Cops and Robbers on Geometric Graphs and Graphs with a Set of Forbidden Subgraphs

by

Masood Masjoody

M.Sc., Pure Mathematics, University of Tehran
M.Sc., Geotechnical Engineering, Tarbiat Modares University
B.Sc., Civil Engineering, University of Tehran

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Summer 2019

Approval

Name: **Masood Masjoody**

Degree: **Doctor of Philosophy (Mathematics)**

Title: **Cops and Robbers on Geometric Graphs and Graphs with a Set of Forbidden Subgraphs**

Examining Committee: **Chair:** Amarpreet Rattan
Associate Professor

Ladislav Stacho
Senior Supervisor
Associate Professor

Bojan Mohar
Co-Supervisor
Professor

Pavol Hell
Supervisor
Professor

Luis Goddyn
Internal Examiner
Professor

Gary MacGillivray
External Examiner
Professor
Department of Mathematics and Statistics
University of Victoria

Date Defended: **June 18, 2019**

Abstract

In this thesis we study the game of cops and robber on some special class of graphs, including planar graphs and geometric graphs. Moreover, under some conditions on graph diameter, we characterize all sets $\mathscr{H}$ of graphs with bounded diameter for which $\mathscr{H}$-free graphs are cop-bounded. Furthermore, we extend our characterization to the case of cop-bounded classes of graphs defined by a set $\mathscr{H}$ of forbidden graphs such that the components of members of $\mathscr{H}$ have bounded diameter.

Keywords: Combinatorial games on graphs, the game of cops and robbers, graph classes; planar graphs; geometric graphs; forbidden induced subgraphs; generalized claw; generalized net; cop-bounded

Table of Contents

List of Figures

Chapter 1

Introduction

This thesis concerns the combinatorial game of *cops and robbers* on graphs. The game of cops and robbers was introduced by Nowakowski and Wrinkler [12] as a pursuit game on vertices of a graph with two players, a cop and a robber. Since then, several variations of the game have been introduced, among which this thesis focuses on the variation introduced by Aigner and Fromme [1] where one player controls a set of cops and the other controls a robber. At the start of the game every agent (i.e. a cop or the robber) occupies a vertex of a graph G with the cops taking positions ahead of the robber, each agent knows the position of every other agent in the game, and in each step of the game an agent can choose to either stay in its current position or move to a neighboring vertex. We assume that the cops and the robber have perfect knowledge about the game and that they play their best strategies. The game ends when a cop is put on top of the robber in which case we say that the cops have *won* (or the robber has been *captured*). For the game of cops and robbers on a graph, or a class of graphs, a major question is to determine the minimum number of cops which can guarantee to win the game. Since we consider the game of cops and robbers on finite graphs, that minimum exists and is bounded above by the number of vertices of the graph on which the game is played. A finite set of cops which guarantee the capture of the robber are said to have a *winning strategy*.

Definition 1.0.1. Given a graph G, the minimum number of cops with a winning strategy on G is called the *cop number* of G denoted $C(G)$. We call G a *copwin* graph if $C(G) = 1$. A class $\mathscr{G}$ of graphs the cop-number of whose members is bounded is siad to br *cop-bounded*. We say that $\mathscr{G}$ is *cop-bounded by* k if $C(G) \leq k$ for every $G \in \mathscr{G}$.

Since the cop number of a graph is equal to the sum of the cop numbers of its components, whenever the game of cops and robbers is concerned, we restrict ourselves to connected graphs unless specified otherwise.

Copwin graphs have been characterized by the existence of a specific elimination ordering among all of their vertices. See [1, 5] for a complete discussion.

A major result about the cop number of classes of graph is by Aigner and Fromme [1] who showed that on every (connected) planar graph, three cops have a winning strategy:

Theorem A. $C(G) \leq 3$ *for every connected planar graph* G*.*

Our contributions in this thesis are on the game of cops and robbers on some cop-bounded classes of graphs, including planar graphs and geometric graphs, and also on the game of cops and robbers on classes of graphs defined by a set of forbidden induced subgraphs. In the later sections, we will list some of the major results in each of the directions this thesis is concentrating.

1.1 Planar graphs

The original proof of Theorem A presented in [1] was rather complicated and since its appearance several attempts have been made to reformulate it into a simpler proof. (See, for instance, the proof given in Bonato and Nowakowski's book [5].) In Section 2.1 we present a new proof for Theorem A which is simpler than the existing proofs. Our argument is based on Fáry's Theorem [7] that states every planar graph has a straight-line embedding in the plane. Given any straight-line embedding of a planar graph, we present a 3-cop winning strategy which, unlike the other proofs suggested for Theorem A, is of an algorithmic nature and provides a more straightforward argument for Theorem A.

1.2 Geometric graphs

Geometric graphs are intersection graphs of unit disks in the plane and, as such, form a subclass of *string graphs* [6]. It has been shown that string graphs are cop-bounded by 15 [9] and, hence, so are geometric graphs. In [4], authors construct a geometric graph on 1440 vertices whose cop number is three. Furthermore, they suggest a proof for the claim that geometric graphs are cop-bounded by nine. In Section 2.2 we show why the latter argument is incomplete. In Section 2.3, utilizing the operations of *clique substitution* (Definition 1.5.4) and *k-subdivision* (Definition 1.5.5), we present two constructions for geometric graphs of cop number at least three, either of which can be used to produce an infinite family of such geometric graphs. We also use our techniques to generate a geometric graph of cop number three on 440 vertices, improving the example provided in [4].

1.3 Classes of graphs defined by a set of forbidden induced subgraph

It is known that the class of graphs with a forbidden induced subgraph H (also called H*-free* graphs) is cop-bounded iff H is a forest of paths [10]. In particular, restricting H to be connected implies that H-free graphs are cop-bounded iff H is a path. In Chapter 3 we extend these results. In Theorem 3.4.1 we characterize all sets $\mathscr{H}$ of graphs with bounded diameter such that the class of $\mathscr{H}$-free graphs (Definition 3.1.1) is cop-bounded, by showing that $\mathscr{H}$-free graphs are cop-bounded if and only if $\mathscr{H}$ contains a path, or contains a generalized claw and a generalized net (Definition 3.2.1). As a special case, this result provides a characterization of all finite sets $\mathscr{H}$ of connected graphs such that the class of $\mathscr{H}$-free graphs is cop-bounded. We also extend our characterization to sets $\mathscr{H}$ of graphs with diameter of components of members of $\mathscr{H}$ bounded (Theorem 3.4.3).

1.4 Structures inspired by the game of cops and robbers

Inspired by the game of cops and robbers, in Chapter 4 we showed that the set of connected claw- and bull-free graphs is the union of the set of connected graphs which are complements of triangle-free graphs, the set of extensions of paths, and the set of extensions of cycles, where an extension of a graph G is obtained by replacing its vertices with disjoint cliques and adding all edges between cliques corresponding to adjacent vertices of G. It turned out that this structure was proposed as a byproduct of another paper published in 1991 with a strategy for a proof briefly sketched out [16]. Nevertheless, we found this structure independently and proposed a complete proof using a substantially different approach.

1.5 General definitions and notation

In this thesis we shall be using standard terminology and notation from [17]. In the following we list some of the less common ones to be used in this work.

Notation.

a. We denote the cardinality of a set A by $|A|$.

b. We denote the maximum order of an induced cycle in G by $\ell(G)$.

c. If u, v are vertices of a graph G, we denote their graph distance in G by $d_G(u, v)$. We occasionally use the notation $u \leftrightarrow_G v$, or simply $u \leftrightarrow v$ if G is understood from the context, to mean that u and v are adjacent in G.

d. If u, v are points in the plane, we denote their Euclidean distance by $d_E(u, v)$.

e. We shall call a path P on n vertices an *n-path* and define its *length* by $l(P) = n-1$.

f. Given a walk $W:\ w_0, w_1, \ldots, w_k$ and $i, j \in [0 \cdot\cdot k]$ with $i \leq j$, we denote the subwalk of W from w_i to w_j by $W(w_i, w_j)$.

Definition 1.5.1. Let $G = (V, E)$ be a graph. For each $v \in V$ we define the *open neighborhood* $N_G(v)$ of v to be $\{w \in V : vw \in E\}$ and the *closed neighborhood* $N_G[v]$ to be $N_G(v) \cup \{v\}$. Also for each $V' \subseteq V$ we define

$$N_G[V'] = \bigcup_{v \in V'} N_G[v], \qquad \text{and} \qquad N_G(V') = N_G[V'] \setminus V'.$$

For every subgraph $G' = (V', E')$ of G we set $N_G[G'] := N_G[V']$ and $N_G(G') = N_G(V')$.

Remark. We might drop the subscript G when the graph is understood from the context.

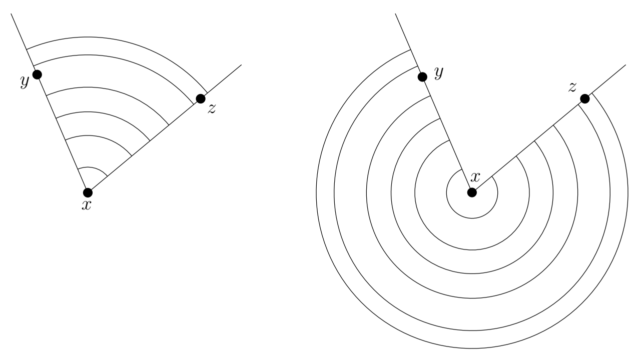
Definition 1.5.2. A *geometric graph* is a graph that has a finite set V of points in the plane with no collinear triples as its vertex set, and the set of all line-segments between pairs of distinct points in V with Euclidean distance less than or equal to a positive constant r, called the *parameter* of the geometric graphs, as its edge set. Two distinct edges of a geometric graph are said to *cross* if they have a common interior point.

Definition 1.5.3. Suppose G and H are graphs. We say that G is *H-free* if it does not contain an isomorphic copy of H as an induced subgraph.

Definition 1.5.4. Let $G = (V, E)$ be a graph and $v \in V(G)$. The *clique substitution at v* is the graph obtain from G by replacing v with a clique of size $|N_G(v)|$ and matching vertices in $N_G(v)$ with the vertices of that clique. The *clique substitution* *of* G, denoted $K(G)$, is the graph obtained from G by sequentially performing clique substitutions at all vertices of G. We refer to a clique substituted for a vertex of G as a *knot* (of $K(G)$).

Definition 1.5.5. Let G be a graph and $k \in \mathbb{N}$. The operation of introducing k vertices in the interior of every edge of G, making edges of G into internally disjoint $(k+2)$-paths, is called *k-subdivision* of G.

Definition 1.5.6 (Cone)**.** Let x, y and z be non-collinear points in the plane. We denote by $\text{cone}_{\text{CW}}(y, x, z)$ (resp. $\text{cone}_{\text{CCW}}(y, x, z)$) the cone with apex x and supporting rays through y and z and clockwise (resp. counter-clockwise) interior angle from the supporting ray through y to the supporting ray through z.



(a) (b)

Figure 1.1: (a) $\text{cone}_{\text{CW}}(y, x, z)$, **(b)** $\text{cone}_{\text{CCW}}(y, x, z)$

Chapter 2

Cops and Robbers on Geometric Graphs

2.1 The Game of Cops and Robbers

A *game of cops and robbers* is a pursuit game on graphs, or a class of graphs, in which a set of agents called the *cops* try to get to the same position as another agent, called the *robber*. Several variants of such a game has been introduced and studied, such as *fast robber* [8], *cops and drunk robbers* [11], *lazy cops and robbers* [3], just to name a few.

Since our focus in this thesis will be on the original variant introduced in [1], which is played on finite undirected graphs; we shall simply refer to this variant as “the" game of cops and robbers. Furthermore, a graph in this thesis always means a finite simple graph. The description of the game is as follows. Let G be a graph and consider a finite set of cops and a robber which are to play on G. At the beginning of the game (step 1) each cop will be positioned at a vertex of the graph and then the robber will be positioned at some vertex. In each of the subsequent steps each agent either moves to a vertex adjacent to its current position or stays still, with the robber taking its turn after all of the cops. The cops win in a step i of the game if in that step one of the cops gets to the vertex where the robber is located. The minimum number of cops that are guaranteed to capture the robber on G in a finite number of steps is called the *cop number* of G and denoted $C(G)$. Observe that the cop number of a graph is equal to the sum of the cop numbers of its components. Hence, in each of the results on the cop number of a graph set forth in this chapter we consider the graph under consideration connected. A graph is said to be *k-copwin* ($k \in \mathbb{N}$) if its cop number is bounded above by k. We call a 1-copwin graph simply a *copwin* graph.

Copwin graphs have been fully characterized as *dismantlable* graphs [12, 15]. The class of copwin graphs include all trees:

Proposition 2.1.1 ([5])**.** *Trees are copwin.*

Sketch of Proof. Consider the following strategy for the cop: In every step of the game the cop moves toward the robber along the unique path joining the present position of the cop to that of the robber. This way, the distance between the robber and the cop will never increase and will surely decrease when the robber moves toward the cop or the cop moves toward the robber while the robber is located in a leaf of the tree. Hence, as T is finite, the cop is sure to successively decrease its distance from the robber to zero. □

Given a class of graphs we call it *cop-bounded* if there is $k \in \mathbb{N}$ such that every graph in the class is k-copwin. For instance, it can be easily seen that the class of cycles is cop-bounded, for two cops always have the winning strategy of moving toward the robber from opposite directions. On the other hand, the class of bipartite graphs is not cop-bounded; a fact that can be established in light of the following result:

Theorem 2.1.2 ([1])**.** *For a graph G with minimum degree δ one has $C(G) \geq \delta$ provided the girth of G is at least 5.*

Theorem 2.1.2 and its variants have been widely used to obtain lower bounds for the cop number of different classes of graphs and also to show that certain graph classes are not cop-bounded. For a result of the latter type see [14] where the author shows that the class of bipartite graphs is cop-unbounded. On the other hand, [1] provides a property of geodesic paths in graphs (Proposition 2.1.3) that has proven handy in many situations where finding an upper-bound for the cop number of a graph or class of graphs is concerned.

Definition 2.1.1. Let P be a path with end vertices u and v.

a. We shall write $P = P(u, v)$ to mean that P is being considered with the orientation from u to v.

b. For every pair u', v' of vertices of P we shall write $Q = P(u', v')$ to mean that Q is the sub-path of P with end vertices u' and v' considered with the orientation from u' to v'.

c. If P is a subgraph of a graph G such that $l(P) = d_G(u, v)$, then we say that P is *geodesic* in G.

d. If $P = P(u, v)$ is geodesic in G, for every $r \in V(G)$ we denote by $S_{P,G}(r)$, or occasionally $S_P(r)$ if G is understood from the context, the *shadow* in G of r on P, defined in terms of the graph distance $k := d_G(r, u)$ by:

$$S_{P,G}(r) = \begin{cases} v_k & \text{if} \quad k \leq l(P), \\ v & \text{if} \quad k > l(P), \end{cases}$$

where v_k is the kth vertex of P starting from u.

Remark. With P as in part **d.** of Definition 2.1.1 and r_1 and r_2 being two adjacent vertices of G, $S_{P,G}(r_1)$ and $S_{P,G}(r_2)$ are either identical or adjacent. In particular, if the game of cops and robbers is being played on G and at the end of a step i of the game the robber is at some $r \in V(G)$ and a cop, say C, is in $S_{P,G}(r)$, C can keep occupying the shadow of the position of the robber with respect to P in all steps to come. When such a condition is met, we simply say that C is *shadowing* the robber on P, or P is being *1-guarded* by C.

Proposition 2.1.3. *[1] Let $P = P(u, v)$ be a geodesic path in a graph G where the game of cops and robbers is being played. Then any cop C can get to the shadow of the robber on P in a finite number of steps. Moreover, if the robber steps onto a vertex of P while being shadowed by C, it will be caught by C in the very next step of the game.*

As shown in [1], using Proposition 2.1.3 one can provide a winning strategy for three cops on any planar graph where the graph under consideration is identified by any of its planar imbeddings. Here, we provide an alternative proof for this result based on Proposition 2.1.3 and the following theorem by Fáry:

Theorem 2.1.4 (Fáry's Theorem)**.** *[7] Every simple planar graph has a straight-line plane representation with no three vertices collinear.*

Remark. Even though the proof of Theorem 2.1.5 in this section is developed for a straight-line embedding of a given planar graph G, the argument essentially works for any planar embedding of G and the rotation system associated with it. Considering a straight-line embedding, on the other hand, enables us to simplify the presentation of the proof, for example by using the simple notion of a cone, given by Definition 1.5.6.

Theorem 2.1.5. *[1] Planar graphs are 3-copwin.*

Proof. Let G be a planar graph on more than three vertices and identify it with any of its straight-line plane drawings where no three vertices are collinear. With three cops

in play, in the first step of the game we position all of them in the same vertex of the graph. We consider one of the cops *active* and the remaining two *dormant*. After the robber chooses its first position we will have the following configuration which defines *stage* 0 of the game: Let $v_0 \in V(G)$ and $r_0 \in V(G) \setminus \{v_0\}$ be the initial position of the active cop and the robber, respectively. Set $R_0 = V(G)$ and $G_0 = G$. Let R_1 be the set of all vertices of G which are reachable along paths in G from r_0 without stepping onto v_0, and set $G_1 = G[R_1]$.
In general, proceeding from a stage $i-1$ ($i \in \mathbb{N}$), our strategy is to get, after a finite number of steps, to stage i with one of the following *scenarios*:

A_i: There is only one active cop, which is located in a vertex $v_i \in R_j$ for some $j \in [0 \cdot\cdot i]$; the robber is in a vertex $r_i \in R_i \setminus \{v_i\}$; R_{i+1} is the set of all vertices of G which are reachable along paths from r_i without stepping onto v_i; and we have $G_{i+1} := G[R_{i+1}]$.

B_i: Two of the cops are active and the remaining one is dormant; there are four paths $P_i = P_i(a_i, b_i), Q_i = Q_i(c_i, d_i), X_i = X_i(a_i, c_i), Y_i = Y_i(b_i, d_i)$ that form a simple closed polygonal curve Γ_i; there are $k_i, l_i \in [0 \cdot\cdot i]$ such that P_i and Q_i are geodesic and being shadowed in G_{k_i} and G_{l_i}, respectively; each of the paths X_i and Y_i has at least one edge; the robber is at a vertex $r_i \in R_i \setminus V(\Gamma_i)$; R_{i+1} is the set of all vertices reachable along paths from r_i without stepping onto Γ_i; R_{i+1} does not contain any vertex in the neighborhood of an internal vertex of X_i or Y_i; and we have $G_{i+1} := G[R_{i+1}]$.

Observe that the scenario in stage 0 is A_0. Moreover, in a stage i of the game with either scenario the robber cannot leave R_{i+1} without being captured by an active cop. In addition, if $|R_{i+1}|$ is less than or equal to the number of the dormant cops, then the simple strategy of saturating R_{i+1} will lead to the robber's capture. Based on these observations, it suffices to show the following:

Claim 1. *Suppose there is a total of 3 cops in play. Let $i \in \mathbb{N} \cup \{0\}$ and suppose the game is in stage i (with scenario A_i or B_i) such that $|R_{i+1}|$ is greater than the number of the dormant cops. Then in a finite number of steps the cops can either capture the robber or get the game to stage $i+1$ (with scenario A_{i+1} or B_{i+1}) such that $R_{i+2} \subseteq R_{i+1}$, where the inclusion is proper unless the change of scenarios is from B_i to A_{i+1}.*

To prove the claim and, thereby, the theorem we consider the possible cases as follows:

Case 1: The scenario in stage i is A_i, and v_i has only one neighbor x in R_{i+1}.

Case 2: The scenario in stage i is A_i, and v_i has at least two neighbors in R_{i+1}.

Case 3: The scenario in stage i is B_i, P_i has a unique vertex x with a neighbor in R_{i+1}, and Q_i has a unique vertex y with a neighbor in R_{i+1}.

Case 4: The scenario in stage i is B_i, and there is a unique vertex x in $V(P_i) \cup V(Q_i)$ with a neighbor in R_{i+1}.

Case 5: The scenario in stage i is B_i, and one of the paths P_i and Q_i has at least two vertices with a neighbor in R_{i+1}.

Case 1: In the next step of the game move the active cop from v_i to x. Let $y \neq x$ be the position the robber assumes in the same step. We have $y \in R_{i+1} \setminus \{x\} \subsetneq R_{i+1}$. Moreover, since every path from y to v_i has to contain x, the set of all vertices reachable from y without stepping onto x is a subset of $R_{i+1} \setminus \{x\}$. Hence, by setting $v_{i+1} := x$ and $r_{i+1} := y$, we will reach at stage $i+1$ with scenario A_{i+1} and $R_{i+2} \subsetneq R_{i+1}$.

Case 2: Choose distinct vertices $x, y \in N(v_i) \cap R_{i+1}$, and choose a geodesic path $P = P(x, y)$ in G_{i+1} (Figure 2.1).

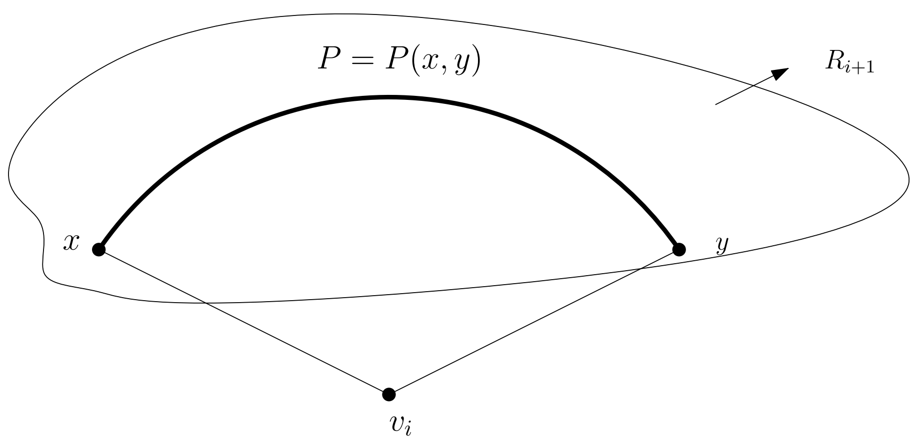


Figure 2.1: Case 2: scenario A_i, with $|N(v_i) \cap R_{i+1}| \geq 2$

Then move a dormant cop in stage i so that it reaches the shadow of the robber on P in G_{i+1} in some step of the game. We may assume that the position, say z of the robber in that step of the game is not in $V(P)$, for otherwise the game would be already over. As such, let Q be the 1-path at v_i, R be the set of all vertices reachable from z without stepping onto P or Q, and X and Y be the 2-paths from x to v_i and from y to v_i, respectively. Let Γ be the simple closed curve formed by P, Q, X, and Y. Then by setting $r_{i+1} := z$, $X_{i+1} := X$, $Y_{i+1} := Y$, $P_{i+1} := P$, and $Q_{i+1} := Q$, the game reaches at stage $i+1$ with scenario B_{i+1} such that $\Gamma_{i+1} = \Gamma$ and $R_{i+2} = R \subsetneq R_{i+1}$.

Case 3: Since Γ_i is closed and according to the definition of R_{i+1}, one of the two planar regions with boundary Γ_i contains all vertices in R_{i+1}. We name this region Λ. Consider the neighbors x' and y' of x and y along the counter-clockwise orientation of Γ_i and assume, without loss of generality, that x' belongs to the path $P_i(a_i, x)$. Set the symbol $\sigma \in \{\text{CCW}, \text{CW}\}$ by putting $\sigma = \text{CCW}$ or $\sigma = \text{CW}$ according as Λ is bounded or unbounded. (See Figure 2.2 for the case that Λ is bounded.)

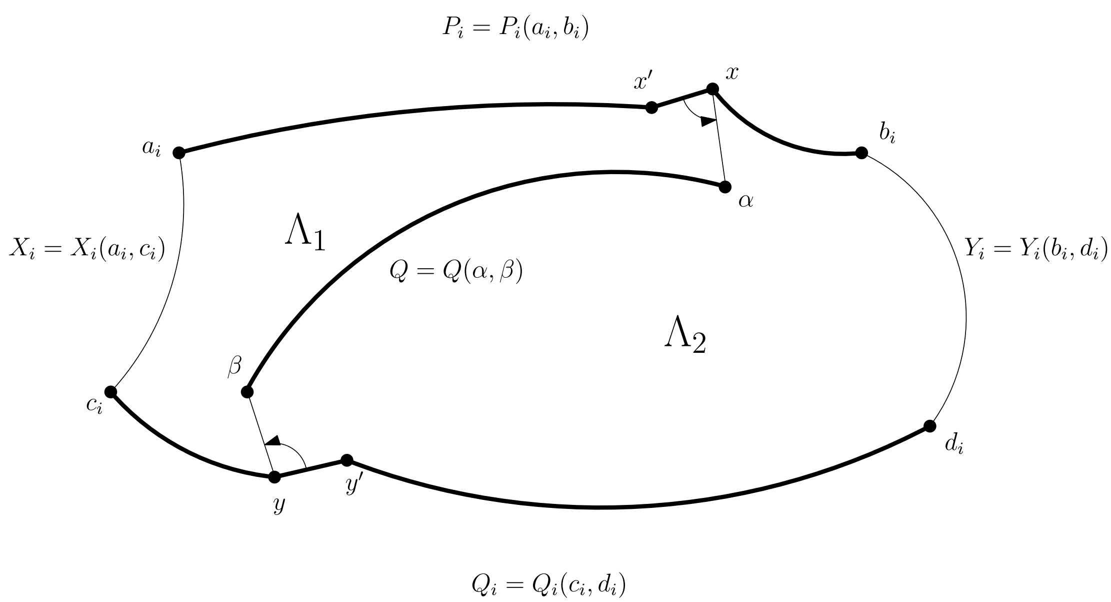


Figure 2.2: Case 3: scenario B_i, with each of the paths P_i and Q_i having a unique vertex with a neighbor in R_{i+1}

Consider $\alpha \in R_{i+1} \cap N(x)$ and $\beta \in R_{i+1} \cap N(y)$ such that $\text{cone}_\sigma(x', x, \alpha) \cap (R_{i+1} \cap N(x)) = \varnothing$ and $\text{cone}_\sigma(y', y, \beta) \cap (R_{i+1} \cap N(y)) = \varnothing$. Choose a geodesic path $Q = Q(\alpha, \beta)$ in G_{i+1} and let Q' be the path obtained by adjoining $x\alpha$ and $y\beta$ to Q. Move the dormant cop in stage i so that it reaches the shadow of the robber on Q in G_{i+1} in some step of the game. As in case 2, we may assume the position z the robber assumes in that step of the game is not in $V(Q)$. Let Λ_1 and Λ_2 be the two regions which are obtained by cutting Λ with Q', so that the boundaries of Λ_1 and Λ_2 include X_i and Y_i, respectively. Since x and y are the only vertices on Γ_i with a neighbor in R_{i+1}, and no vertex of R_{i+1} is adjacent to an internal vertex of X_i or Y_i, any path from z to a vertex of Γ_i has to contain a vertex of Q'. Hence, by letting R be the set of all vertices reachable from z without stepping onto $V(Q')$, we will have

$$R \subseteq R_{i+1} \setminus V(Q) \subsetneq R_{i+1}. \tag{2.1}$$

Without loss of generality assume z is located in Λ_1. As such, let P be the 1-path at y, X be the 2-path from y to β, and Y be the path from y to α obtained by adjoining αx to the intersection of Γ_i with the boundary of Λ_1. Let Γ be the simple closed curve formed by P, Q, X, and Y. Move the cop on P_i to y and then make the cop originally on Q_i dormant. Then, by setting $a_{i+1} := y =: b_{i+1}$, $c_{i+1} := \alpha$, $d_{i+1} := \beta$, $P_{i+1} := P$, $Q_{i+1} := Q(\beta, \alpha)$, $X_{i+1} := X$, and $Y_{i+1} := Y$, and according to the defining property of α we will reach at stage $i + 1$ with scenario B_{i+1}. Moreover, by (2.1) and the defining property of α, we will have $R_{i+2} \subseteq R \subsetneq R_{i+1}$, as desired.

Case 4: Suppose, without loss of generality, that $x \in V(P_i)$. Move one of the dormant cops in stage i so that in some step of the game it reaches x and make all the other cops dormant. Let $y \neq x$ be the position the robber assumes in the same step. As in case 1, we have $y \in R_{i+1} \setminus \{x\} \subsetneq R_{i+1}$, and the set of all vertices reachable from y without stepping onto x is a subset of $R_{i+1} \setminus \{x\}$. Hence, by setting $v_{i+1} := x$ and $r_{i+1} := y$, we will reach at stage $i + 1$ with scenario A_{i+1} and $R_{i+2} \subsetneq R_{i+1}$.

Case 5: Suppose P_i has at least two vertices with a neighbor in R_{i+1}, and let x and y be the first and last such vertices of P_i. Let x' (resp. y') be the vertex of P_i immediately succeeding (resp. preceding) x (resp. y) along P_i. Furthermore, suppose, without loss of generality, that the orientation of P_i, i.e. from a_i to b_i, is consistent with the clockwise orientation of Γ_i. As in Case 3, let Λ be the planar region with boundary Γ_i that contains the vertices in R_{i+1}. Let $\{\sigma, \tau\} = \{\mathrm{CCW}, \mathrm{CW}\}$ where σ and τ are determined by putting $\sigma = \mathrm{CW}$ or $\sigma = \mathrm{CCW}$ according as Λ is bounded or unbounded. Consider $\alpha \in R_{i+1} \cap N(x)$ and $\beta \in R_{i+1} \cap N(y)$ such that all elements of $N(x) \cap R_{i+1}$ are in $\mathrm{cone}_\sigma(x', x, \alpha)$ and all elements of $N(y) \cap R_{i+1}$ are in $\mathrm{cone}_\tau(y', y, \beta)$.

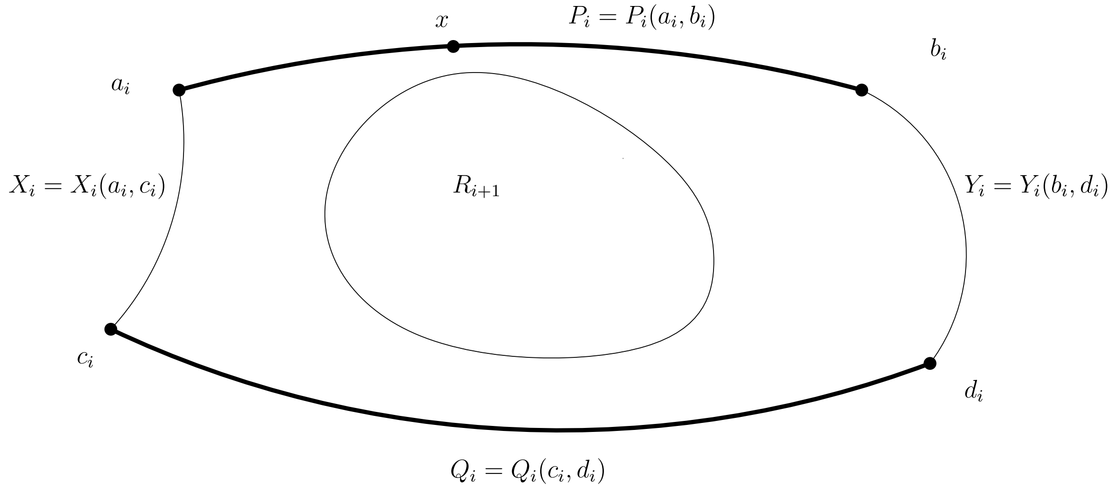


Figure 2.3: Case 4: scenario B_i, with a unique vertex in $V(P_i) \cup V(Q_i)$ having a neighbor in R_{i+1}

Let $Q = Q(\alpha, \beta)$ be a geodesic path in G_{i+1} and let Q' be the path obtained by adjoining the edges $x\alpha$ and $y\beta$ to Q. As in Case 3, if $V(Q) = R_{i+1}$ the dormant cop in stage i can capture the robber; otherwise, that cop can be moved so that in some step of the game it reaches the shadow of the robber on Q in G_{i+1}. Let z be the position the robber assumes in that step of the game. Let Λ_1 and Λ_2 be the regions obtained by cutting Λ with Q', such that Λ_1 is the one whose boundary contains Q_i.
Whether $z \in \Lambda_1$ or $z \in \Lambda_2$ we will be able to get to stage $i+1$ as follows:

Case 5a: $z \in \Lambda_1$.

Make the cop on P_i dormant. Let $X = X(\alpha, c_i)$ be the path consisting of X_i, $P_i(a_i, x)$, and the edge $x\alpha$, and let $Y = Y(\beta, d_i)$ be the path consisting of Y_i, $P_i(y, b_i)$, and the edge $y\beta$. Let R be the set of all vertices of G reachable from z without stepping onto Q_i or Q. According to the defining properties of α and β, R has no vertex in the neighborhood of x and y. Therefore, R has no vertex in the neighborhood of an internal vertex of X or Y, and hence is a subset of $R_{i+1} \setminus V(Q)$. By setting $a_{i+1} := \alpha$, $b_{i+1} := \beta$, $c_{i+1} := c_i$, $d_{i+1} := d_i$, $P_{i+1} := Q$, $Q_{i+1} := Q_i$, $X_{i+1} := X$, and $Y_{i+1} := Y$ we will reach at stage $i+1$ with scenario B_{i+1} and $R_{i+2} \subsetneq R_{i+1}$.

Case 5b: $z \in \Lambda_2$.

Move the cop on Q_i to the shadow of the robber on $P_i(x, y)$ in G_{k_i}, and make dormant the cop which was active on P_i in stage i. Let X'' and Y'' be the 2-paths from

x to α and from y to β, respectively. Then, by setting $a_{i+1} := x$, $b_{i+1} := y$, $c_{i+1} := \alpha$, $d_{i+1} := \beta$, $P_{i+1} := P_i(x, y)$, $Q_{i+1} := Q$, $X_{i+1} := X$, and $Y_{i+1} := Y$ we will reach at stage $i + 1$ with scenario B_{i+1} and $R_{i+2} \subseteq (R_{i+1} \setminus V(Q_{i+1})) \subsetneq R_{i+1}$.

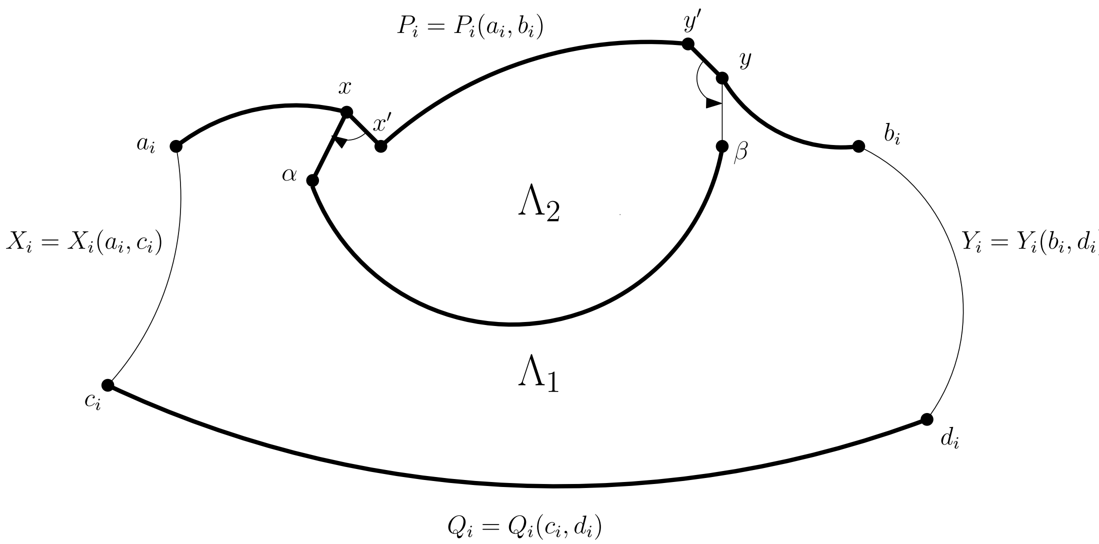


Figure 2.4: Case 5: scenario B_i, with P_i having at least two vertices with a neighbor in R_{i+1}

□

2.2 Cops and Robbers on Geometric Graphs

In this section we discuss the difficulties and pitfalls in trying to extend the idea of the proof of Theorem 2.1.5 presented in [1] or [5] to the class of (all) geometric graphs. In view of Proposition 2.1.3, in an instance of the game played on a plane graph, shadowing the robber on a geodesic path prevents the robber from traversing any edge that has a point in common with the path. However, the same might not be true if the game is played on a generic geometric graph. The reason is that in a geometric graph, by traversing an edge crossing an edge of a 1-guarded geodesic path the robber does not necessarily end up in the neighborhood of the shadowing cop (See Fig 2.5). Hence, implementing the shadowing strategy of Definition 2.1.1d does not always prevent the robber from both stepping onto or crossing an edge of a geodesic path in a geometric graph. One might hope that using more shadowing cops could resolve the issue. In that regard, it was shown in [4] that tripling the cops can work for geodesic paths in geometric graphs (Observation 2.2.1).

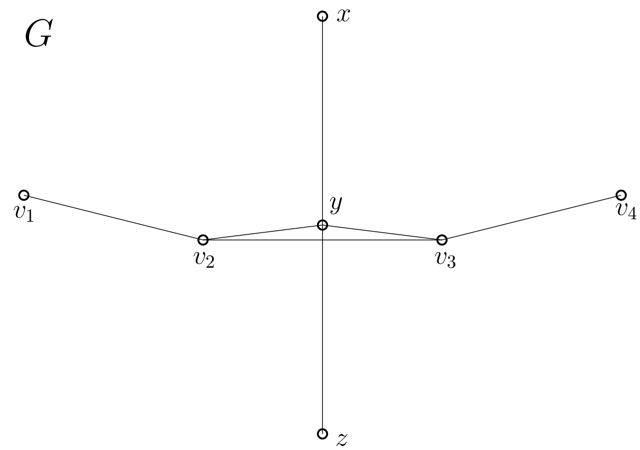


Figure 2.5: $P := v_1v_2v_3v_4$ is a geodesic path in a geometric graph G. We have $S_P(x) = v_4$, $S_P(y) = v_3$, and $S_P(z) = v_4$. While yz crosses edge v_2v_3 of P, z is not adjacent to $S_P(y)$.

Definition 2.2.1. Let $P = P(u, v)$ be a path oriented from u to v. For every $w \in V(P)$ we denote the immediate predecessor and the immediate successor of w along P by w_P^- and w_P^+, respectively; setting $u_P^- := u$ and $v_P^+ := v$. If P is a geodesic path in a graph H on which the game of cops and robbers with at least three cops is being played, in a step i of the game a group of three cops are said to be ***3-guarding*** P in H if right before robber's turn the cops in the group are at w_P^-, w, w_P^+ where w is the shadow of the position of the robber at the end of step $i - 1$ on P.

Remark. If P is a geodesic path in a graph H where the game of cops and robbers is being played, any set $\{C_1, C_2, C_3\}$ of three cops can eventually, i.e. after a finite number of steps, reach their 3-guarding positions on P. One strategy, for instance, is to get C_1 in u^+, and C_2 and C_3 both in u (phase 1), and then move C_2 toward the present shadow of the position of the robber on P while keeping C_1 and C_3 in the immediate predecessor and immediate successor of the position of C_2 (phase 2), until C_2 gets to the shadow in some step i of the game. Then, before robber's move in step i, cops C_1, C_2, and C_3 will be in their 3-guarding positions on P. Note that a set of cops 3-guarding a geodesic path P in a step i of the game can retain their 3-guarding positions in all subsequent steps, since the shadows of the robber on P in two consecutive steps are within distance 1 along P; a fact that one can easily verify using geodesicity of P.

Observation 2.2.1. *[4] Let P be a geodesic path in a geometric graph G where the game of cops and robbers is being played. In a finite number of steps any group of* *three cops can be put in the 3-guarding position for P in G. Moreover, if the robber steps onto P or crosses P while it is being 3-guarded, the robber will be caught in the next step of the game.*

In view of Observation 2.2.1, it was claimed in [4] that "three cops in a geometric graph play the role of one cop in a planar graph guarding a geodesic path", based on which the authors concluded that every geometric graph was 9-copwin. The main problem with this argument is that in an adaptation of a proof of Theorem 2.1.5 to get a possible upper bound of nine for the cop number of geometric graphs, one needs to shrink the present *playground*, i.e. the subgraph of the original graph to which the game is restricted, and the latter in general requires the removal of not only some vertices but also some edges of the present playground. As such, a playground obtained from a geometric playground (i.e. a playground which is a geometric graph) might not be always geometric. Therefore, in order for the idea in [4] to work, one would need to show that geodesic paths in not only geometric graphs but also in subgraphs of geometric graphs were 3-guardable.

2.3 Constructing Geometric Graphs of Cop Number Three

A geometric graph can be viewed as a straight-line drawing of a graph in the plane where two points representing two vertices are adjacent if and only if their Euclidean distance is less than or equal to a positive parameter r. As such, in accordance with Definition 1.5.2, we call r the *parameter* of the geometric graphs. In this setup, ahead of determining whether a given drawing of a graph is geometric we need to fix r. A path drawn in the plane as a geometric graph is called a *geometric path*. Geometric graphs constitute a proper subclass of *string graphs*, as the intersection graphs of strings (or curves) in the plane. It has been shown that $C(G) \leq 15$ for every string graph G [9], but a geometric graph with cop number ≥ 4 is yet to be found. In [4] the authors provide one geometric graph on 1440 vertices with cop number three. Indeed, they present a planar graph with girth five and minimum degree 3 as a geometric graph. That such a graph has cop number three simply follows from Theorem 2.1.2. In Section 2.4 we improve this result by providing a representation of a graph on 440 vertices and cop number three as a geometric graph (Theorem 2.4.1). Indeed, the technique utilized in the proof of Theorem 2.4.1 leads to the following general results:

Theorem 2.3.1. *Every planar graph G with maximum degree $\Delta \leq 5$ has a subdivision into a planar geometric graph with cop number $C(G)$ or $C(G)+1$.*

Theorem 2.3.2. *For every planar graph G with maximum degree $\Delta \leq 9$, there is a subdivision of the clique substitution $K(G)$ of G (Definition 1.5.4) having a geometric representation and cop number at least $C(G)$.*

Note that either result can be used to provide geometric graphs of cop number three. The construction provided in the proof of Theorem 2.3.1 is based on obtaining a polygonal-curve embedding from a given straight-line embedding of G and then subdividing the *edge-curves*, i.e. the polygonal curves representing edges of G, equally many times such that with an appropriate parameter for the geometric graphs, the resulting embedding is geometric. The latter, for example, requires that no subdividing vertex on an edge-curve be adjacent to a vertex belonging to another edge-polygonal curve. The latter, in particular, requires the angle between any two segments incident with a vertex of G be greater than $\pi/3$. The idea for the construction in the proof of Theorem 2.3.2 is similar.

2.3.1 Proof of Theorem 2.3.1

Given a planar graph G with $\Delta(G) \leq 5$, we first identify it with any of its straight-line embeddings, which exist according to Fáry's Theorem [7]. Then, if necessary, we replace the endings of edge-segments with a polygonal curve of at most five segments in such a way that in the resulting polygonal-curve planar graph the angles between any pair of consecutive edges at a vertex is greater than $\pi/3$ and in every edge-curve, both of the angles between any two consecutive segments are also greater than $\pi/3$ - see Lemma 2.3.6 for a justification of such adjustments. Finally, we shall show that in such a polygonal-curve embedding of G all edge-curves can be subdivided into paths of some fixed length so that the resulting graph is geometric. Thus, we can use the following lemma to establish Theorem 2.3.1.

Lemma 2.3.3. *[10] Let G' be the subdivision of a graph G obtained by replacing every edge of G with a path of length l for some fixed $l \in \mathbb{N}$. Then,*

$$C(G) \leq C(G') \leq C(G)+1. \tag{2.2}$$

Lemma 2.3.4. *Let A and B be two points in the plane having Euclidean distance 1, and let S be the square having AB as a diagonal. Then, given $k \in \mathbb{N}$, the parameter r*

of geometric graphs can be set so that for each integer l between $5k+1$ and $4k^2+6k+1$ there exists a geometric path (i.e. a path drawn as a geometric graph) of length l between A and B having no vertex outside of S.

Proof. Consider the Cartesian coordinate system where $A=(0,0)$ and $B=(1,0)$, and let $C=(1/2,1/2)$ and $D=(1/2,-1/2)$ be the other corners of S. Given $k \in \mathbb{N}$ let

$$\alpha_k = \sin^{-1}\left(\frac{1}{2(2k+1)^2}\right) \qquad (0<\alpha_k<\pi/2). \tag{2.3}$$

Observe that since sum of the squares of $(4k+1)/(4k+2)$ and $1/(2(2k+1)^2)$ is less than one, we have $\cos\alpha_k > (4k+1)/(4k+2)$ and, hence, $(2k+1)\tan\alpha_k < 1/(4k+1)$. We let the parameter of geometric graphs be r_k given by

$$r_k = \frac{\sqrt{2}}{4k}\left(1-(2k+1)\tan\alpha_k\right). \tag{2.4}$$

As such, we will have

$$r_k > \frac{\sqrt{2}}{4k+1}. \tag{2.5}$$

Moreover, we set the vectors

$$\vec{X_k} = \langle \cos(\pi/4-\alpha_k), \sin(\pi/4-\alpha_k)\rangle, \tag{2.6}$$

$$\vec{Y_k} = -\langle \cos(\pi/4+\alpha_k), \sin(\pi/4+\alpha_k)\rangle, \tag{2.7}$$

and let γ_k be the polygonal curve from A to B consisting of $k+1$ line-segments parallel to $\vec{X_k}$, called $\vec{X_k}$*-segments*, each from a point on AD to a point on BC, k line-segments parallel to $\vec{Y_k}$, called $\vec{Y_k}$*-segments*, each from a point on BC to a point on AD, and $2k$ line-segments of length r_k parallel to the directed line-segment from B to C, called *flat segments*, such that the initial and terminal segments of γ_k are $\vec{X_k}$-segments, and each of the first k $\vec{X_k}$-segments in γ_k is followed by exactly one flat segment which itself is followed by a $\vec{Y_k}$-segment, and, likewise, each of the Y_k-segments in γ_k is followed by exactly one flat segment which is itself followed by an $\vec{X_k}$-segment- see Figures 2.6 and 2.7 for examples. We call any of the k three-segment subcurves of γ_k starting with an X_k-segment a *dent* of γ_k. By (2.4), we have $2rk < \sqrt{2}/2$. Moreover, with l being the common length of X_k-segments and Y_k-segments (which can reasonably be referred to as *slant* segments) we have $l = \sqrt{2}/(2\cos\alpha_k) < (2k+1)\sqrt{2}/(4k+1)$.

Hence, according to (2.5), we obtain

$$2r_k k < \frac{\sqrt{2}}{2} < l < (2k+1)r_k. \tag{2.8}$$

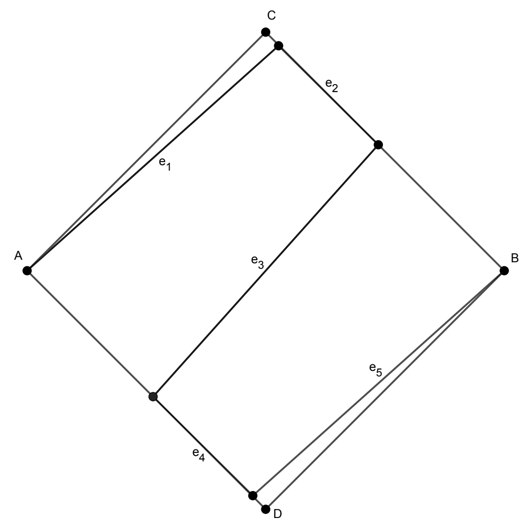


Figure 2.6: The polygonal curve γ_1 consisting of five line segments: e_1 and e_5 parallel to X_1, e_3 parallel to Y_1, and e_2 and e_4 of length r_1.

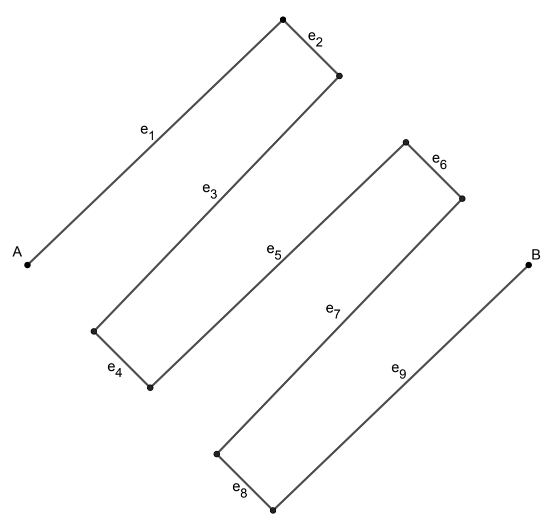


Figure 2.7: The polygonal curve γ_2 consisting of nine line segments: X_2-segments e_1, e_5, and e_9, Y_2-segments e_3 and e_7, and flat-segments e_2, e_4, e_6 and e_8 of length r_2.

Consequently, with r_k as the parameter of geometric graphs, each of the slant segments of γ_k can be subdivided into a geometric path of length $2k+1$. Thereby, as γ_k consists of $2k$ flat segments of length r_k and $2k+1$ slant segments, it can be subdivided into a geometric path of length $(2k+1)^2 + 2k = 4k^2 + 6k + 1$. We shall denote such a geometric path also by γ_k. Hence, to complete the proof it suffices to show the following:

Claim 1. Given any dent D of γ_k and for each $s \in [2 \cdot\cdot (4k+2)]$, one can delete some vertices of D and then introduce one or two new vertices inside D to make D into a geometric path of length s. Moreover, such a change can be made for each of the dents of γ_k so that the entire resulting path will stay geometric. In particular, any integer between $5k+1$ and $4k^2+6k+1$ can be attained as the length of a geometric path between A and B having no vertex outside of S.

Proof of Claim 1. Let $p_0, \ldots, p_{2k+1}, q_{2k+1}, q_{2k}, \ldots, q_0$ be the sequence of vertices in D and let $s = 2t+1$ or $s = 2t+2$ for some $t \in [0 \cdot\cdot k]$. Furthermore, let w be the intersection of the segments $p_t q_{t+1}$ and $q_t p_{t+1}$ and pick points p' and q' on the segments $p_{t+1}w$ and $q_{t+1}w$ such that $d_E(p_t, q') = d_E(q_t, p') > r_k$ and $d_E(p', q') \leq r_k$ (Figure 2.8).

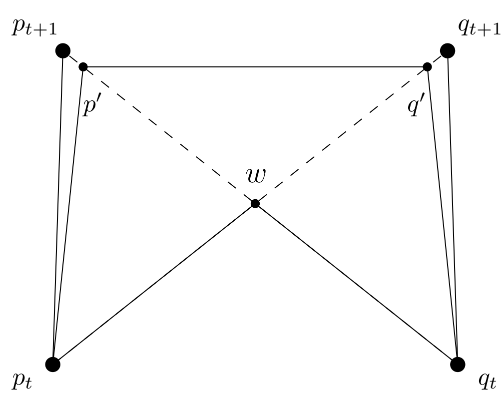


Figure 2.8: Adjusting the length of a dent

Observe that the paths $p_0, \ldots, p_t, w, q_t, \ldots, q_0$ and $p_0, \ldots, p_t, p', q', q_t, \ldots, q_0$ are geometric. Hence, to obtain the desired length s it suffices to remove all vertices p_i, q_i with $i > t$ from D, and according as $s = 2t+1$ or $s = 2t+2$, add w or both p' and q'. Note that the newly added vertices will be at a distance greater than r_k from vertices of γ_k which are not in D. Hence, all of the dents of γ_k can be adjusted this way while keeping the resulting path geometric. Since the last Y_k-segment of γ_k and k of the flat segments of γ_k are in no dent and any dent can be reduced to a geometric path of any length between 2 and $4k+2$ or kept at the original length of $4k+3$, γ_k can be adjusted to a geometric path of any length between $2k+(2k+1)+k=5k+1$ and $4k^2+6k+1$.

□ Claim 1

□

The following two technical lemmas will serve to justify that the proposed alterations of the terminal parts of edges of G in the proof of Theorem 2.3.1 will keep the graph geometric without adding too many segments.

Lemma 2.3.5. *Let O be a vertex of degree at most five in a straight-line plane graph G. Let $\mathscr{H}$ be the collection of all sets H of six distinct rays emanating from O such that the angle between any consecutive pair of rays in H is $\pi/3$. For every $H \in \mathscr{H}$ let $\sigma(H)$ be the number of edges of G incident with O making an angle $\leq \pi/37$ with a ray in H. Then* $\min\{\sigma(H) : H \in \mathscr{H}\} = 0$.

Proof. Let $x = \min\{\sigma(H) : H \in \mathscr{H}\}$. Note that one can rotate the rays of any $H \in \mathscr{H}$ to obtain some $H' \in \mathscr{H}$ satisfying $\sigma(H') \leq 5 - \sigma(H)$; hence, $x \leq 2$. Consider some $L \in \mathscr{H}$ with $\sigma(L) = x$, and set $\alpha = \pi/18$. Note that if an edge incident with O makes an angle $\leq \pi/37$ with a ray in L, then any of the 10 rotations of H about O by $\pm\alpha$, $\pm 2\alpha, \pm 3\alpha$, $\pm 4\alpha$, and $\pm 5\alpha$ puts that edge in an angular distance greater than $\pi/37$ from any ray in L. According to this observation,

- we have $x \neq 2$, for otherwise rotating the rays of L by 2α would leave no more than one edge incident with O within angular distance of $\pi/37$ to a ray of H, a contradiction; and
- we also have $x \neq 1$, for otherwise if one applies five consecutive rotations of all rays in L about O by $\pi/18$, after each of the rotations at least one new edge has to be placed within angular distance of $\pi/37$ to a ray in L, requiring $deg(O) \geq 6$, a contradiction.

Hence, $x = 0$. □

Lemma 2.3.6. *With the assumptions and notation of Lemma 2.3.5, let $L \in \mathscr{H}$ such that $\sigma(L) = 0$; and let H_1 and H_2 be distinct regular hexagons with center O and side length l_1 and l_2 (with $l_1 > l_2$) and corners on the rays of L. Then for every edge*

e incident with O one can pick $\pi(e) \in \{1,2\}$ and replace the segment of e inside $H_{\pi(e)}$ with a simple polygonal curve α_e consisting of at most four line segments: a line segment C_e that connects O to a point Q_e on the boundary of $H_{\pi(e)}$, together with a connected portion of the boundary of $H_{\pi(e)}$ comprising at most three line-segments, in such a way that the following hold (see Figure 2.9 and Figure 2.10):

a. *Every segment of each α_e is longer than*

$$s_{\pi(e)} := \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\tan(31\pi/222)\right) l_{\pi(e)} (> 0.093 l_{\pi(e)}). \tag{2.9}$$

b. *After replacing the ending of each e with α_e, no two consecutive segments on the polygonal curve representing e have an angle $\leq \pi/3$.*

c. *For every pair e, e' of distinct edges incident with O both angles (clockwise and counterclockwise) between C_e and $C_{e'}$ are greater than $\pi/3$. Moreover, the minimum distance between α_e and $\alpha_{e'}$ is at least* $\min\{(\sqrt{3}/2)(l_1 - l_2), l_2, (\sqrt{3}/2)s_1, \delta\}$ *where s_1 is given by* (2.9)*, and δ is the minimum distance between $e_1, \ldots, e_5$ outside the smaller hexagon H_2.*

Proof. For each $i \in \{1,2\}$ let the $A_{i,k}$ $(k = 1, \ldots, 6)$ be the corners of H_i , say, clockwise around O such that for every k, $A_{1,k}$ and $A_{2,k}$ belong to the same ray of L. We shall establish the lemma in the following extreme cases; the other cases can be dealt with in a similar fashion.

Case I: $\deg(O) = 5$ and all edges incident with O are between two consecutive rays in L; in other words, all such edges intersect the same side of H_j $(j = 1, 2)$.

Case II: $\deg(O) = 5$ and no two consecutive rays in L enclose two of the edges incident with O; in other words, no to edges incident with O cross the same side of H_j $(j = 1, 2)$.

Let $e_1, \ldots, e_5$ be the edges incident with O in the clockwise order around O, and for each $i \in [1 \cdot\cdot 5]$ and $j \in [1 \cdot\cdot 2]$ let $C_{i,j}$ be the intersection of e_i with the boundary of H_j. We also denote every Q_{e_i} simply with Q_i.

Handling of Case I: Suppose $e_1, \ldots, e_5$ cross sides $A_{1,i}A_{2,i}$ of H_i $(i = 1, 2)$. Set $\pi(e_1) = \pi(e_3) = \pi(e_5) = 1$ and $\pi(e_2) = \pi(e_4) = 2$. Also, set $Q_1, \ldots, Q_5$ as follows:

- $Q_1 = A_{1,5}$

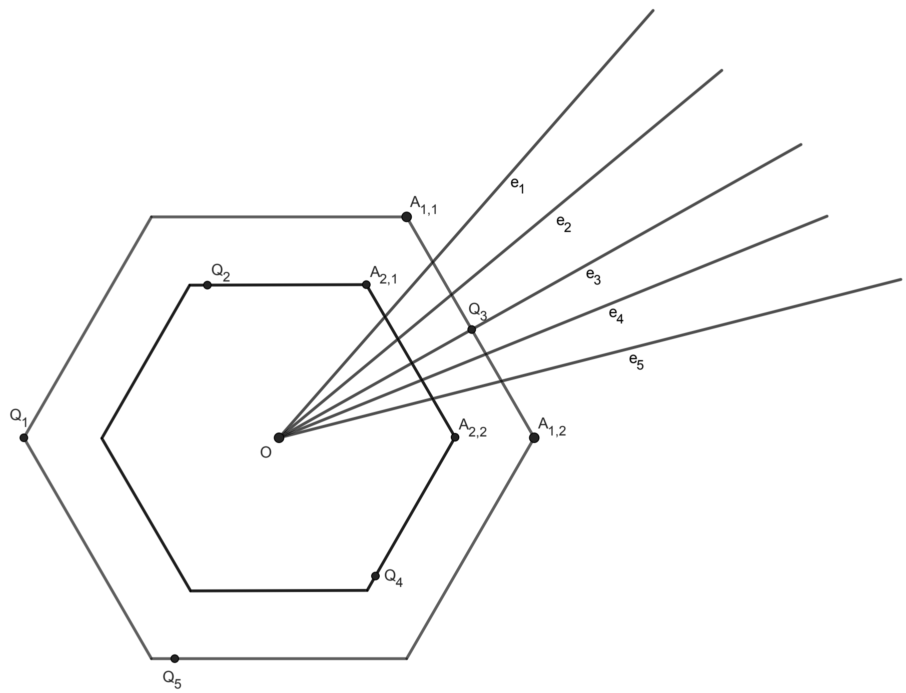


Figure 2.9: Case I: $\deg_G(O) = 5$ and all edges incident with O intersect the sides $A_{1,1}A_{1,2}$ of H_1 and $A_{2,1}A_{2,2}$ of H_2

- Q_2: the point on the segment $A_{2,1}A_{2,6}$ with $\measuredangle(Q_2, O, A_{2,6}) = \pi/37$;
- $Q_3 = C_{3,1}$;
- Q_4: the point on the segment $A_{2,2}A_{2,3}$ with $\measuredangle(Q_4, O, A_{2,3}) = \pi/37$; and
- Q_5: the point on the segment $A_{1,3}A_{1,4}$ with $\measuredangle(Q_5, O, A_{1,4}) = \pi/38$.

Furthermore, from O let α_{e_1} and α_{e_2} be clockwise around H_1 and H_2 , and α_{e_4} and α_{e_5} be counterclockwise around H_2 and H_1. Then, one can easily check that properties **(a)**-**(c)** are satisfied by α_{e_i}s.

Handling of Case II: Suppose $C_{i,1}$ belongs to the side $A_{1,i}A_{1,i+1}$ of H_1 for each $i \in [1 \cdot\cdot 5]$. As $\sigma(H_1) = 0$, we have $\measuredangle(C_{i,1}, O, A_{1,i}) > \pi/37$ or, equivalently, $d_E(A_{1,i}, C_{i,1}) > s_1$ for each $i \in [1 \cdot\cdot 5]$, where s_1 is given by (2.9). Let

$$a = \min\{d_E(A_{1,i}, C_{i,1}) - s_1 : i \in [1 \cdot\cdot 5]\}.$$

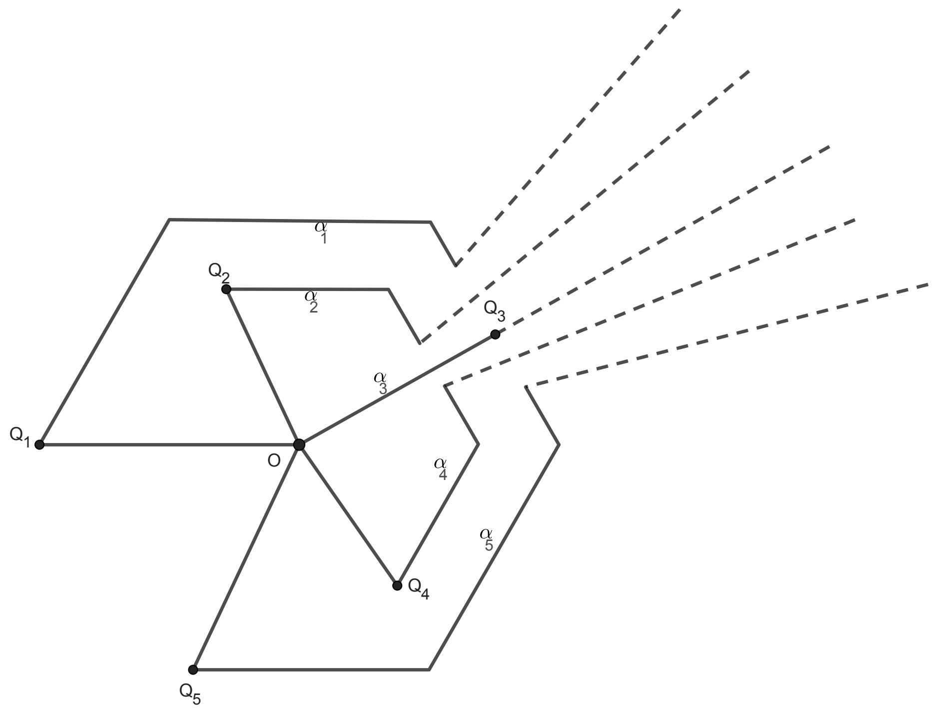


Figure 2.10: Replacing the endings in case I with polygonal curves $\alpha_1, \ldots, \alpha_5$ to gain the desired properties

Furthermore, to satisfy **(a)**-**(c)**, for each $i \in [1 \cdot\cdot 5]$ let α_i be clockwise around H_1 and pick Q_i on the segment $C_i A_{1,i}$ such that

$$d_E(Q_i, A_{1,i}) = \frac{a(i-1)}{5}.$$

□

Proof of Theorem 2.3.1. Let α be the minimum angle between pairs of edges of G with a common endpoint, and choose $a > 0$ small enough so that

- the (minimum) Euclidean distance between any pair of non-incident edges is greater than $3a$; and
- for every $v \in V(G)$, the ball of radius $6a$ centered at v does not contain any vertex in $V(G) \setminus \{v\}$.

Fixing a as such, we break up every edge-segment $e_i = uv$ of length, say, l_i into three parts, an initial part of length $2a$ starting, say, at u, a middle part of length $\lambda_i a$ where $\lambda_i \in \lfloor l_i/a \rfloor - 4$, and a terminal part ending at v which is (necessarily) of a length between $2a$ and $3a$. Suppose $\lambda_1 \leq \cdots \leq \lambda_m$ where m is the number of edges of G. Next, we apply Lemma 2.3.6 to each vertex of the graph and with regular hexagons of side lengths a and $1.5a$ to replace each of the initial and terminal parts of edge-segments with polygonal curves of at most five segments, one segment outside the hexagon associated with the edge-ending and up to four more segments, according to Lemma 2.3.6. Note that for each resulting edge polygonal-curve, the endings will consist of at most 10 segments of a total length less than $15a$. Therefore, for each $k \in \mathbb{N}$, by using $r_k a$ as the parameter of geometric graphs one can replace the endings of each edge with a total of not more than $10 + 15\lceil 1/r_k \rceil$ segments, which is bounded above by $10 + 60k$, according to (2.5). We choose $k \in \mathbb{N}$ large enough so that

$$\lambda_1 k^2 \geq 10 + 60k, \qquad \lambda_1(3k^2 + 6k + 1) \geq \lambda_m(5k + 1),$$

and, from (2.9),

$$r_k \leq \min\{2\sin(\alpha/2), \sqrt{3}/2(0.093)\}.$$

Then according to Lemma 2.3.4, the middle part of every edge polygonal curve can be replaced with a geometric path of appropriate lengths (between $(5k+1)\lambda_i$ and $4k^2 + 6k + 1)\lambda_i$ for each part of an initial (Euclidean) length $\lambda_i a$, so that the resulting graph G' is a graph obtained from G by replacing edges with paths of the same (graph) length. Moreover, according to Lemmas 2.3.4 and 2.3.6 and our choices for a and k, G' is a geometric plane graph. Finally, $C(G') \in \{C(G), C(G) + 1\}$ according to Lemma 2.3.3. □

Corollary 2.3.7. *There is an infinite family of geometric graphs of cop number three.*

Sketch of proof. Let G be any straight-line embedding of the dodecahedron (or any other planar graph of cop number three). Applying the construction described in the proof of Theorem 2.3.1 gives a planar geometric graph G' with a parameter r, chosen as in the proof of the theorem. Given any $k \in \mathbb{N}$ replace every edge e of G' with a path of k equal-length segments, without changing the geometry of e. The resulting plane graph G_k will be a geometric graph with parameter r/k (by construction) and the cop number of three. □

2.3.2 Proof of Theorem 2.3.2

Let G be a planar graph such that $\Delta(G) \leq 9$, identified with any of its straight-line embeddings in the plane. We shall show how to construct a subdivision of (a drawing of) $K(G)$ which is geometric and has cop number $\geq C(G)$, where the latter will be established using Lemma 2.3.3 alongside the following result:

Lemma 2.3.8. *[10] The operation of clique substitution does not decrease the cop number.*

Proof of Theorem 2.3.2 (Sketch). The techniques are similar to those in the proof of Theorem 2.3.1 and related lemmas. The construction is carried out in two main phases:
Phase I: At each vertex v of G we pick a partition of the plane into cones with apex v and angle $2\pi/9$, such that one of the edges incident with v lies on one of the rays of the cones. We also consider four regular 9-gons $\Pi_{i,v}$ ($i \in [1 \cdot\cdot 4]$) centered at v corresponding to the chosen partition of the plane at v, with distinct side lengths s_i (independent of v) such that $\min\{s_i\}$ is substantially less that the shortest edge in G. Then, for each edge incident with v, we replace its end at v with a polygonal curve consisting of a portion of the boundary of one of the 9-gons and a segment from v that lies on one of the rays in the decomposition. This phase is implemented in a similar fashion to the adjustments of the edge endings in Lemma 2.3.6, except that with up to nine possible edges incident with v, one needs to use the boundaries of four, rather than two, polygons.

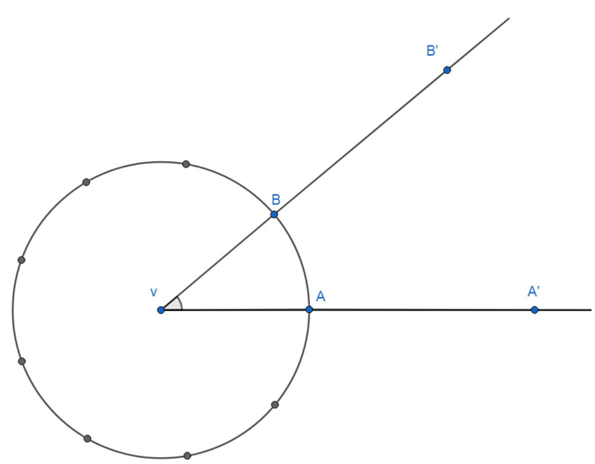


Figure 2.11: A and B are two consecutive vertices of a regular n-gon circumscribed by the circle of radius $r/2$ centered at v, and $|AA'| = |BB'| = r$. We need $|A'B'| > r$ or, equivalently, $n < \pi/(\sin^{-1}(1/3))$.

Phase II: We pick the parameter r of the geometric graphs such that $r \ll \min\{s_i\}$ and $r \ll \min_{i \neq j} |s_i - s_j|$. Then, we subdivide the edges so that the second vertices of endings at every vertex form a clique of size $deg_G(v)$. As such, by removing the original vertices of G, what we obtain is a drawing of a graph obtained from $K(G)$ by subdividing edges outside the knots. Finally, using the technique of Lemma 2.3.4 we can adjust the latter graph to a geometric graph where the edges outsides the knots are subdivided into an equal number of edges. Note that we need $\Delta(G) \leq 9$ in order to make sure that subdividing vertices for different edges incident to a vertex of G do not lie within distance r from each other (See Figure 2.11).

□

2.4 A small geometric graph requiring three cops

Theorem 2.4.1. *The graph G obtained by subdividing every edge of the dodecahedron into 15 edges has got cop number 3; moreover, it admits a geometric representation.*

Proof. Since the dodecahedron has cop number 3, we have $C(G) \in \{3, 4\}$, according to Lemma 2.3.3. But being planar G has cop number ≤ 3. Hence, $C(G) = 3$.

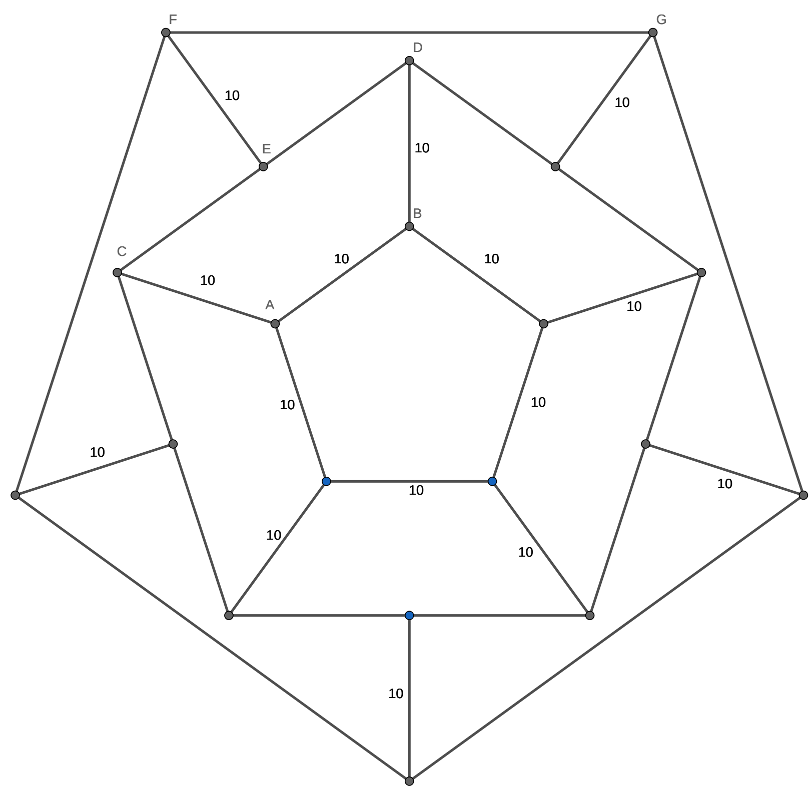


Figure 2.12: An embedding of the dodecahedron

To complete the proof, we provide a geometric representation of G derived from a specific straight-line embedding shown in Figure 2.12. In this embedding, where number 10 next to some edges refers to their length, we consider five different edge types represented by AB, AC, CE, DE, and FG. Next, we replace each of these edge types with appropriate polygonal curves, as shown in Figure 2.13, so that the resultant graph remains planar and can be made into a geometric graph by introducing 14 new vertices along each curve.

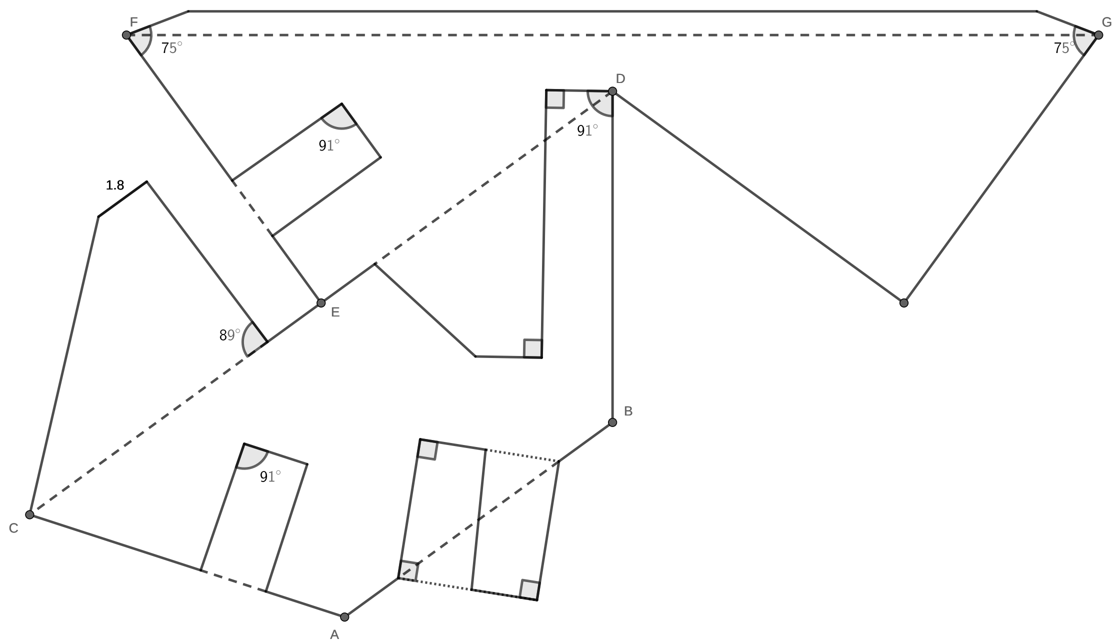
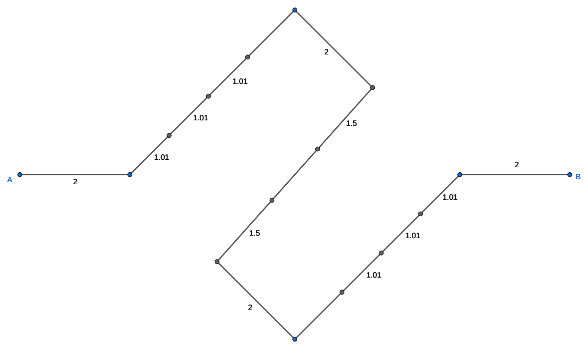
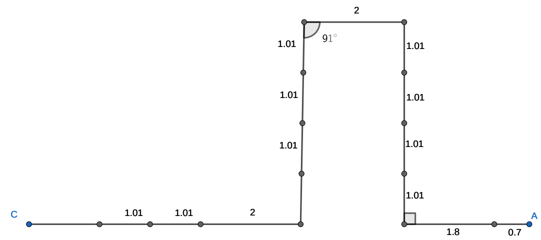


Figure 2.13: Replacing edges in Figure 2.12 with polygonal-curve embedding of the Dodecahedron

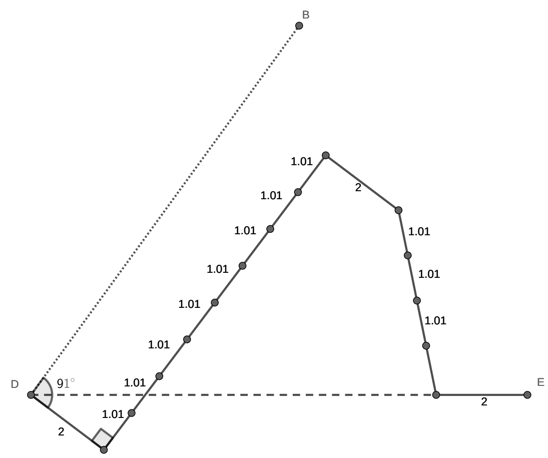
The details of the latter operations are shown in Figure 2.14. One can easily check that the final embedding is indeed a geometric representation of G with parameter $r = 2$.



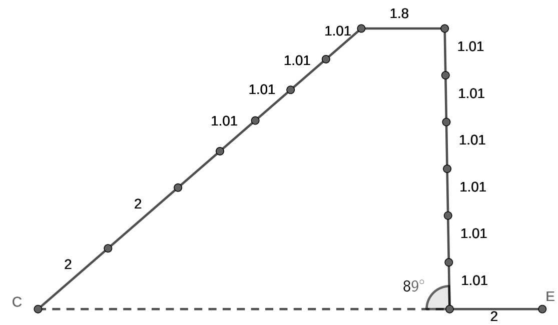
(a) Subdividing AB



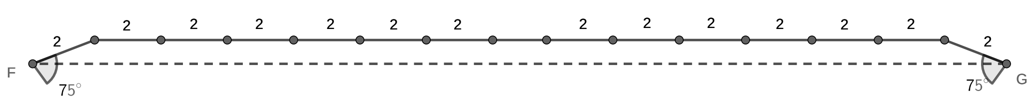
(b) Subdividing AC and EF



(c) Subdividing DE



(d) Subdividing CE



(d) Subdividing FG

Figure 2.14: Subdividing the graph obtained as in Figure 2.13

□

Chapter 3

Cops and Robbers on Graphs with Forbidden Induced Subgraphs

In this chapter we study the game of cops and robbers on classes of graphs defined by a set of forbidden induced subgraphs and aim to characterize such classes which are cop-bounded.

3.1 Known results

Definition 3.1.1. Let $\mathscr{H}$ be a set of graphs. A graph G is called *$\mathscr{H}$-free* if no graph in $\mathscr{H}$ is an induced subgraph of G. If $\mathscr{H}$ is a singleton, say $\{H\}$, we shall use $\{H\}$-free and H-free interchangeably.

Our point of departure is the following theorem which characterizes graphs H such that the class of H-free graphs is cop-bounded.

Theorem 3.1.1 ([10]). *The class of H-free graphs has bounded cop number if and only if every connected component of H is a path.*

Theorem 3.1.1 in particular implies that the class of claw-free graphs is not cop-bounded. In this regard, in Section 3.2 we present some subclasses of claw-free graphs which are cop-bounded and show winning strategies for some constant number of cops on each class. Section 3.3 is devoted to presenting a tool (Lemma 3.3.2) that plays a part in establishing necessary and sufficient conditions on $\mathscr{H}$ for the class of $\mathscr{H}$-free graphs be cop-bounded, in case the diameters of the members of $\mathscr{H}$ have a bound, as presented in Theorem 3.4.1, Section 3.4. In Section 3.4.1 we extend Theorem 3.4.1 to the case that the diameters of components of members of $\mathscr{H}$ have a bound (Theorem 3.4.3). Finally, we study some particular classes of graphs without a generalized claw and a generalized net, providing customized winning strategies and better upper bounds than the general setup used in Theorem 3.4.1.

Notation. Let U and W be disjoint subsets of the vertex set of a graph G. Then we write $U \Leftrightarrow_G W$ (or simply $U \Leftrightarrow W$ if the graph G is understood from the context) to mean that every vertex in U is adjacent to every vertex in W.

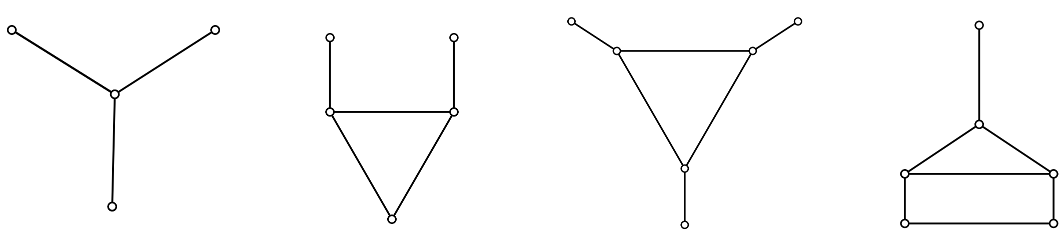
The two operations of clique substitution (Definition 1.5.4) and k-subdivision (Definition 1.5.5) were used in [10] and play a crucial role in the main results of this chapter as well. The aforementioned operations are specially useful in this chapter, since according to Lemmas 2.3.3 and 2.3.8 neither can reduce the cop number of a graph.

In addition, we shall be using the fact that cubic graphs are not cop-bounded:

Theorem 3.1.2. *[2] For every $k \geq 3$ the class of k-regular graphs is cop-unbounded.*

3.2 Cops and Robbers on Some Classes of Claw-free Graphs

We shall use H_C, H_B, H_N, and H_A to denote claw, bull, net, and antenna, as shown in Figure 3.1.



(a) Claw (H_C) **(b)** Bull (H_B) **(c)** Net (H_N) **(d)** Antenna (H_A)

Figure 3.1: Claw, Bull, Net, and Antenna

Definition 3.2.1. [Generalized claw and net] With $n_1, n_2, n_3 \in \mathbb{N} \cup \{0\}$ and $X \in \{H_C, H_N\}$, we denote the graph obtained by replacing the pendent edges in X with paths of length n_1, n_2, and n_3 by $X(n_1, n_2, n_3)$, calling it a generalized claw or net, according as $X = H_C$ or $X = H_N$. For $n \in \mathbb{N} \cup \{0\}$, we denote $X(n, n, n)$ simply by $X(n)$, calling it the n-claw or n-net, according as $X = H_C$ or $X = H_N$.

Note that by Theorem 3.1.1, for each $X \in \{H_C, H_B, H_N, H_A\}$ the class of X-free graphs is cop-unbounded. We shall consider the game of cops and robbers on the following classes of graphs and for each of them provide an upper bound for the cop number of its members. Note that we are not particularly interested in optimizing these upper bounds.

- $\mathbf{Cl}_1$: the class of connected claw- and bull-free graphs;
- $\mathbf{Cl}_2$: the class of connected claw-, net-, and antenna-free graphs;
- $\mathbf{Cl}_3$: the class of connected claw- and net-free graphs.

In what follows we also denote the class of connected claw-free graphs by $\mathbf{Cl}$.

Lemma 3.2.1. *Let G be a connected graph of order ≥ 2. Pick any two adjacent vertices u_0 and u_1 of G and let U be the set of neighbors of u_0 in $G - u_1$. Let H be the graph obtained from G by removing U. Set $N_0 = \{u_0\}$, and for each $i \in \mathbb{N}$ let N_i be the ith neighborhood of u_0 in H. In other words, for every $i \in \mathbb{N} \cup \{0\}$ let $N_i = \{v \in V(H) : d_H(u_0, v) = i\}$. Then:*

- ***a.*** *If $G \in \mathbf{Cl}$ and i an integer ≥ 2 any pair of distinct vertices in N_i with a common neighbor in N_{i-1} are adjacent. In particular, N_2 is a clique.*
- ***b.*** *If $G \in \mathbf{Cl}_1$ then each N_i is a clique. Moreover, $N_i \Leftrightarrow N_{i-1}$ for $i \geq 1$.*
- ***c.*** *If $G \in \mathbf{Cl}_2$ then every N_i is a clique and contains a vertex that dominates N_{i+1}.*
- ***d.*** *If $G \in \mathbf{Cl}_3$ each N_i has independence number ≤ 2.*

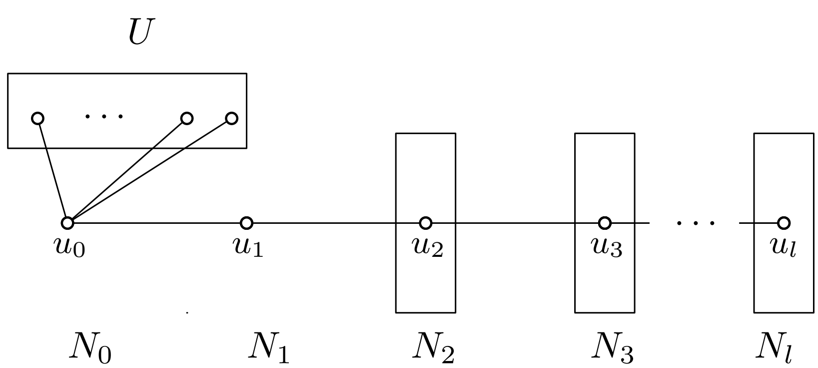


Figure 3.2: The structure of the graph H obtained by excluding every but one neighbor of a vertex $u_0 \in V(G)$. We have $U = N_G(u_0) \setminus \{u_1\}$, $H = G - U$, and $u_0, u_1, \ldots, u_l$ is a longest geodesic path in H starting at u_0. Each N_i is the bag corresponding to u_i, defined as in Lemma 3.2.1

Proof. **(a)** Let $i \geq 2$ and u, v be distinct vertices in N_i with a common neighbor $w \in N_{i-1}$. By definition, there is a vertex $z \in N_{i-2}$ such that $wz \in E(G)$. As such, one has $zu \notin E(G)$ and $zv \notin E(G)$. Therefore, $uv \in E(G)$, for otherwise $G[\{u, v, w, z\}]$ would be a claw, a contradiction. Furthermore, for every pair x, y of distinct vertices in N_2 one must have $xy \in E(G)$, for otherwise $G[\{u_0, u_1, x, y\}]$ would be a claw, a contradiction. Hence, N_2 is a clique.

(b) According to (a), the claim holds for $i \leq 2$. Proceeding by induction on i, suppose $j \geq 2$ and that the claim holds for all $i \in [1 \cdot\cdot j]$. If $N_{j+1} = \varnothing$, the claim vacuously hold for $i = j + 1$. Otherwise, for each $k \in [j - 2 \cdot\cdot j + 1]$ choose any $x_k \in N_k$. Note that by (a) it suffices to show that $x_j x_{j+1} \in E(G)$. To this end, suppose, toward a contradiction, that $x_j x_{j+1} \notin E(G)$. Pick any $y_j \in N_j$ such that $y_j x_{j+1} \in E(G)$. Then, by the induction hypothesis we have $x_{j-1} y_j, x_j y_j \in E(G)$ and $x_{k-1} x_k \in E(G)$ for each $k \in [j - 2 \cdot\cdot j]$. As such, $G[\{x_{j-2}, x_{j-1}, x_j, y_j, x_{j+1}\}]$ will be a bull, a contradiction (Figure 3.3).

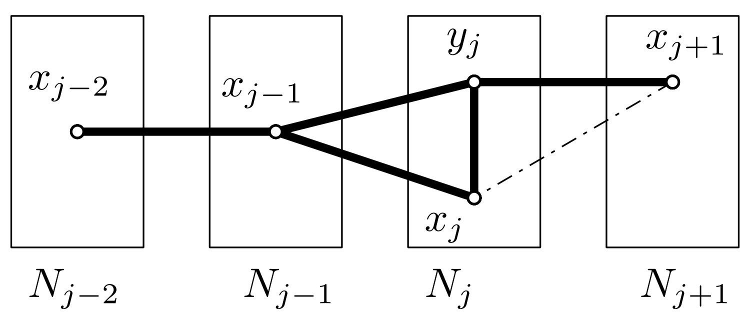


Figure 3.3: Demonstration of the proof of Lemma 3.2.1(b), by induction on i: We have $j \geq 2$ and $x_k \in N_k$ for each k. Moreover, $x_{j+1} x_j \notin E(G)$ and y_j is any neighbor of x_{j+1} in N_j. With this arrangement, if (b) holds for every $i \leq j$, the graph $G[\{x_{j-2}, x_{j-1}, x_j, y_j, x_{j+1}\}]$ will be a bull.

(c) According to (a) it suffices to show that every N_i ($i \geq 2$) contains a vertex that dominates N_{i+1}. We shall proceed by induction on i, noting that this is true for $i \in \{0, 1\}$ since N_0, N_1 are singletons. Suppose $i \geq 2$ and assume N_k contains a vertex that dominates N_{k+1} whenever $k < i$. Let $z \in N_{i-2}$ and $z' \in N_{i-1}$ such that $N_G(z) \supseteq N_{i-1}$ and $N_G(z') \supseteq N_i$. Then, assume, toward a contradiction, that no vertex in N_i is adjacent to all vertices in N_{i+1}. The latter implies there exist $x, y \in N_i$ and $x', y' \in N_{i+1}$ such that $E[G\{x, x', y, y'\}] = \{xx', yy', xy\}$. But then $G[\{x, x', y, y', z, z'\}]$

would be a net or an antenna, according as $xy \notin E(G)$ or $xy \in E(G)$, a contradiction (Figure 3.4).

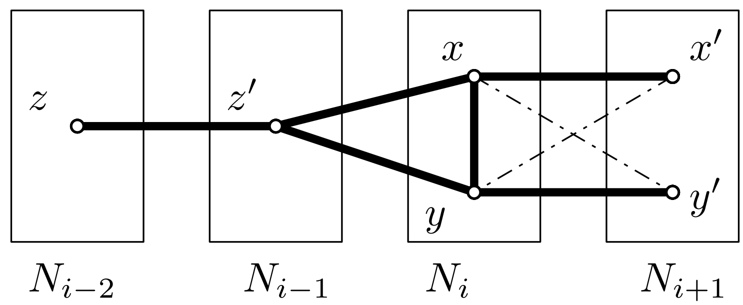
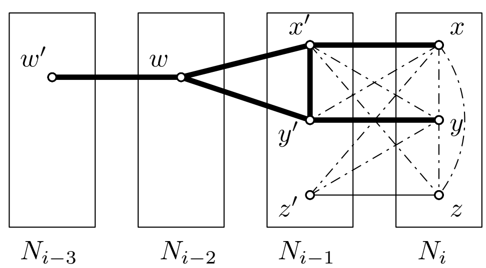
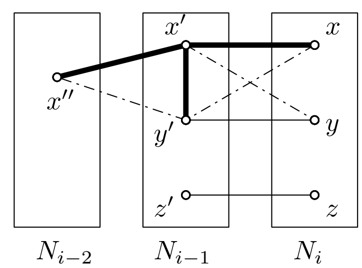


Figure 3.4: Demonstration of the proof of Lemma 3.2.1(c), by induction on i. We have $i \geq 2$, and $z \in N_{i-2}$ and $z' \in N_{i-1}$ dominating N_{i-1} and N_i, respectively. Moreover, we have $x, y \in N_i$ and $x', y' \in N_{i+1}$ such that $E(\{x, y\}, \{x', y'\} = \{xx', yy'\}$. With this arrangement the graph $G[\{x, x', y, y', z, z'\}]$ will be a net or an antenna.



If x', y' have a common neighbor $w \in N_{i-1}$ and if $w' \in N_{i-3}$ is adjacent to w then $G[\{x, x', y, y', w, w'\}]$ is a net.



If x', y' have no common neighbor $w \in N_{i-1}$ then for every neighbor x'' of x' in N_{i-2} the graph $G[\{x, x', x'', y'\}]$ is a claw.

Figure 3.5: Demonstration of the proof of Lemma 3.2.1(d), by proceeding toward a contradiction. Assume $i \geq 3$, N_{i-1} has independence number ≤ 2, but N_i contains three independent vertices x, y, z with neighbors x', y', z' in N_{i-1} such that $x'y' \in E(G)$.

(d) In light of (a) $G[N_i]$ has independence number ≤ 2 whenever $i \leq 2$. Proceeding by induction on i, suppose $i \geq 3$ and that $G[N_{i-1}]$ has independence number ≤ 2. Toward a contradiction, suppose $G[N_i]$ had independence number ≥ 3. As such, let $\{x, y, z\}$ be a set of three independent vertices in N_i. By (a), no two of x, y, z have a common neighbor in N_{i-1}. Therefore, there exist distinct vertices $x', y', z' \in N_{i-1}$ such that $E(\{x, y, z\}, \{x', y', z'\}) = \{xx', yy', zz'\}$. Moreover, by induction hypothesis,

$|E(G[\{x', y', z'\}])| \geq 1$. Assume, without loss of generality, that $x'y' \in E(G)$. If x', y' had a common neighbor $w \in N_{i-2}$, then for every neighbor w' of z in N_{i-3} the graph $G[\{x, x', y, y', w, w'\}]$ would be a net, a contradiction. Hence for any neighbor x'' of x in N_{i-1} one has $x''y' \notin E(G)$; thereby, $G[\{x, x', x'', y\}]$ has to be a claw, a contradiction. (See Figure 3.5 for a demonstration of this proof.)

□

Theorem 3.2.2.

- ***a.*** *If $G \in \mathbf{Cl}_1$ then G is two-copwin.*
- ***b.*** *If $G \in \mathbf{Cl}_2$ then G is three-copwin.*
- ***c.*** *If $G \in \mathbf{Cl}_3$ then G is five-copwin.*

Proof. In each case, we shall put all the cops in hand initially at the same vertex, say, u_0. When the robber takes its first position, say, r, consider a geodesic path P in G from u_0 to r. With u_1 being the vertex of P following u_0, set $U = N_G(u_0) \setminus \{u_1\}$ and, then, define H and the N_i as in Lemma 3.2.1. Furthermore, let H' be the component of u_0 in H. For the entire duration of the game keep one cop, say C_1, at u_0. This, forces the robber to stay in H' and, hence, to the sets N_i, according to Lemma 3.2.1. Since H' is finite, there is a unique $k \in \mathbb{N}$ such that $N_k \neq \varnothing$ and $N_{k+1} = \varnothing$.
(a) Let C_2 be the other cop in play. By the strategy of moving C_2 in H' along any shortest path from N_0 to N_k, in (at most) k steps C_2 either captures the robber or gets to an N_i where the robber is located. In the latter case, note that since the robber's next position has to be in $N_{i-1} \cup N_i \cup N_{i+1}$. Moreover, according to Lemma 3.2.1(b), $N_{i-1} \cup N_i \cup N_{i+1}$ is a subset of the closed neighborhood of every vertex in N_i, including the position of C_2. Hence, irrespective of the the robber's next move, C_2 will remain within distance one from the robber. Consequently, C_2 will be able to capture the robber in its very next move.
(b) Let C_2, C_3 be the other cops in play. According to Lemma 3.2.1(c), there is an induced path $x_0, \ldots, x_{k-1}$ in G from N_0 to N_{k-1} such that $N[x_j] \supseteq N_{j+1}$ for each $j \leq k-1$. Hence, the strategy of moving C_2 to x_{i-1} and C_3 to x_i in every step i of the game, either leads to the robber's capture in no more than $k-1$ steps, or to having a cop in a vertex neighboring the position of the robber on the cops' turn after exactly k steps. Hence, either way, the cops can always capture the robber in at most k steps.
(c) Let C_2, C_3, C_4, C_5 be the other cops in play. By Lemma 3.2.1(d), for each $j \in [1 \cdot\cdot k]$ one can choose $A_j \subseteq N_j$ such that $|A_j| \leq 2$ and $N[A_j] \subseteq N_j$. Now consider the following

strategy for the cops. Initially, we will have C_2, C_3 and C_4, C_5 saturate A_0 and A_1, respectively. As such, in order to avoid being captured, the robber has to stay in $\bigcup_2^k N_j$. We proceed in the following recursive fashion. With two groups of two cops saturating A_{i-1} and A_i for some $i \geq 1$ and the robber being in $\bigcup_{i+1}^k N_j$, in a finite number of steps we will move the cops in A_{i-1} to A_{i+1}, while keeping the cops in A_i fixed in their position. Keeping cops in A_i will force to robber to stay in $\bigcup_{i+1}^k N_j$, since the only way for the robber to leave $\bigcup_{i+1}^k N_j$ is to enter N_i. In addition and for the same reason, having cops saturating A_{i+1} will force the robber further to stay in $\bigcup_{i+1}^k N_j$. Hence, by following this strategy, the cops can iteratively reduce the robber's territory and eventually win.

□

3.3 Train-chasing the robber

In the game of cops and robbers, initial positioning of the cops will not affect the existence of a cop-winning strategy. Hence, any vertex of the graph can be chosen as the common initial position of all of the cops. With this fact under consideration, in this section we will provide a Lemma for establishing the main results of this thesis, Theorems 3.4.1, 3.4.3, and the special cases discussed in Section 3.5.
Roughly speaking, Lemma 3.3.2 proposes a tool for placing a *train* of cops along an induced path in the graph with a vertex, say, v as an end-point such that the train of cops blocks the robber from escaping and can be recursively elongated, as long as enough cops are available, so as to force the robber further and further away from v. The proof of Lemma 3.3.2 follows essentially from Proposition 3.3.1

Proposition 3.3.1. *Consider an instance of the game of cops and robbers on a graph G. Suppose on the cops' turn there are at least two cops C_1, C_2 in a vertex v of the graph while the robber is in a vertex w. Let P be any (w, v)-geodesic path in G. Let u be the second last vertex of P and set $X = N_G(v) \setminus \{u\}$. Moreover, let H be the component of v in $G - X$. Then moving C_2 to u and keeping C_1 and C_2 on v and u for the rest of the game forces the robber to stay in H.*

Proof. The statement follows since the robber can leave H only by moving on a vertex in X or u, in which case it would be caught by C_1 or C_2. □

We shall use the following definition to formulate the proof of Lemma 3.3.2 based on Proposition 3.3.1.

Definition 3.3.1. Let G be a graph and U be the set of all triples (u, v, H) where H is a connected subgraph of G, and $u, v \in V(H)$ with $d_H(u, v) \geq 2$. A *chasing function for G* is a function θ mapping every triple $(u, v, H) \in U$ onto a neighbor of u which belongs to a (u, v)-geodesic path in H.

Lemma 3.3.2. *Consider an instance of the game of cops and robber on a graph G. Let θ be a chasing function for G. Let $k \in \mathbb{N}$ and suppose on the cops' turn in step one there are k cops $C_1, \ldots, C_k$ in a vertex v_1 of the graph while the robber is located in a vertex w_1. Further suppose the robber has and will use a strategy to survive the next k moves of each of the cops $C_1, \ldots, C_k$. Denote the following $k-1$ robber's positions with $w_2, \ldots, w_k$. Further, recursively define H_i ($i \in [1 \cdot\cdot k]$) and v_i $i \in [2 \cdot\cdot k]$ as follows:*

- $H_1 = G$;
- $v_{i+1} = \theta(v_i, w_i, H_i)$ *for each* $i \in [1 \cdot\cdot k]$;
- $X_i = N_{H_i}(v_i) \setminus \{v_{i+1}\}$ *for each* $i \in [1 \cdot\cdot k]$;
- H_{i+1} : *the component of* v_1 *in* $H_i - X_i$ *for each* $i \in [1 \cdot\cdot k]$.

Then the following hold:

a. *Every H_i is an induced subgraph of G.*

b. *If $uv \in E(G) \setminus E(H_{k+1})$ such that $u \in V(H_{k+1})$, then $v \in \bigcup_1^k X_i$.*

c. *Vertices $v_1, \ldots, v_{k+1}$, in that order, induce a path in H_k.*

d. *The cops can play so than on the cops' turn in step k every C_i, $i \in [1 \cdot\cdot k]$, is located in vertex v_i.*

e. *Keeping every C_i in v_i for the rest of the game forces the robber to stay in H_{k+1}.*

Proof. All of the parts follow from Proposition 3.3.1 and the fact that every H_{i+1} is an induced subgraph of H_i and, hence, of $H_1 = G$. (See Figure 3.6.)

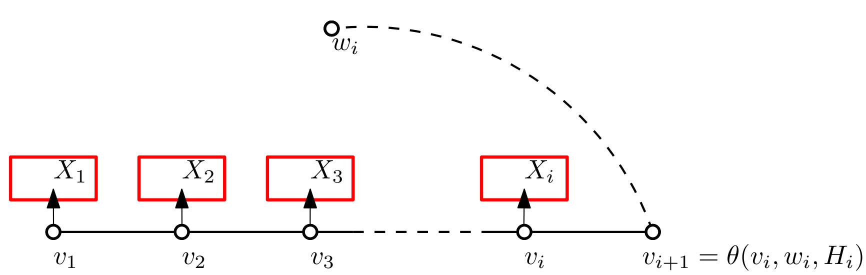


Figure 3.6: Train-chasing the robber according to Lemma 3.3.2

□

The first part of the following result was originally shown in [10]. Here, we provide a simple short proof for a stronger result based on Lemma 3.3.2.

Corollary 3.3.3. *For every integer $k \geq 3$ the class of P_k-free graphs is cop-bounded by $k-2$ [10]. Indeed, the cops need no more than $k-1$ steps to capture the robber on a P_k-free graph. Moreover, on a P_k-free graph ($k \geq 3$) there is a one-active-cop winning strategy for $k-2$ cops.*

Remark. See [13] for the definition of the *one-active-cop* version of the game of cops and robbers.

Proof. Acoording to Lemma 3.3.2, starting from every vertex v_1 a set of $k-2$ cops can either capture the robber by the end of step $k-2$, or be positioned on $k-2$ vertices that induce a path P in G and restrict the robber to stay in an induced subgraph H of G that contains P such that the degree of every internal vertex of P in H is 2 and $\deg_H(v_1) = 1$. In the latter case, since G is P_k-free so will be H; thereby, P will be a dominating path for H. Hence, the robber will be captured in the very next move of the cops. □

3.4 The main result

In this section we shall prove the following generalization of Theorem 3.1.1 proved in [10].

Theorem 3.4.1. *Let $\mathscr{H}$ be a class of graphs such that there is $k \in \mathbb{N}$ bounding the diameter of every element in $\mathscr{H}$. Then the class of $\mathscr{H}$-free graphs is cop bounded iff*

- ***a.*** *$\mathscr{H}$ contains a path, or*
- ***b.*** *$\mathscr{H}$ contains a generalized claw and a generalized net.*

Observe that a graph free of a fixed generalized claw and a fixed generalized net, will be free of $H_C(n)$ and $H_N(n)$ for every large enough n. Indeed, one can simply choose n to be larger than the length of a longest pendant path between the forbidden generalized claw and generalized net. In that regard, our following result, which is also of independent interest, plays a large role in establishing Theorem 3.4.1.

Theorem 3.4.2. *If $\mathscr{H} = \{H_C(n), H_N(n)\}$ for some $n \in \mathbb{N}$ then every $\mathscr{H}$-free graph G is $4n$-copwin.*

Proof. By Lemma 3.3.2 we may assume that we have the cops initially covering all the vertices of an induced $4n$-path $P^{[4]} : v_1, \ldots, v_{4n}$ and that there is path $Q^{[4]}$ in G from robber's position to v_{4n} with a second last vertex α_4 such that

$$\left(V\left(Q^{[4]}\right) \setminus \{\alpha_4, v_{4n}\}\right) \cap N\left(P^{[4]}\right) = \varnothing.$$

Note that the robber is forced to stay in the component, say G_4, of v_1 in $G - (N(V(P^{[4]})) \setminus \{\alpha_4\})$.

We define the current robber's territory by $\mathscr{R}_4 := V(G_4) \setminus N[V(P^{[4]}]$, put the vertices of $P^{[4]}$ in four disjoint sets of n vertices $V_1, \ldots, V_4$, such that $V_i := \{v_j : j \in [n(i-1) + 1 \cdot\cdot ni]\}$, and denote the cops covering V_i by $\mathscr{C}_i$. Since G is $\mathscr{H}$-free, for every $w \in \mathscr{R}_4$ we must have

$$N(w) \cap N(V_2) \subseteq N(V_1) \cup N(v_3) \cup N(v_4). \tag{3.1}$$

For a demonstration see Figure 3.7: If there is an edge $wx \in E(G)$ such that $w \in \mathscr{R}_4$ and $x \in N(V_2) \setminus N(V_1 \cup V_3 \cup V_4)$, then for each path Q from w to α_4 through $\mathscr{R}_4$ then vertices of $G[(\cup_1^4 V_i) \cup V(Q) \cup \{x\}]$ contains an induced n-claw or an induced n-net.

By (3.1), cops in $\mathscr{C}_2$ can be freed and, hence, moved in the next $3n$ steps to either capture the robber or cover another set V_5 of n vertices to be augmented with $V(P^{[4]})$ such that

a. $V_5 \cap N[V(P^{[4]}] = \{\alpha_4\}$, and

b. $V_1, \ldots, V_5$ form a $5n$-path $P^{[5]}$ induced in G_4 from v_1 to, say, v_{5n}.

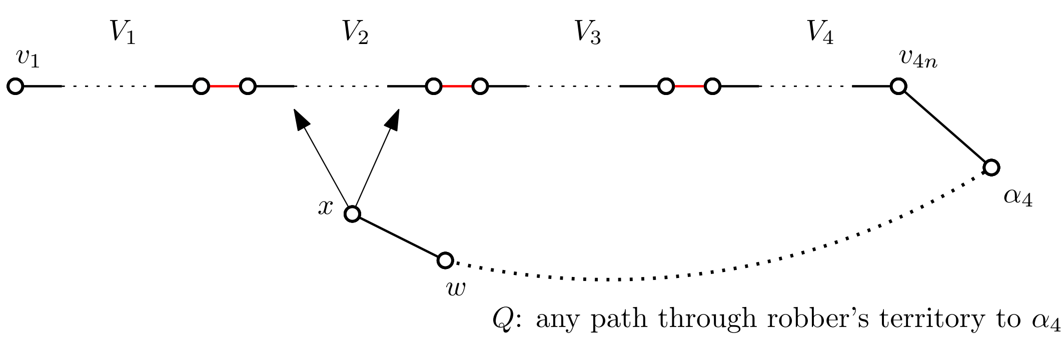


Figure 3.7: Demonstration of the inclusion given by (3.1)

One can define the subsequent robber's territories $\mathscr{R}_j$, positions of the cops $\mathscr{C}_j$, paths $P^{[j]}$, $Q^{[j]}$, and vertices α_j $(j > 4)$ in the obvious recursive way. Again, as G is $\mathscr{H}$-free, the following claim is established in a similar way as before:

Claim 1. Let $w' \in \mathscr{R}_j$ $(j \geq 4)$ and suppose we have the sets of cops $\mathscr{C}_1$, $\mathscr{C}_{j-1}$, and $\mathscr{C}_j$ covering V_1, V_{j-1} and V_j. Then for each $i \in [2 \cdot\cdot (j-2)]$ we have

$$N(w') \cap N(V_i) \subseteq N(V_1) \cup N(V_{j-1}) \cup N(V_j).$$

With $4n$ cops in play, Claim 1 allows us to free n cops after a finite number of steps, and then, in light of Lemma 3.3.2, use them to either capture the robber or augment the current path, say $P^{[q]}$, with a set V_{q+1} of n new vertices. The latter shrinks the robber's territory. Hence, $4n$ cops have a winning strategy on G.

□

Proof of Theorem 3.4.1. **(Necessity.)** Note that by Theorem 3.1.2 the class of cubic graphs is cop-unbounded. Therefore, one can utilize Lemmas 2.3.3 and 2.3.8 to derive other cop-unbounded classes of graphs from the class of cubic graphs. We consider two such classes: the class $\mathscr{G}_1$ of graphs obtained by k-subdividing the cubic graphs, and the class $\mathscr{G}_2$ obtained by applying clique substitution to the graphs in $\mathscr{G}_1$. Observe that the only possible induced subgraphs of graphs in $\mathscr{G}_1$ (resp. $\mathscr{G}_2$) with diameter $< k$ are paths and generalized claws (reps. paths and generalized nets). Since $\mathscr{G}_1$ and $\mathscr{G}_2$ are both cop-unbounded, in case the class $\mathscr{G}$ of $\mathscr{H}$-free graphs is cop-bounded and $\mathscr{H}$ does not contain a path, then $\mathscr{H}$ must contain both a generalized claw and a generalized net.
(Sufficiency.) If $\mathscr{H}$ contains a path then we are done, according to Corollary 3.3.3. Hence, we may assume $\mathscr{H}$ contains an $H_C(n_1, n_2, n_3)$ and an $H_N(m_1, m_2, m_3)$ for some $n_j, m_j \in \mathbb{N} \cup \{0\}$. Let n be the maximum of n_j, m_j's. As such, the class of $\mathscr{H}$-free graphs will be a subclass of the class of $\{H_C(n), H_N(n)\}$-free graphs. But the latter class is $4n$-copwin, according to Theorem 3.4.2; thereby the class of $\mathscr{H}$-free graphs is also $4n$-copwin .

□

3.4.1 A generalization

In light of Lemma 3.3.2, one can show the following generalization of Theorem 3.4.1, extending it to the case where the elements of $\mathscr{H}$ might be disconnected:

Theorem 3.4.3. *Let $\mathscr{H}$ be a class of graphs such that the diameter of every component of each element in $\mathscr{H}$ is bounded by some absolute constant. Then the class of $\mathscr{H}$-free graphs is cop bounded iff*

- ***a.*** *$\mathscr{H}$ contains a forest of paths, or*
- ***b.*** *$\mathscr{H}$ contains two graphs F_1, F_2 each having at least one degree three vertex such that every component of F_1 is a path or a generalized claw, and every component of F_2 is a path or a generalized net.*

Proof. Let $k \in \mathbb{N}$ be an upper bound for the diameter of every component of a graph in $\mathscr{H}$. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be the cop-unbounded classes of graphs as in the proof of Theorem 3.4.1. Then the only induced subgraphs of $\mathscr{G}_1$ with each component of diameter $< k$ are forests with each component either a path or a generalized claw. Similarly, the only induced subgraphs of $\mathscr{G}_2$ with each component of diameter $< k$ are graphs with each component either a path or a generalized net. Hence, if no member of $\mathscr{H}$ is a forest of paths, $\mathscr{H}$ must contain a forest with at least one degree three vertex having every component a path or a generalized claw. In addition, $\mathscr{H}$ must contain a graph with maximum degree three, every component of which is a path or a generalized net. This establishes the necessity of the conditions. For the converse, as in the proof of Theorem 3.4.1 it suffices to assume $\mathscr{H}$ is of the form $\{l \cdot P_n + s \cdot H_C(n), l \cdot P_n + t \cdot H_N(n)\}$ where $l \in \mathbb{N} \cup \{0\}$ and $n, s, t \in \mathbb{N}$. As such, in light of Lemma 3.3.2 with $(n+1)l$ cops and within as many steps of the game, the cops can either capture the robber or get positioned to cover all vertices of an $(n+1)k$-path to restrict the robber to a territory which is $\{s \cdot H_C(n), t \cdot H_N(n)\}$-free. Hence, to complete the proof it suffices to show the following:

Claim 1. Let $n \in \mathbb{N}$ be fixed. For each pair $(s,t) \in \mathbb{N}^2$ denote the class of $\{s \cdot H_C(n), t \cdot H_N(n)\}$-free graphs simply by $\mathscr{G}_{s,t}$. Then $\mathscr{G}_{s,t}$ is cop-bounded by $3(n+1)(s+t-2)+4n$.

Proof of Claim 1. Let $G \in \mathscr{G}_{s,t}$. We shall use induction on $s+t$ to show that G is $4(n+1)(s+t-1)$-copwin. The base case $s+t=2$, i.e. when $s=t=1$, follows from Theorem 3.4.1. If $s+t \geq 3$, using $4n$ cops and starting from any vertex v_1, we follow the strategy and notation described in the proof of Theorem 3.4.2 until capturing the robber or getting to a path $v_1, \ldots, v_{kn}$ with cops on V_1, V_{k-2}, V_{k-1} ,V_k such that there is a vertex in robber's territory which for some $1 < i < k-1$ is adjacent to an element of

$$N[V_i] \setminus (N[V_1] \cup N[V_{k-1} \cup N[V_k]).$$

In the latter case, G must contain an induced graph H which is isomorphic to an n-claw or an n-net. Note that $|V(H)| \in \{3n, 3(n+1)\}$. Moreover, we must have $s \geq 2$ or $t \geq 2$ according as H is an n-claw or an n-net. Therefore, using (at most) $3(n+1)$ cops to be kept on the vertices of H, one can restrict the robber to a territory in $\mathscr{G}_{s-1,t}$ (with $s \geq 2$) or $\mathscr{G}_{s,t-1}$ (with $t \geq 2$). The new territory will be $3(n+1)(s+t-2)+4n$-copwin, by the induction hypothesis. Hence, G is copbounded by $3(n+1)+3(n+1)(s+t-3)+4n = 3(n+1)(s+t-2)+4n$, as desired.□ Claim 1

□

3.5 Some special cases

In this section we consider some particular classes of graphs defined by two forbidden induced subgraphs and provide upper bounds for their cop number. Likewise the general cases covered in Section 3.4, we shall “spend" a constant number of cops to saturate the vertices of some induced path in the graph; thereby, reduce the game to an induced subgraph of the original graph. We then try to show that the unspent cops suffice to capture the robber in the reduced graph. For the latter task, we often establish some structural properties for the reduced graph which will be used in providing a winning strategy for the cops in hands.

3.5.1 Graphs without induced $H_C(1,1,n)$ or $H_N(0,0,n)$

Theorem 3.5.1. *If $\mathscr{H} = \{H_C(1,1,n), H_N(0,0,n)\}$ for some $n \in \mathbb{N}$, then the class of $\mathscr{H}$-free graphs is cop-bounded by $n+1$.*

In light of Lemma 3.3.2, it suffices to show the following:

Lemma 3.5.2. *Let $\mathscr{H}$ be as in Theorem 3.5.1 and G be $\mathscr{H}$-free. Then under the additional assumptions that*

- *there exist distinct vertices $v_0, \ldots, v_{n-1}$ inducing an n-path in G in that order such that $\deg_G(v_0) = 1$ and $\deg_G(v_i) = 2$ for each $i \in [1 \cdot\cdot (n-1)]$;*
- *initially there is a cop C in v_{n-1}; and*
- *the robber starts from a vertex $w \in V(G) \setminus N_G[\{v_i : i \in [0 \cdot\cdot (n-1)\}]$,*

C has a winning strategy.

Proof. As in Lemma 3.2.1 set $N_0 = \{v_0\}$, and for each $i \in \mathbb{N}$ let N_i be the ith neighborhood of v_0 in G. By the structure of G, we have $N_i = \{v_i\}$ for each $i \in [0 \cdot\cdot (n-1)\}]$. Then, as G is $\mathscr{H}$-free and using a basic inductive argument it can be shown than every N_i with $i > n$ has at most one element; thereby, G is simply a path. Hence, starting from any vertex C can capture the robber in no more than $\operatorname{diam}(G)$ steps. □

3.5.2 Graphs without induced $H_C(1,1,n)$ or $H_N(1,1,n)$

Theorem 3.5.3. *If $\mathscr{H} = \{H_C(1,1,n), H_N(1,1,n)\}$ for some $n \in \mathbb{N}$, then the class of $\mathscr{H}$-free graphs is cop-bounded by $n+4$.*

In light of Lemma 3.3.2, it suffices to show the following:

Lemma 3.5.4. *Let $\mathscr{H}$ be as in Theorem 3.5.3 and G be an $\mathscr{H}$-free graph. Then under the additional assumptions that*

- *there exist distinct vertices $v_0, \ldots, v_n$ inducing an $(n+1)$-path in G, in that order, such that $\deg_G(v_0) = 1$ and $\deg_G(v_i) = 2$ for each $i \in [1 \cdot\cdot (n-1)]$;*
- *all of the cops are initially in $v := v_n$; and*
- *the robber starts from a vertex in $w \in V(G) \setminus \{v_i : i \in [0 \cdot\cdot n]\}$,*

four cops $C_1, \ldots, C_4$ have a winning strategy on G.

Proof. We shall consider different levels of neighborhoods of v in $G - \{v_j : j \in [0 \cdot\cdot (n-1)]\}$. For each $i \in \mathbb{N} \cup \{0\}$ let N_i be the ith neighborhood of v in $G - \{v_j : j \in [1 \cdot\cdot (n-1)]\}$, and let $l = \max\{i : N_i \neq \varnothing\}$. As such we will have

$$V(G) = \{v_j : j \in [0 \cdot\cdot (n-1)]\} \cup \left(\bigcup_0^l N_j \right).$$

Claim 1. *For each i we have*

$$\alpha(G[N_i]) \leq 2. \tag{3.2}$$

Proof of Claim 1. Since N_0 is a singleton, (3.2) holds for $i = 0$. Moreover, since $N_G(v) \supset N_1$, any pair of nonadjacent vertices in N_1 together with the vertices of the (unique) (v, v_0)-path in G would induce a $(1,1,n)$-claw, a contradiction. Hence, N_1

has to be a clique; in particular$\alpha(G[N_1]) = 1$. Proceeding by induction, let $j \geq 2$ and suppose (3.2) holds for each $i \leq j-1$. Toward a contradiction let $\{x_j, y_j, z_j\}$ be an independent set of three vertices in $G[N_j]$. If two of these vertices, say x_j, y_j, had a common neighbor, say x, in N_{j-1} then x_j, y_j together with the first $n+1$ vertices of any geodesic (x, v_0)-path in G would induce a $(1,1,n)$-claw in G, again a contradiction. Hence, there are distinct vertices $x_{j-1}, y_{j-1}, z_{j-1}$ in N_{j-1} such that

$$E_G\left(\{x_j, y_j, z_j\}, \{x_{j-1}, y_{j-1}, z_{j-1}\}\right) = \{x_j x_{j-1}, y_j y_{j-1}, z_j z_{j-1}\}. \tag{3.3}$$

But by induction hypothesis the vertices $x_{j-1}, y_{j-1}, z_{j-1}$ are not independent. Hence, we may assume, without loss of generality, that $x_{j-1}y_{j-1} \in E(G)$. Now, if x_{j-1}, y_{j-1} had a common neighbor $x' \in N_{j-2}$ then $x_{j-1}, x_j, y_{j-1}, y_j$ together with the first $n+1$ vertices of any geodesic (x', v_0)-path in G would induce a $(1,1,n)$-net in G, a contradiction. Hence, x_{j-1}, y_{j-1} have no common neighbor in N_{j-2}. But then for every neighbor x'' of x_{j-1} in N_{j-2}, the vertices x_{j-1}, x_j, y_{j-1} together with the first n vertices of any geodesic (x'', v_0)-path in G would induce a $(1,1,n)$-claw in G, also a contradiction. Therefore, we must have $\alpha(G[N_j]) \leq 2$, as desired. □ Claim 1

For each $i \leq l$ let I_i be an independent set of the maximum cardinality in $G[N_i]$. According to Claim 1 we have $|I_i| \leq 2$ for every i. In particular, one can saturate every I_i with two cops to force the robber out of N_i and also prevent the robber from entering N_i. This suggests the following winning strategy for the cops:

Right after the first robber's turn, we define the location of the cops and the robber as *stage* zero of the game. In the next step we keep C_1, C_2 in v and move C_3, C_4 to the single element of I_1. After robber's turn, we define the location of the cops and the robber as stage one of the game. In general, proceeding from a stage i ($i \in \mathbb{N}$) of the game with two of the cops, say C_{i_1}, C_{i_2}, saturating I_{i-1} and the other two, say C_{i_3}, C_{i_4} saturating I_i, we take a finite number of steps to move C_{i_1}, C_{i_2} to saturate I_{i+1}. Then, after the robber's turn we define the locations of the cops and the robber as stage $i+1$. Note that the strategy described here forces the robber further and further away from v. But since $N_{l+1} = \varnothing$, the robber cannot escape the cops beyond stage l. Since transition between consecutive stages takes only a finite number of steps, this strategy leads to the capture of the robber in a finite number of steps. □

3.5.3 Graphs without induced $H_C(1,n,n)$ or $H_N(0,n,n)$

Theorem 3.5.5. *If $\mathscr{H} = \{H_C(1,n,n), H_N(0,n,n)\}$ for some $n \in \mathbb{N}$, then the class of $\mathscr{H}$-free graphs is $(2n+5)$-copwin.*

With all cops initially positioned at a vertex v_0 and assuming that the robber can survive the first $n+4$ steps of the game, we can place $n+4$ cops along the vertices of an induced path $P:\ v_0,\ldots,v_{n+4}$ in G in the sense of Lemma 3.3.2. As such, keeping the cops in $v_0,\ldots,v_n$ fixed in their position will reduce the game to an induced connected subgraph of G where no vertex v_j $(j \in [0 \cdot\cdot (n-1)]$ has a neighbor outside P. Hence, it suffices to show that the remaining cops have a winning strategy on the resulting reduced graph. More precisely, in light of Lemma 3.3.2 it suffices to show the following:

Lemma 3.5.6. *Let $\mathscr{H}$ be as in Theorem 3.5.5 and G be $\mathscr{H}$-free. Let $v_0 \in V(G)$, and suppose $N_j := \{v \in V(G) : d_G(v_0, v) = j\}$ $(j \in [0 \cdot\cdot k])$ are all of the nonempty neighborhood levels of v_0 in G, where $k \geq n+3$. Let $v_0,\ldots,v_k$, in that order, be the vertices of a path from N_0 to N_k. Additionally, suppose*

- $\deg_G(v_0) = 1$ *and* $\deg_G(v_i) = 2$ *for each* $i \in [1 \cdot\cdot n-1]$*;*
- *there are $n+5$ cops in play, of which four cops C_1, C_2, C_3, C_4 initially saturate $\{v_n, v_{n+1}, v_{n+2}, v_{n+3}\}$ and the remaining $n+1$ are free; and*
- *the robber starts from a vertex in $V(G) \setminus N_G[\{v_i : i \in [0 \cdot\cdot (n+3)\}]$.*

Then starting from the initial positioning of the cops and the robber, there is a winning strategy for the cops.

Proof. For each $j \in [(n+2) \cdot\cdot k]$ set

$$A_j = \begin{cases} \{v_{j-2}, v_{j-1}, v_j, v_{j+1}\} & \text{if} \quad n+2 \leq j < k, \\ \{v_{k-2}, v_{k-1}, v_k\} & \text{if} \quad j = k. \end{cases} \tag{3.4}$$

Hence, initially we have C_1, C_2, C_3, C_4 saturating A_{n+2} while the robber is initially in some vertex

$$w_{n+2} \in \bigcup_{n+2}^{k} N_j \setminus N[A_{n+2}].$$

From this initial configuration, to which we refer to as *stage* $n{+}2$, we move C_1, C_2, C_3, C_4 recursively as explained below:

Suppose C_1, C_2, C_3, C_4 saturate A_l for some $l \geq n+2$ and the robber is in a vertex $w_l \in \bigcup_l^k N_l \setminus N[A_l]$ (stage l). Let B_l be the set consisting of every vertex in N_l which is reachable from w_l through some path with no vertex in $N[A_l]$. If $B_l = \varnothing$, we shift the cops saturating A_l to saturate A_{l+1}. This forces the robber to be located in a vertex $w_{l+1} \in \bigcup_{l+1}^{k} N_j \setminus N[A_{l+1}]$ (stage $l+1$). Note that if no stage j with $B_j \neq \varnothing$ is ever reached, this strategy will eventually lead to the robber's capture by only using the four cops C_1, C_2, C_3, C_4. Hence, we may assume the game is in some stage i (with C_1, C_2, C_3, C_4 saturating A_i and the robber in $w_i \in \bigcup_{i}^{k} N_j \setminus N[A_i]$) such that

$$B_i \neq \varnothing. \tag{3.5}$$

Note that i is not predetermined and in general depends on how the robber plays. That is, i is the minimum j with $B_j \neq \varnothing$, only for the instance of the game under consideration. To proceed, pick any $y_i \in B_i$ and choose vertices $y_j \in N_j$, $j \in \{i-3, i-2, i-1\}$, such that $y_{i-3}, \ldots y_i$, in that order induce a path in G. Note that as $y_i \in B_i$, we have $y_i \notin N[A_i]$; in particular, $y_{i-1} \neq v_{i-1}$. Hence, as $y_i y_{i-1} \in E(G)$ we must have

$$y_{i-1} \in N(A_{i-1}), \tag{3.6}$$

for otherwise B_{i-1} would be nonempty, contradicting the choice of i.

We use two of the free cops, say C_5, C_6 to saturate $Q_0 := \{y_i, y_{i-1}\}$, and denote by z_1 the next position the robber will assume. Keep $C_1, \ldots, C_6$ in their positions in $A_i \cup Q_0$ for the rest of the game. This restricts the robber to stay in the component, say (V_1, E_1), of w_i in

$$G\left[\bigcup_{i}^{k} N_j \setminus N[A_i \cup Q_0]\right].$$

Let

$$G' = G[V_1 \cup Q_0].$$

Claim 1. G' is connected.

Proof of Claim 1. Since $y_{i-1} \in N(A_{i-1})$ and (V_1, E_1) is connected, every vertex of G' is in the component of G' containing y_i; i.e. G' is a connected. □ Claim 1

Let P be a path of the shortest length from z_1 to Q_0, (i.e. to a vertex in Q_0) and let q_1 be the second last vertex of P. With $\mathscr{C}$ being the set of all free cops, in a finite number of steps we place all members of $\mathscr{C}$ in q_1.

Suppose, toward a contradiction, that the robber can invariably escape the cops. Then, in light of Lemma 3.3.2, in the following $n-2$ steps the game can be brought to the following configuration:

- there are vertices $q_2, \ldots, q_{n+1}$ such that $q_1, \ldots q_{n+1}$, in that order, induce a path in the subgraph (V_1, E_1) of G;
- each of the vertices $q_1, \ldots, q_{n-1}$ contains a cop in $\mathscr{C}$;
- with $X = (\bigcup_1^{n-1} N(\{q_j\}) \setminus \{q_1, \ldots, q_{n+1}\}$, the robber is located in the component, say G'', of q_1 in $(V_1, E_1) - X$.

The rest of our proof is devoted to showing that this configuration forces either an induced $H_C(1, n, n)$ or $H_N(0, n, n)$ in G, contradicting the assumptions of the theorem. Let

$$j^* = \min\{j : v_j \in A_{i-1} \,\&\, v_j \leftrightarrow y_{i-1}\}.$$

Note that as $j^* \geq i - 2$, for $y_{i-1} \in N_{i-1}$, and $i \geq n + 2$, we have $j^* \geq n$.

Claim 2. $|Q_0 \cap N(q_1)| = 1$.

Proof of Claim 2. If $Q_0 \subseteq N(q_1)$ then $G[\{v_j : j \in [j^* - n + 1 \cdot\cdot j^*]\} \cup Q_0 \cup \{q_j : j \in [1 \cdot\cdot n + 1]\}]$ will be an $H_N(0, n, n)$. □ Claim 2

Claim 3. $\{v_{i-2}, v_{i-1}\} \not\subseteq N(y_{i-1})$.

Proof of Claim 3. If $\{v_{i-2}, v_{i-1}\} \subseteq N(y_{i-1})$ then according as $y_{i-1} \leftrightarrow q_1$ or $y_i \leftrightarrow q_1$ the graph

$$G[\{v_j : j \in [i - n - 2 \cdot\cdot i - 1]\} \cup \{y_{i-1}\} \cup \{q_j : j \in [1 \cdot\cdot n]\}]$$

or the graph

$$G[\{v_j : j \in [i - n - 2 \cdot\cdot i - 1]\} \cup Q_0 \cup \{q_j : j \in [1 \cdot\cdot n - 1]\}]$$

will be an $H_N(0, n, n)$. □ Claim 3

Claim 4. $v_{i-2} \notin N(y_{i-1})$.

Proof of Claim 4. If $v_{i-2} \leftrightarrow y_{i-1}$ then by Claim 3 we will have $v_{i-1} \not\leftrightarrow y_{i-1}$; thereby, according as $y_{i-1} \leftrightarrow q_1$ or $y_i \leftrightarrow q_1$ the graph

$$G[\{v_j : j \in [i - n - 2 \cdot\cdot i - 1]\} \cup \{y_{i-1}\} \cup \{q_j : j \in [1 \cdot\cdot n - 1]\}]$$

or the graph

$$G[\{v_j : j \in [i - n - 2 \cdot\cdot i - 1]\} \cup Q_0 \cup \{q_j : j \in [1 \cdot\cdot n - 2\}]$$

will be an $H_C(1,n,n)$. $\square$ Claim 4

Claim 5. $\{v_{i-1}, v_i\} \not\subseteq N(y_{i-1})$.

Proof of Claim 5. If $\{v_{i-1}, v_i\} \subseteq N(y_{i-1})$ then according as $y_{i-1} \leftrightarrow q_1$ or $y_i \leftrightarrow q_1$ the graph

$$G[\{v_j : j \in [i-n-1 \cdot\cdot i]\} \cup \{y_{i-1}\} \cup \{q_j : j \in [1 \cdot\cdot n]\}]$$

or the graph

$$G[\{v_j : j \in [i-n-1 \cdot\cdot i]\} \cup Q_0 \cup \{q_j : j \in [1 \cdot\cdot n-1\}]$$

will be an $H_N(0,n,n)$. $\square$ Claim 5

Claim 6. $v_{i-1} \notin N(y_{i-1})$.

Proof of Claim 6. If $v_{i-1} \leftrightarrow y_{i-1}$ then by Claim 5 we will have $v_i \not\leftrightarrow y_i$; thereby, according as $y_{i-1} \leftrightarrow q_1$ or $y_i \leftrightarrow q_1$ the graph

$$G[\{v_j : j \in [i-n-1 \cdot\cdot i]\} \cup \{y_{i-1}\} \cup \{q_j : j \in [1 \cdot\cdot n-1]\}]$$

or the graph

$$G[\{v_j : j \in [i-n-1 \cdot\cdot i]\} \cup Q_0 \cup \{q_j : j \in [1 \cdot\cdot n-2\}]$$

will be an $H_C(1,n,n)$. $\square$ Claim 6

Claim 7. We have $A_{i-1} \cap N(y_{i-1}) = \{v_i\}$. Moreover, $i = k$.

Proof of Claim 7. That $A_{i-1} \cap N(y_{i-1}) = \{v_i\}$ simply follows from (3.6) and Claims 4 and 6. As such, if $i < k$ then according as $y_{i-1} \leftrightarrow q_1$ or $y_i \leftrightarrow q_1$ the graph

$$G[\{v_j : j \in [i-n-1 \cdot\cdot i+1]\} \cup \{y_{i-1}\} \cup \{q_j : j \in [1 \cdot\cdot n-1]\}]$$

or the graph

$$G[\{v_j : j \in [i-n-1 \cdot\cdot i+1]\} \cup Q_0 \cup \{q_j : j \in [1 \cdot\cdot n-2\}]$$

will be an $H_C(1,n,n)$. $\square$ Claim 7

Now consider any path $y_0, \ldots, y_{k-1}, y_k$ from N_0 to N_k. (Note that we will have $y_j = v_j$ for $j \leq n$, but this will not affect our arguments.) By Claim 7, according as

$y_{k-1} \leftrightarrow q_1$ or $y_k \leftrightarrow q_1$ the graph

$$G[\{y_j : j \in [k-n-1 \mathinner{\ldotp\ldotp} k-1]\} \cup \{v_k\} \cup \{q_j : j \in [1 \mathinner{\ldotp\ldotp} n]\}]$$

or the graph

$$G[\{y_j : j \in [k-n-1 \mathinner{\ldotp\ldotp} k]\} \cup \{v_k\} \cup \{q_j : j \in [1 \mathinner{\ldotp\ldotp} n-1\}]$$

will be an $H_C(1, n, n)$, a contradiction. $\square$

Chapter 4

The Structure of Claw- and Bull-free Graphs

In Chapter 3, alongside the general results on cop bounded classes of graphs defined by a set of forbidden induced subgraphs, we also dealt with some special classes, including three sub-classes of claw-free graphs in Theorem 3.2.2, together with some more general special classes in Theorem 3.5.1, Theorem 3.5.3, and Theorem 3.5.5. In proving an upper bound for the cop number of each of these special classes we based our argument on presenting some sort of structural properties of the graphs under consideration. The aim of the current chapter is to show the structure of claw- and bull-free graphs (Theorem 4.1.1) using a method inspired by the game of cops and robbers. It should be mentioned that this structure turned out to have been essentially stated as a byproduct of a research on Hamiltonicity of claw-free graphs published in 1991 with a strategy for a proof briefly sketched out [16]. Nevertheless, we found this structure independently and provided a complete proof for it which is in line with the main theme of this thesis.

4.1 Formulation of the main result

Definition 4.1.1 (Expansion)**.** An *expansion* of a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is any graph H obtained from G by substituting its vertices with disjoint cliques $K^{[i]}$, $i = 1, \ldots, n$, (called the *bags* of the expansion) and adding the edges of the complete bipartite graphs with the partite sets $V(K^{[i]})$ and $V(K^{[j]})$ for each $v_i v_j \in E(G)$.

Remark. If G is claw- and bull-free, then according to Lemma 3.2.1, for every $u_0 u_1 \in E(G)$ the component of u_0 in $G - (N_G(u_0) \setminus \{u_1\})$ is an expansion of a path.

We shall prove the following theorem on the structure of claw- and bull-free graphs:

Theorem 4.1.1. *A connected graph G is claw- and bull-free if and only if it is either*

- *a connected graph of independence number at most two,*
- *a graph which is an expansion of a cycle of length at least six, or*
- *a graph which is an expansions of a path of length at least four.*

Remark. Note that graphs of independence number at most two are exactly the graphs which are complements of triangle-free graphs.

4.2 Proof of the main result

We will consider the separate classes of claw- and bull-free graphs based on the maximum length of an induced cycle, denoted $\ell(\cdot)$. In the end, we combine our results on these classes into a complete proof of Theorem 4.1.1.

4.2.1 The case $\ell(G) \geq 6$.

Lemma 4.2.1. *Let G be a claw- and bull-free graph, C an induced cycle of length $k \geq 4$ and $x \in N(C)$. Then $N(x)$ contains two consecutive vertices of C. Moreover, if $k \geq 5$ then $N(x)$ contains three consecutive vertices of C.*

Proof. Let $V(C) = \{v_1, \ldots, v_k\}$ and suppose $xv_1 \in V(G)$. Since G is claw-free, we must have $xv_2 \in E(G)$ or $xv_k \in E(G)$, establishing the first claim. Suppose, without loss of generality, that $xv_2 \in E(G)$. Then, in case $k \geq 5$ one must have $xv_3 \in E(G)$ or $xv_k \in E(G)$, for otherwise $G[\{x, v_1, v_2, v_3, v_k\}]$ would be a bull. (See Figure 4.1.)

□

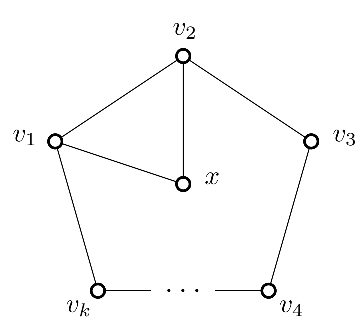


Figure 4.1: Consecutive neighbors for vertices in $N(C)$ where C has length ≥ 4

Lemma 4.2.2. *Let G be a connected claw- and bull-free graph, and C an induced cycle of a length ≥ 4. Then $N[C] = V(G)$.*

Proof. Suppose $N[C] \neq V(G)$. Choose a vertex y at distance two from C and a vertex $x \in N(C) \cap N(y)$. If $N(x) \cap V(C) \geq 3$, then there exist two vertices $u, v \in N(x) \cap V(C)$ which are not adjacent, in which case $\{x, y, u, v\}$ induces a claw, a contradiction (See Figure 4.2). Therefore, according to Lemma 4.2.1, C is a cycle of length 4 such that $N(x) \cap V(C)$ consists of two consecutive vertices of C. Then for each $z \in V(C) \setminus N(x)$ the graph $G[\{N(x) \cap V(C)\} \cup \{x, y, z\}]$ is a bull, a contradiction. (See Figure 4.3.) □

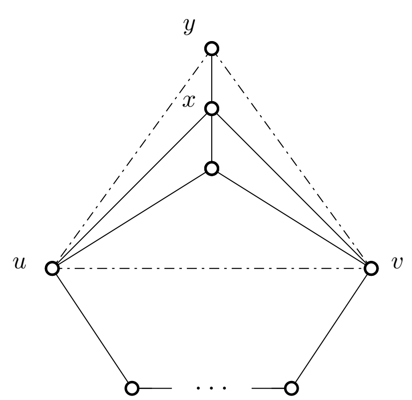


Figure 4.2: Proof of Lemma 4.2.2; the case $|N(x) \cap V(C)| \geq 3$.

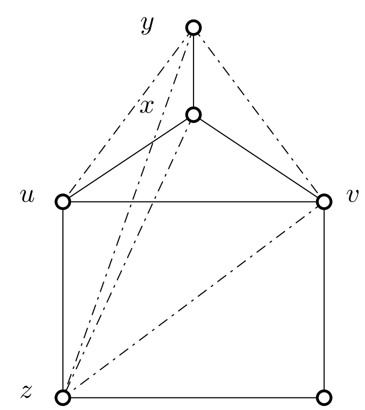


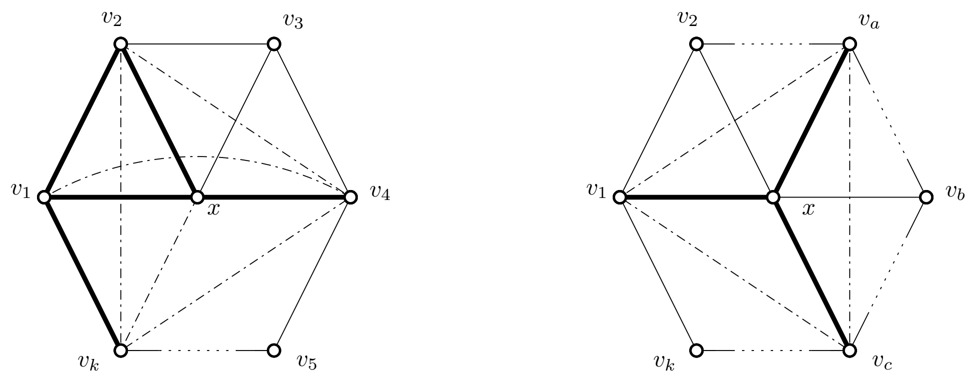
Figure 4.3: Proof of Lemma 4.2.2; the case $|N(x) \cap V(C)| = 2$.

Lemma 4.2.3. *Let G be a connected claw- and bull-free graph and C an induced cycle of G of length k. If $k \geq 6$, then G is an expansion of C.*

Proof. Let $V(C) = \{v_1, v_2, \ldots, v_k\}$. By Lemma 4.2.2 we know that $N[C] = V(G)$ and every vertex outside of C has at least three neighbours in C.

Claim 1. Let $x \in V(G) \setminus V(C)$. Then $|N(x) \cap V(C)| = 3$.

Proof of Claim 1. If $|N(x) \cap V(C)| \geq 5$, then $N(x)$ would contain an independent set of size three, i.e. $G[N[x]]$ would contain a claw. Hence, proceeding by the way of contradiction and in light of Lemma 4.2.1 we may assume $|N(x) \cap V(C)| = 4$. As such, without loss of generality we may assume $N(x) = \{v_1, v_a, v_b, v_c\}$ where $1 < a < b < c < k$. (See Figure 4.4.) Note that $1, a, b, c$ cannot be consecutive for otherwise $G[\{x, v_1, v_2, v_4, v_k\}]$ would be bull. Moreover, if $a > 2$ (resp. $c > b + 1$) then $G[\{x, v_1, v_a, v_c\}]$ (resp. $G[\{x, v_1, v_b, v_c\}]$) would be a claw, a contradiction. Hence, one



(a) The case $a = 2$, $b = 3$, $c = 4$

(b) The case $a > 2$ (the case $c > b + 1$ is similar)

Figure 4.4: Ruling out the case $|N(x) \cap V(C)| = 4$ in Claim 1, Lemma 4.2.3, by considering $N(x) = \{v_1, v_a, v_b, v_c\}$ where $1 < a < b < c < k$

must have $a = 2$, $b > 3$ and $c = b + 1$. But then $G[\{v_1, v_2, v_b, v_k, x\}]$ would be a bull, a contradiction. ☐ Claim 1

For the rest of the proof, set $N_x := N(x) \cap V(C)$ for each $x \in V(G)$.

Claim 2. Let x, y be distinct vertices of G such that $|N_x \cap N_y| \geq 2$. Then $xy \in E(G)$.

Proof of Claim 2. If $x \in V(C)$ then $y \in V(G) \setminus V(C)$; thereby, according to Lemma 4.2.1, N_y consists of three consecutive vertices of C. Hence by Claim 1 we have $N_y = N_x \cup \{x\}$ and; in particular, $xy \in E(G)$. Hence, we may assume $x, y \in V(G) \setminus V(C)$. Suppose, contrary to the claim, that $xy \notin E(G)$.

Case I: $|N_x \cap N_y| = 2$**.**

Let $N_x = \{v_1, v_2, v_3\}$ and $N_y = \{v_2, v_3, v_4\}$. As such, unless $xy \in E(G)$, $G[\{x, y, v_3, v_4, v_5\}]$ would be a bull. (See Figure 4.5(a).)

Case II: $|N_x \cap N_y| = 3$**.**

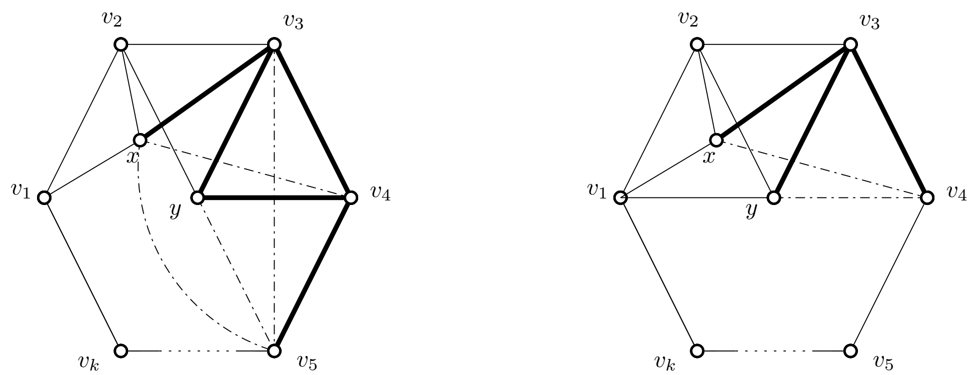
In this case, N_x and N_y are the same set, say, $\{v_1, v_2, v_3\}$. As such, unless $xy \in E(G)$, $G[\{x, y, v_3, v_4\}]$ would be a claw. (See Figure 4.5(b).) ☐ Claim 2

For each $i \in [1 \cdot\cdot k]$ set $C_i = \{x \in V(G) : N_x \supseteq N_{v_i}\}$. According to Lemma 4.2.1 C_is partition $V(G)$. Furthermore, in light of Claim 2 it follows that:

- each C_i is a clique,
- $E[C_i, C_j]$ is a complete bipartite graph if v_i and v_j are consecutive vertices of C, and

- $E[C_i, C_j]$ has no edge if v_i and v_j are distinct nonconsecutive vertices of C;

from which it follows that G is an expansion of C, as desired.



(a) The case $|N_x \cap N_y| = 2$. With $N_x = \{v_1, v_2, v_3\}$ and $N_y = \{v_2, v_3, v_4\}$, $G[\{x, y, v_3, v_4, v_5\}]$ would be a bull unless $xy \in E(G)$.

(b) The case $|N_x \cap N_y| = 3$. With $N_x = N_y = \{v_1, v_2, v_3\}$, $G[\{x, y, v_3, v_4\}]$ would be a claw unless $xy \in E(G)$.

Figure 4.5: Cases I and II in Claim 2, Lemma 4.2.3

□

4.2.2 The cases $\ell(G) \in \{4, 5\}$.

Lemma 4.2.4. *Let G be a claw- and bull-free graph. If $\ell(G) = 4$ or 5, then the maximum size of an independent set of vertices of G is at most 2.*

Proof. Let I be a largest independent set in G with $|I| \geq 3$.

Case 1: $\ell(G) = 4$.

Let $C = v_1, v_2, v_3, v_4, v_1$ be an induced cycle in G. Since $\alpha(C) = 2$, $|I \cap V(C)| \in \{0, 1, 2\}$.

Case 1.1: $|I \cap V(C)| = 2$.

According to Lemmas 4.2.1 and 4.2.2 every vertex $x \in I \setminus V(C)$ is adjacent to two consecutive vertices of C. Hence, $I \cap V(C)$ also has to consist of two consecutive vertices of C, a contradiction.

Case 1.2: $|I \cap V(C)| = 1$.

Let $I \cap V(C) = \{v_1\}$, and let x, y be distinct vertices in $I \setminus V(C)$. Without loss of generality, suppose $v_2, v_3 \in N(x)$. Note that if $v_2 y \in E(G)$, then $G[\{v_1, v_2, x, y\}]$ would be a claw. Hence, we must have $v_3, v_4 \in N(y)$. But then $G[\{v_1, v_2, v_3, x, y\}]$ would be a bull, a contradiction. (See Figure 4.6.)

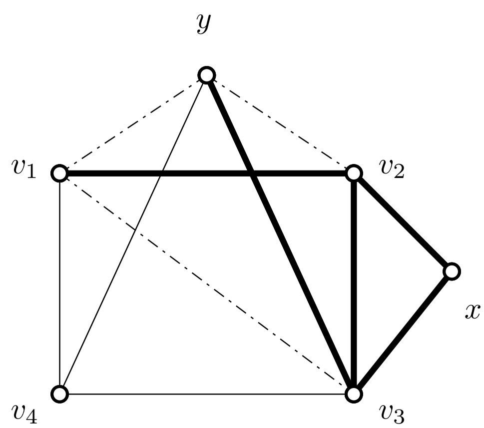


Figure 4.6: Case 1.2, Lemma 4.2.4. With $\{x, y, v_1\} \subseteq I$ one must have $v_2y \notin E(G)$; thereby, $\{v_3, v_4\} \subseteq N(y)$. Then $G[\{v_1, v_2, v_3, x, y\}]$ will be a bull.

Case 1.3: $|I \cap V(C)| = 0$**.**

Let x, y, z be distinct vertices in I. Since G is claw-free, no vertex of C is adjacent to all three of x, y, z. Hence, by the pigeonhole principle and Lemmas 4.2.1 and 4.2.2, we may assume $v_3 \notin N(x)$ and $v_4 \notin N(x)$, which imply $xv_1, xv_2 \in E(G)$. Furthermore, we may assume $v_1 \notin N(y)$. If in addition $v_4 \notin N(y)$, we would have $v_2y, v_3y \in E(G)$ in which case $G[\{v_2, v_3, v_4, x, y\}]$ would be a bull. Hence, $v_4y \in E(G)$, which in turns implies $v_3y \in E(G)$ (according to Lemma 4.2.1). (See Figure 4.7.)

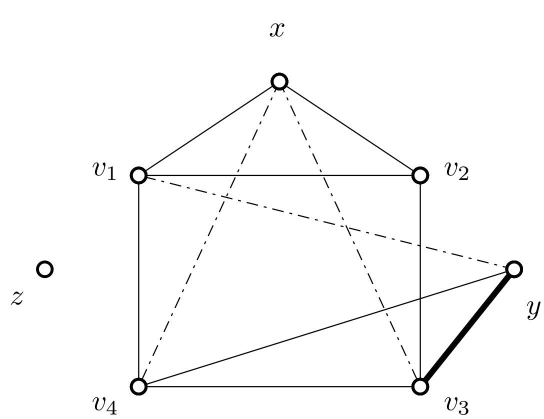
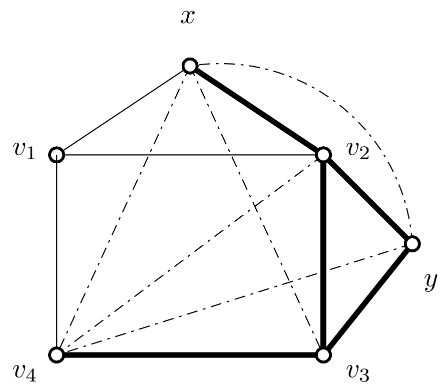
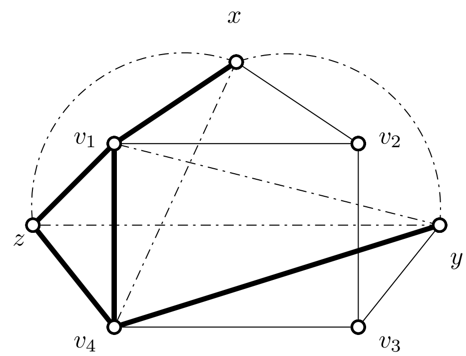


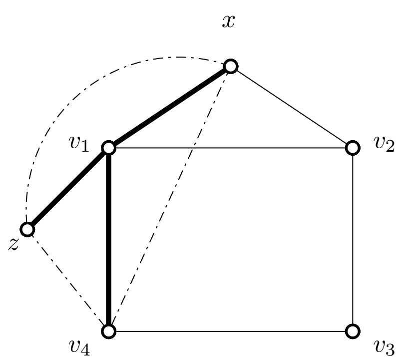
Figure 4.7: General situation in Case 1.3, Lemma 4.2.4. With $\{x, y, z\} \subseteq I$, one may assume $xv_3 \notin E(G)$, $xv_4 \notin E(G)$, implying $xv_1, xv_2 \in E(G)$. One may further assume $yv_1 \notin E(G)$. As such, Lemmas 4.2.1 and 4.2.2 imply $yv_3 \in E(G)$.



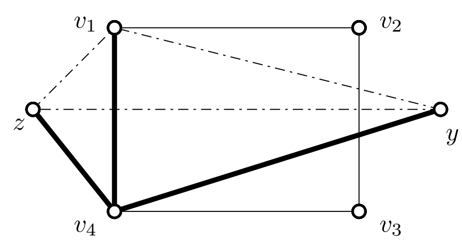
(a) If $v_4y \notin E(G)$, then $yv_2 \in E(G)$; thereby, $G[\{v_2, v_3, v_4, x, y\}]$ would be a bull.



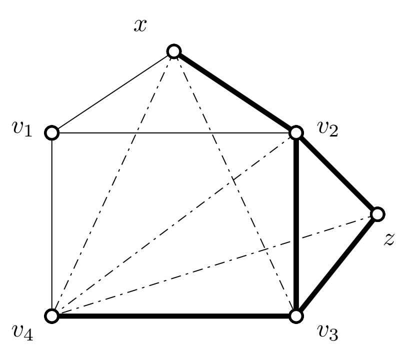
(b) $G[\{v_1, v_4, x, y, z\}]$ would be a bull if $v_1z, v_4z \in E(G)$.



(c) If $v_1z \in E(G)$ and $v_4z \notin E(G)$ then $G[\{v_1, v_4, x, z\}]$ would be a claw.



(d) If $v_4z \in E(G)$ and $v_1z \notin E(G)$ then $G[\{v_1, v_4, y, z\}]$ would be a claw.



(e) Since $v_2z, v_3z \in E(G)$ and $v_4z \notin E(G)$, $G[\{v_2, v_3, v_4, x, z\}]$ is a bull, a contradiction.

Figure 4.8: Lemma 4.2.4 with $\ell(G) = 4$. From the general situation described in Figure 4.7 one gets $\{v_3, v_4\} \subseteq N(y)$ and $N(z) \cap V(C) = \{v_2, v_3\}$, leading to the bull in **(e)**

Now observe that if $v_1, v_4 \in N(z)$ then $G[\{v_1, v_4, x, y, z\}]$ would be a bull, and if only one of v_1, v_4 is in $N(z)$ then $G[\{v_1, v_4, x, z\}]$ or $G[\{v_1, v_4, y, z\}]$ would be a claw. Hence, $v_1 \notin N(z)$ and $v_4 \notin N(z)$; thereby $v_2, v_3 \in N(z)$. But then $G[\{v_2, v_3, v_4, x, z\}]$ would be a bull, a contradiction. (See Figure 4.8 for a demonstration of this argument.)

Case 2: $\ell(G) = 5$.

Let $C = v_1, v_2, v_3, v_4, v_5, v_1$ be an induced cycle in G. As in Case 1, we have $|I \cap V(C)| \in \{0, 1, 2\}$.

Case 2.1: $|I \cap V(C)| = 2$.

By Lemma 4.2.1, every vertex in $I \setminus V(C)$ is adjacent to three consecutive vertices of C. Hence, likewise Case 1.1, I has to contain two consecutive vertices of C, a contradiction.

Case 2.2: $|I \cap V(C)| = 1$.

Let $I \cap V(C) = \{v_1\}$, and let x, y be distinct vertices in $I \setminus V(C)$. Without loss of generality, suppose $v_2, v_3, v_4 \in N(x)$. If $v_2 y \in E(G)$, then $G[\{v_1, v_2, x, y\}]$ would be a claw. Hence, $N(y) \cap V(C) = \{v_3, v_4, v_5\}$. But then $G[\{v_1, v_2, v_3, x, y\}]$ would be a bull, a contradiction. (See Figure 4.9.)

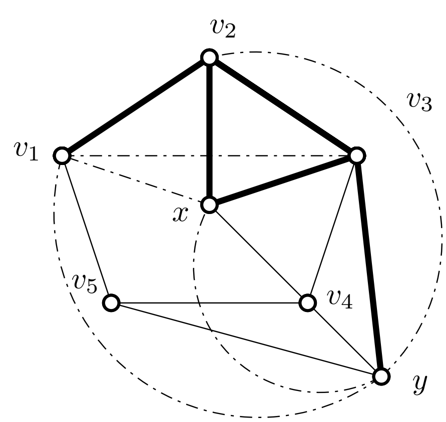


Figure 4.9: Case 2.2, Lemma 4.2.4. With $\{x, y, v_1\} \subseteq I$ and $\{v_2, v_3, v_4\} \subseteq N(x)$, one gets $v_2 y \notin E(G)$, since G is claw-free. But then $G[\{v_1, v_2, v_3, x, y\}]$ will be a bull, a contradiction

Case 2.3: $|I \cap V(C)| = 0$.

Let x, y, z be distinct vertices in I. Since G is claw-free, no vertex of C is adjacent to all three of x, y, z. Hence, by the pigeonhole principle and Lemmas 4.2.1 and 4.2.2, we may assume $v_4 \notin N(x)$ and $v_5 \notin N(x)$, which imply $xv_1, xv_2, xv_3 \in E(G)$. Furthermore, we may assume $v_1 \notin N(y)$; thereby, $yv_3, yv_4 \in E(G)$. But then we must have $yv_5 \in E(G)$ for otherwise $G[\{v_3, v_4, v_5, x, y\}]$ would be a bull. (See Figure 4.10.)

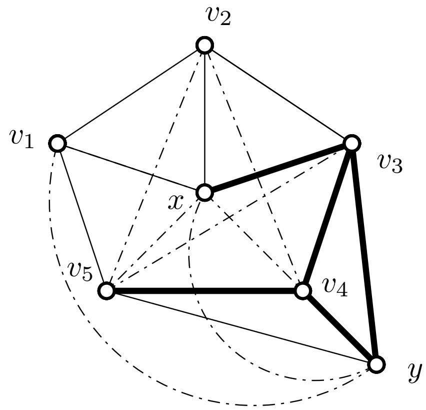
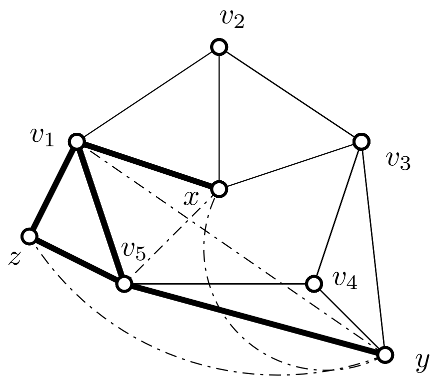
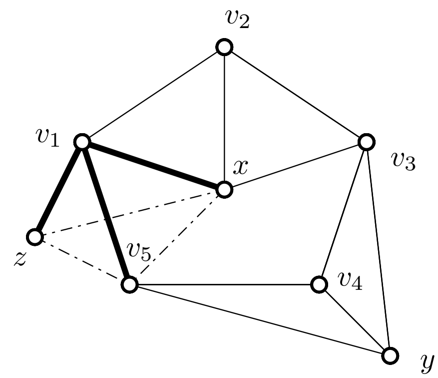


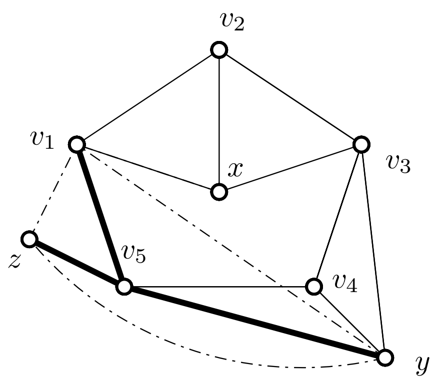
Figure 4.10: General situation in Case 2.3, Lemma 4.2.4. With $\{x, y, z\} \subseteq I$, one may assume $N(x) \cap V(C) = \{v_1, v_2, v_3\}$ and $yv_1 \notin E(G)$, implying that $yv_5 \in E(G)$.



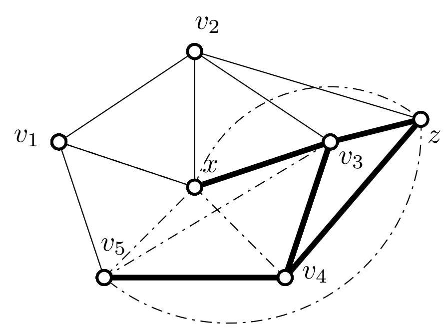
(a) $G[\{v_1, v_5, x, y, z\}]$ would be a bull if $v_1z, v_5z \in E(G)$.



(b) If $v_1z \in E(G)$ and $v_5z \notin E(G)$ then $G[\{v_1, v_5, x, z\}]$ would be a claw.



(c) If $v_5z \in E(G)$ and $v_1z \notin E(G)$ then $G[\{v_1, v_5, y, z\}]$ would be a claw.



(d) Since $v_3z, v_4z \in E(G)$ and $v_5z \notin E(G)$, $G[\{v_3, v_4, v_5, x, z\}]$ is a bull, a contradiction.

Figure 4.11: Lemma 4.2.4 with $\ell(G) = 5$. From the general situation described in Figure 4.10 one gets $\{v_3, v_4\} \subseteq N(y)$ and $N(z) \cap V(C) = \{v_1, v_2, v_3\}$, leading to the bull in **(d)**.

If $v_1, v_5 \in N(z)$ then $G[\{v_1, v_5, x, y, z\}]$ would be a bull, and if only one of v_1, v_5 is in $N(z)$ then $G[\{v_1, v_5, x, z\}]$ or $G[\{v_1, v_5, y, z\}]$ would be a claw. Hence, $v_1 \notin N(z)$ and $v_5 \notin N(z)$; thereby $v_2, v_3, v_4 \in N(z)$. But then $G[\{v_3, v_4, v_5, x, z\}]$ would be a bull, a contradiction. (See Figure 4.11 for a demonstration of this argument.)

□

4.2.3 The case $\ell(G) \leq 3$ with $\alpha(G) \geq 3$

Proposition 4.2.5 and, in multiple occasions, Proposition 4.2.6 will be used in the proof of Lemma 4.2.7 which is the main result of this subsection.

Proposition 4.2.5. *Let G be a claw- and bull-free graph with $\alpha(G) \geq 3$ and $diam(G) = 2$. Then $\ell(G) \geq 6$.*

Proof. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be an independent set of vertices in G. Since $diam(G) = 2$, for each $i \in [1 \cdot\cdot 3]$ there is a common neighbor $w_i \in V(G)$ of the α_js for $j \in [1 \cdot\cdot 3] \setminus \{i\}$. Moreover, for each $i \in [1 \cdot\cdot 3]$ we have $w_i\alpha_i \notin E(G)$, for otherwise $G[\{\alpha_1, \alpha_2, \alpha_3\} \cup \{w_i\}]$ would be a claw. We shall show that the 6-cycle $C: \ \alpha_1 w_3 \alpha_2 w_1 \alpha_3 w_2 \alpha_1$ is induced; thereby $\ell(G) \geq 6$. To this end, suppose on the contrary that C has a chord. As such, without loss of generality we may assume $w_2 w_3 \in E(G)$. But then $G[\{\alpha_1, \alpha_2, \alpha_3, w_2, w_3\}]$ will be a bull, a contradiction. Hence, C is an induced cycle, as desired.

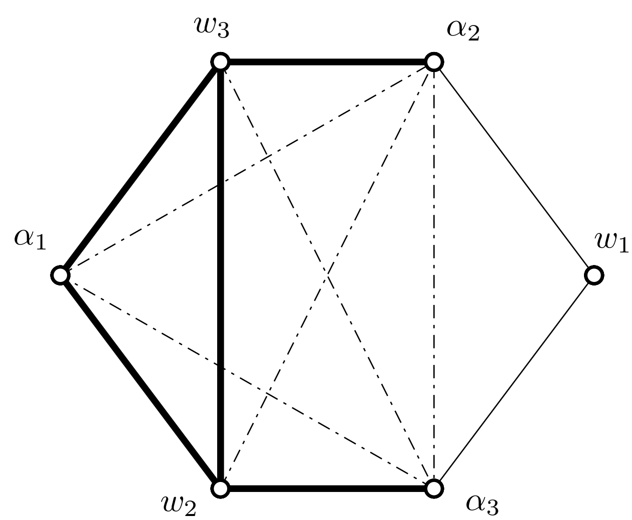


Figure 4.12: Cycle C introduced in the proof of Proposition 4.2.5 is induced: If $w_2 w_3 \in E(G)$ then $G[\{\alpha_1, \alpha_2, \alpha_3, w_2, w_3\}]$ would be a bull, a contradiction.

□

Proposition 4.2.6. *Let $C:\ u, v_0, v_1, \ldots, v_k, u$ be a cycle in a graph G with $\ell(G) \leq 3$, such that $v_0, v_1, \ldots, v_k$ is an induced path in G. Then $uv_i \in E(G)$ for each $i \in [0 \cdot\cdot k]$.*

Proof. Suppose, on the contrary that there is j such that $uv_j \notin E(G)$. Then, setting

$$a := \max\{s:\ s < j \ \&\ uv_s \in E(G)\},$$
$$b := \min\{s:\ s > j \ \&\ uv_s \in E(G)\},$$

one has $b - a \geq (j+1) - (j-1) = 2$. Moreover, the vertices $\{v_t : t \in ([a \cdot\cdot b] \setminus \{j\})\} \cup \{w_j, u\}$ induce a cycle C of length $b - a - 2$, which is greater than or equal to 4. This contradicts the assumption that $\ell(G) \leq 3$. □

The following lemma is the main result of this subsection.

Lemma 4.2.7. *Let G be a claw- and bull-free graph with $\ell(G) \leq 3$ and $\alpha(G) \geq 3$. Then:*

a. *$diam(G) \geq 4$; and*

b. *G is an expansion of a path.*

Proof. Let $k = diam(G)$. According to Proposition 4.2.5 we have $k \geq 3$. Let $v_0, v_k \in V(G)$ such that $d_G(v_0, v_k) = k$, and let $P = v_0, \ldots, v_k$ be a geodesic path between them. Moreover, let $U = N_G(v_0) \setminus \{v_1\}$, set $H = G - U$ and, as in Lemma 3.2.1, let N_is be the neighborhood levels of v_0 in H.

Claim 1. No vertex in U is adjacent to v_3 or a vertex in any N_i with $3 < i \leq k$. Moreover, a vertex of U adjacent to a vertex in some N_i is adjacent to every vertex in every N_j with $j < i$.

Proof of Claim 1. If the first part does not hold, then one has $d_G(v_0, v_k) < 2 + k - 3 < k$, a contradiction. As for the second part of the claim, consider a vertex $u \in U$ which is adjacent to a vertex $w_i \in N_i$ and for each $j \in [0 \cdot\cdot (i-1)]$ choose a vertex $w_j \in N_j$. Then, by the definition of the N_js, $w_0, w_1, \ldots, w_i$ is an induced path. Since $uw_0 = uv_0, uw_i \in E(G)$, every uw_j is an edge of G according to Proposition 4.2.6. This establishes the second part of the claim. □ Claim 1

Claim 2. $N_i = \varnothing$ for $i > k$.

Proof of Claim 2. It suffices to show that $N_{k+1} = \varnothing$. To this end, by the way of contradiction suppose $N_{k+1} \neq \varnothing$ and choose a vertex $w_{k+1} \in N_{k+1}$. Let Q be a geodesic path in G from w_{k+1} to v_0. Considering the fact that $d_H(w_{k+1}, v_0) = k+1 > k$, we conclude that Q must contain exactly one vertex, say u, from U. As such, we must also have $uv_0 \in E(Q)$, i.e. uv_0 must be the last edge of Q. Moreover, since $V(Q) \setminus \{u\} \subseteq V(H)$, every vertex in $V(Q) \setminus \{u\}$ must be in some N_j. Suppose the vertex of Q preceding u is in N_i and call it w_i.

Case I: $i > k$.

Set $w_3 = v_3$ and for each $j \in ([0 \cdot\cdot (i-1)] \setminus \{3\})$ choose $w_j \in N_j$. Note that as $i\, k \geq 3$, the induced path $w_0, w_1, \ldots, w_i$ contains v_3. Moreover, since $uw_0 = uv_0 \in E(G)$ and $uw_i \in E(G)$, we must have $uw_j \in E(G)$ for each $j \in [0 \cdot\cdot i]$; in particular, $uv_3 \in E(G)$. But the latter contradicts Claim 1. Hence, this case does not happen.

Case II: $i \leq k$.

Q will be of the form $w_{k+1}w_k, \ldots, w_i, u, v_0$ where each w_j $(j \in [i \cdot\cdot (k+1)]$ is in N_j. In particular the length of Q, which is bounded above by the diameter k of G, is $k+3-i$. Hence, $i \geq 3$. On the other hand, by Claim 1, we must have $i < 4$ (since u is not adjacent to $v_3 \in N_3$). Therefore, $i = 3$ and, hence, $uv_1, uv_2 \in E(G)$, according to Claim 1. Moreover, we have $uw_4 \notin E(G)$, by Claim 1, whereas $v_2w_3, w_3w_4 \in E(G)$ and $v_0v_2, v_0w_3, v_0w_4, v_2w_4 \notin E(G)$. Thus, $G[\{v_0, v_2, w_3, w_4, u\}]$ will be a bull, a contradiction.

□ Claim 2

Claim 3. $V(G) = (\bigcup_1^k N_i) \cup U$.

Proof of Claim 3. Contrary to the claim, assume $(\bigcup_1^k N_i) \cup U \subsetneq V(G)$ or, equivalently, $W := N(U) \setminus (\bigcup_1^k N_j) \neq \varnothing$. Let $\mathscr{R}$ be the set of paths of the shortest length from a vertex in W to v_k. Note that every path in $\mathscr{R}$ has at least one vertex in common with U, for otherwise w would be in $\bigcup_1^k N_j$, a contradiction. Choose $R \in \mathscr{R}$ such that $|V(R) \cap U|$ is minimum. Furthermore, let w be the initial vertex of R and u the last vertex of R which is in U. Observe that every vertex of R that follows u is in some N_j with $j \in [1 \cdot\cdot k]$ and, according to Claim 1, the immediate successor of R is in $\bigcup_1^3 N_j$. Let the latter be $w_i \in Ni$. Then, we must have

$$R(u, v_k) = \begin{cases} u, w_i, \ldots, w_{k-1}, v_k & \text{if } i < k-1; \\ u, w_{k-1}, v_k & \text{if } i = k-1; \\ u, w_3, v_3 & \text{if } i = k = 3; \end{cases}$$

where each w_j is in N_j and $w_3 \neq v_3$. (Recall that $uv_3 \notin E(G)$; thereby, in case $i = 3$ we must have $w_3 \neq v_3$.) As a result, the length of $R(u, v_k)$ is the $\max\{k - i + 1, 2\}$. Then. from the facts that

- R has at least one edge more than $R(u, v_k)$,
- length of R is bounded above by the diameter k, and
- $i \leq 3$,

it follows that $i \in \{2, 3\}$. In particular, $v_0 w_i \notin E(G)$.

Consequently, if $i = 2$ or $i = k = 3$ then we must have $wu \in E(G)$ (for otherwise the length of R would be grater than k); hence, $G[\{w, u, v_0, v_i\}]$ would be a claw, a contradiction. Also, $G[\{w, u, v_0, v_i\}]$ would be a claw if $i = 3$, $k > 3$ and $wu \in E(G)$. Hence, the only case to examine is when $i = 3 < k$ and $wu \notin E(G)$. As such, that R has length $\leq k$ implies

$$R = \begin{cases} w, u', u, w_3, v_4 & \text{if } k = 4; \\ w, u', u, w_3, \ldots, w_{k-1}, v_k & \text{if } k > 4; \end{cases}$$

Note that we must have $u' \in U$, for otherwise $R(u', v_k)$, would be in $\mathscr{R}$, contradicting

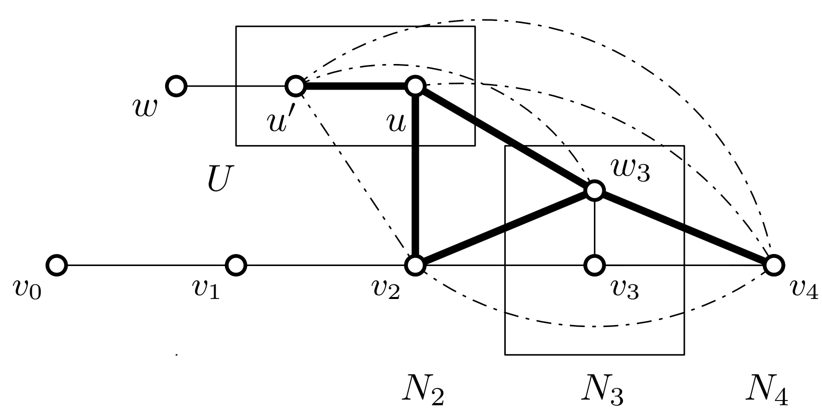


Figure 4.13: Ruling out the case $i = 3 < k$ and $wu \notin E(G)$ in the proof of Claim 3, Lemma 4.2.7. With $R(w, w_i) = w, u', u, w_3$, $G[\{u'u, v_2, v_4, w_3\}]$ will be a bull.

the choice of R as a path of the shortest length in $\mathscr{R}$. Likewise, we must have $u'w_3 \notin E(G)$, for otherwise $wu' + u'w_3 + R(w_3, v_k)$ would be a path in $\mathscr{R}$ yet shorter than R. Furthermore, we must have $u'v_2 \notin E(G)$, for otherwise the path $R' :=$

$wu' + u'v_2 + v_2w_3 + R(w_3, v_k)$ would have the same length as R, implying $R' \in \mathscr{R}$, with the property that

$$|V(R') \cap U| < |V(R) \cap U|,$$

contradicting the choice of R as an element in $\mathscr{R}$ with minimum size intersection with U. But then $G[\{u'u, v_2, v_4, w_3\}]$ will be a bull, a contradiction. (See Figure 4.13.) □ Claim 3

Claim 4. Let $U' \subseteq U$ such that any two vertices in U' have a common neighbor in N_2. Then U' is a clique.

Proof of Claim 4. According to Claim 1 no vertex in U' is adjacent to v_3. Hence, for any pair x, y of distinct vertices in U' with $xy \notin E(G)$, and for every common neighbor $w_2 \in N_2$ of x, y the graph $G[\{x, y, w_2, v_3\}]$ is a claw. Therefore, U' must be a clique. □ Claim 4

Claim 5. If there is a vertex $u \in U$ such that $N_G(u) \cap N_3 \neq \varnothing$ then $diam(G) = 3$.

Proof of Claim 5. Let $u \in U$ and $w_3 \in N_G(u) \cap N_3$ such that $uw_3 \in E(G)$. Then, according to Claim 1 we have $w_3 \neq v_3$ and $uv_2 \in E(G)$. If, in addition, $diam(G) \geq 4$, i.e. if $N_4 \neq \varnothing$, then $G[\{u, v_0, v_2, w_3, v_4\}]$ would a bull, a contradiction. Hence, we must have $diam(G) = 3$. □ Claim 5

Claim 6. $U \Leftrightarrow \{v_1\}$.

Proof of Claim 6. Let $u \in U$ such that $uv_1 \notin E(G)$. Then, by Claim 1 u is adjacent to no vertex in an N_i with $i > 0$; in other words, we have

$$N_G(u) \subseteq U \cup \{v_0\}.$$

Let $\mathscr{Q}$ be the set of paths of the shortest length from u to v_k. Note that every path in $\mathscr{Q}$ has at least two vertices in U (one of which is of course u), for otherwise one would have $uv_0 \in Q$, implying that $l(Q) = l(Q(v_0, v_k)) + 1 > k$, a contradiction. Hence,

$$|V(Q) \cap U| \geq 2 \qquad \forall Q \in \mathscr{Q}.$$

Choose $Q' \in \mathscr{Q}$ such that $|V(Q') \cap U|$ is the minimum and let u' be the last vertex of Q' which is in U. Note that

$$l(Q'(u', v_k)) \geq \begin{cases} k-2 & \text{if } k > 3; \\ k-1 & \text{if } k = 3; \end{cases} \tag{4.1}$$

where the second inequality follows from the fact that $u'v_3 \notin E(G)$. Note that u' must be adjacent to some vertex $w_2 \in N_2$, for otherwise one would have $l(Q) > k+1$, a contradiction. As such, we must have

$$uu' \notin E(G),$$

for otherwise $G[\{u, u', v_1, v_3, w_2\}]$ would be a bull. (See Figure 4.14.)

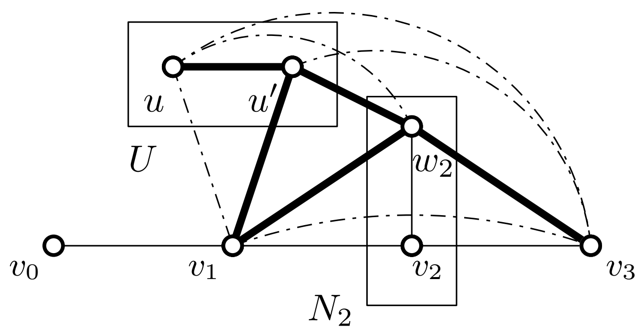


Figure 4.14: Ruling out the case that $|V(Q') \cap U| = 1$ in the proof of Claim 6, Lemma 4.2.7. For every $w_2 \in N_2 \cap N_G(u'')$, the graph $G[\{u, u', v_1, v_3, w_2\}]$ will be a bull.

Thus, according to (4.1), we have $k > 3$. Moreover, there is $u'' \in U$ such that $Q(u, u')$ is the path u, u'', u', and u' is followed by a vertex $w_3 \in N_3$. Note that

$$u''w_3 \notin E(G), \tag{4.2}$$

for otherwise the path from u to v_k obtained by augmenting the path u, u', w_3 to $Q'(w_3, v_k)$ would be shorter than Q, a contradiction. Moreover, as such, we must have

$$u''v_2 \notin E(G), \tag{4.3}$$

for otherwise the path Q'' obtained by augmenting the path u, u'', v_2, w_3 to $Q(w_3, v_k)$ will have the same length as Q' whereas

$$|V(Q'') \cap U| < |V(Q') \cap U|,$$

contradicting the choice of Q'. Finally, as shown in Figure 4.15, $G[\{u', u'', v_2, v_4, w_3\}]$ will be a bull, a contradiction. Hence, $U \Leftrightarrow \{v_1\}$, as desired. □ Claim 6

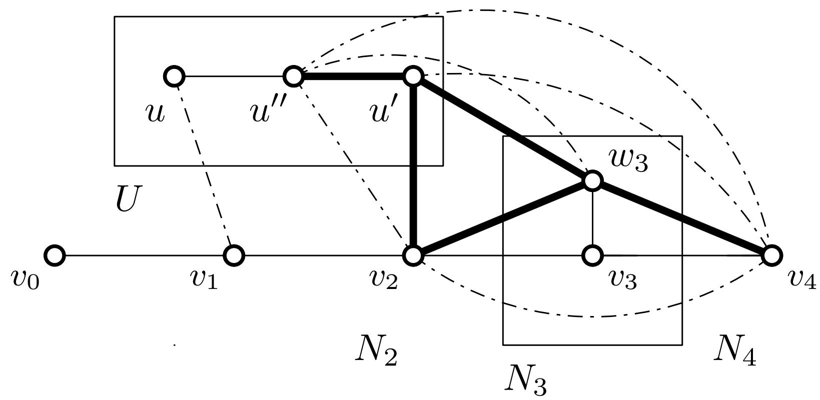


Figure 4.15: Ruling out the case that $|V(Q') \cap U| \geq 2$ in the proof of Claim 6, Lemma 4.2.7. By (4.2) and (4.3) the graph $G[\{u', u'', v_2, v_4, w_3\}]$ will be a bull.

Claim 7. U is a clique.

Proof of Claim 7. Suppose, contrary to the claim, that x, y are distinct vertices in U such that $xy \notin E(G)$. By Claim 6 we have

$$xv_1, yv_1 \in E(G).$$

Moreover, we have $xv_2 \in E(G)$ or $yv_2 \in E(G)$, for otherwise $G[\{x, yv_1, v_2\}]$ would be a claw. In addition, according to Claim 4, v_2 cannot be adjacent to both x and y. Hence, we may assume

$$xv_2 \notin E(G) \quad \& \quad yv_2 \in E(G).$$

But then, $G[\{x, y, v_1, v_2, v_3]$ would be a bull, a contradiction. Hence, U is a clique.

□ Claim 7

(a) By Claim 7 we have $|A \cap U| \leq 1$. Moreover, we have $N_{i-1} \Leftrightarrow N_i$ for every $i \in [1 \cdot\cdot k]$. Hence, as $|A| = 3$, we must have $k \geq 4$.

(b) As $k \geq 4$ and according to Claims 1 and 5, no vertex in U is adjacent to a vertex in any N_i with $i \geq 3$. Note that by Claim 6, we have $uv_1 \in E(G)$ for every $u \in U$. We shall show that every vertex in U is either adjacent to every vertex in N_2 or non-adjacent to every vertex in N_2. To this end, by the way of contradiction, let there be $u \in U$ and $s_2, t_2 \in N_2$ such that $us_2 \in E(G)$ and $ut_2 \notin E(G)$. Then $G[\{s_2, t_2, u, v_3, v_4\}]$ will be a bull, a contradiction. (See Figure 4.16.) Therefore, U is the disjoint union of the sets $V_0 := \{u \in U : \{u\} \Leftrightarrow N_2\}$ and $V_1 := \{u \in U : \nexists w \in N_2 : \ uw \in E(G)\}$, and

G is the expansion of the path $v_0, \ldots, v_k$ where each vertex v_i is replaced by the bag M_i defined by $M_i = N_i \cup V_i$ for $i = 0, 1$ and $M_i = N_i$ for each $i \in [2 \cdot\cdot k]$.

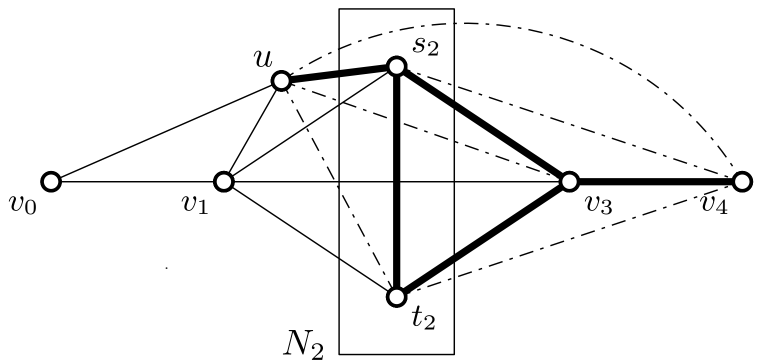


Figure 4.16: proof of part **(b)** of Lemma 4.2.7; showing that $V_0 \cap V_1 = \varnothing$: If $u \in U$ and $s_2, t_2 \in N_2$ with $us_2 \in E(G)$ and $ut_2 \notin E(G)$, then $G[\{s_2, t_2, u, v_3, v_4\}]$ will be a bull.

□

4.2.4 Proof of Theorem 4.1.1

Proof of Theorem 4.1.1. It is easy to check that an expansion of a path, that of a cycle, and the complement of a triangle-free graph are all claw- and bull-free. Conversely, by Lemmas 4.2.3, 4.2.4, and 4.2.7, every claw- and bull-free graph is either an expansion of a cycle of length ≥ 3, or the complement of a triangle-free graph, or an expansion of a path of length ≥ 4. □

Bibliography

[1] Martin Aigner and Michael Fromme. A game of cops and robbers. *Discrete Applied Mathematics*, 8(1):1–12, 1984.

[2] Thomas Andreae. Note on a pursuit game played on graphs. *Discrete Applied Mathematics*, 9(2):111–115, 1984.

[3] Deepak Bal, Anthony Bonato, William B Kinnersley, and Paweł Prałat. Lazy cops and robbers played on graphs. *arXiv preprint arXiv:1312.1750*, 2013.

[4] Andrew Beveridge, Andrzej Dudek, Alan Frieze, and Tobias Müller. Cops and robbers on geometric graphs. *Combinatorics, Probability and Computing*, 21(6):816–834, 2012.

[5] Anthony Bonato and Richard J Nowakowski. The game of cops and robbers on graphs, student mathematical library, vol. 61. *American Mathematical Society*, 2011.

[6] Gideon Ehrlich, Shimon Even, and Robert Endre Tarjan. Intersection graphs of curves in the plane. *Journal of Combinatorial Theory, Series B*, 21(1):8–20, 1976.

[7] István Fáry. On straight-line representation of planar graphs. *Acta Sci. Math.*, 11:229–233, 1948.

[8] Fedor V Fomin, Petr A Golovach, Jan Kratochvíl, Nicolas Nisse, and Karol Suchan. Pursuing a fast robber on a graph. *Theoretical Computer Science*, 411(7-9):1167–1181, 2010.

[9] Tomáš Gavenčiak, Przemysław Gordinowicz, Vít Jelínek, Pavel Klavík, and Jan Kratochvíl. Cops and robbers on intersection graphs. *European Journal of Combinatorics*, 72:45–69, 2018.

[10] Gwenaël Joret, Marcin Kamiński, and Dirk Theis. The cops and robber game on graphs with forbidden (induced) subgraphs. *Contributions to Discrete Mathematics*, 5(2), 2010.

[11] Athanasios Kehagias and Paweł Prałat. Some remarks on cops and drunk robbers. *Theoretical Computer Science*, 463:133–147, 2012.

[12] Richard Nowakowski and Peter Winkler. Vertex-to-vertex pursuit in a graph. *Discrete Mathematics*, 43(2-3):235–239, 1983.

[13] David Offner and Kerry Ojakian. Variations of cops and robber on the hypercube. *Australasian J. Combinatorics*, 59:229–250, 2014.

[14] Pawel Pralat. When does a random graph have constant cop number? In *Australasian Journal of Combinatorics*. Citeseer, 2010.

[15] A. Quillot. Jeux et pointes fixes sur les graphes. *Thèse de 3ème cycle,* Université de Paris VI, 1978.

[16] F Bruce Shepherd. Hamiltonicity in claw-free graphs. *Journal of Combinatorial Theory, Series B*, 53(2):173–194, 1991.

[17] Douglas B. West. *Introduction to graph theory.* Prentice Hall, Inc., Upper Saddle River, NJ, 1996.