ARTICULAR ANALYSIS

by

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Abstract

In this thesis I explore a new *representational* approach to relevance and paraconsistency. This approach is distinguished from truth-conditional, preservational and algebraic approaches in that it exploits the representation of inferentially significant structural features of sentences. The approach introduces a new *hypergraphic* idiom for the study of entailments. Historically the representation has roots in Leibnizian analyses, which, in an articular model, are represented as simple hypergraphs. The set of simple hypergraphs together with meet, join and complementation form a De Morgan lattice, in which the partial ordering is a relation called subsumption. One hypergraph subsumes another when every edge of the former finds a sub-edge in the other. The class of all such structures characterises a binary entailment system in which subsumption interprets entailment. Since a subsumption can itself be represented by a hypergraph, higher-degree systems also emerge with principles that depend solely upon properties of subsumption. In fact such systems arise for any theory of any item, including individuals, concepts, *n*-ary properties and so on, that are representable as hypergraphs. It follows, therefore, that a clear understanding of paraconsistent relations can be obtained between objects of many kinds.

To Ray Jennings, for the lovely rides he so often gave me to and from the skytrain station over the years in his beloved Maisie Miata I and II

"We cannot browse over the field of nature like cows at pasture."

— *Mainly about induction*, INDUCTION AND INTUITION IN SCIENTIFIC THOUGHT BY PETER MEDAWAR, *Jayne Lectures for 1968*

Acknowledgments

This thesis lies under heavy obligations, from which derive the greatest part of such weight as it has.

Professor R. E. Jennings has been most generous in giving me his time for a large number of informal discussions of logical problems, and I have received much from him in the way of valuable suggestions and advice, in addition to what I owe to his lectures and writings.

I have had the advantage of following this line of research, initiated by Professor Jennings, whose original work on hypergraphs in general, dated more than a decade ago, has opened the area of research which led to the current investigations, and without whose insight and creative understanding none of the work presented here would have existed. It was through him that I was first acquainted with hypergraph semantics, and I have benefited as much from his specific suggestions as from his general viewpoint, which had started and guided me along the path I have taken toward the work in the thesis.

Our contact with Professor J. M. Dunn from Indiana has been stimulating. His observations on the relation between first-degree articular inference and *first degree entailment* (*FDE*) started a fruitful field of investigations into other systems in the region of *FDE* by employment of hypergraph semantics.

This has contributed to my conviction that an inquiry into first-degree systems using hypergraph semantics is best advanced by a detailed study of the formal consequences derived from various hypergraphic relations. As the research proceeds, generalizations, mostly of an algebraic nature,

readily present themselves, so much so that from the vantage point we now possess, it is almost a matter of certainty that we have just embarked on a long journey that may lead to various destinations, which on its way would adjoin different fields of logic, and new light may be shone on old problems such as the semantic status of contradiction.

I shall like to express my thanks to Julian Sahasrabudhe and Andrew Withy for various valuable conversations, from whom came many ideas elaborated in the thesis, and to Jeremy Seligman for his technical help and comments that have taught me a great deal. I am most grateful to Andrew Hartline, whose delicate help and encouragement have influenced the present shape of the work.

I should like to develop the following investigations along several lines, and there are definitely more unsolved and unfinished problems presented in this thesis than can properly be called closed; but perhaps it is better to put them forward in their present stage, so that they may profit from criticism.

A remark on the Index: all the important subjects involved in this thesis can be found in the index except hypergraph and first degree entailment; I did not include them because they are everywhere in the thesis.

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Introduction: An historical survey

0.1 Technical matters

In what follows I shall use ‘classical model’ to refer to a ‘full’ propositional model, which is an ordered pair

$$\mathscr{M} = \langle\, U,\ V\,\rangle$$

where U is a nonempty set of ‘possible worlds’ and V

$$V : U \times At \rightarrow \{0, 1\}$$

assigns a truth value to every atom at every element (point) of U. A related notion can be found in the literature. A *singular* model sets U at a singleton, which can be suppressed. Hence the usual notion of a classical model as an assignment:

$$V : At \rightarrow \{0, 1\}.$$

This assignment is what Carnap called a ‘state description’, a complete description of the ‘possible world’, here the sole element of the universe, in a presumably ideal language [8]. A full propositional model $\mathscr{M}$, therefore, can be understood as a class of singular models. In a full model, $[\![\cdot]\!]^{\mathscr{M}}$ extends V to the set of formulae Φ as follows:

$$
\begin{aligned}
[\![P_i]\!]^{\mathscr{M}} &= V(P_i) \\
[\![\neg\alpha]\!]^{\mathscr{M}} &= U - [\![\alpha]\!]^{\mathscr{M}} \\
[\![\alpha \wedge \beta]\!]^{\mathscr{M}} &= [\![\alpha]\!]^{\mathscr{M}} \cap [\![\beta]\!]^{\mathscr{M}} \\
[\![\alpha \vee \beta]\!]^{\mathscr{M}} &= [\![\alpha]\!]^{\mathscr{M}} \cup [\![\beta]\!]^{\mathscr{M}}
\end{aligned}
$$

The valuation V is a function $V : At \to \wp(U)$. Therefore $[\![\cdot]\!]^{\mathscr{M}}$-image of the propositional language is a Boolean Algebra $\mathscr{B}(CL)$ of 'UCLA propositions'[1] on $\langle \wp(U), \subseteq \rangle$ in which the classical connectives '$\neg$', '$\wedge$' and '$\vee$' are interpreted as set-theoretic (relative) complement, intersection, and union respectively. Classical entailment is represented by the inclusion relation between members of $\wp(U)$.

Because the language is semantically compositional, the valuation function V in effect biparts U at each formula α, via the set $[\![\alpha]\!]^{\mathscr{M}}$ and its complement. Classical models have a 'philosophical' appeal in that they seem to provide applications of both the vocabulary of 'truth' and that of 'meaning'. Sentences can be regarded as true everywhere in their meanings and false elsewhere. Moreover, semantic entailment can be regarded as truth-preserving.

The applicability of 'truth' and 'meaning' lend verisimilitude to the claim that classical models provide a semantic account of propositional language. But for the logician 'truth' need be no more than a *façon de parler*. A formula α is satisfied at points in $[\![\alpha]\!]^{\mathscr{M}}$,

$$\models^{\mathscr{M}}_{x} \alpha \Leftrightarrow x \in [\![\alpha]\!]^{\mathscr{M}};$$

a set Σ of formulae is satisfied at points in $[\![\Sigma]\!]^{\mathscr{M}}$ ($\{x \mid \forall \sigma \in \Sigma, \models^{\mathscr{M}}_{x} \sigma\}$)

$$\models^{\mathscr{M}}_{x} \Sigma \Leftrightarrow x \in [\![\Sigma]\!]^{\mathscr{M}}.$$

The preservation of satisfaction by semantic entailment is a consequence of the monotonicity of membership along inclusion. The mention of $\{0, 1\}$ prompts some to describe these models in the

[1] It is unclear who originated the term, but the development of the concept can be traced through the works of Boole, Frege, Carnap, Montague, Kripke and others. It is presumably called a UCLA proposition because both Carnap and Montague were at the University of California at Los Angeles. A clear account of it can be found in [18].

language of truth and falsity at 'worlds'. It is as well to remember that neither 'truth' nor 'world' can be regarded as anything more than a *reading* of features of a model, therefore the valuation V can be equivalently defined as a function from the set of atoms At to $\wp(U)$.

To sum up, for classical logic the model-theoretic counterpart of truth-preservation is the monotonicity of membership over inclusion. Therefore, closure under classical entailment can be regarded as a principled augmentation of a set of formulae under some preservational criterion. Classical entailment is an instance of a more general notion of (semantic) entailment to which we now turn.

Definition 1. $\Sigma \cup \{\alpha\}$ *R-augments* Σ *iff* $\langle \Sigma, \ \alpha \rangle \in R$.

Now we are in a position to introduce THE GENERAL PRINCIPLE OF PRESERVATION
If $E \subseteq \wp(\Phi) \times \Phi$ *is an entailment relation, then* $\exists \phi$ *such that* ϕ *is not* $\subseteq$*-monotonic but is monotonic along E-augmentation.*

One principal proof-theoretic task is to define, by a set of rules, a consequence relation that permits all and only semantically permitted augmentations. This understanding of logical consequence as a set of preservationally constrained augmentations frees logical thinking from the confines of reasoning and laws of thought, going further along the route opened by George Boole, one of whose 'chief titles to fame', as Kneale & Kneale emphatically put it in [29], is to have 'freed logic from the dominion of epistemology and so brought about its revival as an independent science'.

0.1.1 Sets and points

A material conditional in a classical model is interpreted by

$$[\![\alpha \supset \beta]\!]^{\mathscr{M}} = (U - [\![\alpha]\!]^{\mathscr{M}}) \cup [\![\beta]\!]^{\mathscr{M}}$$

where the material horseshoe like any other classical operator is to be represented as a truth-function. This representation, along with those for the other logical operators, during the early days since its discovery by George Boole [5], have been regarded as the formal idiom, if not of human thought, at least of human thinking[2]. But this static set-based account readily gives way to a point-based account. Since the classical horseshoe satisfies the deduction theorem, we naturally seek understanding of the entailment relation from the semantic representation of the horseshoe. The purely set-theoretic character of classical entailment enables us to derive a conditional connective of preservational value. To say that material implication $\alpha \supset \beta$ is satisfied at x is to say that x confirms the inclusion of $[\![\alpha]\!]^{\mathscr{M}}$ in $[\![\beta]\!]^{\mathscr{M}}$, that is

$$[\![\alpha \supset \beta]\!]^{\mathscr{M}} = \{x \mid \{x\} \cap [\![\alpha]\!]^{\mathscr{M}} \subseteq [\![\beta]\!]^{\mathscr{M}}\}$$

The material implication preserves satisfaction from antecedent to consequent. The century after Boole witnessed many innovations in algebraic semantics that provide ample illustration of the truth in the remark made by Anderson and Belnap (A & B) [2], that formal systems can be investigated without explicit recourse to their intended interpretations. As Boole remarked in [5],

> 'They who are acquainted with the present state of the theory of Symbolical Algebra, are aware that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed, is equally admissible, and it is thus that the same processes may, under one scheme of interpretation, represent the solution of the question on the properties of numbers, under another, that of a geometrical problem, and under a third, that of a problem of dynamics or optics…'

Formal logic therefore can be taken as the study of certain mathematically well-defined structures and the properties that these structures enable us to represent, combined with the preservation of these properties over a consequence relation. In fact so far as research is concerned, mathematical structures usually present themselves first before any calculus can be speculated upon, contrary to the order of the formal presentation of a logic system. Boole again [5]:

[2]There is a distinction between the two expressions. Logic at the time of Boole was generally assumed to have psychological functions and aims, and the theorems were the laws of thought, which, as other laws for that generation of thinkers, were at once unchanging and unchangeable, under whose ruthless government thinking was conducted; while a modern logician would more likely reverse the order between thinking and logic and consider *a* logic to be the product of thinking as a dynamic creative process.

> 'That to the existing forms of Analysis a quantitative interpretation is assigned, is the result of circumstances by which those forms where determined, and is not to be construed into a universal condition of Analysis.'

Classical logic, understood in this light, is open to generalization in many ways. Various of these have inspired attempts to 'redefine the classical preservation of truth'. This vague expression has suggested at least two attempts at re-definition. Some have redefined the kind of 'truth' to include the truth of sentences that have more than one truth value; others have liberalized the range of preservation to accommodate mathematically desirable properties other than truth. Subdivisions within these attempts are characterized by their different methods of redefining truth or enriching the preservational repertoire, and very roughly sketched, they fall into three categories: dialethic, preservational, and representational, apart from which there also exist axiomatic proof systems based upon syntactic rather than semantic accounts of truth, falsehood, and contingencies respectively [24, 36, 37, 38].

0.2 Approaches to paraconsistency

The motives for redefining the classical preservationn of truth may be as diverse as the logicians who attempt to do so. In classical (full propositional) models, the truth-set representation of formulae generates such peculiar properties as manifest themselves in a series of famous paradoxes often referred to as the 'paradoxes of material implication'. They were noted by C. I. Lewis in 1912 [31]. It has been suggested that the paradoxical properties of classical entailment revealed in *ex falso quodlibet* arise from the vacuous truth preservation in the argument from $p \wedge \neg p$ to q, since the truth-set of $p \wedge \neg p$ in classical models is $\varnothing$. Moved by this diagnosis, logicians have fashioned various ways in accordance with their interests to avoid this unpalatable instance of classical inference. Among them are those who revised the valuation function V, redefining "truth" by assigning a paradoxical value P, read 'both truth and false', to some formula such as that of the form $p \wedge \neg p$. Dialetheism, which allows satisfaction of an inconsistent set, finds favour among those who adopt this method. Priest in 'The Logic of Paradox' (*LP*) [42] provides a three-valued semantics, adding a paradoxical value to classical truth as one of the designated values and requiring the consequence relation to preserve both. Since the paradoxical value has truth as a component, it generalizes the criterion of truth preservation in such a way as to accommodate the case in which both truth and falsity appear on the left of the consequence relation. Hence truth is preserved from T to P, but not from P to F. In a paradoxical model, the inference from $p \wedge \neg p$ to q no longer holds vacuously, and therefore can be falsified.

Concerning the classical truth-set representation there are other more obscure peculiarities that cannot be fixed by reforming our intuitions about contradictions. Truth for classical logic as part of the semantic idiom[3] is nothing but a property acquired by a formula α at a particular point x in the universe of a classical model $\mathscr{M}$; in other words, α is classically represented as a two-partitioned universe where the truth or falsity of α serve as a means of distinguishing the two parts. By the same token, as a formula can be represented in a three-valued model as a three-partitioned universe with three distinct properties of the formula assigned to each part, such as in Łukasiewicz's three-valued logic [32]; so on for the four-valued model, as in Dunn-Belnap four-valued logic [3][17]; and other many-valued logics, and so forth. In the classical case, at least, the familiar truth tables can be obtained from the canonical model by well-known filtration methods. In general, by specifying the number of truth-values and by *designating* values, $p \wedge \neg p$ need not always assume the non-designated value. Of the three approaches outlined earlier, the semantic approach presented in this thesis is the product of a search for a regular semantic *representation* of formulae that is at once classical and sufficiently discriminating to distinguish $p \wedge \neg\, p$ from $q \wedge \neg q$.

The story has also a preservational dimension. Formulae, as they present themselves to the innocent eye, are packets of information rather than mere forms for truth-value computations. Informally there is an innocent inclination to suppose that uniform substitution preserves form but not content, even for theorems and contradictions. To the same innocent observer, whatever reason moves us to allow distinct semantic representations for $p \supset q$ and $r \supset s$ ought also to support distinct semantic representations for $p \supset p$ and $q \supset q$. Similarly whatever consideration permits a semantic distinction between $p \wedge \neg q$ and $q \wedge \neg p$ should permit a semantic distinction between $p \wedge \neg p$ and $q \wedge \neg q$. Information (however dimly grasped that notion may be) seems to be lost when the pairs are semantically identified. Preservationists have long sought mathematically well-defined properties worthy of preserving, for which a variety of consequence relations were defined. Some of those properties carry with them a mark of philosophical significance, such as truth, falsity, contingency, level of coherence or incoherence [48, 28]; even hierarchies of properties [45]. Analytic implication of W. T. Parry, [41] axiomatized by K. Fine [21], combines strict implication with a variable sharing

[3]By semantic idiom I mean a class of structures together with a truth theory.

criterion and presents a system of implication that preserves analytic dependence. J. M. Dunn [16] presents a system of demodalized analytic implication, keeping the variable sharing principle while demodalizing Parry's original system by adding one principle that trivializes modality. The Dunn extension thus preserves only lexical dependence.

Analytic implication prompts one to look for a way of making analytic components of sentences explicit so that the question as to whether one implies the other can be determined by inspection. Anderson and Belnap began such an investigation of 'tautological entailments' in [2]. The decades that followed the publication of [1] in 1961 witnessed the emergence of various formal semantics for first-degree entailment, (also called 'tautological entailment' in [1] and [2], the specific definition of which will be presented in the first chapter) such as the intuitive semantics given by J. M. Dunn in [17], an algebraic semantics in terms of 'intensional lattices' and the eight-element matrix also given in [2]. However, none of these made direct use of the original normal form representation.

Far removed from the original interests of Anderson and Belnap , the direct use of conjunctive and disjunctive normal form (CNF and DNF) in our semantic representation is achieved through the use of hypergraphs. Explorations into relations among and operations on various types of hypergraph, as presented in this thesis, lie at the heart of hypergraph semantics. However, the experiment with hypergraphic representations started earlier than the current line of research; it goes back to the work of R. E. Jennings and D. Nicholson [40] where they applied the properties of hypergraphs, such as harmonic number and chromatic number, to a formal investigation of family resemblance. It is worth noting that hypergraph in the early work of Jennings and Nicholson ranges more widely over the hierarchy of hypergraph types. In this thesis, unless otherwise defined, hypergraph representation corresponds to the CNF of a formula, with literals (propositional variables or their negations) being interpreted as subsets of the universe, as in classical models; therefore it is hypergraphs of uniform type upon which I here concentrate, i.e. hypergraphs on the power-set of a universe U, having as (hyper)edges subsets of $\wp(U)$. The class of all such hypergraphs for a universe U is denoted by $H(\wp(U))$. As our investigations proceed, it will become clear that the entailment of *FDE* is nothing but a particular kind of relation on $H(\wp(U))$.

Hypergraph semantics also has its algebraic characteristics, and therefore affords a non-trivial contrast with various known algebraic semantics. $H(\wp(U))$ forms an intensional lattice (an involution with an identity element); it follows that $H(\wp(U))$ is a De Morgan lattice[4].

Such lattices are also generated by variant systems as we shall note in later chapters, for purposes of comparison and illustration. No algebraic proof of completeness is given in this thesis. There is a close connection between hypergraph semantics and other non-Boolean algebras used in logic, particularly intensional algebras, a full investigation of which provides a major commitment for future research.

So far I have only roughly sketched three approaches to revising the classical model: dialethic, preservational, and representational.

I adopt the definition of paraconsistency given by G. Priest in [43].

Definition 2. *A logic system* $\mathbf{L}$ *is paraconsistent if and only if* $\forall\alpha \in \Phi_{\mathbf{L}}$, $\exists\beta \in \Phi_{\mathbf{L}}$ *such that* $\{\alpha, \neg\alpha\} \nvdash \beta$.

Though it is neither meaningful nor useful to commit *a priori* to any semantics introduced or invented in this work, for the sake of summary, it may be said that hypergraph semantics, the main theme of the thesis, belongs to the last category. That said, the three are not entirely separable. They complement one another on some occasions and converge on others. Preservationists, who diligently sought out mathematically well-defined properties of a set of formulae were also driven in part by their philosophical interest in the segregation of inconsistent beliefs. However their mathematically defined properties clearly admit of multiple philosophical interpretations. Beliefs that are self-inconsistent, such as those of the form $p \wedge \neg p$ are distinguished from inconsistent set of beliefs, such as $\{p, \neg p\}$. By partitioning a set into consistent cells, one can avoid the second situation but not the first. Our representational approach using hypergraphs makes no such distinction: both $p \wedge \neg p$

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[4] A lattice L is called a distributive lattice if it verifies the property

$$\forall x, y, z \in L,\ x \vee (y \wedge z) = (x \wedge z) \vee (x \wedge z)$$

A distributive lattice is called a De Morgan lattice if a unary operation * is defined on it such that $\forall x, y \in L$, the following two properties hold

1. $x^{**} = x$;
2. $(x \vee y)^* = x^* \wedge y^*$.

and $\{p, \neg p\}$ are represented in the same way. In this representation, sentences such as $p \wedge \neg p$ present no threat even to a classicalist. Why? Because the orthographic difference between one contradiction and another is preserved by their hypergraph representations. Structural information lost in the classical truth-sets representation is retained in the hierarchic structure of hypergraphs.

Different approaches are not distinguishable by the distinct systems they inspire. For example, in [6], B. Brown gives a preservationist semantic analysis of *LP*. Between preservationists and dialethists, divergent intuitions are sometimes taken for divergent techniques. Invoking a terminology introduced by J. M. Dunn in [19], preservationists are less concerned with 'ontological' paraconsistency, which takes seriously the possibility that the world is inconsistent, than with 'epistemological' paraconsistency, which only deals with the inconsistency of beliefs. But both embrace the classical notion of explosion and inconsistency. For example, the coherence level of a set Σ is defined as the width of the least partition of Σ into consistent subsets. That width may just be the cardinal of Σ. On the hypergraph approach, as we shall see, aggregation and level measures are not at odds.

Chapter 1

Logic of Articulation

1.1 Towards a leibnizian project

The project so far is not a Leibnizian project. The lower-case initial indicates only a loose connection with Leibniz's own ideas. It is rather a project starting with a blaze of insight inspired by a division Leibniz drew in his metaphysical writings [30] between two notions of analyticity, which, curiously indeed, was nowhere used by the man himself as a theoretical tool for his logic. The distinction is between finite and infinite analyticity, which Leibniz used to define necessary and contingent truth respectively. We will come back to it for a more elaborate discussion in this chapter. While this distinction is not generally taken seriously by logicians in the know, there is another distinction, unknown to Leibniz himself, which seems better to represent his demarcation of analyticity than that between necessary and contingent truth. As it happens, the light that allows us to see the rebirth of Leibniz' distinction was given by science (the modern enterprise of analytic investigation including the mathematical part of it) of which, to add more interest to the case, Leibniz was one of the representative figures while it was still in its cradle.

Theoretical sentences in science are commonly understood as the *laws* as opposed to the *facts* of nature. Given such a dichotomy, and given the golden principle of science since the seventeenth century, that the acquisition of laws should only be achieved through the examination of facts, presumably there must be a link between the two steps. The elusiveness of the link, i,e, the myth of induction, has been a matter of concern for epistemologists over the centuries. Many attempts have been made to clear the myth by way of various simple-hearted rationales generated by many sophisticated minds of such as Bacon, Comenius, and Condorcet, which, two hundred years later, were thus propounded in a summing up by John Stuart Mill in his *System of Logic* [35]:

> "A certain fact invariably occurs when certain circumstances are present, and does not occur when they are absent; the like is true of another fact, and so on.... These various uniformities, when ascertained by what is regarded as a sufficient induction, we call in common parlance, Laws of Nature."

To Mill as to his intellectual predecessors, these are the rules of discovery by which scientists play to gather laws from facts. The ascertained uniformities, or Laws of Nature, according to Mill, are expressed in the great chain of causation, which is a system of knowledge; in other words, a theory expressed in a certain abstract, usually mathematical, language. However, a necessary explanation as to how he managed to step from patterns of observational states to causal inferences couched in a theoretical language, is nowhere to be found in the *System of Logic*. He did realize that this step, however mysterious, demands an explanation in epistemology if not existence in reality. Otherwise, just as Mill pointed out [35], the whole enterprise of science would become a collection of abstract names attributed to various concurrences of observable phenomena, which, unfortunately, is not too different a picture from the one that naturally emerges out of the writings of his intellectual predecessors. But we must put into historical context the comments made by those early philosophic advocates of the empirical method, that is, in relation to other contemporary issues that they were trying to solve so that we will not be misled into thinking that those simply crafted theories convey their whole attitude towards science. In the Arcadian days of science, truth was believed to lie all around us, like crops, waiting to be harvested, if only we could *observe* carefully. Observation was simply supposed to mean the engagement of that innocent perceptiveness of which mankind still thought themselves to be in possession, beyond the corruption of prejudice and sin that happened after the Fall. Besides, these early writers put much of their emphasis on the refutation of the idea, a quite necessary job to be done, without a doubt, that some mental acts alone, deduction for example, can lead to discovery of new truths and enlargement of understanding. It therefore falls upon the successive generations of philosophers to solve the mystery of the apparent dichotomy between imagination and invention (hypotheses or laws depending on the author) on the one side, and stamp collecting (or observation and experiment) on the other.

We take for granted now that the only way to solve such a mystery, in this case as in many others, is to rewrite it in such a way as would strike an ordinary person as being much less mysterious. To do so we must re-examine our understanding of what lies on both sides. I do not mean to give the impression that this re-examination is artificially done for the sake of solving the mystery; rather it is a natural turn in the history of science from which spring the opportunities for the clarification of many longstanding mysteries of similar kind. Galileo first created the sense of the word *experiment* that is most widely used today. It is largely the day-to-day life of every modern scientist, namely, the design, execution, and modification of the procedure for the evaluation of hypotheses, an equivalent of imaginative guesswork that tells stories that suggest, in however small a way, how the world might be. Experiment, in the proper sense of the word, is an expression of thinking; and there is always a good reason to focus on some observations rather than others. Daily scientific life accordingly is a much more dynamic process than suits the picture drawn by Mill and others. The generative act by which hypotheses are born is still a mystery, but it is no more a mystery than that of any other creative thinking. The calculus of discovery that Mill dreams of, if indeed it exists, must follow the logic of creativity, and the only hope of its discovery comes from discoveries about our brain, which leads to the conclusion that there cannot exist a calculus of discovery that is universally applicable. Moreover, we have come to understand that instead of ‘gathering laws from facts’, what actually happens, in a manner of speaking, is ‘gathering facts from laws’. The old-fashioned facts-hunting imagery of a scientist is not entirely out of the picture, but worth noticing only under very special circumstances. Only when we are in full possession of the facts relevant to the case to be solved can we allow ourselves to be conducted to truth through ‘ascertained uniformities’.

Given this understanding of science, to account for the arriving at scientific truth, i.e. those hypotheses that withstand the evaluation by experiment in the Galilean sense, we must answer the question which we may pose simply as how was the link established between simple observational states and theoretical sentences, such as those involving the notion of causality? This problem echoes Hume’s embarrassment about causation. The search for an apparently more promising solution takes us back to the writings of Leibniz himself, where we come across a theory of analysis for concepts whose application in the light of our understanding of science suggests the new distinction we mentioned at the beginning of this chapter, the distinction between observational and theoretical sentences. Though it is worth remarking here that Leibniz’ whole theory is based on the simple assumption that all sentences are of subject-predicate (SP) form, a law not strictly observed in his own writings.

Those of Leibniz's ideas that impinge upon the leibnizian project for this chapter can be summed up as follows:

(1) All truth is analytical.

(2) Necessary truth is finitely analytical whereas contingent truth is infinitely analytical.

(3) On the assumption that all sentences are of SP form, the only way to ascertain the the kind of analytical truth a sentence possesses is by examining the relationship between its subject and predicate terms.

(4) Both the subject and predicate terms of a sentence can be represented by a combination of essential ingredients that constitute the term.

Therefore, following the argument, if, in examining those ingredients of the subject term, one finds all that constitute the predicate term, then the examination can thus be terminated and the sentence is necessarily true; otherwise the examination presumably must go on to infinity, and the sentence can only be contingently true[1]. Since humans are finite beings, we can only verify the truth of a sentence in finitely many steps; when the task is composed of infinitely many steps, only God can finish it in one stroke. So, at any rate, as Leibniz claimed. Thus the necessary conclusion is, so far as human thinking is concerned, only necessary truth can be verified, whereas contingent truth can only be confirmed.

A leibnizian line, therefore, that originally separates the realm of finite from that of infinite analyticity, exists now between observational and theoretical sentences. It brings into light the assumption implicit in Mill's argument, that in the scientific search for 'truth', observational sentences can be verified while theoretical sentences can only be confirmed.

The reason for this claim cuts right through the myth of induction. It starts with a sober reflection on scientific 'truth' in the light of post-Galilean understanding of scientific methodology. Scientific

[1]The notion of infinity here is merely used as in ordinary parlance, it does not echo Leibniz's work on Infinitesimal Calculus where his notion of infinity greatly influences the history of mathematics and leads to the discovery and development of non-standard analysis.

theories, which are logical systems of theoretical sentences, cannot provide us with a more profound understanding of 'truth' than the temporary fitness of a certain theoretical language as a descriptive language for a set of observations. As Tarski observed [4],

> 'although the meaning of semantic concepts as they are used in everyday language seems to be rather clear and understandable, still all attempts to characterize this meaning in a general and exact way miscarried.. . . the languages (either the formalized languages or – what is more frequently the case – the portions of everyday language) which are used in scientific discourse do not have to be semantically closed. This is obvious in case linguistic phenomena and, in particular, semantic notions do not enter in any way into the subject-matter of a science; for in such a case the language of this science does not have to be provided with any semantic terms at all. . . Semantically closed languages can be dispensed with even in those scientific discussions in which semantic notions are essentially involved.'

Moreover, it is by no means uncommon in science for a large assortment of theoretical languages to exist at the same time with seemingly contradictory implications, of which one of the frequently quoted examples is Bohr's theory of the atom. Graham Priest maintained in [43] that since Bohr's theory suggests bound electrons both radiate energy and do not, the first required according to Maxwell's equations, and the second by the fact that the electrons do not spiral inwards towards the nucleus, it is an instance of inconsistent data from which we are required to draw inferences in a sensible manner. Apparently if we are to adopt the Millian notion of induction as the formation of theory from facts, then the process of establishing scientific truth can be accomplished, according to Mill, only through 'sufficient induction', whose abstract representation is echoed by the Priestian inference from inconsistent data to sensible theories. However, to grant the necessity of such an inference is to assume that certain aspects of the world are described by a theory of the atom that is built upon a language to which we must be committed. This is far from the case. Given what we have known about scientific methodology since Galileo, the search for a good scientific theory is in part a search for a suitable theoretical language. It is not merely a matter of checking truth-values of sentences in some language divinely vouchsafed. Moreover, although a theoretical claim might be rejected on observational grounds, it can at best be compatible with observation, and a rejection is as much a rejection of the theoretical language in which the claim is expressed. To revert to Bohr's theory of the atom, its solution does not depend upon drawing overall inferences from contradictions as contradictions, but rather in building models in different theoretical languages for different purposes. Those contradictions therefore, as Priest observed, are taken as contradictory theoretical sentences from which we are urged to draw sensible inferences only in one situation: when we view them in one theoretical model, applying a uniform theoretical language to all. But this situation is not faced squarely by those working in scientific theorization, nor does it have to be. As I said earlier, experimentation is an expression of thinking, couched in a theoretical language; therefore the expression in a particular language forever holds out the promise of modification and integration through a new language. Besides, integration that generates more comprehensive theories does not necessarily imply the abandonment of restricted and apparently contradictory models. For the sake of specific understanding, it does no harm, for example, to take atoms as waves in some models, and particles in others. As Stephen Hawking said [25],

> The theory of quantum mechanics is based on an entirely new type of mathematics that no longer describes the real world in terms of particles and waves; it is only the observations of the world that may be described in those terms.... for some purpose it is helpful to think of particles as waves and for other purposes it is better to think of waves as particles.

Therefore, radiation of energy by electrons is a phenomenon of electrons as particles, whereas the orbit-keeping behavior around the nucleus is another of electrons as waves. If we dig further into still the same example, we would find even more distinctions made by various models to answer for more complex situations whilst the subject matter, the atom, still eludes us. As Richard Feynman said [20]:

> ... say the electrons act like waves, no, they do not exactly; they act like particles, no they do not exactly; they act like a kind of fog around the nucleus, no, they do not exactly; and if you would like to get a clear, sharp picture of an atom, so that you can tell how it is exactly going to behave correctly, and have a good image, in other words, a really good image of reality, I do not know how to do it. ... It would be something like [having] a computer that you put certain numbers in, and you have the formula for what time the car will arrive at different destinations ... but [still one] cannot picture the car.... for a certain kind of approximate situations, a certain approximate picture works, [to say] that it (an atom) is simply a fog around a nucleus that when you squeeze it, it

> repels you, it is very good for understanding the stiffness of material; [or to say] that it is a wave is very good for some other phenomenon. . . . for changing temperature, . . . [it is enough to say that] the atoms are just little balls, it is good enough that it gives you a very nice picture of temperature. . . . If you want a picture of atom that has all of that in it, I cannot do it.

Fortunately, we are not in need of such a 'picture'. The more comprehensive understanding can be found in that 'entirely new type of mathematics', as Hawking puts it, and we are satisfied with it. Theoretical sentences in science cannot be verified[2] by experiments, nor do experiments force us to accept the 'true' theory in any straightforward way. 'Contradictions' are at least as much an indication of the deficiency of a theoretical language as that of a theory. Again this does not imply that the deficiency must be cured, for it is only in a comparative sense that we speak of deficiency as such, compared to the more comprehensive understanding of a certain phenomenon or group of phenomena, an understanding that accommodates more pictures whilst each picture still possesses its own undeniable and indelible theoretical significance.

From a scientific point of view, 'contradiction' is a term whose call for understanding is not much greater than that of 'truth'. On the observational level, what is truth and what contradicts what are fairly straightforward but not interesting; on the theoretical level, it is an interesting but not straightforward matter even to ask a question about contradictions; for if we do, we must convince ourselves of the necessity willingly to confine ourselves to one picture. But as different pictures serve different purposes, sensible understanding would not allow us to favour or impose the theoretical language of one picture over another. Hence, from a logical point of view, if the claims of certain paraconsistent logics[3] are to capture inference patterns demonstrated by situations as in Bohr's theory of the atom, then their motivating concerns with drawing sensible inferences from 'contradictions', which arguably started various strands of paraconsistent logics, beg the question, and need to be reconsidered. Of course this is not to deny the great achievement of paraconsistent logics that up to now have spanned more than half a century and made great contributions to our repertoire of valuable logical calculi; nor is it to deny the mathematical understanding they afford; nor is it a denial that historically some systems of paraconsistent nature did spring from concerns such as these. I merely

[2]Falsifiability is implicated in our discussion of verifiability and omitted for the sake of convenience.

[3]To review its definition, see definition 4 in the preface.

offer this criticism on the philosophical observations that a large portion of literature on paraconsistent logics claim to be associated with. It is upon the foundation of this criticism that I claim a place among the philosophical concerns generally acknowledged as the perspectives taken by paraconsistent logics, for a new perspective focusing on the role played by the evolution of theoretical languages in scientific investigations (including formal sciences).

1.2 Introducing the principle of articulation

Given the context described in the last section, we are now in a position to revive Leibniz's distinction between finite and infinite analyticity. Crudely speaking, there are two kinds of sentences in any system of scientific knowledge, namely, theoretical and observational. The primitive result we gain by observation and experiment can be generally taken as collections of observational states. Suppose each *observational state* is the smallest unit that can be verified by experiment. Let us call it an *atomic state*, *a-tomos* as in the Greek sense, i.e. that which does not admit of further cutting. (It is not to be confused with the atoms in propositional logic.) Theoretical sentences are distinguished from observational ones by the fact that unlike the observational, they cannot be verified by experiment, and therefore can at most be inadequately represented by observational states. This amounts to the assertion that there is a finite procedure by which we can come to a complete representation of every observational sentence o, its representation denoted by $\mathscr{A}(o)$ as a finite set of atomic states; but no theoretical sentence α can be so represented. If we ask for a complete representation of a theoretical sentence α, it can only be an infinite set of atomic states $\mathscr{A}^*(\alpha)$. Let β_i and ν_i be an arbitrary member of $\mathscr{A}^*(\alpha)$ and $\mathscr{A}(o)$ respectively; imagine an ideal formal system that properly answers for the distinction between observational and theoretical sentences, where 'complete representation' is expressed by the consequence relation '$\vdash_{\mathscr{A}}$', the structure of the system requires specification of relations as such

$$\langle \mathscr{A}^*(\alpha),\ \alpha\rangle,\ \langle \alpha,\ \beta_i\rangle,\ \langle \mathscr{A}(o),\ o\rangle,\ \langle o,\ \nu_i\rangle$$

in terms of '$\vdash_{\mathscr{A}}$', where '$\vdash_{\mathscr{A}}$' is assumed to be non-compact.

According to the definition, $\mathscr{A}^*(\alpha)$ satisfies the following four conditions:

(1) $\forall \beta_i \in \mathscr{A}^*(\alpha), \alpha \vdash_{\mathscr{A}} \beta_i$;

(2) $\{\beta_i \mid \beta_i \in \mathscr{A}^*(\alpha)\} \vdash_{\mathscr{A}} \alpha$;

(3) $\forall n \in \mathbb{N}, \bigwedge_{i=1}^{n} \beta_i \nvdash_{\mathscr{A}} \alpha$;

(4) $\forall \beta_i \in \Phi$, if $\beta_i \in \mathscr{A}^*(\alpha)$, then β_i satisfies (1) − (3).

and $\mathscr{A}(o)$ satisfies the following four conditions:

(1) $|\mathscr{A}(o)| \in \mathbb{N}$;

(2) $o \vdash_{\mathscr{A}} \nu_i$;

(3) If $|\mathscr{A}(o)| = n$, then $\bigwedge_{i=1}^{n} \nu_i \vdash_{\mathscr{A}} o$;

(4) $\mathscr{A}(o)$ is the maximal set which satisfies (1) − (3), i.e. $\forall \Sigma \subset \mathscr{A}(o), \Sigma \nvdash_{\mathscr{A}} o$.

Neither $\mathscr{A}^*(\alpha)$ nor $\mathscr{A}(o)$ is unique.

The atomic states that constitutes $\mathscr{A}^*(\alpha)$ are the *atomic necessary conditions* for α. The totality and only the totality of such necessary conditions constitutes a sufficient condition for α. Since atomic states are observational states, a model that properly answers our purpose should recognize that theoretical sentences and their atomic necessary conditions are of different semantic types; one of the inevitable consequences of this recognition, according to the qualities of theoretical sentences enumerated above, is that we are bound to a formal discussion of infinity with all the difficulties that that entails. However, although theoretical sentences cannot be reduced to a finite set of observational states, they can be reduced to a finite set of *atomic theoretical states* (*'atomic'* still in the sense of *'a-tomos'*). Let us call them *articulate states*, for this process of reduction is also that of articulation, which is reminiscent of a case in classical propositional logic where every formula can be syntactically represented by the conjunction of its *atomic necessary conditions*, i.e. its CNF, where each atomic necessary condition is in the form of a disjunction of literals. In the standard way of representing a propositional formula by its CNF, only the literals that occur in α occur in the CNF of α. Since we are to frequently use the standard way of representing α by its CNF in models we construct later, it is necessary to take a brief review of its construction before we go on generalizing it to more complicated representations that better approach the leibnizian idea illustrated by our speculation on an ideal formal system.

In classical propositional logic, every formula α is semantically interpreted as a truth function the inputs and outputs of which are tabulated in the truth table of α. A disjunctive normal form (DNF) of α can be thus defined from a truth table:

Definition 3. *The formulation of the DNF of α is obtained in two steps: first, let the literals* [4] *of α be $p_1, p_2, \ldots, p_n$. For each row i, form a conjunction i' having as its conjuncts p_i or $\neg p_i$ accordingly as p_i receives a 1 or a 0 in that row; second, disjoin the set of such conjunctions that receive a 1 in that row. The formulation of α thus constructed is the DNF of α.*

With n being the number of propositional variables in α, the truth table of α has 2^n rows, of which the DNF of α has as its disjuncts only those for which the truth function of α outputs a 1. There is a finite procedure by which we can obtain from its DNF an *atomic necessary condition* of α. Let R be a set of literals such that for every conjunction of the DNF of α, R contains at least one literal of that conjunction, and no subset of R has that property. Then I call the disjunction of R ($\bigvee R$) an *atomic necessary condition* of α. The conjunction of all *atomic necessary conditions* of α is the CNF of α. Hence, in the classical sense of complete verification, each sentence can be represented by finitely many atomic necessary conditions, where each condition is again represented by finitely many choices that reach out for verification. According to the distinction we drew between theoretical and observational sentences, sentences in propositional logic are interpreted by classical semantics as observational sentences. Now the obvious question is, where does the theoretical sentence enter the picture? A theoretical sentence is one that cannot be so clearly represented as to render it verifiable. It is a commonplace that sentences of propositional logic in non-classical models have such qualities as we assign to theoretical sentences. Disjunction in quantum logic, for example, is not classically interpreted, and the standard interpretation of quantum disjunction in a complex separable infinite dimensional Hilbert space allows $a \vee b$ to be true without *a "or" b*'s being true [22]. In this case, although α can be *articulated* into conjuncts of its CNF that is finite, it is still one step away from verification. This is because each conjunct is an *articular state*, i.e. an *atomic theoretical state* (a disjunction of literals), which cannot be evaluated in the classical sense. Therefore, normal form representation by itself is not inconsistent with our distinction between observational and theoretical sentences. What is an *atomic necessary condition* in the standard model

[4] From now on, atom refers to propositional variable unless otherwise specified, and *'atomic necessary conditions'* is to be continuously italicized as a phrase of special connotation. A literal is an atom or its negation.

becomes an *articular state* in a new setting.

Assuming that every sentence can be articulated into its articular states, the essential core of our 'leibnizian' project can be translated into the hypothesis that for a universal SP sentence to be true, every articular state of P must be 'underwritten' by some articular state of S. The rest of this thesis is devoted to the study and development of various 'underwriting' relations. Let us focus for now on the case of articulation in classical logic where every formula can be articulated into its CNF or DNF. The classical dogma asserts that for α to entail β, β must be 'contained' in α. The notion of 'containment' can be interpreted in various ways. Parry in [41] offered a principle of analyticity as a restraint on 'containment' such that every propositional variable occurring in β must also occur in α. Analyticity interpreted as such goes against Kant's intuition about analytic statements which include propositions such as

$$p \vdash p \vee q$$

Hintikka [26] provided a criterion of entailment such that α entails β if and only if α and β are tautologically equivalent. Two formulae are tautologically equivalent if and only if they either contain occurrences of exactly the same free variables or can be obtained from such formulae by replacing one or more free individual variables by bound ones. However, it is not excluded from his criteria that

$$\neg p \wedge (q \wedge \neg q) \vdash p$$

In the 1960s, Anderson and Belnap in a series of papers starting with [1] gave a notion of entailment on the basis of CNF and DNF. The idea is a straightforward one. Classically speaking,

$$\alpha \vdash \beta \Leftrightarrow DNF(\alpha) \vdash CNF(\beta).^{5}$$

If we de-formulate the DNF of α and CNF of β into the set $D(\alpha)$ of disjuncts of $DNF(\alpha)$ and the

[5] For a precise definition of what the notation $CNF(\alpha)$ stands for, see chapter 2, definition 25 of a standard CNF of α, which also applies, *mutatis mutandis*, to DNF.

set $C(\beta)$ of conjuncts of $CNF(\beta)$ respectively[6], then for each disjunct $D \in D(\alpha)$ and each conjunct $C \in C(\beta)$, the above statement is equivalent with the following

$$D \vdash C$$

where 'D' is a conjunction of literals and 'C' a disjunction of literals. The next step is where the notion of *tautological entailment*, we here denote as '$\vdash_T$', differs from the classical entailment '$\vdash$'. With no extra constraints, classical logic allows for the entailment from D to C where D is of the form $p \wedge \neg p$ and C is of the form $q \vee \neg q$. The constraint put on the next step of entailment by Anderson and Belnap says $D \vdash_T C$ if and only if $D \rightarrow C$ is a *tautological entailment*, the best explanation of which comes from the series of definitions given in the following part of [1][7]:

> An atom is a propositional variable or the negate of a propositional variable. A *primitive disjunction* is a disjunction $A_1 \vee A_2 \ldots \vee A_m$ where each disjunct A_i is an atom. A *primitive conjunction* is a conjunction $B_1 \& B_2 \ldots \& B_n$, each conjunct B_i being an atom. $A \rightarrow B$ is a *primitive entailment* if A is a primitive conjunction and B is a primitive disjunction. We take it as obvious that if A and B are both atoms, then $A \rightarrow B$ should be a valid entailment if and only if A and B are the same atom;... We think it equally obvious that if $A_1 \& A_2 \ldots \& A_m$ is a primitive conjunction and $B_1 \vee B_2 \ldots \vee B_n$ is a primitive disjunction, then $A_1 \& A_2 \ldots \& A_m \rightarrow B_1 \vee B_2 \ldots \vee B_n$ should be a valid argument if and only if some atom A_i is the same as some atom B_j. ...We shall say that a primitive entailment $A \rightarrow B$ is *explicitly tautological*, if some (conjoined) atom of A is identical with some (disjoined) atom of B. Such entailments may be thought of as satisfying the classical dogma that for A to entail B, B must be 'contained' in A.

A literal of the form p or $\neg p$, therefore, can tautologically entail itself and only itself. Intuitively, the proposal of Anderson and Belnap attempts at an understanding of entailment where a certain sort of relevance between the antecedent and consequent is implicated. To present it in parallel with the classical case, it may well be understood that α relevantly entails β in the sense of Anderson and Belnap if and only if for each disjunct $D \in DNF(\alpha)$ and each conjunct $C \in CNF(\beta)$,

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[6]For example, the de-formulated set of $DNF(p \vee q)$ is $\{p, q\}$, and that of $CNF(p \vee q)$ is $\{p \vee q\}$.

[7]Anderson and Belnap use a different terminology. An 'atom' in the quotation is a literal by our usage.

$$D \vdash_T C \tag{1.1}$$

After we de-formulate $DNF(\alpha)$ and $CNF(\beta)$ into the set $D(\alpha)$ and the set $C(\beta)$; each disjunct $D \in D(\alpha)$ and each conjunct $C \in C(\beta)$ can be further de-formulated into a set of literals. Therefore 1.1 is equivalent to the condition such that

$$\forall \alpha, \beta \in \Phi, \alpha \vdash \beta \Leftrightarrow \forall B \in C(\beta), \forall A \in D(\alpha) : \exists a \in A, b \in B : a \vdash_T b \tag{1.2}$$

Since CNF and DNF bear a dual relation to one another, we can rephrase the criterion of tautological entailment using only the CNFs. It is merely a matter of mechanical transformation. Thus (1.2) can be rewritten as $\forall \alpha, \beta \in \Phi$,

$$\alpha \vdash \beta \Leftrightarrow \forall B \in C(\beta), \exists A \in C(\alpha) : \forall a \in A, \exists b \in B : a \vdash_T b \tag{1.3}$$

In other words, it is the underwriting of every articular state of β by an articular state of α that constitutes the notion of 'containment'. In the context of classical propositional logic where every formula α can be syntactically represented by its CNF, i.e. the conjunction of its *articular states*, each of which is in the form of a disjunction of atoms. The entailment relation in [1] is thereby reduced to a relation between two sets of sets of literals .

The resulting formal system is called *first degree* fragment of E, abbreviated as FDE[8].

1. $\vdash \neg\neg p \leftrightarrow p$;
2. $\vdash p \wedge (q \vee r) \rightarrow (p \wedge q) \vee r$;
3. $\vdash p \rightarrow p \vee q$;
4. $\vdash p \wedge q \rightarrow p$.

together with three rules

$$\textit{Transitivity}: \frac{\vdash \alpha \rightarrow \beta \qquad \vdash \beta \rightarrow \gamma}{\vdash \alpha \rightarrow \gamma}$$

[8]This particular axiomatization was given as postulates for the system E_{fde} in [2], chapter 3, §15.2.

$$\textit{Left disjunctivity}: \frac{\vdash \alpha \to \gamma \qquad \vdash \beta \to \gamma}{\vdash \alpha \vee \beta \to \gamma}$$

$$\textit{Contraposition}: \frac{\vdash \alpha \to \beta}{\vdash \neg\beta \to \neg\alpha}$$

The decades following its discovery by Anderson and Belnap witnessed the emergence of various semantics to which it is amenable, among which are the situation semantics given by Dunn [17], possible world semantics with the 'star operation' of Routley & Routley [44], and intensional algebraic semantics given by Belnap [4], and so on. As we now stand, we are naturally inclined to interpret the literals as truth-sets, i.e. as members of $\wp(U)$ where U is the universe in a full propositional model, then the articular representation of a sentence as a set of sets of literals under this interpretation becomes a simple hypergraph on $\wp(U)$[9]. Our argument proceeding up to this point requires some preliminary definitions to which we now turn. We first define a generalization of the well-known notion of a graph. A graph consists of a set of points some pairs of which are connected by "edges." A hypergraph generalizes a graph by allowing more than a pair of points to be connected and so a "hyperedge" can be viewed as a set of points. More formally:

Definition 4. *A hypergraph H is a pair $H = (X, E)$ where X is a set of elements, called nodes or vertices, and E is a non-empty set of subsets of X called hyperedges or links. Therefore, $E \neq \emptyset$ is a subset of $\wp(X)$, where $\wp(X)$ is the power set of X.*

In subsequent discussions, we refer to H so defined as a hypergraph on X. For convenience, we can write H as a collection of (hyper)edges, i.e. $H = \{E_1, E_2, \ldots, E_n\}$ where $\forall i, 1 \leqslant i \leqslant n, E_i \in \wp(X)$.

Definition 5. *A simple hypergraph $H = \{E_1, E_2, \ldots, E_n\}$ is a hypergraph such that if $\forall E_i, E_j \in H$, $E_i \not\subset E_j$.*

Now we can close the section with the principle it has been motivating:

Principle of Articulation: *Every propositional formula α has a classical semantic representation*

[9]The simplicity is derived from the standard CNF construction where no disjunction in its constitution is the sub-disjunction of another.

as a simple hypergraph H_α *on the power set of a set of states.*

which as a principle of representation is at the center of the semantic approach that is gradually to unfold in the subsequent part of the thesis. Hypergraphs and simple hypergraphs will receive fuller consideration later. According to this representation, the leibnizian notion of 'containment', i.e. the underwriting of every β-articular-state by some α-articular-state can be expressed in such a way that every (hyper)edge of H_β (the hyper-edge itself is a hypergraph on U) has some (hyper)edge of H_α (also a hypergraph on U) as its sub-edge; that is, every hypergraph on U that constitutes H_β has a corresponding hypergraph on U in H_α as a subgraph. Therefore, that the β-representation is contained in the α-representation, i.e. that H_α stands in such a relation R to H_β so that α entails β can be summed up step by step as the following chain of conditions go:

$$\alpha \vDash_{fde} \beta \Leftrightarrow H_\alpha R H_\beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B : a = b \tag{1.4}$$

which in turn can be shortened as the condition

$$\alpha \vDash_{fde} \beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : A \subseteq B \tag{1.5}$$

The result of this experiment is a new semantics for *FDE* with which we start the next chapter. In an algebraic setting, the entailment relation can be expressed in similar but more general algebraic terms where '=' in (1.4) is replaced by any partial ordering '$\leqslant$'. The implication of this generalization shall receive further consideration in the rest of the thesis. Entailments based on generalized algebraic orderings generate a class of articular inferences. They are *binary* (in the sense of [23], a formal definition of which will be given at the beginning of the next chapter) sublogics of *PL*, that are at once relevant and paraconsistent, with various properties. This subject will be the main topic of the second chapter.

Chapter 2

A Family of First Degree Articular Relations

2.1 Hypergraph Semantics for first degree systems

We think of "degree" as meaning the degree of arrow nesting. The degree of a formula is determined by the depth of nested arrows in the formula. Therefore, a sentence of the form $p \wedge q$ is of degree zero. A formal definition of the degree of a propositional formula α, $\mathscr{D}(\alpha)$, can be inductively given thus:

Definition 6. *$\forall \alpha \in \Phi$,*

1. *$\mathscr{D}(\alpha) = 0$ if α contains no arrow;*

2. *$\mathscr{D}(\alpha) = \mathscr{D}(\neg\alpha)$;*

3. *$\mathscr{D}(\beta \vee \gamma) = \mathscr{D}(\beta \wedge \gamma) = max\ (\mathscr{D}(\beta), \mathscr{D}(\gamma))$;*

4. *$\mathscr{D}(\beta \rightarrow \gamma) = max\ (\mathscr{D}(\beta), \mathscr{D}(\gamma))+1$.*

The family of first degree systems of inference that I shall present in this chapter all consist entirely of first degree formulae; like FDE, they are *binary* (in the sense of [23]) sublogics of *PL*.

Definition 7. *A unary logic is a set L of formulae closed under the application of certain inferential rules to its members. The members of L are called L-theorems. We write* $\vdash_L \alpha$ *if* $\alpha \in L$*; whereas a binary logic is a collection* S *of ordered pairs* $\langle\alpha, \beta\rangle$*, where* β *is derivable from* α *in* S *satisfying certain closure conditions. We write* $\alpha \vdash_S \beta$ *if* $\langle\alpha, \beta\rangle \in S$*.*

Within the scope of this thesis, we restrict ourselves to binary logics that satisfy the following condition:

Definition 8. *In a binary logic* S*,* $\forall\Gamma \subseteq \Phi$*, and* $\alpha \in \Phi$*, if* $\Gamma \vdash_S \alpha$*, then* $\exists\beta_1, \beta_2, \ldots, \beta_n \in \Gamma$ *such that* $\bigwedge_{i=1}^{n} \beta_i \vdash_S \alpha$*.*

The comparatively rich hypergraph-theoretic idiom reveals other systems in the region of FDE as the ordering relation between simple hypergraphs is varied. What we shall explore in this chapter is the delimitation of this hypergraph-theoretic idiom and its formal application to interpreting the family of systems in the region of FDE, especially their ordering relations. To put it in the more informal language introduced in the last chapter, the leibnizian notion of 'containment', summed up in the following chain of conditions

$$\alpha \vdash_{fde} \beta \Leftrightarrow H_\alpha R H_\beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B : a = b \tag{2.1}$$

is but one of a family $\mathscr{R}_X$ of ordering relations on the set $\mathbb{H}$ of simple hypergraphs on X, $\mathscr{R} \subseteq \mathbb{H} \times \mathbb{H}$. Later when there is no risk of confusion, we omit the subscript of $\mathscr{R}$. As $R \in \mathscr{R}$ varies, the systems it generates differ in important respects. The hypergraph semantics interprets this family of first degree inferences via an articular model to which we now turn.

2.1.1 Articular Model

An articular model (a-model) is an ordered triple $\mathscr{M} = \langle U, \mathbb{H}, \mathbf{H} \rangle$ where

1. $U \neq \varnothing$ is a set;
2. $\mathbb{H} \subseteq \wp\wp\wp(U)$ such that every member of $\mathbb{H}$ is a simple hypergraph.
3. $\mathbf{H} : At \rightarrow \mathbb{H}$.

That is, $\mathbb{H}$ is a set of *simple* hypergraphs, and to each p_i, $\mathbf{H}$ assigns a simple hypergraph on $\wp(U)$, $\mathbf{H}(p_i)$. Simplicity of hypergraphs is generally defined thus:

Definition 9. *H is a simple hypergraph if and only if $\forall E, E' \in H$, $E \not\subset E'$.*

Since not all set-theoretic operations naturally preserve the simplicity of hypergraphs, for present purposes, we obtain from every non-simple hypergraph H a simple hypergraph by a star operation that casts out super-edges of H.

Definition 10. $\star H = H$ - $\{E \in H \mid \exists E' \in H : E' \subset E\}$.

2.1.2 Extending H to Φ

The account of $\mathbf{H}(\cdot)$ which extends $\mathbf{H}$ to Φ, requires some preliminary definitions. For some set S,

Definition 11. *If $A \subseteq \wp(S)$, then b is an intersector of A iff $\forall a \in A, b \cap a \neq \phi$.*
Definition 12. *If $A \subseteq \wp(S)$, then $\tau(A) = \{b \mid b$ is a minimal intersector of $A\}$.*
Definition 13. *If $A \subseteq \wp(S)$, then $\overline{[A]} = \{\overline{a} \mid a \in A\}$.*
Definition 14. *If H is a hypergraph, then $\tau(H)$ is the transverse hypergraph of H.*

A hypergraph understood in the sense of definition 4 and its transverse hypergraph are dual to each other. We will come back to this point at the end of this chapter, where we explore a notion of hypergraph in a more general sense, whose (hyper)edges are not subsets of a base set. In fact, they are not sets at all, which makes a difference with regard to duality between H and $\tau(H)$.

$\mathbf{H}(\cdot)$ extends $\mathbf{H}$ to Φ as follows:
$H_{P_i} = \mathbf{H}(P_i)$
$H_{\neg\alpha} = \{\overline{[B_i]} \mid B_i \in \tau(H_\alpha)\}$
$H_{\alpha\vee\beta} = \star\{a \cup b \mid a \in H_\alpha,\ b \in H_\beta\}$
$H_{\alpha\wedge\beta} = \star(H_\alpha \cup H_\beta)$.

Semantic entailment is therefore defined as a relation between two simple hypergraphs on $\wp(U)$, which we now give.

Definition 15. *$\forall H, H' \in \mathbb{H}, H \sqsubseteq H'$, ($H'$ subsumes H) if and only if $\forall b \in H'$, $\exists a \in H$ such that $a \subseteq b$.*

2.1.3 Hypergraph lattice

Though a slight digression here, there is something to be said of the algebraic structure that naturally emerges from the definition of subsumption relation, i.e. that $\langle \mathbb{H}, \sqsubseteq \rangle$ is a lattice. The demonstration of it is fairly simple. It is easily seen that '$\sqsubseteq$' is a partial ordering; moreover,

$$\forall \alpha, \beta \in \Phi, sup(H_\alpha, H_\beta) = H_{\alpha \vee \beta} \tag{2.2}$$

$$\forall \alpha, \beta \in \Phi, inf(H_\alpha, H_\beta) = H_{\alpha \wedge \beta} \tag{2.3}$$

Such structures as these shall be given more attention later in this chapter with further generalizations. We are now in a position to define our semantic notion of entailment in terms of the subsumption relation.

$\mathbb{H}$ is the set of simple hypergraphs on $\wp(U)$. We have introduced the star function as an operation applicable to all hypergraphs, simplifying a hypergraph by casting out its super-edges. It is obvious that

$$\star : H \rightarrow \tau\tau(H)$$

The set of all hypergraphs on $\wp(U)$ therefore is a set of equivalence classes, each of which is a set of hypergraphs that get mapped to by the above function to the same simple hypergraph. Let H be the set of all hypergraphs on $\wp(U)$, the lattice $\langle [\mathsf{H}], \equiv \rangle$ where

1. $[\mathsf{H}] = \{[\mathsf{H}]_x \mid x \in \mathbb{H}\}$ is the set of equivalence classes; there is a one-to-one correspondence between $[\mathsf{H}]$ and $\mathbb{H}$.
2. $\forall x \in \mathbb{H}$, and $\forall H, H' \in [\mathsf{H}]_x$, $\star(H) = \star(H') \in \mathbb{H}$.

3. $[\mathsf{H}]_x \leqslant [\mathsf{H}]_y$ if and only if $x \sqsubseteq y$.

is isomorphic with $\langle \mathbb{H}, \sqsubseteq \rangle$.

Definition 16. *$\forall \alpha, \beta \in \Phi$, $\alpha \models \beta$ (α semantically entails β) if and only if $\forall \mathscr{M} = \langle U, \mathbf{H} \rangle$, $H_\alpha \sqsubseteq H_\beta$. Alternatively, we say that $\alpha \vdash \beta$ is valid. So,* mutatis mutandis*, for $\Gamma \models \alpha$ (Γ is an arbitrary set of formulae).*

2.2 Introduction of syntax for the Family of first degree systems

The preliminaries are the usual. The language of first degree systems (in BNF) is defined by

$$\alpha ::= p|\neg\alpha|(\alpha \vee \alpha)$$

where p ranges over At, the set of atomic formulae. In the following, α, β, γ are arbitrary formulae and Γ, Σ, and so on are sets of formulae. We present the first degree systems as binary systems whose theorems $\langle \alpha, \beta \rangle$ are written in the form of $\alpha \vdash \beta$.

2.2.1 *FDE*

FDE was first presented by Anderson and Belnap in [2]. What is given here is a binary version of the well-known first degree fragment of the system E.

The system has ten binary axioms:

1. $\neg(p \wedge q) \dashv\vdash \neg p \vee \neg q$;
2. $\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$;
3. $\neg\neg p \dashv\vdash p$;
4. $p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r)$;[1]
5. $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$;

[1] The converse is a theorem that can be proved by axiom 7, [LD] and [RC].

6. $p \vdash p \vee q$;

7. $p \wedge q \vdash p$.

The system satisfies the three structural rules in the sense of [49]:

Mon $\Sigma \vdash \alpha \ / \ \Sigma, \ \Gamma \vdash \alpha$;

Ref $\alpha \in \Sigma \ / \ \Sigma \vdash \alpha$;

Cut $\Sigma, \ \alpha \vdash \beta, \ \Sigma \vdash \alpha \ / \ \Sigma \vdash \beta$.

together with:

LD $\alpha \vdash \beta, \ \gamma \vdash \beta \ / \ \alpha \vee \gamma \vdash \beta$; (left disjunctivity)

RC $\alpha \vdash \beta, \ \alpha \vdash \gamma \ / \ \alpha \vdash \beta \wedge \gamma$. (right conjunctivity)

It is easy to see that all the binary axioms of FDE are valid in the sense of definition 16, and all the rules preserve validity. The relation between subsumption and the account of first degree entailment proposed by Anderson and Belnap in [1] is straightforward. It may well be said that the subsumption relation is a semantic interpretation of the modified entailment results from the replacement of classical with tautological entailment. In chapter 1, we summed up in (1.2) the relevance condition given by Anderson and Belnap for first degree entailment, representing the antecedent as its DNF and the consequent its CNF; which can be converted to another condition using only CNF:

$$\alpha \vdash \beta \Leftrightarrow \forall B \in C(\beta), \exists A \in C(\alpha) : \forall a \in A, \exists b \in B : a \vdash_T b$$

Because of the normal form representation, a and b are literals, so according to the definition of tautological entailment, $a \vdash_T b$ simply means $a = b$. This again amounts to saying $A \subseteq B$, which, under the Principle of Articulation introduced in the last chapter, points to the fact that first degree entailment is an instance of the subsumption relation. In fact, the subsumption relation fully captures the notion of entailment expressed by E in its first degree fragment, because as we shall see later, FDE is sound and complete with respect to the class of articular models together with the subsumption relation. This is the topic of section 2.3.

As we know, in the class of articular models, subsumption between simple hypergraphs determines validity. Accordingly, different members of the family $\mathscr{R}$ of first degree articular relations, defined at the beginning of this chapter, determine validity in other classes of articular models. Some of them generate binary logic systems with interesting properties other than those of FDE. This point is best illustrated by some examples of such ordering relations between simple hypergraphs and their resulting[2] first degree systems.

2.2.2 Strict Subsumption

Definition 17. *$\forall H, H' \in \boldsymbol{H}(U)$, $H \sqsubseteq H'$, (H' strictly subsumes H) iff $\forall b \in H'$, $\exists a \in H$ such that $a \subseteq b$ and $\forall a' \in H$, $\exists b' \in H'$ such that $a' \subseteq b'$.*[3]

The system in consequence is a subsystem of FDE .

1. $p \vdash p$;
2. $\neg(p \wedge q) \dashv\vdash \neg p \vee \neg q$;
3. $\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$;
4. $\neg\neg p \dashv\vdash p$;
5. $p \vee (q \wedge r) \dashv\vdash (p \vee q) \wedge (p \vee r)$.

Of the three structural rules only one remains.

Cut $\Sigma,\ \alpha \vdash \beta,\ \Sigma \vdash \alpha\ /\ \Sigma \vdash \beta$.

But a strengthened rule for conjunction also preserves validity.

C $\alpha \vdash \beta,\ \gamma \vdash \delta\ /\ \alpha \wedge \gamma \vdash \beta \wedge \delta$. (conjunctivity)

[2]The relationship between the ordering relations and their resulting logics is not one of determination as in the case of FDE. Some of the systems do not fully capture the ordering relation in the sense that it has been proved complete with respect to some class of articular models with the relation.

[3]A more general definition has been introduced in the PhD thesis of Tara Nicholson, 2007.

2.2.3 Supersumption

Definition 18. $\forall H, H' \in \boldsymbol{H}(U), H \sqsubseteq^* H'$ *iff* $\forall b \in \tau(H'), \exists a \in \tau(H)$ *such that* $a \subseteq b$.

The definition says in effect that $H \sqsubseteq^* H'$ iff $\tau(H) \sqsubseteq \tau(H')$, which because of the duality property of H and $\tau(H)$ amounts to $H \sqsupseteq H'$ on the condition that the simplicity operation casts out subsets.

The system has a curious reversal of conjunction and disjunction of FDE.

1. $\neg(p \wedge q) \dashv\vdash \neg p \vee \neg q$;
2. $\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$;
3. $\neg\neg p \dashv\vdash p$;
4. $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$;[4]
5. $p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r)$;[5]
6. $p \vdash p \wedge q$; (!)
7. $p \vee q \vdash p$. (!)

RD $\alpha \vdash \beta,\ \alpha \vdash \gamma\ /\ \alpha \vdash \beta \vee \gamma$;

LC $\alpha \vdash \gamma,\ \beta \vdash \gamma\ /\ \alpha \wedge \beta \vdash \gamma$.

One structural rule is satisfied:

Cut $\Sigma,\ \alpha \vdash \beta,\ \Sigma \vdash \alpha\ /\ \Sigma \vdash \beta$.

2.2.4 Subclusion

Definition 19. $\forall H, H' \in \boldsymbol{H}(U), H \sqsubset H'$ *iff* $\forall a \in H, \exists b \in H'$ *such that* $a \subseteq b$.

1. $p \vdash p$;
2. $\neg(p \wedge q) \dashv\vdash \neg p \vee \neg q$;

[4]The converse is provable as a theorem by axiom 7, [LC] and [RD].

[5]Again the converse can be proved by axiom 7, [LC] and [RD].

3. $\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$;

4. $\neg\neg p \dashv\vdash p$;

5. $(p \vee q) \wedge (p \vee r) \dashv\vdash p \vee (q \wedge r)$;

6. $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$.

RC $\alpha \vdash \beta,\ \alpha \vdash \gamma\ /\ \alpha \vdash \beta \wedge \gamma$.

Cut $\Sigma,\ \alpha \vdash \beta,\ \Sigma \vdash \alpha\ /\ \Sigma \vdash \beta$.

2.2.5 Subgraph

Definition 20. $\forall H, H' \in \boldsymbol{H}(U), H \sqsubset' H'$ *iff* $\forall a \in H$, $\exists b \in H'$ *such that* $a = b$.

The following binary axioms and rules constitute the overlap with FDE:

1. $p \ \vdash p$;

2. $\neg(p \wedge q) \dashv\vdash \neg p \vee \neg q$;

3. $\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$;

4. $\neg\neg p \dashv\vdash p$;

5. $p \vee (q \wedge r) \dashv\vdash (p \vee q) \wedge (p \vee r)$;

6. $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$.

RC $\alpha \vdash \beta,\ \alpha \vdash \gamma\ /\ \alpha \vdash \beta \wedge \gamma$.

D $\alpha \vdash \beta,\ \gamma \vdash \delta\ /\ \alpha \vee \gamma \vdash \beta \vee \delta$.

It is easy to verify the structural rule:

Cut $\Sigma,\ \alpha \vdash \beta,\ \Sigma \vdash \alpha\ /\ \Sigma \vdash \beta$.

All of the above systems form a De Morgan lattice. The specific proofs of these relevant metatheorems can be found in the Appendix.

2.3 Metatheory

It is straightforward that *FDE* is sound with respect to a class of articulate models. We demonstrate completeness using the Henkin method , which requires some preliminary definitions and lemmas.

Definition 21. *$\forall\alpha \in \Phi$, At (α) is the collection of all the atoms $p_1,\ p_2, \ldots, p_n$ that occur in α.*

Definition 22. *$\forall\alpha \in \Phi$, if At (α) has n atoms, then the set Lit(α) of literal pairs in the language of α is the collection of literals formed from the atoms in the subwff closure[6] of α, i.e. Lit$(\alpha) = \{\{p_i, \neg p_i\} \mid 1 \leqslant i \leqslant n\}$.*

Definition 23. *An observation set is a maximal consistent set of literals such that exactly one of each literal pair ($p_i/\neg p_i$) is in it.*

Definition 24. *A full theory is a deductive closure of an observation set, viz. if $\Gamma \vdash \gamma$, then $\gamma \in \Gamma$.*

The immediate consequence of which is in the next two lemmas.

Lemma 1. *If Σ is a full theory in FDE, then $\forall\alpha \in \Phi$, exactly one of α and $\neg\alpha$ is in Σ.*

The proof directly follows from the axioms and rules of *FDE*.

Lemma 2. *Every observation set has a unique full theory extension.*

Proof. Assume there are two full theories Σ_1 and Σ_2 extended from the same observation set such that $\exists\alpha$ and $\alpha \in \Sigma_1$ but $\alpha \notin \Sigma_2$. Then by lemma 1, $\neg\alpha \in \Sigma_2$. Suppose p' is a literal in the observation set where $p \in At\ (\alpha)$ (p' can be either an atom p or its negation $\neg p$), then $p' \in \Sigma_1$ by the definition

---

[6] $\forall\alpha \in \Phi$, the subwff closure of α, $S(\alpha)$, is defined inductively as follows:

1. Every formula is in its own subwff closure;
2. If $\neg\beta \in S(\alpha)$, then $\beta \in S(\alpha)$;
3. If $\beta \wedge \gamma \in S(\alpha)$, then $\beta \in S(\alpha)$ and $\gamma \in S(\alpha)$;
4. If $\beta \vee \gamma \in S(\alpha)$, then $\beta \in S(\alpha)$ and $\gamma \in S(\alpha)$.

of deductive closure; but $\neg\alpha \in \Sigma_2$, so $\neg p'$ is in the observation set. (If p' is of the form $\neg p$, then from $\neg p' \in \Sigma_2$, i.e. $\neg\neg p \in \Sigma_2$, we may deduce $p \in \Sigma_2$, and therefore p is in the observation set.) Contrary to the definition of full theory. Therefore, $\Sigma_1 = \Sigma_2$. □

2.4 Fundamental theorem for FDE

Let the canonical model $\mathscr{M}^*$ be the ordered pair $\langle U^*,\ V^*\rangle$ where

1. U^*: A set of full theories.
2. V^*: $V^*(P_i) = \{\{|P_i|\}\}$[7].

We define the CNF of a formula α to be a set $\mathbf{CNF}_\alpha$ of equivalence classes CNF_α modulo permutation[8], where each member of an equivalent class CNF_α is a conjunction of disjunctions of literals provably equivalent to α such that $\forall CNF^m_\alpha, CNF^n_\alpha \in CNF_\alpha$, then $CNF^m_\alpha \dashv\vdash CNF^n_\alpha$. Every class differs from other classes by the set of literals involved in the CNF of the class. For any two CNFs within an equivalence class, CNF^i_α and CNF^j_α,

$$At\,(CNF^i_\alpha) = At\,(CNF^j_\alpha)$$

Among the equivalent classes in $\mathbf{CNF}_\alpha$, we define one particula class as the *standard CNF class* of α.

Definition 25. *CNF_α is the standard CNF class of α if and only if*

1. *$CNF_\alpha \in \mathbf{CNF}_\alpha$;*
2. *$At\,(\alpha) = At\,(CNF_\alpha)$;*
3. *At least one of each literal pair based on the language of α is in every conjunct.*

---

[7] $|P_i|$ is a full theory that contains P_i.

[8] Informally it is understood in the sense of rearrangement, e.g. there are six permutations of the set $\{1, 2, 3\}$, namely $[1, 2, 3]$, $[1, 3, 2]$, $[2, 1, 3]$, $[2, 3, 1]$, $[3, 1, 2]$, and $[3, 2, 1]$. We here take the formal definition that corresponds to this meaning in group theory and algebra. A permutation of a set S is a bijection from S to itself (i.e., a map $S \to S$ for which every element s of S occurs exactly once as image value). To such a map f is associated the rearrangement of S in which each element s takes the place of its image $f(s)$.

We then define $CNF(\alpha)$ to be the standard CNF of α, which is the member of the standard CNF class with the least number of conjuncts and the least number of literals in every conjunct.

Definition 26. $\forall\alpha \in \Phi$, *suppose an arbitrary CNF in the equivalence class $CNF(\alpha)$ is in the form of a conjunction of Δ_i $(1 \leqslant i \leqslant n)$, which are disjunctions of literals δ_j $(1 \leqslant j \leqslant m_i)$*[9]*, then the deformulated set corresponding to $CNF(\alpha)$, denoted by $\mathbb{CNF}(\alpha)$, is a collection of Δ_i $(1 \leqslant i \leqslant n)$ such that $\Delta_i = \{\delta_i \mid 1 \leqslant j \leqslant m_i\}$.*

FUNDAMENTAL THEOREM: $\forall\alpha \in \Phi$, $H^*_\alpha = \{|[\Delta_i]|^{10} \mid \Delta_i \in \mathbb{CNF}(\alpha), 1 \leqslant i \leqslant n\}$.

Proof. See Appendix. □

Representation Theorem Every formula of PL is FDE-provably equivalent with its standard CNF.

Proof. By mathematical induction on the length of α.
Basis: $\alpha = p_i$.
By [Mon], $\alpha \vdash p_i$.

Assume that the proposition holds for all α of length $n < k$. For α of length k, there are three subcases to consider.
[1] α is of the form $\neg\beta$.
By hypothesis of induction, $\beta \vdash \bigwedge\limits_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee\limits_{i=1}^{m} \delta_i$.
There are again three subcases to consider.
(a) Suppose β is of the form $\neg\gamma$.
Then $\alpha = \neg\neg\gamma$.
By hypothesis of induction, $\gamma \vdash \bigwedge\limits_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee\limits_{i=1}^{m} \delta_i$;
By axioms 3($\vdash$) and [Cut], $\alpha \vdash \bigwedge\limits_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee\limits_{i=1}^{m} \delta_i$.
(b) If β is of the form $\gamma \wedge \eta$, then α is of the form $\neg(\gamma \wedge \eta)$.
Therefore by axiom 1 ($\vdash$), $\alpha \vdash \neg\gamma \vee \neg\eta$.

[9] This notation m_i suggests that the number of literals in the ith disjunction is the output of a function m taking i as input.

[10] For any set S and any operation $\dagger$, we use $\dagger[S]$ to denote the set $\{\dagger s \mid s \in S\}$. Given that $\Delta_i = \{\delta_i \mid 1 \leqslant j \leqslant m_i\}$, $|[\Delta_i]|$ denotes the set of proof sets of δ_j $(1 \leqslant j \leqslant m_i)$.

By hypothesis of induction, we have $\neg\gamma \vdash \bigwedge_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee_{i=1}^{m} \delta_i$ and $\neg\eta \vdash \bigwedge_{i=l}^{l} \Delta_i$ where $\Delta_i = \bigvee_{i=1}^{m} \delta_i$;
Then by axioms 6, [LD] and [Cut], $\alpha \vdash \bigwedge_{i=1}^{n} \Delta_i \vee \bigwedge_{i=1}^{l} \Delta_i$.
By repeated applications of axiom 4 and [Cut], we obtain $\alpha \vdash \bigwedge_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee_{i=1}^{m} \delta_i$.
(c) The same argument applies when β is of the form $\gamma \vee \eta$.

[2] α is of the form $\beta \wedge \gamma$.
By hypothesis of induction, $\beta \vdash \bigwedge_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee_{i=1}^{n} \delta_i$ and $\gamma \vdash \bigwedge_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee_{i=1}^{m} \delta_i$.
By [Mon], [RC] and [Cut], we obtain $\alpha \vdash \bigwedge_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee_{i=1}^{l} \delta_i$.

[3] α is of the form $\beta \vee \gamma$.
The proof is similar to that of (b) for the first subcase.
It is easily seen that the other direction can be proved similarly. Hence, $\forall \alpha \in \Phi$, the representation theorem holds. $\square$

2.5 Completeness

The completeness proof for FDE in the familiar Henkin style can be found in the Appendix along with those for the other variant systems.

$$\alpha \models_{FDE} \beta \Rightarrow \alpha \vdash_{FDE} \beta$$

2.6 Generalization of FDE

FDE is determined by the set of hypergraph models in which

$$\alpha \models \beta \Leftrightarrow H_\alpha \sqsubseteq H_\beta$$

that is to say,

$$\alpha \models \beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B \textit{ such that } a = b$$

which is easily generalized to

$$\alpha \models \beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B \textit{ such that } a \leqslant b. \tag{Sub}$$

where '$\leqslant$' is an arbitrary partial ordering[11]. Therefore, the subsumption relation under observation can be weakened to an algebraic relation between vertices.

Lemma 3. *FDE is both sound and complete with respect to a class of articular models where entailment relation is interpreted as (Sub).*

Soundness is obvious, since identity is one of the partial orderings. As to the demonstration of completeness, we use the same canonical model as we did for FDE where each formula is represented as a collection of collections of proof-sets of literals. The partial ordering here is the subset relation between proof sets of literals. For two literals p', q', $|p'| \subseteq |q'|$ if and only if $\vdash p' \to q'$. But both p' and q' are literals, therefore $p' = q'$. Apart from this detail, the rest of the completeness proof is more or less the same as that for FDE. According to (Sub), the macro-relation between hypergraphs is determined by the micro-relation between vertices of hypergraph. This feature of subsumption naturally leads us to hope that by exercising a certain maneuver on the algebraic relation between vertices, a new hypergraph relation relation can be generated that carries with it some desirable characteristics. For example, imagine if we impose that the set $V(H)$ of vertices of the subsumed hypergraph H bear some relation, R to the set $V(H')$ of vertices of the subsuming hypergraph H', i.e.

$$V(H_\beta) R\, V(H_\alpha)$$

Now suppose we prescribe a principle that R is inclusion. The system that is generated by the subsumption relation in the class of general a-models along with this prescriptive principle is FDE without the following principle:

[11]Another generalization is $\alpha \models \beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B \textit{ such that } a \geqslant b$. No effect is generated by reversing the order of '$\leqslant$' so far as FDE is concerned.

$$\alpha \vdash \alpha \vee \beta$$

We call the system $FDAE$, with A for 'analyticity'. The system differs from that of Parry (in Fine's axiomatisation), in notable ways, first in not having the principle:

$$\alpha \vee (\beta \wedge \neg\beta) \vdash \alpha.$$

and second in being based upon a system of entailment rather than upon a system (S4) of strict implication.

$FDAE$ is sound and complete with respect to the class of general a-models in which entailment relation is represented by (Sub) together with the condition that R is inclusion. The smallest lattice capable of illustrating the inferential distinctiveness of the system requires two independent variables.

The best way to illustrate and compare the structures of various system-lattices is by way of pictures. Classical logic when formulated with only finitely many propositional variables may be represented as $\wp(X)$ with some finite set X, and this is of course a finite boolean algebra. By the same token, each of the many systems introduced above, FDE, $FDAE$, etc, has a corresponding lattice which again has a corresponding digraph. Every arbitrary formula is represented by a circle, and $\alpha \vdash \beta$ is represented by an arrow from the circle representing α to that representing β. The arrows in our picture goes from south to north, and so does inference. The following digraph is a fragment of the digraph involving two variables that describes, rather crudely, the inferential behavior of $\wedge$ and $\vee$ in $FDAE$.

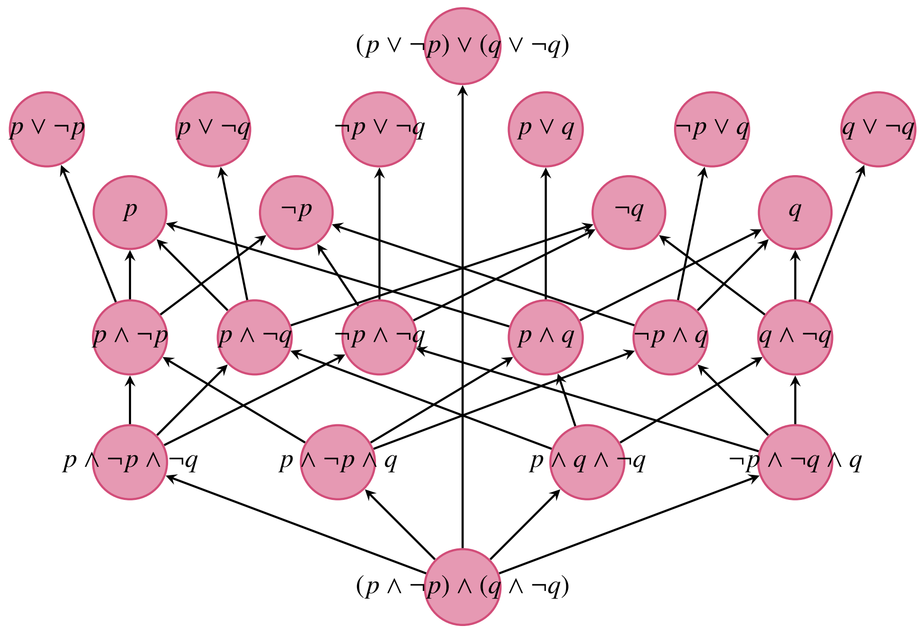


Figure 2.1: A fragment of $FDAE$

The complete two-variable fragment has infinitely many nodes, as is evident from the fact that starting from any two nodes bearing an inference relation, there is always another node that can be inserted vertically between them. Take $p \wedge q$ for example, $p \wedge q \vdash q$, there are infinitely many formulae above $p \wedge q$ with shorter and shorter distance from it. The mechanism for generating the nodes between $p \wedge q$ and q approaching $p \wedge q$ can be described thus:

1. $(p \wedge q) \vee q$ is between $p \wedge q$ and q;
2. If α is the *nth* formula inserted by this method, then the formula between $p \wedge q$ and α is $\alpha \wedge q$ or $\alpha \vee q$, according as whether n is odd or even.

Therefore to avoid the difficulty in representing the infinite pictorially, we confine ourselves to a homogeneous representation at different hemispheres. If we were to take the four literal nodes, p, q, $\neg p$, $\neg q$, as forming the equator of the sphere, then the northern hemisphere is confined to the representation of the inferential behavior of $\vee$ whereas the southern hemisphere is confined to that of $\wedge$.

We may look at R from a slightly different perspective. Put in the orginal context, there are two ways of understanding analyticity, the syntactic and the semantic. The logic of analytic implication [21] that lies behind $FDAE$ was the axiomatisation given by Kit Fine of a system orginally proposed by W. T. Parry [41]. Parry's original idea of analyticity is to impose a condition on logical implication such that the implication itself has nothing to do with information acquisition. Specifically speaking, with two logical formulae α and β separated by a turnstile, the criterion becomes $At\ (\beta) \leqslant At\ (\alpha)$[12]. It is based on this idea of analyticity that we define $FDPE$ where 'P' stands for Parry:

$$V(H_\beta)R^*\ V(H_\alpha)$$

and then consider as a Parry principle:

[P] R^* satisfies:

Definition 27. *$\forall H, H' \in \mathbf{H}(U)$, $\forall x$, if $\exists e \in V(H)$ such that $e \in |x|$, then $\exists e' \in V(H')$ such that $e' \in |x|$ where $|x|$ is the pair $\{x, \overline{x}\}$.*

Since we relax the condition on R, demanding for every vertex in the vertex set of the subsumed hypergraph, instead of inclusion, only the existence of its complement in the vertex set of the subsuming hypergraphs; the corresponding smallest lattice illustrating the inferential behaviour of $\wedge$ and $\vee$ of the system is a superlattice of that for $FDAE$. It has, though not the rule $\vee$-introduction itself, four instances of $\vee$-introduction:

- $p \vdash p \vee \neg p$;
- $\neg p \vdash p \vee \neg p$;
- $q \vdash q \vee \neg q$;
- $\neg q \vdash q \vee \neg q$;

[12] See definition 21.

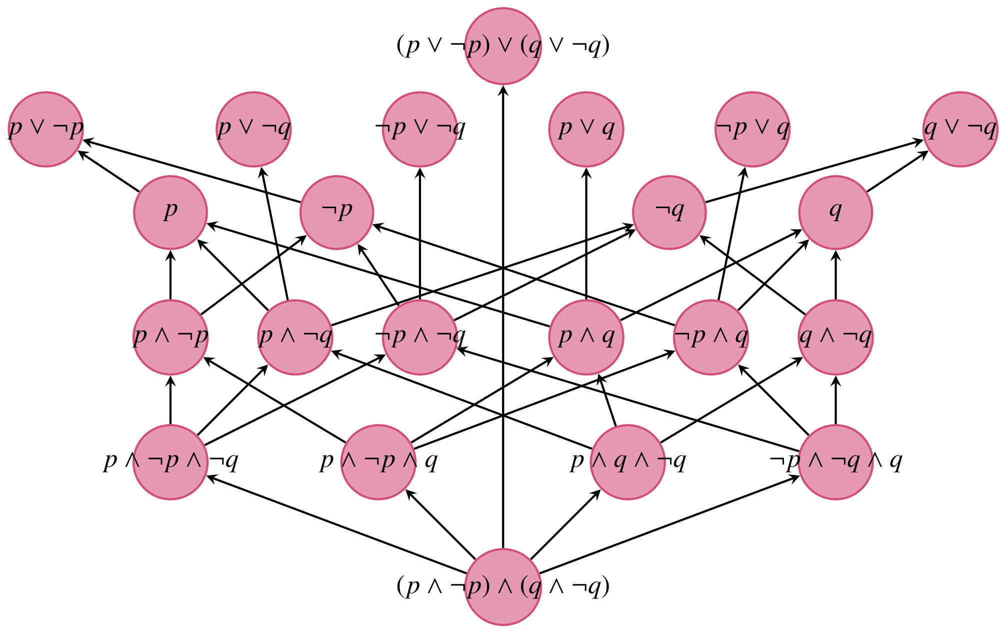


Figure 2.2: A fragment of $FDPE$

The diagram is different from figure 2.1 in that the path from $p \wedge \neg p$ to $p \vee \neg p$ consists of two sub-paths: $p \wedge \neg p \vdash p$ and $p \vdash p \vee \neg p$, because the two instances of $\vee$-introduction, $p \vdash p \vee \neg p$ and $q \vdash q \vee \neg q$ survive the Parry principle, but not the inclusion principle. In the fragment of $FDAE$, all instances of $\vee$-introduction failed. The systems based upon the resulting hypergraph-structures along with their defining hypergraph relations are presented in table 2.1.

Systems	**Entailment relation**
FDE	$\forall e' \in H', \exists e \in H$ such that $e \subseteq e'$
FDAE	$FDE\ +\ (V(H') \leqslant V(H))$
FDPE	$FDE\ +\ (\forall e' \in V(H'),\ if\ e' \in \lvert x \rvert,\ then\ \exists e \in V(H)\ such\ that\ e \in \lvert x \rvert)$
	$(\lvert x \rvert\ is\ the\ pair\ \{x, \overline{x}\})$

Table 2.1: **Semantics table**

For the sake of comparision, the lattice representing, *mutatis mutandis*, FDE is presented thus:

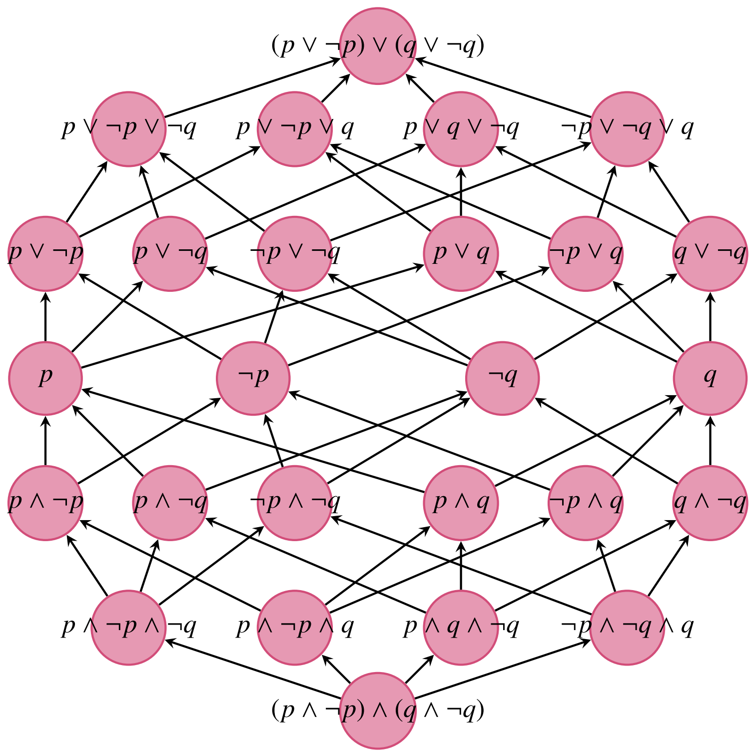


Figure 2.3: A fragment of FDE

So far we have discussed sufficiently many systems that bear syntactic similarities to each other that we can note a valuable comparison. These systems were independently designed for different purposes, mathematical as well as philosophical, and admit to widely different semantics, but still there are traces of similarities which cannot but reveal the subtle connection between them. For example, the system $FDAE$ and AI are both subject to the criterion of analyticity, i.e. all the information on the right hand side of the turnstile can also be found on the left; the difference, with regard to one rule, $p \vee (q \wedge \neg q) \vdash p$, is obviously due to the fact that the criterion is imposed upon strict implication for AI, which allows the rule; and entailment for $FDAE$, which does not.[13]

[13]It is reminiscent of a particularly relevant remark made by A. N. Prior (to R. E. Jennings in 1969) that entailment is strict implication, or rather it is not; for entailment is a muddle, and strict implication is not.

Table 2.2 permits comparisons between codifications of these and binary versions of other well-known systems.[14]

Natural Deductive Rules	PL	FDE	FDAE	FDPE	LP	AI	E
$p \dashv\vdash \neg\neg p$	✓	✓	✓	✓	✓	✓	$\neg\neg p \vdash p$
$p \wedge (a \vee r) \dashv\vdash (p \wedge r) \vee (q \wedge r)$	✓	✓	✓	✓	✓	✓	✓
$\neg(p \vee q) \dashv\vdash \neg p \wedge \neg q$	✓	✓	✓	✓	✓	✓	✓
$p \vdash p \vee q$	✓	✓	✓	✓	✓	✓	✓
$p \wedge q \vdash p$	✓	✓	$p \wedge q \vdash p \vee q$	$p \wedge q \vdash p \vee q$ $p \vdash p \vee \neg p$	✓	✓	✓
$p \vee (q \wedge \neg q) \vdash p$	✓	×	×	×	×	✓	×
$[MP]$	✓	✓	✓	✓	×	✓	✓
$p \vdash q \Rightarrow \neg q \vdash \neg p$	✓	✓	✓	✓	×	✓	✓
$[Mon]$	✓	✓	✓	✓	✓	✓	✓
$[Ref]$	✓	✓	✓	✓	✓	✓	✓
$[Trans]$	✓	✓	✓	✓	×	✓	✓

Table 2.2: **Syntax table**

2.7 Hypergraphs on Lattice

In [9], every formula is assigned a hypergraph on the power set of the universe, $\wp(U)$, which is a boolean algebra. Each edge of the hypergraph representing some formula in an articular model is an element of the boolean algebra $\wp(U)$. We know that the logical principles of the systems introduced so far are validated by the various articular relations we defined, whose properties we have explored to some extent in this chapter. However, all the articular relations we have introduced so far shared one common property: they are relations between hypergraphs on $\wp(U)$. In the last section of this chapter, we generalize $\wp(U)$ to an arbitrary lattice L, and thereby redefine the notion of hypergraph as a collection of abstract mathematical objects, specifically a set of elements $H(L)$ from a certain lattice L. The class of simple hypergraphs $\mathbb{H}(L)$ on L is therefore a subset of $\wp(L)$ satisfying the following condition:

$$\forall H(L) \in \mathbb{H}(L), l,\ l' \in H(L) \Rightarrow l \nleq l'$$

[14] AI refers to Analytic Implication[21].

An algebraic articular model is a triple $M = \langle L, V, 0\rangle$ where

1. $L = \langle X, \leqslant\rangle$ is a lattice;
2. $V : At \to \mathbb{H}(L)$.
3. $\forall l \in L, l > 0$.

That is, to each P_i, V assigns a simple hypergraph on L, denoted by $H(p_i)$. In the subsequent text, where there is no risk of confusion, we omit L for the sake of convenience.

Definition 28. *t is a transversal of H if and only if $\forall l \in H$, $t \wedge l > 0$.*

Definition 29. *$\tau(H)$ is the set of minimal transversals of H if and only if for any two transversals t, t' of $\tau(H)$, $t \nless t'$.*

In extending V to Φ, we encounter the same difficulty as was earlier in the way of extending **H** to Φ in articular model; since not all algebraic operations naturally preserve the simplicity of hypergraphs, we again have to introduce an operation for simplification.

Definition 30. $\bullet H = H$ - $\{l \in H \mid \exists l' \in H : l' \leqslant l\}$.

Definition 31. *With $\wedge$, $\vee$ being the meet and join of the lattice L,*
$H \sqcup H' \ = \ \bullet\{\{E \vee E'\} \mid E \in H,\ E' \in H'\}$
$H \sqcap H' \ = \ \bullet\{E \mid E \in H \text{ or } E \in H'\}$
$\overline{H} \ = \ \{f(l_i) \mid l_i \in \tau(H)\}$.

Definition 32. *$f : L \to L$ is a homomorphic injection of period 2 where 0 is a fixed point. f satisfies the following three conditions:*

1. *$\forall l, l' \in L$, if $l \leqslant l'$, then $f(l) \leqslant f(l')$;*
2. *$ff(l) = l$;*
3. *$f(0) = 0$.*

$V(\cdot)$ extends V to Φ as follows:

$H_{p_i} = V(p_i)$
$H_{\neg\alpha} = \overline{H}$
$H_{\alpha\vee\beta} = H_\alpha \sqcup H_\beta$
$H_{\alpha\wedge\beta} = H_\alpha \sqcap H_\beta$.

Definition 33. *$\forall H, H' \in \mathbb{H}(L), H \sqsubseteq H'$, ($H$ is algebraically subsumed by H') iff $\forall E' \in H'$, $\exists E \in H$ such that $E \leqslant E'$.*

Definition 34. *$\forall\alpha,\beta$, $\alpha \vDash \beta$ (α entails β) iff $\forall\mathscr{M} = \langle L, V, f\rangle$, $H_\alpha \sqsubseteq H_\beta$. Alternatively, we say that $\alpha \vdash \beta$ is valid. So,* mutatis mutandis, *for $\Gamma \vdash \alpha$.*

From these definitions it is easy to prove three lemmas based on basic algebraic properties.

Lemma 4. *$\langle\mathbb{H}(L), \sqsubseteq\rangle$ is a lattice.*
Lemma 5. *$H \sqsubseteq H'$ if and only if $\tau(H') \sqsubseteq \tau(H)$.*
Lemma 6. *$H \subseteq \tau\tau(H)$.*

The proofs are all straightforward, following directly from definition 29 and 31. The resulting system is a subsystem of *FDE*. The only axiom missing is

$$p \vdash \neg\neg p$$

We have already known that if L is a boolean algebra, then $H = \tau\tau(H)$, therefore the lattice L being a boolean algebra is sufficient for $H(L)$ to be identical with its double transversal. It remains only to explore the necessary condition.

Lemma 7. *If $H = \tau\tau(H)$, then L is a boolean algebra.* [15]

Proof. An element l of a lattice L is an *atom* if $0 < l$, and there is no $x \in L$ with $\perp < x < l$. Let the set of atoms in L be $At\ (L)$. Define $At\ (x) = \{y \in At(L) \mid y \leqslant x\}$, $\forall l \in L$, l is either join reducible or

[15]The proof is due to Julian Sahasrabudhe.

join irreducible, i.e. $l \in At\ (L)$.

Suppose $l \in L$ is join irreducible. Consider the hypergraph $\{l\}$, it is easy to see that $\tau(\{l\}) = At\ (l)$, and $\tau(At\ (l)) = \{\bigvee At\ (l)\}$. Given $\{l\} = \tau\tau(\{l\})$, we have $\{l\} = \{\bigvee At\ (l)\}$. Contrary to hypothesis. Since l is an arbitrary element of the lattice, we conclude that every element of L is join reducible. Therefore, every element can be reduced to a finite set of atoms.

There exists a function $f : L \to \wp(At\ (L))$ that maps to each element of the lattice a set of corresponding atoms $At\ (x) = \{y \in At(L) \mid y \leqslant x\}$.

Since any finite boolean algebra is isomorphic to the boolean algebra $\wp(S)$ of all subsets of some finite set S, L is a boolean algebra. □

Therefore $H = \tau\tau(H)$ if and only if L is a boolean algebra.

2.8 A Preservationist Project

The project of preservation is an independent project, placed in the circumstances of the ongoing discussion of hypergraph semantics and representational strategy. The project has its own history. In a series of publications over several decades, Jennings and Schotch promote a conception of inferential correctness as preservation of desirable features of data. Those features include truth as well as various measures of coherence that do not require truth. In Schotch and Jennings [28] one such measure is coherence level: the cardinal of the narrowest partition of a set of data into coherent subsets; Jennings and Schotch [48] discusses another: the cardinal of the largest coherent subset of a set of data. Since neither such preservationist strategy tolerates $\wedge$-introduction, the shared strategy has (misleadingly) been labelled non-aggregative. In Jennings, Chan and Dowad [13], preservation is discussed in more general terms and it is proposed that systems typically given dialethist semantic analysis might also be given non-dialethist preservationist construals, a proposal affirmed by later practice. Brown [6], for example, provided a preservationist interpretation of *LP*.

From the preservational point of view, hypergraphs, taken as special arrangements of data, naturally give rise to a variety of features it can be desirable to preserve. Compared with previous preservationist projects, these representational features share one common quality: that they are aggregation-tolerating within the scope of classicality (so non-dialethist), an apparently paradoxical claim that follows, however, simply from the properties of hypergraphs. As sentences are represented by hypergraphs, the logic operations we introduced on them, i.e. '$\neg$', '$\wedge$' and '$\vee$', render the set of data under those operations quite classically. Together they form a De Morgan lattice[16]. At the same time, the set of data also admits of aggregation-tolerant coherence measures. The process of aggregating hypergraphs is one that leads from coherence to coherence. It does not create contradiction in the dialethic sense. A contingent formula α conjoining with another contingent formula $\neg\alpha$ generates a still contingent formula $\alpha \wedge \neg\alpha$. There is therefore no semantic distinction between the set $\{p, q\}$ and the formula $p \wedge q$; and the segregation strategy used in [28] and [48] for finding the coherence level is no longer needed.

So far we have not proved it possible to give a preservational *characterization* of first degree entailment, but there are several preservational features of hypergraph relations[17] that are worth exploring to which we now turn.

2.8.1 Preservational properties of first degree articular inferences

Hypergraph representation preserves the structural characteristics of sentences masked by the truth-set representation of a standard semantic model. Conversely, a classical truth-set can also be obtained from a hypergraph by performing an operation on its (hyper)edges.

$$[\![\alpha]\!]^{\mathscr{M}} = \bigcap_{i=1}^{n}\{\bigcup E_i \mid E_i \in H_\alpha,\ 1 \leqslant i \leqslant n\}$$

Thus, for all the systems of articular inference presented earlier in this chapter, there exists *Modus Ponens* as a property such that if $x \in [\![\alpha]\!]^{\mathscr{M}}$ & $\alpha \vdash \beta$, then $x \in [\![\beta]\!]^{\mathscr{M}}$. That is, first degree articular inference preserves satisfaction.

Moreover, no first degree articular inference as presented in this thesis admits of increase in the number of contradictions over entailment. Together with the result above, first degree articular inferences preserve satisfaction and the number of contradictions. Both are simple points, the latter

[16] See footnote 2 in Preface.

[17] These are understood as members of $\mathscr{R}_X$, described on page 26 of this thesis.

of which can be illustrated by an example such that $(\alpha \wedge \neg\alpha) \wedge (\beta \wedge \neg\beta) \vdash \alpha \wedge \neg\alpha$ but $\alpha \wedge \neg\alpha \nvdash (\alpha \wedge \neg\alpha) \wedge (\beta \wedge \neg\beta)$.

2.8.2 Harmonic number and chromatic number

In [40], harmonic number $\eta(H)$ of a hypergraph H on the universe U is defined as the least number of edges of H whose intersection is the empty set $\varnothing$. This is expressed in the following two definitions:

Definition 35. $\begin{pmatrix} H \\ n \end{pmatrix}$ *is the set of all n-tuple subsets of H.*

Definition 36.

$$\eta(H) = \begin{cases} min\ n \in \mathbf{Z}^{+} : \exists G \in \begin{pmatrix} H \\ n \end{pmatrix} : \bigcap G = \varnothing & \textit{if this limit exists;} \\ \infty & \textit{otherwise} \end{cases}$$

H is said to be n-harmonic if $\eta(H) > n$.

Definition 37. $\forall H \in \mathbb{H}$, $\mathbb{X}(H)$ *is the set of colourings for the hypergraphs H if*

1. $\mathbb{X}(H) \subset \mathbb{H}$;
2. $\forall c(H) \in \mathbb{X}(H)$, $V(H) = V(c(H))$;[18]
3. $H \cap c(H) = \varnothing$.

Obviously, the idea involved in the definition is the extension of that for graph-colouring, i.e. that no edge should be left monochrome. Thus,

Definition 38. *The chromatic number* $\chi(H)$ *of H is the lowest cardinality of* $c(H)$[19] *in* $\mathbb{X}(H)$*, i.e.* $\chi(H) = min\{x \mid x = |c(H)|,\ c(H) \in \mathbb{X}(H)\}$.

[18] $V(H)$ indicates the set of vertices of the hypergraph H. It was first mentioned in this thesis on page 38.

[19] Cardinality of a hypergraph is the number of edges in the hypergraph.

Therefore $\forall H \in \mathbb{H}$, $\forall n \geqslant \chi(H)$, H may be regarded as n-colourable. Otherwise it is uncolourable. Hence another preservational property of a first degree articular inference, i.e. that strict entailment preserves uncolourability:

Lemma 8. *If A strictly entails B ($H_\alpha \sqsubseteq H_\beta$), then colourability is preserved from H_α to H_β, i.e. $|c(H_\alpha)| \geqslant |c(H_\beta)|$. So uncolourability is preserved from H_β to H_α.*

This is obvious because chromatic number decreases over first degree articular inferences.

Chapter 3

First Degree Logic of Necessity

3.1 Introducing modality

We explore three ways of introducing the language of modality into first degree entailment. The language of first degree inference with modality (in BNF) is defined by

$$\alpha ::= p|\neg\alpha|(\alpha \vee \alpha)|\Box$$

where $\Diamond := \neg\Box\neg$.

3.1.1 Hypergraph and primordial proposition

So far we have come across the notion of proposition in various idioms. In the standard model of classical semantics, a proposition is a subset of the universe of a model. The standard picture of semantics takes a proposition to be the meaning of a sentence in the sense that it partitions the universe into the set of states where it is true and the set of states where it is false. On this understanding of proposition, a binary relational frame $\mathscr{F} = \langle U, R\rangle$ can be described as a structure in which each object, or 'point', in U is assigned a primordially necessary proposition, $R(x)$. In a model on such a frame $R(x)$ might not be expressed by any sentence of the language; nevertheless, the expressed necessities at x in the model are the propositions expressed by sentences of the language and semantically entailed by $R(x)$. The set of necessities at x in a particular model is the theory expressed by the filter of expressed propositions semantically entailed by $R(x)$. In neighbourhood idiom, the function, $N : U \to \wp\wp(U)$ of a neighbourhood frame (N-frame), $\mathscr{F} = \langle U, N\rangle$ merely assigns to a point x a set of primordially necessary propositions called neighbourhoods. In a model, those sentences of the language are necessary at x that express one of x's primordial necessities. There exists some subclass of models on structures as such that all points in the model share a single primordially necessary proposition. For example, we can understand the universality of R as the special case of the neighbourhood account in which N is constant, and assigns to each point the singleton collection, $\{U\}$. In the more general case, a constant N assigning a non-empty set of propositions merely guarantees that propositions expressed by modal sentences are either universal or empty. If the set of neighbourhoods is a filter, then the set of sentences expressing neighbourhoods is a classical theory, that is, a deductively closed set. The system K, (the principles of which are [K], [RM], and [RN]) is determined by the class of all powerset filters. $N(x)$ can be naturally understood as a *hypergraph* $H(x)$ on $\wp(U)$.

In 1975, Schotch proposed[1] a generalization of the neighbourhood idiom in which the truth-condition for a modal formula $\Box\alpha$ in a model $\mathscr{M}$ was weakened to require only that one of the propositions assigned to a point x in the underlying frame *semantically entails* the proposition expressed by α in the model. In effect, this would require that for some edge, e of $H(x)$, $e \subseteq [\![\alpha]\!]^{\mathscr{M}}$. The logic corresponding to this new "environ" idiom is the closure of PL under the rule [RM]. Jennings proposed a further idiomatic restriction by which, in effect, $H(x)$ is always simple. The simplicity restriction not being modally definable, the resulting distinct idiom was called "locale" semantics [47]. A locale frame (L-frame), $\mathscr{F}$ is a pair $\langle U, L\rangle$ in which L is a function assigning to each point, x, a family of sets $L(x)$, the set of x-locales. It is easy to see that if for every x, $L(x) \neq \emptyset$, then [RN] preserves validity, and that if for every x, $L(x)$ is a singleton, then [K] is valid.[2]

Locale semantics is connected in subtle ways both with relational semantics and hypergraph semantics. A moment's reflection reveals that a L-model $\mathscr{M} = \langle U, L, V\rangle$ satisfying the condition that for every x, $|L(x)| = n$ is equivalent to a n-ary relational model $\mathscr{N} = \langle U, R_1, \ldots, R_n, V\rangle$ where $R_i xy$ iff $y \in a_i$ where $a_i \in L(x)$. The interpretation of $\Box\alpha$ is the truth-condition

[1] This word is due to R. E. Jennings.

[2] A more detailed historical discussion can be found in [11]

$$\models^{\mathscr{M}}_{x} \Box\alpha \Leftrightarrow \exists i\ (1 \leqslant i \leqslant n) : \forall y, R_i xy \Rightarrow \models^{\mathscr{M}}_{y} \alpha.$$

It should be equally evident that if, for every x, $L(x)$ is $n-$coherent, that $\tau(L(x))$, possibly with some diagonalisation, will yield a set of n-tuples. Formally, let $a_1, \ldots, a_i \ldots$ be an enumeration of $L(x)$. Then $\langle x, y_1, ...y_n\rangle$ is in R iff $\forall a_i \in L(x), \exists y_j : y_j \in a_i$. Thus a locale frame satisfying the $n-$coherence restriction will generate an equivalent $n+1$-ary relational frame. The class of all $n-$ary relational frames determines the system K_n which has [RM], [RN] and all instances of the schema

$$\Box\alpha_1 \wedge ... \wedge \Box\alpha_n \rightarrow \Box(\bigwedge(\alpha_j \vee \alpha_k)(1 \leqslant j \neq k \leqslant n)).$$

On the other hand, in revised neighbourhood idiom, $L(x)$ is a simple hypergraph $H(x)$ on $\wp(U)$. An arbitrary formula α in an articular model is represented as a simple hypergraph on $\wp(U)$, thus the requirement that for some edge, e of $H(x)$, $e \subseteq [\![\alpha]\!]^{\mathscr{M}}$ is generalized as

$$\forall [\![\alpha]\!] \in H^{\mathscr{M}}_{\alpha},\ \exists e \in H(x) : e \subseteq [\![\alpha]\!]$$

where $[\![\alpha]\!]$ is taken to be a proposition in the classical sense. And the interpretation of $\Box\alpha$ in locale semantics becomes the truth-condition

$$\models^{\mathscr{M}}_{x} \Box\alpha \Leftrightarrow L(x) \sqsubseteq H^{\mathscr{M}}_{\alpha}.$$

We are now in a position to give a formal semantics for first degree modal logic that uses a hypergraph as a primodial necessity.

3.1.2 First degree necessity

Definition 39. *An articular frame is a pair, $\mathscr{F} = \langle U, \mathsf{H}\rangle$ where H is a designated simple hypergraph on $\wp(U)$. A model on $\mathscr{F}$ adds a function* **H***, defined as for hypergraph models and extended to* modally augmented Φ *by the clause:*

$$\mathscr{M} \models \Box\alpha \Leftarrow \mathsf{H} \sqsubseteq \mathbf{H}(\alpha);\ else\ \mathscr{M} \not\models \Box\alpha. \qquad (FDEM\Box)$$

As we have said, the '$\sqsubseteq$' here is that of FDE. It is notable that if the '$\sqsubseteq$' of PL is adopted, then not only does the resulting underlying system revert to PL, but the resulting class of frames is just the class of locale frames satisfying $\forall x, L(x) \neq \emptyset$, modulo the altered representation of propositional wffs. The system $FDEM$ is the system that adjoins to the codification of FDE, the rule of aggregation

$$(\alpha)(\beta)(\Box\alpha \wedge \Box\beta \vdash \Box(\alpha \wedge \beta)) \tag{Ke}$$

and rules of monotonicity and of normality

$$(\alpha)(\beta)(\vdash \alpha \models \beta \Rightarrow \vdash \Box\alpha \models \Box\beta) \tag{RMe}$$

$$(\alpha)(\vdash \alpha \ \Rightarrow \vdash \ \Box\alpha) \tag{RNe}$$

Since $\Diamond = \neg\Box\neg$, we have according to $FDEM$

$$\mathscr{M} \models \Diamond\alpha \Leftrightarrow \exists b \in H_\alpha, \exists a \in \tau(\mathsf{H}) : \ a \subseteq \overline{b}.$$

Since H is simple, $\tau\tau(H) = H$, and we have the duality principle for $\Box$ and $\Diamond$.

$$\Box P \ / \ \neg\Diamond\neg P \tag{3.1}$$

$$\neg\Diamond\neg P \ / \ \Box P \tag{3.2}$$

If H is not a simple hypergraph, $\tau\tau(H) \subset H$, then we have only one direction of the duality principle, namely,

$$\neg\Diamond\neg P \ / \ \Box P \tag{3.3}$$

However, for both cases, $FDEM$ has the rule of monotonicity for $\vdash$

$$\alpha \vdash \beta \,/\, \Box\alpha \vdash \Box\beta \tag{3.4}$$

$$\alpha \vdash \beta \,/\, \Diamond\alpha \vdash \Diamond\beta \tag{3.5}$$

If H is a singleton family of a set of subsets of U, then [K] is valid.

$$\Box\alpha \wedge \Box\beta \vdash \Box(\alpha \wedge \beta) \tag{3.6}$$

In such case, $FDEM$ has a quasi-Scott's rule.

$$\Sigma \vdash \alpha \,/\, \Box[\Sigma] \vdash \Box\alpha \tag{3.7}$$

3.1.3 An observation

We define a hierarchy of hypergraphs recursively:

- $H^0 = H$;
- $\forall n < \omega$, H^{n+1} is a hypergraph on $\wp(H^n)$.

Let H be H^i with $0 \leqslant i \leqslant n$. A Model $M = \langle\, U, \ \mathsf{H}^i, \ \mathbf{H} \,\rangle$ defined on an articular frame[3] can be thus generalized[4]:

1. U is a nonempty set of points;
2. $\mathsf{H}^i \in \mathbb{H}$ is a designated hypergraph on $\wp^i(U)$.
3. $V : At \rightarrow \{\mathsf{H}^i \mid \mathsf{H}^i \in \wp\wp(\wp^n(U))\}$.

The extension of $\mathbf{H}$ in as usual, and the truth condition for $\Box\alpha$ can be specified thus

[3]See definition 39.

[4]For $\mathbb{H}$, see the definition of an articular model in chapter 2.

$$\mathscr{M} \models \Box\alpha \Leftrightarrow \forall b \in H_\alpha, \exists a \in \mathsf{H}^n : \ a \subseteq b$$

3.1.4 Defining necessity

Natural language presents us with a choice among at least three alternative semantic parsings of claims of necessity

1. (Necessarily α) is true;
2. α is (necessarily true);
3. Necessarily (α is true).

(1) is a metalinguistic claim about the sentence necessarily α in the object language; (2) is a metalinguistic claim about the object language sentence α; (3) is a metalinguistic claim about a metalinguistic claim. The usual treatment of necessity adopts the first parsing as in $\models^{\mathscr{M}}_{x} \Box\alpha$; in a first degree system that adopts $\Box$, it seems natural to adopt the third parsing, and to this end we introduce the notation $\Box\alpha$ to mark the necessity of the truth of α in a model. Adopting this parsing creates special considerations, in particular, it creates difficulty for the treatment of nested modalities in higher degree modal systems. However, here we are only concerned with first degree modal systems.

It is in the second of the three senses introduced above that we define necessity in a class of articular models:

Definition 40. $\sqsupset\ \alpha \Leftrightarrow \forall e \in H_\alpha, \exists v \in e$ *such that* $\exists v' \in e : v' = \overline{v}$.

Such a property dictates in an articular model what the hypergraph of α should be like in order to justify α to be necessary. It is not an interpretation for 'necessarily α' which would answer for the symbol $\Box\alpha$. The difference between property and function can be illustrated by a remark made by Kneale & Kneale in [29] on material and strict implication:

> When Whitehead and Russell spoke of $P \supset Q$ as a statement of material implication, they seemed to have confused together two different questions, namely (i) 'What justifies inference from the proposition that-P to the proposition that-Q?' and (ii) What

> is the weakest additional premiss which in conjunction with the premiss that-P suffices for inference to the conclusion that-Q?'... $P \supset Q$, this formula is constructed by use of the propositional signs P and Q, but it is not *about* the propositions which they express.... On the other hand, Lewis's formula $P \mathrel{-\!\!3} Q$ is supposed to be *about* the propositions expressed by P and by Q precisely because it is conceived as a proper answer to the first of the questions noticed above.

Just as Lewis' '$\mathrel{-\!\!3}$', our notion of necessity here is also conceived to answer the first, instead of the second, question.

The corresponding logic system consists of the axioms and rules of FDE in addition to[5]

$$\frac{\sqsupset \alpha,\ \sqsupset \beta}{\sqsupset (\alpha \wedge \beta)} \tag{$K^\wedge$}$$

$$\frac{\sqsupset \alpha,\ \alpha \vdash \beta}{\sqsupset \beta} \tag{RM}$$

and

$$\frac{\alpha \vdash \beta}{\sqsupset (\alpha \rightarrow \beta)} \tag{RN}$$

Again we prove completeness by the Henkin method .

The burden of proof for the fundamental theorem lies in the generalization of H^* to Φ; that for the non-modalized formulae has already been done in [9], what is left to prove is the following:

Lemma 9. *$\forall \alpha \in \Phi$, $\mathscr{M} \mid\!\sqsupset \alpha \Rightarrow\ \sqsupset \alpha$.*

Proof. Since $\sqsupset \alpha$, according to the definition of H^*,

$$H^*_\alpha = \ \{\{|\delta_1|,\ |\delta_2|, \ldots, |\delta_{m_i}|\} = |[\Delta_i]| \mid \forall 1 \leqslant i \leqslant n, \Delta_i \in \mathbb{CNF}(\alpha)\ \&\ \exists i, j:\ \delta_i = \neg\delta_j\}^6$$

[5]The arrow formulae are interpreted as in 4.3.1.

[6]See definition 26.

More specifically,

$$H^*_\alpha = \{\{\ldots, |p_1|,\ |\neg p_1|, \ldots\}, \ldots, \{\ldots, |p_n|,\ |\neg p_n|, \ldots\}\}$$

By definition 13, $\exists \alpha' \in \Phi$ such that

$$H^*_{\alpha'} = \{\{|p_1|,\ |\neg p_1|\}, \ldots, \{|p_n|,\ |\neg p_n|\}\}$$

By definition 13 and metatheorem 2, we have

$$\alpha' \dashv\vdash (\neg p_1 \vee p_1) \wedge \ldots \wedge (\neg p_n \vee p_n)$$

which is equivalent with

$$\alpha' \dashv\vdash (p_1 \rightarrow p_1) \wedge \ldots \wedge (p_n \rightarrow p_n)$$

Since

$$p \vdash p$$

By RN we have

$$|\!\Box\, (p \rightarrow p)$$

By $K^\wedge$ and rules of FDE, we have

$$|\!\Box\, ((p_1 \rightarrow p_1) \wedge \ldots \wedge (p_n \rightarrow p_n))$$

which is simply

$$|\!\Box\, \alpha'$$

But

$$\alpha' \vdash \alpha$$

By RM, we have

$$\models_{\Box} \alpha$$

$\Box$

The rest is standard work.

Chapter 4

Proposition and Entailment

4.1 Hypergraph as a unifying semantic idiom

4.1.1 Coupled trees

In chapter 1, we gave an informal account that decribes the formative stages of hypergraph semantics. It is a refinement of some earlier considerations toward paraconsistency within the framework of classical semantic theory. The structure of a hypergraph representing a formula in a class of articular models reflects the normal form of the formula. A parallel case can be drawn where a classical formula is represented by a truth tree. Each branch of the tree represents a way of making the formula true, and thereby constitutes a conjunction in the DNF of the formula. Given an argument, $\alpha \vdash \beta$ and its truth-tree representation, a test for the validity of the argument requires, according to classical semantics, simply to check, for each branch of the α-tree, whether there exists a branch in the β-tree that is completely covered by it. Such is the basic idea of the coupled tree method introduced by Richard Jeffrey in 1967 [27]. The test of validity was thus given the name *covering criterion*. Since there are two trees involved, they are arranged in some order. The representation of an argument $\alpha \vdash \beta$[1] usually has the α-tree on top, upside down, with branches extending downwards; while the β-tree is situated at the bottom, with branches pointing upwards. Given our understanding of the connection between a truth-tree of a formula and its corresponding DNF, the covering criterion for an arbitrary binary formula $\alpha \vdash \beta$ (in the sense of definition 7) can be restated as a relation

[1] $\Gamma \vdash \alpha$ is understood in the sense of definition 8.

between $DNF(\alpha)$ and $DNF(\beta)$:

$$\alpha \vDash \beta \Leftrightarrow DNF(\alpha)\ R\ DNF(\beta) \Leftrightarrow \forall A \in D(\alpha), \exists B \in D(\beta) : A \supseteq B$$

As in (1.2), $D(\alpha)$ denotes the de-formulated set[2] of $DNF(\alpha)$[3], i.e. a collection of collection of literals. Classically we interpret literals as members of $\wp(U)$, therefore there exists a natural interpretation of α as a simple hypergraph corresponding to $D(\alpha)$, and the relation R stated above is the dual relation to subsumption introduced in chapter 2[4]. Since H_α interprets α in articular models, the covering criterion of Jeffrey's coupled tree method can be thus expressed in terms of hypergraphs:

$$\alpha \vDash \beta \Leftrightarrow \forall A \in \tau(H_\alpha), \exists B \in \tau(H_\beta) : A \supseteq B \tag{4.1}$$

$\tau(H_\alpha)$ and $\tau(H_\beta)$ are the transverse hypergraphs of H_α and H_β respectively, and condition (4.1) is the dual condition of that expressed in (1.5), i.e. subsumption.

However, Jeffrey realized that the covering criterion as it is cannot cover all cases of classical validity. Accordingly he allowed for the two exceptional cases that escape the rule, and added them to the class of valid inferences. What makes the case interesting is that the two exceptional cases are none other than those that make classical logic fail the criterion of paraconsistency. It was J. Michael Dunn [17] who first realized that the covering criterion itself suffices to give us a paraconsistent logic that is non-classical only in the sense of being paraconsistent, i.e. FDE. Had Jeffrey not added the two exceptional cases that violated the covering criterion in order that his coupled tree method validates all classical inference, he would have arrived at the same system as Anderson and Belnap. A closer look at the cases added by Jeffry will make the point.

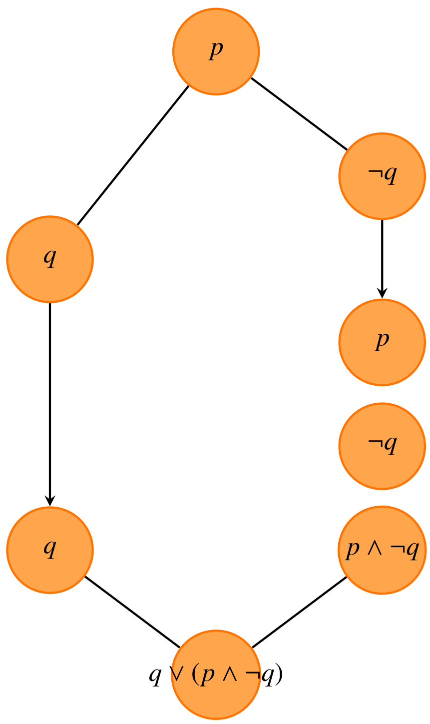
Classical inferences such as $p \wedge \neg p \vdash q$ and $q \vdash p \vee \neg p$ are invalid on the covering criterion. So Jeffrey contrived two exceptional cases where they could be valid outside that criterion. To allow for the first one, Jeffrey added to his coupled tree a closed path, i.e. a branch that has both p and $\neg p$ in

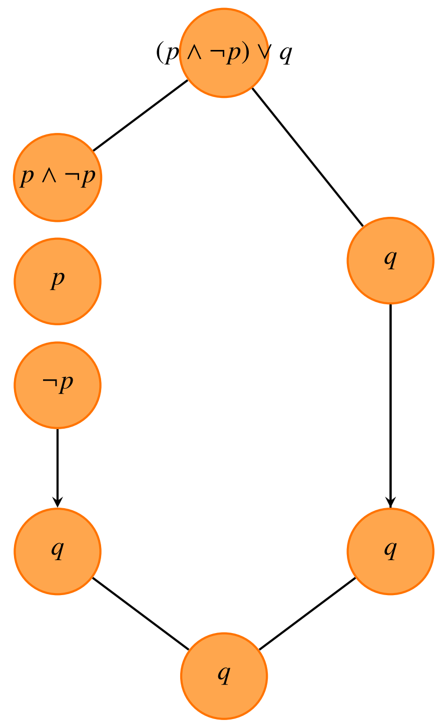
[2]See footnote 6 of chapter 1.

[3]This symbol is used to denote the *standard* DNF of α, as the dual of the standard CNF of α in definition 26.

[4]To see the point, simply examine the connection between (1.3) and (1.4).

it, from which a branch with any sentence q can be drawn; to allow for the second inference, Jeffrey introduced a notion of 'punt', i.e. the generation from any branch of two split branches, one with p, the other with $\neg p$. The exceptional cases for Jeffrey's coupled tree method are best illustrated by diagrams. Below there are two of them representing the argument $p \vdash q \vee (p \wedge \neg q)$ (left) and $(p \wedge \neg p) \vee q \vdash q$ (right) corresponding to the two cases[5]:





It is clear how they violate the covering criterion. The removal of the two cases from coupled tree semantics, leaving us only with the covering criterion, yields the system of first degree entailment. Dunn, inspired by the consequence of removing the two exceptional cases, came up with his own covering criterion, which he called *the relevance criterion*, allowing for the validity of those inferences that exactly constitute FDE. The covering criterion standing on its own, as we have seen in the last subsection, corresponds to the dual of subsumption, which is a member of the family $\mathscr{R}$ of articular relations discussed in the second chapter. However, the covering criterion represented as a relation between two simple hypergraphs[6] has implications that stretch beyond Dunn's predictions.

[5]The arrows represent the covering relation. The left diagram represents a case of punt whereas the right a case of closed path.

[6]See 3.1.

We will give a detailed explication of this point after taking a closer look at situation semantics in the next subsection.

4.1.2 Situation semantics

The relevance criterion defines validity for first degree entailment in a situation model, a semantic idiom that we shall look more closely in the subsequent text. Its approach toward paraconsistency is somewhat different from hypergraph semantics, especially with respect to their treatment of higher degree entailments. These differences are reflections of their distinct understandings of paraconsistency. Though both semantic idioms allow us to arrive at one and the same system in the first degree case, in their more ambitious attempts toward paraconsistent entailment of all degrees, it becomes clear that they issue from undeniably distinct understandings of entailment . Their disagreements are manifested in numerous ways. In the next subsection, after we are furnished with necessary materials on situation semantics, I will give special attention to one particular point that betrays this difference of understanding, i.e. the contrast of the station given first degree entailment in the two semantic approaches. Roughly speaking, in the one case it is a byproduct of semantic modeling for a relevance logic; in the other it serves as a free-standing base system for a family of articular systems.

The definitive status of first degree entailment in hypergraph semantics is largely due to some properties of hypergraph relations. As entailments are represented by relations between hypergraphs, the *degree* of entailments corresponds to the order of hypergraph relations. However, the degree of entailment is not linked to the degree of formulae defined in the second chapter[7], as we shall see later with more precise definitions. Entailment degree is a metatheoretic property whereas the degree of a propositional formula is a property of its syntax. And the higher the degree, the more obvious the discrepancy. Hence the eventual achievement of a complete representation of entailment in relational terms will tell a story about paraconsistency distinct from the well-known system *E* of entailment.

We start the story through a detailed examination of their respective treatment of contradiction, the topic that lies at the very heart of of paraconsistency. Here we can see some similarity of attitudes

[7]See definition 6.

between the two approaches, as well as the burgeoning of dissent in the respective understandings of entailment .

The goal in the two cases is to make contradictions and tautologies semantically contingent. We here omit discussion of tautologies on the assumption that their treatment can be readily inferred from our discussion of contradictions. Classical truth-functional semantics assigns the same truth value to all contradictions. As a consequence, one contradiction, for example, $\alpha \wedge \neg\alpha$ is indistinguishable from another, $\beta \wedge \neg\beta$. In this realization we are not alone. As Dunn made explicit in [17]:

> The question bluntly then is whether the condition that p is true and p is false is the same condition as that q is true and q is false. I think it is not.

A hypergraph representation restores contradiction to its orthographical variety. This approach also rejects a single universal representation of all contradictions. Dunn [17] proposed a slight modification to the classical notion of proposition as a characteristic function, by introducing a notion of situation to the standard model. A situation need not be realizable; thus the addition of unrealizable situations expands the interpretive power of the model. A single situation may simultaneously admit some contradictions that are true and others that are false. And a contradiction true in one situation may be false in another. Contradictions are therefore distinguished from one another by situations, and the more situations there are, the more contradictions we can distinguish.

Both the hypergraph and the situation account revise the notion of proposition. Dunn identifies a proposition with a relation from a set of situations into $\{\mathbf{T}, \mathbf{F}\}$. A contradictory proposition is therefore a relation such that $\mathbf{F}$ is in the image of every situation. On this account, we can distinguish two contradictions in one situation when one of them has only one $\mathbf{F}$ in its image and the other has two. However, within one situation, we cannot distinguish more than two contradictory propositions because the image set has at most two elements and there are only two truth values.

A situation model $\langle K,\ \phi\rangle$ is a pair where K is a non-empty set of situations and ϕ a three-place relation[8] relating sentences, situations, and truth values in a natural recursive manner. The proposition

[8]not function

corresponding to a formula α remains its truth condition, which, as in the classical semantic idiom, associates a situation with the truth value assigned to α. Only what assigns a truth value here is not a function, but a relation. Equivalently we may say that Dunn takes a proposition to be a function from situations to 2^2. Hence, as Dunn observes, the truth condition of $p \wedge \neg p$ and that of $q \wedge \neg q$ are independent. As regards the controversial classical inference $p \wedge \neg p \vdash q$, its validity fails when ϕ assigns $\{\mathbf{T}, \mathbf{F}\}$ to p and $\{\mathbf{F}\}$ to q.

As Dunn was aware, the valuation of first degree entailments in a situation model functions in the same way as that of V in a four-valued model $\mathscr{M} = \langle K, V\rangle$ where K is a non-empty set of situations and V is a valuation function assigning subsets of $\{\mathbf{T}, \mathbf{F}\}$ to formula. The validity of the entailment of β by α is determined by the preservation of $\{\mathbf{T}\}$ from $V(\alpha)$ to $V(\beta)$ at each situation. The set of valid first degree entailments thus corresponds to the set of logical entailments, residing at the top of a hierarchy of validity conditions given by Dunn . Below it are two other levels of validity, viz. universal entailment in a set of situations and entailment in a situation model. α entails β in a situation model $\langle K, \phi\rangle$ iff for all situations in K, if $\mathbf{T}$ is in the set of truth values assigned by ϕ to α, then $\mathbf{T}$ is also in that assigned to β. α universally entails β in a set of situations K iff α entails β in $\langle K, \phi\rangle$ for all situation models. α logically entails β iff α universally entails β in all sets of situations. It is easily seen that among the three levels of validity conditions, only the top level does not require the notion of situation in its semantic modeling. The entailments that are valid on the top level form FDE. The following representation of a four-element lattice that evaluates first degree entailments in situation semantics, originated by Dunn, was also used by Belnap [3] in his four-valued semantics:

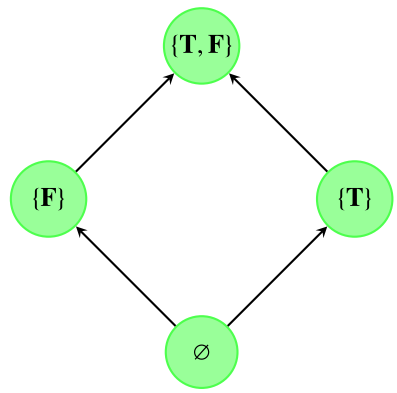


Figure 4.1: How a computer should think?

Brown [7] compared Priest's three-valued semantics of *LP* and Dunn's four-valued semantics of *FDE*. Situation models block such undesirable inferences as $p \wedge \neg p \vdash q$ or $p \vdash q \vee \neg q$ in a way similar to Priest's three-valued logic. Both allow situations in which the undesirable inference fails because there is a **T** on the left hand side of '$\vdash$' but only **F** on the right, a condition which characterizes the criterion of *relevance validity* independently of situations. Given the stipulation that a contradiction must have an **F** in its image for every situation, the whole class of contradictions is thus split into two equivalence classes with respect to a single situation, i.e. those that have both **T** and **F** in the image of the contradiction in that situation and those that have only **F**. A proposition in a situation model is a truth condition , and the truth conditions of at most 2^n contradictions can be distinguished in a situation model $\langle K, \phi \rangle$ where n is the cardinality of K. Our earlier observation that the truth condition of at most two contradictions can be distinguished in one situation can therefore be understood as but an instance of this rule. On the other hand, it may be said of hypergraph semantics that a proposition is a hypergraph representation, and different contradictions can be represented by different hypergraphs.

Taking propositions as truth conditions limits the number of contradictions that can be distinguished in a model, but it also gives an advantage of situation semantics over hypergraph semantics. It allows a situation model to work for higher degree entailment, thus solving the problem of first and higher degree systems in one stroke. This is because it keeps a universal standard for entailments of all degrees, i.e. the preservation of **T** over logical entailment. An important point that I hope to make by comparing the two semantic idioms in the first degree case is to see the scope and limits of hypergraph representation in application to higher degree entailments. The hope of progress in hypergraph semantics should lie in a language-sensitive representation whose representational power can be universally applied to entailment of all degrees. This is a question beyond the scope of the research presented in this thesis, but some hints will be provided later this chapter.

4.1.3 Proposition as truth condition ?

The differences manifested in the way hypergraph and situation semantics treat FDE is the epitome of a deeper bifurcation between two formal understandings of 'meaning'. As Dunn remarked, the proposition expressed by a sentence in situation models is equated with its truth condition, which in this case is a relation and not a function. Therefore situation semantics does not espouse determinate bivalence, consequently, the logic it gives is non-classical. Although two contradictions may have distinct propositions in one situation, the situation still recognizes the meaning of a sentence as its truth condition. The grasp of meaning is the grasp of the truth condition. However, like other semantics that share this meaning theory, it does not give an explanation of this understanding or grasp of truth. Michael Dummett [14] was the first person to formulate a challenge to this theory of meaning, which he called the 'manifestation challenge'. It was later thus summed up by him [15]:

> ... [the meaning theory as described above] forces so large a gap between what makes a statement true and that on the basis of which we are able to recognize it as true, the theory has difficulty in explaining how we derive our grasp of the latter from a knowledge of the former.

This view was endorsed by Crispin Wright [53]:

> Could that knowledge [of truth conditions] consist... in any ability whose proper exercise is tied to appreciable situations?

Neil Tennant [51], though disputing Dummett's manifestation argument for anti-realism, acknowledged the efficiency of the challenge against the notion of recognition-transcendent truth. If logic, so to speak, is justified by a semantics, and the semantics is justified by a meaning theory, then one cannot avoid the Dummett's question as to how the meaning theory is to be justified. On Dummett's account, a meaning theory is judged to be successful accordingly as it provides or does not provide us with a satisfactory explanation of what it is to understand a language. Dummett's expression, 'Understanding a language' was intended to suggest that a genuine account of meaning, apart from giving a truth condition that says a sentence is true or false under a certain condition, must provide us with an understanding of what it is to know that a sentence is true or false. On the other hand, an unsatisfactory meaning theory where truth is central has the feature that grasp of truth conditions is not explained in terms of any more fundamental notion: we are just told that to understand the meaning is to understand the truth conditions without explaining what it is for a sentence to be true [39]. A full-blooded[9] theory should not only offer a genuine account of meaning in terms of understanding of the language, but also an explanation of understanding, which does not rely on a prior grasp of concepts such as 'understanding', or 'knowing the truth conditions'.

We are not here engaged in any debate on realism and anti-realism, but we take the challenge and provide a genuine explanation for the understanding of propositions, which does not involve the notion of 'truth'. So instead of asking: what it is for a sentence to be true? we ask more directly, what is the proposition (the hypergraph that represents it) of the sentence that tells us what it means? The meaning of a sentence, by our account, is richer than its truth condition. Our semantics is therefore spared the embarrassment of a theory of meaning without being accompanied by a genuine truth-theory, as happens to those where truth is the central notion to the understanding of a proposition. It echoes a point we made in the last subsection, that a proposition for an entailment in hypergraph semantics has a relational character, because entailment is a relation.

4.2 Toward a general representation of entailment

At the end of the first subsection in this chapter, we remarked upon the representational potential of hypergraph semantics. We have investigated situation semantics. Through it we gained a more detailed understanding of the essential status of first degree entailment as a base system for hypergraph representation of entailment. Unlike situation semantics, it does not have a universal account of proposition for all formulae. The requirement that consistently holds throughout the degree of formulae is for entailment to be a relation. As first degree entailments are relations between hypergraphs, second degree entailments should be relations of relations between hypergraphs, and so

[9]The terminology is Dummett's.

on and so forth. The degree of an entailment, by this calculation, is the order of the relation representing it, a notion which is different from the degree of a formula.[10] An entailment, so long as it is represented as a hypergraph, can be substituted for an atom; therefore we can conduct a kind of uniform substitution along this line of thinking, free of degrees, that generates a system beyond the scope of the language. The result of such a substitution, in FDE for example, i.e. the set of binary formulae that are valid in the class of articular models, are higher-degree (binary) formulae that remain first degree entailments.

Entailments between formulae of mixed degrees are represented as relations between representations, but their corresponding degrees cannot be determined according to the highest degree of the relata as the degree of formulae.[11] Our ultimate goal, which is not fulfilled within the scope of this thesis, is to give a thorough discussion of the representational approach to entailment in general.[12]

This goal gives the theme for all the discussions of entailments in this thesis. For this reason, before we enter into the more specific discussion of first degree entailment with free substitution, I will clarify a few things about the theoretical motives behind this goal in such a way that it may help us to get a picture of the more complete structure of our project of entailment in general, of which we have only explored a small fraction. First degree entailment (FDE) is generally regarded, and rightfully so considering its historical background, as the first degree fragment of the system E. Naturally, given this connexion, we think it a reasonable expectation to carry out the representational approach we have applied to FDE throughout the system E. But we cannot stop here. For various good reasons due to our understanding of entailments as relations of changing order, this expectation cannot be satisfied by one stroke of generalization, leading to an understanding of the whole project of E in light of hypergraphs and their orderings. Historically, Anderson and Belnap did not carry out their constructive project of interpreting a formula through its normal form. For them, though the standard of relevance and necessity was satisfied by E as a system of entailment, it was not achieved, as in the first degree case, by imposing the relevance condition on the necessary components of the constituent formulae, but by other non-constructive means.

[10] See definition 6.

[11] It may be best illustrated by an example. $p \vdash p \vee (q \rightarrow (r \rightarrow s))$ is still a first degree entailment, but of mixed-degree formulae.

[12] Abstracts of later research can be found at the very end of the thesis.

E, as is well known, consists of R and $S4$, which respectively provided relevance and necessity. Both R and $S4$ were axiomatised in various ways before E. Since the axiomatisation by Moh (1950) and Church (1951) of the positive implicational fragment of R, a Henkin proof was given for an axiomatisation of R by Urquhart in [52] with a semantics using information models. Since higher degree formulae, except in the classical case, cannot be represented as normal forms, their contribution to a semantic understanding of E is confined to the first degree fragment. This, however, is not the case with the generalized representation we introduced. A higher degree system of entailment, is undeniably a system that involves embedded entailment, understood consistently as in the first degree case as a relation that satisfies the properties of relevance and necessity. A genuine system, therefore, should require an understanding of higher degree entailment as the embedding of relations, rather than the embedding of operators "→"; and any attempt to give a semantics of entailment as if it were a kind of logical operator would be a compromise of this requirement.

4.2.1 Free substitution of FDE

FDE as the first degree fragment of E admits of various semantic modelings. Apart from the four-valued semantics given by Dunn, there are also Routley's possible worlds semantics, Anderson and Belnap's constructive semantics mentioned in the first chapter, and so on. These semantics, in contrast with hypergraph semantics, have a trait in common: with each newly introduced element in the language, the interpretive expansions depend upon the introduction of new truth conditions. Take situation semantics for example. The validity of $p \vdash p \vee (q \rightarrow r)$ cannot be determined by the situation model $\langle K, \phi \rangle$ that validates $p \vdash p \vee q$, unless ϕ is extended recursively to interpret $q \rightarrow r$ along with $\neg q$, $p \wedge q$ and $p \vee q$. That is, ϕ relates an arbitrary conditional $\alpha \rightarrow \beta$ to one of the subsets of $\{\mathbf{T}, \mathbf{F}\}$ in a situation. The satisfaction of such a condition would qualify a situation model as a semantic model for some higher degree entailments. Hence, the validity of higher degree substitutional instances of first degree entailments can only be determined in a new semantic model for higher degree entailment. This, however, is not the case for hypergraph semantics.

As we know from 4.1.3, hypergraphs interpret sentences without reference to truth conditions. In this sense of interpretation, the proposition of a sentence *is* its corresponding hypergraph. This feature of hypergraph semantics permits the introduction of the language of proposition independent of the language of truth condition. The valuation function $\mathbf{H}(\cdot)$ in an articular model $\mathscr{M} = \langle U, \mathbb{H}, \mathbf{H} \rangle$, introduced in the second chapter, extends $\mathbf{H}$ to Φ by hypergraph operations in a recursive manner. The positive fragment of FDE corresponds to the hypergraph lattice, $\langle \mathbb{H}, \sqsubseteq \rangle$ while negation is interpreted in terms of functions on hypergraphs.[13] As it happens, any combination of the language, be it an atom or an entailment of degree n, that is interpreted as a hypergraph of some sort, is an element in the lattice. Thus by representing formulae as hypergraphs, binary articular logic with subsumption serves as a base system for the interpretation of entailments.

4.3 Entailment

Entailment in the binary articular logics was represented as a relation between simple hypergraphs. It can be summed up in a simple statement to the effect that α entails β if and only if every (hyper)edge of H_β stands in an ordering relation to some (hyper)edge of H_α. Adding entailment to our language, $\alpha \to \beta$ (read as α entails β) acquires the representation such that each necessary component of the entailment of β from α can be represented as a collection of pairs of edges $\langle E^i_\alpha, E^j_\beta \rangle$ with a fixed j to signify a particular β-edge whereas i ranges over the entire set of α-edges. Collections as such with E^j_β ranging over the entire set of β-edges constitute the representation of $\alpha \to \beta$. Suppose there are m edges of H_α and n edges of H_β, then the representation of $\alpha \to \beta$ can be written as the set

$$\{\{\langle E^1_\alpha, E^1_\beta \rangle, \ldots, \langle E^m_\alpha, E^1_\beta \rangle\}, \ldots \{\langle E^1_\alpha, E^n_\beta \rangle, \ldots, \langle E^m_\alpha, E^n_\beta \rangle\}\}$$

which is a hypergraph $H = (R, E)$ built on a certain relation R between H_α and H_β. We call it a hypergraph on $\wp(R)$, according to definition 4. Each edge of the hypergraph is a sub-relation of the relation R. The dual of this construction is the set of projections from H_β to H_α:

$$\tau(H_{\alpha\to\beta}) = \{f \mid H_\beta \xrightarrow{f} H_\alpha\}.$$

Therefore $H_\alpha \sqsubseteq H_\beta$ is a property of $H_{\alpha\to\beta}$ given that subsumption is interpreted in the general way

[13]The interpretation of negation given in the second chapter, $H_{\neg\alpha} = \{\overline{[B_i]} \mid B_i \in \tau(H_\alpha)$ is based on the fact that the vertices of hypergraphs involved in the definition are sets. It can be thus generalized: suppose H_α is a simple hypergraph on an arbitrary set X, $H_{\neg\alpha} = \{f[B_i] \mid B_i \in \tau(H_\alpha)\}$ such that $\forall a, b \in B_i$, $f^2(a) = a$ and $a \leqslant b \Rightarrow f(a) \geqslant f(b)$, i.e. $f[B_i]$ forms a DeMorgan lattice. See footnote 8 in chapter 2 for the reading we adopt of $f[B_i]$.

as defined in (Sub) on page 38. To repeat it here for the convenience of the reader,

$$H_\alpha \sqsubseteq H_\beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B \textit{ such that } a \leqslant b. \tag{Sub}$$

where '$\leqslant$' is an arbitrary partial ordering. Now with ordered pairs as vertices of some hypergraph, we give a specific parsing of (Sub) where the partial ordering is subsumption all the way down:

$$\alpha \models \beta \Leftrightarrow \forall B \in H_\beta, \exists A \in H_\alpha : \forall a \in A, \exists b \in B \textit{ such that } a \sqsubseteq b.$$

The reasons that motivate this specification are technical, largely due to the understanding of ordered pairs as sets with inner structures. For example, $\langle a,\ b\rangle$ is understood as the higher order set $\{\{a, b\}, a\}$. As we know, subsumption satisfies the three structural rules. For two sets S and x not necessarily of the same order,

Ref $x \in S \Rightarrow S \sqsubseteq \{x\}$;

Mon $S \leqslant S',\ S \sqsubseteq x \Rightarrow S' \sqsubseteq x$;

Cut $S \cup S' \sqsubseteq x,\ S \sqsubseteq S' \Rightarrow S \sqsubseteq x$.

Thus we can work out the entailment layer by layer, subsumption by subsumption, until we arrive at a clear set-theoretic inequality. The definition is incomplete without the qualification that in the extreme case, when a and b are sets, $a \sqsubseteq b$ amounts to '$a \supseteq b$'. This is so because we can see that the inner clause of (Sub): $\forall a \in A, \exists b \in B$ *such that* $a \leqslant b$ represents a partial ordering, which in the extreme case is the identity relation. It is routine work to verify that the following binary formulae of higher-degree entailments are valid under the interpretation of hypergraphs on relation.

1. $p \to q \vdash p \wedge r \to q$;
2. $(p \to q) \wedge (p \to r) \dashv\vdash (p \to q \wedge r)$;
3. $(p \vee q \to r) \vdash (p \to r) \wedge (q \to r)$.

Applying the same technique as many times as necessary, we will find that the generalized subsumption finally comes down to an ordinary subsumption of the representations of some zero-degree formulae, as in FDE. Here again it reinforces the idea that FDE serves as a free-standing base system for entailment.

4.3.1 A simple Baconian experiment

What we shall present is a Baconian experiment, that is, a contrived experience intended to enlarge our knowledge and satisfy our curiosity regarding what goes on. Though in the case of entailment it is more creative play than advancement of learning, there are some interesting features worth recording. $FDEM$, as mentioned in the third chapter, bears a family resemblance to the first degree fragment of NR, one of the modal systems built upon the relevance logic system R [33]. Presented in the axiomatic manner, the first degree modal fragment of NR is the same as $FDEM$ where entailment is a relation between two semantic representations. However, in R, $\rightarrow$ is an operator, as is $\wedge$ and $\vee$. In this sense we are more inclined to call it an implication than an entailment. Implication is the reading of a logical operator interpreted as a function, that like any other functions we meet in semantics, it takes propositions as inputs, be it hypergraphs or truth-sets, and generates outputs of the same sort. To facilitate further comparison, we must define $\alpha \rightarrow \beta$ in the language of FDE.

We interpret $\rightarrow$ as the following binary function taking hypergraphs as inputs and generating a hypergraph as output,

$$H_{\alpha\rightarrow\beta} = H_{\neg\alpha} \cup H_{\beta}.$$

Then the following R-theorems are valid:

W_I1 $p \rightarrow p$;

W_I2 $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$;

W_I3 $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$;

W_I4 $(p \rightarrow q) \rightarrow (r \rightarrow p) \rightarrow (r \rightarrow q)$.

The axioms of R are presented in the manner of Alonzo Church [12]. For the sake of convenience, we refer to the collection of all the binary theorems of FDE involving '→' as $FDE^{\#}$ (read FDE sharp). It is not difficult to detect the material nature of the → given, along with which come the following CL-theorems[14]:

$$p \to (q \to p) \tag{II}$$

$$\neg p \to (p \to q) \tag{III}$$

II, III are theorems of $FDE^{\#}$ of which R is also a subsystem. Now it may be tempting to think that we are taking a detour via $FDE^{\#}$ to classical logic, but a moment's reflection reveals that the following two well-formed formulae are disqualified as theorems of FDE, and thereby that of $FDE^{\#}$:

$$p \wedge \neg p \to q \tag{IV}$$

$$p \to (q \vee \neg q) \tag{V}$$

Note that in FDE [9], we do not have $\top$ or $\bot$ in the language, for their hypergraph representations are excluded by the definition of hypergraph in this thesis.[15]

It can be demonstrated that $FDE^{\#}$ as a binary logic has the following principles:

Materiality $p \to q \dashv\vdash \neg p \vee q$;

RC $((p \to q) \wedge (p \to r)) \dashv\vdash (p \to (q \wedge r))$;

LD $(p \to r) \wedge (q \to r) \dashv\vdash (p \vee q \to r)$.

[14] CL refers to classical logic.

[15] See definition 4.

together with the rule of adjunction.

Since it is not in a new language that we have worked so far, having simply introduced a derivative operator, the following result is implicated in the completeness proof of FDE:

Lemma 10. *In the canonical model* $\mathscr{M}^*$, $H^*_{\alpha\to\beta} \sqsubseteq H^*_{\gamma\to\delta} \Rightarrow \alpha \to \beta \vdash \gamma \to \delta$.

Proof. See Appendix. □

4.4 Abstracts of papers for later research

4.4.1 Abstract for *Two Semantic Analyses of a Logic of Entailments*

FDE can be interpreted by a semantics entirely based on the properties of hypergraphs. Its status as a free-standing base system for logics of all entailments can be rendered even clearer by the two semantic extensions given in this paper, treating entailments as hypergraphs on $\wp\wp(U)\times\wp\wp(U)$[16]. It is an interpretation that semantically combines two essential properties of entailment: a hypergraph and a relation (on $\wp\wp\wp(U)$). Two ways of representing entailments as hypergraphs on relations are to be introduced, and an entailment on this account is represented either as a collection of collections of ordered pairs of subsets of $\wp(U)$, or as a collection of projections from the hypergraph representing the consequent to that representing the antecedent, that is, as a subset of $(H')^H$ where both H and H' are collections of collections of subsets of U. It shall be demonstrated that the two treatments of entailments are dual to each other, leading to an understanding of higher-degree entailments embodied in systems of entailment as sublogics of E. Three systems emerge accordingly as we use '$\leqslant$', '$\geqslant$' and '$=$' as the ordering on the vertex level[17]. A table is made to better comparison. An entailment $\alpha \to \beta$ is valid if and only if one of the projections from H_β to H_α, i.e. a member of $(H_\alpha)^{H_\beta}$ in the hypergraph representing the entailment, is a generalized subsumption.

[16]Here we use the definition of a hypergraph H on a set S in the sense different from that in definition 4. According to this definition, a hypergraph H on a set S is a finite family of subsets of S. It is more general as a definition and we shall from now on refer to this definition when we talk about hypergraphs on sets.

[17]See (Sub) on page 38.

4.4.2 Abstract for *A Modal Logic of Entailment*

A logic of entailment can be extended to include modality as we represent $\Box\alpha$ as the union of hypergraphs entailed by a primordial hypergraph. In this paper we will introduce a new method of modalizing logics of entailment. Just as the various systems of normal modal logics are modalizations of the propositional logic, the modal systems thus obtained are modalizations of entailment logics. Our main purpose here is to demonstrate the process of obtaining various modal systems from their corresponding entailment logics. A case can be made in the familiar territory of $S4$. It will be shown that $S4$ is the extension of a logic of entailment with the principles $(\alpha \to \beta) \to (\alpha \to (\alpha \to \beta))$ and $((\alpha \to \alpha) \to \beta) \to \beta$. Also an entailment logic with the principle of contraction $(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$ can be extended to include the modal principle T^* $(\Box\Box\alpha \to \Box\alpha)$.

Appendix A

Appendices, sectioning

Appendices appear in the Contents table on the level of chapters and are numbered starting with A. 1.

A.1 Proof of fundamental theorem

Fundamental Theorem: $\forall \alpha \in \Phi$, $H^*_\alpha = \{|[\Delta_i]| \mid \Delta_i \in \mathbb{CNF}\,(\alpha), 1 \leqslant i \leqslant n\}$.

Proof strategy: The proof that follows is notationally cumbrous, but the idea is simple. The deformulated set $\mathbb{CNF}\,(\alpha)$ corresponding to the standard CNF of α is a hypergraph on the union of $Lit(\alpha)$, the set of literal pairs[1] in the language of α,[2] and what we want to demonstrate here is that the hypergraph H^*_α, i.e. the representation of α in the canonical model, is structurally the same as $\mathbb{CNF}\,(\alpha)$, in fact it is exactly the same as $\mathbb{CNF}\,(\alpha)$ except that in place of each literal p_i, we put its proof set $|p_i|$ instead. Therefore H^*_α is a hypergraph on the set of proof sets of literals based on the language of α, where the vertices are all proof sets of literals and edges are collections of proof sets of literals.

Proof. By induction on the length of α.

[1] S is a set of sets s, the union of the set S, $\cup S$, is the union of its element sets, i.e. $\cup S = \{s' | s' \in s$ for some $s\}$.

[2] For the precise definition of conceptions as hypergraph on a set and the set of literal pairs in a language, please see definitions 4, 21 and 22.

Basis: $\alpha \;=\; p_i$
then $\mathbb{CNF}\ (\alpha) = \{\{p_i\}\}$
By the definition of the canonical model, $H^*_{p_i} \;=\; V^*(p_i) \;=\; \{\{|p_i|\}\}$.

Assume that the proposition holds for all α of length strictly less than k. Then to prove that it holds for α of length k, we have three subcases to consider.

[1] α is of the form $\neg\beta$.
$\forall B_i \in H^*_\beta, B_i \;=\; \{|\delta_1|,\ |\delta_2|, \ldots, |\delta_{n_i}|\}$.
And $\forall A_i \in H^*_{\neg\beta}, A_i \;=\; \overline{[B_i]}$ where $B_i \in \tau(H^*_\alpha)$.
Therefore, $A_i \;=\; \{\overline{|\delta_1|},\ \overline{|\delta_2|}, \ldots, \overline{|\delta_{n_i}|}\}$
So $\forall A_i \in H^*_{\neg\beta}, A_i \;=\; \{|\neg\delta_1|,\ |\neg\delta_2|, \ldots, |\neg\delta_{n_i}|\}$.
By the hypothesis of induction (HI), $CNF(\beta) \;=\; \bigwedge_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee_{j=1}^{m} \delta_j$;
Then by axioms 1, 2, 3, $CNF(\alpha) \;=\; \bigwedge_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigvee_{j=1}^{m} \neg\delta_j$.
Henceforth, $\forall i \in I,\ H^*_{\neg\beta} \;=\; \{\{|\delta_1|,\ |\delta_2|, \ldots, |\delta_{n_i}|\} \mid \delta_j \in \Delta_i\ \&\ \Delta_i \in \mathbb{CNF}\ (\neg\beta)\}$.

[2] α is of the form $\beta \vee \gamma$.
By HI, $H^*_\beta \;=\; \{|\delta_1|,\ |\delta_2|, \ldots, |\delta_{n_i}|\} \mid \delta_g \in \Delta_i\ \&\ \Delta_i \in \mathbb{CNF}\ (\beta),\ 1 \leqslant i \leqslant m\}$
And $H^*_\gamma \;=\; \{|\theta_1|,\ |\theta_2|, \ldots, |\theta_{h_j}|\} \,|\theta_k \in \Theta_j\ \&\ \Theta_j \in \mathbb{CNF}\ (\gamma),\ 1 \leqslant j \leqslant l\}$.
But $H_{\alpha\vee\beta} = \ \{\{a,\ b\} \mid a \in H_\alpha,\ b \in H_\beta\}$.
If $CNF(\beta) \;=\; \bigwedge_{i=1}^{m} \Delta_i$ where $\Delta_i = \bigvee_{g=1}^{n_i} \delta_g$ and $CNF(\gamma) \;=\; \bigwedge_{j=1}^{l} \Theta_j$ where $\Theta_j = \bigvee_{k=1}^{h_j} \theta_k$,
then by axioms 4, 6 and [RC], $CNF(\beta \vee \gamma) \;=\; \bigwedge_{s=1}^{lm} \Sigma_s$ where $\Sigma_s = \bigvee_{f=1}^{n_i+h_j} (\delta_g \vee \theta_k)_f$.
Therefore, $H^*_{\beta\vee\gamma} \;=\; \{\{|\delta_1 \vee \theta_1|, \ldots,\ |\delta_g \vee \theta_k|, \ldots, |\delta_{n_i} \vee \theta_{h_j}|\} \mid \delta_g \vee \theta_k \in \Sigma_s\ \&\ \Sigma_s \in \mathbb{CNF}\ (\beta \vee \gamma),\ 1 \leqslant s \leqslant lm\}$.

[3] α is of the form $\beta \wedge \gamma$.
By HI, $H^*_\beta \;=\; \{|\delta_1|,\ |\delta_2|, \ldots, |\delta_{n_i}|\} \mid \delta_g \in \Delta_i\ \&\ \Delta_i \in \mathbb{CNF}\ (\beta),\ 1 \leqslant i \leqslant m\}$
And $H^*_\gamma \;=\; \{|\theta_1|,\ |\theta_2|, \ldots, |\theta_{h_j}|\} \,|\theta_k \in \Theta_j\ \&\ \Theta_j \in \mathbb{CNF}\ (\gamma),\ 1 \leqslant j \leqslant l\}$.
But $H_{\alpha\wedge\beta} = \ H_\alpha \cup\ H_\beta$.

If $CNF(\beta) = \bigwedge_{i=1}^{m} \Delta_i$ where $\Delta_i = \bigvee_{g=1}^{n_i} \delta_g$ and $CNF(\gamma) = \bigwedge_{j=1}^{l} \Theta_j$ where $\Theta_j = \bigvee_{k=1}^{h_j} \theta_k$,
then by [Mon] and [RC], $CNF(\beta \wedge \gamma) = \bigwedge_{i=1}^{l+m} \Sigma_s$ where Σ_s is either Δ_i or Θ_j.
Therefore, $H^*_{\beta\wedge\gamma} = \{\{|\epsilon_1|, |\epsilon_2|, \ldots, |\epsilon_{t_s}|\} \mid |\epsilon_{t_s}| \in \Sigma_s, 1 \leqslant t_s \leqslant max\ (n_i, h_j)\ \&\ \Sigma_s \in \mathbb{CNF}\ (\beta \wedge \gamma), 1 \leqslant s \leqslant l + m\}$.

This concludes the proof of the fundamental theorem. From now on, we use a special notion to denote the truth representation of α in the canonical model, the articular set of α. □

A.2 Proof of completeness

A.2.1 *FDE*

$\alpha \vDash_{FDE} \beta \Rightarrow \alpha \vdash_{FDE} \beta$

Proof. Assume $\alpha \vDash \beta$, then $H^*_\alpha \sqsubseteq H^*_\beta$.
By the Fundamental Theorem, we have $H^*_\alpha = \{\{|\sigma_1|, |\sigma_2|, \ldots, |\sigma_{n_i}|\} \mid \sigma_k \in \Sigma_i\ \&\ \Sigma_i \in \mathbb{CNF}\ (\alpha), i \in I\}$ and $H^*_\beta = \{\{|\delta_1|, |\delta_2|, \ldots, |\delta_{m_j}|\} \mid \delta_l \in \Delta_j\ \&\ \Delta_j \in \mathbb{CNF}\ (\beta), j \in J\}$.
Therefore, $H^*_\alpha \sqsubseteq H^*_\beta$ says to the effect that $\forall j \in J$, $\exists i \in I$ such that $\Sigma_i \subseteq \Delta_j$.
By axiom 6 it is obtained that $\forall j \in J$, $\exists i \in I$ such that $\bigvee_{k=1}^{n_i} \sigma_k \vdash \bigvee_{l=1}^{m_j} \delta_l$.
By [Mon], we have $\forall j \in J$, $CNF(\alpha) \vdash \bigvee_{l=1}^{m_j} \delta_l$.
By [RC], $CNF(\alpha) \vdash CNF(\beta)$
Then by the Representation Theorem, $\alpha \vdash \beta$. Therefore, if $\alpha \vDash \beta$, then $\alpha \vdash \beta$, i.e. *FDE* is complete with respect to the class of articular models with subsumption. □

In the following each metatheorem applies to the system of the corresponding subsection.

A.2.2 Supersumption

$\alpha \vDash_S \beta \Rightarrow \alpha \vdash_S \beta$

In this particular system, DNF plays the role that was previously played by CNF for FDE. The situation can be thus summarized in simple terms: disjunction obeys the rules of conjunction and conjunction disjunction, so both the Representation Theorem and the completeness proof are mirror images of those for FDE.

Representation Theorem Every formula of PL is provably equivalent with its standard DNF.

Proof. We proceed by mathematical induction on the length of α.
Basis: $\alpha \ = \ p_i$.
By [Mon], $\alpha \ \vdash p_i$.

Assume that the proposition holds for all α of length strictly less than k, then to prove that it holds for α of length k, we have three subcases to consider.
[1] α is of the form $\neg\beta$.
By HI, $\beta \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{m} \delta_i$.
There again are three subcases to consider.
(a) Suppose β is of the form $\neg\gamma$.
Then $\alpha \ = \ \neg\neg\gamma$.
By HI, $\gamma \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{m} \delta_i$;
By axioms 3($\vdash$) and [Cut], $\alpha \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{m} \delta_i$.
(b) If β is of the form $\gamma \wedge \eta$, then α is of the form $\neg(\gamma \wedge \eta)$.
Therefore by axiom 1 ($\vdash$), $\alpha \ \vdash \neg\gamma \vee \neg\eta$.
By HI, we have $\neg\gamma \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{m} \delta_i$ and $\neg\eta \ \vdash \bigvee_{i=l}^{l} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{m} \delta_i$;
Then by axioms 6, [RD] and [Cut], $\alpha \ \vdash \bigvee_{i=1}^{n} \Delta_i \vee \bigvee_{i=1}^{l} \Delta_i$.
Therefore $\alpha \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{m} \delta_i$.
(c) The same argument applies when β is of the form $\gamma \vee \eta$.

[2] α is of the form $\beta \wedge \gamma$.
By HI, $\beta \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{n} \delta_i$ and $\gamma \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{m} \delta_i$.
By By repeated applications of the theorem suggested in the footnote on page 29, i.e.$p \wedge (q \vee r) \ \vdash (p \wedge q) \vee (p \wedge r)$ and [Cut], we obtain $\alpha \ \vdash \bigvee_{i=1}^{n} \Delta_i$ where $\Delta_i = \bigwedge_{i=1}^{l} \delta_i$.

[3] α is of the form $\beta \vee \gamma$.

The argument goes in a similar way to that of (b) for the first subcase.
It is easily seen that the other direction of the proposition can be proved accordingly. Hence, $\forall\alpha \in \Phi$, the representation theorem holds. $\square$

The proof of completeness is omitted, for it can be readily inferred from the completeness proof of FDE by uniformly replacing CNF with DNF.

A.2.3 Strict subsumption

For this system as for the next two, subclusion and subgraph, the Representation Theorem fails because of the failure of $\vee$-introduction and $\wedge$-elimination.

A.3 A remark on $FDE^{\#}$

Lemma 11. *In the canonical model $\mathscr{M}^*$, $H^*_{\alpha\to\beta} \sqsubseteq H^*_{\gamma\to\delta} \Rightarrow \alpha \to \beta \vdash \gamma \to \delta$.*

Suppose $\alpha \dashv\vdash \bigvee_{i=1}^{n} E_i$ and $\beta \dashv\vdash \bigwedge_{j=1}^{m} \Lambda_j$, where $E_i = \bigwedge_{k=1}^{k_i} \epsilon_k$ and $\Lambda_j = \bigvee_{l=1}^{l_j} \lambda_l$.

It is straightforward to verify that

$$H^*_{\alpha\to\beta} = \{\{|\epsilon_1 \to \lambda_1|, \ldots, |\epsilon_{k_1} \to \lambda_1|\}, \ldots, \{|\epsilon_{k_i} \to \lambda_{l_j}|, \ldots, |\epsilon_{k_n} \to \lambda_{l_m}|\}\}$$

The same procedure can be applied to γ and δ.

Now suppose $\gamma \dashv\vdash \bigvee_{i'=1}^{n'} Z_{i'}$ and $\delta \dashv\vdash \bigwedge_{j'=1}^{m'} \Theta_{j'}$, where $Z_{i'} = \bigwedge_{k'=1}^{k'_{i'}} \zeta_{k'}$ and $\Theta_{j'} = \bigvee_{l'=1}^{l'_{j'}} \theta_{l'}$.

To avoid using too many letters, we use n', m', k', l', i' and j' instead of n, m, k, l, i and j deliberately to indicate that the disjunctions and conjunctions in both the CNF and DNF representations may be different. The structure of $H^*_{\gamma\to\delta}$ bears strong similarity to that of $H^*_{\alpha\to\beta}$:

$$H^*_{\gamma\to\delta} = \{\{|\zeta_1 \to \theta_1|, \ldots, |\zeta_{k'_1} \to \theta_1|\}, \ldots, \{|\zeta_{k'_{i'}} \to \theta_{l'_{j'}}|, \ldots, |\zeta_{k'_{n'}} \to \theta_{l'_{m'}}|\}\}$$

It is easy to infer from $H^*_{\alpha\to\beta} \sqsubseteq H^*_{\gamma\to\delta}$ that

$\forall e \in H^*_{\gamma\to\delta}, \exists e' \in H^*_{\alpha\to\beta}$,

$$e' = |\epsilon_1 \wedge \ldots \wedge \epsilon_{k_n} \to \lambda_1 \vee \ldots \vee \lambda_{l_m}| \subseteq |\zeta_1 \wedge \ldots \wedge \zeta_{k'_{n'}} \to \theta_1 \vee \ldots \vee \theta_{l'_{m'}}| = e$$

that is,

$$\epsilon_1 \wedge \ldots \wedge \epsilon_{k_n} \to \lambda_1 \vee \ldots \vee \lambda_{l_m} \vdash \zeta_1 \wedge \ldots \wedge \zeta_{k'_{n'}} \to \theta_1 \vee \ldots \vee \theta_{l'_{m'}}.$$

By [Mon] and [RC] it follows that

$$\bigwedge_{i=1}^{n}(\epsilon_1 \wedge \ldots \wedge \epsilon_{k_n} \to \lambda_1 \vee \ldots \vee \lambda_{l_m}) \vdash \bigwedge_{i'=1}^{n'}(\zeta_1 \wedge \ldots \wedge \zeta_{k'_{n'}} \to \theta_1 \vee \ldots \vee \theta_{l'_{m'}})$$

which in $FDE^{\#}$ is equivalent to

$$\bigvee_{i=1}^{n}(\epsilon_1 \wedge \ldots \wedge \epsilon_{k_n}) \to \lambda_1 \vee \ldots \vee \lambda_{l_m} \vdash \bigvee_{i'=1}^{n'}(\zeta_1 \wedge \ldots \wedge \zeta_{k'_{n'}}) \to \theta_1 \vee \ldots \vee \theta_{l'_{m'}}.$$

Again by [Mon] and [RC] it follows that

$$\bigwedge_{j=1}^{m}(\bigvee_{i=1}^{n}(\epsilon_1 \wedge \ldots \wedge \epsilon_{k_n}) \to \lambda_1 \vee \ldots \vee \lambda_{l_m}) \vdash \bigwedge_{j'=1}^{m'}(\bigvee_{i'=1}^{n'}(\zeta_1 \wedge \ldots \wedge \zeta_{k'_{n'}}) \to \theta_1 \vee \ldots \vee \theta_{l'_{m'}})$$

Therefore,

$$\bigvee_{i=1}^{n}(\epsilon_1 \wedge \ldots \wedge \epsilon_{k_n}) \to \bigwedge_{j=1}^{m}(\lambda_1 \vee \ldots \vee \lambda_{l_m}) \vdash \bigvee_{i'=1}^{n'}(\zeta_1 \wedge \ldots \wedge \zeta_{k'_{n'}}) \to \bigwedge_{j'=1}^{m'}(\theta_1 \vee \ldots \vee \theta_{l'_{m'}})$$

which is equivalent to

$$DNF(\alpha) \to CNF(\beta) \vdash DNF(\gamma) \to CNF(\delta)$$

Hence,

$$\alpha \to \beta \vdash \gamma \to \delta$$

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