SCIENTIFIC PATRONAGE AND RIVALRY IN LAPLACE'S EARLY CAREER

by

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Abstract

Before the French Revolution, access into the elite echelon of French society, including academic society, was almost entirely restricted to those of noble birth. Pierre Simon Laplace (1749-1827) was one of the exceptions to this rule; he entered the French scientific community based on his merit as a scientist and his ability to obtain powerful patrons. We will discuss the influence of Laplace's patrons on his personal, professional and scholarly life by looking at primary documents from his early career, mostly on differential equations. While Jean le Rond d'Alembert (1717-1783) was one of Laplace's most influential patrons, he was not the only. We will look at the impact of not only d'Alembert but also the other patrons in Laplace's early life.

Laplace's career was equally shaped by the influence of scientific rivalry. Rivalry can be seen as having positive or negative effects. Positive rivalry can lead an individual to produce work of the highest calibre, in reaction to the work of a competitor. The friendships that Laplace developed with contemporaries, such as Joseph-Louis Lagrange (1736-1813), can be seen as containing rivalries leading to innovative work. As a negative influence, rivalry can lead to priority disputes and result in rifts between those involved. Laplace's early career contained both positive and negative rivalries: both of which we will examine.

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Chapter 1

Introduction

Before the French Revolution, access into the elite echelon of French society, including academic society, was almost entirely restricted to those of noble birth. Pierre Simon Laplace (1749-1827) was one of the exceptions to this rule; he entered the French scientific community based on his merit as a scientist and his ability to obtain powerful patrons.

Laplace arrived in Paris in 1769 and quickly began to make a name for himself. Laplace showed himself to be an adept mathematician, but this was not often enough to guarantee the meteoric rise in position that Laplace experienced. Within five years of moving to Paris, Laplace had obtained employment as professor of mathematics at the École Militaire, been elected to the Académie des Sciences and made himself financially self-sufficient. Laplace himself did not believe the rise was as quick as it should have been, as he perceived himself to have been repeatedly snubbed by the Académie while people who Laplace believed to be less talented and deserving were promoted before himself. With this in mind, when he was elected, he was much younger than the average age of newly elected members. Laplace did have a sudden rise to fame and this can be attributed to the networking that he did just as much as his inherent talents as a mathematician. This thesis will look at the dual influence that early patronage and rivalry played in Laplace's career and how the results of these influences can be seen in the early work that Laplace produced.

Patronage can take many forms and can describe many relationships. The Académie des Sciences was founded by Louis XIV in 1666, and for this reason, early in the history of the Académie, patronage was usually in the form of royal patronage, where the King was the patron of the society and his ministers were in charge of funding, housing and

recruitment.[1] This is not the form of patronage that we will be looking at in this paper. The Académie des Sciences was a means for a scientist to show his merit, but to do this he would first have to submit a paper through an academician, who would then present it to the Académie. While the academician was not required to present the other scientist's work and presentation did not guarantee publication, this route was the one most likely to allow a young scientist to have their work successfully published. Also, even if the report by a committee of academicians recommended publication, this could take years. Overall, before being able to submit a paper, a scientist would have to become acquainted with an member of the Académie who deemed his work worthy of going forward to the Académie. The role that a senior scientist would play in aiding and mentoring a novice scientist is the one that will be looked at in this paper. We will also divide this role into professional patronage and academic patronage. By professional patronage, we mean when a senior scientist assists the junior scientist in finding employment. By academic patronage, we mean the effect that the patron scientist has in guiding the younger academic to study certain areas.

Jean le Rond d'Alembert (1717-1783) was at this time one of the leading figures in the Académie and someone in a position to help Laplace both professionally and academically. As a young man who had been financially cut off by his family due to his sudden decision to discard his path towards a career in the church, Laplace had to set his sights high. He quickly made his introduction to d'Alembert when he first arrived in Paris with a letter from Pierre Le Canu of the University of Caen. D'Alembert had been elected to the Académie in 1741 and had been *sous-directeur* and *directeur* respectively in 1768 and 1769.[2] D'Alembert corresponded with many of the great scientific names of the time such as Leonhard Euler (1707-1783), Gabriel Cramer (1704-1752) and Joseph-Louis Lagrange (1736-1813). This made d'Alembert the perfect choice for both a professional and academic patron to Laplace. He was someone who would be able to provide Laplace with employment and to help him in his goal of becoming a member of the Académie.[3] Laplace, though, was an unknown man

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[1] For more information on the early Académie look at Alice Stroup, *A Company of Scientists: Botany, Patronage, and Community at the Seventeenth-Century Parisian Académie des Sciences*, (Berkley: University of California Press, 1990). http://ark.cdlib.org/ark:/13030/ft587006gh/

[2] Michelle Chapront-Touzé, "D'Alembert (Dalembert), Jean le Rond", in *Dictionary of Scientific Biography* vol. 20, (Detroit: Charles Scribner's Sons, 2009/02 2008), 229

[3] This is somewhat speculative. Hahn argues that Laplace wanted to leave a mark in the mathematical sciences and that a major step was to have his work recognised by the Académie and to gain membership into the group. Roger Hahn *Pierre Simon Laplace, 1749-1827 : a determined scientist*, (Cambridge, Mass.: Harvard University Press, 2005), 37-40

who showed up without invitation and asked d'Alembert for assistance. While Laplace had every reason to seek help from d'Alembert, d'Alembert had absolutely no reason, initially, to help the young man who was soliciting assistance. However, talent prevailed and Laplace was able to prove his worth to the older academician. Professionally, d'Alembert was able to obtain employment for Laplace, and thus allow the recent graduate to be financially able to stay in Paris. D'Alembert also appears to have had an influence on Laplace's area of study, as the young scientist took up topics that had previously been investigated by d'Alembert, such as physical astronomy. During the four year period between Laplace arriving in Paris and becoming elected to the Académie des Sciences, he wrote thirteen papers that were representative of his interests for the rest of his career; one of these interests was physical astronomy, to use d'Alembert's term for what we would now call celestial mechanics. Chapront-Touzé states that

> d'Alembert... names the science which we call celestial mechanics *physical astronomy* as opposed to *astronomy*, which does not seek to establish a link between the observed phenomena and their physical causes.[4]

Laplace later coined the term celestial mechanics when he wrote the five volumes making up the *Mécanique céleste*. During the period (1769-1783) when d'Alembert acted as mentor to and protector of Laplace and his influence over the younger man is apparent in Laplace's work as we will see below; we will therefore retain the term physical astronomy in describing his work in this time period.

We can see that the choice of studying physical astronomy may be linked to d'Alembert for at least two reasons. In studying physical astronomy, Laplace would be able to learn potentially from d'Alembert and for this reason d'Alembert may have suggested it. Also we can see that if Laplace was working in a field in which d'Alembert had previously published work, Laplace would be able to show his gratitude to d'Alembert by looking favourably on his prior researches.

Besides d'Alembert, Laplace had other patrons in Antoine-Laurent Lavoisier (1743-1794) and the people who helped Laplace succeed Étienne Bezout (1739-1783) as examiner of the

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[4]"d'Alembert ... désigne la science que nous appelons mécanique céleste par l'expression *astronomie physique*, en opposition à l'*astro-nomie* qui ne cherche à établir aucun lien entre les phénomènes observés et leur cause physique." Jean le Rond d'Alembert, Michelle Chapront-Touzé, editor *Jean le Rond d'Alembert, Oeuvres complètes: Premiers textes de méchanique celeste 1747-1749*, vol 6 of *Traités et mémoirs mathématique 1736-1756* (Paris: CNRS Editions, 2002) xiii.

artillery. The influence of these people over Laplace's academic and professional pursuits respectively will also be briefly addressed.

The first part of this thesis will address the role of patronage in Laplace's early career. We will investigate this theme mainly by looking at Laplace's early employment at the École militaire and as the examiner of the artillery. The role of examiner involved testing students at the artillery school.

Scientific rivalry can be a negative influence as well as a positive one. On the negative side, we can see the rivalries that developed frequently between d'Alembert and his contemporaries, such as Clairaut, Daniel Bernoulli and even Euler. These rivalries often involved priority and frequently resulted in rifts between those involved. The relationship between these rivalries and Laplace's duties to his patron will be discussed in Chapter 3. As a positive influence, such rivalry can spur competitors on to produce better work, which can lead to new developments. Also, when there is a friendly element to the rivalry, those involved can provide insight into each other's works. This is the case in the rivalry between Laplace and Lagrange. Unlike in the other case, there were rarely public debates between them regarding priority and each was willing to cite the other. Laplace, who rarely cited his predecessors, can be seen as frequently referencing the work of Lagrange, even including parts of their correspondence.

As we shall see in what follows, Lagrange did not have the direct influence on Laplace's early career that d'Alembert had, but due to his immense standing in the academic world he appears to have been still able to shape Laplace's scientific interests. Laplace, taking the initiative as he had with d'Alembert, began and continued a correspondence with Lagrange from 1773 until 1785,[5] which was around the time that Lagrange moved to Paris (making correspondence unnecessary).[6] Although d'Alembert was the main person who was in a position to bring the most immediate benefit to Laplace's career, Laplace still recognised the importance of cultivating a network of powerful friends. Lagrange's first role was that of a potential professional or academic patron for Laplace, but we can see through their correspondence that this relationship soon developed into more of a friendly rivalry. This rivalry can be seen in the published works of Laplace and Lagrange, which often investigated

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[5] Roger Hahn, *Calendar of the Correspondence of Pierre Simon Laplace*, (Berkeley: Office for History of Science and Technology, University of California, Berkeley, 1982), 1-8

[6] Lagrange arrived in Paris in 1786. Since the correspondence was about a letter or so a year at this point, it seems reasonable that this was the reason for the end of their correspondence.

similar subjects. We will investigate the relationship between Laplace and Lagrange by looking at their correspondence and their published works in solving ordinary differential equations, especially in relation to perturbation theory. Here we will look specifically at the time period from 1772 until 1776.

While the rivalry between Laplace and Lagrange was a positive one, not all of Laplace's relations were as harmonious. Arguments similar to those between d'Alembert and his rivals emerged between Laplace and both Condorcet and Legendre. These rivalries will be investigated further in Chapter 8.

The second part of this thesis will discuss the specific influence of rivalry on Laplace's early scientific work. To this end, we will compare the published work of Laplace and Lagrange in the areas of solutions to ordinary differential equations and potential theory as well as looking at the negative rivalries with Condorcet and Legendre.

While Laplace is a well known scientist, there are not many biographies devoted to him. Hahn has written an excellent book on Laplace's life,[7] but this book serves as a general guide to Laplace. Here, we hope to address specifically the roles of patronage and rivalry when looking at Laplace's scientific output. Andoyer has also provided an account of Laplace's life and work in his *L'oeuvre scientifique de Laplace*.[8] While Andoyer did provide a sketch of Laplace's life, his book concentrated more on the scientific output of Laplace. Andoyer also dealt generally with all of Laplace's work while we will concentrate on his early work that relates to physical astronomy.

There is some doubt about many details of the early life of Laplace. Bigourdan has investigated the discrepancies that exist in his article on Laplace's early life.[9] Here Bigourdan looks specifically at Laplace's life and does not include much information on Laplace as a scientist.

The section of the *Dictionary of Scientific Biography* devoted to Laplace is the longest article.[10] In this section, Gillispie, Fox and Grattan-Guinness provide insight into Laplace's life, but concentrate on giving a thorough overview of his scientific work. Here, we will only

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[7]Roger Hahn, *Pierre Simon Laplace, 1749-1827 : a determined scientist,* (Cambridge, Mass.: Harvard University Press, 2005).

[8]H. Andoyer, *L'oeuvre scientifique de Laplace*, (Paris: Payot & Cié., 1922).

[9]G. Bigourdan, "La Jeunesse de P.S. Laplace," in *La Science Moderne*, vol 8:1931.

[10]Charles Coulston Gillispie, Robert Fox, and Ivor Grattan-Guinness, "Laplace, Pierre-Simon, Marquis De," vol 15 of *Complete Dictionary of Scientific Biography* chapter 6, (Detroit: Charles Scribner's Sons, 2008) 273-403.

discuss Laplace's early life and work and hopefully provide greater detail than was possible in the DSB.

The chronology and content of the first thirteen papers that Laplace presented to the Académie before he was elected have been investigated by Stigler.[11] Not all of these memoirs were published, though the unpublished works were often incorporated into later published memoirs. One of these publications was the first memoir which we will investigate, *Sur le principe de la gravitation universelle et sur les inégalités séculaire des planètes qui en dépendent*[12] cited in what follows as *Sur le principe.* Stigler contends that four of Laplace's earlier unpublished works were incorporated into this memoir, which may explain the complicated structure of this paper and its breadth of topics. While physical astronomy was a frequent subject for these first thirteen scientific efforts, they also dealt with other fields notably probability and methods for solving particular problems in differential and integral calculus. In Stigler's investigation, he looks at who Laplace was citing and the frequency of citation. We will concentrate more on the mathematical content of the papers we investigate than on the issue of citations.

While we will only discuss d'Alembert's work in passing, when it is cited by Laplace and others, we note that at present the d'Alembert edition is underway. One volume of this, edited by Chapront-Touzé, has already been mentioned.

While the role of patronage in shaping Laplace's life has not been looked at in general, Guerlac has written on the collaborative work of Lavoisier and Laplace.[13] At present, we will concentrate only on the role of patronage in their relationship and not discuss the chemistry, which is Guerlac's focus. Duveen and Hahn have written a case study on the succession of Laplace to Bezout's post as examiner of the artillery.[14] While this paper investigated the method that a person might use to obtain such a position, we will concentrate on the shift in this situation from an applicant's ability being the key element in obtaining employment to powerful supporters becoming the sole means of securing a job. As well, the biographies

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[11]Stephen M. Stigler, "Laplace's Early Work: Chronology and Citations" published in *Isis* vol. 69, No 2 (Chicago: The University of Chicago Press, 1978) 249-254.

[12]Pierre Simon Laplace, "Sur le principe de la gravitation universelle et sur les inégalités séculaire des planètes qui en dépendent" in *Ouevres de Laplace* vol VIII (Paris: Gautier-Vallier, 1878-1912 [1773]).

[13]Henry Guerlac, "Chemistry as a Branch of Physics: Laplace's Collaboration with Lavoisier," in *Historical studies in the physical sciences* 7:193.

[14]Denis I. Duveen and Roger Hahn, "Laplace's Succession to Bézout's Post of Examinateur des Élèves de l'Artillerie" in *Isis* 28(4):416-427, Dec. 1957.

of Laplace all touch on the subject of patronage, though we hope to confront this issue in greater detail.

Besides the biographies that have already been mentioned, Sarton also briefly investigates the relationship between Lagrange and Laplace.[15] Here Sarton gives an interpretation of Lagrange's opinion of Laplace which appears to be somewhat short lived. We will discuss both Sarton's beliefs and how we have interpreted the documentation that exists regarding their relationship in an attempt to determine its true nature.

While Laplace's contribution to perturbation theory and differential equations has not been well investigated, many articles address the field in general and Lagrange's contributions in particular. Archibald has given a survey of differential equations in the eighteenth century which addresses the role that Lagrange and others played in the development of the field.[16] Engelsman looks specifically at the contribution made by Lagrange to the theory of differential equations, specifically first-order partial differential equations.[17] Kline has provided a brief introduction to the development of variation of parameters upon which we will expand upon.[18] Wilson has given a description of the development of perturbation theory looking at the contributions of both Laplace and Lagrange. For out analysis, we will look at Laplace's development of perturbation theory and the links between perturbation theory and the development of the method of variation of parameters.[19]

Overall, we aim to provide a new interpretation of the role of patronage and rivalry in Laplace's early career. We also seek to provide an account of aspects of Laplace's early work in the fields of perturbation theory and differential equations that is more detailed than those of earlier writers, in particular, concentrating on methods of solving ordinary differential equations, notably variation of parameters.

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[15]George Sarton, "Lagrange's Personality (1736-1813)," *Proceedings of the American Philosophical Society,* 88(6):457-496, 1944. Copied from the Archive de l'Académie des Science, Paris.

[16]Tom Archibald, "Differential Equations: A Historical Overview to circa 1900," in *A History of Analysis: editor Hans Jahnke,* (Providence, RI: The American Mathematical Society, 2003) 325-353.

[17]S.B. Engelsman, "Lagrange's Early Contribution to the Theory of First-Order Partial Differential Equations," in *Historia Mathematica*, 7 (1980), 7-23.

[18]Morris Kline, *Mathematical thought from Ancient to Modern Times,* vol 2 (New York: Oxford University Press, 1990) 494-499.

[19]Curtis A. Wilson, "Perturbations and Solar Tables from Lacaille to Delambre: the Rapprochement of Observation and Theory, Part II," in *Archive for the History of Exact Sciences,* vol 22 (Springer-Verlag, 1980) 189-304.

Chapter 2

Laplace's Biography from 1749-1789

Pierre Simon de Laplace was born on 23 March, 1749, the son of Pierre de Laplace and Marie-Anne Sochon.[1] Pierre Laplace was a merchant but after his marriage took on the role of tavern and inn owner, which was a respected role at the time. Marie-Anne was from a family of landowners of moderate wealth. Pierre Simon was the fourth of five children born to the couple in the town of Beaumont-au-Auge. He had an older sister who survived into adulthood and older twin brother and sister who died shortly after their baptisms. This left Pierre Simon as the eldest son and thus with the responsibility of being head of the household after his father.

While the family was not wealthy they were fairly well off and had substantial standing in the town where they lived. Pierre de Laplace was later mayor of the town and Marie-Anne acted as godmother for many local children. Pierre Simon himself had as his godparents local notables, showing that the family wanted and was able to guarantee that the boy would be looked after in the event that he was orphaned. By the time of Laplace's infancy, a notable was no longer necessarily a member of the nobility. While Laplace's father was not nobility, his role as tavern owner placed him in the role as local notable, as the tavern was a place of gathering for a community.

Pierre Simon's uncle Louis de Laplace was a local notable and may have assisted the young Laplace from an early age. His uncle was deacon of Beaumont and obtained a sinecure as chaplain in nearby Criqueville in 1752. Since Uncle Louis's position left him with ample

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[1] For more biographical information on Laplace look in Hahn's *Pierre Simon Laplace, 1749-1827 : a determined scientist.* The majority of the biographical information here can be found in this work and the *Dictionary of Scientific Biography* article on Laplace.

free time, it can be assumed that he was placed in charge of the young Laplace's early education. While Pierre Simon was not born of the aristocracy, there was an avenue available so that a young man such as Laplace would be able to raise his status in life. Being of relative wealth and social standing, the best hope that Pierre Simon had was to obtain an education and find a career in either law, the clergy or the military.

Pierre Simon's formal education started at the Benedictine Collège de Beaumont where he was a day student. The school was understaffed and his uncle Louis had been asked to assist at the school even though he was not a Benedictine. The Duke of Orléans allowed for six scholarships to needy noblemen and also allowed the sons of local inhabitants to attend for free.[2] Being a local resident, Pierre Simon was able to obtain a free early education. The duke did require that, in exchange for his kindness, the students were to make a daily prayer for him and on public occasions there was to be a formal address given to him.

After Laplace finished his early education at Beaumont, he continued at the University of Caen. This would seem an obvious choice as being a nearby university, but had the young Laplace shown his aptitude for science at this stage, it would seem more likely that he would have moved to Paris for his education. Hahn points out that many of Laplace's scientific contemporaries did move to Paris at this stage in their development. This points to the fact that Laplace was not recognized as the scientist that he was to become and that his destiny at this point still seemed to lie with the church.

The records from the University of Caen show that Laplace paid his masters' fees in mid 1768 and passed his examination in 1769.[3] The examiners were Jean Adam, Pierre Le Canu, Pierre Lelièvre, Louvel and Moysant. Laplace is listed as having studied with Jean Adam of Collège du Bois, but it appears that Laplace was influenced more by Le Canu and Christophe Gadbled. Contemporaries note that these men were "more than mentors, friends."[4] While all three men are notable mainly for their influence on the man who was to outshine all of them in importance, they did have some significance at the time and at the University of Caen.

Adam was a figure of some importance in Caen during this period. He was an ordained priest and canon of the church of Saint Sépulcre which was a religious institution unattached to any religious order. He maintained a conservative view which served him badly during

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[2]Hahn, *Pierre Simon Laplace*, 11

[3]Hahn, *Pierre Simon Laplace*, 18

[4]Hahn, *Pierre Simon Laplace*, 18

the revolution. When he began his career as an educator at his alma mater of the University of Caen, he was "an orthodox supporter of the Jesuit traditions in theology and metaphyics,"[5] despite being unattached to any specific order. When the Jesuits were disbanded in 1762 and thrown out of the universities, Adam took advantage of the situation and bought demonstration apparatus at low cost from the Jesuits. This and his reputation as a teacher in the Jesuit tradition ensured him large classes, since the Jesuits had a reputation as providing a solid scientific education. Adam's lectures were full of demonstrations but lacked scientific grounding. This made them nearly useless to the student who was interested in learning such as the young Laplace. While the demonstrations made the lectures interesting to watch, without the background information, the students were not really learning science, they were more attending a show.

Gadbled stood as a rival academically to Adam and espoused values completely different from him. While both were ordained priests and canons of the church of Saint Sépulcre, Gadbled was a moderate. The Paris-educated Gadbled introduced higher mathematics at Caen which appears to have appealed to Laplace more than the mathematics-free instruction of Adam. Pierre Le Canu became assistant to Gadbled after having studied with him from 1760-1762. Gadbled later was accused, by none other than Adam, of spreading irreligion among his students. Gadbled retorted that his teaching was in accordance with church doctrine.[6] Still, he and his circle taught students to question science. While Gadbled's sophisticated lectures may have appealed to the young Laplace, his decision to abandon the easy route of a church career in favour of an uncertain future as a scientist could not have been so simple a choice. Still Gadbled, and by extension Le Canu, could only have assisted in this decision by opening the young Laplace's eyes to the inherent mystery of the universe. Gadbled's attitude towards questioning science, also led students to challenge the basic attitude regarding the ideals of religion. Laplace did abandon theology and he was decidedly non-religious from this time onward. Even after the efforts of close friends, including his wife, Laplace never returned to the Christian philosophy.

Gadbled may have initiated Laplace into the idea of abandoning the cloth but it was with a letter from Le Canu that Laplace arrived at the door of d'Alembert in 1769. The story of Laplace's first meeting with d'Alembert has gained a level of notoriety and for this reason the truth behind the story becomes more difficult to decipher. There are at least two

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[5]Hahn, *Pierre Simon Laplace*, 20

[6]Hahn, *Pierre Simon Laplace*, 26

versions of the story written by Laplace's contemporaries. Neither author was acquainted with Laplace at the time; one of the authors was still a very young man and the other not yet born. We will discuss both versions of the story as well as other takes on the meeting in Chapter 3. No matter what happened in this meeting, Laplace found a powerful patron in d'Alembert and one who was able to propel the young scientist's career.

D'Alembert was able to find Laplace a job almost immediately as professor at the relatively newly formed École militaire; Laplace began to work for the École on 20 September 1769.[7] The details of Laplace's employment at the École Militaire will also be discussed in Chapter 3. Laplace remained an instructor at the École Militaire until the curriculum was changed in 1776 and he was pensioned off.[8] Laplace became examiner at the artillery corp in 1783 and later the naval engineering school, as we will discuss further in Chapter 3.[9]

While the École militaire provided Laplace with the ability to stay in Paris, he had a much bigger goal in mind. Laplace wished to obtain recognition by and even membership in the Academy of Science. For such a young man this was a somewhat lofty goal, but Laplace believed it to be well within his grasp and there was some precedent for young members. It took Laplace three years to obtain membership in the Academy which appears to be a relatively quick progression from unknown scientist to member of prestigious organization. For Laplace this rise was not quick enough as he felt that those who were elected were his scientific inferior. Laplace's election will be discussed in further detail in Chapter 3. The fact remains that Laplace was elected, and his status and prospects as a scientist were greatly improved.

In 1773, Laplace began his correspondence with Lagrange. While there is some debate as to the relations between the two men. We will argue that their relationship can be best described as a friendly rivalry, though there are some that believe that Lagrange found Laplace vain and that this vanity tinged their relationship. We will address this issue by looking directly at both Lagrange and Laplace's written research and the letters that they sent eachother. We can see from the letters that they sent each other their recent work and commented on each other's contributions. This debate will be discussed in Chapter 5. We can also see that they often worked in similar areas of study. We will look further into one of these areas in Chapters 6.

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[7]Hahn, *Pierre Simon Laplace*, 35.

[8]Hahn, *Pierre Simon Laplace,* 85.

[9]Hahn, *Pierre Simon Laplace*, 98

Overall, it was in this early period of Laplace's career that he showed the ambition and talent necessary for a man of non-aristicratic background to reach the pinnacle of French scientific society.

Part I

The Impact of Patronage

Chapter 3

D'Alembert

By the time that Laplace arrived in Paris, Jean le Rond d'Alembert (1717-1783) was one of the most powerful men in science, but he had a somewhat more humble beginning. D'Alembert was the illegitimate son of Mme de Tencin, a salon hostess, and the Chevalier Destouches-Canon, a cavalry officer. The baby was abandoned on the steps of Saint-Jean-le-Rond church in Paris, which is the origin of d'Alembert's given names. While his mother never acknowledged d'Alembert, his father looked after the young boy financially and left him in the care of a glazier named Rousseau. D'Alembert was given a prestigious education, which, similar to Laplace, was initially intended to send him into theology; he instead studied medicine and law. The young man began sending papers to the Académie des Sciences in 1739 and by 1741, he was elected as an *adjoint* member. Within the Académie, the professional members (as opposed to those who obtained their election due to their status as nobility or clergy and those non-resident members) were divided, in descending order of privilege, into the *pensionnaires*, *associés* and, until 1785, the *élèves*. After 1785, the section called *élèves* was renamed *adjoints*. A member was elected to the lowest rung and then slowly worked his way up to *pensionnaire*, if he was lucky. The roles of influence within the Académie, such as secretary, were given to members from this branch. By 1769, when Laplace arrived in Paris, Jean le Rond d'Alembert was one of the leading figures in the Académie and someone in a position to help Laplace both professionally and academically.

As stated before, there is some discrepancy in accounts of the initial meeting between Laplace and d'Alembert. One version is given by Joseph Fourier (1768-1830) in his eulogy for Laplace. The only way that Fourier could have known the story is second hand, since he was only a year old when Laplace made his way to Paris. In Fourier's version of the

story, Laplace arrived in Paris with numerous letters of introduction,[1] but these proved to be useless as the great d'Alembert refused to meet the young man. Fourier recounted that Laplace decided to attempt a different tactic and instead wrote d'Alembert a letter, from which, Fourier asserted, "M. Laplace, many times, had cited to me different passages."[2] This letter caused d'Alembert to call upon the young Laplace the very day that he received it. As Fourier explained, it was only a few days later that d'Alembert helped Laplace become professor of mathematics at the École militaire de Paris. While we cannot be sure of the validity of Fourier's story, Hahn believes that the point that Fourier was trying to make was that from around this time "In all walks of French life, merit was beginning to be a factor, but only if recognized and reinforced by strong supporters."[3] Laplace would not have been able to gain d'Alembert as a patron had he not shown talent, but he would have been completely unable to make a success of his career as a scientist had he not had d'Alembert's backing.

A second account of the meeting was given by Dominique François Jean Arago (1786-1853), who was not yet born when Laplace made his entrance into Parisian academic society. Still, Arago might well have heard the story of Laplace's first meeting with d'Alembert from Laplace himself when Arago was a member of the informal Society of Arcueil. Arago's version of the story had a different tone then did Fourier's but it also presented a somewhat different scenario. Arago explained that d'Alembert had, earlier in his career, welcomed the people who came to him for help, but he slowly had become disgusted by the visits from these "thieves of his time."[4] In an attempt to dissuade people from staying too long in his presence, Arago described, d'Alembert had started not allowing the majority of his visitors to sit. When the young Laplace entered his study, d'Alembert stood up. Laplace handed the senior scientist his recommendation letter and d'Alembert proceeded to attempt to deter the nervous young man from pursuing science as a career by calling it very arduous and not very lucrative. Before sending the young man on his way, Arago suggested that d'Alembert gave the young Laplace a very difficult problem saying "Come back and see me... when you have found the solution."[5] Arago explained that while d'Alembert expected never to see the

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[1]Fourier did not provide the names of any individuals who may have written these letters.

[2]"M. Laplace m'a, plusieurs fois, cité divers fragments." Archive de l'Académie de Sciences.

[3]Hahn, *Pierre Simon Laplace*, 32

[4]"voleurs du temps" Archives de l'Académie des Sciences.

[5]"Revenez me voir... quand vous en aurez trouvé la solution." Archives de l'Académie des Sciences.

young man again, Laplace returned just a few days later with his solution. After looking at the solution, d'Alembert offered the man a seat. Arago ended the story dramatically with "then, he took the hand of the young man and assured him of his protection and predicted a great destiny for him."[6] This story was found at the Archives of the Paris Académie des Sciences and unfortunately did not contain the date of writing. This means that, we cannot be sure of the validity of Arago's version. Arago's romantic style of writing does lead the reader to question the validity of Arago's claims. It seems that details may have been added for dramatic effect.

Looking at three different biographies of Laplace, we continue to see inconsistency in the story. Hahn follows Fourier's version of events adding more details of the treatise that Laplace produced.[7] This four page paper now rests in the Archives of the Académie des Sciences and its existence is incontestable. In this paper, Laplace questioned d'Alembert's examination of the principle of inertia. This bold move could have backfired, but d'Alembert instead saw this as a sign of the scientific worth of the young man.

In an article examining Laplace's early life, Bigourdan tells the story that Laplace presented himself modestly to d'Alembert with a letter from le Canu and d'Alembert set him a problem, telling him to return in eight days. Laplace returned the next day and d'Alembert gave him a second, more difficult, problem. Laplace gave the solution to this problem that very night. After this response, d'Alembert sent Laplace a letter stating

> Monsieur, you see that I have little regard for recommendations; you have no need of them. You have made yourself known, and this is sufficient for me. I will support you.
>
> D'Alembert[8]

This version appears to follow Fourier's, except that Fourier had stated that Laplace had not been able to meet d'Alembert until he had sent him a letter and Fourier does not mention a second meeting. Also, this version agrees with Arago's suggestion that d'Alembert himself

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[6]"puis il tendit le main au jeune homme, l'assura de sa protection et lui prédit de hautes destinées." Archives de l'Académie de Sciences.

[7]Hahn, *Pierre Simon Laplace*, 33

[8]"Monsieur, Vous voyez que je fais assez peu de cas des recommendations; vous n'en aviez pas besoin. Vous vous êtes fait mieux connaître et cela me suffit. Mon appui vous est du. D'Alembert." As quoted in G. Bigourdan, "La Jeunesse de P.S. Laplace," in *La Science Moderne*, vol 8:1931, 381

had set Laplace the problem. Gillispie appears to follow Bigourdan's version entirely.[9]

We may never know exactly what happened in this first meeting. What we do know is that Laplace impressed d'Alembert, and from this time the younger man had his situation secured.

3.1 Professional Patronage

3.1.1 Employment at École Militaire

Laplace's first hurdle, after finding a patron, was to find a livelihood. When Laplace had turned away from a career with the church, he had effectly cut himself off from any monetary support from his family. This meant that becoming self-sufficient was imperative. D'Alembert was able to quickly secure Laplace a position at the École militaire. The École Militaire was established in 1753 as an institute of secular learning for the sons of impoverished nobles seeking military careers. The military was a career sought by such nobles as a means of maintaining status while obtaining financial stability. Here Laplace was able to hone his own knowledge while educating the future generation of military officers. The École had a substantial library, of which it is assumed that Laplace took full advantage. Laplace's salary allowed him financially to stay in Paris. Even though Laplace did not turn to a career in the church, he did find some use for his early study in theology. When he began work at the École militaire, d'Alembert suggested that he represent himself as a member of the church. As such, d'Alembert referred to him as the abbé Laplace and told him to dress in clerical garb. Hahn points out that since the professors were expected to be of high moral standing, the guise of the clergy could only be beneficial. As well, there were many other examples of ordained priests acting as scientists (such as Laplace's teachers at the University of Caen).

When Laplace was given this position, d'Alembert did not send a letter to Laplace himself but rather to Le Canu. First, it should be noted that while this letter was addressed to Le Canu, it was obviously meant for Laplace himself. The information given in this letter would have been completely useless for Le Canu. Had d'Alembert wanted to simply alleviate any fears that Le Canu may have had about the future of his student, d'Alembert would not have included the details that he did.

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[9]Gillispie, 276

In order to discuss this letter fully, we will reproduce it here in its entirety

> I write you to leave you the satisfaction of announcing to Mr. l'abbé de la
> Place his good fortune; you may tell him that he is assured the placement of
> professor of mathematics at the École militaire also you may repeat to him the
> conditions: funished accommodations with 6 sets of wood, 1800 livres for the
> appointment, therefore 1400 livres net because we retain 400 livres for food. If
> these conditions are suitable to him, it is necessary 1° that he writes this to
> me right away, because I am leaving September 7 for the country where I will
> stay for 3 weeks. 2° that he writes also to M. Bizot, rue du Temple near rue
> des Gravilliers in Paris. M. Bizot will be the director of studies from October 1.
> He should therefore inform M. Bizot that he may count on him and add to that
> expressions of honesty and of suitable gratitude. 3° It will be necessary that he
> arrives in Paris on September 20 at the latest or, if he can, some days before. On
> arriving he should seek M. Bizot, who can be found every morning from 10:00
> until 2:00 and every evening from 5:00 until 8:00 at rue St. Louis at the marsh,
> home of M. Paris du Verney. I hope that Mr. l'abbé de la Place, by his zeal, his
> punctuality and his good behaviour will bring honour to my recommendation. I
> forgot to tell you that he will only have to give three to 4 hours of class every
> morning; the rest of the time will be his own. I have the honour of being, sir,
> with my respectful attachment, your very humble and obedient servant
>
> d'Alembert at Paris August 25, 1769.[10]

The first question that this letter raises is why d'Alembert is writing this to le Canu.
The answer appears to lie in the first sentence that d'Alembert writes. It seems that Laplace
was not in Paris but in Caen. The address on the letter is given as the university. If Laplace
was in Caen, the easiest way to find him would be through le Canu and the easiest way
to have a letter reach le Canu would be by sending it to the University. Therefore, this
letter may have been how Laplace first learned that he had employment in Paris. This
reasoning makes sense, but it seems odd that after Laplace had arrived in Paris and found
a supporter that he would return to Caen immediately. He might have returned to Caen for
work, since while he had found a champion, he had not yet found a job in Paris. He might

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[10]The original French is quoted in appendix A, Archive de l'Académie des Sciences, dossier de Laplace.

also have returned to visit family. Otherwise, Laplace may have been preceded by his letter of recommendation and his subsequent paper on inertia while he himself was still in Caen; this seems somewhat unlikely and against all the stories presented. Alternatively, this letter could have been standard practice to let le Canu know that his student would be taken care of in Paris, though the extent of the information in this letter makes this appear unlikely.

Whatever the reason behind this letter, it signifies Laplace's entrance into Parisian scientific society and the beginning of d'Alembert's role as Laplace's scientific patron. It also appears to signify the end of le Canu's role as patron to Laplace.

3.1.2   Membership in Paris Académie des Sciences

Since Laplace's post at the École militaire included room and board, his salary seems generous.[11] Laplace, though, appeared to want more than simply monetary gain; he wanted to be acknowledged as a scientist. To do this, Laplace would need to gain membership into one of the leading academies of science. Laplace first set his sights on the Paris Académie des Sciences.

The Académie only appointed new members when an existing member died, retired or was promoted. This made the opportunities for nomination irregular. The basic method of obtaining nomination was to have a powerful benefactor - which Laplace had in d'Alembert - and to write worthwhile papers that the Académie recognised as being significant. Starting in 1770 Laplace began writing academic papers at a feverish rate, completing thirteen papers in the three years prior to his election. Hahn points to six occasions when Laplace was in the running for election and was overlooked. These included three separate occasions in 1771, possibly twice in 1772 and twice in 1773. On many occasions, Laplace believed that the person who was elected before him was his inferior. While this may have been a usual reaction, it does provide evidence of Laplace's attitude at this age and perhaps in general.

Laplace may have been correct in his assessment that he was being neglected, as can be seen by d'Alembert's response. While Laplace may have suggested the idea, it was d'Alembert who wrote on his behalf to Lagrange. In this letter, which will be discussed further in Chapter 5, d'Alembert requested that Lagrange consider Laplace for admission into the Berlin Academy.[12] This letter argues that an election to the Paris Académie could

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[11]Noting that 400 livres was enough for food, the remaining 1400 livres can be put into perspective.

[12]Hahn *Pierre Simon Laplace* 42-43

take as long as twenty-five years. In this letter, d'Alembert contended that despite his own vote, someone who was vastly inferior to Laplace was elected ahead of him.[13] This letter shows d'Alembert acting as Laplace's champion. While his solicitation did not lead to Laplace's election in Berlin, the effort that d'Alembert put into attempting to find Laplace a suitable placement shows the regard that he had for the younger man's abilities.

Similar efforts were made at the St Petersburg academy. Here Laplace was able to induce Lalande to write an introduction for him to Euler.[14] Laplace, himself, also wrote to Euler but there is no record of any response. While these requests did not materialize into offers, thankfully, Laplace gained membership in the Académie in 1773. While Laplace may have felt overlooked during this period, the fact remains that opportunities to join the Académie were few. This meant that older, more established scientists had an advantage over younger nominees in the argument that they would have fewer chances to gain membership. The average age for a new academician in decade of the 1730's was about 28, while in the 1780's this age had risen to just over 39.[15] Laplace was granted membership at the age of twenty-four, making him still very young to receive this honour.[16]

Overall, in both of these cases, we can see the role that patronage played in furthering the career of the young scientist. While we can see that d'Alembert was able to create opportunities for Laplace, we can wonder if Laplace really needed d'Alembert's help. Below, we will examine a case where patronage was not successful and the results, in the case of the young Jean-Jacques Rousseau (1712-1778).

3.1.3   Comparison to Rousseau

In Rousseau's *Confessions*, he discusses how he made his entrance into Parisian academic society. Similar to Laplace, Rousseau procured a selection of letters of introduction to various people in Paris who might have been able to help him. While it is only in Fourier's account that Laplace had more than one letter of introduction, it does seem possible. When Rousseau arrived in Paris in Autumn of 1741, he had fifteen *louis*,[17] a system of musical notation and

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[13]In an editor's note, it is suggested that d'Alembert is referring to Cousin's election in March, 1772.

[14]Hahn, *Pierre Simon Laplace*, 42.

[15]Hahn, *The Anatomy of a Scientific Institution: The Paris Scademy of Sciences, 1666- 1803*, (Berkeley: University of California Press, 1971), 97

[16]D'Alembert was also twenty-four when he gained membership to the Académie.

[17]A *louis* was worth about 24 *livres*, therefore he had about 360 *livres*.

his letters of introduction. Rousseau comments that "A man who arrives in Paris with a decent appearance and advertises himself by his talents is always sure of a welcome. My good reception procured me some pleasures but did not lead to anything much."[18] While Rousseau had obtained several introductions, he only found three useful. Through one of these introductions, Rousseau met René-Antoine Ferchault de Réaumur (1683-1757), a member of the Académie des Sciences. Rousseau discussed with de Réaumur his notation scheme and his desire to present it for examination before the Académie. He luckily found a supporter in de Réaumur, though it seems not a powerful supporter. After this introduction, de Réamur undertook the negotiation and Rousseau was accepted to present his work. While he had arrived almost a year before, it was not until August 22, 1742 that Rousseau was able to present his work.[19] He commented that "The paper was a success, and brought me compliments that surprised me as much as they flattered me."[20] Rousseau's optimism was short lived. When the Académie set to examine the scheme, they found that there had been a similar method developed before, thus making Rousseau's not original. Rousseau argued that his was both simpler and more convenient. When they investigated the system itself they did not understand it well enough to find in it any virtues. Rousseau went so far as to say "I was astonished by the ease with which they refuted my arguments with the help of a few high-sounding phrases, without in the least understanding them."[21] In the end, Rousseau was granted a certificate from the Académie with praises thinly masked by criticisms.

The example of Rousseau shows several features of eighteenth century patronage. First, the patron that was chosen did matter. Rousseau found a sponsor, but one without the resources of d'Alembert. While it may have been ambitious for Laplace to seek out the most powerful patron that he could, the fact remains that d'Alembert's endorsement of Laplace's work was often enough, while de Réaumur did not appear to have this power. Second, while powerful supporters can prove the difference between success and failure, powerful detractors can make the same difference. The three individuals who were asked to examine Rousseau's work knew little about music, Jean-Philippe Rameau (1683-1764) was one of the

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[18]Jean-Jacques Rousseau *The Confessions: translated by J.M. Cohen*, (Harmondsworth, England: Penguin Press Ltd, 1971), 266.

[19]Laplace presented his first paper to the Académie about six months after his arrival in Paris.

[20]Rousseau, 267.

[21]Rousseau, 268.

most important individuals in French music of all time. Rousseau was able to brush aside the criticisms of the committee examining him, but he himself admitted that "The only serious objection to be made against me system was made by Rameau."[22] The objection that Rameau gave to Rousseau's work may have had the opposite effect as the praise that d'Alembert showed Laplace. Third, while a letter of introduction is necessary, it becomes useless if a person can not show their own merit. Rousseau's method of notation may have been as important and useful as he felt it was, but he was not able to make this point. Without having been able to show his merit, he was left with nothing. Last, it appears that Rousseau relied entirely on his letters to open doors for him. If Fourier's account is correct, in Laplace's case, the letters were less important than Laplace being able to prove to his future patron that he had the skills to deserve d'Alembert's support. Overall, while there was some benefit to having letters of introduction, as the example of Rousseau shows, this is not often enough.

3.2 Academic Patronage

So far, it seems as if the relationship between d'Alembert and Laplace was only beneficial to Laplace. While d'Alembert may have had entirely altruistic intentions, this seems unlikely. Laplace was given the opportunity to compensate d'Alembert through his scientific output. The areas that Laplace studied may also have been a direct consequence of d'Alembert. Stigler notes that Laplace's first memoirs were purely mathematical, but in 1770 there was an abrupt change to physical astronomy.[23] While this switch may have been due to Laplace's ambition and therefore a desire to work in the field that occupied the great names of the time including Euler, Lagrange and d'Alembert, Stigler also note that "The most direct influence upon Laplace... may have been his patron, d'Alembert."[24] He continues that d'Alembert was publishing related works exactly when Laplace started to show as interest in the field. Laplace pursuing research in d'Alembert's field is similar to the modern practice of a graduate student assisting in the research of his or her supervisor. In Laplace's case, though, his own work often proved superior to his patron's.

Besides simply working in the same areas, Laplace was able to show his appreciation

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[22]Rousseau, 268.

[23]Stigler, 238.

[24]Stigler, 239.

to d'Alembert by citing his patron and praising his research. While this may have been forced on Laplace, he did frequently write highly of d'Alembert, especially in the papers investigated here. The issue of Laplace's choice in the matter can be seen in a letter that Laplace wrote d'Alembert where he apologized for not citing d'Alembert thoroughly enough and showed his patron the addition he is adding to a paper to correct the matter.

In the first memoir that we will investigate, *Sur le principe,* Laplace acknowledges the work of the great scientists who had completed prior work in this subject matter. This memoir itself will be discussed in further detail in Chapter 6. Even though Laplace acknowledges other scientists' work, he is not always complementary. Besides the obligatory reference to the "illustre géomètre," he has some harsh criticisms and even jabs at his contemporaries. Laplace did look favourably on the work of d'Alembert and Lagrange; he regarded the work of Euler somewhat ambiguously; and Laplace found fault with the work of Daniel Bernoulli. The reasons for Laplace's treatment of other scholarly research were not always about the academic worth of the products. He seems to have either accepted or rejected other scholarly research for different reason, which will be discussed. Though Laplace mentioned many great names, he also left out some figures whom he could have mentioned. This aspect of Laplace's memoir will also be discussed. While there may not have been a choice in citing d'Alembert, he did have a choice in how he represented the work of his other contemporaries. When Laplace found an opportunity, he often derided the work of d'Alembert's rivals. The way that Laplace showed his gratitude to d'Alembert by simultaneously praising his work and slighting that of d'Alembert's rivals will be discussed next.

3.2.1 Citing d'Alembert

Based on the relationship between Laplace and d'Alembert, it would seem like the most obvious choice, if there was a choice in the matter, to repay d'Alembert by commenting favourably on his work when Laplace was given the opportunity. Laplace starts the second part *Sur le principe* by stating that

> Mr d'Alembert has given the first general solution of this problems, the most direct method to arrive at it and, at the same time, the most useful application that can be made in his treatise *On the precession of the equinoxes* an original work which shines throughout with the genius of invention and that we can see as containing the germ of

all that has been done since in mechanics of solid bodies.[25]

This passage could not be more flattering to d'Alembert, but does it hold true? Wilson addresses the issue of priority in "D'Alembert versus Euler on the Precession of the Equinoxes and the Mechanics of Rigid Bodies." He argues that, while d'Alembert may have developed the ideas first on paper, "only through compensating errors of sign did he emerge with a correct solution."[26] By contrast, according to Wilson, Euler made his arguments clearly and correctly. Laplace referenced this work no less than three times but does not present any direct examples from d'Alembert's *Recherches sur la Précession des Equinoxes*. Here Laplace might have felt obliged to praise the work of his mentor who was soon to assist in Laplace's successful election to the Académie, but did not wish to repeat any errors that d'Alembert may have made in his presentation. In that view, because Laplace knew that he could not use d'Alembert's work verbatim, he instead referred to the conclusions but derived his own equations of motion and calculated for himself where he could have used results from d'Alembert. A full comparison with d'Alembert's *Recherches* would be required to comment further on this matter, which is beyond our present scope. Still, it can be noted that Laplace did find cause to praise the work of d'Alembert and in this way showed his gratitude even if he may have found fault with the actual work.

While Laplace did praise the work of d'Alembert, he may not have had a choice in the matter. In a letter to d'Alembert, written November 15, 1777, Laplace explained the passage that he was adding to one of his works to give greater credit to d'Alembert. After stating first what he had originally included, which stated his indebtedness to d'Alembert and explained the contents of d'Alembert's paper, Laplace stated, in full, the material that he was adding, which included more praises of d'Alembert. What Laplace said at the end of this letter shows the reaction that d'Alembert must have had when Laplace did not properly cite his mentor's work. Laplace ended his letter as follows:

> I have always cultivated Mathematics due to taste rather than a desire to obtain
> a reputation, of which I have none. My greatest joy is in studying the work of

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[25]"M. d'Alembert a donné, le premier, la solution générale de ce problème, le méthode la plus directe pour y parvenir, et tout à la fois l'application la plus utile et la plus heureuse que l'on en puisse faire, dans son excellent Traité *Sur la précession des équinoxes*, Ouvrage original, qui brille partout du génie de l'invention, et qu'on peut regarder comme renfermant le germe de tout ce qu'on a fait depuis dan la Mécanique des corps solides." Laplace, "Sur le principe," 201.

[26]Curtis Wilson, "d'Alembert versus Euler on the precession of the equinoxes and the mechanics of rigid bodies," in *Archive for the Histroy of Exact Sciences*, 37(3), 1987, 234.

inventors, of seeing their genius gripping the obstacles in which they encounter and of which they overcome; I put myself in their place and ask myself how I would surmount these same obstacles and although this substitution has never been anything except humiliating for myself, the pleasure of enjoying their success amply compensates for my humiliation. If I am lucky enough to add something to their work, I attribute it to their previous efforts, well persuaded that in my position they would have gone further than myself. You see from this, my dear colleague, that no one reads your works with greater attention, nor does anyone find greater profit in them than myself; also, no one is greater disposed to render to you a justice more entire, and I pray that you see me as one of those who loves and admires you the most. It is with these sentiments that I have the honour of being, Monsieur and illustrious Colleague,

Your very humble and very obedient servant,

Laplace.[27]

This letter shows Laplace apologizing for neglecting to cite d'Alembert. In the article that Laplace mentioned, he had already written glowing praise for d'Alembert, but it seems as though this was not sufficient. Here it seems that Laplace was not simply citing d'Alembert because he felt that credit was due to d'Alembert, or even because of the conventional courtesy due a patron by his protégé, but because if Laplace did not sufficiently cite d'Alembert, he would be asked to rewrite his work and add a further reference. Overall, Laplace was able to show his gratitude to d'Alembert by citing his mentor's work; he did not have a choice in the matter.

3.3   Praising d'Alembert by Deriding his Rivals

While Laplace praises the work of d'Alembert, every instance appears to present Laplace with an opportunity to show how the work could be improved. When Laplace discusses the work of d'Alembert's rivals, any comment that he makes against their work can only bring further credit to d'Alembert's. The method that Laplace takes in addressing the work of d'Alembert's rivals will be addressed below.

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[27]The original French is reproduced in Appendix A, "Lettres Inédites de Laplace" in *Oeuvres de Laplace* vol 14 (Paris: Gautier-Vallier), 348.

3.3.1 Daniel Bernoulli

Laplace picks at least two points in *Sur le principe* to take jabs at the work of Daniel Bernoulli: first when discussing gravitation being a consequence of it working on all parts of a body; second when discussing the instant propagation of gravity. In neither case does Laplace go into detail about Bernoulli's research but does question the validity of his work with respect to these cases. In this first case, Laplace begins his discussion by saying that "Quelques illustres géomètres, M. Daniel Bernoulli" believe that it is only a probability that weight is the result of the attraction of all the parts of the body, but observation has led to the truth that attraction works on the smallest parts of a body.[28] Laplace did not need to say any names but he took the effort to point out the errors that Bernoulli had made. Also while Laplace states that this is a belief of some geometers he only names Daniel Bernoulli. This could be taken as Laplace saying that even someone of the stature of Bernoulli could make mistakes, but Laplace does not go into details about Bernoulli's arguments and, after mentioning that Bernoulli was erroneous in his belief, he goes on to discuss this assumption without further reference to Bernoulli. Basically, Laplace has gone out of his way to point out that Bernoulli had made a mistake and continues his analysis as a point of departure from Bernoulli.

While Laplace calls Bernoulli an illustrious geometer and refers to his research on the tides as excellent, he still takes the extra effort to show the shortcomings of Bernoulli. The question is raised as to why Laplace would take the pains to antagonise a respected member of the scientific community? The answer may lie in d'Alembert's rivalry with Daniel Bernoulli.

Briggs points out that d'Alembert was a man who developed many scientific rivalries over his career; one of them involving Daniel Bernoulli. D'Alembert and Bernoulli at least twice had differing approaches which led to public debates; once with respect to the controversy involving the vibrating string problem and another involving probability with respect to inoculation. The vibrating string problem brought into question the nature of a function and set d'Alembert, Euler and Daniel Bernoulli as mutual opponents.[29] The debate about inoculation involved calculating how long a person was expected to live, from any given age,

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[28]Laplace, "Sur le principe," 215

[29]Hans Niels Jahnke, "Limitations of the analytic function concept", *A History of Analysis*, vol. 24, ch. 4.6, pp. 123-127, (Providence, RI; London: American Mathematical Society, London Mathematical Society, 2003) 123-127, for more information on the vibrating string controversy.

with or without being inoculated against smallpox.[30] D'Alembert and Bernoulli had what Briggs calls "extensive arguments" in regard to this problem.

Given Bernoulli's relationship to d'Alembert it would seem a wise move on Laplace's part to find fault with Bernoulli's work in combination with praising d'Alembert's. For this reason, Laplace may have taken the extra effort to point out contentious aspects in Bernoulli's work. Whatever the reason, there is a notable dismissiveness in this memoir with regard to the work of Daniel Bernoulli.

3.3.2 Clairaut

Clairaut represents another of d'Alembert's academic rivals. Briggs discusses that d'Alembert's "rivalry with Clairaut... continued until Clairaut's death."[31] Laplace had taken the opportunity to criticize the work of Daniel Bernoulli, who was also a rival of d'Alembert's; it seems odd that given the chance, he did not proceed in the same way with Clairaut who was a more long term rival than Bernoulli.[32] Laplace investigated the shape of the Earth when discussing his second assumption of universal gravitation which was a topic of rivalry between the two men. In the 1730's, expeditions had been launched to measure the Earth to determine whether Newton or Descartes had been correct about the shape of the Earth.[33] Newton believed that the Earth was shaped like an onion while Descartes thought that the Earth was shaped like a lemon. Clairaut accompanied the trip to the Arctic. Clairaut, d'Alembert and Euler had all been involved in looking at the data and evaluating the validity of Newton universal law of gravitation. Surprisingly, using differing methods all three men found that the data did not follow the theory. Clairaut later announced that Newton was correct after all and a simplification that all three men had made was the reason for the discrepancy. D'Alembert quickly agreed. The difference between this case and that of Bernoulli was that Clairaut seems to have been correct rather than d'Alembert in this case

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[30]At this time, inoculation involved a person being injected with a small amount of fluid from a person infected with smallpox and the result would be either a mild case of the disease followed by immunity or a more severe case often resulting in death. D'Alembert is noted as pointing out that the probability is little comfort to a person who has just lost his or her child. J. Morton Briggs, *Alembert, Jean le Rond d'*, vol. 1, (Detroit: Charles Scribner's Sons, 2009/04 2008), 116

[31]Briggs, 111.

[32]Briggs discusses that d'Alembert and Clairaut's rivalry began shortly after d'Alembert entered the Académie and as has been noted extended until Clairaut died.

[33]This controversy is discussed in Thomas L. Hankins, *Science and the enlightenment*, (Cambridge, Mass.: Cambridge University Press, 1985) 37-41.

while with Bernoulli, d'Alembert had not written about the same topic and there was less room for comparison. Since Bernoulli's work was separate from d'Alembert's in the case that was mentioned, it made it easier to point out Bernoulli's failings while not tempting any backlash towards d'Alembert. Had Laplace placed a comment about Clairaut in this memoir, there may have been an opportunity for Clairaut to turn the tables and attack d'Alembert which was definitely not what Laplace wanted. A close reading of Clairaut's work could have found some error as Laplace had been able to find with Bernoulli, but the result of any comment might not have been positive. Still, the fact remains that the discussion of Clairaut and this controversy were left out of this paper entirely.

3.3.3 Euler

D'Alembert and Euler were often involved in scientific debates and a rift developed between the two men which caused them to cease communication for over a decade.[34] Still, Euler was the most respected man of science in Europe then and some may argue of all time. For this reason, criticizing Euler could be seen as tantamount to career suicide. In the part of the memoir under discussion, Laplace only mentioned Euler once and does so in general terms. When he had the option to mention Euler in regard to his work on the precession of the equinoxes, he did not take this route, favouring d'Alembert's work. Here Laplace appears to be attempting to pay tribute to d'Alembert's contribution by neglecting the superiority of Euler's. He later looked at Euler's work more directly. This section of the memoir, involving secular inequalities, did not factor into the current paper and looking more in depth into Laplace's treatment of Euler could be a subject of further research.

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[34]Wilson, "D'Alembert vs Euler on the Precession of the Equinoxes and the Mechanics of Rigid Bodies," 237

Chapter 4

Other Patrons

As we have seen, d'Alembert was a powerful patron for Laplace, but he was not Laplace's only supporter. We will look at two cases that illustrate Laplace's other patrons. Like d'Alembert, Lavoisier acted as patron for Laplace because of the latter's scientific ability. In fact, Lavoisier appears to have sought out Laplace because of his interest in having Laplace help him in his work. Laplace in his turn was able to see the value in working with someone as respected and distinguished as Lavoisier. The second situation that we will discuss is completely different. In the case of Laplace's appointment as examiner of the artillery, references meant everything and the ability of the candidate was rarely discussed. The influence of other patrons in Laplace's early career will be discussed below.

4.1   Lavoisier

Antoine-Laurent Lavoisier initially followed the family tradition of studying law, but found his interests lay more in science. Lavoisier presented research to the Académie des Sciences on hydrometry, or the measuring of components of the water cycle, such as rainfall. After presenting such a paper in 1768, Lavoisier was elected as a member of the Académie. It was shortly after this election that he bought membership into the *Ferme Générale*. This private consortium, which collected taxes for the king, would, in the end, lead to Lavoisier's demise.

While Lavoisier was elected to the Académie in 1768, it was an experiment started in the summer of 1768 and presented to the Académie in 1770 which gave him the reputation as a first rate experimentalist and scientist. At the time, there was some debate as to the nature of water; many scientists believed that water could transmute itself into earth. Lavoisier

showed that the solid material that was found after distillation was not water turning into earth, but rather was due to the distillation process and was associated with the glass used.[1] This experiment brought Lavoisier public fame and recognition. By the time that Laplace had arrived in Paris, Lavoisier was already a famous man.

Laplace began collaboration with Lavoisier in early 1777, on work related to vaporization and evaporative cooling.[2] As Gillispie points out, Laplace's work with Lavoisier allowed Laplace to advance as more than just a mathematician.

> It associated him with the one person who was clearly emerging as the scientific leader
> of the Academy in their generation, the newly appointed administrator of the Arsenal
> and reformer of the munitions industry with influential connections in the worlds of
> government and finance[3]

Guerlac states that "[t]he partnership, as [Laplace] later acknowledged, was distinctly to his professional advantage."[4] Here, Guerlac does not give any hints as to where Laplace acknowledged this. Though it can be assumed that Laplace would have recognised the professional advantages in working with a scientist as distinguished as Lavoisier was at the time, it would be interesting to see this in Laplace's own words.

Laplace's interest in working with Lavoisier seems apparent, but Lavoisier must have had a reason for wanting to collaborate with Laplace. Laplace had by this time shown himself an adept scientist (not simply a mathematician) with a considerable breadth of interest. His main work at the time was in the fields of probability and the gravitational physics of the solar system.[5] Laplace's work on probability and its relation to the theory of error is most likely what interested Lavoisier and it appears a likely reason why Laplace was a good candidate for experimental collaboration with Lavoisier. Guerlac notes that Lavoisier was interested in experimentation that would require considerable precision and therefore "he could well have seen the advantage of working with a man like Laplace, sensitive to the

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[1] Henry Guerlac, "Lavoisier, Antoine-Laurent," in *Complete Dictionary of Scientific Biography*, vol 8, (Detroit: Charles Scribner's Sons, 2008), 71.

[2] Gillispie, et al., 312

[3] Gillispie, et al., 312

[4] Henry Guerlac, "Chemistry as a Branch of Physics: Laplace's Collaboration with Lavoisier" *Historical Studies in the Physical Sciences* 7th annual volume Edited by Russell McCormmach (Princeton, NJ: Princeton University Press, 1976) 197

[5] Guerlac, "Chemistry as a Branch of Physics: Laplace's Collaboration with Lavoisier," 197

theoretical difficulties of accurate experimentation."[6]

Laplace and Lavoisier worked together on a number of experiments, but it should be noted that it was Laplace who first sought to end the collaboration. In 1782, Laplace sent a letter to Lavoisier politely asking to end his experimental work, explaining that he wanted more time to devote to mathematics.[7] Lavoisier was able to convince Laplace to continue with the experimental work. It may say something about Lavoisier's power of persuasion that he was able to change Laplace's mind, but it may also have been that Lavoisier was in a position, one in which Laplace was later to find himself, of being able to compel Laplace into continuing research that he himself no longer wished to pursue. Whatever the case, there is no evidence of further experimental collaboration after 1784.[8]

This example shows how Laplace was able to use his scientific ability to work with someone who could aid him in rising in position professionally. Neither Guerlac nor Gillispie provide specific examples of how Lavoisier helped Laplace professionally, but Gillispie does mention that they served together on committees for the Academy. It may have been Lavoisier who put forth Laplace's name when positions were available though this can not be said for sure. We do know that when Laplace was lobbying for the positions left vacant after the death of Bézout, it was to Lavoisier that he turned. This will be discussed below. Lavoisier had power and prestige, as a leading member of the Académie and a celebrated scientist, and at the time of the beginning of their collaboration, Laplace had only recently been elected to the Academy. Sadly, Lavoisier was executed during the Reign of Terror in 1794. Lagrange is noted to have said the day after Lavoisier's execution "It took them only an instant to cut off his head, and a hundred years may not produce another like it."[9]

4.2   Becoming Examiner of the Artillery

Étienne Bézout accelerated quickly in the Académie des Sciences. By 1758, he had already been elected as *adjoint*. By 1763, Bézout was a father and in need of further financial opportunities. This made his appointment as teacher and examiner for student naval officers quite timely. In 1768, he was given a similar appointment at the artillery. Besides being

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[6]Guerlac, "Chemistry as a Branch of Physics: Laplace's Collaboration with Lavoisier," 205

[7]Guerlac, "Chemistry as a Branch of Physics: Laplace's Collaboration with Lavoisier," 240

[8]Guerlac, "Chemistry as a Branch of Physics: Laplace's Collaboration with Lavoisier," 274

[9]Guerlac, "Lavoisier, Antoine-Laurent," 85.

teacher and examiner, Bézout also wrote the textbooks for his students. These textbooks were used widely in France, and were later translated and used in American schools, including Harvard.[10] When Bézout died in 1783, his positions as examiner became prized commodities, coveted by the new generation of scientists.

In 1776, the École militaire had been reorganized and Laplace was not longer required to teach, but continued to receive a pension. His total income at the time of Bézout's death was most likely only 600 livres from his yearly pension and 500 livres from the Académie. Laplace's limited earnings probably made the positions vacated by Bézout more desirable to him. Laplace acted quickly to gain the support of the necessary people; all within the month between September 27, 1783, when Bézout died and October 23, 1783, when Laplace was appointed examiner of the artillery.

In a letter written to Lavoisier, Laplace outlined the steps he had taken to obtain both positions, as well as the potential ones of his opponents. Bézout died about a month before d'Alembert and while d'Alembert had been the most influential person for Laplace to turn to, it appears that by 1783, this person was now Lavoisier. While Lavoisier may not have been directly linked to Laplace obtaining the position as examiner, we know at least that he turned to Lavoisier during this time. While appointments of this kind were ultimately made by the king, he would follow the advice of his minister on these matters.[11] Laplace, therefore, discusses with Lavoisier his tactics to be introduced to the Ministers of War and of the Navy: Philippe Henri, Marquis de Ségur (1724-1801) and Eugène-Gabriel de la Croix, Marquis de Castries (1727-1806) respectively. Laplace pointed out five people who he hoped would either recommend him to one of the ministers or introduce him to one of the ministers. Most likely the majority of these recommendations and introductions occurred in person, which would account for there being no record left. At the military archives in Vincennes, just outside of Paris, the remaining documentation is sparse. In the dossier for Laplace there are: a letter from a mathematician Mauduit to "Monseigneur," a letter from Vaudreuil to "Monseigneur," a letter sent from Jean-Baptiste Vacquette de Gribeauval to Ségur, a letter from Castellane de Berghes to Ségur, the announcement of Laplace's appointment as examiner of the artillery and a one page *mémoire* from members of the academy supporting

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[10]Judith V. Grabiner, "Bezout, Étienne," in *Dictionary of Scientific Biography* vol 2 (Detroit: Charles Scribner's Sons, 2008), 112.

[11]Denis I. Duveen and Roger Hahn, "Laplace's Succession to Bézout's Post" *Isis*, 491.

Laplace.[12] Duveen and Hahn suggest that the letters to "Monseigneur" were all for Ségur.

The letter that may have been the most useful was likely the one from Gribeauval. Besides the contents of this letter, which we will not discuss in detail, a letter from Gribeauval would be helpful simply because it was from Gribeauval. He had experienced a rapid rise, which Alder states he attributed to being one of the few French heroes of the Seven Years War, and to his ability to acquire influential patrons.[13] While Gribeauval signed his letter simply as "gribeauval," at the time of the writing he was first inspector of the artillery. Any positive letter from such a man could only help Laplace's chances.

The document that stands out among those in Laplace's dossier is the letter from Mauduit to Ségur. This letter is not discussed in detail by Duveen and Hahn and besides relating to the post that Laplace eventually obtained, does not involve Laplace himself. In this letter, Mauduit puts himself forward for the position. This is in stark contrast to Laplace's tactics which involved having others support him. As well, in the other letters the mathematical abilities of Laplace are absent. In Mauduit's letter, he outlined his own mathematical abilities as well as his interest in artillery. Since Mauduit did not succeed Bézout, this tactic did not work. The other noteworthy point in Mauduit's letter is the date: September 26, 1783. This was the day before Bézout's death. While the date alone raises more questions than answers, we can only hope that this was a misprint.

Duveen and Hahn conclude that the main points taken into consideration in the succession to Bézout's post were: the desires of Bézout, any agreements between Bézout's family and Laplace and the concerns of the students.[14] It appears though that these points were just as important as those making them. Laplace had found strong champions and the people themselves might have been as important as the points they made. This situation is in stark contrast to Laplace's previous encounters with patronage. Nowhere in the letters that remain does it appear that Laplace's mathematical or teaching abilities were important. The point that Gribeauval made regarding the students was that a successor should be appointed soon and one who would not change the requirements, not that Laplace would make a good examiner for the students. Laplace's abilities as a mathematician had allowed him to find patrons, but it appears that once he had patrons, finding employment was more about who supported him than about his own abilities. Mauduit, it seems, had only himself

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[12]Archives Militaires Y^h 165 Dossier Laplace

[13]Ken Alder, *Engineering the Revolution: Arms and Enlightenment in France, 1763-1815*, 36.

[14]Duveen and Hahn, 426.

to recommend himself and only his abilities to help him obtain the position. Laplace had patrons to speak up for him and the wishes of the deceased and his family to aid him. In the end, Bézout's posts were split and Laplace was appointed examiner of the artillery and Gaspard Monge (1746-1818) was appointed examiner of the navy.

Part II

The Impact of Rivalry

Chapter 5

Lagrange

Lagrange and Laplace had very different experiences growing up and entering the scientific community. On January 30, 1736, Giuseppe Lodovico Lagrangia was baptized in the parish church of Sant Eusebio in Turin, Piedmont. Lagrangia was an italianized version of the French name Lagrange. While Lagrange's family was relatively well-to-do, they lived modestly. Lagrange is noted to have said that if he had had the finances, he, most likely, would not have become a mathematician.[1] Lagrange quickly showed his mathematical dexterity in the Athenaeum of Turin, where he was educated. It was there that the young man set to master the works of Newton, Leibniz, Euler, the Bernoullis and d'Alembert. It was at the young age of eighteen that Lagrange began his correspondence with Euler and Fagnano. In 1755, Lagrange was appointed teacher at the Royal School of Mathematics and Artillery in Turin and by 1757, he had helped to found the Academy of Turin. It was around this time that Lagrange began his correspondence with d'Alembert. George Sarton argues that while Lagrange's first scientific correspondence was with Euler and Fagnano, his first friendship was with d'Alembert.[2]

In 1763, Lagrange was invited to accompany the Marchese Domenico Caràccioli to London where the latter had been appointed ambassador of the King of Naples.[3] It was on this trip that Lagrange first introduced himself to Parisian scientific society. On this trip he met

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[1]Jean Itard,"Lagrange, Joseph Louis," in *The Dictionary of Scientific Biography* vol 7, (Detroit: Charles Scribner's Sons, 2008) 560.

[2]George Sarton "Lagrange's Personality (1736-1813)" in *Proceedings of the American Philosophical Society*, 88 (6) 1944, Copied from the Archive de l'Académie des Sciences, 476.

[3]Sarton, 458.

d'Alembert, Fontaine, and Clairaut, to name a few. Sadly, Lagrange was struck by illness and was unable to continue to London.

Lagrange returned to Turin where he found his financial and scientific situation somewhat poor. Before Lagrange left Paris, d'Alembert attempted to aid the young scientist by writing to Madame Geoffrin, who was acquainted with the ambassador of Sardinia. While d'Alembert's intervention appeared to have a positive response from the king and minister, by 1765 the promises had still not come to fruition.

In the autumn of 1765, d'Alembert solicited Frederick II of Prussia to offer Lagrange a position in Berlin. Lagrange had previously declined posts in Prussia, apparently because of shyness,[4] and replied to d'Alembert's efforts with "It seems to me that Berlin would not be at all suitable for me while Mr Euler is there."[5] When d'Alembert informed Lagrange of Euler accepting a position in St. Petersburg, Lagrange opened up to the idea and accepted the proposal of the King of Prussia to be director of the mathematics section of the Berlin Academy. Lagrange was to stay in Berlin until 1787, when he moved to Paris, where he stayed for the rest of his life.

It was during Lagrange's stay in Berlin that he first became acquainted with Laplace. When Laplace arrived in Paris, he went in search of employment and scientific recognition. As already seen, d'Alembert was quickly able to find Laplace employment at the École militaire, but Laplace wanted, and maybe needed, more. On January 1, 1773, d'Alembert sent Lagrange a letter asking four questions:

> 1° if there might now be a position open at the Academy of Berlin; 2° if he [Laplace] would be able to enjoy, on his entrance, a sufficient salary, such as 3000 or 4000 livres, in French money; 3° if you [Lagrange] are in a position to interest yourself in this matter without causing you worries; 4° if, in the case where you do not want to become involved, I may write to the King and propose to him M. de la Place as a subject who I know, who I esteem and of whom you may yourself also testify.[6]

While Lagrange may have been indebted to d'Alembert for the role the older scientist played in Lagrange's early career, Lagrange's response did not show this. Lagrange said that he

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[4]Itard, 561.

[5]As quoted in Itard, 563.

[6]In original French is found in appendix A, Correspondance de Lagrange avec d'Alembert in *Oeuvre de Lagrange* vol 13, 255

was not in a position to make a request to the King. Rather, he argues that d'Alembert should write the King himself, though not mention Lagrange at all. Lagrange did say that he saw the merit in the young man and thought that the Academy would find an excellent acquisition in him. Overall, this exchange came to little as Laplace was elected to the Paris Académie des Science three months later.

In d'Alembert's letter to Lagrange regarding Laplace, he stated that he was asking for Lagrange's recommendation for Laplace because "[Laplace] showed me a letter from you in which it appears to me that you were pleased with something that he had sent you."[7] The letter from Laplace and Lagrange's response have both been lost. The first remaining letter from the correspondence between Laplace and Lagrange is a letter from Lagrange dated March 15, 1773. It appears that when Lagrange wrote this letter Laplace was not yet a member of the Académie des Sciences. In this letter, Lagrange attempted to persuade Laplace to follow the advise he had given d'Alembert, for Laplace would surely soon be elected to the Paris Académie des Sciences and he himself would not be able to procure a position for Laplace at the Berlin Academy.

It is possible that this initial interaction with Laplace left Lagrange with mixed feelings about the young man. Sarton argues that Laplace's "vanity" annoyed Lagrange.[8] Sarton's proof for this seems to rest exclusively in a letter from Lagrange to Condorcet in 1774, where he said

> I am a little surprised by what you say of M. de la Place; it is methinks the weakness of young men to be puffed up with their first successes, but presumption diminishes later in proportion as science increases.[9]

While this letter does not appear to put Laplace in an entirely positive light, it seems unlikely that Lagrange continued in his original assessment of Laplace. Shortly after the series of exchanges regarding Laplace finding employment, their relationship seemed to change into one of mutual respect and admiration. This can be seen clearly in the correspondence between the two men. On April 10, 1775 Lagrange responded to Laplace's research on the determination of the secular inequalities of the planets by writing Laplace a letter. Here,

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[7]"[Laplace] m'a montré une Lettre de vous par laquelle il me paraît que vous êtes content de quelque chose qu'il vous a envoyé." Correspondence de Lagrange avec d'Alembert, 255

[8]Sarton, 478

[9]Sarton, 479

Lagrange explains that he had planned to send further memoirs to the Académie which would expand on his previous work on the theory of Jupiter and Saturn. After seeing the work of Laplace, Lagrange continued that

> As I see that you have embarked upon this research yourself, I abandon it happily and I know that you, similarly, would dispense with this work persuaded that science would only gain much from doing so.[10]

As we will see later in Chapter 7, Laplace may have been faced with this situation in the context of singular solutions of differential equations. Andoyer contends that "the respect of Lagrange for Laplace is... perfectly sincere."[11] Overall, while Lagrange might have been initially unimpressed with the ambition of the young man, he grew to respect the scientific ability that he showed.

There appears to be something of a dichotomy in the relationship between Laplace and Lagrange. On the one hand, there was the respect that caused them to praise each other's work, though there are times when this respect could be simply out of protocol. On the other hand, there was the comment that followed the majority of their compliments that no matter how perfect the work that had just been completed, it could still be improved. Hahn argues that "Each tried to upstage the other, such that their relationship was always tinged with both mutual admiration and jealousy."[12]

In the following chapters, we will investigate this relationship by looking at the memoirs of the two men, as well as the letters that they sent each other. In these memoirs, we can see the results of this friendly rivalry. The two scientists worked in similar areas and sometimes one had greater success and sometimes the other. We will look at a time when Lagrange had the greater success in the area of solving ordinary differential equations, in which his methods are still used today. Also, while we will not investigate when Laplace's work is recognized as better and more influential, we can point out that this is the case when looking at potential theory. Here, while Lagrange had laid the groundwork, the key early result is attributed to Laplace and the equation, $\nabla^2 V = 0$, still bears his name. In their correspondence, we will see the opinions that they gave each other on the other's work.

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[10]"Comme je vois que vous avez entrepris vous-même cette recherche, j'y renonce volontiers, et je vous sais même très bon gré de me dispenser de ce travail, persuadé que les sciences ne pourront qu'y gagner beaucoup." Correspondence de Lagrange avec Laplace, 60

[11]"L'estime de Lagrange pour Laplace est... partfaitement sincère." Andoyer. 24

[12]Hahn, *Pierre-Simon Laplace*, 74

These remarks are sometimes strong enough to even be included in the published versions. Overall, we will investigate the role that this friendly rivalry played in the development of Laplace as a scientist, concentrating on the period between 1772 and 1776, by looking at both the correspondence between Laplace and Lagrange as well as the content of both of their work.

We will start by looking at how Laplace went from using perturbation techniques, in *Sur le principe*, to developing them into what he called a general method, in *Recherches sur le calcul intégral et sur le système du monde*[13] (cited as *Sur le calcul intégral*). We will look at these two papers to discuss not only Laplace's method but his presentation of the ideas. We will discuss some of the drawbacks to Laplace's presentation which will be highlighted when we discuss Lagrange's work.

The first two papers by Laplace that we will discuss appear very different in flavour: the first appears more physical while the second can be seen as more mathematical. We mean physical in the modern sense; the study of the physical world. In Laplace's time, "physical" meant the combination of computations with measurment. In the first memoir that we will investigate, Laplace uses a method to aid him in his analysis of a specific example. Here, the mathematics are secondary to the results that they produce. In the second paper, Laplace is showing how to use the mathematics and then bringing these methods to the physical setting, thus allowing the mathematics to appear as the main topic rather than the physics. Some argue that Laplace's style, in general, places physics as the primary concern while the mathematics used to describe the physics is simply a means to an ends. This section will concentrate on the mathematical methods that Laplace developed and employed in his early work to solve problems in physical astronomy.

Laplace, instead, often omits the detailed procedures using words such as "clearly" or "easily" rather than including his steps. Morris Kline briefly compares the attitudes of Laplace to Lagrange with respect to their scientific endeavors.[14] Kline describes Lagrange as a mathematician and argues that "Laplace created a number of new mathematical methods that were subsequently expanded into branches of mathematics, but he never cared for mathematics except as it helped him to study nature."[15] Poisson went so far as to note,

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[13]Pierre-Simon de Laplace, "Recherches sur le calcul intégral et sur le système du monde" in *Oeuvres complètes de Laplace* vol viii (Paris: Gauthier-Villars, 1878-1912), 369-477.

[14]Morris Kline *Mathematical Thought from Ancient to Modern Times*, vol 2 (New York: Oxford University Press, 1990) 495

[15]Kline, 495

"In the issues he treated, Lagrange seemed more often to notice only the mathematics that were involved so that he prized the elegance of formulae and the generality of his methods; for Laplace, on the contrary, analysis was an instrument he wielded for the most varied of applications, always subordinating the specific method to the goal he pursued."[16] The tone that Poisson used is most likely more due to his relationship as protégé of Laplace than to a superiority of Laplace's methods. This view expressed by both Kline and Poisson appears consistent with the content and approach in both of the memoirs that we investigate here based on Laplace's approach to the mathematics required: he still skips the majority of the explanation that would make his method easy to follow; this is especially true in *Sur le principe.*

Still, it is interesting to note that the examples that Laplace explains in *Recherches* do not necessarily have a physical application: they are simply general equations that illustrate his method. In this context, Laplace is acting as a "mathematician" in Kline's sense and we can see Laplace citing Lagrange's work in both memoirs that we will discuss. Overall, Laplace keeps the mathematical explanation to a minimum, giving preference to a physical explanation. Laplace's habit of failing to give detailed explanations is one of our criticisms with his calling his method in *Recherches* general. We will address this further in Chapter 6.

This is in stark contrast to Joseph-Louis Lagrange. In looking at a small sampling of Lagrange's memoirs, it becomes apparent that he is trying to make the mathematics as accessible as possible. He appears to help the reader to understand the sometimes complicated procedures that he uses, especially when these procedures are not commonly used. We will next turn to Lagrange who began by looking at variation of constants as a method to find singular solutions, which he called particular solutions in his 1774 *Sur les intégrales particulières des équations différentielles*[17] or *Sur les intégrales particulières.* This paper continued work that Laplace had began with his 1772 *Mémoire sur les solutions particulières des équations différentielles et sur les inégalités seculaires des planètes*[18] or *Sur les solutions*

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[16] Siméon Denis Poisson, *Discours [prononcé aux funérailles de M. le Marquis de Laplace, le 7 mars 1827]* as quoted in Hahn, 256.

[17] Joseph-Louis Lagrange, "Sur les intégrales particulières des équations différentielles" in *Oeuvres de Lagrange* vol 4 (Paris: Gauthier-Villars, 1867-1892 [1774]) 5-108.

[18] Pierre-Simon de Laplace, "Mémoire sur les solutions particulières des équations différentielles et sur les inégalités seculaires des planètes," in *Oeuvres complètes de Laplace,* vol 8 (Paris: Gauthier-Villars, 1878-1912 [1772]) 352-367.

particulières. By looking at these two papers, in particular, we can see the strong differences in the styles of the two men. Lagrange then used variation of parameters to solve inhomogeneous ordinary differential equations in his 1775 *Recherches sur les suites récurrentes dont les terms varient de plusieurs manières différentes ou sur l'intégration des équations linéaires aux différences finies et partielles; et sur l'usage de ces équations dans la théories des hasards*[19] or *Sur les suite.* While this long paper dealt with other issues, we will focus on his presentation of variation of parameters. Next, Lagrange used the method of variation of parameters to solve problems relating to the mean movement of the planets in his 1776 *Sur l'altération des moyen mouvements des planètes,*[20] or *Sur l'altération* Lagrange continued to use these methods in several other papers, which we will not discuss here. In looking at these papers by Lagrange, we can see the evolution of the idea and the parallels between Lagrange's work and Laplace's. This will be investigated fully in Chapter 7.

By comparing Laplace's approach to asymptotics, and solving ODE's in general, with Lagrange's not only in method but in presentation, we can gain a different perspective on their relationship.

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[19]Joseph-Louis Lagrange, "Recherches sur les suites récurrentes dont les terms varient de plusieurs manières différentes ou sur l'intégration des équations linéaires aux différences finies et partielles; et sur l'usage de ces équations dans la théories des hasards" in *Oeuvres de Lagrange* vol 4 (Paris: Gauthier-Villars, 1867-1892 [1775]) 151-251.

[20]Joseph-Louis Lagrange, "Sur l'altération des moyen mouvements des planètes," in *Oeuvres de Lagrange* vol 4 (Paris: Gauthier-Villars, 1867-1892 [1776]) 255-271.

Chapter 6

Laplace and Asymptotics

Overall, *Sur le Principe de la gravitation universelle et sur les inégalités séculaires des planètes qui en dépendent* addresses, much as its title states, the principles of universal gravitation and the secular inequalities of the planets which depend on it. Here, we will only discuss the first half of the paper, which investigates universal gravitation. We are limiting ourselves to this portion of the paper as it illustrates Laplace's method of approaching a problem and introduces Laplace's use of asymptotics. He began his investigation by deriving equations for the force and for the moment of force of an attracting mass on a point, then looking at four assumptions regarding universal gravitation:

1. Attraction is directly proportional to mass and inversely proportional to the square of the distance,

2. The attractive force of a body is the result of the attraction of each part of which the body is composed,

3. This force is instantaneously propagated from the body attracting to the one being attracted,

4. This force behaves the same whether the body is at rest or in motion.

We will investigate his work investigating the assumption that gravitational force is propagated instantaneously, or that gravity acts instantaneously at a distance. In questioning this assumption, Laplace arrived at a differential equation that he did not have the methods to solve. Laplace dealt with this problem by making a series of simplifications until he arrived at an equation that he could unravel: a linear second order ordinary differential equation.

The simplification that Laplace used is based on the idea that the orbit of each planet is only slightly different from a circle. Therefore, the orbit can be modeled on an ellipse produced by slightly perturbing a circle. Since this perturbation is small, Laplace was able to neglect terms involving the square of this perturbation. This idea itself was not new, but Laplace was able to use it to find new results based on the final equation that he found.

Wilson discusses those who had contributed to this nascent field of perturbation theory before Laplace such as Euler, Clairaut and d'Alembert.[1] In this paper Wilson does not explicitly show how perturbation theory was used by Euler or Clairaut and therefore it is difficult to compare the usage made by these men with that of Laplace without looking directly at their work. Since our present topic is not a history of perturbation theory but rather a look at Laplace's methodology, we will not look deeply at these other works. However, we can note from Wilson that Laplace had used the same substitution to introduce his perturbation as Lagrange had used in an earlier work. The way that Laplace then maneuvers his small values is different from Lagrange, and in fact Laplace presented his method explicitly as a general method. Wilson also looks explicitly, though briefly, at *Sur le principe* when discussing Laplace's own contributions to the field.

Sur le principe represents Laplace's first published work in physical astronomy and in this memoir Laplace walks the reader through his analysis. While Laplace helps the reader follow his progression, he leaves out many steps, in contrast to Lagrange's more effective expository style. In the sampling of Lagrange's work that we will investigate, he showed the reader exactly what he was thinking and rarely required the reader to interpolate steps. In the present paper, we will show steps that can fill in those missing in Laplace's simplification and investigate the reasoning that Laplace may have used to make this progression. In looking at this simplification, we will attempt to address why Laplace proceeded in the way that he did and whether he believed himself to be justified in what he was doing; we will see that Laplace's apparent path to his simplified equation involves steps for which rigorous justification would be difficult.

Shortly after this memoir was written, Laplace wrote a further memoir, *Recherches sur le calcul intégral et sur le système du monde*. Stigler provides some clues about this memoir by suggesting that the first portion of *Sur le calcul intégral* may have been presented to the

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[1]Curtis A. Wilson "Perturbations and Solar Tables from Lacaille to Delambre: the Rapprochement of Observation and Theory, Part II" published in *Archive for the History of the Exact Sciences*, vol 22 (Springer-Verlag: 1980) 198-304

Academy in 1774 under the name "Sur le calcul intégral."[2] In this paper, Laplace presents the method he used in *Sur le principe* as a general method for simplifying differential equations. Here Laplace again walks the reader through his steps but with specific examples to illustrate his points. In what follows, we will look at how Laplace moved from merely using his method in *Sur le Principe* to presenting this method as a general one for solving ordinary differential equations. Again, we will discuss whether this claim of generality was justified, and identify potential problems with regarding this as a general method.

6.1 Laplace's Initial Use of Perturbation Theory

In the first example we examine, Laplace was trying to show that gravity need not be propagated instantaneously as well as trying to determine the speed at which gravity is propagated.[3] Laplace began by assuming that gravity is produced by the impulse of a corpuscle infinitely smaller than the body on which it is acting: this is depicted in Figure 6.1 based on the diagram that Laplace included when describing the system under discussion. In this diagram, N is the corpuscle, p is the attracted body and S is the body causing the attraction. This is a confusing model. Laplace did not explain the physical meaning behind this model at all and therefore we are left only to guess at his possible assumptions. It appears that Laplace was assuming that some corpuscle is sent from each body and pushes these bodies together upon arrival at its "target." This does not seem to make sense and appears counter-intuitive because these corpuscles would be required to travel around the bodies to push them together. In any case the appearance of this corpuscle would seem to imply that Laplace is dealing with a three body problem since there is now the attracted body, the attracting body and the corpuscle causing the attraction. Laplace did not set up the problem in this way and neglects any gravitational influence produced by the mass of the corpuscle. Laplace may have believed this to be reasonable because he had supposed the corpuscle to be of a mass that would be inconsequential, or he may have considered it to be free of mass. In a modern interpretation, we could think of this corpuscle as being massless and only acting to communicate information, ie attract in this direction. Whatever the reason, Laplace proceeded without altering his equations as would be necessary if he were

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[2]Stigler, 253.

[3]All primary information presented in this section is found in Pierre Simon Laplace, *Sur le principe de la gravitation universelle et sur les inégalités séculaire des planètes qui en dépendent*, 221-224

including the gravitational impact of the corpuscle and we can only note that we cannot tell how this corpuscle acts.

While Laplace did not reference George-Louis Lesage (1724-1803), the resemblance of Laplace's corpuscles to Lesage's "otherworldly particles" (*particules ultramondaines*) warrants some comparison. Lesage is best known for his work explaining gravity mechanistically.[4] This theory relies on particles that were extremely small and moved at very high frequency and were called "otherworldly" because of their exemption from universal gravitation. An isolated mass would not be affected by the particles, because this mass would be hit by a balanced amount of particles from all directions. But, in the case of two masses, each mass would create a shadow blocking the particles from reaching the other mass causing an attraction. Lesage was easily able to argue that the relation would vary inversely with respect to distance. But to show that the relation was directly with respect to the mass instead of the surface area, he argued that this was due to the porosity of the body creating the shadow. This appears to represent Laplace's model. Laplace did not always reference his sources and therefore it is not unusual here that he did not give credit to Lesage. Lesage's theory was not universally accepted, which makes Laplace's rather offhanded mention of the corpuscle seem reasonable. Still, Lesage was a member of the Académie which Laplace hoped to gain membership to and therefore using the method of a more senior academician would seem to make sense politically. It should also be noted that while Laplace did correspond with Lesage, this began well after the writing of this memoir.

6.2 Brief Note on the Nature of a Function

There is some confusion when reading both Laplace and Lagrange about the nature of an equation. We are often told, for example, that the equation $F = 0$ is to be considered a function of certain variables. Today, this would normally be written as $F(x, y) = 0$, where F is a function of x and y. Then if we were to take the derivative of F, we would need to use partial derivatives. For Laplace and Lagrange, we are told often that dt is to be kept constant. This means that t is to be taken as the independent variable, and if we take derivatives, they would be total differentials with respect to t. This confusion is somewhat clarified when we look at the definition of a function as given in the *Encyclopédie*. Here, we

---

[4]J.B. Gough, *Lesage, George-Louis*, in *Complete Dictionary of Scientific Biography* vol 8 (Detroit: Charles Scribner's Sons, 2008) 261.

are told

> Today, we call a function of x, an algebraic quantity composed of a many terms
> as we could want and of which x finds itself in any manner, mixed or not, with
> constants. All the terms of a function of x are supposed to have the same
> dimension.[5]

Therefore, we can see that when both Laplace and Lagrange are writing, they are using a
definition of function which was not so different from Euler.[6] Since set theory did not exist
at this time, we can expect not to find any trace of it here. Still, the definition used brings
some level of understanding into the usage that we encounter. In general, we will follow the
original authors' naming of functions and equations.

6.2.1   Remarks on Notation

Laplace began by introducing the notation Ψ, Ψ', Ψ'' as the components of the forces parallel
to the z, y and x directions respectively. He also named M the attracted body. He does
not call this the mass of the body but the body itself. At times, such as when Laplace
derived his equations of motion, M does appear to represent mass, but Laplace seems to use
M differently depending on the context. This can be seen later when he uses the notation
$\frac{\Psi''}{M}$. While this appears to be the force divided by the body, it seems more likely that
this notation is similar to the modern subscript and instead represents the force on the
body M. We see this notation again in the section under investigation here. Here, we
have a body p and it appears that $\frac{\Psi''}{p}$ is used in the same manner. As well, there is some
confusion because Laplace uses p to represent the body itself as well as its position in space.
Again, Laplace appears to use p differently depending on the context making his notation
somewhat confusing. Throughout, we have remained loyal to Laplace's notation but have
made comments when this notation is troublesome.

---

[5]"Aujourd'hui on appelle fonction de x, une quantité algébrique composée de tant de termes qu'on voudra
& dans lesquelles x se trouve d'une maniere quelconque, mêlée ou non, avec des constantes. Tous les
termes d'une fonction de x sont censés avoir la même dimension." Editors M. Diderot and M d'Alembert,
Encyclopédie, "Table analytique," (Paris: Panckoucke, 1780), 1177.

[6]Victor J. Katz *A History of Mathematics: An Introduction, 3rd Edition* (Boston: Addison-Wesley, 2009),
618.

6.2.2 Setting up the Physical Model

Laplace began by setting up the situation as described in Figure 6.1, where there is an infinitely small body p moving around S, which is considered as immobile, with an orbit on the fixed plane pSM. He then labeled r and ϕ as in the diagram.[7] Next, Laplace used the notation $\frac{\Psi'}{p}$ to denote the force perpendicular to Sp and acting in the same sense as the movement of the planet. The planet is assumed to be the infinitely small body p though he did not here or otherwise say that p refers to a planet. While this seems odd to call a planet infinitely small, nevertheless, in comparison to a more massive body, such as the sun, this may be assumed to be the case, which is the example that Laplace used later. He further defined $\frac{\Psi''}{p}$ as the force acting along Sp directed from S to p. The notation is curious to our eyes, but, as mentioned above, division does not appear to be implied. Based on how Laplace has defined Ψ'' and Ψ' previously in this paper, the reader can assume that Laplace wished to have NO be the x-axis and qO be the y-axis. Laplace later stated that these two are perpendicular but does not here or anywhere define these as his axes. This could represent a style of the time, but appears odd to the modern reader.
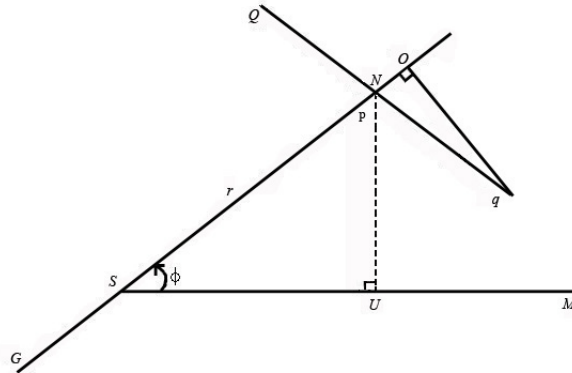
Figure 6.1: Reproduction of Laplace's diagram to show his system involving a corpuscle transmitting gravity

Also, based on Laplace's previous definition of Ψ'' and Ψ' it becomes difficult to determine what form he was giving to his variable p, which has been used to describe both the infinitely

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[7]In Laplace's version of this diagram neither these variables, the right angle, the point U nor the perpendicular from p to U are included.

small body and a point on his diagram. At the beginning of the paper, Laplace stated that Ψ'' is the sum of the forces perpendicular to the x'-axis, which in turn is perpendicular to the x-axis and similarly for Ψ'. If this is still the case, then p must be a dimensionless quantity. Laplace did not describe this and it is left to the reader to assume. Later, we can infer the nature of these variables based on how Laplace replaced them in the equations that he had previously derived.

6.2.3   Setting up the Equations

After Laplace had explained the configuration of his system, he moved on to look at the equations of motions. This is the first place that we see him use his form of simplification to arrive at a result that he could solve exactly. Previously in this paper, Laplace had derived a system of equations to determine the force on a body caused by the gravitational attraction of another body.

Laplace derived these equations by starting with the analytical version of Newton's second law

$$\Psi'' - M\frac{d^2x}{dt^2} = 0$$
$$\Psi' - M\frac{d^2y}{dt^2} = 0 \text{ and}$$
$$\Psi - M\frac{d^2z}{dt^2} = 0.$$

Laplace then substituted

$$x = r\cos\phi, \ y = r\sin\phi, \ \text{and} \ z = rs$$

thereby changing to cylindrical coordinates. Following through with this substitution Laplace found

$$\frac{d\phi}{dt} = \frac{1}{r^2}\left(C + \int \frac{\Psi' r dt}{M}\right)$$
$$0 = \frac{d^2r}{dt^2} - \frac{1}{r^3}\left(C + \int \frac{\Psi' r dt}{M}\right)^2 - \frac{\Psi''}{M}$$

and

$$0 = \frac{d^2s}{dt^2} + \frac{2dsdr}{rdt^2} + \frac{s}{r^4}\left(C + \int \frac{\Psi' r dt}{M}\right)^2 - \frac{S\Psi'' - \Psi}{M}.$$

For the physical system described in Figure (6.1), Laplace replaced the first two of these equations with ones more specific to his current system:

$$\frac{d\phi}{dt} = \frac{1}{r^2}\left(c + \int \frac{\Psi'}{p} r\, dt\right)$$
(6.1)

and

$$0 = \frac{d^2 r}{dt^2} - \frac{1}{r^3}\left(c + \int \frac{\Psi'}{p} r\, dt\right)^2 - \frac{\Psi''}{p}.^{8}$$
(6.2)

The appearance of c represents an integration constant. Laplace used the capital and lower case c and C and we have simply followed his example.

Laplace wanted to be able to solve (6.1) and (6.2) for ϕ and r, but before he was able to do this he must first determine Ψ'' and Ψ'. At this point, he named N the corpuscle that gravitates p towards S. We should note that both N and p are located at the same point. This makes sense as N must be at the same place as p in order to communicate to it. He then argued that if p is at rest, N will communicate to it towards S the force $\frac{S}{r^2}$. Here again, it becomes difficult to determine what quantity the variable S is describing. For simplicity, we will consider S to be a placeholder, which we can investigate further later. Next, Laplace let pG be the distance that the corpuscle travels in the time that the body p travels pQ, which is tangent to the orbit of the body. Laplace set pQ equal to pq. Here again, there is some confusion with labeling, as p represents both the body and its position. As well, Laplace called the quantities pG, pQ and pq both distances and speeds. For simplicity, we will consider these to be distances. When Laplace talked of the speed pQ, for example, this will be considered to be the speed of the body p moving from the point p to G. He next argued that p is animated by a force $\frac{S}{r^2}$ directed from p to S and a force $\frac{S}{r^2}\frac{pQ}{pG}$ directed from p to q. He now let $\frac{\theta}{\alpha}$ be the distance that the corpuscle travels in the time T, where T and α are constants, α being an extremely small numerical coefficient and θ being a variable depending on any function of the distance from p to S. Laplace then was able to produce the relation

$$\frac{pQ}{pG} = \alpha T \frac{\sqrt{dr^2 + r^2 d\phi^2}}{\theta\, dt}.$$

This equation itself seems to warrant some further explanation.

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[8]We have retained Laplace's notation here.

First this equation appears to make more sense when written as follows

$$\frac{pQ}{pG} = \overbrace{\frac{\alpha}{\theta}T}^{(*)} \underbrace{\frac{\sqrt{dr^2 + r^2 d\phi^2}}{dt}}_{(**)}.$$

Based on Laplace's definitions, we can see that $(*)$ and $(**)$ respectively represent the inverse of the speed of a body moving the distance pG and the speed of a body moving the distance pQ. Looking first at $(*)$, we can see this is simply the inverse of $\frac{\theta/\alpha}{T}$, which represents the speed of the corpuscle from p to G based on Laplace's previous explanation where $\frac{\theta}{\alpha}$ is the distance the corpuscle moves in time T. Moving on to $(**)$ is somewhat more complicated. If the triangle NOq is similar to the triangle pUS, which has been added in the diagram, and if, as it appears, Laplace was setting NO to be dr then using a small angle approximation, Laplace would be able to determine the change in the length of pq to be $\sqrt{dr^2 + r^2 dq^2}$. The term $(**)$ follows directly from this.

Laplace then noted that the force $\frac{S}{r^2}\frac{pQ}{pG}$ is given by $\frac{S}{r^2}\alpha T\frac{\sqrt{dr^2+r^2d\phi^2}}{\theta dt}$. He decomposed this term into components, along what we have here called the x and y axes, which are respectively

$$-\frac{\alpha S T dr}{\theta r^2 dt}$$

and

$$\frac{S}{r^2}\frac{T\alpha r d\phi}{\theta dt}.$$

Laplace was now able to conclude "easily," as he puts it, that

$$\frac{\Psi''}{p} = -\frac{S}{r^2} - \frac{\alpha S T dr}{\theta r^2 dt}$$

and

$$\frac{\Psi'}{p} = -\frac{\alpha S T d\phi}{\theta r dt}.$$

This can be seen as simply summing up the components in the x-direction and y-direction respectively, noting that Laplace had already stated that there was a force from p to S (the x-direction) of $\frac{S}{r^2}$. Laplace has now reached his first goal of determining the values of $\frac{\Psi''}{p}$ and $\frac{\Psi'}{p}$, therefore he substituted these values into (6.1) and (6.2) and found

$$\frac{d\phi}{dt} = \frac{1}{r^2}\left(c - \int \frac{\alpha S T d\phi}{\theta}\right) \tag{6.3}$$

and

$$0 = \frac{d^2r}{dt^2} - \frac{1}{r^3}\left(c - \int \frac{\alpha S T d\phi}{\theta}\right)^2 + \frac{S}{r^2} + \frac{\alpha S T dr}{\theta r^2 dt}. \tag{6.4}$$

6.2.4   The Simplification

We will now concentrate on equation (6.4), and Laplace's simplification of this equation. Since this equation is still not solvable as is, Laplace now assumed that the orbit of the planet is nearly circular and has nearly uniform angular velocity. This allowed him to define

$$r = a(1 + \alpha y) \text{ and } \phi = nt + \alpha x. \tag{6.5}$$

This represents a small perturbation from a circle which would have had radius, $r = a$ and polar angle, $\phi = nt$. He defined α to be a small constant such that quantities of order α^2 can be neglected. It should be noted that this is the same α that Laplace was using before.[9] This turns out to be the key feature of Laplace's simplification technique though, as we shall see, his methodology is somewhat surprising.

Laplace began by looking at (6.1) and assumed θ to be constant. This allowed him to conclude that

$$\int \frac{\alpha S T d\phi}{\theta} = \frac{\alpha S T n t}{\theta}. \tag{6.6}$$

If we break this step up we can easily see how Laplace arrived at this conclusion. First, after substituting Laplace's definition of ϕ, he found

$$\int \frac{\alpha S T d\phi}{\theta} = \int \frac{\alpha S T d(nt + \alpha x)}{\theta}.$$

Since θ is assumed constant we can move most of the variables outside of the integral

$$\int \frac{\alpha S T d\phi}{\theta} = \frac{\alpha S T}{\theta} \int d(nt + \alpha x).$$

Therefore, we end up with
$$\int \frac{\alpha S T d\phi}{\theta} = \frac{\alpha S T}{\theta}(nt + \alpha x).$$

Since Laplace has told us that we can neglect terms of order α^2, we can neglect the second term here and we end up with (6.6).

This is the first example of Laplace neglecting terms involving α^2, but he was, in general, not consistent in this usage. As we shall see, it is impossible to determine his methods exactly from the steps that he included, which makes checking the validity of his steps difficult.

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[9]This is practically the same substitution that Lagrange made, though Lagrange used different notation as seen in Wilson, "Perturbations and solar tables from Lacaille to Delambre: the rapprochement of observation and theory," 198.

After substituting the expressions from (6.5) for r and ϕ and using (6.6), Laplace now quickly arrived at the simplification that (6.4) is the same as

$$0 = \frac{d^2y}{dt^2} + \frac{1}{\alpha}\left(\frac{S}{a^3} - \frac{c^2}{a^4}\right) + \left(\frac{3c^2}{a^4} - \frac{2S}{a^3}\right)y + \frac{2S}{a^3}\frac{cT}{a\theta}nt \tag{6.7}$$

which again appears to require more explanation than Laplace provides. This is very close to the final differential equation that Laplace will solve. We will suggest intermediate steps that Laplace did not include so as to decipher Laplace's simplification strategy to arrive at a solvable second order differential equation.

If we start from the initial substitutions of (6.6) and the expressions from (6.5) for r and ϕ into (6.4), we have

$$\begin{aligned}
0 =& a\alpha\frac{d^2y}{dt^2} - \frac{1}{a^3(1+\alpha y)^3}\left[c^2 - \alpha\frac{2cSTnt}{\theta} + \alpha^2\left(\frac{STnt}{\theta}\right)^2\right] \\
& + \frac{S}{a^2(1+\alpha y)^2} + \frac{\alpha ST}{\theta a^2(1+\alpha y)^2}a\alpha\frac{dy}{dt}.
\end{aligned}$$

The terms involving $\frac{1}{1+\alpha y}$ are problematic. We will now examine two possible ways that Laplace may have dealt with these terms, both of which have difficulties.

Method 1: Geometric series expansion

We can start by using a geometric series expansion for this contentious term:

$$\frac{1}{1+\alpha y} = 1 - \alpha y + \alpha^2 y^2 - \cdots.$$

This expansion is only true for $-1 < \alpha y < 1$. Since y is derived from the orbit of the planet, we can start by assuming that y is bounded and therefore, given α small enough, we can use this approximation.

This means that we have

$$\left(\frac{1}{1+\alpha y}\right)^2 = 1 - 2\alpha y + 3\alpha^2 y^2 - \cdots \text{ and}$$

$$\left(\frac{1}{1+\alpha y}\right)^3 = 1 - 3\alpha y + 6\alpha^2 y^2 - \cdots.$$

Returning to our original DE, we have

$$
\begin{aligned}
0 =& a\alpha \frac{d^2y}{dt^2} \\
& - \frac{1}{a^3}\left(1 - 3\alpha y + 6\alpha^2 y^2 - \cdots\right)\left[c^2 - \alpha\frac{2cSTnt}{\theta} + \alpha^2\left(\frac{STnt}{\theta}\right)^2\right] \\
& + \frac{S}{a^2}\left(1 - 2\alpha y + 3\alpha^2 y^2 - \cdots\right) \\
& + \alpha^2\frac{ST}{a^2\theta}\left(1 - 2\alpha y + 3\alpha^2 y^2 - \cdots\right)\frac{dy}{dt}.
\end{aligned}
$$

Laplace told us to remove all terms containing α^2 or higher. Therefore, we are left with

$$
0 = a\alpha\frac{d^2y}{dt^2} - \frac{c^2}{a^3} + 3\alpha\frac{c^2}{a^3}y + \alpha\frac{2cSTnt}{a^3\theta} + \frac{S}{a^2} - 2\alpha\frac{S}{a^2}y, \tag{6.8}
$$

which, after rearranging, can be written as equation (6.7).

But there are problems with this method. First, we made the geometric expansion based on the assumption that y is bounded. This matter will be discussed when we look at the general solution that Laplace obtains and compare the two methods.

Method 2: Partial fractions

To be able to use this method, we must start by rewriting the ODE as

$$
\begin{aligned}
0 =& \frac{d^2y}{dt^2} - \frac{1}{a^4}\frac{1}{\alpha(1+\alpha y)^3}\left[c^2 - \alpha\frac{2cSTnt}{\theta} + \alpha^2\left(\frac{STnt}{\theta}\right)^2\right] \\
& + \frac{S}{a^3}\frac{1}{\alpha(1+\alpha y)^2} + \frac{\alpha^2 ST}{\theta a^2}\frac{1}{\alpha(1+\alpha y)^2}\frac{dy}{dt}.
\end{aligned} \tag{6.9}
$$

Let us first examine the term, $\frac{1}{\alpha(1+\alpha y)^3}$. We can expand and write

$$
\frac{1}{\alpha(1+\alpha y)^3} = \frac{1}{\alpha(1 + 3\alpha y + 3\alpha^2 y^2 + \alpha^3\alpha^3 y^3)}.
$$

Laplace told us to remove all terms containing α^2 or higher powers. This means we have

$$
\frac{1}{\alpha(1+\alpha y)^3} \approx \frac{1}{\alpha(1 + 3\alpha y)}, \tag{6.10}
$$

or, using partial fractions

$$
\frac{1}{\alpha(1+\alpha y)^3} \approx \frac{1}{\alpha} - \frac{3y}{1 + 3\alpha y}.
$$

Similarly, we can say,

$$\frac{1}{\alpha(1+\alpha y)^2} \approx \frac{1}{\alpha} - \frac{2y}{1+2\alpha y}.$$

This is still somewhat problematic because y in the denominator does not appear anywhere in equation (6.7). Therefore, if this method was used by Laplace, there must be some way of dealing with these terms. Unfortunately, Laplace did not give any indication of what to do in this case because there are no α^2 terms. If, in modern terms, we could say that $1 \gg \alpha y$, then we could write

$$\frac{1}{\alpha(1+\alpha y)^3} \approx \frac{1}{\alpha} - 3y \text{ and}$$
$$\frac{1}{\alpha(1+\alpha y)^2} \approx \frac{1}{\alpha} - 2y,$$

but there is no indication that Laplace would have thought this way. What we might also assume is that if Laplace had proceeded this way, he may have thought that since the y term in equation (6.10) had an α^2, we can remove this term after we no longer require it. We could not have removed it earlier because we would not have been able to use partial fractions without it.

Substituting into equation (6.9), we have

$$0 = \frac{d^2 y}{dt^2} - \frac{1}{a^4}\left(\frac{1}{\alpha} - 3y\right)\left[c^2 - \alpha\frac{2cSTnt}{\theta} + \alpha^2\left(\frac{STnt}{\theta}\right)^2\right]$$
$$+ \frac{S}{a^3}\left(\frac{1}{\alpha} - 2y\right) + \frac{\alpha^2 ST}{\theta a^2}\left(\frac{1}{\alpha} - 2y\right)\frac{dy}{dt}.$$

Again, Laplace has told us to remove all terms involving α^2. After doing this we find,

$$0 = \frac{d^2 y}{dt^2} - \frac{1}{\alpha}\frac{c^2}{a^4} - \frac{2cSTnt}{a^4\theta} - \alpha\left(\frac{STnt}{\theta}\right)^2 + 3y\frac{c^2}{a^4} + 3\alpha y\frac{2cSTnt}{a^4\theta} \qquad (6.11)$$
$$+ \frac{1}{\alpha}\frac{S}{a^3} - 2y\frac{S}{a^3} + \alpha\frac{aST}{\theta a^3}\frac{dy}{dt},$$

but this is not the same as equation (6.7).

We can find Laplace's answer in two ways: either we can remove terms containing α^2 before expanding or we can multiply through by α and then remove the terms that contain α^2.

Proceeding in the first way, we write our ODE as

$$0 = \frac{d^2 y}{dt^2} - \frac{1}{a^4}\left(\frac{1}{\alpha} - 3y\right)\left(c^2 - \alpha\frac{2cSTnt}{\theta}\right) + \frac{S}{a^3}\left(\frac{1}{\alpha} - 2y\right),$$

which after expanding and rearranging is the same as equation (6.7).

Otherwise, we can multiply equation (6.11) by α and proceed with the substitution and we find

$$
\begin{aligned}
0 =& \alpha \frac{d^2 y}{dt^2} - \frac{c^2}{a^4} - \alpha \frac{2cSTnt}{a^4\theta} - \alpha^2 \left(\frac{STnt}{\theta}\right)^2 + 3\alpha y \frac{c^2}{a^4} + 3\alpha^2 y \frac{2cSTnt}{a^4\theta} \\
& + \frac{S}{a^3} - 2\alpha y \frac{S}{a^3} + \alpha^2 \frac{aST}{\theta a^3} \frac{dy}{dt}.
\end{aligned}
$$

If we now removed all terms involving α^2 and rearrange our terms, we find equation (6.7). No matter which way Laplace may have proceeded, there remains that question of where the expansion is valid. This question will be addressed below.

6.2.5 General Solution and Potential Problems

Beginning with the simplified ODE, equation (6.7)

$$
0 = \frac{d^2 y}{dt^2} - \frac{1}{\alpha}\left(\frac{S}{a^3} - \frac{c^2}{a^4}\right) + \left(\frac{3c^2}{a^4} - \frac{2S}{a^3}\right) y + \frac{2S}{a^3}\frac{cT}{a\theta}nt,
$$

the easiest way to proceed, which is most likely different to how Laplace would have, is to first multiply through by α and set $\alpha = 0$. This gives

$$
0 = \frac{S}{a^3} - \frac{c^2}{a^4}.
$$

Laplace does not show any intermediate steps but rather states that "It is clear that $\frac{S}{a^3} - \frac{c^2}{a^4}$ must be of the order of α, and, since a is arbitrary, I will suppose $\frac{S}{a^3} = \frac{c^2}{a^4}$." [10] Since no other terms contain α, this does seem clear and does not require extra explanation as long as Laplace's intended audience would have understood.

Laplace then arrived as the simple ODE

$$
0 = \frac{d^2 y}{dt^2} + \frac{c^2}{a^4}y + \frac{2c^3}{a^6}\frac{a}{\theta}Tnt, \tag{6.12}
$$

His next step was to integrate equation (6.12). Using methods available to Laplace, the answer can be written as

$$
y = c_1 \cos\frac{c}{a^2}t + c_2 \sin\frac{c}{a^2}t - \frac{2c}{a^2}\frac{a}{\theta}Tnt,
$$

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[10]"Il est clair que $\frac{S}{a^3} - \frac{c^2}{a^4}$ doit être de l'ordre de α, et, comme a est arbitraire, je supposerai $\frac{S}{a^3} = \frac{c^2}{a^4}$," Laplace, 223.

where c_1 and c_2 are constants which can be determined from the conditions of the problem. Laplace instead writes

$$y = K \cos{(\frac{c}{a^2}t + \epsilon)} - \frac{2c}{a^2}\frac{a}{\theta}Tnt$$

where K and ϵ are arbitrary constants. These two can be easily seen as equivalent by looking at the trigonometric identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos{(\alpha + \beta)}$. Therefore, $c_1 = K\cos\epsilon$ and $c_2 = -K\sin\epsilon$. This substitution could also be made using sine in place of cosine. Laplace uses sine for the x-term which he solves next though, theoretically, either could be used in each case.

Here is where we see the problem with both methods of reaching Laplace's simplified ODE: both relied on y being bounded. This means that Laplace is working with a non-converging infinite series using either method. But, because Laplace has shown none of his steps in reaching his simplified equation, equation (6.7), we do not know for sure if this is the case. There may be another method which does not rely on a non-converging infinite series, but based on the information that Laplace has given, it is only possible to guess at his method. The use of a non-convergent series is not unlikely. Jahnke says that

> Investigations of convergence did not play a systematic part in 18th-century analysis. However, most mathematicians at that time were usually well aware of for which values and how fast the series under consideration converged or diverged.[11]

This may well have been true for Laplace, but the series that we think that Laplace has used would only be useful for values of $y < \frac{1}{\alpha}$. This would imply that it would also only be useful on small time scales, but Laplace's main purpose, in the end, is to make a conclusion for large time scales. Therefore, there can be some doubt as to whether he knew or cared that he was using a series which was divergent over large time scales.

Returning to Laplace's explanation, he now supposed that the line SM is where the body p is placed at the first instant of movement which is also the place of the aphelion of the orbit of p, if $\frac{a}{\theta} = 0$. He then named αe the relation of the primitive eccentricity to the mean radius of the orbit and concluded that $K = e$ and $\epsilon = 0$.

Laplace now investigated equation (6.3), which he simplified in a method similar to the one he used to simplify (6.4). He set $n = \frac{c}{a^2}$ and ended up finally with

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[11]H.N. Jahnke, "Algebraic Analysis in the 18th Century" in *A History of Analysis* (Providence, RI: American Mathematical Society, 2003) 121.

$$r = a(1 + \alpha e \cos nt - 2\frac{a\alpha}{\theta}Tn^2t)$$

and

$$\phi = nt - 2\alpha e \sin nt + \frac{3}{2}\frac{a\alpha}{\theta}Tn^3t^2.$$

From the last term in the equation for ϕ, Laplace was able to conclude that the body p is subjected to a secular equation proportional to the square of time. A secular equation is simply one that varies with time. This is also how the term "secular inequality" can be interpreted. He argued that in "ordinary" investigations $\frac{a\alpha}{\theta}$ is considered to be infinitely small and this last term disappears. Laplace stated that if this term cannot be neglected, then it would cause a change in the mean movement of the planets. Laplace proceeded to use this part of the equation to determine the speed of the propagation of gravity. In the end, Laplace calculated the distance that the corpuscle travelled in one minute and compared this distance to the speed of light. Laplace was then able to find that the speed of the corpuscle was 7 680 000 times greater than the speed of light.

Overall, we see that Laplace has not provided enough information for the reader to understand what he is doing. If Laplace used the expansions that we have here suggested, there should be some limitation placed on their validity and this limitation would hinder the physical interpretation that Laplace next made. Basically, here would be a place that more explanation would not only be helpful; it would seem to be necessary. As we shall see, even when Laplace was trying to explain this method, he still did not provide the necessary explanation to make himself clear.

6.3 Generalization

In the second paper that we will investigate, *Sur le calcul intégral*, Laplace attempted to generalize the simplification method that he used in *Sur le principe*. This long memoir - over 130 pages in total - investigates specific differential equations, which Laplace solves using his methods. These examples do not necessarily have an astronomical context and for the most part simply represent different forms of D.E's. After having explained his method, Laplace turns to the theory of planets and attempts to show the applicability of his investigation. In this paper, we will concentrate only on the first example that Laplace

explained and examine what he is doing and why.[12]

Laplace began this example by simply stating, "Let it be proposed to integrate the differential equation

$$\frac{d^2y}{dt^2} + y = \alpha y \cos 2t."^{13} \tag{6.13}$$

Laplace did not provide any motivation for looking at this specific equation, nor does this equation appear to represent the perturbation situation that the above example does. Therefore, the physical context that Laplace had before does not appear, but this example illustrates his method. It appears that this is the reason that it is used rather than any applicability that it might have. While he was still not being as systematic as Lagrange was when describing his method in *Sur l'attraction*, we can see this as resembling Lagrange's approach simply because Laplace has chosen an equation for its mathematical properties rather than its physical applicability.

Returning to Laplace's presentation of his method, he next stated that α is a very small quantity and t will be the independent variable.[14] Laplace then turned to integrating the simple D.E.

$$\frac{d^2y}{dt^2} + y = 0.$$

This is the normal approach to this type of problem in the modern context: to first solve the problem by setting $\alpha = 0$. Laplace solved this easier equation, and found

$$y = p \sin t + q \cos t$$

where p and q are found from the initial conditions.

Laplace now inserted what could easily be described as a perturbation. He set the solution to (6.13) to be a perturbation of the simpler equation, or

$$y = p \sin t + q \cos t + \alpha z. \tag{6.14}$$

Laplace now substituted this into (6.13) and found a new differential equation with z as the dependent variable

$$\frac{d^2z}{dt^2} + z = \frac{p}{2} \sin 3t - \frac{p}{2} \sin t + \frac{q}{2} \cos 3t + \frac{q}{2} \cos t. \tag{6.15}$$

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[12]All primary information presented in this section is from Pierre Simon Laplace, "Recherches sur le calcul intégral et sur le système du monde" vol 8 of *Oeuvres de Laplace*, 369-37

[13]"Soit proposé d'intégrer l'équation différentielle $\frac{d^2y}{dt^2} + y = \alpha y \cos 2t$."

[14]Laplace makes this point by stating that dt is a constant. This seems an odd method of pointing to the independent variable to the modern reader, but seems to represent a pattern in Laplace's writing and was quite normal in the eighteenth century. This method is also seen in the writings of Euler and the Bernoullis

To obtain this result, Laplace neglected terms involving α^2, even if this meant making terms contain α^2. Following through with this example, we can only find one slightly contentious simplification. When we make the substitution into (6.13), we find

$$\alpha \frac{d^2 z}{dt^2} + \alpha z = \alpha p \sin t \cos 2t + \alpha q \cos t \cos 2t + \alpha^2 z \cos 2t.$$

We can see that using a trigonometric identity, this can be written as

$$\alpha \frac{d^2 z}{dt^2} + \alpha z = \alpha \frac{p}{2}[\sin 3t - \sin t] + \alpha \frac{q}{2}[\cos 3t + \cos t] + \alpha^2 z \cos 2t.$$

At this stage, it would seem reasonable to divide through by α at which point there would be no term involving α^2. Laplace instead removed the last term which contains an α^2 term and then divides through by α. In the modern context, this seems natural, but there is today the terminology to describe what Laplace is doing.[15] However, this terminology was not contemporary and Laplace seemed to have needed some way to describe what he was doing. His description could have been more transparent had he included a sentence along the lines of "If there are no terms containing α^2, multiply by α." Here again, we see Laplace as being less of a mathematician in the sense that Lagrange was. Even in a work where he was attempting to produce a new method for solving equations, he did not include the relevant steps to make his procedure easily followed. Nor did he make it clear when it would work and when it would not.

Laplace immediately integrated (6.15) and, after substituting into (6.14) found

$$y = \left(p + \frac{\alpha}{4}qt\right)\sin t + \left(q + \frac{\alpha}{4}pt\right)\cos t - \frac{\alpha p}{16}\sin 3t - \frac{\alpha q}{16}\cos 3t. \tag{6.16}$$

This result appears to warrant some further explanation.

When this type of expansion is done in the modern context, the most general solution would be found by starting with the initial value problem

$$\begin{cases} \frac{d^2 y}{dt^2} + y = \alpha y \cos 2t \\ y(0) = y_0, \ y'(0) = y_0'. \end{cases}$$

When we set $\alpha = 0$, we have

$$\begin{cases} \frac{d^2 y}{dt^2} + y = 0 \\ y(0) = y_0, \ y'(0) = y_0'. \end{cases}$$

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[15]He is taking the order one ($O(1)$) solution and therefore removes all terms containing higher orders of α ($O(\alpha)$).

Looking at the next order of α equation, we set

$$\begin{cases} \frac{d^2 z}{dt^2} + z = \frac{p}{2}\left[\sin 3t - \sin t\right] + \frac{q}{2}\left[\cos 3t + \cos t\right] \\ z(0) = 0, \ z'(0) = 0. \end{cases}$$

From Laplace's solution, this is not what he did. Instead when he was solving for z, he set the homogeneous solution equal to zero. Normally, when solving an inhomogeneous equation, the first step is to solve the homogeneous equation, here this would mean solving

$$\frac{d^2 z_c}{dt^2} + z_c = 0,$$

where z_c is the homogeneous solution or

$$z_c = \bar{p}\sin t + \bar{q}\cos t.$$

Using the initial condition on z, we can find $\bar{p}$ and $\bar{q}$ in terms of p and q. Based on equation (6.16), this is not what Laplace did. Instead, it appears that Laplace set both $\bar{p}$ and $\bar{q}$ to be identically zero. Proceeding in this way, Laplace was able to find a simpler equation for p and q so as to solve for these values, as we will investigate next.

Laplace could have stopped at equation (6.16) but instead continued by attempting to find a relationship between p and q. He did this using two methods. First, Laplace looked at (6.13) with $t = T + t_1$ and compared this result with (6.16). Later, Laplace used a method based more on intuition. While Laplace argued that the first appears more direct, we will look at both methods.

Looking at the first method, Laplace started by rewriting (6.13) as

$$\frac{d^2 y}{dt^2} + y = \alpha y \cos\left(2T + 2t_1\right).$$

Laplace can proceed in this way if he believed his solution was periodic, though Laplace did not make this stipulation. As it turns out, the solution is not periodic, it is growing, but the method does appear to work. In following through with this integration, Laplace found that equation (6.16) is replaced by

$$y = \left(p' + \frac{\alpha}{4}q't_1\right)\sin\left(T + t_1\right) + \left(q' + \frac{\alpha}{4}p't_1\right)\cos\left(T + t_1\right) \tag{6.17}$$
$$- \frac{\alpha p'}{16}\sin\left(3T + 3t_1\right) - \frac{\alpha q'}{16}\cos\left(3T + 3t_1\right).^{16}$$

Laplace stated that by comparing equations (6.16) to (6.17), we can see that the difference between p and p' or q and q' will be of order α, therefore he set

$$p' = p + \delta p$$
$$q' = q + \delta q$$

where both δp and δq are considered to be single variables of order α.

Laplace was now able to find the equation

$$0 = \left(\delta p - \frac{\alpha}{4}Tq\right)\sin t + \left(\delta q - \frac{\alpha}{4}Tq\right)\cos t$$

by substituting p', q' and setting $t_1 = t - T$. From this equation Laplace argued that since t is a variable and T is a constant, we know

$$\delta p = \frac{\alpha}{4}Tq \tag{6.18}$$

$$\delta q = \frac{\alpha}{4}Tp. \tag{6.19}$$

He set $x = \frac{\alpha}{4}T$ and argued that

$$p' = p + \delta p = p + x\frac{dp}{dx} + \frac{x^2}{1\cdot 2}\frac{d^2p}{dx^2}\cdots \tag{6.20}$$

$$q' = q + \delta q = q + x\frac{dq}{dx} + \frac{x^2}{1\cdot 2}\frac{d^2q}{dx^2}\cdots. \tag{6.21}$$

This resembles a Taylor expansion around $x = 0$, but in that case Laplace would have been required to use the values of the successive derivatives at $x = 0$. Jahnke discusses the use of the Maclaurin series, or a Taylor series around $x = 0$, and states that in 1742, when Maclaurin published his *Treatise on Fluxions*, he stated that the function which is being expanded is evaluated at $x = 0$. Jahnke also points out that this form of expansion had been used by Newton and Euler before being published by Taylor or Maclaurin, though he does not state explicitly that Euler had stipulated that the function be evaluated at $x = 0$.[17] By the way he proceeded, we can tell that Laplace did not set $x = 0$. The expansion itself makes sense because α is a small constant, but there can be some question as to its validity in the form that Laplace wrote it.

While the presentation seems flawed to the modern audience, it may have had a historical precedent. Fraser points out that the symbol δ was given new meaning by Lagrange in a

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[17]Jahnke, 113.

work first published in 1762.[18] With Lagrange's definition of δ, it seems that

$$\delta = x\frac{d}{dx} + \frac{x^2}{1 \cdot 2}\frac{d^2}{dx^2} \cdots .$$

While Laplace's interpretation still seems erroneous, it would appear that this notation would have been clear to contemporary readers.

By comparing equations (6.18) to (6.20), and equations (6.19) to (6.21) Laplace determined that p and q are related by

$$\frac{dp}{dx} = q \text{ and}$$
$$\frac{dq}{dx} = p.$$

After solving and substituting back the value for x, Laplace found[19]

$$p = fe^{\frac{\alpha}{4}T} + f'e^{-\frac{\alpha}{4}T} \text{and} p = fe^{\frac{\alpha}{4}T} - f'e^{-\frac{\alpha}{4}T}.$$

Laplace was then able to reach the general solution

$$\begin{aligned} y = &fe^{\frac{\alpha}{4}T}(\sin T + \cos T - \frac{\alpha}{16}\sin 3t - \frac{\alpha}{16}\cos 3T) \\ &+ f'e^{-\frac{\alpha}{4}T}(\sin T - \cos T - \frac{\alpha}{16}\sin 3t + \frac{\alpha}{16}\cos 3T), \end{aligned} \tag{6.22}$$

where f and f' are constants. Laplace proceeded with his second method and reaches the same result.

Laplace continued *Sur le calcul intégral* by explaining his method through a series of other examples. We will not investigate these other examples in this paper.

The question now arises as to whether Laplace was justified in thinking he had found a general way to find approximate solutions to certain O.D.E.'s and whether he himself thought what he was doing constituted a general method. To look at whether Laplace was justified, we need to consider first, whether the method itself was correct and second, whether it was indeed a general method.

We can at least determine whether the method itself seems reasonable by looking at how well the approximation fits an exact solution. In this case, we can solve this O.D.E. exactly using a computer. After solving the differential equation and comparing it to Laplace's

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[18]Craig Fraser "The Calculus of Variations: A Historical Survey" in *A History of Analysis*, (Providence, RI: American Mathematical Society, 2003), 361.

[19]Laplace again uses his odd notation. In his version the f' is replaced with ^1f.

approximation, we see that Laplace's results show a strong relationship to the exact values. This will be discussed in detail in the Appendix. Therefore, we can say that Laplace's method was justified (for this example at least) because he was able to obtain results that would not have been possible using exact methods.

Still, was this a general method? As expressed before, a general method must be something that is repeatable. In this example, there are two steps which were somewhat contentious: arriving at equation (6.15) and the expansion in equations (6.20) and (6.21). The first was the only step that involved an α^2 term. Here, this was not too difficult to follow, but it could have been explained in a more transparent manner. In the second problematic step, there is no explanation and it is to the modern reader, there seems little reason to proceed in this manner. Overall this means that Laplace's method is difficult to follow and again, it appears as if the answer must be known before the approximation is started. As seen in the previous section, Laplace removed some terms and left others in an apparent ad hoc fashion that is only discernible when the end result is already known.[20] A person attempting to use the method without knowing the answer by simply following Laplace's explanation of neglecting terms containing α^2, would only arrive at the same answer as Laplace if he had the same intuition as Laplace or were working with the man himself. In this respect, this method does not appear to represent a general method. Thus, we are left with a method that produces wonderful results, but is not general.

Regardless of whether we can accept Laplace's method as general, did he himself believe what he had done was a general method? Also, what did it even mean to have a general method at the time? By the way Laplace presented his method, he seemed to believe that the steps that he took were perfectly consistent and therefore reproducible. The fact alone that Laplace had written this memoir makes it apparent that he was sure that what he was doing constituted a general method. Laplace could easily have thought because he himself could follow the method and could easily maneuver the variation that it presented that this would make it easy for others to follow. Whatever the reason, Laplace did appear to accept his method as rigorous enough and general.

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[20]The level of accuracy that is seen in Laplace's approximation raises the question as to whether Laplace already knew the answer before attempting to show his generalization.

Chapter 7

The Evolution of the Method of Variation of Parameters

The method of variation of parameters is today a useful method for solving ordinary differential equations as well as problems in perturbation theory,[1] but it was not a method that was developed in only one paper. Lagrange first used a similar method, variation of constants, in *Sur les intégrales particulières* published in 1774. Here the method was used to find what are now called singular solutions. While the application is different, the method itself is very similar to variation of parameters. This paper appears to have been inspired by Laplace's memoir published in 1772, *Sur les solutions particulières.* We will look at Laplace's version first and then compare this to Lagrange's. Lagrange then used variation of parameters in a more modern context in *Sur les suites,* published in 1775. Here, the method is used to solve inhomogeneous ordinary differential equations, in a manner that is still used today. Lastly, we will look at Lagrange's 1776 paper, *Sur l'altération,* where Lagrange brings this method back to the context of physical astronomy, where Laplace had introduced the need of such a method. This method was used in many more of Lagrange's papers on physical astronomy, but these were the first cases and we will concentrate entirely on them. After looking at the work of both authors, we will investigate the influence of friendship as well as the role that rivalry played in the evolution of this useful method. While a rivalry may have initiated Lagrange in the work that he is still remembered for today, the value he saw

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[1]See William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems, 9th Edition,* (Hoboken, NJ: John Wiley & Sons, Inc., 2009) for more information on solving ordinary differential equations.

in his friendship with Laplace set the tone for much of their later interaction.

7.1   Laplace's Method for Finding Singular Solutions: 1772

Laplace began *Sur les solutions particulières* by arguing that the general solution to a differential equation contains an arbitrary constant, and we can give this constant any value. Next, Laplace said that it would be natural to assume that any solution to the differential equation could be found by changing the value of the constant; but this is not the case. He then said that it was Euler in the first volume of his *Calcul intégral* who first determined this, but Euler's method made an assumption about the form of these "other" solutions and therefore is "impractical." To be practical, a method would not rely on assumptions about the form of the solution. It should be noted that Euler's work had been only recently published in 1768. Laplace then argued that he had found a better method to find these solutions.

Laplace next defined the terms that he would use. He said that a "solution" to a differential equation of any degree was the collection of all equations of a lower degree than the differential equation which satisfy the differential equation. By how Laplace proceeded, we can see that Laplace meant that a solution to a differential equations was all equations of a lower order that satisfied the original equation. Therefore, the solution of a third order differential equation, or one where the largest differential is of the form d^3, would be a second order differential or a lower-order differential. This is different to how the modern reader would view a solution. Today, we consider a solution to no longer contain any differentials. He called a "intégrale particulière" or "particular integral" all solutions which are contained in the general integral and a "solution particulière" or "particular solution" all solutions which are not contained in it. This naming appears to coincide with Euler's.[2] Euler named a particular integral one which does not contain a constant and is "an integral" of the differential equation. Euler did not have a name for singular solutions themselves. Euler also called a complete integral one where a constant not contained in the differential is involved. Laplace has not defined what he means by "general integral," but it seems to be what we would call the general solution. While he did define these terms, Laplace later introduced other terms which he did not here or later define. We will see an example of this immediately. In modern

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[2]Euler, *Institutiones Calculi Integralis* vol 1 (1768), 341.

terminology, the terms that Laplace defined would be one of the meanings of a particular solution and a singular solution, respectively. In general, we will use the modern equivalents to avoid confusion. We will use quotation marks to indicate when we are using the original author's terminology.

Laplace showed the existence of singular solutions by means of an example:

$$ydy + xdx = dy\sqrt{x^2 + y^2 - a^2}. \tag{7.1}$$

This has the singular solution

$$x^2 + y^2 - a^2 = 0 \tag{7.2}$$

and general solution

$$y + C = \sqrt{x^2 + y^2 - a^2}.^3 \tag{7.3}$$

After this example, Laplace argued that there are really only two problems to be addressed in this sort of analysis. Given a differential equation of any order and of any number of variables, where the "complete integral" is not known

1. Determine if an equation of a lower order which satisfies the differential equation is contained in the general solution

2. Determine all the singular solutions of this differential equation.

Laplace had here added a new term which he has not defined: complete integral. Laplace could mean the same as he meant by a general integral, but already Laplace's naming has become troublesome. When Euler had addressed this, he stated the problem simply as seeking to determine if a solution were singular or not.[4]

Laplace spent the majority of the rest of the paper solving six problems: determine if a solution of the differential equation $dy = p\,dx$ is contained in its "general integral," without knowing the general integral; determine all the singular solutions of a given differential equation $dy = p\,dx$; these same questions for a second order differential equation $d^2y = p\,dx^2$; and these same questions for a three variable differential equation $dz = p\,dx + q\,dy$. Since all the other problems are variations on the first two and have similar solutions, which Laplace acknowledges, we will only look at the first two problems. These problems follow a similar

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[3]Laplace, *Sur les solutions particulières*, 327.

[4]Euler, *Institutiones Calculi Integralis* vol 1 (1768), 342.

approach. Laplace worked through some theory, arrived at a solution and then wrote this as a theorem. While Laplace was not always clear when he worked through the theory, we will show that the theorems at the end concisely summarized his methodology.

The last section of *Sur les solutions particulières* deals with secular inequalities. This section appears to have been added on to the rest at a later date and does not match the rest of the paper. While this seems odd, it would seem to follow one of Laplace's patterns. As pointed out by Stigler, Laplace's published papers were often combinations of previous work.[5] While Lagrange was able to find a link between these two topics in the method that he developed, they are not presented in this way by Laplace. While the juxtaposition of two unrelated topics seems odd to the reader, it does not appear so in light of Laplace's other works.

7.1.1 Problem 1: Determine if a given solution of a differential equation is contained or not in the general integral without knowing this integral

Laplace's notation has certain peculiarities as well. He employed both d and δ to refer to differentials, and used the different notation denote derivatives of different functions. He also used the ∂ notation in the same way that it is used today. This is different from Lagrange, who only used total differential notation. The use of notation will also be noted when it is used.

Laplace used a graphical argument to describe how to determine if a solution to a given differential equation is a singular solution. He did this by defining two solutions: $\mu = 0$, the given solution and $\phi = 0$, the "complete integral." It was common to describe functions of one variables as level sets of a function of two variables. He then argued that these could be used to define two curves: HCM and LCN, respectively. This is shown in figure 7.1.

We can determine the constant yielding ϕ by the condition that the curve LCN passes through the point C on curve HCM. Now, Laplace said that μ will be a contained in ϕ if the two curves coincide at all points. Using modern terminology, Laplace next sets C to be the point (x, y), N to be $(x + \alpha, Y')$ and M to be $(x + \alpha, y')$. Laplace then performed a series

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[5]Stigler, 237.
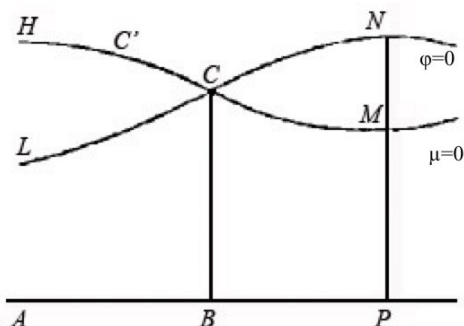
Figure 7.1: Laplace's diagram describing a singular solution

expansion and found

$$y' = y + \frac{\alpha\delta y}{\delta x} + \frac{\alpha^2}{1\cdot 2}\frac{\delta^2 y}{\delta x^2} + \frac{\alpha^3}{1\cdot 2\cdot 3}\frac{\delta^3 y}{\delta x^3} + \cdots \tag{7.4}$$

$$Y' = y + \frac{\alpha dy}{dx} + \frac{\alpha^2}{1\cdot 2}\frac{d^2 y}{dx^2} + \frac{\alpha^3}{1\cdot 2\cdot 3}\frac{d^3 y}{dx^3} + \cdots . \tag{7.5}$$

Here we can see Laplace using the different differential notation to point out that these corresponded to different curves. We should also note that y here does not appear to correspond to the function y, but to the coordinate y. Since this must be true at all points, Laplace was left with the condition that for the solution to be contained in the general solution, the following equations must be satisfied:

$$\frac{\delta y}{\delta x} = \frac{dy}{dx}, \ \frac{\delta^2 y}{\delta x^2} = \frac{d^2 y}{dx^2}, \ \frac{\delta^3 y}{\delta x^3} = \frac{d^3 y}{dx^3}, \cdots . \tag{7.6}$$

Laplace next renamed his differentials, setting $\frac{\delta y}{\delta x}, \frac{\delta^2 y}{\delta x^2}, \cdots$ to be $\nu, \nu', \cdots$. Since $\mu = 0$ is the given solution, we can easily determine these differentials by simply taking the derivatives of μ. He then noted that $\frac{dy}{dx} = p$, therefore he called p'

$$\frac{d^2 y}{dx^2} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\frac{dy}{dx} = \frac{\partial p}{\partial x} + p\frac{\partial p}{\partial y}, \tag{7.7}$$

and similarly for p'' and so forth. Laplace used the partial notation in the original. He also defined its usage using the modern interpretation. This is in contrast to Lagrange, who used only total differential notation in *Sur les intégrales particulières*. Laplace was now able to write the condition as

$$\nu - p = 0, \ \nu' - p' = 0, \nu'' - p'' = 0, \cdots . \tag{7.8}$$

Since $\mu = 0$ is a solution of the differential equation, the first equation is necessarily true. Therefore, all we must do to determine whether the given solution is or is not a singular solution, is to determine if the other conditions hold.

Laplace began by looking for a necessary condition on a particular solution that would mean that the solution was singular. First, he investigated the trivial solution, $y = 0$. In this section, Laplace relied heavily on series expansion. He argued that since $y = 0$, $\nu = 0$, $\nu' = 0$ and so forth, we need only concern ourselves with the values of p, p' and so forth. If these values do not disappear for $y = 0$ than the solution is singular. Laplace therefore set

$$p = fy^n + f'y^{n'} + f''y^{n''} + \cdots . \tag{7.9}$$

Laplace stipulated that this series was written in ascending order (ie $n' > n$, $n'' > n'$ etc) and that f, f', f'', $\cdots$ are functions of x. He argued that this means that p has the form $y^n q$, where q becomes neither zero nor infinite when $y = 0$. Laplace then found

$$\frac{dp}{dx} = \frac{d^2y}{dx^2} = \frac{dy}{dx}(nfy^{n-1} + n'f'y^{n'-1} + \cdots) + y^n\frac{df}{dx} + y^{n'}\frac{df'}{dx} + \cdots \text{ or,}$$

$$\frac{d^2y}{dx^2} = nf^2y^{2n-1} + (n+n')ff'y^{n+n'-1} + \cdots + y^n\frac{df}{dx} \text{ and}$$

$$\frac{d^2p}{dx^2} = \frac{d^3y}{dx^3} = n(2n-1)f^3y^{2n-2} + \cdots .$$

Laplace contended that these would all disappear, given $y = 0$, only if $n \geq 1$. Therefore, for p in the form $y^n q$, the solution is singular when $n < 1$.

Laplace ended this section by briefly looking at what it would mean if p were instead in the form $\frac{q}{\ln y^r}$,[6] or $e^{-\frac{1}{y}}q$.[7] These examples are all similar to the one for $y^n q$.

After Laplace investigated the trivial case, he generalized this to a particular solution of the form $\mu = 0$. Laplace set $\mu = 0$ to be a solution to $dy = p\,dx$. As he had already argued, this means that $\nu - p = 0$. He therefore asserted that μ and $\nu - p$ must have a common factor. Laplace then defined what he meant by a common factor: a factor of a quantity is a function that when set to zero makes the quantity disappear.

Laplace next allowed that μ is in a form such that $\frac{\partial\mu}{\partial y}$ and $\frac{\partial\mu}{\partial x}$ are neither zero nor infinite for $\mu = 0$. This, he said, is the case when the factors of μ are never raised to a power other

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[6]Laplace wrote $\ln y$ in the form (l.y). For simplicity, we will use the common notation.

[7]Here e is simply a number larger than unity and is not necessarily the inverse of the natural logarithm. Also, the small section involving the exponential was added later as pointed out in an editor's note.

than unity. He, therefore, let $y = M$, $y = M'$, $y = M''$, $\cdots$ be the factors of μ, where M, M', M'', $\cdots$ are functions of x. Therefore, we have

$$\mu = (y - M)(y - M')(y - M'') \cdots , \tag{7.10}$$

from which we can show

$$\frac{\partial \mu}{\partial y} = (y - M')(y - M'') \cdots + (y - M)(y - M'') \cdots + (y - M)(y - M') \cdots \text{ and } \tag{7.11}$$

$$-\frac{\partial \mu}{\partial x} = \frac{dM}{dx}(y - M')(y - M'') \cdots + \frac{dM'}{dx}(y - M)(y - M'') \cdots . \tag{7.12}$$

We can see that none of the factors of μ (ie. $y - M = 0$, $y - M' = 0$ etc) cause these partial derivatives to be either zero or infinite.

Laplace next set

$$\nu - p = \mu^n q. \tag{7.13}$$

This is similar to his argument in the trivial case and, again, q is such that it is neither zero nor infinite when $\mu = 0$ and n is positive. Laplace allowed from the definition that

$$\nu = -\frac{\frac{\partial \mu}{\partial x}}{\frac{\partial \mu}{\partial y}} \quad \text{and} \quad \frac{dy}{dx} = p. \tag{7.14}$$

This means that 7.13 can be written as

$$-\mu^n q \, dx = \frac{\frac{\partial \mu}{\partial x}}{\frac{\partial \mu}{\partial y}} dx + dy. \tag{7.15}$$

Using total differentials, this is

$$d\mu = -\mu^n q \frac{\partial \mu}{\partial y} dx. \tag{7.16}$$

This, Laplace asserted, is the same as $dy = p \, dx$. Laplace next set $h = -q\frac{\partial \mu}{\partial y}$, where h is a function of x and μ, which is always finite for $\mu = 0$. Laplace now had the result

$$d\mu = \mu^n h \, dx. \tag{7.17}$$

From the previous article, Laplace was now able to say that $\mu = 0$ is a singular solution if $n < 1$.

While what Laplace had done up to this point appears to have made his point, there is still some doubt about his method. Does it work in all cases? Is the method still valid when transcendental functions are involved? Would the practitioner need to perform a series

expansion if transcendental functions were involved? If there are exceptions, what are they? Laplace later noted that there would be difficulties if μ were not a function of both x and y, but nothing about the form of the functions. By "a function of," Laplace meant that the value appeared explicitly in the the expression. Laplace did not answer any of these questions, but he did provide an example here to show what he meant by his method. Returning to his first example, Laplace used

$$\frac{dy}{dx} = \frac{-x}{y - \sqrt{x^2 + y^2 - a^2}},$$

with the given solution

$$x^2 + y^2 - a^2 = 0 = \mu. \tag{7.18}$$

Laplace showed that

$$
\begin{aligned}
\mu^n q &= -\frac{\frac{\partial \mu}{\partial x}}{\frac{\partial \mu}{\partial y}} - \frac{dy}{dx} \\
&= -\frac{x}{y} + \frac{x}{y - \sqrt{x^2 + y^2 - a^2}} \\
&= \frac{x\sqrt{x^2 + y^2 - a^2}}{y^2 - y\sqrt{x^2 + y^2 - a^2}} \\
&= \mu^{1/2} \frac{x}{y^2 - y\sqrt{x^2 + y^2 - a^2}} .^8
\end{aligned}
$$

Laplace then asserted that since $n = 1/2$, $x^2 + y^2 - a^2 = 0$ must be a singular solution. In this case, Laplace could easily split the function to find μ^n, but is this always the case? Here, the reader might be more sure of Laplace's method if he had provided more than one example. As it stands, we can see that this method works in this situation, but there is some question about generality, in spite of his claims concerning the impracticality of Euler's method.

Euler's method is very similar to Laplace's up until this point. This may have been the reason that Laplace felt compelled to generalize his method into a theorem; this was Laplace's next step. Again, Laplace relied on series expansion. First he set

$$\mu^n h = \mu^n l + \mu^{n'} l' + \cdots,$$

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[8]Laplace left out some of these steps and they were only added for clarity

where l, l', $\cdots$ are functions of x. Laplace next differentiated this with respect to x and found

$$\frac{d}{dx}\mu^n h = n\mu^{n-1}l\frac{\partial\mu}{\partial x} + \mu^n\frac{\partial l}{\partial x} + \cdots .\tag{7.19}$$

When Laplace performed this step he stated that he was dividing by dx and did not write this as a function. This, Laplace claimed becomes infinite when $\mu = 0$ for $n < 1$. Here, the modern author would use limits. By Laplace's definition

$$\mu^n h = \frac{d\mu}{dx} = \frac{\partial\mu}{\partial x} + \frac{\partial\mu}{\partial y}\frac{dy}{dx} = \frac{\partial\mu}{\partial x} + \frac{\partial\mu}{\partial y}p.$$

Here, Laplace argued that $\mu = 0$ would be a singular solution if the derivative of $\frac{\partial\mu}{\partial x} + \frac{\partial\mu}{\partial y}p$ with respect to x is infinite. Laplace then generalized his method into a theorem.

Theorem 1 *If the equation $\mu = 0$ is a solution to the differential equation $dy = p\,dx$, it will be a singular solution whenever it renders zero the quantity*

$$\frac{1}{\frac{\partial^2\mu}{\partial x^2} + 2p\frac{\partial^2\mu}{\partial x\partial y} + \frac{\partial p}{\partial x}\frac{\partial\mu}{\partial y}},\tag{7.20}$$

otherwise, it will be a particular solution.[9]

In modern terminology, we would again use limits to explain Laplace's theorem.

After giving his theorem, Laplace did not show how to use the general method with an example. It seems that an example would be better placed here, with the theorem, than where it is, the step before the conclusion of Laplace's method.

7.1.2 Problem 2: Determine all singular solutions when the differential $dy = p\,dx$ is given

While Euler had already looked at how to determine if a given solution was in fact a singular solution, he did not look into how to find a singular solution itself. While Laplace's explanation of the solution to this problem is difficult to follow and involves several variables that Laplace did not define, it was something new. We will attempt to explain Laplace's work and show the points of confusion.

Laplace began by assuming that the complete integral was known. He defined β to be the factor by which $dy - p\,dx$ must be multiplied in order to make it an exact difference, or

$$\beta dy - \beta p dx = d\phi(x,y).$$

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[9]Laplace, "Sur les solutions particulières," 334. The original French is given in Appendix A

Here, we find the first point of confusion. Initially Laplace used the symbol β in his definition, but when he inserted this into the equation, he changed it to the symbol 6, as well as all subsequent times. This may have had nothing to do with Laplace and be due to the typesetter, but it remains a point of confusion. To avoid such confusion, we will only use β. Also, the variable ϕ has been introduced with no explanation.

Laplace next pointed out that since μ was a function of both x and y, we can write y as a function of x and μ. Therefore, ϕ is also a function of x and μ. Laplace made an argument at the end of this problem to address when μ is a function of only x or of only y. He next set $\psi + C = 0$ to be the complete integral of the equation $dy = p\,dx$, C being an arbitrary constant. Laplace next made his most contentious and confusing argument. He said that since $\mu = 0$ is not a particular solution - it is instead a singular solution - it will not make $\psi(x, \mu) + C$ disappear, no matter what the value of C is. This means that it will also not make the difference $\beta(dy - p\,dx)$ disappear. But, by assumption that it renders $dy - p\,dx$ zero, we require that $\mu = 0$ renders β infinite. Therefore, we find that μ is a factor of $\frac{1}{\beta}$. While this argument seems somewhat absurd to the modern reader, it was a common technique in the eighteenth century. Laplace asserted that this result was found also by Condorcet, but by a different method, which seems to be a confirmation of the validity of the approach. Laplace states that this can be found in Condorcet's *Calcul intégral*,[10] though he did not provide any information about Condorcet's version, simply that Condorcet had found the same result in an earlier work. This cursory mention of Condorcet will be investigated in Chapter 8.

Now Laplace must find which of the factors of $\frac{1}{\beta}$ are singular solutions. If we differentiate $\frac{1}{\beta}$, we have $dy = \gamma\,dx$ which is satisfied by $\mu = 0$. By assumption $\mu = 0$ also satisfies $dy - p\,dx = 0$, therefore $\mu = 0$ also makes $\gamma - p$ disappear. Therefore, μ must be a common factor of both $\frac{1}{\beta}$ and $\gamma - p$. Now, Laplace noted that all common factors of these two are solutions of $dy - p\,dx = 0$, therefore, we must distinguish which are singular solutions, which we can "easily" do. This most likely refers to the solution to problem 1.

Laplace argued that it is not always possible to do this exactly and therefore, we may have to resort to approximate methods. Here, we might see an indication of perturbation theory, but Laplace did not proceed with this line of thought. Instead, Laplace said that he

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[10]Pierre-Simon de Laplace, "Sur les solutions particulières des équations différentielles et sur les inégalités séculaires des planètes," in *Oeuvres complètes de Laplace* vol 8(Paris: Gauthier-Villars, 1878-1912), 336

had a solution, which again relied on series expansion. From the last problem we know

$$d\mu = \mu^n h dx \text{ or}$$

$$\frac{\partial \mu}{\partial y} dy = \mu^n h \, dx - \frac{\partial \mu}{\partial x} dx$$

Since, $dy = p \, dx$, we can write

$$p = \frac{\mu^n h}{\frac{\partial \mu}{\partial y}} - \frac{\frac{\partial \mu}{\partial x}}{\frac{\partial \mu}{\partial y}}. \tag{7.21}$$

Laplace then set $\mu = y - X$, which he asserted is always possible. We can believe this because we can always put μ in this form through a series expansion. With this substitution, we have

$$\frac{\partial \mu}{\partial y} = 1 \text{ and } -\frac{\partial \mu}{\partial x} = \frac{dX}{dx}.$$

Laplace next used an increasing series expansion to write

$$h = l + \mu^{n'} l' + \cdots$$

where l, l', $\cdots$ are functions of x. Using 7.21, Laplace now stated that

$$p = \frac{dX}{dx} + \mu^n l + \mu^{n+n'} l' + \cdots ,$$

which means that

$$\frac{\partial p}{\partial y} = n\mu^{n-1} l + (n + n')\mu^{n+n'-1} l' + \cdots . \tag{7.22}$$

Laplace then explained that 7.22 became infinite by the assumption that $\mu = 0$, since $n < 1$. Therefore $\mu = 0$ must render $\frac{1}{\frac{\partial p}{\partial y}}$ zero. This means that μ must be a factor of $\frac{1}{\frac{\partial p}{\partial y}}$. Here again, we see the same faulty logic that Laplace used before.

Laplace proceeded by differentiating $\frac{1}{\frac{\partial p}{\partial y}}$ and supposing that this gave $dy = \beta \, dx$. He argued that $\mu = 0$ must satisfy this, without any reasoning. Since, we already know that $\mu = 0$ satisfies $dy = p \, dx$, it must also render $p - \beta$ zero. Therefore, $\mu = 0$ is a common factor of both $\frac{1}{\frac{\partial p}{\partial y}}$ and $p - \beta$. Moreover, all common factors of these two quantities must be singular solutions. Laplace's reasoning was that if μ is such a factor, it will make both the quantities $dy - \beta dx$ and $p \, dx - \beta \, dx$ disappear. The first point is true because $\frac{dy}{dx} = p$. The second point appears to also follow from $\mu = 0$ making $p - \beta$ disappear. Laplace ended by noting that it is "easy" to see that

$$\beta = -\frac{\frac{\partial^2 p}{\partial x \partial y}}{\frac{\partial^2 p}{\partial y^2}},$$

which appears to follow from Laplace's statement that by differentiating $\frac{1}{\frac{\partial p}{\partial y}}$ we obtain $dy = \beta\, dx$. Laplace was then able to generalize his work into the following theorem:

Theorem 2 *If $\mu = 0$ is a singular solution of the differential equations $dy = p\, dx$, μ is a common factor of the two quantities*

$$p + \frac{\frac{\partial^2 p}{\partial x \partial y}}{\frac{\partial^2 p}{\partial y^2}} \ \ and \ \ \frac{1}{\frac{\partial p}{\partial y}}; \tag{7.23}$$

and reciprocally, all common factors of these two quantities, equalled to zero, are singular solutions of the differential equation $dy = p\, dx$.[11]

Laplace next stated that he had previously given this theorem, though without proof.

Laplace then provided an example to explain his method. For his example, Laplace again returned to equation 7.1,

$$\frac{dy}{dx} = \frac{-x}{y - \sqrt{x^2 + y^2 - a^2}} = p.$$

Therefore, we can see

$$\frac{1}{\frac{\partial p}{\partial y}} = -\frac{\sqrt{x^2 + y^2 - a^2}(y - \sqrt{x^2 + y^2 - a^2})}{x} \qquad \text{and}$$

$$p + \frac{\frac{\partial^2 p}{\partial x \partial y}}{\frac{\partial^2 p}{\partial y^2}} = \frac{\sqrt{x^2 + y^2 - a^2}(y^2 - a^2 - y\sqrt{x^2 + y^2 - a^2})}{x(y - \sqrt{x^2 + y^2 - a^2})^2}$$

It is "easy to see" that the only common factor is $\sqrt{x^2 + y^2 - a^2}$, therefore, this is the singular solution and it is unique.

Overall, we have seen that Laplace's method relies on series expansion and a faulty usage of common factors. While the reader can believe that Laplace's method works in certain situations, because he has shown through an example that it at least sometimes works, there is some doubt as to whether it is general. Besides the problems pointed out already, when we get to the end, is this method really easy to use? If a person can convince themselves that this method is general, which it is not, then will the person even be able to use the method?

Laplace ended this problem by discussing what to do if μ does not contain both x and y. We will not discuss this section. Overall, we can see that there are many deficiencies in

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[11]Laplace, "Sur les solutions particulières," 340. The original French is given in Appendix A.

Laplace's method such as claimed generality without theorems that support such generality, proofs which are only true in special cases, the question of the usefulness of this method and the seeming use of tricks to reach important results. These weaknesses together are most likely why this paper by Laplace is not often quoted. What is frequently referred to is Lagrange's method, which does not have the glaring difficulties that Laplace's does.

7.1.3 Discussion of secular inequalities

In this final section, Laplace discussed an extension to his previous investigation of secular inequalities in *Sur le principe.* Laplace reminded the reader that in this memoir he had determined that there were no secular inequalities in the mean movement or the mean distance of the planets, by virtue of each planet's mutual action towards each other. But the formulas that he used were approximations and this conclusion only holds over a limited time period. He believed that these equations were sufficient for all time, but it would be interesting from the point of view of analysis to have exact equations. While he said that he had wanted to solve these equations exactly, he has since given this up because of the difficulty and the lack of utility. All is not lost because Lagrange had worked on this in *Sur les inégalités seculaires du mouvement des noeuds et de l'inclinaison des orbites des planètes.* Laplace then proceeded to show how this work of Lagrange could be extended to examine the secular inequalities of the eccentricities and the movement of the aphelion of the planets.

At this point the reader may ask themselves "What does any of this have to do with singular solutions of differential equations?" The answer is not much. Laplace developed some differential equations, which he didn't solve, but said they were the same as the ones that Lagrange had found. Still there does not seem to be any degree of continuity with the rest of this memoir. Overall, this section appears to be separate from the rest of the memoir and almost appears to be just added on. We have added some discussion of it here because Lagrange later extended his variation of parameters method to physical astronomy.

7.2 Comparison with Lagrange's Method for Finding Singular Solution: Variations of Constants: 1774

Lagrange began *Sur les intégrales particulières* similarly by discussing the problem: some solutions to a given differential equation are not found by giving a certain value to the

constant in the general solution. After discussing the problem, Lagrange gives a history of the problem, including the current literature. Besides Euler, who Laplace also mentioned, Lagrange also gave credit to d'Alembert, Condorcet and Laplace himself. It seems odd, given what we have already seen about Laplace citing d'Alembert, that Laplace did not mention him in this context.

Lagrange defined his terms differently than Laplace. He called a "complete integral of a first order differential equation" one that satisfies the differential equation and contains an arbitrary constant. When a value is given to the constant, the integral becomes "incomplete." "Particular integrals," which he will look at, are ones that do not have an arbitrary constant, but there is no value that can be given to the constant in the "complete solution" which will lead to this equation. In modern terminology these are general solutions, particular solutions and singular solutions respectively. While there is nothing wrong with Laplace's naming scheme, it does seem more complicated than Lagrange's. The designations particular solution and particular integral appear to be easily confused, while Lagrange's appear less ambiguous. Since Laplace's article did appear first, Lagrange might have realized this and decided on the naming accordingly. Again, we will use the modern terminology to avoid confusion.

Next, Lagrange set up the majority of *Sur les intégrales particulières* in an almost identical manner to Laplace's *Sur les solutions particulières*. Lagrange divided this memoir into five articles:

1. Of the singular solutions of two variable first order differential equations and of the manner of deducing them from the general solution,

2. Of the extension of the singular solutions of first order differential equations and of the manner of finding these solutions without knowing the general solutions,

3. In which we deduce the theory of the singular solutions from the consideration of the curves,

4. Of the singular solutions of second and higher order differential equations,

5. Of the singular solutions of partial differential equations with some new remarks of the nature and the integration of these sorts of equations.

The first three articles appear in scope strikingly similar to the first two problems that Laplace proposed in *Sur les solutions particulières*, though they appear to be in a different

order. Here we will examine only these first three articles. Laplace first gave a graphical argument, then discussed how to determine if a given solution was singular without knowing the general solution and then set out to find singular solutions from the differential equation without knowing the general solution. The last two articles, which we will not discuss, also have the same feel as Laplace's other problems. When looking closely at these two memoirs, we will continue to see strong similarities.

Before starting our discussion of Lagrange's memoir, as we already noted, unlike Laplace, Lagrange did not use partial differential notation. We will be loyal to Lagrange and therefore only use total differential notation.

7.2.1   Article 1

After making his definitions, Lagrange's next step was to show an example. This example bears a very strong resemblance to the one used by Laplace. Lagrange said that the general solution of

$$x\,dx + y\,dy = dy\sqrt{x^2 + y^2 - b^2} \tag{7.24}$$

is

$$\sqrt{x^2 + y^2 - b^2} = y + a \text{ or} \tag{7.25}$$
$$x^2 - 2ay - a^2 - b^2 = 0.$$

These are exactly Laplace's equations 7.1 and 7.3, except that the names for the constants have been changed. This was also the example used by Euler and therefore it may have been the general example that was used at the time for singular solutions. Lagrange next showed that different values could be given to a to make the solution particular. Then, he showed that this differential equation was also solved by

$$x^2 + y^2 - b^2 = 0. \tag{7.26}$$

This, he asserted, could not possibly be contained in equation 7.25 since equation 7.26 represents a circle of radius b and equation 7.25 represents a parabola with a parameter of $2a$. While Laplace did show this example, his explanation of why one was not contained in the other was that there is no constant that will allow equation 7.3 to equal equation 7.2. We can see already that Lagrange is clarifying Laplace's work as well as making it

more rigorous. It is also interesting, given Lagrange's reputation, that he used a geometric criterion. Lagrange continued by looking at a second example.

After showing the existence of singular solutions *à posteriori*, Lagrange set out to find them *à priori* and using only the principles of integral calculus. This turns out to be the introduction to variation of constants. Lagrange started this by showing how to begin with the general solution and recover the differential equation. Lagrange called the differential equation $Z = 0$ and the general solution $V = 0$. If we differentiate $V = 0$, we will have $\frac{dy}{dx} = p$. Lagrange now contended that by using the two equations $V = 0$ and $\frac{dy}{dx} = p$, we can solve for the arbitrary constant a and recover the differential equation. To show this Lagrange turned to an example.

Starting with equation 7.25, Lagrange took the derivative and found

$$\frac{dy}{dx} = \frac{x}{a}. \tag{7.27}$$

Solving for a from equation 7.25 and substituting this into equation 7.27, we find equation 7.24. Now, Lagrange was able to do this based on the assumption that a was a constant. What if he treated it as a variable? This would mean that when he took the derivative of $V = 0$, he would now get,

$$dy = p\,dx + q\,da. \tag{7.28}$$

We still want this to reduce to $\frac{dy}{dx} = p$, therefore, we must require that $q = 0$. By varying only y and a in equation 7.28, Lagrange found that $q = \frac{dy}{da}$. Therefore if we solve $q = 0$ or $\frac{dy}{da} = 0$, we will find a value for a in terms of x and y. If we then substitute this value back into $V = 0$, Lagrange argued we will have a singular solution. Already, this method seems substantially easier than Laplace's. While we can see definite similarities in the approaches of the two men, in the end, the methods are completely different. Unlike Laplace, Lagrange did not use any series expansion and did not rely on infinite values.

Lagrange next addressed two potential problems: that $\frac{dy}{da} = 0$ contains only constants and neither x nor y or that $\frac{dy}{da} = 0$ contains only x and y and not a. In the first case, there is no singular solution. In the second case, $\frac{dy}{da} = 0$ will itself be a solution of the differential equation and further analysis will be required to determine if it is a singular solution or not.

Lagrange's next step was to clarify his method by means of an example. Again, he used equation 7.24. Unlike Laplace who used one example and not at the end of his analysis, with his complete method, but near the end with almost the finished method, Lagrange showed examples along the way and at the end of his analysis provided four examples, each

of varying difficulty and complexity. Reading this article is similar to reading a textbook with worked examples; by the end the method is clear and easily replicated. When reading Laplace, the theorem is clear, but the method itself is less grounded. While the reader can understand the proof, there is a persistent question about the rigor of the proof. Even though Lagrange has not given a rigorous proof, we can still understand his method and the examples he showed allow for easy replication.

Before Lagrange moved on to the next article, he made the point that "we must regard the theorem that we have just given, less as an exception than as a necessary supplement to the general rules of integral calculus."[12]

7.2.2 Article 2

Lagrange began addressing the issue of how to find a singular solution of a differential equation without knowing the general solution by looking at what he called "differentio-differential" equations. While Lagrange did not provide a definition of differentio-differential equations, they appear to be the derivative of a differential equation. So that the first solution of the differentio-differential equation is the differential equation. If the differential equation is given by $Z = 0$, the differentio-differential equation is given by $Z' = 0$. While the general solution of the differentio-differential equation might be difficult to find, he argued that it may be easier to integrate to the differential equation. Then, we can find the singular solutions of the differential equation. Under certain conditions, these are also singular solutions of the differentio-differential equation. This introduction is followed by two examples: one where the solution is not singular in the case of the differentio-differential equation and one where it is.

Lagrange extended this method and said that since $Z = 0$ is independent of a, we have $\frac{dZ}{da} = 0$. Since Z is a function of x, y and $\frac{dy}{dx}$, we have

$$dZ = Ad\frac{dy}{dx} + Bdy + Cdx \text{ or} \tag{7.29}$$

$$\frac{dZ}{da} = A\frac{d^2y}{dxda} + B\frac{dy}{da} = 0, \tag{7.30}$$

where A, B and C are rational, entire functions, as polynomials were called at the time. Since we are looking for singular solutions, we need, from before, $\frac{dy}{da} = 0$. Therefore, we are

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[12]"on doit regarder la théorie que nous venons de donner, moins comme une exception que comme un supplément nécessaire à la règle générale du Calcul intégral." Lagrange, "Sur les intégrales particulières," 17.

left with the condition

$$A\frac{d^2y}{dxda} = 0.$$

Lagrange reasoned that A must be zero, which implies, from 7.29 that for $Z = 0$,

$$B\frac{dy}{dx} + C = 0. \qquad (7.31)$$

Lagrange now asserted that from 7.29

$$\frac{d^2y}{dx^2} = -\frac{B\frac{dy}{dx} + C}{A}. \qquad (7.32)$$

From the analysis that he just did, he knows that this must be indeterminate or in the form "$\frac{0}{0}$" for a singular solution. Solving the equations 7.31 and $A = 0$, Lagrange will be able to determine any singular solution. Before looking at some examples, Lagrange examined how to find singular solutions when the differential equation is in a certain form: what is now known as Clairaut's equation. Lagrange investigated the procedure for solving this form of differential equation and cited the work of Euler and Clairaut with respect to this solution.

The last part of this article contains the four examples that Lagrange investigated before when he showed how to find singular solutions if the general solution is known. Overall, Lagrange's method is entirely self contained and not as complicated as Laplace's. Also, as Lagrange points out, while he first assumed that his differential equation did not contain any transcendental functions, it would work equally well if there were. While this is not a rigorous proof, Lagrange can satisfy the reader that his method works in all situations, while this is not the case with Laplace's explanation or his method.

7.2.3   Article 3

This section stands out by itself because Lagrange, in general, attempted to avoid graphical interpretations. After a full discussion about the nature of a curve which touches all the members of the "family" of curves formed from giving different values to the constant in the general solution, Lagrange stated that

> [F]rom what we have demonstrated above... we must conclude that the singular
> solution of a first order differential equation is represented by the curve which
> touches all the different curves represented by varying the arbitrary constant in

general solution of this equation, that is to say all the different curves which may
be represented at the same time by the same differential equation.[13]

As Archibald points out, this is not entirely correct as a singular solution may contain
a branch which is a particular solution, but this was not found out for another hundred
years.[14]

Next, Lagrange described how to find these singular solution, which we will not discuss.
We can note that while both Laplace and Lagrange use graphical arguments, not only are the
arguments completely different and for different purposes but they are presented differently.
Laplace began his investigation with a graphical argument. He used a figure to give a
graphical definition of a singular solution but a negative one. He said that if we had the
general solution and another solution, we could determine whether the other solution was a
singular solution by comparing the two solutions. Following Laplace, we would need to solve
the general solution for the constant by setting the general solution to overlap the other
solution at a given point. If the solutions then overlapped at all points, the other solution
was not singular. Laplace then expanded this to set forth his method. Here, we can see that
Laplace's purpose was to give a graphical basis for his analysis.

Lagrange, on the other hand, gave the graphical definition as something of an after
thought after he had given a complete method for finding singular solutions. Rather than
giving a negative definition, Lagrange asserted how a singular solution can be found from the
set of all particular solutions, rather than how can we see after graphing whether we have
a singular solution. While the definition that he provided is solid, it seems that the main
method is not the graphical one, but variation of constants. Lagrange described a graphical
argument, with examples, but without any actual graphs. This alone seems unusual, but
it is in line with Lagrange's habits. Lagrange stated in the introduction to his *Méchanique
Analitique* "On ne trouvera point de Figures dans cet Ouvrage."[15] In reading Lagrange's
work, there are rarely figures, so the lack of one here is not surprising. While Laplace does
have a figure, it is more of a general example: he does not use it to demonstrate a specific
example.

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[13]The original French is given in appendix A. Lagrange, "Sur les intégrales particulières," 38.

[14]Tom Archibald, "Differential Equations: A Historical Overview to circa 1900" in *A History of Analysis*,
335.

[15]"We will not find any figures in this work." Lagrange, *MA*, vi

Overall, while these two papers are glaringly similar in presentation, the actual content is completely different. Laplace may have written his memoir first, but Lagrange had the simpler method and in the end, one that is still valid, while Laplace's has been left in obscurity.

7.3 Lagrange's Extension to the Theory of Chance: Variation of Parameters: 1775

In *Sur les suites*, Lagrange stated that he wanted to push the theory of chance further than he had before. We will discuss the comments that Lagrange made regarding Laplace's prior work below, when we investigate the role that rivalry played in this research.

We will only look at the first article of this memoir which Lagrange called "Of the simple recursive series or of the integration of ordinary linear differential equations between two variables."[16] Lagrange began with an equation of the form

$$Ay_x + By_{x+1} + Cy_{x+2} + \cdots + Ny_{x+n} = 0,$$

where he let A, B, $\cdots$, N be constants and he defined $y_x = a\alpha^x$, a and α being unknown constants. We will not look directly at Lagrange's procedure for solving this.[17] He next examined how to solve recursive equations of the form

$$A_x y_x + B_x y_{x+1} + C_x y_{x+2} + \cdots + N_x y_{x+n} = 0, \tag{7.33}$$

where A_x, B_x, $\cdots$ are functions of x rather than being constants. The solution to this equation would be of the form

$$y_x = a\alpha_x + b\beta_x + c\gamma_x + \cdots. \tag{7.34}$$

Now, if, instead of 7.33, we had

$$Ay_x + By_{x+1} + Cy_{x+2} + \cdots + Ny_{x+n} = X_x,$$

the solution would be 7.34 only if $X_x = 0$. If $X_x \neq 0$, then the solution is given by

$$y_x = a_x\alpha_x + b_x\beta_x + c_x\gamma_x + \cdots.$$

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[16]"Des suites récurrentes simples, ou de l'intégration des équations linéaires aux différences finies entre deux variables." Lagrange, *Sur les suites,* 152.

[17]Lagrange, *Sur les suites,* 152-153

This is solved by varying y with respect to x and setting the variations of the "constants" equal to zero.

After discussing this method, Lagrange included a remark where he developed what we now call variation of parameters. He began by stating that the methods that he had just developed could be extended to the study in integral calculus.[18] Lagrange started by looking at the differential equation

$$Py + Q\frac{dy}{dx} + R\frac{d^2y}{dx^2} + \cdots + V\frac{d^ny}{dx^n} = X,$$

where P, Q, $\cdots$, V and X are functions of x. Also, Lagrange stipulated that the solution for the differential equation be known if $X = 0$. For simplicity, we will follow the example of Kline and only look at the second order case

$$Py + Q\frac{dy}{dx} + R\frac{d^2y}{dx^2} = X. \tag{7.35}$$

In the case where $X = 0$, Lagrange knew the solution to be of the form

$$y = ap + bq, \tag{7.36}$$

where a and b are constants and p and q are functions of x which form the solution to the homogeneous differential equation. If instead, $X \neq 0$, we can treat the constants in 7.36 as undetermined variables and assume that the sum of the derivatives of these variables are zero. Therefore, we find

$$dy = a\,dp + b\,dq \text{ and} \tag{7.37}$$

$$0 = p\,da + q\,db. \tag{7.38}$$

We would generally write this today as

$$\frac{dy}{dx} = a\frac{dp}{dx} + b\frac{dq}{dx} \text{ and}$$

$$0 = p\frac{da}{dx} + q\frac{db}{dx},$$

but we will use Lagrange's original notation.

Taking the second derivative of 7.36 and taking differentials as Lagrange did, we find

$$d^2y = a\,d^2p + b\,d^2q + dp\,da + dp\,db. \tag{7.39}$$

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[18]Lagrange, *Sur les suites*, 159.

The first two terms of 7.39 combined with 7.37 satisfy 7.35 when $X = 0$, therefore if we substitute 7.37 and 7.39 into 7.35, we find, after dividing by V

$$dp\,da + dp\,db = \frac{X}{V}dx^2, \tag{7.40}$$

or as we would write it today,

$$\frac{dp}{dx}\frac{da}{dx} + \frac{dp}{dx}\frac{db}{dx} = \frac{X}{V},$$

By using 7.38 and 7.40, we can solve for a and b and therefore resolve 7.36.

Lagrange then argued that any time we know the general solution of

$$\frac{d^n y}{dx^n} + P = 0, \tag{7.41}$$

where P is a function of x, y, $\frac{dy}{dx}$, $\cdots$, and $\frac{d^{n-1}y}{dx^{n-1}}$, we can solve equations of the form

$$\frac{d^n y}{dx^n} + P = \Pi, \tag{7.42}$$

where Π is a function of x, y, $\frac{dy}{dx}$, $\cdots$, and $\frac{d^{n-1}y}{dx^{n-1}}$, using a similar method as that outlined above. Unfortunately, this is often very difficult and we must resort to approximate methods. This is the first place where we can see the application of variation of parameters to perturbation theory. Shortly after this comment, Lagrange was able to tie perturbation theory, singular solution, physical astronomy to the method of variation of parameters.

If we were given an equation of the form of 7.42, where Π is very small compared to P, and we knew the general solution when $\Pi = 0$, we can find n equations

$$da = A\,\Pi\,dx, \quad db = B\,\Pi\,dx, \quad dc = C\,\Pi\,dx, \cdots,$$

where A, B, C, $\cdots$ are functions of x, a, b, c, $\cdots$ and Π is a very small function of these two quantities. This means that $\frac{da}{dx}$, $\frac{db}{dx}$, $\frac{dc}{dx}$, $\cdots$ will also be very small. By first regarding a, b, c, $\cdots$ as constants, we can find solutions that approach the true values of these quantities. Lagrange did not here show an example or go into any great detail about how to do this, but we can assume the method would be similar to the one above.

We are told not to fear if the functions A, B, C, $\cdots$ become infinite, because this is the condition that the general solution of 7.41 is a singular solution. Lagrange here self-cites *Sur les intégrales particulières*. While Lagrange did not provide an example, he here made a claim about the power of his prior work.

Lagrange's last point was that this method will be particularly useful in calculating the variation of the elements of the planets. This, he said, he has looked at before, but this method is more direct and more general. He also stated that he will extend this work.

This memoir is the first time where all the elements have come together for our present study. While we already had an idea that singular solutions were related to variation of constants; we now have them related to variation of parameters. We have a method that involves perturbation theory and most importantly we have an indication of how Laplace's earlier work in perturbation theory can be related to both singular solution and variation of parameters. Lagrange has answered all the questions that have so far been raised.

7.4 Lagrange's Extension to Perturbation Theory: Variation of Parameters: 1776

While Lagrange had previously mentioned that the method of variation of parameters would be useful for finding the variation in the elements of the planets, he did not provide any details before about how to go about doing this. In *Sur l'altération*, we come full circle and arrive back at perturbation theory with respect to astronomical physics.

Lagrange began by saying that for every planet in the solar system, there was a central force resulting from the gravitation pull of the sun. In addition to this there was a perturbing force. This force was much smaller than the central force which means that the perturbing force does not have a large effect on the orbit of the planets. Instead, "we may suppose that this orbit is a real ellipse, but of which the dimensions and the position vary from one instant to the next."[19] Lagrange mentioned six elements of the elliptical orbit of the planets which can be altered by perturbing forces: the semi-major axis, the eccentricity, the position of the axis or the line of the apsis, the inclination of the plane of the ellipse with another given plane, the position of the intersection of the two planes or the line of nodes, and the period of the mean movement or the value of the mean longitude for a given time. Since these six are considered independent, from them we can determine the evolution of the system. Lagrange argued that all six variations can be found by looking at the variations in the semi-major axis.

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[19]"on peut supposer que cette orbite est une véritable ellipse, mais dont les dimensions et le position varient d'un instant à l'autre." Lagrange, *Sur l'altération*, 258.

Lagrange defined the origin of his system as being the centre of the principal attracting mass. He used the term principal force to distinguish the solar attraction from the perturbing forces. He named F the value of the principle force at a distance of 1. He next called r the distance from the body to the centre, which could be defined in rectangular coordinates as

$$r = \sqrt{x^2 + y^2 + z^2}.$$

He next called the general expression of the force $\frac{F}{r^2}$. If the force is decomposed into its x, y and z components respectively, we have

$$\frac{Fx}{r^3}, \quad \frac{Fy}{r^3}, \quad \frac{Fz}{r^3}.$$

Now, Lagrange asserted that he could also decompose the perturbing force into X, Y and Z. Therefore, he ended up with the general equations

$$\frac{d^2x}{dt^2} + \frac{Fx}{r^3} + X = 0$$
$$\frac{d^2y}{dt^2} + \frac{Fy}{r^3} + Y = 0$$
$$\frac{d^2z}{dt^2} + \frac{Fz}{r^3} + Z = 0.$$

Now, Lagrange moved on to his general variation of parameters method. He first assumed that X, Y and Z were zero. This means that the planet is only being acted on by the principle force. Therefore, using known formulae, we may find the three coordinates x, y and z, with respect to t. This allowed him to make three remarks:

1. These values must be the complete and finite integral of the three differentio-differential equations

$$\frac{d^2x}{dt^2} + \frac{Fx}{r^3} = 0, \quad \frac{d^2y}{dt^2} + \frac{Fy}{r^3} = 0, \quad \frac{d^2z}{dt^2} + \frac{Fz}{r^3} = 0, \qquad (7.43)$$

   and that they must consequently contain six arbitrary constants;

2. That these constants will be precisely the six elements of the elliptical orbit that were spoken of before;

3. That if we differentiate these three integrals, we will have six equations which will help us to determine the six arbitrary constants in terms of x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$. We will then have six first order differential equations,

of which each one contains an arbitrary constant and will be consequently

a first integral of the three proposed differentio-differential equations.[20]

Lagrange now looked at one of these first order equations

$$V = k \tag{7.44}$$

where V is a function of x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ and k is an arbitrary constant. This k will be one of the elements of the elliptical orbit of the planet. When we take the derivative, we find

$$dV = 0.$$

This equation will no longer contain an arbitrary constant and will itself be equal to the differentio-differential equation that we started with.

If we now include the perturbing forces, we will still have equation 7.44, but we can now think of k as a variable instead of a constant. This mean that

$$dV = dk.$$

Following through with the calculations, Lagrange found

$$dk = - \left(\frac{dV}{d\frac{dx}{dt}} X + \frac{dV}{d\frac{dy}{dt}} Y + \frac{dV}{d\frac{dz}{dt}} Z \right).$$

There would also be similar expression for the other elements.

Lagrange was now able to make a conclusion about the form of the six differential equations. He stated that whether or not there was a perturbing force, the equations, such as $V = k$, would take the same form. The only difference would be that in one case k would be a constant whereas in the other k would be a variable. This means that x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ will all also be expressed in the same manner with respect to t and the six elements of the elliptical orbit regardless of whether or not there is a perturbing force. This means that we can always regard these six elements as constant, if we are looking at very short periods of time. Lagrange proceeded by showing how to calculate the variation in the semi-major axis of the elliptical orbit of a body moved by a central force.

We have now moved full circle and returned to something similar to what Laplace was investigating in *Sur le principe*. Now, we can ask ourself, what influence did the relationship between Laplace and Lagrange have on the development of both of their work?

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[20]The original French is given in appendix A. Lagrange, *Sur l'altération*, 260-261.

7.5 The Role of Friendship and Rivalry

There appears little doubt that Laplace's earlier memoir on singular solutions spurred Lagrange on to work on the same subject. When we look at the correspondence between the two men as well as the papers themselves we can see both sides of friendship and rivalry in the relationship.

Laplace appears to have started sending Lagrange his work in his first letters to the older scientist, which date back as far at 1772. Lagrange commented on the work that Laplace had undertaken in the study of singular solutions in a letter dated 10 April, 1775, saying

> You have sorted the matter of the singular solutions; I have only had enough time to skim through your beautiful memoir on this subject as well as the one on the theory of chance; I propose to come back to it and to study it in detail. They appear to me very appropriate to maintain the high opinion that your other works have already given to your genius.[21]

This appears to refer to *Sur les solutions particulières*. It seems that Lagrange was true to his word and not only did he read the work in detail, he wrote himself on the subject and sent this memoir to Laplace. In a letter from Lagrange to Laplace dated 10 May, 1776, Lagrange wrote

> I ask you, as a mark of friendship, for which I will be infinitely appreciative, to tell me your opinion of these memoirs [that I am sending you], and especially on the first which concerns singular integrals. It appears to me that we may still glean more from you and I will be very flattered to have been able to add something to your work.[22]

An editor's note indicates that Lagrange is referring to *Sur les intégrales particulières*. In this paper, Lagrange credited Laplace by stating

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[21]"Vous avez bien nettoyé la matière des intégrales particulières; je n'ai encore eu le temps que de parcourir votre beau Mémoire sur ce sujet, ainsi que celui sur la théorie des hasards; je me propose d'y revenir et de les étudier à fond. Ils me paraissent très propres à soutenir la haute opinion que vos autres Ouvrages ont déjà donnée de votre génie." "Correspondence de Lagrange avec Laplace," 63.

[22]"je vous demande comme une marque d'amitié à laquelle je serai infiniment sensible de me dire votre avis sur ces Mémoires, et surtout sur le premier qui concerne les intégrale particulière. Il m'a paru qu'on pouvait encore glaner après vous, et je serai bien flatté d'avoir pu ajouter quelque chose à votre travail." "Correspondence de Lagrange avec Laplace," 64.

> I just read a *Mémoire sur les solutions particulières des équations différentielles*, that M. de Laplace has recently given at the Académie des Sciences and which will appear in the 1772 volume; the author had kindly given me an advanced copy of this work. In this memoir, M. de Laplace perfects and extends further the known theory of singular solutions, but also did what no one had before, he gives methods for finding directly all the singular solutions which may satisfy a given differential equation and which are not at all contained in the general solution of this equation. [23]

Here we see Lagrange praising Laplace, but more than this, he has not pointed out any fault in Laplace's work. This seems odd and leaves the reader asking "If there is nothing wrong with what Laplace did, then why is this paper necessary?" While, Lagrange never admitted any defects in Laplace's prior memoir, he did end his introduction by saying that he is presenting a new and complete theory from the point of view of analysis. In this way, Lagrange is able to present his paper without ever belittling Laplace. This act alone shows the friendly aspect in the relationship of Laplace and Lagrange. We can see that Lagrange's method is superior to Laplace's and is the one that is remembered today, but Lagrange was willing, at the time, to shyly ask Laplace for his opinion and not directly point to the problems with what Laplace had produced.

Later in the same letter from 10 May, 1776, Lagrange wrote of *Sur les suites*, as pointed out again by an editor's note, saying

> I just read to the Academy two memoirs on the integration of linear partial differential equations and on their usage in the theory of chance. You can see that it is your beautiful work on this subject that has led me to also occupy myself with it. I flatter myself to have also been fortunate enough to add something; to the rest, my work on this subject differs from yours as much as the one on singular solutions differs from yours on the same subject; the only thing in common is

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[23]"je viens de lire un *Mémoire sur les solutions particulières des équations différentielles*, que M. de Laplace a donné depuis peu à l'Académie des Sciences, et qui doit paraître dans le volume de 1772, mais dont l'Auteur a bien voulu m'envoyer d'avance un exemplaire imprimé. Dans ce Mémoire, M. de Laplace perfectionne et étend plus loin la théorie déjà connue des solutions particulières, et ce que personnes n'avait encore fait, il donnes des méthodes pour trouver directement toutes les solutions particulières qui peuvent satisfaire à une équation differentielle donné, et qui seraient point comprises dans la solution générale de cette équation." Lagrange, *Sur les intégrales particulières*, 6

the subject.[24]

This letter shows Lagrange giving Laplace credit for initiating Lagrange in research, but he made the point, clearly, that he was not impinging on Laplace priority and that he was approaching the subject completely differently. There is a level of flattery in Lagrange working on the same subject as Laplace, and working on it because of Laplace, but there is also the ever present issue of priority. Lagrange emphasized that he was not taking away from Laplace's priority, he was just trying to add something new. As the case stands, while it may not have been Lagrange's intention, few people today look at the work that Laplace did in the study of singular solution, mainly because of the superiority of Lagrange's attempts.

In *Sur les suites*, Lagrange acknowledged that Laplace had made progress in this field but he stated directly after noting two works by Laplace that

> I believe however that we may still add something to the work of this illustrious Geometer, and treat the same subject in a manner more direct, simpler and, overall, more general.[25]

Here, we can see the competitive nature of the relationship between Lagrange and Laplace in stark contrast to how Lagrange referred to Laplace's work above. Overall, through this series of memoirs, we can see how Lagrange used Laplace's initial memoir as a springboard, but was able to extend his own work far more than Laplace had been able to. We also see that Laplace appears to have abandoned his research in singular solutions after Lagrange produced his work. If we remember the letter (quoted in Chapter 5) that Lagrange wrote Laplace saying that he was ceasing his work because he could see that Laplace was working in this field and he knew that Laplace would do the same if he thought that science would prosper. Here, we see that Laplace was put in this situation, and as it seems, he did give up in favor of the friendship he had for Lagrange and the benefit science would see in Lagrange's work. Competition may have led Lagrange to work on this subject, but the friendship he had for Laplace set the tone for how he would go about it. In the end, Laplace moved from

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[24]Je viens de lire à l'Académie deux Mémoires sur l'intégration des équations linéaires à différences partielles et sur leur usage dans la théorie des hasards; vous jugez bien que c'est votre beau travail sur cette matière qui m'a engagé à m'en occuper; je me flatte d'avoir aussi été assez heureux pour y ajouter quelque chose; au reste, mon Ouvrage sur cette matière diffère autant du vôtre que celui sur les intégrales particulières diffères autant du vôtre sur le même sujet; il n'y a guère entre eux que le sujet de commun." "Correspondence de Lagrange avec Laplace," 65.

[25]"Je crois cependant qu'on peut ajouter quelque chose au travail de cet illustre Géomètre, et traiter le même sujet d'une manière plus directe, plus simple et surtout plus générale." Lagrange "Sur les suites," 152.

the study of singular solutions to an application of similar methods in secular inequalities. This may also explain the odd juxtaposition seen in the paper investigated here by Laplace.

Chapter 8

Other Rivals

8.1 Condorcet

Marie-Jean-Antoine-Nicolas Caritat Marquis de Condorcet (1743-1794) was born in the small village of Ribemont where Antoine, his father was stationed as a cavalry captain; his father died shortly after Condorcet's birth.[1] Condorcet obtained his degree in philosophy in Paris at the Collège de Navarre. His thesis was examined by, among others, d'Alembert. While his family would have seen him in a military career, Condorcet moved to Paris to follow a career in mathematics. While Laplace had to show his worth to obtain acceptance by the scientific elite, Condorcet had certain advantages. Since Condorcet was educated in Paris, he was known by the scientific community before he had moved to Paris. Unlike Laplace who had to quickly find a source of income when he moved to Paris, Condorcet was given financial support by his mother, even though he was going against the wishes of his family.

Condorcet began publishing memoirs in integral calculus in 1765 with his *Essai sur le calcul intégral*, which he submitted to the Académie des Sciences himself. This memoir brought Condorcet to the attention of d'Alembert. While d'Alembert must have been aware of Condorcet before, this paper showed the older scientist Condorcet's talent as a mathematician. As Goodell notes, it was from this time that d'Alembert acted as "mentor and friend" to Condorcet.[2] Herein lies one of the main differences between the relationships of d'Alembert and Laplace and that of d'Alembert and Condorcet: friendship. D'Alembert

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[1] Different sources have the Condorcet's father's death at between days and years after the child's birth.

[2] Edward Goodell, *The Noble Philosopher*, (Prometheus Books: New York, 1994), 10.

may appear to have done more as a mentor and patron for Laplace in the realm of science, but their relationship is never described as one of friendship. While d'Alembert was able to assist Condorcet's career, especially when the younger man was attempting to and succeeded in becoming permanent secretary of the Académie des Sciences or when he was elected to the Académie Française, d'Alembert also introduced Condorcet into salon society and when d'Alembert's health became poor in 1770, he had Condorcet accompany him on a vacation to recover.[3] This is a relationship completely different from that of d'Alembert and Laplace.

Condorcet was elected to the Académie des Sciences in 1769, the same year that Laplace moved to Paris. While Condorcet has the reputation of a mathematician, Granger notes that "It must be acknowledged that reading Condorcet's mathematical works is a thankless task and often a disappointing one."[4] Gilain asserts in agreement "The imprecision of the language, the instability of the terminology and the overlapping technical errors, on the one hand, and the priority given to general methods as opposed to specific examples, on the other hand, has made these [Condorcet's] writings both obscure and non-rigorous as well as difficult to read and understand."[5] Condorcet may not have been the clearest of mathematical writers and Hahn argues that Laplace used this to his advantage, saying that it was his strategy to "enlist himself in a productive enterprise that d'Alembert admired, while also showing his technical superiority over figures like Condorcet."[6] Shortly after being elected to the Académie des Sciences, Condorcet moved his concentration away from mathematics and towards philosophy and politics. Hahn asserts that this was in response to the superiority of Laplace's abilities.

While there is little evidence of the exact relationship of Laplace and Condorcet, we do know that, at least initially, the two were less than on the best of terms. From the response that Lagrange gave to Condorcet's letter mentioning Laplace,[7] we can tell that Condorcet

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[3]Goodell, 128-129.

[4]Gilles Gaston Granger, "Condorcet, Marie-Jean-Antoine-Nicolas Caritat Marquis de" in *Complete Dictionary of Scientific Biogaphy*, vol 3 (Charles Scribner's Sons: Detroit, 2009/02 2008), 384.

[5]"L'imprécision du langue, l'instabilité de la terminologie et la multiplication des erreurs techniques d'une part, le privilège accordé aux méthodes générales et aux vues programmatiques, au détriment de l'étude détaillée de cas précis, d'autre part, conduisent souvent, en effet, à des écrits obscurs et peu rigoureux, difficiles à lire et à compredre." Christian Gilain, "Condorcet et le calcul intégral," in *Sciences à l'époque de la Révolution Française*, (Lib. Sci. Tech. Albert Blanchand: Paris, 1988), 88.

[6]Hahn, *Pierre-Simon Laplace*, 45.

[7]While Condorcet's letter is lost, we have quoted the pertinent section from Lagrange's reply in Chapter 5.

found Laplace vain. Any rivalry that existed between the two men would appear to be less than friendly simply from this response. The rivalry that existed between the two men may have been linked to their shared patron. D'Alembert assisted Laplace in his career as a scientist and Condorcet as a person. Laplace did mention Condorcet in his written works, but any mention is usually cursory, simply a note that Condorcet had written on the subject before. From these notes, there is little we can say for sure about their relationship; Laplace neither criticized nor praised Condorcet's research. Overall, the evidence is scarce, and little more can be said about the matter at this point. All that can be said is that when Laplace entered the mathematical scene, Condorcet seems to have started to leave.

8.2 Legendre

Adrien-Marie Legendre (1752-1833) was born into an affluent family. Unlike the other scientists that we have looked at, Legendre was given the opportunity to study science directly, and appears to have been supported in this endeavor by his family. Legendre graduated from the Collège Mazarim after defending his thesis in mathematics and physics in 1770. His family provided him with a small fortune in which to live off, though he did supplement his income with teaching responsibilities at the École Militaire. By 1782, Legendre had attracted the attention of Lagrange by winning a Berlin Academy prize competition. Lagrange wrote Laplace asking

> Do you know M. Legendre? He just won our Prize concerning Ballistics. His work appears to me as good can be done on this subject and it introduces its author, if he is still young, as a man of talent and of knowledge which amy take him far; I pray you to tell him the part that I take in his success.[8]

Laplace responded that he was acquainted with the young man who had made himself known to the Académie through his merit as a mathematician.[9] Itard notes that it was through Laplace that Legendre submitted articles on various subject matter. When Laplace was promoted to *associé* in the Académie, it was Legendre who took his place as *adjoint*. While

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[8]"Connaissez-vous M. Legendre? Il vient de remporter notre Prix sur la Balistique. Sa pièce m'a paru aussi bonne que le sujet peut le comporter, et elle annonce dans son auteur, s'il est jeune encore, des talents et des connaissance qui pourront le mener loin; je vous prie de lui dire la part que je prends à son succès." Lagrange à Laplace 15 septembre 1782, "Correspondence de Lagrange avec Laplace," 116.

[9]Laplace à Lagrange 10 février 1783, "Correspondence de Lagrange avec Laplace," 121.

the association of Laplace with Legendre appears to have been one that began early, it may not have always been harmonious. It seems that Laplace believed that after a piece of research had been presented to the Académie, it became public knowledge and therefore is was unnecessary to cite the original author. This may have been common practice, but it was a procedure which could cause insult. Since Legendre submitted articles through Laplace, Laplace had the opportunity to investigate the younger man's work even before it had been submitted. Hahn remarks that "Legendre ... was bitter about the apparent usurpation of his priority, pointedly remarking in his published paper that he had presented his ideas to the Académie before Laplace had developed his."[10] Todhunter contends that "we shall find indications that Legendre was not quite satisfied with Laplace's silence as to the matter of priority."[11]

This issue of proper citing and priority between Laplace and Legendre can be seen in the example of their respective work in the theory of the shape of the planets. After discussing both a work by d'Alembert and a 1772 memoir by Laplace, Legendre began his *Recherches sur la Figure des Planètes*, originally presented to the Académie in 1784, by noting that while Laplace had presented his work published in 1784 in a "scholarly and general manner," Legendre had presented the same work prior to Laplace.[12] While Legendre did not make any criticisms of the quality of Laplace's research, he did make the obvious point that Laplace had failed to recognize Legendre's priority and Legendre was rectifying the situation. Overall, while Laplace may not have meant any disrespect, this does seem to have been the response.

We can compare the reaction of Legendre in this case with that of Lagrange in the case of the secular inequalities of Jupiter and Saturn. When Lagrange thought that Laplace was using his previous work, he first said that he would forego his own research so that Laplace could continue his own. Lagrange also asked if Laplace had read Lagrange's previous work instead of insinuating that there was a priority issue. As well, Lagrange wrote this in a letter instead of making the issue public by including such a comment in his published work. At the stage that Lagrange was at in his career, he was already well established and this may have made him feel less attached to each individual piece of work. Legendre, in contrast, was a young scholar hoping to make a name for himself and therefore less willing to ignore what

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[10]Hahn, *Pierre-Simon Laplace*, 75

[11]I. Todhunter, *A History of the Mathematical Theories of Attraction and the Figure of the Earth*, vol 2, (Dover Publishing: New York, 1962), 19.

[12]Todhunter, 43.

he felt was an insult to his priority. The reason may also have been in the personality of the two men. In general, Lagrange tended to attack matters in a diplomatic manner. While Legendre's reaction may be understandable, it was not in Lagrange's temperament to have a public debate with a fellow scholar.

Again, the documentation regarding the relationship between Legendre and Laplace is paltry. While we can see that there was some form of rivalry between the two men and this rivalry does not appear to have been a friendly one, we can say little more at this point without research that goes beyond the scope of the current work.

Chapter 9

Conclusion

Laplace was an ambitious man who arrived in Paris with the intention of making a name for himself. He would have most likely had a better chance at a successful reception had he made his introduction to a lesser known figure, but the rewards would also be less certain. Laplace instead picked the man in France who he thought best able to advance his career: d'Alembert. While d'Alembert was Laplace's first powerful patron, he was not Laplace's only supporter; Laplace's ability to network can be seen in multiple cases in his career and may have contributed to his longevity. Overall, while Laplace's ability to promote himself was key, he would not have been to make any headway scientifically had he not had a genuine talent. D'Alembert was able to open the doors for Laplace after he had proved himself worthy, but Laplace also had to show his worth to the greater scientific community.

The papers by Laplace analyzed here show his attitude towards mathematics. Laplace was more interested in developing physical theory than in developing mathematical theory. For this reason, his mathematical work is somewhat difficult to follow, but as we have seen, even the physical model is sometimes difficult to understand. In the section of *Sur le Principe* that was investigated, a work from 1773, Laplace began by introducing a physical system which he did not fully explain. After setting up a confusing model, Laplace continued by using notation which is somewhat ambiguous. After Laplace had set up his physical model, he continued by setting up his system of equations. Here again Laplace missed what appear to be vital steps that would have made the memoir easier to follow. As it stands, the reader is forced to make assumptions in order to be able to follow what Laplace has done. Whether these assumptions are the same as Laplace's is impossible to know, because he did not help the reader along. Nonetheless, they seem plausible and illuminate the content.

When we reach the simplification that Laplace had made, if the reader has followed this far, Laplace skipped steps that are necessary to follow his simplification. This is fine if we can trust that Laplace had not made any errors, but as it stands, we can not even check the validity of his work because we can not be guaranteed that what we believe he did was actually what he did. The question can even be asked if the secular equation that Laplace found at the end of his calculations was simply the result of flawed methodology. Again, we can not be sure because Laplace has not told the reader enough information.

While Laplace only mentioned d'Alembert's contributions to the field and did not show any specific examples, the way that Laplace presented his work provides some indication of his relationship to d'Alembert. D'Alembert was constantly concerned about issues of priority, which caused him to publish work that was incomplete or rushed. This made his results, while often correct, difficult to follow and sometimes correct due to multiple errors. With this example as a mentor, there is some precedent for the way that Laplace presents his own work. While Laplace did not seem as concerned about issues of priority, he was producing work at a phenomenal rate, which can lead a person to be sloppy.

When we arrive at *Sur le calcul intégral*, from 1772, we are faced with the same problem: what exactly did Laplace do? Here though we can check his results. In this example, we can show that his answer strongly resembles the correct result, but we cannot say with certainty how he obtained it. There are enough questionable aspects in the method to cause alarm and yet the method provides the right answer.

Laplace had a good mathematical intuition and was therefore able to obtain mathematically sound results even though his steps along the way were confusing and difficult to generalize. Overall, while Laplace did attempt to build general methods, his generalization is often less than satisfactory. Here, we see Laplace claiming generality without actually reaching a generalization.

This pattern appears again when Laplace looked at singular solutions of ordinary differential equations. We still see confusing notation and flawed methodology, but in this case, we see that Laplace had a competitor whose work did not show these pitfalls. When we turn to the work of Lagrange, we see a similar set-up, but the problems that we found with Laplace have disappeared. Lagrange initially believed Laplace to be brash and self-serving, but Laplace was still able to make his acquaintance and gain the respect of the more senior academic. This respect is shown not only through the correspondence between the two men, but also in the way they approached each other's work in their published research.

While Lagrange proved to be a long time friend and scientific ally, Laplace's relations with the scientific community were not always harmonious. Even though there is evidence for negative rivalries between Laplace and some of his contemporaries, the information is scarce compared with that showing the positive relationship between Laplace and Lagrange.

We can see that Laplace was able to work with the tool kit that d'Alembert had assembled and build on his mentor's work. While Laplace may be difficult to follow at times, his writing is still less complicated than d'Alembert. We can see Laplace refining d'Alembert's work and always giving credit, whether he wanted to or not, to the mentor who had given him his start in the Parisian scientific community. When we see the same mathematical areas investigated by both Laplace and Lagrange, namely singular solutions, we can immediately see the superiority of Lagrange's presentation. Still, the two were able to maintain a friendship and assist each other in their mathematical pursuits. Later, Laplace was able to improve on Lagrange's accomplishments in the field of potential theory.

While Laplace entered the scientific community because of the powerful patrons that he found, his career was shaped as much, if not more, by the role that rivalry, both positive and negative, played in his scientific endeavours.

Appendix A

Appendix: Original French of Translated Material

A.1 Letter from d'Alembert to le Canu: August 25, 1769

Il est juste de vous laisser la satisfaction d'annoncer à Mr l'abbé de la Place la bonne fortune vous pouver lui dire qu'il est sur d'une place de professeur de mathématique à l'école militaire, et lui repeter les conditions; logé meublé 6 voies de bois, 1800# d'appointements, donc 1400# de net parce qu'on retient 400# pour la nourriture si ces conditions lui conviennent, il faut 1° qu'il ma l'écrive sur le champ; car je pars le 7 septembre pour la campagne ou je resterai 3 semaines. 2° qu'il écrive aussi sur le champ à M Bizot, rue du Temple près la rue des Gravilliers à Paris ce Mr Bizot doit etre le directeur des études au 1er octobre. Il mandera donc à Mr Bizot qu'il peut compter sur lui, et il y ajoutera les expressions d'honnêteté et de reconnaissance convenables. 3° Il faudra qu'il se rende à Paris le 20 septembre au plus tard et meme s'il se peut quelque jours avant et qu'en arrivant il aille trouver Mr Bizot, qu'il pourra voir tous les jour rue St. Louis au Marais chez Mr Paris du Verney, le matin depuis 10h jusqu'à 2 et le soir depuis 5 jusqu'à 8. j'espere que Mr l'abbé de la Place, par son zèle, son assiduité, & la bonne conduite fera honneur à ma recommandation, j'oublie de vous dire qu:il n'aura que trois à 4 heures a donner tous les matin à la classe, ce que le reste du temps sera à lui j'ai l'honneur d'etre avec mes respectueux attachemens, Monsieur

Votre très humble et très obeissant serviteur

d'Alembert à Paris, le 25 aout 1769

A.2   Letter from Laplace to d'Alembert: November 15, 1777

J'ai toujours cultivé les Mathématiques par goût plutôt que par le désir d'une vaine réputation, dont je ne fait aucun cas. Mon plus grand amusement est d'étudier la marche des inventeur, de voir leur génie aux prises avec les obstacles qu'ils ont rencontrés et qu'ils ont su franchir; je me mets alors à leur place et je me demande comment je m'y serais pris pour surmonter ces même obstacles, et quoique cette substitution n'ait, le plus souvent, rien que d'humiliant pour mon amour-propre, cependant le plaisir de jouir de leur succès me dédommage amplement de cette petite humiliation. Si je suis assez heureux pour ajouter quelque chose à leur travaux, j'en attribue tout le mérite à leur premiers efforts, bien persuadé que dans ma position ils auraient été beaucoup plus loin que moi. Vous voyez par là, mn cher Confrère, que personne ne lit vos Ouvrages avec plus d'attention et ne cherche mieux à en faire son profit que moi; aussi personne n'est plus disposé à vous rendre une justice plus entière, et je vous prie de me regarder comme un de ceux qui vous aiment et qui vous admirent le plus. C'est dans ces sentiments que j'ai l'honneur d'être, Monsieur et illustre Confrère,

Votre très humble et très obéissant serviteur,

Laplace.

A.3   Letter from d'Alembert to Lagrange: January 1, 1773

1 ° s'il peut actuellement être placé à l'Académie de Berlin; 2 ° s'il pourrait y jouir, dès son entrée, d'un revenu suffisant pour vivre, comme 3000 ou 4000 livre, argent de France; 3 ° si vous êtes dans une position à vous intéresser pour lui sans vous faire de tracasseries; 4 ° si, dans la supposition où vous ne voudriez pas vous en mêler, je pourrais écrire au Roi et lui proposer M. de la Place comme un sujet que je connais, que j'estime, et dont vous pourrez vous-même lui rendre témoignage.

A.4   Laplace's Theorem 1 from *Sur les solutions particulières*

Si l'équation $\mu = 0$ est une solution de l'équation différentielle $dy = p\,dx$, elle sera une solution particulière, toutes les fois qu'elle rendra nulle la quantité

$$\frac{1}{\frac{\partial^2 \mu}{\partial x^2} + 2p\frac{\partial^2 \mu}{\partial x \partial y} + \frac{\partial p}{\partial x}\frac{\partial \mu}{\partial y}}, \tag{A.1}$$

autrement, elle sera une intégrale particulière.

A.5   Laplace's Theorem 2 from *Sur les solutions particulières*

Si $\mu = 0$ est une solution particulière de l'équation différentielle $dy = p\,dx$, μ est facteur commun aux deux quantités

$$p + \frac{\frac{\partial^2 p}{\partial x \partial y}}{\frac{\partial^2 p}{\partial y^2}} \text{ et } \frac{1}{\frac{\partial p}{\partial y}}; \tag{A.2}$$

et, réciproquement, tout facteur commun à ces deux quantités, égalé à zéro, est une solution particulière de l'èquation différentielle $dy = p\,dx$.

A.6   Lagrange's graphical definition of singular solution

[d]e ce que nous avons démontré plus haut... on doit conclure que l'intégrale particulière d'une équation différentielle du premier ordre est représentée par la courbe qui touche toutes les différentes courbes représentées par l'intégrale complète de cette équation, en faisant varier la constante arbitraire, c'est-à-dire toutes les différentes courbes qui peuvent être représentées à la fois par la même équation différentielle.

A.7   Lagrange's three remarks when the perturbing forces are zero

Il nous suffit de remarquer:

1. Que ces valeurs doivent être des intégrales complètes et finies des trois équations différentio-différentielles

$$\frac{d^2 x}{dt^2} + \frac{Fx}{r^3} = 0, \ \ \frac{d^2 y}{dt^2} + \frac{Fy}{r^3} = 0, \ \ \frac{d^2 z}{dt^2} + \frac{Fz}{r^3} = 0,$$

   et qu'elles doivent par conséquent renfermer six constantes arbitraires;

2. Que ces constantes seront précisément les six éléments de l'orbite elliptique dont nous avons parlé plus haut;

3. Que, si l'on différentie les trois intégrales dont il s'agit, on aura six équations à l'aide desquelles on pourra déterminer les six constantes arbitraire en x, y, z, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$; de sorte qu'on aura ainsi six équation différentielles du premier ordre, dont chacune

renfermera une constante arbitraire, et sera par conséquent une intégrale première des trois équations différentio-différentielles proposées.

Appendix B

Appendix: Comparison of Laplace's Approximation

The example looked at from the second memoir investigated can be solved numerically. Here, we would like to briefly discuss this solution and the comparison of results.

The solution was found using both Maple and Matlab. After finding the numerical solution, an IVP was set up to compare Laplace's solution to this numerical solution. The values used were arbitrarily chosen. These values were used to solve for Laplace's constants f and f'. It was noticed when this was first plotted, that there was a shift introduced in either Laplace's approximation or the exact solution. To compensate for this shift, the following alteration on Laplace's equation was used

$$y = f e^{\frac{\alpha}{4}(-\delta)} \left[\sin(t-\delta) + \cos(t-\delta) - \frac{\alpha}{16} \sin 3(t-\delta) - \frac{\alpha}{16} \cos 3(t-\delta) \right] \qquad \text{(B.1)}$$
$$+ f' e^{-\frac{\alpha}{4}(t-\delta)} \left[\sin(t-\delta) - \cos(t-\delta) - \frac{\alpha}{16} \sin 3(t-\delta) + \frac{\alpha}{16} \cos 3(t-\delta) \right].$$

In this case, $\alpha = 0.01$ and initial conditions $y(0) = 1$ and $y'(0) = -2$ were used. The constants f, f' and δ were then found by minimizing the difference between Laplace's approximation and the numerical result that Matlab found. These values are

$$f = 1.4824$$
$$f' = -.547 \text{ and}$$
$$\delta = 4.7139.$$

The plot showing the approximation and the numerical solution are given in Figure B.1.

This plot shows that the results seem similar, but to see how good the approximation is,
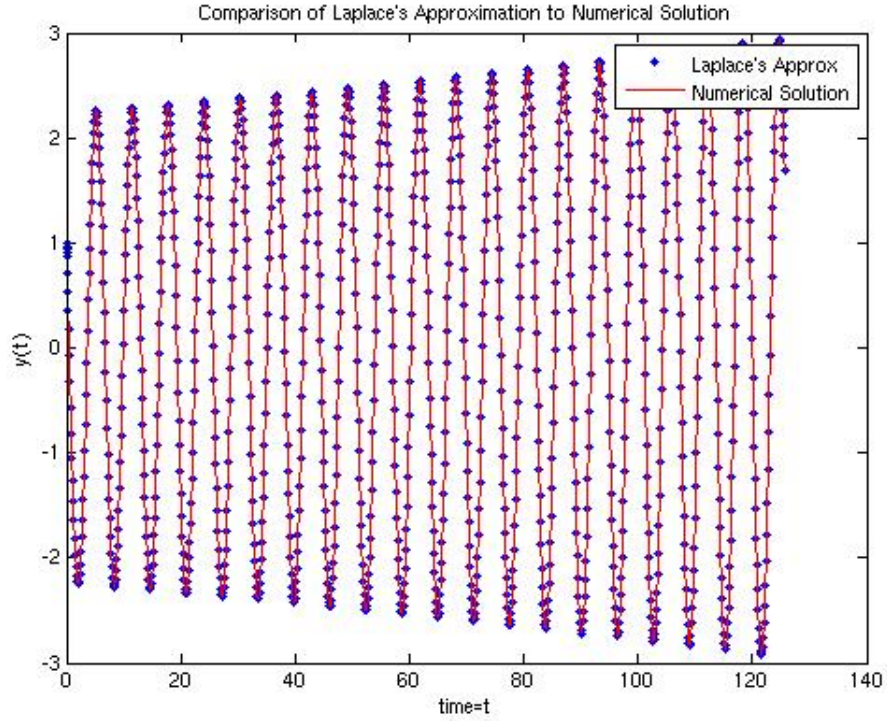

Figure B.1: Comparison of Laplace's Approximation to Numerical Solution

the error should also be investigated. The plot showing the absolute value of the difference between the numerical solution and the approximation is shown in Figure B.2. The standard deviation of the approximation for values of time less then $t = 40\pi$ was found to be less than 5×10^{-4}. Overall, this shows that Laplace's approximation represents the exact value almost as well as possible. It should be noted that when the shift was removed, the error increased by orders of magnitude. This could be due to the flawed use of the Maclaurin expansion which is discussed above.
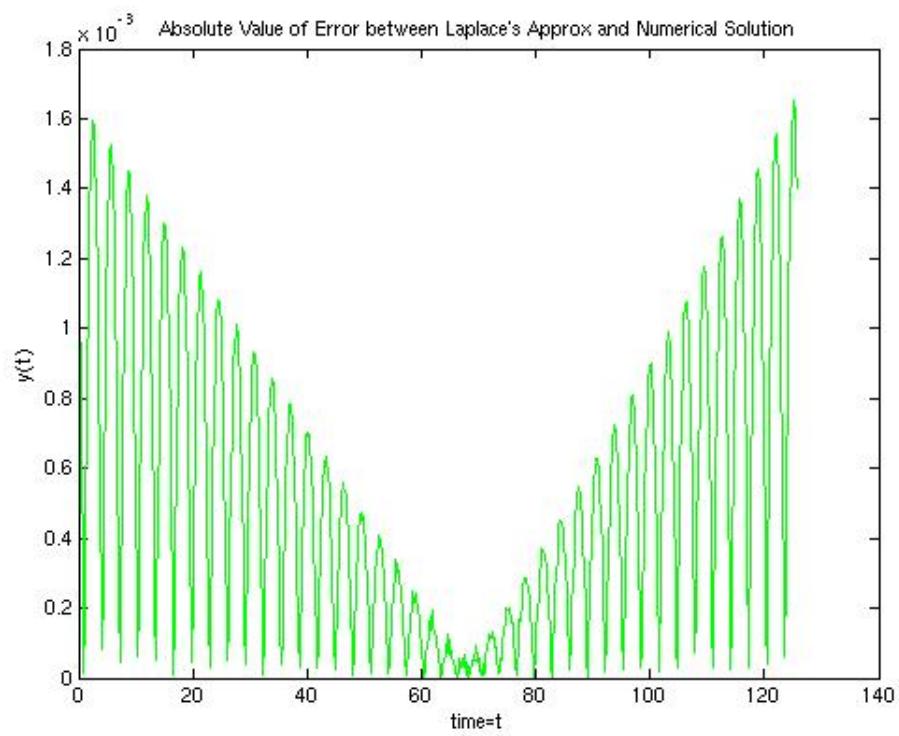
Figure B.2: Error between Laplace's Approximation and Numerical Solution

Bibliography

[1] Archives de l'Académie des Sciences, Dossier de Laplace.

[2] Archives de l'Académie des Sciences, Dossier Lagrange.

[3] Archives Militaires Y^h 165 Dossier Laplace.

[4] Archives Nationale de France, Dossier de Laplace.

[5] Ken Alder. *Engineering the Revolution : Arms and Enlightenment in France, 1763-1815.* Princeton University Press, Princeton, N.J. ; Chichester, U.K., 1997.

[6] H. Andoyer. *L'oeuvre scientifique de Laplace.* Payot, Paris, 1922.

[7] Thomas Archibald. Energy and the Mathematization of Electrodynamics in Germany, 1845-1875. *Archives Internationales d'Histoires des Sciences*, 39(123):276–308, 1989.

[8] Tom Archibald. *Differential Equations: A Historical Overview to circa 1900*, volume 24 of *A History of Analysis: Editor Hans Niels Jahnke*, chapter 11, pages 325–353. American Mathematical Society, Providence, RI, 2003.

[9] Marco Beretta. *Lavoisier, Antoine-Laurent*, volume 22 of *Complete Dictionary of Scientific Biography*, chapter 22, pages 213–220. Charles Scribner's Sons, Detroit, 2008.

[10] G. Bigourdan. La Jeunesse de Laplace. *Science Moderne*, 8:377–384, 1931.

[11] U. Bottazzini. *The higher calculus : a history of real and complex analysis from Euler to Weierstrass.* Springer-Verlag, New York, 1986. ID: 13395859.

[12] William E. Boyce and Richard C. DiPrima. *Elementary Differential Equations and Boundary Value Problems, 9th Edition.* John Wiley & Sons, Inc., Hoboken, NJ, 2009.

[13] Robert E. Bradley and Charles E. Sandifer. *Leonhard Euler : life, work and legacy*, volume 5. Elsevier, Amsterdam, 2007.

[14] J. Morton Briggs. *Alembert, Jean Le Rond D'*, volume 1 of *Dictionary of Scientific Biography*, chapter 1, pages 110–117. Charles Scribner's Sons, Detroit, 2008.

[15] Michelle Chapront-Touzé. *D'Alembert (Dalembert), Jean Le Rond*, volume 20 of *Dictionary of Scientific Biography*, chapter 20, pages 229–233. Charles Scribner's Sons, Detroit, 2008.

[16] M. P. Crosland. *The Society of Arcueil; a view of French science at the time of Napoleon I.* Harvard University Press, Cambridge, Massachusetts, 1967.

[17] Jean Le Rond d'Alembert. *Oeuvres complètes: Premiers textes de méchanique céleste 1747-1749, editor Michelle Chapront-Touzé*, volume 6. CNRS, Paris, 2002.

[18] Marquis de La Place. *Celestial mechanics. Vol. V.* Chelsea Publishing Co, Inc., Bronx, N.Y., 1969. 01.60; MR0265116 (42 #29).

[19] Pierre-Simon de Laplace. *Mémoire sur les solutions particulières des équations différentielles et sur les inégalités séculaires des planètes*, volume 8 of *Oeuvres complètes de Laplace*, pages 325–367. Gauthier-Villars, Paris, 1878-1912.

[20] Pierre-Simon de Laplace. *Recherches sur le calcul intégral et sur le système du monde*, volume 8 of *Oeuvres complètes de Laplace*, pages 369–477. Gauthier-Villars, Paris, 1878-1912.

[21] Pierre-Simon de Laplace. *Théorie des attractions des sphéroïdes et de la figure des planètes*, volume 10 of *Oeuvres complètes de Laplace*, pages 341–419. Gauthier-Villars, Paris, 1878-1912.

[22] Denis Diderot and Jean le Rond d'Alembert, editors. *Table analytique et raisonnée des matières contenues dans les XXXIII Volumes in-folio du dictionaire des sciences, des arts et des métiers et dans son supplément tome premier*, page 1177. Encyclopédie. Panckoucke, Paris, 1780.

[23] Denis I. Duveen and Roger Hahn. Laplace's Succession to Bézout's Post of Examinateur des Elèves de l'Artillerie. *Isis*, 48(4):416–427, Dec. 1957.

[24] S. B. Engelsmann. Lagrange's Early Contributions to the Theory of First-Order Partial Differential Equations. *Historia Mathematica*, 7:7–23, 1980.

[25] Eric G. Forbes. *Mayer, Johann Tobias*, volume 9 of *Dictionary of Scientific Biography*, chapter 9, pages 232–235. Charles Scribner's Sons, Detroit, 2008.

[26] Craig Fraser. J.L. Lagrange's early contributions to the principles and methods of mechanics. *Archive for the History of Exact Sciences*, 28(3 September):197–241, 1983.

[27] Craig Fraser. *The Calculus of Variations: A Historical Survey*, volume 24 of *A History of Analysis*, chapter 10, pages 355–383. American Mathematical Society, editor Hans N. Jahnke, Providence, RI, 2003.

[28] Craig G. Fraser. Joseph Louis Lagrange's Algebraic Vision of the Calculus. *Historia Mathematica*, 14:38–53, 1987.

[29] Christian Gilain. *Condorcet et le calcul intégral*, pages 87–147. Sciences à l'époque de la Révolution Française. Lib. Sci. Tech. Albert Blanchand, Paris, 1988.

[30] Charles Coulston Gillispie, editor. *Complete Dictionary of Scientific Biography*. Charles Scribner's Sons, Detroit, 2008.

[31] Charles Coulston Gillispie, Robert Fox, and Ivor Grattan-Guinness. *Laplace, Pierre-Simon, Marquis De*, volume 15 of *Complete Dictionary of Scientific Biography*, pages 273–403. Charles Scribner's Sons, Detroit, 2008.

[32] J. B. Gough. *Lesage, George-Louis*, volume 8 of *Complete Dictionary of Scientific Biography*, chapter 8, pages 259–260. Charles Scribner's Sons, Detroit, 2008.

[33] Judith V. Grabiner. *Bezout, Étienne*, volume 2 of *Complete Dictionary of Scientific Biography*, chapter 2, pages 111–114. Charles Scribner's Sons, Detroit, 2008.

[34] Gilles Granger. *Condorcet, Marie-Jean-Antoine-Nicolas Caritat, Marquis De*, volume 3 of *Complete Dictionary of Scientific Biography;*, chapter 3, pages 383–388. Charles Scribner's Sons, Detroit, 2008.

[35] Henry Guerlac. Chemistry as a Branch of Physics: Laplace's Collaboration with Lavoisier. *Historical studies in the physical sciences*, 7:193, 1976.

[36] Henry Guerlac. *Lavoisier, Antoine-Laurent*, volume 8 of *Complete Dictionary of Scientific Biography;*, chapter 8, pages 66–91. Charles Scribner's Sons, Detroit, 2008.

[37] Roger Hahn. *The anatomy of a scientific institution: the Paris Academy of Sciences, 1666-1803*. University of California Press, Berkeley, 1971. ID: 147970.

[38] Roger Hahn. *Calendar of the correspondence of Pierre Simon Laplace*. Office for History of Science and Technology, University of California, Berkeley, Berkeley, 1982. ID: 8877709.

[39] Roger Hahn. *Pierre Simon Laplace, 1749-1827 : a determined scientist*. Harvard University Press, Cambridge, Mass., 2005. ID: 58457459.

[40] Thomas L. Hankins. *Science and the Enlightenment*. Cambridge University Press, Cambridge; New York, 1985. ID: 11030787.

[41] Jean Itard. *Lagrange, Joseph Louis*, volume 7 of *Complete Dictionary of Scientific Biography*, chapter 7, pages 559–573. Charles Scribner's Sons, Detroit, 2008.

[42] Jean Itard. *Legendre, Adrien-Marie*, volume 8 of *Complete Dictionary of Scientific Biography*, chapter 8, pages 135–143. Charles Scribner's Sons, Detroit, 2008.

[43] Hans Niels Jahnke. *Algebraic Analysis in the 18th Century*, volume 23 of *A History of Analysis: editor Hans N. Jahnke*, chapter 4, pages 105–136. American Mathematical Society, Providence, RI, 2003.

[44] Hans Niels Jahnke. *Limitations of the analytic function concept*, volume 24 of *A History of Analysis: editor Hans N. Jahnke*, chapter 4.6, pages 123–127. American Mathematical Society, London Mathematical Society, Providence, RI; London, 2003.

[45] D. W. Jordan and Peter Smith. *Nonlinear ordinary differential equations : an introduction to dynamical systems*, volume 2. Oxford University Press, Oxford, 1999.

[46] Victor J. Katz. *A History of Mathematics: An Introduction, 3rd Edition.* Addison-Wesley, Boston, 2009.

[47] Morris Kline. *Mathematical thought from Ancient to Modern Times*, volume 2. Oxford University Press, New York, 1990.

[48] Joseph-Louis Lagrange. *Recherches sur les suites récurrentes dont les termes varient de plusieurs manieres différentes ou sur l'intégration des équations linéaires aux différences finies et partielles et sur l'usage de ces équations dans la théorie des hasards*, volume 4, pages 151–251. Gauthier-Villars, Paris, 1776 1775.

[49] Joseph-Louis Lagrange. *Méchanique analitique.* Vve Desaint, Paris, 1788.

[50] Joseph-Louis Lagrange. *Correspondance de Lagrange avec Laplace*, volume 14 of *Oeuvres de Lagrange*, pages 56–132. Gauthier-Villars, Paris, 1867-1892.

[51] Joseph-Louis Lagrange. *Correspondance inédite de Lagrange et d'Alembert publiée d'après les manuscrits autographes et annotée par Ladovic Lalanne*, volume 13 of *Oeuvre de Lagrange.* Gauthier-Villars, Paris, 1867-1892.

[52] Joseph-Louis Lagrange. *Mémoire sur la théorie des variations des éléments des planètes et en particulier des variations des grands axes de leurs orbites*, volume 6 of *Oeuvres de Lagrange*, pages 713–768. Gauthier-Villars, Paris, 1867-1892.

[53] Joseph-Louis Lagrange. *Remarques générales sur le mouvement de plusieurs corps qui s'attirent mutuellement en raison inverse des carrés des distances*, volume 4 of *Oeuvres de Lagrange.*, pages 401–418. Gauthier-Villars, Paris, 1867-1892.

[54] Joseph-Louis Lagrange. *Sur l'altération des moyens mouvements des planètes*, volume 4 of *Oeuvres de Lagrange*, pages 255–271. Gauthier-Villars, Paris, 1867-1892.

[55] Joseph-Louis Lagrange. *Sur les intégrales particulières des équations différentielles*, volume 4 of *Oeuvres de Lagrange*, pages 5–108. Gauthier-Villars, Paris, 1867-1892.

[56] Louis-Joseph Lagrange. *Sur l'attraction des sphéroïdes elliptiques*, volume 3 of *Oeuvres de Lagrange publiées par les soins de M. J.-A. Serret [et G. Darboux]*, chapter XX, pages 619–694. Gauthier-Villars, Paris, 1867-1892.

[57] Pierre S. Laplace and Nathaniel Bowditch. *Celestial mechanics*, volume II. Chelsea Pub. Co, 1966.

[58] Pierre-Simon Laplace. *Lettres inédites de Laplace*, volume 14 of *Oeuvres de Laplace*, chapter 4, pages 346–348. Gauthier-Villars, Paris, 1878-1912.

[59] Pierre-Simon Laplace. *Sur le principe de la gravitation universelle et sur les inégalités seculaires des planètes qui en dépendent*, volume VIII of *Oeuvres de Laplace*, chapter XXXVI, pages 201–275. Gauthier-Villars, Paris, 1878-1912.

[60] Jean-Jacques Rousseau. *The Confessions: translated by J.M. Cohen.* Penguin Books Ltd., Harmondsworth, Middlesex, England, 1954 1971.

[61] Jacob E. Safra and Ilan Yeshua, editors. *Alembert, Jean le Rond D'*, volume 1 of *The New Encyclopedia Britannica*, pages 235–236. Encyclopedia Britannica Inc, Chicago, 2002.

[62] George Sarton. Lagrange's personality (1736-1813). *Proceedings of the American Philosophical Society*, 88(6):457–496, 1944. Copied from the Archive de l'Académie des Sciences, Paris.

[63] L'Abbé G. A. Simon, le Comte de Colbert-Laplace, and Karl Pearson. Laplace, notes on his ancestry and life. *Biometrika, A Journal for the Statistical study of Biological Problems*, XXI(1-4):202–230, 1929. Copied for the Archive de l'Académie des Sciences, Paris.

[64] Stephen M. Stigler. Laplace's early work: Chronology and citations. *Isis*, 69(2):234–254, Jun. 1978.

[65] Alice Stroup. *The Institute and its Patronage*, chapter II, page 14. A Company of Scientists: Botany, Patronage, and Community at the Seventeenth-Century Parisian Royal Academy of Sciences. University of California Press, Berkley, 1990.

[66] I. Todhunter. *A History of the Mathematical Theories of Attraction and the Figure of the Earth From the time of Newton to that of Laplace.* Dover Publications, New York, 1962.

[67] Clifford Truesdell. *An idiot's fugitive essays on science : methods, criticism, training, circumstances / C. Truesdell.*, pages xvii, 661 p. : ill., ports. ; 25 cm. Springer-Verlag,, New York :, c1984. c1984.

[68] Curtis Wilson. d'Alembert versus Euler on the precession of the equinoxes and the mechanics of rigid bodies. *Archive for History of Exact Sciences*, 37(3):233–273, 1987. CODEN: AHESAN; MRCLASS: 01A50; MRNUMBER: MR904328 (88j:01015); MR-REVIEWER: V. E. Thoren.

[69] Curtis Wilson. *The Dynamics of the Solar System*, volume II of *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences: editor I. Gratten-Guinness*, chapter 8.8, pages 1044–1053. Routledge, London, 1994.

[70] Curtis A. Wilson. Perturbations and solar tables from Lacaille to Delambre: the rapprochement of observation and theory, part ii. *Archive for History of Exact Sciences*, 22(3):189–304, 1980.