Jacobians of Curves in Abelian Surfaces

by

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M.Math., Indian Statistical Institute, 2021
B.Sc. (Hons., Mathematics), Tezpur University, 2019

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Summer 2024

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Abstract

In this thesis we decompose the Jacobians of certain smooth curves into smaller abelian varieties up to isogeny. The curves lie in the linear systems of $(1, d)$-polarizations of simple abelian surfaces. This goal is motivated by Poincaré's reducibility theorem which states that any abelian variety is isogenous to a product of simple abelian varieties such that the simple factors are unique up to isogeny. We construct curves inside simple abelian surfaces and determine the isogenous decomposition of their Jacobians into simple factors when the degrees of polarizations are $(1, 2)$, $(1, 3)$ and $(1, 4)$. We establish isogeny relations for Jacobians of similar curves lying in the linear systems of polarizations of higher degrees. Sufficient conditions for these curves to cover elliptic curves and thus have elliptic factors in the decomposition of their Jacobians have also been established.

Keywords: Jacobians of curves; abelian surfaces; isogenies

Dedication

To my guru, Om Swami. Words cannot express what my heart knows. Your grace and blessings have made my life beautiful.

Acknowledgements

I am grateful to my parents and my sister for their unconditional love and support. I am here today because of them. I am thankful to Neela and my roommate Taha, whose friendship, care and support in the last one year have made Vancouver feel like home.

Sincere thanks to my local as well as non-local colleagues/friends: Nawal, Jaskaran, Aniket, Ahmad, Leo, Chi Ki, Kshitij, Harshil, Hitesh, Shiva, Juganta, Deniz, Kuldeep Sarma, David Sun, Zhe Xu and Sharon Robins. Sharon has been a brotherly figure and has shared experiences from his own journey as a graduate student which helped me navigate some of my challenges.

I would like to thank Dr. Nils Bruin for his valuable comments during the graduate progress report meeting and both Dr. Nils Bruin and Dr. Imin Chen for their interesting questions and suggested improvements of the thesis. I am thankful to Dr. Jake Levinson for his immaculate course in Scheme Theory and our conversations during his time at SFU. His door was always open and he went the extra mile in helping students. Dr. Levinson's attitude towards math and academic life – both inside and outside classrooms/seminars – is something I deeply admire and find inspiring.

Thanks to my football club, Real Madrid, for countless moments of joy.

Finally, I would like to convey my deepest gratitude to my supervisor, Dr. Katrina Honigs. Her support has been invaluable since the very first day: from helping me select courses as well as a research problem of my interest, to beautifully explaining complex topics in algebraic geometry. In the initial stages of my research, she patiently repeated the general outline of the problem and many important related facts, which helped me get used to them and made me confident. Later, Dr. Honigs gave me a lot of freedom and stepped in at various stages to help me move forward, which also includes thoroughly reading earlier drafts of this thesis. Her guidance, kindness and patience have given me a sense of belonging in math and I consider myself fortunate for the opportunity to work with her.

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Chapter 1

Introduction

An abelian variety A over the field of complex numbers $\mathbb{C}$ is a complete connected variety whose points form an abelian group. The group operation is denoted by $+$, where $+ : A \times A \longrightarrow A$ is a morphism of varieties. Abelian varieties are projective, non-singular and two dimensional abelian varieties are called abelian surfaces. An abelian variety is called *simple* if its only abelian subvarieties are 0 and itself. Maps between abelian varieties that are surjective morphisms on the level of varieties, preserve corresponding group structures and have finite kernels are called *isogenies*. Being isogenous is an equivalence relation on abelian varieties, often denoted by $\sim$. The following is a decomposition theorem for abelian varieties.

Theorem 1.1 (Poincaré's reducibility)**.** *Let A be an abelian variety. Then $A \sim \prod_{i=1}^{n} A_i^{d_i}$, where each A_i is a simple abelian variety and $A_i \nsim A_j$ for $i \neq j$. Furthermore, each d_i is a uniquely determined non-negative integer and each A_i is unique up to isogeny.*

In this thesis, we are interested to know this decomposition for Jacobians of curves lying inside abelian surfaces. As an abelian variety, the Jacobian of a non-singular curve C is the connected component of the identity in the Picard group of C and is denoted by J_C.

Any abelian variety A carries a *polarization*, which is an ample line bundle L on A that induces an isogeny $\varphi_L : A \longrightarrow A^\vee$ from A to its dual. The kernel of φ_L is denoted by $K(L)$ and it is known that $K(L) \cong (\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z})^2$, where $(d_1, \ldots, d_n)$ is a unique tuple of positive integers with $d_i \mid d_{i+1}$ for $1 \leq i \leq n-1$, called the *type of* L. When A is an abelian surface, any polarization L on A has type (d_1, d_2) and the linear system of L, denoted by $|L|$, contains curves in A. The following theorem will be our main tool to decompose the Jacobian of a smooth curve in the linear system of a polarization on a simple abelian surface.

Theorem 1.2 ([KR89], Theorem B)**.** *Let C be a curve and G be a finite subgroup of* $\mathrm{Aut}(C)$ *such that $G = H_1 \cup \cdots \cup H_t$, where the subgroups $H_i \leq G$ satisfy $H_i \cap H_j = 1$ if $i \neq j$. Then* *we have the isogeny relation*

$$J_C^{t-1} \times J_{C/G}^{g} \sim J_{C/H_1}^{h_1} \times \cdots \times J_{C/H_t}^{h_t} \tag{1.1}$$

where $|G| = g$, $|H_i| = h_i$ *and, as usual,* $J^n = J \times \cdots \times J$ *(n times).*

We will use this theorem on curves obtained from the following construction (see Construction 3.3): for a simple abelian surface A with a $(1, d)$-polarization L, where L is a symmetric line bundle of characteristic zero, and an order d subgroup X of $K(L) \cong \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, consider the isogeny $\pi : A \longrightarrow A/X$. The principally polarized abelian surface A/X is isomorphic to J_H for some smooth genus two curve H. The pre-image $C := \pi^{-1}(H)$ is a smooth curve in the linear system $|L|$ such that -1 is an involution of C and translations induced by elements of X are automorphisms of C. The following question is the central theme of this thesis:

Question 1.3. How does J_C decompose into smaller abelian varieties up to isogeny and what properties of C can be captured from this decomposition?

Using the theory of the Heisenberg group and theta functions, the Jacobians of curves arising from similar constructions were studied by Birkenhake and Lange for $(1, 3)$-polarized abelian surfaces (see [BL94]), and by Borówka and Ortega for $(1, 4)$-polarized abelian surfaces (see [BO19]). Although the curve C and the isogeny class of J_C will depend on the structure of the group X, it is known that A will always be a simple factor in the decomposition of such Jacobians. We want to know what other factors can appear. Taking a different approach compared to the methods of previously done examples, we will show that the automorphism groups of these curves have suitable subgroups that allow us to use our main tool (Theorem 1.2) to decompose their Jacobians. Our first main result is the following.

Theorem 1.4 (Proposition 3.19)**.** *If the curve C is obtained from a cyclic subgroup $X = \langle x \rangle$ of $K(L)$, then we have the following isogeny relations:*

1. *When d is odd:* $J_C \sim A \times J^2_{C/\langle -1 \rangle}$
2. *When d is even:* $J_C \sim A \times J_{C/\langle -1 \rangle} \times J_{C/\langle -1 \circ t_x \rangle}$

where -1 is the involution of C descending from the $[-1]$-involution of A, and t_x denotes the translation by x.

As an immediate consequence of this result, we give the complete decomposition of J_C into its simple factors for the cases $d = 2, 3$. When A is $(1, 4)$-polarized, another choice of X is the Klein four-group. In this case, π will be called the Klein cover/isogeny. We establish the following two results about the curve coming from this case and thus recover the two main results proved in [BO19].

Theorem 1.5 (Theorem 3.21)**.** *The Jacobian of a curve arising from a Klein cover of an abelian surface A with $(1,4)$-polarization is isogenous to the product of A and three elliptic curves.*

Theorem 1.6 (Theorem 3.22)**.** *The curve C arising from a Klein cover of a $(1,4)$-polarized abelian surface is hyperelliptic.*

We also give a sufficient condition on the type of polarization for the curve C of the construction to cover an ellliptic curve:

Theorem 1.7 (Theorem 3.26)**.** *When $d = 2k$ (resp. $d = 3k$) and $X \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$ (resp. $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$), the curve C covers an elliptic curve. Further, the degree of the cover is d (resp. $\frac{2d}{3}$) and J_C has an elliptic factor in its isogeny class.*

In Chapter 4, we look at polarizations of type $(1,p^2)$, where p is a prime. This is motivated by the classification theorem for abelian groups which can be written as a union of proper subgroups that interesect trivially (Theorem 2.13). For curves obtained from the construction above with X non-cyclic, we establish isogeny relations of their Jacobians in the cases of $(1,9)$-polarized and $(1,25)$-polarized simple abelian surfaces and generalize our findings with the following result.

Theorem 1.8 (Theorem 4.1)**.** *Let d be the square of a prime p, and X be the subgroup $\langle px, py\rangle$ of $K(L) = \langle x, y\rangle \cong (\mathbb{Z}/p^2\mathbb{Z})^2$. If C is the curve arising from the construction with this choice of X, then we have the following isogeny relations of Jacobians:*

$$J_C \sim A \times \prod_{i=1}^{p+1} J_{C_i}^2$$
$$J_{C/\langle -1\rangle} \sim \prod_{i=1}^{p+1} J_{C_i},$$

where each C_i is a curve of genus $\frac{p-1}{2}$.

Plan of the thesis:

Chapter 2 discusses relevant background material on solvable groups and partitions, abelian varieties and isogenies, the dual and polarization of an abelian variety, curves and their Jacobians, and our main tool: the theorem of Kani and Rosen.

We begin Chapter 3 by laying out the setup of our problem, where we mention the construction of the curve C. Section 3.2 contains some known results related to our construction. We discuss fixed points of automorphisms of our curves in Section 3.3 and use these observations repeatedly in the later sections. We prove our main result regarding cyclic isogenies in Section 3.4 and the two main results on the Klein cover in Section 3.5. In Section 3.6, we look at when C can be hyperelliptic or can cover an elliptic curve.

Chapter 4 discusses Jacobian decompositions of curves coming from polarizations of higher degrees. We end this thesis with some questions that can serve as future directions of research related to this project.

Chapter 2

Background

2.1 Prelude

This chapter presents a terse review of the theory of solvable groups, curves and abelian varieties. The purpose of this chapter is to refresh the definitions and basic properties for the reader so that the main question can be well formulated and then answered using the tools established here. Readers interested in the details are encouraged to refer to [Sil86], [BL04], [Mil08], [Mum70] and [Sch94].

2.2 Solvable groups and partitions

The aim of this section is to discuss the theory of solvable groups that admit partitions. Such groups play an important role in the application of the main tool in answering the central question of this thesis. We start by introducing solvable groups.

Definition 2.1. A group G is called ***solvable*** if it has a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_s = G$$

such that G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is abelian for all $i = 0, \ldots, s-1$.

Example 2.2. Abelian groups are trivially solvable. Non-abelian solvable groups include D_6 (the dihedral group of order 6) and S_4 (group of permutations on 4 letters).

Solvability of a group can be deduced by combining information about its normal subgroup and the resulting quotient in the following way:

Lemma 2.3. *If $N \trianglelefteq G$, then G is solvable if and only if N and G/N are solvable.*

Proof. Suppose G is solvable and $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_s = G$ is a chain of subgroups of G. Define $G_i' := G_i \cap N$ for $1 \leq i \leq s$. Let $x \in G_i'$ and $y \in G_{i+1}'$. Then $x \in G_i$ and $y \in G_{i+1}$. By normality of G_i in G_{i+1}, we have $yxy^{-1} \in G_i$. Since x, y also lie in N, so does yxy^{-1}. Therefore, $yxy^{-1} \in G_i'$ and hence $G_i' \trianglelefteq G_{i+1}'$. To see that G_{i+1}'/G_i' is abelian, observe that the group homomorphism $G'_{i+1} \longrightarrow G_{i+1}/G_i$ given by $g_{i+1} \mapsto \overline{g_{i+1}}$ has kernel G'_i and hence induces an isomorphism of G'_{i+1}/G'_i into a subgroup of G_{i+1}/G_i, which is abelian since G is solvable. Therefore, $1 = G'_0 \trianglelefteq G'_1 \trianglelefteq G'_2 \trianglelefteq \ldots \trianglelefteq G'_s = N$ is a desired chain of subgroups making N solvable.

For solvability of G/N, consider the subgroups H_i of G given by $H_i = G_iN$ for $i = 0, \ldots, s$. Clearly, each H_i contains N and an element in H_i is of the form g_in where $g_i \in G_i$ and $n \in N$. For $1 \leq i \leq s$, the map $H_{i+1} \longrightarrow G_{i+1}/G_i$, $g_{i+1}n \mapsto \overline{g_{i+1}}$ is a group homomorphism (since $g_{i+1}N = Ng_{i+1}$) with H_i as kernel. Thus, H_i is a normal subgroup of H_{i+1} and, by the first isomorphism theorem, H_{i+1}/H_i is isomorphic to a subgroup of G_{i+1}/G_i and hence is abelian. Now, let $\overline{H_i} := H_i/N$ for $1 \leq i \leq s$. Each $\overline{H_i}$ is a subgroup of G/N. Since $H_i \trianglelefteq H_{i+1}$, by the correspondence theorem $\overline{H_i} \trianglelefteq \overline{H_{i+1}}$. Moreover, as H_i/H_{i+1} is abelian, by the third isomorphism theorem $\overline{H_i}/\overline{H_{i+1}}$ is abelian. Therefore, we have a desired chain of subgroups $1 = \overline{H_0} \trianglelefteq \overline{H_1} \trianglelefteq \overline{H_2} \trianglelefteq \ldots \trianglelefteq \overline{H_s} = G/N$, making G/N solvable.

Conversely, assume N and G/N are solvable. Then we have chains of subgroups:

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \ldots \trianglelefteq N_r = N \text{ such that } N_{i+1}/N_i \text{ is abelian}$$
$$1 = \overline{N} = \overline{H_0} \trianglelefteq \overline{H_1} \trianglelefteq \overline{H_2} \ldots \trianglelefteq \overline{H_s} = G/N \text{ such that } \overline{H_{i+1}}/\overline{H_i} \text{ is abelian.}$$

The chain $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_{r+s} = G$, where $G_i = N_i$ for $0 \leq i \leq r$ and $G_i = H_{i-r}$ for $r+1 \leq i \leq r+s$, gives a chain of subgroups of G making it solvable. □

Definition 2.4. A group G is called a ***Frobenius group*** if it has a proper non-trivial subgroup H such that $H \cap gHg^{-1} = 1_G$ for all $g \in G \setminus H$.

Example 2.5. The dihedral group of order $2d$ is a Frobenius group when d is odd but not when d is even.

Definition 2.6. Let G be a group and p be a prime dividing $\#G$. The ***Hughes subgroup of*** G ***relative to*** p, denoted by $H_p(G)$, is the subgroup generated by all elements whose order is not p, i.e., $H_p(G) = \langle x \in G \mid x^p \neq 1 \rangle$.

Definition 2.7. A finite group G is said to be a group of ***Hughes–Thompson type*** if it is not a p-group and $H_p(G) \neq G$ for some prime divisor p of $\#G$.

Example 2.8. Dihedral groups are of Hughes–Thompson type (take $p = 2$ in the definition).

Definition 2.9. Let G be a group. A set of non-trivial subgroups $\mathcal{P} = \{H_1, H_2, \ldots, H_n\}$ is said to be a ***partition of*** G if $n \geq 2$ and each non-identity element of G lies in exactly one H_i. Equivalently, $G = H_1 \cup H_2 \cup \cdots \cup H_n$ where $H_i \cap H_j = 1$ for $i \neq j$. The members of $\mathcal{P}$ are called ***components of the partition*** or simply ***components*** when the partition is clear from the context.

Example 2.10. The dihedral group $D_{2d} = \langle r, s \mid r^d = s^2 = 1,\ srs^{-1} = r^{-1}\rangle$ has a partition $\{\langle r\rangle, \langle s\rangle, \langle sr\rangle,\ \langle sr^2\rangle, \ldots, \langle sr^{d-1}\rangle\}$. Cyclic groups cannot have partitions since the subgroup containing a generator will be the whole group.

The study of partitions of groups was initiated by G. A. Miller in 1906. He characterized finite abelian groups having partitions.

Definition 2.11. A finite ***elementary abelian group*** is an abelian group where all non-identity elements have the same order.

Remark 2.12. The order of each non-identity element in an elementary abelian group must be a prime number. By the fundamental theorem of abelian groups, any elementary abelian group is of the form $\mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$ for some prime p. Thus, an elementary abelian group is a finite-dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$ and, in particular, a p-group.

Theorem 2.13. *A finite abelian group G has a partition $\mathcal{P} = \{H_1, H_2, \ldots, H_n\}$ if and only if it is an elementary abelian group whose order is not a prime.*

Proof. Let G be a finite abelian group having a partition $\mathcal{P} = \{H_1, H_2, \ldots, H_n\}$. To prove that G is an elementary abelian group, we need to show that all non-identity elements in G will have the same order. If not, pick $x \in G$ of maximal order and let $y \in G$ be such that $\operatorname{ord}(y) < \operatorname{ord}(x)$. If we denote the group operation of G additively, then the non-identity element $\operatorname{ord}(y) \cdot (x + y) = \operatorname{ord}(y) \cdot x$ is a multiple of both $x + y$ and x. Since a component contains an element if and only if it contains all multiples of that element, therefore, x and $x+y$ will be in the same component and consequently, this component will also contain y. Thus, the component of $\mathcal{P}$ containing an element x of maximal order should also contain all elements of order less than that of x. This implies that all elements lie in the same component, as any two components are either equal or have trivial intersection. This contradicts the hypothesis that $\mathcal{P}$ is a partition of G. Therefore, all non-identity elements of G must have the same order and thus G is an elementary abelian group. Finally, the order of G should not be a prime otherwise G will be cyclic and cannot admit a partition.

Conversely, if G is an elementary abelian p-group of order greater than p, then $G = \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$ (n times). This group has $p^n - 1$ elements of order p, each lying in exactly one cyclic group of order p. These $\frac{p^n-1}{p-1}$ subgroups of G form a partition of G. $\square$

Proposition 2.14. *Let G be an elementary abelian group of order p^m. If $\mathcal{P} = \{H_1, \ldots, H_n\}$ is a partition of G such that each component in the partition has order p^a, then $a \mid m$. Conversely, if $a \mid m$ and $a \neq m$, then there is a partition of G by subgroups of order p^a.*

Proof. If such a partition exists, then comparing the number of non-identity elements, we get:

$$p^m - 1 = n \cdot (p^a - 1) \Longrightarrow p^a - 1 \mid p^m - 1 \Longrightarrow a \mid m.$$

Conversely, let a be a proper divisor of m. Since G is a $\mathbb{Z}/p\mathbb{Z}$-vector space of dimension m, we can consider the cannonical basis $\{e_1, \ldots, e_m\}$ of G. Since $a \mid m$, we can divide this basis into $\frac{m}{a}$ sets each containing a elements. The subgroups of G generated by each of these sets are of order p^a and form a partition of G. □

The following result classifies solvable groups with partitions.

Theorem 2.15 ([Sch94], Theorem 3.5.10)**.** *A solvable group G has a partition $\mathcal{P}$ if and only if it is exactly one of the following:*

(i) S_4.

(ii) a Frobenius group.

(iii) a group of Hughes–Thompson type, where $H_p(G) \in \mathcal{P}$, $[G : H_p(G)] = p$ and every element not in $H_p(G)$ has order p.

(iv) a p-group with $\mathcal{P}$ containing a component H such that every element in $G \setminus H$ has order p; furthermore $\#\mathcal{P} \equiv 1 \pmod{p}$.

To illustrate the theorem, we give an example of each type of group with a partition.

Example 2.16. For S_4, consider the set of maximal cyclic subgroups, i.e., the cyclic subgroups that are not contained in any other proper cyclic subgroup. The union of these maximal cyclic subgroups is S_4 because any element will lie in a maximal cyclic subgroup. Further, if any two maximal cyclic subgroups X and Y of S_4 intersect non-trivially, then the only possibility is $\#X = 4$, $\#Y = 4$ and $\#(X \cap Y) = 2$, which is a contradiction because S_4 has three cyclic subgroups of order 4 and any two of them intersect trivially. For a Frobenius group, consider S_3 which admits the partition $S_3 = \langle(123)\rangle \cup \langle(12)\rangle \cup \langle(13)\rangle \cup \langle(23)\rangle$. For a group of Hughes–Thompson type, consider the dihedral group of order $2d$, denoted by $D_{2d} = \langle r, s \mid r^d = s^2 = (sr)^2 = 1\rangle$. When $p = 2$, we have $H_p(D_{2d}) = \langle r\rangle \neq D_{2d}$. Moreover, $\{\langle r\rangle, \langle s\rangle, \langle sr\rangle, \langle sr^2\rangle, \ldots, \langle sr^{d-1}\rangle\}$ is a partition of D_{2d}. Finally, the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle(1,0)\rangle \cup \langle(0,1)\rangle \cup \langle(1,1)\rangle$ is an example of the fourth type in the theorem above.

2.3 Curves and divisors

The following is a brief discussion of curves and Weil divisors. We are interested in non-singular curves, so we will define them to be smooth for convenience. We introduce genus of a curve, Picard group of divisors and also state the Riemann–Hurwitz formula which is helpful in computing genera of curves. Our base field k will be the field of complex numbers $\mathbb{C}$.

Definition 2.17. A ***curve*** is a smooth projective variety of dimension one. A ***map of curves*** $\pi : C_1 \longrightarrow C_2$ is a morphism at the level of varieties.

Any map of curves is either constant or surjective. When it is surjective, we call it a ***cover***.

Definition 2.18. Let $\pi : C_1 \longrightarrow C_2$ be a map of curves. The ***degree*** of π is defined to be 0 if π is constant or the degree $[k(C_1) : \pi^* k(C_2)]$ of the extension of function fields otherwise. In the latter case, π is said to be a ***finite map***.

Definition 2.19. A map from a curve C to itself that is also a bijection is called an ***automorphism of*** C. The set of all automorphisms of C form a group under composition, denoted by $\mathrm{Aut}(C)$. An element of order two in this group is called an ***involution***. For $\tau \in \mathrm{Aut}(C)$, a point $p \in C$ is said to be a ***fixed point*** of τ if $\tau(p) = p$. The set of fixed points of an automorphism τ of a curve C will be denoted by $\mathrm{Fix}_C[\tau]$.

Definition 2.20. If G is a finite subgroup of $\mathrm{Aut}(C)$, then the quotient $\widetilde{C} := C/G$ admits a unique structure of a curve such that the quotient map $\pi : C \longrightarrow \widetilde{C}$ is a cover of degree $d = \#G$. We call it a ***Galois cover*** with group G. Equivalently, G is the Galois group of the function field extension $k(C)/k(\widetilde{C})$.

Lemma 2.21. *If X_1 and X_2 are conjugate subgroups of* $\mathrm{Aut}(C)$, *then* $C/X_1 \cong C/X_2$.

Proof. Consider the extensions $k \subset k(C/X_1) \subset k(C)$ and $k \subset k(C/X_2) \subset k(C)$. Then $k(C)/k(C/X_i)$ is a Galois extension with X_i as the Galois group. We know from Galois theory that $k(C/X_1) \cong k(C/X_2)$ are isomorphic as extensions of k if and only if X_1 and X_2 are conjugate subgroups. The isomorphism $k(C/X_1) \cong k(C/X_2)$ of function fields implies that C/X_1 and C/X_2 are isomorphic as curves. □

Definition 2.22. Let $\pi : C_1 \longrightarrow C_2$ be a cover of degree d of curves and $P \in C_1$. Then we have an induced inclusion of function fields $\pi^* : k(C_2) \longrightarrow k(C_1)$. We define the ***ramification index*** of π at P by

$$e_{\pi,P} := \mathrm{ord}_P(\pi^*(t))$$

where $t \in k(C_2)$ is a uniformizer of $\pi(P)$. We say that π is ***ramified at*** P if $e_{\pi,P} > 1$ and ***unramified at*** P otherwise. We say that the cover π is ramified if there is a ramified point $P \in C_1$ and unramified otherwise. If the map π is clear from the context then we use the notation e_P for the ramification index of π at P.

Proposition 2.23 ([Sil86], Proposition II.2.6)**.** *Let $\pi : C_1 \longrightarrow C_2$ be a cover of curves. Then:*

(a) For all $Q \in C_2$ we have

$$\sum_{P \in \pi^{-1}(Q)} e_P = \deg(\pi).$$

(b) For all but finitely many $P \in C_1$ we have $e_P = 1$.

Definition 2.24. The ***genus*** of a curve C, denoted by g_C, is an invariant of the curve that corresponds to the dimension of the k-vector space of global sections of the cannonical sheaf on C.

Remark 2.25. A curve has genus 0 if and only if it is isomorphic to $\mathbb{P}^1$.

Definition 2.26. A curve of genus one is called an ***elliptic curve***. A curve of genus at least two with a double cover onto $\mathbb{P}^1$ is called a ***hyperelliptic curve***. A hyperelliptic curve has an affine equation of the form $y^2 = f(x)$ for some separable polynomial $f(x)$. The involution τ of a hyperelliptic curve that sends any point (x, y) to the point $(x, -y)$ is called the ***hyperelliptic involution*** of C and it satisfies $C/\langle\tau\rangle \cong \mathbb{P}^1$.

Theorem 2.27 (Riemann–Hurwitz)**.** *Let $\pi : C_1 \longrightarrow C_2$ be a cover of degree d such that the extension $k(C_1)/k(C_2)$ of function fields is separable. Then*

$$2g_{C_1} - 2 = d(2g_{C_2} - 2) + \sum_{P \in C_1} (e_P - 1).$$

Definition 2.28. Let X be a smooth projective variety. A ***divisor*** of X is a formal $\mathbb{Z}$-linear combination $D = \sum_Y n_Y \cdot Y$ where all but finitely many n_Y are zero and each Y is a co-dimension 1 subvariety of X. The free abelian group of all divisors is denoted by $\mathrm{Div}(X)$. The sum $\sum_Y n_Y$ is called the ***degree of the divisor*** D. When $X = C$ is a curve, a divisor on C is of the form $D = \sum_{P \in C} n_P \cdot P$ where P is a point in C and $n_P = 0$ for all but finitely many P.

Definition 2.29. For a smooth projective variety X and any function $f \in k(X)$ we define

$$\mathrm{div}(f) = \sum_{Z \in X} \mathrm{ord}_Z(f) \cdot Z,$$

where Z is a subvariety of codimension 1. This is an element in $\mathrm{Div}(X)$ because any such function has finitely many zeroes and poles. Moreover, this divisor has degree 0. A divisor of the form $\mathrm{div}(f)$ is called a ***principal divisor***. The set of principal divisors, denoted by $\mathrm{Princ}(X)$, form a subgroup of $\mathrm{Div}^0(X)$, the group of divisors of degree 0. We define the ***class group*** as the quotient $\mathrm{Cl}(X) := \mathrm{Div}(X)/\mathrm{Princ}(X)$. The ***Picard group*** of X, denoted by $\mathrm{Pic}(X)$, is defined to be the set of isomorphism classes of line bundles on X. It is an abelian group where the group operation is the tensor product. The set of isomorphism classes corresponding to degree zero line bundles form a subgroup, denoted by $\mathrm{Pic}^0(X)$. The quotient group $\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is called the ***Néron–Severi*** group of X. For a smooth variety X, the class group $\mathrm{Cl}(X)$ is isomorphic to the Picard group $\mathrm{Pic}(X)$ and $\mathrm{Pic}^0(X) \cong \mathrm{Div}^0(X)/\mathrm{Princ}(X)$. For a divisor $D \in \mathrm{Div}(X)$, the corresponding element in $\mathrm{Pic}(X)$ is called the ***divisor class*** of D and denoted by $[D]$.

Remark 2.30. Points on an elliptic curve E form an abelian group that is naturally isomorphic to $\mathrm{Pic}^0(E)$.

Definition 2.31. For a cover of curves $\pi : C_1 \longrightarrow C_2$ we define the ***pullback***

$$\pi^* : \mathrm{Div}(C_2) \longrightarrow \mathrm{Div}(C_1)$$

as follows. For $Q \in C_2$ define $\pi^*(Q) := \sum_{\pi(P)=Q} e_P \cdot P$. Extend π^* to $\mathrm{Div}(C_2)$ by linearity, i.e.,

$$\pi^* \left(\sum_{Q \in C_2} n_Q \cdot Q \right) = \sum_{Q \in C_2} n_Q \cdot \pi^*(Q).$$

Similarly, we define the map $\pi_* : \mathrm{Div}^0(C_1) \longrightarrow \mathrm{Div}^0(C_2)$ by

$$\pi_* \left(\sum_{P \in C_1} n_P \cdot P \right) = \sum_{P \in C_1} n_P \cdot \pi(P)$$

We call π_* the ***norm*** or the ***pushforward***.

Proposition 2.32 ([Gal12], Corollary 8.3.10)**.** *For a cover $\pi : C_1 \longrightarrow C_2$ of curves the induced maps $\pi_* : \mathrm{Pic}^0(C_1) \longrightarrow \mathrm{Pic}^0(C_2)$ and $\pi^* : \mathrm{Pic}^0(C_2) \longrightarrow \mathrm{Pic}^0(C_1)$ are well-defined group homomorphisms.*

2.4 Abelian varieties: definitions and properties

In this section we introduce the single most important object of this thesis: abelian variety. Although we define them algebraically, they can also be defined complex analytically.

Definition 2.33. A ***group variety*** over k is an algebraic variety over k together with regular maps

$$m : V \times_k V \longrightarrow V \text{ (multiplication)}$$
$$i : V \longrightarrow V \text{ (inverse)}$$

and an element $e \in V(k)$ such that the structure on $V(\bar{k})$ defined by m and i is a group with identity element e. Here $V \times_k V$ denotes the fiber product of V with itself as k-varieties.

A group variety (V, m, i, e) satisfies the following:

(i) The maps $V \xrightarrow{(\mathrm{id},\, e)} V \times_k V \xrightarrow{m} V$ and $V \xrightarrow{(e,\, \mathrm{id})} V \times_k V \xrightarrow{m} V$ are identity maps. Thus, e is the identity.

(ii) The maps $V \xrightarrow{\Delta} V \times_k V \underset{\mathrm{id} \times i}{\overset{i \times \mathrm{id}}{\rightrightarrows}} V \times_k V \xrightarrow{m} V$ are both equal to the composition $V \longrightarrow \mathrm{Spec}(k) \xrightarrow{e} V$. Thus, every element in V has an inverse with respect to m.

(iii) The following diagram commutes (and thus associativity holds).

$$\begin{array}{ccc} V \times_k V \times_k V & \xrightarrow{1\times m} & V \times_k V \\ \Big\downarrow{\scriptstyle m\times 1} & & \Big\downarrow{\scriptstyle m} \\ V \times_k V & \xrightarrow[m]{} & V \end{array}$$

Thus, a group variety has a group structure.

Definition 2.34. An ***abelian variety*** over a field k is a complete connected group variety over k.

The group law for an abelian variety will be written additively, the identity element will be denoted by 0 and $-a$ will denote the inverse of the element a.

Example 2.35. An elliptic curve is an abelian variety. In fact, any abelian variety of dimension one is an elliptic curve. An abelian variety of dimension two is called an ***abelian surface***. The product $E_1 \times E_2$ of two elliptic curves is an abelian surface.

Definition 2.36. A ***homomorphism*** $f : A \longrightarrow B$ of abelian varieties is a morphism of varieties that preserves the group structure.

The aim of the following lemma and the proposition thereafter is to describe regular maps between abelian varieties in terms of homomorphisms.

Lemma 2.37 ([Mil08], Theorem 1.1)**.** *Consider a regular map $\alpha : V \times W \longrightarrow U$, and assume that V is complete and that $V \times W$ is geometrically irreducible. If there are points $u_0 \in U(K),\ v_0 \in V(k),$ and $w_0 \in W(k)$ such that*

$$\alpha(V \times \{w_0\}) = \{u_0\} = \alpha(\{v_0\} \times W)$$

then $\alpha(V \times W) = \{u_0\}$.

Proposition 2.38 ([Mil08], Corollary 1.2)**.** *Every regular map of abelian varieties is the composite of a homomorphism with a translation.*

Proof. Let $\alpha : A \longrightarrow B$ be a regular map of abelian varieties that sends 0 to b. Composing α with the translation by $-b$, we can assume that $\alpha(0) = 0$. Consider the map

$$\varphi : A \times A \longrightarrow B,\ \varphi(a, a') = \alpha(a + a') - \alpha(a) - \alpha(a').$$

This is a regular map as it is the difference of the two regular maps

$$A \times A \xrightarrow{m} A \xrightarrow{\alpha} B$$

and

$$A \times A \xrightarrow{\alpha \times \alpha} B \times B \xrightarrow{m} B.$$

Since $\varphi(A \times \{0\}) = \{0\} = \varphi(\{0\} \times A)$, Lemma 2.37 implies $\varphi = 0$ and hence α is a homomorphism. □

Corollary 2.39. *The group law on an abelian variety is commutative.*

Proof. A group is commutative if and only if the map on it that takes any element to its inverse is a homomorphism. Let A be an abelian variety. Consider the regular map $A \longrightarrow A$, $a \mapsto -a$. Since this map takes 0 to 0, it is a homomorphism by Proposition 2.38. Thus, the group law on an abelian variety is commutative □

Theorem 2.40 ([Mil08], Theorem 6.4)**.** *Abelian varieties are projective. Equivalently, any abelian variety has an ample line bundle.*

2.5 Isogenies

Isogenies are an interesting class of maps between abelian varieties. They induce equivalence relations on abelian varieties. The purpose of this section is to introduce isogenies and show some of their basic properties. Our base field k will be the field of complex numbers $\mathbb{C}$.

Definition 2.41. Let $f : A \longrightarrow B$ be a finite surjective morphism between algebraic varieties over a field k. The ***degree of*** f is the degree of the finite field extension of the function field $k(A)$ over $f^*k(B)$.

Definition 2.42. An ***isogeny*** between abelian varieties A and B is a surjective homomorphism with finite kernel. The ***degree of an isogeny*** is its degree in the sense of the previous definition and for an isogeny it is the same as the size of its kernel. We say that A ***is isogenous to*** B, and write $A \sim B$, if there exists an isogeny $f : A \longrightarrow B$.

Isogenies can be characterised in many useful equivalent ways. The following result states some of them. Recall that a morphism of schemes $f : A \longrightarrow B$ is called finite if every $b \in B$ has an affine neighborhood $V = \mathrm{Spec}(R)$ such that $U = f^{-1}(V)$ is an affine open set $\mathrm{Spec}(S)$ in A and the ring map $R \longrightarrow S$ makes S a finitely generated R-module. The map f is flat if the local ring $\mathcal{O}_{A,a}$ is flat over the local ring $\mathcal{O}_{B,f(a)}$, where a ring S over a ring R is said to be flat if $- \otimes_R S$ preserves exact sequences of R-modules.

Proposition 2.43 ([Mil08], Proposition 7.1)**.** *For a homomorphism $f : A \longrightarrow B$ of abelian varieties, the following statements are equivalent:*

1. *f is an isogeny;*

2. *$\dim A = \dim B$ and f is surjective;*

3. $\dim A = \dim B$ *and* $\operatorname{Ker} f$ *is a finite group;*

4. f *is finite, flat and surjective.*

For every positive integer n, the map

$$[n] : A \longrightarrow A, \quad x \mapsto \underbrace{x + \cdots + x}_{\text{n times}}$$

is an endomorphism of A and an isogeny. We define the $[-1]$ endomorphism as the homomorphism on A that associates an element to its inverse under the group law, and for $n > 1$ we define $[-n]$ as the composition of $[n]$ and $[-1]$. This identifies $\mathbb{Z}$ as a subring of $\operatorname{End}_k(A)$.

Definition 2.44. Let A be an abelian variety over k. For a positive integer n we define the ***group of n-torsion points*** of A as the kernel of the endomorphism $[n] : A \longrightarrow A$ and denote it by $A[n]$. It is known that $A[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ where $g = \dim A$.

Proposition 2.45. *Being isogenous is an equivalence relation on abelian varieties.*

Proof. Being isogenous will be denoted by $\sim$. The relation $\sim$ is reflexive because the identity morphism on an abelian variety is an isogeny. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be isogenies. By Proposition 2.43, $g \circ f$ is surjective and $\dim A = \dim B = \dim C$. Therefore, $A \sim C$ and $\sim$ is transitive. It remains to prove that $\sim$ is symmetric. Let $\phi : A \longrightarrow B$ be an isogeny. We want to construct an isogeny $B \longrightarrow A$. Let N be the exponent of $\operatorname{Ker} \phi$. Then $\operatorname{Ker} \phi \subset \operatorname{Ker}[N]$. This induces a map $\pi : A/\operatorname{Ker} \phi \longrightarrow A/\operatorname{Ker}[N]$. Since ϕ is an isogeny, we have an isomorphism $\psi : A/\operatorname{Ker} \phi \longrightarrow B$. Then $\alpha := \pi \circ \psi^{-1}$ is an isogeny from B to $A/\operatorname{Ker}[N]$. Now, the isogeny $[N] : A \longrightarrow A$ gives us an isomorphism $\beta : A/\operatorname{Ker}[N] \longrightarrow A$. The composition $\beta \circ \alpha : B \longrightarrow A$ is an isogeny (see Figure 2.1). □

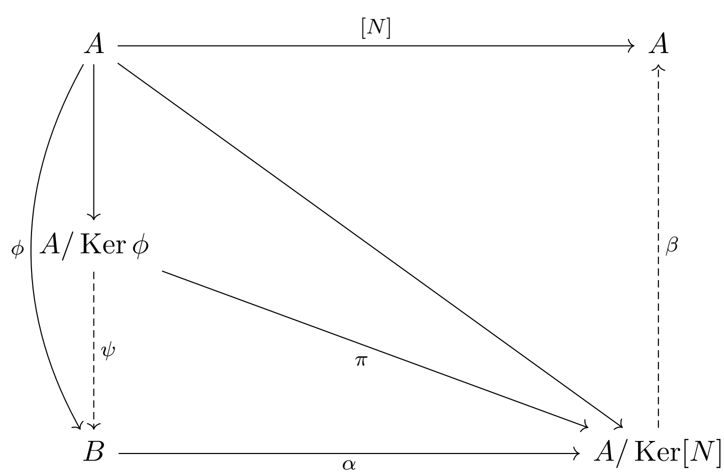

Figure 2.1

Lemma 2.46. *Let $f_1 : A_1 \longrightarrow A_2,\ f_2 : A_2 \longrightarrow A_3,\ \dots,\ f_n : A_n \longrightarrow A_{n+1}$ be isogenies of abelian varieties. Then*

$$\#\operatorname{Ker}(f_n \circ \cdots \circ f_2 \circ f_1) = \#\operatorname{Ker}(f_1)\#\operatorname{Ker}(f_2)\cdots\#\operatorname{Ker}(f_n).$$

Proof. By induction, it is sufficient to prove for $n = 2$. Suppose $f_1 : A_1 \longrightarrow A_2$ and $f_2 : A_2 \longrightarrow A_3$ are isogenies of abelian varieties with $\operatorname{Ker}(f_1) = \{x_1, \dots,\ x_m \mid x_i \in A_1\}$ and $\operatorname{Ker}(f_2) = \{y_1, \dots, y_n \mid y_i \in A_2\}$. Since isogenies are surjective, we can write $y_i = f_1(x_i')$ for $x_i \in A_1$ with $1 \leq i \leq n$. Then, $\operatorname{Ker}(f_2 \circ f_1) = \{x_i + x_j' \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and hence $\#\operatorname{Ker}(f_2 \circ f_1) = \#\operatorname{Ker}(f_1)\#\operatorname{Ker}(f_2)$. □

2.6 Dual abelian variety and polarizations

In this section we introduce the dual of an abelian variety. We then discuss about polarization on an abelian variety and outline the choice of the polarizing line bundle, which will play a key role in our reults in the next chapter.

2.6.1 Line bundles on abelian varieties

We mention two important theorems about line bundles on abelian varieties which will be required in our construction of the dual abelian variety. We start with the theorem of the cube.

Theorem 2.47 ([Mil08], Theorem 5.1)**.** *Let X, Y and Z be varieties such that X and Y are complete. Let L be a line bundle on $X \times Y \times Z$ and $x_0 \in X$, $y_0 \in Y$ and $z_0 \in Z$ be points such that the restrictions of L on $X \times Y \times \{z_0\}, X \times \{y_0\} \times Z$, and $\{x_0\} \times Y \times Z$ are trivial. Then L is trivial.*

The following theorem, known as the theorem of the square, is a consequence of the theorem of the cube.

Theorem 2.48 ([Mil08], Theorem 5.5)**.** *Let A be an abelian variety, $x, y \in A$ and L a line bundle on A. Then $t_{x+y}^*L \otimes L \cong t_x^*L \otimes t_y^*L$.*

Theorem 2.49 ([Mil08], Remark 8.7)**.** *The group $\operatorname{Pic}^0(A)$ naturally has the structure of an abelian variety, called the **dual of** A and denoted by $A^\vee$.*

Remark 2.50. If E is an elliptic curve, then $E^\vee \cong E$.

2.6.2 Construction of the dual

Since abelian varieties are projective, they have ample line bundles. Let L be an ample line bundle on A. Then we have the map:

$$\begin{aligned} \varphi_L : A &\longrightarrow \mathrm{Pic}^0(A) \subset \mathrm{Pic}(A) \\ x &\longmapsto [t_x^* L \otimes L^{-1}]. \end{aligned}$$

The image of φ_L is inside $\mathrm{Pic}^0(A)$ by the theorem of the square, and it can be shown that the image is precisely $\mathrm{Pic}^0(A)$. The kernel of φ_L, denoted by $K(L)$, is a finite group scheme. Thus, $A^\vee = \mathrm{Pic}^0(A) \cong A/K(L)$ and $\dim A^\vee = \dim A$.

Theorem 2.51 ([Mil08], Theorem 9.1)**.** *An isogeny $f : A \longrightarrow B$ induces a dual isogeny $f^\vee : B^\vee \longrightarrow A^\vee$ with $\#\,\mathrm{Ker}(f) = \#\,\mathrm{Ker}(f^\vee)$.*

2.6.3 Polarization

Definition 2.52. A ***polarization*** on an abelian variety A is the choice of an element in $\mathrm{NS}(A)$ corresponding to an ample line bundle L. It gives an isogeny $\varphi_L : A \longrightarrow A^\vee$, which is independent of the choice of the line bundle, i.e., if L and L' are representatives of the same class in $\mathrm{NS}(A)$ then $\varphi_L = \varphi_{L'}$ (see [BL04] Proposition 2.5.3). The ***degree of the polarization*** is the degree of the isogeny φ_L.

Definition 2.53. If L is a polarization then $\mathrm{Ker}(\varphi_L) \cong (\mathbb{Z}/d_1\mathbb{Z} \times \dots \mathbb{Z}/d_n\mathbb{Z})^2$, where each d_i is a positive integer and $d_i \mid d_{i+1}$ (see [BM16], Section 1). The n-tuple $(d_1, \dots, d_n)$ is called the ***type*** or ***degree*** of the polarization L, and L is also called a $(d_1, \dots, d_n)$-polarization. A polarization of type $(1, \dots, 1)$ is called a ***principal polarization***.

Definition 2.54 ([Mil08], Chapter 1, Section 13)**.** For an abelian variety A and each integer $n \geq 1$, the isogeny $[n] : A \longrightarrow A$ gives a non-degenerate bilinear pairing $e_n : A[n] \times A^\vee[n] \longrightarrow \mu_n$, where μ_n is the cyclic group of n^{th}-roots of unity. This is called the ***Weil pairing***. When combined with a polarization L with the isogeny $\varphi_L : A \longrightarrow A^\vee$, this gives us the pairing

$$e_n^L : A[n] \times A[n] \longrightarrow \mu_n, \quad (a, b) \longmapsto e_n(a, \varphi_L(b)).$$

Definition 2.55. Let $\pi : A \longrightarrow B$ be an isogeny and M be a polarization on B that induces the isogeny $\varphi_M : B \longrightarrow B^\vee$. The ***pullback polarization*** of M is the polarization on A associated to the ample line bundle $\pi^* M$ on A and that induces the isogeny $\varphi_{\pi^* M} := \pi^\vee \circ \varphi_M \circ \pi$ on A (see Figure 2.2).

In the opposite direction, starting with an isogeny $\pi : A \longrightarrow B$ and a polarization L on A with $\mathrm{Ker}(\pi) \subset \mathrm{Ker}(\varphi_L)$, we can find a polarization M on B such that $\pi^* M \cong L$. Generally this requires an isotropy condition such that $e_n^L(a, b)$ is trivial for all $a, b \in \mathrm{Ker}(\pi)$. In the

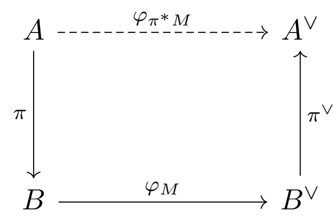

Figure 2.2

case $\mathrm{Ker}(\pi) \subset \mathrm{Ker}(\varphi_L)$ the isotropy condition is indeed satisfied because the pairing acts on the second argument by the isogeny φ_L, and hence for $a, b \in \mathrm{Ker}(\pi)$ we will have that $e_n^L(a, b)$ is trivial for all $n \geq 1$.

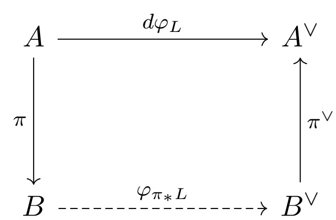

Figure 2.3

Definition 2.56 ([Mum70], Corollary Page 231)**.** Suppose $\pi : A \longrightarrow B$ is an isogeny of abelian varieties and L is a polarization on A that induces an isogeny $\varphi_L : A \longrightarrow A^\vee$. If $\mathrm{Ker}(\pi)$ is isotropic under the pairing induced by the polarization L and d is the minimum positive integer such that $\mathrm{Ker}(\pi) \subset \mathrm{Ker}(d\varphi_L)$, then the ***pushforward polarization*** of L by π is a polarization denoted by $\pi_* L$ which fills the diagram in Figure 2.3. The polarization corresponding to $d\varphi_L$ is also the pullback of $\pi_* L$ in these circumstances.

Proposition 2.57. *Let A be an abelian surface and L be a $(1, d)$-polarization on A. Suppose X is a subgroup of $\mathrm{Ker}(\varphi_L) = K(L) \cong \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ of order dividing d and $\pi : A \longrightarrow A/X$ is the quotient isogeny. Then $\pi_* L$ is a $\left(1, \frac{d}{\#X}\right)$-polarization on the abelian surface A/X. In particular, if $\#X = d$ then A/X is principally polarized.*

Proof. The quotient isogeny $\pi : A \longrightarrow A/X$ has kernel $X \subset \mathrm{Ker}(\varphi_L)$, which is an isotropic subgroup. Therefore, π induces a pushforward polarization $\pi_* L$ on A/X and we have the following commutative diagram.

Lemma 2.46 gives $d^2 = \#\,\mathrm{Ker}(\varphi_L) = (\#\,\mathrm{Ker}(\pi))^2 \#\,\mathrm{Ker}(\varphi_{\pi_* L}) = (\#X)^2 \#\,\mathrm{Ker}(\varphi_{\pi_* L})$, which implies $\#\,\mathrm{Ker}(\varphi_{\pi_* L}) = \left(\frac{d}{\#X}\right)^2$ and hence the type of polarization of $\pi_* L$ is $\left(1, \frac{d}{\#X}\right)$. When $\#X = d$, the polarization is of type $(1, 1)$, which is a principal polarization. □

Although the choice of the line bundle does not change the induced isogeny of the polarization, it does affect the behavior of automorphisms of the curves inside the corresponding

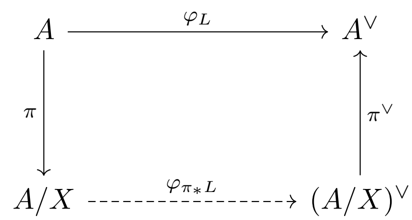

Figure 2.4

linear system. As our main tool to decompose the Jacobians of curves lying in the linear systems of abelian surfaces is dependent on their automorphisms, the choice of the polarizing line bundle plays an important role in our approach.

Choice of the polarizing line bundle

- **Symmetric:** A line bundle L on an abelian variety A is called *symmetric* if $[-1]^*L \cong L$, i.e., they correspond to the same element in $\mathrm{Pic}(A)$. By Corollary 2.3.7 in [BL04], it is possible to choose the polarizing line bundle L to be symmetric. When L is symmetric, the $[-1]$-involution of A induces a linear operator on $|L|$ with eiegnvalues ± 1 and the curves lying in these eigenspaces inherit the $[-1]$-involution of A as their involutions.

- **Symmetric theta structure:** If L is an arbitrary line bundle in an abelian variety A, then the set of all *automorphisms of* L *over points of* X form a group, called the theta group of L and denoted by $\mathcal{G}(L)$. In addition, if L is symmetric, then -1 is an automorphism of L and the extended theta group $\mathcal{G}^{\mathrm{e}}(L)$ is defined as the semi-direct product $\mathcal{G}^{\mathrm{e}}(L) := \mathcal{G}(L) \rtimes \langle -1 \rangle$. Let $D = (d_1, \ldots, d_g)$ be the type of L. The set $\mathcal{H}(L) := \mathbb{C}^* \times K(D)$, where $K(D) = \mathbb{Z}^g/d_1\mathbb{Z}^g \oplus \cdots \oplus \mathbb{Z}^g/d_g\mathbb{Z}^g$, forms a group and is called the *Heisenberg group of type* D. The semi-direct product $\mathcal{H}^{\mathrm{e}}(D) := \mathcal{H}(D) \rtimes \langle -1 \rangle$ is called the *extended Heisenberg group of* D. The line bundle L is said to have a *symmetric theta structure* if there is an isomorphism between $\mathcal{G}^{\mathrm{e}}(L)$ and $\mathcal{H}^{\mathrm{e}}(D)$. If C is a curve in the linear system of a line bundle on an abelian surface such that the line bundle has a symmetric theta structure, then the number of fixed points on C by the $[-1]$-involution is known (see Theorem 3.14). For a detailed exposition of symmetric theta structures, see Section 6.9 of [BL04].

- **Characteristic zero:** A *characteristic* of a line bundle L corresponds to the list of all line bundles in its Néron–Severi class $[L]$. The connected component of $\mathrm{Pic}(A)$ containing $[L]$, denoted by $\mathrm{Pic}^{[L]}(A)$, forms a torsor over $\mathrm{Pic}^0(A)$. Any choice of an identity element in $\mathrm{Pic}^{[L]}(A)$ determines an isomorphism $\mathrm{Pic}^{[L]}(A) \cong \mathrm{Pic}^0(A)$, where the identity element is said to have *characteristic zero* in the class $[L]$. The importance of a characteristic 0 line bundle on an abelian variety is briefly the following: If L

is symmetric and has characteristic zero, then $\{t_x^*L : x \in A[2]\}$ is the set of all line bundles with symmetric theta structures in the Néron–Severi class of L (see Theorem 6.9.5, [BL04]). Therefore, a characteristic zero line bundle has a symmetric theta structure and this gives information on the number of fixed points of the $[-1]$-involution of curves in the linear system of such a line bundle. We will not require any more information about characteristics of line bundles for the purpose of this thesis. The interested reader can find more about it in Chapter 3 of [BL04].

To answer the central question of this thesis, we will choose the polarizing line bundle of a $(1, d)$-polarization on a simple abelian surface to be symmetric and of characteristic zero, which will also imply that it has a symmetric theta structure.

2.7 Poincaré's reducibility

The purpose of this section is to establish Poincaré's reducibility theorem which shows the power of isogenies. The following theorem is the first step towards this goal. It roughly says that any abelian subvariety of an abelian variety is complemented by another abelian subvariety such that their product is isogenous to the parent abelian variety.

Theorem 2.58. *Let A be an abelian variety and $B \subset A$ a proper abelian subvariety. Then there exists an abelian variety $C \subset A$ such that the map*

$$B \times C \longrightarrow A, \quad (b, c) \mapsto b + c$$

is an isogeny.

Proof. Let f be the inclusion $B \hookrightarrow A$. Since abelian varieties are projective, we can choose an ample line bundle L on A. This gives an isogeny $\varphi_L : A \longrightarrow A^\vee$ (this comes from polarization, see Definition 2.52). Consider the homomorphism $\psi : A \longrightarrow B^\vee$ defined as $\psi := f^\vee \circ \varphi_L$ and let $C := \operatorname{Ker}(\psi)_0$, the connected component of the kernel containing 0_A. It is an abelian subvariety of A and we want to show that $B \times C$ is isogenous to A. Note,

$$\dim C \geq \dim \operatorname{Ker} f^\vee \geq \dim A^\vee - \dim B^\vee = \dim A - \dim B.$$

The restriction (pullback) to B of the ample line bundle L is again an ample line bundle, call it M. The restriction $\psi|_B$ is the isogeny φ_M obtained from M, and hence has finite kernel. Observe that, $B \cap C \subset B \cap \operatorname{Ker} \psi = \operatorname{Ker} \psi|_B = \operatorname{Ker} \varphi_M$, which is finite. Therefore, the map

$$B \times C \longrightarrow A, \ (b, c) \mapsto b + c$$

has finite kernel and hence is an isogeny onto its image. This gives, $\dim B + \dim C \leq \dim A$. Combined with the previous inequality, we get, $\dim B + \dim C = \dim A$. Since the homomorphism $B \times C \longrightarrow A$ of abelian varieties preserves dimensions and has finite kernel, therefore, by Proposition 2.43 it is an isogeny. $\square$

We now introduce the building blocks of abelian varieties: the simple abelian varieties.

Definition 2.59. An abelian variety is called ***simple*** if it has no abelian subvarieties other than the zero-subvariety and itself.

The following result classifies all homomorphisms between simple abelian varieties.

Lemma 2.60. *Any homomorphism $A \longrightarrow B$ between simple abelian varieties is either an isogeny or the* 0*-morphism.*

Proof. Let $\phi : A \longrightarrow B$ be a homomorphism of simple abelian varieties. Then, $\operatorname{Ker}\phi$ is a closed subset of A and the connected components of $\operatorname{Ker}\phi$ are closed and mutually disjoint subsets of $\operatorname{Ker}\phi$. Since any closed set is a finite union of subvarieties, we must have only finitely many connected components of $\operatorname{Ker}\phi$. Now, consider the connected component of $\operatorname{Ker}\phi$ containing 0. It is an abelian subvariety of A and has to be either A or 0 since A is simple. As there are only finitely many connected components of $\operatorname{Ker}\phi$ and since each component is a translation of the identity component, we get $\operatorname{Ker}\phi = A$ or $\#\operatorname{Ker}\phi < \infty$. On the other hand, $\mathrm{Im}\phi$ is an abelian subvariety of B, so either $\mathrm{Im}\phi = 0$ or $\mathrm{Im}\phi = B$. If ϕ is not the 0-morphism, we must have $\#\operatorname{Ker}\phi < \infty$ and $\mathrm{Im}\phi = B$. This implies ϕ is an isogeny. $\square$

We are now in a position to state and prove a result due to Poincaré, which is similar in spirit to the fundamental theorem of arithmetic and makes the central question of this thesis meaningful.

Theorem 2.61 (Poincaré's reducibility)**.** *Let A be an abelian variety. Then $A \sim \prod_{i=1}^{n} A_i^{d_i}$, where each A_i is a simple abelian variety and $A_i \nsim A_j$ for $i \neq j$. Furthermore, each d_i is a uniquely determined non-negative integer and each A_i is unique up to isogeny.*

Proof. If A is not simple then it has a proper non-trivial abelian subvariety A_1 of A and by Theorem 2.58, there exists another proper non-trivial abelian subvariety B_1 of A such that $A \sim A_1 \times B_1$. If both A_1 and B_1 are simple abelian varieties, then we are done. Otherwise, suppose B_1 is not simple. Then there exist proper abelian subvarieties A_2 and B_2 of B_1 such that $B_1 \sim A_2 \times B_2$ and hence $A \sim A_1 \times A_2 \times B_2$. Note that the dimensions of A_1, A_2, B_1 and B_2 are strictly smaller than that of A, and further splitting will shrink dimension more. Similarly, if A_1 is also not simple then we can break it as a product of abelian subvarieties of smaller dimensions. Thus, after finitely many steps we will have A isogenous to a product of simple abelian subvarieties. Since being isogenous is an equivalence relation, by grouping together the mutually isogenous terms we can write $A \sim \prod_{i=1}^{n} A_i^{d_i}$, where each A_i is a simple abelian variety with $A_i \nsim A_j$ for $i \neq j$ and each d_i is a positive integer.

For the uniqueness, it is sufficient to show that if $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ are simple abelian varieties such that $A_1 \times \cdots \times A_n \sim B_1 \times \cdots \times B_m$ then $n = m$ and $A_i \sim B_i$ up to a permutation of indices. First, we want to show that A_1 is isogenous to some B_j. If this is not true, then $A_1 \nsim B_j$ for all $1 \leq j \leq m$ and we claim that the only homomorphism from A_1 to $B_1 \times B_2$ is the zero morphism. If not, let $f : A_1 \longrightarrow B_1 \times B_2$ be a non-zero homomorphism. The composition, $\pi_2 \circ f : A_1 \longrightarrow B_2$ is a homomorphism and since we have assumed $A_1 \nsim B_j$, it must be the zero homomorphism by Lemma 2.60. Therefore, $\mathrm{Im}(f) \subset \mathrm{Ker}(\pi_2) = B_1$ and f is a non-zero homomorphism from A_1 whose image is contained in B_1, which is simple. Lemma 2.60 will imply $A_1 \sim B_1$, a contradiction to $A_1 \nsim B_j$. Thus, if $A_1 \nsim B_j$ for $1 \leq j \leq m$, then any homomorphism $A_1 \longrightarrow B_1 \times B_2$ must be the zero homomorphism. Now, consider a homomorphism $g : A_1 \longrightarrow B_1 \times B_2 \times B_3$. Composing g with the projection $\pi_{12} : B_1 \times B_2 \times B_3 \longrightarrow B_1 \times B_2$ again gives us a homomorphism $\pi_{12} \circ g : A_1 \longrightarrow B_1 \times B_2$, which must be the zero morphism as shown above. Therefore, $\mathrm{Im}(g) \subset \mathrm{Ker}(\pi_{12}) = B_3$. If g is non-zero, then $\pi_{12} \circ g : A_1 \longrightarrow B_3$ will be an isogeny by Lemma 2.60, a contradiction to $A_1 \nsim B_j$. Therefore, any morphism $A_1 \longrightarrow B_1 \times B_2 \times B_3$ must be the zero morphism.

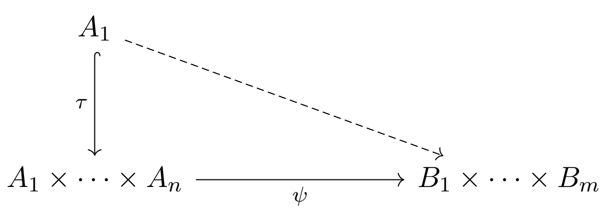

Figure 2.5

Continuing in this manner, we can show that if $A_1 \nsim B_j$ for $1 \leq j \leq m$ then the only possible homomorphism $A_1 \longrightarrow B_1 \times \cdots \times B_m$ is the zero morphism. Now, if $\psi : A_1 \times \cdots \times A_n \longrightarrow B_1 \times \cdots \times B_m$ is an isogeny, then the morphism $\psi \circ \tau : A_1 \longrightarrow B_1 \times \cdots \times B_m$, where $\tau : A_1 \longrightarrow A_1 \times \cdots \times A_n$ is the canonical embedding, will be the zero morphism. This will imply $\dim(\mathrm{Ker}(\psi \circ \tau)) = \dim(A_1) > 0$, which is a contradiction because $\mathrm{Ker}(\psi \circ \tau)$ must be finite since ψ is an isogeny and τ is injective. Therefore, $A_1 \nsim B_j$ for $1 \leq j \leq m$ cannot be true and hence $A_1 \sim B_j$ for some j. Similar argument with any other factor of $A_1 \times \cdots \times A_n$ will imply that for each $1 \leq i \leq n$ there exists j with $1 \leq j \leq m$ such that $A_i \sim B_j$. Interchanging the roles of $A_1 \times \cdots \times A_n$ and $B_1 \times \cdots \times B_m$, we will get that each B_j is isogenous to some A_k.

Now, for a moment, assume that in the products $A_1 \times \cdots \times A_n$ and $B_1 \times \cdots \times B_m$ the factors are mutually non-isogenous, i.e, assume $A_p \nsim A_q$ for $p \neq q$ and $B_u \nsim B_v$ for $u \neq v$. In this case, each A_i will be isogenous to exactly one B_j and each B_j will be isogenous to exactly one A_k, since being isogenous is an equivalence relation. Thus, there is a one-to-one correspondence between the sets $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_m\}$, which gives $n = m$

and $A_i \sim B_i$ for $1 \le i \le n$ up to a permutation of indices. For the general case where some factors in the products can be mutually isogenous, we group together the isogenous factors and write the products as $A_1^{d_1} \times \cdots \times A_s^{d_s}$ and $B_1^{e_1} \times \cdots \times B_t^{e_t}$ to get the isogeny $\psi : A_1^{d_1} \times \cdots \times A_s^{d_s} \longrightarrow B_1^{e_1} \times \cdots \times B_t^{e_t}$, where $A_p \nsim A_q$ for $p \neq q$ and $B_u \nsim B_v$ for $u \neq v$. Again, just as in the previous case, we will get $s = t$ and $A_i \sim B_i$ for $1 \le i \le s$, which simplifies the isogeny to $\psi : A_1^{d_1} \times \cdots \times A_s^{d_s} \longrightarrow B_1^{e_1} \times \cdots \times B_s^{e_s}$ and in particular, implies that

$$\psi(A_1^{d_1} \times \cdots \times A_s^{d_s}) = B_1^{e_1} \times \cdots \times B_s^{e_s}. \tag{2.1}$$

Finally, to show that $d_i = e_i$, note that the restriction $\psi|_{A_i}$ is again an isogeny from A_i onto its image. Since A_i is simple, this image should also be a simple factor of $B_1^{e_1} \times \cdots \times B_s^{e_s}$. As we have already established that $A_i \nsim B_j$ for $i \neq j$, this image is precisely B_i. Thus, the image of $\psi|_{A_i^{d_i}}$ should be $B_i^{d_i}$ for all i and hence $\psi(A_1^{d_1} \times \cdots \times A_s^{d_s}) = B_1^{d_1} \times \cdots \times B_t^{d_s}$. Comparing with (2.1), we must have $d_i = e_i$ for all $1 \le i \le s$. □

Lemma 2.62. *If* $0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0$ *is a short exact sequence of abelian varieties, then* $A_1 \times A_3 \sim A_2$*.*

Proof. Since f is an injection, we can identify A_1 with its image under f and consider A_1 to be an abelian subvariety of A_2. By Theorem 2.58, there exists an abelian subvariety C of A_2 such that $A_1 \times C \sim A_2$. Moreover, as in the proof of Theorem 2.58, we can take C to be $\mathrm{Ker}(\psi)_0$, where $\psi := f^\vee \circ \varphi_L$ and $\varphi_L : A_2 \longrightarrow A_2^\vee$ is the isogeny coming from an ample line bundle of A_2. It is sufficient for us to show that $C \sim A_3$. For this, we will prove that the restriction $g|_C \colon C \longrightarrow A_3$ is an isogeny.

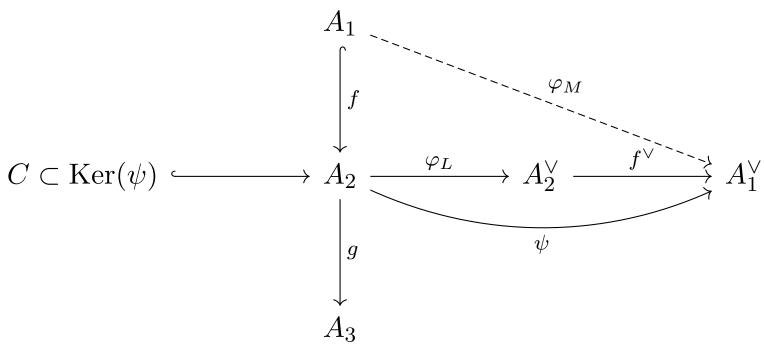

Figure 2.6

First, we want to show that $g|_C$ has a finite kernel. Let $c \in C$ be such that $g(c) = 0_{A_3}$. Since $C = \mathrm{Ker}(\psi)_0$ is a subset of $\mathrm{Ker}\,\psi$, therefore $\psi(c) = f^\vee \circ \varphi_L(c) = 0$, from which we get $f^\vee \circ \varphi_L \circ f(a) = 0$ since the short exact sequence allows us to write $c = f(a)$ for some $a \in A_1$. Observe that $f^\vee \circ \varphi_L \circ f : A_1 \longrightarrow A_1^\vee$ is the isogeny φ_M, where M is the line bundle obtained by restricting the ample line bundle L to A_1. Thus, $c \in \mathrm{Ker}(g|_C)$ if and only if $a = f^{-1}(c) \in \mathrm{Ker}(\varphi_M)$. Since f is an injection, we get a one-to-one correspondence between

Ker($g|_C$) and Ker(φ_M). The latter is finite as φ_M is an isogeny. Therefore, Ker($g|_C$) is also finite.

It remains to show that $g|_C : C \longrightarrow A_3$ is a surjection. Let y be an element in A_3. By the short exact sequence, there exists $a \in A_2$ such that $g(a) = y$. Again, by Theorem 2.58, there exists $b \in A_1,\ c \in C$ such that $a = b + c$ (as stated earlier, we are identifying A_1 with its image inside A_2 under the injection f and hence considering A_1 as an abelian subvariety of A_2). This implies $y = g(a) = g(b + c) = g(b) + g(c) = g(c)$ by the short exact sequence. Thus, $g|_C$ is surjective and hence is an isogeny. □

2.8 Jacobians of curves

Finally, we discuss the specific class of abelian varieties that we are interested in: Jacobians of curves.

Definition 2.63. The ***Jacobian*** of a smooth curve C, denoted by J_C, is a principally polarized abelian variety isomorphic to $\mathrm{Pic}^0(C)$ as a group.

Example 2.64. If E is an elliptic curve, then $J_E \cong E$.

Theorem 2.65 (Albanese property of the Jacobian)**.** *Let C be a smooth curve and $p \in C$. Then there is an injective morphism $\alpha_p : C \longrightarrow J_C$ of varieties. Further, if $f : C \longrightarrow A$ is any morphism of varieties from C into an abelian variety A such that $f(p) = 0$ then there exists a unique homomorphism $g : J_C \longrightarrow A$ of abelian varieties such that the following diagram commutes.*

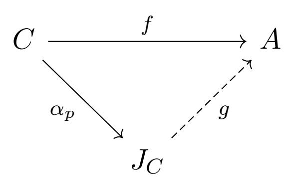

Figure 2.7

The map α_p is called the ***Abel–Jacobi map***. Now suppose C is a curve embedded in an abelian surface A. By the theorem above, after choosing a point $p \in C$ we get the Abel–Jacobi map and an induced map $g : J_C \longrightarrow A$. Denote the kernel of g by $K(C, A)$. This kernel is connected (see [BS17], Lemma 2.6) and hence is an abelian variety. If g is surjective then we have an exact sequence:

$$0 \longrightarrow K(C, A) \hookrightarrow J_C \longrightarrow A \longrightarrow 0.$$

The following result proves that g is indeed surjective and this gives us a decomposition of J_C.

Lemma 2.66. *The map $g : J_C \longrightarrow A$ is surjective and $J_C \sim A \times K(C, A)$.*

Proof. We have a short exact sequence

$$0 \longrightarrow \mathcal{O}_A(-C) \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

This induces the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(A, \mathcal{O}_A(-C)) \longrightarrow H^0(A, \mathcal{O}_A) \longrightarrow H^0(C, \mathcal{O}_C) \\ \longrightarrow H^1(A, \mathcal{O}_A(-C)) \longrightarrow H^1(A, \mathcal{O}_A) \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow \dots \end{aligned}$$

By Kodaira vanishing, $H^1(A, \mathcal{O}_A(-C)) = 0$ and hence $H^1(A, \mathcal{O}_A) \longrightarrow H^1(C, \mathcal{O}_C)$ is injective. Thus, the morphism $i^* : \mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(C) = J_C$ induced by $i : C \hookrightarrow A$ is injective up to isogeny, i.e., has finite kernel. Since this map is dual to g (up to isogeny), we get that g is surjective. Lemma 2.62 now implies $J_C \sim A \times K(C, A)$ □

Proposition 2.67 ([Mil08], Page 86)**.** *If g denotes the genus of C, then* $\dim(J_C) = g$. *In particular, $J_{\mathbb{P}^1}$ is trivial.*

Definition 2.68. For a cover $\pi : C \longrightarrow \widetilde{C}$ of curves, the ***Prym variety*** of π is the complementary abelian subvariety of $\mathrm{Im}(\pi^*)$ in J_C. It is denoted by $P(\pi)$. Equivalenty, it is the defined as the connected component of $\mathrm{Ker}(\pi_*)$ containing the identity.

Remark 2.69. Let $\pi : C \longrightarrow \widetilde{C}$ be a surjective map of smooth curves. The group homomorphisms

$$\mathrm{Pic}^0(C) \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} \mathrm{Pic}^0(\widetilde{C})$$

correspond to homomorphisms of abelian varieties between the Jacobians

$$J_C \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} J_{\widetilde{C}}.$$

Further, $\pi_* \circ \pi^* = [n]$, multiplication-by-n map on $J_{\widetilde{C}}$ where $n = \deg \pi$. Since $[n]$ is an isogeny, we get that $\mathrm{Ker}\, \pi^*$ is finite and hence $\pi^*(J_{\widetilde{C}}) \sim J_{\widetilde{C}}$. This gives $J_C \sim J_{\widetilde{C}} \times P(\pi)$.

The following theorem classifies principally polarized abelian surfaces. We shall use it in the construction of our curves whose Jacobians we want to decompose.

Theorem 2.70 ([BL04], Page 341)**.** *Over an algebraically closed base field, a principally polarized abelian surface is either the Jacobian of a smooth curve of genus 2 or the canonically polarized product of two elliptic curves.*

2.9 The main tool

The main goal of this thesis is to decompose the Jacobian of a smooth curve inside an abelian surface into smaller abelian varieties. As mentioned in Chapter 1, we will be interested in certain curves that are obtained from a construction. Our main tool will be the following theorem by Kani and Rosen:

Theorem 2.71 ([KR89], Theorem B). *Let C be a curve and G be a finite subgroup of* $\mathrm{Aut}(C)$ *such that $G = H_1 \cup \cdots \cup H_t$, where the subgroups $H_i \leq G$ satisfy $H_i \cap H_j = 1$ if $i \neq j$. Then we have the isogeny relation*

$$J_C^{t-1} \times J_{C/G}^{g} \sim J_{C/H_1}^{h_1} \times \cdots \times J_{C/H_t}^{h_t} \tag{2.2}$$

where $|G| = g$, $|H_i| = h_i$ and, as usual, $J^n = J \times \cdots \times J$ (n times).

It is clear from the statement of the theorem that one needs information on the automorphism group of the curve to use this theorem to get an isogeny relation on its Jacobian. More precisely, for a curve C, we will need a subgroup G of $\mathrm{Aut}(C)$ that can be covered by subgroups $H_i \leq G$ such that any two of these H_i intersect only at the identity element, i.e., G should have a partition. The curves obtained from our construction will have such subgroups in their automorphism groups. Moreover, the theory of partitions discussed in section 2.2 will also allow us to look at suitable polarizations of higher degrees.

Our first main step in using this tool will be to lay out the construction from which we will obtain our curves and study the automorphism groups of these curves.

Chapter 3

Decomposition of Jacobians

In this chapter, we will establish the Jacobian decomposition of certain curves lying in the linear system of a simple $(1, d)$-polarized abelian surface A. These curves arise from quotients of the abelian surface A. We give a general construction, from which we will get different curves based on the choice of a subgroup of $K(L)$. Section 3.2 discusses known results related to abelian surfaces and curves lying in their linear systems. We make some observations on the fixed points of automorphisms of our curves in section 3.3. In section 3.4 we establish the isogeny relations of Jacobians of curves arising from cyclic covers, and in section 3.5 we give results for the curve arising from the Klein cover. Section 3.6 investigates when our curves can be hyperelliptic or have covers onto elliptic curves.

3.1 Setup

In this section, we lay out the setup in which we will be working and mention a few well-known facts. The main reference for this section is [BM16].

Let A be a simple abelian surface. Every abelian variety carries a polarization and therefore so does A. Polarizations on abelian surfaces are of type (d_1, d_2) where d_1, d_2 are positive integers and $d_1 \mid d_2$. A polarization of type $(1, d)$ is called a *primitive polarization.* We will focus on a $(1, d)$-polarization L of A. We can take L to be a symmetric line bundle, i.e., $[-1]^*L \cong L$. This polarization induces an isogeny:

$$\begin{aligned} \varphi_L : A &\longrightarrow A^\vee \\ x &\longmapsto t_x^* L \otimes L^{-1} \end{aligned}$$

where t_x denotes the translation by x.

Lemma 3.1 ([BM16], section 1)**.** *The isogeny φ_L has kernel $K(L) := \langle x, y \rangle \cong (\mathbb{Z}/d\mathbb{Z})^2$, where x and y are elements of order d in $K(L)$.*

From now on, X will be a subgroup of $K(L)$ of order d. Since X is a finite subgroup of A, the quotient A/X is again an abelian surface and the quotient map $\pi : A \longrightarrow A/X$ is an isogeny. We can say more about the polarization type of A/X:

Lemma 3.2. *The abelian surface A/X is principally polarized by a line bundle P with $P \cong \pi_* L$. Moreover, $A/X = J_H$ where H is a smooth curve of genus 2.*

Proof. The first statement follows from Proposition 2.57 on pushforward polarization. The second part follows from Theorem 2.70 on the classification of principally polarized abelian surfaces and the fact that A is a simple abelian surface. □

Construction 3.3. Let A be a simple abelian surface with a $(1, d)$-polarization, such that the polarizing line bundle L is symmetric, is of characteristic 0 and admits a symmetric theta structure. Suppose X is a subgroup of $K(L)$ of order d. By Lemma 3.2, $A/X \cong J_H$ for a smooth genus two curve H. Since every smooth curve is embedded in its Jacobian, we have $H \subset J_H \cong A/X$. Consider the pre-image $C := \pi^{-1}(H)$, where $\pi : A \longrightarrow A/X$ is the quotient isogeny. Then C is a smooth curve in the linear system $|L|$.

The main goal of this thesis is to answer the following :

Question 3.4. How does J_C decompose into smaller abelian varieties up to isogeny and what properties of C does this decomposition reveal?

Recall that the complete linear system $|L|$ is the projectivization of the vector space of global sections of L. Since L is symmetric, the $[-1]$-involution on A induces a map $[-1]^* : |L| \longrightarrow |L|$ that satisfies $[-1]^* \circ [-1]^* = \mathrm{id}_{|L|}$. Therefore, $|L|$ has two eigenspaces: corresponding to the eigenvalues 1 and -1.

Lemma 3.5. *The translations t_x for any $x \in X$ and the restriction $-1|_C$ are automorphisms of C. Moreover, C has genus $d + 1$.*

Proof. Consider the commutative diagram below.

$$\begin{array}{ccccc} C & \xrightarrow{\ \pi\ } & H & \xrightarrow{\ -1|_H\ } & H \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ A & \xrightarrow{\ \pi\ } & J_H \cong A/X & \xrightarrow{\ -1\ } & J_H \end{array}$$

Figure 3.1

Let $p \in C$. Then $\pi(p) \in H$ and hence $(-1|_H \circ \pi)(p) \in H$ as $-1|_H$ is the hyperelliptic involution of H. When viewed as an element in J_H, we have $(-1|_H \circ \pi)(p) = -\pi(p) = \pi(-p)$ as π is a homomorphism on A (here $-p$ is the inverse of p with respect to the group law of the abelian surface A). Since $\pi(-p)$ is an element of H, we have $-p \in \pi^{-1}(H) = C$. Thus, $-1|_C$ is an automorphism of C.

To conclude that for any $x \in X$, the translation $t_x : A \longrightarrow A$ when restricted to C is also an automorphism of C, it is sufficient to show that for any point p on the curve C, the point $\pi(p + x)$ is in H. This is indeed true, since $\pi(x) = 0$ implies $\pi(p + x) = \pi(p) \in \pi(C) = H$ by definition of C.

Finally, for the genus of C, we apply the Riemann–Hurwitz formula on the curve morphism $\pi|_C$: $C \longrightarrow H$. Note that this map passes C onto its quotient by all translations induced by elements of the group X, and hence it is unramified. The Riemann–Hurwitz formula gives

$$2 \cdot g_C - 2 = d(2 \cdot 2 - 2) + 0,$$

implying $g_C = d + 1$. Thus, C is a curve of genus $d + 1$ in the linear system $|L|$. □

Remark 3.6. The element $[C]$ of $|L|$ corresponding to the curve C lies in an eigenspace of $[-1]^*$ because $[-1]$ is an involution of C.

3.2 Related examples

This section briefly discusses a few examples previously treated by others that are similar in taste to our main problem.

(1,2)-polarization:

W. Barth studied abelian surfaces with $(1, 2)$-polarization and proved the following results.

Lemma 3.7 ([Bar87], Page 47, 1.5)**.** *If C is a smooth genus three curve in the linear system of a $(1, 2)$-polarization on an abelian surface A, then $\# \operatorname{Fix}_C[-1] = 4$.*

Theorem 3.8 ([Bar87], Proposition 1.8)**.** *For a smooth genus three curve D the following properties are equivalent:*

- *D admits an elliptic involution, i.e., there is an involution τ of D such that $D/\langle\tau\rangle$ is an elliptic curve.*
- *D admits an embedding into an abelian surface A.*

Although he did not explicitly compute the Jacobian decomposition of these curves, we will make use of his results and decompose the Jacobian for such a curve arising from our construction.

(1,3)-polarization:

For a $(1,3)$-polarization L on an abelian surface A that is not necessarily simple, Birkenhake and Lange constructed a curve similar to ours and decomposed its Jacobian as follows: The group $K(L) := \{x \in X \mid t_x^* L \cong L\}$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. There are four cyclic subgroups of $K(L)$ isomorphic to $\mathbb{Z}/3\mathbb{Z}$, each of which gives a cyclic isogeny of degree 3 onto a principally polarized abelian surface. If the principally polarized abelian surface is taken to be the Jacobian J_H of a smooth genus two curve H and if π denotes the cyclic isogeny, then $C := \pi^{-1}(H)$ is a smooth curve of genus four in the linear system $|L|$. They established the following result.

Theorem 3.9 ([BL94], Proposition 2.2)**.** *The $[-1]$-involution of A restricted to C is an elliptic involution on C and the embedding $C \hookrightarrow A$ induces an exact sequence of abelian varieties:*

$$0 \longrightarrow E \times E \longrightarrow J_C \longrightarrow A \longrightarrow 0,$$

with $E = C/\langle -1 \rangle$.

From the exact sequence, it follows that $J_C \sim A \times E \times E$.

In the same paper, the authors also showed an example of a non-simple abelian surface $A = F \times F$ for an elliptic curve F, with the $(1,3)$-polarization $L = \mathcal{O}_A(F \times \{0\} + \{0\} \times F + \Delta')$ where Δ' is the anti-diagonal. If $\pi : A \longrightarrow J_H$ is the isogeny onto the Jacobian of a smooth curve H of genus two, then $C = \pi^{-1}(H)$ is a genus four curve in A with an étale cyclic 3-fold covering $C \longrightarrow H$ induced by π. The family of these curves have been studied and it has been shown that they admit a large group of automorphisms. If the elliptic curve F is identified with its dual, then the polarization L induces an isogeny $\varphi_L : F \times F \longrightarrow F \times F$ which can also be represented as the matrix:

$$\varphi_L = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

This gives, $K(L) = \text{Ker}(\varphi_L) = \{(x, x) \mid 3x = 0\}$. The automorphisms

$$T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

of $A = F \times F$ generate the symmetric group S_3 of order 6. The following result gives information about the automorphism group of C.

Theorem 3.10 ([BL94], Proposition 4.1)**.** *The automorphism group* $\text{Aut}(C)$ *of C contains the group $S_3 \times S_3 = \langle T, J \rangle \times \langle t_x, -1 \rangle$ where t_x denotes the translation by an order three element $x \in K(L)$ and -1 is the automorphism on the curve descending from the $[-1]$-involution of A.*

Further, it has been shown that the involutions J and -1 of C are elliptic. Moreover,

$$C/\langle J\rangle \cong C/\langle J\circ T\rangle \cong C/\langle J\circ T^2\rangle = F$$
$$C/\langle -1\rangle \cong C/\langle -1\circ t_x\rangle \cong C/\langle -1\circ t_{2x}\rangle = F',$$

where F is the elliptic curve that gives the abelian surface $A = F\times F$ and F' is another elliptic curve. Finally, they compute the isogeny type of the Jacobian of C with the following result:

Theorem 3.11 ([BL94], Proposition 5.1)**.** *The embedding $C\hookrightarrow A = F\times F$ induces an exact sequence:*

$$0\longrightarrow F'\times F'\longrightarrow J_C\longrightarrow F\times F\longrightarrow 0,$$

which gives $J_C\sim F\times F\times F'\times F'$.

(1,4)-polarization:

Borówka and Ortega have studied hyperelliptic curves that can be embedded in an abelian surface ([BO19]). They give a necessary condition on the genus of such curves as well as on the degree of polarization of the surface. Further, they give a necessary and sufficient condition for a hyperelliptic curve to be embedded in a $(1,4)$-polarized abelian surface.

Theorem 3.12. *A smooth hyperelliptic curve of genus* 5 *can be embedded in a* $(1,4)$*-polarized abelian surface if and only if it is a non-isotropic étale Klein covering of a genus* 2 *curve.*

As a consequence of this result, they also decompose the Jacobian of this curve into abelian subvarieties.

3.3 Fixed points of automorphisms

In this section, we will study the $[-1]$-involution of C of Construction 3.3 in more detail. First, we mention a couple of known results about the number of points fixed by this involution and then investigate the same for involutions that are obtained by shifting the $[-1]$-involution by translations.

Theorem 3.13 ([BM16], Proposition 2.1)**.** *For the curve C of Construction 3.3, we have:*

$$\#\,\mathrm{Fix}_C[-1] = \begin{cases} 6 \text{ or } 10, & \text{if } d \text{ is odd} \\ 4 \text{ or } 12, & \text{if } d \text{ is even.} \end{cases}$$

Theorem 3.14 ([Bar87], Page 47, 1.5)**.** *If C is a genus* 3 *curve in the linear system of a* $(1,2)$*-polarized abelian surface A such that the polarizing line bundle is of characteristic* 0 *and admits a symmetric theta structure, then $\#\,\mathrm{Fix}_C[-1] = 4$.*

As we shall use Theorem 3.13 throughout this thesis, let us briefly look at an example illustrating how the number of these fixed points are computed. Fix an abelian surface A with $(1, d)$-polarization L where the polarizing line bundle is symmetric and is of characteristic zero. This polarization gives a matrix:

$$T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \\ -1 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}$$

By Definition 2.54, we also have non-degenerate bilinear pairings $e_n^L : A[n] \times A[n] \longrightarrow \mu_n$ of commutative group schemes. As $\mathrm{Fix}_C[-1] \subseteq A[2]$, we want to look at the pairing on 2-torsion points. Denote $e_2^L : A[2] \times A[2] \longrightarrow \mathbb{Z}/2\mathbb{Z}$ by $\langle \ , \ \rangle$. This pairing satisfies $\langle x, x' \rangle = x^t T x'$, where x^t denotes the transpose of the vector x. A subgroup $X \leq A[2]$ will be ***isotropic*** with respect to the pairing if $\langle x, x' \rangle = 0$ for all $x, x' \in X$. Since the polarizing line bundle L is of characteristic zero, it is possible to write

$$A[2] = X \oplus Y$$

as groups where X, Y are isotropic subgroups and $X \cap Y = 0$. Let us fix such a decomposition. We know that $A[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$ as $\mathbb{Z}/2\mathbb{Z}$-vector spaces. Take $X = \mathrm{Span}\{v_1, v_2\}$ and $Y = \mathrm{Span}\{v_3, v_4\}$ where v_1, v_2, v_3, v_4 are the canonical vectors of $(\mathbb{Z}/2\mathbb{Z})^4$. Then for $x = (a, b, 0, 0)$ and $x' = (a', b', 0, 0)$ in X, we have

$$\langle x, x' \rangle = x^t T x' = \begin{bmatrix} a & b & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \\ -1 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{bmatrix} \begin{bmatrix} a' \\ b' \\ 0 \\ 0 \end{bmatrix} = 0.$$

Thus, X is isotropic and similarly Y is also isotropic. This is one such decomposition of $A[2]$ into isotropic subgroups.

Once we have the pairing $\langle \ , \ \rangle$ and the isotropic decomposition $A[2] = X \oplus Y$, we can define a quadratic form on $A[2]$. Since any element $z \in A[2]$ can be uniquely written as $z = x + y$ with $x \in X, \ y \in Y$, we get the quadratic form

$$\begin{aligned} q : A[2] &\longrightarrow \mathbb{Z}/2\mathbb{Z} \\ z &\longmapsto \langle x, y \rangle. \end{aligned}$$

As the matrix multiplication $\langle x, y\rangle = x^tTy$ is being done over over the field $\mathbb{Z}/2\mathbb{Z}$, we can take the matrix T to be

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

depending on whether d is odd or even respectively. When d is even, then $A[2]^+ := \{a \in A[2] \mid q(a) = 0\}$ is the set $\{\mathbf{0}, v_1, v_2, v_3, v_4, v_1+v_2, v_3+v_4, v_2+v_4, v_2+v_3, v_1+v_4, v_2+v_3+v_4, v_1+v_2+v_4\}$ of 12 elements, and the 4-element set $\{v_1+v_2+v_3, v_1+v_3, v_1+v_3+v_4, v_1+v_2+v_3+v_4\}$ is precisely $A[2]^- := \{a \in A[2] \mid q(a) = 1\}$. Clearly, $A[2]^+$ and $A[2]^-$ together contain all 16 elements of $A[2]$. We say that the multiplicity at a point $a \in A$ is 1 if a lies on the curve C and 0 otherwise, and denote the multipilicity as $\operatorname{mult}(a)$. It is known that for a point $a \in A[2]$, the curve C contains a if and only if $\operatorname{mult}(a) - \operatorname{mult}(0) = q(a)$. Thus, the curve C contains all of $A[2]^+$ or $A[2]^-$, which implies that $\#\operatorname{Fix}_C[-1] = 4$ or 12 when d is even. Similarly, $\#\operatorname{Fix}_C[-1] = 6$ or 10 when d is odd.

We now want to look at the number of points fixed by automorphisms other than the $[-1]$-involution on the curve C. Recall that the translations t_x for $x \in X$ do not have any fixed points on C. Once we have information on the number of fixed points of the $[-1]$-involution of C, it is natural to investigate the same for the involutions obtained by composing -1 with translations t_x, where $x \in X$. Observe that, $t_x \circ [-1] = [-1] \circ t_{(k-1)x}$, where k is the order of the element x. Therefore, it is sufficient to consider the involutions of the form $-1 \circ t_x$ to exhaust all possible compositions that can be obtained from $[-1]$-involution and translations by elements of the group X. For a subgroup $Y \leq X \leq K(L)$, the subgroup $\langle t_y \mid y \in Y\rangle$ of translations will be denoted by $t(Y)$. It is easy to see that $t(Y) \leq t(X) \leq \operatorname{Aut}(C)$. Consider the isogenies $\pi_1 : A \longrightarrow A/Y$ and $\pi_2 : A/Y \longrightarrow A/X$, and the curve $\widetilde{C} := C/t(Y)$. Observe that the composition $\pi_2 \circ \pi_1$ is the isogeny $\pi : A \longrightarrow A/X$ of Construction 3.3. The following diagram, where the vertical arrows are inclusions, is commutative.

$$\begin{array}{ccccc} C & \xrightarrow{\pi_1} & \widetilde{C} = C/t(Y) & \xrightarrow{\pi_2} & H = C/t(X) \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{\pi_1} & A/Y & \xrightarrow{\pi_2} & A/X \end{array}$$

Figure 3.2

The curve morphism $\pi_1 : C \longrightarrow \widetilde{C}$ does not have any fixed points since it is a cover induced by translations. We have the following result relating the fixed points of -1 on $\widetilde{C}$ with points fixed by shifts of -1 on C.

Proposition 3.15. *If $C_Y := \cup_{y\in Y} \operatorname{Fix}_C[-1\circ t_y]$, then π_1 restricts to a $\#Y$-to-1 surjection from C_Y onto $\operatorname{Fix}_{\widetilde{C}}[-1]$.*

Proof. Suppose $a \in C_Y$. Then a is fixed by $-1\circ t_y$ for some $y \in Y$, i.e., $-a-y = a$. Therefore, $-\bar{a} = \bar{a}$ as elements of A/Y, implying $\pi_1(a) \in \operatorname{Fix}_{\widetilde{C}}[-1]$ and proving $\pi_1(C_Y) \subset \operatorname{Fix}_{\widetilde{C}}[-1]$. For the other inclusion, start with $b = \bar{a} \in \operatorname{Fix}_{\widetilde{C}}[-1]$. Note that, $a \in C$ because $\widetilde{C}$ is the image of C in A/Y under π_1. Now, $\bar{a} \in \operatorname{Fix}_{\widetilde{C}}[-1]$ implies $-\bar{a} = \bar{a}$ as elements of A/Y. Thus, $-a = a+y$ in A for some $y \in Y$, implying $-a-y = a$, which is equivalent to $-1\circ t_y(a) = a$. This gives $a \in C_Y$, proving $\operatorname{Fix}_{\widetilde{C}}[-1] \subset \pi_1(C)$. Therefore, $\pi_1(C_Y) = \operatorname{Fix}_{\widetilde{C}}[-1]$. This is a $\#Y$-to-1 map because each element in A/Y has exactly $\#Y$ pre-images in A under π_1 and this holds even when π_1 is restricted to C. □

Corollary 3.16. $\sum_{x\in X} \# \operatorname{Fix}_C[-1\circ t_x] = 6\cdot \#X$

Proof. If $x \neq x'$ in X, then $\operatorname{Fix}_C[-1\circ t_x] \cap \operatorname{Fix}_C[-1\circ t_{x'}] = \phi$. Therefore, $\sum_{x\in X} \# \operatorname{Fix}_C[-1\circ t_x] = \#C_X$, where $C_X = \cup_{x\in X} \operatorname{Fix}_C[-1\circ t_x]$. Moreover, in Construction 3.3, we had the isogeny $\pi : A \longrightarrow A/X$ whose restriction to C gave the curve morphism $\pi : C \longrightarrow H$. Equivalently, $H = C/t(X)$. Since, H is a smooth hyperelliptic curve of genus 2 and -1 is the hyperelliptic involution of H, we get that $\operatorname{Fix}_H[-1] = 2\cdot g_H + 2 = 2\cdot 2+2 = 6$. Using Proposition 3.15 with $Y = X$ and $\widetilde{C} = H$, we get $\sum_{x\in X} \# \operatorname{Fix}_C[-1\circ t_x] = 6\cdot \#X$. □

We can say more about the number of points fixed by automorphisms of the form $-1\circ t_x$ when $x \in X$. Based on the type of the polarization and the point x, we can relate the number of fixed points of the automorphism $-1\circ t_x$ with that of -1.

Lemma 3.17. *$\# \operatorname{Fix}_C[-1\circ t_x] = \# \operatorname{Fix}_{t_y(C)}[-1]$, where $y \in A$ is such that $2y = x$ and $t_y(C)$ denotes the translation of the curve C by the element y.*

Proof. Let p be a point on the curve C. Then $p \in \operatorname{Fix}_C[-1\circ t_x]$ if and only if $-1\circ t_x(p) = p$, or equivalently, $-p-x = p$. If $y \in A$ is such that $2y = x$, then the equality $-p-x = p$ is equivalent to $-p-y = p+y$, which is again equivalent to $[-1](p+y) = p+y$. This is true if and only if the point $p+y$ of the curve $t_y(C)$ is fixed by -1. Therefore, we have a one-to-one correspondence between $\operatorname{Fix}_C[-1\circ t_x]$ and $\operatorname{Fix}_{t_y(C)}[-1]$, where $2y = x$. □

Theorem 3.18. *Let C be the curve of Construction 3.3 and $x \in X$. If d is odd, then $\# \operatorname{Fix}_C[-1\circ t_x] = 6$ or 10. If d is even and there exists $y \in K(L)$ such that $2y = x$, then $\# \operatorname{Fix}_C[-1\circ t_x] = 4$ or 12.*

Proof. In Construction 3.3 we assumed that the $(1, d)$-polarization L on A is symmetric and has a symmetric theta structure. Since the curve C is in the linear system of the polarizing line bundle, therefore $L \cong \mathcal{O}(C)$. For any $y \in A$, we have $\mathcal{O}(t_y(C)) = \mathcal{O}(C + y) \cong t_{-y}^* L$.

When d is odd, the multiplication-by-2 map is an isomorphism on the subgroup $X \leq K(L)$ (as X has order d). Thus, there exists $y \in X \leq K(L)$ such that $2y = x$. Since X is a subgroup, $-y$ is also in $X \leq K(L)$ and hence $t_{-y}^* L \cong L$. This gives $\mathcal{O}(t_y(C)) \cong L$, and by Theorem 3.14 and Lemma 3.17 we will have $\# \operatorname{Fix}_C[-1 \circ t_x] = 6$ or 10.

When d is even, the multiplication-by-2 map from $A[2]$ to itself is no longer surjective. So, with the added hypothesis of existence of $y \in K(L)$ satisfying $2y = x$, we will have $\mathcal{O}(t_y(C)) \cong L$ and the rest of the proof follows similarly as in the previous case. □

3.4 Cyclic cover

In this section, we want to decompose (up to isogeny) the Jacobian of the curve C of Construction 3.3 obtained from a cyclic subgroup X of $K(L)$.

Write $X = \langle x \rangle$ for some $x \in K(L)$ of order d. By Lemma 3.5, we have $G := \langle -1, t_x \rangle$ is a subgroup of $\operatorname{Aut}(C)$, where the elements t_x and -1 are of orders d and 2 respectively. They satisfy the relation $(-1 \circ t_x)^2 = 1$. Therefore, $G \cong D_{2d}$, the dihedral group of order $2d$, which admits a partition as

$$G = \langle t_x \rangle \cup \langle -1 \rangle \cup \langle -1 \circ t_x \rangle \cup \cdots \cup \langle -1 \circ t_{(d-1)x} \rangle. \tag{3.1}$$

Theorem 2.71 gives the isogeny relation:

$$J_C^d \times J_{C/G}^{2d} \sim J_{C/\langle t_x \rangle}^d \times J_{C/\langle -1 \rangle}^2 \times J_{C/\langle -1 \circ t_x \rangle}^2 \times \cdots \times J_{C/\langle -1 \circ t_{(d-1)x} \rangle}^2. \tag{3.2}$$

Note that the map $C \longrightarrow C/G$ factors as $C \longrightarrow C/\langle t_x \rangle \longrightarrow C/G$. Now, $C/\langle t_x \rangle = H$ and hence $C/G = C/\langle t_x, -1 \rangle = H/\langle -1 \rangle$. Since H is hyperelliptic and the restriction of -1 on H is the hyperelliptic involution, we will get $C/G = H/\langle -1 \rangle = \mathbb{P}^1$. By Proposition 2.67, we have $J_{C/G}^{2d}$ is trivial. Also, when d is odd, the subgroups $\langle -1 \rangle, \langle -1 \circ t_x \rangle, \ldots, \langle -1 \circ t_{(d-1)x} \rangle$ are precisely all the 2-Sylow subgroups of D_{2d} and hence are conjugates. Consequently, by Lemma 2.21, we will have:

$$J_{C/\langle -1 \rangle} \cong J_{C/\langle -1 \circ t_x \rangle} \cong \ldots \cong J_{C/\langle -1 \circ t_{(d-1)x} \rangle}.$$

When d is even, $\langle -1 \rangle$, $\langle -1 \circ t_{2x} \rangle$, $\langle -1 \circ t_{4x} \rangle, \ldots, \langle -1 \circ t_{(d-2)x} \rangle$ are conjugates and the remaining two order subgroups are conjugates to $\langle -1 \circ t_x \rangle$. In this case, we will have two isomorphism classes of Jacobians:

$$J_{C/\langle -1\rangle} \cong J_{C/\langle -1\circ t_{2x}\rangle} \cong \ldots \cong J_{C/\langle -1\circ t_{(d-2)x}\rangle}$$
$$J_{C/\langle -1\circ t_x\rangle} \cong J_{C/\langle -1\circ t_{3x}\rangle} \cong \ldots \cong J_{C/\langle -1\circ t_{(d-1)x}\rangle}.$$

Thus, when d is odd, the isogeny relation (3.2) reduces to

$$J_C^d \sim J_H^d \times J_{C/\langle -1\rangle}^{2d}.$$

By Construction 3.3, we have $J_H \cong A/X$ and the quotient isogeny $A \longrightarrow A/X$ gives that $A/X \sim A$. Thus, for odd d we get

$$J_C^d \sim A^d \times J_{C/\langle -1\rangle}^{2d}.$$

Similarly, for even d we get

$$J_C^d \sim A^d \times J_{C/\langle -1\rangle}^d \times J_{C/\langle -1\circ t_x\rangle}^d.$$

Therefore, we conclude the following.

Proposition 3.19. *For the curve C arising from a cyclic subgroup $X \leq K(L)$ of order d in Construction* 3.3*, we have the following isogeny relations:*

1. *When d is odd: $J_C \sim A \times J_{C/\langle -1\rangle}^2$*
2. *When d is even: $J_C \sim A \times J_{C/\langle -1\rangle} \times J_{C/\langle -1\circ t_x\rangle}$.*

As an immediate consequence of this proposition, we give the complete decomposition of J_C into simple factors in the cases of $d = 2$ and $d = 3$. Recall that $\dim(J_C) = g_C = d + 1$. As isogenies preserve dimension, the right hand side of the two isogeny relations in Proposition 3.19 must also have dimension $d + 1$. Since A is a surface, for $d = 3$ we must have $\dim(J_{C/\langle -1\rangle}) = 1$ which implies that $C/\langle -1\rangle$ is an elliptic curve. Similarly, for $d = 2$ we have $\dim(J_C/\langle -1\rangle) + \dim(J_{C/\langle -1\circ t_x\rangle}) = 1$. Further, Theorem 3.14 implies that $C/\langle -1\rangle$ is elliptic.

Corollary 3.20. *When the curve C of Construction 3.3 is obtained from a cyclic isogeny, the Jacobian of C decomposes as:*

$$J_C \sim \begin{cases} A \times E & d = 2 \\ A \times E \times E & d = 3 \end{cases}$$

where E is the elliptic curve $C/\langle -1\rangle$.

When d is prime, the only possibility of an order d subgroup X of $K(L)$ is the cyclic group. Hence, Proposition 3.19 decomposes (into smaller abelian varieties up to isogeny) the Jacobians of all curves coming from Construction 3.3 when d is a prime. It is natural to ask: what are the genera of the curves $C/\langle -1 \rangle,\ C/\langle -1 \circ t_x \rangle$ in the cyclic case? As a direct consequence of Proposition 3.19, by comparing dimensions we get: $g_{C/\langle -1 \rangle} = \frac{d-1}{2}$ when d is odd, and $g_{C/\langle -1 \rangle} + g_{C/\langle -1 \circ t_x \rangle} = d - 1$ when d is even.

3.5 The Klein cover

We shall now focus on the curve C of Construction 3.3 when $d = 4$ and the subgroup X of $K(L) = \langle x, y \rangle \cong (\mathbb{Z}/4\mathbb{Z})^2$ is the Klein four-group, i.e., $X = \langle 2x, 2y \rangle$. We will decompose the Jacobian of C up to isogeny and prove that C is hyperelliptic.

We start with a $(1, 4)$-polarized simple abelian surface A. Consider the subgroup $X = \langle 2x, 2y \rangle$ of $K(L) = \langle x, y \rangle$. Then X is the Klein four-group and Construction 3.3 gives the curve C arising from the Klein cover. The group $G = \langle t_{2x},\ t_{2y} \rangle \leq \mathrm{Aut}(C)$ admits a partition as

$$G = \langle t_{2x} \rangle \cup \langle t_{2y} \rangle \cup \langle t_{2x+2y} \rangle.$$

Theorem 2.71 gives the isogeny relation

$$J_C^2 \times J_{C/G}^4 \sim J_{C/\langle t_{2x} \rangle}^2 \times J_{C/\langle t_{2y} \rangle}^2 \times J_{C/\langle t_{2x+2y} \rangle}^2$$

which simplifies to

$$J_C \times J_{C/G}^2 \sim J_{C/\langle t_{2x} \rangle} \times J_{C/\langle t_{2y} \rangle} \times J_{C/\langle t_{2x+2y} \rangle} \tag{3.3}$$

Note that, $C/G \cong H$ and $J_H \cong A/X \sim A$. Figure 3.3 below shows that $C/\langle t_{2x} \rangle$ is a smooth curve in the abelian surface $A/\langle 2x \rangle$. The isogeny π_1 pushes forward the $(1, 4)$-polarization L

$$\begin{array}{ccccc} C & \xrightarrow{\pi_1} & C/\langle t_{2x} \rangle & \xrightarrow{\pi_2} & H = C/G \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{\pi_1} & A/\langle 2x \rangle & \xrightarrow{\pi_2} & A/X \end{array}$$

Figure 3.3

on A to the $(1, 2)$-polarization $(\pi_1)_* L$ on $A/\langle 2x \rangle$. Moreover, applying the Riemann–Hurwitz formula on the map $\pi_1 : C \longrightarrow C/\langle t_{2x} \rangle$ will give us that genus of $C/\langle t_{2x} \rangle$ is three. Thus, $C/\langle t_{2x} \rangle$ is a smooth genus three curve in the linear system of the $(1, 2)$-polarization of the simple abelian surface $A/\langle 2x \rangle$. Similarly, $C/\langle t_{2y} \rangle$ and $C/\langle t_{2x+2y} \rangle$ are smooth genus three curves in the linear systems of the $(1,2)$-polarizations of the corresponding simple abelian surfaces $A/\langle 2y\rangle$ and $A/\langle 2x+2y\rangle$ respectively.

Now, consider the cover $\pi_2 : C/\langle t_{2x}\rangle \longrightarrow H$. For convenience, denote $C/\langle t_{2x}\rangle$ by $\widetilde{C}$. This is a cyclic cover by the order two subgroup $\langle t_{2y}\rangle$ and decomposing $J_{\widetilde{C}}$ with the group $D_4 \cong \langle -1, t_{2y}\rangle$ gives:

$$J_{\widetilde{C}} \sim A/\langle 2x\rangle \times J_{\widetilde{C}/\langle -1\rangle} \times J_{\widetilde{C}/\langle -1\circ t_{2y}\rangle}.$$

Since $\widetilde{C}$ is a genus three curve, comparing dimensions in this isogeny relation gives that one of $J_{\widetilde{C}/\langle -1\rangle}$ and $J_{\widetilde{C}/\langle -1\circ t_{2y}\rangle}$ must be an elliptic curve and the other must be trivial. Therefore, we can write:

$$J_{C/\langle t_{2x}\rangle} \sim A/\langle 2x\rangle \times E_1,$$

where E_1 is an elliptic curve. Similarly, for $C/\langle t_{2y}\rangle$ and $C/\langle t_{2x+2y}\rangle$ with corresponding elliptic curves E_2 and E_3 respectively, we can write:

$$J_{C/\langle t_{2y}\rangle} \sim A/\langle 2y\rangle \times E_2,$$
$$J_{C/\langle t_{2x+2y}\rangle} \sim A/\langle 2x+2y\rangle \times E_3.$$

Thus, the isogeny relation (3.3) reduces to

$$J_C \sim A \times E_1 \times E_2 \times E_3. \tag{3.4}$$

We have established the following:

Theorem 3.21. *If $d = 4$ and X is the Klein four-group, then the Jacobian of the curve C in Construction 3.3 is isogenous to the product of A and three elliptic curves.*

A natural question to ask at this point will be: is the Jacobian decomposition of the curve in Klein cover different from that in the cyclic cover? For the the curve C coming from the cyclic isogeny in the case of $(1,4)$-polarization, we will have the following decomposition by Proposition 3.19:

$$J_C \sim A \times J_{H'} \times E', \tag{3.5}$$

where $H' = C/\langle -1\rangle$ is a curve of genus two and $E' = C/\langle -1 \circ t_x\rangle$ is elliptic. In the very general case, the Jacobian of a smooth genus two curve is a simple abelian surface. Therefore, (3.4) and (3.5) correspond to two different isogeny classes.

In their paper [BO19], Borówka and Ortega showed that every $(1,4)$-polarized abelian surface has a hyperelliptic curve embedded in it. Below, we prove that the curve C coming from the Klein cover is hyperelliptic.

Theorem 3.22. *If $d = 4$ and X is the Klein four-group, then the curve C in Construction 3.3 is hyperelliptic.*

Proof. Since t_{2x}, t_{2y} are degree two automorphisms, they commute with the -1 automorphism. Therefore, the subgroup $G = \langle t_{2x}, t_{2y}, -1\rangle$ of $\mathrm{Aut}(C)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ and hence admits a partition as

$$G = \langle -1\rangle \cup \langle t_{2x}\rangle \cup \langle t_{2y}\rangle \cup \langle t_{2x+2y}\rangle \cup \langle -1\circ t_{2x}\rangle \cup \langle -1\circ t_{2y}\rangle \cup \langle -1\circ t_{2x+2y}\rangle.$$

By Theorem 2.71, we get the isogeny relation:

$$J_C^6 \times J_{C/G}^8 \sim J^2_{C/\langle t_{2x}\rangle} \times J^2_{C/\langle t_{2y}\rangle} \times J^2_{C/\langle t_{2x+2y}\rangle} \times J^2_{C/\langle -1\rangle} \times J^2_{C/\langle -1\circ t_{2x}\rangle} \times J^2_{C/\langle -1\circ t_{2y}\rangle} \times J^2_{C/\langle -1\circ t_{2x+2y}\rangle}.$$

This simplifies to

$$J_C^3 \sim J_{C/\langle t_{2x}\rangle} \times J_{C/\langle t_{2y}\rangle} \times J_{C/\langle t_{2x+2y}\rangle} \times J_{C/\langle -1\rangle} \times J_{C/\langle -1\circ t_{2x}\rangle} \times J_{C/\langle -1\circ t_{2y}\rangle} \times J_{C/\langle -1\circ t_{2x+2y}\rangle}. \quad (3.6)$$

Since the genus of C is 5, the left hand side of the isogeny relation above has dimension 15. The curves $C/\langle t_{2x}\rangle, C/\langle t_{2y}\rangle$ and $C/\langle t_{2x+2y}\rangle$ are each of genus three and hence the product of their Jacobians contribute dimension 9 in the right hand side. Consequently, we must have

$$g_{C/\langle -1\rangle} + g_{C/\langle -1\circ t_{2x}\rangle} + g_{C/\langle -1\circ t_{2y}\rangle} + g_{C/\langle -1\circ t_{2x+2y}\rangle} = 6.$$

To conclude that C is hyperelliptic, it is sufficient to show that one of the genera in the equation above is zero. Observe that the degree four cover $C \longrightarrow C/G = H$ induces a four-to-one map from $\cup_{z\in X} \mathrm{Fix}_C[-1\circ t_z] = \mathrm{Fix}_C[-1] \cup \mathrm{Fix}_C[-1\circ t_{2x}] \cup \mathrm{Fix}_C[-1\circ t_{2y}] \cup \mathrm{Fix}_C[-1\circ t_{2x+2y}]$ onto $\mathrm{Fix}_H[-1]$ by Proposition 3.15. Since, H is hyperelliptic of genus two with -1 as the hyperelliptic involution, we will have $\#\,\mathrm{Fix}_H[-1] = 6$. Therefore, by Corollary 3.16, we get:

$$\#\,\mathrm{Fix}_C[-1] + \#\,\mathrm{Fix}_C[-1\circ t_{2x}] + \#\,\mathrm{Fix}_C[-1\circ t_{2y}] + \#\,\mathrm{Fix}_C[-1\circ t_{2x+2y}] = 24. \quad (3.7)$$

By Theorem 3.18, each of $\#\,\mathrm{Fix}_C[-1], \#\,\mathrm{Fix}_C[-1\circ t_{2x}], \#\,\mathrm{Fix}_C[-1\circ t_{2y}]$ and $\#\,\mathrm{Fix}_C[-1\circ t_{2x+2y}]$ can either be 4 or 12. Therefore, for (3.7) to hold, exactly three of them have to be 4 and the remaining one has to be 12. The involution that has 12 fixed points will be the hyperelliptic involution. Thus, C is hyperelliptic. □

3.6 Hyperelliptic curves and covers on elliptic curves

It is clear from Construction 3.3 that the curve C always admits a degree d cover onto the hyperelliptic curve H. We also saw in the case of the Klein cover that C itself is hyperelliptic. These observations motivate the following questions:

1. Is C always hyperelliptic?

2. Under what conditions does C cover an elliptic curve?

The answer to the first question is negative. Borówka and Ortega have given necessary conditions to have embeddings of hyperelliptic curves into abelian surfaces (section 2, [BO19]). The aim of this section is to study their results and conclude when the curve C in Construction 3.3 can be hyperelliptic. We also answer the second question by giving a sufficient condition on the type of the polarization. We begin with the following result by Borówka–Ortega that rules out the possibility of C being hyperelliptic when C arises from a cyclic cover and $d \neq 2$.

Theorem 3.23 ([BO19], Proposition 2.3)**.** *Let $\alpha \in \operatorname{Aut}(C)$, $\widetilde{C} = C/\langle\alpha\rangle$ and $f : C \longrightarrow \widetilde{C}$ be an étale cyclic covering of degree n. If C is hyperelliptic then $\widetilde{C}$ is also hyperelliptic and $n = 2$.*

Proof. Let τ be the hyperelliptic involution of C. Then τ commutes with α and descends to an involution $\widetilde{\tau}$ of $\widetilde{C}$. Let g and $\widetilde{g}$ denote the genera of C and $\widetilde{C}$ respectively. As étale morphisms of curves are unramified, the Riemann–Hurwitz formula on $f : C \longrightarrow \widetilde{C}$ gives:

$$2g - 2 = n(2\widetilde{g} - 2) + 0, \quad \text{i.e.,} \quad g = n(\widetilde{g} - 1) + 1.$$

Since τ is the hyperelliptic involution of C, we will get $\#\operatorname{Fix}_C[\tau] = 2g + 2 = 2n(\widetilde{g} - 1) + 4$. Note that, $\#\operatorname{Fix}_{\widetilde{C}}[\widetilde{\tau}] \geq \frac{\#\operatorname{Fix}_C[\tau]}{n}$ because τ descends to $\widetilde{\tau}$. This will give $\#\operatorname{Fix}_{\widetilde{C}}[\widetilde{\tau}] \geq 2(\widetilde{g} - 1) + \frac{4}{n}$, so we can write $\#\operatorname{Fix}_{\widetilde{C}}[\widetilde{\tau}] = 2(\widetilde{g} - 1) + b$ for some positive integer b. Consider the quotient curve $\widetilde{C}/\langle\widetilde{\tau}\rangle$ and denote its genus by g'. To show that $\widetilde{C}$ is hyperelliptic, it is sufficient to show $g' = 0$. The Riemann–Hurwitz formula on the quotient map $\widetilde{C} \longrightarrow \widetilde{C}/\langle\widetilde{\tau}\rangle$ gives:

$$2\widetilde{g} - 2 = 2(2g' - 2) + \#\operatorname{Fix}_{\widetilde{C}}[\widetilde{\tau}] = 2(2g' - 2) + 2\widetilde{g} - 2 + b.$$

Since $b > 0$, the only possibility is $g' = 0$. Thus, $b = 4$ and $\widetilde{C}$ is hyperelliptic with $\widetilde{\tau}$ as the hyperelliptic involution.

For the second part, note that the composition $C \longrightarrow \widetilde{C} \longrightarrow \widetilde{C}/\langle\widetilde{\tau}\rangle \cong \mathbb{P}^1$ is a Galois covering with Galois group D_{2n} generated by α and τ. Since τ is the hyperelliptic involution of C, it commutes with every other automorphism of C. Hence, D_{2n} must be abelian. This is possible only for $n = 2$. □

We now give a necessary condition on the genus of the curve C of Construction 3.3 to be hyperelliptic. The following lemma will be used in the proof of the main result.

Lemma 3.24 ([BO19], Lemma 2.6)**.** *If C is a smooth hyperelliptic curve inside an abelian surface A and W is the set of Weierstrass points of C, then C can be embedded in A such that the $[-1]$-involution of A descends to the hyperelliptic involution of C and $W = C \cap A[2]$.*

Proof. Let $i : C \longrightarrow A$ be the embedding of the smooth hyperelliptic curve C into the abelian surface A. After composing by a translation if required, we can assume $i(p_0) = 0$ for some $p_0 \in W$. Let $\alpha_{p_0} : C \longrightarrow J_C$ be the Abel–Jacobi map. By the universal property of the Abel–Jacobi map, we have the commutative diagram in Figure 3.4, where f is the extension of i to J_C.

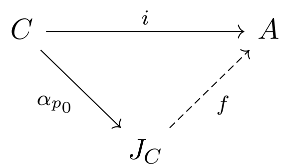

Figure 3.4

Let τ be the hyperelliptic involution of C. First, we claim that the extension of τ to J_C is the $[-1]$-involution of J_C, i.e., $\alpha_{p_0} \circ \tau = -1_{|C}$. The Abel–Jacobi map α_{p_0} sends any point p on the curve C to the element $[p - p_0] \in J_C$ (the divisor class of the degree zero divisor $p - p_0$). So, $\tau(p)$ is mapped to $[\tau(p) - \tau(p_0)] = [\tau(p) - p_0]$ (since p_0 is a Weierstrass point and hence is fixed by τ). Observe that $[p + \tau(p)] - [2p_0] = [p + \tau(p) - 2p_0]$ is the pullback of a degree zero divisor on $\mathbb{P}^1$ under the map $C \longrightarrow \mathbb{P}^1$, and hence is a divisor of a function. Thus, it is linearly equivalent to zero and as an element of $J_C \cong \mathrm{Pic}^0(C)$, we have:

$$
\begin{aligned}
[p + \tau(p)] - [2p_0] &= 0 \\
[p - p_0] + [\tau(p) - p_0] &= 0 \\
[\tau(p) - p_0] &= -[p - p_0] \\
\alpha_{p_0}(\tau(p)) &= -\alpha_{p_0}(p).
\end{aligned}
$$

This finishes the claim. Next, we want to show that the $[-1]$-involution of A descends to the hyperelliptic involution of C. We start with the embedding $i' := i \circ \tau$ of C into A. Note that $i'(p_0) = 0$ and induces a map $f' : J_C \longrightarrow A$ by the universal property of the Abel–Jacobi map. We have shown above that for any point $p \in C$, $\alpha_{p_0}(\tau(p)) = -\alpha_{p_0}(p)$. This implies

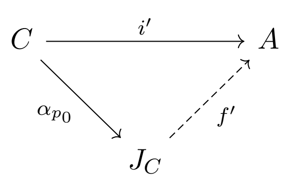

Figure 3.5

$f'(\alpha_{p_0}(\tau(p))) = -f'(\alpha_{p_0}(p))$ as elements in A, which is equivalent to $\tau^2(p) = -\tau(p)$. Since τ is an involution, $\tau^2(p) = p$ for all points $p \in C$. Thus, $\tau(p) = -p = -1|_C(p)$, which shows that the hyperelliptic involution τ of C descends from the $[-1]$-involution of A. Finally, to conclude that $W = C \cap A[2]$, observe that the surjectivity of the map $f : J_C \longrightarrow A$ implies $f(J_C[2]) = A[2]$. It is sufficient to show that $\alpha_{p_0}(W) = \alpha_{p_0}(C) \cap J_C[2]$, because this will imply $f(\alpha_{p_0}(W)) = f(\alpha_{p_0}(C)) \cap f(J_C[2]) = f(\alpha_{p_0}(C)) \cap A[2]$, which is the same as the assertion. These equalities hold because $f|_{\alpha_{p_0}}$ equals the injective map (inclusion) i and hence will preserve intersections of subsets.

To show that $\alpha_{p_0}(W) = \alpha_{p_0}(C) \cap J_C[2]$, we start with a point $q \in C$ such that $\alpha_{p_0}(q) \in \alpha_{p_0}(C) \cap J_C[2]$. Now, $\alpha_{p_0}(q) \in J_C[2]$ means $\alpha_{p_0}(q) = -\alpha_{p_0}(q)$ and the latter is equal to $\alpha_{p_0}(\tau(q))$ as we saw above. By injectivity of α_{p_0}, we get $q = \tau(q)$, i.e., $q \in W$ and hence $\alpha_{p_0}(q) \in \alpha_{p_0}(W)$. Conversely, $q \in W$ implies $\tau(q) = q$, which gives $\alpha_{p_0}(\tau(q)) = \alpha_{p_0}(q)$ which implies $\alpha_{p_0}(q) = -\alpha_{p_0}(q)$ (since $\alpha_{p_0}(\tau(q)) = -\alpha_{p_0}(q)$), and this finally gives $\alpha_{p_0}(q) \in J_C[2]$. Thus, $\alpha_{p_0}(W) = \alpha_{p_0}(C) \cap J_C[2]$. □

Proposition 3.25. *If the curve C of Construction 3.3 is hyperelliptic, then $g_C \in \{2, 3, 4, 5\}$ and the polarization L is of type $(1, g_C - 1)$.*

Proof. Let C be hyperelliptic, τ denote the hyperelliptic involution of C and W denote the set of Weierstrass points on C, i.e., $W = \mathrm{Fix}_C[\tau]$. Recall that $g_C = d+1$, where the type of polarization is $(1, d)$. The Riemann–Hurwitz formula on the map $C \longrightarrow C/\langle\tau\rangle \cong \mathbb{P}^1$ gives:

$$2(d+1) - 2 = 2(2 \cdot 0 - 2) + \#\,\mathrm{Fix}_C[\tau] \implies \#\,\mathrm{Fix}_C[\tau] = 2d + 4.$$

If $d \geq 5$, then $\#W = \#\,\mathrm{Fix}_C[\tau] = 2d+4 \geq 14$. But, Lemma 3.24 implies $\#W = \#(C \cap A[2])$, which can be at most 12 by Theorem 3.13. Therefore, $d \leq 4$ and hence $g_C = d+1 \in \{2, 3, 4, 5\}$. □

Now, we consider the second question in the beginning of this section- when does the curve C of Construction 3.3 cover an elliptic curve? Note that C itself cannot be elliptic, otherwise the abelian surface A will contain C as a factor in its isogeny class and this is not possible since A is simple. However, in the cases of cyclic covers of $(1, 3)$-polarization and the Klein cover of $(1, 4)$-polarization, we saw that C covers an elliptic curve. Below, we show that whenever d is a multiple of 2 or 3, C admits a cover onto an elliptic curve.

Theorem 3.26. *When $d = 2k$ (resp. $d = 3k$) and $X \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$ (resp. $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$), the curve C of Construction 3.3 covers an elliptic curve. Further, the degree of the cover is d (resp. $\frac{2d}{3}$) and J_C has an elliptic factor in its isogeny class.*

Proof. For $d = 2k$, consider two elements x and y in $K(L)$ of orders k and 2 respectively such that $X = \langle x, y\rangle$ is a subgroup of order $2k$ and C be the curve obtained from Construction 3.3 with this choice of the subgroup X. By the Riemann–Hurwitz formula, $\widetilde{C} := C/\langle t_x\rangle$ is of genus three. Further, $\widetilde{C}$ lies in the linear system of the $(1, 2)$-polarization of $A/\langle x\rangle$ by Proposition 2.57. From the figure below it is clear that $\widetilde{C}$ has a cyclic cover onto H induced by the cyclic isogeny $\pi_y : A/\langle x\rangle \longrightarrow A/\langle x, y\rangle$. With a similar argument as in Proposition 3.19, we can decompose the Jacobian of $\widetilde{C}$ using the group $D_4 \cong \langle -1, t_y\rangle$ to get the

$$
\begin{array}{ccccc}
C & \xrightarrow{\pi_x} & \widetilde{C} = C/\langle t_x \rangle & \xrightarrow{\pi_y} & H = \widetilde{C}/\langle t_y \rangle \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\pi_x} & A/\langle x \rangle & \xrightarrow{\pi_y} & A/X
\end{array}
$$

Figure 3.6

following isogeny relation for $J_{\widetilde{C}}$:

$$J_{\widetilde{C}} \sim A/\langle x \rangle \times J_{\widetilde{C}/\langle -1 \rangle} \times J_{\widetilde{C}/\langle -1 \circ t_y \rangle}.$$

Comparing dimensions of both sides, we will get that either $J_{\widetilde{C}/\langle -1 \rangle}$ or $J_{\widetilde{C}/\langle -1 \circ t_y \rangle}$ is elliptic and that C has a degree $2k$ cover onto that elliptic curve. This elliptic curve will be a factor in the isogeny class of J_C since the cover $C \longrightarrow \widetilde{C}$ of curves implies the isogeny relation $P(C/\widetilde{C}) \times J_{\widetilde{C}} \sim J_C$, where $P(C/\widetilde{C})$ denotes the Prym variety.

For $d = 3k$, we consider $x, y \in K(L)$ of orders k and 3 respectively such that $X = \langle x, y \rangle$ is of order $3k$. We proceed just like the previous case, where $\widetilde{C} := C/\langle t_x \rangle$ is now a curve of genus four in the simple $(1, 3)$-polarized abelian surface $A/\langle x \rangle$. Decomposing $J_{\widetilde{C}}$ with the cyclic cover $\pi_y : \widetilde{C} \longrightarrow H$ and the group $D_6 \cong \langle -1, t_y \rangle$ gives:

$$J_{\widetilde{C}} \sim A/\langle x \rangle \times J^2_{\widetilde{C}/\langle -1 \rangle}.$$

Comparing dimensions, we get that $\widetilde{C}/\langle -1 \rangle$ is an elliptic curve with a degree $2k$ cover from C and that J_C contains this elliptic factor in its isogeny class.

□

Chapter 4

Polarizations of higher degrees

In the previous chapter, we established the isogeny classes of Jacobians of curves lying in the linear system of $(1, d)$-polarization and arising from cyclic isogenies (Proposition 3.19). We gave a complete decomposition into simple factors for $d = 2, 3, 4$. We also decomposed the Jacobian of the curve coming from the Klein cover in the case of $(1, 4)$-polarization. In this chapter we shall give isogeny relations for Jacobians of curves coming from non-cyclic covers for other interesting cases of d. Our main tool will again be the result of Kani and Rosen (Theorem 2.71). The prime ingredient in this theorem, as we saw, is a subgroup of the automorphism group of the curve that admits a partition. The previous chapter had two main candidates for this subgroup: for the cyclic case we used $G = \langle t(X), -1\rangle \cong D_{2d}$ where X is a cyclic subgroup of $K(L)$ of order d, and for the Klein case we used the Klein-four group $G = \langle t_{2x}, t_{2y}\rangle = t(X)$ where $X = \langle 2x, 2y\rangle \leq \langle x, y\rangle = K(L) \cong (\mathbb{Z}/4\mathbb{Z})^2$. In this chapter, we will look at other polarizations of higher degrees that admit suitable subgroups of automorphism groups of curves coming from similar construction.

4.1 Automorphism groups other than the dihedral group

The aim of this section is to use the classification of abelian groups admitting a partition to obtain isogeny relation on Jacobians of curves arising from polarizations of higher type.

We are interested in simple abelian surfaces with $(1, d)$ polarizations. Given such an abelian surface A, Construction 3.3 gives a curve C of genus $d + 1$ lying in the linear system $|L|$ of the polarization L. The curve will vary depending on the choice of the order d subgroup $X \leq K(L)$ which gives the isogeny $\pi : A \longrightarrow A/X \cong J_H$ and hence determines the curve $C = \pi^{-1}(H)$. We also saw that $t_x, -1 \in \mathrm{Aut}(C)$ for all elements $x \in X$. For any d, if we take $X \leq K(L) \cong (\mathbb{Z}/d\mathbb{Z})^2$ to be the cyclic group of order d, then the resulting curve C will have the automorphism group $G = \langle t_x, -1\rangle \cong D_{2d}$. For $d = 2^2$, we also considered the case of X being the Klein four-group and $G = t(X)$ which led to a decomposition that was different from the cyclic case of $d = 4$. This motivates us to look at the cases of the next two squares of primes, i.e., $d = 3^2$ and $d = 5^2$ and generalize our findings.

$(1, 9)$-polarization

When C is the curve in construction Construction 3.3 arising from a $(1, 9)$-polarization with $X = \langle 3x, 3y\rangle \leq \langle x, y\rangle = K(L)$, then $G = \langle t_{3x},\ t_{3y}, -1\rangle$ is a subgroup of $\mathrm{Aut}(C)$. Moreover, G can be partition into four subgroups of order 3 and 9 subgroups of order two (and thus covering all elements of G):

$$\begin{aligned} G = \langle t_{3x}\rangle \cup \langle t_{3y}\rangle \cup \langle t_{3x+3y}\rangle \cup \langle t_{3x+6y}\rangle \cup \langle -1\rangle \cup \langle -1 \circ t_{3x}\rangle \cup \langle -1 \circ t_{6x}\rangle \cup \langle -1 \circ t_{3y}\rangle \\ \cup \langle -1 \circ t_{6y}\rangle \cup \langle -1 \circ t_{3x+3y}\rangle \cup \langle -1 \circ t_{6x+6y}\rangle \cup \langle -1 \circ t_{3x+6y}\rangle \cup \langle -1 \circ t_{6x+3y}\rangle \end{aligned}$$

Theorem 2.71 gives the isogeny relation:

$$\begin{aligned} J_C^{12} \times J_{C/G}^{18} \sim J^3_{C/\langle t_{3x}\rangle} \times J^3_{C/\langle t_{3y}\rangle} \times J^3_{C/\langle t_{3x+3y}\rangle} \times J^3_{C/\langle t_{3x+6y}\rangle} \times J^2_{C/\langle -1\rangle} \times J^2_{C/\langle -1\circ t_{3x}\rangle} \\ \times J^2_{C/\langle -1\circ t_{6x}\rangle} \times J^2_{C/\langle -1\circ t_{3y}\rangle} \times J^2_{C/\langle -1\circ t_{6y}\rangle} \times J^2_{C/\langle -1\circ t_{3x+3y}\rangle} \times J^2_{C/\langle -1\circ t_{6x+6y}\rangle} \\ \times J^2_{C/\langle -1\circ t_{3x+6y}\rangle} \times J^2_{C/\langle -1\circ t_{6x+3y}\rangle}. \end{aligned}$$

Since G is a group of order 18, all subgroups of G of order 2 are 2-Sylow and hence will be conjugates of $\langle -1\rangle$. Therefore, $J_{C/\langle -1\circ t_{3x}\rangle}, J_{C/\langle -1\circ t_{6x}\rangle}, J_{C/\langle -1\circ t_{3y}\rangle}, J_{C/\langle -1\circ t_{6y}\rangle}, J_{C/\langle -1\circ t_{3x+3y}\rangle}$, $J_{C/\langle -1\circ t_{6x+6y}\rangle}, J_{C/\langle -1\circ t_{3x+6y}\rangle}$ and $J_{C/\langle -1\circ t_{6x+3y}\rangle}$ are isomorphic to $J_{C/\langle -1\rangle}$. The isogeny relation now reduces to:

$$J_C^4 \sim J_{C/\langle t_{3x}\rangle} \times J_{C/\langle t_{3y}\rangle} \times J_{C/\langle t_{3x+3y}\rangle} \times J_{C/\langle t_{3x+6y}\rangle} \times J^6_{C/\langle -1\rangle}. \tag{4.1}$$

The curve $C/\langle t_{3x}\rangle$ has genus four by Riemann–Hurwitz formula. Moreover, it lies in the linear system of the pushed-forward $(1, 3)$-polarization of the simple abelian surface $A/\langle 3x\rangle$. Figure 4.1 shows that $C/\langle t_{3x}\rangle$ has a cyclic cover onto H induced by the cyclic isogeny π_{3y} :

$$\begin{array}{ccccc} C & \xrightarrow{\pi_{3x}} & C/\langle t_{3x}\rangle & \xrightarrow{\pi_{3y}} & H = C/\langle t_{3x}, t_{3y}\rangle \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{\pi_{3x}} & A/\langle 3x\rangle & \xrightarrow{\pi_{3y}} & A/X \end{array}$$

Figure 4.1

$A/\langle 3x\rangle \longrightarrow A/X$. As in the proof of Theorem 3.26, we get $J_{C/\langle t_{3x}\rangle} \sim A/\langle 3x\rangle \times E_1^2 \sim A \times E_1^2$

for an elliptic curve E_1. Similarly, we will have:

$$\begin{aligned} J_{C/\langle t_{3y}\rangle} &\sim A \times E_2^2, \\ J_{C/\langle t_{3x+3y}\rangle} &\sim A \times E_3^2, \\ J_{C/\langle t_{3x+6y}\rangle} &\sim A \times E_4^2, \end{aligned}$$

for elliptic curves E_2, E_3 and E_4. Thus, (4.1) simplifies to:

$$J_C^4 \sim A^4 \times E_1^2 \times E_2^2 \times E_3^2 \times E_4^2 \times J_{C/\langle -1\rangle}^6. \tag{4.2}$$

Since isogenies preserve dimensions and $\dim(J_C) = 10$, we must have $\dim(J_{C/\langle -1\rangle}) = 4$. Poincaré's reducibility will imply that the exponent of any simple factor in the right hand side of the isogeny relation must be a multiple of four. We will have the following cases based on the elliptic factors E_i:

Case 1: The elliptic factors E_i are mutually non-isogenous.
By Poincaré's reducibility, $J_{C/\langle -1\rangle}$ must be isogenous to $E_1 \times E_2 \times E_3 \times E_4$. Therefore, we get the complete decomposition of J_C into simple factors as:

$$J_C \sim A \times E_1^2 \times E_2^2 \times E_3^2 \times E_4^2.$$

Case 2: Exactly two elliptic factors are isogenous.
Suppose $E_1 \sim E_2$. Then the isogeny relation (4.2) reduces to

$$J_C^4 \sim A^4 \times E_1^4 \times E_3^2 \times E_4^2 \times J_{C/\langle -1\rangle}^6.$$

By Poincaré's reducibility, $J_{C/\langle -1\rangle} \sim E_1^2 \times E_3 \times E_4$ or $J_{C/\langle -1\rangle} \sim E_3 \times E_4 \times F^2$ where F is an elliptic curve that is not isogenous to E_i. Accordingly, we will have the isogeny relations:

$$J_{C/\langle -1\rangle} \sim E_1^2 \times E_3 \times E_4 \Longrightarrow J_C^4 \sim A^4 \times E_1^{16} \times E_3^8 \times E_4^8 \Longrightarrow J_C \sim A \times E_1^4 \times E_3^2 \times E_4^2$$

or

$$J_{C/\langle -1\rangle} \sim E_3 \times E_4 \times F^2 \Longrightarrow J_C^4 \sim A^4 \times E_1^4 \times E_3^8 \times E_4^8 \times F^{12} \Longrightarrow J_C \sim A \times E_1 \times E_3^2 \times E_4^2 \times F^3.$$

Case 3: Exactly three elliptic factors are isogenous.
Suppose $E_1 \sim E_2 \sim E_3$. Then (4.2) reduces to

$$J_C^4 \sim A^4 \times E_1^6 \times E_4^2 \times J_{C/\langle -1\rangle}^6.$$

Then $J_{C/\langle -1\rangle} \sim E_1 \times E_4 \times F^2$ for some elliptic curve F that is not isogenous to E_1 or E_4. This will give

$$J_C^4 \sim A^4 \times E_1^{12} \times E_4^8 \times F^{12} \Longrightarrow J_C \sim A \times E_1^3 \times E_4^2 \times F^3.$$

Case 4: $E_1 \sim E_2 \sim E_3 \sim E_4$.
The isogeny relation (4.2) reduces to:

$$J_C^4 \sim A^4 \times E_1^8 \times J_{C/\langle -1\rangle}^6.$$

Again by Poincaré's reducibility, $J_{C/\langle -1\rangle}$ cannot be simple. If it has an elliptic factor in its isogeny class, the power of that elliptic factor must be two or four. Accordingly, $J_{C/\langle -1\rangle} \sim E^4$ or $J_{C/\langle -1\rangle} \sim E^2 \times F^2$. Also, $J_{C/\langle -1\rangle}$ may not have any elliptic factor and instead may be a product of two simple abelian surfaces. In that case, the two abelian surfaces must be isogenous and if we denote them by B then we will have $J_{C/\langle -1\rangle} \sim B^2$. Thus, we will have the following isogeny relations:

$$\begin{aligned} J_{C/\langle -1\rangle} \sim B^2 &\Longrightarrow J_C^4 \sim A^4 \times E_1^8 \times B^{12} \Longrightarrow J_C \sim A \times E_1^2 \times B^3, \text{ or} \\ J_{C/\langle -1\rangle} \sim E^4 &\Longrightarrow J_C^4 \sim A^4 \times E_1^8 \times E^{24} \Longrightarrow J_C \sim A \times E_1^2 \times E^6, \text{ or} \\ J_{C/\langle -1\rangle} \sim E^2 \times F^2 &\Longrightarrow J_C^4 \sim A^4 \times E_1^8 \times E^{12} \times F^{12} \Longrightarrow J_C \sim A \times E_1^2 \times E^3 \times F^3. \end{aligned}$$

Note that, we may have the possibilities $E_1 \sim E$ or $E_1 \sim F$ or $A \sim B$, in which case these decompositions can be simplified further.

This exhausts all possibilities and we get decompositions of J_C into simple factors. However, a few of these can be ruled out. This is achieved with the help of the subgroup $t(X) = \langle t_{3x}, t_{3y}\rangle$ of $\mathrm{Aut}(C)$. From Construction 3.3 it is clear that $C/t(X) = H$. We have the partition

$$\langle t_{3x}, t_{3y}\rangle = \langle t_{3x}\rangle \cup \langle t_{3y}\rangle \cup \langle t_{3x+3y}\rangle \cup \langle t_{3x+6y}\rangle.$$

This gives the isogeny relation

$$J_C^3 \times J_{C/t(X)}^9 \sim J_{C/\langle t_{3x}\rangle}^3 \times J_{C/\langle t_{3y}\rangle}^3 \times J_{C/\langle t_{3x+3y}\rangle}^3 \times J_{C/\langle t_{3x+6y}\rangle}^3. \tag{4.3}$$

We saw above that each factor in the right hand side can be decomposed as the product of the abelian surface A and the square of an elliptic curve E_i. Further, using $J_{C/t(X)} = J_H \cong A/X \sim A$, the isogeny relation (4.3) reduces to:

$$J_C^3 \times A^9 \sim A^{12} \times E_1^6 \times E_2^6 \times E_3^6 \times E_4^6$$

which is equivalent to

$$J_C \sim A \times E_1^2 \times E_2^2 \times E_3^2 \times E_4^2.$$

Comparing this with (4.2), we get:

$$J_{C/\langle -1\rangle} \sim E_1 \times E_2 \times E_3 \times E_4.$$

This relation says that up to isogeny $J_{C/\langle -1\rangle}$ cannot have any simple factor other than the elliptic curves $E_1, \ldots, E_4$. This rules out some of the possibilities in the four different cases we saw above and thus reduces them to:

$$\begin{aligned}
&\text{Case 1:} \quad J_C \sim A \times E_1^2 \times E_2^2 \times E_3^2 \times E_4^2.\\
&\text{Case 2:} \quad J_C \sim A \times E_1^4 \times E_3^2 \times E_4^2.\\
&\text{Case 3:} \quad J_C \sim A \times E_1^6 \times E_4^2.\\
&\text{Case 4:} \quad J_C \sim A \times E_1^8.
\end{aligned}$$

How does this compare to J_C when C comes from the cyclic isogeny, i.e., when X is a cyclic group of order 9? By Proposition 3.19, we know that when X is the cyclic group of order 9 and C is the curve coming from the cyclic isogeny $\pi : A \longrightarrow A/X$, then $J_C \sim A \times J^2_{C/\langle -1\rangle}$. Unlike the non-cyclic case which we saw above, we do not have further information on how $J_{C/\langle -1\rangle}$ splits in the cyclic case. Although its dimension will still be 4 in the cyclic case, it may be a simple abelian variety which is not possible in the non-cyclic case as we saw above.

(1, 25)-polarization

In this case, $K(L) = \langle x, y\rangle \cong \mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$. Let X be the subgroup $\langle 5x, 5y\rangle \leq K(L)$ and C be the corresponding curve as in Construction 3.3. Then the group $G = \langle t_{5x}, t_{5y}\rangle$ is a subgroup of $\mathrm{Aut}(C)$ and has the partition:

$$G = \langle t_{5x}\rangle \cup \langle t_{5y}\rangle \cup \langle t_{5x+5y}\rangle \cup \langle t_{5x+10y}\rangle \cup \langle t_{5x+15y}\rangle \cup \langle t_{5x+20y}\rangle.$$

By Theorem 2.71 we have the isogeny relation:

$$J_C^5 \times J_{C/G}^{25} \sim J^5_{C/\langle t_{5x}\rangle} \times J^5_{C/\langle t_{5y}\rangle} \times J^5_{C/\langle t_{5x+5y}\rangle} \times J^5_{C/\langle t_{5x+15y}\rangle} \times J^5_{C/\langle t_{5x+20y}\rangle}. \tag{4.4}$$

Note, $J_{C/G} \sim A$. The curve $C/\langle t_{5x}\rangle$ is of genus six and it is in the linear system of the $(1, 5)$-polarization of the simple abelian surface $A/\langle 5x\rangle$. Figure 4.2 shows that $C/\langle t_{5x}\rangle$ has a cyclic cover onto H induced by the cyclic isogeny $\pi_{5y} : A/\langle 5x\rangle \longrightarrow A/X$. Using the cyclic isogeny and the the group $D_{10} \cong \langle -1, t_{5y}\rangle$ to decompose the Jacobian of $C/\langle t_{5x}\rangle$, we get $J_{C/\langle t_{5x}\rangle} \sim A/\langle 5x\rangle \times J^2_{H_1} \sim A \times J^2_{H_1}$ for a genus two hyperelliptic curve H_1. Similarly, we

$$
\begin{array}{ccccc}
C & \xrightarrow{\pi_{5x}} & C/\langle t_{5x}\rangle & \xrightarrow{\pi_{5y}} & H = C/\langle t_{5x}, t_{5y}\rangle \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\pi_{5x}} & A/\langle 5x\rangle & \xrightarrow{\pi_{5y}} & A/X
\end{array}
$$

Figure 4.2

will have:

$$
\begin{aligned}
J_{C/\langle t_{5y}\rangle} &\sim A \times J^2_{H_2}, \\
J_{C/\langle t_{5x+5y}\rangle} &\sim A \times J^2_{H_3}, \\
J_{C/\langle t_{5x+10y}\rangle} &\sim A \times J^2_{H_4}, \\
J_{C/\langle t_{5x+15y}\rangle} &\sim A \times J^2_{H_5}, \\
J_{C/\langle t_{5x+20y}\rangle} &\sim A \times J^2_{H_6},
\end{aligned}
$$

Where H_2, H_3, H_4, H_5 and H_6 are genus two hyperelliptic curves. Therefore, (4.4) reduces to :

$$J_C \sim A \times J^2_{H_1} \times J^2_{H_2} \times J^2_{H_3} \times J^2_{H_4} \times J^2_{H_5} \times J^2_{H_6}.$$

On the other hand, if X is the cyclic group of order 25 and C is the curve coming from the cyclic isogeny, then $J_C \sim A \times J^2_{C\langle -1\rangle}$ (Proposition 3.19), where $C/\langle -1\rangle$ is now a curve of genus 12. We do not know if $J_{C/\langle -1\rangle}$ will decompose further, unlike the non-cyclic case. Moreover, the decomposition in the non-cyclic case is a complete decomposition into simple factors in the very general case, as Jacobians of very general curves of genus two are simple abelian surfaces.

$(1, p^2)$- polarization

We saw that when the type of polarization is $(1, 4)$ or $(1, 9)$ or $(1, 25)$, then there are choices of X other than the cyclic groups which result in different curves whose automorphism groups again admit subgroups with partitions. Thus, we can get isogeny relations of Jacobians of more curves in such cases. The classification of abelian groups that admit partitions can be used to generalize this findings for polarizations of type $(1, p^2)$, where p is a prime. The key observation here is that for a polarization L on a simple abelian surface A of type $(1, p^2)$, $K(L) \cong \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z}$ has the subgroup $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, which is an elementary abelian p-group and hence has a partition. If x and y generate $K(L)$, then the group $\langle px, py\rangle$ is a suitable choice for X to get a curve C by Construction 3.3, whose Jacobian can be decomposed using our main tool.

Theorem 4.1. *Let d be the square of a prime p, and X be the subgroup $\langle px, py\rangle$ of $K(L) = \langle x, y\rangle \cong (\mathbb{Z}/p^2\mathbb{Z})^2$. If C is the curve arising from the construction with this choice of X, then we have the following isogeny relations of Jacobians:*

$$J_C \sim A \times \prod_{i=1}^{p+1} J_{C_i}^2$$
$$J_{C/\langle -1\rangle} \sim \prod_{i=1}^{p+1} J_{C_i},$$

where each C_i is a curve of genus $\frac{p-1}{2}$.

Proof. The subgroup $G = \langle t_{px}, t_{py}\rangle \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ of $\mathrm{Aut}(C)$ has the partition:

$$G = \langle t_{px}\rangle \cup \langle t_{py}\rangle \cup \langle t_{px+py}\rangle \cup \langle t_{px+2py}\rangle \cup \cdots \cup \langle t_{px+(p-1)py}\rangle.$$

Theorem 2.71 then gives the isogeny relation:

$$J_C^p \times J_{C/G}^{p^2} \sim J_{C/\langle t_{px}\rangle}^p \times J_{C/\langle t_{py}\rangle}^p \times J_{C/\langle t_{px+py}\rangle}^p \times J_{C/\langle t_{px+2py}\rangle}^p \times \cdots \times J_{C/\langle t_{px+(p-1)py}\rangle}^p. \quad (4.5)$$

Note, $J_{C/G} \sim A$. The curve $C/\langle t_{px}\rangle$ is of genus $p+1$ and it is in the linear system of the $(1,p)$-polarization (pushforward of $(1,p^2)$-polarization) on the simple abelian surface $A/\langle px\rangle$. Figure 4.3 shows that $C/\langle t_{px}\rangle$ has a cyclic cover onto H induced by the cyclic

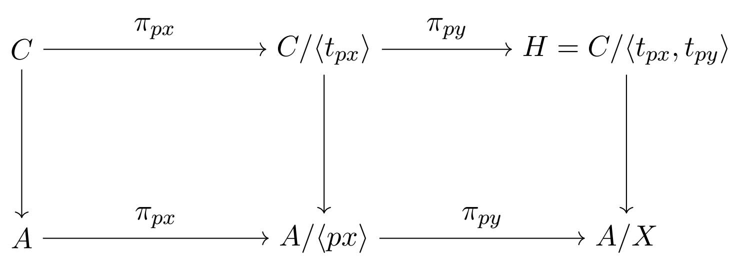

Figure 4.3

isogeny $\pi_{py} : A/\langle px\rangle \longrightarrow A/X$. Using this isogeny and the group $D_{2p} \cong \langle -1, t_{py}\rangle$ to decompose the Jacobian of $C/\langle t_{px}\rangle$, we get $J_{C/\langle t_{px}\rangle} \sim A/\langle px\rangle \times J_{C_1}^2 \sim A \times J_{C_1}^2$ where $C_1 = C/\langle t_{px}, -1\rangle$ is of genus $\frac{p-1}{2}$. Similarly, we will have:

$$J_{C/\langle t_{py}\rangle} \sim A \times J_{C_2}^2,$$
$$J_{C/\langle t_{px+py}\rangle} \sim A \times J_{C_3}^2,$$
$$J_{C/\langle t_{px+2py}\rangle} \sim A \times J_{C_4}^2,$$
$$\vdots$$
$$J_{C/\langle t_{px+(p-1)py}\rangle} \sim A \times J_{C_{p+1}}^2,$$

where $C_2 := C/\langle t_{py}, -1\rangle$ and $C_i := C/\langle t_{px+(i-2)py}\rangle$ for $3 \leq i \leq p+1$. Note that $g_{C_i} = \frac{p-1}{2}$. Substituting these in (4.5), we get:

$$\begin{aligned}
&J_C^p \times A^{p^2} \sim (A \times J_{C_1}^2)^p \times (A \times J_{C_2}^2)^p \times (A \times J_{C_3}^2)^p \times \cdots \times (A \times J_{C_{p+1}}^2)^p \\
\text{i.e.,}\quad &J_C^p \times A^{p^2} \sim A^{p(p+1)} \times J_{C_1}^{2p} \times J_{C_2}^{2p} \times J_{C_3}^{2p} \times \cdots \times J_{C_{p+1}}^{2p} \\
\text{i.e.,}\quad &J_C^p \sim A^p \times J_{C_1}^{2p} \times J_{C_2}^{2p} \times J_{C_3}^{2p} \times \cdots \times J_{C_{p+1}}^{2p} \\
\text{i.e.,}\quad &J_C \sim A \times J_{C_1}^2 \times J_{C_2}^2 \times J_{C_3}^2 \times \cdots \times J_{C_{p+1}}^2 \\
\text{i.e.,}\quad &J_C = A \times \prod_{i=1}^{p+1} J_{C_i}^2.
\end{aligned}$$

To get the second isogeny relation in the statement, let us consider the subgroup $G = \langle t_{px}, t_{py}, -1\rangle$ of $\mathrm{Aut}(C)$ and decompose it as:

$$G = \bigcup_{m,n=0}^{p-1} \langle -1 \circ t_{mpx+npy}\rangle \bigcup \langle t_{px}, t_{py}\rangle$$

This gives the isogeny relation:

$$J_C^{p^2} \times J_{C/G}^{2p^2} \sim J_{C/\langle t_{px}, t_{py}\rangle}^{p^2} \times \prod_{m,n=0}^{p-1} J_{C/\langle -1\circ t_{mpx+npy}\rangle}^2 \tag{4.6}$$

Now, $C/\langle t_{px}, t_{py}\rangle = H$ and hence $C/G \cong \mathbb{P}^1$. So, $J_{C/\langle t_{px}, t_{py}\rangle} \sim A$ and $J_{C/G}$ is trivial. Also, all order two subgroups of G are Sylow subgroups and are conjugate to $\langle -1\rangle$. Therefore, the isogeny relation (4.6) now reduces to:

$$J_C^{p^2} \sim A^{p^2} \times J_{C/\langle -1\rangle}^{2p^2}.$$

Using $J_C \sim A \times \prod_{i=1}^{p+1} J_{C_i}^2$, which we proved above, we get:

$$\begin{aligned}
\left(A \times \prod_{i=1}^{p+1} J_{C_i}^2\right)^{p^2} &\sim A^{p^2} \times J_{C/\langle -1\rangle}^{2p^2} \\
\text{i.e.,}\quad J_{C/\langle -1\rangle} &\sim \prod_{i=1}^{p+1} J_{C_i}.
\end{aligned}$$

This finishes the proof. □

4.2 Future directions:

Throughout this project, our main tool for decomposing the Jacobians was the powerful theorem by Kani and Rosen. More subgroups of $\mathrm{Aut}(C)$ with partitions will give more isogeny relations of J_C, taking us closer to determining the complete decomposition of J_C into simple factors. Hence it will be interesting to know more about the group $\mathrm{Aut}(C)$ and get complete decomposition of J_C into simple factors. For example, for the curve C coming from the cyclic isogeny in the case of $(1, d)$-polarization (Proposition 3.19), one can try to get further decompositions of $J_{C/\langle -1 \rangle}$.

Conversely, we can look at what information about the group $\mathrm{Aut}(C)$ is captured by the isogeny class of J_C. These are some interesting questions to ponder. Moreover, we can imitate Construction 3.3 to obtain curves in the linear systems of polarizations that are not primitive, i.e., polarizations of type (d_1, d_2) where $d_1 > 1$. It will be interesting to look at the automorphism groups of such curves and to know if they have subgroups with partitions which will enable us to apply the theorem of Kani and Rosen to decompose their Jacobians. Another interesting problem will be to determine the eigenspace of $[-1]^* : |L| \longrightarrow |L|$ where the constructed curve lies and to investigate if there is any relation between the eigenspace containing the curve and its Jacobian decomposition.

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