The Cahn-Hilliard Equation as a Gradient Flow

by

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Abstract

Some evolution equations can be interpreted as gradient flows. Mathematically this is subtle as the flow depends on the choice of a functional and an inner product (different functionals or inner products give rise to different dynamics). The Cahn-Hilliard equation is a simple model for the process of phase separation of a binary alloy at a fixed temperature. This equation was first derived using physical principles but can also be obtained as a specific gradient flow of a free energy. Having these two viewpoints is quite common in physics and often one prefers to work with the variational formulation. For example, a variational formulation allows one to obtain many possible evolutionary models for the system.

For a gradient flow, the basic idea is to start with an energy functional (F) defined on a Hilbert space. One then writes out the gradient flow associated with the functional and the Hilbert space:

$$\begin{cases} u'(t) = -K grad_H F(u(t) & t > 0 \\ u(0) = u_0. \end{cases}$$

The above becomes an evolution equation which will be dependent on the Hilbert space. Whether it is a good model for the dynamics of the system is another question as it is not based upon any dynamic physical law (eg. a force balance law).

In this thesis we will examine the above ideas focusing on the Cahn-Hilliard equation. We will develop the necessary tools from functional analysis and PDE theory.

Dedication

This thesis is dedicated to the love of my life, Lil' Karin, whose support and love made the completion of this work possible. I would also like to dedicate this thesis to Peppy, who lived more in his short life than most in a lifetime.

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I would like to thank Dr. Brian Thomson for introducing me to analysis and Dr. Ralf Wittenberg for taking the time to listen to my sometimes skewed point of view. Finally, I would like to thank my supervisor Dr. Rustum Choksi for all his great research ideas and for giving me a gentle kick when needed.

Contents

Chapter 1

Introduction

Typically evolution equations which model physical processes are derived using the physics related to the problem. Another way is to use a variational approach which lets the process evolve such that a certain energy functional decreases in time. This variational approach is what we will investigate in this thesis. To showcase this method we will concentrate on the Cahn-Hilliard equation which can be derived using standard physics and chemistry, but can also be derived using this variational approach. Let's look at the following example of the heat equation to illustrate these two methods.

Example 1.0.1. (Linear Heat) *Suppose we are given a region Ω of $\mathbb{R}^n$ where the boundary of Ω is held at the fixed temperature zero and with an initial temperature distribution $\phi(x)$. If $u(x,t)$ denotes the temperature at x and time t, we can show that u should evolve according to:*

$$\begin{cases} u_t &= \Delta u \qquad \Omega \times (0,\infty) \\ u &= 0 \qquad \partial\Omega \times [0,\infty) \\ u &= \phi \qquad \Omega \times \{t=0\}. \end{cases} \tag{1.1}$$

Typically one uses ideas of flux and conservation of energy to arrive at (1.1).

We will now switch to a variational viewpoint. Let F denote the following energy functional:

$$F(u) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx & u \in H_0^1(\Omega) \\ \infty & otherwise. \end{cases} \tag{1.2}$$

Now suppose we knew that u should evolve in such a way as to have F decrease in time (This could be motivated by a least action principle from physics.) One obvious way to do this would be to let u evolve in the opposite direction to the gradient of F at u. To calculate the gradient we will have to indicate what Hilbert space we are working on. If we use the Hilbert space $L^2(\Omega)$, then we can show that

$$grad_{L^2}F(u) = -\Delta u. \qquad (\textit{gradient of } F \textit{ at } u \textit{ over } L^2(\Omega))$$

So we arrive at (1.1) *if we let u evolve according to the following gradient flow:*

$$\begin{cases} u_t &= -grad_{L^2}F(u) \qquad t > 0 \\ u(0) &= \phi. \end{cases} \tag{1.3}$$

We can also arrive at (1.1) *if we interchange F_0 for F and $H^{-1}(\Omega)$ for $L^2(\Omega)$, where $H^{-1}(\Omega)$ denotes the dual of $H_0^1(\Omega)$ and F_0 is defined as follows:*

$$F_0(u) := \begin{cases} \frac{1}{2}\int_\Omega u^2 dx & u \in L^2(\Omega) \\ \infty & otherwise. \end{cases}$$

1.1 Cahn-Hilliard Basics

The Cahn-Hilliard equation was originally proposed as a simple model for the process of phase separation of a binary alloy at a fixed temperature, by Cahn and Hilliard. If one is interested in the history related to the Cahn-Hilliard equation one should consult [Fife] and the references within.

If we let $u(x,t)$ denote the concentration of one of the two metallic components of the alloy, and if we assume the total density is constant, then the composition of the mixture may be adequately expressed by the single function u. If we let $\Omega \subset \mathbb{R}^n$ denote the vessel containing the alloy and if we assume that there is no alloy entering or leaving the vessel, then we will have conservation of mass. ie. $\int_\Omega u(x,t)dx =$ constant. Let $W : \mathbb{R} \to \mathbb{R}$ denote a non-negative double well potential with equal minima at $u = u_1$ and u_2 where u_1 and u_2 are preferred states of u. If we define the "free energy" functional F_0 by

$$F_0(u) := \int_\Omega W(u(x))dx,$$

then one could try and model the evolution of u by letting u evolve such that F_0 decreases in time while conserving mass. One objection to this approach is that u could oscillate wildly in the spatial sense between u_1 and u_2 but not raise the energy. We expect the energy to penalize this transition in phase. One way of doing this is to add a term which penalizes spatial oscillation. The most obvious way to do this is to add a small gradient term. So for $\epsilon > 0$ but small, let F_ϵ denote the "gradient-corrected free energy" which is defined as follows:

$$F_\epsilon(u) := \int_\Omega \left\{ W(u(x)) + \frac{\epsilon^2}{2}|\nabla u(x)|^2 \right\} dx.$$

Now we want u to evolve such that F_ϵ decreases in time. As mentioned earlier, a standard way of doing this is to let u evolve in the direction opposite to the gradient of $F_\epsilon(u)$, where the gradient is calculated over some Hilbert space. So we see it is natural to seek a law of evolution of the form

$$u_t = -K grad F_\epsilon(u),$$

while imposing the conservation of mass constraint, where $K > 0$. To simplify the problem let's suppose we knew that $\int_\Omega u(x,0)dx = 0$, hence by conservation of mass we will want $\int_\Omega u(x,t)dx = 0$ for all $t > 0$. (This zero mass constraint can be obtained by using the shifted density $\hat{u} := u - (u)_\Omega$ where $(u)_\Omega$ denotes the average of u over Ω.)

Let's try and write the gradient flow of F_ϵ over $L^2(\Omega)$. One way to impose the mass constraint is to instead write the gradient flow over $\dot{L}^2(\Omega)$, which is the zero-average subspace of $L^2(\Omega)$. If one does the calculations one will arrive at the following evolution equation:

$$u_t = \epsilon^2 \Delta u - W'(u) + (W'(u))_\Omega . \tag{1.4}$$

Typically (1.4) is rejected as a good model on the grounds that $(W'(u))_\Omega$ is an integral operator, hence is not local in nature. If one tries to write the gradient flow over $H^k(\Omega)$ for $k \geq 0$ there will again be objections to the evolution equation that is arrived at.

We can use the physics of the problem to impose certain boundary conditions on u but we will not take this approach. We will show that if we minimize F_ϵ over the zero-average subspace of $H^1(\Omega)^*$ (denoted by H_0^{-1}), then we will see that a minimizer u is in fact smooth and satisfies $\partial_\nu u = 0 = \partial_\nu \Delta u$ on $\partial\Omega$. (It is understood that if $u \notin \dot{H}^1(\Omega)$ or $W(u) \notin L^1(\Omega)$ then $F_\epsilon(u) = \infty$). This will serve as our motivation for the imposed boundary conditions.

We will see that if we use H_0^{-1} with an appropriate inner product, as our Hilbert space, then we will arrive at

$$u_t = -\epsilon^2 \Delta^2 u + \Delta W'(u),$$

which is local in nature, therefore more realistic. If we let u denote a solution to the above evolution equation with the imposed boundary conditions then, without employing some uniqueness, it is not entirely obvious that u will conserve mass. In other words if $u(t)$ evolves in H_0^{-1} then clearly we have conservation of mass, but if $u(x,t)$ solves the above evolution equation it is not entirely obvious we have conservation of mass. But to see this is the case note:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u dx &= \int_\Omega \left\{-\epsilon^2 \Delta^2 u + \Delta W'(u)\right\} dx \\
&= \int_{\partial\Omega} \partial_\nu \left\{-\epsilon^2 \Delta u + W'(u)\right\} dS(x) \\
&= \int_{\partial\Omega} \left\{-\epsilon^2 \partial_\nu \Delta u + W''(u)\partial_\nu u\right\} dS(x) \\
&= 0
\end{aligned}$$

where we have used the boundary conditions to get the surface integral equal to zero. So we see with these boundary conditions we arrive at the Cahn-Hilliard equation:

$$\begin{cases} u_t = -\epsilon^2 \Delta^2 u + \Delta f(u) & \Omega \times (0,\infty) \\ \partial_\nu u = \partial_\nu \Delta u = 0 & \partial\Omega \times [0,\infty) \end{cases} \tag{1.5}$$

where $f(u) := W'(u)$. As noted earlier, the Cahn-Hilliard equation was not originally derived using the gradient flow approach but by using sound physical arguments.

1.2 Thesis Layout

In Chapter 2 will examine various elements of functional analysis that we will need if we hope to write the Cahn-Hilliard equation as a gradient flow. This will include distributions, Hilbert spaces and Sobolev spaces.

In Chapter 3 we will look at the duals of various Sobolev spaces. In particular we will examine H^{-1} and $(H^1)^*$. We will introduce the non-standard Hilbert space H_0^{-1} (zero average of $(H^1)^*$), which we will use to write the Cahn-Hilliard equation as a gradient flow over.

In Chapter 4 we will examine the standard notions of a gradient, namely, classical gradients (Gâteaux), subdifferentials-subgradients and constrained gradients. The constrained gradient is the one that we will use. We will develop some very elementary properties of this constrained gradient. This will be sufficient to allow us to write the Cahn-Hilliard equation as a gradient flow. We will also look at examples of constrained gradients of various functionals over various Hilbert spaces. We will also look at a simple non-linear evolution equation and will obtain a global solution using both the semigroup method and the subdifferential method. This example is to showcase the two methods.

In Chapter 5 we will write the Cahn-Hilliard equation as a gradient flow over the specific Hilbert space that we defined in Chapter 3. We will argue why this Hilbert space is a physically reasonable one to use. We will also examine certain properties of the functional F. In particular we will show that the minimizers of F, which correspond to steady states of the Cahn-Hilliard equation, are smooth and satisfy certain boundary conditions. We will obtain a local solution to the Cahn-Hilliard equation when $W'(u) = u^3 - u$ and when Ω is an open, bounded and connected subset of $\mathbb{R}^3$ with a smooth boundary.

Chapter 2

Mathematical Tools

2.1 Distributions

Distribution theory allows one to put objects like the "Dirac δ function" on rigorous footing and also allows one to develop properties of certain function spaces in a systematic way. Here we will essentially just define what a distribution is and also define what we mean by a partial derivative of a distribution.

Take $\Omega \subseteq \mathbb{R}^n$ to be an open set. We define the space of test functions by $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$, where $C_c^\infty(\Omega)$ is the set of C^∞ functions with compact support in Ω.

We say $\phi_m \to \phi$ in $\mathcal{D}(\Omega)$ if there exists a compact $K \subseteq \Omega$ with $supp(\phi_m) \subseteq K$ and $\partial^\alpha \phi_m \to \partial^\alpha \phi$ uniformly on K for all multi-indices α. It is possible to describe this topology but for our purposes the above characterization of convergent sequences will be enough.

$\mathcal{D}'(\Omega)$ will denote the set of real continuous linear functionals on $\mathcal{D}(\Omega)$ which we call the space of distributions on Ω. We will denote the $\mathcal{D}'(\Omega)$, $\mathcal{D}(\Omega)$ pairing by $\langle \cdot, \cdot \rangle_{\mathcal{D}',\mathcal{D}}$.

To define a partial derivative for a distribution u we will use integration by parts as motivation.

Definition 2.1.1. *For $u \in \mathcal{D}'(\Omega)$ and for any multi-index α we define $\partial^\alpha u$ by*

$$\langle \partial^\alpha u, \phi \rangle_{\mathcal{D}',\mathcal{D}} := (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle_{\mathcal{D}',\mathcal{D}} \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega). \tag{2.1}$$

It is easily seen that $\partial^\alpha u \in \mathcal{D}'(\Omega)$, which is the real power behind distribution theory. Even when one requires some classical smoothness it is often easier to work with the given functions as distributions and later show that the distributions have the required classical smoothness.

We will keep with the standard practice of identifying $f \in L^1_{loc}(\Omega)$ and the distribution $\phi \mapsto \int_\Omega f\phi$. The set of compactly supported distributions in Ω is defined by

$$\mathcal{E}'(\Omega) := \{u \in \mathcal{D}'(\Omega) : supp(u) \subset \Omega\}. \tag{2.2}$$

In order to make sense of (2.2) we must define what we mean by the support of a distribution. Given $u \in \mathcal{D}'(\Omega)$, we say $u = 0$ on an open set $V \subseteq \Omega$ if for all $\phi \in C_c^\infty(V)$ we have $\langle u, \phi \rangle_{\mathcal{D}',\mathcal{D}} = 0$. Let V denote the maximal open subset of Ω with $u = 0$ on V. Then we define $supp(u)$ to be the complement of V in Ω. Now for some notation that we will use later.

$$\mathcal{D}'_{-m}(\Omega) := \left\{ u \in \mathcal{D}'(\Omega) : \sup_{\phi \in C_c^\infty(\Omega), \|\phi\|_{H^m} \le 1} \langle u, \phi \rangle_{\mathcal{D}',\mathcal{D}} < \infty \right\} \tag{2.3}$$

$$\mathcal{E}'_{-m}(\Omega) := \left\{ u \in \mathcal{E}'(\Omega) : \sup_{\phi \in C_c^\infty(\Omega), \|\phi\|_{H^m} \le 1} \langle u, \phi \rangle_{\mathcal{D}',\mathcal{D}} < \infty \right\} \tag{2.4}$$

where m is a non-negative integer and where $\|\cdot\|_{H^m}$ is defined in Section (2.3). See [Folland] for more details on distribution theory.

2.2 Banach Spaces, Hilbert Spaces and Complete Metric Spaces.

In this section we will summarize various results from functional analysis that we will need later. All vector spaces will be over the scalar field $\mathbb{R}$.

Given $1 \leq p, q \leq \infty$ we call p and q conjugate if $1/p + 1/q = 1$ where $1/0 = \infty$ and $1/\infty = 0$.

L^p inequalities.

Cauchy's inequality with ϵ.

$$\|fg\|_{L^1} \leq \epsilon \|f\|_{L^2}^2 + \frac{1}{4\epsilon}\|g\|_{L^2}^2$$

for all $\epsilon > 0$.

Hölder's inequality.

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$$

where p and q are conjugate.

Young's inequality with ϵ.

$$\|fg\|_{L^1} \leq \epsilon \|f\|_{L^p}^p + \frac{1}{(\epsilon p)^{q/p}}\frac{1}{q}\|g\|_{L^q}^q$$

for all $\epsilon > 0$ where p and q are conjugate.

Banach Spaces

Given a metric space (X, d) and a mapping $A : X \to X$, we say A is a contraction mapping if there exists some $\gamma < 1$ such that for all $x, y \in X$ we have

$$d(Ax, Ay) \leq \gamma d(x, y).$$

Theorem 2.2.1. (Banach's Fixed Point Theorem) *Given (X, d) a complete metric space with $A : X \to X$ a contraction mapping, there exists a unique $x \in X$ with $Ax = x$.*

Proof. See [Thomson] page 399. □

Definition 2.2.1. *Given a linear space X and two norms on X say $\|\cdot\|_1$ and $\|\cdot\|_2$, we say that the norms are equivalent on X if there exist $a, b > 0$ such that*

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \quad \text{for all } x \in X.$$

Theorem 2.2.2. (Open Mapping Theorem) *Assume X is a linear space which is complete w.r.t. the two norms $\|\cdot\|_1$, $\|\cdot\|_2$. If there exists some $a > 0$ such that*

$$\|x\|_1 \leq a\|x\|_2 \quad \text{for all } x \in X,$$

then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on X.

Proof. See [Thomson] page 563. □

Definition 2.2.2. (Dual Spaces) *Given a normed linear space $(X, \|\cdot\|)$, we define X^* to be the set of continuous linear functionals on X. X^* will be endowed with the operator norm $\|\cdot\|_{X^*}$, which is defined by*

$$\|x^*\|_{X^*} := \sup_{\|x\|\leq 1} \langle x^*, x \rangle \,.$$

Definition 2.2.3. (Locally Lipschitz) *Let X, Y be normed linear spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$. Given $f : X \to Y$, we say f is locally Lipschitz from X to Y, written $f \in Lip^{loc}(X, Y)$, if for all $R > 0$ there exists some $L(R) > 0$ such that*

$$\|f(z) - f(x)\|_Y \leq L(R)\|z - x\|_X$$

whenever $\|x\|_X$, $\|z\|_X \leq R$.

2.2.1 Hilbert Spaces

H will denote a Hilbert space with inner product and norm given by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ respectively. H^* will denote the dual of H and will be a Hilbert space with inner product $(\cdot,\cdot)_{H^*}$, which be defined in a moment. The H^*,H pairing will be given by $\langle\cdot,\cdot\rangle_{H^*,H}$.

Theorem 2.2.3. (Riesz Representation Theorem) *Given $u^* \in H^*$ there exists a unique $u \in H$ such that*

$$\langle u^*, v\rangle_{H^*,H} = (u, v)_H \quad \forall v \in H.$$

In particular, we have $\|u^*\|_{H^*} = \|u\|_H$.

Proof. See [Thomson] page 621. □

Definition 2.2.4. *Given u^* and u as above, we will call u the associate of u^*.*

Define $\Phi : H^* \to H$ by $\Phi(u^*) := u$ where u^* and u are defined as above. Define $\Psi := \Phi^{-1}$.

We will use Φ to induce a inner product on H^* and this inner product will induce the operator norm. So towards this let $u^*, v^* \in H^*$ and let $u = \Phi(u^*)$, $v = \Phi(v^*)$. Then we define

$$(u^*, v^*)_{H^*} := (u, v)_H.$$

So we see that, by construction, Φ and Ψ are unitary maps.

Theorem 2.2.4. (Element of Least Norm) *Given a Hilbert space H and a non-empty closed convex set $C \subseteq H$, there exists a unique $x \in C$ such that*

$$\|x\|_H = \inf_{y \in C} \|y\|_H.$$

Proof. See [Thomson] page 620. □

Definition 2.2.5. (Orthogonal Complement) *Given a set $X \subseteq H$, where H is a Hilbert space, we define the orthogonal complement of X in H by*

$$X^\perp := \{y \in H : (y, x)_H = 0 \quad \forall x \in X\}.$$

Theorem 2.2.5. (Decomposition) *Given a closed subspace X of H, we have the decomposition*

$$H = X \oplus X^\perp,$$

in the sense that for all $z \in H$ there exist unique $x \in X$, $y \in X^\perp$ such that $z = x + y$. Moreover we have $\|z\|_H^2 = \|x\|_H^2 + \|y\|_H^2$.

Proof. See [Thomson] page 621. □

Definition 2.2.6. (Weak Sequential Convergence) *Given $u_n \in H$, we will say u_n convergences weakly to u in H, written*

$$u_n \rightharpoonup u \quad in \quad H$$

if $\quad (u_n, v)_H \to (u, v)_H \quad$ *for all $v \in H$.*

In a finite dimensional Hilbert space we will quite frequently use the fact that a closed, bounded set is compact. In an infinite dimensional Hilbert space we do not have this compactness result, but we do have the following which will turn out to be extremely useful.

Theorem 2.2.6. (Weak Sequential Compactness)

(i) *Given u_n bounded in H, there exists a subsequence u_{n_k}* (which generally won't be renamed) *and $u \in H$ such that*

$$u_{n_k} \rightharpoonup u \quad \text{in} \quad H.$$

(ii) *If $u_n \rightharpoonup u$ in H then*

$$\|u\|_H \leq \liminf_n \|u_n\|_H.$$

Proof. For (i) see [Thomson] page 631.

(ii) Let $u_n \rightharpoonup u$. Then

$$(u_n, u)_H \leq \|u_n\|_H \|u\|_H.$$

Now take a lim inf of both sides to get

$$\|u\|_H^2 \leq \|u\|_H \liminf_n \|u_n\|_H.$$

□

Theorem 2.2.7. (Mazur's Theorem) *Assume C is convex and closed in H. Then C is weakly closed in H.*

Proof. See [Evans] page 639. □

Definition 2.2.7. (Lower Semicontinuous) *Given $F : H \to (-\infty, \infty]$, we say:*

(i) *F is lower semicontinuous on H if*

$$u_n \to u \quad \text{in } H \qquad \text{implies} \qquad F(u) \leq \liminf_n F(u_n).$$

(ii) *F is weakly lower semicontinuous on H if*

$$u_n \rightharpoonup u \quad \text{in } H \qquad \text{implies} \qquad F(u) \leq \liminf_n F(u_n).$$

Quite often one will be interested in minimizing some function $F : H \to (-\infty, \infty]$ over some $\mathcal{A} \subseteq H$. The above ideas will be extremely useful for accomplishing this.

2.3 Sobolev Spaces

I will define various Sobolev spaces and list various theorems that we will need later. Most of the theorems we will be using can be found in [Evans]. Note that the theorems quoted are typically not the most general. If one requires these one should consult [Adams].

Definition 2.3.1. *For m a non-negative integer and $p \in [1, \infty)$, we define*

$$\|u\|_{W^{m,p}} := \left\{ \sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right\}^{1/p}$$

whenever the right hand side makes sense and where the derivatives are taken in the sense of $\mathcal{D}'(\Omega)$.

Now let's define various function spaces:

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \ \text{ for all } \ |\alpha| \leq m\}.$$

Then $W^{m,p}(\Omega)$ is a Banach space with the above defined norm.

$$W_0^{m,p}(\Omega) := \ \text{closure of } \ C_c^\infty(\Omega) \ \text{ in } \ W^{m,p}(\Omega).$$

We will typically denote $W^{m,2}(\Omega)$, $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$, $(H_0^m(\Omega))$ respectively, since it will be seen that they are Hilbert spaces.

We define a inner product on $H^m(\Omega)$ by

$$(u,v)_{H^m} := \sum_{0\leq|\alpha|\leq m} (\partial^\alpha u, \partial^\alpha v)_{L^2}$$

and this inner product induces the above norm.

Define the semi-inner product and semi-norm on $H^1(\Omega)$ by

$$(u,v)_{H_0^1} := (\nabla u, \nabla v)_{L^2} \quad \text{and} \quad \|u\|_{H_0^1} := \|\nabla u\|_{L^2}.$$

This semi-norm will turn out to be a norm on $H_0^1(\Omega)$ and $\dot{H}^1(\Omega)$ under suitable conditions, where the "dot" denotes the zero-average subspace.

Definition 2.3.2. (First Eigenvalue)

$$\lambda_1 := \inf\left\{\frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} : u \in H_0^1(\Omega), u \neq 0\right\}.$$

Later we will show that if $c \in L^\infty(\Omega)$ with $\inf_\Omega c > -\lambda_1$, then

$$(u,v) := \int_\Omega \{\nabla u \cdot \nabla v + cuv\}\, dx$$

and $(\cdot,\cdot)_{H^1}$ are equivalent inner products on $H_0^1(\Omega)$. In particular, if $\lambda_1 > 0$, then $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H_0^1}$ are equivalent on H_0^1. To have $\lambda_1 > 0$ it is sufficient that Ω be bounded. Generally λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

When $\lambda_1 > 0$, it is understood that $H_0^1(\Omega)$ has inner product and norm given by above with $c = 0$, unless otherwise mentioned.

To see that we do require some sort of restriction on Ω for λ_1 to be positive, examine the following.

Example 2.3.1. *Take $n = 1$ and let $\phi \in C_c^\infty(\mathbb{R})$ with $\phi \neq 0$. Define $\phi_m(x) := \phi\left(\frac{x}{m}\right)$. Then easily seen that*

$$\frac{\|\phi_m'\|_{L^2}^2}{\|\phi_m\|_{L^2}^2} = \frac{\|\phi'\|_{L^2}^2}{m^2\|\phi\|_{L^2}^2} \to 0.$$

Hence $\lambda_1 = 0$.

Let's now state various Sobolev space theorems that we will continuously use. In what follows Ω will always be a subset of $\mathbb{R}^n$.

Theorem 2.3.1. (Global Approximation by Smooth Functions)

Let Ω be an open and bounded set with a C^1 boundary. Then

$$C^\infty(\overline{\Omega}) \quad \textit{is dense in} \quad W^{1,p}(\Omega)$$

provided $1 \leq p < \infty$.

Proof. See [Evans] page 252. □

Definition 2.3.3. (Hölder spaces) *Let Ω be an open and bounded subset of $\mathbb{R}^n$. Let $0 < \gamma \leq 1$. For $u \in C(\overline{\Omega})$, we define the γ^{th}-Hölder seminorm of u by*

$$[u]_{C^{0,\gamma}(\overline{\Omega})} := \sup\left\{\frac{|u(x) - u(y)|}{|x-y|^\gamma} : x, y \in \Omega, x \neq y\right\}.$$

The γ^{th}-Hölder norm of u is defined by

$$||u||_{C^{0,\gamma}(\overline{\Omega})} := ||u||_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}.$$

The Hölder space $C^{k,\gamma}(\overline{\Omega})$ consists of all $u \in C^k(\overline{\Omega})$ for which the norm

$$||u||_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha| \leq k} ||\partial^\alpha u||_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [\partial^\alpha u]_{C^{0,\gamma}(\overline{\Omega})}$$

is finite.

Theorem 2.3.2. (Estimates for $W^{1,p}, n < p \leq \infty$) *Let Ω be an open and bounded subset of $\mathbb{R}^n$ with a C^1 boundary. Then for $u \in W^{1,p}(\Omega)$, we have $u \in C^{0,\gamma}(\overline{\Omega})$ and*

$$||u||_{C^{0,\gamma}(\overline{\Omega})} \leq C||u||_{W^{1,p}(\Omega)}$$

where $C = C(p, n, \Omega)$ and $\gamma := 1 - \frac{n}{p}$. Note that we are identifying functions that agree a.e.

Proof. See [Evans] page 269. □

Theorem 2.3.3. (Poincaré's Inequality)

Let Ω be open, bounded and connected with a C^1 boundary. Take $1 \le p \le \infty$. Then

$$\|u - (u)_\Omega\|_{L^p} \le C\|\nabla u\|_{L^p}$$

for all $u \in W^{1,p}(\Omega)$, where $C = C(p, n, \Omega)$. Here $(u)_\Omega$ denotes the average of u over Ω.

Proof. See [Evans] page 275. □

The space in the following definition will take a pivotal role when we define H_0^{-1}.

Definition 2.3.4. (H_A^1) *Let Ω be as in Poincaré's Inequality. Define*

$$H_A^1 := \dot{H}^1(\Omega),\ \|u\|_{H_A^1} := \|\nabla u\|_{L^2},\ (u, v)_{H_A^1} := (\nabla u, \nabla v)_{L^2}.$$

Using Poincaré's inequality we see that H_A^1 is a Hilbert space and the norms $\|\cdot\|_{H^1}$, $\|\cdot\|_{H_A^1}$ are equivalent on $\dot{H}^1(\Omega)$.

Theorem 2.3.4. (Rellich-Kondrachov Compactness Theorem) *Assume that Ω is a bounded open set with a C^1 boundary. Then*

$$W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{and} \quad W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

where $1 \le p \le \infty$ and $\hookrightarrow$ denotes a compact imbedding. Note that for $W_0^{1,p}$ case we can drop assumption on smoothness of the boundary. Also we have for $1 \le p < n$ that

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for} \quad 1 \le q < p^* := \frac{np}{n-p}$$

and a continuous imbedding when $q = p^*$.

For $kp > n$ we have

$$W^{k,p}(\Omega) \to C(\overline{\Omega}),$$

where $\to$ denotes a continuous imbedding.

Since we will typically be working with $\Omega \subseteq \mathbb{R}^3$, it will be helpful to remember the following imbeddings.

Let $\Omega \subseteq R^3$ be open, bounded and with smooth boundary. Then we have the following continuous imbeddings:

$$
\begin{array}{lcll}
H^2(\Omega) & \to & C(\overline{\Omega}), L^q(\Omega) & 1 \le q \le \infty \\
H^2(\Omega) & \to & W^{1,q}(\Omega) & 2 \le q \le 6 \\
H^1(\Omega) & \to & L^q(\Omega) & 2 \le q \le 6 \\
H^4(\Omega) & \to & W^{2,\infty}(\Omega). &
\end{array}
$$

Proof. See [Evans] page 272. □

Theorem 2.3.5. (General Sobolev inequalities) *Let Ω be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary and also assume $k > \frac{n}{p}$. Then we have the following continuous imbedding:*

$$W^{k,p}(\Omega) \to C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(\overline{\Omega})$$

where $\gamma := \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}$ if $\frac{n}{p}$ is not an integer. If $\frac{n}{p}$ an integer than γ can be any number in $(0,1)$. Here $\lfloor \cdot \rfloor$ denotes the floor function. In addition the imbedding is continuous and the constant depends only on k, p, n, γ and Ω.

In particular for $n = 3, p = 2$ and $k \ge 2$ we have for $u \in W^{k,2}(\Omega)$ that

$$u \in C^{k-2,\frac{1}{2}}(\overline{\Omega}).$$

Proof. See [Evans] page 270. □

Definition 2.3.5. (Banach Algebra under pointwise multiplication) *Given a Sobolev space of functions X on Ω with $X \subseteq L^1(\Omega)$, we say X a Banach Algebra under pointwise multiplication if*

$$u, v \in X \qquad \textit{implies} \qquad uv \in X,$$

where $(uv)(x) := u(x)v(x)$.

Theorem 2.3.6. ($W^{k,p}(\Omega)$ as a Banach Algebra) *Let Ω be an open and bounded set in $\mathbb{R}^n$ with a sufficiently smooth boundary. Then if $kp > n$ we have that $W^{k,p}(\Omega)$ is a Banach algebra, under pointwise multiplication.*

Proof. See [Adams] page 115. □

We will need a way to assign boundary values to elements of various Sobolev spaces. Since Sobolev functions are only defined up to sets of measure zero and "nice" open sets will typically have boundaries with n-dimensional Lebesgue measure equal to zero, we see there is no obvious way to define what one means by $u = 0$ or $\partial_\nu u = 0$ on $\partial\Omega$. One way around this apparent problem is to use what is called a trace operator.

A comment on notation. Given $\Omega \subseteq \mathbb{R}^n$, $|\Omega|$ will denote the n-dimensional Lebesgue measure of Ω and $|\partial\Omega|$ will typically denote the $(n-1)$-dimensional Lebesgue measure of $\partial\Omega$.

Theorem 2.3.7. (Trace Theorem) *Assume Ω is an open and bounded subset of $\mathbb{R}^n$ with a C^1 boundary. Then there exists*

$$T_0 \in \mathcal{L}(W^{1,p}(\Omega), L^p(\partial\Omega))$$

such that $T_0(u) = u|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.

Proof. See [Evans] page 258. □

We call T_0 the trace operator.

Theorem 2.3.8. (Trace-zero functions in $W^{1,p}$) *Assuming the same hypotheses as* Theorem 2.3.7, *we have for $u \in W^{1,p}(\Omega)$ that*

$$u \in W_0^{1,p}(\Omega) \qquad \textit{iff} \qquad T_0(u) = 0 \quad \textit{on} \quad \partial\Omega,$$

where $T_0(u) = 0$ on $\partial\Omega$ means that $T_0(u) = 0$ in $L^p(\partial\Omega)$.

Proof. See [Evans] page 259. □

To talk about the normal derivative of u on $\partial\Omega$ we can also use the idea of a trace operator. We will not give this operator its own notation but let's just say that to make sense of $\partial_\nu u$ on $\partial\Omega$ we will require $u \in W^{2,p}(\Omega)$.

Theorem 2.3.9. (Green's formulas) *For $u \in H^2(\Omega)$, $v \in H^1(\Omega)$ and $\partial\Omega$ sufficiently smooth, we have*

$$\begin{aligned} \int_\Omega \Delta u \, dx &= \int_{\partial\Omega} \partial_\nu u \, dS(x) \\ \int_\Omega \nabla u \cdot \nabla v \, dx &= -\int_\Omega v\Delta u + \int_{\partial\Omega} v\partial_\nu u \, dS(x), \end{aligned}$$

where we have used the trace operator to interpret the boundary terms.

Proof. Use classical Green's and then use a density / continuity argument.

□

For the remainder of this thesis we will use the following convention.

Convention 2.3.1. (Domains) *A Domain in $\mathbb{R}^n$ will denote a bounded open set in $\mathbb{R}^n$ with a C^∞ boundary.*

From here on we will always take the "best constant" when using various Sobolev inequalities. For the remainder of this thesis we will assume that $\lambda_1 > 0$.

Chapter 3

Duals of Sobolev Spaces and H_0^{-1}

In this chapter we will examine the duals of various Sobolev spaces and in particular the duals of $H^1(\Omega)$, $H_0^1(\Omega)$ denoted by $(H^1(\Omega))^*$, $H^{-1}(\Omega)$. More specifically we will look at the associates related to the spaces $(H^1(\Omega))^*$, $H^{-1}(\Omega)$ and the related elliptic boundary value problems. We will quote a standard representation theorem for H^{-1}, which is more compatible with distribution theory than the Riesz Representation Theorem. We will obtain weak solutions to these elliptic boundary value problems using the Riesz Representation Theorem, (no need for Lax-Milgram), and also using a variational approach. Standard regularity theorems will also be presented. Eventually we will introduce the Hilbert space (H_0^{-1}) over which we will write the Cahn-Hilliard equation as a gradient flow.

3.0.1 The dual of $H_0^m(\Omega)$

Definition 3.0.6. *For m a non-negative integer we define $H^{-m}(\Omega) := H_0^m(\Omega)^*$.*

In this section we will show that $H^{-m}(\Omega)$ can be identified in a natural way with a subspace of $\mathcal{D}'(\Omega)$. We will also introduce a standard representation of $H^{-1}(\Omega)$.

Let's now look at the identification mentioned above.

$\mathcal{D}'_{-m}(\Omega)$ (see (2.3)) can naturally be identified with $H^{-m}(\Omega)$ in the following sense:
(i) Given $u \in H^{-m}(\Omega)$ we have $u|_{\mathcal{D}(\Omega)} \in \mathcal{D}'_{-m}(\Omega)$.

(ii) Given $u \in \mathcal{D}'_{-m}(\Omega)$, u can be extended uniquely to some $\bar{u} \in H^{-m}(\Omega)$.

(i) follows directly from the definition of $\mathcal{D}'_{-m}(\Omega)$ and the definition of the operator norm.

To see (ii) we will use following fact:

Given X, Y metric spaces with Y complete and $A : S \subseteq X \to Y$ uniformly continuous with S dense, then A posesses a unique continuous extension to all of X and this extension is uniformly continuous.

So take $S := \mathcal{D}(\Omega)$ which is dense in $H_0^m(\Omega)$ and apply the above result. It is easily seen the extension is linear.

So from the above what we see is that when working in $H^{-m}(\Omega)$ there is no loss of information if we take the distributional viewpoint. Later we will see this is not the case in general for the dual of $H^m(\Omega)$.

The most obvious way to examine $H^{-1}(\Omega)$ is to use the Riesz Representation Theorem. If we do this then we see that we will identify

$$f \in H_0^1(\Omega) \quad \text{and} \quad T_f$$

where $\langle T_f, v\rangle := (f, v)_{H_0^1}$, but we typically do not use the above identification since it does not agree with the convention that we already set forth in distribution theory.

So we identify sufficiently regular functions f (see next page), and $T_f \in H^{-1}(\Omega)$ where T_f is given by

$$\langle T_f, v\rangle := \int_\Omega fv.$$

So if we identify using the L^2 pairing then we see that $H_0^1(\Omega) \subseteq L^2(\Omega) \subseteq H^{-1}(\Omega)$.

Let's now look at a characterization of $H^{-1}(\Omega)$ where we are identifying using the L^2 pairing.

Theorem 3.0.10. *Given $f \in H^{-1}(\Omega)$ we have*

$$\|f\|_{H^{-1}}^2 = \inf_{E(f)} \sum_{k=0}^{n} \|f^k\|_{L^2}^2$$

where $E(f)$ is the set of $(f^0, f^1, ..., f^n) \in \Pi_{k=0}^{n} L^2$ such that

$$\langle f, v\rangle_{H^{-1},H_0^1} = (f^0, v)_{L^2} + \sum_{k=1}^{n}(f^k, \partial^k v)_{L^2} \quad \forall v \in H_0^1(\Omega).$$

Proof. Let $f \in H^{-1}(\Omega)$. By the Riesz Representation Theorem we know there exists a unique $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v + uv = \langle f, v\rangle_{H^{-1},H_0^1} \quad \forall v \in H_0^1(\Omega).$$

(Note we are using the $H^1(\Omega)$ inner product.)

From this we see $(u, \partial^1 u, ..., \partial^n u) \in E(F)$. For the rest of the proof see [Evans] page 283.

□

A basic but useful fact to keep in mind is that if $H^1(\Omega) \to L^p(\Omega)$ then we have $L^{p'}(\Omega) \to H^1(\Omega)^*$, where $\to$ denotes either a continuous or compact imbedding and where p' denotes the conjugate of p.

Here we are identifying $f \in L^{p'}$ with the linear functional $\langle f, v\rangle = \int_\Omega fv$. To see this functional is continuous on H^1 note that we have

$$\langle f, v\rangle \leq \|f\|_{L^{p'}}\|v\|_{L^p} \leq \|f\|_{L^{p'}}C\|v\|_{H^1}. \tag{3.1}$$

From this we see $\|f\|_{(H^1)^*} \leq C\|f\|_{L^{p'}}$ which gives us the continuous imbedding. To see the compact version we need a slightly more advanced argument.

With above facts, we are now in a position to see what constitutes a "sufficiently regular function" f, from the previous page.

Recall: $p^* := \frac{np}{n-p}$ for $n > p$. Take $n \geq 3$ and examine (3.1). Since $H_0^1 \to L^{2^*}$ we have

$$L^{\frac{2n}{n+2}} = L^{(2^*)'} = (L^{2^*})^* \to H^{-1}.$$

So "sufficiently regular" is at least $L^{\frac{2n}{n+2}}$. I suspect "sufficiently regular" is at most $L^{\frac{2n}{n+2}}$ since if it was bigger than we could extend the Sobolev Imbedding $H^1 \to L^q$ to q values bigger than 2^*.

3.0.2 Dirichlet's Problem and $H_0^1(\Omega)$, $H^{-1}(\Omega)$ associates.

In this section we will examine the $H_0^1(\Omega)$, $H^{-1}(\Omega)$ associates and the related elliptic boundary value problem. Standard methods of obtaining solutions to these boundary value problems will be examined and we will quote a regularity result.

Let's introduce what we mean by a weak solution to the following elliptic problem:

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{3.2}$$

where $f \in H^{-1}(\Omega)$.

Definition 3.0.7. (Weak solution to Dirichlet) *We will say $u \in H_0^1(\Omega)$ is a weak solution to* (3.2) *if*

$$(u,v)_{H_0^1} = \langle f, v\rangle_{H^{-1},H_0^1} \qquad \forall v \in H_0^1(\Omega).$$

Using our associate notation we see $\Phi(f) = u$ and $\Psi(u) = f$.

Using Green's formulas we see that a weak solution of (3.2) is compatible with a classical solution of (3.2).

We can obtain a unique weak solution to (3.2) directly by the Riesz Representation Theorem.

A Variational Formulation of (3.2)

Let's examine a variational method of obtaining a solution to (3.2). Define
$J : H_0^1(\Omega) \to \mathbb{R}$ by

$$J(w) := \frac{1}{2}\|w\|_{H_0^1}^2 - \langle f, w\rangle_{H^{-1},H_0^1}.$$

So we see that

$$J(w) \geq \frac{1}{2}\|w\|_{H_0^1}^2 - \|f\|_{H^{-1}}\|w\|_{H_0^1},$$

hence J is bounded from below. If we let w_m denote a minimizing sequence for J then clearly w_m is bounded in $H_0^1(\Omega)$. By passing to a suitable subsequence we can assume that $w_m \rightharpoonup u$ in $H_0^1(\Omega)$ for some u. Now using the fact that a norm is sequentially weakly l.s.c. on a Hilbert space, we see that

$$J(u) \leq \lim_m J(w_m) \quad \text{therefore} \quad \inf_{H_0^1} J(w) = J(u).$$

Now fix $\phi \in H_0^1(\Omega)$ and define g on $\mathbb{R}$ by

$$g(t) := J(u + t\phi) = J(u) + t\,(u,\phi)_{H_0^1} + t^2 J(\phi) - \langle f, u\rangle_{H^{-1},H_0^1} - t\,\langle f,\phi\rangle_{H^{-1},H_0^1}.$$

Since g obtains its minimum at $t=0$ we have $g'(0)=0$, hence

$$(u,\phi)_{H_0^1} = \langle f,\phi\rangle_{H^{-1},H_0^1} \qquad \text{for all } \phi \in H_0^1(\Omega).$$

Therefore u is a weak solution to (3.2).

For a linear problem like above we will typically not use this variational method of obtaining a solution, but we will use this method when confronted with certain nonlinear elliptic B.V. problems.

Let's now examine Dirichlets problem but with non-zero boundary conditions.

$$\begin{cases} -\Delta u = f & in \ \Omega \\ u = g & on \ \partial\Omega \end{cases} \tag{3.3}$$

where $f \in H^{-1}(\Omega)$ and $g = T_0(w_0)$ for some $w_0 \in H^1(\Omega)$.

We say $u \in H^1(\Omega)$ a solution to (3.3) if

(i) $u \in \mathcal{A} := w_0 + H_0^1(\Omega)$

(ii) $(u,v)_{H_0^1} = \langle f,v\rangle_{H^{-1},H_0^1}$ for all $v \in H_0^1(\Omega)$.

We can solve (3.3) by using (3.2) along with a change of dependent variables. To see this let $\tilde{f} := f + \Delta w_0 \in H^{-1}(\Omega)$ (where Δw_0 is viewed as an element of $H^{-1}(\Omega)$), and let $\tilde{u} \in H_0^1(\Omega)$ solve (3.2) with f replaced with $\tilde{f}$. Then we take $u := \tilde{u} + w_0 \in \mathcal{A}$ and we see that u solves (3.3).

We can use the variational method to solve (3.3) just as we did for (3.2).

Let's first examine (3.3) when $f \in L^2(\Omega)$. Take w_0 and $\mathcal{A}$ as above. Define $J : \mathcal{A} \to \mathbb{R}$ by

$$J(w) := \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \int_\Omega f w\,dx.$$

It is easily seen that J is bounded below on $\mathcal{A}$. Let $w_m \in \mathcal{A}$ be such that

$$J(w_m) \longrightarrow \inf_{\mathcal{A}} J.$$

Since $J(w_m)$ bounded we get an inequality of the form

$$\int_\Omega |\nabla u_m|^2 dx \leq C_0 + C_1 \|u_m\|_{L^2} + C_2 \|\nabla u_m\|_{L^2}$$

where $w_m = w_0 + u_m$ and $u_m \in H_0^1(\Omega)$. This shows $\|\nabla u_m\|_{L^2}$ bounded. After using a Poincaré type inequality and passing to a suitable subsequence, we have

$$w_m \rightharpoonup w \quad \text{in} \quad H^1(\Omega).$$

Since $\mathcal{A}$ is weakly closed in $H^1(\Omega)$ we have $w \in \mathcal{A}$. It is possible to show that $J(w) = \inf_{\mathcal{A}} J$.

Now let $\phi \in H_0^1(\Omega)$. So $w + t\phi \in \mathcal{A}$ for all $t \in \mathbb{R}$. Define g on $\mathbb{R}$ by $g(t) := J(w + t\phi)$. Since $g'(0) = 0$ we get

$$\int_\Omega \nabla w \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx.$$

From this we see w solves (3.3).

Let's now try and solve (3.3) when $f \in H^{-1}(\Omega)$. One obvious problem is that f is not defined on all of $\mathcal{A}$ and so the above approach that worked for $f \in L^2(\Omega)$, will have to be modified.

Without loss of generality we can take $w_0 \in \mathcal{A} \cap H_0^1(\Omega)^\perp$ (here $H_0^1(\Omega)^\perp$ is w.r.t. $H^1(\Omega)$). To see this take $w_0 \in \mathcal{A}$ such that

$$\|w_0\|_{H^1}^2 = \inf_{w \in \mathcal{A}} \|w\|_{H^1}^2.$$

Now define $J : \mathcal{A} \to \mathbb{R}$ by

$$J(w) := \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx - \langle f, Pv \rangle_{H^{-1}, H_0^1}$$

where $P : H^1(\Omega) \to H_0^1(\Omega)$ is the projection operator. Now the proof goes as in the case $f \in L^2(\Omega)$.

Let's now quote a standard regularity result.

Theorem 3.0.11. (Dirichlet Regularity) *Let Ω be a domain in $\mathbb{R}^n$ and $f \in H^m(\Omega)$. If $u \in H_0^1(\Omega)$ is a weak solution to*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

then we have $u \in H^{m+2}(\Omega)$.

Proof. See [Barbu] page 148. ☐

3.0.3 Neumann's Problem with L^2 data.

In this section we will be interested in solving the following:

$$\begin{cases} -\Delta u = f & in \ \Omega \\ \partial_\nu u = g & on \ \partial\Omega \end{cases} \tag{3.5}$$

where $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and where Ω is a domain in $\mathbb{R}^n$. We will introduce the notion of a weak solution to (3.5) and we will obtain a weak solution using two different methods (as we did in the last section). Again a standard regularity result will be quoted.

Definition 3.0.8. (Weak solution to Neumann.) *We say $u \in H^1(\Omega)$ is a weak solution to (3.5) if*

$$(u, v)_{H_0^1} = \int_\Omega fv \, dx + \int_{\partial\Omega} gv \, dS \qquad \forall v \in H^1(\Omega).$$

Use Green's formula to see this notion of a weak solution is compatible with a classical solution.

One thing to notice is we have a compatibility constraint imposed on us. Taking $v = 1$ we see that we need $\int_\Omega f + \int_{\partial\Omega} gdS = 0$.

Let's now obtain a solution to (3.5) when $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ and where the compatibility constraint is satisfied. We will also need to assume Ω connected.

Define $F : \dot{H}^1(\Omega) \to \mathbb{R}$ by $\langle F, v\rangle := \int_\Omega fv + \int_{\partial\Omega} gvdS$.

Clearly $F \in \dot{H}^1(\Omega)^*$ where $\dot{H}^1(\Omega)$ has the H^1 norm. But by Poincaré's inequality we know that $\|\cdot\|_{H_0^1}$ and $\|\cdot\|_{H^1}$ are equivalent on $\dot{H}^1(\Omega)$. So by an application of the Riesz Representation Theorem applied to $\dot{H}^1(\Omega)$ with the $H_0^1(\Omega)$ inner product, we see there exists a unique $u \in \dot{H}^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\partial\Omega} gv \qquad \forall v \in \dot{H}^1(\Omega).$$

Now let $v \in H^1(\Omega)$ and define $\tilde{v} := v - (v)_\Omega \in \dot{H}^1(\Omega)$. Then we see

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla v &= \int_\Omega \nabla u \cdot \nabla \tilde{v} \\ &= \int_\Omega f\tilde{v} + \int_{\partial\Omega} g\tilde{v} \\ &= \int_\Omega f\{v - (v)_\Omega\} + \int_{\partial\Omega} g\{v - (v)_\Omega\} \\ &= \int_\Omega fv + \int_{\partial\Omega} gv - (v)_\Omega \left\{\int_\Omega f + \int_{\partial\Omega} g\right\} \\ &= \int_\Omega fv + \int_{\partial\Omega} gv.\end{aligned}$$

So we have a solution to (3.5) and we see this solution is unique if we restrict ourselves to functions with zero-average.

We can also obtain a solution to (3.5) using the variational method. To do this define $J : \dot{H}^1(\Omega) \to \mathbb{R}$ by

$$J(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 - \left\{\int_\Omega fw + \int_{\partial\Omega} gw\right\}.$$

So we see that

$$J(w) \geq \frac{1}{2}\|\nabla w\|_{L^2}^2 - \|f\|_{L^2}\|w\|_{L^2} - C\|g\|_{L^2(\partial\Omega)}\|w\|_{H^1} \qquad \forall w \in \dot{H}^1$$

where C is obtained from the trace operator. Now using Poincaré's inequality we see that J is bounded below on $\dot{H}^1$. Using arguments similar to the previous sections, we see that J obtains a minimum over $\dot{H}^1$ at say u, and u satisfies

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\partial\Omega} gv \qquad \forall v \in \dot{H}^1.$$

If we assume f and g satisfy the compatibility constraint then we can as usual extend above integral equality to hold for all $v \in H^1$ and we are done. One thing to notice is that the Neumann boundary condition is worked into the functional to be minimized but the Dirichlet boundary condition is worked into the space we minimize over.

Let's now quote a standard regularity result.

Theorem 3.0.12. (Neumann Regularity) *Let Ω be a domain in $\mathbb{R}^n$. Take c to be the constant* 0 *or* 1 *and let $f \in H^m(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution to*

$$\begin{cases} -\Delta u + cu = f & \text{in} \quad \Omega \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.6}$$

then we have $u \in H^{m+2}(\Omega)$.

Proof. See [Barbu] page 152. □

3.0.4 $H^1(\Omega)^*$, $H^1(\Omega)$ associates

Let $u^* \in H^1(\Omega)^*$ and let $u \in H^1(\Omega)$ denote the associate of u^*. Then by definition we know

$$\int_\Omega \{\nabla u \cdot \nabla v + uv\} = \langle u^*, v\rangle_{(H^1)^*,H^1} \qquad \forall v \in H^1(\Omega).$$

Now suppose u^* was of the form

$$\langle u^*, v\rangle_{(H^1)^*,H^1} = \int_\Omega fv + \int_{\partial\Omega} gvdS$$

for $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. (Clearly right hand side is an element of $H^1(\Omega)^*$.) Then we see that $u \in H^1(\Omega)$ would be a weak solution to

$$\begin{cases} -\Delta u + u = f & \text{in} \quad \Omega \\ \partial_\nu u = g & \text{on} \quad \partial\Omega \end{cases}$$

where we are using a slight variation of Definition 3.0.8 to interpret above. Note also that f and g need not satisfy the compatibility condition.

Let's now weaken slightly the regularity of the data.

Take $u^* \in \mathcal{E}'_{-1}(\Omega) + L^2(\Omega) \subseteq H^1(\Omega)^*$, and let $u \in H^1(\Omega)$ denote its associate (see (2.4) for definition of $\mathcal{E}'_{-1}(\Omega)$). Then we see that u should be a weak solution to

$$\begin{cases} -\Delta u + u = u^* & \text{in} \quad \Omega \\ \partial_\nu u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and this turns out to be a suitable interpretation. The only potential problem is when u^* is too singular near $\partial\Omega$ and it "grabs boundary values". An example of this is any u^* of form $v \mapsto \int_{\partial\Omega} gv$. If u^* was of the same form but over $\Omega' \subset\subset \Omega$ then we would not run into above problem. Note that for arbitrary $u^* \in H^1(\Omega)^*$ we have

$$-\Delta u + u = u^* \quad \text{in } H^{-1}(\Omega) \text{ or in } \mathcal{D}'(\Omega) \tag{3.7}$$

where we are taking suitable restrictions.

3.1 The δ Function

We define the Dirac delta function, denoted δ, on $\mathcal{D}(\Omega)$ by

$$\langle \delta, \phi \rangle_{\mathcal{D}',\mathcal{D}} := \phi(0) \qquad \text{for} \quad \phi \in \mathcal{D}(\Omega).$$

If X is a Sobolev space of functions defined on Ω with $\mathcal{D}(\Omega) \subseteq X$, then a natural question to ask is if δ can be extended to some $\overline{\delta} \in X^*$.

To see this is even plausible note that given $W^{k,p}(\Omega)$, we can smooth the space out and increase the norm, therefore enlarge the dual in two obvious ways:

1) Increase k.

2) Or smooth it out in the L^p sense. ie. increase p.

Theorem 3.1.1. *If $0 \in \Omega$ where Ω is a domain in $\mathbb{R}^n$ and if $k > \frac{n}{p}$ then*

$$\overline{\delta} \in W^{k,p}(\Omega)^*$$

Proof. By theorem 2.3.5 we have

$$W^{k,p}(\Omega) \to C^{m,\gamma}(\overline{\Omega}) \qquad \text{is a continuous embedding}$$

where γ is given by theorem 2.3.5 and $m := k - \left\lfloor \frac{n}{p} \right\rfloor - 1$.

Let $\phi \in \mathcal{D}(\Omega)$. Then we have

$$|\langle \delta, \phi \rangle_{\mathcal{D}',\mathcal{D}}| = |\phi(0)| \leq \|\phi\|_{C^{m,\gamma}} \leq C\|\phi\|_{W^{k,p}}$$

Since $\mathcal{D}(\Omega)$ is dense in $W_0^{k,p}(\Omega)$ we see that δ can be uniquely extended to $\overline{\delta} \in W_0^{k,p}(\Omega)^*$. If we want to extend δ to some $\overline{\delta} \in W^{k,p}(\Omega)^*$ where $\mathcal{D}(\Omega)$ not dense in $W^{k,p}(\Omega)$ then we will have to use Hahn-Banach to non-uniquely extend δ.

□

Now note the above proof won't work for $k = 1$ if $p = n$, but we really don't need the full power of theorem 2.3.5. Let's try a borderline case : $k = 1$, $p = n = 2$ and see what happens.

By using extension methods we see if $\delta \notin (W^{1,2}(\mathbb{R}^2))^*$ then $\delta \notin W^{1,2}(\Omega)^*$. Now let's switch to Fourier transform methods. (See [Folland] for details of this method.) Let $\hat{\delta}$ denote the Fourier transform of δ. It can be shown that $\hat{\delta} = 1$ and

$$\begin{aligned} \|\delta\|_{(-1)}^2 := \int_{\mathbb{R}^2} |\hat{\delta}(\xi)|^2 \left\{1 + |\xi|^2\right\}^{-1} d\xi &= \int_{\mathbb{R}^2} \frac{1}{1 + |\xi|^2} d\xi \\ &= \sigma(S^1) \int_0^\infty \frac{r}{1 + r^2} dr = \infty \end{aligned}$$

where $\sigma(S^1)$ is the surface measure of the unit sphere in $\mathbb{R}^2$ and where $\|\cdot\|_{(-1)}$ denotes the $H^{-1}(\mathbb{R}^2)$ norm using the Fourier Transform method.

Hence we see $\delta \notin W^{1,2}(\Omega)^*$. So when $k = 1$ and $p = n = 2$ we see that δ cannot be extended to some $\overline{\delta} \in W^{1,2}(\Omega)^*$.

Let's examine the case $k = 1$ and $p = 2$. From above we see that δ can be extended to an element of $H^{-1}(\Omega)$ iff $n = 1$.

3.2 Hilbert Space related to Cahn-Hilliard Equation

In this section we will define the non-standard Hilbert space (H_0^{-1}) that we will eventually use to write the Cahn-Hilliard equation as a gradient flow over. Throughout this section take Ω to be a connected domain in $\mathbb{R}^n$. The notation we will use might cause some confusion with the dual of H_0^1, which is denoted by H^{-1}, but I believe this is somewhat standard notation for this space.

So let H_0^{-1} denote $\left\{u^* \in H^1(\Omega)^* : \langle u^*, 1\rangle_{(H^1)^*,H^1} = 0\right\}$. Since Ω is bounded we know $1 \in H^1(\Omega)$, hence H_0^{-1} is well defined.

Before we define a norm and inner product on H_0^{-1} we need to define a few spaces. Recall Definition 2.3.4 where we defined the Hilbert space H_A^1.

Let $(H_A^1)^*$ denote the dual of H_A^1. We will use $(H_A^1)^*$ to induce an inner product on H_0^{-1} and so let's examine this space a bit.

Given $u^*, v^* \in (H_A^1)^*$ with associates $u, v \in H_A^1$, we have by definition

$$\langle u^*, \phi\rangle_{(H_A^1)^*,H_A^1} = (u, \phi)_{H_A^1} = (\nabla u, \nabla \phi)_{L^2} \quad \forall \phi \in H_A^1$$

and

$$(u^*, v^*)_{(H_A^1)^*} = (u, v)_{H_A^1} \quad , \quad \|u^*\|_{(H_A^1)^*} = \|u\|_{H_A^1}.$$

Let's now define a norm / inner product on H_0^{-1} and then later we can verify that everything is valid.

Definition 3.2.1. (H_0^{-1} norm / inner product)

$$\begin{aligned} H_0^{-1} &:= \left\{u^* \in (H^1)^* : \langle u^*, 1\rangle_{(H^1)^*,H^1} = 0\right\} \\ \|u^*\|_{H_0^{-1}} &:= \|u^*|H_A^1\|_{(H_A^1)^*} \\ (u^*, v^*)_{H_0^{-1}} &:= (\nabla u, \nabla v)_{L^2} \end{aligned}$$

where $u^*, v^* \in H_0^{-1}$ and where $u, v \in H_A^1$ are the associates of $u^*|H_A^1, v^*|H_A^1 \in (H_A^1)^*$. So we have

$$\int_\Omega \nabla u \cdot \nabla \phi = \langle u^*, \phi \rangle_{(H^1)^*, H^1} \quad \forall \phi \in H_A^1. \tag{3.8}$$

Since $\langle u^*, 1 \rangle_{(H^1)^*, H^1} = 0$ *we easily see that* (3.8) *extends to all* $\phi \in H^1(\Omega)$.

Now suppose $u^* \in H_0^{-1} \cap \left\{ \mathcal{E}'_{-1}(\Omega) + L^2(\Omega) \right\}$, then $u \in H_A^1$ is the unique weak solution to

$$\begin{cases} -\Delta u &= u^* \quad \text{in } \Omega \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

with zero-average.

Remark 3.2.1. *At this point it is not entirely obvious that given* $u^* \in H_0^{-1}$, *that we have* $u^* | H_A^1 \in (H_A^1)^*$. *We will see this is the case though.*

Theorem 3.2.1. H_0^{-1} *is closed in* $H^1(\Omega)^*$.

Proof. Let $f_n \in H_0^{-1}$ and $f_n \to f$ in $H^1(\Omega)^*$. There exists a $\delta = \delta(\Omega) > 0$ such that for all $c \in \mathbb{R}$ with $|c| \le \delta$ we have $\|c\|_{H^1} \le 1$. (Take $\delta(\Omega) := 1/\sqrt{|\Omega|}$). So

$$\begin{aligned} \|f_n - f\|_{(H^1)^*} &= \sup \left\{ \langle f_n - f, v \rangle_{(H^1)^*, H^1} : \|v\|_{H^1} \le 1 \right\} \\ &\ge \langle f_n - f, c \rangle_{(H^1)^*, H^1} \\ &= c \langle f_n, 1 \rangle_{(H^1)^*, H^1} - c \langle f, 1 \rangle_{(H^1)^*, H^1} \\ &= -c \langle f, 1 \rangle_{(H^1)^*, H^1} \end{aligned}$$

But $\|f_n - f\|_{(H^1)^*} \to 0$.

Hence we see $c \langle f, 1 \rangle_{(H^1)^*, H^1} \ge 0$ for all $|c| \le \delta$. From this we can conclude that $\langle f, 1 \rangle_{(H^1)^*, H^1} = 0$ or $f \in H_0^{-1}$.

□

Now we have used $(H_A^1)^*$ to define our norm and inner product for H_0^{-1}. In particular we defined the inner product / norm on H_0^{-1} such that $(H_A^1)^*$ and H_0^{-1}

are essentially the same spaces, as far as Hilbert spaces are concerned (there will be a unitary map between the two). Even though these spaces are "the same", we will give them their own notation to avoid any confusion on which domains the linear functionals are defined.

We already have H_0^{-1} a Hilbert space w.r.t. the $(H^1)^*$ norm and so if we can show that $\|\cdot\|_{H_0^{-1}}$ and $\|\cdot\|_{(H^1)^*}$ are equivalent on H_0^{-1} then we'd have H_0^{-1} a Hilbert space w.r.t. $\|\cdot\|_{H_0^{-1}}$.

Theorem 3.2.2. *$\|\cdot\|_{H_0^{-1}}, \|\cdot\|_{(H^1)^*}$ are equivalent on H_0^{-1}.*

Proof. Let C denote one of the constants from the fact that the H_A^1, H^1 norms are equivalent on H_A^1. Let $u^* \in H_0^{-1}$ and $v \in H_A^1$. Then

$$\begin{aligned}
\left\langle u^*|H_A^1, v\right\rangle_{(H_A^1)^*, H_A^1} &= \langle u^*, v\rangle_{(H^1)^*, H^1} \\
&\leq \|u^*\|_{(H^1)^*}\|v\|_{H^1} \\
&\leq C\|u^*\|_{(H^1)^*}\|v\|_{H_A^1}.
\end{aligned}$$

Hence $u^*|H_A^1 \in (H_A^1)^*$ and $\|u^*\|_{H_0^{-1}} \leq C\|u^*\|_{(H^1)^*}$.
Now let $u^* \in H_0^{-1}$ and $v \in H^1$ with $\|v\|_{H^1} \leq 1$. So $v - (v)_\Omega \in H_A^1$ and

$$\begin{aligned}
\langle u^*, v\rangle_{(H^1)^*, H^1} &= \langle u^*, v - (v)_\Omega\rangle_{(H^1)^*, H^1} \\
&= \left\langle u^*|H_A^1, v - (v)_\Omega\right\rangle_{(H_A^1)^*, H_A^1} \\
&\leq \|u^*\|_{H_0^{-1}}\|v - (v)_\Omega\|_{H_A^1} \\
&= \|u^*\|_{H_0^{-1}}\|\nabla v\|_{L^2} \\
&\leq \|u^*\|_{H_0^{-1}}\|v\|_{H^1} \\
&\leq \|u^*\|_{H_0^{-1}}.
\end{aligned}$$

Now sup over v to see $\|u^*\|_{(H^1)^*} \leq \|u^*\|_{H_0^{-1}}$. Hence we see $\|\cdot\|_{H_0^{-1}}$ and $\|\cdot\|_{(H^1)^*}$ are equivalent on H_0^{-1}.

□

Definition 3.2.2. *So as to not keep saying associate, let's define a couple of mappings. From here on Φ, Ψ will be defined in following way: Given $u^* \in H_0^{-1}$ with $u \in H_A^1$ as described in* Definition (3.2.1), *define*

$$\Phi : H_0^{-1} \to H_A^1 \quad by \quad \Phi(u^*) := u.$$

Define $\Psi := \Phi^{-1}$. Both Φ, Ψ are unitary.

One might ask do we really need to consider "singular" elements of $H^1(\Omega)^*$ when we define H_0^{-1} or can we just use the subspace $M := \dot{L}^2(\Omega) \subseteq H_0^{-1}$, ie.

$$M := \left\{ f \in L^2(\Omega) : \int_\Omega f = 0 \right\}$$

where M has the $H^1(\Omega)^*$ norm. If we hope to use Hilbert space theory then the answer is YES we need to consider the "singular" elements. To see this let's show that M, as defined above, is not complete w.r.t. the $H^1(\Omega)^*$ norm.

Theorem 3.2.3. *M is not complete w.r.t. the $H^1(\Omega)^*$ norm.*

Proof. Let $f \in M, v \in H^1(\Omega)$ with $\|v\|_{H^1} \leq 1$. Then we have

$$\langle f, v \rangle_{(H^1)^*, H^1} = \int_\Omega fv \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2}$$

Hence for all $f \in M$ we have $\|f\|_{(H^1)^*} \leq \|f\|_{L^2}$.

Now suppose we have M complete w.r.t. the $H^1(\Omega)^*$ norm. Then by theorem 2.2.2 (Open Mapping Theorem) we know there exists a $C > 0$ such that

$$\|f\|_{L^2} \leq C \|f\|_{(H^1)^*} \qquad \forall f \in M. \tag{3.10}$$

Now let's try and show no such C can exist.

Let $\{e_n\}_n \subset M$ denote an orthonormal system in $L^2(\Omega)$ and a orthogonal system in $H_0^1(\Omega)$ (here by orthonormal / orthogonal system we do not mean a basis). Then by standard Hilbert space theory we know

$$e_n \rightharpoonup 0 \quad \text{in} \quad L^2.$$

Fix n and let $\{v_n^m\}_m \subseteq H^1$ be such that $\|v_n^m\|_{H^1} \leq 1$ and

$$\langle e_n, v_n^m \rangle_{(H^1)^*,H^1} \rightarrow \|e_n\|_{(H^1)^*}.$$

By theorem 2.2.6 (Weak Sequential Convergence) along with theorem 2.3.4 ($H^1 \hookrightarrow L^2$) and after passing to a suitable subsequence (without renaming), there exists a $\|v_n\|_{H^1} \leq 1$ such that

$$\begin{aligned} v_n^m &\rightharpoonup v_n \quad \text{in} \quad H^1 \\ v_n^m &\rightarrow v_n \quad \text{in} \quad L^2. \end{aligned}$$

Hence we have

$$\|e_n\|_{(H^1)^*} = \langle e_n, v_n \rangle_{(H^1)^*,H^1} = (e_n, v_n)_{L^2}.$$

Again by theorem 2.2.6 (Weak Sequential Convergence) along with theorem 2.3.4 ($H^1 \hookrightarrow L^2$) and after passing to a suitable subsequence (without renaming), there exists $\|v\|_{H^1} \leq 1$ such that

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{in} \quad H^1 \\ v_n &\rightarrow v \quad \text{in} \quad L^2. \end{aligned}$$

From this we see

$$\|e_n\|_{(H^1)^*} = (e_n, v_n - v)_{L^2} + (e_n, v)_{L^2} \leq \|v_n - v\|_{L^2} + (e_n, v)_{L^2} \rightarrow 0$$

since $v_n \rightarrow v$ in L^2 and $e_n \rightharpoonup 0$ in L^2.

But this contradicts (3.10). Hence by contradiction we have proven the theorem.

□

Remark 3.2.2. *We could have eliminated a few steps from the above proof if we had used the fact that every continuous linear functional on a reflexive Banach space is norm obtaining on the closed unit ball.*

Chapter 4

Gradients and Gradient Flows

In the next few sections we will introduce various notions of a gradient on a Hilbert space. These will include the classical gradient, sub-differential (sub-gradient) and the constrained gradient.

Take H to be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let $\langle\cdot,\cdot\rangle$ denote the H^*, H pairing.

4.1 Classical Gradients

Given $F : H \to \mathbb{R}$ and $u \in H$ we say F is G-differentiable, in honor of Gâteaux, at $u \in H$ with derivative $F'(u) \in H^*$ if

$$\frac{d}{dt}F(u+tv)\bigg|_{t=0} = \langle F'(u), v\rangle \quad \forall v \in H. \tag{4.1}$$

If this limit converges uniformly for $\|v\| = 1$ then we say F is Fréchet differentiable at u.

So if F is G-differentiable a $u \in H$ then by the Riesz Representation Theorem we know there exists a unique $w \in H$ such that

$$\langle F'(u), v\rangle = (w, v) \qquad \forall v \in H.$$

We will denote w by $gradF(u)$. This is what we will call the classical gradient of F at u. Note that this agrees with our usual notion of a gradient of F when $H = \mathbb{R}^n$. Let us now move on to the notion of a gradient that is, perhaps, the most widely used when F is convex.

4.2 Subgradients and Subdifferentials

Take $F : H \to (-\infty, \infty]$ to be convex and define

$$\begin{aligned} D(F) &:= \{u \in H : F(u) \in \mathbb{R}\} \\ \partial F(u) &:= \{v \in H : F(w) \geq F(u) + (v, w-u), \forall w \in H\} \\ D(\partial F) &:= \{u \in H : \partial F(u) \neq \emptyset\}. \end{aligned}$$

$\partial F(u)$ is what we call the subdifferential of F at u. Note that $\partial F(u)$ is set valued. We call $v \in \partial F(u)$ a subgradient of F at u. The geometric interpretation of $\partial F(u)$ is that $v \in \partial F(u)$ if$v$ is the "slope" of a affine functional touching the graph of F from below at u. We will say F is proper if it is not identically ∞.

Let's look at a simple example of a subgradient.

Example 4.2.1. $F(x) := |x|$ *where* $H := \mathbb{R}$. *Then we have*

$$\partial F(u) = \{-1\} \quad \forall u < 0 \quad , \quad \partial F(0) = [-1, 1] \quad , \quad \partial F(u) = \{1\} \quad \forall u > 0.$$

The following theorem shows that the notion of a subdifferential is a suitable generalization of a gradient.

Theorem 4.2.1. $F : H \to \mathbb{R}$ *convex and G-differentiable at* $u \in H$. *Then*

$$\partial F(u) = \{gradF(u)\}.$$

Proof. Let $w \in H, t \in (0, 1)$ and define

$$\begin{aligned} L(t) := \frac{F(u+t(w-u))-F(u)}{t} &= \frac{F(tw+(1-t)u)-F(u)}{t} \\ &\leq \frac{tF(w)+(1-t)F(u)-F(u)}{t} \\ &= F(w)-F(u) \end{aligned}$$

where the inequality follows from convexity of F. But $L(t) \to (gradF(u), w-u)$ as $t \to 0^+$. Hence we get

$$F(w)-F(u) \geq (gradF(u), w-u)$$

and so $gradF(u) \in \partial F(u)$.

Now let $v \in \partial F(u)$ and let $g \in H$ be such that $v = gradF(u) + g$. Hence for all $t > 0$ we have

$$F(u+tg) \geq F(u) + (gradF(u)+g, tg) = F(u) + t\,(gradF(u), g) + t\|g\|^2.$$

After some rearrangement we get

$$\frac{F(u+tg)-F(u)}{t} \geq (gradF(u), g) + \|g\|^2.$$

Now letting $t \to 0^+$ we get

$$(gradF(u), g) \geq (gradF(u), g) + \|g\|^2$$

so we see $g = 0$. Hence we find

$$\partial F(u) \subseteq \{gradF(u)\}.$$

□

So the subgradient of F at u allows us to make some sense of $gradF(u)$ even if $gradF(u)$ does not exist in the G sense.

Theorem 4.2.2. (Basic Properties of Subdifferentials) *Take* $F : H \to (-\infty, \infty]$ *to be convex, proper and lower semicontinuous. Then*

(*i*) $D(\partial F) \subseteq D(F)$

(*ii*) *For all* $w \in H$ *and* $\lambda > 0$ *the problem*

$$u + \lambda \partial F(u) \ni w$$

has a unique solution $u \in D(\partial F)$.

Assertion (ii) means that there exists $u \in D(\partial F)$ and $v \in \partial F(u)$ such that

$$w = u + \lambda v.$$

Proof. See [Evans] page 524. □

Let's now move on to the generalization of a gradient which we will use almost exclusively.

4.3 Constrained Gradient

The idea of a constrained gradient is to limit the directions v in (4.1). Let's look at an example to see why this might help.

Example 4.3.1. *Define* $F : L^2(\Omega) \to (-\infty, \infty]$ *by*

$$F(u) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx & u \in H^1(\Omega) \\ \infty & \textit{otherwise.} \end{cases}$$

Fix $u \in H^2(\Omega)$. *Since* $F = \infty$ *on a dense set of* $L^2(\Omega)$, *we see that there is no chance of* $gradF(u)$ *existing in the classical sense. Now let* $v \in C_c^\infty(\Omega)$. *Then we have*

$$\begin{aligned} \frac{F(u+tv) - F(u)}{t} &= (\nabla u, \nabla v)_{L^2} + \frac{t}{2}\|\nabla v\|_{L^2}^2 \\ &\to (\nabla u, \nabla v)_{L^2} \qquad (4.2) \end{aligned}$$

as $t \longrightarrow 0$. *Now using Green's formula we get*

$$(\nabla u, \nabla v)_{L^2} = -\int_\Omega \Delta u v = (-\Delta u, v)_{L^2}.$$

Hence, if we limit the directions v *to* $C_c^\infty(\Omega)$, *where* $H = L^2(\Omega)$, *in (4.1) then we see that* $gradF(u)$ *should be* $-\Delta u$.

Let's use the above example as motivation to try and generalize this idea of a constrained gradient.

Again take $F : H \longrightarrow (-\infty, \infty]$ and let $X \subseteq H$ denote a subspace. Take $u \in H$ with $F(u) \in \mathbb{R}$.

Definition 4.3.1.

$$G(F, X, u) := \left\{ f \in H : \quad \lim_{t \to 0} \frac{F(u + tv) - F(u)}{t} = (f, v) \qquad \forall v \in X \right\}.$$

So note X *is limiting the directions* v *and* $G(F, X, u)$ *is a set of good candidates to be called* $gradF(u)$.

Now note that if F was G-differentiable at u then $G(F, X, u) = gradF(u) + X^\perp$.

Now we some how want to define our constrained gradient using $G(F, X, u)$. Let's, for the time being, assume we know that $G(F, X, u)$ is closed, convex and non-empty. Then by theorem 2.2.4 (Element of Least Norm), $G(F, X, u)$ has a unique element of least norm. So this is how we will define our constrained gradient.

Definition 4.3.2. (X Constrained Gradient of F at u)

Assuming $G(F, X, u)$ *is closed, convex and non-empty then define* $grad^X F(u)$ *to be the unique element of* $G(F, X, u)$ *of least norm.*

When necessary to indicate the Hilbert space H *we will write it as* $grad_H^X F(u)$.

Let's borrow some notation from convex analysis.

Definition 4.3.3. *Take* $F : H \longrightarrow [-\infty, \infty]$. (Note that we are not restricting F to be convex). *Take* X *a subspace of* H. *Now define*

$$D(grad_H^X F) \;:=\; \{u \in H : G(F, X, u) \neq \emptyset\}. \qquad \text{(domain of } X \text{ constrained gradient of } F\text{)}$$

Lemma 4.3.1. *$G(F, X, u)$ is closed and convex.*

Proof. This almost follows by definition of $G(F, X, u)$.

□

The next Theorem will generally allow us not to worry about taking the element of $G(F, X, u)$ with least norm.

Theorem 4.3.1. *If X is a dense subspace of H then $G(F, X, u)$ is empty or a singleton.*

Proof. Take $G(F, X, u)$ non-empty and let $f_1, f_2 \in G(F, X, u)$. Hence we have $(f_1, v) = (f_2, v)$ for all $v \in X$. So we see $f_1 - f_2 \in X^{\perp}$ but X dense hence $X^{\perp} = \{0\}$ so $f_1 = f_2$.

□

This idea of a constrained gradient will allow us to handle certain functionals that a subgradient will not and will be the generalization of a gradient we will use when looking at the Cahn-Hilliard equation.

4.4 Examples of Gradients

Let's examine some typical functionals and see what their gradients are over various spaces. Take Ω a domain in $\mathbb{R}^n$.

Define

$$F_0(u) := \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx \qquad F_1(u) := \int_{\Omega} W(u(x))dx$$

where $W \in C^{\infty}(\mathbb{R}, \mathbb{R})$. Then we have for $X := C_c^{\infty}(\Omega)$ and $u \in C_c^{\infty}(\Omega)$ that

$$grad_{L^2}^X F_0(u) = -\Delta u \tag{4.3}$$

$$grad_{H_0^1}^X F_0(u) = u \tag{4.4}$$

$$grad_{L^2}^X F_1(u) = W'(u) \tag{4.5}$$

$$grad_{H^{-1}}^X F_0(u) = \Delta^2 u \tag{4.6}$$

$$grad_{H^{-1}}^X F_1(u) = -\Delta W'(u) \qquad \text{if} \quad W'(0) = 0. \tag{4.7}$$

We will justify the above claims in a moment.

We will use the following convention since we will be looking at the above two functionals over various Hilbert spaces.

Remark 4.4.1. **(Notational Convention)** *If a functional F on a Hilbert space H is given by a formula then, unless otherwise mentioned, the domain of F will be understood to be the biggest subset of H where the "formula" is well defined and finite. To clarify this let F_0 be given as above and $H := L^2(\Omega)$. Then it is understood that F_0 is in fact given by*

$$F_0(u) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx & u \in H^1(\Omega) \\ \infty & \text{otherwise,} \end{cases}$$

where as on occasion we might use the functional

$$F(u) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx & u \in H_0^1(\Omega) \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 4.4.1. *For $u, v \in C_c^\infty(\Omega)$ we have*

$$\frac{F_0(u+tv) - F_0(u)}{t} \to \int_\Omega \nabla u \cdot \nabla v \qquad \text{and} \qquad \frac{F_1(u+tv) - F_1(u)}{t} \to \int_\Omega W'(u)v.$$

Proof. Trivial. □

Now let's check above claims of gradients.

(4.3) Since

$$\int_\Omega \nabla u \cdot \nabla v = (-\Delta u, v)_{L^2}$$

for all $v \in X$ we are done.

(4.4) Follows since

$$\int_\Omega \nabla u \cdot \nabla v = (u, v)_{H_0^1}$$

for all $v \in X$.

(4.5) This follows directly from Lemma 4.4.1.

For the next two examples we will switch to * notation to agree with previous sections on associates for $H^{-1}(\Omega)$, $H_0^1(\Omega)$. Starred elements will be viewed as belonging to $H^{-1}(\Omega)$ and non-starred elements will be viewed as belonging to $H_0^1(\Omega)$.

(4.6) Let $u^* \in C_c^\infty(\Omega)$ and $v^* \in X$. Define $w := -\Delta u^* \in X \subseteq H_0^1$. Let $w^* \in H^{-1}(\Omega)$, $v \in H_0^1(\Omega)$ denote the associates of w and v^* respectively. By elliptic regularity we have $v \in H_0^1(\Omega) \cap C^\infty(\overline{\Omega})$. Then we have

$$\begin{aligned}
(w^*, v^*)_{H^{-1}} = (w, v)_{H_0^1} &= \int_\Omega (-\Delta v) w \\
&= \int_\Omega v^* w \\
&= \int_\Omega v^*(-\Delta u^*) \\
&= \int_\Omega \nabla v^* \cdot \nabla u^*
\end{aligned}$$

Since this holds all $v^* \in X$ we see $grad_{H^{-1}}^X F_0(u^*) = w^* = -\Delta w = \Delta^2 u^*$.

(4.7) Let $u^* \in C_c^\infty(\Omega)$, $v^* \in X$ and define $w^* := -\Delta W'(u^*) \in C^\infty(\overline{\Omega})$. Let $v, w \in H_0^1(\Omega)$ denote the associates of v^*, w^* respectively. By elliptic regularity we have $v, w \in H_0^1(\Omega) \cap C^\infty(\overline{\Omega})$. So we have

$$\begin{aligned}
(w^*, v^*)_{H^{-1}} &= (w, v)_{H_0^1} \\
&= \int_\Omega \nabla w \cdot \nabla v \\
&= \int_\Omega v(-\Delta w) \\
&= \int_\Omega w^* v \\
&= \int_\Omega (-\Delta W'(u^*)) v \\
&= \int_\Omega (-\Delta v) W'(u^*) + \int_{\partial\Omega} W'(u^*) \partial_\nu v \\
&= \int_\Omega v^* W'(u^*) + \int_{\partial\Omega} W'(u^*) \partial_\nu v \\
&= \int_\Omega v^* W'(u^*) + W'(0) \int_{\partial\Omega} \partial_\nu v \\
&= \int_\Omega v^* W'(u^*) \qquad \text{since } W'(0) = 0.
\end{aligned}$$

So we see that

$$grad^X_{H^{-1}} F_1(u^*) = w^* = -\Delta W'(u^*).$$

Let's examine what happens when $W'(0) \neq 0$. Let $u^* \in C_c^\infty(\Omega)$ and $v^* \in X$. So as before we have

$$\lim_{t \to 0} \frac{F_1(u^* + tv^*) - F_1(u^*)}{t} = \int_\Omega W'(u^*)v^*.$$

If we assume $w^* = grad^X_{H^{-1}} F_1(u^*) \in H^{-1}$ exists and $w, v \in H_0^1$ denote the associates of w^*, v^* then as usual we have

$$\begin{aligned} \int_\Omega W'(u^*)v^* &= (w^*, v^*)_{H^{-1}} \\ &= \int_\Omega \nabla w \cdot \nabla v \\ &= \int_\Omega w(-\Delta v) \\ &= \int_\Omega w v^* \end{aligned}$$

Since this holds for all $v^* \in X := C_c^\infty(\Omega)$ then we see that $W'(u^*) = w \in H_0^1$. But since $W'(u^*) = W'(0) \neq 0$ in a neighborhood of $\partial\Omega$ we see $W'(u^*) \notin H_0^1$, hence by contradiction we see that $grad^X_{H^{-1}} F_1(u^*)$ does not exist.

We would hope that if we shrink the subspace of directions (X) to say (Y), then $grad^Y_{H^{-1}} F_1(u^*)$ might exist. Towards this take $u^* \in C_c^\infty(\Omega)$ and we will use the subspace of directions given by $Y := \dot{C}_c^\infty(\Omega)$. Take $w := W'(u^*) - W'(0) \in H_0^1$ and let w^* denote the associate in H^{-1}. So $w^* = -\Delta W'(u^*)$ and we have

$$\begin{aligned}(w^*, v^*)_{H^{-1}} &= \int_\Omega \nabla w \cdot \nabla v \\ &= \int_\Omega w(-\Delta v) \\ &= \int_\Omega w v^* \\ &= \int_\Omega v^* \{W'(u^*) - W'(0)\} \\ &= \int_\Omega v^* W'(u^*)\end{aligned}$$

for all $v^* \in Y$. Hence we see that $w^* \in G(F, Y, u^*) \neq \emptyset$ and so $grad^Y_{H^{-1}} F_1(u^*)$ exists. We are NOT claiming that $w^* = grad^Y_{H^{-1}} F_1(u^*)$ since we will have to take the element of least norm in $G(F, Y, u^*)$.

Let's examine yet one more example.

Example 4.4.1. *Take $F_0(u) := \frac{1}{2}\int_\Omega |\nabla u|^2$ and we will use H^1_0 as our Hilbert space but we will use a different inner product. Before we do this let's pick our inner product.*

Lemma 4.4.2. *Let $p \in L^\infty$ but with $\underline{p} := \inf_\Omega p > -\lambda_1$ where λ_1 is defined in Definition (2.3.2). Now define*

$$(u, v)_p := \int_\Omega \{\nabla u \cdot \nabla v + puv\}.$$

Let's show this an equivalent inner product on H^1_0.

Proof. Suppose we could show for $c \in \mathbb{R}$ with $c > -\lambda_1$ that $(\cdot, \cdot)_c$ was an equivalent inner product on H^1_0 then since

$$\|u\|_{\underline{p}} \leq \|u\|_p \leq \|u\|_{\overline{p}}$$

we'd have desired result where $\overline{p} := \sup_\Omega p$. Now let $c \in \mathbb{R}$ be as above. Then

$$\|u\|_c^2 = \int_\Omega |\nabla u|^2 + c\int_\Omega u^2 \leq \{|c| + 1\} \|u\|^2_{H^1} \leq C\{|c| + 1\} \|u\|^2_{H^1_0}$$

where C is obtained from Theorem 2.3.3 (Poincaré's inequality).

To finish the proof of equivalence we need to show there exists some $a > 0$ such that

$$a \|u\|_{H_0^1}^2 \leq \|u\|_c^2 \qquad \forall u \in H_0^1(\Omega). \tag{4.8}$$

If $c \geq 0$ then (4.8) holds trivially with $a = 1$. Take $-\lambda_1 < c < 0$ and suppose no $a > 0$ exists as in (4.8).

Then for all positive integers m there exists a $u_m \in H_0^1(\Omega)$ such that

$$\frac{1}{m}\int_\Omega |\nabla u_m|^2 > \int_\Omega |\nabla u_m|^2 + c\int_\Omega u_m^2.$$

After L^2 normalizing we obtain

$$\left\{1 - \frac{1}{m}\right\}\int_\Omega |\nabla u_m|^2 < -c$$

for some $u_m \in H_0^1(\Omega)$ with $\|u_m\|_{L^2} = 1$. Since $-c < \lambda_1$ we see by taking m sufficiently large that there exists some $u_m \in H_0^1(\Omega)$ with $\|u_m\|_{L^2} = 1$ and $\int_\Omega |\nabla u_m|^2 < \lambda_1$, but this contradicts the definition of λ_1.

□

Now for gradient calculation. Let $u, v \in X := C_c^\infty(\Omega)$. As before we have

$$\frac{F_0(u+tv) - F_0(u)}{t} \to \int_\Omega \nabla u \cdot \nabla v.$$

Now we know $\{\phi \mapsto \int_\Omega \nabla u \cdot \nabla \phi\} \in H^{-1}(\Omega)$ so by Theorem 2.2.3 (Riesz Representation Theorem), applied to $H_0^1(\Omega)$ but with the $(\cdot,\cdot)_p$ inner product, call this space $H_{0,p}^1$, we know there exists a unique $w \in H_0^1$ with

$$(w, v)_p = \int_\Omega \nabla u \cdot \nabla v$$

all $v \in X$. Hence we have $grad_{H_{0,p}^1}^X F_0(u) = w$. Also clearly $w \in H_0^1$ is a weak solution to

$$\left\{\begin{array}{rclll} -\Delta w + pw & = & -\Delta u & & \text{in } \Omega \\ w & = & 0 & & \text{on } \partial\Omega. \end{array}\right. \tag{4.9}$$

4.5 Gradient Flows and Abstract ODE's

A gradient flow is a special abstract ODE over a Hilbert space. An example of a abstract ODE would be the following:

$$\begin{cases} u'(t) = Au(t) & t > 0 \\ u(0) = u_0 \end{cases} \tag{4.10}$$

where X is a Banach space, $u_0 \in X$, $u : [0, \infty) \to X$, and $A : X \to X$ is an operator. One must define what one means by (4.10). The most obvious way is to define $u'(t)$ to be the limit of

$$\frac{u(t+h) - u(t)}{h}$$

as $h \to 0$, where convergence is taken in the strong or weak sense in X. There are other interpretations which utilize distribution theory but we will not examine any of these.

Typically X is some function space and A is some partial differential operator so an abstract ODE typically becomes a PDE.

Definition 4.5.1. (Gradient Flow) *We will think of a gradient flow as the following abstract ODE over a Hilbert space:*

$$\begin{cases} u'(t) = -KgradF(u(t)) & t > 0 \\ u(0) = u_0 \in H \end{cases} \tag{4.11}$$

where $F : H \to \mathbb{R}$ *is some functional and* $K > 0$.

(There are more general notions of a gradient flow but this will suffice for our purposes.)

Gradient Flows Decrease the Energy Functional Along Solutions.

As mentioned in the introduction, a standard way of letting u evolve such that a certain energy (F) decreases in time, is to let u evolve in the direction opposite to

$gradF(u)$. To see this let u denote a solution (in some sense) to (4.11) with $K = 1$. Then, assuming some regularity in both F and u, we have

$$\frac{d}{dt}F(u(t)) = \langle F'(u(t)), u'(t)\rangle_{H^*,H} = (gradF(u(t)), u'(t))_H = -\|gradF(u(t))\|_H^2 \leq 0.$$

So we see F will decrease in time along a solution. This property makes gradient flows fairly attractive for modeling physical processes since typically we will have some energy that, according to the physics, should decrease in time.

PDE's Induced from a Gradient Flow

Let's now return to the section "Examples of Gradient" and see what PDE's the gradient flows induce.

Define $F(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 + W(u(x))\, dx$ where $W'(0) = 0$. If we take $X := C_c^\infty(\Omega)$ we see the induced PDE's from (4.11) are

$$u_t = \Delta u - W'(u) \qquad \text{when} \qquad H = L^2(\Omega)$$

and

$$u_t = -\Delta^2 u + \Delta W'(u) \qquad \text{when} \qquad H = H^{-1}(\Omega)$$

where we have taken $K = 1$.

Note 4.5.1. *For the calculations we assumed $u(t) \in C_c^\infty(\Omega)$ for all $t > 0$, which we expect not to be the case in general for parabolic equations, but the above examples were to show how the resulting evolution equation depends on the choice of the Hilbert space.*

4.6 A Simple Evolution Equation

In this section we will obtain a global solution to a nonlinear parabolic equation which is much simpler than the Cahn-Hilliard equation, therefore it will give us a gentle introduction to the various methods involved. The equation we will look at will be a nonlinear heat equation given by the following:

$$\begin{cases} u_t - \Delta u &= -u^3 =: f(u) \qquad \Omega \times (0, \infty) \\ u(x,t) &= 0 \qquad \partial\Omega \times [0, \infty) \\ u(x,0) &= \phi \qquad \Omega \times \{t = 0\}, \end{cases} \tag{4.12}$$

where Ω is a domain in $\mathbb{R}^3$ and ϕ denotes some function on Ω.

The first method will be a semigroup approach which utilizes Banach's Fixed Point Theorem to obtain a local solution and then we will apply a blow-up alternative along with some a priori bounds to extend this local solution to a global solution. This will be the method we will use later to obtain a local solution to the Cahn-Hilliard equation.

The second method we will use will be a sub-differential method which will give us a global solution directly. We cannot apply this method directly to the Cahn-Hilliard equation since we will lack convexity. (There may be more advanced methods which can handle the lack of convexity.)

We will not start from basics with either of these methods. For the second method we will choose the "correct" functional F and the "correct" Hilbert space H and then apply standard theory. The details can be found in [Evans].

For the first method we will go into more details. We will assume the reader is familiar with linear semigroups, Banach valued integrals and the spaces $L^p(0, T; X)$, where X is a Banach space. We will start from a variation of parameters type formula (Duhamel's Formula) and then use fixed point theory on an appropriate space to obtain a local solution. When we obtain a local solution to Cahn-Hilliard we will not see a fixed point theorem, but be assured that it is hidden in a local existence theorem we will apply.

For more details on either of these methods one should consult [Evans].

4.6.1 Semigroup Approach

Define the operator B on $L^2(\Omega)$ by

$$\begin{aligned} D(B) &:= \left\{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\right\} \qquad \text{(domain of } B) \\ Bu &:= \Delta u \qquad \text{for} \quad u \in D(B). \end{aligned}$$

Note 4.6.1. *Since we have $\partial\Omega$ sufficiently smooth then by Elliptic Regularity we know that $D(B) = H_0^1 \cap H^2$.*

Let $\{S(t)\}_{t\geq 0}$ denote the semigroup generated by B in $L^2(\Omega)$. It can be shown that if $\phi \in L^2(\Omega)$ and $u(t) := S(t)\phi$, then u is the unique solution of the following problem (linear heat):

$$u \in C([0,\infty); L^2) \cap C^1((0,\infty); L^2) \ , \ \ \Delta u \in C((0,\infty); L^2)$$

$$\left\{ \begin{aligned} u'(t) &= \Delta u(t) \qquad \forall t > 0 \\ u(0) &= \phi. \end{aligned} \right. \tag{4.13}$$

In addition we have the following decay estimates :

$$u \in C((0,\infty); H_0^1) \tag{4.14}$$

$$\|\Delta u\|_{L^2} \leq \frac{1}{t\sqrt{2}}\|\phi\|_{L^2} \qquad \forall t > 0 \tag{4.15}$$

$$\|\nabla u\|_{L^2} \leq \frac{1}{\sqrt{2t}}\|\phi\|_{L^2} \qquad \forall t > 0. \tag{4.16}$$

If we assume ϕ has more regularity, then u will also have more regularity. If $\phi \in H_0^1$ then we also have $u \in C([0,\infty); H_0^1)$ and

$$\|\Delta u\|_{L^2} \leq \frac{1}{\sqrt{2}\sqrt{t}}\|\nabla\phi\|_{L^2} \qquad \forall t > 0.$$

See [Cazenave] for details of above estimates.

We will obtain a local solution to (4.12) in a weak sense, which we will call a mild solution. Before we introduce what a mild solution is, let's examine the nonlinear term.

Define $F(u)(x) := f(u(x)) := (u(x))^3$. Then using the fact that H^1 is continuously imbedded in L^6 (theorem 2.3.4), we see

$$\|F(u)\|_{L^2} = \|u\|_{L^6}^3 \leq C_0\|u\|_{H^1}^3.$$

Lemma 4.6.1. *$F \in Lip^{loc}(H^1(\Omega), L^2(\Omega))$.*

Proof. Let $u, v \in H^1(\Omega)$. Then we have

$$\begin{aligned} \|F(u) - F(v)\|_{L^2}^2 &= \int_\Omega |u-v|^2 \left\{u^2 + uv + v^2\right\}^2 dx \\ &\leq \||u-v|^2\|_{L^p} \| \left\{u^2+uv+v^2\right\}^2 \|_{L^q} \end{aligned}$$

where p and q are conjugate. Take $p = 3, q = 3/2$ and apply theorem 2.3.4 to get

$$\||u-v|^2\|_{L^3} \leq C\|u-v\|_{H^1}^2 \qquad \text{and}$$
$$\| \left\{u^2+uv+v^2\right\}^2 \|_{L^{3/2}}^{3/2} = \int_\Omega \left\{u^2+uv+v^2\right\}^3 dx.$$

Define L_0 on $[0, \infty)$ by

$$L_0(R) := \sup\left\{\int_\Omega \left\{u^2+uv+v^2\right\}^3 dx : \|u\|_{H^1}, \|v\|_{H^1} \leq R\right\}.$$

Using Hölder's inequality along with theorem 2.3.4 we easily see that L_0 is increasing and real valued. Combining the above results we see that

$$\begin{aligned} \|F(u) - F(v)\|_{L^2}^2 &\leq \||u-v|^2\|_{L^3} \| \left\{u^2+uv+v^2\right\}^2 \|_{L^{3/2}} \\ &\leq CL_0(R)^{\frac{2}{3}} \|u-v\|_{H^1}^2. \end{aligned}$$

So we see that $F \in Lip^{loc}(H^1, L^2)$, hence $F \in Lip^{loc}(H_0^1, L^2)$.

□

From here on let $L(R)$ be defined as follows:

$$L(R) := \sup\left\{\frac{\|F(u)-F(v)\|_{L^2}}{\|u-v\|_{H_0^1}} : \|u\|_{H_0^1}, \|v\|_{H_0^1} \leq R, u \neq v\right\}.$$

Let's now introduce what we will call a mild solution to (4.12).

Definition 4.6.1. Local Mild Solution to (4.12).

Given $0 < T < \infty$ *and* $u \in C([0,T]; H_0^1(\Omega))$, *we will call* u *a mild solution to* (4.12) *if we have*

$$u(t) \;=\; M_T(u)(t) \qquad \forall t \in [0,T], \tag{4.17}$$

where equality holds in $H_0^1(\Omega)$ *and where* $M_T(u)$ *is defined as follows:*

$$M_T(u)(t) := S(t)\phi + \int_0^t S(t-s)F(u(s))ds. \qquad \text{(Banach valued integral)}$$

Using the fact that $u \in C([0,T]; H_0^1)$ *we see we are looking for* u *such that* $u = M_T(u)$, *where equality holds in* $C([0,T]; H_0^1)$.

We will call $u \in C([0,\infty); H_0^1(\Omega))$ a global mild solution to (4.12) if u satisfies (4.17) for all $0 < T < \infty$.

Let's now show that (4.12) has a unique local mild solution. Before we do this let's introduce some notation and carry out a few calculations.

To ease notation define $X_T := C([0,T]; H_0^1)$ and let B_R denote the closed ball of radius R centered at the origin in X_T. We will show that by picking R sufficiently large and T sufficiently small that M_T will be a contraction mapping on B_R, hence we can apply Banach's Fixed Point theorem to B_R to obtain a unique $u \in B_R$ which satisfies (4.17).

We will not show that M_T maps X_T into itself. One can avoid this by working in $L^\infty(0,T; H_0^1)$, but then one loses some apparent temporal regularity.

Let's now do some calculations. We will use the standard convention of letting C denote a changing constant that does not depend on u or v.

Let $\phi \in H_0^1(\Omega)$, $u, v \in B_R$ and $T > 0$. Then we have

$$\begin{aligned}
\|M_T(u)(t)\|_{H_0^1} &\leq \|S(t)\phi\|_{H_0^1} + \int_0^t \|S(t-s)F(u(s)\|_{H_0^1}\,ds \\
&\leq \|\phi\|_{H_0^1} + \int_0^t \|\nabla S(t-s)F(u(s))\|_{L^2} ds \\
&\leq \|\phi\|_{H_0^1} + \int_0^t \frac{C}{\sqrt{t-s}} \|S(t-s)F(u(s))\|_{L^2} ds \\
&\leq \|\phi\|_{H_0^1} + C\int_0^t \frac{1}{\sqrt{t-s}} \|F(u(s))\|_{L^2} ds \\
&\leq \|\phi\|_{H_0^1} + C\int_0^t \frac{1}{\sqrt{t-s}} \|u(s)\|_{H_0^1}^3\,ds \\
&\leq \|\phi\|_{H_0^1} + CR^3\int_0^t \frac{1}{\sqrt{t-s}} ds \\
&= \|\phi\|_{H_0^1} + CR^3\sqrt{t} \\
&\leq \|\phi\|_{H_0^1} + CR^3\sqrt{T}.
\end{aligned}$$

Also we have

$$\begin{aligned}
\|M_T(u)(t) - M_T(v)(t)\|_{H_0^1} &\leq \int_0^t \|S(t-s)\{F(u(s)) - F(v(s))\}\|_{H_0^1} ds \\
&= \int_0^t \|\nabla S(t-s)\{F(u(s)) - F(v(s))\}\|_{L^2} ds \\
&\leq C\int_0^t \frac{1}{\sqrt{t-s}} \|F(u(s)) - F(v(s))\|_{L^2} ds \\
&\leq C\int_0^t \frac{1}{\sqrt{t-s}} L(R)\|u(s) - v(s)\|_{H_0^1} ds \\
&\leq C\int_0^t \frac{L(R)}{\sqrt{t-s}} \|u-v\|_{X_T} ds \\
&\leq CL(R)\sqrt{T}\|u-v\|_{X_T}
\end{aligned}$$

for all $0 \leq t \leq T$.

From the above estimates we see that for $T > 0$ and $u, v \in B_R$ we have

$$\begin{aligned}
\|M_T(u)\|_{X_T} &\leq \|\phi\|_{H_0^1} + CR^3\sqrt{T} && (4.18) \\
\|M_T(u) - M_T(v)\|_{X_T} &\leq CL(R)\sqrt{T}\|u-v\|_{X_T}. && (4.19)
\end{aligned}$$

Theorem 4.6.1. (Local Mild Solution) *Assume* $\phi \in H_0^1(\Omega)$. *Then* (4.12) *has a unique local solution.*

Proof. Let $\phi \in H_0^1(\Omega)$ and define $R := 2\|\phi\|_{H^1}$. Define T by

$$T = T(\phi) := \min\left\{\frac{1}{1+(2CL(R))^2}, \frac{1}{1+(2CR^2)^2}\right\} > 0. \tag{4.20}$$

Then we see, using (4.18) and (4.19), that M_T maps B_R into itself and $\|M_T(u) - M_T(v)\|_{X_T} \leq \frac{1}{2}\|u-v\|_{X_T}$ for $u, v \in B_R$. So by Banach's Fixed Point Theorem (Theorem 2.2.1), applied to B_R, there exists a unique $u \in B_R$ with $M_T(u) = u$.

From this we see that (4.12) has a unique local mild solution that stays within B_R. Note this does not give us the uniqueness we desire. Let's now obtain uniqueness.

Now let $u, v \in C([0,T]; H_0^1)$ denote mild solutions to (4.12) with $\phi \in H_0^1(\Omega)$. Define

$$t_0 := \sup\{0 \leq t \leq T : u(s) = v(s) \ , \ \forall \ s \in [0,t]\}.$$

If $t_0 = T$ then we are done. Suppose $t_0 < T$. For $\delta > 0$ define $M^\delta : C([0,\delta]; H_0^1) \to C([0,\delta]; H_0^1)$ by

$$M^\delta(w)(t) := S(t)u(t_0) + \int_0^t S(t-s)F(w(s))ds.$$

Fix $R > \|u(t_0)\|_{H_0^1}$. Let B_R^δ denote the closed ball in $C([0,\delta]; H_0^1)$ centered at 0 with radius R. Now pick $\delta > 0$ (small) such that

(i) $\delta \leq T - t_0$

(ii) $\|u(t_0+t)\|_{H_0^1}, \|v(t_0+t)\|_{H_0^1} \leq R$ for all $0 \leq t \leq \delta$

(iii) M^δ maps B_R^δ into itself

(iv) M^δ is a contraction mapping on B_R^δ.

By continuity of u and v we see (ii) will not pose a problem. For (iii) and (iv) we will use estimates of the form (4.18) and (4.19) to pick the required δ. Now by Banach's Fixed Point Theorem we know there exists a unique $w \in B_R^\delta$ with $M^\delta(w) = w$.

Let's now show that $t \mapsto u(t_0+t)$ and $t \mapsto v(t_0+t)$ are fixed points of M^δ. By (ii) we have that both are elements of B_R^δ. To see they are fixed points note that

$$
\begin{aligned}
u(t_0+t) &= S(t+t_0)\phi + \int_0^{t_0+t} S(t_0+t-s)F(u(s))ds \\
&= S(t)\left\{S(t_0)\phi + \int_0^{t_0} S(t_0-s)F(u(s))ds\right\} + \int_{t_0}^{t_0+t} S(t_0+t-s)F(u(s))ds \\
&= S(t)u(t_0) + \int_0^t S(t-\tau)F(u(t_0+\tau))d\tau
\end{aligned}
$$

and similarly for v.

So from uniqueness we have $u(t_0+t) = v(t_0+t)$ for all $0 \leq t \leq \delta$, contradicting the definition of t_0. Hence by contradiction we have uniqueness of local mild solutions.

□

From Local to Global Mild Solution

By examining the function $T : H_0^1 \to (0, \infty)$, one is naturally led to what is called a blow-up alternative.

Theorem 4.6.2. (Blow-Up Alternative) *For $\phi \in H_0^1(\Omega)$ we have the following:*

Either (4.12) has a global mild solution

or

There exists a $T_{max} \in (0, \infty)$ and $u \in C([0, T_{max}); H_0^1)$, which satisfies (4.17) for all $0 < T < T_{max}$ and $\lim_{t \to T_{max}^-} \|u(t)\|_{H_0^1} = \infty$.

Proof. See [Cazenave] page 70.

□

Let's now argue that (4.17) has a global solution. Instead of dealing directly with the mild solution (4.17) we will use (4.12) to argue for a global solution. This is not a problem since it can be shown that these solutions are equivalent under suitable conditions.

Theorem 4.6.3. (4.12) *admits a global solution.*

Proof. Let u denote a solution to (4.12) on $[0,T]$, where $T < \infty$. Multiply (4.12) by $-\Delta u$ and integrate over Ω to get

$$\frac{d}{dt}\frac{1}{2}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + 3\int_\Omega u^2|\nabla u|^2 = 0 \qquad t \in [0,T).$$

Integrate over time up to $t < T$ to get

$$\frac{1}{2}\|\nabla u\|_{L^2}^2 + \int_0^t \|\Delta u\|_{L^2}^2 + 3\int_0^t \|u|\nabla u|\|_{L^2}^2 = \frac{1}{2}\|\nabla \phi\|_{L^2}^2 \qquad \forall t \in (0,T).$$

From this we see that

$$\begin{array}{rl} u & \text{uniformly bounded in } L^2(0,T;H^2) \\ u & \text{uniformly bounded in } L^\infty(0,T;H_0^1) \\ u|\nabla u|. & \text{uniformly bounded in } L^2(0,T;L^2). \end{array}$$

Here u uniformly bounded in $L^2(0,T;H^2)$ is taken to mean the following: there exists an $M > 0$ such that

$$\|u\|_{L^2(0,t;H^2)} \leq M$$

for all $0 < t < T$.

In particular we see that

$$\limsup_{t \to T^-} \|u(t)\|_{H_0^1} < \infty.$$

So we get desired result from the blow-up alternative.

□

4.6.2 Gradient Flow approach using Subdifferentials

In this section we will attempt to get a global solution to (4.12) using the following theorem.

Theorem 4.6.4. (Solution of gradient flow) *Take* $F : H \to (-\infty, \infty]$ *to be convex, proper and lower semicontinuous. Then for* $\phi \in D(\partial F)$ *there exists a unique*

$$u \in C([0,\infty); H) \qquad \textit{with} \qquad u' \in L^{\infty}(0,\infty; H)$$

such that

$$\begin{array}{rrcll} (i) & u(0) & = & \phi & \\ (ii) & u(t) & \in & D(\partial F) & \forall t > 0 \\ (iii) & u'(t) & \in & -\partial F(u(t)) & a.e. \quad t > 0. \end{array}$$

Proof. See [Evans] page 529.

□

Let's re-arrange (4.12) slightly to get $u_t = -\{-\Delta u + u^3\}$.

So if we can find a functional F defined on a Hilbert space H with $-\Delta u + u^3 \in \partial F(u)$ for $u \in D(\partial F)$ (or more preferably $\partial F(u) = \{-\Delta u + u^3\}$ for $u \in D(\partial F)$), then we could try and apply theorem (4.6.4) to obtain a global solution to (4.12).

The u^3 term suggests a functional of the form $\frac{1}{4}\int_\Omega u^4 \, dx$ and the Hilbert space $L^2(\Omega)$. The $-\Delta u$ term suggests the Hilbert space $L^2(\Omega)$ and a functional of the form $\frac{1}{2}\int_\Omega |\nabla u|^2 \, dx$, or more precisely

$$F(u) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx & u \in H^1(\Omega) \\ \infty & \text{otherwise.} \end{cases}$$

To satisfy our boundary conditions we will want $u(t) \in H_0^1(\Omega)$ for all $t > 0$ but the above F gives no incentive for the flow to stay in $H_0^1(\Omega)$. To fix this we will modify F slightly (see F_0 below for modifications).

Let's now define the functional F and the space H.

Take $H = L^2(\Omega)$ and take F as defined below.

Define

$$
\begin{aligned}
F_0(u) &:= \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx & u \in H_0^1(\Omega) \\ \infty & otherwise \end{cases} \\
F_1(u) &:= \begin{cases} \frac{1}{4}\int_\Omega u^4 dx & u \in L^4(\Omega) \\ \infty & otherwise \end{cases} \\
F(u) &:= F_0(u) + F_1(u).
\end{aligned}
$$

Note carefully that $F_0(u) = \infty$ for $u \in H^1(\Omega) \setminus H_0^1(\Omega)$ (This is the slight modification mentioned above).

It is easily seen that F, F_0, F_1 are all convex, proper and lower semicontinuous on $L^2(\Omega)$.

Theorem 4.6.5. *$D(\partial F) = H_0^1 \cap H^2$ and for $u \in H_0^1 \cap H^2$ we have*

$$\partial F(u) = \left\{-\Delta u + u^3\right\}.$$

Before we prove theorem 4.6.5 we will need a few results.

Lemma 4.6.2. *For $u \in L^6(\Omega)$ we have*

$$u^3 \in \partial F_1(u).$$

Proof. To see that $u^3 \in \partial F_1(u)$ we will need to show

$$\int_\Omega w^4 \geq \int_\Omega u^4 + 4\int_\Omega u^3(w-u) \qquad \forall w \in L^2(\Omega).$$

If $w \notin L^4(\Omega)$ then we are done trivially. Take $w \in L^4(\Omega)$, then using Young's ϵ inequality we get

$$
\begin{aligned}
\int_\Omega u^3 w &\leq \epsilon \int_\Omega |u|^{3p} + \frac{1}{q(\epsilon p)^{p/q}} \int_\Omega |w|^q \\
&= \frac{3}{4}\int_\Omega u^4 + \frac{1}{4}\int_\Omega w^4,
\end{aligned}
$$

where we have taken $q = 4, p = 4/3, \epsilon = 3/4$. Re-arranging this we see that $u^3 \in \partial F_1(u)$.

□

Lemma 4.6.3. *$D(\partial F_0) = H^2(\Omega) \cap H_0^1(\Omega)$ and for $u \in D(\partial F_0)$ we have*

$$\partial F_0(u) = \{-\Delta u\}\,.$$

Proof. See [Evans] page 534. □

Proof of Theorem 4.6.5

Define

$$\begin{aligned} D(A) &:= H_0^1 \cap H^2 \\ Au &:= -\Delta u + u^3 \qquad \text{for } u \in D(A). \end{aligned}$$

Let $u \in D(A)$. Using theorem 2.3.4 along with lemma 4.6.2 and lemma 4.6.3 we have

$$-\Delta u \in \partial F_0(u) \quad \text{and} \quad u^3 \in \partial F_1(u).$$

But it is easily seen, using the definition, that

$$\partial F_0(u) + \partial F_1(u) \subseteq \partial F(u).$$

So we have

$$Au := -\Delta u + u^3 \in \partial F(u) \quad \text{and} \quad D(A) \subseteq D(\partial F).$$

Let's now prove the other direction. To do this we will first show that $Range(I + A) = L^2$.

Let $f \in L^2(\Omega)$ and define J on $L^2(\Omega)$ by

$$J(w) := \begin{cases} \int_\Omega \left\{\frac{1}{2}|\nabla w|^2 + \frac{1}{4}w^4 + \frac{1}{2}w^2 - fw\right\} dx & w \in H_0^1(\Omega) \\ \infty & \textit{otherwise}. \end{cases}$$

We easily see that

$$w \mapsto \int_\Omega \left\{\frac{1}{2}w^2 - fw\right\} dx$$

is bounded below on L^2. Hence we see that J is bounded below on H_0^1. From standard lower semicontinuity arguments we see there exists a $u \in H_0^1$ such that

$$J(u) = \inf_{w \in H_0^1} J(w).$$

Let $\phi \in H_0^1$ and define g on $\mathbb{R}$ by

$$g(t) := J(u + t\phi).$$

Since g has minimum at $t = 0$ we easily see that

$$0 = g'(0) = \int_\Omega \{\nabla u \cdot \nabla \phi + u^3\phi + u\phi - f\phi\}\, dx$$

or

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega \{f - u^3 - u\}\, \phi$$

for all $\phi \in H_0^1$. Hence $u \in H_0^1$ is a weak solution to

$$\begin{cases} -\Delta u = f - u^3 - u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

But since $u \in H_0^1$ we can use theorem 2.3.4 to see $f - u^3 - u \in L^2$, hence by theorem 3.0.11 (elliptic regularity), we see that $u \in H^2 \cap H_0^1 =: D(A)$. So $u + Au = f$. So we see that $Range(I + A) = L^2$.

Now let $\hat{u} \in D(\partial F)$, $\hat{v} \in \partial F(\hat{u})$. So

$$\hat{u} + \hat{v} \in \hat{u} + \partial F(\hat{u}).$$

But $Range(I + A) = L^2$, hence there exists a $u \in D(A)$ such that

$$u + Au = \hat{u} + \hat{v}.$$

But $D(A) \subseteq D(\partial F)$ and $Au \in \partial F(u)$ so $\hat{u} + \hat{v} \in u + \partial F(u)$.

Noting that we already have $\hat{u} + \hat{v} \in \hat{u} + \partial F(\hat{u})$ and then using uniqueness from theorem 4.2.2 we see that

$$\hat{u} = u \in D(A) \qquad \text{and} \qquad \hat{v} = Au.$$

Combining with previous result we see that $D(A) = D(\partial F)$ and for $u \in D(A)$ we have

$$\partial F(u) = \{Au\},$$

which completes the proof of theorem 4.6.5.

To use theorem 4.6.4 with $H = L^2(\Omega)$, we see that we will have to add some regularity to ϕ. We will require $\phi \in H^2 \cap H_0^1$ as opposed to just $\phi \in H_0^1$.

Now applying theorem 4.6.4 we see there exists a unique $u \in C([0,\infty); L^2)$ with $u' \in L^\infty(0,\infty; L^2)$ such that

$$\begin{array}{llll} (i) & u(0) = \phi & & \\ (ii) & u(t) \in D(\partial F) = H^2 \cap H_0^1 & & \forall t > 0 \\ (iii) & u'(t) \in -\partial F(u(t)) = \{\Delta u(t) - u(t)^3\} & a.e. & t > 0. \end{array}$$

So in particular we have $u'(t) = \Delta u(t) - u(t)^3$ for a.e. $t > 0$ where equality holds in $L^2(\Omega)$.

Chapter 5

The Cahn-Hilliard Equation

The Cahn-Hilliard equation is the following evolution equation:

$$\begin{cases} u_t &= -\epsilon^2\Delta^2 u + \Delta W'(u) \quad \Omega\times(0,\infty) \\ \partial_\nu u &= \partial_\nu \Delta u = 0 \qquad\quad \partial\Omega\times[0,\infty) \end{cases} \tag{5.1}$$

with some initial condition. As noted in the introduction, a solution u will conserve mass. We will show that the Cahn-Hilliard equation can be written as a gradient flow, using the Hilbert space H_0^{-1} (see definition (3.2.1)), and the Cahn-Hilliard Energy Functional F, as defined below.

Definition 5.0.2. *Define the Cahn-Hilliard Energy Functional F on H_0^{-1} by*

$$F(u) := \int_\Omega \left\{ W(u(x)) + \frac{\epsilon^2}{2}|\nabla u|^2 \right\} dx$$

where $W:\mathbb{R}\to\mathbb{R}$ is some non-negative smooth double well potential and where it is understood that $F(u)=\infty$ if $u \notin \dot{H}^1(\Omega)$ or if $W(u)\notin L^1(\Omega)$.

5.1 Minimizing the Cahn-Hilliard Energy Functional.

In this section we will minimize the Cahn-Hilliard Energy Functional over H_0^{-1} with a standard double well potential W given by $W(u) := \frac{1}{4}(u^2-1)^2$ and where Ω is a connected domain in $\mathbb{R}^3$. We will show that minimizers will be smooth ($C^\infty(\overline{\Omega})$) and satisfy $\partial_\nu u = \partial_\nu \Delta u = 0$ on $\partial\Omega$. This will serve as our motivation for imposing the two boundary conditions in the Cahn-Hilliard equation.

Before we attempt to minimize the Cahn-Hilliard Energy Functional over H_0^{-1} let's examine the double well term. Using theorem 2.3.4 we have $H^1(\Omega)$ continuously imbedded in $L^p(\Omega)$ for $1 \leq p \leq 6$, hence we have

$$F(u) := \begin{cases} \int_\Omega \left\{\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(u^2-1)^2\right\} dx & u \in \dot{H}^1(\Omega) \\ \infty & u \in H_0^{-1} \backslash \dot{H}^1(\Omega), \end{cases}$$

where we have taken $\epsilon = 1$ in definition of F for simplicity. From here on we will just work with F over $\dot{H}^1(\Omega)$ since if F has a minimizer u over H_0^{-1} then $u \in \dot{H}^1(\Omega)$.

Remark 5.1.1. *(Size of ϵ) Taking $\epsilon = 1$ will be immaterial to showing existence and regularity of minimizers of F in this section and in the next, where we consider higher power nonlinearities. But we want to take ϵ sufficiently small such that we have a non-trivial case. So towards this let $0 < \epsilon < C$ where C is from Poincaré's inequality with $p = 2$. Then we have*

$$F(u) = \frac{1}{2}\left\{\epsilon^2 \int_\Omega |\nabla u|^2 dx - \int_\Omega u^2 dx\right\} + \frac{1}{4}\int_\Omega u^4 dx + \frac{|\Omega|}{4}.$$

If we let $u \in \dot{H}^1(\Omega)$ witness the fact that $\epsilon < C$ then we have

$$\epsilon^2 \int_\Omega |\nabla u|^2 dx - \int_\Omega u^2 dx < 0.$$

Let's now examine $F(ru)$ for $r > 0$.

$$F(ru) = \frac{r^2}{2}\left\{\epsilon^2 \int_\Omega |\nabla u|^2 dx - \int_\Omega u^2 dx + \frac{r^2}{2}\int_\Omega u^4 dx\right\} + \frac{|\Omega|}{4}.$$

By taking r sufficiently small we see that

$$F(ru) < \frac{|\Omega|}{4} = F(0)$$

and so we have nontrivial minimizers.

Theorem 5.1.1. *F has a minimum over $\dot{H}^1(\Omega)$.*

Proof. Let $u_m \in \dot{H}^1(\Omega)$ be such that

$$\lim_m F(u_m) = \inf \left\{ F(u) : u \in \dot{H}^1(\Omega) \right\} \geq 0.$$

Since $W \geq 0$ and after passing to a suitable subsequence (not relabeled), we have

$$\begin{array}{llll} u_m & \rightharpoonup & u & \text{in} \quad H^1(\Omega) \\ u_m & \rightarrow & u & \text{in} \quad L^2(\Omega) \\ u_m & \rightarrow & u & \text{a.e.} \end{array}$$

for some $u \in \dot{H}^1(\Omega)$.

By a standard weakly l.s.c. argument we have

$$\int_\Omega |\nabla u|^2 dx \leq \liminf_m \int_\Omega |\nabla u_m|^2 dx$$

or put another way, the $H_0^1(\Omega)$ norm is weakly l.s.c. on $\dot{H}^1(\Omega)$, which follows directly from theorem 2.2.6.

Also we have $\int_\Omega u_m^2 \rightarrow \int_\Omega u^2$ since $u_m \rightarrow u$ in L^2. Since $u_m \rightarrow u$ a.e. we have by Fatou's Lemma that

$$\int_\Omega u^4 \leq \liminf_m \int_\Omega u_m^4$$

Combining all the above and using properties of lim inf we see that

$$F(u) \leq \liminf_m F(u_m) = \inf_{\dot{H}^1} F(v).$$

□

Theorem 5.1.2. *Assume u a local minimizer of F over $\dot{H}^1(\Omega)$. Then $u \in C^\infty(\overline{\Omega})$ and $\partial_\nu u = \partial_\nu \Delta u = 0$ on $\partial\Omega$.* (Here local is w.r.t. the strong $H^1(\Omega)$ topology.)

Proof. Before we prove Theorem 5.1.2 we will need a few results.

□

Then next lemma will essentially be a particular case of theorem 2.3.6 along with a particular imbedding. We are putting it in lemma form so as to simplify the proof of theorem 5.1.2.

Lemma 5.1.1. (An Algebra type result.)

$$u \in H^1(\Omega) \quad \Rightarrow \quad u^3 \in L^2(\Omega) \tag{5.2}$$

$$u \in H^k(\Omega) \quad \Rightarrow \quad u^3 \in H^k(\Omega) \qquad \forall k \geq 2. \tag{5.3}$$

Proof. (5.2) follows directly from the fact $H^1(\Omega)$ is continuously imbedded in $L^6(\Omega)$. (5.3) follows from the fact $W^{k,p}(\Omega)$ is a Banach Algebra for $kp > n$ (see theorem 2.3.6).

□

Proof of Theorem 5.1.2

Let $u \in \dot{H}^1(\Omega)$ denote a local minimizer of F over $\dot{H}^1(\Omega)$. Fix $v \in \dot{H}^1(\Omega)$ and define g on $\mathbb{R}$ by

$$g(t) := F(u + tv).$$

Since g has a local minimum at $t = 0$ we have $g'(0) = 0$. From this we obtain

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega (u - u^3)v \qquad \forall v \in \dot{H}^1(\Omega).$$

Now define $f := u - u^3$. By lemma (5.1.1) we have $f \in L^2(\Omega)$, hence $f_0 := f - (f)_\Omega \in \dot{L}^2(\Omega)$.

Let $v \in \dot{H}^1(\Omega)$. Then we have

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv = \int_\Omega f_0 v.$$

Now let $v \in H^1(\Omega)$. Then $v - (v)_\Omega \in \dot{H}^1(\Omega)$ and

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla v &= \int_\Omega \nabla u \cdot \nabla \{v - (v)_\Omega\} \\ &= \int_\Omega f_0 \{v - (v)_\Omega\} \\ &= \int_\Omega f_0 v - (v)_\Omega \int_\Omega f_0 \\ &= \int_\Omega f_0 v.\end{aligned}$$

So we see $u \in \dot{H}^1(\Omega)$ is a weak solution to

$$\begin{cases} -\Delta u = f_0 & \Omega \\ \partial_\nu u = 0 & \partial\Omega. \end{cases}$$

Now by theorem (3.0.12) we have $u \in \dot{H}^2(\Omega)$. Now apply lemma 5.1.1 to get $f_0 \in H^2$. Then by theorem 3.0.12 we have $u \in H^4$. Keep bootstrapping to see $u \in \dot{H}^m(\Omega)$ for all $m \geq 1$. From this we can conclude that $u \in \dot{C}^\infty(\overline{\Omega})$.

Since u solves a Neumann problem and has enough regularity to check the first boundary condition, we know that $\partial_\nu u = 0$ on $\partial\Omega$ in the sense of trace.

Let's now check the second boundary condition. We have enough regularity to use classical derivatives. So we have

$$-\nabla(\Delta u) = \nabla f_0(u) = f_0'(u)\nabla u \qquad \text{where equality holds in} \quad C(\overline{\Omega}; \mathbb{R}^3)$$

Now use continuity to extend this to the boundary and dot this with the normal vector ν. We then arrive at

$$-\partial_\nu \Delta u = f_0'(u)\partial_\nu u = 0 \qquad \text{on} \quad \partial\Omega,$$

which completes the proof of theorem 5.1.2.

5.1.1 Minimizing the Cahn-Hilliard Energy Functional with Higher Power Non-linearities

This section is more of a curiosity than anything else since we will not attempt at getting a local or global solution to the modified Cahn-Hilliard equation where the double well potential involves higher powers than seen in the last section.

Again take Ω to be a connected domain in $\mathbb{R}^3$, but now take the double-well potential to be

$$W(u) := \frac{1}{6}u^6 - \frac{1}{2}u^2 + \frac{1}{3}.$$

Define $F : \dot{H}^1(\Omega) \to \mathbb{R}$ by

$$F(u) := \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + W(u) \right\} dx.$$

In this section we will show that F has minimizers over $\dot{H}^1(\Omega)$ and if u a minimizer then $u \in C^\infty(\overline{\Omega})$ with $\partial_\nu u = \partial_\nu \Delta u = 0$ on $\partial\Omega$, which is the same conclusion as when W was the standard double-well potential as in the last section.

As before it is easily seen that F will be minimum obtaining over $\dot{H}^1(\Omega)$ at say u. Let $v \in \dot{H}^1(\Omega)$ and define $g(t) := F(u + tv)$. Using the fact that $g'(0) = 0$, we see

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega \left\{ u - u^5 \right\} v \qquad \forall v \in \dot{H}^1(\Omega).$$

Now define $f := u - u^5 - (u - u^5)_\Omega$. Then we see

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall v \in H^1(\Omega),$$

so u is a weak solution to

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Using the fact that $H^1(\Omega) \subseteq L^6$ we see that $f \in L^{6/5}$, hence by elliptic L^p regularity theory we have

$$u \in W^{1,2}(\Omega) \cap W^{2,\frac{6}{5}}(\Omega).$$

This is not enough regularity to allow us to bootstrap as we did before. I suspect if one knew more L^p regularity than I, then one could proceed from here. So we will proceed with a different approach, which will not be fully justified (a priori, we don't have enough regularity to justify the following computations.) Take a gradient of both sides of the above PDE to obtain

$$\nabla(\Delta u) = (5u^4 - 1)\nabla u \qquad \text{in} \quad \Omega$$

and then dot both sides with ∇u to find

$$\nabla(\Delta u) \cdot \nabla u = (5u^4 - 1)|\nabla u|^2 \qquad \text{in} \quad \Omega.$$

Now integrate over Ω and use Green's formula to obtain

$$\begin{aligned} \int_\Omega (5u^4 - 1)|\nabla u|^2 &= \int_\Omega \nabla(\Delta u) \cdot \nabla u \\ &= -\int_\Omega (\Delta u)^2 + \int_{\partial\Omega} (\partial_\nu u)\Delta u \\ &= -\int_\Omega (\Delta u)^2 \qquad \text{since} \quad \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Re-arranging we arrive at

$$\int_\Omega (\Delta u)^2 + 5\int_\Omega u^4|\nabla u|^2 = \int_\Omega |\nabla u|^2,$$

hence we see that $\Delta u,\ u^2|\nabla u| \in L^2(\Omega)$. Using the fact that $\Delta u \in L^2$ and $\partial_\nu u = 0$ on $\partial\Omega$ we see $u \in H^2(\Omega) = W^{2,2}(\Omega)$ (use elliptic regularity). Since $n = 3$ we have H^k an algebra for $k \geq 2$ and so we can proceed as in last section. So after some bootstrapping we will obtain

$$u \in \dot{H}^k(\Omega) \quad \forall k \geq 1,$$

hence

$$u \in C^\infty(\overline{\Omega}).$$

As before we can argue that $\partial_\nu \Delta u = 0$ on $\partial\Omega$.

5.2 Motivation for the Choice of H_0^{-1}

The most obvious (and easiest) Hilbert space to try and write the gradient flow of the Cahn-Hilliard Energy Functional over would be H^k for $k \geq 0$. There are some physical objections to this. Let's write it over $L^2(\Omega)$ to see what we get. Using (4.3) and (4.6) we see that we arrive at the following evolution equation:

$$u_t = \epsilon^2 \Delta u - W'(u).$$

From this and from the imposed boundary conditions we get

$$\begin{aligned} \frac{d}{dt}\int_\Omega u &= \epsilon^2 \int_\Omega \Delta u - \int_\Omega W'(u) \\ &= \epsilon^2 \int_{\partial\Omega} \partial_\nu u - \int_\Omega W'(u) \\ &= -\int_\Omega W'(u). \end{aligned}$$

So it appears we won't have conservation of mass in general. We can impose the conservation of mass constraint by using $\dot{L}^2(\Omega)$ instead of $L^2(\Omega)$ as our Hilbert space.

Let $X := \dot{C}_c^\infty(\Omega)$. Later we will show X dense in $\dot{L}^2(\Omega)$ (see lemma (5.3.1). Now let's write the gradient flow

$$u_t := -grad_{\dot{L}^2}^X F(u).$$

It is easily seen that $grad_{\dot{L}^2}^X F_0(u) = -\Delta u$ for $u \in \dot{H}^2(\Omega)$. Similarly we find that for $u \in \dot{H}^2(\Omega)$ we have

$$grad_{\dot{L}^2}^X F_1(u) = W'(u) - (W'(u))_\Omega .$$

So the evolution equation we arrive at over $\dot{L}^2(\Omega)$ is

$$u_t = \epsilon^2 \Delta u - W'(u) + (W'(u))_\Omega .$$

Now u will conserve mass but we have this extra average term. Typically this average term will be non-local. We reject this evolution equation as a good model for the physical process because of its non-local nature (ie. no action at a distance). Let's look at an example to see what we mean by non-local. Typically $W'(u) := u^3 - u$. So we see

$$|\Omega| \left(W'(u)\right)_\Omega = \int_\Omega u^3 - \int_\Omega u = \int_\Omega u^3.$$

This integral operator is non-local.

There are various other Hilbert spaces we could choose from to write our gradient flow, $u_t = -grad_H F(u)$, over and obtain a local evolution equation with the correct boundary conditions, but H_0^{-1} will be the simplest.

5.3 Cahn-Hilliard gradient calculations

Take Ω to be a connected domain in $\mathbb{R}^n$ and on occasion we will add the restriction that $n = 3$.

Before we calculate the X constrained gradient of the Cahn-Hilliard Energy Functional (F) over H_0^{-1}, let's try and pick a suitable subspace X. We would prefer if X was dense in H_0^{-1} since then we wouldn't have to worry about picking the element of least norm in $G(F, X, u)$.

In this direction let's show $X := \dot{C}_c^\infty(\Omega)$ is dense in H_0^{-1}.

Remark 5.3.1. *X will denote $\dot{C}_c^\infty(\Omega)$ for the remainder of this thesis.*

Lemma 5.3.1. *X is dense in $\dot{L}^2(\Omega)$.*

Proof. Let $u \in \dot{L}^2(\Omega)$ and let $\phi_m \in C_c^\infty(\Omega)$ be such that $\phi_m \to u$ in L^2. By Hölder's inequality we have $\int_\Omega \phi_m \to 0$. Now fix $0 \leq \phi \in C_c^\infty(\Omega)$ such that $\int_\Omega \phi = 1$. Let $t_m \in \mathbb{R}$ be such that $2t_m - 1 = -\int_\Omega \phi_m$. So $t_m \to 1/2$. Now define

$$\psi_m := (2t_m - 1)\phi + \phi_m \in X.$$

It is easily seen that $\psi_m \to u$ in L^2, hence X dense in $\dot{L}^2(\Omega)$.

□

Theorem 5.3.1. *X dense in H_0^{-1}*

Proof. We will first show $\dot{L}^2$ is dense in H_0^{-1}. Towards this define $E := \{u \in H^2 \cap H_A^1 : \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega\}$ and let's show that $\Phi(\dot{L}^2) = E$. Before we do this let's recap the definition of $\Phi : H_0^{-1} \to H_A^1$ and $\Psi := \Phi^{-1}$.

If $f \in H_0^{-1}$ and $u := \Phi(f)$, then we have

$$\int_\Omega \nabla u \cdot \nabla \phi = \langle f, \phi \rangle_{(H^1)^*, H^1} \qquad \forall \phi \in H^1(\Omega).$$

Recall that when f is sufficiently regular, (L^2 will suffice), we have an elliptic formulation relating f and u given by

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Now let $f \in \dot{L}^2$ and let $u := \Phi(f) \in H_A^1$. So u and f satisfy (5.4) and by elliptic regularity we have $u \in E$.

Now take $u \in E$ and define $f := \Psi(u) \in H_0^{-1}$. Let $\tilde{f} := -\Delta u$. Use Green's formula to see $\tilde{f} \in \dot{L}^2$. Then we have

$$\begin{aligned} \langle f, \phi \rangle_{(H^1)^*, H^1} &= \int_\Omega \nabla u \cdot \nabla \phi \\ &= \int_\Omega \phi(-\Delta u) + \int_{\partial\Omega} \phi \partial_\nu u \\ &= \int_\Omega (-\Delta u)\phi \\ &= \int_\Omega \tilde{f}\phi \end{aligned}$$

for all $\phi \in H^1(\Omega)$. So we have $f = \tilde{f}$ in $H^1(\Omega)^*$, but $\tilde{f} \in \dot{L}^2$, hence we are done. So we have $\Phi(\dot{L}^2) = E$.

Now $X \subseteq E$ and X is clearly dense in H_A^1, so E dense in H_A^1 and since Φ is an isometry we see $\dot{L}^2$ is dense in H_0^{-1}.

Now let $u^* \in H_0^{-1}$ and $\epsilon > 0$. There exists a $w^* \in \dot{L}^2$ such that

$$\|u^* - w^*\|_{H_0^{-1}} < \epsilon.$$

By lemma (5.3.1) there exists $w_m^* \in X$ with $w_m^* \to w^*$ in L^2. Hence we get

$$\begin{aligned}
\|u^* - w_m^*\|_{H_0^{-1}} &\leq \|u^* - w^*\|_{H_0^{-1}} + \|w^* - w_m^*\|_{H_0^{-1}} \\
&< \epsilon + \|w^* - w_m^*\|_{H_0^{-1}} \\
&\leq \epsilon + C\|w^* - w_m^*\|_{(H^1)^*} \\
&\leq \epsilon + C\|w^* - w_m^*\|_{L^2}
\end{aligned}$$

where C is obtained from the fact that the H_0^{-1} and $H^1(\Omega)^*$ norms are equivalent on H_0^{-1} (See Theorem 3.2.2). The last inequality follows from the fact that the L^2 norm is bigger than the $(H^1(\Omega))^*$ norm on L^2. By taking m sufficiently large we see

$$\|u^* - w_m^*\|_{H_0^{-1}} < 2\epsilon.$$

□

5.3.1 Formal Gradient Calculations of the Cahn-Hilliard Energy Functional

In this section we will formally obtain the Cahn-Hilliard equation as a gradient flow of the Cahn-Hilliard Energy Functional over H_0^{-1}. In the next section we will carry out the calculations with more rigour.

Let F denote the Cahn-Hilliard Energy Functional and X be defined as before. Let u^* be sufficiently smooth and satisfy $\partial_\nu u^* = 0 = \partial_\nu \Delta u^*$ on $\partial\Omega$. Let $v^* \in X$. Then we have

$$\frac{F(u^* + tv^*) - F(u^*)}{t} = \int_\Omega \left\{ \epsilon^2 \nabla u^* \cdot \nabla v^* + \frac{\epsilon^2}{2} t |\nabla v^*|^2 + \frac{W(u^* + tv^*) - W(u^*)}{t} \right\} dx.$$

So we see

$$\lim_{t\to 0}\frac{F(u^*+tv^*)-F(u^*)}{t}=\int_\Omega \left\{\epsilon^2\nabla u^*\cdot\nabla v^*+W'(u^*)v^*\right\}dx.$$

So if $w^* = grad^X_{H_0^{-1}}F(u^*)$ then we need

$$\int_\Omega \left\{\epsilon^2\nabla u^*\cdot\nabla v^*+W'(u^*)v^*\right\}dx=(w^*,v^*)_{H_0^{-1}} \tag{5.5}$$

for all $v^*\in X$. Let $u,v,w\in H^1_A$ denote the associates of u^*,v^*,w^* respectively. So w satisfies

$$\left\{\begin{array}{rclcc} -\Delta w &=& w^* & \text{in} & \Omega \\ \partial_\nu w &=& 0 & \text{on} & \partial\Omega \end{array}\right. \tag{5.6}$$

and similarly for u,v. Now if we identify H^1_A with the dual of H_0^{-1} then we have

$$(w^*,v^*)_{H_0^{-1}}=\langle w,v^*\rangle_{H^1_A,H_0^{-1}}=\int_\Omega wv^*$$

since v^* and w are sufficiently smooth. Combining this with (5.5) we see that

$$\int_\Omega \epsilon^2\nabla u^*\cdot\nabla v^*=\int_\Omega \left\{w-W'(u^*)\right\}v^*$$

for all $v^*\in X$. If we let $C:=(w-W'(u^*))_\Omega$ then we have

$$\int_\Omega \epsilon^2\nabla u^*\cdot\nabla v^*=\int_\Omega \left\{w-W'(u^*)-C\right\}v^*$$

for all $v^*\in C_c^\infty(\Omega)$. So in particular we have

$$-\epsilon^2\Delta u^*=w-W'(u^*)-C \quad \text{in} \quad \Omega$$

or

$$w=W'(u^*)+C-\epsilon^2\Delta u^* \quad \text{in} \quad \Omega.$$

Now noting that w^* satisfies (5.6) we see

$$w^*=-\Delta w=-\Delta\left\{W'(u^*)+C-\epsilon^2\Delta u^*\right\}=\epsilon^2\Delta^2u^*-\Delta W'(u^*) \quad \text{in} \quad \Omega.$$

So $grad^X_{H_0^{-1}}F(u^*)=\epsilon^2\Delta^2u^*-\Delta W'(u^*)$.

From this we see if we write $u_t=-grad^X_{H_0^{-1}}F(u)$ then we get the Cahn-Hilliard equation.

5.3.2 Gradient Calculations of the Cahn-Hilliard Energy Functional

In this section we will carry out the calculations from the previous section with more rigour. In particular, in the last section, we assumed the existence of $w^* = grad_{H_0^{-1}} F(u^*)$ along with a few other unjustified calculations.

For this section we will assume $W \in C^\infty(\mathbb{R}, \mathbb{R})$ and we will again take $X := \dot{C}_c^\infty(\Omega)$. Now let's calculate the constrained gradients. To do this let's break F into two pieces. Define

$$F_0(u^*) := \begin{cases} \frac{1}{2} \int_\Omega |\nabla u^*|^2 dx & u^* \in \dot{H}^1(\Omega) \\ \infty & u^* \in H_0^{-1} \backslash \dot{H}^1(\Omega) \end{cases}$$

$$F_1(u^*) := \begin{cases} \int_\Omega W(u^*)\, dx & W(u^*) \in L^1(\Omega) \\ \infty & otherwise \end{cases}$$

Now for gradient calculations. Starred elements will be viewed as elements of H_0^{-1} and non-starred as elements of H_A^1. Let's first calculate $grad_{H_0^{-1}} F_0(u^*)$.

Theorem 5.3.2. *For $u^* \in \dot{H}^4(\Omega)$ we have $grad^X_{H_0^{-1}} F_0(u^*)$ given by the following functional:*

$$v \mapsto \int_\Omega (\Delta^2 u^*) v - \int_{\partial\Omega} (\partial_\nu \Delta u^*) v.$$

Proof. Fix $u^* \in \dot{H}^4(\Omega)$ and let $v^* \in X$. As usual we have

$$\frac{d}{dt} F_0(u^* + tv^*)\Big|_{t=0} = \int_\Omega \nabla u^* \cdot \nabla v^*.$$

Define

$$w := -\Delta u^* - (-\Delta u^*)_\Omega \in \dot{H}^2(\Omega) \subseteq H_A^1$$

and $w^* := \Psi(w) \in H_0^{-1}$, $v := \Phi(v^*) \in H_A^1$. Then we have

$$\begin{aligned}
(w^*, v^*)_{H_0^{-1}} &= \int_\Omega \nabla w \cdot \nabla v \\
&= \langle v^*, w \rangle \\
&= \int_\Omega v^* w \\
&= \int_\Omega (-\Delta u^*) v^* \\
&= \int_\Omega \nabla u^* \cdot \nabla v^* - \int_{\partial\Omega} (\partial_\nu u^*) v^* \\
&= \int_\Omega \nabla u^* \cdot \nabla v^* \qquad \text{since } v^* \in X
\end{aligned}$$

and so we see $w^* = grad^X_{H_0^{-1}} F_0(u^*)$. Now let's find w^*.

Let $v \in H^1(\Omega)$ and w^* be as above, with $w \in H_A^1$ the associate of w^*. Then we have

$$\begin{aligned}
\langle w^*, v \rangle &= \langle w^*, v - (v)_\Omega \rangle \\
&= \int_\Omega \nabla w \cdot \nabla v \\
&= \int_\Omega (-\Delta w) v + \int_{\partial\Omega} (\partial_\nu w) v \\
&= \int_\Omega (\Delta^2 u^*)\, v - \int_{\partial\Omega} (\partial_\nu \Delta u^*) v
\end{aligned}$$

and so $w^* \in H_0^{-1}$ is given by the following functional

$$v \mapsto \int_\Omega (\Delta^2 u^*) v - \int_{\partial\Omega} (\partial_\nu \Delta u^*) v.$$

It is easily seen that the integral formulation of w^* has the required continuity and by an application of Green's formula we easily see that it has zero average.

□

From above we see that $\dot{H}^4(\Omega) \subseteq \mathcal{D}(grad^X_{H_0^{-1}} F_0)$.

Let's now move on to calculating $grad^X_{H_0^{-1}} F_1(u^*)$. We will now take Ω to be a connected domain in $\mathbb{R}^3$.

We will need a few technical details to ensure that we have the expected

$$\frac{d}{dt}F_1(u^* + tv^*)\Big|_{t=0} = \int_\Omega W'(u^*)v^*.$$

So we have by theorem 2.3.4 that $H^4(\Omega)$ is continuously imbedded in $C(\overline{\Omega})$.

Lemma 5.3.2. *For $u^* \in \dot{H}^4(\Omega)$, $v^* \in X$ we have*

$$\frac{F_1(u^* + tv^*) - F_1(u^*)}{t} \to \int_\Omega W'(u^*)v^*$$

Proof. It is easily seen that

$$\frac{W(u^* + tv^*) - W(u^*)}{t} \to W'(u^*)v^* \qquad \text{a.e. on} \quad \Omega.$$

Also it is easily seen that for $\forall\, 0 < |t| < 1$ we have

$$\left|\frac{W(u^* + tv^*) - W(u^*)}{t}(x)\right| \le \|v^*\|_{L^\infty} \sup\{|W'(z)| : |z| \le \|u^*\|_{L^\infty} + \|v^*\|_{L^\infty}\}$$

for a.e. $x \in \Omega$. Hence by the dominated convergence theorem we get the desired result.

□

Recall that theorem 2.3.4 gives us the following continuous imbedding

$$H^4(\Omega) \to W^{2,\infty}(\Omega).$$

Lemma 5.3.3. *Assume $F \in C^\infty(\mathbb{R}, \mathbb{R})$ and $u \in H^4(\Omega)$. Then if we define $v := F(u)$, we have $v \in H^4(\Omega)$. Also note that $v \in L^\infty(\Omega)$.*

Proof. For $|\alpha| \le 4$ write out $\partial^\alpha v$ and check that the above imbedding is enough to give us the desired result. After we have $v \in H^4(\Omega)$ then $v \in L^\infty(\Omega)$ by theorem 2.3.4.

□

Theorem 5.3.3. *For $u^* \in \dot{H}^4(\Omega)$ we have $grad^X_{H_0^{-1}} F_1(u^*)$ given by the following functional:*

$$v \mapsto \int_\Omega -\Delta W'(u^*)v + \int_{\partial\Omega} (\partial_\nu W'(u^*))v.$$

Proof. Fix $u^* \in \dot{H}^4(\Omega)$ and let $v^* \in X$. Define

$$w := W'(u^*) - (W'(u^*))_\Omega \in \dot{H}^4(\Omega) \subsetneq H^1_A$$

and $w^* := \Psi(w) \in H_0^{-1}$. Then we have

$$\begin{aligned}\int_\Omega W'(u^*)v^* &= \int_\Omega wv^* \\ &= \langle v^*, w\rangle \\ &= (w^*, v^*)_{H_0^{-1}}.\end{aligned}$$

So we see $grad^X_{H_0^{-1}} F_1(u^*) = w^*$.

Now let $v \in H^1(\Omega)$. Then we have

$$\begin{aligned}\langle w^*, v\rangle &= \langle w^*, v - (v)_\Omega\rangle \\ &= \int_\Omega \nabla w \cdot \nabla v \\ &= \int_\Omega (-\Delta w)v + \int_{\partial\Omega} (\partial_\nu w)v \\ &= \int_\Omega (-\Delta W'(u^*))v + \int_{\partial\Omega} (\partial_\nu W'(u^*))v\end{aligned}$$

and so we see that we have desired result. Again easily seen that integral formulation of w^* has the desired continuity and has zero average.

□

Let's now impose the two boundary conditions and see what the gradients are. Let $u^* \in \dot{H}^4(\Omega)$ and denote the boundary conditions by

(i) $\partial_\nu u = 0$

(ii) $\partial_\nu \Delta u = 0$

on $\partial\Omega$.

Boundary condition (i) will allow us to identify $grad^X_{H_0^{-1}} F_1(u^*)$ with $-\Delta W'(u^*) \in \dot{H}^2(\Omega) \subseteq H_0^{-1}$ and boundary condition (ii) will allow us to identify $grad^X_{H_0^{-1}} F_0(u^*)$ with $\Delta^2 u^* \in \dot{L}^2(\Omega) \subseteq H_0^{-1}$.

So for $u^* \in \dot{H}^4(\Omega)$ and with the two boundary conditions satisfied we have

$$grad^X_{H_0^{-1}} F_0(u^*) = \Delta^2 u^* \qquad grad^X_{H_0^{-1}} F_1(u^*) = -\Delta W'(u^*).$$

As one might note we have not even tried to calculate $\mathcal{D}(grad^X_{H_0^{-1}} F_i)$ for $i = 0, 1$ but we did show $\dot{H}^4(\Omega)$ was contained in both. This apparent laziness can be somewhat justified later when we see that for $W(u) := \frac{1}{4}(u^2-1)^2$ and $n = 3$, that a solution u (in some sense) to the Cahn-Hilliard equation will have $u(t) \in \dot{H}^4_\nu(\Omega)$ for a.e. $t > 0$ provided the initial condition is sufficiently regular, where $\dot{H}^4_\nu(\Omega) := \left\{ v \in \dot{H}^4(\Omega) : \partial_\nu v = 0 = \partial_\nu \Delta v \quad \text{on} \quad \partial\Omega \right\}$.

5.4 Local Existence for the Cahn-Hilliard Equation

In this section we will obtain a local solution to the Cahn-Hilliard equation

$$\begin{cases} u_t + \Delta^2 u &= \Delta\{u^3 - u\} & \Omega \times (0,T] \\ \partial_\nu u &= \partial_\nu \Delta u = 0 & \partial\Omega \times [0,T] \\ u(x,0) &= \phi & \Omega \times \{t=0\} \end{cases} \tag{5.7}$$

where Ω is a connected domain in $\mathbb{R}^3$ and $\phi \in \dot{H}^2(\Omega)$ with $\partial_\nu \phi = 0$ on $\partial\Omega$. Note that we have taken $W'(u) := u^3 - u$.

To do this we will use the the following framework which is from [Zheng].

Take V, H to be separable Hilbert spaces such that V is dense in H and where V is compactly imbedded in H. So we have

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*.$$

Observe that H and its dual are identified but V and its dual are not. (Similar to when we didn't identify H_0^1 and H^{-1}.)

Let $A \in \mathcal{L}(V, V^*)$, ie. A is a continuous linear mapping from V to V^*, and define $b : V \times V \to \mathbb{R}$ by

$$b(u,v) := \langle Au, v \rangle \qquad \text{where } \langle \cdot, \cdot \rangle \text{ denotes the } V^*, V \text{ pairing.}$$

We will say b is coervice if there exists a $\alpha > 0$ such that $\alpha \|u\|_V^2 \leq b(u,u)$ for all $u \in V$.

Define the domain of A by $D(A) := \{u \in V : Au \in H\}$.

We will investigate the following abstract ODE.

$$\begin{cases} u_t = -Au + g(u) & 0 < t \leq T \\ u(0) = \phi \in V. \end{cases} \tag{5.8}$$

Theorem 5.4.1. (Local Existence) *If given $A \in \mathcal{L}(V, V^*)$ with b as defined above and b coercive along with $g \in Lip^{loc}(V, H)$, then* (5.8) *admits a local solution with*

$$u \in C([0,T]; V) \cap L^2(0,T; D(A)) \ , \ u_t \in L^2(0,T; H).$$

(It is understood that $D(A)$ is equipped with the graph norm.
ie. $\|x\|_{D(A)} := \|x\|_H + \|Ax\|_H$ for $x \in D(A)$.)

Proof. See [Zheng] page 21. □

We will want to try and use theorem 5.4.1 to get a local solution to (5.7). To do this we will need to pick the appropriate spaces and mappings. Toward this define

$$\begin{aligned} H &:= \dot{L}^2(\Omega) \\ V &:= \left\{ u \in \dot{H}^2(\Omega) : \partial_\nu u = 0 \quad \text{on} \quad \partial\Omega \right\} \\ \langle A(u), v \rangle &:= \int_\Omega \Delta u \Delta v \end{aligned}$$

where H has the L^2 norm and V has the H^2 norm. Let $\|\cdot\|$ denote the H norm.

Now define $b : V \times V \to \mathbb{R}$ by $b(u, v) := \langle A(u), v \rangle$.

Let's now check that with these choices, the hypothesis of theorem 5.4.1 are satisfied.

Using theorem 2.3.4 we obtain the desired compact imbedding of V into H. To see that V is dense in H note that $\dot{C}_c^\infty(\Omega)$ is dense in $\dot{L}^2(\Omega)$ by lemma 5.3.1, and then use the fact that $\dot{C}_c^\infty(\Omega) \subseteq V$.

Lemma 5.4.1. $A \in \mathcal{L}(V, V^*)$.

Proof. Let $u, v \in V$. Then we have

$$|\langle A(u), v \rangle| \leq ||\Delta u|| ||\Delta v|| \leq ||u||_V ||v||_V.$$

Now sup over $||v||_V \leq 1$ to see

$$||A(u)||_{V^*} \leq ||u||_V.$$

From this we see that $A \in \mathcal{L}(V, V^*)$ and

$$||A||_{\mathcal{L}(V,V^*)} \leq 1.$$

□

Lemma 5.4.2. *b is coercive.*

Proof. If $v \in H^1(\Omega)$ is a weak solution to

$$\begin{cases} -\Delta v + v &= f \quad \text{in } \Omega \\ \partial_\nu v &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$, then we know by elliptic regularity that $v \in H^2(\Omega)$ and we also have the estimate

$$||v||_{H^2} \leq C||f||_{L^2},$$

where C is independent of f. (See [Jost] for details of the estimate.)

Now let $u \in H^1(\Omega)$ denote a weak solution to

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then clearly we don't have the above estimate (if u is a weak solution then $u + c$ is also a weak solution for any constant c), but if we also know that u has zero-average then we do have the estimate. To see this note that we have

$$\|u\|_{H^2} \leq C\|f + u\|_{L^2} \leq C\left\{\|f\|_{L^2} + \|u\|_{L^2}\right\}$$

and

$$\|\nabla u\|_{L^2}^2 \leq \|f\|_{L^2}\|u\|_{L^2}.$$

Combining these and using Poincaré's inequality we see that

$$\|u\|_{H^2} \leq C_0\|f\|_{L^2}$$

where C_0 independent of f.

Now that we have this estimate we see immediately that b is coercive on V.

□

Since we defined A as an operator it is not entirely obvious what $D(A)$ is.

Theorem 5.4.2. *$D(A) = \{u \in H^4(\Omega) \cap V : \partial_\nu \Delta u = 0 \text{ on } \partial\Omega\}$ and $Au = \Delta^2 u$ on $D(A)$.*

Proof. Given $u \in H^4(\Omega)$ and $v \in H^2(\Omega)$ we arrive at

$$\int_\Omega \Delta u \Delta v = \int_\Omega (\Delta^2 u)v + \int_{\partial\Omega} (\partial_\nu v)\Delta u - \int_{\partial\Omega} v(\partial_\nu \Delta u) \tag{5.9}$$

after two applications of Green's formula.

Define $E := \{u \in H^4 \cap V : \partial_\nu \Delta u = 0 \text{ on } \partial\Omega\}$. Let $u \in E$ and $v \in V$. Then we see

$$\langle Au, v\rangle = \int_\Omega \Delta u \Delta v = \int_\Omega (\Delta^2 u)v$$

and so $Au \in L^2(\Omega)$. But $\int_\Omega \Delta^2 u = \int_{\partial\Omega} \partial_\nu \Delta u = 0$ and so $Au \in \dot{L}^2(\Omega) =: H$. From this we see $E \subseteq D(A)$ and for $u \in E$ we have $Au = \Delta^2 u$.

Let's now prove the opposite inclusion.

Let $u \in D(A) \subseteq V$. So there exists some $h \in \dot{L}^2(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v = \int_\Omega hv \qquad \forall v \in V.$$

Using (5.9) as motivation we see that u should be a weak solution to

$$\begin{cases} \Delta^2 u &= h \text{ in } \Omega \\ \partial_\nu u &= 0 \text{ on } \partial\Omega \\ \partial_\nu \Delta u &= 0 \text{ on } \partial\Omega. \end{cases} \tag{5.10}$$

Let's obtain a solution to (5.10). Examine the following system:

$$\begin{cases} \Delta w &= h \quad \text{in } \Omega \\ \partial_\nu w &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{5.11}$$

and

$$\begin{cases} \Delta \tilde{u} &= w \quad \text{in } \Omega \\ \partial_\nu \tilde{u} &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.12}$$

Since $h \in \dot{L}^2(\Omega)$, there exists a unique $w \in \dot{H}^2(\Omega)$ which solves (5.11). Since $w \in \dot{H}^2(\Omega)$, there exists a unique $\tilde{u} \in \dot{H}^4(\Omega)$ that solves (5.12). So we see $\tilde{u}$ a "strong" solution to (5.10) in the sense that $\Delta^2 \tilde{u} = h$ in $L^2(\Omega)$ and the boundary conditions hold in the sense of trace.

Let $v \in V$. Since $\tilde{u} \in \dot{H}^4(\Omega)$ we have enough smoothness to use (5.9) with u replaced with $\tilde{u}$, and when we do this we obtain

$$\int_\Omega \Delta \tilde{u} \Delta v = \int_\Omega hv$$

after taking into account the boundary properties of both $\tilde{u}$ and v. So we see

$$A(u) = A(\tilde{u}) \qquad \text{in} \quad V^*.$$

Since b is coercive we easily see that $A : V \to V^*$ is injective, hence $u = \tilde{u}$ in V, but $\tilde{u} \in E$, so we are done.

$\square$

To apply theorem 5.4.1 we need only check that $\{u \mapsto \Delta f(u)\} \in Lip^{loc}(V,H)$, where $f(u) := u^3 - u$. Toward this define

$$g(u) := \Delta f(u) = 6u|\nabla u|^2 + (3u^2-1)\Delta u =: 6g_1(u) + g_2(u).$$

The next two lemmas will make use of theorem (2.3.4) extensively without making mention to it.

Lemma 5.4.3. *$g_1 \in Lip^{loc}(V,H)$.*

Proof. Let $u, v \in V$. Then we have

$$\begin{aligned} \|g_1(u) - g_1(v)\| &= \|u|\nabla u|^2 - v|\nabla v|^2\| \\ &\leq \|u(|\nabla u|^2 - |\nabla v|^2)\| + \|(u-v)|\nabla v|^2\| \\ &\leq \|u\|_{L^\infty}\||\nabla u|^2 - |\nabla v|^2\| + \|u-v\|_{L^\infty}\||\nabla v|^2.\| \end{aligned}$$

But we also have $\||\nabla v|^2\| \leq \|v\|^2_{W^{1,4}}$. Also we have

$$||\nabla u|^2 - |\nabla v|^2|^2 \leq (|\nabla u| + |\nabla v|)^2 |\nabla u - \nabla v|^2 =: \gamma\beta.$$

Now use Hölder's inequality to get

$$\||\nabla u|^2 - |\nabla v|^2\|^2 \leq \|\gamma\|_{L^{3/2}}\|\beta\|_{L^3}.$$

Now

$$\|\gamma\|^{3/2}_{L^{3/2}} \leq 8\int_\Omega |\nabla u|^3 + 8\int_\Omega |\nabla v|^3 \leq 8\|u\|^3_{W^{1,3}} + 8\|v\|^3_{W^{1,3}}$$

where we have used the fact that $(a+b)^p \leq 2^p(a^p + b^p)$.

$$\|\beta\|^3_{L^3} = \int_\Omega |\nabla u - \nabla v|^6 \leq \|u - v\|^6_{W^{1,6}}$$

Combining the above and using theorem 2.3.4 we see $g_1 \in Lip^{loc}(V,H)$.

□

Lemma 5.4.4. $g_2 \in Lip^{loc}(V, H)$.

Proof. Let $u, v \in V$. Then we have

$$\|g_2(u) - g_2(v)\| \leq \|(3u^2 - 1)(\Delta u - \Delta v)\| + \|(3u^2 - 3v^2)\Delta v\| =: I_1 + I_2.$$

But

$$I_1 \leq \left\{3\|u\|_{L^\infty}^2 + 1\right\} \|\Delta u - \Delta v\|.$$

Also we have

$$\begin{aligned}
I_2^2 &= 9\int_\Omega (u+v)^2(u-v)^2|\Delta u|^2 \\
&\leq 9\left\{\|u\|_{L^\infty} + \|v\|_{L^\infty}\right\}^2 \int_\Omega |\Delta u|^2 |u-v|^2 \\
&\leq 9\left\{\|u\|_{L^\infty} + \|v\|_{L^\infty}\right\}^2 \||\Delta u|^2\|_{L^1} \||u-v|^2\|_{L^\infty} \\
&\leq 9\left\{\|u\|_{L^\infty} + \|v\|_{L^\infty}\right\}^2 \|u\|_{H^2}^2 \|u-v\|_{L^\infty}^2
\end{aligned}$$

Now combining I_1 and I_2 and using theorem 2.3.4 we see $g_2 \in Lip^{loc}(V, H)$. □

Combining the two previous lemmas we see $g \in Lip^{loc}(V, H)$.

Using theorem 5.4.1 we see that if $\phi \in V$ then there exists a $T > 0$ and $u \in C([0,T]; V) \cap L^2(0,T; \mathcal{D}(A)), u_t \in L^2(0,T; H)$ such that u is a solution to (5.8). Let's now translate this abstract solution into a form that is more readable. After some interpretation of spaces we see

$u(t) \in \dot{H}^2(\Omega)$ and $\partial_\nu u(t) = 0$ on $\partial\Omega$ for all $0 \leq t \leq T$

$u(t) \in H^4(\Omega)$, $Au(t) = \Delta^2 u(t)$ and $\partial_\nu \Delta u(t) = 0$ on $\partial\Omega$ for a.e. $0 \leq t \leq T$

$u_t = -\Delta^2 u + \Delta\left\{u^3 - u\right\}$ holds in $L^2(0,T; L^2)$ and so

$u_t = -\Delta^2 u + \Delta\left\{u^3 - u\right\}$ holds in $L^2(\Omega)$ for a.e. $0 \leq t \leq T$.

The Cahn-Hilliard equation does possess a global solution. The interested reader is encouraged to see [Sell] for details.

5.5 Conclusion

In this thesis we have obtained the Cahn-Hilliard equation as a gradient flow over H_0^{-1}, and in doing so we needed to examine the idea of a constrained gradient. In the end we obtained only a local solution to the Cahn-Hilliard equation and hence we did not examine any long term dynamics related to the Cahn-Hilliard equation. If one is interested in more modern aspects of the Cahn-Hilliard equation, one should consult [Fife] (www.math.utah.edu/˜fife/), and the references within.

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