THE COMBINATORIAL STRUCTURE OF THE PRIME SPECTRUM OF QUANTUM MATRICES

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Abstract

Herein we study the prime ideals in the algebra of quantum matrices. The main content of this work is the application of combinatorial methods to the analysis of a special subclass of prime ideals, namely, those invariant under the action of an algebraic torus $\mathcal{H}$. We call such ideals $\mathcal{H}$-primes.

By the $\mathcal{H}$-stratification theory of Goodearl and Letzter, the set of prime ideals can be partitioned (or "stratified") in a manner such the parts (or "strata") are in bijective correspondence with the $\mathcal{H}$-primes. Moreover, each stratum satisfies nice topological properties. Thus, to understand the set of prime ideals, a first step is to understand the $\mathcal{H}$-primes.

The first problem we approach is the question of finding a generating set for a given $\mathcal{H}$-prime. Launois proved that for almost all quantum matrix algebras, the generating sets consist of certain "quantum minors" derived from the generators of the quantum matrix algebra. Launois also provided an algorithm to find such minors, however it is algebraic in nature and somewhat unwieldy. We prove that Launois' algorithm can be considered combinatorial. Specifically, we show that the problem of determining which quantum minors appear in a given $\mathcal{H}$-prime is equivalent to finding sets of non-intersecting paths in a certain graph associated to the $\mathcal{H}$-prime.

Each stratum has a notion of "dimension" attached to it. In particular, the $\mathcal{H}$-primes in strata of dimension zero are members of an important subclass of the prime ideals, namely, the primitive ideals. We give an easy answer to the problem of determining the dimension. The main result is that the dimension is equal to the number of odd cycles in a certain permutation that is easily found given an $\mathcal{H}$-prime. We are then able to give enumeration formulae for d-dimensional strata.

In memory of Ian Stewart

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Foremost, I thank Jason Bell for his guidance during this process. It has been no less than the highest privilege to work with him.

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It is the rare doctorate that could be completed without the help and support of the multitude, and this work is certainly no exception. The totality of such people would be too numerous to mention, but I would like to at least explicitly thank my parents and step-parents: Florence Casteels, Hans Casteels, Sharon Verniero and the late Ian Stewart.

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Chapter 1

Introduction

This thesis uses combinatorial methods to study the set of prime ideals in the algebra of quantum matrices. The importance of understanding this algebra is evident by its many applications to other areas of mathematics and classic problems in physics. Without going into detail, we note that this algebra has connections to knot theory [40], representation theory, Hopf algebras, the quantum Yang-Baxter equation [7], Poisson algebras [18], and, perhaps most surprisingly, to totally non-negative matrices and their generalizations [14, 17, 16, 18, 38] (which in turn have applications to classical problems such as oscillations in mechanical systems and stochastic processes).

The notion of a quantum group is somewhat difficult to describe precisely as there is as yet no formal definition of what constitutes a "quantum group". Indeed this is a fundamental open problem in the field. Instead, the theory is built around a collection of algebraic objects that by general agreement are called quantum groups, which itself is misleading as they are, in fact, algebras. Roughly speaking, these objects are built by combining the definition of a given classical algebra with a parameter q (or multiple parameters) in such a way that when q is set to be 1, the original algebra is recovered. In short, one can say that a quantum group $\mathcal{A}_q$ is built "by analogy" to some algebra $\mathcal{A}$.

In this spirit, we should like to understand the algebraic structure of some quantum group $\mathcal{A}_q$, by finding "quantum analogues" of the structural properties of $\mathcal{A}$. For example, if $\mathcal{A}$ is the coordinate ring of some algebraic variety, a key component in the analysis of the variety is the study of the prime ideals of $\mathcal{A}$. In particular, it is a basic fact in classical algebraic geometry that the prime ideals are in one to one correspondence with the irreducible algebraic subvarieties of the corresponding variety.

Thus it would seem important to have a comprehension of the set of prime ideals (the *spectrum*) of the "quantized coordinate ring" $\mathcal{A}_q$. The goal of this thesis is to further this understanding in the case that $\mathcal{A}_q$ is the quantized coordinate ring of $m \times n$ matrices over a field $\mathbb{K}$ of characteristic zero. For the remainder of this chapter, $\mathcal{A}_q$ refers to this algebra. Under a common abuse of terminology (that we adopt) this algebra itself is called an "$m \times n$ quantum matrix algebra", or simply "$m \times n$ quantum matrices".

Fortunately, we are not starting from scratch. In particular, two developments will be of crucial assistance in our quest. The first observation, due to Goodearl and Letzter [7], is that in many cases there is a natural action of an algebraic torus $\mathcal{H}$ by "nice" automorphisms on the algebra in question. Under this action, the spectrum of $\mathcal{A}_q$ (and indeed, more general algebras) can be partitioned into "$\mathcal{H}$-strata", such that each $\mathcal{H}$-stratum is indexed by a unique prime ideal that is invariant under the action of $\mathcal{H}$ (an "$\mathcal{H}$-prime"). Moreover, there are only finitely many $\mathcal{H}$-strata and each $\mathcal{H}$-stratum is homeomorphic (with respect to the Zariski topology) to a scheme of irreducible subvarieties of a torus.

The second development is due to Cauchon [8, 9]. Applying his "deleting-derivations algorithm" to quantum matrices, he embeds the spectrum of quantum matrix into the spectrum of a simpler algebra, namely, "quantum affine space". Since the spectrum of this latter algebra is understood by results of Goodearl and Letzter [21], a nice description of the $\mathcal{H}$-primes of $m \times n$ quantum matrices is obtained. This description is combinatorial in nature. Specifically, each $\mathcal{H}$-prime in the algebra of $m \times n$ quantum matrices can be associated to a unique combinatorial object now known as an ($m \times n$) *Cauchon diagram.* Such a diagram is simply an $m \times n$ grid of squares that are coloured black or white according to the rule that if a square is coloured black, then either every square to its left or every square above it must also be coloured black. See Figure 1.1, where the left diagram is *not* a Cauchon diagram, while the other two are.

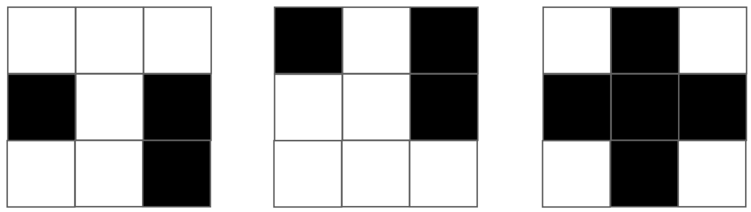


Figure 1.1: Some 3×3 diagrams. The middle and right diagrams are Cauchon diagrams

Based on data obtained for 2×2 and 3×3 quantum matrices, it had been conjectured by Goodearl and Lenagan [20] that generating sets for $\mathcal{H}$–primes always consist of "quantum minors" of the algebra. In the case that the parameter q is transcendental over $\mathbb{Q}$, this was proved by Launois [27, 28], who, moreover, gave an algorithm to find all quantum minors in a given $\mathcal{H}$-prime. While Launois' algorithm begins with a Cauchon diagram, the main step involves repeatedly modifying the entries of a certain matrix to obtain a final matrix whose set of vanishing quantum minors corresponds to a generating set for the $\mathcal{H}$-prime. A downside to this algorithm is that the final matrix has entries possibly consisting of exponentially (in the dimensions of the matrix) many summands.

In Chapter 4, we show that Launois' algorithm can be interpreted as that of finding (or rather, not finding) certain sets of non-intersecting paths in a directed graph that we associate to a Cauchon diagram. This allows the main step of Launois' algorithm to be essentially eliminated, and so the entire problem of finding a generating set for a given $\mathcal{H}$-prime may be considered combinatorial. To achieve this we prove a "quantum analogue" of Lindström's Lemma (Lemma 2.6).

In retrospect, the appearance of Lindström's Lemma in our work is not surprising. Recall that a *totally nonnegative matrix* is a matrix with the property that every minor is nonnegative. Postnikov [38] has developed a theory generalizing this notion to that of the *totally non-negative Grassmannian.* He noted that, in particular, the collection of totally non-negative matrices can be partitioned into cells, where two matrices are in the same cell if and only if they share the same set of vanishing minors. Postnikov's results imply that such cells can be partitioned into so-called "⅃-diagrams". In fact, an ⅃-diagram is nothing but a Cauchon diagram! We note that Talaska [41], independently from us, made this connection explicit by giving a method based on the classical version of Lindström's Lemma. Further details about the connection between $\mathcal{H}$-primes in quantum matrices and totally non-negative matrices can be found in the series of papers by Goodearl, Launois and Lenagan [17, 16, 18].

In representation theory, the prime ideals that are *primitive* correspond to irreducible representations of the ring in question. In infinite dimensional algebras (such as quantum matrices), the problem of classifying the irreducible representations seems to be difficult, if not impossible. Thus Dixmier [12] proposed that one should break down the problem into first finding a method to recognize primitive ideals and then to each primitive ideal P find all irreducible representations with annihilator P. On the other hand, as a consequence of the $\mathcal{H}$-stratification theory, it is known that the primitive ideals are precisely those primes that are maximal within their strata.

The problem is to determine conditions under which a Cauchon diagram corresponds to a primitive $\mathcal{H}$-prime. The utility of Cauchon diagrams in this context has been convincingly demonstrated by several authors. Launois and Lenagan [31] first determined the conditions under which a Cauchon diagram with only white squares corresponds to a primitive ideal. Somewhat surprisingly, the criterion they found is a simple arithmetic condition, depending solely on the number of rows and columns of the Cauchon diagram. This result in fact implies conditions under which $\mathcal{H}$-primes of $1 \times n$ quantum matrices are primitive.

Papers by Bell, Launois & Nguyen [4], and Bell, Launois & Lutley [3] determine primitivity conditions for $\mathcal{H}$-primes in $2 \times n$ quantum matrices and $3 \times n$ quantum matrices respectively. In both cases, they were able to use their condition to give a closed formula that counts the total number of primitive $\mathcal{H}$-primes. The method used to determine primitivity of $\mathcal{H}$-primes in $2 \times n$ quantum matrices depends heavily on the value $m = 2$ and so it is not possible to extend the ideas in that paper. On the other hand, the techniques used in the $3 \times n$ quantum matrix case are based on finite-state automata. While in principle, this strategy can be used for higher values of m, the $m = 3$ case is already highly non-trivial and so another approach is desired.

The second part of this thesis provides this approach. We provide a simple primitivity condition for any m and n. In fact, we do more than this. Each $\mathcal{H}$-stratum has a dimension associated to it, defined to be the longest chain (with respect to inclusion) of prime ideals contained in the stratum. The primitive $\mathcal{H}$-primes correspond to zero-dimensional $\mathcal{H}$-strata. Our method determines, for any $\mathcal{H}$-prime, the exact value of the dimension of the corresponding $\mathcal{H}$-strata.

The condition is essentially as follows. To each Cauchon diagram, we can associate a certain permutation using the device of "pipe-dreams". We prove that the dimension of a given $\mathcal{H}$-stratum is exactly equal to the number of odd cycles in the disjoint cycle decomposition of that permutation. We achieve this result by using a lemma of Bell and Launois [2] that says the dimension of a given $\mathcal{H}$-stratum is precisely the dimension of the kernel of a certain matrix which can be naturally constructed from a Cauchon diagram. We show that the kernel of this matrix is isomorphic to the kernel of another matrix that is built from the permutation provided by pipe-dreams.

As a consequence of our work, we count the number of d-dimensional $\mathcal{H}$-strata. This allows us to settle some conjectures concerning these numbers. In particular, we show that for fixed m, the proportion of primitive $m \times n$ $\mathcal{H}$-primes approaches $\binom{2m}{m}/4^m$ as $n \to \infty$.

This thesis is structured as follows. In Chapter 2 we briefly review some well-known combinatorial concepts that will be needed for our purposes. The more substantial Chapter 3 reviews important algebraic results such as $\mathcal{H}$-stratification and the deleting derivations algorithm. In Chapter 4 we give the vertex-disjoint paths method to calculate generating sets for a given $\mathcal{H}$-prime. In Chapter 5 we give the simple criterion that determines the dimension of a given $\mathcal{H}$-stratum. Finally, Chapter 6 provides a result recently obtained with Bell and Launois: a generalization of the isomorphism between kernels described above to the class of "quantized enveloping algebras"

Chapter 2

Combinatorial Preliminaries

2.1 Introduction

While most readers will be familiar with many of the notions in this chapter, for the sake of completeness (and for the uninitiated), we give some basic definitions from graph theory and combinatorics. There are many good references for this material; we refer the reader to, for example, the books by Van Lint and Wilson [42], and Stanley [39].

Notation 2.1. First we set some basic notation that will be used throughout this work.

1. For a positive integer n, we take $[n] := \{1, 2, \ldots, n\}$.

2. For integers i and j, we set the *Kronecker delta function* $\delta_{i,j}$ to be 1 if $i = j$, and 0 otherwise.

3. If M is a matrix and I and J are subsets of the rows and columns respectively, then we let $M[I, J]$ denote the submatrix of M indexed by I and J. Furthermore, if $I = \{i\}$ and $J = \{j\}$, then we write $M[i, j]$ to be the $(i, j)^{\text{th}}$ entry of M.

4. We will denote sequences or tuples in a ring (e.g., elements of $\mathbb{Z}^n$) in boldface

such as $\boldsymbol{v}$. Entries in a finite tuple will be denoted by attached subscripts. For example, the i^{th} entry of $\boldsymbol{v}$ will be denoted $\boldsymbol{v}_i$.

5. $\mathbb{K}$ will always denote a field. $\mathbb{K}^*$ will always denote the set of non-zero elements of $\mathbb{K}$. In this work we always assume that $\mathbb{K}$ has characteristic zero.

6. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are two functions and $g(x) \neq 0$ for all $x \in \mathbb{R}$, then we write $f \sim g$ ("f is asymptotic to g") to mean that
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

7. If A and B are two isomorphic algebraic structures, then we write $A \simeq B$. On the other hand, if T and S are two homeomorphic topological spaces, then we write $T \simeq S$. The sense of "$\simeq$" we mean should be clear from context.

2.2 Graphs

Definition 2.2. A *directed graph* $G = (V, E)$ consists of a set V of vertices and a set $E \subseteq V \times V$ of *directed edges*. If $e = (v, w)$ is a directed edge, we call v the *tail* of e and w the *head* of e.

Definition 2.3. Given a directed graph $G = (V, E)$, an *embedding* of G is a set of two injective maps ι from V to the (Euclidean) plane, and ϵ from E to the set of simple oriented curves in the plane, such that for each edge $e = (v, w)$, $\epsilon(e)$ is a simple oriented curve that begins at $\iota(v)$, ends at $\iota(w)$ and does not contain $\iota(y)$ for any $y \in V \setminus \{v, w\}$. An embedding is said to be *planar* if for all distinct edges e_1 and e_2, the interiors of $\epsilon(e_1)$ and $\epsilon(e_2)$ are disjoint.

Of course, it is common practice to simply draw $\iota(V)$ and $\epsilon(E)$ in the plane, or at least to describe such a drawing, rather than explicitly giving the mappings. For this reason, we abuse notation by writing v for $\iota(v)$ and e for $\epsilon(e)$ when the embedding is understood. We will often drop the adjective "directed" when it is understood that we are discussing a directed graph (which is always the case in this thesis).

Definition 2.4. Let $G = (V, E)$ be a directed graph. A *directed path* P is an alternating sequence of vertices and edges $P = (v_0, e_1, v_1, \ldots, e_n, v_n)$ such that:

1. For all $i \neq j$, $v_i \neq v_j$;
2. For all i, $e_i = (v_{i-1}, v_i)$.

Note that a single vertex is a directed path. When we wish to emphasize the "starting" vertex v_0 and "ending" vertex v_n of a path P, we will write $P : v_o \Rightarrow v_n$. Two paths are *vertex-disjoint* if they have no vertices in common.

A *directed cycle* is defined similarly except that we take $v_0 = v_n$. We call a directed graph *acyclic* if it contains no directed cycles.

The following result is fundamental in graph theory and can be found, for example, in [10]. Given a set of vertices A and B in a directed graph G, an (A, B)-*cut* is a set of vertices $X \subseteq V$ whose removal from G leaves no directed paths starting in A and ending in B.

Theorem 2.5 (Menger's Theorem)**.** *Let $G = (V, E)$ be a directed graph and let $A, B \subseteq V$. Then the maximum number of mutually vertex-disjoint paths starting in A and ending in B is equal to the minimum size of an (A, B)-cut.*

It is also well-known that the required cut set in Menger's Theorem can be found in polynomial time. Again, see [10] for details.

2.3 Lindström's Lemma

In this section we state a classic result of Lindström [34] that relates the determinant of a certain matrix to sets of vertex-disjoint paths in a graph. The result is often attached to Gessel and Viennot [15] who more convincingly demonstrated the utility of the lemma. The book [1] contains an excellent exposition of the result and its applications, and we refer the reader to it for the beautiful proof.

Let R be a *commutative* ring with unity. Let $G = (V, E)$ be an acyclic, directed graph. Let us also assign, to each edge $e \in E$, a *weight* $w(e) \in R$. Given a path

$$P = (v_0, e_1, \ldots, e_n, v_n),$$

we define the weight of the path to be

$$w(P) = w(e_0)w(e_1)\cdots w(e_n).$$

A set of paths $\mathcal{P}$ is an (I, J)-*path system* if I is the set of starting vertices for the paths in P, and J is the set of ending vertices. We define the weight of a path system $\mathcal{P}$ to be

$$w(\mathcal{P}) = \prod_{P\in\mathcal{P}} w(P).$$

Finally, a path system is *vertex-disjoint* if no two paths in the path system have a common vertex.

Now suppose that we are given two sets $I = \{v_1, \dots, v_k\}$ and $J = \{w_1, \dots, w_k\}$ of vertices. Notice that an (I, J)-path system $\mathcal{P}$ implies the existence of a permutation $\sigma_{\mathcal{P}}$ by $\mathcal{P} = \{P_i : v_i \to w_{\sigma_{\mathcal{P}}(i)}$. Denote by $\mathrm{sgn}(\mathcal{P})$ the sign of $\sigma_{\mathcal{P}}$. The *path matrix* M is the matrix over R with rows indexed by I and columns indexed by J defined by

$$M[i, j] = \sum_{P} w(P),$$

where the sum is over all directed paths in G from $i \in I$ to $j \in J$. Note that

Lemma 2.6 (Lindström's Lemma [34])**.** *Let $G = (V, E)$ be an acyclic directed graph. Let M be the path matrix as defined above. Then*

$$\det(M) = \sum_{\mathcal{P}} \mathrm{sgn}(\mathcal{P})w(\mathcal{P}),$$

where the sum is over all **vertex-disjoint** *(I, J)-path systems.*

2.4 Generating Functions and Stirling numbers

Definition 2.7. Let $(d_n)_{n\geq 0}$ be some sequence with $d_n \in R$ for some ring R. The *ordinary generating function* $F(x)$ and *exponential generating function* $G(x)$ for d_n are, respectively, the formal power series

$$F(x) = \sum_{n\geq 0} d_n x^n$$

and

$$G(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!}.$$

By convention, if $F(x)$ is a generating function for the sequence d_n, then we write the coefficient operator as $[x^n]F(x) := d_n$. The theory of generating functions is of vital importance in many combinatorial applications. Good references for the subject are Wilf [43] and Stanley [39]. Notice that we may consider multivariate generating functions by taking R itself to be a formal power series ring. For example if $d_n = d_n(x) = \sum_m f_{m,n}x^m$ then we may write $F(y) = \sum d_n y^n$ as $F(x,y) = \sum_{m,n} f_{m,n}x^m y^n$. We shall not require more than a basic understanding of the theory in this work, and so we refrain from exploring it deeply. However, we now collect some well-known (and useful) facts. Before we give the statement,

Conventions 2.8. Let k be a nonnegative integer and α be an element in some ring.

1. Define the k^{th} *falling factorial* $(x)_k$ by

   $$(x)_k := x(x-1)\cdots(x-k+1)$$

   and

   $$\binom{x}{k} := \frac{(x)_k}{k!}.$$

2. For any x,

   $$e^x = \sum_{n\geq 0} \frac{x^n}{n!}.$$

3. $$(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k.$$

4. If $F(x,y)$ and $G(x,y)$ are exponential generating functions for the sequences $f(m,n)$ and $g(m,n)$ respectively, then

   $$\left[\frac{x^m}{m!}\frac{y^n}{n!}\right] F(x,y)G(x,y) = \sum_{m'=0}^{m}\sum_{n'=0}^{n} \binom{m}{m'}\binom{n}{n'} f(m',n')g(m-m',n-n').$$

A special sequence of numbers that will be of particular importance to us are the Stirling numbers of the second kind.

Definition 2.9. Let n and k be non-negative integers. The *Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ is, for $n \geq 1$, the number of partitions of $[n]$ into exactly k parts. For $n = 0$ we define $\left\{ {0 \atop k} \right\}$ to be 1 if $k = 0$, and 0 otherwise.

This number is often denoted by $S(n, k)$, but we use the first notation to save some space later on. The following formulae are well-known. For example, they can be found in [42].

Proposition 2.10. *If n and k are non-negative integers, then the following hold:*

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n; \tag{2.1}$$

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k = x^n; \tag{2.2}$$

$$\frac{1}{k!}(e^x - 1)^k = \sum_{m=k}^{\infty} \left\{ {m \atop k} \right\} \frac{x^m}{m!}. \tag{2.3}$$

Chapter 3

Algebraic Preliminaries

3.1 Introduction

This chapter outlines some algebraic concepts needed to understand the original results of this thesis. Unfortunately, it would be impossible to give a rigorous account of the required theory in a reasonably short chapter. Therefore, we resort to only providing statements of the most important theorems. The reader who is interested in the details is referred to the books by Goodearl and Warfield [22] and Brown and Goodearl [7].

3.2 Algebraic Preliminaries

In this section, we give basic definitions and theorems from non-commutative algebra which will be relevant in the presentation of this work. We assume that the reader is familiar with the elementary notions of a ring, a field, an algebra, automorphisms, left/right/two-sided ideals, etc. Moreover, we always assume that a ring has unity.

A ring is *noetherian* if for every ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ there exists some j such that $I_j = I_k$ for all $k \geq j$. Equivalently, a ring is noetherian if all left ideals and right ideals are finitely generated. For an ideal I with generating set X, we write $I = (X)$. A ring is a *domain* if it does not contain any zero divisors.

Given a ring R and a ring automorphism α of R, an α-*derivation* is an additive function $\delta : R \to R$ such that $\delta(rs) = \delta(r)\alpha(s) + r\delta(s)$. Of course, this rule resembles the product rule for the classical derivative (particularly when α is the identity function), whence the name α-derivation.

Definition 3.1. Let R be a ring together with a ring automorphism α and an α-derivation δ of R. An *Ore extension* of R is a ring $S = R[x; \alpha, \delta]$ that satisfies the following properties:

1. R is a subring of S;

2. $x \in S$;

3. S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

4. $xr = \alpha(r)x + \delta(r)$ for all $r \in R$.;

Remark 3.2. We give three brief comments about the previous definition.

1. In practice, one should think of an Ore extension simply as a polynomial ring in the indeterminate x such that commutativity between x and the ring elements is given by Condition 4.

2. Although it is not quite obvious from the definition, it can be shown that given a ring R, a ring automorphism α and α-derivation δ of R, there always exists a unique (up to isomorphism) Ore extension of R.

3. One may take an Ore extension of an Ore extension (and so forth) to obtain an *iterated Ore extension*.

Given a ring R and a subset $X \subseteq R$, it is sometimes desirable to have at hand a ring $S \supseteq R$ such that every element of X is a unit in S. Naturally this is not always possible, however under certain conditions on X such a ring S does exist.

Definition 3.3. Let R be a ring and $X \subseteq R$. We say X is an *Ore set* if the following conditions on X hold:

1. Every element of X is *regular*, i.e., not a zero-divisor;

2. X contains 1 and is closed under multiplication;

3. For all $x \in X$ and $r \in R$, $xR \cap rX \neq \emptyset$ and $Rx \cap Xr \neq \emptyset$.

For example, $R \setminus \{0\}$ is always an Ore set when R is a noetherian domain [22].

Definition 3.4. Let R be a ring and $X \subseteq R$ be an Ore set. A *ring of fractions* for R with respect to X is a ring $S \supseteq R$ such that:

1. Every $x \in X$ is a unit in S (as usual, we write its inverse as x^{-1});

2. Every element in S can be expressed as rx^{-1} for some $r \in R$ and $x \in X$;

3. Every element in S can be expressed as $x^{-1}r$ for some $r \in R$ and $x \in X$.

Furthermore, S is guaranteed both to exist and be unique (with respect to a suitable universal property) [22]. We write $S = RX^{-1} = X^{-1}R$.

As a matter of convenience, if $x \in R$, and X is the multiplicative set generated by x, then we write $RX^{-1} = R[x^{-1}]$. Furthermore, the phrase "the localization of R with respect to X" is often used to mean the ring of fractions RX^{-1}.

In the case that R is a noetherian domain and $X = R \setminus \{0\}$, RX^{-1} is called the *skew field of fractions*, denoted by $\mathrm{Frac}(R)$. Next, we state some facts regarding Ore extensions and localizations. In essence, the following says that both inherit "niceness".

Proposition 3.5. *Let R be a ring, $X \subseteq R$ an Ore set, α an* automorphism *and δ an α-derivation of R.*

1. *If R is a noetherian ring, then so are $R[x; \alpha, \delta]$ and RX^{-1}.*

2. *If R is a domain, then so are $R[x; \alpha, \delta]$ and RX^{-1}.*

3.3 Some Quantum Algebras

While there is as yet no axiomatic definition of a "quantum algebra" (also called a "quantum group"), there are many examples of algebraic objects that by consensus form the basis of study for the theory. In this section we give several definitions of such algebras. In most cases, we give a group $\mathcal{H}$ that acts *rationally* on the algebra by $\mathbb{K}$-algebra automorphisms. What it means for an automorphism to act rationally shall not concern us here, suffice it to say such automorphisms act in a particularly nice manner. See Section II.2.6 of [7] for details.

Definition 3.6. Let $\mathbb{K}$ be a field. With respect to the definitions provided below, we denote by $\mathfrak{Mat}_q(\mathbb{K})$ the collection of algebras of the form $\mathcal{O}_q(\mathbb{K}^{m\times n})$, $\mathcal{O}_q((\mathbb{K}^\times)^{m\times n})$, $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$, $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))$, and $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$. To each such algebra, we always associate the automorphism subgroup $\mathcal{H}$ described below ($\mathcal{H}'$ in the case of $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))$).

3.3.1 Quantum Affine Spaces I

Definition 3.7. Let $\mathbb{K}$ be a field and Q an $r\times r$ *multiplicatively antisymmetrix* matrix with $Q[i,j] := q_{i,j}$. In other words, for all $1 \leq i,j \leq r$, Q satisfies the relations $q_{i,i} = 1$ and $q_{i,j}q_{j,i} = 1$. A *multiparameter quantum affine space of rank* r is the $\mathbb{K}$-algebra $\mathcal{O}_Q(\mathbb{K}^r)$ generated by $t_1, t_2, \ldots, t_r$ satisfying $t_it_j = q_{i,j}t_jt_i$, for all $1 \leq i,j \leq r$.

A multiparameter quantum affine space can be written as an iterated Ore extension $\mathcal{O}_Q(\mathbb{K}^r) = \mathbb{K}[t_1][t_2;\alpha_2,\delta_2]\cdots[t_r;\alpha_2,\delta_r]$ as follows. Let us set $R_1 := \mathbb{K}[t_1]$ and $R_j := R_{j-1}[t_j;\alpha_j,\delta_j]$ where α_j is defined by $\alpha_j(t_i) := q_{i,j}^{-1}t_i$ for $i < j$, and setting $\delta_j \equiv 0$ for all j. Thus $\mathcal{O}_Q(\mathbb{K}^r)$ is a noetherian domain, and we can construct its skew field of fractions. Furthermore, it can be shown that the monomials in the t_i form a basis for the space [7].

Definition 3.8. The *(multiparameter) quantum torus* is defined to be the localization of $\mathcal{O}_Q(\mathbb{K}^r)$ with respect to the Ore set generated by $t_1, t_2, \ldots, t_r$. We shall denote this algebra by $\mathcal{O}_Q((\mathbb{K}^\times)^r)$. This algebra is also sometimes called the *McConnell-Petit algebra*. Again, it can be shown that the monomials in the t_i form a basis (where negative exponents are allowed now).

In Chapter 5, we will be interested in calculating the *center* of $\mathcal{T} = \mathcal{O}_Q((\mathbb{K}^\times)^r)$, that is, the set $Z(\mathcal{T}) = \{z \in \mathcal{T} \mid zr = rz, \forall\, r \in \mathcal{T}\}$. Fortunately, this is easily calculated, and was done so by Goodearl and Letzter [21]. We record their result here. Denote the $(i,j)^{\text{th}}$ entry of Q by $q_{i,j}$. For an r-tuple of integers $\boldsymbol{k} = (k_1, \ldots, k_r)$, let $\boldsymbol{t}^{\boldsymbol{k}} := t_1^{k_1} t_2^{k_2} \cdots t_r^{k_r}$.

For $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ and $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r) \in \mathbb{Z}^r$, notice that

$$\boldsymbol{t}^{\boldsymbol{k}} \boldsymbol{t}^{\boldsymbol{\ell}} = \left(\prod_{i,j=1}^{n} q_{i,j}^{k_i \ell_j} \right) \boldsymbol{t}^{\boldsymbol{\ell}} \boldsymbol{t}^{\boldsymbol{k}}.$$

If we set

$$\sigma(\boldsymbol{k}, \boldsymbol{\ell}) := \prod_{i,j=1}^{n} q_{i,j}^{k_i \ell_j},$$

then the map $\sigma : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{K}^*$ is an *alternating bicharacter*. In other words, σ is a group homomorphism such that for all integer r-tuples $\boldsymbol{k}$ and $\boldsymbol{\ell}$,

$$\begin{aligned} \sigma(\boldsymbol{k}, \boldsymbol{k}) &= 1, \text{ and} \\ \sigma(\boldsymbol{k}, \boldsymbol{\ell}) &= \sigma(\boldsymbol{\ell}, \boldsymbol{k})^{-1}. \end{aligned}$$

Lemma 3.9 (Goodearl and Letzter [21])**.** *Let $\mathcal{T}$ and σ be as above. If*

$$S := \{\boldsymbol{k} \mid \sigma(\boldsymbol{k}, \boldsymbol{\ell}) = 1, \forall\, \boldsymbol{\ell} \in \mathbb{Z}^r\},$$

then

$$Z(\mathcal{T}) = \mathbb{K}[\boldsymbol{t}^{\boldsymbol{k}} \mid \boldsymbol{k} \in S].$$

Proof. It is clear from the definition of S that $Z(\mathcal{T}) \supseteq \mathbb{K}[\boldsymbol{t}^{\boldsymbol{k}} \mid \boldsymbol{k} \in S]$. On the other hand, let $z = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^r} \alpha_{\boldsymbol{\ell}} \boldsymbol{t}^{\boldsymbol{\ell}} \in Z(\mathcal{T})$, where all but finitely many of the $\alpha_{\boldsymbol{\ell}}$ are zero. Note that for any $\boldsymbol{j} \in \mathbb{Z}^r$ we have $z\boldsymbol{t}^{\boldsymbol{j}} - \boldsymbol{t}^{\boldsymbol{j}} z = 0$. On the other hand,

$$\begin{aligned} 0 &= z\boldsymbol{t}^{\boldsymbol{j}} - \boldsymbol{t}^{\boldsymbol{j}} z \\ &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^r} \alpha_{\boldsymbol{\ell}} \boldsymbol{t}^{\boldsymbol{\ell}} \boldsymbol{t}^{\boldsymbol{j}} - \sum_{\boldsymbol{\ell} \in \mathbb{Z}^r} \alpha_{\boldsymbol{\ell}} \boldsymbol{t}^{\boldsymbol{j}} \boldsymbol{t}^{\boldsymbol{\ell}} \\ &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^r} \alpha_{\boldsymbol{\ell}} \left(1 - \sigma(\boldsymbol{j}, \boldsymbol{\ell})\right) \boldsymbol{t}^{\boldsymbol{\ell}} \boldsymbol{t}^{\boldsymbol{j}} \end{aligned}$$

Since the monomials are linearly independent, it follows that $\sigma(\boldsymbol{j}, \boldsymbol{\ell}) = 1$ whenever $\alpha_{\boldsymbol{\ell}} \neq 0$. In other words, $z \in \mathbb{K}[\boldsymbol{t}^{\boldsymbol{k}} \mid \boldsymbol{k} \in S]$. □

3.3.2 Quantum Affine Spaces II

It is worth isolating a special case of Definition 3.7.

Definition 3.10. Let $m, n \geq 1$ and fix a $q \in \mathbb{K}^*$. Let Q be the multiplicatively antisymmetric $mn \times mn$ matrix with rows and columns indexed by $[m] \times [n]$, defined by taking

$$Q[(i,j),(k,\ell)] = \begin{cases} q & \text{if either } i = k \text{ and } j < \ell, \text{ or } j = \ell \text{ and } i < k, \\ q^{-1} & \text{if either } i = k \text{ and } j > \ell, \text{ or } j = \ell \text{ and } i > k, \\ 1 & \text{otherwise.} \end{cases}$$

The $m \times n$ *(uniparameter) quantum affine space* is the $\mathbb{K}$-algebra

$$\mathcal{O}_q(\mathbb{K}^{m \times n}) := \mathcal{O}_Q(\mathbb{K}^{mn}).$$

In other words, $\mathcal{O}_q(\mathbb{K}^{m \times n})$ is the $\mathbb{K}$-algebra generated by $t_{i,j}$ for $(i,j) \in [m] \times [n]$ (the *canonical generators*), according to the following commutativity relations. If T is the $m \times n$ matrix with $T[i,j] := t_{i,j}$, then:

1. If $t_{i,j}$ and $t_{i,k}$ are in the same row of T and $j < k$, then $t_{i,j}t_{i,k} = qt_{i,k}t_{i,k}$. We say that these two elements *q-commute*. Note that we are obliged to take $t_{i,k}t_{i,k} = q^{-1}t_{i,j}t_{i,k}$ for this to make sense.

2. If $t_{i,j}$ and $t_{\ell,j}$ are in the same column of T and $i < \ell$, then $t_{i,j}t_{\ell,j} = qt_{\ell,j}t_{i,j}$ (and $t_{\ell,j}t_{i,j} = q^{-1}t_{i,j}t_{\ell,j}$).

3. In any other case, $t_{i,j}$ and $t_{\ell,k}$ commute.

We call T the *matrix of canonical generators* for $\mathcal{O}_q(\mathbb{K}^{m \times n})$.

Given positive integers m and n, consider the *algebraic torus* $\mathcal{H} = (\mathbb{K}^*)^{m+n}$. Notice each $h = (\rho_1, \ldots, \rho_m, \gamma_1, \ldots, \gamma_n) \in \mathcal{H}$ acts on $\mathcal{O}_q(\mathbb{K}^{m \times n})$ by automorphisms by $h(t_{i,j}) = \rho_i \gamma_j t_{i,j}$. One can think of such an action as one that multiplies each row i of T by ρ_i and each column j by γ_j. This action can be extended in the obvious way to the quantum torus $\mathcal{O}_q((\mathbb{K}^{\times})^{m \times n})$.

3.3.3 Quantum Matrices

The algebra which will be of particular importance to us is the following.

Definition 3.11. Fix $q \in \mathbb{K}^*$ and two positive integers m and n. The *quantized coordinate ring of $m \times n$ matrices over* $\mathbb{K}$. is defined as follows. If $m \geq 2$ and $n \geq 2$, then $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is the $\mathbb{K}$-algebra generated by $x_{i,j}$ for $i, j \in [m] \times [n]$ (the *canonical generators*) that satisfy the following relations. If X is the $m \times n$ matrix with $X[i,j] = x_{i,j}$, then for any 2×2 submatrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of X the following hold:

1. $ab = q\,ba$ and $cd = q\,dc$;

2. $ac = q\,ca$ and $bd = q\,db$;

3. $bc = cb$;

4. $ad - da = (q - q^{-1})\,bc$.

Finally, for any integers $m, n \geq 1$, we set

$$\mathcal{O}_q(\mathrm{M}_{1,n}(\mathbb{K})) := \mathcal{O}_q(\mathbb{K}^{1\times n})$$

and

$$\mathcal{O}_q(\mathrm{M}_{m,1}(\mathbb{K})) := \mathcal{O}_q(\mathbb{K}^{m\times 1}).$$

To unify notation, we will write the matrix of canonical generators for these latter two cases as X rather than T.

Before we continue, let us make several remarks concerning this definition. Most authors, including this one, will often abuse nomenclature by talking about the "algebra of $m \times n$ quantum matrices", or even more colloquially, "$m \times n$ quantum matrices", to mean the algebra $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. It should be emphasized to the reader not familiar with this field that the algebra does not actually consist of matrices. Rather, it is simply an algebra of polynomials built in analogy to the classical coordinate ring of $m \times n$ matrices.

Evidently, it is only Condition 4 in Definition 3.11 that separates an algebra of quantum matrices from an $m \times n$ quantum affine space. The reason for the difference is partly historical, arising from a construction of solutions to the quantum Yang-Baxter equation, and also algebraic as the condition ensures that $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is a *bialgebra*. Since we will have no use for this level of algebraic sophistication, we leave the details of this latter property to the interested reader (see [7]).

The exact value of q in Definition 3.11 may be important, depending on the application. Note that when $q = 1$, $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is the usual coordinate ring of $m \times n$ matrices, $\mathcal{O}(\mathrm{M}_{m,n}(\mathbb{K}))$. Generally, however, the theory of quantum matrices (and quantum algebras in general) is split in two: the *generic* case where q is *not* a root of unity, and the non-generic case when q *is* a root-of unity. The methods employed to study either of these two cases differ greatly. In this thesis, we always assume that q **is *not* a root of unity.** Furthermore, some of the theory we develop in this thesis is only valid when q is transcendental over $\mathbb{Q}$. We will indicate when such a restriction is needed.

Definition 3.12. Linearly order the set $([m] \times [n]) \cup (m, n+1)$ as follows. Set $(i, j) <_L (k, \ell)$ if and only if either $i < k$, or $i = k$ and $j < \ell$. Notice that this ordering (from smallest to largest) can be thought of as reading the entries of an $m \times n$ matrix (plus $(m, n+1)$ from left to right and top to bottom, as one would read a page in the English language. For this reason, we call this the *lexicographic ordering.*

Given $(i, j) \in [m] \times [n]$, let $(i, j)^-$ be the largest element (with respect to $<_L$) of $[m] \times [n]$ less than (i, j). We also set $(m, n+1)^- := (m, n)$.

Using the lexicographic ordering on the canonical generators, we may write $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ as an iterated Ore extension $\mathcal{A} = \mathbb{K}[x_{1,1}][x_{1,2}; \alpha_{1,2}, \delta_{1,2}] \cdots [x_{n,m}; \alpha_{n,m}, \delta_{n,m}]$ where $\mathcal{A}_{1,1} := \mathbb{K}[x_{1,1}]$ and $\mathcal{A}_{k,\ell} := \mathcal{A}_{(k,\ell)^-}[x_{k,\ell}; \alpha_{k,\ell}, \delta_{k,\ell}]$, with $\alpha_{k,\ell}$ and $\delta_{k,\ell}$ defined as follows. For $(i, j) <_L (k, \ell)$,

$$\alpha_{k,\ell}(x_{i,j}) = \begin{cases} q^{-1} x_{i,j} & \text{if } x_{i,j} \text{ and } x_{k,\ell} \text{ are in the same row or column of } X, \\ x_{i,j} & \text{otherwise.} \end{cases}$$

Next we take

$$\delta_{k,\ell}(x_{i,j}) = \begin{cases} -(q-q^{-1})x_{i,\ell}x_{k,j} & \text{if both } i<k \text{ and } j<\ell, \\ 0 & \text{otherwise.} \end{cases}$$

We may conclude that $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is a noetherian domain so that we may consider its skew field of fractions. As in the $m \times n$ quantum affine space, we note that the algebraic torus $\mathcal{H} = (\mathbb{K}^*)^{m+n}$ acts (rationally) on $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ by automorphisms as follows. If $h = (\rho_1, \ldots, \rho_m, \gamma_1, \ldots, \gamma_n) \in \mathcal{H}$, then h acts on $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ by sending $x_{i,j}$ to $\rho_i\gamma_j x_{i,j}$.

Finally, notice that there is a simple isomorphism between $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ and $\mathcal{O}_q(\mathrm{M}_{n,m}(\mathbb{K}))$ obtained by sending $X_{i,j}$ to $X_{j,i}$. Thus any result we prove for the former algebra is automatically true for the latter.

3.3.4 The Quantum Determinant

Definition 3.13. Let $\mathbb{K}$ be a field and let Y be an $n \times n$ matrix with $Y[i,j] = y_{i,j}$. Let $\mathcal{Y}$ be the $\mathbb{K}$-algebra generated by the $y_{i,j}$, possibly subject to some commutativity relations. For a fixed $q \in \mathbb{K}$, the *quantum determinant*, or *q-determinant*, is the element

$$\det_q(Y) := \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} y_{1,\sigma(1)} y_{2,\sigma(2)} \cdots y_{n,\sigma(n)} \in \mathcal{Y},$$

where $\ell(\sigma)$ is the usual length function on permutations, i.e., $\ell(\sigma)$ is the number of pairs $i < j$ with $\sigma(i) > \sigma(j)$. A *quantum minor* (or *q-minor*) is a quantum determinant of a square submatrix of Y.

It is easy to see that we may always write σ as a product of adjacent transpositions, i.e., transpositions of the form $(i\ i+1)$. It is an established fact [39] that $\ell(\sigma)$ is the minimum number of such transpositions required in such a product. In the case that $Y = X$ is the matrix of canonical generators for $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$, it is known [40] that $\det_q(X)$ is *central*, i.e., it commutes with every element in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$.

Using the quantum determinant, one may define the *quantized coordinate ring of the general linear group* as

$$\mathcal{O}_q(\mathrm{GL}_n(\mathbb{K})) := \mathcal{O}_q(\mathrm{M}_{n,n}(\mathbb{K}))[(\det_q(X))^{-1}],$$

and the *quantized coordinate ring of the special linear group* as

$$\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K})) := \mathcal{O}_q(\mathrm{M}_{n,n}(\mathbb{K}))/(\det_q(X) - 1).$$

Both algebras are noetherian domains by Proposition 3.5 and the basic fact that quotients of noetherian rings by ideals are noetherian. In fact, they are both *Hopf* algebras [7], but again we have no use for that level of sophistication. Notice that both algebras can be presented by using the matrix of canonical generators for $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$, together with an extra condition. We will abuse notation here and continue to refer to such matrices as "matrices of canonical generators" despite there being an extra generator in the $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$ case.

Recall the action of the group $\mathcal{H}$ on $\mathcal{O}_q(\mathrm{M}_{n,n}(\mathbb{K}))$ from the previous subsection. It is easy to see that $\det_q(X)$ is an *eigenvector* for $\mathcal{H}$: there exists a homomorphism $\lambda : \mathcal{H} \to \mathbb{K}^*$ such that for all $h \in \mathcal{H}$, $h(\det_q(X)) = \lambda(h)\det_q(X)$. Therefore, $\mathcal{H}$ can be uniquely extended to a rational action by automorphisms on $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$.

In the case $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))$ we need to be a bit more careful. Since in this algebra, $\det_q(X) = 1$, we must restrict ourselves to the subgroup $\mathcal{H}'$ of $\mathcal{H}$ consisting of those $h = (\rho_1, \ldots, \rho_n, \gamma_1, \ldots, \gamma_n)$ with $\rho_1 \cdots \rho_n \gamma_1 \cdots \gamma_n = 1$. This choice of $\mathcal{H}'$ acts rationally on $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))$ by automorphisms. In the sequel we will abuse notation and write $\mathcal{H}$ for the action of $\mathcal{H}'$ when used in the context of $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))$.

3.4 The Prime Spectrum of an Algebra

The main purpose of this thesis is to study the set of prime ideals of $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. Naturally then, we should review some basic definitions and facts concerning prime ideals.

Definition 3.14. Let R be a ring. An ideal P is *prime* if $P \neq R$ and whenever I and J are such that $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

Alternatively, the following are easily shown to be equivalent:

1. P is a prime ideal in R;

2. (0) is a prime ideal in the ring R/P;

3. If $x, y \in R$ and $xRy \subseteq P$ then either $x \in P$ or $y \in P$.

An ideal P in R is *completely prime* if whenever $xy \in P$, either $x \in P$ or $y \in P$.

Definition 3.15. The set of all prime ideals of a ring R is called the *spectrum* of R, denoted $\mathrm{spec}(R)$. Furthermore, we always endow $\mathrm{spec}(R)$ with the *Zariski topology* defined by taking the closed sets to be those of the form

$$V(I) = \{P \in \mathrm{spec}(R) \mid P \supseteq I\}$$

for any ideal I of R.

A ring is *simple* if its only ideals are the zero ideal (0) and the ring itself. We now present the first part of an example that will run through the remainder of this chapter.

Example 3.16. Let us sketch an example of calculating, for q not a root of unity, the spectrum of

$$\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{1,2}(\mathbb{K})) \simeq \mathbb{K}[x, y]/(xy - qyx).$$

For simplicity, we add the further assumption that the base field $\mathbb{K}$ is algebraically closed, although the result remains true for $\mathbb{K}$ of characteristic zero [17].

Now since $\mathcal{A}/(y) \simeq \mathbb{K}[x]$ we see that the prime ideals containing y are (y) and, since $\mathbb{K}[$ is algebraically closed, those of the form $(y, x - \alpha)$ for $\alpha \in \mathbb{K}[$. Similarly the prime ideals containing x are (x) and $(y - \alpha, x)$, for $\alpha \in \mathbb{K}[$.

In fact the ideals in the previous paragraph, together with (0), describe the entire prime spectrum of $\mathcal{A}$. To see this we must show that any non-zero prime ideal must contain either x or y. Let X be the multiplicative set generated by x and y. We show first that the localization $\mathcal{A}X^{-1}$ is a simple ring which, by a well-known theorem, will give us that every prime ideal in $\mathcal{A}$ contains either x or y.

First notice that, since $\mathcal{A}$ is a domain, X contains only regular elements. Moreover, since x and y q-commute, it is easy to see that any element $z \in X$ satisfies $z\mathcal{A} = \mathcal{A}z$ (such elements are called *normal*). Since trivially $1 \in X$, it follows that X is an Ore set so that we may consider the localization $\mathcal{B} = \mathcal{A}X^{-1}$.

Suppose that I is a non-zero ideal of $\mathcal{B}$. We now show that we must have $1 \in I$ which implies $I = \mathcal{B}$, i.e., that $\mathcal{B}$ is simple. For $z \in I$, we can write $z = \sum_{(i,j)\in S} \alpha_{i,j} x^i y^j$, where $S \subseteq \mathbb{Z}^2$ is finite and chosen to be of minimal size, and $\alpha_{i,j} \neq 0$ for $(i,j) \in S$. Furthermore, since we are working in the localization of $\mathcal{A}$ by x and y and since I is an ideal, we may assume, without loss of generality, that $(0,0) \in S$ and $\alpha_{0,0} = 1$. Thus we may write $z = 1 + z'$.

Since I is an ideal we know that the commutator $[z,x] \in I$. On the other hand,

$$\begin{aligned} [z,x] &= [1+z', x] \\ &= z'x - xz' \\ &= \sum_{(i,j)\in S\setminus(0,0)} \alpha_{i,j}(q^j - 1)x^i y^j. \end{aligned}$$

By minimality of S, this implies $\alpha_{i,j}(q^j - 1) = 0$ for all i, j. Since q is not a root of unity, we must have $\alpha_{i,j} = 0$ for all $j \neq 0$. Similarly, $\alpha_{i,j} = 0$ for all $i \neq 0$. Thus $z' = 0$ and so $z = 1 \in I$ as desired. We have shown that $\mathcal{B}$ is a simple ring. By Theorem

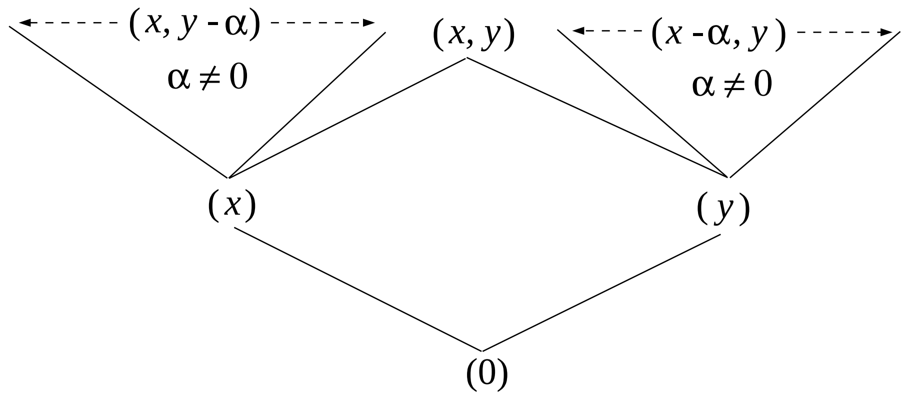


Figure 3.1: The inclusion lattice of spec($\mathcal{O}_q(\mathbb{K}^{1\times 2})$)

10.20 in [22], there is a bijection between prime ideals of $\mathcal{A}X^{-1}$ and prime ideals of $\mathcal{A}$ that are disjoint from X. In particular, the prime ideal (0) in $\mathcal{A}$ corresponds to the prime ideal (0) in $\mathcal{B}$. Hence if P is a non-zero prime ideal of $\mathcal{A}$ then P is not disjoint from X and so $x^i y^j \in P$ for some i and j.

Now for the choice of i and j in the previous paragraph, $x^i y^j \mathcal{A}$ is a subideal of P. On the other hand, since $x^i y^j$ is normal in $\mathcal{A}$, we have $x^i y^j \mathcal{A} = (x\mathcal{A})^i (y\mathcal{A})^j \subseteq P$. Now the ideal P is prime so either $(x\mathcal{A})^i \subseteq P$ or $(y\mathcal{A})^j \subseteq P$. Suppose that $(x\mathcal{A})^i \subseteq P$. Again by normality of x, either $x\mathcal{A} \subseteq P$ or $(x\mathcal{A})^{i-1} \in P$. Continuing in this manner we must eventually conclude that either $x \in P$ or $y \in P$. The situation is pictured in Figure 3.1.

An important subset of spec($\mathcal{A}$) which will be of special interest to us are the *primitive* ideals.

Definition 3.17. A ring R is *primitive* if there exists simple left and, respectively, right R-modules M_1 and M_2 such that both $rM_1 = 0$ if and only if $r = 0$, and $M_2 r = 0$ if and only if $r = 0$. An ideal P of a ring R is *primitive* if and only if R/P is a primitive ring.

It is well known that a primitive ideal is always prime [22]. We will see later that in the context of the algebra of quantum matrices, primitive ideals have a nice description via $\mathcal{H}$-stratification.

3.5 The Theory of $\mathcal{H}$-Stratification

Let $\mathbb{K}$ be a field of characteristic zero and $q \in \mathbb{K}^*$ not a root of a unity. Given an algebra $\mathcal{A} \in \mathfrak{Mat}_q(\mathbb{K})$ and its corresponding group of rational actions $\mathcal{H}$, this section describes a useful partition of spec($\mathcal{A}$) due to Goodearl and Letzter called the *$\mathcal{H}$-stratification.*

An ideal I of $\mathcal{A}$ is said to be an *$\mathcal{H}$-ideal* if $h(I) = I$ for all $h \in \mathcal{H}$. In analogy to the definition of a prime ideal, we say P is an *$\mathcal{H}$-prime ideal*, or *$\mathcal{H}$-prime* for short, if P is a proper $\mathcal{H}$-ideal of $\mathcal{A}$ and if I and J are any two $\mathcal{H}$-ideals with $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. It is not immediately obvious from the definition but it can be shown that every $\mathcal{H}$-prime ideal is in fact a prime ideal [7].

Now given an ideal I, let $(I : \mathcal{H})$ denote the largest $\mathcal{H}$-ideal contained in I. In other words,

$$(I : \mathcal{H}) = \bigcap_{h \in \mathcal{H}} h(I).$$

Let $\mathcal{H}(I) = \{h(I) \mid h \in \mathcal{H}\}$ be the orbit of I with respect to $\mathcal{H}$. Notice that if I and I' have the same orbit, then clearly $(I : \mathcal{H}) = (I' : \mathcal{H})$.

Definition 3.18. Let J be an $\mathcal{H}$-prime in $\mathcal{A}$. The *$\mathcal{H}$-stratum* associated to J is the set

$$\operatorname{spec}_J(\mathcal{A}) := \{P \in \operatorname{spec}(\mathcal{A}) \mid (P : \mathcal{H}) = J\}.$$

By the observation immediately preceding this definition, the collection of all $\mathcal{H}$-strata partition $\operatorname{spec}(\mathcal{A})$. This partition is called the *$\mathcal{H}$-stratification* of $\mathcal{A}$.

An *$\mathcal{H}$-eigenvector* of $\mathcal{A}$ is an element $x \in \mathcal{A}$ such that for all $h \in \mathcal{H}$, there exists a homomorphism $\lambda : \mathcal{H} \to \mathbb{K}^*$ with $h(x) = \lambda(h)x$. For example, it is not difficult to check that in the case $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{2,2}(\mathbb{K}))$, the monomials in the canonical generators are $\mathcal{H}$-eigenvectors. Now if J is an $\mathcal{H}$-ideal then $\mathcal{A}/J$ inherits the action of $\mathcal{H}$ on $\mathcal{A}$ in a natural way. Goodearl and Letzter [7] have shown that the set E_J of regular $\mathcal{H}$-eigenvectors of $\mathcal{A}/J$ forms an Ore set. Let

$$\mathcal{A}_J = (\mathcal{A}/J)[E_J]^{-1}.$$

The proof of the following theorem due to Goodearl and Letzter can be found split into several different parts in Chapter II of [7]. It should be noted that the full theory is applicable to algebras other than those in $\mathfrak{Mat}_q(\mathbb{K})$ and, in fact, is deeper than the parts we present here, we have only collected those aspects of the theory that are of relevance to us.

Theorem 3.19 ($\mathcal{H}$-stratification of algebras in $\mathfrak{Mat}_q(\mathbb{K})$)**.** *Let $\mathbb{K}$ be a field of characteristic zero and let q be a non-root of unity in $\mathbb{K}$. Let $\mathcal{A} \in \mathfrak{Mat}_q(\mathbb{K})$, and let $\mathcal{H}$ be the torus which acts on $\mathcal{A}$ as defined in Section 3.2. Let J be an $\mathcal{H}$-prime ideal of $\mathcal{A}$. The following hold:*

1. $\mathcal{H}$-$\operatorname{spec}(\mathcal{A})$ *is a finite set;*

2. *J is a completely prime ideal;*

3. $\operatorname{spec}_J(\mathcal{A}) \simeq \operatorname{spec}(\mathcal{A}_J)$*;*

4. $\operatorname{spec}(\mathcal{A}_J) \simeq \operatorname{spec}(Z(\mathcal{A}_J))$*;*

5. $Z(\mathcal{A}_J)$ *is a Laurent polynomial ring over* $\mathbb{K}$ *in a finite number of indeterminates (cf. Lemma 3.9).*

For a given $\mathcal{H}$-prime ideal J, we define the number of indeterminates arising in Part (5) of the $\mathcal{H}$-stratification theorem to be the *(Krull) dimension* of the $\mathcal{H}$-stratum corresponding to J. It is the maximum possible length of a chain of prime ideals

$$J = P_0 \subset P_1 \subset \cdots \subset P_d$$

contained in the $\mathcal{H}$-stratum. It is well-known that the dimension is invariant under homeomorphism.

Another theorem of Goodearl and Letzter tells us that *the primitive ideals of* $\mathcal{A}$ *are precisely those ideals that are maximal within their strata.* We provide an easy method in Chapter 5 to calculate the dimension of a given $\mathcal{H}$-stratum.

Example 3.20. Consider again Example 3.16. In this case, it is clear that $(0), (x), (y)$ and (x, y) are $\mathcal{H}$-invariant. On the other hand, for $\alpha \neq 0$ and $h = (\rho, 1) \in \mathcal{H}$, we have $h((x, y-\alpha)) = (\rho x, \rho y - \alpha) = (x, y - \alpha/\rho)$. Since we may choose $\rho \neq 1$, we see that $(x, y-\alpha)$ is not an $\mathcal{H}$-prime ideal for any $\alpha \neq 0$. Similarly, $(x-\alpha, y)$ is not an $\mathcal{H}$-prime for $\alpha \neq 0$. In conclusion, $\mathcal{O}_q(\mathrm{M}_{1,2}(\mathbb{K}))$ contains exactly four $\mathcal{H}$-prime ideals: $(0), (x), (y)$ and (x, y). See Figure 3.2. The primitive ideals are (0) and those of the form $(x-\alpha, y-\beta)$ where at least one of α and β is zero.

The results contained in Examples 3.16 and 3.20 have been generalized by Goodearl and Letzter [21] to describe the $\mathcal{H}$-prime ideals of any multiparameter quantum affine space. We record their result here.

Theorem 3.21. *Let* $\mathcal{O}_Q(\mathbb{K}^r)$ *be a multiparameter quantum affine space. For each* $D \subseteq [r]$*, the ideal* $K_D = (t_i \mid i \in D)$ *is an* $\mathcal{H}$*-prime ideal. Moreover, all* $\mathcal{H}$*-prime ideals are of the form* K_D *for some* $D \subseteq [r]$*.*

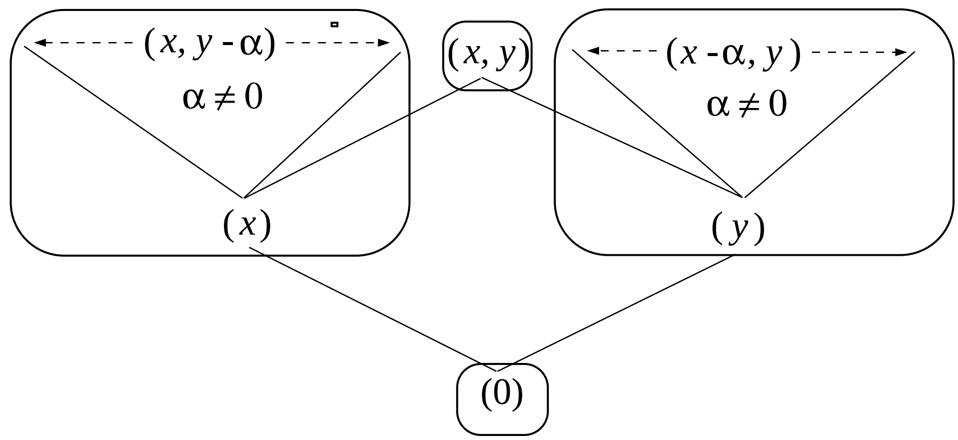


Figure 3.2: The $\mathcal{H}$-strata of spec$(\mathcal{O}_q(\mathbb{K}^{1\times 2}))$ outlined in rounded boxes.

3.6 The Deleting Deriviations Algorithm

Besides the $\mathcal{H}$-stratification theory itself, the key step forward in the structural theory of $\mathcal{H}$-spec$(\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K})))$ was given by Cauchon and his *deleting derivations algorithm* [8, 9]. In this section we describe this algorithm as applied to $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ and the resulting combinatorial objects which have come to be known as *Cauchon diagrams.*

In general, the algorithm can be applied to any Ore extension of a ring satisfying some additional assumptions. Nevertheless, for our purposes it will suffice to describe the procedure as applied to the special case which is of interest to us, namely, $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$.

Notation 3.22. Let us fix the following terminology for our description of the deleting derivations algorithm as applied to $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$.

1. Recall that Frac$(\mathcal{A})$ denotes the skew field of fractions of $\mathcal{A}$.

2. We place the lexicographic ordering $<_L$ on the set $([m] \times [n]) \cup \{(m, n+1)\}$. Recall that given $(i, j) \in [m] \times [n]$, we let $(i, j)^-$ be the largest element (with respect to $<_L$) of $[m] \times [n]$ less than (i, j).

3. If $X^{(k,\ell)}$ is a matrix over $\mathrm{Frac}(\mathcal{A})$, then we denote its $(i,j)^{\text{th}}$ entry by $x_{i,j}^{(k,\ell)}$.

We now describe the deleting derivations algorithm as applied to $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. Begin by setting $X^{(m,n+1)}$ to be the $m \times n$ matrix over $\mathrm{Frac}(\mathcal{A})$ defined by taking $x_{i,j}^{(m,n+1)} := x_{i,j}$, where the $x_{i,j}$ are the canonical generators of $\mathcal{A}$. Now given the matrix $X^{(k,\ell)}$, we define $X^{(k,\ell)^-}$ as follows.

$$x_{i,j}^{(k,\ell)^-} = \begin{cases} x_{i,j}^{(k,\ell)} - x_{i,l}^{(k,\ell)}(x_{k,\ell}^{(k,\ell)})^{-1}x_{k,j}^{(k,\ell)} & \text{if } i < k \text{ and } j < \ell, \\ x_{i,j}^{(k,\ell)} & \text{otherwise.} \end{cases}$$

On a computational level, the algorithm modifies, at each step (k,ℓ), the entries which are "north-west" of the $(k,\ell)^{\text{th}}$ entry. Let us give a brief example of applying the algorithm to a 3×3 matrix.

Example 3.23. We initialize by setting

$$X^{(3,4)} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}.$$

The first two steps of the algorithm give us the following two matrices:

$$X^{3,3} = \begin{bmatrix} x_{1,1} - x_{1,3}x_{3,3}^{-1}x_{3,1} & x_{1,2} - x_{1,3}x_{3,3}^{-1}x_{3,2} & x_{1,3} \\ x_{2,1} - x_{2,3}x_{3,3}^{-1}x_{3,1} & x_{2,2} - x_{2,3}x_{3,3}^{-1}x_{3,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$$

and

$$\begin{aligned} X^{3,2} &= \begin{bmatrix} (x_{1,1} - x_{1,3}x_{3,3}^{-1}x_{3,1}) - (x_{1,2} - x_{1,3}x_{3,3}^{-1}x_{3,2})x_{3,2}^{-1}x_{3,1} & x_{1,2} - x_{1,3}x_{3,3}^{-1}x_{3,2} & x_{1,3} \\ (x_{2,1} - x_{2,3}x_{3,3}^{-1}x_{3,1}) - (x_{2,2} - x_{2,3}x_{3,3}^{-1}x_{3,2})x_{3,2}^{-1}x_{3,1} & x_{2,2} - x_{2,3}x_{3,3}^{-1}x_{3,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix} \\ &= \begin{bmatrix} x_{1,1} - x_{1,2}x_{3,2}^{-1}x_{3,1} & x_{1,2} - x_{1,3}x_{3,3}^{-1}x_{3,2} & x_{1,3} \\ x_{2,1} - x_{2,2}x_{3,2}^{-1}x_{3,1} & x_{2,2} - x_{2,3}x_{3,3}^{-1}x_{3,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}. \end{aligned}$$

The final matrix has entries which are too long to display properly on this page, but if one so wishes, the reader can check that

$$\begin{aligned} x_{1,1}^{(1,1)} &= x_{1,1} - x_{1,2}x_{3,2}^{-1}x_{3,1} - x_{3,1}x_{2,3}^{-1}x_{2,1} + x_{3,1}x_{2,3}^{-1}x_{2,2}x_{3,2}^{-1}x_{3,1} \\ &\quad -(x_{1,2} - x_{1,3}x_{3,3}^{-1}x_{3,2} - x_{3,1}x_{2,3}^{-1}x_{2,2})(x_{2,2} - x_{2,3}x_{3,3}^{-1}x_{3,2})^{-1}(x_{2,1} - x_{2,2}x_{3,2}^{-1}x_{3,1}). \end{aligned}$$

As is evident from the previous example, the algorithm does not in general produce particularly "nice" matrices. Nevertheless, there are some quite useful algebraic consequences of this algorithm which we detail in the remainder of this section.

Theorem 3.24 (Cauchon [8, 9])**.** *Let $X^{(k,\ell)}$ denote the matrix produced at the $(k,\ell)^{\text{th}}$ step of the deleting derivations algorithm applied to $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. Let $\mathrm{Frac}^{(k,\ell)}(\mathcal{A})$ be the subalgebra of $\mathrm{Frac}(\mathcal{A})$ generated by the entries of $X^{(k,\ell)}$ (hence $\mathrm{Frac}^{(m,n+1)}(\mathcal{A}) \simeq \mathcal{A}$).*

1. *There exists an embedding $\phi_{k,\ell} : \mathrm{spec}(\mathrm{Frac}^{(k,\ell)^-}(\mathcal{A})) \to \mathrm{spec}(\mathrm{Frac}^{(k,\ell)}(\mathcal{A}))$. Thus, by composition, there exists an embedding $\phi : \mathrm{spec}(\mathcal{A}) \to \mathrm{spec}(\mathrm{Frac}^{(1,1)}(\mathcal{A}))$.*

2. *$\mathrm{Frac}^{(1,1)}(\mathcal{A}) \simeq \mathcal{O}_q(\mathbb{K}^{m\times n})$, the $m \times n$ quantum affine space.*

Set $\mathcal{B} = \mathcal{O}_q(\mathbb{K}^{m\times n})$. By the previous theorem it is evident that in order to understand $\mathcal{H}$-$\mathrm{spec}(\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K})))$, a good understanding of its image under ϕ in $\mathrm{spec}(\mathcal{B})$ would be useful. Fortunately, Cauchon was able to refine his results in this special case. The next definition will be of crucial importance to us.

Definition 3.25. Fix positive integers m and n. An $m \times n$ *diagram* is an $m \times n$ grid of squares, each square coloured black or white. An $m \times n$ diagram is an $m \times n$ *Cauchon diagram* if the squares are coloured according to the following rule: If a square is black, then either all squares above or all squares to its left must also be black. See Figure 3.3.

Remark 3.26. Given an $m \times n$ diagram D, index the squares by elements of $[m] \times [n]$ as one would index the entries of an $m \times n$ matrix. Therefore, we always identify a diagram with subsets $B(D) \subseteq [m] \times [n]$ (the black squares) and $W(D) = ([m] \times [n]) \setminus B(D)$ (the white squares).

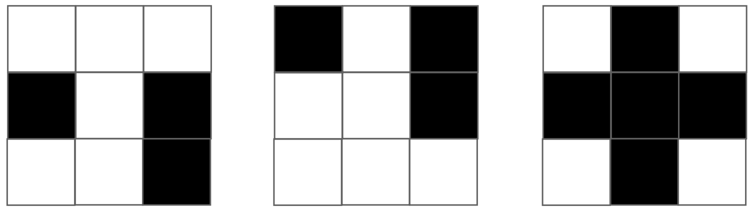


Figure 3.3: Some 3×3 diagrams. The middle and right diagrams are Cauchon diagrams

Notice that the complete primeness of $\mathcal{H}$-ideals in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ gives an intuitive motivation for the Cauchon diagram colouring condition. If, say, we have a 2×2 submatrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of the matrix of canonical generators, then

$$ad - da = (q - q^{-1})bc.$$

Therefore, if $d \in P$, then $bc \in P$, and so either $b \in P$ or $c \in P$. In other words, either the entry above d or the entry to the left of d is in P. This motivation is not to be taken too far however since, as we shall see in Chapter 3, generating sets for $\mathcal{H}$-primes do not in general consist of the "entries of X corresponding to black squares."

We conclude this section by giving some more results of Cauchon. Let $X^{(1,1)}$ be the final matrix obtained from applying the Deleting Derivations Algorithm to the matrix of canonical generators of $\mathcal{A}$. For notational convenience, relabel the entries of $X^{(1,1)}$ by setting $x_{i,j}^{(1,1)} := t_{i,j}$. Given a Cauchon diagram D, let

$$\mathrm{spec}_D(\mathcal{B}) := \{P \in \mathrm{spec}(\mathcal{B}) \mid P \cap \{t_{i,j} \mid (i,j) \in [m] \times [n]\} = \{t_{i,j} \mid (i,j) \in B(D)\}\}.$$

By Theorem 3.21, $\mathrm{spec}_D(\mathcal{B})$ is the $\mathcal{H}$-stratum of $\mathrm{spec}(\mathcal{B})$ associated to the $\mathcal{H}$-prime ideal $(t_{i,j} \mid (i,j) \in B(D))$.

Theorem 3.27 (Cauchon [8, 9]). *For the algebras* $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ *and* $\mathcal{B} = \mathcal{O}_q(\mathbb{K}^{m \times n})$ *the following hold:*

1. *For every* $m \times n$ *Cauchon diagram* D, $\mathrm{spec}_D(\mathcal{B})$ *is non-empty;*

2. $\operatorname{spec}(\mathcal{B})$ *is the disjoint union* $\bigcup \operatorname{spec}_D(\mathcal{B})$ *over all* $m \times n$ *Cauchon diagrams* D*;*

3. *Under the injection* ϕ *from Theorem 3.24 (1), each* $\mathcal{H}$*-stratum in the* $\mathcal{H}$*-stratification of* $\operatorname{spec}(\mathcal{A})$ *is homeomorphic to some* $\mathcal{H}$*-stratum* $\operatorname{spec}_D(\mathcal{B})$ *of* $\mathcal{B}$*, for some* $m \times n$ *Cauchon diagram* D*. Hence the* $\mathcal{H}$*-strata of* $\mathcal{A}$ *are parametrized by Cauchon diagrams.*

If we have fixed the m and n in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$, then we will often drop the qualification "$m \times n$" in the phrase "$m \times n$ Cauchon diagram" when it is clear that we are referring to the $m \times n$ Cauchon diagram associated to an $\mathcal{H}$-prime ideal in $\mathcal{H}$-$\operatorname{spec}(\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K})))$. Returning once again to Examples 3.16 and 3.20, we note that in Figure 3.4, we have drawn the Cauchon diagrams associated to each of the $\mathcal{H}$-primes.

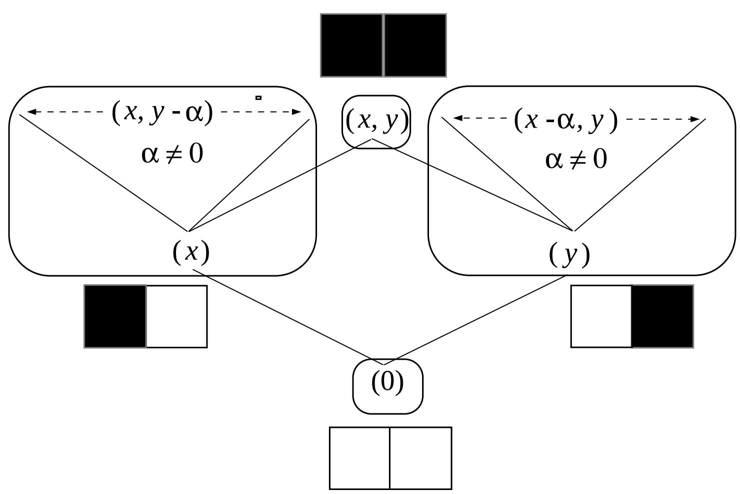


Figure 3.4: The Cauchon diagrams corresponding to $\mathcal{H}$-strata of $\operatorname{spec}(\mathcal{O}_q(\mathbb{K}^{1\times 2}))$

Chapter 4

Generating sets for $\mathcal{H}$-primes

4.1 Introduction

Given an algebraic structure, an immediate problem is to find or describe a generating set for the structure. In the case of 2×2 and 3×3 quantum matrices, Goodearl and Lenagan [20, 19] were able to calculate generating sets for all $\mathcal{H}$-prime ideals. In each case, a generating set consists of a certain collection of quantum minors of the matrix of canonical generators. For this reason they conjectured that this property held for all $\mathcal{H}$-prime ideals in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. Their question was answered in the affirmative by Launois [27, 28] in the case that $\mathbb{K} = \mathbb{C}$ and q is transcendental over $\mathbb{Q}$. In this chapter, we begin by reviewing Launois' work which includes an algorithm that determines the desired generating set. In fact the algorithm determines *all* quantum minors in a given $\mathcal{H}$-prime ideal. The main content of the remainder of the chapter is to show that Launois' algorithm can be interpreted as that of finding sets of non-vertex-disjoint paths in a Cauchon diagram. In order to do this, we establish a quantum analogue of Lindström's Lemma (Lemma 2.6).

We note that Yakimov [44] has recently, and independently from us, also approached the issue. Yakimov's method gives an alternate description of a generating set of a given $\mathcal{H}$-prime when q is transcendental. His methods, which involve *restricted permutations* (cf. Chapter 5), will, in general, give a different generating set than our method. It is not clear if Yakimov's generating set is minimal.

4.2 Launois' Algorithm

Launois originally proved the following result for $\mathbb{K} = \mathbb{C}$, but by results of Goodearl, Launois and Lenagan [18] it suffices to set $\mathbb{K}$ to be any field of characteristic zero.

Theorem 4.1 (Launois [28])**.** *For $q \in \mathbb{K}^*$ transcendental over $\mathbb{Q}$, the $\mathcal{H}$-invariant prime ideals of $\mathcal{A}$ are generated by quantum minors of the matrix of canonical generators X.*

The proof is long and quite technical, so once again we refrain from providing a full proof here. However, for the sake of at least approaching completeness, we sketch out the steps. First we state a lemma due to Levasseur and Stafford [32]. Let X and Y be matrices of canonical generators for $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))$ and $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$ respectively. Let

$$\theta : \mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))[z, z^{-1}] \to \mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$$

be the map defined by

$$\begin{aligned} \theta(X_{i,j}) &= Y_{i,j} \text{ for } i > 1, \\ \theta(X_{1,j}) &= Y_{1,j} \det_q(Y)^{-1} \\ \theta(z) &= \det_q(Y). \end{aligned}$$

Lemma 4.2. *The map θ described above is an isomorphism from $\mathcal{O}_q(\mathrm{SL}_n(\mathbb{K}))[z, z^{-1}]$ to $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$.*

Outline of the Proof of Theorem 4.1. Set $N = m + n$. The crucial point (and where the assumption that q is transcendental is used) is a complete description of the generating sets of $\mathcal{H}$-prime ideals of the algebra $\mathcal{O}_q(\mathrm{SL}_N(\mathbb{K}))$ obtained by Hodges and Levasseur [23] (see also [6]). Such generating sets consist of quantum minors of the matrix X of canonical generators of $\mathcal{O}_q(\mathrm{SL}_N(\mathbb{K}))$. Let Y be the matrix of canonical generators for $\mathcal{O}_q(\mathrm{GL}_N(\mathbb{K}))$.

Launois notes that the above map θ sends quantum minors of $\mathcal{O}_q(\mathrm{SL}_N(\mathbb{K}))$ (considered as elements in $\mathcal{O}_q(\mathrm{SL}_N(\mathbb{K}))[z, z^{-1}]$) either to a quantum minor of $\mathcal{O}_q(\mathrm{GL}_N(\mathbb{K}))$ or a quantum minor of $\mathcal{O}_q(\mathrm{GL}_N(\mathbb{K}))$ multiplied by $\det_q(Y)^{-1}$.

Next, Launois shows that every ideal I of $\mathcal{O}_q(\mathrm{SL}_N(\mathbb{K}))$ can be extended uniquely to an ideal $\hat{I}$ of $\mathcal{O}_q(\mathrm{SL}_N(\mathbb{K}))[z, z^{-1}]$. This induces a bijection between the sets $\mathcal{H}'$-spec$(\mathcal{O}_q(\mathrm{SL}_N(\mathbb{K})))$ and $\mathcal{H}$-spec$(\mathcal{O}_q(\mathrm{GL}_N(\mathbb{K})))$ given by sending I to $\theta(\hat{I})$. It follows that the $\mathcal{H}$-ideals of $\mathcal{O}_q(\mathrm{GL}_N(\mathbb{K}))$ are generated by quantum minors of the matrix of canonical generators X' for $\mathcal{O}_q(\mathrm{M}_{N,N}(\mathbb{K}))$.

The main part of Launois' paper, which is unfortunately too long to give in detail here, is as follows. Given an $\mathcal{H}$-prime ideal J in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$, we attach an $\mathcal{H}$-prime ideal K in $\mathcal{O}_q(\mathrm{GL}_N(\mathbb{K}))$, which we already know is generated by quantum minors of X'. By using a modification of the deleting-derivations algorithm, Launois shows that we can transform the generating set for K into a generating set for J so that quantum minors of X' are transformed into quantum minors of the matrix of canonical generators for $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. □

The above result and its proof were translated by Launois [27] into an algorithm for finding a generating set for a given $\mathcal{H}$-prime J in $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. The idea is as follows. Let T be the matrix of canonical generators for the algebra $\mathcal{B} = \mathcal{O}_q(\mathbb{K}^{m\times n})$. We consider the entries of T to be in the skew field of fractions $\mathrm{Frac}(\mathcal{B})$. Now given the Cauchon diagram D corresponding to J, we set those entries in T that correspond to black squares in D to zero. Next, we essentially apply the deleting derivations algorithms in reverse to this new matrix to obtain a matrix T'. Finally we determine all vanishing quantum minors in T'. Launois' results tell us that the corresponding quantum minors in the matrix X of canonical generators for $\mathcal{A}$ form a generating set for the $\mathcal{H}$-prime ideal corresponding to D.

We now make the preceding discussion more precise.

Algorithm 4.3. (Launois [27])

Input An $m \times n$ Cauchon diagram D.

Output A matrix $T^{(m,n)}$ with entries from the algebra $\mathrm{Frac}(\mathcal{B})$.

Initialization Let $T^{(1,1)}$ be an $m \times n$ matrix defined by

$$T^{(1,1)}[i,j] = \begin{cases} t_{i,j} & \text{if } (i,j) \in W(D), \\ 0 & \text{if } (i,j) \in B(D). \end{cases}$$

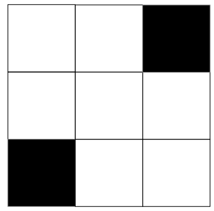


Figure 4.1: The Cauchon diagram used in Example 4.4

For any (s,t) write $T^{(s,t)}[i,j] := t_{i,j}^{(s,t)}$. Begin by setting $(s,t) = (1,2)$.

While $(s,t) \neq (m, n+1)$**,** do the following:

1. Construct the matrix $T^{(s,t)}$ by

$$t_{i,j}^{(s,t)} = \begin{cases} t_{i,j}^{(s,t)^-} + t_{i,s}^{(s,t)^-} \left(t_{s,t}^{(s,t)^-}\right)^{-1} t_{r,j}^{(s,t)^-} & \text{if } (i,j) \leq (s-1, t-1) \text{ and } t_{s,t} \neq 0, \\ t_{i,j}^{(s,t)^-} & \text{otherwise.} \end{cases}$$

2. Let $(s,t) \mapsto (s,t)^+$.

Example 4.4. Consider the 3×3 Cauchon diagram D in Figure 4.1. The initialization step of Launois' algorithm gives

$$T^{(1,1)} = \begin{bmatrix} t_{1,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix}.$$

Now at step (s,t) of the algorithm, the only entries of $T^{(s,t)}$ that are modified from the previous step are those which are "north-west" of (s,t). In particular, steps (s,t) with either $s = 1$ or $t = 1$ do not change the previous matrix. In our example, we thus have $T^{(1,1)} = T^{(1,2)} = T^{(1,3)} = T^{(2,1)}$.

At step $(2,2)$, the only entry north-west of this entry is $(1,1)$. We therefore obtain

$$T^{(2,2)} = \begin{bmatrix} t_{1,1} + t_{1,2} t_{2,2}^{-1} t_{2,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix}.$$

The next step is $(2,3)$ and entry $(2,3)$ of $T^{(2,2)}$ is non-zero. However, the entries north-west of $(2,3)$ are $(1,1)$ and $(1,2)$, and since $t_{1,3}^{(2,2)} = 0$, the net effect of the algorithm at this step is to change nothing. Thus $T^{(2,3)} = T^{(2,2)}$.

We also have $T^{(3,1)} = T^{(2,3)}$, so consider step $(3,2)$. By similar reasoning as in step $(2,3)$, we find that $T^{(3,1)} = T^{(3,2)}$. The last step is $(3,3)$. Applying the algorithm we get

$$T^{(3,3)} = \begin{bmatrix} t_{1,1} + t_{1,2}t_{2,2}^{-1}t_{2,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} + t_{2,3}t_{3,3}^{-1}t_{3,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix}.$$

The next result simply collects the above discussion. Its proof follows from the detailed proof of Theorem 4.1 found in [28].

Theorem 4.5 (Launois [27, 28])**.** *Let J be an $\mathcal{H}$-invariant prime ideal of $\mathcal{A} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$. Let D be the Cauchon diagram associated to J. Apply Algorithm 4.3 to D to obtain the matrix $T^{(m,n)}$. If a square submatrix in $T^{(m,n)}$ has a vanishing quantum minor, then the corresponding quantum minor in the matrix of canonical generators of $\mathcal{A}$ is a generator for J. Furthermore, J is generated by all such quantum minors.*

4.3 Cauchon graphs

Let D be a Cauchon diagram, and recall that $W(D)$ and $B(D)$ denote the sets of white and black squares respectively. We essentially follow Postnikov [38] by placing upon D an edge-weighted, directed graph. If (i,j) is a white square in D, then let $(i,j^-), (i,j^+)$ and (i^+,j) be the first white squares to the left, right and below (i,j) respectively (if they exist). As in the previous section, we let the set of $t_{i,j}$ be the canonical generators for $\mathcal{B} = \mathcal{O}_q(\mathbb{K}^{m\times n})$, which we consider as elements of $\mathrm{Frac}(\mathcal{B})$.

Definition 4.6. Let D be an $m \times n$ Cauchon diagram. The *Cauchon (directed) graph* $G_D = (V, E, w)$ is the edge-weighted directed graph defined as follows. The vertices consist of the set $V = W(D) \cup \{r_1, \ldots, r_m\} \cup \{c_1, \ldots, c_n\} := W(D) \cup R \cup C$. The set E of directed edges and a weight function $w : E \to \text{Frac}(\mathcal{B})$ are constructed as follows.

1. For every $i \in [m]$, put a directed edge from r_i to the right-most white square in row i (if one exists), say (i, k). Give these edges the weight $t_{i,k}$.

2. For every column $j \in [n]$, make a directed edge from the bottom-most white square in column j (if one exists) to the vertex c_j. Give these edges weight 1.

3. For every $(i, j) \in W(D)$, make a directed edge from (i, j^-) to (i, j) (if (i, j^-) exists). Give these edges a weight $t_{i,j}^{-1} t_{i,j^-}$.

4. For every $(i, j) \in W(D)$, make a directed edge from (i, j) to (i^+, j) (if (i^+, j) exists). Give each of these edges a weight of 1.

For convenience, we always assume that a Cauchon graph is embedded in the plane in the following way. First place a vertex in each white square of D and label the vertex by the entry coordinates of the white square. Next, place a vertex to the right of each row and below each column. The vertex to the right of row i is labelled r_i, and the vertex below column j is c_j. See Figure 4.2. Under this embedding of the Cauchon graph we may unambiguously use directional terms such as horizontal, vertical, above, below, left and right in the sequel.

Notation 4.7. We list some useful notation and conventions for the Cauchon graph corresponding to an $m \times n$ Cauchon diagram D.

1. If $I \subseteq [m]$ and $J \subseteq [n]$, then $R_I := \{r_i \in R \mid i \in I\}$ and $C_J := \{c_j \in C \mid j \in J\}$.

2. Let $e = ((i, k), (i, j))$ be a horizontal edge with both endpoints in $W(D)$ (thus $k > j$). Set $\text{row}(e) = i$, $\text{r.col}(e) = k$ and $\text{l.col}(e) = j$. In other words, $\text{l.col}(e)$ is the column containing the head of e and $\text{r.col}(e)$ is the column containing the tail of e. The pair $\{k, j\}$ of column indices will be denoted by $\text{col}(e)$.

3. As we are working in a noncommutative algebra, we should clarify that the weight $w(P)$ of a path $P = (v_0, e_1, v_1, \ldots, e_n, v_n)$ is the product of the weights

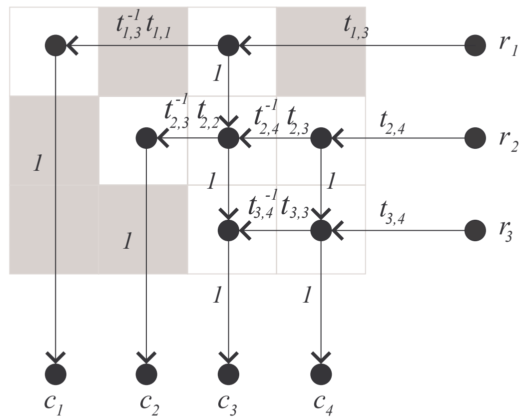


Figure 4.2: A Cauchon graph superimposed on top of its Cauchon diagram.

of its edges, multiplied from left to right in the order they appear in P. In other words,

$$w(P) = w(e_1)w(e_2)\cdots w(e_n) = \prod_{i=1}^{n} w(e_i).$$

4. If $K : v_0 \Rightarrow v$ and $L : v \Rightarrow v_n$ are two paths in a Cauchon graph, then we write $KL : v_0 \Rightarrow v_n$ for the path obtained by appending L to K in the obvious way. That is, KL is the path which travels along K from v_0 to v, and then continues on L from v to v_n. Note that KL is a directed path since, by the next proposition, Cauchon graphs are acyclic.

Proposition 4.8. *Let D be a Cauchon diagram.*

1. *The Cauchon graph G_D is acyclic.*

2. *The embedding described above is a planar embedding.*

3. *If the path $P : (i, j_2) \Rightarrow (i, j_1)$ consists only of horizontal edges, then*

$$w(P) = t^{-1}_{i,j_2} t_{i,j_1}.$$

Proof. As all edges are directed either right to left or top to bottom, the first property is obvious. To see planarity, if two edges cross, then these edges must consist of one vertical and one horizontal edge, and their intersection point corresponds to a black square. This implies that we have a black square in C with both a white square above and a white square to its left, contradicting the definition of a Cauchon diagram.

Finally, if (i,k) is an internal vertex of $P : (i,j_1) \Rightarrow (i,j_2)$ (i.e., $k \neq j_1, j_2$), then (i,k^-) and (i,k^+) exist. Now the edge $e_1 := ((i,k^+),(i,k)) \in P$ has weight $t_{i,k^+}^{-1}t_{i,k}$ and the edge $e_2 = ((i,k),(i,k^-)) \in P$ has weight $t_{i,k}^{-1}t_{i,k^-}$. Therefore, $w(e_1)w(e_2) = t_{i,k^+}^{-1}t_{i,k^-}$. It follows that $w(P)$ is a telescoping product and so clearly $w(P) = t_{i,j_2}^{-1}t_{i,j_1}$.
□

4.4 Some Technical Lemmas

This section exclusively deals with some lemmas concerning the commutativity between sets of edge weights. These lemmas are to be used in the proof of the quantum analogue of Lindström's Lemma.

Lemma 4.9. *Let D be a Cauchon diagram. Let e and f be distinct horizontal edges in G_D with both endpoints in $W(D)$ and such that* $\mathrm{row}(f) \leq \mathrm{row}(e)$.

1. *If* $\mathrm{col}(e) \cap \mathrm{col}(f) = \emptyset$, *then* $w(f)w(e) = w(e)w(f)$.

2. *If* $|\mathrm{col}(e) \cap \mathrm{col}(f)| = 1$, *then:*

   i. $w(f)w(e) = q\, w(e)w(f)$, *if* $\mathrm{i.col}(e) = \mathrm{i.col}(f)$ *for* $\mathrm{i} = \mathrm{l}$ *or* $\mathrm{i} = \mathrm{r}$*;*

   ii. $w(f)w(e) = q^{-1}\, w(e)w(f)$ *otherwise.*

3. *If* $|\mathrm{col}(e) \cap \mathrm{col}(f)| = 2$, *then* $w(f)w(e) = q^2 w(e)w(f)$.

Proof. We verify the case $\mathrm{col}(e) \cap \mathrm{col}(f) = \emptyset$. The other cases can be disposed of in a similar manner by checking the possibilities (see Figure 4.3). So suppose that $\mathrm{col}(e) \cap \mathrm{col}(f) = \emptyset$.

First note that if j is such that $\mathrm{l.col}(e) < j < \mathrm{r.col}(e)$, then the square $(\mathrm{row}(e), j)$ is black in D and, since $(\mathrm{row}(e), \mathrm{l.col}(e))$ is a white square to its left, we must have that (i,j) is black for every $i \leq \mathrm{row}(e)$. In other words, no horizontal edge in G_D in a

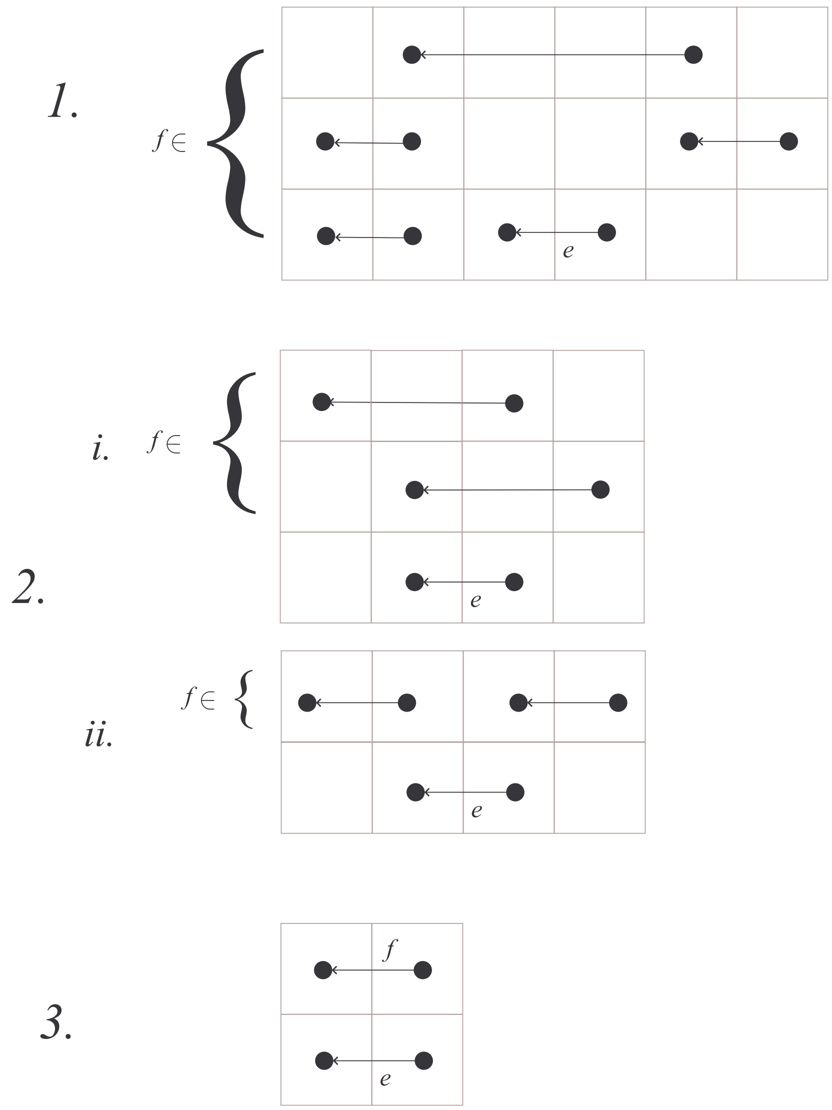


Figure 4.3: Examples of the different cases in Lemma 4.9.

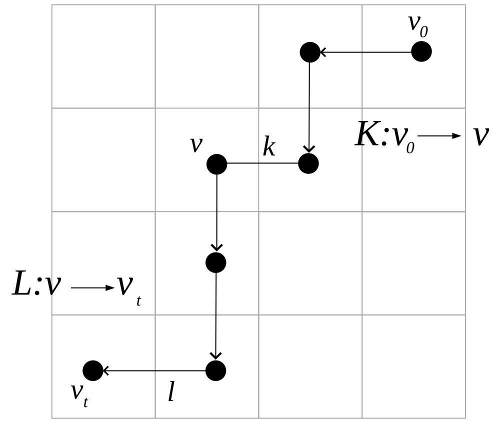


Figure 4.4: Example of a situation in Lemma 4.11.

row above row(e) has an endpoint whose column coordinate lies strictly between the column coordinates of e.

Now if row(e) $>$ row(f), then $w(e)$ and $w(f)$ clearly commute by the definition of the algebra $\mathcal{B}$. Suppose that row(e) $=$ row(f). Say $w(e) = t_{i,j_1}^{-1} t_{i,j_2}$ and $w(f) = t_{i,j_3}^{-1} t_{i,j_4}$ where $j_1 > j_2 > j_3 > j_4$. Therefore,

$$\begin{aligned} w(e)w(f) &= (t_{i,j_1}^{-1} t_{i,j_2})(t_{i,j_3}^{-1} t_{i,j_4}) \\ &= (q^{-1}q)\ t_{i,j_3}^{-1}(t_{i,j_1}^{-1} t_{i,j_2}) t_{i,j_4} \\ &= (q^{-1}q)(qq^{-1})\ (t_{i,j_3}^{-1} t_{i,j_4})(t_{i,j_1}^{-1} t_{i,j_2}) \\ &= w(f)w(e). \end{aligned}$$

$\square$

Remark 4.10. Note that Lemma 4.9, parts (1) and (2) remain true whenever e or f is an edge which has an endpoint in R.

Lemma 4.11. *Let $K : v_0 \Rightarrow v$ and $L : v \Rightarrow v_t$ be directed paths in a Cauchon graph.*

1. *If either K or L contain only vertical edges, then $w(K)w(L) = w(L)w(K)$.*

2. *If both K and L contain a horizontal edge, then $w(K)w(L) = q^{-1}\, w(L)w(K)$.*

Proof. We need only consider the horizontal edges of K and L since all vertical edge weights are 1 and, of course, 1 is central in $\mathrm{Frac}(\mathcal{B})$. Now if either K or L contain only vertical edges, then $w(K) = 1$ or $w(L) = 1$ and so $w(K)$ and $w(L)$ commute.

Suppose that both K and L contain horizontal edges. Let k be the last horizontal edge in K and let ℓ be the first horizontal edge in L. By the embedding of a Cauchon graph, the horizontal edges of L are always to the left of horizontal edges of K, or "south-west" of K. When computing $w(K)w(L)$, the only edge weights which do not commute are $w(k)$ and $w(\ell)$ by Lemma 4.9 (1). By Lemma 4.9 $(2ii)$ we obtain

$$\begin{aligned}
w(K)w(L) &= w(K \setminus \{k\})w(k)w(\ell)w(L \setminus \{\ell\}) \\
&= q^{-1}\, w(K \setminus \{k\})w(\ell)w(k)w(L \setminus \{\ell\}) \\
&= q^{-1}\, w(\ell)w(L \setminus \{\ell\})w(K \setminus \{k\})w(k) \\
&= q^{-1}\, w(L)w(K).
\end{aligned}$$

□

Lemma 4.12. *Let G_D be a Cauchon graph and let $K : v \Rightarrow c_i$ and $L : v \Rightarrow c_j$ be two directed paths with only their initial vertex in common. Let K be the path that starts with a horizontal edge and L be the path that starts with a vertical edge.*

1. *If L consists only of vertical edges (or no edges at all), then $w(K)w(L) = w(L)w(K)$.*
2. *If L has a horizontal edge, then $w(K)w(L) = q\, w(L)w(K)$.*

Proof. Case 1 is obvious since here we have $w(L) = 1$, so we may suppose that L has at least one horizontal edge. By Lemma 4.9, any horizontal edge e in L commutes with any edge in K except those whose column coordinates intersect the set $\mathrm{col}(e)$. Recall from the proof of Lemma 4.9 that no edge in K has an endpoint in between (with respect to column coordinates) the endpoints of edges in L.

If f_1 is the first horizontal edge in K and e_1 is the first horizontal edge in L, then we have $\mathrm{r.col}(e_1) = \mathrm{r.col}(f_1)$. There are two cases to consider (see Figure 4.5).

Case (i)**:** $\mathrm{l.col}(f_1) < \mathrm{l.col}(e_1)$. Here we have that, by Lemma 4.9 $(2i)$,

$$w(f_1)w(e_1) = q\, w(e_1)w(f_1).$$

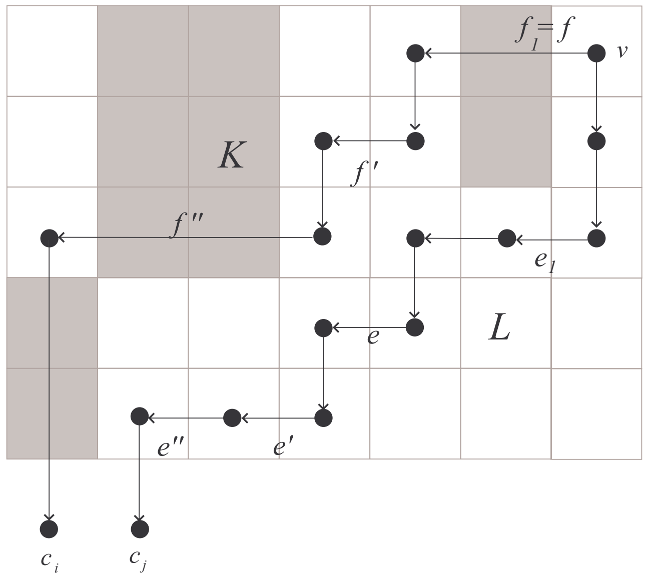


Figure 4.5: Example of a typical situation of Lemma 4.12.

Case (ii)**:** l.col(f_1) = l.col(e_1). In this case, the second horizontal edge f_2 of K satisfies r.col(f_2) = l.col(e_1) and l.col(f_2) < l.col(e_1). Applying Lemma 4.9 twice we find

$$\begin{aligned} w(f_1)w(f_2)w(e_1) &= w(f_1)q^{-1}w(e_1)w(f_2), \text{ by Lemma 4.9 (2.i)}, \\ &= (q^{-1}q^2)\, w(e_1)w(f_1)w(f_2), \text{ by Lemma 4.9 (3)}, \\ &= q\, w(e_1)w(f_1)w(f_2). \end{aligned}$$

It follows that $w(K)w(e_1) = q\, w(e_1)w(K)$.

Now suppose that e is *not* the first horizontal edge in L. We show that $w(e)w(K) = w(K)w(e)$. There are exactly three possibilities for the edge e.

Case (a): Every edge f in K satisfies col$(f) \cap$ col$(e) = \emptyset$. For an example of an edge which falls in this case, see edge e'' in Figure 4.5. By Lemma 4.9 (1), it follows that $w(e)w(K) = w(K)w(e)$.

Case (b): There exist two distinct edges f' and f'' in K such that $|$col$(f')\cap$col$(e)| = 1$ and $|$col$(f'')\cap$col$(e)| = 1$, and col$(g)\cap$col$(e) = \emptyset$ for all other edges g in K. For an example of an edge which falls in this case, see edge e' in Figure 4.5. Now together with e, both f' and f'' fall under case (2) of Lemma 4.9. Furthermore, exactly one is in subcase ($2i$) of that lemma, while the other is in subcase ($2ii$). Without loss of generality, we suppose that f' is in case ($2ii$) of Lemma 4.9. We have

$$\begin{aligned} w(f'')w(f')w(e) &= w(f'')q^{-1}\,w(e)w(f') \\ &= q\,w(e)w(f'')q^{-1}\,w(f') \\ &= w(e)w(f'')w(f'). \end{aligned}$$

Since by Lemma 4.9 (1) we have that $w(e)$ commutes with $w(g)$ for every edge g in K with $\mathrm{col}(g) \cap \mathrm{col}(e) = \emptyset$, it follows that we again have $w(e)w(K) = w(K)w(e)$, as desired.

Case (c): There exist three distinct edges f,f' and f'' in K such that e and f' fall under Lemma 4.9 (3), while e, together with either f or f'' fall into Lemma 4.9 ($2ii$). Furthermore, $\mathrm{col}(g)\cap\mathrm{col}(e) = \emptyset$ for all other edges g in K. For an example of this case, see edge e in Figure 4.5. We have

$$\begin{aligned} w(f'')w(f')w(f)w(e) &= w(f'')w(f')q^{-1}w(e)w(f) \\ &= w(f'')q^{2}\,w(e)w(f')q^{-1}w(f) \\ &= q^{-1}\,w(e)w(f'')q^{2}\,w(f')q^{-1}w(f) \\ &= w(e)w(f'')w(f')w(f). \end{aligned}$$

Since by Lemma 4.9 (1) we know $w(e)$ commutes with $w(g)$ for every edge g in K with $\mathrm{col}(g) \cap \mathrm{col}(e) = \emptyset$, it follows that we again have $w(e)w(K) = w(K)w(e)$, as desired. This completes the analysis of all possibilities for e not being the first horizontal edge in L, and so we conclude that $w(e)w(K) = w(K)w(e)$ for all such edges e. Hence

$$\begin{aligned} w(K)w(L) &= w(K)w(e_1)w(L \setminus e_1) \\ &= q\,w(e_1)w(K)w(L \setminus e_1) \\ &= q\,w(e_1)w(L \setminus e_1)w(K) \\ &= q\,w(L)w(K). \end{aligned}$$

□

4.5 A Quantum Analogue of Lindström's Lemma

In this section we derive one of the main tools used in the proof of Theorem 4.20. We first slightly refine the definition of a path system from Section 2.3.

Definition 4.13. Let D be an $m \times n$ Cauchon diagram. Let $I = \{i_1, \ldots, i_k\} \subseteq [m]$ and $J = \{j_1, \ldots, j_k\} \subseteq [n]$ be two subsets of equal size with $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots j_k$.

An *(R_I, C_J)-path system* $\mathcal{P}$ is a set of k directed paths in G_D, each starting at different a vertex in R_I and each ending at a different vertex in C_J. Note that:

- There exists a permutation $\sigma_{\mathcal{P}} \in S_k$ such that we can write $\mathcal{P}$ as the set

$$\mathcal{P} = \{P_\ell : r_{i_\ell} \Rightarrow c_{j_{\sigma_{\mathcal{P}}(\ell)}} \mid \ell \in [k]\}.$$

- The *q-sign* of $\mathcal{P}$ is the quantity

$$\mathrm{sgn}_q(\mathcal{P}) = (-q)^{\ell(\sigma_{\mathcal{P}})},$$

  where we recall that $\ell(\sigma_{\mathcal{P}})$ is the length of the permutation $\sigma_{\mathcal{P}}$ as in Definition 3.13.

- Recall that a path system is vertex-disjoint if no two paths share a vertex.

- The weight of $\mathcal{P}$ is the (ordered) product $\prod_{\ell=1}^{k} w(P_\ell) = w(P_1)w(P_2)\cdots w(P_k)$.

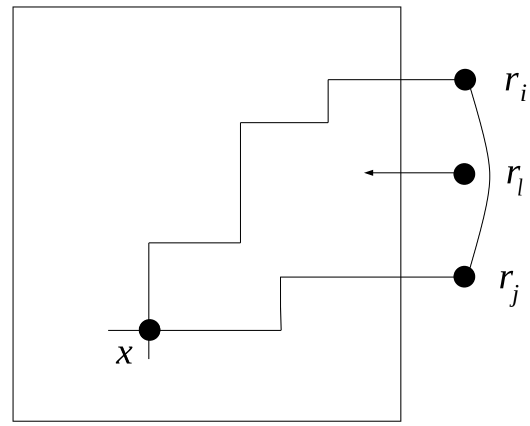


Figure 4.6: Visualization of the proof of Lemma 4.14. Here we see that the path starting at r_ℓ must intersect either r_i or r_j in order to reach $c_{\sigma_{\mathcal{P}}(\ell)}$

Lemma 4.14. *Let D be a Cauchon diagram and let $I \subseteq [m]$ and $J \subseteq [n]$ be two sets of cardinality k. If $\mathcal{P} = \{P_1, \ldots, P_k\}$ is a* non-*vertex-disjoint (R_I, C_J)-path system in G_D, then there exists an i such that P_i and P_{i+1} share a vertex.*

Proof. Let

$$d := \min\{|i-j| \mid i \neq j \text{ and } P_i \text{ and } P_j \text{ share a vertex}\}.$$

Observe that d is well-defined and at least 1 since, by assumption, there exists at least one pair of intersecting paths in $\mathcal{P}$. Let P_i and P_j be two paths which achieve this minimum. Hence there is a first vertex $x \in W(D)$ at which they intersect. If r_i and r_j are the first vertices on the paths P_i and P_j respectively, then the two subpaths $P_i' : r_i \Rightarrow x$ and $P_j' : r_j \Rightarrow x$ together with a new vertical edge (r_i, r_j) form a closed simple loop L in the plane. See Figure 4.6.

If $d > 1$, then there exists an $\ell \in I$ such that r_ℓ lies between r_i and r_j in G_D. But in order for P_ℓ to reach its endpoint in C, we must have that an internal vertex of P_ℓ intersects L. This intersection point occurs at a vertex by planarity. Hence P_ℓ shares a vertex with either P_i or P_j, which contradicts the minimality of d.

□

Definition 4.15. Let D be an $m \times n$ Cauchon diagram. The *path matrix* of G_D is the $m \times n$ matrix M_D with

$$M_D[i,j] = \sum_{P:r_i \Rightarrow c_j} w(P),$$

where the sum is over all directed paths in G_D starting at r_i and ending at c_j. If no such path exists, then $M_D[i,j] := 0$.

Now we are ready to prove the quantum analogue of Lindström's Lemma. We note that the basic idea of the proof is only a slight modification of the proof found in [1]. The main difficulty is working around the noncommutativity of the algebra $\mathrm{Frac}(\mathcal{B})$, but this is taken care of by the lemmas of the previous section.

Theorem 4.16. *Let D be an $m \times n$ Cauchon diagram. If $I \subseteq [m]$ and $J \subseteq [n]$ are two sets of size k, then*

$$\det_q(M_D[I,J]) = \sum_{\mathcal{P}} w(\mathcal{P}),$$

where the sum is over all ***vertex-disjoint*** *(R_I, C_J)-path systems in G_D.*

Proof. In order to simplify the presentation of this proof we take $I = J = \{1, \ldots, k\}$. The proof of the general case is essentially the same but notationally more cumbersome.

To begin, note that

$$\begin{aligned} \det_q(M_D[I,J]) &= \sum_{\sigma \in S_k} \mathrm{sgn}_q(\sigma) \left(\prod_{i=1}^{k} M_D[i, \sigma(i)] \right) \\ &= \sum_{\sigma} \mathrm{sgn}_q(\sigma) \left(\prod_{i=1}^{k} \left(\sum_{P:r_i \Rightarrow c_{\sigma(i)}} w(P) \right) \right) \\ &= \sum_{(R_I, C_J)\text{-path systems } \mathcal{P}} \mathrm{sgn}_q(\mathcal{P}) w(\mathcal{P}). \end{aligned}$$

Let $\mathcal{N}$ be the set of *non*-vertex-disjoint (R_I, C_J)-path systems. We claim that

$$\sum_{\mathcal{P} \in \mathcal{N}} \mathrm{sgn}_q(\mathcal{P}) w(\mathcal{P}) = 0.$$

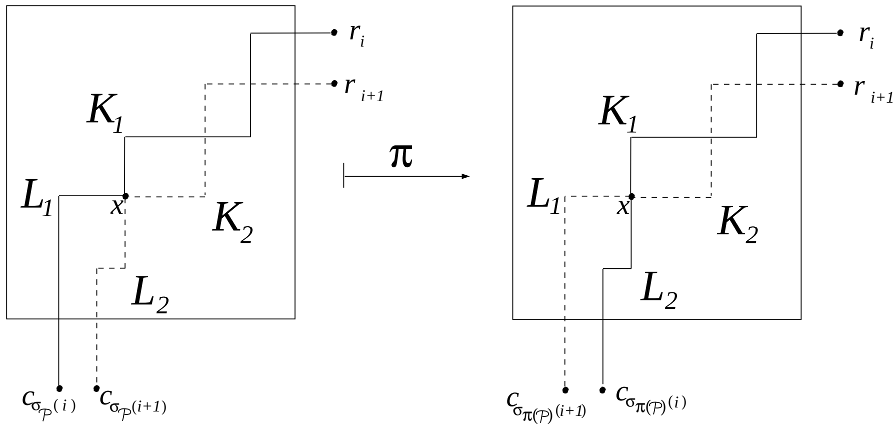


Figure 4.7: Example of how π acts on two intersecting paths. On the left hand side, P_i is the solid path and P_{i+1} is the dotted path. On the right hand side, $\pi(P_i)$ is the solid path, $\pi(P_{i+1})$ is the dotted path.

To show this, we find a fixed-point free involution $\pi : \mathcal{N} \to \mathcal{N}$ with the property that for every $\mathcal{P} \in \mathcal{N}$,

$$\operatorname{sgn}_q(\mathcal{P})w(\mathcal{P}) \quad = \quad -\operatorname{sgn}_q(\pi(\mathcal{P}))w(\pi(\mathcal{P})), \tag{4.1}$$

where $\pi(\mathcal{P}) := \{\pi(P_1), \ldots, \pi(P_k)\}$.

So suppose that $\mathcal{P} = \{P_1, \ldots, P_k\} \in \mathcal{N}$. Define π as follows. Let i be the minimum index of I such that P_i and P_{i+1} intersect (which exists by Lemma 4.14). Let x be the *last* vertex which they have in common. Let $K_1 : r_i \Rightarrow x$ and $L_1 : x \Rightarrow c_{\sigma_{\mathcal{P}}(i)}$ be the two subpaths of P_i such that $P_i = K_1L_1$. Define K_2 and L_2 from P_{i+1} similarly. Now we set

$$\pi(P_\ell) = \begin{cases} K_1L_2 & \text{if } \ell = i, \\ K_2L_1 & \text{if } \ell = i+1, \\ P_\ell & \text{otherwise.} \end{cases}$$

It is clear from the definition that π is an involution without fixed points so it remains to prove Equation 4.1. Since π is an involution, we may assume, without loss of generality, that $\sigma_{\mathcal{P}}(i) < \sigma_{\mathcal{P}}(i+1)$. Thus $\sigma_{\pi(\mathcal{P})} = \sigma_{\mathcal{P}} \circ (i \;\; i+1)$ and so $\sigma_{\pi(\mathcal{P})}$ has exactly one more inversion, i.e.,

$$\ell(\sigma_{\pi(\mathcal{P})}) \quad = \quad \ell(\sigma_{\mathcal{P}}) + 1. \tag{4.2}$$

Now consider $w(P_i)w(P_{i+1})$. There are two cases to consider. First suppose that L_2 has a horizontal edge. In this case, we find that

$$\begin{aligned} w(P_i)w(P_{i+1}) &= w(K_1)w(L_1)w(K_2)w(L_2) \\ &= w(K_1)\, q\, w(K_2)w(L_1)w(L_2) \text{ (Lemma 4.11)} \\ &= w(K_1)\, q\, q\; w(K_2)w(L_2)w(L_1) \text{ (Lemma 4.12)} \\ &= w(K_1)\, q\, q\, q^{-1}\, w(L_2)w(K_2)w(L_1) \text{ (Lemma 4.11)} \\ &= q\, w(\pi(P_i))w(\pi(P_{i+1})). \end{aligned} \tag{4.3}$$

If L_2 has only vertical edges, then a similar calculation shows again that $w(P_i)w(P_{i+1}) = q\, w(\pi(P_i))w(\pi(P_{i+1}))$. Therefore,

$$\begin{aligned} w(\mathcal{P}) &= \left(\prod_{j=1}^{i-1} w(P_j)\right) w(P_i)w(P_{i+1}) \left(\prod_{j=i+2}^{k} w(P_j)\right) \\ &= \left(\prod_{j=1}^{i-1} w(\pi(P_j))\right) q\, w(\pi(P_i))w(\pi(P_{i+1})) \left(\prod_{j=i+2}^{k} w(\pi(P_j))\right) \\ &= q\, w(\pi(\mathcal{P})). \end{aligned}$$

Finally, by Equations (4.2) and (4.3), we obtain

$$\begin{aligned} \operatorname{sgn}_q(\mathcal{P})w(\mathcal{P}) + \operatorname{sgn}_q(\pi(\mathcal{P}))w(\pi(\mathcal{P})) &= \\ (-q)^{\ell(\sigma_{\mathcal{P}})}\, q\, w(\pi(\mathcal{P})) + (-q)^{\ell(\sigma_{\mathcal{P}})+1} w(\pi(\mathcal{P})) &= 0. \end{aligned}$$

This proves Equation 4.1 and shows that

$$\det_q(M_D[I,J]) = \sum_{\mathcal{P}} \operatorname{sgn}_q(\mathcal{P})w(\mathcal{P}),$$

where the sum is over all vertex-disjoint (R_I, C_J) path systems.

By Proposition 4.8, we know that G_D is planar and so a vertex-disjoint path system $\mathcal{P}$ cannot have any edge crossings. This implies that we necessarily must have

$\mathcal{P} = \{P_\ell : r_\ell \Rightarrow c_\ell \mid \ell = 1, \dots, k\}$. Thus $\sigma_{\mathcal{P}}$ is always the identity permutation, i.e., $\mathrm{sgn}_q(\mathcal{P}) = 1$. Therefore, we obtain the desired equation in the statement of the theorem, namely,

$$\det_q(M_D[I, J]) = \sum_{\mathcal{P}} w(\mathcal{P}),$$

where the sum is over all vertex-disjoint (R_I, C_J) path systems. □

Corollary 4.17. *Let q be transcendental over $\mathbb{Q}$, D a Cauchon diagram and $I \subseteq [m]$ and $J \subseteq [n]$ be two subsets of the same size. Then $\det_q(M_D[I, J]) = 0$ if and only if there does not exist a vertex-disjoint (R_I, C_J)-path system.*

Proof. If there does not exist a vertex-disjoint (R_I, C_J)-path system, then by Theorem 4.16, $\det_q(M_D[I, J])$ is the empty sum and so $\det_q(M_D[I, J]) = 0$

Conversely, suppose that there exists at least one vertex-disjoint (R_I, C_J)-path system. If $\mathcal{P}$ is one, then $w(\mathcal{P})$ consists of a nonempty sum of elements of the algebra $\mathcal{B}$ where each summand is a product of $\mathcal{B}$-generators and their inverses. By arranging each such product so the generators appear from left to right in lexicographic order, it follows that we can uniquely write

$$\begin{aligned} \det_q(M_D[I, J]) &= \sum_{\mathcal{P}} w(\mathcal{P}) \\ &= \sum_{Q \subseteq [m] \times [n]} P_Q(q) \prod_{\alpha \in Q} t_\alpha^{r(\alpha, Q)}, \end{aligned} \tag{4.4}$$

where $P_Q(q)$ is some polynomial in $\mathbb{Z}_{\geq 0}[q, q^{-1}]$, and $r(\alpha, Q)$ is an integer.

Since at least one vertex-disjoint (R_I, C_J)-path system exists, the sum in Equation 4.4 is non-empty and so there exists at least one $Q \subseteq [m] \times [n]$ such that $P_Q \not\equiv 0$. Since q is transcendental over $\mathbb{Q}$, we know that $P_Q(q) \neq 0$ for any $P_Q \not\equiv 0$. Thus $\det_q(M_D[I, J]) \neq 0$.

□

4.6 Finding Vanishing Quantum Minors

The next lemma provides the critical connection between Launois' results and the q-analogue of Lindström's Lemma.

Lemma 4.18. *For a Cauchon diagram D, the path matrix M_D is equal to the matrix obtained at the end of Algorithm 4.3.*

Before we give the proof, let us briefly return to Example 4.4.

Example 4.19. Consider Figure 4.8, which is the diagram from Example 4.4. In this simple case, the path matrix is easy to calculate. For example, there are clearly two different directed paths from r_1 to c_1. The sum of their weights is $t_{1,1} + t_{1,2}t_{2,2}^{-1}t_{2,1}$. Continuing in this manner, we find that the path matrix is

$$M_D = \begin{bmatrix} t_{1,1} + t_{1,2}t_{2,2}^{-1}t_{2,1} & t_{1,2} & 0 \\ t_{2,1} & t_{2,2} + t_{2,3}t_{3,3}^{-1}t_{3,2} & t_{2,3} \\ 0 & t_{3,2} & t_{3,3} \end{bmatrix},$$

which is precisely the matrix $T^{(3,3)}$ obtained at the end of Example 4.4.

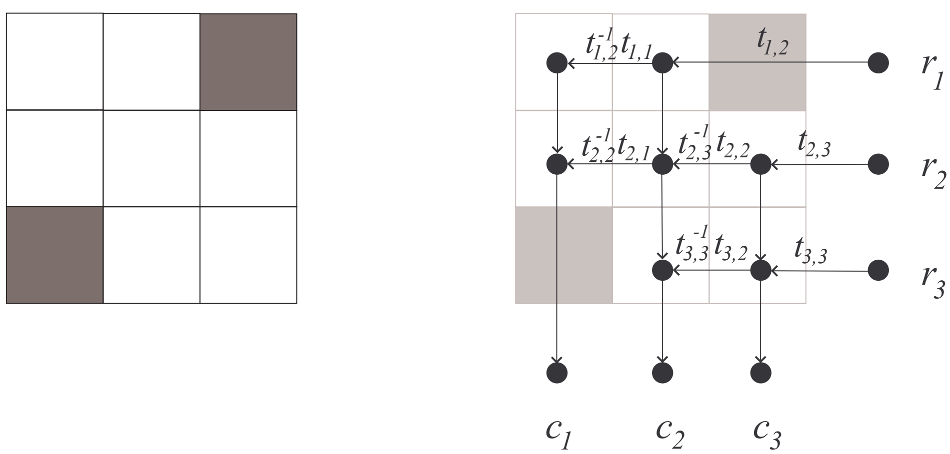


Figure 4.8: Cauchon diagram and its Cauchon graph

Proof of Lemma 4.18. Fix n. We prove the lemma by induction on the number of rows m. As in Algorithm 4.3, we will denote $T^{(s,t)}[i,j]$ by $t_{i,j}^{(s,t)} \in \operatorname{Frac}(\mathcal{B})$ where $\mathcal{B} = \mathcal{O}_q(\mathbb{K}^{m\times n})$.

First note that since we only modify entries which are north-west of the entry corresponding to the current step, the algorithm will always leave the mth row unmodified. That is, $T^{(s,t)}[m,[n]] = T^{(1,1)}[m,[n]]$ for all (s,t). By the algorithm we have that, for $k \in [n]$,

$$t_{m,k}^{(m,n)} = t_{m,k}^{(1,1)} = \begin{cases} t_{m,k} & \text{if } (m,k) \in W(D), \\ 0 & \text{if } (m,k) \in B(D). \end{cases}$$

Now in the mth row of G_D, there is clearly at most one possible path from r_m to c_k. This path exists if and only if (m,k) is a white square, and by Proposition 4.8, this path has weight exactly $t_{m,k}$.

From these two observations we see that the mth row in $T^{(m,n)}$ is exactly the same as the mth row in M_D. Similarly, the nth column of $T^{(m,n)}$ is equal to the nth column of M_D. In particular, the lemma is true when $m = 1$.

Suppose that the lemma is true for all Cauchon diagrams with less than m rows. If we obtain the Cauchon diagram D' from D by deleting the mth row, then by induction we have $T^{(m-1,n)}\left[[m-1],[n]\right] = M_{D'}$. An equivalent way of stating this is that for $i < m$, $t_{i,j}^{(m-1,n)}$ is the total of the weights of all paths in G_D from r_i to c_j which *do not* use a horizontal edge in row m.

As we already noted, $T^{(m,n)}[m,[n]] = M_D[m,[n]]$ and $T^{(m,n)}[[m],n] = M_D[[m],n]$ so to complete the proof, we establish the following claim by induction on $k \in [n]$ where k will denote the kth column of D. It will follow from this that $T^{(m,n)} = M_D$. At this point, the reader may wish to review the while loop in Algorithm 4.3.

Claim. If $(i,j) \leq (m-1,k-1)$, then $t_{i,j}^{(m,k)}$ is obtained from $t_{i,j}^{(m,k)^-}$ by adding the weights of all paths P that satisfy the following properties:

1. P is a path from r_i to c_j;

2. P contains the subpath $K_j : (m,k) \Rightarrow c_j$. Note that K_j consists of horizontal edges from (m,k) to (m,j) and then the vertical edge $((m,j),c_j)$;

3. P contains a vertical edge $((\ell,k),(m,k))$ for some $\ell < m$. In other words, if $k' > k$, then vertex (m,k') (if it exists) is *not* an internal vertex of P.

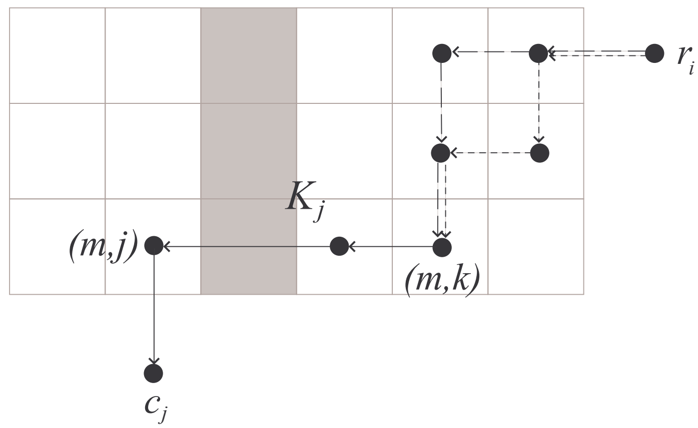


Figure 4.9: K_j is the path consisting of the edges drawn with a solid line. Concatenating K_j with either of the paths drawn with dotted edges, gives a path which satisfies the properties in the claim.

For $k = 1$, we know that $T^{(m,1)} = T^{(m-1,n)}$. On the other hand, since there are no $j < k$ the claim is trivially true. So let $k > 1$ and assume that the claim is true for step $(m, k-1)$.

If (m, k) is black in D, then according to Algorithm 4.3 we set $t_{i,j}^{(m,k)} = t_{i,j}^{(m,k-1)}$. On the other hand, if (m, k) is black, then K_j can not exist for any j and so there are no paths which satisfy the properties in the claim. This proves the claim in the case that (m, k) is black.

Suppose that (m, k) is white. If (m, j) is black for some $j < k$, then $t_{m,j}^{(m,k-1)} = t_{m,j}^{(1,1)} = 0$ and so by the algorithm we again have $t_{i,j}^{(m,k)} = t_{i,j}^{(m,k-1)}$. On the other hand, if (m, j) is black, then K_j can not exist. This proves the statement in the claim for those $j < k$ such that (m, j) is black.

Finally, if (m, k) and (m, j) are white squares, then K_j exists in G_D. Note that $w(K_j) = t_{m,k}^{-1} t_{m,j}$ by Proposition 4.8. On the other hand, if P is a path satisfying the properties in the claim, then there is a path $P' : r_i \Rightarrow (m, k)$ such that $P = P'K_j$. By Property 3, the last edge in P' is vertical. So if we concatenate P' with the vertical path $L_k : (m, k) \Rightarrow c_k$, then we get a path (with the same weight as P') from r_i to c_k which does not use any horizontal edge in the last row.

By induction, the set of all such P' has total weight $t_{i,k}^{(m-1,k)}$. This entry has not been modified at step (m, ℓ) of the algorithm for any $\ell < k$ so the set of all such P' has total weight $t_{i,k}^{(m,k-1)}$. Hence the total weight of all P that satisfy the properties in the claim is exactly

$$t_{i,k}^{(m,k-1)} w(K_j) = t_{i,k}^{(m,k-1)} t_{m,k}^{-1} t_{m,j}.$$

On the other hand, by Algorithm 4.3,

$$t_{i,j}^{(m,k)} = t_{i,j}^{(m,k-1)} + t_{i,k}^{(m,k-1)} t_{m,k}^{-1} t_{m,j}.$$

This finishes the proof of the claim and the lemma.

□

Now we state the main result of this chapter, which follows immediately from Theorem 4.17, Theorem 4.5 and Lemma 4.18.

Theorem 4.20. *Let D be an $m \times n$ Cauchon diagram corresponding to the $\mathcal{H}$-invariant prime ideal P in the algebra $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. A quantum minor* $\det_q(X[I, J])$ *of X is in P if and only if there does not exist a vertex-disjoint (R_I, C_J)-path system in the Cauchon graph G_D.* □

A computational downside of Launois' algorithm is that, in general, the entries of the final matrix may have entries with exponentially (in m and n) many terms. Therefore, for large m or n, it may be difficult or infeasible to apply the algorithm in practice. By Theorem 4.20 however, we are able to get around this problem. Given subsets R_I and C_J of k rows and columns in a Cauchon graph, we would like to decide if there exist k vertex-disjoint paths from R_I to C_J. But this is precisely the problem solved in Menger's Theorem (Theorem 2.5)! In particular, we can simply look for an (R_I, C_J)-cut in the Cauchon graph. If we find one of size less than k, then there does not exist a vertex-disjoint path system, and we may conclude that the corresponding minor in the matrix of canonical generators of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is in a generating set for the $\mathcal{H}$-prime corresponding to our Cauchon diagram.

4.7 Some Applications

The point of this section is to show that Theorem 4.20 can be used to reprove some old results of Launois [27] and Launois, Lenagan and Rigal [26] in a more efficient and intuitive manner.

Let X be an $m \times n$ matrix (e.g., the matrix of canonical generators of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$). If $I \subseteq [m]$ and $J \subseteq [n]$ are two sets of size k, then we call the submatrix $X[I, J]$ *contiguous* if $I = \{i, i+1, i+2, \ldots, i+k-1\}$ and $J = \{j, j+1, \ldots, j+k-1\}$ for some i and j. Notice that (i, j) is the upper-left entry in this submatrix. For the list of cases in the next theorem, the reader may wish to look at Figure 4.10).

Theorem 4.21. *Let m and n be positive integers and $\{u, s\} \subseteq [\min(m, n)]$. Let D be the Cauchon diagram corresponding to an $\mathcal{H}$-prime ideal J of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$.*

1. *(Launois, Lenagan, Rigal [26]) If D has exactly one black square in position (i, j) (thus either $i = 1$ or $j = 1$), then J is generated by the single quantum minor of X consisting of the largest contiguous square submatrix of X with (i, j) in the upper-left entry.*

2. *(Launois [27]) Let D be the Cauchon diagram such that the square (i, j) is black if and only if both $i \leq m-u$ and $j \leq n-u$. In this case, J is generated precisely by all $(u+1) \times (u+1)$ minors of X.*

3. *(Launois [27]) Let $u \leq n-1$ and let D be the diagram with the square (i, j) black if and only if either both $i \leq m-u$ and $j \leq n-u$, or $j = n$ and $1 \leq i \leq s$. In this case, J is generated by:*

   (a) The $(u+1) \times (u+1)$ quantum minors of X;

   (b) The $u \times u$ quantum minors of X that do not include any of the rows $s+1, \ldots, m$;

   (c) Those $x_{i,n}$ for $1 \leq i \leq s$.

Proof. Part one is trivial by inspection of the Cauchon diagram and an application of Theorem 4.20. Part two requires a little more work. It is clear by inspection that

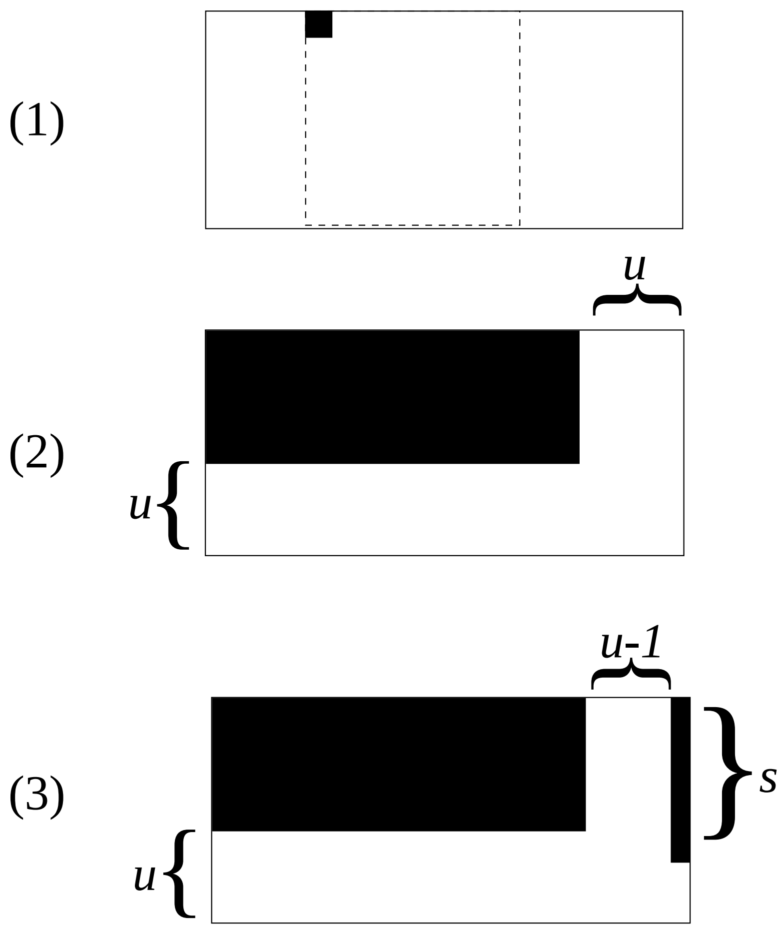


Figure 4.10: Examples of the Cauchon diagrams from Theorem 4.21. The relevant largest contiguous square submatrix in (1) is outlined as a dashed square.

a generating set of J consists of all $t \times t$ minors for appropriate $t \geq u + 1$. So to prove this result, we must show that the presence of all $(u + 1) \times (u + 1)$ minors in the generating set implies the presence of all $t \times t$ minors for $t \geq u + 1$, from which it follows that the set of all $(u + 1) \times (u + 1)$ minors generate J.

Let $t = u + 2$, and let $I = \{i_1 < i_2 \ldots\}$ and $J = \{j_1 < j_2, \ldots\}$ be two t-subsets of $[m]$ and $[n]$. By an appeal to Lemma 4.16 we obtain

$$\det_q(M_D[I, J]) = \sum_{\mathcal{P}} w(\mathcal{P}) \tag{4.5}$$

$$= \sum_{P: r_{i_1} \Rightarrow c_{j_1}} w(P) \left(\sum_{\mathcal{P}'} w(\mathcal{P}') \right), \tag{4.6}$$

where the right-hand sum in Equation 4.5 is over all vertex-disjoint (R_I, C_J) path systems, and the right-most sum in Equation 4.6 is over all vertex-disjoint $(R_{I \setminus i_1} C_{J \setminus j_1})$ path systems. Since this latter sum is over vertex-disjoint path systems of size $u + 1$, and none exist, we see that $\det_q(M_D[I, J]) = 0$ as desired. Proceeding in this manner, we see inductively that the presence of all $(u + 1) \times (u + 1)$ minors in J implies the existence of all $t \times t$ minors for $t \geq u + 1$, as desired.

Part (3a) is similarly argued, while Part (3c) is trivial by inspection. As we now show, Part (3b) is a simple application of Menger's Theorem. Let R_I be a set of u rows with $i \leq s$ for all $i \in I$. If $s \leq m - u$, then the vertices (i.e., white squares) $\{(s, n-u), (s, n-u+1), \ldots, (s, n-1)\}$ form a cut set of size $u-1$. If $s > m-u$ then the vertices $\{(m-u+1, n-u+1), (m-u+2, n-u+2), \ldots, (s, z), (s, z+1), \ldots, (s, n-1)\}$ forms a cut set of size $u - 1$, where $z = n - u + (s - (m - u))$. Thus for any J, there cannot exist a (R_I, C_J) vertex-disjoint path system.

On the other hand, it is easy to see that if R_I is a set of u rows with at least one $i \in I$, $i > s$, then we can always find a vertex-disjoint path system. Similarly, any $t \times t$ minor with $t < u$ (except the cases in $(3b)$) is not in J as we can find a vertex-disjoint path system corresponding to that minor.

□

4.8 Open Questions

The first question we ask is the obvious one: can the condition that q is transcendental be weakened to q being a non-root of unity? The work of Launois [27] fully depends on the description by Hodges and Levasseur [23] of prime ideals in the quantized special linear group. This description is only known to be true when q is transcendental over $\mathbb{Q}$. If that description can be weakened so that q is only a non-root of unity, then Launois' results would hold. Furthermore, our quantum analogue of Lindström's Lemma is valid for any q. On the other hand, we use the transcendence of q to show that the quantum determinant is zero if and only if there is no vertex-disjoint path system. However, we believe that more subtle reasoning should allow this latter result to remain true for q not a root of unity.

Next consider the issue of finding a *minimal* generating set for a given $\mathcal{H}$-prime P in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. Consider the quantum minor $z = \det_q(X[I, J])$ for some sets I and J with $|I| = |J| = k$. We know that if this minor is in P, then there does not exist a vertex-disjoint (R_I, C_J)-path system in the Cauchon graph G_D corresponding to P. On the other hand, by Menger's Theorem 2.5, there must exist an (R_I, C_J)-cut in G_D of size $\ell < k$. Now if $\ell < k - 1$, then we can use the cofactor expansion trick used in the proof of Theorem 4.21 to conclude that z is a generated by other quantum minors in P. Unfortunately, this is not the end of the story as the $\mathcal{H}$-prime ideal in $\mathcal{O}_q(\mathrm{M}_{2,2}(\mathbb{K}))$ corresponding to the Cauchon diagram  is generated by $x_{1,1}$ and $x_{1,2}$ which implies that determinant $x_{1,1}x_{2,2} - qx_{1,2}x_{2,1}$ is in the ideal. However, this minor has a maximum cut of size 1 in the associated Cauchon graph.

Problem 4.22. *Find a characterization of a minimal generating set using cut sets for a given $\mathcal{H}$-prime ideal in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$.*

We believe that a solution to the above problem could be used to approach the following question that appears as Conjecture II.10.9 in [7].

Conjecture 4.23. *Every $\mathcal{H}$-prime J of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ can be generated by a* polynormal *sequence of quantum minors. That is, $J = (d_1, d_2, \ldots, d_k)$ where for all $1 \le j \le k$, the image of d_j in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))/(d_1, \ldots, d_{j-1})$ is normal.*

Chapter 5

The Dimensions of $\mathcal{H}$-strata

5.1 Introduction

In this chapter we shall be working exclusively in the algebra $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. Thus all references to $\mathcal{H}$-prime, $\mathcal{H}$-strata etc. are in this context. Recall from Theorem 3.19 that each $\mathcal{H}$-stratum is homeomorphic to the spectrum of a Laurent polynomial ring in a certain number of indeterminates, called the Krull dimension of the stratum. In particular, primitive $\mathcal{H}$-ideals are precisely those $\mathcal{H}$-primes whose $\mathcal{H}$-strata are zero-dimensional [7]. In general, one would like simple criteria that decide if a given $\mathcal{H}$-strata is d-dimensional.

Launois and Lenagan [31] completely solved the problem for any m and n in the special case that the given $\mathcal{H}$-prime is (0) (corresponding to the all-white Cauchon diagram). They showed that the dimension can easily be given by an arithmetic condition on m and n. As a consequence of the work in this chapter, we give an elementary proof of this condition in Corollary 5.18.

A special case of Launois and Lenagan's result is the determination of the dimension in the $1 \times n$ case for any n. Using their result, it is a simple matter to check that an $\mathcal{H}$-prime is primitive if and only if the number of white squares is even, and otherwise the dimension is 1.

Bell, Launois & Nguyen [4] solved the dimension zero case for $m = 2$ and any n. Their condition is again arithmetic, but more complicated than the condition for the all-white diagram. However, using their condition they were able to give an enumeration formula for the total number of primitive $\mathcal{H}$-primes in $\mathcal{O}_q(\mathrm{M}_{2,n}(\mathbb{K}))$.

Bell, Launois & Lutley [3] analyzed the $m = 3$ case for any n. They translated the problem of determining primitivity of $\mathcal{H}$-primes into a problem involving finite state automata. A description of their method would be far out of the scope of this work, but we note they were able to give an enumeration formula for the total number of primitive $\mathcal{H}$-primes in $\mathcal{O}_q(\mathrm{M}_{3,n}(\mathbb{K}))$. While their method could, in theory, be extended to the general case, the difficulty increases exponentially for even small values of m. On the other hand, it was the careful analysis of one of the proofs in that paper which eventually led us to the main result of this chapter, namely Corollary 5.16 which, for any m, n and d, gives a simple criterion for a given $\mathcal{H}$-stratum of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ to be d-dimensional.

5.2 $\mathcal{H}$-Strata Dimensions in Quantum Affine Spaces

In this section, we give a lemma due to Bell and Launois [2], which will prove crucial to the results in this chapter. Let us recall the $m \times n$ quantum affine space $\mathcal{B} = \mathcal{O}_q(\mathbb{K}^{m\times n})$ of Definition 3.10. In particular, this algebra is defined with respect to the multiplicatively antisymmetric matrix Q given in the definition. Now each entry of Q is one of q^1, q^{-1} or q^0. It is therefore natural to consider the skew-symmetric $mn \times mn$ matrix M with entries in $\{1, -1, 0\}$, obtained by considering the relation $Q[i, j] = q^{M[i,j]}$.

Definition 5.1. Let D be an $m \times n$ diagram (not necessarily Cauchon). Denote by $Q(D)$ and $M(D)$ respectively, the matrices obtained from Q and M by deleting the rows and columns corresponding to those $(i, j) \in B(D)$. Let K_D be the $\mathcal{H}$-ideal $(t_{i,j} \mid (i, j) \in B(D))$ in $\mathcal{B} = \mathcal{O}_Q(\mathbb{K}^{mn}) = \mathcal{O}_q(\mathbb{K}^{m\times n})$. Let $\mathcal{B}_D := \mathcal{B}/K_D$. Note that if D has N white squares, then

$$\mathcal{B}_D \simeq \mathcal{O}_{Q(D)}(\mathbb{K}^N) \subseteq \mathcal{O}_q(\mathbb{K}^{m\times n}).$$

More specifically, if one labels the white squares of a diagram D (with N white squares) in lexicographic order by the elements of $[N]$, then $M(D)$ can be defined by setting: $M(D)[i, j] = 1$, if square i is to the left or above square j; $M(D)[i, j] = -1$, if square i is to the right or below square j; and taking $M(D)[i, j] = 0$ otherwise.

Example 5.2. Consider the diagram D in Figure 5.1. The matrix $M(D)$ is

$$M(D) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

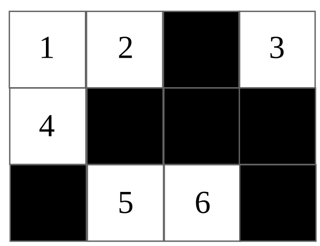


Figure 5.1: A labelled Cauchon diagram

Lemma 5.3 (Bell and Launois [2])**.** *Let J be an $\mathcal{H}$-prime ideal of $\mathcal{A} = \mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ with corresponding Cauchon diagram D. The dimension of the $\mathcal{H}$-stratum corresponding to J is equal to* $\dim(\ker(M(D)))$.

Proof. Let K_D, $\mathcal{B}$ and $\mathcal{B}_D$ be as in Definition 5.1. By Theorem 3.27 (together with the remark above it),

$$\begin{aligned} \mathrm{spec}_J(\mathcal{A}) &\simeq \mathrm{spec}_D(\mathcal{B}) \\ &\simeq \mathrm{spec}_{(0)}(\mathcal{B}_D). \end{aligned}$$

It is easy to check that the set E of regular $\mathcal{H}$-eigenvectors of $\mathcal{B}_D$ are precisely the monomials in the canonical generators. Hence $(\mathcal{B}_D)E^{-1} \simeq \mathcal{O}_{Q(D)}((\mathbb{K}^{\times})^N)$, the quantum affine torus. Therefore, by Theorem 3.19, $\mathrm{spec}_{(0)}(\mathcal{B}_D)$ is homeomorphic to $\mathrm{spec}(Z(\mathcal{O}_{Q(D)}((\mathbb{K}^\times)^N)))$. This latter set has already been calculated in Lemma 3.9. Recall that for $\boldsymbol{k}, \boldsymbol{\ell} \in \mathbb{Z}^N$, we set

$$\begin{aligned} \sigma(\boldsymbol{k}, \boldsymbol{\ell}) &= \prod_{i,j=1}^{n} (Q(D)[i,j])^{\boldsymbol{k}_i \boldsymbol{\ell}_j} \\ &= q^{\sum_{i,j=1}^{n}(M(D)[i,j])\boldsymbol{k}_i \boldsymbol{\ell}_j}. \end{aligned}$$

Then $Z(\mathcal{O}_{Q(D)}((\mathbb{K}^\times)^N)) = \mathbb{K}[\boldsymbol{t}^{\boldsymbol{k}} \mid \boldsymbol{k} \in S]$, where $S = \{\boldsymbol{k} \mid \sigma(\boldsymbol{k}, \boldsymbol{\ell}) = 1, \forall\ \boldsymbol{\ell} \in \mathbb{Z}^N\}$.

Now q is not a root of unity, so $q^{\sum_{i,j=1}^{n}(M(D)[i,j])\boldsymbol{k}_i \boldsymbol{\ell}_j} = 1$ for all $\boldsymbol{\ell} \in \mathbb{Z}^N$ if and only if for all $j = 1, \ldots, N$, $\sum_{i=1}^{N}(M(D)[i,j])\boldsymbol{k}_i = 0$. In other words, S (considered as a $\mathbb{Z}$-module) has a basis consisting of those $\boldsymbol{k} \in \mathbb{Z}^N$ that are in $\ker(M(D))$. Tracing back through the homeomorphisms, we see that the dimension of $\mathrm{spec}_J(\mathcal{A})$ is equal to $\dim(\ker(M(D)))$, as desired.

□

5.3 Pipe dreams

For two permutations $\sigma, \sigma' \in S_p$, let us set $\sigma \leq \sigma'$ if the following condition holds for each $j \in [p]$: If we write $\sigma([j]) = \{a_1, \ldots, a_j\}$ (ordered so that $a_i < a_{i+1}$) and $\sigma'([j]) = \{a'_1, \ldots, a'_j\}$ (ordered so that $a'_i < a'_{i+1}$), then $a_i \leq a'_i$ for every $i \in [j]$.

This partial ordering is called the (reverse) Bruhat order on S_p. Launois [29] proved that, in the case q is transcendental over $\mathbb{Q}$, the partially ordered set (ordered by inclusion) of $\mathcal{H}$-strata of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is isomorphic to the sub-poset $\mathcal{S}$ of permutations where

$$\mathcal{S} = \{\sigma \in S_{m+n} \mid -n \leq \sigma(i) - i \leq m, \forall\ i \in [m+n]\}.$$

Let us call a permutation in $\mathcal{S}$ *restricted* (cf. Definition 5.5).

Launois achieved part of this result by proving that the number of $m \times n$ Cauchon diagrams is equal to the *poly-Bernoulli number* B_m^{-n} [25], where

$$B_n^{-m} = \sum_{k=0}^{\min(m,n)} (k!)^2 \left\{ {n+1 \atop k+1} \right\} \left\{ {m+1 \atop k+1} \right\}.$$

The generating function for B_m^{-n} (and hence Cauchon diagrams) has been derived and we record it here for future use.

Theorem 5.4 (Kaneko [25]). *Let $C(m,n)$ be the number of $m \times n$ Cauchon diagrams. Set $C(m,0) = C(0,n) = C(0,0) := 1$ for all $m, n \geq 1$. If*

$$\mathcal{C}(x,y) = \sum_{m,n \geq 0} C(m,n) \frac{x^m}{m!} \frac{y^n}{n!},$$

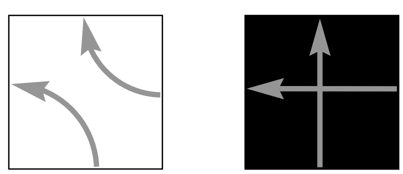
then

$$\mathcal{C}(x,y) = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

$\square$

A direct connection between Cauchon diagrams and restricted permutations remained unclear until the work of Postnikov [38] was noticed. He pointed out that one may apply the so-called *pipe-dreams* construction of Fomin and Kirillov [13] to Ⅎ/Cauchon diagrams. This method allows the removal of the condition that q transcendental over $\mathbb{Q}$. The remainder of this section describes this construction.

Let D be a diagram (either Cauchon or non-Cauchon). Place a "directed hyperbola" on every white square, and a "directed cross" on every black square. See Figure 5.2. We will omit the arrowheads when drawing a picture of pipe dreams on a diagram.



(a) (b)

Figure 5.2: (a) a "directed hyperbola" placed on a white square, (b) a "directed cross" placed on a black square.

If D is a diagram with pipes placed as explained above, then label the bottom and right sides of the columns and rows respectively by distinct elements of $[n+m]$, and label the top and left sides of the columns and rows respectively also by distinct elements of $[n+m]$. This defines a permutation $\sigma \in S_{n+m}$ obtained by sending label i (on the bottom or right side of D) to the label $\sigma(i)$, where $\sigma(i)$ is the label on the left or top side of D that one reaches by following the pipe starting at i. It should be clarified that pipes always travel straight through a black square (i.e., they do not turn at the intersection point of the cross). Notice that σ^{-1} is obtained simply by reversing this process.

Definition 5.5. Let D be an $m \times n$ diagram. The *standard labelling* of D is as follows. Starting from the left, label the bottom side of each column by the numbers $\{1, 2, \ldots, n\}$, and the top side of each column by $\{m+1, m+2, \ldots, m+n\}$. Starting from the bottom, label the left side of each row by $\{1, 2, \ldots m\}$ and the right side of each row by $\{n+1, n+2, \ldots n+m\}$.

The permutation obtained from the standard labelling is, in fact, exactly the restricted permutation in the sense of Launois' results.

Example 5.6. Figure 5.3 gives an example of a 3×4 diagram with both pipes laid on it, and the standard labelling. This example gives rise to the restricted permutation $(2\,3\,5\,4)(6\,7) \in S_{3+4}$.

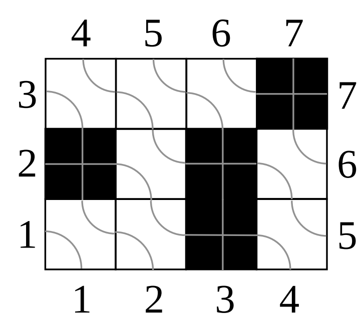


Figure 5.3: An example of the standard labelling of a diagram

Under the standard labelling of the sides of the collection of all $m \times n$ diagrams (i.e., not necessarily Cauchon diagrams), the map from the collection of all diagrams to the permutations arising from diagrams with this fixed labelling is not, in general, injective. For example, the diagrams  and  both give the restricted permutation (2 3). On the other hand, as a consequence of Launois' results [29], the map *is* bijective if one restricts attention only to the $m \times n$ Cauchon diagrams.

Definition 5.7. Let D be an $m \times n$ diagram. The *toric labelling* of D is defined as follows. Starting from the left, label both the bottom and top sides of each column by the numbers $\{m+1, m+2, \ldots, m+n\}$. Starting from the bottom, label both the left and right sides of each row by $\{1, 2, \ldots m\}$.

We call the permutation obtained from this labelling the *toric* permutation of D. Since the labels on opposite sides of each row and column match, one may think of the diagram as a torus and each cycle in the disjoint-cycle decomposition of this permutation is found by following the pipes around the torus.

Example 5.8. Figure 5.4 gives an example of the toric labelling on a diagram D, from which one obtains the toric permutation $(1\,4)(2\,7)(3\,6\,5)$.

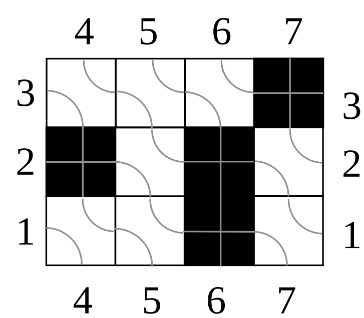


Figure 5.4: An example of the toric labelling of a diagram

Definition 5.9. Let ω be the (restricted) permutation obtained by pipe-dreams from the standard-labelling on the all-black diagram. Thus ω can be described by

$$\omega(i) = \begin{cases} m+i & \text{if } 1 \leq i \leq n, \\ i-n & \text{if } n+1 \leq i \leq n+m. \end{cases}$$

The following proposition follows immediately from the definitions of the permutations involved.

Proposition 5.10. *For D an $m \times n$ diagram, σ the restricted permutation of D and τ the toric permutation of D, we have $\tau = \sigma\omega^{-1}$.* □

For any permutation δ, denote the corresponding permutation matrix by P_δ.

Lemma 5.11. *If D is a diagram and $\tau = \sigma\omega^{-1}$ is the toric permutation of D (where σ is the restricted permutation of D), then $\boldsymbol{v} \in \ker(P_\omega + P_\sigma)$ if and only if $\boldsymbol{v}_b = -\boldsymbol{v}_{\tau(b)}$ for every label b.*

Proof. Consider the pipe in D corresponding to the cycle in τ containing b and $\tau(b)$. Now $\boldsymbol{v} \in \ker(P_\omega + P_\sigma)$ if and only if for all a we have $\boldsymbol{v}_{\omega(a)} + \boldsymbol{v}_{\sigma(a)} = 0$. Taking $a = \omega^{-1}(b)$ we obtain $\boldsymbol{v}_b = -\boldsymbol{v}_{\tau(b)}$ as desired. □

Given a permutation σ, recall that σ is said to be *even* if $\ell(\sigma)$ is even, and *odd* if $\ell(\sigma)$ is odd, where $\ell(\sigma)$ is the length function as described in Definition 3.13. While it may be confusing, note that a cycle is odd if and only if it permutes an *even* number of elements. For example, any transposition is odd.

Lemma 5.12. *Let D be an $m \times n$ diagram and consider the toric permutation $\tau = \sigma\omega^{-1}$, where σ is the restricted permutation of D. The dimension of $\ker(P_\omega + P_\sigma)$ is the number of odd cycles in the disjoint-cycle decomposition of τ.*

Proof. Let $\gamma_1, \ldots, \gamma_\ell$ be the set of odd cycles in the disjoint cycle decomposition of $\sigma\omega^{-1}$. Given the odd cycle $\gamma = (a_1\, a_2\, \ldots\, a_{2k})$ define the vector $\boldsymbol{v}^\gamma \in \mathbb{Z}^{m+n}$ by

$$\boldsymbol{v}^\gamma_b = \begin{cases} 1 & \text{if } b = a_i \text{ and } i \text{ is odd}, \\ -1 & \text{if } b = a_i \text{ and } i \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

We claim that the set $B = \{\boldsymbol{v}^{\gamma_i} \mid i \in [\ell]\}$ forms a basis for $\ker(P_\omega + P_\sigma)$. Since the odd cycles are mutually disjoint, it is clear that the members of B form an independent set in the module $\mathbb{Z}^{m+n}$.

Suppose that $\boldsymbol{v} \in \ker(P_\omega + P_\sigma)$. By Lemma 5.11, we know that if b is in an even cycle of τ, then $\boldsymbol{v}_b = 0$. Moreover, the values of the entries corresponding to an odd cycle agree up to multiplication by -1. Thus we see that $\boldsymbol{v}$ can be written as a linear combination of elements of B. $\square$

5.4 The Isomorphism

This section is devoted to proving Theorem 5.15, which describes the kernel of $M(D)$ for any diagram D using toric permutations. Thereafter we obtain the main result of this Chapter, Corollary 5.18, which says that the dimension of an $\mathcal{H}$-stratum is exactly equal to the number of odd cycles in the toric permutation.

Notation 5.13. Fix an $m \times n$ diagram D with N white squares labelled by distinct elements of the set $[N]$. Let $\tau = \sigma\omega^{-1}$ be the toric permutation of D. Let $\boldsymbol{w}$ be in the column space of $M(D)$ and $\boldsymbol{v}$ in the column space of $P_\omega + P_\sigma$. We refer to the entries of $\boldsymbol{w}$ by $\boldsymbol{w}_j$ for $j \in [N]$ and the entries of $\boldsymbol{v}$ by $\boldsymbol{v}_a$ where $a \in [m+n]$.

1. Since, in the toric labelling, each side of a row or column is given the same label, we may unambiguously refer to a row or column by this label.

2. Given a diagram D with the toric labelling and a white square i of D, let $\mathrm{left}(i)$ and $\mathrm{up}(i)$ be, respectively, the labels of the rows or columns reached by following the bottom and top pipes of the hyperbola pipe placed on i.

3. For $S \subseteq [N]$, let $\boldsymbol{w}_S = \sum_{j \in S} \boldsymbol{w}_j$.

4. For a given white square i let $A(i), R(i), B(i)$ and $L(i)$ be the sets of white squares that are, respectively, strictly above, strictly to the right, strictly below and strictly to the left of square i.

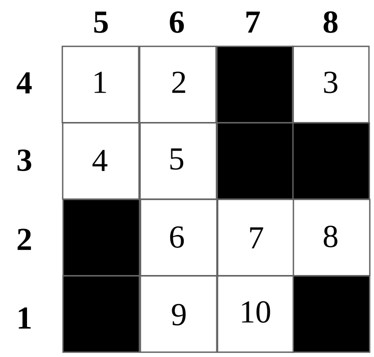


Figure 5.5: The diagram used in Example 5.14

Example 5.14. Consider the diagram in Figure 5.5. For this example, we have labelled the white squares in a regular font, and the row and column labels in bold font. We have, for example, $\text{left}(7) = \mathbf{4}$ and $\text{up}(7) = \mathbf{7}$, while $\text{left}(8) = \mathbf{7}$ and $\text{up}(8) = \mathbf{6}$. On the other hand $A(5) = \{2\}, R(5) = \emptyset, B(5) = \{6, 9\}$ and $L(5) = \{4\}$.

Before proving the main theorem of this section, we note that by the definition of $M(D)$ we have $\boldsymbol{w} \in \ker(M(D))$ if and only if for every white square i,

$$\boldsymbol{w}_{A(i)} + \boldsymbol{w}_{L(i)} = \boldsymbol{w}_{B(i)} + \boldsymbol{w}_{R(i)}.$$

Theorem 5.15. *If D is a diagram and σ is the restricted permutation obtained from D, then*

$$\ker(P_\omega + P_\sigma) \simeq \ker(M(D)).$$

Proof. We place the toric labelling on D. Suppose that D has N white squares labelled $1, 2, \ldots, N$. The theorem is proved once we exhibit injective functions $\phi : \ker(P_\omega + P_\sigma) \to \ker(M(D))$ and $\psi : \ker(M(D)) \to \ker(P_\omega + P_\sigma)$.

Given $\boldsymbol{v} \in \ker(P_\omega + P_\sigma)$, we take $\boldsymbol{w} := \phi(\boldsymbol{v})$ to be the vector whose entries are defined by $\boldsymbol{w}_i = \boldsymbol{v}_{\text{left}(i)} - \boldsymbol{v}_{\text{up}(i)}$. To show $\boldsymbol{w} \in \ker(M(D))$ we need only check that for all white squares i the relation $\boldsymbol{w}_{A(i)} + \boldsymbol{w}_{L(i)} - \boldsymbol{w}_{B(i)} - \boldsymbol{w}_{R(i)} = 0$ holds.

Notice that if $S = \{s_1, s_2, \ldots, s_k\}$ is any block of consecutive white squares in the same row (where s_1 is the left-most white square, and s_k is the right-most), then we have $\boldsymbol{w}_S = \boldsymbol{v}_{\mathrm{left}(s_1)} - \boldsymbol{v}_{\mathrm{up}(s_k)}$. Similarly, if $S = \{s_1, s_2, \ldots, s_k\}$ is any block of consecutive white squares in the same column (where s_1 is the bottom-most white square, and s_k is the top-most white square), then $\boldsymbol{w}_S = \boldsymbol{v}_{\mathrm{left}(s_1)} - \boldsymbol{v}_{\mathrm{up}(s_k)}$.

Fix a white square i. For the sake of brevity, we assume that $A(i), L(i), B(i)$ and $R(i)$ are all non-empty, but otherwise an argument similar to what follows can be used. Suppose that the label of the row containing i is r and the label of the column containing i is c. Let $s_{r,1}$ be the left-most white square in row r, $s_{r,k}$ the right-most white square in row r, $s_{c,1}$ the bottom-most white square in column c and $s_{c,\ell}$ the top-most white square in column c. We have

$$\begin{aligned}
\boldsymbol{w}_{A(i)} + \boldsymbol{w}_{L(i)} - \boldsymbol{w}_{B(i)} - \boldsymbol{w}_{R(i)} &= \big(\boldsymbol{v}_{\mathrm{up}(i)} - \boldsymbol{v}_{\mathrm{up}(s_{c,\ell})}\big) + \big(\boldsymbol{v}_{\mathrm{left}(s_{r,1})} - \boldsymbol{v}_{\mathrm{left}(i)}\big) \\
&\quad -\big(\boldsymbol{v}_{\mathrm{left}(s_{c,1})} - \boldsymbol{v}_{\mathrm{left}(i)}\big) - \big(\boldsymbol{v}_{\mathrm{up}(i)} - \boldsymbol{v}_{\mathrm{up}(s_{r,k})}\big) \\
&= \big(\boldsymbol{v}_{\mathrm{left}(s_{r,1})} + \boldsymbol{v}_{\mathrm{up}(s_{r,k})}\big) - \big(\boldsymbol{v}_{\mathrm{up}(s_{c,1})} + \boldsymbol{v}_{\mathrm{left}(s_{c,\ell})}\big) \\
&= \big(\boldsymbol{v}_r + \boldsymbol{v}_{\tau(r)}\big) - \big(\boldsymbol{v}_c + \boldsymbol{v}_{\tau(c)}\big) \\
&= 0,
\end{aligned}$$

where the last equality follows by Lemma 5.11.

Now define $\psi : \ker(M(D)) \to \ker(P_\omega + P_\sigma)$ as follows. Let $\boldsymbol{w} \in \ker(M(D))$ and write $\boldsymbol{v} = \psi(\boldsymbol{w})$. If a is a column we take $\boldsymbol{v}_a$ to be the sum of all white squares in column a, but if a is a row, then we take $\boldsymbol{v}_a$ to be the negative of the sum of all white squares in row a. By Lemma 5.11, we want to show that for all labels a,

$$\boldsymbol{v}_a + \boldsymbol{v}_{\tau(a)} = 0. \tag{5.1}$$

Fix a label a and let $(s_1, s_2, \ldots, s_k)$ be the sequence of white squares used in the pipe from a to $\tau(a)$. We assume that a is a column as the case in which a is a row is argued similarly. We first prove by induction that, for every $i = 1, \ldots, k$,

$$\begin{aligned}
\boldsymbol{v}_a &= \boldsymbol{w}_{s_1} + \boldsymbol{w}_{A(s_1)} && (5.2) \\
&= (-1)^{i+1}\boldsymbol{w}_{s_i} + \boldsymbol{w}_{A(s_i)} - \boldsymbol{w}_{B(s_i)}. && (5.3)
\end{aligned}$$

If $i = 1$, then Equation (5.3) is trivially true since $B(s_i) = \emptyset$. So suppose that it is true for all values less than $i > 1$. There are two cases to consider. If i is even, then

s_i is the first white square to the left of s_{i-1}. Note that $\boldsymbol{w}_{R(s_i)} = \boldsymbol{w}_{R(s_{i-1})} + \boldsymbol{w}_{s_{i-1}}$ and $\boldsymbol{w}_{L(s_{i-1})} = \boldsymbol{w}_{L(s_i)} + \boldsymbol{w}_{s_i}$. Since $\boldsymbol{w} \in \ker(M(D))$ we have the two equations

$$\boldsymbol{w}_{A(s_{i-1})} + \boldsymbol{w}_{L(s_i)} + \boldsymbol{w}_{s_i} - \boldsymbol{w}_{B(s_{i-1})} - \boldsymbol{w}_{R(s_{i-1})} = 0,$$

and

$$\boldsymbol{w}_{A(s_i)} + \boldsymbol{w}_{L(s_i)} - \boldsymbol{w}_{B(s_i)} - \boldsymbol{w}_{R(s_{i-1})} - \boldsymbol{w}_{s_{i-1}} = 0.$$

Subtracting the second equation from the first we obtain

$$\boldsymbol{w}_{s_{i-1}} + \boldsymbol{w}_{A(s_{i-1})} - \boldsymbol{w}_{B(s_{i-1})} \quad = \quad -\boldsymbol{w}_{s_i} + \boldsymbol{w}_{A(s_i)} - \boldsymbol{w}_{B(s_i)}. \tag{5.4}$$

But since $i-1$ is odd, the left hand side of (5.4) is equal, by induction, to $\boldsymbol{w}_{s_1} + \boldsymbol{w}_{A(s_1)}$ and thus (5.4) implies (5.3) for the case that i is even.

Now for i odd, s_i is the first white square above s_{i-1}. It is easy to check that we have the two equations

$$\boldsymbol{w}_{A(s_{i-1})} = \boldsymbol{w}_{A(s_i)} + \boldsymbol{w}_{s_i}$$

and

$$\boldsymbol{w}_{B(s_i)} = \boldsymbol{w}_{B(s_{i-1})} + \boldsymbol{w}_{s_{i-1}}.$$

By induction we obtain

$$\begin{aligned} \boldsymbol{w}_{s_1} + \boldsymbol{w}_{A(s_1)} &= -\boldsymbol{w}_{s_{i-1}} + \boldsymbol{w}_{A(s_{i-1})} - \boldsymbol{w}_{B(s_{i-1})} \\ &= -\boldsymbol{w}_{s_{i-1}} + \boldsymbol{w}_{A(s_i)} + \boldsymbol{w}_{s_i} - (\boldsymbol{w}_{B(s_i)} - \boldsymbol{w}_{s_{i-1}}) \\ &= \boldsymbol{w}_{s_i} + \boldsymbol{w}_{A(s_i)} - \boldsymbol{w}_{B(s_i)}. \end{aligned}$$

This finishes the proof of Equation (5.3).

Now we verify Equation (5.1). If k is even, then $\tau(a)$ is a column and so $A(s_k) = \emptyset$. By (5.3),

$$\begin{aligned} \boldsymbol{v}_a &= \boldsymbol{w}_{s_1} + \boldsymbol{w}_{A(s_1)} \\ &= -\boldsymbol{w}_{s_k} + \boldsymbol{w}_{A(s_k)} - \boldsymbol{w}_{B(s_k)} \\ &= -\boldsymbol{w}_{s_k} - \boldsymbol{w}_{B(s_k)} \\ &= -\boldsymbol{v}_{\tau(a)}. \end{aligned}$$

On the other hand, if k is odd, then $\tau(a)$ is a row and so $L(s_k) = \emptyset$. Since $\boldsymbol{w} \in \ker(M(D))$, we must have $\boldsymbol{w}_{A(s_k)} - \boldsymbol{w}_{B(s_k)} = \boldsymbol{w}_{R(s_k)}$. Hence

$$\begin{aligned} \boldsymbol{v}_a &= \boldsymbol{w}_{s_1} + \boldsymbol{w}_{A(s_1)} \\ &= \boldsymbol{w}_{s_k} + \boldsymbol{w}_{A(s_k)} - \boldsymbol{w}_{B(s_k)} \\ &= \boldsymbol{w}_{s_k} + \boldsymbol{w}_{R(s_k)} \\ &= -\boldsymbol{v}_{\tau(a)}. \end{aligned}$$

The final step is to prove that the functions ϕ and ψ are injective. First, suppose that $\psi(\boldsymbol{w}) = \mathbf{0}$ with $\boldsymbol{w} \neq \mathbf{0}$. There must, therefore, exist a white square i (with i minimum) such that $\boldsymbol{w}_i \neq 0$ but $\boldsymbol{w}_{A(i)} = 0$ and $\boldsymbol{w}_{L(i)} = 0$. Since $\boldsymbol{w} \in \ker(M(D))$ we have $\boldsymbol{w}_{B(i)} + \boldsymbol{w}_{R(i)} = 0$. On the other hand, since $\psi(\boldsymbol{w}) = \mathbf{0}$ we have, by the construction of $\psi(\boldsymbol{w})$, that $\boldsymbol{w}_{B(i)} + \boldsymbol{w}_i = 0$ and $0 = \boldsymbol{w}_{R(i)} + \boldsymbol{w}_i = -\boldsymbol{w}_{B(i)} + \boldsymbol{w}_i$. Thus we must have $\boldsymbol{w}_i = 0$, contradicting the choice of i. Therefore ψ is injective.

To show that ϕ is injective we show that $\psi(\phi(\boldsymbol{v})) = -2\boldsymbol{v}$ for every $\boldsymbol{v} \in \ker(P_\omega + P_\sigma)$. First, if a is a row then $\psi(\phi(\boldsymbol{v}))_a = -(\boldsymbol{v}_a - \boldsymbol{v}_{\tau(a)})$. On the other hand, by Lemma 5.11, we know that $\boldsymbol{v}_{\tau(a)} = -\boldsymbol{v}_a$ and thus $\psi(\phi(\boldsymbol{v}))_a = -2\boldsymbol{v}_a$. A similar argument shows that if a is a column then $\psi(\phi(\boldsymbol{v}))_a = -2\boldsymbol{v}_a$. Hence ϕ is injective. □

Corollary 5.16. *For a Cauchon diagram D, the dimension of the $\mathcal{H}$-stratum corresponding to D is equal to the number of odd cycles in the disjoint cycle decomposition of the toric permutation of D.*

Proof. This follows immediately from Theorems 5.3 and 5.15. □

Example 5.17. We return to the diagram in Figure 5.5. The toric permutation of this diagram is $\tau = \sigma\omega^{-1} = (1\,4\,8\,7\,2\,6)(3\,5)$. By Lemma 5.12 we can find a basis for $\ker(P_\sigma + P_\omega)$, namely, $\boldsymbol{v}^1 = (1, 1, 0, -1, 0, -1, -1, 1)$ and $\boldsymbol{v}^2 = (0, 0, 1, 0, -1, 0, 0, 0)$.

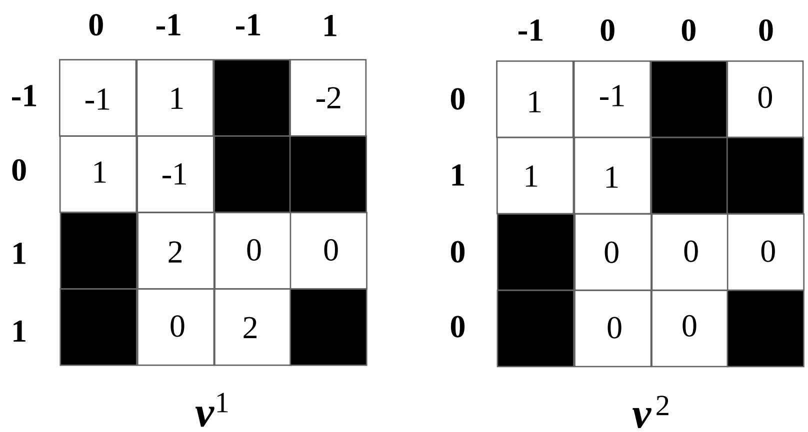
Under the map ϕ defined in the proof of Theorem 5.15, we have

$$\phi(\boldsymbol{v}^1) = (-1, 1, -2, 1, -1, 2, 0, 0, 0, 2)$$

and

$$\phi(\boldsymbol{v}^2) = (1, -1, 0, 1, 1, 0, 0, 0, 0, 0).$$

In Figure 5.17 we have twice redrawn the diagram from Figure 5.5. In each drawing we have replaced the row and column labels by the corresponding entry in $\boldsymbol{v}^i$, and the white square labels by the corresponding entry in $\phi(\boldsymbol{v}^i)$.



As an application of Corollary 5.16 we provide an elementary proof of a result of Launois and Lenagan [31]. They calculated the dimension of the $\mathcal{H}$-stratum corresponding to an all-white diagram, i.e., the $\mathcal{H}$-stratum of $\operatorname{spec}(\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K})))$ corresponding to the $\mathcal{H}$-prime (0). Their condition gives an arithmetic criterion to decide when $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is a primitive ring. Let $\nu_2(k)$ be the *2-valuation* of a positive integer k. That is, $\nu_2(k)$ is the largest integer α such that $2^\alpha | k$.

Corollary 5.18. *For positive integers m and n, the dimension d of the $\mathcal{H}$-stratum corresponding to* (0) *is*

$$d = \begin{cases} 0 & \text{if } \nu_2(m) \neq \nu_2(n), \\ \gcd(m,n) & \text{otherwise.} \end{cases}$$

Proof. Let D be the all-white $m \times n$ diagram. Observe that, for $m = n$, the toric permutation always consists of transpositions of the form $(k \;\; n+k)$ with $1 \leq k \leq n$. In particular, notice that every cycle has exactly one row and one column label and that there are exactly m cycles.

Now suppose we are in the general case, i.e., possibly $m \neq n$. Let $c(D)$ be the number of cycles in the toric permutation of D. Let us suppose we adjoin an $m \times m$ all white square diagram D' to the right side of D to obtain an $m \times (n+m)$ all white diagram D''. Notice that the equation $c(D'') = c(D)$ follows immediately from the above observation about square all-white diagrams: any pipe that contains label k in D will enter row k in D'' and then use the label k in D'' without using any other row labels.

Thus we can peel off $m \times m$ subdiagrams from D. If $m|n$, then we end at an $m \times m$ square subdiagram of D and we stop. Otherwise, we end up with an $m \times n'$ diagram with $1 \leq n' < m$. We can then peel off $n' \times n'$ subdiagrams and so forth. Eventually, we must end at some square subdiagram. In fact notice that this is the largest square subdiagram that partitions D. The number of cycles in this final square subdiagram (which by the above observation is its size) is equal to $c(D)$. In fact this procedure is really just the Euclidean algorithm in a thin disguise! Hence the number of cycles $c(D)$ in D is the greatest common divisor of m and n.

Now since D is invariant under horizontal and vertical rotations, we see that all cycles must have the same length as the cycle containing 1. In other words, each cycle in D has the same length and by the above paragraph, this length is $(m+n)/\gcd(m,n)$.

If $\nu_2(m) = \nu_2(n)$, then for some nonnegative integer α we can write $m = 2^\alpha m'$ and $n = 2^\alpha n'$ where m' and n' are odd. Notice that

$$\frac{m+n}{\gcd(m,n)} = \frac{m'+n'}{\gcd(m',n')}.$$

Since $\gcd(m',n')$ is odd while $m'+n'$ is even it follows that $(m+n)/\gcd(m,n)$ is even. By a similar argument, if $\nu_2(m) \neq \nu_2(n)$, then $(m+n)/\gcd(m,n)$ can be reduced to an odd number divided by an odd number, and thus is odd. The statement of the result then follows by Corollary 5.18.

$\square$

5.5 Enumeration of $\mathcal{H}$-strata

In this section we give enumeration formulae for the total number of $m \times n$ $\mathcal{H}$-strata of any given dimension. First suppose that we have an $m \times n$ diagram D such that the disjoint cycle decomposition of the toric permutation τ consists of exactly one cycle. Notice that there exists some $k \geq 1$ such that we can write $\tau = (R_1, C_1, R_2, \ldots, R_k, C_k)$ where the R_i form a partition of $[m]$ and each R_i is written in increasing order, while the C_i form a partition of $m + [n]$ and each C_i is written in decreasing order. On the other hand, by Lemma 5.10 and the work of Launois [29], there exists a unique Cauchon diagram (and hence $\mathcal{H}$-stratum) for every such permutation. Without loss of generality we always assume that $1 \in R_1$. Recall that the Stirling number of the second kind $\left\{ {n \atop k} \right\}$ counts the number of partitions of an n-set into k non-empty parts. Thus the number $d_{m,n}$ of $m \times n$ diagrams whose toric permutation τ consists of exactly one cycle is exactly

$$d_{m,n} = \sum_{k=1}^{\min(m,n)} k!(k-1)! \left\{ {m \atop k} \right\} \left\{ {n \atop k} \right\}.$$

If we let $\mathcal{D}(x,y) := \sum_{m,n \geq 1} d_{m,n} \frac{x^m}{m!} \frac{y^n}{n!}$ be the exponential generating function of $d_{m,n}$, then a direct application of the well-known exponential formula [39, 43] yields that the exponential generating function $C(x,y)$ of toric permutations (and hence of Cauchon diagrams) satisfies

$$C(x,y) = \exp(x+y)\exp(\mathcal{D}(x,y)), \tag{5.5}$$

where the extra $\exp(x+y)$ is the exponential generating function for the all-black diagrams (i.e., the identity toric permutation).

We may refine $\mathcal{D}(x,y)$ further by noting that

$$\mathcal{D}_e(x,y) = \frac{1}{2}(\mathcal{D}(x,y) - \mathcal{D}(-x,-y))$$

is the generating function for the even toric cycles, while

$$\mathcal{D}_o(x,y) = \frac{1}{2}(\mathcal{D}(x,y) + \mathcal{D}(-x,-y))$$

is the generating function for the odd toric cycles.

Therefore, if $C(x,y,t)$ is the generating function where the coefficient of $\frac{x^n}{n!}\frac{y^m}{m!}t^k$ is the number of $m \times n$ Cauchon diagrams with k odd cycles in the toric permutation, then

$$\begin{aligned} C(x,y,t) &= \exp(x+y+\mathcal{D}_e(x,y)+t\mathcal{D}_o(x,y)) \\ &= \exp(x+y)\exp(\mathcal{D}(x,y))^{\frac{t+1}{2}}\exp(\mathcal{D}(-x,-y))^{\frac{t-1}{2}}. \end{aligned} \tag{5.6}$$

On the other hand, by Theorem 5.4,

$$C(x,y) = \frac{e^{x+y}}{e^x+e^y-e^{x+y}}. \tag{5.7}$$

Theorem 5.19. *If $h(m,n,d)$ is the number of $m \times n$ Cauchon diagrams whose toric permutation has d odd cycles, then $C(x,y,t) = \sum_{m,n,d} h(m,n,d)t^d \frac{x^m}{m!}\frac{y^n}{n!}$ satisfies*

$$C(x,y,t) = (e^{-y}+e^{-x}-1)^{\frac{-1-t}{2}}(e^x+e^y-1)^{\frac{1-t}{2}}. \tag{5.8}$$

Proof. Comparing Equations 5.5 and 5.7 we see that

$$\begin{aligned} \exp(D(x,y)) &= (e^x+e^y-e^{x+y})^{-1} \\ &= e^{-x-y}(e^{-y}+e^{-x}-1)^{-1}. \end{aligned}$$

Substituting the latter equality into Equation 5.6 leads to Equation 5.9, as desired. □

Corollaries 5.16 and 5.19 immediately give the next result.

Corollary 5.20. *If $h(m,n,d)$ is the number of d-dimensional $\mathcal{H}$-strata in the prime spectrum of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$, then $C(x,y,t) = \sum_{m,n,d} h(m,n,d)t^d \frac{x^m}{m!}\frac{y^n}{n!}$ satisfies*

$$C(x,y,t) = (e^{-y}+e^{-x}-1)^{\frac{-1-t}{2}}(e^x+e^y-1)^{\frac{1-t}{2}}. \tag{5.9}$$

□

Bell, Launois & Nguyen [4] and Bell, Launois & Lutley [3] have given exact formula for the number of $2 \times n$ and $3 \times n$ primitive $\mathcal{H}$-primes respectively. Furthermore, Bell and Launois [2] conjectured the following.

Conjecture 5.21. *Let m be a natural number. Recall that $h(m,n,d)$ is the number of d-dimensional $\mathcal{H}$-strata in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$, while $C(m,n)$ is the total number of $\mathcal{H}$-strata in $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$. For all $1 \le d \le m$,*

$$\lim_{n\to\infty} \frac{h(m,n,d)}{C(m,n)} \to 2^{1-\delta_{d,0}} \binom{2m}{m+d} 4^{-m}.$$

We are now in a position to show that this conjecture is true for $d=0$ while, for $d>0$, it is not true in general. Fortunately, we are able to give the correct value of the limit (Corollary 5.23) for all m and d. Let us first give the following result. We note that the first part of the following theorem is a generalization of Conjecture 2.9 in [4] (where it is conjectured in the case $d=0$).

Theorem 5.22. *If $m,n>0$ and $d \ge 0$ are integers, then for all integers $k \in \{1-m\dots m+1\}$ there exist rational numbers $c_k(m,d)$ such that the number $h(m,n,d)$ of $m \times n$ d-dimensional $\mathcal{H}$-strata in the prime spectrum of $\mathcal{O}_q(\mathrm{M}_{m,n}(\mathbb{K}))$ is given by*

$$h(m,n,d) = \sum_{k=1-m}^{m+1} c_k(m,d) k^n.$$

Moreover, if $a(d) = [t^d](t+1)(t+3)\cdots(t+2m-1)$, then

$$c_{m+1}(m,d) = 2^{-m} a(d).$$

Proof. Before we begin, the reader may wish to review Proposition 2.10 and Conventions 2.8. Note that we have

$$\begin{aligned}
(e^x+e^y-1)^{\frac{1-t}{2}} &= (1+e^x-1+e^y-1)^{\frac{1-t}{2}}\\
&= \sum_{k\ge 0} \binom{\frac{1}{2}(1-t)}{k} (e^x-1+e^y-1)^k\\
&= \sum_{k\ge 0} \binom{\frac{1}{2}(1-t)}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} (e^x-1)^\ell (e^y-1)^{k-\ell}\\
&= \sum_{k\ge 0} \binom{\frac{1}{2}(1-t)}{k} \sum_{\ell=0}^{k} k! \left(\sum_{m=\ell}^{\infty} \left\{ {m \atop \ell} \right\} \frac{x^m}{m!}\right) \left(\sum_{n=k-\ell}^{\infty} \left\{ {n \atop k-\ell} \right\} \frac{y^n}{n!}\right)\\
&= \sum_{k\ge 0} \sum_{\ell=0}^{k} \sum_{m=\ell}^{\infty} \sum_{n=k-\ell}^{\infty} \left(\frac{1}{2}(1-t)\right)_k \left\{ {m \atop \ell} \right\} \left\{ {n \atop k-\ell} \right\} \frac{x^m}{m!} \frac{y^n}{n!}.
\end{aligned}$$

Thus if we set $f(m,n) = \left[\frac{x^m}{m!}\frac{y^n}{n!}\right](e^x + e^y - 1)^{\frac{1-t}{2}}$, then

$$\begin{aligned}
f(m,n) &= \sum_{\ell=0}^{m}\sum_{k=\ell}^{n+\ell}\left(\frac{1}{2}(1-t)\right)_k \left\{ m \atop \ell \right\}\left\{ n \atop k-\ell \right\} \\
&= \sum_{\ell=0}^{m}\sum_{k=0}^{n}\left(\frac{1}{2}(1-t)\right)_{\ell+k} \left\{ m \atop \ell \right\}\left\{ n \atop k \right\} \\
&= \sum_{\ell=0}^{m}\sum_{k=0}^{n}\left(\frac{1}{2}(1-t)\right)_{\ell}\left(\frac{1}{2}(1-t)-\ell\right)_k \left\{ m \atop \ell \right\}\left\{ n \atop k \right\} \\
&= \sum_{\ell=0}^{m}\left(\frac{1}{2}(1-t)\right)_{\ell}\left\{ m \atop \ell \right\}\sum_{k=0}^{n}\left(\frac{1}{2}(1-t)-\ell\right)_k \left\{ n \atop k \right\} \\
&= \sum_{\ell=0}^{m}\left(\frac{1}{2}(1-t)\right)_{\ell}\left\{ m \atop \ell \right\}\left(\frac{1}{2}(1-t)-\ell\right)^n .
\end{aligned}$$

Similarly, if we set $g(m,n) = \left[\frac{x^m}{m!}\frac{y^n}{n!}\right](e^{-x} + e^{-y} - 1)^{\frac{-1-t}{2}}$, then

$$g(m,n) = \sum_{\ell=0}^{m}\left(\frac{-1}{2}(1+t)\right)_{\ell}\left\{ m \atop \ell \right\}(-1)^{m+n}\left(\frac{-1}{2}(1+t)-\ell\right)^n .$$

Hence

$$
\begin{aligned}
\left[\frac{x^m}{m!}\frac{y^n}{n!}\right] C(x,y,t) &= \sum_{m'=0}^{m}\sum_{n'=0}^{n}\binom{m}{m'}\binom{n}{n'}\sum_{\ell_1=0}^{m'}\left(\frac{1}{2}(1-t)\right)_{\ell_1}\left\{ {m' \atop \ell_1} \right\}\left(\frac{1}{2}(1-t)-\ell_1\right)^{n'} \\
&\quad \cdot \sum_{\ell_2=0}^{m-m'}\left(\frac{-1}{2}(1+t)\right)_{\ell_2}\left\{ {m-m' \atop \ell_2} \right\}(-1)^{m-m'+n-n'} \\
&\quad \cdot \left(\frac{-1}{2}(1+t)-\ell_2\right)^{n-n'} \\
&= \sum_{m'=0}^{m}\sum_{\ell_1=0}^{m'}\sum_{\ell_2=0}^{m-m'}\binom{m}{m'}\left\{ {m' \atop \ell_1} \right\}\left\{ {m-m' \atop \ell_2} \right\}(-1)^{m-m'}\left(\frac{1}{2}(1-t)\right)_{\ell_1} \\
&\quad \cdot \left(\frac{-1}{2}(1+t)\right)_{\ell_2}\sum_{n'=0}^{n}\binom{n}{n'}\left(\frac{1}{2}(1-t)-\ell_1\right)^{n'} \\
&\quad \cdot \left(\frac{1}{2}(1+t)+\ell_2\right)^{n-n'} \\
&= \sum_{m'=0}^{m}\sum_{\ell_1=0}^{m'}\sum_{\ell_2=0}^{m-m'}\binom{m}{m'}\left\{ {m' \atop \ell_1} \right\}\left\{ {m-m' \atop \ell_2} \right\}(-1)^{m-m'} \\
&\quad \cdot \left(\frac{1}{2}(1-t)\right)_{\ell_1}\left(\frac{-1}{2}(1+t)\right)_{\ell_2} \\
&\quad \cdot \left(\frac{1}{2}(1-t)-\ell_1+\frac{1}{2}(1+t)+\ell_2\right)^{n} \\
&= \sum_{m'=0}^{m}\sum_{\ell_1=0}^{m'}\sum_{\ell_2=0}^{m-m'}\binom{m}{m'}\left\{ {m' \atop \ell_1} \right\}\left\{ {m-m' \atop \ell_2} \right\}(-1)^{m-m'} \\
&\quad \cdot \left(\frac{1}{2}(1-t)\right)_{\ell_1}\left(\frac{-1}{2}(1+t)\right)_{\ell_2}(1-\ell_1+\ell_2)^{n}. \qquad (5.10)
\end{aligned}
$$

Note that within the indices of summation we have the bounds $0 \le \ell_1 \le m$ and $0 \le \ell_2 \le m$ and so $1-m \le 1-\ell_1+\ell_2 \le 1+m$. Therefore, since $h(m,n,d) = [t^d]\left[\frac{x^m}{m!}\frac{y^n}{n!}\right] C(x,y,t)$, the first conclusion in the theorem statement follows from 5.10.

Now notice that $1-\ell_1+\ell_2 = 1+m$ if and only if $m'=0$ (and thus $\ell_1 = 0$) and

$\ell_2 = m$. Therefore,

$$\begin{aligned} c_{m+1}(m,d) &= [t^d](-1)^m\left(\frac{-1}{2}(1+t)\right)_m \\ &= (-1)^m\left(\frac{-1}{2}\right)^m[t^d](t+1)(t+3)\cdots(t+2m-1) \\ &= 2^{-m}[t^d](t+1)(t+3)\cdots(t+2m-1). \end{aligned}$$

$\square$

Formula 5.10 arrived at the in the proof of Theorem 5.22 is amenable to computation in Maple. For posterity we give some examples. Note that $h(2,n,0)$ appears in [4], while $h(3,n,0)$ appears in [3].

(m,d)	$h(m,n,d)$
$(2,0)$	$\frac{3}{4}3^n - \frac{1}{2}2^n + \frac{1}{2} - \frac{1}{4}(-1)^n$
$(2,1)$	$3^n - \frac{1}{2}2^n$
$(2,2)$	$\frac{1}{4}3^n - \frac{1}{2} + \frac{1}{4}(-1)^n$
$(3,0)$	$\frac{15}{8}\,4^n - \frac{9}{4}\,3^n + \frac{13}{8}\,2^n - \frac{3}{4}\,(-1)^n + \frac{3}{8}\,(-2)^n$
$(3,3)$	$\frac{1}{8}4^n - \frac{3}{8}2^n - \frac{1}{8}(-2)^n$
$(4,0)$	$\frac{105}{16}5^n - \frac{45}{4}4^n + 93^n - \frac{11}{4}2^n - \frac{5}{8} - (-1)^n\,(-2)^n\,\frac{15}{16}\,(-3)^n$
$(4,4)$	$\frac{1}{16}5^n - \frac{1}{4}3^n + \frac{3}{8} - \frac{1}{4}(-1)^n + \frac{1}{16}(-3)^n$
$(5,0)$	$\frac{945}{32}6^n - \frac{525}{8}5^n + \frac{2025}{32}\,4^n - (30)3^n + \frac{23}{16}2^n + \frac{225}{32}\,(-2)^n - \frac{75}{8}\,(-3)^n + \frac{105}{32}\,(-4)^n$

Corollary 5.23. *Fix a positive integer m and let $a(d) = [t^d](t+1)(t+3)\cdots(t+2m-1)$. The proportion of d-dimensional $\mathcal{H}$-strata in $\mathcal{O}_q(M_{m,n})$ tends to $a(d)/(m!2^m)$ as $n \to \infty$.*

Proof. Note that for fixed m, it is easily seen from Equation 2.1 that the Stirling number of the second kind satisfies $\left\{ {n \atop m} \right\} \sim m^n/m!$ as $n \to \infty$. From this we deduce that for $k < m$, $\left\{ {n \atop k} \right\}/\left\{ {n \atop m} \right\} \to 0$ as $n \to \infty$. Now recall that, for $n \geq m$, the total number of $\mathcal{H}$-primes is the poly-Bernoulli number

$$B_n^{-m} = \sum_{k=0}^{m}(k!)^2\left\{ {n+1 \atop k+1} \right\}\left\{ {m+1 \atop k+1} \right\}.$$

Thus as $n \to \infty$,

$$\begin{aligned} B_n^{-m} &\sim (m!)^2 \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \\ &\sim (m!)(m+1)^n. \end{aligned}$$

Therefore, by Theorem 5.22, we have

$$\lim_{n\to\infty} \frac{\#\ d\text{-dimensional } \mathcal{H}\text{-strata in } \mathcal{O}_q(M_{m,n})}{\#\text{-}\mathcal{H}\text{-strata in } \mathcal{O}_q(M_{m,n})} = \frac{a(d)}{m!2^m}.$$

$\square$

Note that for $(m,d) = (3,1)$, Corollary 5.23 gives the asymptotic proportion as $23/48$, while Conjecture 5.21 gives the value $15/32$, thus refuting the conjecture for $d > 0$.

5.6 Open Question

While the question of how to determine the dimension of an $\mathcal{H}$-stratum seems to be settled, there are still some intriguing open problems. For example, a more sophisticated analysis of the statistics of toric permutations would be of interest. In particular, what is

$$\lim_{n\to\infty} \frac{\#\ d\text{-dimensional } \mathcal{H}\text{-primes in } \mathcal{O}_q(\mathrm{M}_{n,n}(\mathbb{K}))}{\#\ \mathcal{H}\text{-primes in } \mathcal{O}_q(\mathrm{M}_{n,n}(\mathbb{K}))}?$$

We believe that for $d = 0$ the limit should be zero.

Chapter 6

A Generalization of Theorem 5.15

6.1 Introduction

The purpose of this chapter is to give a recently obtained generalization (Theorem 6.11) of Theorem 5.15 in the context of *quantized enveloping algebras* of simple Lie algebras. The proof of this joint work with Jason Bell and Stéphane Launois [5, 30]. While we do not present any direct applications here, it is an ongoing project to apply this theorem to obtain new enumeration formulae for various quantum algebras. The details will be presented in a forthcoming paper.

For this chapter, it may be helpful (although not strictly necessary) if the reader is familiar with the elementary theory of Lie algebras. In particular, it would be most beneficial if the reader is practiced with calculations in root systems and their associated Weyl groups. In any case, we begin by giving a quick review of these concepts. See the book by Humphreys [24] for a good exposition of the theory. We then quickly define the quantized enveloping algebra, and a subclass of these algebras which will be of interest. Finally, we prove the generalization in the final two sections.

6.2 Root Systems

Let E^d be Euclidean d-space, i.e., the vector space $\mathbb{R}^d$ equipped with the standard inner product $\langle \cdot \mid \cdot \rangle$.

Definition 6.1. Let $\mathbf{\Phi}$ be a finite set of vectors (which we'll call *roots*) in E^d. We call $\mathbf{\Phi}$ a *root system* if it satisfies the following properties:

1. $\text{span}(\mathbf{\Phi}) = E^d$;

2. If $\alpha \in \mathbf{\Phi}$ then $k\alpha \in \mathbf{\Phi}$ if and only if $k = \pm 1$;

3. For any roots $\alpha, \beta \in \mathbf{\Phi}$, the reflection $s_\alpha(\beta)$ of β through the hyperplane perpendicular to α is in $\mathbf{\Phi}$. In other words,
$$s_\alpha(\beta) = \beta - \frac{2\langle \alpha \mid \beta \rangle}{\langle \alpha \mid \alpha \rangle}\alpha \in \mathbf{\Phi};$$

4. For any $\alpha, \beta \in \mathbf{\Phi}$,
$$\frac{2\langle \alpha \mid \beta \rangle}{\langle \alpha \mid \alpha \rangle} \in \mathbb{Z}.$$

Next we collect some elementary and well-known data about root systems.

Definitions and Facts 6.2. Suppose that $\mathbf{\Phi}$ is a root system of E^d.

1. The group consisting of the reflections through hyperplanes perpendicular to a root is the *Weyl group* of $\mathbf{\Phi}$. Note that these reflections are isometries, i.e., $\langle \alpha \mid \alpha' \rangle = \langle s(\alpha) \mid s(\alpha') \rangle$ for all roots α and α' and reflections s.

2. A root system $\mathbf{\Phi}$ is *irreducible* if there does not exist a partition of $\mathbf{\Phi}$ into two sets such that their span are mutually orthogonal subspaces of E^d.

3. Let $\mathbf{\Phi}_+$ be a subset of $\mathbf{\Phi}$ that is closed under addition and such that $\alpha \in \mathbf{\Phi}_+$ if and only if $-\alpha \notin \mathbf{\Phi}_+$. Such a subset (which always exists, but is usually not unique) is called a set of *positive roots* of $\mathbf{\Phi}$. The compliment $\mathbf{\Phi} \setminus \mathbf{\Phi}_+ = \mathbf{\Phi}_-$ is a set of *negative roots*.

4. Given a set $\mathbf{\Phi}_+$ of positive roots, we call $\alpha \in \mathbf{\Phi}_+$ *simple* if it cannot be written as a linear combination of at least two elements in $\mathbf{\Phi}_+$. The simple roots form a basis of E^d such that every element of $\mathbf{\Phi}_+$ can be written as a linear combination of simple roots with positive coefficients, and every element of $\mathbf{\Phi}_-$ can be written as a linear combination of simple roots with negative coefficients.

5. Given a set of simple roots $\{\alpha_1, \ldots, \alpha_d\}$ of $\mathbf{\Phi}_+$, the *Cartan matrix* C is defined by
$$C[i,j] := \frac{2\langle \alpha_i \mid \alpha_j \rangle}{\langle \alpha_i \mid \alpha_i \rangle}.$$
It is known that the Cartan matrix always satisfies $c_{i,j}c_{j,i} = 0, 1, 2$ or 3 for all $i \neq j$.

6. If $\alpha \in \mathbf{\Phi}$, then the *coroot* $\alpha^\vee$ is the element
$$\alpha^\vee = \frac{2}{\langle \alpha \mid \alpha \rangle} \alpha.$$

7. The *root lattice* of a set $\{\alpha_1, \ldots, \alpha_d\}$ of simple roots is
$$Q = \bigoplus_{i=1}^{d} \mathbb{Z}\alpha_i.$$

It is known that to any simple Lie algebra there corresponds an irreducible root system. The rather strict conditions in Definition 6.1 allows the complete classification of irreducible root systems (and thus simple Lie algebras) into eight types: A_n ($n \geq 1$); B_n ($n \geq 2$); C_n ($n \geq 3$); D_n ($n \geq 4$); E_6; E_7; E_8; F_4 and G_2.

Remark 6.3. Given a set of simple roots $\{\alpha_1, \ldots, \alpha_d\}$ of a root system, let s_i denote the reflection map that corresponds to α_i. The Weyl group W can be presented as
$$W = \langle s_1, \ldots, s_d \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1, i \neq j \rangle,$$
where $m_{ij} = 2, 3, 4$ or 6, ($i \neq j$) according to the respective value of $c_{i,j}c_{j,i} = 0, 1, 2$ or 3. We call the generators in this presentation *simple reflections*.

The *length* of a word $w \in W$ is the minimum number $\ell(w)$ such that we can write $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ as a product of simple reflections. It is known that this is a well-defined notion, and that in W there is a unique longest element whose length is the number of positive roots.

Fix a set $\{\alpha_1, \ldots, \alpha_d\}$ of simple roots. The *fundamental weights* are defined to be those elements $\{\omega_1, \ldots, \omega_d\}$ in the dual space of Q defined by
$$\langle \omega_j \mid \alpha_i^\vee \rangle = \delta_{i,j}.$$

The *weight lattice* is

$$P = \bigoplus_{i=1}^{d} \mathbb{Z}\omega_i.$$

It is known that for every α_j, we may write

$$\alpha_j = \sum_i c_{i,j}\omega_i.$$

Thus $P \subseteq Q$.

6.3 The Quantized Enveloping Algebra

For this section, we'll need a notion of q-factorials and q-binomial coefficients. For an integer n and nonzero $q \neq \pm 1$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The q-factorial is

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q.$$

For a nonnegative integer k, the q-binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Let us fix a complex simple Lie algebra $\mathfrak{g}$, with simple roots $\{\alpha_1, \ldots, \alpha_d\}$, Weyl algebra W, and Cartan matrix C, where we let $C[i,j] := c_{i,j}$. Let $\mathbb{K}$ be a field of charateristic zero and take $q \in \mathbb{K}^*$. For each $1 \in [d]$, let $d_i := \frac{1}{2}\langle \alpha_i, \alpha_i \rangle$ and set

$$q_i = q^{d_i}.$$

We assume, in addition, that $q_i \neq \pm 1$ for any $i \in [d]$.

Definition 6.4. Fix a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ of a simple Lie algebra $\mathfrak{g}$. In the following, $X_i := X_{\alpha_i}$ for $X = E, F$ or K. The *quantized enveloping algebra* $U_q(\mathfrak{g})$ is the $\mathbb{K}$-algebra with generators $E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm}1, \ldots, K_n^{\pm 1}$ which satisfy the following relations for all i, j:

1. $K_i E_j K_i^{-1} = q_i^{c_{i,j}} E_j$ and $K_i F_j K_i^{-1} = q_i^{-c_{i,j}} F_j$;

2. $K_i K_j = K_j K_i$;

3. $$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}};$$

4. (The quantized Serre relations) For $i \neq j$,
$$\sum_{l=0}^{1-c_{i,j}} (-1)^l \begin{bmatrix} 1 - c_{i,j} \\ l \end{bmatrix}_{q_i} E_i^{1-c_{i,j}-l} E_j E_i^l = 0$$
and
$$\sum_{l=0}^{1-c_{i,j}} (-1)^l \begin{bmatrix} 1 - c_{i,j} \\ l \end{bmatrix}_{q_i} F_i^{1-c_{i,j}-l} F_j F_i^l = 0.$$

Denote by $U_q^+(\mathfrak{g})$ the subalgebra generated by the E_i.

Let
$$E_i^{(k)} := \frac{E_i^k}{[k]_{q_i}!} \quad \text{and} \quad F_i^{(k)} := \frac{F_i^k}{[k]_{q_i}!}.$$

Definition 6.5. The *braid group* $\mathcal{T}_W$ is the group generated by $T_1, \ldots, T_d$ that satisfy the same relations as in the presentation of W in Remark 6.3, except we omit the involution relation. We set an action of $\mathcal{T}_W$ on $U_q(\mathfrak{g})$ by the following rules:

$$\begin{aligned}
T_i(E_j) &= \sum_{s=0}^{c_{i,j}} (-1)^{s-c_{i,j}} q_i^{-s} E_i^{(-c_{i,j}-s)} E_j E_i^{(s)} (j \neq i); \\
T_i(F_j) &= \sum_{s=0}^{c_{i,j}} (-1)^{s-c_{i,j}} q_i^{-s} F_i^{(s)} F_j F_i^{(-c_{i,j}-s)} (j \neq i); \\
T_i(E_i) &= -F_i K_1; \\
T_i(F_i) &= -K_i^{-1} E_i; \\
T_i(K_j) &= K_{s_i(\alpha_j)},
\end{aligned}$$

where, for $\beta = \sum_\ell m_\ell \alpha_\ell \in Q$,
$$K_\beta := \prod_\ell K_\ell^{m_\ell}.$$

We now give the definition of the algebra which is the main concern of this chapter.

Definition 6.6. Let $w = s_{i_1}s_{i_2}\cdots s_{i_t}$ be a *reduced* word in the Weyl algebra. The algebra $U_q^w(\mathfrak{g})$, which we'll write as $U^+(w)$, is the subalgebra of $U_q^+(\mathfrak{g})$ generated by the E_{β_j} defined as follows. Set $E_{\beta_1} = E_{\alpha_1}$ and for $j > 1$, $E_{\beta_j} = T_{i_1}\cdots T_{i_{j-1}}(E_{\alpha_j})$.

Let us define $\beta_i \in Q$ as follows. Set $\beta_1 := \alpha_{i_1}$ and for $j > 1$ we set

$$\beta_j = s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j}).$$

Theorem 6.7 (Lusztig [35], Levendorskii-Soibelman [33])**.** *With regards to Definition 6.6 the following hold:*

1. *For all j, $E_{\beta_j} \in U_q^+(\mathfrak{g})$. Thus $U^+(w)$ is a subalgebra of $U_q^+(\mathfrak{g})$;*

2. *The algebra $U^+(w)$ does not depend on the reduced expression for w;*

3. *If w_0 is the longest word in the Weyl group, $U^+(w_0) = U_q^+(\mathfrak{g})$;*

4. *For all $i < j$,*
$$E_{\beta_i}E_{\beta_j} - q^{\langle\beta_i|\beta_j\rangle} = \sum z_{\boldsymbol{k}}\boldsymbol{E}^{\boldsymbol{k}},$$
where $\boldsymbol{E}^{\boldsymbol{k}} := E_{\beta_1}^{k_1}\cdots E_{\beta_t}^{k_t}$, $z_{\boldsymbol{k}} \in \mathbb{Q}[q^{\pm 1}]$ and $z_{\boldsymbol{k}} = 0$ unless $\boldsymbol{k}_r = 0$ for $r \leq i$ and $r \geq j$;

5. *The algebra $U^+(w)$ can be written as an iterated Ore extension. There is a rational action of an algebraic torus $\mathcal{H}$ by automorphisms.*

Mériaux, in his PhD thesis (see [37, 36]), considers the case that $w = w_0$ is the longest word written in a particular way (a so-called "good" ordering). In this case, one may apply Cauchon's deleting derivations algorithm and the $\mathcal{H}$-stratification theory to $U_q(\mathfrak{g})$ to obtain information about the prime spectrum. The "Cauchon diagrams" now correspond simply to *subwords* of w. For this reason we will call *any* subword of w a *diagram*.

Example 6.8. Let $\mathfrak{g}$ be of type A_{n+m-1}, with a corresponding set of simple roots $\{\alpha_1, \ldots, \alpha_{n+m-1}\}$. Arrange the roots in an $m \times n$ grid as in Figure 6.1). We let

$$w_0 = s_m(s_{m-1}s_{m+1})(s_{m-2}s_m s_{m+2}) \cdots (s_{n-1})(s_{n+1})s_n,$$

which is essentially obtained by starting at the upper-left corner of the grid in Figure 6.1, and then tracing down through the consecutive anti-diagonals.

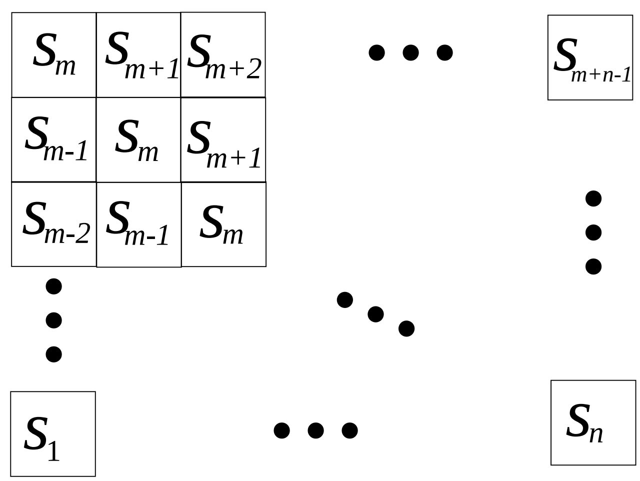


Figure 6.1: Special ordering of Weyl group generators

Notice that a diagram corresponds to a subset of squares (the "black" squares).

6.4 Some Lemmas

Fix a root system $\mathbf{\Phi}$, a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$, and the corresponding Weyl group W generated by the simple reflections $\{s_1, \ldots, s_n\}$ where $s_i := s_{\alpha_i}$. Let

$$w = s_{i_1}s_{i_2} \cdots s_{i_k}$$

be any word in W.

Notation 6.9. Throughout this section and the next, we set the following notation.

1. Fix $\Delta \subseteq \{i_1, i_2, \ldots, i_k\}$ and $\{t_1, \ldots, t_m\} = B \setminus \Delta$.

2. For all $j \in [k]$, we set
$$s_{i_j}^{\Delta} := \begin{cases} s_{i_j} & \text{if } j \in \Delta, \\ \text{id} & \text{otherwise.} \end{cases}$$

3. $w^{\Delta} := s_{i_1}^{\Delta} \cdots s_{i_k}^{\Delta} \in W$.

4. $\beta_1^{\Delta} := \alpha_{i_1}$ and for $j > 1$,
$$\beta_j^{\Delta} = s_{i_1}^{\Delta} \cdots s_{i_{j-1}}^{\Delta}(\alpha_{i_j}).$$

Now consider the map $w + w^{\Delta} : Q \to Q$. Notice that there exist (possibly empty) words $w_0, \ldots, w_m \in W$ such that

$$w^{\Delta} = w_0 w_1 \cdots w_m$$

and

$$w = w_0 s_{j_1} w_1 s_{j_2} w_2 \cdots s_{j_m} w_m,$$

where we write $j_\ell := i_{t_\ell}$.

Notation 6.10. For $w = w_0 s_{j_1} w_1 s_{j_2} w_2 \cdots s_{j_m} w_m$ we fix the following notation.

1. For all $1 \leq \ell \leq m$,
$$u_\ell^{\Delta} := w_0 s_{j_1} w_1 s_{j_2} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1}(s_{j_\ell} - \text{id}) w_\ell w_{\ell+1} \cdots w_m.$$

2. For all $1 \leq \ell \leq m$,
$$v_\ell^{\Delta} := w_0 s_{j_1} w_1 s_{j_2} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1}(s_{j_\ell} + \text{id}) w_\ell w_{\ell+1} \cdots w_m.$$

3. We abuse notation and write, for each j_ℓ, β_{j_ℓ} to be that β_i where i is the position of j_ℓ in the presentation of w as simple reflections. We define $\beta_{j_\ell}^{\Delta}$ similarly.

Let $M(w^\Delta)$ be the $m \times m$ skew-symmetric matrix defined by setting the (i, ℓ)-entry ($i < \ell$) to be $\langle \beta_{j_i} | \beta_{j_\ell} \rangle$. We regard $M(w^\Delta)$ as a map from $\mathbb{Q}^m$ to itself. The main result of this chapter is a generalization of Theorem 5.15 which was obtained in close collaboration with Jason Bell and Stéphane Launois.

Theorem 6.11. *For any word w in the Weyl group and any diagram w^Δ of w,*

$$\ker(M(w^\Delta)) \simeq \ker(w + w^\Delta).$$

The remainder of this section consists of some technical lemmas used in the construction of the isomorphism needed to prove Theorem 6.11. We note that De Concini and Procesi [11] have proved the special case when $\Delta = \emptyset$, i.e., $w^\Delta = \mathrm{id}$, the identity map.

Lemma 6.12. *For $\ell \in [m]$ we have*

$$w + w^\Delta \;=\; v_\ell^\Delta - \sum_{h<\ell} u_h^\Delta + \sum_{h>\ell} u_h^\Delta.$$

Proof. Note that

$$\begin{aligned}
& \sum_{h<\ell} u_h^\Delta \\
= \;& \sum_{h<\ell} \big(w_0 s_{j_1} w_1 \cdots w_{h-1} s_{j_h} w_h \cdots w_m - w_0 s_{j_1} w_1 \cdots w_{h-2} s_{j_{h-1}} w_{h-1} w_h \cdots w_m \big) \\
= \;& w_0 s_{j_1} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1} w_\ell \cdots w_m - w_0 \cdots w_m \\
= \;& w_0 s_{j_1} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1} w_\ell \cdots w_m + w^\Delta.
\end{aligned}$$

Similarly,

$$\sum_{h>\ell} u_h^\Delta = w - w_0 s_{j_1} \cdots w_{\ell-1} s_{j_\ell} w_\ell w_{\ell+1} \cdots w_m.$$

Thus

$$\begin{aligned}
& \sum_{h>\ell} u_h^\Delta - \sum_{h<\ell} u_h^\Delta \\
= \;& w + w^\Delta - w_0 s_{j_1} \cdots w_{\ell-1} s_{j_\ell} w_\ell w_{\ell+1} \cdots w_m - w_0 s_{j_1} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1} w_\ell \cdots w_m \\
= \;& w + w^\Delta - v_\ell^\Delta.
\end{aligned}$$

The result follows. $\square$

Let $V := \sum_{i=1}^{n} \mathbb{Q} \cdot \omega_i \;=\; P \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 6.13. *If $\gamma \in V$, then for all $\ell \in [m]$ we have*

$$u_\ell^\Delta(\gamma) = -\langle (\beta_{j_\ell}^\Delta)^\vee | w^\Delta(\gamma) \rangle \beta_{j_\ell}$$

and

$$\langle v_\ell^\Delta(\gamma) | \beta_{j_\ell} \rangle = 0.$$

Proof. Recall that

$$u_\ell^\Delta = w_0 s_{j_1} w_1 s_{j_2} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1}(s_{j_\ell} - \mathrm{id}) w_\ell w_{\ell+1} \cdots w_m.$$

Let

$$\gamma' = w_\ell w_{\ell+1} \cdots w_m(\gamma).$$

Notice that $s_{j_\ell} - \mathrm{id}$ has kernel spanned by $\{\omega_i \mid i \neq j_\ell\}$. Also, notice that since $\langle \alpha_j^\vee \mid \gamma' \rangle$ is equal to the coefficient of ω_j in γ', we may write

$$\gamma' = \langle \alpha_{j_\ell}^\vee | \gamma' \rangle \omega_{j_\ell} + \delta,$$

with δ in the kernel of $s_{j_\ell} - \mathrm{id}$. Since $s_{j_\ell} - \mathrm{id}$ sends ω_{j_ℓ} to $-\alpha_{j_\ell}$, we see that

$$u_\ell^\Delta(\gamma) = -\langle \alpha_{j_\ell}^\vee | \gamma' \rangle w_0 s_{j_1} w_1 s_{j_2} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1}(\alpha_{j_\ell}),$$

which is just

$$-\langle \alpha_{j_\ell}^\vee | \gamma' \rangle \beta_{j_\ell}.$$

Since the operators s_{i_j} are isometries of V and its dual space, we see that

$$\begin{aligned} & \langle \alpha_{j_\ell}^\vee | \gamma' \rangle \\ = \;& \langle w_0 \cdots w_{\ell-1}(\alpha_{j_\ell}^\vee) | w_0 \cdots w_{\ell-1}(\gamma') \rangle \\ = \;& \langle (\beta_{j_\ell}^\Delta)^\vee | w_0 \cdots w_m(\gamma) \rangle \\ = \;& \langle (\beta_{j_\ell}^\Delta)^\vee | w^\Delta(\gamma) \rangle. \end{aligned}$$

Thus

$$u_\ell^\Delta(\gamma) := -\langle (\beta_{j_\ell}^\Delta)^\vee | w^\Delta(\gamma)\rangle \beta_{j_\ell}.$$

Observe that

$$v_\ell^\Delta(\gamma) \in w_0 s_{j_1} w_1 s_{j_2} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1}(s_{j_\ell} + \mathrm{id})(V).$$

If we consider the map $s_{j_\ell} + \mathrm{id} : Q \to Q$, then, for all k,

$$\begin{aligned} \langle (s_{j_\ell} + \mathrm{id})(\alpha_k) \mid \alpha_{j_l}\rangle &= \langle s_{j_l}(\alpha_k) \mid \alpha_{j_l}\rangle + \langle \alpha_k \mid \alpha_{j_l}\rangle \\ &= \langle \alpha_k \mid s_{j_l}(\alpha_{j_l})\rangle + \langle \alpha_k \mid \alpha_{j_l}\rangle \\ &= -\langle \alpha_k \mid \alpha_{j_l}\rangle + \langle \alpha_k \mid \alpha_{j_l}\rangle \\ &= 0. \end{aligned}$$

In other words, $(s_{j_\ell} + \mathrm{id})$ is simply a projection onto the orthogonal complement of $\mathbb{Q}\alpha_{j_\ell}$. It follows that

$$v_\ell^\Delta(\gamma) \in w_0 s_{j_1} w_1 s_{j_2} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1}(\langle \alpha_{j_\ell}\rangle^\perp).$$

In particular, $v_\ell^\Delta(\gamma)$ is orthogonal to $w_0 s_{j_1} w_1 s_{j_2} \cdots w_{\ell-2} s_{j_{\ell-1}} w_{\ell-1}(\alpha_{j_\ell}) = \beta_{j_\ell}$. This completes the proof. □

Let $\boldsymbol{e}_\ell$ be the vector in $\mathbb{Q}^m$ whose ℓ^{th} coordinate is 1 and all other coordinates are zero. Now for some statement X, we let $\chi(X) = 1$ if X is true, and 0 otherwise. As a result of the preceding lemmas, we obtain the next result.

Corollary 6.14. *For $1 \leq \ell \leq m$ and $\gamma \in V$ we have*

$$\langle (w + w^\Delta)(\gamma) | \beta_{j_\ell}\rangle \;=\; \sum_{h \neq \ell} (-1)^{\chi(h>\ell)} \langle \beta_{j_h} | \beta_{j_\ell}\rangle \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\gamma)\rangle.$$

Proof. Recall that Lemma 6.12 gives

$$w + w^\Delta = v_\ell^\Delta + \sum_{h \neq \ell} (-1)^{\chi(h<\ell)} u_h^\Delta.$$

We apply both sides to $\gamma \in V$, and then apply the operator $\langle \cdot | \beta_{j_\ell} \rangle$. By Lemma 6.13, this operator annihilates $v_\ell^\Delta(\gamma)$ and sends $u_h^\Delta(\gamma)$ to

$$-\langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\gamma) \rangle \langle \beta_{j_h} | \beta_{j_\ell} \rangle.$$

Thus

$$\langle (w + w^\Delta)(\gamma) | \beta_{j_\ell} \rangle = \sum_{h \neq \ell} (-1)^{\chi(h>\ell)} \langle \beta_{j_h} | \beta_{j_\ell} \rangle \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\gamma) \rangle,$$

as required. □

We consider the map $B : V \to \mathbb{Q}^m$ defined by

$$B(\gamma) = \sum_{\ell=1}^{m} \langle \gamma | \beta_{j_\ell} \rangle \boldsymbol{e}_\ell.$$

The bulk of the work needed to prove Theorem 6.11 is contained in the following result.

Theorem 6.15. *Let $f : \mathbb{Q}^m \oplus V \to \mathbb{Q}^m$ be defined by*

$$f(x, y) := \left(M(w^\Delta) \right)(x) + B(y).$$

The function f is surjective and a basis for the kernel is given by the elements of the form

$$\left(\sum_{h=1}^{m} \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\omega_j) \rangle \boldsymbol{e}_h, (w + w^\Delta)(\omega_j) \right),$$

for $j \in [n]$.

Proof. Note that

$$f(\boldsymbol{e}_h, \beta_{j_h}) = \sum_{\ell \neq h} (-1)^{\chi(\ell>h)} \langle \beta_{j_\ell}, \beta_{j_h} \rangle \boldsymbol{e}_\ell + \sum_{\ell=1}^{m} \langle \beta_{j_\ell}, \beta_{j_h} \rangle \boldsymbol{e}_\ell = \langle \beta_{j_h}, \beta_{j_h} \rangle \boldsymbol{e}_h + 2 \sum_{\ell<h} \langle \beta_{j_\ell}, \beta_{j_h} \rangle \boldsymbol{e}_\ell.$$

Since $\alpha \neq 0$ for any $\alpha \in \boldsymbol{\Phi}$, these elements span $\mathbb{Q}^m$ as h ranges from 1 to m. Therefore, f is onto. For this reason, we know that the dimension of the kernel of f is n. Thus to see that the elements

$$\left(\sum_{h=1}^{m} \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\omega_j) \rangle \boldsymbol{e}_h, (w + w^\Delta)(\omega_j) \right),$$

for $j = 1, 2, \dots, n$, form a basis, it is enough to show that they are linearly independent and are in the kernel. Note that by Corollary 6.14 we have that for any $\gamma \in V$,

$$B((w + w^{\Delta})(\gamma)) = \sum_{\ell=1}^{m} \left(\sum_{h \neq \ell} (-1)^{\chi(h>\ell)} \langle \beta_{j_h} | \beta_{j_\ell} \rangle \langle (\beta_{j_h}^{\Delta})^{\vee} | w^{\Delta}(\gamma) \rangle \right) \boldsymbol{e}_\ell.$$

But this is just

$$-M(w^{\Delta}) \left(\sum_{h=1}^{m} \langle (\beta_{j_h}^{\Delta})^{\vee} | w^{\Delta}(\gamma) \rangle \boldsymbol{e}_h \right),$$

and so these elements are indeed in the kernel.

Now for $a_i \in \mathbb{Q}$, $i \in [n]$, consider the linear combination

$$\begin{aligned} &\sum_{i=1}^{n} a_i \left(\sum_{h=1}^{m} \langle (\beta_{j_h}^{\Delta})^{\vee} | w^{\Delta}(\omega_j) \rangle \boldsymbol{e}_h, (w + w^{\Delta})(\omega_j) \right) \\ &= \left(\sum_{h=1}^{m} \langle (\beta_{j_h}^{\Delta})^{\vee} | w^{\Delta} \left(\sum_{i=1}^{n} a_i \omega_j \right) \rangle \boldsymbol{e}_h, (w + w^{\Delta}) \left(\sum_{i=1}^{n} a_i \omega_j \right) \right) \\ &= \left(\sum_{h=1}^{m} \langle (\beta_{j_h}^{\Delta})^{\vee} | w^{\Delta}(\gamma) \rangle \boldsymbol{e}_h, (w + w^{\Delta})(\gamma) \right) \end{aligned}$$

Therefore, to check that the kernel elements in the statement of the theorem are linearly independent, it suffices to show that if $\gamma \in \ker(w+w^{\Delta})$ and $\langle (\beta_{j_h}^{\Delta})^{\vee} | w^{\Delta}(\gamma) \rangle = 0$ for $h \in [m]$, then γ must be zero.

Suppose that $\gamma \in \ker(w + w^{\Delta})$ satisfies

$$\langle (\beta_{j_h}^{\Delta})^{\vee} | w^{\Delta}(\gamma) \rangle = 0$$

for $h \in [m]$. Equivalently,

$$\langle w_0 \cdots w_{h-1}(\alpha_{j_h}^{\vee}) | w_0 \cdots w_m(\gamma) \rangle = 0$$

for $h \in [m]$. Since the w_i are isometries, we see that $w_h \cdots w_m(\gamma)$ is annihilated by the linear functional $\alpha_{j_h}^{\vee}$ and hence

$$w_h \cdots w_m(\gamma) \in \mathrm{Span}\{\omega_i \mid i \neq j_h\} = \ker(s_{j_h} - \mathrm{id}).$$

In particular, $s_{j_h} w_h \cdots w_m(\gamma) = w_h \cdots w_m(\gamma)$ for $h \in [m]$.

Claim. $w(\gamma) = w^\Delta(\gamma)$.

Linear independence will follow since we already know that $w(\gamma) = -w^\Delta(\gamma)$ by assumption. For $h \in [m]$ we let

$$\gamma_h^\Delta = w_h \cdots w_m(\gamma)$$

and

$$\gamma_h = s_{j_h} w_h s_{j_{h+1}} w_{h+1} \cdots s_{j_m} w_m(\gamma).$$

We now show that for any $h \in [m]$, we have $\gamma_h^\Delta = \gamma_h$.

To this end, note that $s_{j_m} w_m(\gamma) = w_m(\gamma)$, and so $\gamma_m = \gamma_m^\Delta$. Suppose that there exists some h such that $\gamma_h^\Delta \neq \gamma_h$. If h is chosen to be maximal with respect to the property $\gamma_h^\Delta \neq \gamma_h$, then

$$\begin{aligned}
\gamma_h^\Delta &= w_h \cdots w_m(\gamma) \\
&= s_{j_h} w_h \cdots w_m(\gamma) \\
&= s_{j_h} w_h(\gamma_{h+1}^\Delta) \\
&= s_{j_h} w_h(\gamma_{h+1}) \\
&= \gamma_h.
\end{aligned}$$

This is a contradiction. Thus $\gamma_h^\Delta = \gamma_h$ for $1 \leq h \leq m$. In particular, if we take $h = 1$, then we see that

$$w^\Delta(\gamma) = w_0(\gamma_1^\Delta) = w_0(\gamma_1) = w(\gamma),$$

which proves the claim and the theorem.

□

6.5 Proof of Theorem 6.11

Proof of Theorem 6.11. Let $\Psi : \ker(w + w^\Delta) \to \ker(M(w^\Delta))$ be defined by setting

$$\Psi(\gamma) := \sum_{h=1}^{m} \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\gamma) \rangle \boldsymbol{e}_h.$$

Note that if γ is in the kernel of $w + w^\Delta$, then $\Psi(\gamma)$ is in $\ker(M(w^\Delta))$ by Theorem 6.15. Moreover, Ψ is surjective, since if x is in the kernel of $M(w^\Delta)$, then $f(x, 0)$ must be zero and by assumption $(x, 0)$ is in the span of the elements

$$\left(\sum_{h=1}^{m} \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\omega_j)\rangle \boldsymbol{e}_h, (w + w^\Delta)(\omega_j)\right),$$

for $j = 1, 2, \ldots, m$, which means that $x = \Psi(\gamma)$ for some γ in the kernel of $w + w^\Delta$. We show that Ψ is injective, which will complete the proof of Theorem 6.11.

For $\gamma \in \ker(w + w^\Delta)$ in the kernel of Ψ we have

$$\langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\gamma)\rangle = 0$$

for $1 \leq h \leq m$. We showed that this could not occur in our proof of the linear dependence of the kernel of the map f.

□

For completeness, we construct an inverse Φ to the Ψ from the previous proof. Define $\Phi : \mathbb{Q}^m \to V$ by

$$\Phi(\boldsymbol{e}_h) = \frac{1}{2}(w^\Delta)^{-1}(\beta_{j_h})$$

for $h = 1, \ldots, m$. To check that this is indeed an inverse map, note that since the map Ψ is onto, if $x = \sum_{h=1}^{m} c_h \boldsymbol{e}_h \in \ker(M(w^\Delta))$, then there is some γ in $\ker(w + w^\Delta)$ such that

$$c_h = \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\gamma)\rangle.$$

Now one can easily use the definitions to check that $u_1^\Delta = v_1^\Delta - 2w^\Delta$. But we know

$(w + w^\Delta)(\gamma) = 0$. These facts, together with Lemma 6.12, give us

$$\begin{aligned}
-2w^\Delta(\gamma) &= (w + w^\Delta - 2w^\Delta)(\gamma) \\
&= (v_1^\Delta + \sum_{h>1} u_h^\Delta - 2w^\Delta)(\gamma) \\
&= \sum_{h=1}^{m} u_h^\Delta(\gamma) \\
&= -\sum_{h=1}^{m} \langle (\beta_{j_h}^\Delta)^\vee | w^\Delta(\gamma) \rangle \beta_{j_h} \\
&= -\sum_{h=1}^{m} c_h \beta_{j_h}.
\end{aligned}$$

Thus

$$\gamma = \frac{1}{2}\left((w^\Delta)^{-1} \left(\sum_{h=1}^{m} c_h \beta_{j_h} \right) \right) = \Phi(x).$$

Thus we have shown that

$$\Psi \circ \Phi = \mathrm{id}|_{\ker(w+w^\Delta)},$$

and so we immediately obtain the following theorem.

Theorem 6.16. *For the functions Ψ and Φ defined above, $\Psi \circ \Phi = \mathrm{id}|_{\ker(w+w^\Delta)}$ and $\Phi \circ \Psi = \mathrm{id}|_{\ker(M(w^\Delta))}$.*

Example 6.17. Suppose that $\mathfrak{g}$ is of type A_4 with

$$w = s_2 s_1 s_3 s_2 s_4 s_3.$$

For the calculations in this example, we'll use the Cartan matrix

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Thus we are working in an algebra isomorphic to $\mathcal{O}_q(\mathrm{M}_{2,3}(\mathbb{K}))$. We have

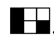
$$\begin{aligned}
\beta_1 &= \alpha_2; \\
\beta_2 &= \alpha_1 + \alpha_2; \\
\beta_3 &= \alpha_2 + \alpha_3; \\
\beta_4 &= \alpha_1 + \alpha_2 + \alpha_3; \\
\beta_5 &= \alpha_2 + \alpha_3 + \alpha_4; \\
\beta_6 &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.
\end{aligned}$$

Let $\Delta = \{1, 2, 6\}$. then $w^\Delta = s_2 s_1 s_3$. We may write

$$w = w_0 s_3 w_1 s_2 w_2 s_4 w_3$$

where $w_0 = s_1 s_2$, $w_1 = w_2 = \mathrm{id}$ and $w_3 = s_3$. In this case,

$$M(w^\Delta) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

The matrix $M(w^\Delta)$ is, perhaps unsurprisingly, the matrix $M(D)$ from Chapter 5 corresponding to the diagram . We now record the β_i^Δ:

$$\begin{aligned}
\beta_1^\Delta &= \alpha_2; \\
\beta_2^\Delta &= \alpha_1 + \alpha_2; \\
\beta_3^\Delta &= \alpha_2 + \alpha_3; \\
\beta_4^\Delta &= \alpha_1; \\
\beta_5^\Delta &= \alpha_4; \\
\beta_6^\Delta &= \alpha_2 + \alpha_3.
\end{aligned}$$

Note that

$$\beta_{j_1}^\Delta = \beta_3^\Delta = \alpha_2 + \alpha_3,$$

$$\beta_{j_2}^\Delta = \beta_4^\Delta = \alpha_1$$

and

$$\beta^{\Delta}_{j_3} = \beta^{\Delta}_{5} = \alpha_4.$$

Now it is clear that a basis for $\ker(M(w^{\Delta}))$ is the singleton $\{\boldsymbol{e}_2 - \boldsymbol{e}_3\}$. Thus

$$\begin{aligned} 2\Phi(\boldsymbol{e}_2 - \boldsymbol{e}_3) &= (w^{\Delta})^{-1}(\beta_{j_2}) - (w^{\Delta})^{-1}(\beta_{j_3}) \\ &= s_3 s_1 s_2(\alpha_1) - s_3 s_1 s_2(\alpha_4) \\ &= \alpha_2 + \alpha_3 - \alpha_3 - \alpha_4 \\ &= \alpha_2 - \alpha_4 \end{aligned}$$

Indeed, one can check that $(w + w^{\Delta})(\alpha_2 - \alpha_4) = 0$. On the other hand,

$$\begin{aligned} \Psi(\alpha_2 - \alpha_4) &= \sum_{h=1}^{3} \langle (\beta^{\Delta}_{j_h})^{\vee} \mid w^{\Delta}(\alpha_2 - \alpha_4)\rangle \boldsymbol{e}_h \\ &= \langle (\alpha_2 + \alpha_3)^{\vee} \mid \alpha_1 - \alpha_4\rangle \boldsymbol{e}_1 + \langle (\alpha_1)^{\vee} \mid \alpha_1 - \alpha_4\rangle \boldsymbol{e}_2 + \langle (\alpha_4)^{\vee} \mid \alpha_1 - \alpha_4\rangle \boldsymbol{e}_3 \\ &= \frac{1}{2}\left(\langle \alpha_2 - \alpha_3 \mid \alpha_1 - \alpha_4\rangle \boldsymbol{e}_1 + \langle \alpha_1 \mid \alpha_1 - \alpha_4\rangle \boldsymbol{e}_2 + \langle \alpha_4 \mid \alpha_1 - \alpha_4\rangle \boldsymbol{e}_3\right) \\ &= \boldsymbol{e}_2 - \boldsymbol{e}_4. \end{aligned}$$

Of course, this is unsurprising in view of Theorem 6.16.

One may check that, for Example 6.17, the maps ϕ and ψ given in Theorem 5.15 give the same bases for the respective kernels as the maps Φ and Ψ do. This is true in general and thus Theorem 6.11 is a generalization of Theorem 5.15.

Since Theorem 5.15 was so useful in the derivation of the enumeration formulae in Section 5.5, we hope to be able to use Theorem 6.11 to deduce new enumeration results for the algebras $U_q^w(\mathfrak{g})$ for various choices of w and $\mathfrak{g}$.

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