ESSAYS ON ARTIFICIAL STOCK MARKET METHODS

by

Yiping XU

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Name: **Yiping Xu**

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Title of Thesis **Essays on Artificial Stock Market Methods**

Examining Committee:

Chair: Gordon Myers

Jasmina Arifovic
Senior Supervisor

Robbie Jones
Supervisor

Ken Kasa
Supervisor

Ramo Gencay
Internal Examiner

Thomas Lux University of Kiel, Germany
External Examiner

Date Approved: Thursday April 28, 2005

Abstract

This dissertation proposes a two-risky-asset Artificial Stock Market Model and investigates its applications in financial markets. In the first essay, this model is applied to the stock market. Simulation results show that within some range of the parameters, the model can replicate many stylized facts of real financial data and some financial anomalies. This essay also finds that the dynamics of the model and the simulated results can be explained well by two approximation equations: the bubble pricing equation and the mean difference equation of the market share.

The second essay applies the noise trader version of this model to the foreign exchange market and aims at solving the equilibria selection dilemma in the context of Kareken and Wallace (1981). The simulation results show that if agents have full memory, the average portfolio fraction will converge and the initial equilibrium that it converges to is history dependent. However under the lasting evolutionary pressure brought by the noise trader, the asymptotical outcome will be history independent. The model will converge to the neighborhood of an equilibrium with agents equally putting their savings into two currencies. If the agents do not have full memory, the foreign exchange market will show periodic crises. Before and after a market crisis, the exchange rate will converge to different stationary equilibria. A mean difference equation of the average portfolio fraction is also given to describe the dynamics of the

model.

The third essay aims at revealing the role played by the self-referential process inside the artificial stock models, and studying how it is related to the model performance. Three potential dangers that can make a GA learning model degenerate to a pure numerical optimization process are identified. It is also found that although the strength of the self-referential process may not change the convergence property of a GA model, it may lead to substantial differences in the model dynamics before the convergence is achieved.

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Chapter 1

Financial Market Bubbles: An Agent-based Learning Model

A two-risky-asset artificial stock market model is proposed in this essay. Simulation results show that within some range of the parameters, the model can replicate many stylized facts of real financial data and some financial anomalies. This essay also finds that the dynamics of the model and the simulated results can be explained well by two approximation equations: the bubble pricing equation and the mean difference equation of the market share.

1.1 Introduction

The return distributions of financial assets have three typical features: fat tails, skewness and volatility clustering (Engle (1982) and Bollerslev (1986)). There also exist well known financial market anomalies such as short run cross-sectional return momentum (Jegadeesh and Titman (1993)), abnormal return to contrarian strategies

(Lakonishok et al. (1994)), and long-term return reversal (De Bondt and Thaler (1985)). In addition, in contrast to no trade theorem, we can observe large trading volumes in the financial market. How to explain these properties of financial markets is a challenge to academic researchers.

Parke and Waters (2003) explained the source of ARCH effect in an evolutionary game framework. Scheinkman and Xiong (2003) used a heterogeneous belief model to interpret the relationship between trading volume, price volatility and pricing bubbles. Rational bubble models are used by Santos and Woodford (1997) and Blanchard and Watson (1982). Barberis et al. (1998), and Daniel et al. (1998) used behavioral approaches to explain the return momentum and reversal. Bansal and Yaron (2004) explained the predicting power of dividend price ratio and risk premium under the assumption that there is a persistent component in the growth rate of dividend and the dividend process itself follows the ARCH process. In general these models can only explain some aspects of financial market anomalies or stylized facts.

Cochrane, Longstaff and Santa-Clara (2003) [CLS] proposed a model that can give a coherent explanation for almost all the stylized facts and financial anomalies mentioned above except that their model can not generate trading. CLS's model is based on the Lucas tree model with two risky assets and log utility. Starting from two geometric Brownian motion dividends, they show that the dividend share is a complex process with cubic drift and quadratic diffusion. Since asset returns are functions of the dividend share in the rational expectations equilibrium, the rich dynamics of the dividend share in their model lead to the rich dynamics of asset returns.

As a compliment to CLS' rational expectations approach, this paper demonstrates that adaptive learning together with heterogeneity can be an alternative source of the financial stylized facts and anomalies. Same as CLS' model, the context of this paper's approach is also the Lucas tree model with two risky assets and log utilities. However in this paper's approach, the expected dividend share is assumed to be constant, so that in the rational expectations equilibrium, the market share and the expected returns of assets are all constants. The purpose of a setup like this is to provide an easy way to isolate the effect of the evolution of heterogeneous beliefs. Other alternative specifications of dividend processes are not denied by this paper, however, it may cause agents' beliefs to interact in a complicated way with the dynamics of the dividend share, which may make the effect of the evolution of agents' beliefs unidentifiable.

Besides providing a unified explanation for the financial stylized facts and anomalies, this paper's approach also provides a platform to study the relationship between the trading volumes and the movement of asset prices. In particular, it can successfully replicate the positive relationship between the trading volume and the return volatility.

In this paper, the evolution of heterogeneous beliefs is simulated on the platform of an artificial stock market which is an extension of LeBaron (2001), extending the model with one risky asset to the two-risky-asset model. The artificial stock market model is a type of agent based learning model. It was developed at the Santa Fe Institute (SFI) in the late 1980's and early 1990's. Its applications are recorded in Plamer et al. (1994), Arthur et al. (1997), LeBaron et al. (1999), Tay and Linn (2001), and LeBaron (2000,2001,2002) among others. These applications successfully replicate stylized facts of financial data including skewness, fat tails, and volatility clustering, but they do not explain how these facts are generated. This paper finds that the dynamics of the model and the simulated results can be explained well by two equations: the bubble pricing equation, and the mean difference equation of the market share.

Summers (1986) pointed out that if there is a slow decaying component in the asset price, the long-horizon return will show reversal. Following the insight of Summers (1986), this paper gives the source of this slow decaying component, the evolution of the market mood. In addition, it is shown that Summers (1986)'s pricing equation is a special case of the bubble pricing equation proposed in this paper. On the one hand, as predicted by Summers (1986), the short run return shows little autocorrelation, while the long-horizon return shows large negative autocorrelation, i.e., return reversal. On the other hand, and in contrast to Summers, this paper's approach can also generate positive autocorrelation in short run returns and negative cross sectional correlation, i.e, momentum.

The organization of the paper is as follows: The artificial stock market model is given in Section 1.2; Section 1.3 illustrates the design of the simulation; Section 1.4 summarizes the simulation results; Section 1.5 derives the bubble pricing equation and the mean difference equation of the market share, and then discusses the dynamics of the artificial stock market model and how the stylized facts are generated in the model. The sensitivity of the simulation results to the experiment design is discussed as well; Section 1.6 relates the model output to the return momentum and reversal; Section 1.7 concludes.

1.2 Model

1.2.1 The Benchmark Lucas Tree Model

The benchmark model is borrowed from Sargent (1987). The agent has log utility function, and tries to maximize her lifetime utility:

$$u_t = E_t \sum_{\tau=0}^{\infty} \lambda^{\tau} \ln c_{t+\tau}, \tag{1.1}$$

subject to the intertemporal budget constraint

$$w_t = p_{1,t}s_{1,t} + p_{2,t}s_{2,t} + c_t = (p_{1,t} + d_{1,t})s_{1,t-1} + (p_{2,t} + d_{2,t})s_{2,t-1},$$

where $s_{1,t}$, $s_{2,t}$ are the agent's holding of risky asset 1 and asset 2 at period t respectively, w_t is the wealth of the agent at period t, $d_{1,t}$ and $d_{2,t}$ are dividend of asset 1 and 2 respectively. The supplies of asset 1 and 2 are assumed to be fixed at one unit respectively.

The Euler equation for the ith stock is

$$p_{i,t} = \lambda E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} (p_{i,t+1} + d_{i,t+1}) \right] \quad i = 1, 2. \tag{1.2}$$

Denote the aggregate dividend as d_t, which is equal to $d_{1,t} + d_{2,t}$. Substitute the equilibrium consumption $c_t = d_t$ and $u(c_t) = \ln(c_t)$ into (1.2), we get

$$p_{i,t} = \lambda E_t \left[\frac{d_t}{d_{t+1}} (p_{i,t+1} + d_{i,t+1}) \right]. \tag{1.3}$$

In equilibrium,

$$p_{i,t} = \phi_{i,t} d_t, \quad i = 1, 2. \tag{1.4}$$

Substituting (1.4) into (1.3) gives

$$\phi_{i,t} = \lambda E_t \phi_{i,t+1} + \lambda E_t (\frac{d_{i,t+1}}{d_{t+1}}), \tag{1.5}$$

which implies

$$\phi_{i,t} = \sum_{\tau=1}^{\infty} \lambda^{\tau} E_t(\frac{d_{i,t+\tau}}{d_{t+\tau}}). \tag{1.6}$$

Let us consider a special case of equation (1.6), where

$$E_t(\frac{d_{1,t+\tau}}{d_{t+\tau}}) \equiv E_t(\epsilon_{t+\tau}) = \alpha, \quad \tau = 1, 2, \cdots, \infty$$

Substitute it into equation (1.6) and (1.4), we get

$$p_{1,t} = \frac{\alpha\lambda}{1-\lambda} d_t, \tag{1.7}$$

$$p_{2,t} = \frac{(1-\alpha)\lambda}{1-\lambda} d_t. \tag{1.8}$$

The gross returns of asset 1 and asset 2 are

$$R_{1,t+1} = \frac{p_{1,t+1} + d_{1,t+1}}{p_{1,t}} = (1 + \epsilon_t \frac{1-\lambda}{\alpha\lambda}) \frac{d_{t+1}}{d_t}, \tag{1.9}$$

and

$$R_{2,t+1} = \frac{p_{2,t+1} + d_{2,t+1}}{p_{2,t}} = (1 + (1-\epsilon_t) \frac{1-\lambda}{(1-\alpha)\lambda}) \frac{d_{t+1}}{d_t}, \tag{1.10}$$

respectively.

In the rational expectations equilibrium, the market share, defined as the share of asset 1 in the total market value, is $p_{1,t}/(p_{1,t} + p_{2,t}) = \alpha$. It is constant over the time.

In equilibrium, $s_{1,t} = s_{2,t} = 1$, and the saving rate can be shown is equal to λ, which is also a constant. This means that the agent's saving decision is separated from her portfolio decision. Therefore if we fix the saving rate at the optimal solution λ, then the expected logarithm of the next period portfolio return,

$$E_t \ln[\tilde{\alpha} R_{1,t+1} + (1-\tilde{\alpha}) R_{2,t+1}], \tag{1.11}$$

should also be maximized at $\tilde{\alpha} = \alpha$, where $R_{1,t+1}$ and $R_{1,t+1}$ are given by equation (1.9) and (1.10), $\tilde{\alpha}$ is the fraction of asset 1 in the portfolio held by the agent.

1.2.2 The Artificial Stock Market Model

Instead of a representative agent, in the artificial stock market, there is a population of agents of size N, and each agent j, $j = 1, \cdots, N$, holds $s_{1,j,t}$, $s_{2,j,t}$ shares of risky asset 1 and asset 2 respectively at period t. There are also a population of strategies of size N_s. Each strategy α_n defines what proportion of savings is going to be put in asset 1. Each rule is coded in a genetic string with length 20. The genetic string is a string which is composed of 0 or 1. α_n is decoded as follows: $\alpha_n = \sum_{k=1}^{20} a_n^k 2^{k-1}/K$, where $K = 2^{20} - 1$, a_n^k is the value 0 or 1 taken at the kth position in the nth string. In each period, agents search for optimal strategies in the strategy population. The relation between strategies and agents is analogous to that between investors and investor advisers or mutual fund managers.

Agents do not know what the fundamental values of risky assets are and what the actions of other agents will be. They make their investment decisions based on observable realized returns. Each agent j has a memory length of T_j, so they can memorize the past T_j periods of the realized returns. In each period, agent j randomly selects a sample of realized returns with length L from her memory to evaluate the performances of strategies in the strategy population. Thus the performance measure of strategy α_n is

$$\frac{1}{L}\sum_{\tau=0}^{L-1} \ln(\alpha_n(1 + r_{1,t-T_j+nn+\tau}) + (1 - \alpha_n)(1 + r_{2,t-T_j+nn+\tau})), \qquad n = 1, \cdots, N_s \quad (1.12)$$

where $0 < \alpha_n < 1$, and $0 \leq nn \leq T_j - L$ is the distance between the starting point of the sample and the earliest period that agent j remembers. If $nn = 0$, agent j uses $r_{i,t-T_j}$, $\cdots$, $r_{i,t-T_j+L-1}$, i.e., the first L returns in her memory to evaluate the strategies; if $nn = T - L$, agent j uses $r_{i,t-L}$, $\cdots$, $r_{i,t-1}$, i.e. last L returns in her memory to evaluate the strategies.

This setup of the performance measure is to capture the fact that different agents use different information sets to form their decisions. And even for the agents having the same memory length, they may still put different weight to the data in making decisions.

The supplies of asset 1 and 2 are still fixed at one unit respectively, hence the market clearing condition is

$$\sum_{j=1}^{N} s_{1,j,t} = \sum_{j=1}^{N} s_{2,j,t} = 1.$$

The saving rates of all agents are fixed at λ. Agents only make investment decisions. Agent j's investment decision is denoted as $\alpha_{j,t}$, which means agent j will put $\alpha_{j,t}\lambda w_{j,t}$ in asset 1, and $(1-\alpha_{j,t})\lambda w_{j,t}$ in asset 2, where $w_{j,t} \equiv (p_{1,t}+d_{1,t})s_{1,j,t-1} + (p_{2,t}+d_{2,t})s_{2,j,t-1}$ is the wealth of agent j at period t, $\lambda w_{j,t}$ is the savings of agent j. So the demand of agent j for asset 1 is

$$s_{1,j,t} = \frac{\alpha_{j,t}\lambda w_{j,t}}{p_{1,t}}.$$

Summing the demands of all agents, and substituting into the market clear condition, we get

$$p_{1,t} = \sum_{j=1}^{N} \alpha_{j,t}\lambda w_{j,t}. \tag{1.13}$$

In the similar way, we can obtain the price for asset 2,

$$p_{2,t} = \sum_{j=1}^{N} (1-\alpha_{j,t})\lambda w_{j,t}. \tag{1.14}$$

Since $w_{j,t}$ is a function of $p_{1,t}$ and $p_{2,t}$, the asset prices of period t have to be solved jointly from the above two equations.

Suppose agents finally coordinate at the optimal portfolio implied in the rational expectations solutions, then agents will consume $1-\lambda$ of their wealth and put α of their savings in asset 1 and $1-\alpha$ in asset 2. Then the prices of two assets are the solutions to the following two equations

$$p_{1,t} = \alpha\lambda(p_{1,t} + p_{2,t} + d_t),$$

$$p_{2,t} = (1-\alpha)\lambda(p_{1,t} + p_{2,t} + d_t).$$

It is trivial to check that the solutions are indeed equations (1.7) and (1.8).

Note that the condition $\sum_{j=1}^{N} c_{j,t} = d_{1,t} + d_{2,t}$ is satisfied automatically no matter the market is in rational expectations equilibrium or not. For each agent, the following budget constraint must be binding,

$$w_{j,t} = p_{1,t}s_{1,j,t} + p_{2,t}s_{2,j,t} + c_{j,t} = (p_{1,t} + d_{1,t})s_{1,j,t-1} + (p_{2,t} + d_{2,t})s_{2,j,t-1},$$

where $s_{i,j,t}$ and $s_{i,j,t-1}$ are agent j's holding of asset i, $i = 1, 2$, in period t and $t-1$. Summing the above equation across the agents, and using the market clearing condition, we get

$$\sum_{j=1}^{N} c_{j,t} = d_{1,t} + d_{2,t}.$$

The ex post net returns of asset 1 and 2 are calculated as

$$r_{i,t+1} = \ln(\frac{p_{i,t+1} + d_{i,t+1}}{p_{i,t}}), \quad i = 1, 2. \tag{1.15}$$

The trading volumes are recorded as follows

$$V_{i,t} = \sum_{j=1}^{N} I_{i,t}(s_{i,j,t} - s_{i,j,t-1}), \quad i = 1, 2, \tag{1.16}$$

where $I_{i,t} = 1$ if $s_{i,j,t} - s_{i,j,t-1} > 0$, otherwise it equals zero.

1.3 The Design of the Simulation

The design of the GA experiment follows the approach used by LeBaron (2001).

At the beginning of each period, half of the population of agents will be randomly selected to try new rules. They will evaluate the performance of each rule in the strategy population using equation (1.12) and randomly choose one rule from the candidate strategy set, which is a subset of the strategy population having best performance. The parameter 'candidate', which stands for the size of the candidate strategy set, is set at the beginning of the program. For example, if the parameter 'candidate' is set as 1, then it means agents always choose the rule with the best performance; setting the parameter 'candidate' to one half of the size of the strategy population means each agent will randomly choose one rule from the best half of the strategy population. If this rule has better performance than the one she currently uses, she replaces the old one with the new one, otherwise she still uses the old one.

In each period, the timeline of the market is as follows:

1. Dividends are revealed and paid.

2. Each agent evaluates the performance of the strategies using her memory of ex post returns of assets, and chooses one rule as her current strategy.

3. Each agent reports her decision to a 'market maker'. The market maker sets the prices to clear the market.

4. The ex post returns and trading volumes of assets are calculated and stored.

5. Rules evolve.

6. Agents evolve.

Evolution is divided into two parts: the evolution of strategy rules, and the evolution of agents. The evolution of strategy rules is simulated by genetic algorithm: A strategy can be selected as a member of the parent set if at least one agent has used it over the last 10 periods. Strategies that have not been used for 10 periods are marked for replacement. This is a mimic of the real life where investment advisors with no customers will quit from the market.

The sequence of genetic updating is as follows: First, form the set of strategies to be eliminated, then the algorithm chooses among three methods with equal probability to generate new strategies to replace them:

Crossover A pair of strings is selected randomly from the parent set, which are called parent strings. With probability of p_{cross}, crossover will happen. If it happens, an integer k is selected from [1,20], again at random. Two offspring strings are formed by swapping the set of values to the right of position k of the parent strings. If crossover does not happen, these two offsprings are the same as the parent strings. We use the offsprings to replace the rules that have no customers.

Mutation Choose one string from the parent set, and the value of each position within a string is altered with probability p_{mut}. If it is originally 0 then switch to 1, and vice versa. The mutation probability p_{mut} is the same across the population.

New rule Randomly generates a new strategy, which is similar to the arrival of a new mutual fund manager.

The evolution of agents is quite simple. In each period, one randomly selected agent is eliminated from the agent population. Its asset holdings are redistributed equally to other agents. At the same time, a new agent is added into the population with the average asset holding transferred from other agents. The memory length is also randomly generated. In this way, the net impact on the total resources of the market is neutral. This process is designed to represent random arrivals and departures in the stock market, and make the market participators more homogenous in wealth. Specifically the procedure of asset transfer is as follows: suppose agent 1 is chosen to be eliminated, then its asset holdings will be redistributed equally to other agents, at the same time each remaining agent contributes $\frac{1}{N-1}$ of her assets to the new arrival, in this way after redistribution, the asset holding of the new arrival will be:

$$s_{i,1,t} = \frac{\sum_{j=2}^{N} \hat{s}_{i,j,t}}{N-1} \qquad i = 1, 2,$$

where $\hat{s}_{i,j,t}$ is agent j's holding of asset i before the asset redistribution in period t. The asset holdings of the agents in the remaining pool are

$$s_{i,j,t} = \frac{N-2}{N-1}\hat{s}_{i,j,t} + \frac{\hat{s}_{i,1,t}}{N-1}.$$

1.4 Simulation

The exogenous aggregate dividend of assets is assumed to follow

$$\ln d_t = \ln d_{t-1} + g + v_t, \tag{1.17}$$

where $v_t \sim N(0.00, 0.07^{1/2})$, $g = 0.0015$. The value of g here is approximately equal to the average monthly growth rate of dividend in the US stock market. The $\epsilon_t = \dfrac{d_{1,t}}{d_t}$ is generated from beta distribution so that $0 \leq \epsilon_t \leq 1$. The density function of the dividend share is as follows

$$f(\epsilon_t \mid a, b) = \frac{1}{B(a,b)}\epsilon_t^{a-1}(1-\epsilon_t)^{b-1}, \quad 0 \leq \epsilon_t \leq 1,$$

where $B(a, b)$ is the beta function. In the simulation, a and b are set 100 respectively. The mean and variance of ϵ_t are 0.5 and 0.0012437 respectively. The reason that we specify the parameters in such a way for the distributions of v_t and ϵ_t is trying to get two individual dividend processes with little correlation. The correlation between the two dividend sequences is computed numerically at about -0.017. Thus the implied rational expectations equilibrium market share is 0.5. The plot of the density function is shown in figure 1.1.

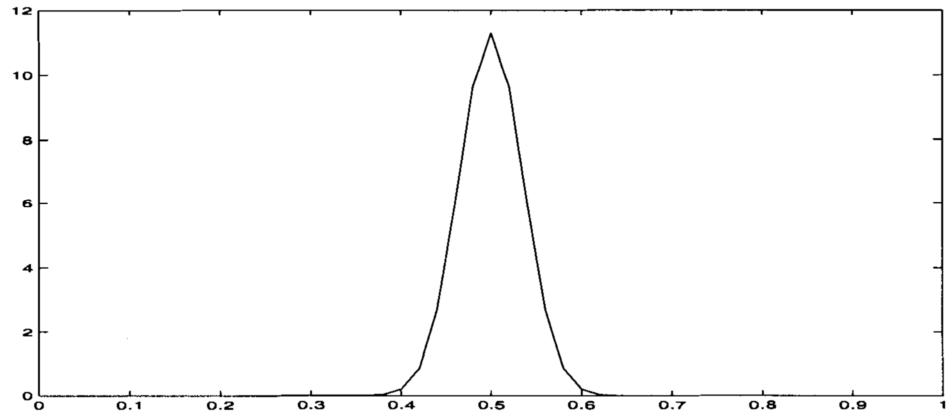

Figure 1.1: The density function of the dividend share: a=b=100

Following Lebaron (2001), the number of periods that is used to identify the strategies to be eliminated is 10. The probability of crossover and mutation is 0.6 and 0.03 respectively. These are the common values used in the literature. The length of the genetic string of strategy is 20. The parameter 'candidate' is set at 75. The number of agents and strategy rules are 300 and 150 respectively. The range of investment strategies is set as (0.00001, 0.99999). This is a mechanism that prevent the model converge to the boundaries, although it may only happen for the simulation with extremely large λ. In evaluating the performance of an investment strategy α_n, if realized returns of $(r_{1,\tau}, r_{2,\tau})$ make $\alpha_n(1+r_{1,\tau})+(1-\alpha_n)(1+r_{2,\tau}) < 0$, its log value is set as -30, which will ensure this strategy will not be used by the agents.

In period 1, strategy rules are generated randomly, agent holdings of asset 1 and asset 2 are equal across the population, and the strategy rule each agent uses is randomly drawn from the strategy population.

Table 1.1: Summary statistics for the simulation with different λ: full memory full sample case

$\lambda =$	μ_{MKT}	std. Dev.	σ_{MKT}	std.Dev	μ_{Vol1}	std.Dev	$\mu_{LM(4)}$	std.Dev
0.005	0.500002	0.002159	0.014511	0.00105	0.01450	0.00105	1.6240	1.7047
0.02	0.499574	0.001240	0.013462	0.00227	0.01342	0.00227	0.9502	0.5952
0.05	0.500583	0.001210	0.011357	0.00214	0.01337	0.00214	0.6328	0.5930
0.1	0.499736	0.002055	0.012487	0.00218	0.01248	0.00218	1.2237	0.8067
0.3	0.499145	0.001734	0.011197	0.00161	0.01120	0.00161	0.7445	0.4837
0.5	0.499832	0.001280	0.012167	0.00216	0.01217	0.00216	0.9924	0.6192
0.7	0.499647	0.001355	0.016730	0.01625	0.01673	0.01625	1.4647	1.6835
0.9	0.499926	0.000953	0.010203	0.00172	0.01020	0.00172	0.7645	0.7548
0.95	0.500012	0.001343	0.006904	0.00180	0.00690	0.00180	0.9774	0.89164
0.98	0.499681	0.001834	0.002843	0.00131	0.00284	0.00131	1.0688	0.4650
0.995	0.493452	0.024109	0.012450	0.00304	0.00124	0.0030	0.8024	0.4917

*Each simulation has 1000 periods. For each set of parameters, experiments are run 10 times using different seeds. The statistics of simulation results are calculated using the sample from the last 500 periods, then the average of the statistics across these 10 runs are reported. Column 3,5,7,9 are standard deviations of the last column's statistics over 10 runs respectively

1.4.1 Benchmark

In the benchmark simulation, agents all have full memory, which means agents remember all past realized returns, and, in addition, they use the full sample to evaluate the performance of the strategies. The patience parameter λs are set as 0.005, 0.02, 0.05, 0.1, 0.3, 0.5, 0.7, 0.9, 0.95, 0.98, 0.995, respectively, and each simulation has 1000 periods. For each set of parameters, experiments are run 10 times using different seeds. The statistics of the simulation results are calculated using the sample from the last 500 periods, then the average of the statistics across these 10 runs are reported in Table 1.1. Under the setup of the experiments, in the rational expectations equilibrium, the market share equals $\alpha = E(\epsilon_t) = 0.5$. Column 2 of Table 1.1 shows the averages of the mean of the market share over the 10 runs. Column 3 shows the standard deviations of the mean of the market share over the 10 runs, column 4 shows the averages of standard deviations of the market share over 10 runs, column

5 reports its standard deviation over the 10 runs. Column 6 reports the averages of mean trading volume and column 8 reports the average of ARCH LM(4) statistics on the return of asset 1. The tests of ARCH are on the residual of the following regression:

$$r_{i,t} = c + \rho_1 r_{i,t-1} + \rho_2 r_{i,t-2} + \rho_3 r_{i,t-3} + \rho_4 r_{i,t-4} + \varepsilon_{i,t}, \qquad i = 1, 2.$$

Although this specification may include too many lags, especially for the full memory case, its purpose is to make the statistics comparable with the short memory cases, in which it seems necessary to reduce the autocorrelation in the residual, so that to reduce the risk that a missing autocorrelation term in the mean equation leads to a false ARCH effect in the residual. We can see that the market share converges to the rational expectations solution very well for all λs, the mean of the market share is very close to 0.5 and shows little fluctuation. As expected, the trading volumes are low and usually below 1.5%, and the ARCH LM statistics are low and insignificant. The intuition of these outcomes is straightforward. Converging to the rational expectations solution means that each agent uses the same strategy, and the strategy they use is the rational expectations equilibrium strategy, thus there is zero trading volume. Due to the stochastic nature of the algorithm, the above claims will not hold exactly, so we can still see some low level of trading activities. The rational expectations solution of asset prices is a linear transformation of the dividend process, thus if there is no ARCH effect in the dividend process, the rational expectations equilibrium return series will not have it either.

1.4.2 Short Memory Case

In these simulations, agents memory lengths are randomly generated from the uniform distribution of [10, 40], and agents randomly pick up a sample of realized returns with length 10 from their memory window to evaluate the performance of the strategies. The patience parameter λs are set as 0.1, 0.4, 0.8, 0.9, 0.95, 0.96, 0.97, 0.98, 0.99, respectively, and each simulation has 10000 periods. The statistical results are obtained using the results from period 9000 to 10000. For each set of parameters, experiments are run 10 times using different seeds, the average of the statistics across these 10 runs are reported in table 1.2. The ARCH LM test is calculated under the same specification as the long memory case.

The features of the simulated data are as follows:

1) The behavior of the return of asset 1 is very similar to that of asset 2. This is because under the setup of parameters, there is not much difference between asset 1 and asset 2 in each aspect. Thus similar behavior of these two assets is expected, and we will concentrate on the behavior of the return of asset 1.

2) There are strong upward trends in the measures of skewness and kurtosis as λ increases. For the return of asset 1, the skewness and kurtosis increase from 0.455 to 4.166, and 3.457 to 35.120, respectively, when λ increases from 0.1 to 0.99.

3) The first order autocorrelation coefficients of the return of asset 1 is moderate, generally below 0.15. However when λ is bigger than 0.97, they become quite large. The model can generate both positive autocorrelation and negative autocorrelation. Whether the autocorrelation is positive or negative seems mainly to depend on the value of λ. Different seeds seldom change the sign of the autocorrelation. For example, for λ equal to 0.4, 0.8, 0.9, we have 30 runs in total, but only one of them shows positive first order autocorrelation, while all others are negative.

Table 1.2: Summary statistics for the simulation with different λ: short memory cases

$\lambda =$	0.1	0.4	0.8	0.9	0.95	0.96	0.97	0.98	0.99
Panel A:	statistics for r_1								
Skewness	0.455	0.468	0.337	0.392	0.764	0.845	1.303	2.447	4.166
	(0.106)	(0.074)	(0.111)	(0.165)	(0.253)	(0.249)	(0.350)	(0.709)	(1.033)
Kurtosis	3.457	3.452	3.542	4.193	5.967	6.849	9.781	17.996	35.120
	(0.285)	(0.270)	(0.313)	(0.809)	(1.161)	(1.490)	(2.270)	(6.665)	(13.28)
ρ_1	0.096	-0.067	-0.133	-0.050	0.064	0.093	0.147	0.265	0.398
	(0.036)	(0.040)	(0.022)	(0.050)	(0.048)	(0.047)	(0.050)	(0.115)	(0.100)
ARCH	1.141	1.328	4.620	6.463	7.667	6.805	4.941	3.370	5.347
LM(4)	[1]	[1]	[8]	[7]	[9]	[9]	[5]	[4]	[6]
ARCH(1)	0.002	0.025	0.083	0.132	0.306	0.330	0.456	0.902	1.104
	(0.050)	(0.030)	(0.051)	(0.067)	(0.056)	(0.097)	[10]	[10]	[10]
GARCH(1)	0.590	0.293	0.376	0.522	0.526	0.479	0.408	0.249	0.134
	(0.130)	(0.560)	(0.468)	(0.104)	(0.064)	(0.103)	[10]	[10]	[9]
Mean of v_1	0.268	0.253	0.196	0.168	0.147	0.142	0.131	0.101	0.070
	(0.005)	(0.003)	(0.002)	(0.006)	(0.006)	(0.005)	(0.007)	(0.018)	(0.016)
Panel B:	statistics for r_2								
Skewness	0.537	0.440	0.391	0.407	0.795	0.750	1.217	2.279	3.954
	(0.131)	(0.105)	(0.112)	(0.128)	(0.236)	(0.189)	(0.520)	(0.814)	(0.871)
Kurtosis	3.814	3.374	3.521	4.297	6.283	6.199	10.130	17.740	31.489
	(0.351)	(0.433)	(0.293)	(0.512)	(0.891)	(0.509)	(3.491)	(6.542)	(9.047)
ρ_1	0.100	-0.064	-0.107	-0.054	0.093	0.088	0.123	0.235	0.417
	(0.040)	(0.040)	(0.034)	(0.041)	(0.038)	(0.037)	(0.033)	(0.094)	(0.094)
ARCH	1.327	1.487	4.569	6.354	8.988	7.270	3.431	2.322	6.062
LM(4)	[1]	[4]	[9]	[7]	[10]	[10]	[6]	[4]	[7]
ARCH(1)	0.014	0.009	0.102	0.147	0.334	0.328	0.353	0.795	1.223
	(0.048)	(0.043)	(0.048)	(0.064)	(0.087)	(0.088)	[9]	[10]	[10]
GARCH(1)	0.283	0.231	0.347	0.541	0.521	0.409	0.456	0.256	0.155
	(0.583)	(0.240)	(0.463)	(0.085)	(0.073)	(0.185)	[10]	[9]	[9]
Mean of v_2	0.269	0.253	0.196	0.168	0.145	0.143	0.134	0.104	0.071
	(0.004)	(0.003)	(0.001)	(0.006)	(0.005)	(0.005)	(0.009)	(0.021)	(0.020)
Panel C:									
Correl(r1,r2)	-0.768	-0.762	-0.670	-0.606	-0.716	-0.517	-0.498	-0.367	-0.170
	(0.013)	(0.012)	(0.021)	(0.022)	(0.036)	(0.022)	(0.032)	(0.104)	(0.091)

*Each simulation has 10000 periods. The statistical results are obtained using the results from period 9000 to 10000. For each set of parameters, experiments are run 10 times using different seeds. The statistics of simulation results are calculated using the sample from the last 1000 periods, then the average of the statistics across these 10 runs are reported. Numbers in parenthesis are standard deviations using 10 runs. Numbers in brackets are the times that reject the null hypothesis in the 10 runs. v_i, $i = 1, 2$ are the trading volumes of asset 1 and 2.

Table 1.3: Summary statistics for the stock returns of different industries

$\lambda =$	Cnsmr	Manuf	Hitec	Hlth	Other
Skewness	0.109	0.405	0.152	0.155	1.020
Kurtosis	10.008	11.455	6.605	9.955	16.787
ρ_1	0.125	0.066	0.091	0.074	0.162
	(0.000)	(0.042)	(0.005)	(0.023)	(0.000)
ARCH	14.527	22.571	38.708	32.551	48.681
LM(4)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
ARCH(1)	0.106	0.128	0.120	0.094	0.127
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
GARCH(1)	0.876	0.853	0.859	0.823	0.839
	(0.000)	(0.000)	(0.000)	(0.324)	(0.000)

*Numbers in square brackets are the significant level.
Data range from July 1926 to December 2003, average value weighted monthly return, sample size: 2915. Data are provided by Fama and French, and can be found at Kenneth French's website, http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/index.html.

4) For the ARCH and GARCH effects, the results can be divided into three regimes: regime 1, λ is 0.1 and 0.4; regime 2, λ is 0.90, 0.95 and 0.96; regime 3, λ is 0.98 and 0.99. In regime 1, there is almost no ARCH effect. Among 20 runs, only two of them marginally reject the null hypothesis of no ARCH effect. On average, the statistics of ARCH LM test are very small. In regime 2, the ARCH effect is strong, and is the GARCH effect. In regime 3, the ARCH LM test can only detect half of the runs having ARCH effect. However GARCH estimation show significant ARCH component and GARCH component in almost all runs.

5) The trading volumes are much bigger than those of the full memory full sample cases. There are downward trends in the measure of trading volume with the increasing of λ. The trading volumes of asset 1 decrease from 26.8 percent to 7 percent when λ increases from 0.1 to 0.99.

6) The correlation coefficients between the return of asset 1 and the return of asset 2 all are negative with large absolute values.

Comparing the statistics of the simulated return data with those of the returns of 5 portfolios organized by industries reported in table 1.3, we can see that the simulated data in regime 2 match the real data best: the skewness, kurtosis and ρ_1 are compatible, the GARCH component is bigger than the ARCH component, ARCH LM tests are significant etc.. The regime 1 data fail to replicate all other features of the real data except for ρ_1. Regime 3 data lose in three aspects: First, ρ_1 is bigger than the real data. Second, in its GARCH specification, the ARCH component is bigger than the GARCH component. Third, the sum of the ARCH component and the GARCH component is bigger than 1, which implies non-stationary process. Although regime 3 data don't match these portfolio data very well, it doesn't necessary mean we can not observe the real return series having the same pattern. For example, Campbell and Hamao (1992) reported Japanese value-weighted index monthly returns from 1971 to 1990 show ρ_1 nearly 0.22. Tatsuyoshi (2002) reports the daily returns of Sony company and 7-11 Japan have the GARCH 0.069, ARCH 2.007, GARCH 0.219, ARCH 1.781, respectively, sample period from January 1, 1998 to April 30, 2000.

1.4.3 The Jump-diffusion Representation of Regime 3 Data

The AR(4)-GARCH(1,1) specification can be expressed as

$$r_{i,t} = c + \rho_{i,1}r_{i,t-1} + \rho_{i,2}r_{i,t-2} + \rho_{i,3}r_{i,t-3} + \rho_{i,4}r_{i,t-4} + \varepsilon_{i,t}, \quad i = 1,2 \tag{1.18}$$

$$\varepsilon_{i,t}|(\varepsilon_{i,t-1}, \varepsilon_{i,t-2}, \cdots) \sim N(0, \sigma_{i,t}^2),$$

$$\sigma_{i,t}^2 = \omega_i + \gamma_{i,1}\varepsilon_{i,t-1}^2 + \gamma_{i,2}\sigma_{i,t-1}^2, \quad \gamma_{i,1}, \gamma_{i,2} \geq 0, \tag{1.19}$$

$$\varepsilon_{i,t} \equiv \sigma_{i,t}z_t, \quad z_t \sim N(0.1),$$

where equation (1.18), (1.19) are the mean equation and variance equation respectively, $\gamma_{i,1}$ and $\gamma_{i,2}$ are coefficients of ARCH and GARCH components respectively. The variance equation says that the conditional variance of $\varepsilon_{i,t}$ at period t, equals the long run variance, ω_i, plus the lag of the squared residual from the mean equation, the ARCH term $\varepsilon_{i,t-1}^2$, which is news about volatility from the previous period, plus last period's forecast variance, the GARCH term $\sigma_{i,t-1}^2$.

Let $v_{i,t} = \varepsilon_{i,t}^2 - \sigma_{i,t}^2$, and after recursively substituting out the variance in the variance equation, we get

$$\varepsilon_{i,t}^2 = \omega_i + (\gamma_{i,1} + \gamma_{i,2})\varepsilon_{i,t-1}^2 + v_{i,t} - \gamma_{i,2}v_{i,t-1}.$$

For $\varepsilon_{i,t}^2$ to be stationary, the sum of coefficients of ARCH term, $\gamma_{i,1}$ and GARCH term, $\gamma_{i,2}$, have to be less than 1. In regime 3, it is common that this sum is greater than 1. Thus it appears that the GARCH specification does not well describe the behavior of the return series in regime 3.

One special feature of the return series of regime 3 is that they have a large measure of skewness. Although the condition distribution of the residual of ARCH or GARCH is normally distributed, its unconditional distribution has fat tails. Thus ARCH or GARCH model can handle some degree of the fat tails problem. However it is not so good in dealing with the distribution with big skewness. An alternative approach is the jump-diffusion model. In continuous time, the jump-diffusion process of stock price can be expressed as follows:

$$\frac{dP_t}{P_t} = \mu dt + \sigma_t dz_t + Y_t dN_t, \tag{1.20}$$

where z_t is a standard Wiener process (diffusion part), N_t is a standard Poisson process (jump part), which represents the total number of extreme shocks that occur until time t. For a wiener process, $var(\Delta z_t) = \Delta t$, thus $\sigma_t \Delta z_t \sim N(0, \sigma_t^2 \Delta t)$. If there is no jump part, then as Δt goes to zero, the size of the change in the price will become smaller and smaller, and prices will become less and less volatile. For the Poisson process, let ν denote the jump arrival intensity, and assume during an

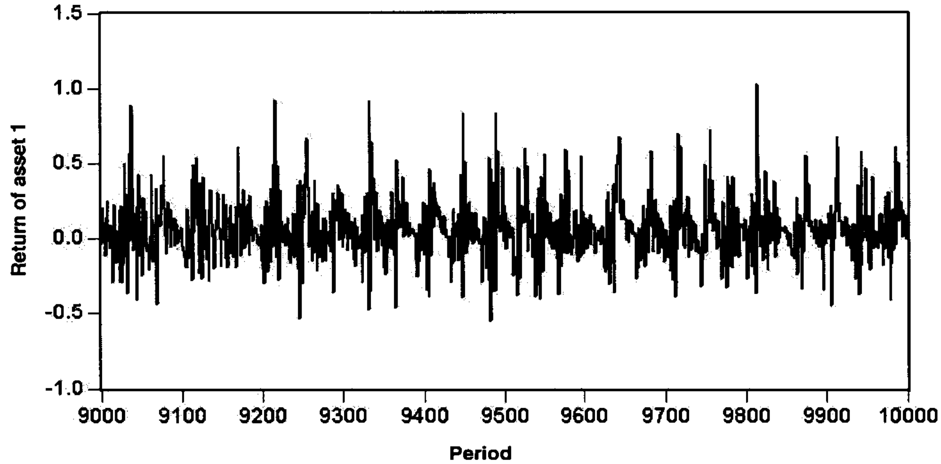

Figure 1.2: Return example in regime 2: $\lambda = 0.95$

infinitesimal time interval Δt, the increment in the N_t only has two possibilities, 0 or 1. Then $dN_t = 1$ with probability $\nu\Delta t$, and $dN_t = 0$ with probability $1 - \nu\Delta t$, so as Δt goes to zero, the probability of observing the occurrence of jump will go to zero, however, once a jump occurs, the size of change, y_t, will be independent of Δt, which is the critical difference between the Poisson process and Wiener process. The Poisson process has the feature of discontinuity, which is used to capture extreme shocks to the stock price, such as an overnight big change in the stock price etc.

To make a comparison, two typical return series from regime 2 and regime 3 are shown in Figure 1.2 and Figure 1.3. The return of run 3 in regime 3 shown in Figure 1.3 displays severe spikes, which is different from the pattern shown in figure 1.2, suggesting that we should use different models to describe them. The jump-diffusion may be a good choice to capture the spikes.

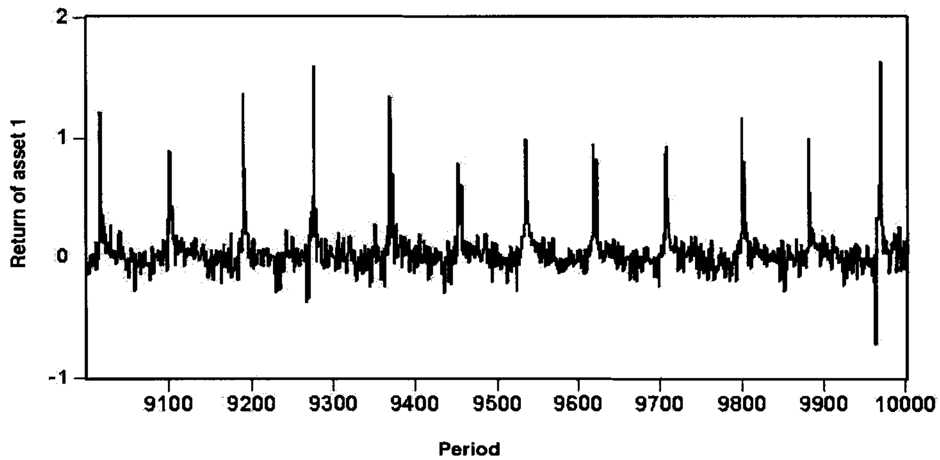

Figure 1.3: Return example in regime 3: $\lambda = 0.98$, run 3, LM(4)=2.06, ARCH=1.352, GARCH=0.163

The jump-AR(4)-GARCH(1,1) model can be represented as

$$r_{i,t} = c+\rho_{i,1}r_{i,t-1}+\rho_{i,2}r_{i,t-2}+\rho_{i,3}r_{i,t-3}+\rho_{i,4}r_{i,t-4}+\varepsilon_{i,t}+\sum_{n_t=1}^{N_t}\ln(Y_{i,n_t}), \quad i = 1,2, \tag{1.21}$$

$$\varepsilon_{i,t}|(\varepsilon_{i,t-1}, \varepsilon_{i,t-2}, \cdots) \sim N(0, \sigma_{i,t}^2),$$

$$res_{i,t} = r_{i,t} - c - \rho_{i,1}r_{i,t-1} - \rho_{i,2}r_{i,t-2} - \rho_{i,3}r_{i,t-3} - \rho_{i,4}r_{i,t-4},$$

$$\sigma_{i,t}^2 = \omega_i + \gamma_{i,1}res_{i,t-1}^2 + \gamma_{i,2}\sigma_{i,t-1}^2,$$

$$\varepsilon_{i,t} \equiv \sigma_{i,t}z_t, \quad z_t \sim N(0,1), \quad Prob(N_t = n_t) = \frac{e^{-\nu_i}\nu_i^{n_t}}{n_t!}, \quad \ln Y_{i,n_t} \sim N(\theta_i, \delta_i^2),$$

where, $\varepsilon_{i,t} \equiv \sigma_{i,t}z_t$ is the diffusion term, $\sum_{n_t=1}^{N_t}\ln(Y_{i,n_t})$ is the jump term (here we assume that the jump size follows a lognormal distribution). This model can be estimated by MLE following the idea of Jorion (1988), with the likelihood function as

$$L = -T\nu_i - \frac{T}{2}\ln 2\pi + \sum_{t=1}^{T}\ln(\sum_{n_t=0}^{\infty}\frac{\nu_i^{n_t}}{n_t!}\frac{1}{\sqrt{\sigma_{i,t}^2 + \delta_i^2 n_t}}exp(\frac{-(res_{i,t} - \theta_i n_t)^2}{2(\sigma_{i,t}^2 + \delta_i^2 n_t)})). \tag{1.22}$$

Table 1.4: Summary statistics for the GARCH(1,1) model against the jump-GARCH(1,1) model

GARCH(1,1)		jump				
γ_1	γ_2	ν	θ	δ	likehood	LRtestχ_3^2
Run 3						
1.353 (0.070)	0.164 (0.020)				598.05	
0.112 (0.009)	0.386 (0.091)	0.026 (0.008)	0.500 (0.168)	0.593 (0.160)	813.33	428.56
Run 2						
1.379 (0.070)	0.110 (0.023)				178.50	
0.199 (0.034)	0.465 (0.053)	0.127 (0.021)	0.113 (0.060)	0.489 (0.047)	378.05	399.90

*Numbers in parenthesis are standard errors. $LR = -(\ln L_r - \ln L_u)$.The 5% critical value for χ_3^2 is 7.81

The log likelihood function (1.22) is a weighted sum of normal distributions, with the weight equal to the occurring 0, 1, 2, 3, $\cdots$, times of jump. In order to optimize it numerically, the infinite sum has to be truncated after some value of N_t. This paper choose $N_t = 5$ in its estimation. The reason is as follows: the contribution of n_tth jump to the likelihood is weighted by $\frac{\nu_i^{n_t}}{n_t!}$. Since the value of ν_i is typically very small, when it is powered by n_t, it will make the contribution of the n_tth jump become trivial for large n_t. In our estimation, there is no much difference in the results between the case when $N_t = 5$ and those for even larger values.

A series of nested model hypotheses can be tested by the LR test. Putting restriction $\nu_i = \theta_i = \delta_i = 0$, jump-GARCH(1,1) model is reduced to GARCH(1,1) model. Pure jump processes can be tested against the jump-GARCH model by restricting $\gamma_{i,1} = \gamma_{i,2} = 0$, etc.. Here we are interested in testing the jump-GARCH model against the GARCH(1,1) model. The estimation results use returns of asset 1 in run 2, 3, with $\lambda = 0.98$ are listed in Table 1.4. We can see, all coefficients of the GARCH(1,1) part and jump part are significant at usual significant levels. With the introducing of the jump part, the sum of coefficients of GARCH(1,1) become less than one, and the relative magnitude of coefficients shows the usual pattern with GARCH coefficient bigger than the ARCH coefficient. The LR test results are in favor of jump-GARCH(1,1) model. This suggests, for the return series in regime 3, it is possible to use the jump-GARCH model to describe the data.

1.5 The Dynamics of the Market Share and the Stylized Facts

In order to go further in analyzing the dynamics of the model and the stylized facts reported above, we simplify the model, while retaining the essential features. In particular we assume that wealth is evenly distributed, thus ignoring the wealth distribution effect.

1.5.1 The Rational Expectations Solution Benchmark

The benchmark is the case when the market share converges to the rational expectations solutions. It will happen when agents have full memory and use full sample. In this case, we know

$$p_{1,t} = \frac{\alpha\lambda}{1-\lambda} d_t.$$

Substituting it into the definition of the gross return of asset 1, equation (1.15), we get

$$R_{1,t} = (1 + \epsilon_t \frac{1-\lambda}{\alpha\lambda}) \frac{d_t}{d_{t-1}}.$$

Taking logs of both sides,

$$r_{1,t} = \ln(R_{1,t}) = \ln(1 + \epsilon_t \frac{1-\lambda}{\alpha\lambda}) + \ln d_t - \ln d_{t-1}.$$

Substituting equation (1.17) into it, we get

$$r_{1,t} = \ln(1 + \epsilon_t \frac{1-\lambda}{\alpha\lambda}) + g + v_t.$$

Expanding it around $\epsilon_t = E(\epsilon_t) = \alpha$ [1], we get

$$r_{1,t} = g - \ln\lambda + (1-\lambda)\ln\frac{\epsilon_t}{\alpha} + v_t. \tag{1.23}$$

Since ϵ_t and v_t are independent to each other across time, the conditional variance of $r_{1,t}$

$$var_{t-1}(r_{1,t}) = (1-\lambda)^2 var(\ln\frac{\epsilon_t}{\alpha}) + var(v_t), \tag{1.24}$$

is constant in this case. Therefore in the setup of our learning model, if the algorithm converges to the rational expectations solution, there will be no ARCH effect.

1.5.2 The Generalized Bubble Pricing Equation

In general, the prices of the two assets are determined by

$$p_{1,t} = \lambda \sum_{j=1}^{N} \alpha_{j,t}[(p_{1,t} + d_{1,t})s_{1,j,t-1} + (p_{2,t} + d_{2,t})s_{2,j,t-1}], \tag{1.25}$$

$$p_{2,t} = \lambda \sum_{j=1}^{N} (1-\alpha_{j,t})[(p_{1,t} + d_{1,t})s_{1,j,t-1} + (p_{2,t} + d_{2,t})s_{2,j,t-1}]. \tag{1.26}$$

Based on the simplifying assumption that wealth is evenly distributed, $s_{1,j,t-1} = 1/N$ and $s_{2,j,t-1} = 1/N$. Then we get

$$p_{1,t} = \frac{\lambda\bar{\alpha}_t d_t}{1-\lambda}, \tag{1.27}$$

[1] See appendix

$$p_{2,t} = \frac{\lambda(1-\bar{\alpha}_t)d_t}{1-\lambda}.$$

Therefore, the market share is $\bar{\alpha}_t$, and

$$R_{1,t} = \frac{\frac{\lambda\bar{\alpha}_t d_t}{1-\lambda} + d_{1,t}}{\frac{\lambda\bar{\alpha}_{t-1}d_{t-1}}{1-\lambda}},$$

i.e.,

$$R_{1,t} = (\frac{\bar{\alpha}_t}{\bar{\alpha}_{t-1}} + \frac{\epsilon_t(1-\lambda)}{\bar{\alpha}_{t-1}\lambda})\frac{d_t}{d_{t-1}}.$$

Thus

$$r_{1,t} = \ln(R_{1,t}) = \ln(\frac{\bar{\alpha}_t}{\bar{\alpha}_{t-1}} + \frac{\epsilon_t(1-\lambda)}{\bar{\alpha}_{t-1}\lambda}) + \ln(d_t) - \ln(d_{t-1}). \quad (1.28)$$

Following Campbell and Shiller (1988), equation (1.28) can be approximated by a first order Taylor series expansion along the deterministic rational expectations equilibrium path[2], where $\ln(d_t/d_{t-1}) = \ln(p_{1,t}/p_{1,t-1}) = g$, $\epsilon_t = E(\epsilon_t) = \alpha$, so that

$$r_{1,t} \approx g - \ln\lambda + (1-\lambda)\ln(\frac{\epsilon_t}{\alpha}) + v_t + \lambda(\ln\bar{\alpha}_t - \ln\bar{\alpha}_{t-1}) - (1-\lambda)(\ln(\bar{\alpha}_{t-1}) - \ln\alpha). \quad (1.29)$$

Equation (1.29) is the bubble pricing equation. Comparing equation (1.29) with equation (1.23), we can see the first four terms reflect the fundamental value of the asset. The fifth is the market mood term. If investors have an optimistic view of asset 1, they will increase their holdings of this asset, so that the market mood term, $\ln\bar{\alpha}_t - \ln\bar{\alpha}_{t-1} > 0$, has positive impact on the return of asset 1. For the same reason, if investors are pessimistic, then it will have negative impact. The sixth term is the level effect of asset holdings. If investors hold less of asset 1 than what's implied in the rational expectations equilibrium, the level effect term will be positive, and vice versa. The sum of the market mood term and the level effect term is the bubble in the asset prices.

[2]See appendix

1.5.3 The Mean Difference Equation of the Market Share

The strategy updating procedure in each period can be divided into two stages: in the first stage, half of the agents will be randomly chosen to try new strategies; in the second stage, the selected agents randomly choose strategies from the candidate set and compare them with the strategies they currently use. If the new ones are better, they switch to new ones. Otherwise, they keep the old ones.

Without loss of generality, let's assume, after stage 1, the index of agents are reassigned, so that, the first half agents try new strategies and the second half agents keep the old ones. Denote $\alpha_{j,t}$ as the strategy used by agent j in period t, then

$$\bar{\alpha}_t = \frac{1}{N}(\sum_{j=1}^{N/2} \alpha_{j,t} + \sum_{i=N/2+1}^{N} \alpha^e_{j,t}),$$

where $\alpha^e_{j,t}$ is the strategy used by agent j who keeps the old one. $\alpha^e_{j,t} = \alpha_{j,t-1}$ and each strategy $\alpha_{m,t-1}$, $m = 1 \cdots N$ has the same probability $\frac{1}{N}$ to be selected. Thus

$$E_{t-1}(\alpha^e_{j,t}) = \frac{1}{N}\sum_{m=1}^{N} \alpha_{m,t-1} = \bar{\alpha}_{t-1}.$$

Since each strategy $\alpha_{m,t-1}$ also has the same probability $\frac{1}{N}$ to be selected to be updated,

$$E_{t-1}(\alpha_{j,t}) = \frac{1}{N}\sum_{m=1}^{N} E_{t-1}(\alpha_{m,t}|\alpha_{m,t-1}),$$

where $E_{t-1}(\alpha_{m,t}|\alpha_{m,t-1})$ is the expected value of agent m's next period strategy, when agent m is selected to try new strategies. Thus

$$\begin{aligned} E_{t-1}(\bar{\alpha}_t) &= \tfrac{1}{N}(\textstyle\sum_{j=1}^{N/2} E_{t-1}(\alpha_{j,t}) + \sum_{j=N/2+1}^{N} E_{t-1}(\alpha^e_{j,t-1})) \\ &= \tfrac{1}{2}\tfrac{1}{N}\textstyle\sum_{m=1}^{N} E_{t-1}(\alpha_{m,t}|\alpha_{m,t-1}) + \tfrac{1}{2}\bar{\alpha}_{t-1}, \end{aligned}$$

At period $t-1$, agent m evaluates the fitness of strategies by

$$\sum_{\tau=0}^{L-1} \ln(\alpha(1 + r_{1,t-T_m+nn+\tau-1}) + (1-\alpha)(1 + r_{2,t-T_m+nn+\tau-1})). \tag{1.30}$$

Denote the optimal solution as $\alpha^*_{m,t}$. Let's first pick up one agent m who uses strategy $\alpha_{m,t-1}$ in period $t-1$, and see what the expected value of next period strategy will be conditional on $\alpha_{m,t-1}$. There are two possible outcomes: first with probability $p(\alpha_{m,t-1})$, this strategy will be replaced by the new strategy. We do not know the explicit form of $p(\alpha_{m,t-1})$, but when the genetic algorithm converges to the rational expectations equilibrium, $p(\alpha_{m,t-1})$ will shrink to zero. The second outcome is that this strategy wins in the competition and will still be used in the next period, with probability $1-p(\alpha_{m,t-1})$. In the first case, the conditional expectation of the new strategy is $E_{t-1}(\alpha_{new}|\alpha_{m,t-1})$. In the second case the conditional expectation is $\alpha_{m,t-1}$. Denote $p(\alpha_{m,t-1})$ as $\tilde{p}_{m,t}$, $E_{t-1}(\alpha_{new}|\alpha_{m,t-1})$ as $b_{m,t}$. Then

$$\begin{aligned} E_{t-1}(\alpha_{m,t}|\alpha_{m,t-1}) &= \tilde{p}_{m,t}b_{m,t} + (1-\tilde{p}_{m,t})\alpha_{m,t-1} \\ &= b_{m,t} + (1-\tilde{p}_{m,t})(\alpha_{m,t-1} - b_{m,t}) \\ &= \alpha^*_{m,t} + (b_{m,t} - \alpha^*_{m,t}) + (1-\tilde{p}_{m,t})(\alpha_{m,t-1} - b_{m,t}). \end{aligned}$$

So for $j < N/2+1$

$$E_{t-1}(\alpha_{j,t}) = \frac{1}{N}\sum_{m=1}^{N}(\alpha^*_{m,t} + (b_{m,t} - \alpha^*_{m,t}) + (1-\tilde{p}_{m,t})(\alpha_{m,t-1} - b_{m,t}))$$

Then

$$\begin{aligned} E_{t-1}(\bar{\alpha}_t) - \bar{\alpha}_{t-1} &= \tfrac{1}{2N}(\textstyle\sum_{m=1}^{N}((\alpha^*_{m,t} - \bar{\alpha}_{t-1}) + (b_{m,t} - \alpha^*_{m,t}) \\ &\quad +(1-\tilde{p_{m,t}})(\alpha_{m,t-1} - b_{m,t}))) \\ &= \tfrac{1}{2}(\bar{\alpha}^*_t - \bar{\alpha}_{t-1}) + \tfrac{1}{2N}(\textstyle\sum_{m=1}^{N}(b_{m,t} - \alpha^*_{m,t})) \\ &\quad +\tfrac{1}{2N}(\textstyle\sum_{m=1}^{N}(1-\tilde{p}_{m,t})(\alpha_{m,t-1} - b_{m,t})), \end{aligned} \tag{1.31}$$

where $\bar{\alpha}^*_t = \frac{1}{N}\sum_{m=1}^{N}\alpha^*_{m,t}$. From the mean difference equation of the market share (1.31), we can see that the updating of the mean $\bar{\alpha}_t$ can be decomposed into three parts: the first is the gap between the average of α^*_t and $\bar{\alpha}_{t-1}$, which is the main determinant of the movement of the mean $\bar{\alpha}_t$; the second term is the average of $b_{m,t} - \alpha^*_{m,t}$, which is determined by the shape of the fitness function and the probability distribution of the strategy in the candidate set. It measures how close between the optimal portfolio and the mean of the admitted new strings, thus can be called admission noise; The third term is the average of a weighted average of $\alpha_{m,t-1} - b_{m,t}$, which embodies the contribution of wrong "pick up". The sum of the second term and the third term is a measurement of the irrationality of agents, since this sum will become zero if agents always pick the the strategy with best fitness measure.

1.5.4 The Explanation for the Simulated Results

The Dynamics of the Market Share

The implication of the mean difference equation is that if $\bar{\alpha}^*_t > \bar{\alpha}_{t-1}$, i.e., if agents on average are willing to increase their holding of asset 1, the market share will have a tendency to increase, and vice versa. The rational expectations equilibrium will be achieved if $\bar{\alpha}^*_t = \bar{\alpha}_{t-1} = \alpha$ for all $t > \tilde{t}$, where $\tilde{t}$ is an arbitrary number.

The story behind the dynamics of the market share is as follows: Suppose in period 1, the realized returns have the relation of $r_{1,1} > r_{2,1}$. Then at the end of period 1, all agents will just use this one observation to evaluate the performance of the strategies, consequently the optimal strategies for all agents are also the same, which is to put all savings into asset 1. Since only half of the agents try the new strategies, and also due to the effect of admission noise and wrong picks, $\bar{\alpha}_2$ will not reach 1, but on average it will increase, which will have a positive effect on the return of asset 1 and a negative effect on the return of asset 2, based on the bubble pricing equation (1.29). If the level effect is small and is dominated by the market mood term, on average it is more likely to see $r_{1,2} > r_{2,2}$. If it is so, then the optimal strategy still will be $\bar{\alpha}_t^* = 1$ for period 3, so that the upward movement of the market share will continue.

However after the market share passes the rational expectations equilibrium level, as the market share becomes bigger and bigger, on the one hand, the increasing potential of the market share becomes smaller and smaller, on the other hand, the level effect which has negative effect on the return of asset 1 becomes bigger and bigger. So eventually we will see more realized returns with the relation of $r_{1,t} < r_{2,t}$. When the market reaches such a point that the accumulated effect of bigger returns of asset 2 leads to $\bar{\alpha}_t{}^* < \bar{\alpha}_{t-1}$, the reversal movement of the market share will be observed. When the market share is decreasing, as long as $\bar{\alpha}_t$ is still bigger than the rational expectations equilibrium level α, the market mood term and the level term will work in the same direction, which will generate sequence of returns in favor of asset 2, so that the movement of market share will not stop at the rational expectations equilibrium value, but go to somewhere below it, then the effect of the market mood and the level effect will diverge again, the downward movement slows down, and then reverses again.

In the full memory full sample case, this overshooting procedure will continue forever due to the stochastic nature of the problem, but the size of the overshoot will settle down to a very small magnitude, and the market share will converge to the neighborhood of the rational expectations equilibrium level α. There are two reasons for it. First, the market share cannot converge to other values. Suppose the market share converges to a value η which is bigger than α. After the convergence, the market mood effect will disappear and the level effect will dominate. When $\eta > \alpha$, the level effect has negative impact on the return of asset 1 and makes it lower than the fair return. Therefore the market share will move towards the rational expectations

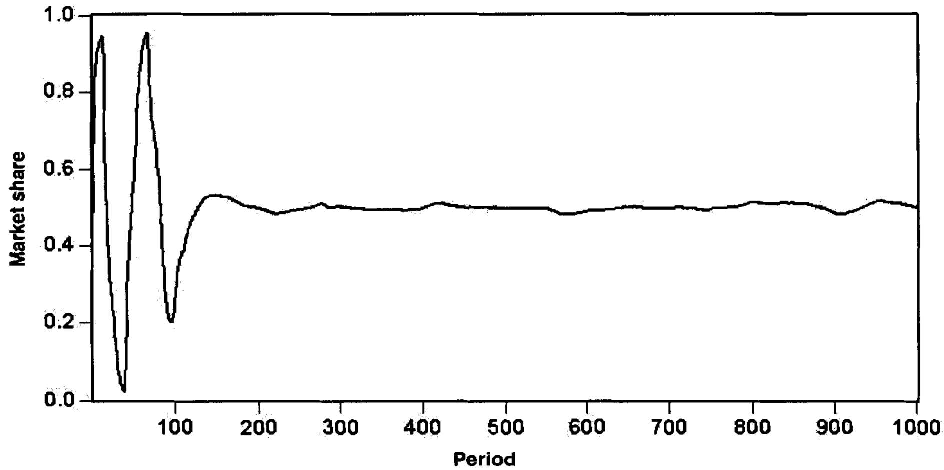

Figure 1.4: Movement of the market share: $\lambda = 0.95$, run 10, full memory, 1000 periods

equilibrium value α, and vice versa. Second, as the overshooting procedure goes on, the realized returns will become more and more balanced, and α_t^* will eventually settle down, and become more and more difficult to be changed, which in turn reduces the fluctuation in $\bar{\alpha}_t$. Therefore, the level effect will dominate asymptotically, and make the market share move to the equilibrium value. An example of the movement of the market share in the full memory full sample case is shown in Figure 1.4.

In the non-full memory case, the story is different. Let's first assume all agents have identical long memory T, but not full, and do not use random sampling. If T is large enough, after the initial turbulence, the market share will settle down to the neighborhood of the rational expectations equilibrium market share value, just like the full memory full sample cases. However, when the market operates longer than T periods, agents will begin to forget the realized returns in the initial periods. Suppose the initial realized returns are in favor of asset 1, then forgetting this part

Table 1.5: The autocorrelation of the log market share

$\lambda =$	0.1	0.4	0.8	0.9	0.95	0.96	0.97	0.98	0.99
ρ_{lgmkt}	0.274	0.386	0.728	0.861	0.919	0.926	0.938	0.959	0.977
	(0.028)	(0.031)	(0.009)	(0.011)	(0.007)	(0.007)	(0.005)	(0.013)	(0.007)

*The statistical results are obtained using the results from period 9000 to 10000. Means over 10 runs are reported. Numbers in parentheses are standard errors over 10 runs.

of the realized returns will destroy the balance of the realized return distribution, leading the remaining realized return distribution in favor of asset 2, so that we will see the market share has a trend to go down. When the number of periods exceeds the length of agents' memory, the movement of the market share will be approximately a replication of what happened in the initial periods but in an opposite direction. The periodic length is approximately the memory length of the agents. If the memory lengths of agents is sufficiently small, for example, in the short memory cases, we can not observe the market share settle down, and can only observe the periodic swing of the market share.

In the simulations of the short memory cases, agents' memory lengthes are not identical and they use random sampling. However, it does not change the qualitative results of the above analysis. There are two reasons for it. First, although agents memory lengthes are not identical, they are all short and cluster in a small range; Second, the sampling size 10 is large relative to the expected memory length 25. Both of these make the heterogeneity in agents' actions not as big as it seems to be. An example of the movement of the market share in the short memory case ($\lambda = 0.3$, run 3) is shown in figure 1.5.

The driving forces of the movement of the market share are the market mood term and the level effect term. The market mood term is a self-sustaining force. The optimistic mood can self-validate itself by generating positive excess return so as to

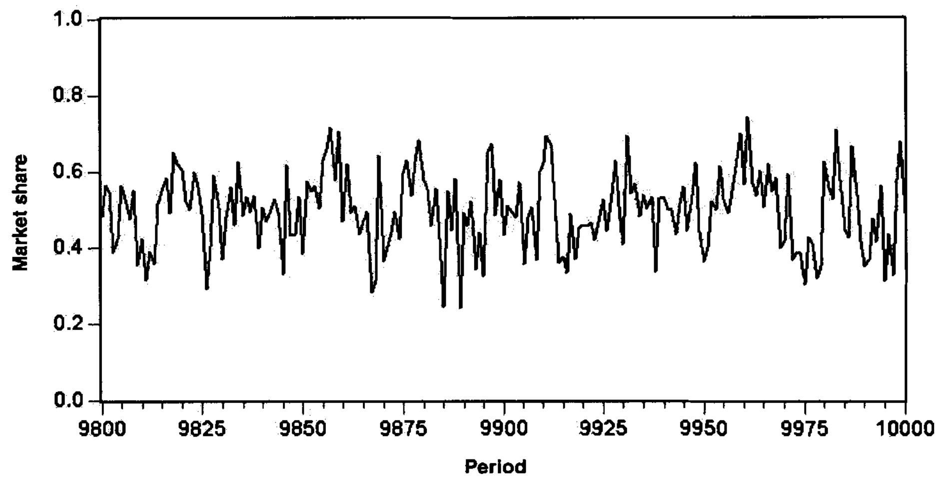

Figure 1.5: Movement of the market share: $\lambda = 0.3$, run 2, short memory, periods: 9800-10000

make the optimistic view stand. On the contrary, the level effect term is an error adjusting term, whose direction is always making the market share move towards the rational expectations equilibrium value. Thus without the level effect term, the market share can deviate far from the rational expectations equilibrium value. The strength of the market mood term and the level effect term are weighted by λ and $1-\lambda$, respectively. Thus if λ is really small, the market mood term will be dominated by the level effect term, which leads to two effects: First, the market share will fluctuate in a relatively smaller range around the rational expectations equilibrium value; second, the deviation will not be so persistent so that we will see periodic movement of the market share with higher frequency, which can be seen from the comparison of figures 1.5 and 1.6. The first order autocorrelation of the log market share is reported in table 1.5. As the value of λ increases,

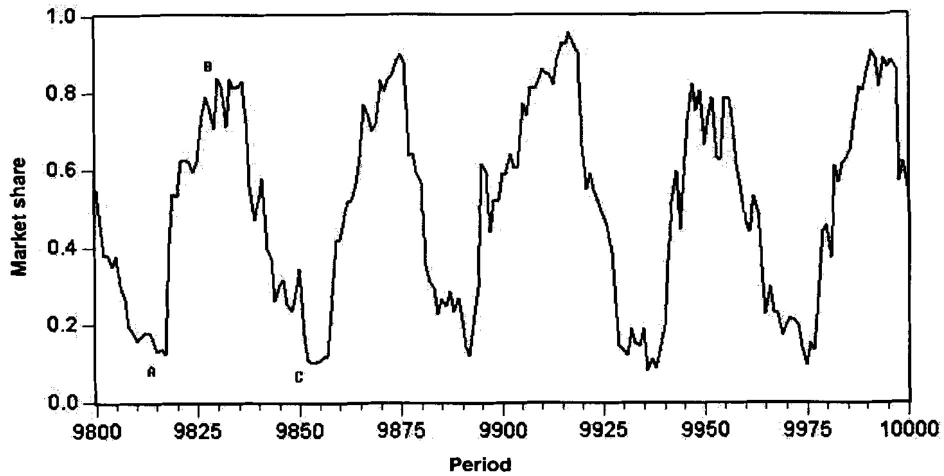

Figure 1.6: Movement of the market share: $\lambda = 0.96$, run 2, short memory, periods: 9800-10000

1.5.5 The Sources of ARCH Effect, Skewness and Jump

In section 1.4.2, we have shown that when λ is small (in regime 1), there is almost no ARCH effect; however, when λ is large (in regime 2), ARCH effect becomes strong; when λ is extremely large (in regime 3), the data will not only exhibit ARCH effect, but also display the properties of jump process. From figures 1.5 and 1.6, we also notice that the main change in the dynamics of the model as λ increases is that the market share show more and more persistent deviation from the equilibrium value. Therefore, the natural starting point to study the sources of ARCH effect is to study how assets returns are related to the movement of the market share. Let's see how the ARCH effect is generated following this idea.

In figure 1.6, after approaching point A, the local minimum, the market share begins to rise. The market mood effect in equation (1.29) can be approximately written as $\frac{\Delta\bar{\alpha}_t}{\bar{\alpha}_{t-1}}$. At point A, $\bar{\alpha}_{t-1}$ is very small. A small increase in the market share will lead to very large positive capital gain for asset 1. For example, for a change in the market share of size Δ, at $\bar{\alpha}_{t-1} = 0.1$, its contribution to the return of asset 1 will be as big as 10Δ, but at $\bar{\alpha}_{t-1} = 0.9$, it only leads to 1.1Δ. Let us call this as the *dilution effect.* In addition, the level effect also has strongest positive effect on the return of asset 1 at point A. Therefore in the right neighborhood of point A, asset 1 has biggest excess return over the fair return on average. Denote it as $r^+_{1,max}$. From point A to point B, the market share has an upwards trend, and the market mood effect has a positive effect on the return of asset 1. As long as the market share is below 0.5 which is the rational expectations equilibrium value, the level effect will also have positive effect on the return of asset 1, so that on average the return of asset 1 will show positive excess return over the fair return. When the market share moves past the value of 0.5, the market mood effect is still positive, but the level effect will become negative. With the market share approaching closer and closer to point B, the upward potential is eventually exhausted, hence the market mood effect will become smaller and smaller, but the level effect will become larger and larger. So that the realized return of asset 1 will eventually go from above the fair return to below the fair return. When the market share passes point B and goes from B to C, the opposite story will happen. Thus we see, along the periodic path of the market share, the excess return over the fair value will show periodic movement. It means volatility of the return, i.e., the square of excess return, will show the ARCH effect.

The implication of the above analysis is that the more persistent the movement of the market share, the stronger the ARCH effect.

When the movement of the market share becomes extremely persistent, the market share will approach close to the boundaries of 0 or 1, and each periodic movement will

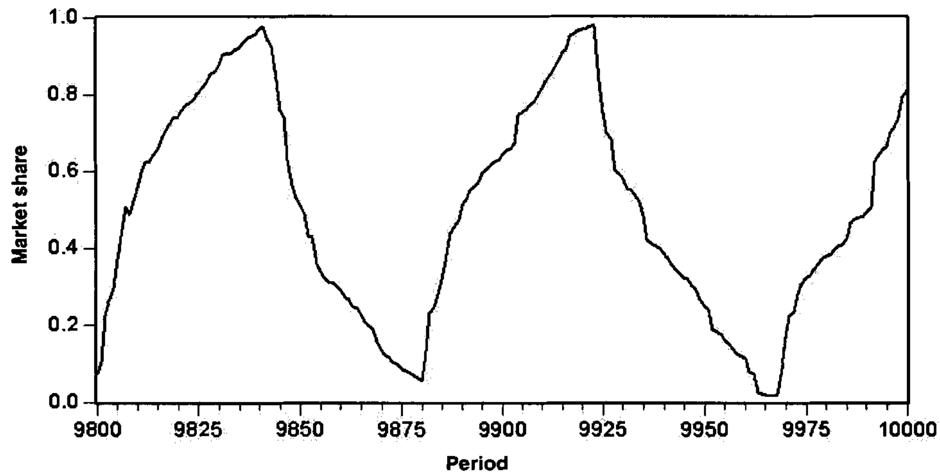

Figure 1.7: Movement of the market share: $\lambda = 0.98$, run 3, short memory, periods: 9800-10000

last for more periods. When the market share reverses its movement around zero, it is quite easy to generate extremely large value of the return of asset 1. Thus we have the two conditions of the jump-diffusion process: extreme big returns and infrequency. An example of the movement of the market share in regime 3 ($\lambda = 0.98$, run 3) is shown in figure 1.7,the first order autocorrelation is 0.982.

We also notice that the smallest negative returns of asset 1 are generated in the right neighborhood of point B. Denote it as $r^{-}_{1,min}$. Since in this area, $\bar{\alpha}_{t-1}$ is larger, due to the dilution effect, $\mid r^{+}_{1,max} \mid > \mid r^{-}_{1,min} \mid$ on average. Therefore the realized distributions of assets will be skew to the right. Since the closer the market share approach the boundary 0 or 1, the stronger the dilution effect, the measurement of the skewness will increase as λ increases.

1.5.6 The ARCH Effect and Trading Volume

Generally, the simulated trading volume has a positive relationship with the return volatility. Let us see why. The trading is caused by the heterogeneity in agents' beliefs. Obviously, if agents are homogeneous in the strategies they are using, there will be no trade. And the more discrepancy in the strategies that agents use, the bigger the trading volume. Intuitively, when the movement of the market share begins to reverse, the largest discrepancy will occur. This is because at this time agents' beliefs conflict with each other most: some still want to follow the old track, but others have already decided to go in the opposite direction. We also know that the agents' beliefs are most homogeneous when the market share approaches close to local minima or maxima. The reason is that at these times, the upward or downward potential has almost been exhausted, therefore there is little room left for agents to diversify their beliefs.

An example of the joint-movement of the market share and the trading volume is shown is figure 1.8. We can see when the market share pass the point A, local minimum, there is a spike of trading volume. Then as the market share moves from point A to point B, the trading volume decreases. After the market share passes point B, the trading volume sharply increases again. Since the trading volume shows the same pattern as that of the return volatility, it means that we can use the trading volume to explain the ARCH effect. From table 1.6, we can see, after adding the trading volume to the variance equation, the magnitude of the coefficients of the ARCH component and the GARCH component is greatly reduced, and most of them become insignificant. These results are consistent with the empirical findings of Lamoureux and Lastrapes (1990), Gallent et al. (1992), Jones et al. (1994), Miyakoshi (2002), etc.. We also notice that the regression coefficients of trading volume in regime 3

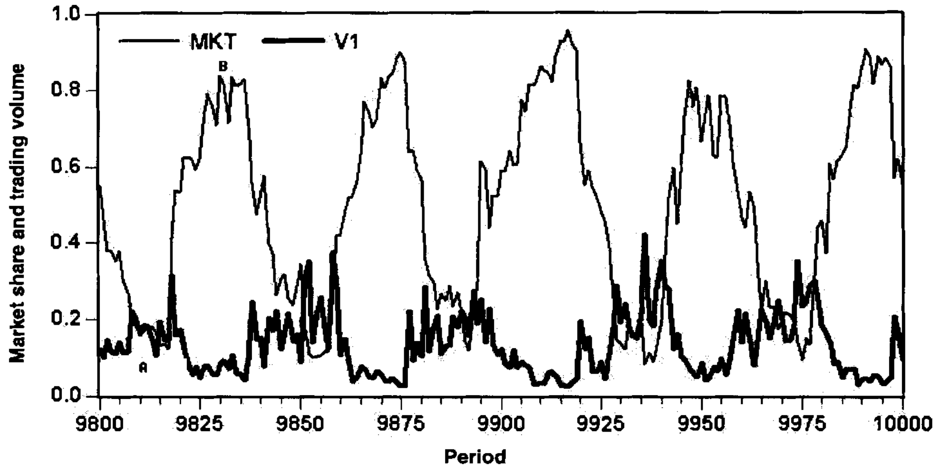

Figure 1.8: Movement of the market share and trading volume: $\lambda = 0.96$, run 2, short memory, periods: 9800-10000

are bigger than those in regime 2 by a substantial margin, this is because the more persistent the movement of the market share, the better match between the pattern of the trading volume and the market share.

Since the more persistent the movement of the market share, the more homogenous agent's beliefs can become. Therefore as λ increases, the average trading volume will decrease, which can be seen from table 1.2.

1.5.7 Other Parameters that Can Affect the Dynamics of the Model

From the above analysis, we know that the stronger the market mood effect, the easier it is to generate the pricing bubbles. The market mood effect depends on the distance between $\bar{\alpha}_t$ and $\bar{\alpha}_{t-1}$. From equation (1.31), we can see it is affected by

Table 1.6: ARCH effect after adding the trading volume

$\lambda =$	0.95	0.96	0.97	0.98	0.99
Panel A:	without	trading	volume		
ARCH(1)	0.306	0.330	0.456	0.902	1.104
	(0.056)	(0.097)	(0.190)	(0.403)	(0.460)
GARCH(1)	0.526	0.479	0.408	0.249	0.134
	(0.064)	(0.103)	(0.159)	(0.170)	(0.151)
Panel B:	with	trading	volume		
ARCH(1)	0.087	0.023	0.033	0.038	0.072
	(0.048)	(0.033)	(0.031)	(0.033)	(0.046)
GARCH(1)	0.097	0.015	0.0	0.0	0.0
	(0.175)	(0.047)	(0.0)	(0.0)	(0.0)
volume	0.155	0.184	0.283	0.263	0.272
	(0.057)	(0.146)	(0.113)	(0.114)	(0.071)

*The statistical results are obtained using the results from period 9000 to 10000. Means over 10 runs are reported. Numbers in parentheses are standard errors using 10 runs.

three parameters: The first is how many agents choose to try new strategies. In the simulation, we fix this number as one half of the agent population. As this number increases, it is expected that the market mood effect will be strengthened. The second is how aggressively the agents change their strategies, which is measured by the size of the candidate set. If the size of the candidate set is small, the probability of a wrong pick will also be small, hence there will be relatively less agents keeping their old strategies. Thus the smaller the size of the candidate set, the more aggressive the agents are. The more aggressive the agents, the stronger the market mood term. The third is how agents utilize the realized returns to evaluate the performance of strategies. If all agents use most recently realized returns, instead of doing random sampling, it is expected that the market mood effect will be strengthened.

1.6 Short Run Momentum and Long Run Reversal

1.6.1 The Short Run Momentum: A Comparison with a Bubble Pricing Model

One interesting observation of the simulated data is that the model can generate both positive and negative autocorrelation in returns, and whether the autocorrelation is positive or negative seems mainly depend on the value of λ. Now let's see why this is the case.

The rational expectations equilibrium return of equation (1.23) can be written as

$$r_{1,t} = r_1 + e_t, \tag{1.32}$$

where $r_1 = g - \ln\lambda + (1-\lambda)E(\ln\frac{\epsilon_t}{\alpha})$, $e_t = v_t + (1-\lambda)(\ln\frac{\epsilon_t}{\alpha} - E(\ln\frac{\epsilon_t}{\alpha}))$. Thus, our model is the one in which assets have constant expected return, perhaps the most intensively used model in testing market efficiency hypothesis.

In general, the price of asset 1, equation (1.27), can be written as

$$P_{1,t} = P^{*}_{1,t} + u_t, \tag{1.33}$$

where $P_{1,t} = \ln p_{1,t}$, $P^{*}_{1,t} = \ln(\frac{\alpha\lambda}{1-\lambda}) + \ln d_t$, $u_t = \ln\frac{\bar{\alpha}_t}{\alpha}$. Let's also represent the movement of u_t in the simulation as an AR(1) process

$$u_t = \rho_u u_{t-1} + \xi_t, \tag{1.34}$$

where ξ_t is a random shock. If we ignore the self-referential nature of our learning model, then equation (1.32), (1.33) and (1.34) will form a model which is exactly the bubble model that Summers (1986) proposed, although L.Summers didn't point out the economic interpretation of u_t. Analogous to Summers (1986), the return of asset

1, equation (1.29) can be written as

$$r_{1,t} = r_1 + e_t + \lambda u_t - u_{t-1}. \tag{1.35}$$

In Summers (1986)'s model, there is no parameter λ. His results correspond to the case where $\lambda = 1$. Let $z_t = r_{1,t} - r_1$ denote the excess return of asset 1 over the fair return, then

$$\sigma_z^2 = \sigma_e^2 + (1 + \lambda^2 - 2\lambda\rho_u)\sigma_u^2, \tag{1.36}$$

$$\rho_k^z = \frac{\rho_u^{k-1}((1+\lambda^2)\rho_u - \lambda - \lambda\rho_u^2)\sigma_u^2}{\sigma_e^2 + (1 + \lambda^2 - 2\lambda\rho_u)\sigma_u^2}. \tag{1.37}$$

The changes in the signs of ρ_k^zs along with the change of parameters λ and ρ_u are as follows: taking the first derivative of $(1+\lambda^2)\rho_u - \lambda - \lambda\rho_u^2$ with respect to λ, we get $2\lambda\rho_u - 1 - \rho_u^2$, which is less than $-(1-\rho_u)^2$, thus the numerator of equation (1.37) is monotonously decreasing in λ. Since when $\lambda = 0$, the nominator is $\rho_u^k\sigma_u^2 > 0$, when $\lambda = 1$, the nominator is $-\rho_u^{k-1}(1-\rho_u)^2\sigma_u^2 < 0$. Thus as λ goes from 0 to 1, ρ_k^z will go from positive to negative. Taking the first derivative of $(1+\lambda^2)\rho_u - \lambda - \lambda\rho_u^2$ with respect to ρ_u, we get $1 + \lambda^2 - 2\rho_u\lambda$, which is bigger than $(1-\lambda)^2$, thus the numerator of equation (1.37) is monotonously increasing in ρ_u. $(1+\lambda^2)\rho_u - \lambda - \lambda\rho_u^2$ is $-\lambda < 0$ when $\rho_u = 0$, and it is $(1-\lambda)^2 > 0$ when $\rho_u = 1$. Hence as ρ_u goes from 0 to 1, ρ_k^z will go from negative to positive. When $\rho_u = \lambda$, $(1+\lambda^2)\rho_u - \lambda - \lambda\rho_u^2$ equals zero. Therefore, when $\rho_u > \lambda$, ρ_k^z will be positive, and vice versa. Thereby, different from L.Summers (1986)'s model, which only allows negative autocorrelation, this paper's approach allows both positive and negative autocorrelation.

The autocorrelation coefficients of the log market share are listed in table 1.5. When λ equals to 0.1, ρ_u, i.e., the autocorrelation of log market share, is bigger than λ, equation (1.37) predicts that returns should show positive autocorrelation. Since all ρ_us are less than λs when λ is bigger than 0.1, equation (1.36) predicts that returns should all show negative autocorrelations. The simulation results shown in table 1.2 show that when λ equals 0.1, the first order autocorrelations of returns of asset 1 and asset 2 are positive; when λ is within the range [0.4, 0.90], they become negative; when $\lambda > 0.95$, they become positive. Therefore, in the cases of $\lambda > 0.95$, the simulation results are different from the prediction of the approximate equation (1.37). This may be due to the fact that when $\lambda > 0.95$, AR(1) process is not a good representation of u_ts. Actually, significant positive autocorrelation in the residue ξ_t can be detected in these cases. Therefore although there is some discrepancy between the predictions of the approximation equation (1.37) and the simulated data, there is one thing confirmed: this paper's approach has the potential to generated positive autocorrelation in returns, i.e., the return momentum, both in simulations and in the approximate model.

1.6.2 The Long Run Return Reversal

Although the bubble pricing model composed of equations (1.32), (1.33) and (1.34) implies that it is possible for the returns to show little autocorrelations even if the stock prices deviate far from their fundamental values. It also implies the long-horizon return may show large negative autocorrelations. The kth order autocorrelation of n-period return is

$$\rho_k^z = \frac{\rho_u^{n(k-1)}((1+\lambda^2)\rho_u^n - \lambda - \lambda\rho_u^{2n})\sigma_u^2}{\sigma_e^2 + (1+\lambda^2 - 2\lambda\rho_u)\sigma_u^2}. \tag{1.38}$$

Equation (1.38) has the same form as equation (1.37) except that ρ_u enters the equation with power n. When n is sufficiently big, even though ρ_u is bigger than λ, ρ_u^n will be less than λ. Thus we will observe negative autocorrelation for long-horizon returns. The simulation results are consistent with this prediction. ρ_1 of returns of 1 period, 12 periods and 24 periods are listed in table 1.7, which shows long-horizon returns

Table 1.7: Long-horizon return autocorrelation

$\lambda =$	0.1	0.4	0.8	0.9	0.95	0.96	0.97	0.98	0.99
1	0.096	-0.077	-0.127	-0.060	0.062	0.099	0.161	0.239	0.382
	(0.027)	(0.029)	(0.012)	(0.030)	(0.016)	(0.031)	(0.039)	(0.044)	(0.092)
12	-0.622	-0.628	-0.551	-0.482	-0.395	-0.330	-0.277	-0.122	0.010
	(0.044)	(0.025)	(0.036)	(0.027)	(0.029)	(0.029)	(0.036)	(0.056)	(0.055)
24	-0.143	-0.107	-0.185	-0.281	-0.588	-0.594	-0.623	-0.528	-0.352
	(0.119)	(0.086)	(0.070)	(0.057)	(0.043)	(0.051)	(0.080)	(0.108)	(0.136)

*The statistical results are obtained using the results from period 8000 to 10000. Means over 10 runs are reported. Numbers in parentheses are standard errors using 10 runs.

usually have large negative autocorrelation. The statistics are calculated using the sample from period 8000 to 10000. The continuous compound n-period returns are calculated as $r_{1,n,t} = \sum_{i=1}^{n} r_{1,t-n+i}$, using non-overlapping samples. From Table 1.5, we can see that for large value of λ, ρ_u is also larger. Thus it will need more periods for return to reverse, which can be seen from the cases with $\lambda > 0.95$.

1.7 Conclusion

The main results of this paper are divided into two categories: In the full memory full sample cases, there will be no bubbles in the asset prices, and the artificial stock market converges to the rational expectations equilibrium; However,when agents have short memories, it is possible for bubbles to be generated with the evolution of agent beliefs.

This paper also shows that, if agents are sufficiently patient, the market mood effect can make the market share persistently deviate from the rational expectations equilibrium level. Along with the persistent movement of the market share comes the clustering of volatility of returns, and its positive relationship with the trading volume. When the movement of the market share becomes extremely persistent, the jump-diffusion process will appear. The persistent movement of the market share also adds a slow decaying component to the asset price, which make short run return show momentum and long horizon return show reversal.

To capture the dynamics of the model, this paper proposes two approximating equations: the mean difference equation of the market share and the bubble pricing equation. They provide coherent explanations for almost all results. More generally, the methodology of mean difference equation analysis can be applied to a variety of genetic algorithm models with the selection operator. Bubble pricing equations can be applied to other models as well, as long as the movement of the market share is defined.

This paper demonstrates that adaptive learning and heterogeneous beliefs can lead to the reproduction of stylized facts, and return momentum and reversal on its own even if the rational expectations equilibrium returns do not have these properties. On the other hand, Cochrane, Longstaff and Santa-Clara (2003) can also explain these facts by a rational expectations model under a different dividend structure. Therefore, theoretically, whether there are bubbles in the asset prices is till an open question, which calls for more empirical test. One of the possible ways to do it is to test whether technical trading can make excess returns conditional on the portfolios showing financial anomalies. Charters often claim that technical rules can capture the market trend or mood. So if financial anomalies are indeed caused by the market mood, technical rules should be able to pick them up. But if financial anomalies is just a reflection of fundamental values, technical rules will not lead to excess returns.

One advantage of this paper's approach is that it can generate continuous trading. Continuous trading is stemming from agents' heterogeneous beliefs. In the model, the evolution of heterogeneous is such a process: when the market begins to reverse, agents' beliefs show greatest variety, then as the reversal becomes more and more confirmed, agents' beliefs become more and more homogenous. Thereby the trading volume in the model has such a pattern that when asset prices approach the local minima or maxima, the trading volumes will be the lowest; after asset prices pass the local minima or maxima, the trading volumes will be the highest. In the model, the ARCH effect can be explained by the trading volume which is consistent with many empirical studies. However the question of whether the real trading volume has the same pattern as predicted by the model or not calls for more detailed empirical studies, which can provide valuable information about how to modify the modelling of heterogeneous beliefs in the future.

In the model, asset pricing bubbles are not significant when agents are extremely impatient. The implication of this is that the financial bubbles, if exist, are mainly caused by long-term investors instead of short-term investors. The usual argument that the speculation will cause bubbles is to some extent misleading in the sense that only long term speculation can cause bubbles.

Since the pricing bubbles only show up when agents are sufficiently patient and have short memory, it suggests that the bubble pricing model should not be applied to all assets.

The ARCH effect is not a reliable indicator for whether there are pricing bubbles, although the ARCH effect is created along with the pricing bubble in this paper's approach. The reason is that this paper doesn't deny the possibility that the rational expectations equilibrium returns can have ARCH effects. No ARCH effect assumption is made for the convenience of isolating the effect of adaptive learning.

1.8 Appendix

Lemma 1: if $x \approx 1$, then $\ln x \approx x - 1$.

Proof: Expand $\ln x$ around 1 by first order Taylor series:

$$\ln x \approx \ln(1) + 1 \cdot (x-1) = x - 1,$$

Done.

Result 1: In rational expectation rational expectations equilibrium solution,

$$r_{1,t} \approx g - \ln\lambda + (1-\lambda)\ln\frac{\epsilon_t}{\alpha} + v_t.$$

Proof: In rational expectation rational expectations equilibrium,

$$r_{1,t} = \ln(1 + \epsilon_t \frac{1-\lambda}{\alpha\lambda}) + \ln d_t - \ln d_{t-1}. \tag{1.39}$$

Expand $\ln(1 + \epsilon_t \frac{1-\lambda}{\alpha\lambda})$ around $\epsilon_t = \alpha$,

$$\begin{aligned} \ln(1 + \epsilon_t \tfrac{1-\lambda}{\alpha\lambda}) &\approx \ln(1 + \tfrac{1-\lambda}{\lambda}) + \lambda(1 + \epsilon_t \tfrac{1-\lambda}{\alpha\lambda} - 1 - \alpha\tfrac{1-\lambda}{\alpha\lambda}) \\ &= -\ln\lambda + (1-\lambda)(\tfrac{\epsilon_t}{\alpha} - 1) \\ &= -\ln\lambda + (1-\lambda)\ln\tfrac{\epsilon_t}{\alpha}. \quad (\textit{use Lemma } 1) \end{aligned} \tag{1.40}$$

Substitute equation (1.40) and $\ln d_t = \ln d_{t-1} + g + v_t$ into equation (1.39), we get

$$r_{1,t} \approx g - \ln\lambda + (1-\lambda)\ln\frac{\epsilon_t}{\alpha} + v_t.$$

Done.

Result 2: On the off rational expectations equilibrium path, the approximation of the return of asset 1 is:

$$r_{1,t} \approx g - \ln\lambda + (1-\lambda)\ln\frac{\epsilon_t}{\alpha} + v_t + \lambda(\ln\bar{\alpha}_t - \ln\bar{\alpha}_{t-1}) - (1-\lambda)(\ln\bar{\alpha}_{t-1} - \ln\alpha).$$

Proof: Based on the definition,

$$r_{1,t} = \ln(\frac{p_{1,t} + d_{1,t}}{p_{1,t-1}}) = \ln(\delta_t + \Phi_t h_t), \tag{1.41}$$

where $\delta_t = p_{1,t}/p_{1,t-1}$, $\Phi_t = d_{1,t}/d_{1,t-1}$, and $h_t = d_{1,t-1}/p_{1,t-1}$. On the determinstic rational expectations equilibrium growth path where $\Phi^0 = e^g$ and $\epsilon_t = \alpha$, we have

$$\delta^0 = \frac{p^0_{1,t}}{p^0_{1,t-1}} = \Phi^0 = e^g,$$

where $p^0_{1,t}$ and $p^0_{1,t-1}$ are prices of asset 1 at period t and $t-1$ on the determinstic rational expectations equilibrium path. Define

$$\rho \equiv \frac{p^0_{1,t}}{p^0_{1,t} + d^0_{1,t}} = \frac{\frac{\alpha\lambda}{1-\lambda} d_t}{\frac{\alpha\lambda}{1-\lambda} d_t + \alpha d_t} = \lambda, \tag{1.42}$$

and

$$h^0 = \frac{d^0_{1,t}}{p^0_{1,t}} = \frac{1-\rho}{\rho}. \tag{1.43}$$

Expand equation (1.41) around h^0, Φ^0 and δ^0 by Taylor series, and denote $\Gamma = \delta^0 + \Phi^0 h^0$, we get

$$\begin{aligned} r_t = \ln(\delta_t + \Phi_t h_t) &\approx \ln\Gamma + \Gamma^{-1}(\delta_t - \delta^0) + \Gamma^{-1}h^0(\Phi_t - \Phi^0) + \Gamma^{-1}\Phi^0(h_t - h^0) \\ &= g - \ln\rho + \rho(e^{-g}\delta - 1) + (1-\rho)(e^{-g}\Phi_t - 1) + (1-\rho)(h_t/h^0 - 1) \\ &\approx g - \ln\rho + \rho\ln(e^{-g}\delta) + (1-\rho)(\ln(e^{-g}\Phi_t) + \ln(h_t/h^0)) \\ &= k + \rho\ln p_{1,t} + (1-\rho)\ln d_{1,t} - \ln p_{1,t-1}, \end{aligned} \tag{1.44}$$

where $k = -\ln\rho + (1-\rho)\ln\frac{\rho}{1-\rho}$. Approximation formula (1.44) was actually proposed by Campbell and Shiller(1988). Substitute $p_{1,t} = \frac{\lambda\bar{\alpha}_t d_t}{1-\lambda}$, $\rho = \lambda$ and $d_{1,t} = \epsilon_t d_t$ into equation (1.44), we get

$$\begin{aligned} r_{1,t} &\approx k + \rho\ln(\tfrac{\lambda\bar{\alpha}_t d_t}{1-\lambda}) + (1-\rho)\ln(\epsilon_t d_t) - \ln(\tfrac{\lambda\bar{\alpha}_{t-1}d_{t-1}}{1-\lambda}) \\ &= k - (1-\lambda)\ln(\tfrac{\lambda}{1-\lambda}) + \lambda\ln\bar{\alpha}_t - \ln\bar{\alpha}_{t-1} + (1-\lambda)\ln\epsilon_t + \ln d_t - \ln d_{t-1} \\ &= g - \ln\lambda + (1-\lambda)\ln\tfrac{\epsilon_t}{\alpha} + v_t + \lambda(\ln\bar{\alpha}_t - \ln\bar{\alpha}_{t-1}) - (1-\lambda)(\ln\bar{\alpha}_{t-1} - \ln\alpha). \end{aligned}$$

Done.

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Chapter 2

The Behavior of the Exchange Rate in the Genetic Algorithm with Agents Having Long Memory

This paper studies the behavior of exchange rate in Kareken and Wallace (1981)'s model under the genetic algorithm adaptation with agents having long memory. The simulation results show that if agents have full memory, the average portfolio fraction will converge, and the initial equilibrium that it converges to is history dependent. Under the lasting evolutionary pressure of the noise trader, the market will eventually drift from one equilibrium to another, and asymptotically it will converge to the neighborhood of an equilibrium with agents equally putting their savings into two currencies. If the agents do not have full memory, the foreign exchange market will show periodic crisis. Before and after a market crisis, the average portfolio fraction will converge to different stationary equilibria. A mean difference equation of the average portfolio fraction is also given to describe the dynamics of the model.

2.1 Introduction

In the perfect-foresight equilibria of Kareken and Wallace (1981)'s model, the no arbitrage condition requires that the rates of return on two currencies have to be equal, and thus the exchange rate will be constant over time. The properties that make Kareken and Wallace (1981)'s model so interesting lie in the facts that it leaves exchange rate totally unrestricted, any number bigger than zero is consistent with the perfect-foresight equilibria. Since there are infinite number of equilibria consistent with the rational expectation, rational expectation itself cannot tell us much about which of them are more appropriate to serve as equilibria. On the other hand, adaptive learning approach does not help us out of this difficulty so far as well. In Arifovic (1996), Arifovic and Gençay (2000), and Lux and Schornstein (2004), it has been shown with the genetic algorithm (GA) that the exchange rate will fluctuate forever and never converge to a stationary equilibrium, and any exchange rate level can be reached by the economy. Sargent (1993) shows that under stochastic approximation, the exchange rate will converge to a constant. However, the solution is history-dependent because different initial conditions will lead to different stationary exchange rates. The purpose of this paper is to provide a GA framework that may solve the equilibria selection dilemma in Kareken and Wallace (1981)'s model.

In Arifovic (1996), Arifovic and Gençay (2000), and Lux and Schornstein (2004), a special assumption about the agents in the economy is that agents only have one period memory, and all realized returns older than one period is irrelevant in agents' decision-making procedure. It makes the equilibrium investment decision unrestricted in the sense that all investment decisions will yield the same return. Therefore even if the GA can converge to a stationary equilibrium, it cannot prevent the invasion of strategies that only change the portfolio composition. The inequality in the rates of returns of two currencies caused by the invasion, no matter how small, will be sufficient to change the optimal portfolio fraction to 0 or 1 and destroy the equilibrium. This is one of the main reasons why in their models, the foreign exchange market is persistently turbulent and cannot be settled down.

The GA application used in this paper is based on the framework of LeBaron (2001) where agents have long memory. Long memory is an important ingredient in human beings' learning process, which makes it possible for human beings to draw lessons from their past experience and reduce the probability of making the same mistake again in the future. This paper is going to show that with long memory, agents can eventually realize that the fluctuation in the exchange rate unnecessarily adds the risk to themselves and learn to coordinate around some desired exchange rates. When the memory is long, agents' portfolio decisions will be determined by a sequence of realized returns instead of just one period realized returns. Past failures in the coordination will make agents become conservative in changing their investment decisions, and make the coordination robust against small shocks in the market.

Another property of this paper's GA design is that a noise trader is added into the agent population. She is allowed to freely choose her strategy without using the fitness based election. Fitness based election only allows strategies winning in the fitness competition to be used by agents, which makes agents' actions become more homogeneous and similar. However, in reality, besides the mainstream beliefs, it is common to see there exist some people who have different or completely opposite opinions even when they have the same information as other people. To capture this, two groups of investors are used to represent them. The dominating group is the "rational" traders, who choose strategies based on the stated fitness measure. The other group of investors are noise traders, whose size is much smaller. Within its group, members's behavior is similar. But the rules guiding their behavior are unobservable. Thus the random draw is used to represent their decision-making procedure. The impact of noise traders on the market is determined by the ratio between the population sizes of the noise traders and the "rational" traders. For simplicity, the number of noise traders is normalized as one in the model.

The simulation results show that if agents have full memory, the GA will converge, and the initial equilibrium that the average portfolio fraction converges to is different from one experiment to another, just like what happened in Sargent (1993), which is history dependent. Under the lasting evolutionary pressure generated by the noise trader, the market will eventually drift from one equilibrium to another, and asymptotically it will converge to the neighborhood of an equilibrium with agents equally putting their savings into two currencies. If the agents do not have full memory, then the market will show periodic crisis. Before and after a market crisis, the average portfolio fraction will converge to different stationary equilibria.

This paper is organized as follows: The description of the Kareken and Wallace (1981)'s model is given in Section 2.2. The design of the GA application is given in Section 2.3. The simulation results are given in Section 2.4. The mean difference equation of the average portfolio fraction and the role of the noise trader are discussed in Section 2.5. We will discuss the dynamics of the simulation in Section 2.6, and provide the sensitivity analysis in Section 2.7. Section 2.8 concludes.

2.2 Kareken and Wallace (1981)'s Model

The economy is a version of the two-country over-lapping generations (OLG) model with two currencies. In each period t, there are N young people born. Their two periods' endowments are w^1, w^2 respectively. Agents's preference is given by $u_t[c_t(t), c_t(t+1)] = \ln c_t(t) + \ln c_t(t+1)$, where $c_t(t)$ and $c_t(t+1)$ are the consumption of generation t in period t and $t+1$ respectively.

Thus an agent of generation t solves the following maximization problem:

$$\max \quad \ln c_t(t) + \ln c_t(t+1)$$

subject to

$$c_t(t) \leq w^1 - \frac{m_1(t)}{p_1(t)} - \frac{m_2(t)}{p_2(t)},$$
$$c_t(t+1) \leq w^2 + \frac{m_1(t)}{p_1(t+1)} + \frac{m_2(t)}{p_2(t+1)},$$

where $m_l(t)$ is the agent's nominal holdings of currency l, $l = 1, 2$ in period t, $p_l(t)$ is the nominal price of the good in terms of currency l in period t.

The exchange rate $e(t)$ between the two currencies is defined as $e(t) = p_1(t)/p_2(t)$. The no arbitrage condition requires that in a stationary equilibrium, the rates of returns on both currencies should be equal, i.e., $\frac{p_1(t)}{p_1(t+1)} = \frac{p_2(t)}{p_2(t+1)}$, which implies $e(t+1) = e(t) = e$. The market clearing condition is,

$$\frac{1}{2}[w^1 - w^2\frac{p_1(t+1)}{p_1(t)}] = \frac{H_1(t)}{p_1(t)} + \frac{H_2(t)e}{p_1(t)}, \tag{2.1}$$

where $H_l(t)$ is the nominal supply of currency l in period t. It states that the world real money demand $\frac{1}{2}[w^1 - w^2\frac{p_1(t+1)}{p_1(t)}]$ should equal the world real money supply $\frac{H_1(t)}{p_1(t)} + \frac{H_2(t)e}{p_1(t)}$.

The indeterminacy of the exchange rate in this model results from the fact that there is only one equation for the world real money demand (equation (2.1)). The individual real demands for each currency are therefore not well defined. The indeterminacy of exchange rate proposition states that if there exists an exchange rate e which solves the agents maximization problem, then any other $\hat{e} \in (0, \infty)$ can achieve identical values of savings in a monetary equilibrium with the exchange e.

2.3 GA Application

The design of the GA experiment follows the ideas introduced in chapter 1. The whole evolutionary process is divided into two parts: the evolution of the agents and the evolution of the strategies. Since in OLG model, in each period t, only the actions of generation t affect the market outcomes, therefore, in our GA application, the economy only consists of one agent population, the young population at period t, with size N. There also exist a population of strategies with size N_s as an analogy to the investor advisor in the real foreign exchange market. Although each generation of agents only have two periods of life, the new-born agents can consult investor advisers to get the information about the past information of the market. The strategy j, $j = 1 \cdots N_s$, is composed of two parts: the first one is $\beta_{j,t}$, defined as the fraction of w^1 being consumed in period t. The second one is $\alpha_{j,t}$, which is the portfolio fraction, defining the fraction of agent's savings that is put into currency 1 in period t. Each strategy is encoded in a genetic string with the length of 20. The first 10 bits of string j denotes $\beta_{j,t}$. The last 10 bits denotes $\alpha_{j,t}$. Both of them are decoded in the same way as in chapter 1.

For the sake of convenience, the strategy that agent i chooses is referred as i. Thus the consumption of agent i in period t is

$$c_i(t) = w^1 \beta_{i,t}. \tag{2.2}$$

The difference between w^1 and $c_i(t)$ gives the savings $s_i(t)$ of agent i in period t.

$$s_i(t) = w^1(1 - \beta_{i,t}). \tag{2.3}$$

Agent i places the fraction $\alpha_{i,t}$ of the savings into currency 1 and the fraction $1 - \alpha_{i,t}$ into currency 2. Thus the prices of the consumption goods in terms of currency 1 and currency 2 can be calculated as:

$$p_1(t) = \frac{H_1}{\sum_1^N \alpha_{i,t} s_i(t)}, \tag{2.4}$$

$$p_2(t) = \frac{H_2}{\sum_1^N (1-\alpha_{i,t}) s_i(t)}. \tag{2.5}$$

The ex post return of investment in currency l from period t to period $t+1$ is calculated as

$$R_l(t+1) = \frac{p_l(t)}{p_l(t+1)}, \qquad l = 1, 2. \tag{2.6}$$

The portfolio return of agent i is

$$R_i(t+1) = \alpha_{i,t} R_1(t+1) + (1-\alpha_{i,t}) R_2(t+1). \tag{2.7}$$

And the second-period consumption of agent i of generation t is as follows:

$$c_i(t+1) = w^2 + s_i(t) R_i(t+1). \tag{2.8}$$

Suppose that the memory length of agents is T, then the potential performance measure for strategy j is as follows:

$$\frac{1}{T}\sum_{\tau=1}^{T}(\log(w^1\beta_j) + \log(w^2 + w^1(1-\beta_j)(\alpha_j R_1(t-\tau) + (1-\alpha_j)R_2(t-\tau)))). \tag{2.9}$$

The timing of the market is as follows:

1. Rules are evolved.

2. Agents choose their current strategies.

3. Agents are evolved.

4, Each agent reports her decision to a 'market maker'. The market maker will set the prices of assets to clear the market.

5. The ex post returns of assets are calculated and stored.

In step 1, the evolution of strategy rules is simulated in the same way by genetic algorithm as in chapter 1. First, form the set of strategies to be eliminated, then the algorithm chooses one method from crossover, mutation or new rule to generate new strategies to replace them.

In step 2, each agent will evaluate the performance of every single rule in the strategy population and randomly choose one rule from the candidate set. If the selected rule has a higher performance than the one she currently uses, the current one will be replaced with the new one. Otherwise, she still uses the old one.

The noise trader is generated in step 3. In step 2, all agents have already chosen their strategies, and all these strategies have been filtered by the fitness based election. However in step 3, one agent will be randomly chosen. The strategy that she will use will be replaced by a strategy randomly selected from the whole strategy population. T.

2.4 Simulation Results

The model parameters are set as follows: $w^1 = 10$, $w^2 = 4$, $H_1 = H_2 = 300$. Thus the implied optimal consumption rate is 0.7. The sizes of the population of agents and strategies are set as $N = 60$, $N_s = 150$, respectively. The parameter 'candidate' is set as 75, i.e. agents will randomly choose one rule from the best half of the strategy population. The probabilities of crossover and mutation are $p_{cross} = 0.6$, $p_{mut} = 0.3$, respectively. The range of the portfolio fraction is set as (0.0001, 0.9999). The initial strategy strings are generated randomly, and all agents randomly choose their strategies from the strategy population.

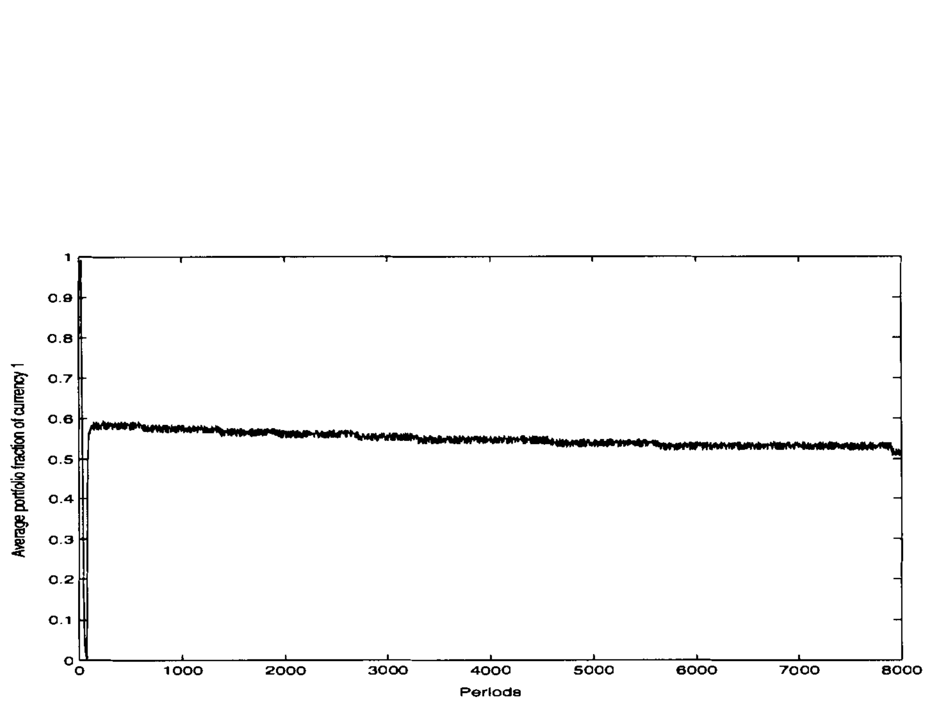

Figure 2.1: Average portfolio fraction of currency 1: full memory.

2.4.1 Benchmark Simulation: the Full Memory Case

The benchmark simulation is conducted by setting the memory length of all agents at 8000, which is the same as the number of iterations in the experiment. Therefore agents have full memory. The simulation results are different from those of Sargent (1993) and also different from those of Arifovic (1996). Sargent (1993) shows that the exchange rate will converge to a constant value using the stochastic approximation algorithm. The value that the exchange rate converges towards depends on the initial conditions. Arifovic (1996) shows that the exchange rate will display large range fluctuation and never converge to a constant value.

From figure 2.1 we can see that in the benchmark simulation the average portfolio fraction of currency 1 quickly settles down to the neighborhood of 0.6, which however is different from one experiment to another however. Thus, to some extent, it is similar to the results of Sargent (1993), which is history dependent. After the market staying

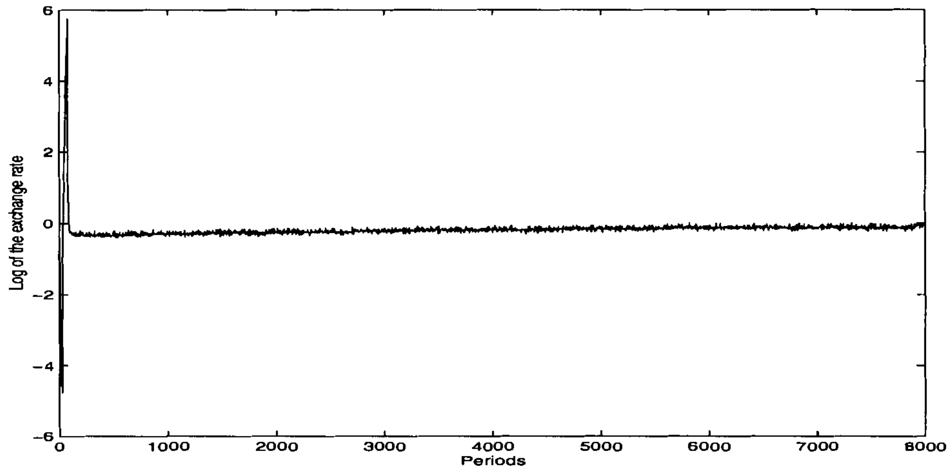

Figure 2.2: Log of exchange rate: full memory.

there for some periods, the average portfolio fraction drifts down a little bit. And after another while, the average portfolio fraction drifts down again. Although figure 2.1 only shows downward drifts, upwards drifts are observed in other experiments as well if the initial value that the average portfolio fraction converges to is below 0.5. Actually, the equilibrium with average portfolio fraction of 0.5 is the attractor of all simulations. The reason will be discussed in the section 2.5.

The dynamics of the log exchange rates is shown in figure 2.2. It can be seen that just like the behavior of the average fraction of the portfolio, for most of time the exchange rate is stable and some small deviations appear and disappear from time to time. Corresponding to the downward drift in average portfolio fraction, there is an upward drift in the exchange rate.

Same as the results of Arifovic (1996), the consumption ratio converges to the optimal solution 0.7 very quickly, which is shown in figure 2.3.

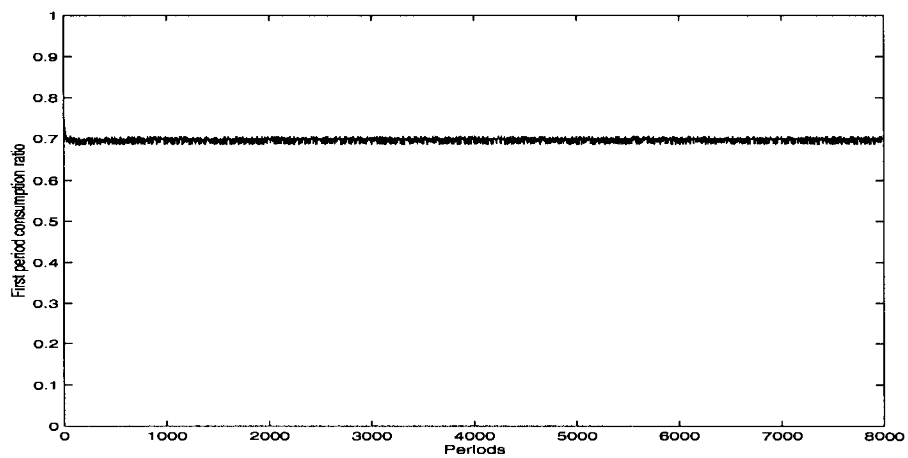

Figure 2.3: Average first period consumption ratio: full memory.

2.4.2 The Incomplete Memory Case

Set the memory length of agents as 400 [1] and do the simulation again. The behavior of the average portfolio fraction and the log exchange rate are shown in figure 2.4 and 2.5. It can be observed that the market exhibits periodic movements. First, after initial turbulence, the average portfolio fraction will settle down to a relatively stable level, then the market will become turbulent again and gradually settle down to another stable level. The cycle of the market is approximately 400 periods, which is the memory length of the agents. Of course, together with the movements of the average portfolio fraction are the corresponding change in the exchange rate, which is shown in figure 2.5 . Although there are violent fluctuations in the market, the first period consumption still remains stable after reaching the optimal consumption level,

[1]The reason that we set the memory length to be at 400 is that we want to give GA enough time to settle down. However if this number is too big, huge number of iterations will be needed in order to show the whole picture of the dynamics.

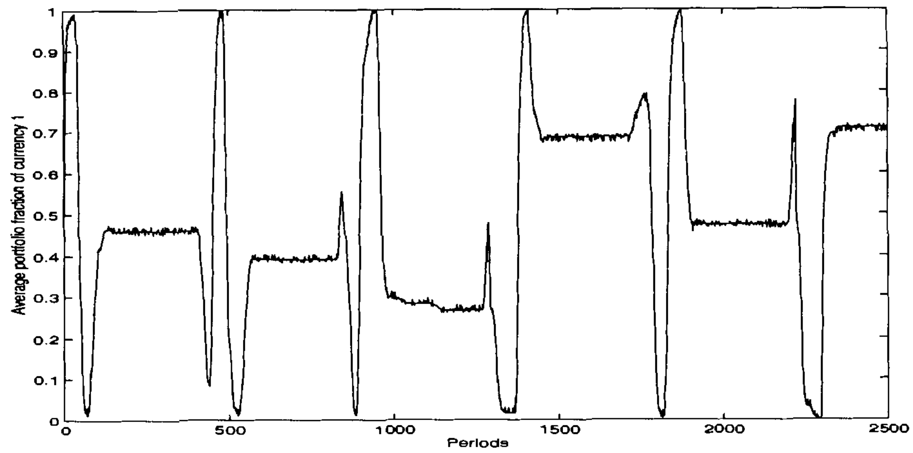

Figure 2.4: Average portfolio fraction: memory length=400

which is shown in figure 2.6.

Thus an important property of the incomplete memory case is that it can generate cyclical financial crisis in the foreign exchange market. The occurrence of the financial crises is not because of the change in fundamental variables, but purely because of the change in the agents' beliefs. Between financial crises, the exchange rates that the market converges to are not the same. In fact, it will drift from one to another. Due to the indeterminacy of the exchange rate, any fixed level of exchange rate can play the same transaction role as others. Thus it is natural to see that after one financial crisis, agents coordinate around another exchange rate level.

2.4.3 Short Memory Case

In addition, the behavior with agents having very short memory is shown in figures 2.7 and 2.8. From figure 2.8, we can see that with memory length 10, the consumption

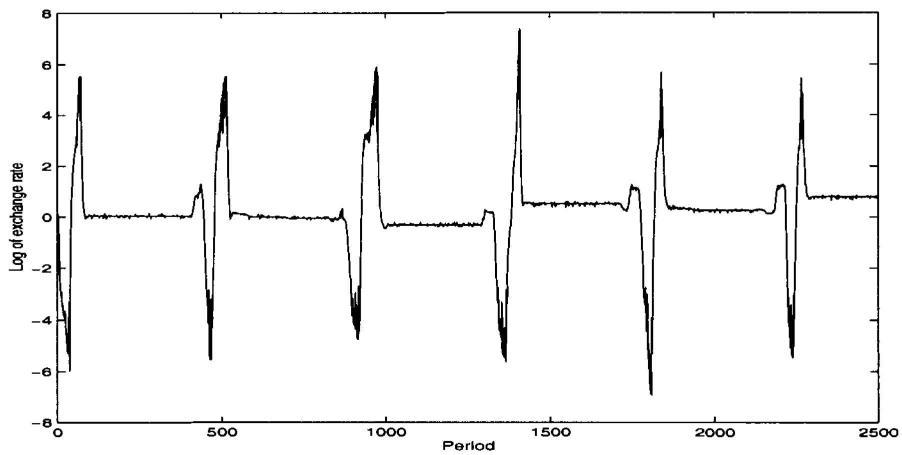

Figure 2.5: Log of exchange rate: memory length=400

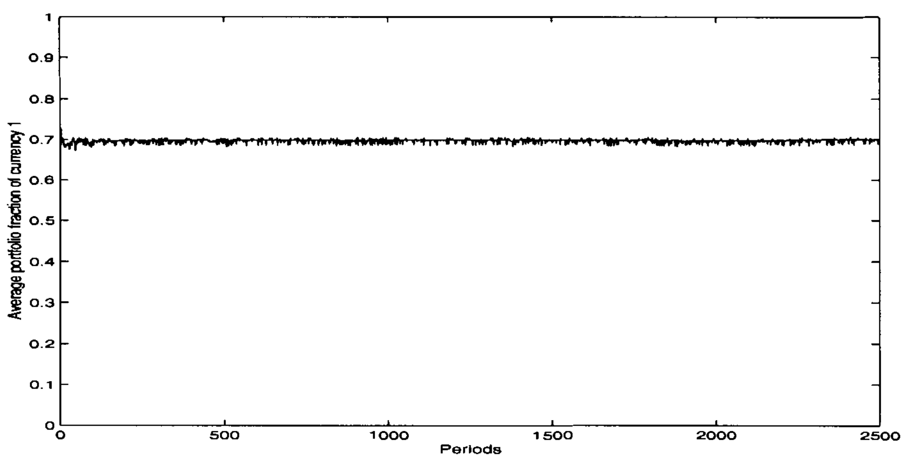

Figure 2.6: Average first period consumption ratio: memory length=400

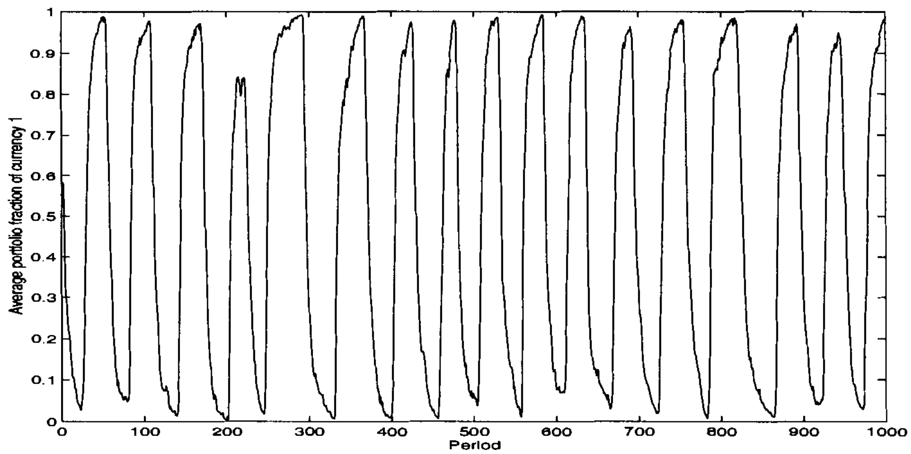

Figure 2.7: Average portfolio fraction of currency 1: memory length=10.

level can still converge to the neighborhood of the optimal level. However, the average market portfolio will have large oscillations, and show no trend of settling down to an equilibrium.

2.5 Why the Average Portfolio Fraction 0.5 is the Only Equilibrium that is Stable under the Evolution?

In the simulation, the consumption ratio always converges and stays near the optimal solution no matter what values the average portfolio fraction converges to. Thus once the optimal consumption ratio is reached, the evolution of the average portfolio fraction will be mainly determined by the isolated genetic change in the portfolio decision part instead of the consumption part. To study this, substitute the optimal

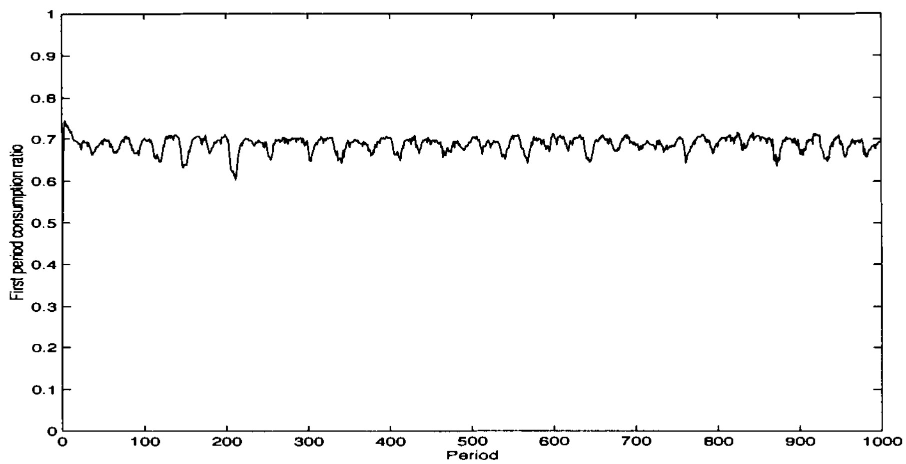

Figure 2.8: Average first period consumption ratio: memory length=10

consumption ratio 0.7 into the performance measure equation (2.9), and ignore the first term, the performance measure equation for strategy α becomes:

$$\frac{1}{T}\sum_{\tau=1}^{T}(\log(w^2 + 0.3w^1(\alpha R_1(t-\tau) + (1-\alpha)R_2(t-\tau))). \tag{2.10}$$

Let α_t^* be the optimal solution that maximizes the performance measure (2.10), subject to the condition $0 < \alpha < 1$. And denote the average portfolio fraction as $\hat{\alpha}_t$. The question is to find the equation that controls the movement of $\hat{\alpha}_t$.

2.5.1 The Mean Difference Equation of the Average Portfolio Fraction

Without loss of generality, let us assume the index of agents is assigned as such, so that, the first agent is the noise trader who randomly picks up the strategy from the strategy population, and all remaining agents are the ones who try new strategies from the candidate set. Denote $\alpha_{i,t}$, $i = 1, \cdots, N$ as the investment strategy used by agent i in period t, then

$$E_{t-1}(\bar{\alpha}_t) = \frac{1}{N}(E_{t-1}(\alpha_{1,t}) + \sum_{i=2}^{N} E_{t-1}(\alpha_{i,t})), \tag{2.11}$$

where $E_{t-1}(\cdot)$ is the expectation conditional on the information set of period $t-1$.

To find $E_{t-1}(\alpha_{1,t})$, let's assume the noise trader has the probability p_1 to pick up a strategy from the parent set, and probability $1 - p_1$ from the non-parent set, then

$$\begin{aligned} E_{t-1}(\alpha_{1,t}) &= (1-p_1)(E_{t-1}(\alpha_t^{nc})) + p_1 E_{t-1}(\alpha_t^c)) \\ &= (1-p_1)(E_{t-1}(\alpha_t^{nc}) - E_{t-1}(\alpha_t^c)) + E_{t-1}(\alpha_t^c), \end{aligned} \tag{2.12}$$

where α_t^c denotes the strategy from the parent set, and α_t^{nc} denotes the strategy from the non-parent set.

For $i \geq 2$, the procedure to get $\alpha_{i,t}$ is as follows: At the end of period $t-1$, agent i picks up a strategy from the candidate set, and compares the fitness of the strategies with that of $\alpha_{i,t-1}$, the one she currently uses. If the new one has higher fitness, she uses the new one. Otherwise she keeps the old one. Thus, there are two possible outcomes: The first outcome, with probability $p(\alpha_{i,t-1})$, is that the string $\alpha_{i,t-1}$ will be replaced by a new string. The second one is that string $\alpha_{i,t-1}$ wins in the election and enters the next generation, with probability $1 - p(\alpha_{i,t-1})$. In the first case, the expectation of the new string is $E_{t-1}(\alpha_{new}(\alpha_{i,t-1}))$, which is the mean of strategies in the candidate set whose fitness is bigger than that of $\alpha_{i,t-1}$. In the second case the expectation is $\alpha_{i,t-1}$. Denote $p(\alpha_{i,t-1})$ as $\tilde{p}_{i,t}$, $E_{t-1}(\alpha_{new}(\alpha_{i,t-1}))$ as $b_{i,t}$. Then

$$\begin{aligned} E_{t-1}(\alpha_{i,t}) &= \tilde{p}_{i,t} b_{i,t} + (1-\tilde{p}_{i,t})\alpha_{i,t-1} \\ &= b_{i,t} + (1-\tilde{p}_{i,t})(\alpha_{i,t-1} - b_{i,t}) \\ &= \alpha_t^* + (b_{i,t} - \alpha_t^*) + (1-\tilde{p}_{i,t})(\alpha_{i,t-1} - b_{i,t}), \end{aligned} \tag{2.13}$$

Substituting equation (2.12) and (2.13) into (2.11), and subtracting $\bar{\alpha}_{t-1}$ from both sides, we get

$$\begin{aligned} E_{t-1}(\bar{\alpha}_t) - \bar{\alpha}_{t-1} &= \tfrac{1}{N}((1-p_1)(E_{t-1}(\alpha_t^{nc}) - E_{t-1}(\alpha_t^c)) + E_{t-1}(\alpha_t^c) - \bar{\alpha}_{t-1}) \\ &\quad + \tfrac{N-1}{N}(\alpha_t^* - \bar{\alpha}_{t-1}) + \tfrac{1}{N}(\textstyle\sum_{i=2}^N (b_{i,t} - \alpha_t^*)) \\ &\quad + \tfrac{1}{N}(\textstyle\sum_{i=2}^N ((1-\tilde{p}_{i,t})(\alpha_{i,t-1} - b_{i,t})). \end{aligned} \tag{2.14}$$

From equation (2.14), we can see that the updating of the mean $\bar{\alpha}_t$ can be decomposed into four parts: the first is the noise trader term $\frac{1}{N}((1-p_1)(E_{t-1}(\alpha_t^{nc}) - E_{t-1}(\alpha_t^c)) + E_{t-1}(\alpha_t^c) - \bar{\alpha}_{t-1})$; the second is the gap between α_t^* and $\bar{\alpha}_t$, which is the main determinant of the movement of the average portfolio fraction $\bar{\alpha}_t$; the third term is the average of $b_{i,t} - \alpha_t^*$, which is determined by the shape of the fitness function and the probability distribution of the strategy in the candidate set. It measures how close the optimal portfolio fraction is to the mean of the admitted new string, thus can be called admission noise; The fourth term is the weighted average of $\alpha_{i,t-1} - b_{i,t}$, which embodies the contribution of wrong pickup. The sum of the third term and the fourth term is a measurement of the irrationality of agents. If agents always pick the strategy with best fitness measure, this sum will become zero, since $\tilde{p}_{i,t}$ will be 1 and $b_{i,t} = \alpha_t*$.

2.5.2 The Role of the Noise Trader

In the full memory case, it has been claimed that the equilibrium with average portfolio fraction 0.5 is the attractor of all simulations. Now let us see why.

Suppose the average portfolio fraction converges to a constant number $\hat{\alpha}$. To maintain this equilibrium, it requires: first, all agents use the same investment strategy $\hat{\alpha}$ thereafter, so $\bar{\alpha}_t = \bar{\alpha}_{t-1} = \alpha_{i,t} = \hat{\alpha}$, and the left side of equation (2.14) is zero; second, α_t^* should be a uniquely determined constant α^* thereafter; third, $\alpha^* = \hat{\alpha}$, which implies that the second term of equation (2.14), $\alpha^* - \bar{\alpha}_{t-1}$, is also equal to zero. Since all agents use the optimal strategy α^*, the probability of the current strategy being replaced by a new one, $\tilde{p}_{i,t}$, will become zero. Substitute this into the third and fourth terms of equation (2.14) and simplify, it can be shown that the sum of these two terms is also zero. Therefore, if we ignore the first noise trader term, then both sides of equation (2.14) will become zero, and the average portfolio fraction can stay at $\hat{\alpha}$ forever. However, when the first noise trader term is considered, the equilibrium with average portfolio fraction equal to 0.5 will be the only one that is stable under the evolution.

To see this, suppose the average portfolio fraction converges to a value $\hat{\alpha}$ which is bigger than 0.5. After the convergence, only the strategy with investment part equal to $\hat{\alpha}$ will be used and can enter the parent set. Since all strategies in the parent set are $\hat{\alpha}$, thus $E_{t-1}(\alpha_t^c) = \hat{\alpha}$. In this case, the second noise trader term, $(E_{t-1}\alpha_t^c - \bar{\alpha}_{t-1})$ will become zero. But the first term $(1 - p_1)(E_{t-1}(\alpha_t^{nc}) - E_{t-1}(\alpha_t^c))$ will not, unless $\hat{\alpha} = 0.5$. The reason is as follows: For the strategies not in the parent set, the probabilities of being generated by crossover, new rule operator, and mutation are all $\frac{1}{3}$. Since all strategies in the parent set are $\hat{\alpha}$, the expectation of the strategies generated by crossover is also $\hat{\alpha}$. The expectation of the strategies generated by new rule operator is 0.5. It can be shown that the expectation of the strategies generated by mutation falls into the range $[1 - \hat{\alpha}, \hat{\alpha}]$[2] for a parent string $\hat{\alpha} > 0.5$. Therefore the expectation of the strategies generated by mutation is less than $\hat{\alpha}$. Combining the results of these three cases, we can see if $\hat{\alpha} > 0.5$, then $E_{t-1}(\alpha_t^{nc})$ is less than $\hat{\alpha}$. Therefore $(1 - p_1)(E_{t-1}(\alpha_t^{nc}) - E_{t-1}(\alpha_t^c))$ will be a negative number. This means the

[2]The expectation of the strategies generated by mutation from a parent string $\hat{\alpha}$ is $\hat{\alpha}(1 - p_{mut}) + (1 - \hat{\alpha})p_{mut}$, where p_{mut} is the probability of the mutation. If $p_{mut} = 0$, no bit in the string will be changed. Therefore the mean of the new strings is $\hat{\alpha}$ itself. If $p_{mut} = 1$, every bit in the string will be changed. Therefore the mean of the new strings is $1 - \hat{\alpha}$.

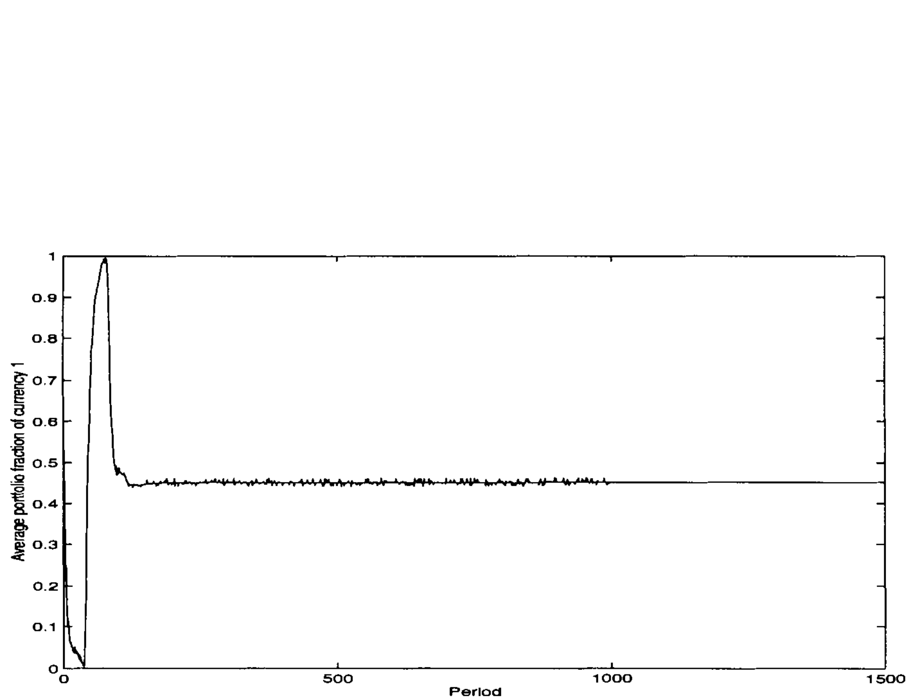

Figure 2.9: Average portfolio fraction of currency 1: full memory with noise trader eliminated at period of 1000.

average portfolio fraction will have the tendency to decrease.

To maintain the downward movement tendency, α_t^* must also show the same trend. Otherwise, any temporary downward drift caused by the noise trader will be offset by the followed actions of rational traders. Actually this is what happens. The downward movement tendency in the average portfolio fraction increases the probability of generating the return pairs with the relationship of $R_{1,t} < R_{2,t}$, which makes the realized returns distributions become more in favor of currency 2. Therefore α_t^* will also show downward movement tendency.

Similarly, if $\hat{\alpha} < 0.5$, then the average portfolio fraction has the tendency of becoming bigger. The only equilibrium that is asymptotically stable under GA is the one with $\hat{\alpha} = 0.5$. In the simulation, due to the stochastic nature of the noise trader, the average portfolio fraction will not converge to a constant number. Instead, there will always exist small fluctuations.

Figure 2.9 shows the dynamics of the average portfolio fraction in the experiment with the noise trader shut down in the period of 1000. It is shown that after the noise trader is eliminated, the fluctuation in the average portfolio fraction will disappear and the drifting of the equilibria will also disappear. It confirms that it is the noise trader who changes the history dependent property and makes it history independent. This kind of behavior is similar to the findings of Young (1993) and Kandori et al. (1993). They found, in repeat games, the introduction of some random behavior can change the results from history dependent to history independent.

We should also notice that the effect of the noise term is quite small. The bigger that N is, the smaller the effect of the noise term. Thus it will only show itself in the long run.

2.6 The Model Dynamics

2.6.1 The Full Memory Cases

The dynamics of the average portfolio fraction is determined by the interaction between the noise trader and the rational traders, whose strengthes are weighted by $\frac{1}{N}$ and $\frac{N-1}{N}$ respectively.

Suppose in period 1, realized returns have the relation of $R_{1,1} > R_{2,1}$. Then at the end of period 1, all rational agents will just use this one observation to evaluate the performance of the strategies, consequently the optimal strategies, α_2^*, for all agents are also the same, which is to put all savings into asset 1 as long as the consumption ratio is not 1.0. Due to the effect of admission noise and wrong picks, $\bar{\alpha}_2$ will not reach 1, but on average it will increase, which will have a positive effect on the return of currency 1 and a negative effect on the return of currency 2 from equation (2.6). If

$\alpha_2^* - \bar{\alpha}_{t-1}$ is big, this portfolio rebalancing effect of the rational agents will dominate the noise trader effect, it is more likely to see $R_{1,2} > R_{2,2}$. If it is so, then the optimal strategy still will be $\bar{\alpha}_3^* = 1$ for period 3, so that the upward movement of market share will continue.

However as the average portfolio fraction becomes higher and higher, on the one hand, the upward potential of the portfolio fraction is eventually exhausted, on the other hand, the probability that the noise trader to pick up a strategy far below the current average portfolio fraction becomes bigger and bigger. Therefore, it becomes more often to see temporary decrease in the average portfolio fraction and relatively bigger realized returns of currency 2. When the market reaches such a point that the accumulated effect of bigger returns of asset 2 leads $\bar{\alpha}_t{}^*$ to become smaller than $\bar{\alpha}_{t-1}$, then rational agents will change their behavior and the reversal movement of the average portfolio will be observed. When $\bar{\alpha}$ becomes small enough, the movement of the average portfolio fraction will reverse again.

During this up and down cyclical movement of the average portfolio fraction, the realized returns of two currencies will become more and more balanced. If agents have full memory, α_t^* will eventually settle down, and become more and more difficult to be changed, which in turn makes the average portfolio also settle down. The value that the average portfolio fraction settles down to is determined by the distribution of the initial realized returns, which is specific to each experiment. Thus it is history dependent.

The dynamics of the average portfolio fraction have following properties: first, the stationary equilibria near the boundary values 0 and 1 are extremely unstable, which makes it almost impossible for agents to coordinate around these levels. There are two reasons: first, when the average portfolio fraction is near 0 or 1, it is very easy for GA to generate big realized returns in currency 1 or 2, which may be big enough to destroy the equilibrium for a single shock. Second, even if a single return peak is not big enough to destroy the equilibrium, nevertheless, it is very easy for the GA to generate a sequence of shocks in short time to destroy the equilibrium. In either case, the GA can not stay around the extreme values for a long period.

Second, the stationary equilibria around some intermediate values can be kept quite long once GA converges to it. The duration that GA spends on these equilibria is negatively correlated with the difference between the average market portfolio fraction of these equilibria and 0.5. It is also negatively correlated to the population size N.

Third, the equilibrium of 0.5 is the attractor of all equilibria. GA cannot stay forever at an equilibrium different from it. The noise trader effect will eventually take control and make the average portfolio move to it.

2.6.2 The Incomplete Memory and Short Memory Cases

In the less than full memory case, the story is quite different. Suppose agents have quite long memory T, but not full, and after the initial turbulence, the average portfolio fraction settles down to the neighborhood of an arbitrary number. When the market operates for longer than T periods, agents will begin to forget the realized returns in the initial periods. Suppose the initial realized returns are in favor of asset 1, then forgetting this part of the realized returns will destroy the balance of the realized return distribution, leading the remaining realized return distribution in favor of asset 2, so that α_t^* will begin to decrease, and we will see that the average portfolio fraction decreases as well. When the number of periods that the market operates exceeds the length of agents memory, the movement of the market share will be approximately a replication of what happened in the previous cycle but in an opposite direction.

When the average portfolio settles down again, it typically will be another value since the process of the equilibrium destruction and establishment is stochastic.

The periodic length is approximately the memory length of agents. If the memory length of agents is sufficient small, we may not observe the average portfolio to settle down, and can only observe the periodic swing of market share, which is shown in the short memory case.

2.7 The Impaction of the Population Size and the Size of the Strategy Set

As illustrated in the introduction, the size of the noise traders is normalized to 1 in the simulation. Therefore, the population size N reflects the relative size of the noise traders and the rational traders. The simulation results with different population sizes are shown in figures 2.10, 2.11, 2.12 and 2.13. It can be seen that when $N = 5$, the effect of the noise trader is strong. The dynamics of the model shows two properties: first, the movement of the average portfolio fraction shows randomness and irregularity; second, the average portfolio fraction will not show momentum toward to boundary 0 or 1. As N increases, the movement of the average portfolio fraction becomes more regular, and it may also take longer for the average portfolio to converge to 0.5.

The pattern of going from randomness to regularity with N increasing is similar to the findings of Lux and Schornstein (2004). They find that as N increases, the randomness will get lost and the dynamics of the average portfolio fraction appears to converge to a regular cycle (like the figure 2.7) under the framework of Arifovic (1996).

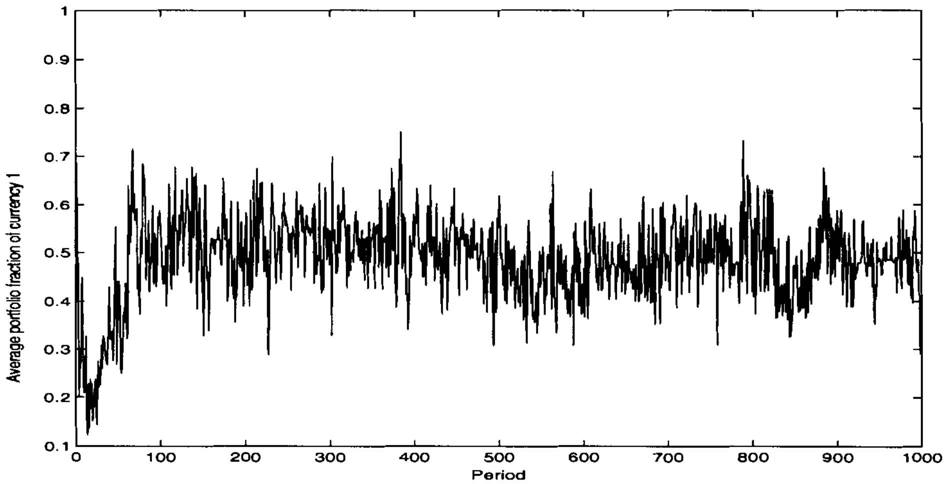

Figure 2.10: Average portfolio fraction of currency 1: $N = 5$.

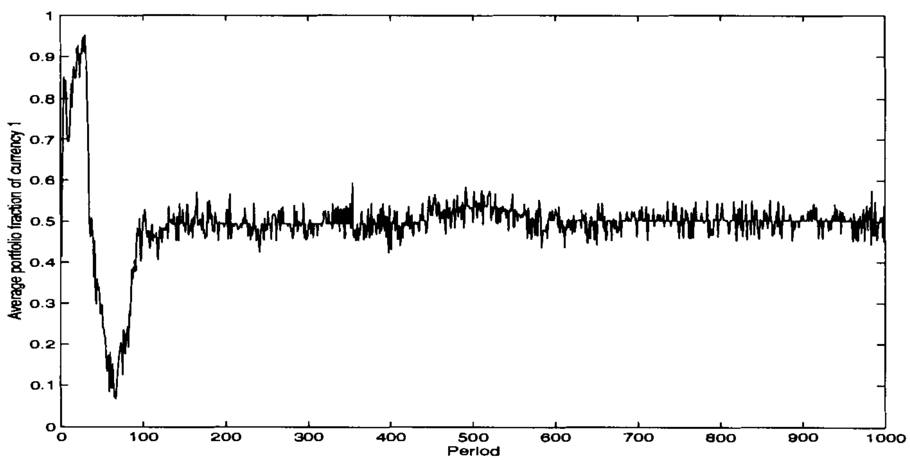

Figure 2.11: Average portfolio fraction of currency 1: $N = 10$.

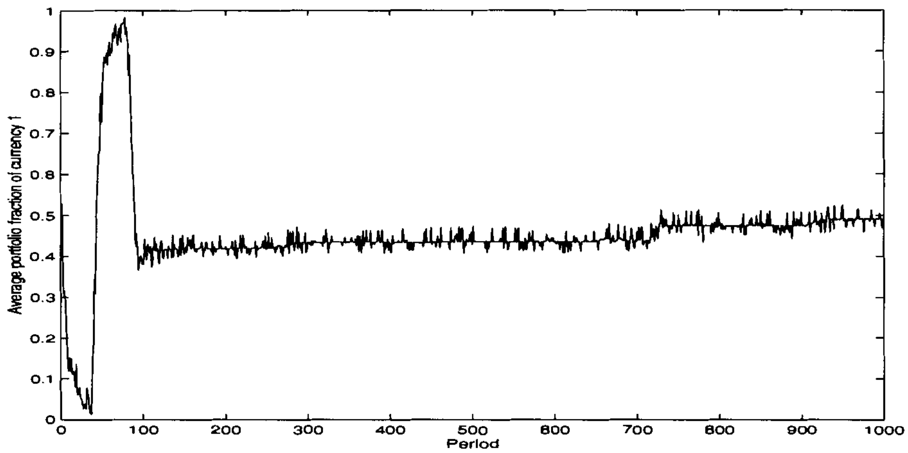

Figure 2.12: Average portfolio fraction of currency 1: $N = 15$.

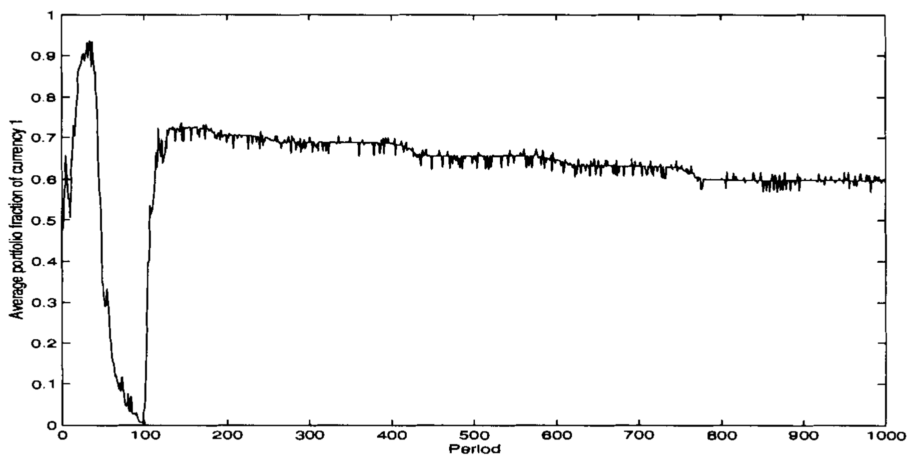

Figure 2.13: Average portfolio fraction of currency 1: $N = 20$.

Another question is whether the results are sensitive to the size of the strategy set. From the analysis of section 2.5.2, we know that the drifting of the average portfolio fraction in the full memory case is mainly determined by the difference between $\hat{\alpha}_t$ and the mean of strategies that are used to replace the dead strategies. As the size of the strategy set increases, the size of the dead strategies will also increase if we fix the ratio between the size of strategy set and the candidate set. This will make the mean difference equation approach more powerful due to the law of large number. To some extent, this is also similar to Lux and Schornstein (2004)'s large N phenomenon. When the size of strategy set is extreme small, the change of the average portfolio fraction will become disrupt, and it generally takes longer for the algorithm to converge. One experiment with strategy population size 4 and candidate size 2, is show in figure 2.14. We also can see that when the size of the strategy population is 20, the behavior of the model is already very similar to our benchmark full memory simulation from figure 2.15.

2.8 Conclusion

Even though agents have long memory in our experiment, it does not imply that agents will remember the initial failure in coordination forever and never coordinate with each other again. On the contrary, simulation results show that agents can learn how to coordinate if they can draw lessons from past failures. However, when agents begin to forget the suffering in the past, the coordination will break down. Thus the reason why people can coordinate with each other is because they can 'remember', not because they will 'forget', past failures. The most destructive power on the exchange rate market is agents' "forgetting".

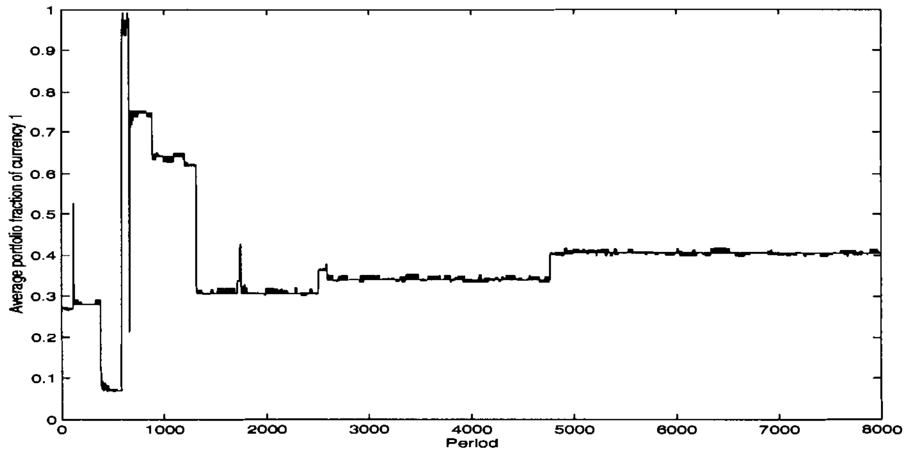

Figure 2.14: Average portfolio fraction of currency 1: $Ns = 4$, candidate=2.

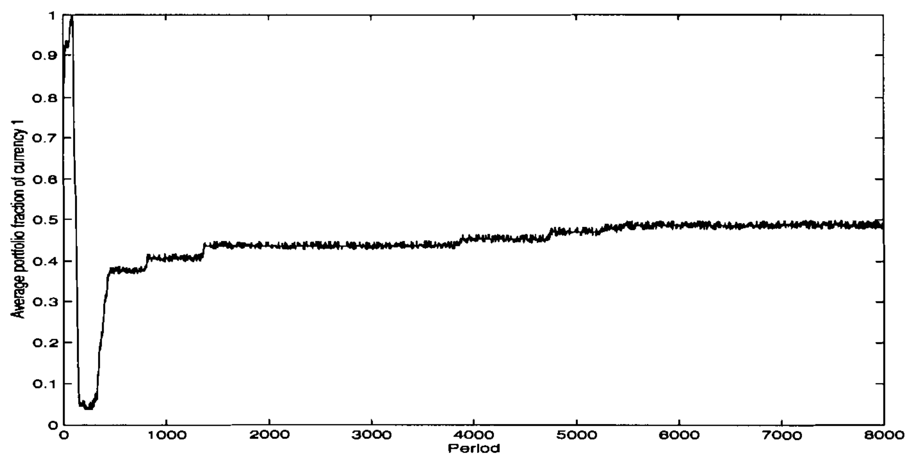

Figure 2.15: Average portfolio fraction of currency 1: $Ns = 20$, candidate=10.

The role of the noise trader is not completely destructive. Comparing this chapter's model with chapter 1's model, we can see that the noise trader is actually playing the role of the level effect term in the equation (1.29). After agents establish the temporary coordination, the portfolio-rebalancing effect (or market mood effect) will become zero, and the activities of the noise trader will take the control. This lead agents asymptotically to coordinate around the equilibrium putting savings equally into currency 1 and 2, no matter what the initial conditions are.

Arifovic (1996) can generate exchange rate series displaying properties remarkably similar to real data, which cannot be observed if all agents having long memory. However when the memory length is short and the population size is small, the behavior of the exchange rate in our model becomes similar to Arifovic (1996).

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Chapter 3

Do Investors' Beliefs Really Matter? An Issue about the Interpretation of GA Models

As a method of modelling the evolution of investors' beliefs, genetic algorithms (GA) have been used extensively in simulating financial markets. Presumably GA models should have self-referential processes as their kernels, where investors' past interaction should affect their future interaction. Otherwise, GA models will degenerate to a pure numerical maximization method. The purpose of this paper is to point out that just putting a population of investors with different beliefs into a GA does not necessarily mean investors' beliefs will really matter. We will also show that whether the self-referential process exists or not, and if exists how strong it is, is a key to understand the dynamics of the GA model.

3.1 Introduction

If there is only one investor in a financial market, the investor's behavior rule will be quite simple. She just needs to maximize her welfare subject to the objective conditions. However, in reality, there are many investors in the market. To make decisions, an investor not only needs to form correct beliefs about the *primary* information (Richardson's (1959) phrase), such as the dividend processes, but also needs to take into consideration the *secondary* information, such as what other investors' projected activities will be. As in Keynes' (1936) phrase, investors need to take into account "what average opinion expects the average opinion to be". In such a context, any outcome is possible if investors' beliefs are free parameters. Therefore, in order to refine the solutions, the restriction of mutual consistent beliefs is usually used in asset pricing models, which means investors mutually know each others' beliefs, and these beliefs should also be consistent with the distributions of exogenous variables. This is, of course, the rational expectations approach.

One of the main reasons why researchers want to simulate a financial market with GAs is because they are not willing to make this restriction. Typically in a GA model, there are almost no restrictions put on investor's beliefs, hence any investment behavior can be observed. The inconsistency between investors' beliefs is resolved by the market. Successful investment strategies are those that happen to be supported by the market, and will have a higher probability of being followed by other investors, and thus survive; poor performing strategies have a lower probability of being followed, and thus eventually die. In this way, researchers can study where financial markets will go without assuming that investors have perfect foresight. There are many successful applications of GA models to the financial markets, see Lux and Schornstein (2004), LeBaron (2000,2001,2002), Tay and Linn (2001), LeBaron et al. (1999), Arthur et al.

(1997), Lettau (1997), Arifovic (1996) and Plamer et al. (1994) among others.

Obviously, what makes GA interesting is not that it is an optimization method, but because it offers a way of putting endogenous heterogenous beliefs into the optimization process, and hence enables us to see what the dynamics of a model will be with the additional uncertainty of the interaction among investors. Therefore, presumably GA models should have self-referential processes as their kernels, where investors' past interaction should affect their future interaction. Otherwise, investor's beliefs will 'not matter', and the GA models will degenerate to a pure numerical maximization method. The purpose of this paper is to point out that just putting a population of investors with different beliefs into GA does not necessarily mean investors' beliefs will really matter. We will also show that whether the self-referential process exist or not, and if exists how strong it is, is a key to understand the dynamics of the GA model.

In GA models, since the survival and death of the strategies are determined by their performance, the question of whether investors' beliefs matter or not is equivalent to the question of whether investors' interaction is important in calculating the performance measure. In the easiest case, if the performance measures of strategies are solely determined by exogenous variables, it is apparent that the GA is purely playing the role of an optimizer. In the second case, when investors are modelled to make multiple decisions simultaneously, it is possible that investors' beliefs matter for some decisions, but do not matter for other decisions. We will show this by using a performance measure function which has a separation property. The third case is the most subtle one, in which although the self-referential process does exist, its strength depends on the parameters of the model.

The GA applications that are used to illustrate our points follow the framework of the artificial stock market proposed by LeBaron (2001). It has been introduced in chapter 1. The theoretical model and the GA design will be described in Section 3.2. Four experiments are analyzed: In experiment 1, the number of risky assets is set to 1, and its return is exogenously given. This example is used to illustrate the function of GA as an optimizer. In experiment 2, the model is an extended Lucas' tree model with one risky asset. We are going to see how investors' beliefs can become irrelevant in forming optimal consumption decisions, although the return of the risky asset is endogenous. This is further illustrated in experiment 3, where two risky assets are included in the model. In experiment 4, although the model is the same as that in experiment 3, our main focus moves from the optimal consumption decision to the optimal portfolio decision. It turns out while investors' beliefs will not change the convergence property of GA in the full memory cases, it will lead to substantial differences in the cases where investors' memory length is bounded.

3.2 The Model and the GA Design

3.2.1 The model

All theoretic models in this paper are taken from Sargent (1987). The basic model is as follows. The representative investor tries to maximize her lifetime log utility:

$$u_t = E_t \sum_{\tau=0}^{\infty} \lambda^{\tau} \log c_{t+\tau}, \tag{3.1}$$

subject to the intertemporal budget constraint,

$$w_{t+1} = (1 - \beta_t) R_t w_t, \tag{3.2}$$

where w_t is the wealth of the investor in period t, λ is the patience parameter, β_t is the proportion of the wealth being consumed in each period, R_t is the gross return from period t to $t+1$. With log utility, the optimal consumption ratio is always $\beta^* = 1 - \lambda$. R_t can be exogenous or endogenous. How R_t is generated will be introduced in each specific experiment.

3.2.2 The GA Application

Like the design of the GA experiments in chapter 1, there is a population of investors of size N in the economy. There is also a population of strategies with size N_s. The strategy s, $s = 1 \cdots N_s$, is composed of two parts: $\beta_{s,t}$, defined as the fraction of w_t being consumed in period t, and $\alpha_{s,t}$, the fraction of investor's savings that is put into asset 1 in period t, if there are two risky assets in the economy. Each strategy is encoded in a genetic string with the length of *clength* + *alength*. The first *clength* bits of string s denotes $\beta_{s,t}$. The last *alength* bits denotes $\alpha_{s,t}$. If *clength* $= 0$ and *alength* $\neq 0$, it means that investors only make portfolio decisions. If *clength* $\neq 0$ and *alength* $= 0$, it means that investors only make consumption decisions.

Assuming that the memory length of investor j is T_j, $j = 1, \cdots, N$, in experiment 1, 2 and 3, investor j evaluates the performance of the rule s at period t using

$$\sum_{\tau=1}^{T_j} \lambda^{\tau-1} \log(\beta_{s,t} w_{t-T_j+\tau-1}), \tag{3.3}$$

where

$$w_{t-T_j+\tau-1} = (1 - \beta_{s,t}) R_{t-T_j+\tau-2} w_{t-T_j+\tau-2}, \quad w_{t-T_j} = 1, \tag{3.4}$$

$R_{t-T_j+\tau-2}$ is the gross return of investment from period $t - T_j + \tau - 2$ to period $t - T_j + \tau - 1$. In experiment 4, a different performance measure will be used. It will be described in the related section.

At the beginning of each period, half of the population of investors will be randomly selected to try new rules. They will evaluate the performance of each rule in the strategy population and randomly choose one rule from the candidate strategy set. If this rule has better performance than the one she currently uses, she replaces the old one with the new one, otherwise she still uses the old one.

In each period, the timeline of the market is as follows:

1) Each investor evaluates the performance of the strategies using her memory of ex post returns of assets, and chooses one rule as her current strategy.

2) The realized returns are generated or determined.

3) Rules evolve.

4) Investors evolve.

The evolutions of strategies and agents are same as those of chapter 1, except that the evolution of agents only occurs when the asset returns are endogenous.

3.3 Experiment 1

There is only one risky asset in the market. The return distribution of the asset is exogenously given and assumed to follow a log normal distribution, $\log R_t \sim N(0.0015, 0.01732^2)$. *clength* and *alength* are set as 10, 0, respectively. This means that investors only make consumption decisions. The patience parameter λ is set as 0.95. Therefore the optimal consumption ratio is 0.05. The period of simulation is 1000. The number of investors and strategy rules are 100 and 60 respectively. Investors have identical memory length of 600. For period 1, strategy rules are generated randomly, and the strategy rule each investor uses is randomly drawn from the strategy population. There is no evolution of investors in experiment 1.

To make comparisons, the simulations are done with two different sizes of candidate set: The first one sets the parameter 'candidate' as 1, i.e., investors always

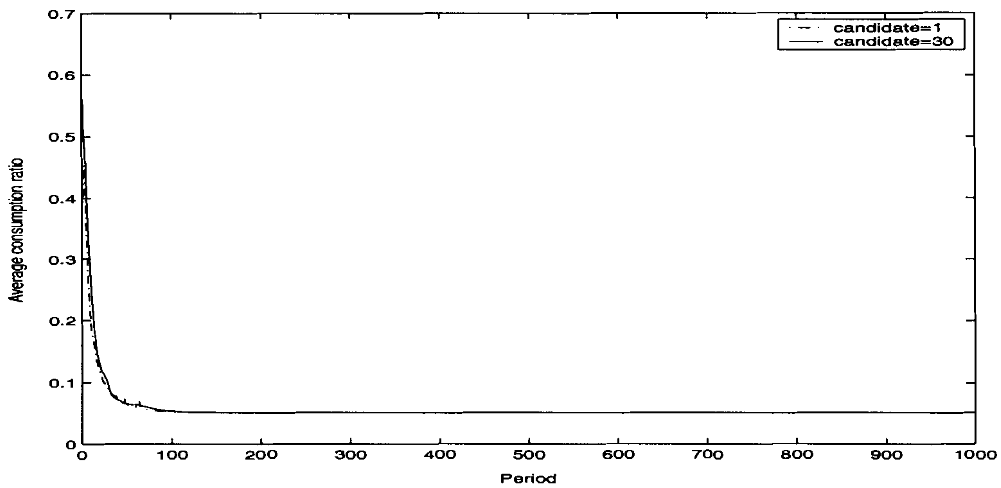

Figure 3.1: Experiment 1: Consumption ratio: candidate=1 and 30

choose the rule with the best performance; the other one sets it as 30, i.e., half of the size of the strategy population. Figure 3.1 shows that, in both experiments, investors can find the optimal consumption rule quickly. In about 100 periods, the average consumption ratio used by the investors converges to optimal solution 0.05 for both experiments.

In this experiment, the interpretation of GA is quite simple. Since the asset return is exogenously given, investors' actions have no impact on the realized returns. Therefore investors' interaction has no impact on the performance measure equation (3.3), which means that there is no self-referential process and GA is acting primarily as an optimization method.

The convergence speed of the average consumption ratio is similar in both simulations. However the distributions of final strategy populations are quite different. When the parameter 'candidate' is 30, the mean of the strategy population is around 0.15, which is much smaller than that of the experiment with 'candidate' 1, 0.27.

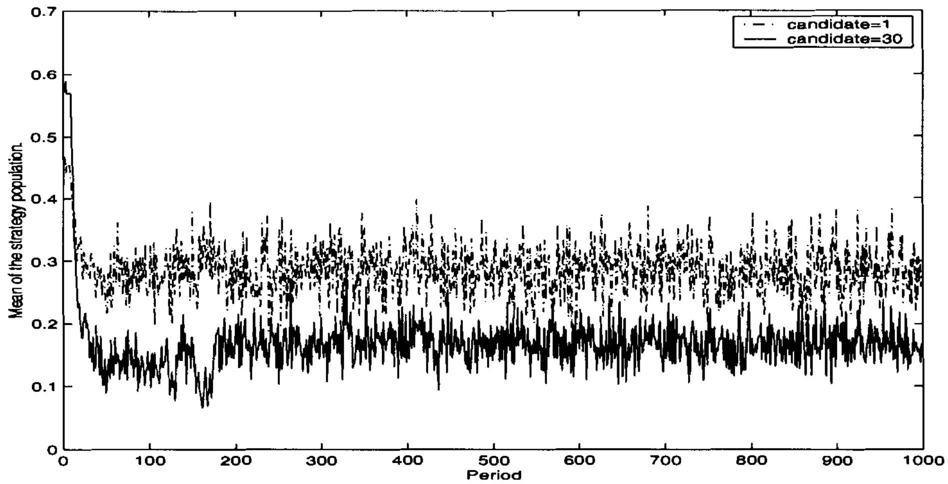

Figure 3.2: Experiment 1: Mean of strategy population: candidate=1 and 30

The reason is as follows: After the average consumption ratio converges to the optimal level, all strategies in the parent set will be identical with value of 0.05. For the strategies not in the parent set, the probabilities of being generated by crossover, new rule operator, and mutation are all $\frac{1}{3}$. Since all strategies in the parent set are 0.05, the expectation of the strategies generated by crossover is also 0.05. The expectation of the strategies generated by new rule operator is 0.5. The expectation of the strategies generated by mutation is larger than 0.05. Thus the mean of the new strategies that are used to replace the dead ones is larger than 0.05. If investors always choose the best one, then, on average, in each period there are more dead strategies than the case when investors diversify their choice over a subset of the strategy population, thus its strategy population has a larger mean. The evolution of the means of the strategy populations is shown in figure 3.2.

3.4 Experiment 2

3.4.1 Model

In this experiment, the Lucas tree model with one asset is used as the simulation base. The return of the asset is endogenous, and the budget constraint of equation (3.2) is changed to be

$$w_t = p_t s_t + c_t = (p_t + d_t)s_{t-1}, \tag{3.5}$$

where s_t is the investor's holding of risky asset in period t. The supply of asset 1 is fixed at one unit. In equilibrium, $s_t = s_{t-1} = 1$, and the consumption ratio is $1 - \lambda$. Substituting these into equation (3.5), gives

$$p_t = \frac{\lambda}{1-\lambda} d_t.$$

In the artificial stock market, the saving rate of investor j is $1 - \beta_{j,t}$. Thus investor j will put $(1 - \beta_{j,t})w_{j,t}$ into the risky asset, where $w_{j,t} = (p_t + d_t)s_{j,t-1}$, $s_{j,t-1}$ is the share of the asset held by investor j in period $t-1$. Thereby the demand for the risky asset of investor j is

$$s_{j,t}(p_t) = \frac{(1 - \beta_{j,t})w_{j,t}}{p_t}.$$

Summing the demand of all investors, plugging this into the market clearing condition $\sum_{j=1}^{N} s_{j,t} = 1$ and solving for p_t, we get

$$p_t = \frac{d_t \sum_{j=1}^{N}(1 - \beta_{j,t})s_{j,t-1}}{1 - \sum_{j=1}^{N}(1 - \beta_{j,t})s_{j,t-1}}.$$

The ex post returns are calculated as

$$R_t = \frac{p_t + d_t}{p_{t-1}}.$$

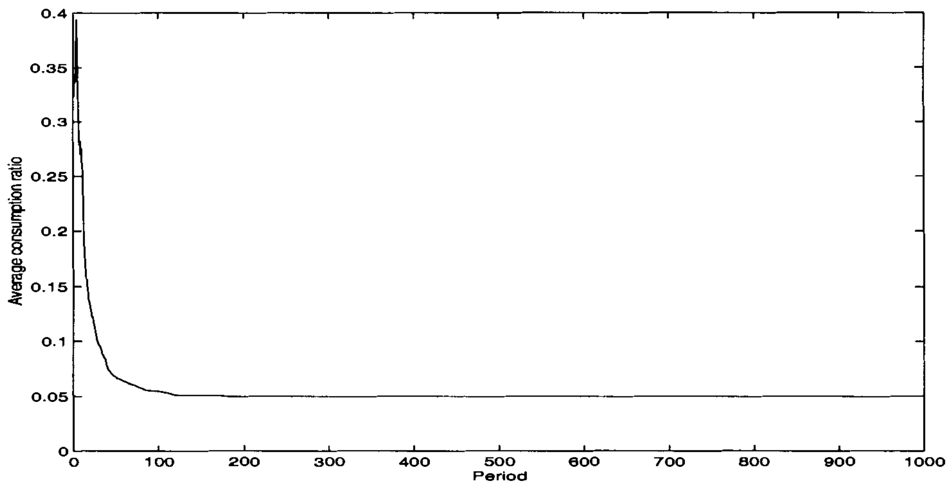

Figure 3.3: Experiment 2: Average consumption ratio: candidate=30

3.4.2 Simulation

The parameter values are the same as those in experiment 1, except that the return distribution is not exogenously given. The exogenous aggregate dividend of assets is assumed to follow

$$\ln d_t = \ln d_{t-1} + g + v_t, \tag{3.6}$$

where $v_t \sim N(0.00, 0.001732^2)$, $g = 0.0015$. This corresponds roughly to the actual dividend properties of the U.S. stock markets (Campbell 2000). The initial asset holdings are assumed to be equal across the investor population. In contrast to experiment 1, the evolution of investors is added to the GA.

From figure 3.3, we can see, the GA evolution does not show much difference from that of experiment 1. The GA still converges to the optimal solution very quickly. This is because, in this experiment, the role of the GA is still mainly as an optimizer. Although the asset returns are endogenous, they are still irrelevant in searching for the optimal consumption decision. This can be shown as follows: For simplicity, we assume investors have infinite memory length and the simulation has operated for a sufficient long period, then the performance of strategy $\beta_{s,t}$ is

$$\sum_{\tau=1}^{\infty} \lambda^{\tau-1} \log(\beta_{s,t} w_\tau) = \log \beta_{s,t} \sum_{\tau=1}^{\infty} \lambda^{\tau-1} + \log(1-\beta_{s,t}) \sum_{\tau=1}^{\infty} ((\tau-1)\lambda^{\tau-1}) + \sum_{\tau=2}^{\infty} \lambda^{\tau-1} \log \prod_{n=1}^{\tau-1} R_n. \tag{3.7}$$

We can see that the asset returns enter the performance measure by the third term in equation (3.7). Since term 3 has the same value across the performance measures of all strategy rules, the relative magnitude of performance measure is determined by the first two terms, which are independent of investors beliefs. Then the thing left is to let GA find the optimal β value to maximize the sum of term 1 and 2,

$$\log \beta \sum_{\tau=1}^{\infty} \lambda^{\tau-1} + \log(1-\beta) \sum_{\tau=1}^{\infty} ((\tau-1)\lambda^{\tau-1}) = \frac{\log \beta}{1-\lambda} + \frac{\lambda \log(1-\beta)}{(1-\lambda)^2}.$$

It is easy to show that it is maximized at $\beta^* = 1 - \lambda$, which is just the optimal solution to the original Lucas tree model.

At first glance, it seems surprising that in such an economy where investors have endogenous heterogenous beliefs, investors, although naive, can still find a way to coordinate with each other and evolve to the optimal equilibrium. However, the real mechanism here is that heterogenous beliefs do not matter at all. There is in fact no fundamental difference between example 1 and 2.

3.5 Experiment 3

3.5.1 Model

In this experiment, the model is extended to two risky assets case. The intertemporal budget constraint is expanded as,

$$w_t = p_{1,t} s_{1,t} + p_{2,t} s_{2,t} + c_t = (p_{1,t} + d_{1,t}) s_{1,t-1} + (p_{2,t} + d_{2,t}) s_{2,t-1},$$

where $s_{1,t}$,$s_{2,t}$ are the investor's holding of risky asset 1 and asset 2 respectively. This model has been discussed in details in chapter 1. Here we still use the special case of this model, where the expected dividend share is a constant. As we know, the equilibrium market share will be a constant as well under this assumption.

In contrast to the experiments in chapter 1, we do not fix the consumption at the optimal level this time, and instead we let agents make both consumption decision and investment decision.

Let the saving rate of investor j, $j = 1 \cdots N$ be $1 - \beta_{j,t}$. Thus investor j will put $\alpha_{j,t}(1 - \beta_{j,t})w_{j,t}$ in asset 1, and $(1 - \alpha_{j,t})(1 - \beta_{j,t})w_{i,t}$ in asset 2, where $w_{j,t} = (p_{1,t} + d_{1,t})s_{1,j,t-1} + (p_{2,t} + d_{2,t})s_{2,j,t-1}$. So the demand for asset 1 is

$$s_{1,j,t}(p_{1,t}) = \frac{\alpha_{j,t}(1 - \beta_{j,t})w_{j,t}}{p_{1,t}}.$$

Summing the demand of all investors, and substitute them into the market clear condition $\sum_{j=1}^{N} s_{1,j,t} = \sum_{j=1}^{N} s_{2,j,t} = 1$, we get

$$p_{1,t} = \sum_{j=1}^{N} \alpha_{j,t}(1 - \beta_{j,t})w_{j,t}. \tag{3.8}$$

In the same way, for asset 2, we have.

$$p_{2,t} = \sum_{j=1}^{N}(1 - \alpha_{j,t})(1 - \beta_{j,t})w_{j,t}. \tag{3.9}$$

Since $w_{j,t}$ is a function of $p_{1,t}$ and $p_{2,t}$, the current prices have to be solved jointly from the above two equations.

The ex post gross returns of asset 1 and 2 are calculated as

$$R_{i,t+1} = \frac{p_{i,t+1} + d_{i,t+1}}{p_{i,t}}, \quad i = 1, 2.$$

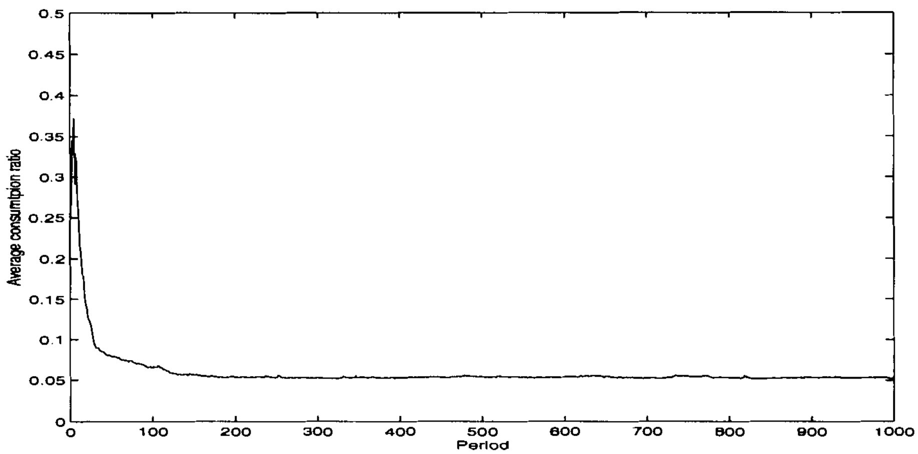

Figure 3.4: Experiment3: Consumption ratio

3.5.2 Simulation

The parameters are set as follows: $alength = 10$, $clength = 10$. Thereby investors make both consumption and investment decisions. The range of investment strategies is set as (0.0001, 0.9999). The initial memory lengths of investors are generated randomly from the uniform distribution of [6,250]. During the evolution of investors, the memory length of new arrivals is generated in the same way. $\epsilon_t = \frac{d_{1,t}}{d_t}$ is generated from the same beta distribution as that in chapter 1. The mean and variance of ϵ_t are 0.5 and 0.0012437, respectively. Accordingly, the implied rational expectations equilibrium market share is 0.5. Other parameters are the same as in experiment 2.

Let us first take a look at the pattern of the consumption ratio, which is shown in figure 3.4. In this relatively complex situation, the consumption ratio can still converge to the neighborhood of the optimal solution 0.05. The fact that the GA successfully converges to the optimal saving ratio, however, does not mean it can

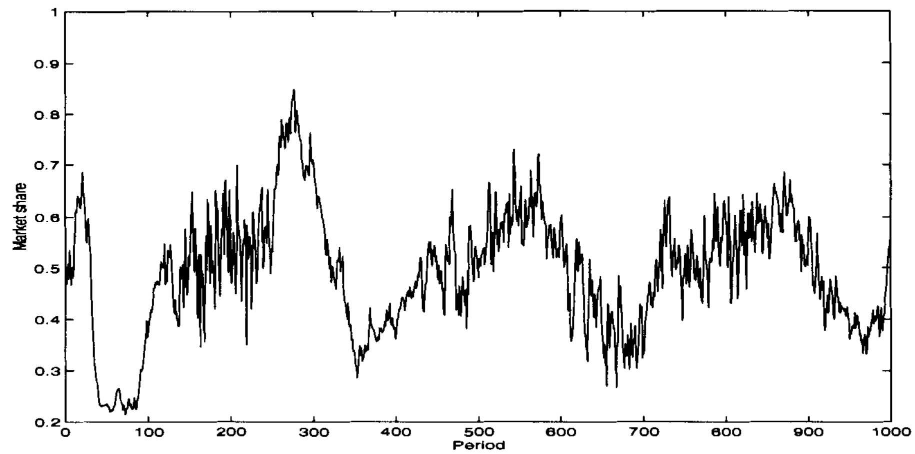

Figure 3.5: Experiment 3: Market share

successfully make the market share converge to the optimal solution. The behavior of the market share is shown in figure 3.5, from which we can see that the market share changes dramatically over time.

What are the forces that lead the GA to converge in search of the optimal consumption ratio, but diverge in search of the optimal investment strategy? To answer this we still need to return to the performance measure function. In equation (3.7), we see that the performance function can be decomposed into three terms, in which the consumption decision β only affects terms 1 and 2, and the investment decision only affects term 3. In experiment 2, there is no investment decision, only the consumption decision, and all investors have the same memory length. Therefore the third term is the same for all strategies. The GA just need to pick the right one to maximize the first two terms. Here in experiment 3, each strategy contains two parts, the consumption strategy and the investment strategy. The investment strategy affects the return of the investment portfolio since $R_n = \alpha_{s,t} R_{1,n} + (1 - \alpha_{s,t}) R_{2,n}$, which depends on $\alpha_{s,t}$. Accordingly, even if each investor has the same memory length T, the third term, $\sum_{\tau=2}^{T} \lambda^{\tau-1} \log(\prod_{n=1}^{\tau-1} R_n)$ is still different for different strategies.

Does this mean investors' beliefs matter in search of the optimal consumption strategy? The answer is still 'no'. For a given memory length T, the strategies can be divided into groups classified by their investment strategies. The strategies having the same investment strategies values are put into the same group. Then among each group, the third term is the same for each member. In the long run, the strategies that can survive in each group have to be the ones that maximize the sum of first two terms, which is the same across the groups. Although GA operators, especially the mutation operator and the new rule operator, will make the strategy population change all the time, it does not change the property that only the strategies maximizing the sum of the first 2 terms can survive. Hence we see the separation property of the performance function still makes investors' interaction not matter in search of the optimal consumption decision.

However, when we turn to the optimal investment decision, the story is totally different. A building block of term 3 is $\log(R_t) = \log(\alpha R_{1,t} + (1 - \alpha) R_{2,t})$, where $R_{1,t}$ and $R_{2,t}$ are determined endogenously by investors' decision. The complexity lies in the fact that when investors optimize α based on the past realization of the assets' returns, they actually change the future behavior of the assets' returns themselves. The interaction between investors' decisions and the information set on which they make decisions forms the self-referential process. It is obvious that in a self-referential process, the investors' past beliefs matter.

However it is premature to conclude that investors cannot find a way to coordinate with each other and converge to the rational expectation equilibrium in this two-risky assets artificial stock market. Taking another look at term 3, we can see that term 3 is actually a weighted average of the past information. At period t, the weight for the period $\tilde{t}$'s information is $\frac{\lambda^{\tilde{t}-t+T_j+1}(1-\lambda^{t-\tilde{t}})}{1-\lambda}$. Thus the information at period $t - T_j$ has the highest weight of $\frac{\lambda(1-\lambda^{T_j})}{1-\lambda}$, and the weight of newer information will decrease exponentially as $\tilde{t}$ goes to $t-1$. For example, when $\lambda = 0.95$, $T_j = 100$, the weight of period $t-1$ information is only $0.95^{100} = 0.00592$, which means it needs extremely large returns in period $t-1$ to match the role of a moderate return in period $t - T_j$ playing in the performance measure function.

Therefore the drawback of this performance measure function is that the effective memory length of investors in term of term 3 is quite short. With short memory length, the investors' behavior is more likely to be affected by the temporary return bubble, and less likely to converge. In addition, investors' behavior will be governed by old information instead of recent information, which is in conflict with common sense.

3.6 Experiment 4

In experiment 4, we will concentrate our efforts on the searching of the optimal investment strategy. We will use the following property of the model: That is if we fix the consumption ratio at the optimal level,

$$\max_{\alpha} E_t \log[\alpha R_{1,t+1} + (1-\alpha)R_{2,t+1}], \tag{3.10}$$

will be maximized at the equilibrium portfolio fraction, given the equilibrium returns $(R_{1,t+1},\ R_{2,t+1})$, where α is the proportion of asset 1 in the portfolio. Using the sample average to replace the expectation operator, we get the performance measure

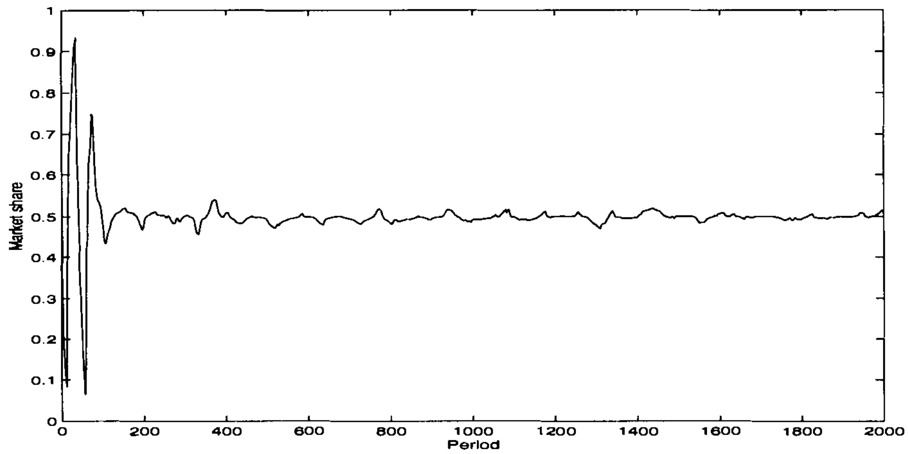

Figure 3.6: Experiment 4: Market share, full memory, $\lambda = 0.9$

as follows,

$$\sum_{\tau=1}^{T} \log(\alpha_{j,t} R_{1,t-T+\tau-1} + (1-\alpha_{j,t}) R_{2,t-T+\tau-1}). \tag{3.11}$$

It should be noted that in contrast to experiment 3, the same weight scheme is applied to the past information in equation (3.11). And this performance measure function is also different from that in chapter 1, where agents will use random sampling.

In the simulation, the parameters are set as follows: *clength* = 0, and *alength* = 10. Thus investors only make investment decisions. The consumption decision is fixed at the optimal level of 0.1 ($\lambda = 0.9$) and 0.9 ($\lambda = 0.1$) respectively.

Figures 3.6 and 3.7 show the dynamics of the market share when investors have full memory, i.e., investors remember all past realized returns. We can see in both simulations, the market shares quickly converge to the rational expectations equilibrium value 0.5, although small fluctuations will exist forever.

On the other hand, when investors have long but not full memory, the dynamics

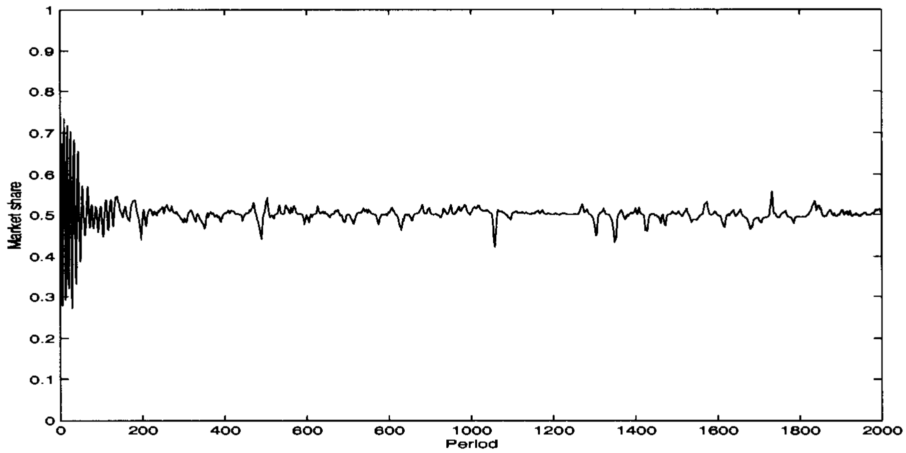

Figure 3.7: Experiment 4: Market share, full memory, $\lambda = 0.1$

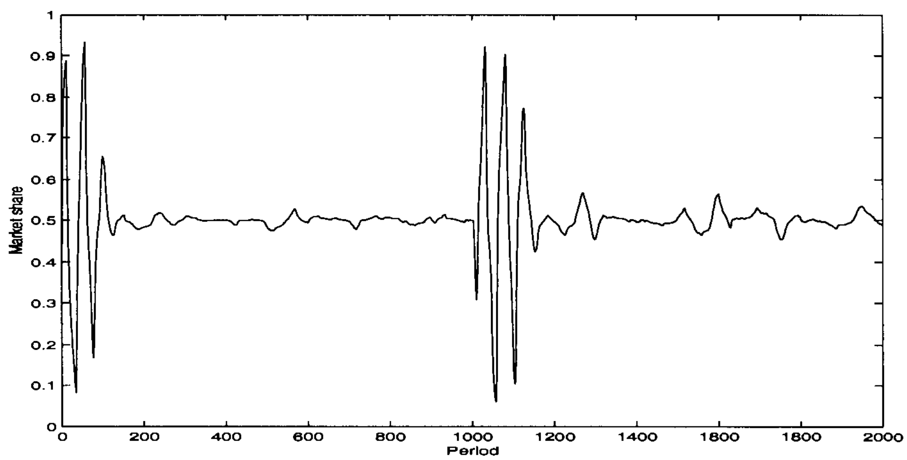

Figure 3.8: Experiment 4: Market share, memory length=600, $\lambda = 0.9$

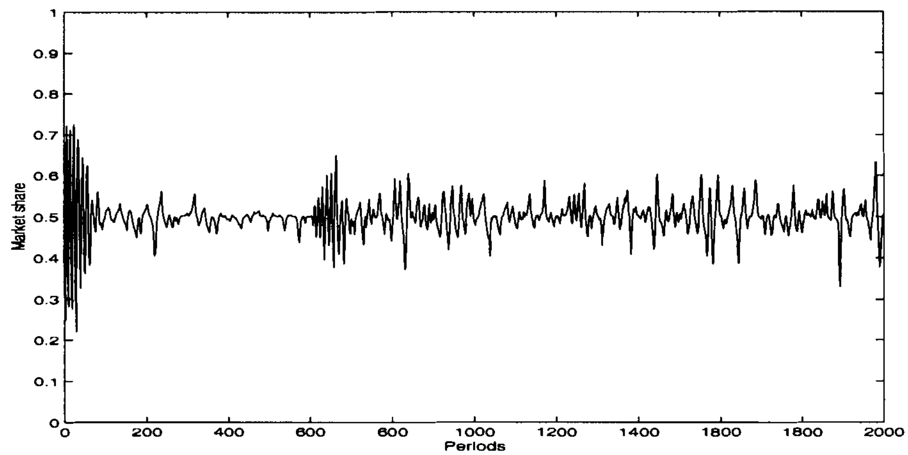

Figure 3.9: Experiment 4: Market share, memory length=600, $\lambda = 0.1$

of the market shares will be quite different, which are shown in figures 3.8 and 3.9, where investors have identical memory length 600. Figure 3.8 shows that, when $\lambda = 0.9$, the market share will display periodic movement. The periodic length is approximately the length of investors' memory. At the start of each cycle, the market share shows persistent deviation from the equilibrium value. When $\lambda = 0.1$, although the market share displays fluctuations around the equilibrium level, it does not have regular pattern. The persistent deviation is not observed.

3.6.1 An Analysis of the Dynamics in Experiment 4

The dynamics of the model has been discussed in details in chapter 1. Here let's give a new interpretation of the bubble pricing equation (1.29) in terms of its self-referential implication. Recall that the bubble pricing equation is as follows,

$$r_{1,t} \approx g - \ln\lambda + (1-\lambda)\ln(\frac{\epsilon_t}{\alpha}) + v_t + \lambda(\ln\bar{\alpha}_t - \ln\bar{\alpha}_{t-1}) - (1-\lambda)(\ln(\bar{\alpha}_{t-1}) - \ln\alpha), \quad (3.12)$$

where $E(\epsilon_t) = \alpha$. As we know, the first four terms reflect the fundamental value of the asset. The fifth is the market mood term. If investors have an optimistic view of asset 1, they will increase their holdings of this asset, so that the market mood term, $\ln \bar{\alpha}_t - \ln \bar{\alpha}_{t-1} > 0$, has a positive effect on the return of asset 1. For the same reason, if investors are pessimistic, then it will have a negative effect. The sixth term is the level effect term. The market mood term and the level effect term are weighted by λ and $1 - \lambda$, respectively.

The question is what the information is conveyed by the simulations across different values of λ. From the point of view of agents' preferences, λ is a parameter of patience. However in the bubble pricing equation, we'v already seen that it is also a parameter that determines the relative strength of market moods and the level effect. From this perspective, we can take it as a measure of market's capacity of amplifying investors' herd behavior. If λ is very small, the market will be inertia in response to investors' activities, and assets' returns will be mainly determined by the fundamentals. In this case, we can hardly say there exist self-referential process in the model. Strong self-referential processes only appear when λ is large, where investors' past interactions will substantially influence realized returns, and hence shape their future behavior. Therefore the magnitude of λ can be taken the measure of how strong the self-referential process is in the model.

In the large λ cases, the self-reference process is strong. Therefore, the market mood can self-validate itself, and makes the market share persistently deviate from the equilibrium value. This will make the movement of the market share show clear pattern, and is easy to be identified, as we see in figure 3.8. However when λ is really small, the assets' returns will be mainly determined by the fundamentals. This in turn makes investors' investment decisions can not deviate far from the equilibrium level. Thereby the movement of the market share will be governed by the random shocks of the fundamentals, and will not display regular pattern, just like what we see in the figure 3.9.

The effect of the market mood term and the level effect term can also be taken as representing two group of investors: bubble followers and fundamentalists, respectively. Bubble followers follow the trend of the movement of price, while fundamentalists always know the fair prices, they will take action against bubble followers and make the asset prices return to its fair value. If the market power of bubble followers is bigger than the fundamentalists, the momentum of prices will be sustained, vice versa. If we interpret the bubble pricing equation in this way, we can see that the simulations with different λs are equivalent to simulations with different combination of bubble followers and fundamentalists.

3.7 Conclusion

This paper suggests that we should be very careful when interpreting the results of GA applications in the financial markets. Although GA is particularly suitable to serve as a platform to model the evolution of the investor's beliefs, it does not necessarily mean that a self-referential process can be formed between the market information and investor's actions. In addition, the strength of the self-referential process may also depend on the parameters of the model. In experiment 1, the strategy performance measure is solely determined by exogenously generated returns, so that investor's activities is irrelevant and GA purely functions as a numerical method to find the optimal solution. In experiment 2 and 3, although the assets' returns are endogenous, they are still irrelevant in searching of the optimal consumption strategy, due to the separation property of the performance measure function.

Experiment 4 shows that if investors' put equal weight to the past information and have full memory, then the investors' investment decisions will converge to the rational expectations equilibrium level, even though the assets' returns are endogenous and relevant.

However the dynamics of the model before the stationary equilibrium is reached is substantially different between the cases where investor's interactions matter and do not matter. Investors' mood effect can self-validating by generating subsequent returns to confirm their mood. Its strength is weighted by investors' patient parameter λ. When λ is large, the self-referential process is strong and the market share can persistently deviate from the equilibrium level. This implies that the model can generate returns significantly different from the equilibrium returns during these period (see chapter 1). However, when λ is small, the self-referential process will become very weak, investors' interaction become trivial. Therefore the market share cannot display persistent deviation and the implied realized returns will be similar to the equilibrium returns.

Comparing experiment 3 and 4, we can see that the weight put on the past information in evaluating performance measure is quite important. Although it is not done in this paper, it is easy to show that if investors put heavy weight to the recent information, then investors will behave like short memory investors, and the stationary equilibrium will not be reached.

As a conclusion, if we are interested in the behavior of the GA model before the stationary equilibrium is reached, it is crucial to study whether investors' beliefs matter or not.

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