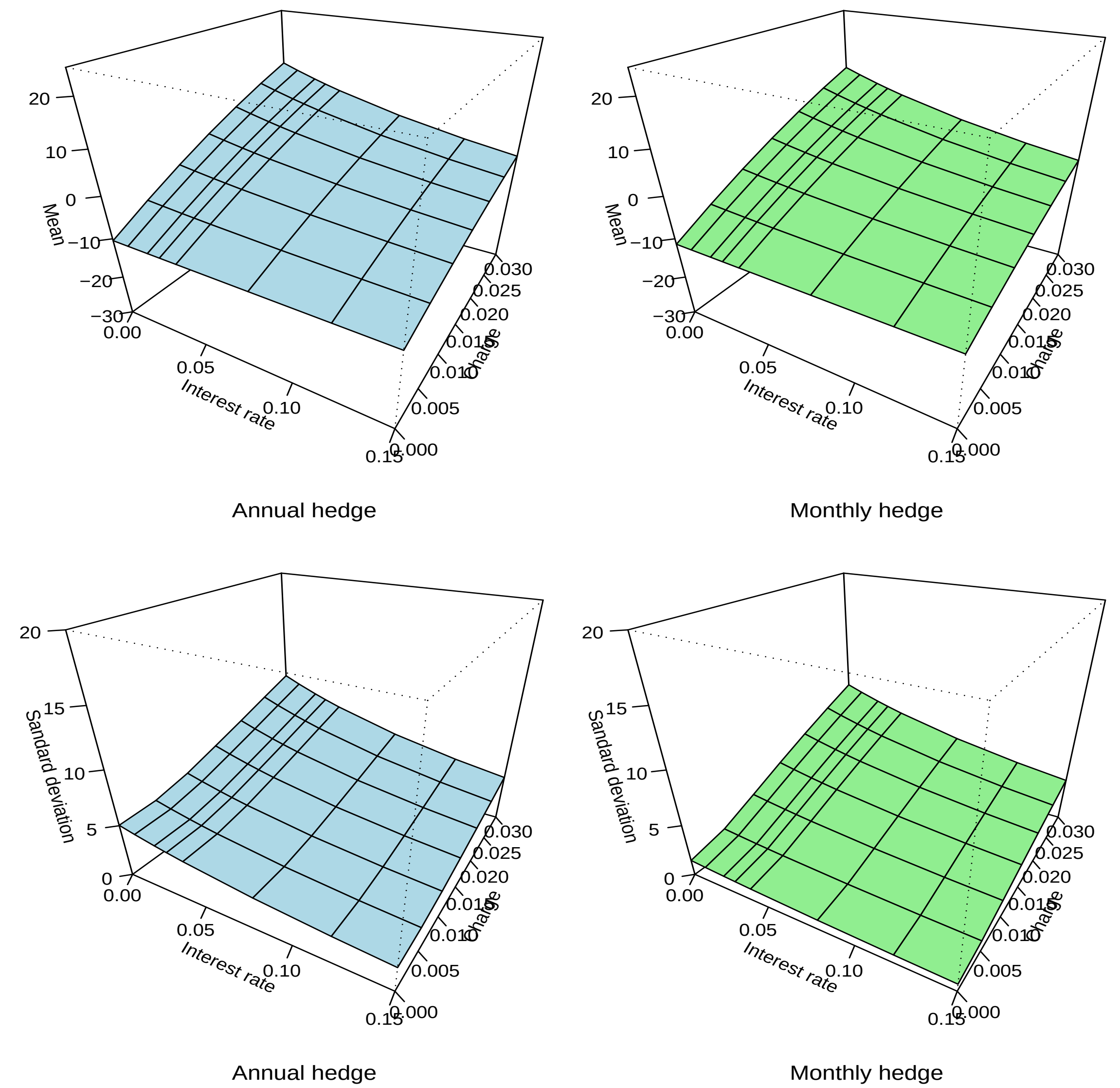

Figure 3.17: The mean and standard deviation of the NPV of the cash flows. Type I, Age 30 and $100 single premium contract is used

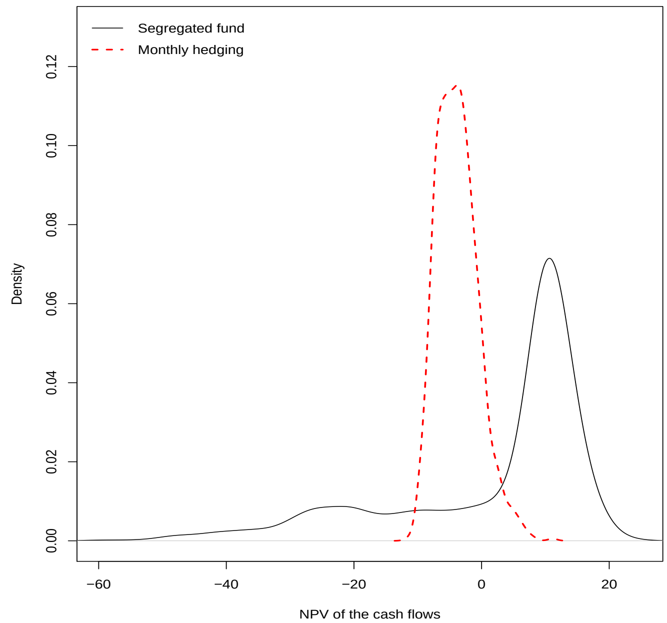

Figure 3.18: Comparison of the distributions of the NPV of the cash flows

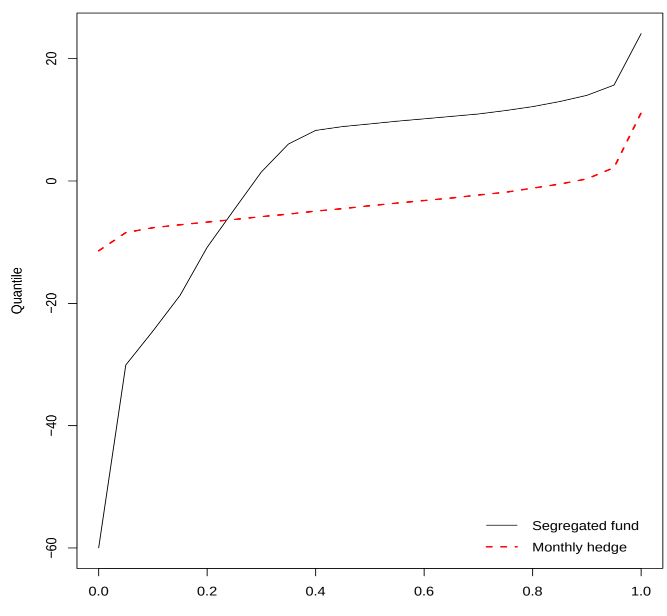

Figure 3.19: Comparison of the percentiles of the NPV of the cash flows

Chapter 4

Valuation of the reset option

Segregated funds may have reset options allowing the policyholders to lock in the investment gains several times per year (http://www.segfundscanada.ca). Once the contract gets reset, the guarantee levels are based on the current fund level and the term of the contract is extended to another 10 years. The reset option adds some complications to the valuation of the segregated fund since both the guarantees and the maturity date are uncertain. *Windcliff*, *Forsyth* and *Vetzal* (2001a, 2001b) discussed a approach to price extendable segregated fund by employing some finite difference methods. In this project, we do not go that far since it needs both a large amount of time and some high speed computer. We only borrow the spirit of this approach and use a simplified case to demonstrate the main idea and further price the added value of the reset option.

4.1 Introduction of Crank-Nicolson method

4.1.1 Discretization techniques

Crank-Nicolson method is one of the implicit finite difference methods solving Black-Scholes PDE numerically for the price of the derivative securities (*Back* (2005)). For simplicity, we use short notations S and P to represent the asset price at time t and the European put price at time t with asset price S, respectively. The PDE is shown by

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \tag{4.1}$$

where $0 \le t \le T$ and $0 \le S \le \infty$. The price of the European put option P is a continuous function of t and S, denoted by $V(S,t)$ (i.e. $P = V(S,t)$). If $\partial t \approx \Delta t$ and $\partial S \approx \Delta S$ for some infinitesimal values of Δt and ΔS as shown in Figure 4.1, the approximate value of P is denoted by

$$V_{i,n} = V\left(i\Delta S, n\Delta t\right).$$

Crank-Nicolson method assumes the existence of the value of P between two grid points $(n\Delta t, i\Delta S)$ and $((n+1)\Delta t, i\Delta S)$ (*Back* (2005)). Let $V'_{i,n}$ denote the Crank-Nicolson value of P, then

$$P \approx V'_{i,n} = \frac{V_{i,n} + V_{i,n+1}}{2}.$$

The derivatives in Equation (4.1) are approximately written as

$$\begin{aligned}
\frac{\partial P}{\partial S} &\approx \frac{V'_{i+1,n} - V'_{i-1,n}}{2\Delta S} \\
&= \frac{V_{i+1,n} - V_{i-1,n} + V_{i+1,n+1} - V_{i-1,n+1}}{4\Delta S}, \\
\frac{\partial^2 P}{\partial S^2} &\approx \frac{\left(V'_{i+1,n} - V'_{i,n}\right) - \left(V'_{i,n} - V'_{i-1,n}\right)}{(\Delta S)^2} \\
&= \frac{V_{i+1,n} - 2V_{i,n} + V_{i-1,n} + V_{i+1,n+1} - 2V_{i,n+1} + V_{i-1,n}}{2(\Delta S)^2}
\end{aligned}$$

and

$$P_t \approx \frac{P_{i,n+1} - P_{i,n}}{\Delta t}.$$

If we let $g = \Delta t$ and $h = \Delta S$, Equation (4.1) can be approximately written as

$$a(i)P_{i+1,n} + b(i)P_{i,n} + c(i)P_{i-1,n} = d(i), \tag{4.2}$$

where

$$\begin{aligned}
a(i) &= -\frac{\sigma^2 S(i)^2}{4h^2} - \frac{rS(i)}{4h}, \\
b(i) &= \frac{1}{g} + \frac{\sigma^2 S(i)^2}{2h^2} + \frac{r}{2}, \\
c(i) &= \frac{rS(i)}{4h} - \frac{\sigma^2 S(i)^2}{4h^2}, \\
d(i) &= -a(i)P_{i+1,n+1} + \left(\frac{1}{g} - \frac{\sigma^2 S(i)^2}{2h^2} - \frac{r}{2}\right) P_{i,n+1} - c(i)P_{i-1,n+1},
\end{aligned}$$

and $S(i)$ denotes the value of S at point $i\Delta S$.

4.1.2 Solving the PDE

As long as we know the values of $P_{i+1,n+1}$, $P_{i,n+1}$ and $P_{i-1,n+1}$, the value of $P_{i+1,n}$, $P_{i,n}$ and $P_{i-1,n}$ can be implicitly derived. As shown in Figure 4.1, assuming a relatively large value S_{max} to represent the case $S \to \infty$, we discretize S from 0 to S_{max} to get N intervals such that $S_{max} = N\Delta S$. Similarly, we disretize t to M intervals such that $T = M\Delta t$. At each time step, $N+1$ equations similar to Equation (4.2) (with $N+1$ discretized asset price from 0 to S_{max}) could be used to solve backward for a collection of P values at those grid points (i.e., $P_{0,n}$, $P_{1,n}$, $\ldots$, $P_{N,n}$). Repeating this algorithm M times gives the price of the European put option at time 0 with the desirable asset price, say S^*.

To make this recursive algorithm work, firstly, we need to set the initial values of P's at maturity time T at which the payoffs of the options are made. That is

$$P_{i,M} = max\left(K - S(i), 0\right),$$

where K is the strike and $i = 0, 1, \ldots, N$.

Secondly, The first and the last of the $N+1$ equations approximately represent the cases $S \to 0$ and $S \to \infty$. Plugging the boundary values of S into equation (4.1), we get

$$P_t = \begin{cases} rP, & \text{if } S \to 0; \\ 0, & \text{if } S \to \infty. \end{cases}$$

This gives us the frist equation

$$\left(\frac{1}{k} + \frac{r}{2}\right) P_{i,n} = \left(\frac{1}{k} - \frac{r}{2}\right) P_{i,n+1}$$

with $S = 0$, and

$$P_{i,n} = P_{i,n+1}$$

with $S = S_{max}$.

The value of ΔS and Δt are quite critical. It also matters to choose the value S_{max} to be far away enough from the desirable asset price. As shown in Table 4.1, for the at-the-money European put option with the strike equal to \$100, setting S_{max} four times as large as the strike, together with at least 3000 asset grids, the values derived via the Crank-Nicolson method converge quite well.

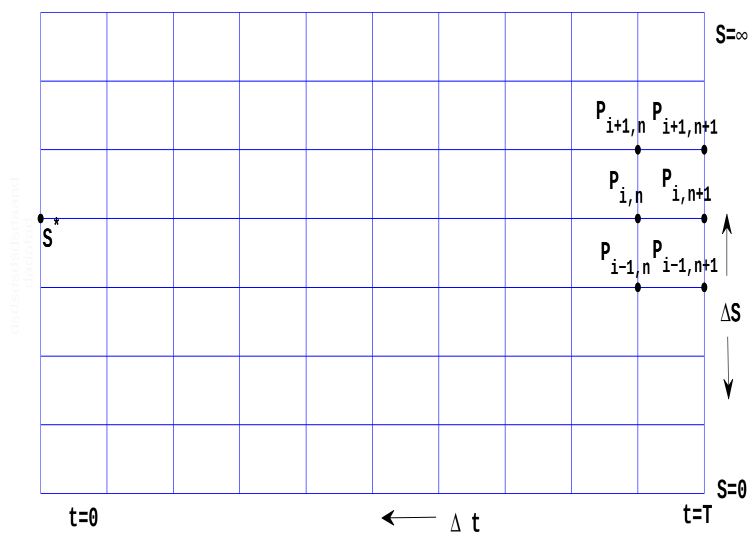

Figure 4.1: Discretization of the Black-Scholes pde

Table 4.1: Solving for European put options with 1000 time grids

		S grids		
	Black-Scholes	1000	3000	3500
$S_{max} = 400$	10.05832	10.05818	10.05830	10.05831
$S_{max} = 300$	10.05832	10.07221	10.05822	10.08268
$S_{max} = 200$	10.05832	10.01914	10.01897	10.01870

4.2 Pricing the segregated fund with reset option

As mentioned in Chapter 2, the present value of future charges can be modeled via an indifference principle. However, reset option makes this approach inapplicable due to the uncertainties of the benefit amount and maturity date. Following the Black-Scholes PDE, a more flexible method can be employed. First we make the following assumptions:

- The value of segregated fund is assumed to follows the stochastic differential equation

$$dS_t = (\mu - f_m)S_t dt + \sigma S_t dB_t;$$

- The value of the segregated fund is a function of the asset price S at time t, guarantee G, maturity date T and current time t, i.e., $P(S_t, G, t, T)$;

- The values of G and T upon reset are denoted by G^* and T^*, respectively, where

$$G^* = \alpha S$$

  and

$$T^* = min(T + 10, T_{max}).$$

  α is either 100% or 75%, and T_{max} is the maximum maturity date of the segregated fund. No reset is allowed in the last ten years of T_{max}. For simplicity, we only consider the case that only one reset is allowed per year;

- Upon reset, the value of the segregated fund is denoted by $P^*(S_t, G, t, T)$, where

$$P^*(S_t, G, t, T) = \begin{cases} P(S_t, G^*, t, T^*), & \text{if reset is allowed;} \\ -\infty, & \text{otherwise.} \end{cases}$$

4.2.1 Optimality statement

When we solve the equations for the value of the segregated fund with certain value of f_m, we need to compare P with P^* at each grid point. From this point of view, the segregated fund can be considered as a collection of American-type options with different strikes and maturity dates. Following the standard statements in *Wilmott*, *Dewynne* and *Howison* (1993), the value of the segregated fund must satisfy

$$P_t + (r - f_m)SP_S + \frac{1}{2}\sigma^2 S^2 P_{SS} - rP + D(t)max(G_d - S, 0) - R(t)f_m S \leq 0, \tag{4.3}$$

and

$$P^* \leq P, \tag{4.4}$$

where one of inequalities (4.3) and (4.4) holds with equality and G_d is the GMDB guarantee. These two inequalities assume that the policyholders exercise the reset options optimally without arbitrage and P^* can be viewed as the lower bound of P ((4.3) states that the return from the segregated fund is not greater than the return on the investment to the risk-free bound. If reset is optimal, (4.4) holds with equality and (4.3) holds with inequality, or vice versa.). $D(t)$ is defined such that the proportion of the policyholders who die between t and $t + dt$ is denoted as $D(t)dt$. R(t) is the proportion of the policyholders who are still alive at time t, i.e. $R(t) = 1 - \int_0^t D(t)dt$. In the following, we use a constant force of mortality in fraction year.

At the end of each year, the number of the used reset is set to zero. The non-arbitrage statement requires the jump condition

$$P(S_{t_i^-}, G, t_i^-, T) = P(S_{t_i^+}, G, t_i^+, T),$$

where t_i^- and t_i^+ represent the moment immediately at the end of the i^{th} year and the beginning of the $i + 1^{th}$ year, respectively.

4.2.2 Boundary values

The terminal values representing the intrinsic values at time T are

$$P_{i,M} = R(T)max\left(G_m - S(i), 0\right), i = 0, 1, \ldots, N$$

where G_m is the GMMB guarantee. The asymptotic condition which reflects the fact that the asset price goes to 0 or ∞ can be written as

$$P_t = \begin{cases} rP - D(t)max(G_d - S, 0), & \text{if } S \to 0; \\ -D(t)max(G_d - S, 0), & \text{if } S \to \infty. \end{cases}$$

4.2.3 Implementation

The inequalities (4.3) and (4.4) are three-dimensional (i.e., S, K and T) and time-dependent, which need very complex computational work. To further programming, we make the following simplifications. First, if we let $X_i = logS(i)$, equality (4.3) can be discretized as

$$a \cdot P_{i+1,n} + b \cdot P_{i,n} + c \cdot P_{i-1,n} = Z(i), \tag{4.5}$$

where

$$\begin{aligned}
a &= -\frac{\sigma^2}{4h^2} - \frac{r - \frac{1}{2}\sigma^2}{4h}, \\
b &= \frac{1}{k} + \frac{\sigma^2}{2h^2} + \frac{r}{2}, \\
c &= \frac{r - \frac{1}{2}\sigma^2}{4h} - \frac{\sigma^2}{4h^2}, \\
Z(i) &= -aP_{i+1,n+1} + \left(\frac{1}{k} - \frac{\sigma^2}{2h^2} - \frac{r}{2}\right) P_{i,n+1} - cP_{i-1,n+1} \\
&\quad + D(t)max\left(K_2 - e^{X_i}, 0\right) - R(t) f_m e^{X_i},
\end{aligned}$$

and $i = 1, 2, \ldots, N$.

The convenience of Equation (4.5) is that all of the N equations except for the first and the last ones have the same coefficients a, b and c which are independent with X_i. Second, the maturity date T varies from 10 to 30. Since the length of the contract can only be extended to anther 10 years upon reset, the possible values of T are constrained by

$$max(t, 10) \leq T \leq min(t + 10, T_{max})$$

for t from 0 to T_{max}. To start the programming, we need to define a three-dimensional array, say $P[D_X][D_G][D_T]$, to represent the collection of American-type options. We set 1001 grids for both S and K dimensions so that $h = \frac{S_{max}}{1001} = \frac{K_{max}}{1001}$, where $S_{max} = G_{max}$ (G_{max} is the maximum guarantee that can be reset to). To avoid interpolation complication, we set the interval of T as the same as time step Δt, i.e. $g = \frac{1}{12}$. Therefore, the maximum number of the values of T is 120. For the last ten years of contract, the segregated fund contract is simply a European put option. At time $t = T_{max}$, the number of T is one (i.e., T can only be T_{max}). The number of T's increases as we solve Equation (4.5) backward until $t = T_{max} - 10$ where the number of T's reaches the maximum (i.e. 120).

For $10 \leq t \leq T_{max} - 10$, the number of T's remains 120. For each time step, we need to compare the value of P with P^*. That is,

$$P[D_X][D_G][D_T] = max(P[D_X][D_G][D_T], P[D_X][D_X][0])$$

for $D_X = 0, 1, \ldots, 1000$, $D_K = 0, 1, \ldots, 1000$ and $D_T = 0, 1, \ldots, 119$. $D_G = D_X$ means that GMMB is reset to the current fund value. $D_T = 0$ represents $T = 10$, $D_T = 1$ represents $T = 10 + \frac{1}{2}$ and so on. For the first 10 years of the contract, the number of T's decreases by one at each time step until $t = 0$ where the value of T is equal to one which represents the case that $T = 10$. At this point, we get the present value of the segregated fund contract with desirable initial fund value, initial guarantee and initial term of 10 years.

4.2.4 The solution of the charge rate

We choose certain value of f_m to solve inequality (4.3) for P at time 0 with initial guarantee and desirable premium. We can try different f_m's to solve Inequality (4.3) until P approaches zero so that the instantaneous charge rate f_m is obtained. However since the computation is quite time-consuming, we can alternatively use the linear interpolation method which is illustrated in Figure 4.2. Some numerical results are displayed in Tables 4.2 and 4.3.

Table 4.2: Instantaneous charge rates of Type I contracts

	Equation (2.2)		Crank-Nicolson	
	Actuarial	Risk-neutral	Standard	Extendable
Age 30	0.7400%	1.5400%	1.5231%	4.4751%
Age 60	0.6900%	1.4100%	1.3876%	3.5356%

Table 4.3: Instantaneous charge rates of Type II contracts

	Equation (2.2)		Crank-Nicolson	
	Actuarial	Risk-neutral	Standard	Extendable
Age 30	0.2000%	0.4100%	0.3993%	1.1255%
Age 60	0.2800%	0.5400%	0.5280%	2.7609%

From the second and the third columns of those two tables, we can see that the analytical solutions of instantaneous charge rates based on Equation (2.2) are close to the numerical ones based on Crank-Nicolson method, which confirms the feasibility of this numerical method. The slight differences may be due to the convergence errors and the interpolation errors. Comparing the values in the last two columns, we can see that segregated fund contracts with the reset options are more than twice as valuable as those without the reset options. Particularly, more dramatic increase occurs to Type II policyholders aged 60.

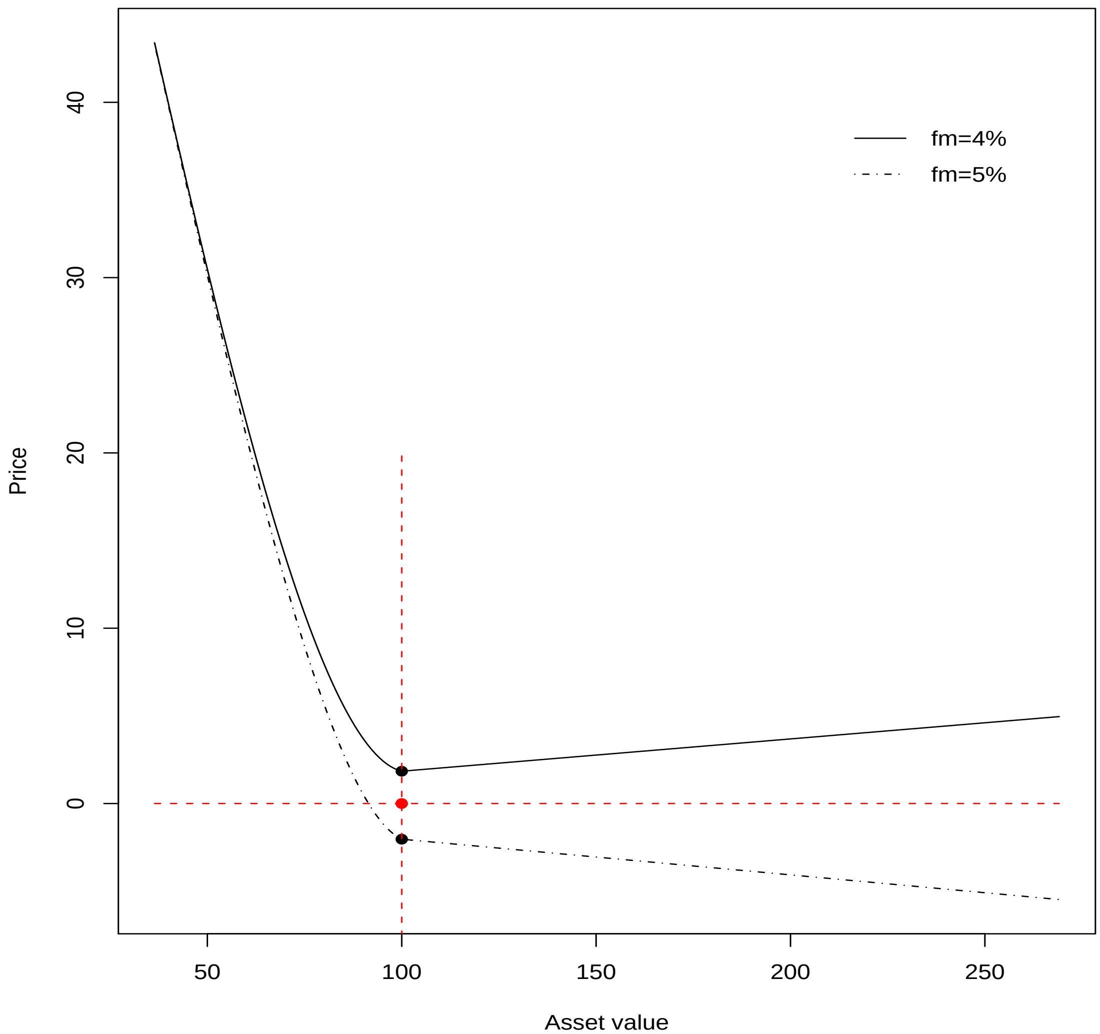

Figure 4.2: The solution of instantaneous charge rate

As been pointed out in Chapter 2, Type II contracts are very sensitive to the mortality risk. The reset options call for longer duration of the contracts, which in turn increase the mortality risk and make this type of contracts relatively expensive.

4.2.5 Further analysis of the charge rate

One of the assumptions the approach above made is that the policyholders optimally exercise their reset options. In practice, however, this situation barely happens. In general, the chance that the rest option is exercised is more or less correlated to the premium level (With higher premium level, the rest options are more frequently used). But the exercise of the reset option is also affected by other factors such as investment preferences (One may only want to hold a segregated fund contract for 10 years so that reset option will not be used) and time issue (When the investment return hits a historical high level, a position at year one is more likely to trigger a reset than that at year nine). The charge rates for the extendable segregated fund contracts in Table 4.2 and 4.3 could be considered as extreme values (Since in reality people do not reset as frequently as they do in optimal case, reset option may not be as valuable as those considered above). In the following we use simulation to further analyze the charge rate for the reset option. In practice a lot of factors and strategies are involved to consider the exercise of the reset option. In this project, however, for simplicity we introduce a constant return threshold to trigger the reset options. Once the return on the premium net of the annual charges goes above the threshold since the beginning of the contracts or last time the reset option is executed, the reset option will be exercised if it is still applicable during that period. For the purpose of illustration, we arbitrarily choose 10% annually compounding (or 0.09531 continuously compounding) as the threshold and only Type I contracts with reset option are considered. Some more assumptions are addressed below:

- The instantaneous returns on the premium are simulated monthly by a normal generator under risk-neutral world (This is to be consistent with the risk-neutral analysis above). If we let $log(\frac{S_m}{S_{m-1}})$ denote the monthly return, then

$$log\left(\frac{S_m}{S_{m-1}}\right) \sim N\left(\frac{1}{12}\left(r-\frac{1}{2}\sigma^2\right), \frac{1}{12}\sigma^2\right).$$

- The maximum duration of the contract is 30 years with no rest option allowed during the last ten years. Only one rest option could be exercised during each year.

- To take mortality effect into account, a pool of 100 30-year-old policyholders with \$1 single premium is considered so that the portfolio is worth \$100 before the charge deduction. The force of interest rate for discount is equal to r. GMDB is payable only at the end of each month. Mortality decrement in faction year is uniformly distributed.

Three cases with annual charge rates 1.5%, 3.0% and 4.5% respectively are tested based on a 10,000-run simulation. In Figure 4.3 and Table 4.4 the fund value and GMMB payoff at the time when the contract matures are examined. The fund value represents the segregated fund investment performance (net of the charges). Since the annual charge rate could be considered as the negative annual return rate, increasing the charge rate pulls down the segregated fund level, which in turn thins the right tail of the distribution and shifts the distribution leftwards. The GMMB payoff is the cost of the insurance company to compensate the difference between the fund value and the guarantee if the fund value is less than the guarantee. The high frequency at value 0 corresponds to the part of the fund distribution where the fund value is beyond 100. It is consistent with the evolution of the fund value. With increasing the charge rate, the chance that the fund value ends up less than 100 is increased and the chance that no GMMB payoff is needed is reduced. In Table 4.4 both the mean and standard deviation of the fund value decrease with charge rate from 1.5% to 4.5%. The mean of the GMMB payoff is around 51 for all three cases. On the other hand, however, there is a significant drop of the standard deviation, which is due to the exercise of the reset option. The GMMB payoff varies between 0 and the guarantee. Without reset option, it is capped with 100. But for the extendable segregated fund contract, the cap is increased every time upon reset. For the large charge rate like 4.5%, the fund return (net of the charges) is less likely hit the 10% threshold and the fund is less frequently reset, which explains the decreasing trend of the standard deviation of the GMMB payoff.

Figure 4.4 and Table 4.5 give further proof that the chance of resetting the fund reduces with charge rate increasing. The duration stands for the time that the contract matures and the maturity guarantee is the accumulative GMMB guarantee at the maturity of the contract. The frequency of the 10-year duration goes up with charge rate increasing. This corresponds with the frequency of the \$100 maturity guarantee. With charge rate 4.5%, it suggests that the about 38% of the contracts do not reset the fund while the proportion drops to about 24% if 1.5% is charged annually. Additionally, the distribution of the duration with charge rate 1.5% displays that some contracts might last up to 30 years while for the 4.5%

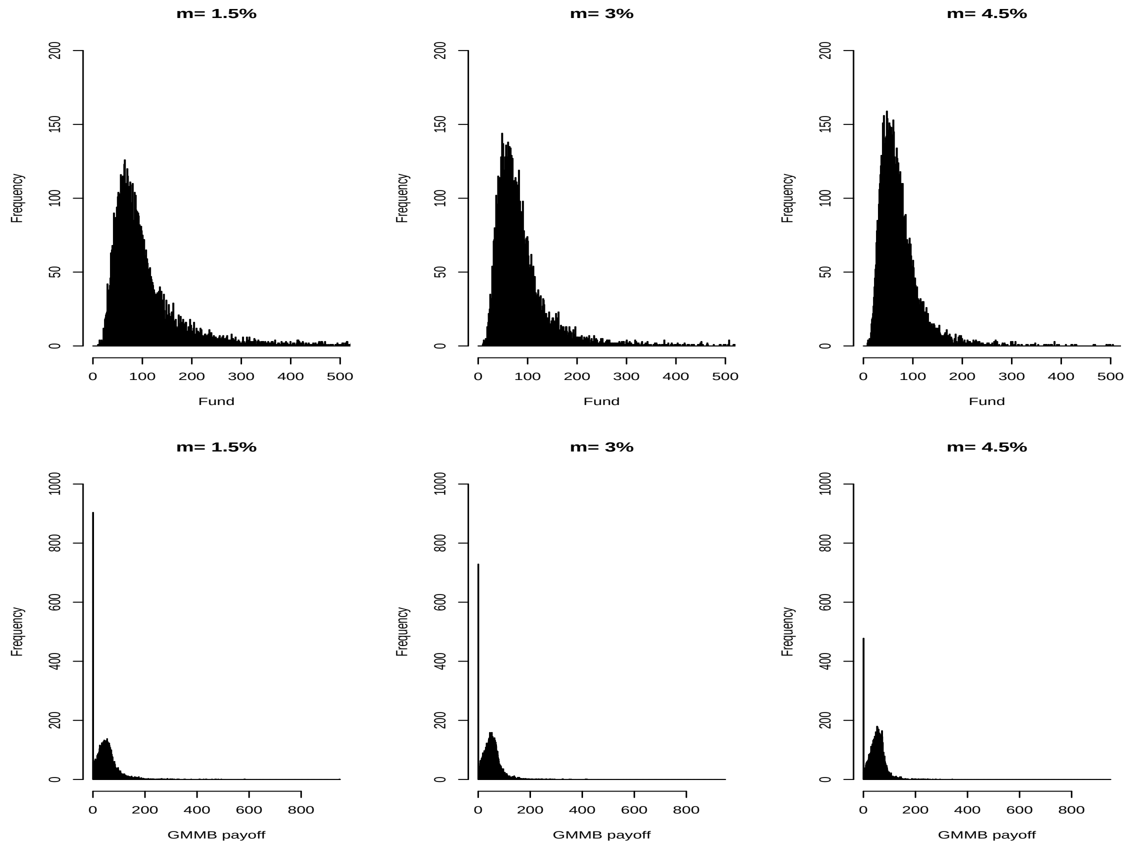

Figure 4.3: The simulated density distributions of the fund value and GMMB payoff at maturity

Table 4.4: The sample mean and standard deviation of the fund value and GMMB payoff at maturity

m=1.5%	Fund	GMMB payoff
Mean	112.4231	51.7606
Standard deviation	105.6241	43.7160

m=3.0%	Fund	GMMB payoff
Mean	87.2771	50.1704
Standard deviation	67.3597	34.5005

m=4.5%	Fund	GMMB payoff
Mean	71.6099	51.5752
Standard deviation	45.7719	29.3519

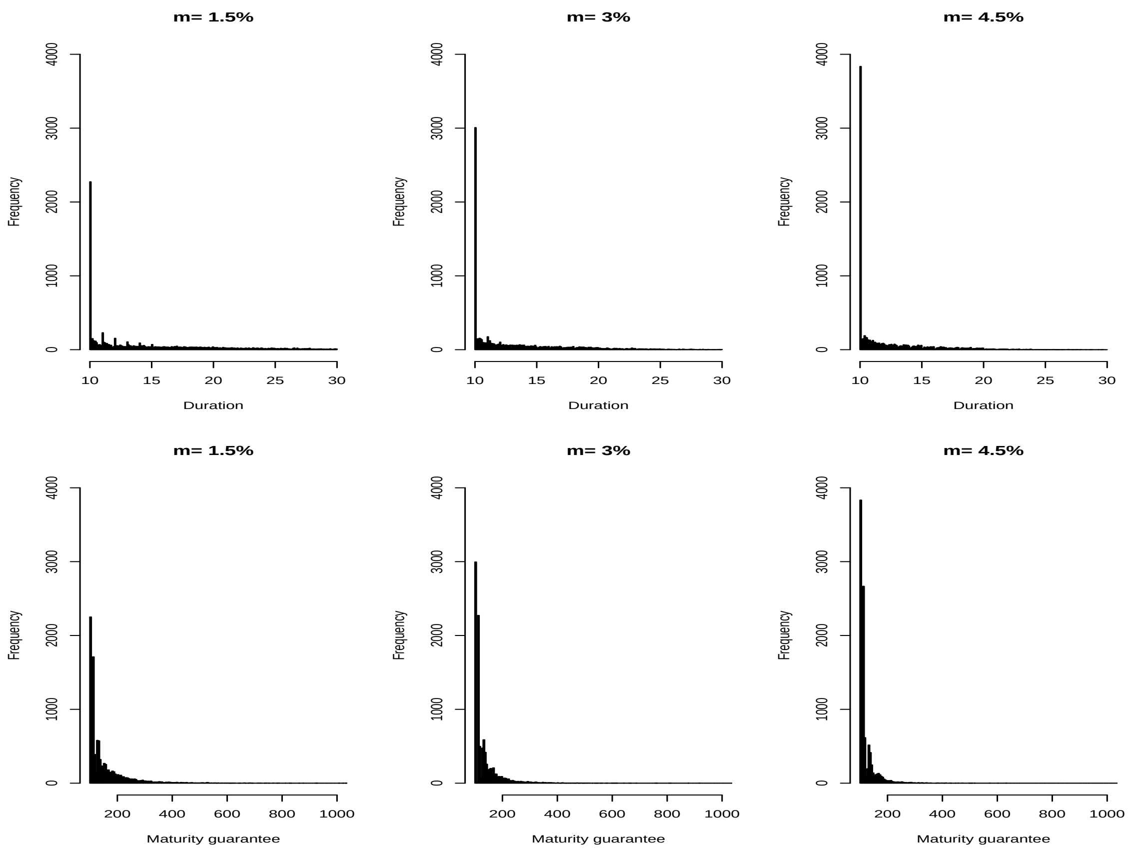

Figure 4.4: The simulated density distributions of the duration and maturity guarantee

Table 4.5: The sample mean and standard deviation of the duration and maturity guarantee

m=1.5%	Duration	Maturity guarantee
Mean	14.9766	159.6207
Standard deviation	5.3272	95.7484

m=3.0%	Duration	Maturity guarantee
Mean	13.4161	135.9271
Standard deviation	4.1863	60.1120

m=4.5%	Duration	Maturity guarantee
Mean	12.3631	122.8358
Standard deviation	3.2954	40.2140

case most of the contracts last around 10 to 15 years. This means that the charge rate 4.5% leads to a less frequency of reset comparing to the case with charge rate 1.5%. Even for those contracts which are reset, a large proportion only gets reset at early times within the first 10 years.

In Figure 4.5 and Table 4.6, The maturity payment is the summation of the fund value and GMMB payoff, and the yield is given by

$$\text{Yield} = \left(\frac{\text{Maturity payment}}{100}\right)^{\frac{1}{\text{Duration}}} - 1.$$

The higher the charge rate is, the more likely the fund value is below the guarantee, and the less likely the reset option is used, which leads to a high chance that the fund matures with \$100 and 0% yield. With a threshold 10%, the illustration above shows that when the charge rate 1.5%, the fund value ends up with \$12 higher than GMMB guarantee in average. The average payment at maturity is \$64 larger than the initial fund value. And the average yield indicates a annual return of 2.14%. When the charge rate increases to 4.5%, in average, the fund value at maturity goes below GMMB guarantee by \$30 and the maturity payment drops to \$123, which yields an average annual return 1.18%. The result is that large charge rate sharply offsets the positive return of the fund investment and deteriorates the fund value when the return is negative. With annual charge deduction of 1.5%, the reset option might be valuable. However, an annual charge rate of 4.5% significantly pulls down the fund value and the value of the reset option is decreased.

Table 4.6: The sample mean and standard deviation of the total payment and yield at maturity

m=1.5%	Maturity payement	Yield at maturity
Mean	164.1838	0.0214
Standard deviation	111.8120	0.0193

m=3.0%	Maturity payement	Yield at maturity
Mean	137.4475	0.0160
Standard deviation	67.4705	0.0166

m=4.5%	Maturity payement	Yield at maturity
Mean	123.1850	0.0118
Standard deviation	43.2121	0.0143

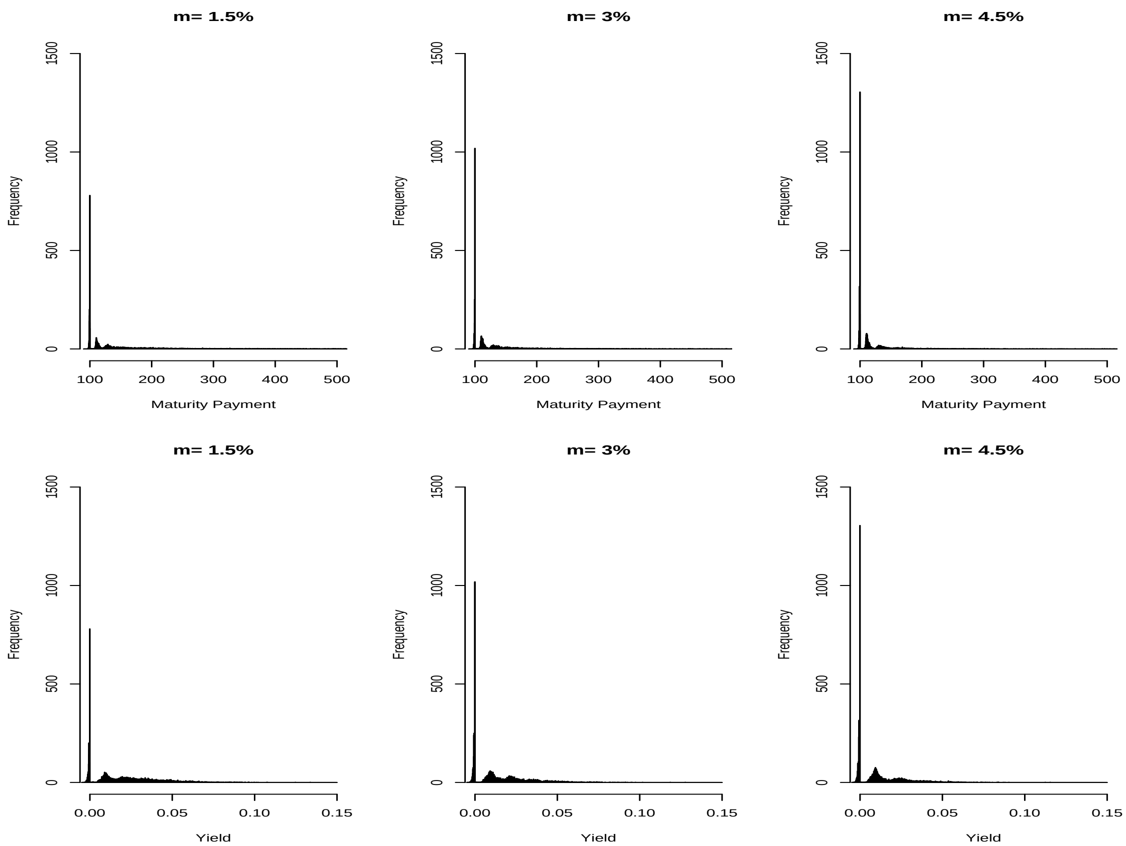

Figure 4.5: The simulated density distributions of the total payment and yield at maturity

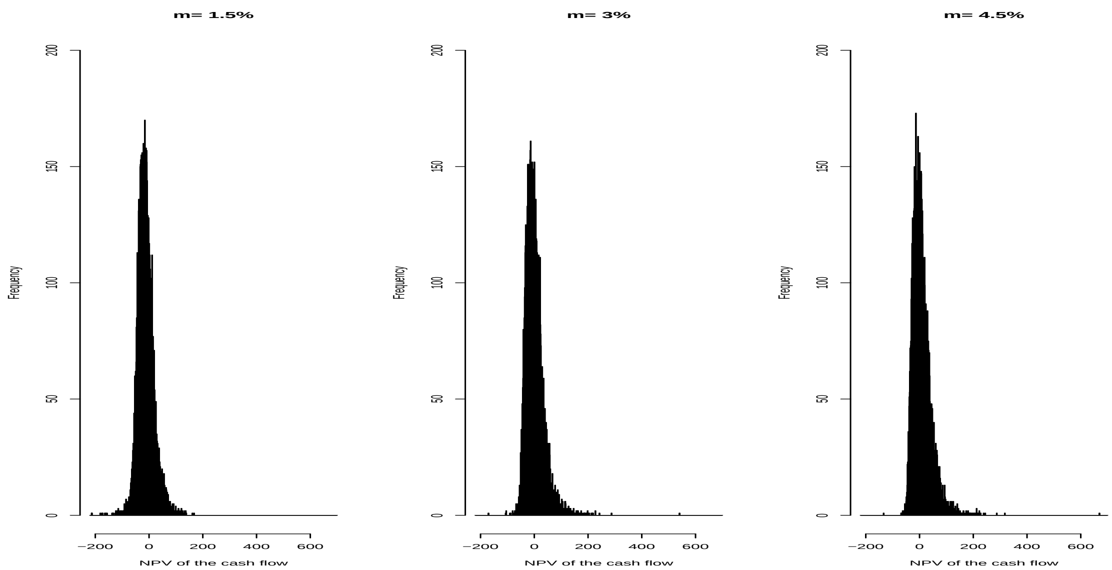

Figure 4.6: The simulated density distributions of the NPV of the cash flows

Figure 4.6 and Table 4.7 show some statistics of the NPV of the cash flows with different charge rates. In the sense of pricing, a negative NPV of the cash flow suggests that 1.5% might not be sufficient while the fund with 4.5% annual charge rate might be deducted (4.5% might be a reasonable charge rate in the case that reset option is treated optimally since the segregated fund is frequently reset). The linear interpolation between 2.9% and 3% gives a charge rate of 2.941% which approximately generates a 0 NPV of the cash flows.

In the optimality case that reset frequently occurs, a large charge of 4.5% from the segregated fund is annually deducted. In the 10%-threshold case, the charge rate is lower, but the reset strategy is still aggressive and the policy is still expensive. Situations in our example might not happen in practice. Nevertheless, for the following illustration, 2.94% is considered as a proper annual charge rate for the segregated fund contract with reset option.

Table 4.7: The sample mean and standard deviation of the NPV of the cash flows

	m=1.5%	m=3.0%	m=4.5%
Mean	-13.0757	0.4222	7.608783
Standard deviation	29.8358	33.4193	34.46849

4.3 Hedging the segregated fund with reset option

Reset option may increase the variation of the benefit payoff and add more risk on the segregated fund. Due to the uncertainty of the length of the contract and the guarantees, the GMMB and GMDB payoffs may be significantly increased. Based on the simulation assumption above, Figure 4.7 displays the distributions and percentiles of both standard and extendable segregated fund contracts, and their means and standard deviations are shown in Table 4.8. For the segregated fund without reset option, we use the same assumption as in Figure 3.18 except we generate the asset price using r instead of μ. From Figure 2.2 the proper charge rate is about 1.5% under risk-neutral method. And the negative mean is due to the insufficiency of 1% of the annual deduction. For the segregated fund with reset option, the slightly positive mean may stem from the interpolation and sample error (since we expect a zero mean).

From the distributions of those two types of contracts, the two side tails from the standard contract are fatter and longer than those from the extendable contract. Specifically, the extreme loss of the extendable contract could be up to \$200. Along with the larger standard deviation from the extendable contract, it is evident that reset option adds considerable risk to the segregated fund.

Table 4.8: The mean and standard deviation of the NPV of the cash flows

	Standard	extendable
Mean	-3.3226	0.0658
Standard deviation	18.3270	33.3381

In the following we will illustrate the risk diversification effect of the dynamic hedging approach. The monthly hedging strategy is employed. At the beginning of the contract, the hedging strategy is set assuming a standard 10-year segregated fund contract. The hedging re-balancing mentioned in Chapters Three is used. Every time upon reset, the previous hedging strategy is terminated. At that point of time, the new capital is established to support another assumed 10-year hedging processes. Note that we do not hedge the reset option. The consideration of hedging the reset option is beyond the scope of this project.

Figure 4.8 gives the simulated results of the extendable segregated fund contracts with or without hedging. The solid lines represent the distribution of the segregated fund without hedging and the dashed lines represent the distribution of the segregated fund under monthly

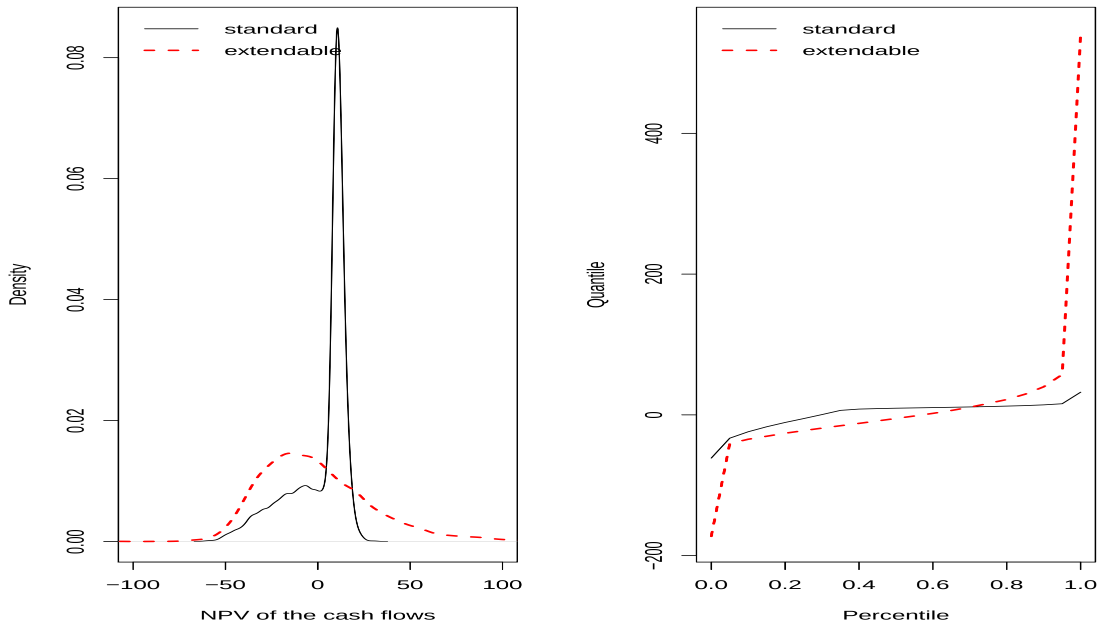

Figure 4.7: The density distributions and percentiles of the NPV of the cash flows

Table 4.9: The mean and standard deviation of the NPV of the cash flows with and without hedging

	Segregated fund	Monthly hedging
Mean	0.0658	15.7677
Standard deviation	33.3381	23.2407

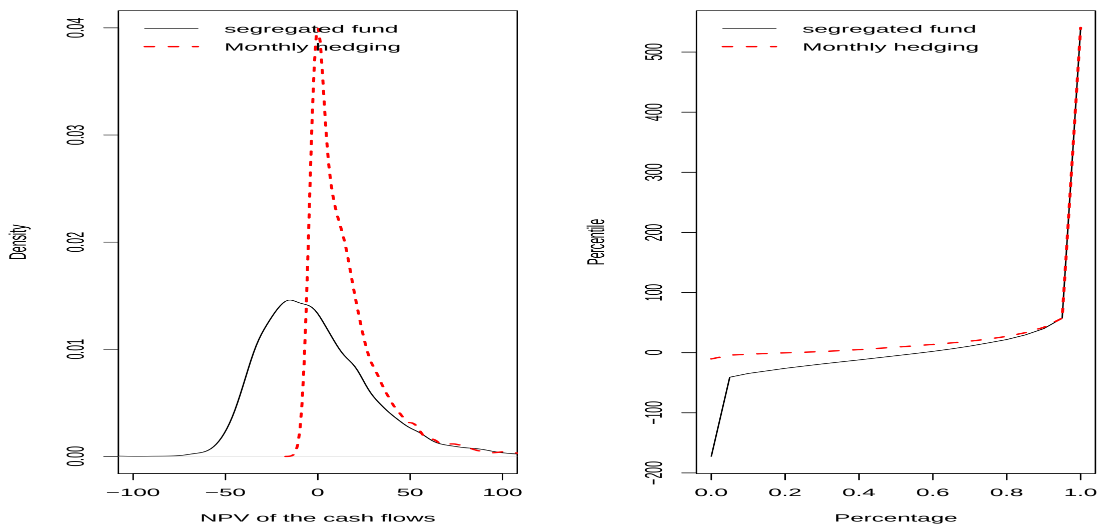

Figure 4.8: The density distributions and percentiles of the NPV of the cash flows with and without hedging

hedging strategy. Comparing their distributions, we can see that the one under monthly hedging strategy is more concentrated with a relatively high density around zero. More importantly, Monthly hedging strategy helps cut off the left tail which caps those extremely bad NPV around \$20. Based on their percentiles, selling the extendable segregated fund contract could generate a poor NPV of the cash flows up to \$200. However,under the hedging approach, the NPV's are quite stable for most percentages (from 0% to 90%). Table 4.9 shows the respective means and standard deviations. With annual charge 2.941%, selling segregated fund with reset option is supposed to get a zero NPV of the cash flows in average. Under monthly hedging strategy, a surplus of \$15.77 is generated. And the volatility of the cash flows is controlled by reduce the standard deviation by 30%.

Chapter 5

Conclusion

Segregated fund is a type of equity-linked insurance contract which offers some financial guarantees. Compared to GMDB, GMMB dominates the risk while the situation might reverse for the relatively old policyholders.

Dynamic hedging approach with an allowance for transaction costs utilizes the Black-Scholes formula to replicate the value of the segregated fund. The H&T reserve is hold to cover the hedge errors net of transaction costs and is assumed to be invested into the zero-coupon bond which provides risk-free interest rate. The hedge errors net of the transaction costs is reduced by incorporating *Leland*'s volatility. Three hedging strategies with their respective re-balancing costs were examined. Under annually hedging strategy the hedging cost is re-balanced only at end of each year, while monthly and weekly hedging strategies capture the details of the asset price movement. Along with the cash flow comparison, it indicated that the NPV of the cash flows under annually hedging approach had greater volatility than the one under monthly hedging approach. Dynamic hedging approach is in favor of risk diversification. In our simulation illustration, Applying monthly hedging approach to the standard segregated fund shortened the left tail of the distribution of the NVP of the cash flows and reduced the cash flow variation.

Extendable segregated fund contracts offer reset options. The reset option adds more complexity to the valuation of the segregated fund due to the uncertainties of the guarantees and maturity date. A simplified form of an finite difference method was used to illustrate the considerable value added by the reset option. The simulation was used to further analyze the value of the reset option. Reset options bring more volatility to the cash flows of the segregated fund. Our example showed that dynamic hedging still had positive effect on controlling the risk.

For the sake of convenience this project made some assumptions to illustrate our points, which leaves much room for future study. For example, we used the Black-Scholes formula to model the asset price. The parameters and interest rate are constant, which is unrealistic for the segregated fund, the term of which is up to ten years or more. Stochastic models such as regime-switching could be employed. Additionally, different approaches could be used to model the reset option. Determining a more realistic reset strategy is another interest of research.

Appendix A

Some proofs

To prove Equations (3.3) and (3.4), a couple of lemmas will be employs. The proofs of those lemmas can be found in *Toft* (1996). To simplify the proofs, some notations are introduced as

$$\tilde{\mu}(t_1, t_2) = (\mu - \frac{1}{2}\sigma^2)(t_2 - t_1),$$
$$\bar{\mu}(t_1, t_2) = (r - \frac{1}{2}\sigma^2)(t_2 - t_1),$$

and

$$\tilde{\sigma}(t_1, t_2) = \sigma\sqrt{t_2 - t_1}.$$

There are some lemmas which will be used:
LEMMA 1 for some constants A, B, C, D and E,

$$\begin{aligned} & \int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\left(Ax^2 + Bx + C\right)\right)\Phi\left(Dx + E\right)dx \\ = \; & \frac{\sqrt{2\pi}}{\sqrt{A}} exp\left(-\frac{1}{2}\left(C - \frac{B^2}{4A}\right)\right)\Phi\left(\sqrt{\frac{A}{A + D^2}}\left(E - \frac{BD}{2A}\right)\right). \end{aligned}$$

LEMMA 2 for some constants A, B, C, D and E,

$$
\begin{aligned}
&\int_{-\infty}^{\bar{x}} exp\left(-\frac{1}{2}\left(Ax^2+Bx+C\right)\right)\Phi\left(Dx+E\right)dx \\
= \;& \frac{\sqrt{2\pi}}{\sqrt{A}} exp\left(-\frac{1}{2}\left(C-\frac{B^2}{4A}\right)\right)\Phi\left(\bar{x}\sqrt{A}+\frac{B}{2\sqrt{A}}, \sqrt{\frac{A}{A+D^2}}\left(E-\frac{BD}{2A}\right), -\frac{D}{\sqrt{A+D^2}}\right).
\end{aligned}
$$

LEMMA 3 for some constants A, B, C, D, E, F, G and ρ,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} exp\left(-\frac{1}{2}\left(Ax^2+Bx+C\right)\right)\Phi\left(Dx+E, Fx+G, \rho\right)dx \\
= \;& \frac{\sqrt{2\pi}}{\sqrt{A}} exp\left(-\frac{1}{2}\left(C-\frac{B^2}{4A}\right)\right) \\
&\times\Phi\left(\sqrt{\frac{A}{A+D^2}}\left(E-\frac{BD}{2A}\right), \sqrt{\frac{A}{A+F^2}}\left(G-\frac{BF}{2A}\right)), \frac{DF+A\rho}{\sqrt{(A+D^2)(A+F^2)}}\right).
\end{aligned}
$$

A.1 The proof of Equation (3.3)

We begin with taking the conditional expectation on Equation (3.1),

$$
\begin{aligned}
\mathbb{E}[H(t_2)|S_{t_1}] \;=\; & Ke^{-r(T-t_2)}\Phi\left(-d_2(t_1)\right) \\
& -Ke^{-r(T-t_2)}\mathbb{E}[\Phi\left(-d_2(t_2)\right)|S_{t_1}] \\
& -\mathbb{E}[S_{t_2}|S_{t_1}]\Phi\left(-d_1(t_1)\right) \\
& -\mathbb{E}[S_{t_2}\Phi\left(-d_1(t_2)\right)|S_{t_1}]. \qquad \text{(A.1)}
\end{aligned}
$$

First,

$$
\mathbb{E}[S_{t_2}|S_{t_1}] = S_{t_1}e^{\mu(t_2-t_1)}. \qquad \text{(A.2)}
$$

Second, using Lemma 1 and taking some transformation,

$$
\begin{aligned}
& \mathbb{E}[\Phi\left(-d_2(t_2)\right)|S_{t_1}] \\
= \;& \int_{-\infty}^{\infty} \frac{1}{x\tilde{\sigma}(t_1,t_2)\sqrt{2\pi}} exp\left(-\frac{\left(log\frac{x}{S_{t_1}} - \tilde{\mu}(t_1,t_2)\right)^2}{2\tilde{\sigma}(t_1,t_2)^2}\right)\Phi\left(-\frac{log\frac{x}{K} + \bar{\mu}(t_2,T)}{\tilde{\sigma}(t_2,T)}\right)dx \\
= \;& \frac{1}{\tilde{\sigma}(t_1,t_2)\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(\frac{z^2 - 2\tilde{\mu}(t_1,t_2) + \tilde{\mu}(t_1,t_2)^2}{\tilde{\sigma}(t_1,t_2)^2}\right)\Phi\left(-\frac{z - logK + \bar{\mu}(t_2,T)}{2\tilde{\sigma}(t_2,T)}\right)dz \\
= \;& \Phi\left(-\frac{log\frac{S_{t_1}}{K} + \bar{\mu}(t_2,T) + \tilde{\mu}(t_1,t_2)}{\tilde{\sigma}(t_1,T)}\right), \qquad \text{(A.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}[S_{t_2}\Phi\left(-d_1(t_2)\right)|S_{t_1}] \\
= \;& \frac{1}{\tilde{\sigma}(t_1,t_2)\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left(-\frac{z^2 - 2(\tilde{\mu}(t_1,t_2) + \tilde{\sigma}(t_1,t_2)^2)z + \tilde{\mu}(t_1,t_2)^2}{2\tilde{\sigma}(t_1,t_2)^2}\right) \\
& \times\Phi\left(-\frac{z - logK + \bar{\mu}(t_2,T)}{\tilde{\sigma}(t_2,T)}\right)dz \\
= \;& S_{t_1}e^{\mu(t_2-t_1)}\Phi\left(-\frac{log\frac{S_{t_1}}{K} + \tilde{\mu}(t_1,t_2) + \bar{\mu}(t_2,T)}{\tilde{\sigma}(t_1,T)}\right). \qquad \text{(A.4)}
\end{aligned}
$$

Then substituting (A.2) (A.3) and (A.4) into (A.1) gives

$$
\begin{aligned}
\mathbb{E}[H(t_2)|S_{t_1}] \;=\;& Ke^{-r(T-t_2)}\left[\Phi(-d_2(t_1)) - \Phi(-d_2^*(t_1,t_2))\right] \\
& -S_{t_1}e^{\mu(t_2-t_1)}\left[\Phi(-d_1(t_1)) - \Phi(-d_1^*(t_1,t_2))\right]. \qquad \text{(A.5)}
\end{aligned}
$$

Taking the conditional expectation on (A.5) given S_{t_0} and discounting it back to t_0 reaches Equation (3.3).

A.2 The proof of Equation (3.4)

Taking the conditional expectation on Equation (3.2), we get

$$
\begin{aligned}
\mathbb{E}[TC(t_2)|S_{t_1}] &= \int_0^\infty \frac{k}{\tilde{\sigma}(t_1,t_2)\sqrt{2\pi}} exp\left(-\frac{\left(log\frac{x}{S_{t_1}}-\tilde{\mu}(t_1,t_2)\right)^2}{2\tilde{\sigma}(t_1,t_2)^2}\right) \\
&\quad \times|\Phi(-d_1(t_2))-\Phi(-d_1(t_1))|dx \\
&= \int_{-\infty}^\infty \frac{k}{\tilde{\sigma}(t_1,t_2)\sqrt{2\pi}} exp\left(z-\frac{(z-\tilde{\mu}(t_1,t_2))^2}{2\tilde{\sigma}(t_1,t_2)^2}\right) \\
&\quad \times|\Phi(-d_1(t_2))-\Phi(-d_1(t_1))|dz \\
&= \int_{-\infty}^{\bar{z}} \frac{k}{\tilde{\sigma}(t_1,t_2)\sqrt{2\pi}} exp\left(z-\frac{(z-\tilde{\mu}(t_1,t_2))^2}{2\tilde{\sigma}(t_1,t_2)^2}\right) \\
&\quad \times|\Phi(-d_1(t_2))-\Phi(-d_1(t_1))|dz \\
&\quad +\int_{\bar{z}}^\infty \frac{k}{\tilde{\sigma}(t_1,t_2)\sqrt{2\pi}} exp\left(z-\frac{(z-\tilde{\mu}(t_1,t_2))^2}{2\tilde{\sigma}(t_1,t_2)^2}\right) \\
&\quad \times|\Phi(-d_1(t_1))-\Phi(-d_1(t_2))|dz, \qquad \text{(A.6)}
\end{aligned}
$$

where

$$\bar{z} = \tilde{\sigma}(t_2,T)d_1(t_1) - \bar{\mu}(t_2,T) - \tilde{\sigma}(t_2,T)^2 + logK.$$

Using Lemma 2, we get a closed form of the solution to Equation (A.6). Now taking the expectation on Equation (A.6) given S_{t_0} and discounting it back to t_0, and with the help of Lemma 3, Equation (3.2) is proved.

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