IMAGE-BASED REASONING IN GEOMETRY

by

Kerry Handscomb

B.Sc. University of London, 1982

B.Ed. University of British Columbia, 2001

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NAME Kerry Handscomb

DEGREE Master of Science

TITLE Image-Based Reasoning in Geometry

EXAMINING COMMITTEE:

Chair Thomas O'Shea

____
Stephen Campbell, Assistant Professor
Senior Supervisor

____
Peter Liljedahl, Assistant Professor
Member

____
Allan MacKinnon, Faculty of Education
Examiner

Date August 9, 2005

ABSTRACT

The goal is to investigate image-based reasoning in school geometry. A theoretical framework is proposed, consisting of a model for conceptualization of image data, four principles of conceptualization, and five specific geometrical skills. An explication of the van Hiele levels in terms of this framework confirms its validity; an interpretation of common errors in geometrical reasoning demonstrates its descriptive utility.

It is argued that an axiomatic treatment of geometry in secondary schools is inappropriate, and therefore geometric arguments in school must utilize image data. Use of image data brings into question the generality of geometric results. The proposed framework is applied to understanding this issue. A pitfall of allowing image-based reasoning is that there must be strict limits on permissible conceptualizations from image data.

The method of this study is theoretical. However, the theoretical model can be used to frame specific research questions in a future empirical study.

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TABLE OF CONTENTS

LIST OF FIGURES

1 Introduction

This thesis concerns the relationship between image and concept, which constitutes one of the great grand problems of the Western intellectual tradition. A comprehensive treatment of this topic would be over-ambitious. Thus, throughout this thesis I delimit the scope of my treatment of this general topic in various ways. My focus here is restricted to outlining a theoretical framework for image-based reasoning in geometry, along with some of the contexts that motivate and constrain it, and to some practical educational implications thereof.

Throughout this thesis *image-based reasoning* will refer to the use in geometrical reasoning of data obtained directly from images. The goal is to understand the necessity, implications, and pitfalls of image-based reasoning in geometry in secondary schools. To achieve this end, it was necessary to investigate various themes in cognitive science, the history of geometry, and mathematics education.

It was found that the existing literature did not provide an adequate framework for achieving the goal of understanding image-based reasoning. In consequence, a new theoretical framework is proposed. This introduction briefly discusses the sources referred to and outlines the development and application of the new model.

1.1 Concept formation in mathematics education

Geometry, at least in secondary schools, is concerned with the investigation of certain spatial properties of objects. These objects may be concrete or imagined, and either way they are usually represented as lines on paper, the geometry diagram. Properties may be given in advance, determined by inspection of the diagram, or established by logical deduction.

These two types of cognitive structure, the diagram and its properties, image and concept, are the raw material for geometrical thinking. The first task of the thesis is to investigate the notions of image and concept and the relationship between them.

Kosslyn (1980, 1983) defined a *percept* as a mental representation of a perceived stimulus and an *image* as mental representation having the characteristics of a percept in the absence of the appropriate visual stimulation. In geometrical reasoning both types of representation are necessary, and both are referred to herein by the single term *image*.

For purposes of this study, a *concept* is manifested by a mental representation consisting of a string of words. Anderson (1978) points out that there may be many surface representations of a single concept, just as there are synonyms and flexible word orders in human languages. It is preferable for *concept* to mean the stable, underlying mental representation rather than the variable surface representation.

Two important theories are discussed of how mental representations are stored in the brain. Paivio (1971) is the originator of *dual-code theory*, which assumes two separate formats for storage of mental representations, visual images and verbal concepts.

According to Anderson (1980), there is evidence that both verbal and visual data are stored in forms that capture the overall gist of the meaning rather than the exact utterance or the exact image. Moreover, it is clearly possible for the mind to translate easily between the two formats.

Factors such as these inspired Pylyshyn's (1973) *propositional theory*, which claims that there is a deep mental structure underlying both images and concepts, and which acts as a common language between the two. According to Pylyshyn, such a common language is essential, for otherwise there would need to be an infinite number of connections between any given concept and all of its particular instantiations.

Anderson (1978) contends that the translation from one representation to another would require a third, intermediate representation, and then the translation from this third representation to one of the first two would require still another representation, and so on. According to Anderson, this regression argument invalidates propositional theory.

A basic mechanism for concept construction in geometry is Piaget's *empirical abstraction*, as described in Dubinsky (1991) and Mitchelmore (2002). A child is shown a number of images of "triangle" and thereafter abstracts the concept "triangle." In *generalization* the domain of a concept is extended. The various forms of generalization are discussed in Mitchelmore (2002). Researchers have identified other forms of abstraction and generalization, but a proper investigation of this complex subject is beyond the scope of this study.

More sophisticated ways of describing the mechanism of concept formation are Sfard's (1991) model of *interiorization-condensation-reification* and the *APOS* model, summarized in Dubinsky (1997) and described more fully in Czarnocha, Dubinsky, Prabhu, and Vidakovic (1999). In both models the starting point for the construction of new conceptual objects is action rather than perception. Tall (1999) critiques the application of action-based models of concept formation to geometry.

The theories of concept formation discussed so far are constructivist. Winsløw (2000) offers a different perspective. He proposes an alternative mechanism for the acquisition of mathematical knowledge, assuming that mathematical concepts are hardwired into the human brain.

The purpose of this investigation is not to judge between the various theories of image and concept in cognitive psychology. Rather, the aim is a functional description of certain phenomena when image data are permitted in geometrical reasoning. A major inspiration in this regard was the work of Fischbein (1993), which comes close to the type of theoretical framework this thesis is reaching for. The prior discussion utilizes many of Fischbein's own references, and is included to provide at least some indication of how to think about these crucial notions of image and concept. The investigation of concept formation finishes with a discussion of Fischbein (1993), which leads naturally into the rest of the thesis.

Fischbein (1993) proposed that image and concept were indivisibly linked in geometrical reasoning, the two together forming a unitary third entity called a *figural concept*. Fischbein and Nachlieli (1998) enriched the notion with further research.

Fischbein (1993) requires his concepts to exist within a formal, axiomatic framework. However, the characteristic feature of image-based reasoning is that it allows informal, non-axiomatic geometrical arguments, and it is necessary, therefore, to allow concepts to exist outside of the formal framework. Secondly, Fischbein does not clearly specify that the image-concept pair should allow flexible conceptualization of image data.

A paradigm example of this desirable flexibility is the image of a square, which may be conceptualized as a rhombus, a rectangle, a parallelogram, and a quadrilateral, to name but a few possibilities. It was a clarification of this second concern in particular that precipitated the proposed theoretical framework.

1.2 Theoretical framework

A *conceptualization* of an image is defined as a set of properties that can be said to be true of the image. As indicated above, with the example of the square, there are typically many possible conceptualizations of an image. Other possibilities for the square are opposite sides parallel, four sides equal, four angles equal, and so on. In order to clarify the interplay of image and concept, four principles of conceptualization were formulated. These are not formal, rigorous principles, but are a functional description of what is possible given a normal reasoning process.

When image-based reasoning is allowed in geometry, a major dilemma is that the image is particular, although any propositions the geometer should wish to prove are general in nature. The framework is applied to resolving this issue. The solution depends on a proper understanding of the Greek schematic diagram and the new notion of schematic case. It was inspired by the work of Netz (1999, 2004), although the new model seems to provide greater clarity.

1.3 Historical overview of image-based reasoning

The proposed framework is used as a lens through which to view the status of image-based reasoning in geometry at various points in the history of geometry. Netz

(1999) demonstrates that the Greeks considered diagrams to be essential to geometrical proofs, and they certainly allowed properties to be conceptualized from the diagram. Greek geometry is the paradigm case of image-based reasoning as a deductive science. According to Netz, the types of properties that can be uploaded reliably from images precisely determine the proper *schematic* interpretation of Greek geometrical diagrams.

Other sources for Greek geometry that the reader may be interested to pursue are Friedman (2000) on the constructible nature of Greek geometrical objects, Heath (1908/1956) and Joyce (1998) on Euclid's *Elements*, Klein (1934-6/1968) on the non-existence of variables in Greek mathematics, and Mueller (1981) on the issue of generality in Greek geometry.

After Netz (1999), the second major historical reference is Greaves (2002). His thesis is that the role of the diagram in geometrical reasoning at any given point in history depends on what is regarded at that time as the ultimate subject matter of geometry. Greaves discusses the history of the diagram from this perspective, from Euclid to Hilbert.

Fowler (1999) clarifies the non-arithmetical nature of Greek geometry, in the sense that Greek geometry did not utilize fixed units for measurement. Additional types of conceptualization are permitted in arithmetized geometry, although it is no longer a deductive science concerned with proving general results. Fowler characterizes Descartes' analytic geometry as the culmination of the arithmetization of geometry begun with Ptolemy, Heron, and Diophantus.

Aside from Netz's (1999) historical evidence, the necessity of image based-reasoning in Greek geometry could also be inferred from the work of Pasch on the inadequacy of the Euclid's axioms for formal reasoning (Greaves, 2002). Hilbert (1899/1971) subsequently developed a complete axiomatization of Euclid's geometry. The importance of the diagram reaches its nadir with Hilbert, in which image-based reasoning is completely disallowed. According to Greaves (2002), deduction in axiomatic geometry is entirely mechanical; there is no ultimate subject matter of axiomatic geometry.

Freudenthal (1973) argues passionately against axiomatic geometry in secondary schools, in favour of a more intuitive approach. Zeitler (1991) surveys axiomatic systems for school geometry and concludes, with Freudenthal, that concrete, intuitive geometry is preferable to axiomatic geometry in school.

1.4 Psychology of image-based reasoning

Curriculum guidelines from British Columbia, Canada, the United Kingdom, and the United States demonstrate that deductive geometry without axioms is prescribed in these three cases.

I then give an example of a typical school geometry proof, from Alexander and Kelly (1998), that uses diagrammatic inferences. This proof is interpreted and analyzed in detail in terms of the proposed framework. The analysis identifies five specific geometrical skills. These five skills greatly increase the descriptive power of the framework and provide a logical, developmental model for geometrical reasoning.

1.5 The van Hiele Levels

The van Hiele levels are an influential didactical theory of geometry. According to van Hiele, students move through distinct, discrete learning stages in geometry, from recognizing simple geometrical figures to formal deductive reasoning.

Primary sources for the van Hiele levels are Van Hiele and van Hiele-Geldof (1958), the works collected in Fuys, Geddes, and Tischler (1984), and van Hiele (1986). A comprehensive secondary source is Fuys, Geddes, and Tischler (1988).

According to Wirszup (1976), the van Hiele levels had a significant impact on Soviet mathematics education in the 1960's and 1970's. It was Wirszup who finally brought the van Hiele levels to the attention of educators in the United States, and subsequently three major research studies were begun, as described by Hoffer (1983). The results of these studies are contained in Usiskin (1982), Burger and Shaughnessy (1986), and Fuys et al. (1988).

Although the theory of van Hiele seems logical and coherent in its original presentation, there are additional factors to consider. Mayberry (1983) contains results that appear to contradict the van Hiele assumption of students moving progressively through discrete levels; Mason (1997) demonstrates that gifted students may skip van Hiele levels; and Clements and Battista (1992) cite findings that connect the van Hiele levels to Piagetian stages of development, indicating that the van Hiele levels should not be considered purely a didactical theory.

The five skills previously identified are compared with the van Hiele levels, using the detailed descriptors from Fuys et al. (1988). It is found that there is indeed an approximate correspondence between the two models, although each van Hiele level covers a broader spectrum of abilities. In the absence of empirical research, the comparison of the two approaches provides a partial validation of the theoretical framework developed in this thesis.

1.6 Pitfalls of image-based reasoning

Further validation is supplied by the ability of the new theoretical model to explain many errors in geometrical reasoning in a simple, comprehensive fashion. These particular types of errors can be characterized as the pitfalls of image-based reasoning.

When image-based reasoning is allowed, there must be limits placed on the permitted inferences from the diagram in order for arguments to make sense. These limits are the schematic properties of the theoretical framework. Many mistakes in geometrical reasoning can be understood to result from non-schematic conceptualizations from the diagram. Descriptions of student errors from Robinson (1976), Fischbein (1993), and Fischbein and Nachlieli (1998) are used to support this notion.

In the first case, a student may assume by appearance alone that a given figure is a square, for example, when it is really a more general quadrilateral. Students need explicit instruction in what they are allowed to infer from the diagram in order to avoid this type of error. Indeed, the pre-deductive measurement geometry taught in schools may mitigate against an appreciation of the schematic diagram.

An even more fundamental case of mistaken diagrammatic inference occurs when a student's understanding of a concept is insufficiently general. For example, a square may not be recognized as such if it is presented standing on a corner. It may be supposed that the orientation of the figure, an incidental element in the student's conceptualization of the diagram, does not match the student's concept of square. Imprecise concept formation in the elementary grades may be at fault.

1.7 Reflection and critique

As mentioned above, the goal throughout the investigation is to understand the necessity, implications, and pitfalls of image-based reasoning, and the proposed theoretical framework goes some way towards achieving this end. A more complete future presentation would further clarify the image-concept foundations of the model and more rigorously define schematic cases and schematic properties. Some areas for future research include a broader geographical investigation of geometry curricula, a study of textbook practice, and a review of computer-aided geometry instruction. More importantly, the theoretical framework may be used to formulate specific questions for empirical research.

1.8 The property lattice

The theoretical model developed emphasizes the conceptualization of properties from image data. Interestingly, an algebraic structure can be imposed on the various conceptualizations of a given image. This structure, the property lattice, is discussed in the appendix. It is separated thus from the main body of the argument because it is somewhat peripheral. Nevertheless, it is an analytical application of the conceptualization model that may interest some readers.

2 Concept Formation in Mathematics Education

Geometrical thinking involves the contemplation and manipulation of spatial images and the conceptual representations that can be held to be true of them. The first step in investigating this process is to clarify the notions of image and concept.

2.1 Images and percepts

The content of thought will be called a *mental representation.* This study will not attempt to describe mental representations at the neural level. The concern will be simply to indicate a way of thinking about mental representations at the functional level and to draw some distinctions.

The notion of mental representation covers the whole gamut of human conscious and unconscious mental activity, including perceptions, ideas, feelings, dreams, language, and so on. For the purposes of investigating geometrical thinking, three types of mental representations are relevant: percepts, images, and concepts. Concerning percepts and images, the focus can be narrowed still further to mental representations that appear to have a visual, spatial content. Thus the apparent sensual content of representations involving other modalities such as hearing or touch is rejected.

A *percept* is defined as "the representation of a perceived stimulus" (Kosslyn, 1983, p. 72). Metaphorically, "we can think of percepts as being projected onto the mental matrix directly from the ocular camera" (ibid., p. 91). Since the percept exists as a representation in the mind, mental operations can adduce properties of the percept such as colours, shapes, and so on. It should be noted that the percept existing in the mind's eye is not the same as the perceived object existing external to the body. In fact, all the observer can ever know of the external object is what he or she can suppose from the mental representation, and the mental representation is facilitated by a jumble of firing neurons in the brain. The external world will remain forever unknowable.

The impenetrable veil separating the external world from human inspection is a view held by philosophers such as Locke and Kant (Wikipedia, 2005, Philosophical ideas about perception, para. 2). It corresponds to the notion of *indirect perception.* Despite the simplicity and clarity of the concept of indirect perception, it seems to entail a paradox. In order to view the content of the screen in the mind's eye, one needs to imagine a little man inside one's head who observes the screen. But then he must have a screen inside his head, and a still smaller little man observing this screen. The process can continue in infinite regression. This absurdity is known as the homunculus problem, after the little man, or alternatively as Ryle's Regress. It is overcome by noting that the percept is not another object that can be viewed through the faculty of sight. Instead, it is a neural structure that only behaves as if the observer "sees" it. The illusion is created by mental functions that make distance comparisons, light intensity comparisons, and so on between different parts of the neural structure. At the neural level the brain has a functional resemblance only to a projection onto a screen. The infinite regress halts, therefore, at the level of the neural architecture of the brain (Kosslyn, 1983, pp. 22-25).

An *image* is defined as a mental representation that "gives rise to the experience of 'seeing' in the absence of the appropriate visual stimulation from the eyes" (Kosslyn, (1983, p. 29). As with percepts, it is possible to think of images as being projected onto the mental matrix, but in this case the projected data has been stored in the mind. In some cases images may have a close resemblance to physically existing objects, as when we recall the face of a loved one, but it is not at all necessary that images have corresponding physical instantiations external to the body—dreams, hallucinations, and creative imaginings are all examples of the latter.

According to Kosslyn (1983, pp. 73-75), images and percepts involve similar mental structures and processes. Hallucinations, dreams, and so on indicate that an image can be mistaken for a percept; moreover, experimental evidence demonstrates that the reverse is true, that percepts can be mistaken for images. The uniform way in which percepts and images can both be treated as functionally similar mental representations is a powerful argument in favour of indirect perception.

It is natural to ask how images and percepts differ, aside from their origin. According to Kosslyn (1983, p. 91), percepts are fixed by the input to the eyes, reflecting the relatively stable reality around us. On the other hand, "images are mutable, at the mercy of the full range of our powers of fantasy" (ibid., p. 91). Parts can be added or deleted from images or images can be transformed in any number of ways, including rotations, dilations, changes of perspective, colour, and so on.

When a geometrical situation is given in verbal form, it may be necessary to envisage a corresponding image in order to provide intuitive input for the reasoning process. This image may be transferred to an external medium, such as a diagram on paper, in order to stabilize it. Further constructions may be visualized and added to the diagram in order to solve a problem, or various transformations may be imagined. In particular, it may be necessary to focus on just part of an image and then to regard this subset of the original image as an image in its own right.

Sometimes, therefore, a stable diagrammatic percept is needed for the reasoning process, and sometimes creative imagining is necessary. Geometrical reasoning involving diagrams, therefore, straddles the perceptual and imaging functions of the brain. This study will use the term *image* consistently for a geometrical diagram, with the understanding that it could be referring to a percept, an image, or both, depending on the context.

2.2 Concepts

Aristotle had the following comments to make about images: "Now we have already discussed imagination in the treatise *On the Soul* and we have concluded there that thought is impossible without an image" (*On Memory and Recollection*, 450a 5, as cited in Kosslyn, 1980, p. 441); "Memory, even the memory of concepts, does not take place without an image" (ibid., 450a 7, p. 441). Can it be true that cognitive activity cannot take place without images?

Bishop Berkeley pointed out that images are of necessity particular—an image of a dog, for example, is a specific breed with a certain size and shape (Kosslyn, 1983, p. 6). It could be supposed that the mind uses a particular image to represent a general concept, as, for example, geometry diagrams do for general propositions. In this case, however, it is not clear which properties of the particular case are supposed to be general and which are particular. In addition, it raises the question of how the brain encodes the information of this particular case given the fact that the particular case has already been earmarked to represent the general concept. (These two points about particular images representing general concepts do not originate in this study, but I cannot trace their origin.)

The mind needs a way of representing abstract, general concepts. Images will not do, but words are available to perform this function. "Dog" does not refer to a particular animal and "square" does not refer to a particular shape. Instead, these words refer to whole classes of objects that share certain similarities. Define *concept* to be a mental representation that manifests in the form of language. A concept is by its nature abstract and general. Examples of concepts in mathematics are "square," "$2+3=5$," and "$y=2x+1$." Note that of course some strings of words will be nonsensical and will not represent concepts.

In natural human languages there are often many ways of saying almost the same thing, such as "John talked to Sarah," "Sarah was talked to by John," "John spoke to Sarah," and so on. It may be supposed that all the many equivalent formulations in natural language will correspond to just one underlying conceptual representation. Anderson (1978, p. 250) describes this feature as "invariance under paraphrase." The situation is analogous to the many particular triangle images corresponding to the one triangle concept.

Although the language of mathematics is more precise than natural language, there are still synonyms and different ways of formulating mathematical statements while preserving the same meaning. A simple example is the equivalence of the two statements "$x=y$" and "$y=x$." As with natural language, it can be supposed that the class of mathematical statements carrying the same meaning will correspond to just one underlying conceptual representation.

Strictly speaking, the term *concept* will refer to the underlying mental representation rather than one of its corresponding surface statements. However, this study will usually refer to the statement itself as the concept.

Humans manipulate concepts by means of deductive reasoning. For example, provided with the statements "All students like geometry" and "Jane is a student," we can deduce the additional third statement "Jane likes geometry." Whether the conclusion is true or not is immaterial where the validity of the deduction is concerned. Modern, axiomatic geometry is entirely concerned with mechanical deduction (Greaves, 2002. p.74).

2.3 The link between image and concept

Images and concepts are two ways the mind stores geometrical information. Which of the two, image or concept, has primacy? Kosslyn (1983, p. 5) points out that language is learned, and the supposition of verbal, conceptual primacy involves a paradox, since the first words must be learned somehow. On the other hand, as has been noted, images are poor carriers of conceptual information.

This study briefly discusses two theories of how mental representations are stored in the brain. The *dual-code theory* of Paivio (1971) maintains that representations are stored in two formats, visual images and verbal concepts.

There is much evidence that verbal information is not necessarily stored in the memory in exact form; humans are often able, for example, to remember the gist of what someone says rather than the exact utterance (Anderson, 1980, pp. 96-98). Likewise, research has shown that visual data is stored by the brain in a more abstract form that captures the picture's meaning; subjects are more likely, therefore, to remember pictures that they are able to interpret meaningfully (ibid., pp. 98-101). Moreover, it is clearly possible to translate image data into verbal data and vice versa. In the first case, we describe what we see in words; in the second case, we draw a diagram to represent a geometrical statement.

These factors inspired the *propositional theory* of Pylyshyn (1973), which claims that there is a deeper, propositional structure underlying both images and concepts and that this propositional structure acts as a common language between the two types of representation. He makes the point that dual-code theory assumes that the only modes of mental representation possible are images and words because these are the only modes of representation consciously available to introspection. There is no reason, however, to exclude the possibility of more abstract mental structures to which humans do not have conscious access. Pylyshyn argues that cognition may be mediated by something quite different from words or pictures.

An image of a square will elicit the word "square." According to Pylyshyn (1973, p. 5), if the connection between image and concept were direct, then there would need to be an infinite number of such connections because of the infinite number of possible squares. The infinitude of connections is cited by Pylyshyn as an argument in favour of an intermediate *propositional representation*, into which the image of the square is translated first. Likewise, there may be many sentences that contain the same basic gist of meaning. These sentences all correspond, according to Pylyshyn, to a single proposition, which may not even need to be expressed in words at the level of deep structure in the mind. Anderson (1980, pp. 384-386), for example, cites empirical evidence that it is possible for humans to make conceptual distinctions even when they lack the words to describe these distinctions.

Anderson (1978), however, challenges the idea that there is a common representation underlying both images and concepts that acts as a common language for translating between the two. He proposes a variation of the dual-code theory, whereby the gist of meaning contained in images and sentences can be encoded in compound representations where imagal fragments are tied together with verbal connections. This model seems to lack the descriptive power of propositional theory, in which different surface representations with the same meaning would correspond to the same proposition at the deep level—it is unclear to me how this desirable feature would be preserved under Anderson's proposal.

Anderson's (1978, p. 256) argument against the Pylyshyn's "lingua franca" idea is the following:

> To translate from Code 1 to Code 2, it is necessary to translate Code 1 into a new code, Code 3, and then from Code 3 to Code 2. However, this argument leads to an infinite regress. To translate from Code 1 to Code 3, a new Code 4 would be needed and so on.

This argument is not entirely convincing. Throughout human history there have been linguae francae used for communication over wide areas between people with different native languages. Latin, English, and Swahili are obvious examples. There is no reason not to suppose that the brain does not use a similar medium for translating between different forms of mental representation.

It is clear that humans are able to translate between types of mental representations, and propositional theory seems to offer a simple, intuitive model for this process. For discussing geometrical reasoning the surface representations are geometrical figures and mathematical statements. It is unclear to what extent the underlying propositional representations would preserve the features of particularity or generality of these surface representations.

Before moving on, it should be noted that there may be many possible translations of an image to general concept. An image of a square, for example, can be both a rhombus and a parallelogram. These multiple translations may only become apparent at the surface level of the conceptual representation, or they may be implicit in the propositional representation. A structure for the conceptual representations of an image is discussed in the Appendix.

2.4 Origin of concepts

According to Plato, concepts exist in the world of forms. Before birth human souls have knowledge of this world, and the role of the educator is to prompt its recollection. (See, for example, Plato's *Meno*, as cited in Fowler, 1999, pp. 3-7.) Opposed to Platonism is the theory of constructivism. Concepts are not "downloaded" from a world of forms, but are progressively constructed by each individual. I will briefly consider constructivist approaches and will finish with a few comments on an alternative both to Platonism and constructivism.

Suppose that an instructor wishes to teach the concept "square" to a child. Then an obvious, and probably universal, approach is to show the child a number of images of squares explicitly associated with the word "square." At some stage the child will have constructed the conceptual representation "square," possibly mediated by a propositional representation.

According to the philosopher John Locke, *abstraction* is the mental process by which general ideas are generated from particular ideas (Kosslyn, 1983). The model of concept formation just presented is known as *empirical abstraction*, as identified by Piaget (Mitchelmore, 2002, pp. 158-159), whose ideas on abstraction are summarized in Dubinsky (1991). Empirical abstraction refers to the study of particular images to determine their similarities, which are then isolated as general concepts. Many other forms of abstraction have been identified, and it is a complex subject that is beyond the scope of this study to deal with adequately.

Generalization is an idea related to abstraction. Mitchelmore's (2002, p. 160) G2 generalization refers to extending the domain of an existing concept, and this simple definition seems appropriate for the purposes of this study. In the context of geometry, generalization would correspond to the child being able to identify certain shapes as triangles spontaneously, even those triangles that differ somewhat from the finite number of cases prompting the original empirical abstraction.

It is common for concept formation to occur imperfectly. Thus, if the only examples of triangles shown to the child are acute-angled, then this incidental property may be abstracted mistakenly as an essential component of the triangle concept, and the child may not be able to generalize "triangle" to obtuse-angled triangles.

In Sfard's (1991) model of concept formation there are two ways of looking at mathematical concepts: the *structural* and *operational* perspectives. Paradigm examples are the natural numbers and functions. A structural conception of a natural number would define it as the class of all sets of the same finite cardinality, whereas an operational conception would regard it as being reached from 0 by the process of repeatedly adding 1; a structural conception of a function would regard it as a set of ordered pairs, whereas an operational conception would regard it as a "machine" that outputs a number for every input number.

The bulk of Sfard's (1991) argument deals with arithmetic and algebra rather than geometry. It seems that an operational interpretation of geometrical concepts lacks the appropriateness of a structural interpretation. Sfard herself acknowledges, "Some kinds of inner representations fit one type of conception better than the other" (ibid., p. 7).

There is, according to Sfard (1991, pp. 18-20), a natural order of concept formation from the purely operational to the purely structural. First comes *interiorization*, in which the student performs the process on already familiar objects, reaching the stage of performing the process mentally without having to carry it out concretely. Secondly, *condensation* refers to the stage at which the process is turned into an autonomous entity, and the student is able to think of the process as a unified whole. Lastly, *reification* occurs when the whole process is seen as a new object. Reification, therefore, corresponds to the formation of structural concepts.

APOS theory is a similar model of concept formation, except that it identifies action prior to process, and ends with schema anterior to conceptualizing structure. APOS theory is discussed in Czarnocha et al. (1999), and the following description is a paraphrase of the summary in Dubinsky (1997).

APOS stands for *action-process-object-schema*. The *action* consists of a physical or mental transformation of a physical or mental object according to some external instructions. Again, the paradigm example is that of the natural numbers, where the *action* corresponds to counting a collection of items, reaching, for example, 4 beans.

When the student can perform the action mentally, or think about it without performing it down to the last detail, then the action has been interiorized to a *process*. Using the example of natural numbers, the student is able to operate efficiently by counting any collection of items, and the identity of the items themselves should slip into the background. Instead of 4 beans, for example, it could be 4 anything, and the action of counting becomes detached from the items themselves.

When the student can see the process as a totality, then the student has *encapsulated* the process to an *object*. Note that even after the process has been encapsulated, the student must be able to operate with the process or the object, as required. The number 4 now exists, for example, as a concept, even though the student, of course, is still able to count 4 items. Encapsualtion is essentially the same as Sfard's (1991) reification.

Finally, the student will group together a coherent collection of actions, processes, and objects into a *schema* for the concept in question. In other words, the number 4 is packaged along with the other natural numbers and counting actions and processes into the natural numbers schema. Note that alongside the development of the concepts of the numbers themselves, operations such as addition, subtraction, and so on, will have developed from primitive to more abstract, complete forms. These operations, too, will be part of the schema.

As was mentioned above, it seems less natural for geometrical concepts to be formed beginning with action or process. Ways can be proposed, of course, for geometrical concepts to be encapsulated with the APOS model. Tall (1999, p. 113) quotes as an example the action of a space transformation that preserves the integrity of a geometrical object. The permanent object can arise by means of encapsulating this set of space transformations. Surely this is not how humans really arrive at the concept of "square." Likewise, the step-by-step construction of a triangle by means of adjoining line segments seems to be a poor substitute for immediate Gestalt recognition.

According to Tall (1999), "Dubinsky and his co-workers have made an impressive effort to formulate everything in action-process-object language. However, the urge to place this sequence to the fore leads to a description that, to me, soon becomes over-prescriptive" (ibid., p. 113). He continues, "The brain observes objects, and what seem to be primitive mathematical and logical concepts in ready-made brain modules. This seriously questions a rigid Action-Process-Object-Schema strategy in every curriculum" (ibid., p. 114). Rhetorically, he asks, "Is it providing a service to necessary diversity in human thought by restricting the learning sequence to one format of building mathematical actions, mathematical processes and mathematical objects?" (ibid., p. 117).

In the three models of concept formation discussed, the simple object-based empirical abstraction model, Sfard's reification, and APOS, the constructivist perspective on mathematical ideas is tacitly assumed. As Winsløw (2000) points out, however, there *is* a common body of mathematical knowledge; each individual does seem to reach the same conclusions when constructing a mathematical understanding; and mathematical truths do appear to be common to all cultures, though the methods of presentation may be very different. Moreover, many mathematicians really believe in the permanence and objective truth of mathematical entities, even while professing formalism (ibid., p. 21, n. 3).

Winsløw (2000, pp. 14-15) summarizes Chomsky's theory of transformational grammar that at a deep structural level all human languages are similar, and there is a universal grammar that is hard-wired into the human brain. If this were not so, according to Chomsky, then natural languages could not be learned by exposure to a finite number of examples of grammatically correct statements.

Winsløw (2000) argues that the acquisition of mathematical knowledge is analogous to the acquisition of language. How can it be that the vast body of mathematical knowledge, applicable to countless physical and social problems, is constructed by individuals on the basis of sensory data and a few simple processes such as counting? Just as humans possess a Chomskyan language acquisition device (LAD), they may also possess a Winsløwan mathematics acquisition device (MAD), and Winsløw gives an outline of how such a MAD should work.

The notion of a mathematics acquisition device has the ring of plausibility. Language, after all, is a relatively new development in human evolutionary terms, whereas animals have had to survive in a Euclidean spatial environment before the first fish crawled from the sea. (I assume, without justification, that human and non-human animals really do inhabit a subjectively Euclidean universe.)

Assimilating geometrical ideas may consist in making connections from imagal representations on the one hand and conceptual representations on the other to innate propositional representations. In fact, in order to accomplish the empirical abstraction necessary for geometry, there must be a faculty whereby humans are able to recognize the similarities between particular images. The philosopher David Hume proposed nearly 300 years ago that this faculty, the principle of association of ideas, was indeed innate.

I have reviewed some of the theories of image and concept and the origin of concepts. However, the aim of the investigation is not to judge between the various theories in cognitive psychology. Instead, a model that is functional rather than foundational is required. Fischbein (1993) supplies an interesting functional model for image-based reasoning. It inspired the theoretical framework discussed later. In addition, many of Fischbein's own references on image and concept have been utilized above. Fischbein's ideas, therefore, have been pivotal in the development of this thesis.

2.5 Fischbein's figural concepts

According to Fischbein (1993), geometers work with image and concept. The image is the result of perception, an intuitive, visual instantiation of a geometric object, typically a geometric diagram. The concept, on the other hand, is a conceptual representation of the same geometric object, consisting of properties "imposed by, or derived from definitions in the realm of a certain axiomatic system" (ibid., p. 141). Fischbein's notion of concept assumes that concepts exist in a formal framework. I do not think this is a necessary assumption, and the theoretical framework developed herein will differ from Fischbein on this point.

Fischbein (1993) claims that both image and concept are necessary for geometrical reasoning. He exemplifies this with what can be identified as a variation of Pappus' proof of Euclid's Proposition 5 from Book I of the *Elements* (Heath, 1908/1956, Vol. 1, pp. 251-255). Also known as the pons asinorum, this theorem states that the angles opposite equal sides of a triangle are themselves equal. Accordingly, an imagined copy of the original triangle is taken and reversed; it can be seen that the new triangle will fit perfectly over the original triangle, demonstrating the required equality of angles.

The image is necessary to the argument, according to Fischbein (1993), because the conceptual triangle cannot be copied and reversed; on the other hand, the conceptual representation is necessary in order to maintain the rigor and generality of the argument.

The particular type of proof chosen by Fischbein (1993) to illustrate his position is known as proof by superposition. As we shall see later, proof by superposition is dubious from a formal perspective, which brings into question its rigor and generality. Even Euclid, who uses proof by superposition elsewhere, preferred a longer, more complex proof of the pons asinorum. Perhaps a different example would more successfully illustrate Fischbein's point. On the other hand, the proof by superposition will work perfectly well if informal reasoning is sufficient.

According to Fischbein (1993), image and concept together form a hybrid third type of entity: "The objects of investigation and manipulation in geometrical reasoning are then mental entities, called by us *figural concepts*, which reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual properties—like ideality, abstractness, generality, perfection" (p. 143, author's italics).

As the dual perspective of image and concept is developed in the proposed theoretical framework, there is interplay between the visual and verbal characteristics of cognition. Indeed, image and concept are inextricably linked as the geometer continually conceptualizes aspects of imagal data. The figural concept approach, on the other hand, does not emphasize this crucial feature of image-based reasoning. For example, the geometer may wish to further the argument by conceptualizing any number of possible properties from the image of a square, such "rhombus," "right angle," "parallel sides," and so on.

Fischbein (1993) interprets some difficulties that students have with geometry as an imperfect fusion of image and concept. For example, a square cannot be seen as a parallelogram because the dominance of the Gestalt image of the square in the figural concept prevents access to the conceptual component of the figural concept that would allow conceptualization of "parallelogram." By applying the model of this thesis, however, this difficulty can be interpreted as a lack of flexibility in conceptualization caused primarily by insufficiently varied instantiations of "parallelogram" that led to the initial empirical abstraction of the concept.

It is interesting to note how Fischbein (1993) handles incidental properties:

> I do not intend to affirm that the representation we have in mind, when imagining a geometrical figure, is devoid of any sensorial quality (like color) except space properties. But I affirm that, while operating with a geometrical figure, we act *as if no other quality counts*. (p. 143, author's italics)

This perspective is none other than Aristotle's *qua* operator, as described by Lear (1982), and discussed later in more detail.

The figural concept, above all, captures the necessary interdependence of conceptual data and image data for geometrical reasoning. It was the starting point for the proposed theoretical framework, which differs from Fischbein (1993) in two main points: firstly, concepts do not require a formal framework; secondly, conceptual interpretations of image data may flexibly change during the course of geometrical reasoning.

3 Theoretical Framework

The core and main contribution of this thesis is a new theoretical framework presented below. It uses the terms *image* and *concept* from the previous discussion, it assumes they are linked, and it assumes that there is a mechanism for translating from one to the other. The details of this mechanism and the exact nature of image and concept will not be dealt with beyond the comments previously made.

The main inspiration for this framework was Fischbein's (1993) figural concepts. It was necessary, however, to move beyond Fischbein's ideas in two respects. Firstly, Fischbein treats mathematical concepts as formal constructs, and I seek a more process oriented approach to conceptualization; secondly, flexible conceptual interpretations of image data are permitted, in that the same image, as I shall illustrate in detail below, can be conceptualized in a variety of different ways. In a sense, this new framework can be thought of as working toward a "logic" of image-based reasoning.

3.1 Instantiation and conceptualization

A *conceptualization* of an image is a conceptual representation that is true of the image. The conceptualization will consist of one or more statements and these statements will be referred to as *properties*. An image will usually have many properties, and any subset of these is a conceptualization.

The idea of conceptualizing images has a pedigree going back to the Ancient Greeks. Lear (1982) interprets Aristotle's *qua* operator as follows:

> Let *b* be a [physical object] and let "*b qua F*" signify that *b* is being considered *as an F*. Then a property is said to be true of *b qua F* if and only if *b* is an *F* and its having that property follows of necessity of its being an *F*. (p. 168, author's italics)

In other words, "door qua rectangle" means that the door is being considered as a rectangle. By this means, the mathematician places himself behind a "veil of ignorance" (ibid., p. 168), in which he can say nothing about the properties of *b* that do not follow from necessity of its being an *F*. Those properties of *b* that do follow of necessity are referred to as *essential* properties of *b qua F*, whereas those properties of *b* that do not follow of necessity are *incidental* properties of *b qua F* (ibid., pp. 168-169).

The door considered as a rectangle has four sides because the door is a rectangle and the fact of its being a rectangle necessitates that it has four sides; "four sides" is therefore an essential property. On the other hand, the door considered as a rectangle is only incidentally brown because "brown" is not a property that follows from the fact of its being a rectangle. Use of the *qua* operator is precisely what is meant by conceptualization, where the image is the door and the concept is "rectangle."

Another perspective on conceptualization is Godfrey's (1910) "geometrical 'eye'" (p. 197). He refers to developing students "power of seeing geometrical properties detach themselves from a figure" (ibid., p. 197).

A conceptualization C of image I is shown in Figure 3.1. In this and other similar diagrams throughout this work, lower ovals will represent images, upper ovals will represent conceptualizations, straight arrows will represent establishment of conceptualizations, and curved arrows, when they appear later, will represent deductions.

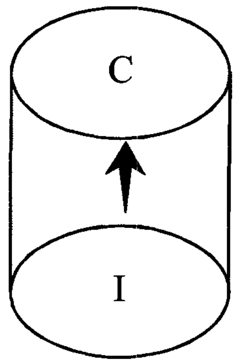


Figure 3.1: **Conceptualization diagram.**

The following four principles of conceptualization are important to note. These principles refer not to a deterministic model of cognition, but what is possible by means of a normal reasoning process. The Appendix will develop this conceptualization framework further in a more analytical direction.

3.1.1 Principle 1 of conceptualization

Suppose that C and D are two conceptualizations of an image I. Then the union of the properties of C and D is also a conceptualization of I.

This principle follows immediately from the definition of conceptualization. It provides a method for combining conceptualizations

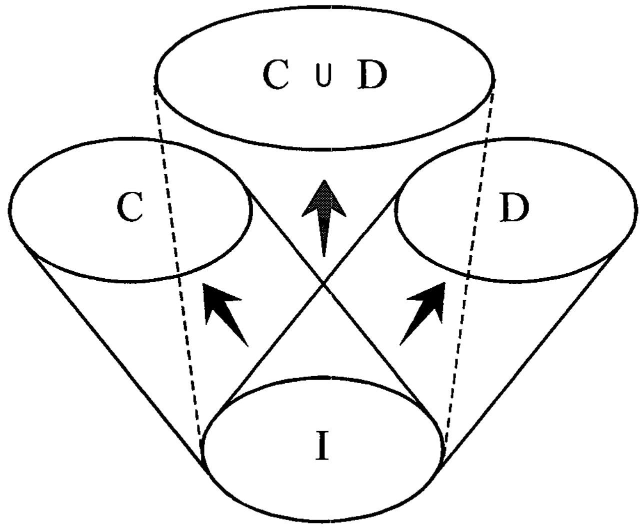


Figure 3.2: Principle 1 of conceptualization.

3.1.2 Principle 2 of conceptualization

If C is a conceptualization of image I, and D is a subset of C, then D is also a conceptualization of I.

This principle also follows immediately from the definition of conceptualization.

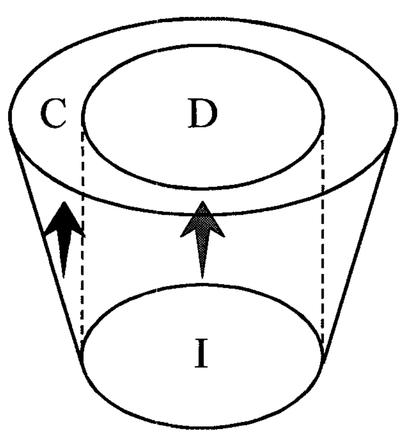


Figure 3.3: Principle 2 of conceptualization.

3.1.3 Principle 3 of conceptualization

If C is a conceptualization of image I, and the properties of D can be deduced from the properties of C, then D is also a conceptualization of I.

The third principle means that any additional properties the geometer deduces from the properties of an image will also be properties of the image. Principle 3 is like a hypothetical syllogism of the form $P \rightarrow Q$ and $Q \rightarrow R$ therefore $P \rightarrow R$, except that "C is a conceptualization of I" is not a logical inference. Principle 3 seems to be such a reasonable supposition, that it will be taken to be self-evidently true for the purposes of this investigation, even though it cannot be rigorously justified.

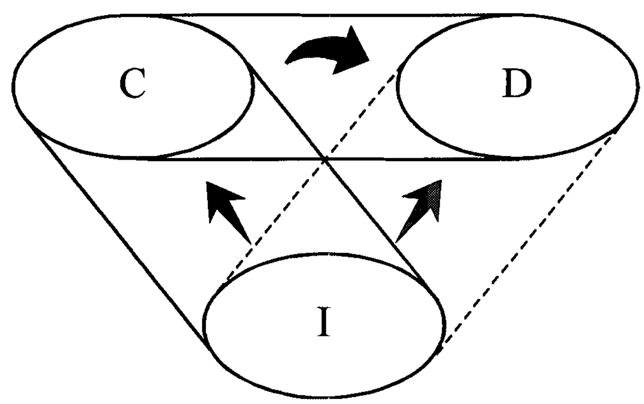


Figure 3.4: Principle 3 of conceptualization.

Principles 1 and 2 can be seen to be special cases of Principle 3. They are discussed separately because they follow immediately from the definition of conceptualization, whereas Principle 3 does not.

3.1.4 Principle 4 of conceptualization

If image I is contained in image J, and C is a conceptualization of I, then C is also a conceptualization of J.

In other words, if certain properties are true of part of an image, then they are true of the whole image. The universal validity of this principle is questionable. However, as will be discussed later, geometrical images can be resolved into finite systems of points, lines, and curves. If a property is true of one such system, then it is surely true of the same system with the addition of a point, line, or curve, and then Principle 4 would follow by induction.

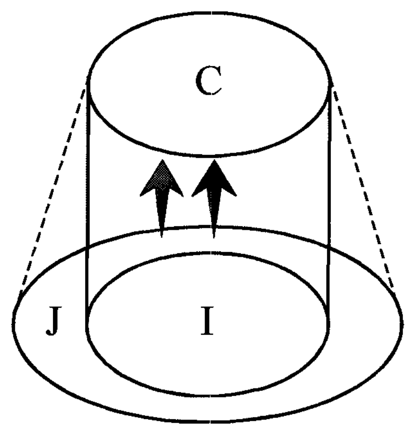


Figure 3.5: Principle 4 of conceptualization.

It is possible to start with a conceptual representation C and then form an image I such that C conceptualizes I. In this case, I is an *instantiation* of C. Thus, given the concept "square," a student may draw a quadrilateral with four equal sides and four equal angles. "Square" conceptualizes the image, and so the image is an instantiation of "square." Conceptualization and instantiation are inverse operations. (Note that the terms “conceptualization” and “instantiation” are certainly not original to this study, but are sprinkled throughout the literature.)

It may occur that an image is given with a list of properties that are supposedly true of the image. Usually these given properties are such that they are not permitted to be conceptualized from the image alone, as will be discussed later. In fact, some properties may even seem untrue on the basis of figural information, particularly if the diagram is not metrically exact, as when a meta-reasoning process such as reductio ad absurdum is to be used.

How do formal notions such as “axiom,” undefined term,” “definition,” and “proposition” fit into the schema being outlined? Euclid’s Postulates 1, 2, and 3 (Heath, 1908/1956, Vol. 1, pp. 195-200) simply govern ways of adding components to a diagram. Thus, Postulate 1, which is present also in Hilbert’s (1899/1971, p. 3) and Birkhoff's (1932, p. 330) formalizations of Euclid’s geometry, states that for any two distinct points there is a straight line that passes through these two points. If image-based reasoning were permitted, this axiom would be a property conceptualized from any diagram containing two points.

The side-angle-side (SAS) congruence theorem cannot be proved in rigorous formulations of geometry, and a weaker version of it becomes an axiom in Hilbert’s (1899/1971, p. 12) system. As explained later, however, the SAS theorem can be demonstrated diagrammatically using proof by superposition. Although in this case imaginative transformation of the diagram is required rather than straightforward conceptualization, the reader may agree that image-based reasoning adequately “proves” the axiom.

Terms that are usually undefined in axiomatic formulations, such as point, line, and so on are Gestalt conceptualizations from the image. Definitions are a shorthand way of referring to groups of Gestalt and relational properties. Thus, a triangle consists of three non-collinear points and the line segments joining the points when taken in pairs.

It is necessary to use terms such as "proposition" or "theorem" whenever geometry is approached deductively, whether on a formal axiomatic basis or on an informal basis allowing diagrammatic conceptualizations. A proposition or theorem simply states that if certain properties are true of an image then certain other properties are also true of the image. Thus, from Principle 3, the geometer can say that these other properties are also a conceptualization of the diagram.

3.2 Schematic cases and generality

When doing geometry, students will often be working from diagrams, usually systems of points, lines, and curves in black ink on white paper. In practical terms, the images will be neither absolutely perfect circles nor absolutely straight lines. It is necessary to consider carefully the allowable conceptualizations from imperfect image data.

3.2.1 Schematic conceptualization and schematic cases

There are many instantiations of the concept "dog"; some of these instantiations have long hair and others have short hair. It is acceptable to conceptualize "long hair" from one of these instantiations, but it is not justified to conclude that all dogs have long hair. But if this were a geometrical example, that is exactly what a geometer would wish to do. The point is that not all instantiations of "dog" have long hair. If the mathematician wishes the conceptualization of an instantiation to be deductively meaningful, then he or she must limit conceptualizations to properties that would be true of all instantiations of the given concept.

Firstly, it is necessary to formulate some conventions to salvage the diagram from complete chaos. Global and local conceptualizations, discussed later, are geometrical reasoning skills. These conventions will enable some fundamental global and local conceptualizations to be made.

The conventions are (a) lines that look reasonably straight must be considered absolutely straight, (b) lines that look curved must be considered to be curves, (c) points must be considered to be dimensionless, (d) lines must be considered to have no thickness, (e) a point that reasonably looks to be on a line segment must be considered actually to be on that line segment, and (f) two line segments that reasonably look as though they meet at a point must be considered really to meet at that point. This list is not supposed to be exhaustive. The idea is that the geometer must be allowed to use the human pattern-recognition faculty to resolve the image into a finite system of points, lines, and curves that are related in spatially obvious ways.

Secondly, it is necessary to eliminate conceptualizations such as orientation, colour, texture, and so on. These incidental properties certainly differ between different instantiations. If the diagram is turned upside down or looked at it in a dim light, they may even vary with the same instantiation. In any case, these kinds of properties are outside the domain of discussion of geometry.

Thirdly, the unavoidable inaccuracy of the image prohibits conceptualization of actual lengths, areas, or angles based on fixed units of measurement. These are referred to as *arithmetical properties*, in the sense of Fowler's (1999, pp. 8-14) "non-arithmetized geometry." Thus given, an image that appears to be a square, it is not permitted in deductive geometry to take a ruler, measure the lengths of the sides, and then use the resulting arithmetical conceptualization in a deductive argument.

Neither is it allowed to conceptualize comparisons of lengths, areas, or angles between features of the diagram. These are referred to as *metrical properties*, in the manner that Netz (1999, p. 18) uses "metric," (although Netz also includes "arithmetical" in his sense of the term.) Thus, given an image that appears to be a square, it is not permitted to assume that the four sides are equal, even though the most accurate ruler would seem to indicate that this is the case.

Metrical conceptualizations include parallel lines because the determination of parallelism within the finite boundaries of the diagram would require a metrical comparison of distances—the geometer cannot follow the lines to infinity to check that they never meet! The property of tangency does not appear to be either arithmetical or metrical, but conceptualization from the diagram is not permitted because unavoidable inaccuracy of the diagram makes it indeterminate whether there are one or two points of intersection of the line and the curve.

The geometer who wishes to conceptualize from the diagram must therefore eliminate incidental, arithmetical, and metrical conceptualizations, because these types of property may vary between particular, imperfect instantiations of a concept. (The geometer would not wish to admit incidental properties even if the diagram were perfect.)

There are two types of allowable conceptualization: global (or Gestalt) properties such as "point," "angle," "triangle," and so on; and local (or relational) properties such as "the point lies on the circle," "the two lines intersect," "the quadrilateral is inscribed in the circle," "the two points can be joined by a straight line," and so on.

These examples of allowed diagrammatic conceptualizations are scattered markers on the boundaries of certitude. I do not know whether a rigorous characterization could be given of all permitted conceptualizations.

Within the domain of geometrical discussion, and provided some reasonable conventions are agreed upon, it can be seen that the permitted conceptualizations are determined by the inability of imperfect diagrams to represent perfect geometrical concepts. This is precisely the Greek understanding of conceptualization of geometrical diagrams, as interpreted by Netz (1999, 2004). Permitted conceptualizations from the diagram are *schematic conceptualizations*, which are composed of *schematic properties*, in the sense that Netz (1999, p. 18) uses the term "schematic." Geometrical situations can involve arithmetical or metrical properties, but these must be given as conceptual representations independently from the diagram.

In accordance with Klein's Erlangen Program (Zeitler, 1990, pp. 20-21), a *schematic transformation* can be defined as a transformation that preserves schematic properties. Schematic transformations, it seems, are similar to projective transformations, except that there is no necessity for the transformation to be globally uniform.

So far the discussion has considered only the necessary conditions on any properties that are allowed to be conceptualized from an instantiation in order for all instantiations of a given concept to have the same set of possible conceptualizations. Are these conditions also sufficient? The short answer is, No. A good way to see this is to look at the different cases of a geometrical proposition. Consider the following: The concept “an angle is subtended at the centre by an arc of a circle, and an angle is inscribed by the same arc at the circumference of the circle” has several different instantiations that look quite different and from which there are different possible conceptualizations, two of which are shown in Figure 3.6.

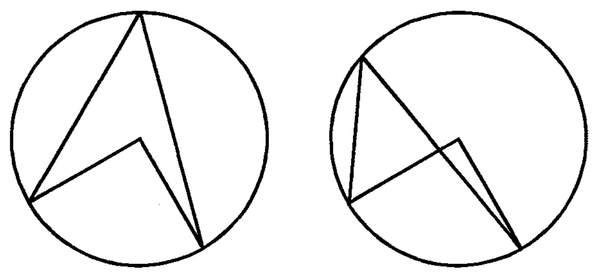


Figure 3.6: Two instantiations that have different conceptualizations.

In every case, the angle subtended at the centre is twice the inscribed angle. Each case requires a different proof because of the different conceptualizations available. Two instantiations of a concept belong to the same *schematic case* if and only if they have the same set of schematic properties.

One final consideration is that components may be added to the instantiation of a concept, making additional schematic properties available, in order to complete a proof. Two instantiations must be considered to belong to the same schematic case provided their *potential* schematic properties also are the same. It seems likely that, within certain constraints, if two instantiations have the same *actual* schematic properties, then they also have the same *potential* schematic properties. A detailed consideration of this issue is beyond the scope of the present study.

3.2.2 Generality

It is important to examine the issue of the generality of geometrical reasoning. After all, if arguments allow concepts uploaded from particular images, how can it be certain that general results are proved? If Fischbein's (1993) figural concepts were the inspiration for the proposed theoretical framework, then a major stimulus behind it was to develop a model within which to understand the issue of generality.

According to Netz (1999, p. 240), the Greeks thought their results were general, and they certainly used image data in their arguments. Netz proposed a solution to the issue of generality in Greek geometry. His analysis motivated the argument below.

It comes down to this: since all the conceptualizations within a given schematic case are identical, it is only necessary to provide one proof for each schematic case in order to guarantee generality.

The difficulty comes when the geometer must identify the various schematic cases. Perhaps the only way of doing this is via spatial intuition—in other words, by playing around with diagrams until the various cases become clear. Another difficulty is that some propositions may require an infinite number of schematic cases, and then it is impossible, without algebra, to provide a general proof.

However, within the constraints necessitated by intuitive verification of a finite number of schematic cases, the geometer can be assured of generality when conceptualizing from image data.

3.2.3 Deducibility

There is another issue. A proposition that is proved by allowing conceptualization of image data may not be provable within an axiomatic system such as Hilbert's (1899/1971). The proposition may use a particular concept uploaded from the diagram that cannot be deduced from the axioms.

If a concept originates in an intuitive domain, there appears to be no way to guarantee its mathematical truth in a formal domain. Things that simply "look true" cannot be squeezed into the narrow constraints of an axiomatic system. Spatial intuition is ultimately irreducible.

The possible non-deducibility of concepts uploaded from diagrams lessens in significance, however, if Gödel's Incompleteness Theorem is taken into account. Gödel showed that there was no axiomatization of basic arithmetic in which every true statement was provable. Surely Euclid's geometry is a more complex structure than basic arithmetic. It seems reasonable to suppose, therefore, that there are indeed true statements in Euclid's geometry that are not provable with *any* axiomatic system. Does it matter, then, if the geometer cannot demonstrate that every proposition proved using concepts uploaded from an image is deducible with Hilbert's (1899/1971) axioms?

The proposed theoretical framework arises out of a desire to understand the functional mechanisms of geometrical reasoning if the geometer is allowed to conceptualize image data. It will provide a foundation for a further discussion of geometrical reasoning from psychological and didactical perspectives. Firstly, however, I wish to look back, through the lens of this theoretical framework, at the genesis of deductive geometry in Ancient Greece.

4 Historical Overview of Image-based Reasoning

The discussion below will utilize the proposed theoretical framework to explicate aspects of the deductive geometry of Ancient Greece. Greek geometry has already been referred to in the investigation of schematic diagrams and generality, in an attempt to relocate some of Netz's (1999, 2004) arguments in a more general context. I will return now to the specific case of Greece, and use this as a launching pad for a historical overview of image-based reasoning. This leads up to Hilbert and a discussion of the necessity of image-based reasoning in school geometry.

4.1 The geometry of Euclid

According to an estimate by Netz (1999, p. 275), it is likely that the first deductive proofs began to appear around 440 BCE. Within 150 years, around the beginning of the third century BCE, Euclid wrote his *Elements*, which can be regarded as the foundation of all mathematics. Two other giants of Greek geometry were Archimedes, active during the middle part of the third century BCE, and Apollonius, active towards the end of the third century BCE. The third century BCE may be regarded as the heyday of Greek deductive mathematics.

Euclid's geometry can be thought of as the first attempt to formalize intuitive understanding of space. Ostensibly, it is a rigorous axiomatic system. From the basis of a set of axioms and definitions a series of propositions are proved by logical deduction.

Despite the genius of the Greek geometers, their axiomatic system was imperfect. It was pointed out by Pasch in 1882 in his *Vorlesungen über Neuere Geometrie* (Greaves, 2002, p. 66; Netz, 1999, p. 27) that the system of postulates and common notions in the *Elements* is inadequate for a logical deduction of all intuitively obvious features of geometrical images. Pasch's Axiom, for example, which cannot be deduced from Euclid's axioms, states, "If a line intersects one side of a triangle and misses the three vertices, then it must intersect one of the other two sides" (Weisstein, 2005b). Pasch's Axiom is immediately available as a conceptualization from the image.

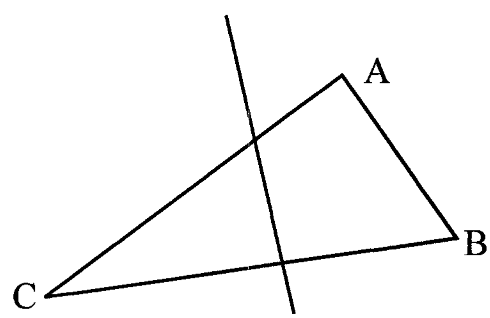


Figure 4.1: Pasch's Axiom.

It is well known, too, that the proof of Euclid's Proposition 1 from Book I, the construction of an equilateral triangle on a given line segment, requires the use of diagrammatic inferences because at least the geometer has to know the two circles of the construction intersect (Heath, 1908/1956, Vol. 1, pp. 241-243).

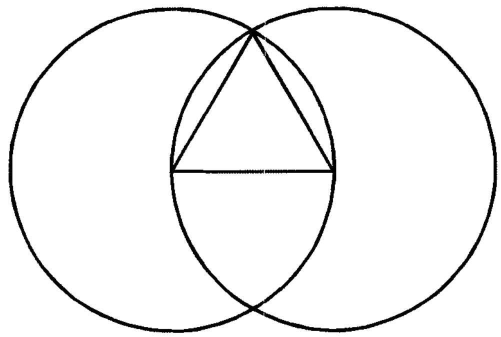


Figure 4.2: Proposition I1 from the *Elements*.

It seems obvious from the diagram that the circles do intersect, but strictly speaking the geometer needs the Continuity Axiom (Weisstein, 2005a).

Other examples where Euclid's reasoning is inadequate by modern standards are the proofs by superposition. According to Joyce (1998, Book I, Proposition 4, The method of superposition, para. 1), superposition occurs three times in the *Elements*: Book I, Proposition 4 (Heath, 1908/1956, Vol. 1, pp. 247-250), Book I, Proposition 8 (ibid., Vol. 1, pp. 261-262), and Book III, Proposition 24 (ibid., Vol. 2, pp. 53-54). In order to show that the two triangles are congruent in Proposition 4 of Book I, one imagines a copy of one superimposed exactly over the other. Greaves (2002, p. 28) argues that Euclid's Postulate 4, all right angles are equal (Heath, 1908/1956, Vol. 1, p. 200), demonstrates that Euclid does not always sanction this kind of maneuver—otherwise why not move one right angle over the other to show equality?

According to Heath (1908/1956, Vol. 1, pp. 224-228), Aristotle, Veronese, Schopenhauer, and Russell are philosophers who had problems with allowing movement in geometry. Greaves (2002, pp. 31-32) suggests that superposition could be reinterpreted to mean constructing another triangle over the first rather than moving the original triangle. Constructing a copy of the triangle is certainly sanctioned by Euclid's ruler and compass methods, but Greaves points out that such exact constructions would seem to rely on arithmetical exactness, and are therefore unreliable.

The geometer cannot defend proof by superposition on the basis that it is a straightforward example of conceptualization of diagrammatic properties. Instead, one *imagines* a new diagram with the two triangles exactly superimposed—the proof is conducted through imaginative manipulation of the diagram. In order to justify the congruence theorems in non-axiomatic geometry, a mechanism such as proof by superposition is essential.

Along with all other types of image-based reasoning, proof by superposition is prohibited by modern standards of rigor. A weaker form of Euclid's Proposition 4 of Book 1, for example, becomes an axiom in Hilbert (1899/1971, p. 12).

There are, therefore, significant gaps in the axiomatic formulation of Euclid. It follows that the Greek geometers would have to use conceptualizations of image data in their arguments. It seems reasonable to suppose that they felt no concern about doing so. I will continue by investigating how the schematic properties and schematic cases discussed previously can be applied to the Greek case.

Firstly, it is necessary to revisit the conventions mentioned for the theoretical framework, which enable the geometer to salvage the diagram from chaos and resolve it into a system of points, lines, and curves. Friedman (2000, p. 186-187) argues that all the objects of Euclid's reasoning are iteratively constructed, with straight edge and compass, by means of Euclid's first three postulates, and are therefore finite systems of points, lines, and circular arcs. It is likely that the Greeks would interpret diagrams in precisely those terms.

Some flexibility must have been allowed. Thus, Netz (2004, p. 115) points out that Archimedes sometimes used curved lines to represent straight lines, as in Figure 4.3, in which a "dodecagon" is inscribed in a circle, presumably to make the sides of the dodecagon more easily visible.

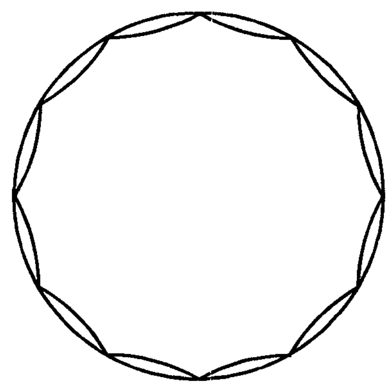


Figure 4.3: Archimedes' "dodecagon" inscribed in a circle.

The permitted schematic conceptualizations of diagrams, as I have already discussed, are precisely determined by the imperfectability of the diagram. Netz (2004, pp. 8-10) confirms the schematic, non-metrical nature of Greek diagrams. It seems reasonable to assume that the analysis of the last chapter corresponds approximately to the Greek view, although the Greeks would not have formulated it in explicit terms.

When metrical information was necessary for a proposition, it was given separately by the Greeks in the text of the proposition. It seems that there were no Greek equivalents of the modern practice of a box to represent a right angle or slashes to denote equality of line segments.

Fowler (1999, pp. 9-10) argues that arithmetical properties were never part of the original geometry of Euclid. Any additional metrical information about a proposition given separately from the diagram could consist only of dimensionless comparisons between objects in the diagram, such as "equal to," "greater than," or a rational ratio of two quantities. Archimedes, for example, in his theorem on the surface area of a sphere, does not give anything like the modern formula, with its irrational π, but instead expresses the surface area of a sphere as four times the area of a great circle (Netz, 2004, p. 144).

Given the schematic nature of Greek diagrams, the arguments for generality of the previous chapter apply, with some clarifications, to the geometry of Euclid. The conclusions, in summary, were as follows: (1) We can assume generality within each schematic case; (2) it is necessary to give a separate proof for each schematic case; (3) the determination of schematic cases must be done visually and intuitively; and (4) the system is inadequate in cases where there are an infinite number of schematic cases.

Netz's (1999, ch. 6) argument for the generality of Greek geometry relies on implicit repeatability of proof for different instantiations of a general situation. If full account is taken of the implications of schematic diagrams, however, and the notion of schematic case is utilized, it seems that repeatability of proof is redundant. The geometer need only supply one proof for each schematic case.

All Greek diagrams are constructible in a finite number of steps. The Greek proof of a proposition would describe the various features of the image in a particular order, and one can imagine the geometer checking for different cases as each feature is added. The cases identified by the Greeks probably would not have corresponded to formal schematic cases, since two diagrams may very well belong to different schematic cases while being identical in conceptualization from the point of view of what is actually needed for the proof.

There are an infinite number of different schematic cases in Archimedes' proposition that the perimeter of a circle inscribed in a polygon is less than the circumference of the circle (Netz, 2004, pp. 41-43). Generality can be achieved for a particular polygon, but in the case of an *n*-gon, with a variable *n*, a single schematic diagram will not do. In fact an infinite number of schematic diagrams is required, one for each value of *n*. Netz (1999, p. 268, n. 59) refers to such diagrams as "doubly particular objects." The only satisfactory way of dealing with situations like this is to use a variable for the number of sides. According to Klein's (1934-6/1968, pp. 122-123) analysis, however, there were no variables in Greek geometry—the objects of investigation were always particular, and only the *method* was general.

According to Mueller (1981, p. 13), the Greeks themselves never provided an answer to the question of generality. It seems probable that the issue of generality did not occur to them. Our speculations, based on the research of Netz (1999, 2004), are simply a retrospective attempt to justify Greek reasoning in the light of modern standards. Does it matter if the Greek style of image-based reasoning lacked rigor? According to Netz (2004), "Archimedes' goal is not axiomatic perfection (where every axiom, and every application of an axiom, must be made explicit), but *truth*" (p. 42, author's italics).

4.2 Arithmetized geometry and the geometry of Descartes

Fowler (1999, p. 172) argues that Greek geometry before the second century BC was completely non-arithmetized, in the sense that no measurements were ever given on the basis of a standard unit. In later centuries, however, with the astronomy of Ptolemy and the mathematics of Heron and Diophantus, Greek geometry was blended with the earlier arithmetical methods from Babylon; these methods were further enriched by Arab astronomers and mathematicians in later centuries (ibid., p. 9). The goal of arithmetized geometry is measurement of particular cases rather than general truths. Arithmetized geometry is not a deductive science proving general results.

In 1638 Rene Descartes built on arithmetized geometry by combining it with algebra to create analytical geometry. Just as algebraic methods transform arithmetic from the study of the particular to the study of the general, analytic geometry transforms arithmetized geometry into a science once again concerned with general properties.

The only axioms required of analytic geometry are those governing the real numbers. We are dealing now with subsets of the real plane that can be identified as the solutions to systems of equations. Geometry becomes manipulation of these equations, and it may properly be regarded as a branch of algebra (Greaves, 2002, p. 39).

Conceptualization of image data is irrelevant for analytic geometry since all conceptual data are developed algebraically from given information. No enumeration of different cases is required because the mechanisms of algebra automatically distinguish between the different cases: the solution of a system of two linear equations either has a single solution or it does not; two circles intersect at 0, 1, or 2 points depending on the number of solutions of a quadratic equation. The variables guarantee generality within the domain of analytic geometry (Greaves, 2002, p. 38).

We should note that the universe of analytic geometry is not Euclid's universe. Freudenthal (1973, pp. 431-432) points out that the algebraic approach leads to what has been referred to as "Euclidean space," in which the automorphisms preserve *distance*. In Euclid's geometry, on the other hand, the automorphisms preserve similarity, or in other words *ratios of distances*. (Because of this ambiguity, the present study is careful to refer to the ancient science as "Euclid's geometry" rather than "Euclidean geometry"; sometimes the term "synthetic geometry" is used for Euclid's geometry to contrast it with analytic geometry. To confuse matters further, Euclid's geometry without axioms will be referred to later as "Euclidean-style geometry.")

Analytic geometry requires a familiarity with basic algebra. This, in itself, limits its applications in school. Freudenthal (1973, pp. 444-445) identified a second, more serious objection to analytic geometry in the classroom: some concepts that are simple and intuitive from the synthetic perspective can seem quite abstruse in analytical geometry. For example, the notion of angle requires some basic trigonometry for its analytical definition, whereas it can be grasped immediately and intuitively in synthetic terms. Freudenthal (1973) writes, "The angle concept is one of the precious gifts of geometry, a gift which should not be refused, a 'transcendent' tool by which extraordinary results are easily obtained, much more easily than by algebraic-analytic methods" (p. 445). Freudenthal makes the point that for analytic geometry to make any sense at all in spatial terms, it is necessary that a student first be acquainted with synthetic geometry in order to be able to reinterpret analytic geometry in synthetic terms. In other words, "angle" would have to be taught first by synthetic means before attempting an analytical approach.

4.3 Poncelet and the Continuity Principle

One of the early nineteenth-century champions of synthetic geometry was French geometer Jean Victor Poncelet. His goal was to provide for the ancient synthetic geometry the same degree of generality that can be achieved through the algebraic methods of analytic geometry.

Poncelet recognized the key function of the schematic nature of Greek diagrams, and he attempted to formalize this notion with his *principle of continuity*. Greaves (2002, p. 45) cites Poncelet as follows:

> The principle of continuity, considered simply from the point of view of geometry, consists in this, that if we suppose a given figure to change its position by having its points undergo a continuous motion without violating the conditions initially assumed to hold between them, the . . . properties which hold for the first position of the figure still hold in a generalized form for all the derived figures.

Poncelet attempted to flesh out the principle of continuity by means of "precisely describing the initial conditions on the diagram and the types of diagrammatic motion which could be guaranteed not to violate them" (Greaves, 2002, p. 46).

According to Greaves (2002, p. 46), Poncelet was unable to accomplish his program with sufficient rigor. Nevertheless, Poncelet is considered to be one of the originators of projective geometry. Schematic transformations can be regarded as a generalization of projective transformations.

4.4 Hilbert's formalization of geometry

Hilbert (1899/1971) devised the first complete axiomatization of Euclidean geometry. In Hilbert's formalization of Euclid's geometry, there is no place for intuitive conceptualization of image data. Formal geometry is mechanical, involving axioms, definitions, and the rules of logical inference.

Greaves' (2002) thesis is the following: "*The possibility of diagrammatic methods in formal proofs . . . in geometry has always been primarily dependent on the characteristics of the metaphysical and ontological theories under which they are carried out*" (pp. 4-5, author's italics). According to the contrapositive of this thesis, the denial of the validity of any diagrammatic inferences implies that there is no ultimate subject matter of geometry in Hilbert's system.

The formal approach dominates mathematics at the university level. The mathematician's attitude, as Freudenthal (1973) puts it, is that the "quicksand of reality is no basis to build a mathematical system; mathematics should be protected against any contamination with non-deductive germs" (p. 403).

In its favour, formal geometry guarantees complete generality within its boundaries. But the formalization does place limits on geometrical truth. As has already been discussed, there is nothing to suggest that every true proposition is deducible from the axioms. Why, therefore, should the geometer follow Hilbert in discounting spatial intuition as a guide to geometrical truth?

Should the perspective of higher mathematics affect curriculum content in secondary schools? If we are to give any credence to the van Hiele levels, discussed later in this thesis, then it is absolutely clear that axiomatic geometry should not and cannot be taught to students unless they first have a strong foundation in spatial, intuitive geometry.

Freudenthal (1973) concurs with van Hiele's analysis that geometrical ideas should be introduced by intuitive means. He claims that it is pointless to introduce geometric objects by definitions, because a definition cannot make sense unless it is known at the outset what is being defined (ibid., pp. 416-418). He argues that the real objective of geometry is grasping the space in which we live and breathe and move (ibid., p. 403). According to Freudenthal,

> Geometry can only be meaningful if it exploits the relation of geometry to the experienced space. If the educator shirks this duty, he throws away an irretrievable chance. Geometry is one of the best opportunities that exists to learn how to mathematize reality. (pp. 406-407)

The reader may agree with Freudenthal (1973, p. 448) that axiomatic geometry is simply too complex for students—there are many axioms to memorize, and often they seem either trivial or obscure. "The whole is so impenetrable that nobody would try working within the axiomatic system; no discoveries can be made within the axiomatic system, and proving propositions is a difficult thing to do" (ibid., p. 448).

Zeitler (1991) analyzes the various axiom systems that could be used for school geometry. He comes to the same conclusion as Freudenthal: "We need less abstraction, less formalism or—better still—no axiomatics" (p. 24).

Given that axioms are inappropriate for school geometry, what exactly is required of students? This topic will be dealt with next.

5 Psychology of Image-based Reasoning

In Chapter Three, I presented a theoretical framework for image-based reasoning in geometry. With this framework in mind, the history of geometry from the origins of image-based reasoning to the formal axiomatization of geometry was explored in Chapter Four. I concluded Chapter Four by arguing that teaching formal axiomatic geometry is inappropriate in a school setting. The first section of this chapter will clarify the kind of geometrical reasoning actually required of students. Then, based on a detailed analysis of a typical school geometrical problem, informed by the theoretical framework presented in Chapter Three, I deconstruct the reasoning required for the problem's solution into a number of specific skills. The geometrical skills identified greatly increase the descriptive power and pedagogical efficacy of the theoretical framework. This is illustrated in Chapter Six by interpreting the classical van Hiele levels pertaining to the development of geometrical reasoning. Then, in Chapter Seven, practical implications of these skills for teaching image-based reasoning in geometry will be explored.

5.1 Geometry in mathematics curricula

The discussion below examines the senior secondary geometry curricula of three organizations. These three were chosen for convenience rather than for their power to represent a variety of geometry teaching. It is hoped, nevertheless, that they can give some general sense of the geometrical knowledge expected of senior secondary school students early in the twenty-first century.

The documents investigated are the following:

- "Shape and Space (3-D Objects and 2-D Shapes)" and "Shape and Space (Measurement)" for Principles of Mathematics 11 in the *Mathematics 10 to 11 Integrated Resource Package* of the Ministry of Education, British Columbia, Canada (Government of British Columbia Ministry of Education [GBCME], 2004a, 2004b)
- "Ma3 Shape, Space and Measures" for Key Stage 4 Higher of the National Curriculum of the Department of Education and Skills, United Kingdom (National Curriculum Online [NCO], n.d.)
- "Geometry Standard for Grades 9-12" in *Principles and Standards for School Mathematics* of the National Council of Teachers of Mathematics of the United States (National Council of Teachers of Mathematics [NCTM], 2004)

The NCTM gives recommendations only, whereas the other two organizations provide mandatory guidelines for the schools within their jurisdictions. GBCME (2004a, 2004b) is specific to Grade 11, whereas NCO (n.d.) covers Grades 10 and 11, and NCTM (2004) gives broad guidelines for Grades 9 to 12.

NCTM (2004) places its expectations under four headings. The first heading is "Analyze characteristics and properties of two- and three-dimensional geometric shapes and *develop mathematical arguments about geometric relationships*" (ibid., 2004, italics added). It covers the traditional Euclidean-style geometry in the curriculum, including congruence and similarity, and students should be able to "establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others" (ibid.). Trigonometry also is included under this heading. The other three headings cover coordinate geometry, transformations, and applications of geometry.

GBCME (2004a) gives a list of theorems. Students are supposed firstly to confirm and apply these theorems by means of geometry software, and secondly to prove them using "established concepts and theorems" (ibid.). These theorems are as follows:

> • the perpendicular bisector of a chord contains the centre of the circle
> • the measure of the central angle is equal to twice the measure of the inscribed angle subtended by the same arc (for the case when the centre of the circle is in the interior of the inscribed angle)
> • the inscribed angles subtended by the same arc are congruent
> • the angle inscribed in a semicircle is a right angle
> • the opposite angles of a cyclic quadrilateral are supplementary
> • a tangent to a circle is perpendicular to the radius at the point of tangency
> • the tangent segments to a circle from any external point are congruent
> • the angle between a tangent and a chord is equal to the inscribed angle on the opposite side of the chord
> • the sum of the interior angles of an n-sided polygon is $180°(n - 2)$.
> (ibid., Prescribed learning outcomes, para. 2)

In addition, students are expected to be able to "solve problems, using a variety of circle properties, and justify the solution strategy used" (ibid., Prescribed learning outcomes, para. 2). Coordinate geometry is included in GBCME (2004b). Measurement, trigonometry, and transformations are not part of the Grade 11 curriculum. The geometry content for Grade 11 in this jurisdiction is entirely, therefore, traditional Euclidean-style geometry (with the aid of technology) plus coordinate geometry. Transformations, trigonometry, and measurement are distributed among the learning expectations for Grades 10 and 12.

NCO (n.d.) is the most detailed of the three documents. Geometry is broken down into four sections. The first covers problem solving, communicating ideas, and general reasoning requirements, and might be equated with the NCTM (2004) applications category. The second section includes traditional Euclidean-style geometry and trigonometry. There is a detailed inventory of all properties of triangles, circles, and other figures that are required. Sometimes the terms "prove" or "understand a proof" are used explicitly of these properties, and at other times terms such as "explain why" or "understand that" are used. Terminology such as this implies that proofs are expected, with varying levels of formality. The third section covers transformations and coordinate geometry; the fourth section comprises measurement and geometrical constructions. The constructions, it should be noted, include traditional Euclidean constructions with straight edge and compasses.

In summary, putting aside possible applications of geometry to other subject areas, it can be seen that the geometry deemed appropriate for senior secondary school students includes five main topics: measurement, trigonometry, coordinate geometry, transformations, and Euclidean-style geometry. Each of these five areas can indeed be seen as the manipulation of properties of visual images, so that each can be interpreted in terms of the image-concept model.

Measurement and trigonometry can be grouped together. Both are concerned with assigning arithmetical values to features of geometrical figures, such as angles, lengths, areas, and volumes. Algebraic and arithmetical manipulations are used to deduce required properties from given properties and known formulae. In lower grades, the student may have to determine approximate values from the diagram by using measuring tools such as a ruler or protractor.

Coordinate (or analytical) geometry requires that the image exists in a plane containing two perpendicular lines, or axes. As with measurement and trigonometry, there needs to be a standard unit, so that coordinate geometry, too, is arithmetical in nature. The genius of coordinate geometry is that the distance of points from the axes can be variable. The properties of the figure, then, are conceptualized in the form of a set of equations, and then these equations are manipulated algebraically to achieve the required result.

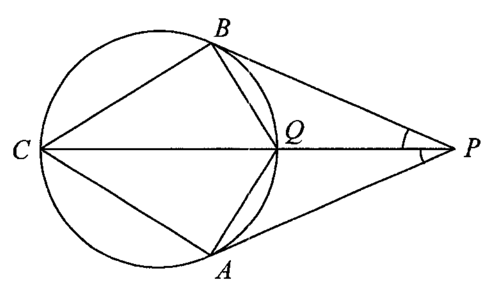
In transformational geometry the property that one figure is the transformation of another figure is represented in functional form, which again relies on a system of coordinates. Transformational geometry, therefore, is another variety of arithmetical geometry, and its conclusions are reached by arithmetical and algebraic manipulations.

All that is left to consider is Euclidean-style synthetic geometry. A close reading of the three documents yields no instances of the term "axiom" or any of its synonyms. Despite references to deduction and proof, therefore, the geometry required is certainly not axiomatic, and it is not even the geometry of Euclid in the sense of Euclid's limited selection of axioms. Note that this study deliberately uses the term "Euclidean-style" for this variety of deductive geometry without axioms.

Geometry as performed by Euclid and his contemporaries does not admit a standard unit of measure, so arithmetical properties are not allowed. Lengths and angles may be compared as ratios with rational values. These have been referred to as metrical properties, which must be given in conceptual form separately from the diagram. The schematic properties of Euclidean-style geometry, on the other hand, may be conceptualized directly from the diagram. This one factor distinguishes Euclidean-style geometry from all other forms of geometry discussed. In consequence, the properties of Euclidean geometry are not generally amenable to arithmetical or algebraic manipulation, and the student is required to use logical deduction. If geometry is regarded as the manipulation of properties conceptualized from images, then Euclidean-style geometry consists of using logical deduction to manipulate schematic and metrical properties.

5.2 Example proof of a geometrical proposition

In order to demonstrate how the image-concept model developed so far works in a full Euclidean-style proof, this study examines a typical student exercise. It is taken from Alexander and Kelly (1998, p. 516), which is one of the two textbooks recommended by the Ministry of Education in British Columbia for their Principles of Mathematics 11 course.



Tangent segments PA and PB are drawn from an external point P to a circle [above]. The bisector of $\angle P$ intersects the circle at Q and C. Prove that $\angle CAQ = 90^0$.

Figure 5.1: Typical problem in deductive geometry.

Note first that the text almost, but not completely, specifies the whole diagram. The construction of line segments CA and AQ is implied, but the statement of the problem makes no mention of line segments CB and BQ, which are not needed for the exercise. Presumably, the added symmetry is supposed to hint at the solution. This omission is similar to the incomplete specification of diagrams identified by Netz (1999, pp. 20-26) in original Greek source material. The small arcs marking $\angle BPQ$ and $\angle APQ$ are not properly part of the diagram, but are a type of shorthand for the metrical property that the two angles are equal.

It would be possible to give this problem without a diagram, leaving it to the student to instantiate the whole situation from the given information. Teaching experience shows, however, that a problem like this without a diagram would be out of reach for any but the most exceptional students.

One may start by randomly conceptualizing properties from the diagram to see what may be deduced. This, however, is like finding one's way to the store by chasing squirrels. One needs to plan a route first, and then set out along that route. It is best to work backwards from the goal.

At this juncture it should be mentioned that the reasoning process is extrapolated from teaching experience and the writer's introspection. This is not a scientific approach for describing student reasoning processes. However, the following discussion is offered as the type of thought process that a student would have to be capable of in order to able to solve this kind of problem. In real life even the most capable student would be unlikely to argue so systematically.

The first observation is that if CQ is a diameter then $\angle CAQ = 90^0$ because "the angle inscribed in a semicircle is a right angle" (Alexander & Kelly, 1998, p. 459).

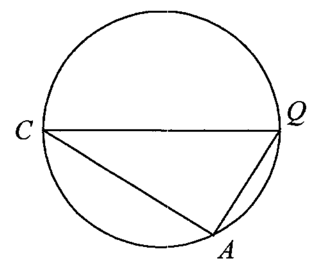


Figure 5.2: Proof (1).

Note that this idea entails focusing on part of the diagram and regarding it as an isolated unit. Implicit is Principle 4 of conceptualization, that any properties that are true of a particular section of the diagram will be true of the whole diagram. Also operative is Principle 3, that if "CQ is a diameter" conceptualizes the diagram, and "CQ is a diameter" implies "$\angle CAQ = 90^0$," then "$\angle CAQ = 90^0$" also conceptualizes the diagram.

The problem reduces, therefore, to showing that CQ is a diameter. "The perpendicular bisector of any chord contains the centre of the circle" (Alexander & Kelly, 1998, p. 441) may occur to an astute student, and by definition a line that passes through the centre of a circle is a diameter. Thus, if CQ can be conceptualized as the perpendicular bisector of a chord, then it will be a diameter, as required. But where is the chord? We have to *instantiate* the concept "CQ is the perpendicular bisector of a chord." The difficult process of creative visualization, linking concept to particular image, may stymie many students. The obvious choice of chords is AB, and so that must be constructed.

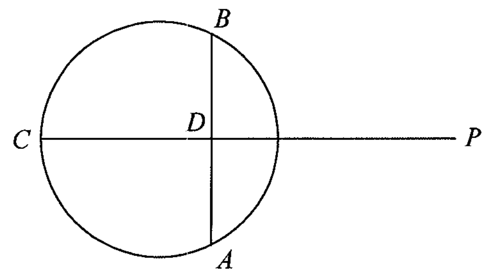


Figure 5.3: Proof (2).

Note that there are some hidden assumptions. The first is that line segment AB actually exists: it is an axiom of formal geometry that any two distinct points can be joined by a straight line. The second is that AB meets CP, which would require an axiom. Secondary school geometry is no longer developed axiomatically, and these assumptions would be taken as self-evidently true based on spatial intuitions. Most teachers would not even draw attention to them. We should point out that again Principle 4 is operative, ensuring that any properties true of the original diagram are also true of the diagram with the additional construction.

Lastly, it is necessary to show that PD is a perpendicular bisector of AB.

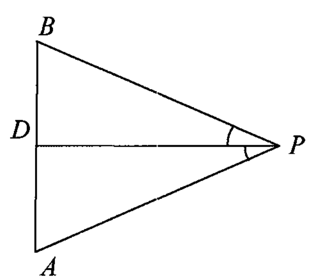


Figure 5.4: Proof (3).

We need to show that $AD = BD$ and that $\angle ADP = \angle BDP = 90^0$. But this will follow if $\Delta APD \cong \Delta BPD$. It is given that $\angle APD = \angle BPD$, and the geometer can conceptualize $DP = DP$. Therefore, if $PA = PB$, then $\Delta APD \cong \Delta BPD$ can be deduced from the SAS theorem. The last (or rather first) link in the deductive chain is showing that $PA = PB$.

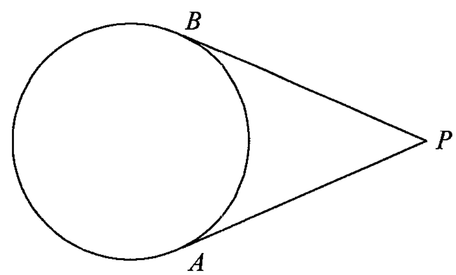


Figure 5.5: Proof (4).

It is given that PA and PB are tangents to the circle, and "the tangent segments to a circle from an external point are equal" (Alexander & Kelly, 1998, p. 490).

Now that the planning process is completed, the whole deductive argument can be put together, starting from the diagram with the additional line segment AB instantiated. The whole process is shown in Figure 5.6.

The argument follows a series of deductions that are local rather than global in that they refer only to isolated parts of the diagram. Each deduction is of the form $R \rightarrow S$. R is a collection of properties that are either (a) given in the statement of the problem, (b) deduced as part of the argument, or (c) inferred directly from the local diagram. An example of the latter is $\angle ADP + \angle BDP = 180^0$. S consists of one or more additional properties that are deduced from the properties of R. The inference often follows as a particular case of a previously proved result.

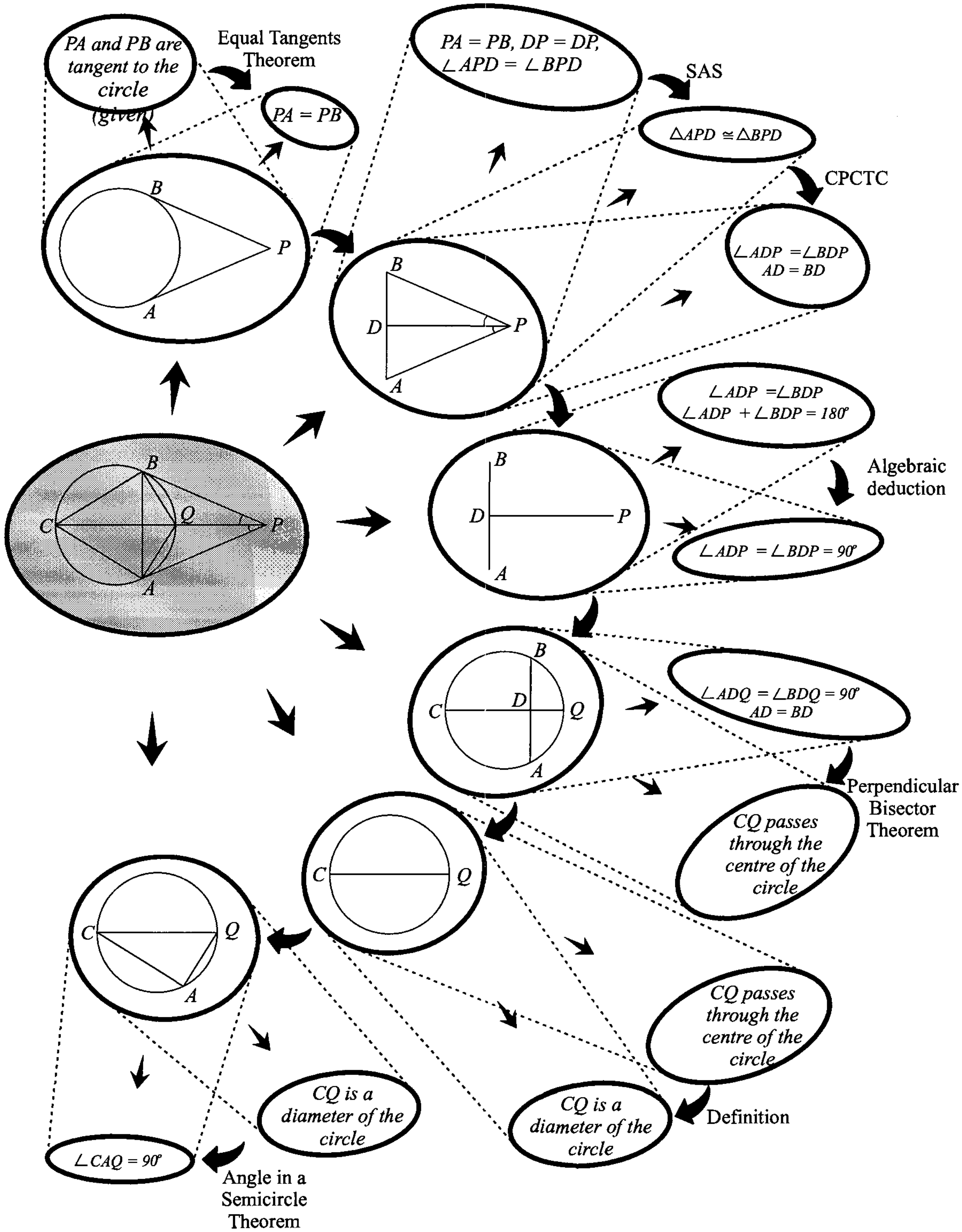


Figure 5.6: Deconstruction of a geometric proof.

However, in the case of "CPCTC" (i.e. "congruent parts of congruent triangles are congruent"), S is a subset of the properties of R since "congruent triangles" is simply a concise way of saying that each pair of corresponding sides and angles is congruent, and Principle 2 is being used. A similar "inference" follows in the case of a definition, where the property is simply restated using different words.

The geometer can apply Principle 3 to justify that any new property is a conceptualization of the locality considered. Then, from Principle 4, it follows that the new property is a conceptualization of the global diagram. Thereafter, the focus can switch to a new part of the diagram for another local deduction. These local deductions are strung together according to a global, or strategic, understanding of the argument. The global perspective includes any additional constructions required—in this case AB—and the overall flow of the argument from locality to locality of the global diagram. Note that the properties for each starting point, R, are collected together from a number of sources specifically for the local deduction. Principle 1 guarantees that the collection of properties is a conceptualization of the local diagram.

It is interesting to note that all of the properties are relational. For example, a line is related to a circle by tangency, two triangles are related by congruence, and so on. These local, or relational, conceptualizations refer to relationships between components of an image. There is another form of conceptualization, without which the local deductive process cannot begin. It is Gestalt, or global, recognition of circles, points, lines, triangles, and so on, which is the raw material for the relations of local conceptualization.

The various components of this complex deductive process have been identified as local and global deduction and local and global conceptualization. If the diagram is not given or if additional constructions have to be added to the diagram, then the imaginative process of instantiation also is required. The discussion turns now to a more detailed investigation of each of these components of the deductive process, followed by an analysis how these ideas relate to an influential model of the development of geometric reasoning, the van Hiele levels.

5.3 Geometrical skills

How can a student achieve the ability to prove geometric propositions? No educator would claim that formal geometric deduction could be taught successfully as an introduction to geometry. Geometric deduction must be approached in stages. There are two issues in this respect, developmental and didactical. From the developmental perspective, the student's perceptual and representational notions of space have to be Euclidean, as described in Piaget and Inhelder (1948/1963), and the following discussion will assume that this is the case.

Even if a student's cognitive development is such that the student is theoretically capable of geometric proof, going straight to global deduction would probably perplex even the most able student. Therefore, certain didactical considerations should be borne in mind.

The levels developed by van Hiele and van Hiele-Geldof are an influential didactical theory of how geometrical understanding can be developed efficiently. These levels will be considered later. For now, the discussion will focus on the specific skills required for formal geometric proof.

5.3.1 Global conceptualization

The ability to assign properties to an image as a whole is referred to as *global conceptualization.* In terms of geometry, this simply means identification of common shapes. A student perceives the image of a square, for example, and conceptualizes it as a square because "it looks like a square." The student has not analyzed the components of the image to determine deductively it is "square" because of it has four equal sides and four equal angles. The result of global conceptualization can be referred to as a global property.

Global conceptualization corresponds to the earliest training that a child receives in Euclidean-style geometry, where the child learns to assign names to common shapes through the process of empirical abstraction and generalization described earlier. The universe of the diagram is grasped holistically and assigned a global property. Thereafter, the student may generalize spontaneously to identify other shapes as satisfying this property.

Examples of global conceptualization are all the usual geometric shapes, such as triangle, square, line, point, circle, and so on. Of course there are many other possible global conceptualizations, and the only limitation is the ability to name images.

In Euclidean-style geometry some global conceptualizations can be made from the diagram, others have to be given conceptually in addition to the diagram or deduced from sufficient properties. It is permissible, for example, because of the diagrammatic conventions discussed earlier, to conceptualize “line,” “point,” and “curve.” It is not allowed to conceptualize “square” simply because the image looks like a square. This property would have to be given in addition to the diagram, or it could be deduced from its being a quadrilateral with four equal sides and four equal angles.

Note that it may be possible to assign more than one global property to an image. Thus, a square may be globally conceptualized also as a rectangle, a rhombus, a parallelogram, a quadrilateral, a polygon, or simply as a geometrical shape.

Primitive shapes such as points and lines are often components of larger objects such as triangles and quadrilaterals, but they may still be regarded as global properties provided they are conceptualized as geometrical objects in themselves. In the same way, geometrical diagrams may consist of more than one triangle, quadrilateral, and so on, each of which has a global conceptualization. Some compound diagrams may even have their own global conceptualizations. For example, a certain arrangement of five interlocking circles may be conceptualized immediately as “Olympic symbol,” without consideration of the individual circles. Some commonly occurring shapes in the environment do have informal names, even if they lack formal designations in geometry. For example, the shape consisting of two rectangles, one centered inside the other, may be thought of as a “frame,” even though “frame” is not an established geometrical term.

The reader may wonder whether an image can be conceptualized without a conceptual statement. Perhaps the object may be encoded at a deep cognitive level in the sense of Pylyshyn’s (1973) propositional theory, even though there is no surface conceptual representation. After all, Anderson (1980, pp. 384-386) cites evidence that conceptual distinctions can be made without the words to describe them. It is beyond the scope of the discussion to deal properly with this issue.

Looking back to the sample proof in Figure 5.6, many global conceptualizations are made during the course of the proof. Circles, lines, points, and triangles have to be recognized as such in order to provide objects that can be related in a deductive structure.

5.3.2 Local conceptualization

A relationship between components of an image is referred to as *local conceptualization.* For example, the properties having four equal sides and four equal angles may be conceptualized from the image of a square—the relationship of equality holds between four different line segments and four different angles, respectively.

Equalities of length, angle, area and volume are metrical properties. More generally, Greek geometry allows proportions involving rational ratios of these quantities. Modern geometry is less concerned with the arithmetic of proportions, but allows equations utilizing the basic arithmetical operations.

Arithmetical properties that measure lengths, areas, and volumes according to standard units are not part of Euclid's geometry. The one exception to this rule is the use of a right angle as a standard unit for measuring angles, and a statement such as $\theta = 60^\circ$ may be interpreted to be a rational ratio of a right angle.

Informal geometry allows direct conceptualization of metrical and arithmetical properties, either by observation or by using measuring tools. In Euclidean-style geometry, however, metrical properties must either be given separately in conceptual form or deduced logically.

Various non-metrical relationships can form the basis for local conceptualizations. For example, one object may be a component, or subset, of another object. Thus, the property of a particular line segment being the side of a square is a local conceptualization. More generally, two objects may have a common subset, so that "two lines intersect at a point" is a local conceptualization. Other examples of this type are "a line passes through the centre of a circle," "a line intersects a circle at two distinct points," "a polygon has three distinct sides," and so on. The last two examples imply the ability to enumerate the instances of a global conceptualization, resulting in a local conceptualization.

These non-metrical relationships are schematic properties, in that they may be uploaded directly from the diagram. The property of a line being tangent to a curve is unusual in that it is non-metrical and non-schematic. The property of tangency must be given conceptually.

This forgoing discussion of possible local conceptualizations is not supposed to be exhaustive. A full examination of all possibilities is beyond the scope of the present study.

All of the conceptual statements in Figure 5.6 are local conceptualizations. Local conceptualizations are the basis of deductive geometry.

5.3.3 Local deduction

An inference from one conceptualization to another is referred to as a *local deduction.* Local deductions may appear in a number of forms, as in the following examples:

Definition. The definition of a square is a quadrilateral with equal sides and equal angles. Therefore, the concept of a quadrilateral with equal sides and equal angles implies the concept of a square.

Elimination. The concept of a square implies four equal sides and four equal angles. Likewise, the concept of congruent triangles implies corresponding pairs of equal sides and equal angles. Essentially, "square" and "congruent triangles" can be thought of as a shorthand notation for a bundle of local properties. A deduction can be made by eliminating some properties from the conceptual representation and keeping others. The CPCTC deduction, for example, is an application of Principle 2.

Algebraic deduction. Metrical relationships may be manipulated by algebraic methods. The perpendicularity of AB and DP in Figure 5.6 is an example.

Proposition. As a result of a previous global deduction, it is known that a certain collection of properties implies some other property. For example, the concept of a point on the perpendicular bisector of a line segment implies the point is equidistant from the endpoints of the line segment.

5.3.4 Global deduction

When local deductions are strung together according to a strategic plan, the result is a *global deduction.* Global deduction implies the ability to prove geometric propositions. An example of global deduction is the complete proof of Figure 5.6.

There is a creative element in global deduction that makes it difficult to grasp and difficult to teach. Typically, a student may have to work backwards from the goal in order to envisage the whole proof. Additional constructions may even be required in order to create conceptualizations to bridge from one part of the proof to another. Line segment BA in the proof of Figure 5.6 is an example of such a construction.

In general terms, it can be seen that global deduction is a relation between local deductions; likewise, local deduction is a relation between local conceptualizations; and in turn, local conceptualization is a relation between global conceptualizations. The model can therefore be regarded as a logical, developmental framework.

5.3.5 Instantiation

In the discussion so far of the various abilities required for a geometrical proof no mention has been made of word problems, in which a student is confronted with a problem set out verbally, without a diagram. The student must produce a diagram by instantiating the concepts present in the verbal presentation of the problem.

Mathematics teachers are aware that many students are reluctant to produce diagrams and are fearful of word problems. Does this fear and reluctance correspond to inability? It seems that instantiation of a concept involves creating an image whose conceptualization is the original concept. Thus instantiation is the inverse of conceptualization. Inverse operations are often difficult to perform. For example, long division is harder than long multiplication and extracting square roots from numbers is harder than squaring numbers. Perhaps the same is true of instantiation.

We distinguished two types of conceptualization, global and local. For the corresponding distinction, "global instantiation" would refer to being able to produce the Gestalt image of a square, whereas "local instantiation" would refer to producing an image representing, for example, "AB is perpendicular to CD."

A square may be immediately instantiated in the mind as a Gestalt whole, and this would correspond to "global instantiation." However, actually producing a concrete image of a square would involve first drawing one side, then another, and so on; as each side is drawn, it must be oriented perpendicularly to the previous side and be the same length. Thus, "global instantiation" of the concrete image of a square of necessity assumes "local instantiation."

Instantiation can be involved as part of a proof, as it was with the construction of line segment AB in Figure 5.6. Additional constructions like these are necessary when a global deduction has the form A→B→C, but concept B is not fully instantiated.

Clearly, creative imagination is involved in instantiation. A through investigation of the phenomenon is beyond the scope of this study.

5.4 Concluding remarks on geometrical reasoning skills

It can be seen that the geometrical reasoning skills identified in this chapter add increased granularity to the framework. In what follows, I will show how these skills cohere with the descriptive power of the van Hiele levels, and also have practical implications with regard to students' geometrical thinking.

I have mentioned above that the deconstruction of the reasoning process required for the solution of a typical geometrical problem, and the resulting geometrical skill set identified in this chapter is based on introspection and analysis of my teaching experience rather than empirical research. The validity of the analysis can be tested, however, by taking an established model for geometrical reasoning and explicating it in terms of the new theoretical framework. The most influential, established model is the van Hiele levels, which will be discussed next.

6 The van Hiele Levels

There are at least four ways to consider the development of geometrical knowledge. The first is historical, and would lead through the arithmetical calculations of Babylon and Egypt, to the deduction of the Greeks, and thence to Descartes, Hilbert, and so on. The second way is to regard geometry as a complete, mature subject, and thence to describe its logical development from first principles. A third approach is to consider development of human cognition from early childhood as the mind becomes capable of increasingly sophisticated spatial processing. This third aspect is dealt with in Piaget and Inhelder (1948/1963); a summary of Piaget's ideas on the development of spatial intuition is given in Smock (1976). Lastly, the didactic approach assumes that the student's cognitive development has already reached the level at which the student is capable of processing complex geometrical information, and considers the most efficient way of achieving actual student facility with geometrical thinking. This latter approach is epitomized by the van Hiele model, which details the ordered stages in which a student must acquire geometrical knowledge in order to be able to apply this knowledge in an insightful way. I will explicate the van Hiele levels in terms of the proposed theoretical framework.

The following summary of the van Hiele levels is taken from Hoffer (1983, p. 207, author's italics):

Level 0:	Students recognize figures by their global appearance. They can say *triangle*, *square*, *cube*, and so forth, but they do not explicitly identify properties of figures.
Level 1:	Students analyze properties of figures: "rectangles have equal diagonals" and "a rhombus has all sides equal," but they do not explicitly interrelate figures or properties.
Level 2:	Students relate figures and their properties: "every square is a rectangle," but they do not organize sequences of statements to justify observations.
Level 3:	Students develop sequences of statements to deduce one statement from another, such as showing how the parallel postulate implies that the angle sum of a triangle is equal to 180^{o}. However, they do not recognize the need for rigor not do they understand relationships between other deductive systems.
Level 4:	Students analyze various deductive systems with a high degree of rigor comparable to Hilbert's approach to the foundations of geometry. They understand such properties of a deductive system as consistency, independence, and completeness of the postulates.

It should be noted that Level 0 is sometimes referred to the Base Level, and that in his later writing Pierre van Hiele changed the numbering of the levels from 1 to 5 to emphasize the importance of the visual level (van Hiele, 1986). Most of the literature, however, refers to them as 0 to 4, and this is the practice this study will follow here for consistency. It should be noted that the idea of stages of development in geometrical thinking goes back at least to Godfrey (1910, p. 200).

In 1957 Pierre van Hiele and Dina van Hiele-Geldof completed dissertations at the University of Utrecht. Pierre van Hiele's work (summarized in Fuys et al., 1984, pp. 237-241) deals with "insight" as a goal of learning, specifically with the example of geometry, and levels of thinking. Dina van Hiele's dissertation (translated in Fuys et al., 1984, pp. 1-206) describes a didactic experiment based on raising the level of students' geometric reasoning. Pierre van Hiele gave a speech at a conference in Sèvres in 1959, which led to the paper "La pensée de l'enfant et la géometrie" (translated as "The Child's

Thought and Geometry" in Fuys et al., 1984, pp. 243-252). Soviet mathematics educators subsequently became interested in the van Hiele levels, resulting in further research and significant changes to the Soviet mathematics curriculum. The impact of the van Hiele levels on Soviet mathematics education is summarized in Wirszup (1976). Hoffer (1983) also is a good reference for the early history of research into the van Hiele ideas. Aside from Freudenthal (1973), however, little interest was shown in the van Hiele levels in the West before the 1980's. Finally, two decades after the original work, it was Izaak Wirszup's lecture at the Closing General Session of the NCTM, and his subsequent paper (Wirszup, 1976) that brought the van Hiele ideas to the attention of educators in the United States. Three major research studies were begun in the United States in 1979; the results of these studies are reported in Usiskin (1982), Burger and Shaughnessy (1986), and Fuys et al. (1988).

A convenient location for many primary sources on the van Hiele model is Fuys et al. (1984). Other primary sources are van Hiele and van Hiele-Geldof (1958) and van Hiele (1986). Secondary sources for van Hiele research are Mayberry (1983), who finds that students may operate at different levels for different concepts; Mason (1997), who claims that gifted students may skip van Hiele levels; and Clements and Battista (1992), who cite findings indicating that the van Hiele levels involve cognitive developmental factors as well as didactical factors.

The most comprehensive explication of the levels themselves, however, is contained in Fuys et al. (1988). The authors identified descriptors for the van Hiele levels, which can be thought of ways in which the levels manifest themselves in student thinking. We will look at each descriptor for Levels 0 to 2, and interpret it in terms of the skills described in the previous section. The descriptors were produced through an analysis of original van Hiele sources on the levels, together with input from Pierre van Hiele and other prominent researchers on the van Hiele levels.

> Level 0 Descriptors
> The student
> 1. identifies instances of a shape by its appearance as a whole
>   a. in a simple drawing, diagram or set of cut-outs.
>   b. in different positions.
>   c. in a shape or other more complex configuration.
> 2. constructs, draws, or copies a shape.
> 3. names or labels shapes and other geometric configurations and uses standard and/or nonstandard names and labels appropriately.
> 4. compares and sorts shapes on the basis of their appearance as a whole.
> 5. verbally describes shapes by their appearance as a whole
>
> (Fuys et al., 1988, pp. 58-59)

Aside from 2, which refers to instantiation, these first five descriptors for Level 0 are simply instances of global conceptualization. These descriptors make it explicit that global conceptualization of a figure should occur in different circumstances and that concept formation is verbal in nature. Moreover, the global concepts can be used to sort objects.

> 6. solves routine problems by operating on shapes rather than by using properties which apply in general.
>
> (ibid. p. 59)

An example of this descriptor given by the authors is the use of concrete manipulatives, such as straight edges, to verify that the sides of a particular parallelogram do not meet and are therefore parallel. This descriptor provides a global conceptualization of a figure (parallelogram) and a local conceptualization (the relation of parallelism between opposite sides), and an apparent implication between the two. However, since the implication is concerned only with a particular case, local deduction is not present from the global concept “parallelogram” to the relational concept “opposite sides parallel.”

> 7. identifies parts of a figure but
> a. does *not* analyze a figure in terms of its components.
> b. does *not* think of properties as characterizing a class of figures.
> c. does *not* make generalizations about shapes or use related language.
> (ibid., p. 59, authors’ emphasis)

Clearly this descriptor refers to global conceptualization when part of a figure is conceptualized with no relationship to the whole. The authors give as an example of this descriptor students measuring the angles of a square to verify that they are all 90°. A byproduct of this process is the local conceptualization that all angles are equal. Again, because of the particularity of the student experiment, local deduction is not present.

In conclusion, the main thrust of Level 0 seems to be global conceptualization, although some local conceptualization is present. Particularity precludes deduction of any type. Perhaps the paradigm Level 0 activity is given by Descriptor 4, sorting figures into classes by overall appearance.

> Level 1 Descriptors
> The student
> 1. identifies and tests relationships among components of figures (e.g., congruence of opposite sides of a parallelogram; congruence of angles in a tiling pattern).
> 2. recalls and uses appropriate vocabulary for components and relationships (e.g., opposite sides, corresponding angles are congruent, diagonals bisect each other).
> (Fuys et al., 1988, p. 60)

The example given by the authors for the first descriptor is of a student spontaneously noting from the image of a square that it has four sides equal and four angles equal. Local conceptualization is present, as it global conceptualization. Any local deduction is informal, as the student is not yet deducing properties from the formal definition of a square. The second descriptor implies increasing familiarity with the conceptual language.

> 3. a. compares two shapes according to relationships among their components.
> b. sorts shapes in different ways according to certain properties, including a sort of all instances of a class from non-instances.
> (ibid., p. 60)

Shapes at Level 0 could be sorted according to their global conceptualization. This descriptor implies now that a student may sort shapes according to possible local conceptualizations.

> 4. a. interprets and uses verbal descriptions of a figure in terms of its properties and uses this description to draw/construct the figure.
> b. interprets verbal or symbolic statements of rules and applies them.
> (ibid., pp. 60-61)

Instantiation of conceptual information is implied by this descriptor. It differs from Descriptor 2 in Level 0 in that shapes are instantiated based on their properties. This form of instantiation is therefore the inverse of local conceptualization.

> 5. discovers properties of specific figures empirically and generalizes properties for that class of figures.
> (ibid., p. 61)

Global and local conceptualizations are present. Unlike Descriptor 6 of Level 0, it can also be concluded that an informal local deduction is present because of the generalization.

> 6. a. describes a class of figures (e.g. parallelograms) in terms of its properties.
> b. tells what shape a figure is, given certain properties.
> (ibid., p. 61)

This descriptor implies an informal local deduction of necessary properties satisfied by a geometrical figure. There is also an informal local deduction from sufficient local properties to the global conceptualization of the figure. As implied by Descriptor 10 below, the deduction from sufficient conditions would be based on guesswork rather than a formal definition.

> 7. identifies which properties used to characterize one class of figures also apply to another class of figures and compares classes of figures according to their properties.
>
> (ibid., p. 62)

A student may note that both squares and rhombuses have four equal sides without making the deduction that all squares are rhombuses. This appears to be another form of Descriptor 3 above, where shapes are sorted according to possible local conceptualizations.

> 8. discovers properties of an unfamiliar class of figures.
>
> (ibid., p. 62)

This is simply local conceptualization.

> 9. solves geometric problems by using known properties of figures or by insightful approaches. (ibid., p. 62)

An example given by the authors is showing the line connecting the centres of two circles of equal radius is perpendicular to the line segment joining their points of intersection. Imaginative instantiation of a quadrilateral is necessary, the figure formed by joining the centres of the circles and their points of intersection. Then a necessary property of the circles is that all radii are equal, thence these equal line segments are a sufficient condition for a rhombus. Lastly a necessary property of a rhombus is that its diagonals are perpendicular. Thus global and local conceptualizations are necessary as is local and even global deduction and instantiation. Certainly an example such as has reached the limits of Level 1 thinking.

10. formulates and uses generalizations about properties of figures (guided by teacher/material or spontaneously on own) and uses related language (e.g. all, every, none) but

a. does *not* explain how certain properties of a figure are interrelated.
b. does *not* formulate and use formal definitions.
c. does *not* explain subclass relationships beyond checking specific instances against given list of properties.
d. does *not* see a need for proof or logical explanations of generalizations discovered empirically and does *not* use related language (e.g., if-then, because) correctly.

(ibid., p. 63, authors' emphasis)

This descriptor places limits on the deductions indicated by Descriptor 9. The fact of a quadrilateral being a rhombus because its sides are equal is an informal deduction rather than an inference from a formal definition. Only by an empirical investigation rather than a formal proof can a student explain why a rhombus has perpendicular diagonals. The fact of all squares being rhombuses could not be deduced rigorously, and would have to be based on a comparison of sets of properties of particular cases, as specified in Descriptor 3.

The important feature of Level 1 thinking is that local conceptualization supplies properties of a particular figure, and then generalization extends these properties to a whole class of figures. The generalization is informal rather than being based on a formal definition, and student reasoning is not yet properly deductive. We can say in broad terms that the primary feature of Level 1 is local conceptualization. The paradigm Level 1 activity is given by Descriptor 5, determining the properties of a whole class of figures.

Level 2 Descriptors
The student

1. a. identifies different sets of properties that characterize a class of figures and tests that these are sufficient.
b. identifies minimum sets of properties that can characterize a figure.
c. formulates and uses a definition for a class of figures.

(Fuys et al., 1988, p. 64)

This descriptor applies to the process of establishing a formal definition. Formal definitions mean that real deductive arguments are possible.

> 2. gives informal arguments (using diagrams, cutout shapes that are folded, or other materials).
>   a. having drawn a conclusion from given information, justifies the conclusion using logical relationships.
>   b. orders classes of shapes
>   c. orders two properties
>   d. discovers new properties by deduction.
>   e. interrelates several properties in a family tree.
>
> (ibid., pp. 64-66)

Descriptors 2 refers to local deduction, based on logical reasoning rather than on empirically verified results. It covers all of the features of local deduction necessary for geometrical proof. It is now possible for a student to see that a square is a rhombus, because a square of necessity has four equal sides, which is a sufficient condition for a rhombus. The authors' use of "informal" in this context presumably means "non-axiomatic."

> 3. gives informal deductive arguments
>   a. follows a deductive argument and can supply parts of the argument.
>   b. gives a summary or variation of a deductive argument.
>   c. gives deductive arguments on own.
>
> (ibid., pp. 66-67)

Descriptors 3 refers presumably to global deduction. According to the authors' examples for a and b, the student is not yet producing global deductive arguments, but is able to follow and summarize the steps of a given argument. An example given for c is showing that opposite sides of a parallelogram are equal.

> 4. gives more than one explanation to prove something and justifies these explanations by using family trees.
> 5. informally recognizes difference between a statement and its converse.
>
> (ibid., p. 67)

Students recognize that there are different routes to reach the same conclusion and that an argument cannot necessarily be reversed. These abilities look toward the strategic thinking necessary for global deduction. They are embellishments from the main new facility of Level 2, which is local deduction.

> 6. identifies and uses strategies or insightful reasoning to solve problems.
> (ibid., p. 67)

The authors give the same example as in Descriptor 9 of Level 1, except that now the two circles have unequal radii, which means that the quadrilateral instantiated is a kite rather than a rhombus. There is little difference between the two examples, except that students may be less familiar with the properties of a kite than a rhombus. The Level 2 deductions, however, have mathematical certainty.

> 7. recognizes the role of deductive argument and approaches problems in a deductive manner but
> a. does *not* grasp the meaning of deduction in an axiomatic sense (e.g., does *not* see the need for definitions and basic assumptions).
> b. does *not* formally distinguish between a statement and its converse (e.g., cannot separate the "Siamese twins"—the statement and its converse).
> c. does *not* yet establish interrelationships between networks of theorems.
> (ibid. p. 68, authors' emphasis)

Lastly, having indicated that students are capable of a full geometric proof, the authors note the limitations on geometrical reasoning ability of students at Level 2. Descriptor 7 refers to a student's lack of appreciation of Euclidean geometry as a formal deductive system.

Level 2, reaching as it does, full geometric proof, with mathematically certain local deductions, has reached the limit necessary for secondary school geometry, assuming the curricula requirements outlined earlier. Level 3 continues with the formal reasoning within an axiomatic system, which is the level at which geometry is actually presented in Euclid's *Elements*. Level 4 takes us up to Hilbert and a comparison of axiomatic systems, which is clearly beyond the secondary school level. We will not analyze the descriptors for Levels 3 and 4, but for the sake of completeness they are given below.

> Level 3 Descriptors
> The student
> 1. recognizes the need for undefined terms, definitions, and basic assumptions (e.g. postulates).
> 2. recognizes characteristics of a formal definition (e.g., necessary and sufficient conditions) and equivalence of definitions.
> 3. proves in an axiomatic setting relationships that were explained informally on level 2.
> 4. proves relationships between a theorem and related statements (e.g., converse, inverse, contrapositive).
> 5. establishes interrelationships among networks of theorems.
> 6. compares and contrasts different proofs of theorems.
> 7. examines effects of changing an initial definition or postulate in a logical sequence.
> 8. establishes a general principle that unifies several different theorems.
> 9. creates proofs from simple sets of axioms frequently using a model to support arguments.
> 10. gives formal deductive arguments but does *not* investigate the axiomatics themselves or compare axiomatic systems.
>
> Level 4 Descriptors
> The student
> 1. rigorously establishes theorems in different axiomatic systems (e.g., Hilbert's approach to foundations of geometry).
> 2. compares axiomatic systems (e.g. Euclidean and non-Euclidean geometries); spontaneously explores how changes in axioms affect the resulting geometry.
> 3. establishes consistency of a set of axioms, independence of an axiom, and equivalency of different sets of axioms; creates an axiomatic system for a geometry.
> 4. invents generalized methods for solving classes of problems.
> 5. searches for the broadest context in which a mathematical theorem/principle will apply.
> 6. does in-depth study of the subject logic to develop new insights and approaches to logical inference.
>
> (Fuys et al., 1988, pp. 69-71, authors' emphasis)

From the discussion above, Level 0 is approximately equal to global conceptualization, Level 1 to local conceptualization, and Level 2 to local deduction. Global deduction straddles Levels 1 and 2. Each van Hiele level, however, covers a broader range of abilities than the respective skills.

Axiomatic thinking belongs to Level 3, whereas teachers at secondary school struggle for their students to achieve competence at Level 2 reasoning. It is clear from the perspective of the van Hiele levels that axioms have no place in secondary school geometry. Yes, it is possible to teach axiomatic geometry to students who have not reached Level 3, and yes, it is possible for students to learn certain facts and procedures by rote, but according to van Hiele and van Hiele-Geldof (1958), "They might accept the explanations of the teacher, but the subject taught will not sink into [the students' minds]. The pupil himself feels helpless" (p. 75).

Comparison of the van Hiele levels with the geometrical skills established through the theoretical framework of this thesis has provided evidence of the validity of the new model. Further evidence of the descriptive power of the framework is given by the ease with which it can be used to interpret many common geometrical errors, which is the next topic.

7 Pitfalls of Image-based Reasoning

Many students find geometry difficult. If the researcher can isolate reasons for common geometrical mistakes, then educators can plan to overcome them. We claim that many geometrical errors by students result from conceptualization of incidental image data. They represent the pitfalls of image-based reasoning. The ease with which various errors in geometrical reasoning can be explained comprehensively within a coherent framework is further evidence of the validity of the proposed theoretical model. The errors fall into two broad categories.

7.1 Concept associated with incidental properties

The student's notion of the concept is burdened with incidental data. For example, a student is unable to conceptualize a square as such if it is represented standing on a corner, or is unable to recognize a quadrilateral as a kite if one of its angles is a reflex angle. As Hoffer (1983, p. 219) puts it, "Misconceptions, once learned, seem to persist, as exemplified by adults who firmly believe that a parallelogram, as displayed in most textbooks, cannot have a right angle."

This problem is presumably caused by inaccurate empirical abstraction at the concept-formation stage because too few instantiations are given for a student to generalize the concept adequately.

Examples of this type of error given by Robinson (1976, p. 23) are (1) a line cannot be straight if it is neither horizontal nor vertical, (2) an angle cannot be a right angle if its vertex is on the left, and (3) an angle cannot be a right angle if its arms are not aligned horizontally and vertically. Clearly, the student has not fully assimilated the concepts “straight line” and “right angle.”

Fischbein and Nachlieli (1998) contains a series of investigations of this type of error. In one experiment 218 students in Grades 9 to 11 were asked to define “parallelogram” and then pick the parallelograms from eight images of quadrilaterals. While 89% were able to give a correct definition, only 72% were able correctly to identify the parallelograms among the quadrilaterals. It is probable that incidental properties such as “obliqueness” have been inadvertently conceptualized in the parallelogram definition by those students who failed, for example, to identify a square as a parallelogram. Another investigation in Fischbein and Nachlieli gives students a number of triangles in different orientations, asking them to identify the right-angled triangles. The right-angled triangle with its hypotenuse horizontal was less frequently identified than right-angled triangles in other orientations, which supports the observation of Robinson (1976). It may be supposed that orientation is an incidental property that has been conceptualized. Lastly, when students were asked to identify the kites from a number of quadrilaterals, the concave kite was identified far less frequently than the others. It is possible that the incidental property of convexity has been conceptualized along with the kite definition. All of these cases deal with faulty geometrical reasoning because incidental properties from particular instantiations are uploaded to the conceptualization.

These types of errors concern global conceptualization, which is the main thrust of van Hiele Level 0. Misconceptions such as these are formed at the beginning of a student's study of geometry and are never corrected. Ensuring proper concept formation at the elementary level would help to minimize these errors. It is possible that elementary textbooks do not contain sufficiently varied particular instantiations of concepts for correct empirical abstraction to occur. A more detailed investigation would refer to textbooks using the methodology of Valverde et al. (2002).

7.2 Conceptualization of incidental properties

A requirement for the validity of image-based reasoning is that diagrams are schematic and only schematic properties are allowed to be conceptualized from diagrams. A misapprehension of this necessity is a source of student error.

An example of local conceptualization of incidental properties is a student assuming that two lines are equal in length because it appears that way in the diagram; an example of the global conceptualization of incidental properties is where the image of a quadrilateral is conceptualized as a square, when it is supposed to represent a more general shape.

Fischbein (1993, pp. 145-148) conducted some research on fundamental errors of this kind. He asked students from Grades 2 to 6 to compare the point at the intersection of two lines and the point at the intersection of four lines. Students were asked if one point was bigger or heavier than the other point. At Grade 2, 68% of students did not reply, perhaps because they did not understand the query; in Grade 3, 45.7% claimed the point at the intersection of four lines was bigger; in Grade 4, 50.9% claimed it was bigger; in Grade 5, 40% claimed it was bigger; in Grade 6 still 20% of students found it was larger. The students are erroneously conceptualizing incidental properties such as the thickness or "weight" of the lines. Students that make this kind of error are far from understanding the basic conventions necessary for utilizing schematic diagrams.

Among older students, conceptualization of incidental properties can take different forms. On the one hand, incidental data may be used for a subsequent deduction.

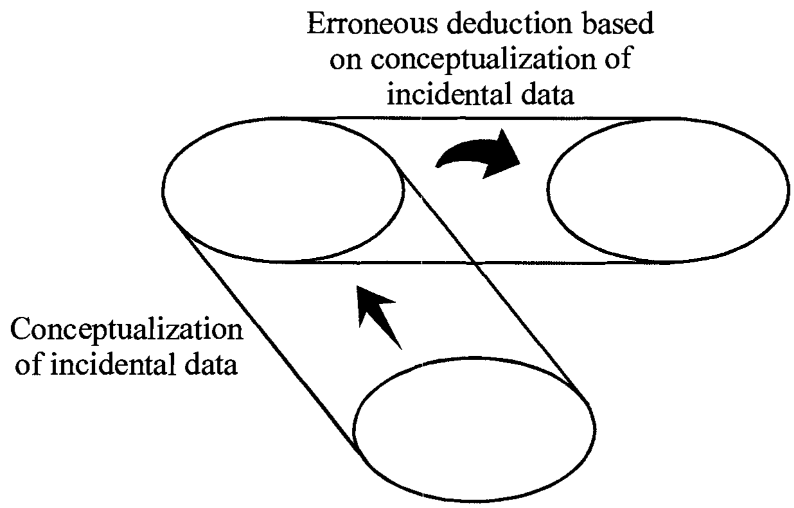


Figure 7.1: Conceptualization of incidental data used for deduction.

The error in conceptualizing four sides of a quadrilateral equal just because they appear that way, for example, is compounded by a deduction that the diagonals are perpendicular. Note that the local deduction itself is correct—it is the initial local conceptualization that is in error.

One the other hand, an error in local deduction may be supported by a conceptualization of incidental data.

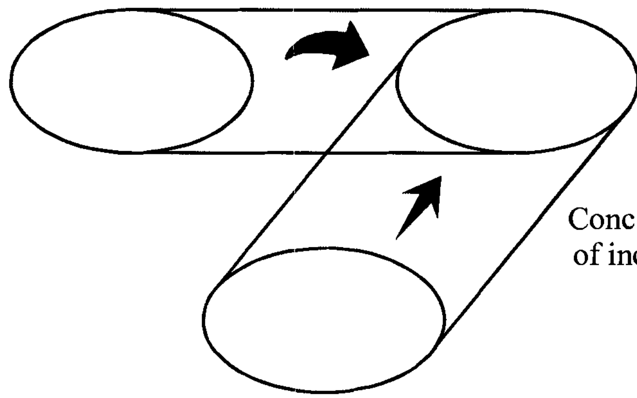


Figure 7.2: **Erroneous deduction supported by incidental data.**

In this case, a theorem may be misapplied or a bogus new theorem invented. For example, a student may conclude that two triangles are congruent because two pairs of sides are equal and a pair of angles is equal, although the angles are not contained by the sides (Robinson, 1976, p. 21). The erroneous deduction is supported by conceptualization of incidental data, provided the two triangles appear to be congruent.

It is possible to characterize this last example alternatively as a simple deductive mistake if the SAS theorem is misremembered or misapplied. Geometrical errors sometimes may be interpreted in more than one way.

Robinson (1976, p. 20) gives an example of a geometrical error that can be interpreted in at least two ways. The researcher shows the student a quadrilateral that looks like a parallelogram. However, the given information is only that one pair of opposite sides are parallel. The student mistakenly deduces that the two transversal line segments are equal because "parallel lines are everywhere equidistant." This erroneous deduction is amply supported, by incidental diagrammatic evidence that the two transversals are of equal length. In another interpretation, the student may have made the inappropriate metrical local conceptualization that the two transversals are parallel. Then a correct deduction can be made that the figure is a parallelogram, from which a second correct deduction can be made that the transversals are equal because opposite sides of parallelograms are equal. In the first interpretation, the erroneous deduction is supported by incidental data; in the second interpretation, an incidental local conceptualization is followed by correct deductions.

7.3 Other types of geometrical error

Of course not all geometric errors can be classified as problems with conceptualization of incidental data. For example, if a student does not correctly understand a concept, then it may result in an incorrect instantiation of the concept, leading to an incorrect deduction. Examples of this type cited by Robinson (1976, p. 20) are misunderstandings of the relations "equidistant," "complementary," and "perpendicular."

Likewise, a student may be unable to understand and apply a theorem. An example of this type is the proportionality between corresponding sides of similar triangles. The student is unable to identify the pairs of corresponding sides, or is unable to write the equation relating their lengths, even if the corresponding sides are identified.

Problems with global deduction may also be classified as geometrical errors. In this case, a student is unable to articulate the series of local deductions necessary to prove a proposition, which may well involve being unable to instantiate any necessary additional constructions. In particular, the reductio ad absurdum type of global deduction is very difficult for students to grasp, since it often involves instantiating concepts that are obviously not true from an arithmetical or metrical reading of the diagram. Difficulties with reductio ad absurdum may also be interpreted as cases of conceptualizing incidental data.

7.4 Concluding remarks about geometrical errors

Although there are different reasons for errors in geometrical reasoning, and some types of errors may even be interpreted in several ways, it is clear that conceptualization of incidental data is a major source of mistakes.

As this investigation argues, geometry without axioms needs schematic diagrams. It is probable, however, that many students do not receive specific instruction in how to recognize and conceptualize schematic properties during their secondary school education. Moreover, the arithmetical, measurement geometry of earlier grades way well mitigate against a proper understanding of schematic diagrams. A conclusion of this study is that specific didactical strategies should be designed and implemented in order that senior secondary students may better appreciate the types of geometrical property that may be successfully uploaded from the diagram.

8 Reflection and Critique

The goal of this study has been to investigate the necessity, implications, and pitfalls of image-based reasoning in school geometry. In order to deal adequately with these issues it was found necessary to develop a new theoretical framework for image-based reasoning in geometry. Thus the conceptualization model, the principles of conceptualization, and the geometrical skills were formulated. The main thrust of this thesis is development of the framework, testing its validity by comparing it with the van Hiele levels, and demonstrating its descriptive power with an interpretation of common errors in geometrical reasoning.

Educational researchers need a framework for image-based reasoning because image-based reasoning is necessary for geometry without axioms. An axiomatic treatment of geometry in secondary schools is inappropriate, as expounded by Freudenthal (1973) and Zeitler (1991) and confirmed by the van Hiele levels.

An important implication of allowing image-based reasoning concerns the fact that the propositions of deductive geometry are general, whereas images are particular. How can the geometer justify arguing from the particular to the general? Finding a satisfactory resolution of this dilemma was a major concern of this study. It relies on a proper understanding of the notion of schematic case, and was inspired by Netz's (1999) analysis of the Greek schematic diagram.

A pitfall of allowing image-based reasoning is that the student must be fully aware of the types of properties that may be uploaded reliably from the diagram. Many errors in geometrical reasoning can be attributed to conceptualization of incidental properties. It is necessary for the schematic diagram and schematic properties to be taught explicitly in the classroom. This task is made all the harder because the measurement geometry of earlier grades would tend to program students with an arithmetical understanding of geometry diagrams.

These conclusions depend firstly on an understanding of image and concept. The limited scope of this study naturally constrained the extent and depth of the investigation. Although image and concept are referred to as mental representations, the discussion does not address the real meaning of "mental representation." Neither does it deal fully with the origin of conceptual representations, aside from a glance at empirical abstraction and a brief overview of two more sophisticated theories. A foundational rather than functional study would attempt to define and distinguish between mental representations at the level of the neural architecture of the brain, and might refer first to the connectionism of Rumelhart et al.(1986) in this regard.

There are some omissions in the argument for the generality of image-based reasoning. In particular, more clarity is required both of the intuitive mechanism for determining schematic cases and the way in which additional diagrammatic constructions can affect schematic cases.

Image-based reasoning was essential in Greek geometry, despite Euclid's axioms. Therefore, Greek geometrical practice is an important case study when trying to understand the implications of geometry without axioms. The historical overview of geometry after Euclid is quite sketchy, although it is not central to the main argument. A more extensive study would review the history of geometry teaching, particularly since Hilbert, which is a topic this thesis does not touch upon.

The current study examines the geometry curricula of only three English-speaking educational jurisdictions. A more extensive investigation of geometry curricula would clarify the relative importance of axioms and schematic diagrams in more cases.

It is conjectured that inaccurate empirical abstraction caused by a paucity of particular examples is the reason for a type of geometrical error related to the conceptualization of incidental properties. One way of investigating this would be to examine textbook practice, for which purpose Valverde et al. (2002) provides a compelling methodology.

This study has suggested that attention to schematic diagrams is essential for successfully teaching deductive geometry without axioms, although it does not propose a concrete didactical program to achieve this end. It may be that the most efficient way to reduce errors caused by incidental conceptualizations would be to teach the schematic diagram explicitly. It would be informative to conduct empirical research on the effect of such a small-scale addition to the senior geometry curriculum.

A gaping hole in the investigation so far is that no mention has been made of computer-aided learning in geometry. Various computer programs allow one to construct a geometrical diagram and then manipulate and measure its various components. A tremendous strength of this approach is that students are easily able to make and test conjectures. In fact, empirical verification of propositions is easy. However, it should be realized that that computer-generated images of this type emphasize arithmetical properties rather than schematic properties. It is still necessary to teach deductive proof. We have suggested that the arithmetical measurement geometry of earlier grades may retard progress in deductive geometry. Computer-generated diagrams may have the same deleterious effect. Nevertheless, computer-aided geometry is a growing, important subject. A further investigation would have to address the role of computer-aided geometry in the classroom.

The most pressing task, however, is to conduct empirical research. The theoretical framework developed through this investigation may suggest specific research questions. The resulting empirical research could provide further validation of the framework developed herein.

In conclusion, the major contributions of this thesis are the theoretical framework, the deconstruction of deductive geometrical reasoning represented by Figure 5.6, and the resulting identification of the five geometrical reasoning skills. This framework for image-based reasoning in geometry and can be seen as an initial attempt to lay a foundation for a "logic" of image-based reasoning. As such, it also sheds light on the van Hiele levels. Moreover, the geometrical reasoning skills I have derived compliment and increase the pedagogical efficacy of this framework and have practical applications that can serve to improve the teaching of geometry.

Appendix

A geometric figure may usually be conceptualized in many ways. A square can be thought of also as a rectangle, a rhombus, a parallelogram, a quadrilateral, or just as a geometrical figure. In fact, a conceptualization of a geometric figure is simply a set of properties of the figure; the set of all conceptualizations is the power set of the set of all properties.

We can impose an algebraic structure on the set of conceptualizations of a figure by means of the binary operation of set union. Thus, if c and d are two conceptualizations of a figure, then define $c \circ d = c \cup d$. However, this is the same structure than can be applied to any power set, and it is uninteresting.

Another approach is needed. Define $\overline{c}$, the *completion* of conceptualization c, to be the set of all properties that can be deduced finitely from the properties of c. (Obviously, $c \subseteq \overline{c}$.) Then define the relation ~ such that $c \sim d \Leftrightarrow \overline{c} = \overline{d}$. The relation ~ is an equivalence relation because it is reflexive, symmetric, and transitive. Thus, ~ defines a set of equivalence classes among the conceptualizations. Write $[c]$ for the equivalence class containing c. In a sense, the members of an equivalence class all represent the same logical information content, and so equality of completions is a natural relation to use. Define *completion classes* to mean this set of equivalence classes of conceptualizations.

Now define a binary operation between the completion classes by means of $[c] \circ [d] = [c \cup d]$. It is not immediately obvious that this operation is well defined, so it is necessary to prove that it is so. Suppose there are conceptualizations c, c', d, and d' such that $\overline{c} = \overline{c'}$ and $\overline{d} = \overline{d'}$. It is necessary to show $[c] \circ [d] = [c'] \circ [d']$. In other words, it is necessary to show $[c \cup d] = [c' \cup d']$, or $\overline{c \cup d} = \overline{c' \cup d'}$. Suppose $p \in \overline{c \cup d}$. (It makes no difference if $p \in \overline{c' \cup d'}$.) Then $\exists p_1, \ldots, p_n \in c \cup d$ such that $\{p_1, \ldots, p_n\} \Rightarrow p$. Assume $p_1 \in c$. (It makes no difference if $p_1 \in d$.) Since $\overline{c} = \overline{c'}$, $\exists q_1, \ldots, q_m \in c'$ such that $\{q_1, \ldots, q_m\} \Rightarrow p_1$. The same is true for every p_i, that it is possible to find a finite set of members of c' or d' from which it can be deduced. Therefore, $p \in \overline{c' \cup d'}$, and so $\overline{c \cup d} \subseteq \overline{c' \cup d'}$. By symmetry, $\overline{c \cup d} \supseteq \overline{c' \cup d'}$. Thus, $[c \cup d] = [c' \cup d']$, as required.

It is easily seen to be true that $([c] \circ [d]) \circ [e] = [c] \circ ([d] \circ [e])$ and $[c] \circ [d] = [d] \circ [c]$ so that the set of completion classes with this operation is a commutative semigroup. The empty set of properties can be referred to as the *null conceptualization*. It can be seen that $[\varnothing]$ is an identity element, so that the set of completion classes is actually a monoid. Write $0 = [\varnothing]$. Every set of completion classes has another special element, which is the completion class of the set of all possible properties, called the *universal conceptualization*. Write 1 for this element. Clearly, $[c] \circ 1 = 1$ for every c. Furthermore, $[c] \circ [c] = [c]$ for every c, so that every completion class is idempotent.

Define the relation $[c] \leq [d] \Leftrightarrow [c] \circ [d] = [d]$, which is easily seen to be reflexive, antisymmetric, and transitive. This relation, therefore, makes the completion classes into a partially ordered set. Clearly 0 is the least element and 1 is the greatest element, so that any two completion classes must have a supremum and an infimum. Therefore, the completion classes form a lattice. This is the *property lattice* of the geometric figure. It is clear that the lattice is more interesting than the semigroup because of the additional structure implied by the supremum. A more detailed analysis would need some lattice theory, which is beyond the scope of this investigation.

The reader may see with the following example that a more interesting structure has been gained by considering the equivalence classes of completions. Consider ΔABC and ΔDEF. Let c be the conceptualization that $AB = DE$ and $AC = DF$. Let d be the conceptualization $\angle A = \angle D$. The completions of c and d will not produce extra properties. However, $\overline{c \cup d}$ generates in addition the properties $BC = EF$, $\angle B = \angle E$, and $\angle C = \angle F$, because the union produces congruent triangles.

There are other benefits to using completion classes. For example, considering the properties of a polygon, it is not necessary to include both interior angles and exterior angles, because, given an interior angle, the exterior angle belongs to the completion class of the interior angle. Similarly, definitions that group together properties are not needed as properties themselves, since they would belong to the completion class of the group of properties that form the definition. In fact, the only properties needed are a list of basic properties from which all other properties are generated. These are the *generating properties*.

It would be very nice to be able to show that each set of generating properties of a given geometric figure contains the same number of elements, and then define this to be the "dimension" of the figure. This does not seem, however, to be possible in the general case.

For ΔABC, assume the six generating properties are the measures of the angles A, B, and C and the lengths of the sides a, b, and c. Then the completion classes are as follows (in which square brackets are omitted to simplify the notation):

0, A, B, C, a, b, c, ABC, Aa, Ab, Ac, Ba, Bb, Bc, Ca,
Cb, Cc, ab, ac, bc, Aab, Aac, Bab, Bbc, Cac, Cbc, 1

It is unnecessary to write out the full multiplication table. Examples are $[A]\circ[a]=[Aa]$, $[A]\circ[cb]=1$, and so on. It is interesting to note that there are 27 completion classes. The same analysis on a quadrilateral produces 160 completion classes.

Consider another situation, with two triangles, and the six generating properties being equality of the three pairs of corresponding angles and the three pairs of corresponding sides. Then there is an obvious bijective mapping between the property sets of the two figures, and this mapping preserves the operation. Therefore, the two property lattices are isomorphic. This is exactly as it should be, because it seems clear that the two situations have precisely the same information structure.

Associating an algebraic structure with a geometric figure in the manner discussed is reminiscent of algebraic topology. It could provide a way of classifying geometrical figures according to the associated lattice structure. It appears, however, that there is no direct link between the property lattice and the invariants of algebraic topology—the former is concerned with information structure, whereas the latter reflects spatial structure.

The property lattice can be applied outside of geometry. Kosslyn (1983) makes the point that "images—like pictures—are susceptible to multiple interpretations. An image of a sitting man could be seen as representing 'bent knees,' 'John's head,' or a 'twentieth century person'" (p. 6). In any given situation, simply define the generating properties. For example, the generating properties "cloudy" (C) and "rainy" (R) determine the following set of completion classes: {0, C, 1}, where 1, of course, is equivalent to CR.

Note that if "sunny" (S) is included in the set of generating properties, the universal conceptualization would represent the very strange weather pattern CRS. It is necessary to be clear that the image is primary and that the conceptualizations are different perspectives on the image. Simply choosing a set of properties without reference to an image can lead to absurdities. For example, if "night" (N) and "day (D)" are chosen, then ND is a ridiculous universal conceptualization.

An application of the property lattice outside of geometry may be the laws of Gestalt, which are summarized by Kosslyn (1983, p. 88). Consider only the first of these, *the law of proximity*, according to which, parts of an image near each other will tend to be grouped together. For example, XXXX is seen as one unit, whereas XX XX is two. In other words, an image is more likely to be conceptualized as a whole if its various components are close together.

Suppose that two regions of an image have conceptualizations c and d, respectively. Then if $\overline{c} \cup \overline{d} = \overline{c \cup d}$, the two regions are *disjoint* with respect to these conceptualizations. Disjointness means that the regions do not have "proximity," and are therefore likely to be conceptualized separately. For example, an image of a man in a car may result in the property "driver," whereas an image of a man standing separately from a car may produce no additional properties other than those supplied by a description of the man and a description of the car.

The property lattice hints at a mathematical definition of the Gestalt law of proximity. Perhaps there are applications elsewhere in cognitive psychology. We can faintly discern that behind and beyond the messy cacophony of impressions, thoughts, and feelings in which we live lies a discrete, pure mathematical structure, the property lattice.

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