Strong Orthogonal Arrays of Strength 2+ with Better Two-Dimensional Projection Properties

by

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Abstract

Space-filling properties are favored in designs of computer experiments. Strong orthogonal arrays of strength 2+ are a class of space-filling designs that guarantee 4×2 and 2×4 two-dimensional space-filling properties. However, the patterns of the two-dimensional projections can be very different with some being notably better than the others. In a strong orthogonal array of strength 2+, we would like to have more of better patterns among the two factor projections. The objective of this study is to identify strong orthogonal arrays of strength 2+ with better two-dimensional projection properties, utilizing two selection criteria. We use second order saturated designs to construct strong orthogonal arrays of strength 2+ and evaluate all available second order saturated designs to find good designs with 16 runs. Designs are identified for the number of factors, $m = 6, 7, 8, 9$, and 10, according to both selection criteria. The study is extended to 32 runs, using 3 out of 12 second order saturated designs to identify designs with better two-dimensional projection properties for $m = 10$ to 21 under both selection criteria.

Keywords: Space-filling designs; Strong orthogonal arrays; Two-dimensional projection properties; Selection criteria; Second order saturated designs

Dedication

To my parents, my sister, and my loving husband. Your unwavering support has made all of this possible. I couldn't have done it without you.

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Chapter 1

Introduction

Design of experiments is a branch of applied statistics focusing on collecting and analyzing data to assess how a set of factors influences a variable of interest, known as the response variable. Factorial designs are a commonly used type of designs. We can study multiple factors, each with specified levels, using a factorial design. Factors in such experiments produce main effects, which quantify their average impact on the response variable across other factors. Additionally, interaction effects occur when a factor's influence on the response variable is affected by other factors under investigation. For a factorial experiment with m factors of 2 levels, denoted by ± 1, there are a total of $2^m - 1$ effects. For example, if a design involves 4 factors, we require at least 16 experimental runs to explore all possible treatment combinations ($2 \times 2 \times 2 \times 2 = 16$). This allows us to study all 15 effects ($2^4 - 1 = 15$) arising from these factors.

As the number of factors increases, the number of experimental runs required of a full factorial design grows exponentially, making it impractical in many real-world scenarios. To address this challenge, fractional factorial designs were introduced. These designs use a fraction of the runs required by a full factorial to study a large number of factors effectively.

With the rapid increase in computer processing power, computer experiments are increasingly replacing conventional physical experiments. For instance, computer experiments are often preferred when the physical system is expensive, the system is difficult to operate, there are safety concerns, or the computer model enables faster exploration.

The response of a computer experiment is often deterministic [7], meaning that it will consistently produce the same output given the same initial input conditions. Consequently, the model we fit to the data from a computer experiment must account for the minimal to nonexistent noise in the response. The absence of random error in computer experiments allows for the exploration of a much larger experimental region than in classical designs.

While changing factors can be challenging in physical experiments, computer experiments easily accommodate factors with a large number of levels.

In classical experimental designs, the three main principles, namely randomization, replication, and blocking are crucial when choosing a design. However, these principles are irrelevant for deterministic models because the response will always be identical for the same factor combination. To effectively use computer experiments, we need space-filling designs.

Space-filling designs are preferred in computer experiments because they ensure that the design points are evenly distributed across the entire design region. Because of their evenly spread-out points, space-filling designs have the potential to capture diverse response behaviors across various regions within the design space. Moreover, in high-dimensional spaces, space-filling designs are particularly valuable because they efficiently distribute points across multiple dimensions without requiring an impractical number of experimental runs.

Strong orthogonal arrays of strength 2+ offer appealing space-filling properties across pairs of factors. In this study, our primary objective is to identify strong orthogonal arrays of strength 2+ that exhibit better two-dimensional projection properties.

In Chapter 2, we briefly introduce fractional factorial designs and orthogonal arrays. Moving to Chapter 3, our focus shifts to strong orthogonal arrays of strength 2+ and their construction, where we explore various patterns observed in two-factor projections within these designs. We then outline the objectives of our study. Chapter 4 delves into the algorithmic implementations undertaken to achieve these objectives, presenting detailed results and emphasizing key findings. Finally, we conclude with a summary of our research and some remarks about possible future work.

Chapter 2

Fractional factorial designs and orthogonal arrays

Consider a design with five factors, each having two levels. To estimate all the effects using a full factorial design, 32 runs are required. Following are the effects which can be studied using this design:

Effect Type		**Number of Effects**
Average effect		1
Main effect		5
Interactions	Two factors	10
	Three factors	10
	Four factors	5
	Five factors	1

Table 2.1: Effects studied in a five-factor full factorial design.

In practical applications, when a large number of factors are studied in an experiment, higher order interactions are usually insignificant [2]. Experimenters typically focus on estimating main effects and some two-factor interactions. Hence, higher order interactions are often not of primary interest, making it unnecessary to conduct all 32 runs to estimate all the effects in Table 2.1. This is where fractional factorial designs become useful.

2.1 Regular fractional factorial designs

An experimental design with a fraction of 2^m runs in a factorial experiment is considered a fractional factorial design. These designs are beneficial for studying the effects of a large number of factors on a response variable with a relatively small number of experimental runs. A regular fractional factorial design, denoted as a 2^{m-p} design, is a $(\frac{1}{2})^p$ fraction of a 2^m design, which uses only 2^{m-p} runs. For example, instead of using 32 runs to study five factors, we can employ a 2^4 full factorial design to study 4 factors and the fifth factor can be studied by confounding it with a higher order interaction. This approach is known as a half-fractional factorial design of a five-factor two-level experiment, denoted as a 2^{5-1} design. By using this design, we only need 16 runs instead of 32 to study all five factors.

Let us examine two more examples where we can effectively study 4 factors and 7 factors using just 8 experimental runs. Note that, we use $\pm$ instead of ± 1 for convenience.

a	**b**	**c**	**d = abc**
+	+	+	+
+	+	−	−
+	−	+	−
+	−	−	+
−	+	+	−
−	+	−	+
−	−	+	+
−	−	−	−

Table 2.2: A 2^{4-1} fractional factorial design

A factorial design with $2^3 = 8$ runs is employed to study three factors. If one wants to study 4 factors using 2^3 factorial design, the *abc* interaction can be used to accommodate the factor *d*. In this design, the main effect of *d* cannot be distinguished from the *abc* interaction because the effects *d* and *abc* are "confounded". They cannot be separately estimated and *abc* is called the "alias" of *d*.

In general, regular fractional factorial designs are constructed using generators. The fractional factorial design in Table 2.2 is a 2^{4-1} design and it is obtained from the following generator

$$d = abc. \tag{2.1}$$

If you multiply the signs of the elements in any column by the signs of those same elements, the result will be a column of ones which is denoted by I, known as identity. Hence, we can say that $a \times a = I$, $b \times b = I$, $c \times c = I$. By multiplying both sides of the generator in equation (2.1) by d, we obtain that

$$d \times d = d^2 = abcd.$$

This will give the defining relation of the design

$$I = abcd,$$

which contains a four letter word *abcd*.

a	**b**	**c**	**d = ab**	**e = ac**	**f = bc**	**g = abc**
+	+	+	+	+	+	+
+	+	−	+	−	−	−
+	−	+	−	+	−	−
+	−	−	−	−	+	+
−	+	+	−	−	+	−
−	+	−	−	+	−	+
−	−	+	+	−	−	+
−	−	−	+	+	+	−

Table 2.3: A 2^{7-4} fractional factorial design

The design in Table 2.3 can be utilized to study 7 factors and it is a 2^{7-4} fractional factorial design. In this design, we use d=ab, $e = ac$, $f = bc$, $g = abc$ as generators. The defining relation for this design starts with

$$I = abd = ace = bcf = abcg. \tag{2.2}$$

In order to complete the defining relation we must add all words which can be created by multiplying the four generators in equation (2.2) in all possible ways. For example, $abd \times ace = a^2bcde = Ibcde = bcde$. The complete defining relation is given by

$$
\begin{aligned}
I &= abd = ace = bcf = abcg = bcde = acdf = cdg = abef \\
&= beg = afg = def = adeg = bdfg = cefg = abcdefg.
\end{aligned}
$$

In regular fractional factorial designs, the effects are either orthogonal or fully confounded, making it impossible to estimate all effects simultaneously, unlike in full factorial designs.

2.2 Orthogonal arrays

Orthogonal arrays (OAs) are a class of fractional factorial designs. A formal definition for OAs with two levels in each factor is, given in, [6] as follows:

Definition 1. *A factorial design with n runs for m factors, each at two levels, is an orthogonal array of strength t if any submatrix with t columns contains all 2^t level combinations with equal frequency. This type of array is denoted by OA($n, m, 2, t$). For a design to be an OA($n, m, 2, t$), n must be a multiple of 2^t, meaning that $n = \lambda 2^t$ for some positive integer λ.*

Let us look at a simple example of an OA:

$$
\begin{pmatrix}
+ & + & + \\
+ & - & - \\
- & + & - \\
- & - & +
\end{pmatrix}.
$$

In this OA, each column represents a factor, and each row represents an experimental run. The two levels are assigned to each factor in a way that every possible combination of levels for any 2 columns (factors) appears equally often. For example, looking at columns 1 and 2, we see that the combinations $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$ each appear once. This is a strength 2 OA, and we use OA(4, 3, 2, 2) to denote this array.

An OA with 8 runs and 4 factors is given below. In this array, each 3-tuple appears with the same frequency in the rows, and the run size n is a multiple of $2^3 = 8$. Thus, the design has strength three and is denoted by OA(8, 4, 2, 3). If a column is removed, the remaining design retains its status as an orthogonal array of strength three.

$$\begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & + & - \\ + & - & - & + \\ - & + & + & - \\ - & + & - & + \\ - & - & + & + \\ - & - & - & - \end{pmatrix}$$

Definition 1 implies that, if a design is an OA of strength t, it must also qualify as an OA of any strength $t' < t$. For example, the above array is not only a strength 3 OA but also has strength 2. In every two columns, each combination of pair of levels appears twice.

The array given below is an orthogonal array with mixed levels, where factors can have different numbers of levels. In this array, the first factor has 4 levels denoted by $0, 1, 2$ and 3, while the remaining two factors have two levels each, denoted by 0 and 1.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

In the orthogonal array given above, all possible combinations of levels in any two columns appear with the same frequency. For instance, each combination of levels in the second and third factors occurs twice, while the combinations in the first and second factors appear once. Therefore, this is a mixed-level orthogonal array of strength 2, which we denote by $\text{OA}(8, 3, 4 \times 2 \times 2, 2)$.

Chapter 3

Finding better strong orthogonal arrays of strength 2+

3.1 Strong orthogonal arrays

Before delving into a formal definition of strong orthogonal arrays (SOAs), let's consider the example below to gather insights so as to better understand the concepts behind SOAs. Consider the following array:

$$\begin{pmatrix} 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 5 & 5 & 4 & 1 & 4 & 0 & 1 \\ 5 & 4 & 1 & 4 & 1 & 4 & 1 \\ 7 & 6 & 2 & 2 & 2 & 3 & 7 \\ 4 & 1 & 5 & 4 & 0 & 1 & 5 \\ 6 & 3 & 6 & 2 & 3 & 6 & 3 \\ 6 & 2 & 3 & 7 & 6 & 2 & 3 \\ 4 & 0 & 0 & 1 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 3 & 6 & 3 & 7 & 6 \\ 2 & 3 & 6 & 3 & 6 & 3 & 6 \\ 0 & 1 & 5 & 5 & 5 & 4 & 0 \\ 3 & 6 & 2 & 3 & 7 & 6 & 2 \\ 1 & 4 & 1 & 5 & 4 & 1 & 4 \\ 1 & 5 & 4 & 0 & 1 & 5 & 4 \\ 3 & 7 & 7 & 6 & 2 & 2 & 2 \end{pmatrix}.$$

The array has 16 runs and 7 factors, each having 8 levels denoted by {0, 1, 2, 3, 4, 5, 6, 7}. It possesses the following interesting properties:

1. If the above array with 8 levels is collapsed into an array with 2 levels using the mapping:

$$\left[\frac{a}{4}\right] = \begin{cases} 0, & \text{if } a = 0, 1, 2, 3, \\ 1, & \text{if } a = 4, 5, 6, 7, \end{cases}$$

where $[x]$ denotes the largest integer not exceeding x, the resulting array becomes an orthogonal array with 2 levels and strength 3, which is an OA(16, 7, 2, 3).

2. If you consider any subarray of two columns and collapse one factor into 2 levels using $[a/4]$, and the other factor into 4 levels using $[a/2]$, the array becomes an orthogonal array with strength 2. This is an OA(16, 2, 2 $\times$ 4, 2) or an OA(16, 2, 4$\times$ 2, 2).

3. Any subarray with one column is an OA(16, 1, 8, 1).

Definition 2. *An $n \times m$ matrix with entries from {0, 1, ..., $s^t - 1$} is called an SOA of n runs, m factors, s^t levels, and strength t if any subarray of g columns for any g with $1 \leq g \leq t$ can be collapsed into an OA$(n, g, s^{u_1} \times \cdots \times s^{u_g}, g)$ for any positive integers $u_1, \ldots, u_g$ with $u_1 + \cdots + u_g = t$. Here, collapsing s^t levels into s^{u_j} levels is done according to $[a/s^{t-u_j}]$ for $a = 0, 1, \ldots, s^t - 1$. We use SOA$(n, m, s^t, t)$ to denote such an array. As an SOA(n, m, s^t, t) can be collapsed into an OA(n, m, s, t), it must have $n = \lambda s^t$ for some integer λ([5]).*

According to Definition 2, the array described above is an SOA$(16, 7, 8, 3)$.

Definition 2 implies that an SOA$(n, m, s^2, 2)$, a strong orthogonal array of strength 2 with s^2 levels in each factor, collapses into an OA$(n, m, s, 2)$ when the s^2 levels are transformed into s levels using the $[a/s]$ transformation. For instance, applying this to an SOA of strength 2 with 4 levels in each factor, SOA$(n, m, 4, 2)$ collapses into an OA$(n, m, 2, 2)$. Therefore, in any two-dimensions, an SOA$(n, m, s^2, 2)$ achieves a stratification on an s $\times$ s grid just as an OA$(n, m, s, 2)$ does. This implies that an SOA$(n, m, s^2, 2)$ fills space to the same extent as an OA$(n, m, s, 2)$ does in two-dimensions, but not better.

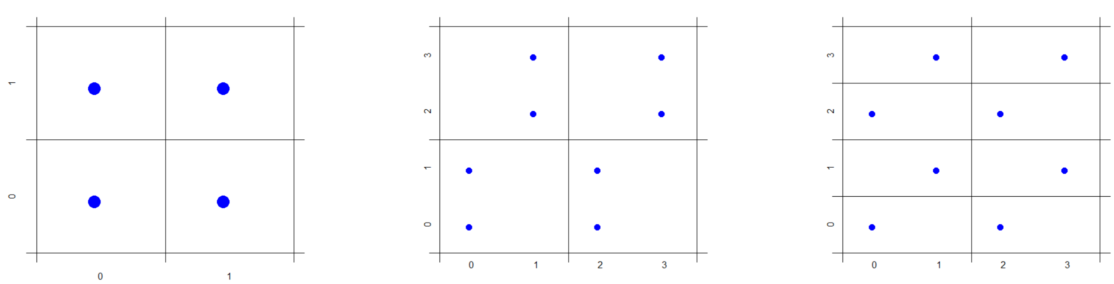
SOAs with strength $t \geq 3$ exhibit better space-filling characteristics compared to comparable OAs. For instance, as we discussed earlier in the example, an SOA$(n, m, 2^3, 3)$ achieves stratifications on $2^2 \times 2$ and 2×2^2 grids in two-dimensions and $2 \times 2 \times 2$ grids in three-dimensions. On the other hand, while an OA$(n, m, 2, 3)$ achieves stratifications on $2 \times 2 \times 2$ grids in three-dimensions, it only ensures stratifications on 2×2 grids in two-dimensions. The better space-filling properties in two-dimensions provided by an SOA$(n, m, s^3, 3)$ is evident.

While the space-filling properties of SOAs with strength 3 are highly attractive, it's worth noting that these designs can become quite costly due to the substantial run sizes needed for various scenarios. In order to address this issue, a novel class of arrays, SOAs of strength 2+ has been introduced in [4]. To put it briefly, SOAs of strength 2+ are SOAs of strength 2 that possess two-dimensional space-filling properties found in SOAs of strength 3.

Let us understand the stratification properties of OAs, SOAs of strength 2, and SOAs of strength 2+ that were discussed earlier, using the following three examples.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 2 \\ 1 & 3 \\ 2 & 0 \\ 2 & 1 \\ 3 & 2 \\ 3 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 1 \\ 1 & 3 \\ 2 & 0 \\ 2 & 2 \\ 3 & 1 \\ 3 & 3 \end{pmatrix}$$

An OA(8, 2, 2, 2) | An SOA(8, 2, 4, 2) | An SOA(8, 2, 4, 2+)



(a) OA(8, 2, 2, 2) (b) SOA(8, 2, 4, 2) (c) SOA(8, 2, 4, 2+)

Figure 3.1: Stratification properties of OAs, SOAs of strength 2 and SOAs of strength 2+

Each of the arrays given above has 2 factors, with all having 4 levels per factor except for the OA. Both the OA(8, 2, 2, 2) and the SOA(8, 2, 4, 2) achieve stratification on a 2×2 grid as shown in Figures 3.1(a) and 3.1(b). Notably, the SOA(8, 2, 4, 2+) achieves a stratification on a 2×4 grid as shown in Figure 3.1(c). It also achieves a stratification on a 4×2 grid. Note that in Figure 3.1(a), the points are relatively bigger compared to the other two figures. This is done to indicate that there are two points on each plotted point, demonstrating that there are 8 runs.

Definition 3. *An SOA of strength 2+ with n runs and m factors of s^2 levels is an $n \times m$ matrix with entries from $\{0, 1, .., s^2 - 1\}$, if any subarray of two columns can be collapsed into an $OA(n, 2, s^2 \times s, 2)$ and an $OA(n, 2, s \times s^2, 2)$. We denote this array by $SOA(n, m, s^2, 2+)$.*

According to Definition 3, an SOA$(n, m, s^2, 2+)$ surpasses an SOA$(n, m, s^2, 2)$ by offering better two-dimensional space-filling properties. While an SOA$(n, m, s^2, 2)$ only guarantees stratifications on $s \times s$ grids in two-dimensions, an SOA$(n, m, s^2, 2+)$ achieves stratifications on both $s^2 \times s$ and $s \times s^2$ grids in two-dimensions. Let us look at an SOA of strength 2+ with 16 runs.

$$\begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 2 \\
2 & 0 & 2 & 0 & 2 & 1 & 2 & 1 & 2 & 2 \\
2 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 \\
0 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\
0 & 2 & 0 & 1 & 3 & 0 & 3 & 0 & 3 & 0 \\
0 & 0 & 2 & 3 & 1 & 1 & 0 & 3 & 3 & 0 \\
0 & 0 & 0 & 3 & 3 & 3 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
1 & 1 & 3 & 2 & 0 & 0 & 3 & 0 & 0 & 3 \\
1 & 3 & 1 & 0 & 2 & 1 & 0 & 3 & 0 & 3 \\
1 & 3 & 3 & 0 & 0 & 3 & 1 & 1 & 2 & 1 \\
3 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 3 & 3 \\
3 & 1 & 3 & 1 & 3 & 0 & 1 & 2 & 1 & 1 \\
3 & 3 & 1 & 3 & 1 & 1 & 2 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3
\end{pmatrix}$$

In this design, any two columns can be collapsed into an OA(16, 2, 2 $\times$ 4, 2) and an OA(16, 2, 4 $\times$ 2, 2). Although the SOA of strength 3 introduced earlier in this section shares the same two-dimensional space-filling property, it is limited to studying 7 factors with 16 runs. In contrast, the SOA of strength 2+ provided in this example can study up to 10 factors. Hence, SOAs of strength 2+ offer economic advantages over SOAs of strength 3. As per [4], Table 3.1 summarizes the maximum number of factors that can be studied in SOAs of strength 3 and 2+.

n	Strength 3	Strength 2+
16	7	10
32	15	22
64	31	50
128	63	≥106
256	127	≥226

Table 3.1: Maximum number of factors studied in SOAs of strength 3 and 2+, where n is the number of runs.

3.2 Construction of strong orthogonal arrays of strength 2+

This section describes the construction of SOAs of strength 2+ using regular 2^{m-p} fractional factorial designs. Specifically, we construct an SOA$(2^k, m, 4, 2+)$, by utilizing designs A and B based on the relationship detailed in the following Lemma, which is Proposition 1 in [4].

Lemma 1. *An SOA$(n, m, s^2, 2+)$, say D, exists if and only if there exist two arrays A and B where $A = (a_1, ..., a_m)$ is an OA$(n, m, s, 2)$ and $B = (b_1, ..., b_m)$ is an OA$(n, m, s, 1)$ such that (a_j, a_k, b_k) is an OA of strength 3 for any $j \neq k$. The three arrays are linked through $D = sA + B$.*

According to Lemma 1, to construct an SOA of strength 2+ with 4 levels in each factor, denoted as SOA$(n, m, 4, 2+)$, the designs A and B should be orthogonal arrays with strengths 2 and 1, respectively. It is important to notice that, this approach constructs SOAs of strength 2+ with 4 levels using OAs with 2 levels in each factor. Furthermore, when selecting any two columns from A and one column from B, where the selected column from B corresponds to either the first or the second column in A, the design created by these three columns should form an OA with strength 3. This requirement ensures that the three columns are independent. The relationship between the three arrays is given by $D = 2A + B$.

Since the levels in SOAs are denoted as $\{0, 1, 2, \ldots, s^t - 1\}$, the levels in the corresponding orthogonal arrays should also be represented using 0 and 1. Regular designs with two levels are often studied using two levels denoted as -1 and $+1$. If A and B have levels ± 1, they can be equivalently represented using levels 0 and 1, respectively, by transforming matrices via $(A+1)/2$ and $(B+1)/2$, where for instance, $(A+1)$ denotes the resulting matrix after adding 1 to every element of A. Therefore, when constructing D from A and B with levels ± 1, we will adjust the relationship $D = 2A + B$ as follows:

$$
\begin{aligned}
D &= 2\frac{(A+1)}{2} + \frac{(B+1)}{2} \\
&= A + \frac{B}{2} + \frac{3}{2}.
\end{aligned}
\tag{3.1}
$$

Now that we understand how to create an SOA of strength 2+ given designs A and B, our focus should shift to the method for constructing A and B. Columns of A and B are selected from a saturated regular design. A saturated design, S, with $n = 2^k$ runs and $m = n - 1$ factors is obtained by first constructing a full factorial design with k factors, and then incorporating every possible interaction column. For instance, the design in Table 2.3 is a saturated design derived from a full factorial design involving 3 factors, resulting in $n = 2^3 = 8$ experimental runs. The saturated design includes $m = n - 1 = 8 - 1 = 7$ factors in total. In other words, it is termed as a saturated design because it cannot be an orthogonal array if any additional factor is introduced. However, if any factor is removed, it remains as an orthogonal array.

Let C be a subset of m columns in S. The set of columns that are not included in C forms the complementary design, and is denoted by $\bar{C} = S \backslash C$. In [1], second-order saturated (SOS) designs were introduced to describe designs in which all degrees of freedom can be utilized to estimate either main effects or two-factor interaction terms. Here, a design C is an SOS design if any $d \in \bar{C}$ can be expressed as $d = ab$ for some $a, b \in C$. According to [4], an SOA$(2^k, m, 4, 2+)$ can be constructed using an SOS design, C, and below are the steps:

1. Take $A = \bar{C}$. Write $A = (a_1, ..., a_m)$.

2. Since C is an SOS design, we must have $a_j = b_j b'_j$ for some $b_j, b'_j \in C$. Take $B = (b_1, ..., b_m)$.

3. Obtain D, an SOA$(2^k, m, 4, 2+)$, using equation (3.1).

We will now examine an example to demonstrate how these three steps can be used to construct D. Consider the SOS design, $C = \{a, b, c, d, abcd\}$, for 16 runs. The saturated design S of 16 runs has $m = 2^4 - 1 = 15$ columns and can be written as

$$S = \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}.$$

Then, the complementary design of C has 10 factors and is given by

$$\bar{C} = \{ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd\}.$$

According to step 1, the above design $\bar{C}$ will allow us to construct an SOA$(16, 10, 4, 2+)$, an SOA of strength 2+ with 10 factors. However, if we aim to construct an SOA$(16, 8, 4, 2+)$, an SOA of strength 2+ with 8 factors, we need to remove two factors from $\bar{C}$ to create a design A with 8 factors. There are $\binom{10}{2} = 45$ ways of doing this. Thus, there are 45 different options for selecting an A with 8 factors. If we opt for

$$A = \{ab, ac, ad, bc, cd, abd, acd, bcd\},$$

our C will be updated as

$$C = \{a, b, c, d, bd, abc, abcd\}.$$

According to Step 2, each term in A can be expressed as a product of two terms in C. There may exist multiple pairs of factors within C that produce each term in A. All possible pairs for the above design A are listed in Table 3.2.

Term No. in A	Term	Pairs of factors
1	ab	(a, b), (c, abc)
2	ac	(a, c), (b, abc), (bd, abcd)
3	ad	(a, d)
4	bc	(b, c), (a, abc)
5	cd	(c, d)
6	abd	(a, bd), (c, abcd)
7	acd	(bd, abc), (b, abcd)
8	bcd	(c, bd), (a, abcd)

Table 3.2: Pairs of factors selected from design C to construct design B

Any factor that appears in a product can be used to construct B. There are

$$4 \times 6 \times 2 \times 4 \times 2 \times 4 \times 4 \times 4 = 24,576$$

choices for B and one of them is given by

$$B = \{a, a, a, b, c, abcd, abcd, abcd\}.$$

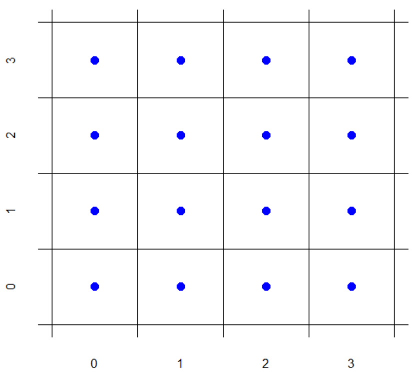
Finally, by substituting the designs A and B obtained above into equation (3.1), we can construct an SOA(16, 8, 4, 2+). The resulting design D is presented in Table 3.3.

factor 1	factor 2	factor 3	factor 4	factor 5	factor 6	factor 7	factor 8
3	3	3	3	3	3	3	3
3	3	1	3	1	0	0	0
3	1	3	1	0	2	0	0
3	1	1	1	2	1	3	3
1	3	3	0	3	0	2	0
1	3	1	0	1	3	1	3
1	1	3	2	0	1	1	3
1	1	1	2	2	2	2	0
0	0	0	3	3	0	0	2
0	0	2	3	1	3	3	1
0	2	0	1	0	1	3	1
0	2	2	1	2	2	0	2
2	0	0	0	3	3	1	1
2	0	2	0	1	0	2	2
2	2	0	2	0	2	2	2
2	2	2	2	2	1	1	1

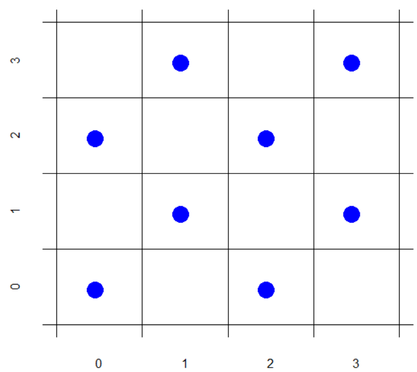
Table 3.3: An SOA(16, 8, 4, 2+)

3.3 Two-factor projection patterns of strong orthogonal arrays of strength 2+

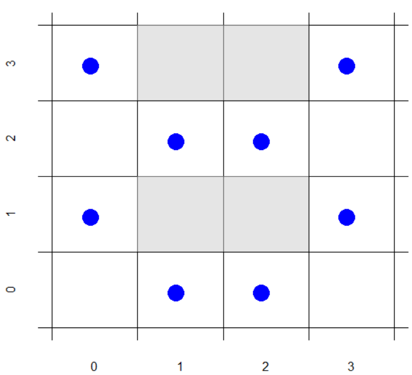
Strong orthogonal arrays of strength 2+ are a class of designs that guarantee stratifications on 4×2 and 2×4 grids in two-dimensional projections. However, the space-filling patterns among the projections can differ. For instance, if we examine the two-factor projections of factors 1 & 5, 1 & 2, 1 & 4, and 1 & 7 in D given in Table 3.3, we will observe four different patterns, as shown in Figures 3.2(a), 3.2(b), 3.2(c), and 3.2(d).



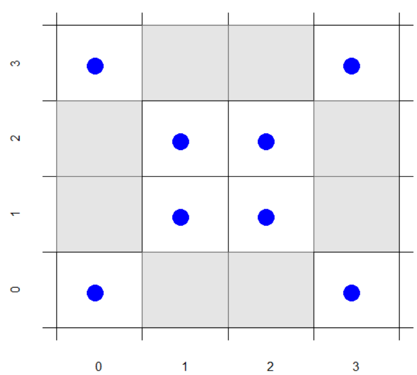
(a) Pattern 1



(b) Pattern 2



(c) Pattern 3



(d) Pattern 4

Figure 3.2: Different patterns of two-factor projections in SOAs of strength 2+

Note that in Figures 3.2(b), 3.2(c), and 3.2(d), the points are relatively bigger compared to the points in Figure 3.2(a). This is done to indicate that there are two replicates coinciding at each plotted point, demonstrating that there are 16 runs in each pattern.

The reason for the difference in the patterns of two-factor projections is due to the different combinations of factors in designs A and B that were used to create D. Let us explore this further using a few examples. Tables 3.4 and 3.5 below show the columns in designs A and B used in the example discussed in Section 3.2.

A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
ab	ac	ad	bc	cd	abd	acd	bcd

Table 3.4: Columns in design A

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
a	a	a	b	c	$abcd$	$abcd$	$abcd$

Table 3.5: Columns in design B

First, we look at Figure 3.2(a), which represents the pattern corresponding to the two-dimensional projection of factors 1 and 5 in D. Let us investigate the four columns corresponding to these factors in A and B.

A_1	A_5
ab	cd

B_1	B_5
a	c

The four columns ab, cd, a, and c are independent, as there is no relation among them. Therefore, the pattern corresponding to two-dimensional projection of the two factors in the design D forms a 4×4 space-filling design as shown in Figure 3.2(a). This pattern is analogous to what we observe in an orthogonal array with 4 levels.

Similarly, we will examine the columns in designs A and B corresponding to the factors 1 & 2 in design D. It can be observed that among the four columns A_1 and A_2 in Table 3.4 and B_1 and B_2 in Table 3.5, there exists a word of length two, $B_1 \times B_2 = a \times a = I$. This results in a distinct pattern as shown in Figure 3.2(b).

In addition, the distinct patterns 3 and 4 in Figures 3.2(c) and 3.2(d) respectively, correspond to the two-dimensional projections of factors 1 & 4 and 1 & 7. The differences in these patterns arise because the corresponding columns in A and B form words of length 3 and 4, respectively, among them. Specifically, $A_1 \times B_1 \times B_4 = ab \times a \times b = I$ and $A_1 \times A_7 \times B_1 \times B_7 = ab \times acd \times a \times abcd = I$, respectively, explain these distinct patterns.

In summary, there are four distinct space-filling patterns in the two-factor projections in SOAs of strength 2+. Pattern 1 exhibits a 4×4 space-filling property, while the other three patterns exhibit 4×2 and 2×4 space-filling properties. The reason for these different patterns is that the corresponding columns in A and B form words of varying lengths. Specifically, Patterns 1, 2, 3, and 4 correspond to all columns being independent, and having words of length 2, 3, and 4, respectively.

3.4 Objective of the study and selection criteria

As discussed in Chapter 1, space-filling properties are favored in designs of computer experiments. Figures 3.2(a), 3.2(b), 3.2(c) and 3.2(d) display various patterns, with some being notably better than the others. So, in an SOA of strength 2+, we would like to have more of better patterns among the two-factor projections. Pattern 1 is the best because it exhibits 4×4 space-filling properties and the points are well spread out in the grid. This makes Pattern 1 the most desirable.

After Pattern 1, Pattern 2 is the next most desirable among the two-factor projections because it does not have large empty regions. The points in Pattern 2 are spread out across the grid more evenly, not as well as in Pattern 1, but significantly better than those in Patterns 3 and 4.

In both Patterns 3 and 4, there are relatively large empty regions (see the shaded regions). However, Pattern 4 has four large empty regions along the edges of the grid, whereas Pattern 3 has only two large empty regions. Therefore, Pattern 3 is preferred over Pattern 4 for two-factor projections due to its more favorable point distribution.

With the above consideration in mind, if we let f_1, f_2, f_3, and f_4 denote the frequencies of Patterns 1, 2, 3, and 4, respectively, we would like to have most occurrences of Pattern 1, followed by Pattern 2, Pattern 3, and Pattern 4 among the two-factor projections for an SOA of strength 2+. In this study our objective is to find good strong orthogonal arrays of strength 2+ with better two-dimensional projection properties, and to do this we employ two selection criteria.

Criterion 1: Sequentially maximize (f_1, f_2, f_3, f_4).

Criterion 2: Sequentially minimize (f_4, f_3, f_2, f_1).

We use Criterion 1 to illustrate. It involves selecting designs by sequentially maximizing f_1, f_2, f_3, and f_4. We first select the designs with the highest f_1 values, from which we select the designs with the highest f_2 values. We then continue this process for f_3 and f_4.

Chapter 4

Algorithmic implementation and results

This chapter first explains the algorithmic approach used to find the best SOAs of strength 2+ for 16 runs and then extends the approach to 32 runs.

4.1 Designs of 16 runs

The construction of SOAs of strength 2+ is based on SOS designs, as detailed in Section 3.2. (See page 13 for a definition of SOS designs). Thus, we should consider all possible SOS designs to identify the best designs. According to [3], there are four minimal SOS designs for 16 runs. A minimal SOS design is one that cannot be reduced to an SOS design by removing any factors. For instance, $C = \{a, b, c, d, abcd\}$ is an SOS design. If we remove any factor from C, it will no longer be an SOS design. For example, if we remove factor a from C, a will move to $\bar{C}$, the complementary design of C. There is no product of pair of factors in C that can produce a in $\bar{C}$. Hence, C is a minimal SOS design. However, if you add any factor to an SOS design, it will be an SOS design.

In our study, we used all the four minimal SOS designs to find good SOAs of strength 2+ with better two-dimensional projection properties. Below are the four SOS designs:

- $\text{SOS}_1(n{=}16)$ with $m = 8 : C = \{a, b, c, d, ab, ac, bc, abc\}$
- $\text{SOS}_2(n{=}16)$ with $m = 8 : C = \{d, ad, bd, cd, abd, acd, bcd, abcd\}$
- $\text{SOS}_3(n{=}16)$ with $m = 5 : C = \{a, b, c, d, abcd\}$
- $\text{SOS}_4(n{=}16)$ with $m = 6 : C = \{a, b, c, d, ab, cd\}$

Remark 1. *Orthogonal arrays with 4 levels for 16 runs exist for up to 5 factors ([6]) and exhibit* 4×4 *two-dimensional projection properties. Since OAs have better projection properties compared to SOAs of strength 2+, we focus our exploration on designs for more than five factors.*

The total number of factors we can study using a factorial design with 16 runs is 15. Since $\text{SOS}_1(n{=}16)$ and $\text{SOS}_2(n{=}16)$ each have 8 factors, the complementary designs $\bar{C}$ of these designs have $15 - 8 = 7$ factors. Since design $A = \bar{C}$ is an orthogonal array, it will remain an orthogonal array even if we remove some factors. This allows us to construct SOAs of strength 2+ containing 7 factors or fewer. However, our focus will be on constructing designs with 6 or more factors (Remark 1). Therefore, we will explore SOAs of strength 2+ with $m = 6, 7$ from both $\text{SOS}_1(n{=}16)$ and $\text{SOS}_2(n{=}16)$. Similarly, using $\text{SOS}_3(n{=}16)$ and $\text{SOS}_4(n{=}16)$, we can study designs with $m = 6, 7, 8, 9, 10$ factors and $m = 6, 7, 8, 9$ factors, respectively.

Note that, according to Table 3.1, SOAs of strength 3 for 16 runs exist for $m = 7$, and they exhibit the same two-dimensional projection properties as SOAs of strength 2+. However, the former offers better space-filling properties than the latter due to their $2 \times 2 \times 2$ projection properties onto three factors. Since our goal is to find SOAs of strength 2+ with better two-dimensional projection properties, we will also consider $m = 6$ and $m = 7$ when identifying the best designs.

We need to find the set of frequencies (f_1, f_2, f_3, f_4) for every SOA of strength 2+ that we could construct using the four minimal SOS designs in order to find the best ones according to the two criteria introduced in Section 3.4. For example, if we consider the design D, an SOA of strength 2+, in Table 3.3, we have to look at all possible two-factor projections of D and compute the frequencies f_1, f_2, f_3 and f_4. In other words, we will examine each and every two-factor projection in D. For each pair of factors selected from D, we will then determine the length of the word formed among the corresponding columns in designs A and B.

When we look at the two-factor projection of the first two factors of D, we can see that the two columns A_1 and A_2 in A (Table 3.4) and two columns B_1 and B_2 in B (Table 3.5) contains a word of length 2 ($B_1 \times B_2 = a \times a = I$) among them. Hence the two-factor projection of first two factors of D will exhibit Pattern 2. Similarly, if we look at the two-factor projection of the second and fourth factor of D, we can see that A_2, A_4, B_2 and B_4 form a word of length four ($A_2 \times A_3 \times B_3 \times B_3 = ac \times bc \times a \times b = I$).

There are $\binom{8}{2} = 28$ two-factor projections in D and Table 4.1 shows the breakdown of the frequencies of those 28 two-factor projections.

Length of the word	Two-factor projections	Frequency
Independent	1 & 5, 1 & 6, 2 & 7, 3 & 4, 3 & 6, 3 & 7, 4 & 8, 5 & 7, 5 & 8	9
Two	1 & 2, 1 & 3, 2 & 3, 6 & 7, 6 & 8, 7 & 8	6
Three	1 & 5, 1 & 8, 2 & 5, 2 & 8, 3 & 8, 4 & 5, 4 & 7, 5 & 6	8
Four	1 & 7, 2 & 4, 2 & 6, 3 & 5, 4 & 6	5

Table 4.1: Length of the words formed among the adjacent factors in designs A and B

Hence the frequencies of observing Pattern 1, 2, 3 and 4 among the two-factor projection in D are $f_1 = 9, f_2 = 6, f_3 = 8$ and $f_4 = 5$ respectively.

In this manner, for each SOS and a given number of factors, we first find the set of frequencies for each SOA of strength 2+ and then find the best one among them based on Criteria 1 and 2 introduced in Section 3.4. Ideally, we would prefer to examine all possible combinations of As and Bs from these four SOSs and compute the frequency sets. However due to computational constraints, we considered all As but only a randomly selected subset of Bs for each A.

Here are the steps involved in finding the best designs for each minimal SOS design for a given number of factors:

1. Consider a design A.
2. Randomly select a subset of 1000 Bs that can be constructed using the design A.
3. Find the set of frequencies (f_1, f_2, f_3, f_4) corresponding to each SOA of strength 2+ that can be constructed using A and B.
4. Repeat steps 2 and 3 for all possible As that can be constructed using the given minimal SOS design.
5. Select the SOA design that performs best according to Criteria 1 or 2.

Since we are randomly selecting a subset of Bs in step 2, we ran this algorithm 5 times. It's reassuring to see that we obtained the same result in each run.

We will now summarize all the findings. Table 4.2 presents the frequency sets associated with good SOAs of strength 2+ that exhibit better two-dimensional projection properties achievable using $SOS_1(n{=}16)$. For instance, among the configurations examined, there exists a design comprising 6 factors where 8 out of its $\binom{6}{2} = 15$ two-factor projections have Pattern 1, and 7 have Pattern 2. Notably, none of these projections exhibit Patterns 3 or 4. The best SOA of strength 2+ identified according to Criterion 2 is the same as the one found according to Criterion 1, in terms of their sets of frequencies.

m	Criterion 1				Criterion 2			
	f_1	f_2	f_3	f_4	f_4	f_3	f_2	f_1
6	8	7	0	0	0	0	7	8
7	0	21	0	0	0	0	21	0

Table 4.2: Frequency sets for best SOAs of strength 2+ constructed using $SOS_1(n{=}16)$

Table 4.3 displays the frequency sets associated with the best SOAs of strength 2+ constructed using $SOS_2(n{=}16)$. It highlights that designs derived from $SOS_2(n{=}16)$ outperform those from $SOS_1(n{=}16)$, according to Criterion 1. For example, for $m = 7$, the best SOA of strength 2+ constructed using $SOS_1(n{=}16)$, according to Criterion 1, exhibits all of its $\binom{7}{2} = 21$ two-factor projections showing Pattern 2. Recall that this pattern exhibit 4×2 and 2×4 space-filling properties. However, for the same criterion, the best design constructed using $SOS_2(n{=}16)$, shows 14 of its two-dimensional projections with Pattern 1, the most favorable pattern with 4×4 space-filling properties.

m	Criterion 1				Criterion 2			
	f_1	f_2	f_3	f_4	f_4	f_3	f_2	f_1
6	11	2	2	0	0	0	7	8
7	14	4	2	1	0	3	6	12

Table 4.3: Frequency sets for best SOAs of strength 2+ constructed using $SOS_2(n{=}16)$

Tables 4.4 and 4.5 show the frequency sets associated with the best SOAs of strength 2+ constructed using $SOS_3(n{=}16)$ and $SOS_4(n{=}16)$, respectively. Although, in a few cases, the configurations provided by both SOSs are the same, $SOS_3(n{=}16)$ generally outperforms $SOS_4(n{=}16)$ and the other two SOS designs. This is because the better designs derived from $SOS_3(n=16)$ include more two-factor projections that exhibit Pattern 1, which is the preferred pattern in space-filling designs.

m	Criterion 1				Criterion 2			
	f_1	f_2	f_3	f_4	f_4	f_3	f_2	f_1
6	12	3	0	0	0	0	3	12
7	15	3	1	2	0	2	11	8
8	16	7	2	3	1	8	3	16
9	18	9	0	9	3	10	7	16
10	15	10	10	10	5	20	5	15

Table 4.4: Frequency sets for best SOAs of strength 2+ constructed using $SOS_3(n=16)$

m	Criterion 1				Criterion 2			
	f_1	f_2	f_3	f_4	f_4	f_3	f_2	f_1
6	12	3	0	0	0	0	3	12
7	15	3	1	2	0	2	11	8
8	14	8	4	2	1	8	5	14
9	18	9	0	9	3	12	3	18

Table 4.5: Frequency sets for best SOAs of strength 2+ constructed using $SOS_4(n=16)$

Table 4.6 shows the frequency sets corresponding to the best SOAs of strength 2+ found from all four SOS designs based on Criterion 1. The designs A and B used to construct these designs are provided alongside their respective frequencies. Notably, $SOS_3(n=16)$ can construct best designs for all $m = 6, 7, 8, 9$, and 10.

m	f_1	f_2	f_3	f_4	A	B
6	12	3	0	0	$\{ab,\ ac,\ ad,\ bc,\ bd,\ abc\}$	$\{cd,\ d,\ d,\ c,\ c,\ cd\}$
7	15	3	1	2	$\{ab,\ ac,\ ad,\ bc,\ bd,\ abd,\ bcd\}$	$\{abcd,\ d,\ acd,\ abc,\ d,\ abc,\ abcd\}$
8	16	7	2	3	$\{ac,\ bc,\ bd,\ cd,\ abc,\ abd,\ acd,\ bcd\}$	$\{c,\ abcd,\ ab,\ c,\ ab,\ ab,\ abcd,\ abcd\}$
9	18	9	0	9	$\{ab,\ ac,\ ad,\ bc,\ bd,\ cd,\ abc,\ abd,\ acd\}$	$\{b,\ c,\ d,\ d,\ c,\ b,\ d,\ c,\ b\}$
10	15	10	10	10	$\{ab,\ ac,\ ad,\ bc,\ bd,\ cd,\ abc,\ abd,\ acd,\ bcd\}$	$\{a,\ a,\ a,\ b,\ b,\ c,\ d,\ c,\ b,\ a\}$

Table 4.6: Frequency sets, designs A and B corresponding to best SOAs of strength 2+ for 16 runs based on Criterion 1

The good SOAs of strength 2+ that are constructed using designs A and B in Table 4.6 through equation (3.1) are provided below.

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 0 & 3 & 1 & 2 \\
2 & 1 & 3 & 0 & 2 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 \\
1 & 3 & 3 & 1 & 1 & 1 \\
0 & 2 & 0 & 1 & 3 & 0 \\
0 & 1 & 3 & 2 & 0 & 2 \\
1 & 0 & 0 & 2 & 2 & 3 \\
1 & 1 & 1 & 3 & 3 & 1 \\
0 & 0 & 2 & 3 & 1 & 0 \\
0 & 3 & 1 & 0 & 2 & 2 \\
1 & 2 & 2 & 0 & 0 & 3 \\
3 & 1 & 1 & 1 & 1 & 3 \\
2 & 0 & 2 & 1 & 3 & 2 \\
2 & 3 & 1 & 2 & 0 & 0 \\
3 & 2 & 2 & 2 & 2 & 1
\end{pmatrix}$$

The best SOA$(16, 6, 4, 2+)$ according to Criterion 1

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 0 & 3 & 0 & 1 & 0 \\
2 & 1 & 2 & 0 & 3 & 2 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 3 \\
0 & 3 & 3 & 0 & 1 & 0 & 0 \\
1 & 2 & 0 & 0 & 2 & 2 & 3 \\
1 & 1 & 2 & 3 & 1 & 1 & 3 \\
0 & 0 & 1 & 3 & 2 & 3 & 0 \\
0 & 1 & 0 & 2 & 3 & 0 & 2 \\
1 & 0 & 3 & 2 & 0 & 2 & 1 \\
1 & 3 & 1 & 1 & 3 & 1 & 1 \\
0 & 2 & 2 & 1 & 0 & 3 & 2 \\
3 & 1 & 0 & 1 & 1 & 3 & 1 \\
2 & 0 & 3 & 1 & 2 & 1 & 2 \\
2 & 3 & 1 & 2 & 1 & 2 & 2 \\
3 & 2 & 2 & 2 & 2 & 0 & 1
\end{pmatrix}$$

The best SOA $(16, 7, 4, 2+)$ according to Criterion 1

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 2 & 1 & 1 & 3 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 & 1 & 3 & 0 & 0 \\
0 & 1 & 1 & 2 & 1 & 1 & 3 & 3 \\
3 & 0 & 0 & 3 & 0 & 0 & 2 & 0 \\
3 & 1 & 2 & 1 & 0 & 2 & 1 & 3 \\
0 & 3 & 0 & 0 & 2 & 0 & 1 & 3 \\
0 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\
1 & 2 & 2 & 3 & 0 & 0 & 0 & 2 \\
1 & 3 & 0 & 1 & 0 & 2 & 3 & 1 \\
2 & 1 & 2 & 0 & 2 & 0 & 3 & 1 \\
2 & 0 & 0 & 2 & 2 & 2 & 0 & 2 \\
1 & 1 & 1 & 3 & 3 & 3 & 1 & 1 \\
1 & 0 & 3 & 1 & 3 & 1 & 2 & 2 \\
2 & 2 & 1 & 0 & 1 & 3 & 2 & 2 \\
2 & 3 & 3 & 2 & 1 & 1 & 1 & 1
\end{pmatrix}$$

The best SOA$(16, 8, 4, 2+)$ according to Criterion 1

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 0 & 2 & 1 & 1 & 2 & 1 & 1 \\
3 & 0 & 3 & 1 & 2 & 1 & 1 & 2 & 1 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 \\
0 & 3 & 3 & 1 & 1 & 2 & 1 & 1 & 2 \\
0 & 3 & 0 & 0 & 3 & 0 & 0 & 3 & 0 \\
0 & 0 & 3 & 3 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 3 & 3 & 3 & 1 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 1 & 0 & 3 & 3 \\
1 & 2 & 1 & 1 & 2 & 1 & 3 & 0 & 3 \\
1 & 2 & 2 & 0 & 0 & 3 & 2 & 2 & 1 \\
2 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 0 \\
2 & 1 & 2 & 0 & 3 & 0 & 2 & 1 & 2 \\
2 & 2 & 1 & 3 & 0 & 0 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0
\end{pmatrix}$$

The best SOA$(16, 9, 4, 2+)$ according to Criterion 1

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 1 & 3 & 1 & 1 & 2 & 1 & 1 & 1 \\
3 & 1 & 3 & 1 & 3 & 0 & 1 & 2 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 3 & 3 \\
1 & 3 & 3 & 0 & 0 & 3 & 1 & 1 & 2 & 1 \\
1 & 3 & 1 & 0 & 2 & 1 & 0 & 3 & 0 & 3 \\
1 & 1 & 3 & 2 & 0 & 0 & 3 & 0 & 0 & 3 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
0 & 0 & 0 & 3 & 3 & 3 & 1 & 1 & 1 & 2 \\
0 & 0 & 2 & 3 & 1 & 1 & 0 & 3 & 3 & 0 \\
0 & 2 & 0 & 1 & 3 & 0 & 3 & 0 & 3 & 0 \\
0 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\
2 & 0 & 0 & 0 & 0 & 3 & 3 & 3 & 0 & 0 \\
2 & 0 & 2 & 0 & 2 & 1 & 2 & 1 & 2 & 2 \\
2 & 2 & 0 & 2 & 0 & 0 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The best SOA$(16, 10, 4, 2+)$ according to Criterion 1

Table 4.7 shows the frequency sets corresponding to the best SOAs of strength 2+ found from all four SOS designs based on Criterion 2. Except for $m = 7$, all the best designs can be constructed using $\mathrm{SOS}_3(n{=}16)$. However, the array with $m = 7$, given by $\mathrm{SOS}_3(n{=}16)$, is also interesting because 8 of the two-factor projections have Pattern 1.

m	f_4	f_3	f_2	f_1	A	B
6	0	0	3	12	$\{ab,\ ad,\ bd,\ abd,\ acd,\ bcd\}$	$\{abc,\ abcd,\ d,\ abc,\ d,\ abcd\}$
7	0	0	21	0	$\{ad,\ bd,\ cd,\ abd,\ acd,\ bcd,\ abcd\}$	$\{d,\ d,\ d,\ d,\ d,\ d,\ d\}$
8	1	8	3	16	$\{ab,\ ac,\ ad,\ bc,\ bd,\ cd,\ abd,\ acd\}$	$\{b,\ b,\ d,\ d,\ bcd,\ c,\ c,\ abcd\}$
9	3	10	7	16	$\{ab,\ ac,\ ad,\ bc,\ bd,\ cd,\ abc,\ abd,\ acd\}$	$\{a,\ c,\ a,\ b,\ c,\ b,\ abcd,\ c,\ b\}$
10	5	20	5	15	$\{ab,\ ac,\ ad,\ bc,\ bd,\ cd,\ abc,\ abd,\ acd,\ bcd\}$	$\{a,\ c,\ d,\ b,\ d,\ c,\ abcd,\ abcd,\ b,\ a\}$

Table 4.7: Frequency sets, designs A and B corresponding to best SOAs of strength 2+ for 16 runs based on Criterion 2

The SOAs of strength 2+ that are constructed using designs A and B in Table 4.7 are provided in below.

$$\begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 & 2 & 3 \\ 0 & 2 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 2 & 0 & 3 \\ 1 & 3 & 1 & 1 & 1 & 3 \\ 1 & 0 & 2 & 3 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 & 2 \\ 0 & 3 & 0 & 2 & 2 & 1 \\ 1 & 1 & 3 & 1 & 3 & 1 \\ 1 & 2 & 0 & 3 & 0 & 2 \\ 3 & 1 & 1 & 3 & 1 & 1 \\ 3 & 2 & 2 & 1 & 2 & 2 \\ 2 & 0 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{pmatrix}$$

The best SOA$(16, 6, 4, 2+)$ according to Criterion 2

$$\begin{pmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 3 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 & 2 & 2 \\ 3 & 1 & 3 & 1 & 3 & 1 & 1 \\ 0 & 2 & 0 & 2 & 0 & 2 & 2 \\ 3 & 1 & 1 & 1 & 1 & 3 & 3 \\ 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 1 & 3 & 3 & 1 & 1 & 3 & 1 \\ 2 & 0 & 0 & 2 & 2 & 0 & 2 \\ 1 & 3 & 1 & 1 & 3 & 1 & 3 \\ 2 & 0 & 2 & 2 & 0 & 2 & 0 \\ 1 & 1 & 3 & 3 & 1 & 1 & 3 \\ 2 & 2 & 0 & 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 3 & 3 & 3 & 1 \\ 2 & 2 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}$$

The best SOA$(16, 7, 4, 2+)$ according to Criterion 2

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 0 & 2 & 0 & 1 & 1 & 0 \\
3 & 1 & 3 & 1 & 2 & 0 & 2 & 0 \\
3 & 1 & 0 & 0 & 1 & 2 & 0 & 3 \\
0 & 2 & 3 & 1 & 0 & 3 & 1 & 2 \\
0 & 2 & 0 & 0 & 3 & 1 & 3 & 1 \\
0 & 0 & 3 & 3 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 3 & 3 & 3 & 1 & 0 \\
1 & 1 & 2 & 2 & 0 & 1 & 3 & 3 \\
1 & 3 & 1 & 1 & 2 & 0 & 0 & 3 \\
1 & 3 & 2 & 0 & 1 & 2 & 2 & 0 \\
2 & 0 & 1 & 1 & 0 & 3 & 3 & 1 \\
2 & 0 & 2 & 0 & 3 & 1 & 1 & 2 \\
2 & 2 & 1 & 3 & 1 & 0 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 0 & 1
\end{pmatrix}$$

The best SOA$(16, 8, 4, 2+)$ according to Criterion 2

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 1 & 3 & 1 & 1 & 2 & 1 & 1 \\
3 & 0 & 3 & 1 & 2 & 1 & 0 & 2 & 1 \\
3 & 0 & 1 & 1 & 0 & 3 & 1 & 0 & 3 \\
1 & 3 & 3 & 0 & 1 & 2 & 0 & 1 & 2 \\
1 & 3 & 1 & 0 & 3 & 0 & 1 & 3 & 0 \\
1 & 0 & 3 & 2 & 0 & 0 & 3 & 0 & 0 \\
1 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 0 & 3 & 3 & 3 & 0 & 1 & 1 \\
0 & 1 & 2 & 3 & 1 & 1 & 1 & 3 & 3 \\
0 & 2 & 0 & 1 & 2 & 1 & 3 & 0 & 3 \\
0 & 2 & 2 & 1 & 0 & 3 & 2 & 2 & 1 \\
2 & 1 & 0 & 0 & 1 & 2 & 3 & 3 & 0 \\
2 & 1 & 2 & 0 & 3 & 0 & 2 & 1 & 2 \\
2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 & 0
\end{pmatrix}$$

The best SOA$(16, 9, 4, 2+)$ according to Criterion 2

$$\begin{pmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 0 & 3 & 0 & 1 & 2 & 0 & 1 & 1 \\
3 & 0 & 3 & 1 & 3 & 0 & 0 & 2 & 1 & 1 \\
3 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 3 & 3 \\
1 & 3 & 3 & 0 & 1 & 3 & 0 & 0 & 2 & 1 \\
1 & 3 & 0 & 0 & 2 & 1 & 1 & 3 & 0 & 3 \\
1 & 0 & 3 & 2 & 1 & 0 & 3 & 1 & 0 & 3 \\
1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
0 & 1 & 1 & 3 & 3 & 3 & 0 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 & 0 & 1 & 1 & 3 & 3 & 0 \\
0 & 2 & 1 & 1 & 3 & 0 & 3 & 1 & 3 & 0 \\
0 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 1 & 2 \\
2 & 1 & 1 & 0 & 1 & 3 & 3 & 3 & 0 & 0 \\
2 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 2 \\
2 & 2 & 1 & 2 & 1 & 0 & 0 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 0 & 0
\end{pmatrix}$$

The best SOA$(16, 10, 4, 2+)$ according to Criterion 2

Note that although the good SOAs of strength 2+ found for $m = 6$ from both Criterion 1 and Criterion 2 have the same set of frequencies, the two designs obtained from the algorithm may differ from each other. This is because there can be multiple designs with the same frequency set corresponding to the best designs, and we have reported one of them.

4.2 Designs of 32 runs

There are 12 minimal SOS designs for 32 runs, but due to computational limitations, we only considered 3 of them.

- $\mathrm{SOS}_1(n{=}32)$ with $m = 10 : C = \{a, b, c, d, abcd, ae, be, ce, de, abcde\}$
- $\mathrm{SOS}_2(n{=}32)$ with $m = 16 : C = \{a, b, c, d, e, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}$
- $\mathrm{SOS}_3(n{=}32)$ with $m = 16 : C = \{e, ae, be, ce, de, abe, ace, ade, bce, bde, cde, abce, abde, acde, bcde, abcde\}$

Remark 2. *Orthogonal arrays with 4 levels for 32 runs exist for up to 9 factors [6]. Therefore, for 32 runs, we only need to study SOAs of strength 2+ with 10 or more factors.*

We can study 31 factors using a factorial design with 32 runs. Since $\mathrm{SOS}_1(n{=}32)$ has 10 factors, we can construct SOAs of strength 2+ with 32 runs for up to $31-10 = 21$ factors. In view of Remark 2, we only need to study designs with $m = 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,$ 20 and 21 factors. Additionally, since both $\mathrm{SOS}_2(n{=}32)$ and $\mathrm{SOS}_3(n{=}32)$ have 16 factors, we can study designs with $m = 10, 11, 12, 13, 14,$ and 15 factors.

According to Table 3.1, SOAs of strength 3 for 32 runs exist for $m = 15$ or less and they offer better space-filling properties than SOAs of strength 2+ due to their three-dimensional projection properties. Since our goal is to find designs of strength 2+ with better two-dimensional projection properties, we also considered $m = 10, 11, 12, 13, 14$ and 15 when identifying the best ones.

Similar to the case with 16 runs, we aimed to examine all possible combinations of As and Bs from these three SOS designs. However, due to increased computational burden caused by the larger run size, we only considered a subset of As and Bs. For 32 runs, in the steps outlined in the previous section, we did not consider all possible As that could be created for a given m and an SOS design. Instead, we randomly selected a subset of 50 As in each of the 5 times the algorithm is run to ensure that a sufficient number of designs is captured.

Next, we will examine the results summarized in Tables 4.8, 4.9 and 4.10 for good SOAs of strength 2+ with 32 runs, which were constructed using $\mathrm{SOS}_1(n{=}32)$, $\mathrm{SOS}_2(n{=}32)$, and $\mathrm{SOS}_3(n{=}32)$ respectively.

m	Criterion 1				Criterion 2			
	f_1	f_2	f_3	f_4	f_4	f_3	f_2	f_1
10	43	2	0	0	0	0	2	43
11	51	2	2	0	0	2	2	51
12	60	2	2	2	0	3	4	59
13	70	2	3	3	0	5	7	66
14	79	4	6	2	1	9	3	78
15	90	6	4	5	1	13	10	81
16	99	10	6	5	2	12	14	92
17	110	7	14	5	3	20	9	104
18	119	9	17	8	5	21	19	108
19	128	15	16	12	7	25	18	121
20	136	13	27	14	9	37	13	131
21	142	16	36	16	12	45	19	134

Table 4.8: Frequency sets for best SOAs of strength 2+ constructed using $\text{SOS}_1(n{=}32)$

Based on Criterion 1, $\text{SOS}_1(n{=}32)$ provides designs with better two-dimensional projection properties in all scenarios, except when $m = 12$, for which $\text{SOS}_3(n{=}32)$ produces the best one. However, the difference is minute even in this case. For $m = 11$, the frequency sets corresponding to the best designs are identical for both $\text{SOS}_1(n{=}32)$ and $\text{SOS}_2(n{=}32)$. Thus, both designs have the potential to yield the best SOA of strength 2+ in this scenario.

For Criterion 2, in $\text{SOS}_1(n{=}32)$, we observe that for $m = 10$ and $m = 11$, the frequency sets corresponding to the best SOAs of strength 2+ are identical to those found under Criterion 1. For $m = 11$ and $m = 14$, $\text{SOS}_3(n{=}32)$ produces better designs compared to other SOS designs. For $m = 15$, $\text{SOS}_2(n{=}32)$ yields the best design, although the designs from the other SOSs are also noteworthy because more of their two-factor projections exhibit Pattern 1.

m	Criterion 1				Criterion 2			
	f_1	f_2	f_3	f_4	f_4	f_3	f_2	f_1
10	41	3	1	0	0	0	5	40
11	47	6	2	0	0	2	6	47
12	53	5	2	6	0	6	10	50
13	56	5	13	4	0	8	15	55
14	48	27	15	1	0	13	46	32
15	0	91	14	0	0	14	91	0

Table 4.9: Frequency sets for best SOAs of strength 2+ constructed using $\mathrm{SOS}_2(n{=}32)$

m	Criterion 1				Criterion 2			
	f_1	f_2	f_3	f_4	f_4	f_3	f_2	f_1
10	43	1	1	0	0	0	5	40
11	51	2	2	0	0	1	5	49
12	60	3	2	1	0	4	3	59
13	68	3	4	3	0	8	3	67
14	78	5	5	3	0	11	5	75
15	89	4	9	3	1	12	10	82

Table 4.10: Frequency sets for best SOAs of strength 2+ constructed using $\mathrm{SOS}_3(n{=}32)$

Overall, $\mathrm{SOS}_1(n{=}32)$ emerges as the preferred choice for providing SOAs with better two-dimensional projection properties based on both criteria. This outcome is not surprising given that $\mathrm{SOS}_1(n{=}32)$ can be considered as the "double" of $\mathrm{SOS}_3(n{=}16)$ [3]. $\mathrm{SOS}_3(n{=}16)$ comprises 5 factors and 16 runs, whereas $\mathrm{SOS}_1(n{=}32)$ expands upon this by incorporating 10 factors and 32 runs. More specifically, $\mathrm{SOS}_1(n{=}32)$ includes all the factors present in $\mathrm{SOS}_3(n{=}16)$ $(a, b, c, d, abcd)$ and augments them with an additional independent factor, denoted as e, resulting in the factors $a, b, c, d, abcd, ae, be, ce, de$ and $abcde$. $\mathrm{SOS}_3(n{=}16)$ has shown to produce better designs for 16 runs, and as expected, $\mathrm{SOS}_1(n{=}32)$ also produces good designs for 32 runs.

Tables 4.11 and 4.12 show the frequency sets corresponding to the best SOAs of strength 2+ found from all three SOS designs based on Criteria 1 and 2 respectively. The designs A and B used to construct these best SOAs are provided alongside their respective frequencies.

m	f_1	f_2	f_3	f_4	A	B
10	43	2	0	0	{*ad, bd, cd, abd, acd, bcd, bce, cde, bde, bcde*}	{*ade, abce, abcde, d, de, abcd, ce, abde, d, de*}
11	51	2	2	0	{*ab, ac, ad, bc, bd, ace, ade, bcd, bde, abce, bcde*}	{*e, acde, abcde, acd, bce, a, ae ,b, acde, abe, ae*}
12	60	3	2	1	{*a, c, d, ac, ad, bc, bd, cd, abc, acd, bcd, abcd*}	{*ab, abde, ab, ce, bde, e, acde, de, be, be, e, bce*}
13	70	2	3	3	{*e, bd, cd, abc, abd, abe, acd, bcd, cde, bde, abce, acde, abde*}	{*bc, ce, d, be, abcd, a, b, ae, ad, ade, d, ac, ade*}
14	79	4	6	2	{*e, ac, ad, bd, abc, abe, acd, ace, bce, cde, bde, abce, bcde, abde*}	{*abcde, ae, a, c, c, abd, cd, d, abcd, bcd, b, ae, cd, abcde*}
15	90	6	4	5	{*e, ab, bc, cd, abc, abe, acd, ace, ade, bcd, cde, bde, acde, bcde, abde*}	{*c, b, ae, ac, abcde, abd, abcd, b, abd, c, d, abcde, abcd, bce, ae*}
16	99	10	6	5	{*e, ab, ac, ad, bc, cd, abc, abd, abe, acd, ade, bcd, bce, bde, acde, abde*}	{*a, abce, be, d, b, be, d, ce, cde, be, a, be, cde, b, d, bd*}
17	110	7	14	5	{*e, ab, ac, bc, bd, cd, abc, abd, abe, acd, ade, bcd, bce, cde, bde, abce, abde*}	{*c, acde, a, ce, abcde, acde, abcde, ad, b, be, a, a, d, ce, ace, bcde, ad*}
18	119	9	17	8	{*e, ab, ad, bc, bd, abc, abd, abe, acd, ace, ade, bcd, bce, cde, bde, abce, acde, abde*}	{*ae, ae, ac, de, ac, abcde, c, b, a, a, ae, b, bcde, c, bcde, be, abcde, ce*}
19	128	15	16	12	{*e, ab, ac, bc, bd, cd, abc, abd, abe, acd, ade, bcd, bce, cde, bde, abce, acde, bcde, abde*}	{*d, b, ce, ad, abcde, c, ace, ad, a, c, d, abcde, c, d, b, de, d, abcde, ce*}
20	136	13	27	14	{*e, ab, ac, ad, bc, bd, cd, abd, abe, acd, ace, ade, bcd, bce, cde, bde, abce, acde, bcde, abde*}	{*ce, b, b, ae, abc, de, d, abcde, abc, be, ae, d, a, a, abc, c, d, de, b, a, ce*}
21	142	16	36	16	{*e, ab, ac, ad, bc, bd, cd, abc, abd, abe, acd, ace, ace, ade, bcd, bce, cde, bde, abce, acde, bcde, abde*}	{*abcde, a, ce, d, c, de, ce, de, abcd, b, abcd, ce, de, ae, c, ce, be, d, abcd, ae, abcde*}

Table 4.11: Frequency sets, designs A and B corresponding to best SOAs of strength 2+ for 32 runs based on Criterion 1

m	f_4	f_3	f_2	f_1	A	B
10	0	0	2	43	$\{ad, bd, cd, abd, acd, bcd, bce, cde, bde, bcde\}$	$\{ade, abce, abcde, d, de, abcd, ce, abde, d, de\}$
11	0	1	5	49	$\{a, b, d, ab, ac, ad, bc, abc, acd, bcd, abcd\}$	$\{ae, abde, cde, ae, acde, bcde, cde, cd, abde, c, ae\}$
12	0	3	4	59	$\{ac, bc, cd, abc, abe, acd, ade, bce, cde, bde, bcde, abde\}$	$\{abd, ae, c, be, ab, b, c, ace, ab, b, ce, be\}$
13	0	5	7	66	$\{e, ad, cd, abd, acd, ace, ade, bce, cde, bde, abce, acde, abde\}$	$\{ce, abc, bc, de, ac, ac, ce, abcde, ce, c, abcde, ce, be, bd\}$
14	0	11	5	75	$\{a, b, c, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}$	$\{bce, bce, ade, bde, abcde, ade, bde, abcde, bcde, cde, abe, bcde, de, ae\}$
15	0	14	91	0	$\{ae, be, ce, de, abe, ace, ade, bce, bde, cde, abce, abde, acde, bcde, abcde\}$	$\{e, e, e, e, e, e, e, e, e, e, e, abd, e, e, e\}$
16	2	12	14	92	$\{e, ab, bc, bd, cd, abc, abd, abe, ace, bcd, cde, bde, abce, acde, bcde, abde\}$	$\{a, a, abcd, ac, a, bce, b, a, a, ae, ac, be, be, ac, d, abcde\}$
17	3	20	9	104	$\{e, ab, ac, bc, bd, cd, abd, abe, acd, ace, ade, bcd, cde, abce, acde, bcde, abde\}$	$\{de, abc, b, abc, de, ce, bde, ae, ad, abc, abcd, abc, d, d, be, c, c\}$
18	5	21	19	108	$\{e, ab, ac, ad, bc, bd, abc, abd, abe, acd, ade, bcd, cde, bde, abce, acde, bcde, abde\}$	$\{a, cd, ae, a, b, b, de, abcde, cd, cd, cd, c, b, a, d, cd, abcde\}$
19	7	25	18	121	$\{e, ab, ad, bc, bd, cd, abc, abd, acd, ace, ade, bcd, bce, cde, bde, abce, acde, bcde, abde\}$	$\{b, b, de, be, de, c, d, de, abcd, c, de, ae, be, abcd, b, abcde, de, a, abcd\}$
20	9	37	13	131	$\{e, ab, ac, ad, bc, bd, cd, abd, abe, acd, ace, ade, bcd, bce, cde, bde, abce, acde, bcde, abde\}$	$\{ae, abc, ce, a, ce, de, ce, c, ce, be, ae, de, a, abc, d, b, abcde, abcde, a, abcd\}$
21	12	45	19	134	$\{e, ab, ac, ad, bc, bd, cd, abc, abd, abe, acd, ace, ade, bcd, bce, cde, bde, abce, acde, bcde, abde\}$	$\{d, be, c, de, b, de, c, d, c, ae, abcde, ae, ae, abcd, ce, d, b, d, b, abcd, abcd\}$

Table 4.12: Frequency sets, designs A and B corresponding to best SOAs of strength 2+ for 32 runs based on Criterion 2

Chapter 5

Concluding remarks

Strong orthogonal arrays of strength 2+ are a class of designs that guarantee 4×2 and 2×4 two-dimensional projection properties. However, the patterns among these projections differ, and some are notably more appealing than others. The objective of our study is to identify designs with good projection properties, utilizing two selection criteria. We employed second order saturated designs to construct SOAs. We have identified and listed the best SOAs of strength 2+ that we obtained from each SOS design for run sizes of 16 and 32. Using our results, experimenters can choose SOAs of strength 2+ with better two-dimensional projection properties rather than selecting one at random.

In this study, we used an algorithm to identify good designs. Due to time constraints and the large number of designs one needs to consider for each SOS design, we were unable to perform a complete search for both run sizes. However, it is possible to conduct a more comprehensive search in future work. Additionally, a theoretical investigation about different patterns and two-factor projections of good designs could be explored. Our study focused on finding regular SOAs of strength 2+. Exploring the use of non-regular designs could also yield valuable insights.

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